UNIVERSIDADE FEDERAL DA PARAÍBA
CENTRO DE INFORMÁTICA
PROGRAMA DE PÓS-GRADUAÇÃO EM
MODELAGEM MATEMÁTICA E COMPUTACIONAL

Correção De Viés Do Modelo De Gumbel Com Censura Tipo I E Tipo II

por

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Dissertação submetida como requisito parcial
para a obtenção do grau de
Mestre em Modelagem Matemática e Computacional

Prof. Dr. Alexandre Simas
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João Pessoa, Março de 2016.

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João Pessoa – PB
Março de 2016

R672c Rocha, Suelana de Souza.

Correção de viés do modelo de Gumbel com censura tipo I e tipo II / Suelana de Souza Rocha.- João Pessoa, 2016.

80f.

Orientador: Alexandre Simas

Dissertação (Mestrado) - UFPB/CI

1. Matemática computacional - modelagem. 2. Modelo de Gumbel. 3. Censura tipo I. 4. Censura tipo II.

UFPB/BC CDU: 519.6(043)

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Orientador: Prof. Dr. Alexandre de Bustamante Simas

Aprovado em: 01 / 03 / 2016

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AGRADECIMENTOS

Agradeço primeiramente a Deus que está sempre comigo e a Nossa Senhora.

À minha família, em especial, aos meus pais, que sempre me incentivaram a estudar e a lutar pelos meus sonhos e a minha irmã Suelen de Souza Rocha.

Ao professor, Alexandre Bustamante Simas, meu orientador, pelos conhecimentos transmitidos, compreensão, atenção, incentivo e pela confiança depositada em mim e à professora Andrea Vanessa Rocha.

Aos professores da Pós-Graduação em Modelagem Matemática Computacional.

À banca examinadora: Prof. Dr. Alberto Masayoshi Faria Ohashi. e Prof. Dr. Wagner Barreto de Souza, por aceitarem participar da avaliação deste trabalho.

Aos meus amigos que estiveram ao meu lado durante o mestrado, compartilhando felicidades e dificuldades, em especial à Josenildo Silva de Lima, Maria Aparecida Silva de Andrade, Ricardo Pinheiro da Costa, Thiago Luis de Oliveira do Rêgo e Victor José Araujo de Carvalho.

Sumário

RESUMO

Nesta dissertação, usamos o modelo de Gumbel; Inicialmente obtivermos a matriz de informação de Fisher e a escrevermos na sua forma matricial, depois calculamos os cumulantes de segunda e terceira ordem, posteriormente derivamos os cumulantes de segunda ordem com respeito aos parâmetros. Em seguida, subtraímos das derivadas dos cumulantes de segunda ordem os cumulantes de terceira ordem e substituímos estes valores na expressão de Cox e Snell para obtenção da correção do Viés. Utilizamos a fórmula encontrada em Cox and Snell (1968), pois a partir de tal resultado, podemos definir um estimador corrigido $\tilde{\theta}_a = \hat{\theta}_a - \hat{B}(\hat{\theta}_a)$, onde $\hat{B}(\hat{\theta}_a)$ é o viés estimado de $\hat{\theta}_a$, onde $\tilde{\theta}_a$ tem viés de ordem $O(n^{-2})$. Logo à medida que o tamanho amostral n aumenta, esperamos que o viés de $\tilde{\theta}_a$ aproxime-se mais rapidamente de zero que o viés de $\hat{\theta}_a$. Faremos a correção do viés para o modelo de Gumbel com censura tipo I e tipo II.

Palavras Chave: Gumbel, censura tipo I, censura tipo II.

ABSTRACT

In this thesis, we use the model of Gumbel; Initially we had the Fisher information matrix and we write in a matrix form, then calculated the second- and third-order cumulants later derive the second order cumulants with respect to the parameters. Then we subtract the derivatives of the second order cumulants the third order cumulants and replace these values in the expression of Cox and Snell to obtain the correction of bias.We use the formula found in Cox and Snell (1968), because from what outcome we can define a corrected estimator $\tilde{\theta}_a = \hat{\theta}_a - \hat{B}(\hat{\theta}_a)$, where $\hat{B}(\hat{\theta}_a)$ it is the estimated bias $\hat{\theta}_a$, where $\tilde{\theta}_a$ has order bias $O(n^{-2})$. As soon as the sample size n increases, we expect the bias $\tilde{\theta}_a$ approaches zero faster than the bias $\hat{\theta}_a$. We will correct the bias for the model with censorship Gumbel type I and type II.

Palavras Chave: Gumbel, censorship type I, censorship type II.

1 INTRODUÇÃO

A distribuição de valor extremo, assim denominado por Emil Julius Gumbel (1891-1966), é conhecida como distribuição de Gumbel e distribuição log-Weibull. Os primeiros problemas envolvendo o valor extremo surgiu das inundações; sua importância econômica foi notável desde o surgimento da economia agrária, essencialmente nos que se baseava no fluxo de água e hidrovias, pois este era o principal sistema de comunicação da época. Sua importância se destacou ainda mais na economia industrial com a construção de usinas hidroelétricas e reservatórios para irrigação e luta contra erosão. O que incentivou o estudo deste tema, foi a importância social do controle do fluxo de água. A natureza estatística desde problema fez com que os procedimentos empíricos usados inicialmente fossem substituídos por métodos derivados da teoria dos valores extremos.

Esta teoria tem atraído cientista das mais diversas áreas, como por exemplos as engenharias (naval, eólica, civil), meteorologia, geologia, estatística populacional e outras. Em Kotz and Nadarajah (2000) podemos encontrar algumas aplicações, como por exemplo, terremotos, corridas de cavalo, tempestade, filas de supermercados, correntes marítimas, velocidade do tempo. Os recentes desastres naturais, por exemplo o furação Katrina, o terremoto do Japão, os deslizamento de terra e as inundações devidas as fortes chuvas ocorridas na Austrália, paquistão e Brasil, além das crises financeiras e vazamentos de óleos demonstram a necessidade de um estudo nas previsões destes fenômenos mais complexos.

Nesta dissertação, apresentamos um modelo geral de regressão de valor extremo, e fórmulas para o vieses de segunda ordem das estimativas de máxima verossimilhança (EMVs) dos parâmetros do nosso modelo. Quando o tamanho da amostra é grande, os vieses da EMVs são desprezíveis, uma vez que em geral, sua ordem é $O(n^{-1})$, enquanto os erros padrão assintóticos são de $O(n^{-\frac{1}{2}})$. Como sempre, porém, a correção do viés é importante enquanto o tamanho da amostra é pequeno. Na literatura, muitos autores tem obtido expressão para os vieses de segunda ordem das EMVs em uma variedade de modelo de regressão. Box (1971) obtém uma expressão geral para o viés n^{-1} em modelos não-lineares multivariados com matrizes de covariância conhecidos. Young and Bakir (1987) mostra a utilidade da correção do viés para o modelo de regressão log-gama generalizado, que tem como caso particular o modelo linear de regressão do valor extremo. Cordeiro and McCullagh (1991) obtém uma fórmula geral matricial para a correção do viés nos Modelos lineares Generalizados (MLGs). Cordeiro and Vasconcellos (1997) fornecem fórmulas matriciais para o viés em modelos não-lineares multivariados com erros seguindo distribuição normal. Este resultado é estendido por Vasconcellos and Gauss (1997) para cobrir modelos heteroscedásticos, enquanto Cordeiro et al. (1998) calcularam o viés de segunda para o modelo não-linear de regressão t-Student univariado. Vasconcellos and Cordeiro (2000) obtiveram uma expressão para o viés de segunda ordem para o modelo de regressão não-linear multivariado t-Student Cordeiro and Botter (2001) obtiveram fórmulas gerais para o viés de segunda ordem das EMVs em MLGs e Modelos Não Lineares Generalizados (MNLGs) com covariáveis de dispersão, respectivamente. Ospina et al. (2006) calcularam o viés de segunda ordem das EMVs para o modelo de regressão linear beta. Recentemente, Simas et al. (2010) obtiveram os vieses de segunda ordem para o modelo geral de regressão beta.

Ao realizar inferência estatística a partir de dados obtidos em testes de confiabilidade, muitas vezes nos deparamos com amostras onde nem todos os tempos de falha desejados são observados. Esses casos são denominados censuras, isto é, são observações parciais em um estudo interrompido por alguma razão, não permitindo que as observações completas do tempo de falha sejam obtidas. Censuras são recorrentes em processos de análise de sobrevivência, onde o tempo e o custo de tais experimentos são limitados, ou por diversos outros motivos alheios ao estudo e às condições impostas sobre o objeto de estudo.

Faremos nosso estudo baseado no artigo de Barreto-Souza and Vasconcellos (2011), onde encontramos a correção do viés da distribuição de Gumbel sem censura.

O Capítulo 1, iniciamos usando o modelo de Gumbel, calculamos a função geradora de momentos, a esperança, variância e temos a construção da verossimilhança com censura tipo I e tipo II.

O capítulo 2, está organizado em seções, primeiramente, usando o modelo de Gumbel com censura tipo I, obtemos a função log-verossimilhança, a função Escore e a matriz informação de Fisher; Na Seção 1, fizermos a correção do viés com censura tipo I de um EMVs, utilizando a fórmula de Cox and Snell (1968), na Seção 2, estudamos a correção do viés com censura tipo I de um EMVs de μ e ϕ.

O capítulo 3, está organizado da mesma forma que o capítulo 2, só trocamos a censura tipo I pela censura tipo II.

2 CONHECIMENTOS PRELIMINARES

Seja Y uma variável aleatória com distribuição de Gumbel, que possui função de densidade de probabilidade dada por

$$g(y;\mu,\phi) = \frac{1}{\phi}\exp\left(\frac{y-\mu}{\phi}\right)\exp\left(-\exp\left(\frac{y-\mu}{\phi}\right)\right);\ y \in \mathbb{R}. \quad (2.1)$$

onde $Y \sim EV(\mu,\phi)$, com $\mu \in \mathbb{R}$ e $\phi > 0$ são parâmetros de localização e escala, respectivamente.

A função geradora de momentos de Y é dada por

$$E(\exp(tY)) = \exp(t\mu)\,\Gamma(1+\phi t),\ \text{com } t > -\phi^{-1}. \quad (2.2)$$

Assim, temos a esperança e a variância de Y dada por

$$E(Y) = \mu - \gamma\phi, \quad (2.3)$$
$$V(Y) = \frac{\pi^2}{6}\phi^2, \quad (2.4)$$

respectivamente, onde γ é a constante de Euler, $\gamma = \lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{1}{k} - \log n\right) \approx 0,5772$, podemos verificar os fatos (2.2), (2.3) e (2.4) no Apêndice(A).

Definição 2.1 (Construção Da Verossimilhança Com Censura Tipo I). *Suponhamos que temos uma amostra $X_1,\ldots,X_n$ aleatória simples da variável aleatória X; Denotaremos a função de densidade de probabilidade por $f(x)$ e a função de distribuição acumulada por $F(x)$. Representamos a censura tipo I, pelo par (y_i,δ_i) com $y_i = min(x_i,T_i)$,*

$$\delta_i = \begin{cases} 0; & x_i > T_i \\ 1; & x_i \leq T_i \end{cases},\ \textit{para } i = 1,\ldots,n. \quad (2.5)$$

onde δ_i é a variável indicadora da censura e T_i é o tempo de censura relacionado a observação. Denotaremos o vetor de parâmetros desconhecidos por $\vartheta = (\vartheta_1,\ldots,\vartheta_p)$.

Assim, a expressão da verossimilhança é:

$$L(X|\vartheta) = \prod_{i=1}^{n} f(x_i).$$

Denotaremos, a parte observada sem censura de $X = (x_1, ..., x_n)$ *por* $W = (w_1, ..., w_m)$ *e com censura por* $Z = (z_{m+1}, ..., z_n)$ *com* $z_i > T_i$*. Integrando* $L(X|\vartheta)$ *com respeito a* Z*, obtemos*

$$\begin{aligned} L(W;\vartheta) = \int L(W,Z;\vartheta)\,dZ &= \int L(W;\vartheta)L(Z;\vartheta)\,dZ \\ &= L(W;\vartheta)\int L(Z;\vartheta)\,dZ \\ &= \prod_{i=1}^{m} f(w_i) \int_{z_j>T_j} \prod_{j=m+1}^{n} f(z_j)dz_j \\ &= \prod_{i=1}^{m} f(w_i) \prod_{j=m+1}^{n} \int_{z_j>T_j} f(z_j)dz_j, \\ &= \prod_{i=1}^{m} f(w_i) \prod_{j=m+1}^{n} \left[1 - \int_{z_j<T_j} f(z_j)dz_j\right] \\ &= \prod_{i=1}^{m} f(w_i) \prod_{j=m+1}^{n} \left[1 - F(T_j)\right], \end{aligned}$$

usando a notação (y_i, δ_i) *com* $y_i = \min(x_i, T_i)$ *e a expressão* (2.5)*, temos a seguinte forma para a verossimilhança com dados censurados (censura tipo I)*

$$L(y,\delta;\vartheta) = \prod_{i=1}^{n} \left[f(y_i)\right]^{\delta_i} \left[1 - F(y_i)\right]^{1-\delta_i}. \tag{2.6}$$

Para maiores informações veja a referência Park and Lee (2012).

Definição 2.2 (Censura tipo II). *Sejam* $T_1, ..., T_n$ *variáveis aleatórias independente e identicamente distribuídas que caracterizam tempos de falhas, com função densidade de probabilidade e função acumulada dadas por* $f(\cdot;\vartheta)$ *e* $F(\cdot;\vartheta)$*, respectivamente, onde* ϑ *é um parâmetro. Seja* $m < n$ *o número pré-fixado de falhas observadas. Uma amostra sob esquema de censura do tipo II é uma amostra* $X_{(1)}, ..., X_{(n)}$

tal que $X_{(1)},...,X_{(n)}$ *são estatísticas de ordem definidas por*

$$X_{(i)} = \begin{cases} T_{(i)}, & se \quad T_{(i)} \leq T_{(m)} \\ T_{(m)}, & se \quad T_{(i)} > T_{(m)}, \end{cases}$$

onde $T_{(m)}$ *é o tempo de vida aleatório da m-ésima falha.*

Obtemos agora a função de verossimilhança para o parâmetro ϑ*. Considerando* $x_{(1)},...,x_{(n)}$ *os valores observados de* $X_{(1)},...,X_{(n)}$*, a função de verossimilhança para este modelo com m falhas observadas é dada por*

$$L(\vartheta) = \frac{n!}{(n-m)!} \prod_{i=1}^{m} \left[f(x_{(i)};\vartheta)\right] \prod_{i=m+1}^{n} \left[1 - F(x_{(m)})\right], \tag{2.7}$$

onde $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(m)}$ *e* $x_{(m+1)} = \cdots = x_{(n)} = x_{(m)}$.

Para maiores informações veja a referência Park and Lee (2012).

3 CORREÇÃO DO VIÉS DO MODELO DE GUMBEL COM CENSURA TIPO I

Seja $Y_1, ..., Y_n$ uma amostra aleatória, onde cada Y_i seja independente com função de densidade de probabilidade dada por (2.1), com parâmetro de localização μ_i e parâmetro de escala ϕ_i, para $i = 1, 2, ..., n$. Suponha que as componentes de ambos os vetores paramétricos $\mu = (\mu_1, ..., \mu_n)^T$ e $\phi = (\phi_1, ..., \phi_n)^T$ variam de acordo com as observações através do modelo de regressão não-linear.

O modelo de Gumbel com covariadas para a localização e escala é definido por (2.1) e por dois componentes sistemáticos dados por

$$g_1(\mu) = \eta_1 = f_1(X; \beta), \quad g_2(\phi) = \eta_2 = f_2(Z; \theta), \tag{3.1}$$

onde $\beta = (\beta_1, ..., \beta p)^T$ e $\theta = (\theta_1, ..., \theta_q)^T$ são vetores de parâmetros de regressão desconhecidos a serem estimados ($\beta \in \mathbb{R}^p$ e $\theta \in \mathbb{R}^q$). Aqui, $f_1(X; \beta)$ e $f_2(Z; \theta)$ são funções de classe C^3 (possivelmente não lineares). Finalmente, $g_1(\cdot)$ e $g_2(\cdot)$ são funções de ligação conhecidas monótonas e três vezes diferenciáveis com domínios $\mathbb{R}$ e $\mathbb{R}^+$, respectivamente. Sejam X e Z matrizes $n \times p$ e $n \times q$ com posto$(X) = p$ e posto$(Z) = q$, respectivamente; X e Z não são necessariamente diferentes.

Sabemos que a função acumulada de (2.1), é

$$G(y) \quad = 1 - \exp\left(-\exp\left(\frac{y-\mu}{\phi}\right)\right), \tag{3.2}$$

isto é verificado no Apêndice(A).

Considere $\vartheta = (\beta, \theta)$ na expressão (2.6), é usando as expressões (2.1) e (3.2) na expressão (2.6), obtemos a seguinte expressão para a verossimilhança

$$L(y, \delta; \beta, \phi) = \prod_{i=1}^{n}\left[\left[\frac{1}{\phi_i}\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\exp\left(-exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right)\right]^{\delta_i}\cdot\left[\exp\left(-exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right)\right]^{1-\delta_i}\right]$$

$$
\begin{aligned}
=&\prod_{i=1}^{n}\left[\left[\frac{1}{\phi_i}\right]^{\delta_i}\left[\exp\left(\frac{y_i-\mu_i}{\phi_i}-exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right)\right]^{\delta_i}\cdot\left[\exp\left(-(1-\delta_i)\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right)\right]\right]\\
=&\prod_{i=1}^{n}\left[\frac{1}{\phi_i^{\delta_i}}\left[\exp\left(\delta_i\left(\left(\frac{y_i-\mu_i}{\phi_i}\right)-exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right)\right)\right]\cdot\left[\exp\left(-(1-\delta_i)\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right)\right]\right].
\end{aligned}
$$

Agora, calcularemos a função log-verossimilhança, basta aplicarmos a função logaritmo na expressão acima, assim

$$
\begin{aligned}
l =&\log(L(y,\delta;\beta,\phi))\\
=&\log\left(\prod_{i=1}^{n}\left[\frac{1}{\phi_i^{\delta_i}}\left[\exp\left(\delta_i\left(\left[\frac{y_i-\mu_i}{\phi_i}\right]-exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right)\right)\right]\cdot\left[\exp\left(-(1-\delta_i)\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right)\right]\right]\right)\\
=&\sum_{i=1}^{n}\log\left(\frac{1}{\phi_i^{\delta_i}}\right)+\sum_{i=1}^{n}\delta_i\left(\frac{y_i-\mu_i}{\phi_i}-exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right)-\sum_{i=1}^{n}(1-\delta_i)exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\\
=&-\sum_{i=1}^{n}\log(\phi_i^{\delta_i})+\sum_{i=1}^{n}\delta_i\left(\frac{y_i-\mu_i}{\phi_i}-exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right)-\sum_{i=1}^{n}(1-\delta_i)exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\\
=&-\sum_{i=1}^{n}\delta_i\cdot\log(\phi_i)+\sum_{i=1}^{n}\delta_i\left(\frac{y_i-\mu_i}{\phi_i}-exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right)-\sum_{i=1}^{n}(1-\delta_i)exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\\
=&\delta_i\left[-\sum_{i=1}^{n}\log(\phi_i)+\sum_{i=1}^{n}\left(\frac{y_i-\mu_i}{\phi_i}\right)-\sum_{i=1}^{n}exp\left(\frac{y_i-\mu_i}{\phi_i}\right)+\sum_{i=1}^{n}exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]-\sum_{i=1}^{n}exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\\
=&-\sum_{i=1}^{n}\delta_i\log(\phi_i)+\sum_{i=1}^{n}\delta_i\left(\frac{y_i-\mu_i}{\phi_i}\right)-\sum_{i=1}^{n}exp\left(\frac{y_i-\mu_i}{\phi_i}\right),
\end{aligned}
$$

com μ_i e ϕ_i definida por (3.1).

A função escore é definida por $U = U(\beta,\theta) = (\partial l/\partial\beta^T, \partial l/\partial\theta^T)^T$. Seja $y_i^\circ = \exp(y_i/\phi_i)$, $\mu_i^\circ = \exp(\mu_i/\phi_i)$ e $v_i = \delta_i(-1-(y_i-\mu_i)/\phi_i)+\exp((y_i-\mu_i)/\phi_i)(y_i-\mu_i)/\phi_i$, para $i = 1, ..., n$. Temos que

$$
\begin{aligned}
U_j(\beta,\theta) &= \frac{\partial l}{\partial\beta_j} = \frac{\partial l}{\partial\mu_i}\frac{d\mu_i}{d\eta_{1i}}\frac{\partial\eta_{1i}}{\partial\beta_j}, \quad j = 1, ..., p.\\
U_J(\beta,\theta) &= \frac{\partial l}{\partial\theta_J} = \frac{\partial l}{\partial\phi_i}\frac{d\phi_i}{d\eta_{2i}}\frac{\partial\eta_{2i}}{\partial\theta_J}, \quad J = 1, ..., q.
\end{aligned}
$$

Logo,

$$
\begin{aligned}
U_j(\beta,\theta) &= \frac{\partial l}{\partial\beta_j} = \sum_{i=1}^{n}\frac{1}{\mu_i^\circ\phi_i}(y_i^\circ-\delta_i\mu_i^\circ)\frac{d\mu_i}{d\eta_{1i}}\frac{\partial\eta_{1i}}{\partial\beta_j}, \quad j = 1, ..., p,\\
U_J(\beta,\theta) &= \frac{\partial l}{\partial\theta_J} = \sum_{i=1}^{n}v_i\frac{1}{\phi_i}\frac{d\phi_i}{d\eta_{2i}}\frac{\partial\eta_{2i}}{\partial\theta_J}, \quad J = 1, ..., q,
\end{aligned}
$$

para mais detalhes ver o Apêndice(B).

Em notação matricial, temos que

$$U_j(\beta,\theta) = \frac{\partial l}{\partial \beta_j} = \widetilde{X}^T\Omega^{-1}M_1u_1,$$

e

$$U_J(\beta,\theta) = \frac{\partial l}{\partial \beta_J} = \widetilde{S}^T\Omega^{-1}M_2v_1,$$

onde os vetores de ordem $n\times 1$, $u_1 = \left(-\delta_1 + \exp\left(\frac{y_1-\mu_1}{\phi_1}\right), ..., -\delta_n + \exp\left(\frac{y_n-\mu_n}{\phi_n}\right)\right)^T$, $v_1 = (v_1, v_2, ..., v_n)$ e as matrizes $\widetilde{X} = \left(\frac{\partial\eta_{1i}}{\partial\beta_j}\right)_{i,j}$ de ordem $n\times p$, $\widetilde{S} = \left(\frac{\partial\eta_{2i}}{\partial\beta_J}\right)_{i,J}$ de ordem $n\times q$, e as matrizes de ordem $n\times n$, $\Omega = \text{diag}(\phi_i)$, $M_1 = \text{diag}\left(\frac{d\mu_i}{d\eta_{1i}}\right)$ e $M_2 = \text{diag}\left(\frac{d\phi_i}{d\eta_{2i}}\right)$.

Os estimadores de máxima verossimilhança para os parâmetros β e θ são obtidos resolvendo o sistema não linear $U = 0$ e não há uma forma fechada para tal solução; Portanto, utilizamos um algoritmo de optimização não linear, como o algoritmo de Newton ou quase-Newton, para encontrar estimadores de máxima verossimilhança.

A matriz informação de Fisher é dada por

$$K' = K'(\beta,\theta) = \begin{pmatrix} K^1_{\beta\beta} & K^1_{\beta\theta} \\ K^1_{\theta\beta} & K^1_{\theta\theta} \end{pmatrix} = \begin{pmatrix} \widetilde{X}^TW^1_{\beta\beta}\widetilde{X} & \widetilde{X}^TW^1_{\beta\theta}\widetilde{S} \\ \widetilde{S}^TW^1_{\theta\beta}\widetilde{X} & \widetilde{S}^TW^1_{\theta\theta}\widetilde{S} \end{pmatrix}.$$

onde, usamos as seguintes matrizes diagonais para obtermos a matriz de Fisher,

$$W^1_{\beta\beta} = \text{diag}\left(\frac{h_{2i}}{\phi_i^2}\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2\right),$$

$$W^1_{\theta\theta} = \text{diag}\left(\frac{(-h_{1i} - 2h_{3i} + h_{5i} + 2h_{4i})}{\phi_i^2}\left(\frac{d\phi_i}{d\eta_{2i}}\right)^2\right)$$

e

$$W^1_{\beta\theta} = \text{diag}\left(\frac{(h_{4i})}{\phi_i^2}\frac{d\mu_i}{d\eta_{1i}}\frac{d\phi_i}{d\eta_{2i}}\right).$$

A inversa da matriz de Fisher é dada por

$$(K')^{-1} = (K')^{-1}(\beta, \theta) = \begin{pmatrix} K^1_{\beta\beta} & K^1_{\beta\theta} \\ K^1_{\theta\beta} & K^1_{\theta\theta} \end{pmatrix}^{-1}.$$

Definimos as matrizes $\mathbb{X}$ e $\widetilde{W}_1$ com dimensões $2n \times (p+q)$ e $2n \times 2n$ respectivamente, com

$$\mathbb{X} = \begin{pmatrix} \widetilde{X} & 0 \\ 0 & \widetilde{S} \end{pmatrix} \quad \text{e} \quad \widetilde{W}_1 = \begin{pmatrix} W^1_{\beta\beta} & W^1_{\beta\theta} \\ W^1_{\theta\beta} & W^1_{\theta\theta} \end{pmatrix}.$$

Então podemos escrever a matriz de Fisher como

$$K' = \mathbb{X}^T \widetilde{W}_1 \mathbb{X}.$$

3.1 Correção Do Viés De Um EMVs

Nesta seção, obtemos uma expressão para os vieses de segunda ordem dos EMVs dos parâmetros do modelo geral de regressão do valor extremo usando Cox e Snell (1968). As derivadas parciais da log-verossimilhança com respeito as componentes dos vetores desconhecidos β e θ são indicados pelos indices $\{j, l, ...\}$ e $\{J, L, ...\}$, respectivamente. Assim, definimos $U_j = \partial l/\partial\beta_j$, $U_J = \partial l/\partial\theta_J$, $U_{jL} = \partial^2 l/\partial\beta_j\partial\theta_L$, $U_{jlM} = \partial^3 l/\partial\beta_j\partial\beta_l\partial\theta_M$ e etc. Para denotar os cumulantes das derivadas parciais acima, usamos a notação introduzida por Lawley (1956): $k_{jl} = E(U_{jl})$, $k_{j,l} = E(U_j U_l)$, $k_{jl,M} = E(U_{jl} U_M)$ e etc, em geral k's são da ordem de $O(n)$. As derivadas destes cumulantes são denotadas da seguinte forma $k^{(m)}_{jl} = \partial k_{jl}/\partial\beta_m$, $k^{(M)}_{jl} = \partial k_{jl}/\partial\theta_M$ e etc. Nem todos os k's são necessariamente independentes. Seja $k^{j,l} = -k^{jl}$ e $k^{J,L} = -k^{JL}$ os elementos de seus respectivos inversos $K_{\beta\beta}$ e $K_{\theta\theta}$ que são $O(n^{-1})$.

A fórmula de Cox e Snell (1968) pode ser usada para obter o viés segunda ordem do EMV para a a-ésima componente do vetor paramétrico $\widehat{\tau} =$

$(\widehat{\tau}_1, ..., \widehat{\tau}_{p+q}) = (\widehat{\beta}^T, \widehat{\theta}^T)$, a qual é dada por

$$\begin{aligned}
B(\widehat{\tau}_a) &= \sum_{j,l,m} k^{aj} k^{lm} \left\{ k_{jl}^{(m)} - \tfrac{1}{2} k_{jlm} \right\} + \sum_{J,l,m} k^{aJ} k^{lm} \left\{ k_{Jl}^{(m)} - \tfrac{1}{2} k_{Jlm} \right\} \\
&+ \sum_{j,L,m} k^{aj} k^{Lm} \left\{ k_{jL}^{(m)} - \tfrac{1}{2} k_{jLm} \right\} + \sum_{j,l,M} k^{aj} k^{lM} \left\{ k_{jl}^{(M)} - \tfrac{1}{2} k_{jlM} \right\} \\
&+ \sum_{J,L,m} k^{aJ} k^{Lm} \left\{ k_{JL}^{(m)} - \tfrac{1}{2} k_{JLm} \right\} + \sum_{J,l,M} k^{aJ} k^{lM} \left\{ k_{Jl}^{(M)} - \tfrac{1}{2} k_{JlM} \right\} \\
&+ \sum_{j,L,M} k^{aj} k^{LM} \left\{ k_{jL}^{(M)} - \tfrac{1}{2} k_{jLM} \right\} + \sum_{J,L,M} k^{aJ} k^{LM} \left\{ k_{JL}^{(M)} - \tfrac{1}{2} k_{JLM} \right\} \qquad (3.3)
\end{aligned}$$

Os parâmetros β e θ não são ortogonais. Assim as entradas da matriz $W_{\beta\theta}$ não são todas nulas. Por isso, todos os termos em (4.2) devem ser considerados.

3.2 Correção do viés dos MLEs para o modelo de Gumbel com censura tipo I

Usando a expressão (4.2) e utilizando algumas manipulações algébricas, que pode ser encontrada no Apêndice (B), podemos obter uma expressão para o viés de segunda ordem de $\widehat{\beta}$ e $\widehat{\theta}$ na forma matricial:

$$\begin{aligned}
B_1(\widehat{\beta}) &= K_1^{\beta\beta} \widetilde{X}^T \left[W_1' Z_{\beta d}' + W_2' D_\beta' + (W_3' + W_5') Z_{\beta\theta d}' + W_4' D_\theta' + W_7' Z_{\theta d}' \right] 1_{n\times 1} \\
&+ K_1^{\beta\theta} \widetilde{S}^T \left[W_3' Z_{\beta d}' + W_4' D_\beta' + (W_6' + W_7') Z_{\beta\theta d}' + W_8' Z_{\theta d}' + W_9' D_\theta' \right] 1_{n\times 1}, \quad (3.4)
\end{aligned}$$

e

$$\begin{aligned}
B_1(\widehat{\theta}) &= K_1^{\theta\beta} \widetilde{X}^T \left[W_1' Z_{\beta d}' + W_2' D_\beta' + (W_3' + W_5') Z_{\beta\theta d}' + W_4' D_\theta' + W_7' Z_{\theta d}' \right] 1_{n\times 1} \\
&+ K_1^{\theta\theta} \widetilde{S}^T \left[W_3' Z_{\beta d}' + W_4' D_\beta' + (W_6' + W_7') Z_{\beta\theta d}' + W_8' Z_{\theta d}' + W_9' D_\theta' \right] 1_{n\times 1}, \quad (3.5)
\end{aligned}$$

onde $W_k' = diag\{w_{k1}', ..., w_{kn}'\}$ para $i = 1, ..., n$ e $k = 1, ..., 9$, $1_{n\times 1}$ denota um vetor com n entradas igual a 1, $Z_{\beta d}' = diag(\widetilde{X} K_1^{\beta\beta} \widetilde{X}^T)$, $Z_{\beta\theta d}' = diag(\widetilde{X} K_1^{\beta\beta} \widetilde{S}^T)$,

$Z'_{\theta d} = diag(\widetilde{S} K_1^{\theta\theta} \widetilde{S}^T)$, $D'_\beta = diag(d'_{1\beta}, ..., d'_{n\beta})$ e $D'_\theta = diag(d'_{1\theta}, ..., d'_{n\theta})$ com $d'_{i\beta} = tr(\widetilde{\widetilde{X}} K_1^{\beta\beta})$, $d'_{i\theta} = tr(\widetilde{\widetilde{S}} K_1^{\theta\theta})$, $\widetilde{\widetilde{X}}_i = (\partial^2 \eta_1 i / \partial\beta_j \beta_l)_{j,l}$ e $\widetilde{\widetilde{S}}_i = (\partial^2 \eta_2 i / \partial\theta_J \theta_L)_{j,l}$ para $i = 1, ..., n$.

Considere os vetores de ordem $(2n \times 1)$, δ'_1 e δ'_2 como

$$\delta'_1 = \begin{pmatrix} [W'_1 Z'_{\beta d} + (W'_3 + W'_5) Z'_{\beta\theta d} + W'_7 Z'_{\theta d}] 1_{n\times 1} \\ [W'_3 Z'_{\beta d} + (W'_6 + W'_7) Z'_{\beta\theta d} + W'_8 Z'_{\theta d}] 1_{n\times 1} \end{pmatrix} \quad (3.6)$$

e

$$\delta'_2 = \begin{pmatrix} [W'_2 D'_\beta + W'_4 D'_\theta] 1_{n\times 1} \\ [W'_4 D'_\beta + W'_9 D'_\theta] 1_{n\times 1} \end{pmatrix} \quad (3.7)$$

os blocos inferiores de ordem $p \times (p+q)$ e superior de ordem $q \times (p+q)$ da matriz $K_1(\tau)^{-1}$ por $K_1^{\beta *} = (k_1^{\beta\beta} k_1^{\beta\theta})$ e $K_1^{\theta *} = (k_1^{\theta\beta} k_1^{\theta\theta})$, respectivamente. Com estas expressões, podemos escrever o viés de segunda ordem de $\widehat{\beta}$ e $\widehat{\theta}$ como

$$B_1(\widehat{\beta}) = K_1^{\beta *} \mathbb{X}^T (\delta'_1 + \delta'_2) \quad \text{e} \quad B_1(\widehat{\theta}) = K_1^{\theta *} \mathbb{X}^T (\delta'_1 + \delta'_2), \quad (3.8)$$

respectivamente. Então, por (3.8) concluímos que o viés de segunda ordem do EMV do vetor conjunto $\widehat{\tau} = (\widehat{\beta}^T, \widehat{\theta}^T)$ possui a forma

$$B_1(\widehat{\tau}) = K_1^{-1} \widetilde{X}^T (\delta'_1 + \delta'_2) = (\mathbb{X}^T \widetilde{W}_1 \mathbb{X})^{-1} \mathbb{X}^T (\delta'_1 + \delta'_2).$$

Definindo $\xi'_1 = \widetilde{W}_1^{-1} \delta'_1$ e $\xi'_2 = \widetilde{W}_1^{-1} \delta'_2$, assim

$$B_1(\widehat{\tau}) = (\mathbb{X}^T \widetilde{W}_1 \mathbb{X})^{-1} \mathbb{X}^T \widetilde{W}_1 (\xi'_1 + \xi'_2). \quad (3.9)$$

A fórmula (4.8) mostra que o viés de segunda ordem de $\widehat{\tau}$ e facilmente obtida com os vetores dos coeficientes de regressão na forma de regressão linear de ξ'_1 e ξ'_2 nas colunas de $\mathbb{X}$ com $\widetilde{W}_1$ sendo a matriz peso. Podemos expressar (4.8) como

$$B_1(\widehat{\tau}) = B'_1(\widehat{\tau}) + B'_2(\widehat{\tau}),$$

com $B_1'(\widehat{\tau}) = (\mathbb{X}^T\widetilde{\widetilde{W}}_1\mathbb{X})^{-1}\mathbb{X}^T\widetilde{\widetilde{W}}_1\xi_1'$ e $B_2'(\widehat{\tau}) = (\mathbb{X}^T\widetilde{\widetilde{W}}_1\mathbb{X})^{-1}\mathbb{X}^T\widetilde{\widetilde{W}}_1\xi_2'$.

Se $\xi_2' = 0$, a fórmula (4.8) dá o viés de segunda ordem para Modelos Lineares de Valores Extremo de Regressão com covariavéis de dispersão linear. Portanto, $B_1'(\widehat{\tau})$ e $B_2'(\widehat{\tau})$ podem ser considerados respectivamente, como a linearidade e não-linearidade em termos do viés total.

3.3 Correção Do Viés De Um MLEs De μ e ϕ

Primeiramente, expandimos as funções $\widehat{\eta_{1i}} = f_1(x_i^T, \widehat{\beta})$ e $\widehat{\eta_{2i}} = f_2(x_i^T, \widehat{\theta})$ dado em (3.1) em série de Taylor até a segunda ordem em torno dos pontos β e θ, respectivamente, obtemos

$$\widehat{\eta_{1i}} - \eta_{1i} = \widetilde{X}_i^T(\widehat{\beta} - \beta) + \frac{1}{2}(\widehat{\beta} - \beta)^T \widetilde{\widetilde{X}}_i(\widehat{\beta} - \beta) + o_p(\|(\widehat{\beta} - \beta)\|^2)$$

e

$$\widehat{\eta_{2i}} - \eta_{2i} = \widetilde{S}_i^T(\widehat{\theta} - \theta) + \frac{1}{2}(\widehat{\theta} - \theta)^T \widetilde{\widetilde{S}}_i(\widehat{\theta} - \theta) + o_p(\|(\widehat{\theta} - \theta)\|^2)$$

onde $\widetilde{X}_i$ e $\widetilde{S}_i$ são a i-ésima linha das matrizes $\widetilde{X}$ e $\widetilde{S}$ respectivamente. Assim, os vieses de segunda ordem de $\widehat{\eta_{1i}}$ e $\widehat{\eta_{2i}}$ na notação matricial são dados por

$$B(\widehat{\eta_{1i}}) = \widetilde{X}B(\widetilde{\beta}) + \frac{1}{2}D_\beta 1_{n\times 1} \text{ e } B(\widehat{\eta_{2i}}) = \widetilde{S}B(\widetilde{\theta}) + \frac{1}{2}D_\theta 1_{n\times 1}$$

Vamos agora expandir as funções $\widehat{\mu}_{1i} = g_1^{-1}(\widetilde{\eta}_{1i})$ e $\widehat{\phi}_{1i} = g_2^{-1}(\widetilde{\eta}_{2i})$ em séries de taylor até a segunda ordem, em torno dos pontos η_{1i} e η_{2i} respectivamente. Com isto, segue que

$$\widehat{\mu}_i - \mu_i = \frac{d\mu_i}{d\eta_{1i}}(\widehat{\eta_{1i}} - \eta_{1i}) + \frac{1}{2}\frac{d^2\mu_i}{d\eta_{1i}^2}(\widehat{\eta_{1i}} - \eta_{1i})^2 + o_p((\widehat{\eta_{1i}} - \eta_{1i})^2)$$

e

$$\widehat{\phi}_i - \phi_i = \frac{d\phi_i}{d\eta_{2i}}(\widehat{\eta_{2i}} - \eta_{2i}) + \frac{1}{2}\frac{d^2\phi_i}{d\eta_{2i}^2}(\widehat{\eta_{2i}} - \eta_{2i})^2 + o_p((\widehat{\eta_{2i}} - \eta_{2i})^2)$$

Assim, obtemos os vieses de segunda ordem de $\widehat{\mu}_i$ e $\widehat{\phi}_i$

$$B(\widehat{\mu}_i) = B(\widehat{\eta_{1i}})\frac{d\mu_i}{d\eta_{1i}} + \frac{1}{2}Var(\widehat{\eta_{1i}})\frac{d^2\mu_i}{d\eta_{1i}^2} \quad \text{e} \quad B(\widehat{\phi}_i) = B(\widehat{\eta_{2i}})\frac{d\phi_i}{d\eta_{2i}} + \frac{1}{2}Var(\widehat{\eta_{2i}})\frac{d^2\phi_i}{d\eta_{2i}^2} \tag{3.10}$$

A fórmula anterior irá nos fornecer uma expressão para os vies de segunda ordem do EMVs de μ e ϕ, em notação matricial, fica como segue

$$B_1(\widehat{\mu}_i) = \frac{1}{2}\left\{M_1[2\widetilde{X}B_1(\widehat{\beta}) + D'_\beta 1_{n\times 1}] + Z'_{\beta d}T_1 1_{n\times 1}\right\}$$

e

$$B_1(\widehat{\phi}_i) = \frac{1}{2}\left\{M_2[2\widetilde{S}B_1(\widehat{\theta}) + D'_\theta 1_{n\times 1}] + Z'_{\theta d}T_2 1_{n\times 1}\right\}$$

Para o modelo de regressão dos valores extremos, usando (3.8) temos,

$$B_1(\widehat{\mu}_i) = \frac{1}{2}\left\{M_1[2\widetilde{X}K_1^{\beta *}\widetilde{X}^T(\delta'_1 + \delta'_2) + D'_\beta 1_{n\times 1}] + Z'_{\beta d}T_1 1_{n\times 1}\right\}$$

e

$$B_1(\widehat{\phi}_i) = \frac{1}{2}\left\{M_2[2\widetilde{S}K_1^{\theta *}\widetilde{X}^T(\delta'_1 + \delta'_2) + D'_\theta 1_{n\times 1}] + Z'_{\theta d}T_2 1_{n\times 1}\right\}$$

Definimos as matrizes diagonais $T_1 = diag\left\{d^2\mu_i/d\eta_{1i}^2\right\}$ e $T_2 = diag\left\{d^2\phi_i/d\eta_{2i}^2\right\}$ de ordem n.

Os estimadores corrigidos $\widetilde{\mu} = \widehat{\mu} - \widehat{B}_1(\widehat{\mu})$ e $\widetilde{\phi} = \widehat{\phi} - \widehat{B}_1(\widehat{\phi})$ de μ e ϕ respectivamente, tem viéses de ordem $O(n^{-2})$, onde $\widehat{B}_1(\cdot)$ denota o EMV de $B_1(\cdot)$, isto é, os parâmetros desconhecidos são substituídos por seus EMVs.

4 CORREÇÃO DO VIÉS DO MODELO DE GUMBEL COM CENSURA TIPO II

Seja $Y_1, Y_2, ..., Y_k$ uma amostra aleatória, onde cada Y_i tem a função de densidade de probabilidade dada por (2.1), com parâmetro de localização μ_i e parâmetro de escala ϕ_i, para $i = 1, 2, ..., k$. Suponha que os componentes de ambos os vetores paramétricos $\mu = (\mu_1, ..., \mu_n)^T$ e $\phi = (\phi_1, ..., \phi_n)^T$ variam de acordo com as observações através do modelo de regressão não-linear.

O modelo de Gumbel com covariadas para a localização e a dispersão é definido por (2.1), seja X e Z as matrizes de dados, onde

$$X_i = (X_{i1}, ..., X_{ip}) \text{ denota a i-ésima linha de } X$$

e

$$Z_i = (Z_{i1}, ..., Z_{ip}) \text{ denota a i-ésima linha de } Z.$$

Definimos

$$g_1(\mu_i) = \eta_{1i} = f_1(X_i; \beta) \quad \text{e} \quad g_2(\phi_i) = \eta_{2i} = f_2(Z_i; \theta), \tag{4.1}$$

onde $\beta = (\beta_1, ..., \beta p)^T$ e $\theta = (\theta_1, ..., \theta_q)^T$ são vetores de parâmetros de regressão desconhecidos a serem estimados ($\beta \in \mathbb{R}^p$ e $\theta \in \mathbb{R}^q$). Aqui, $f_1(X_i; \beta)$ e $f_2(Z_i; \theta)$ são funções de classe C^3 (possivelmente não lineares). Finalmente, $g_1(\cdot)$ e $g_2(\cdot)$ são funções de ligação conhecidas monótonas e três vezes diferenciáveis com domínios $\mathbb{R}$ e $\mathbb{R}^+$, respectivamente. Sejam X e Z matrizes $n \times p$ e $n \times q$ com posto$(X) = p$ e posto$(Z) = q$, respectivamente; X e Z não são necessariamente diferentes.

Obtemos $Y_{1,i}, Y_{2,i}, ..., Y_{n_i,i}$ observações independentes e identicamente distribuídas. A censura tipo II é feita em cima de cada coluna da "matriz" $Y = \{Y_{si}\}$ com $s = 1, 2, ..., n_i$ e $i = 1, 2, ..., k$. Observe que Y como definido anteriormente não é matriz, pois o número de elementos das linhas variam com a coluna.

Assim ordenamos,

$$Y_{(1,n_i)} \leq Y_{(2,n_i)} \leq ... \leq Y_{(r,n_i)} \leq ... \leq Y_{(r,n_i)}.$$

Como $Y_{(1,n_i)}, Y_{(2,n_i)}, ..., Y_{(r,n_i)}, ..., Y_{(r,n_i)}$ estão sob censura do tipo II, a função de verossiminhança de Gumbel com censura tipo II, é abtida tomando $\vartheta = (\beta, \theta)$ na expressão (2.7) é usando (2.1) e (3.2) na expressão (2.7), obtemos

$$\begin{aligned} L_i(\beta,\theta) =& \frac{n_i!}{(n_i - r)} \prod_{s=1}^{r} f(Y_{(s,n_i)}; \beta, \theta) \prod_{s=r+1}^{n_i} (1 - F(Y_{(r,n_i)}; \beta, \theta)) \\ =& \frac{n_i!}{(n_i - r)} \prod_{s=1}^{r} f(Y_{(s,n_i)}; \beta, \theta) \left[1 - F(Y_{(r,n_i)}; \beta, \theta)\right]^{n_i - r}. \end{aligned}$$

Como os $Y_1, Y_2, ..., Y_k$ são independentes, temos

$$L(\beta, \theta) = \prod_{i=1}^{k} L_i(\beta, \theta).$$

Aplicando a função logaritmo na expressão anterior, iremos obter a função log-verossimilhança, dada por

$$l = \log(L(\beta, \theta)) = \sum_{i=1}^{k} l_i(\beta, \theta),$$

onde, $l_i(\beta,\theta) = \log\left(\frac{n_i!}{(n_i - r)!}\right) + \sum_{s=1}^{r} \log(f(Y_{(s,n_i)}; \beta, \theta) + (n_i - r)\log(1 - F(Y_{(r,n_i)}; \beta, \theta)).$

Dessa forma, temos

$$\begin{aligned} l =& \log(L(\beta, \theta)) = \sum_{i=1}^{k} l_i(\beta, \theta) \\ =& \sum_{i=1}^{k} \left[\log\left(\frac{n_i!}{(n_i - r)!}\right) + \sum_{s=1}^{r} \log\left(\left[\frac{1}{\phi_i} \exp\left(\frac{y_{(s,n_i)} - \mu_i}{\phi_i}\right) \exp\left(-\exp\left(\frac{y_{(s,n_i)} - \mu_i}{\phi_i}\right)\right)\right]\right) \right. \\ & \left. + (n_i - r)\log\left(\exp\left(-exp\left(\frac{y_{(r,n_i)} - \mu_i}{\phi_i}\right)\right)\right)\right] \\ =& \sum_{i=1}^{k} \left[\log\left(\frac{n_i!}{(n_i - r)!}\right) + \sum_{s=1}^{r} \log\left(\frac{1}{\phi_i}\right) + \sum_{s=1}^{r} \left(\frac{y_{(s,n_i)} - \mu_i}{\phi_i}\right) - \sum_{s=1}^{r} \exp\left(\frac{y_{(s,n_i)} - \mu_i}{\phi_i}\right) \right. \\ & \left. - (n_i - r)\exp\left(\frac{y_{(r,n_i)} - \mu_i}{\phi_i}\right)\right] \end{aligned}$$

$$= \sum_{i=1}^{k} \left[\log \left(\frac{n_i!}{(n_i - r)!} \right) - r \log(\phi_i) + \sum_{s=1}^{r} \left(\frac{y_{(s,n_i)} - \mu_i}{\phi_i} \right) - \sum_{s=1}^{r} \exp \left(\frac{y_{(s,n_i)} - \mu_i}{\phi_i} \right) \right.$$
$$\left. -(n_i - r) \exp \left(\frac{y_{(r,n_i)} - \mu_i}{\phi_i} \right) \right],$$

com μ_i e ϕ_i definida por (3.1).

A função escore é definida por $U = U(\beta, \theta) = (\partial l / \partial \beta^T, \partial l / \partial \theta^T)^T$. Temos que

$$\begin{aligned} U_j(\beta, \theta) &= \frac{\partial l}{\partial \beta_j} = \frac{\partial l}{\partial \mu_i} \frac{d\mu_i}{d\eta_{1i}} \frac{\partial \eta_{1i}}{\partial \beta_j}, \quad j = 1, ..., p. \\ U_J(\beta, \theta) &= \frac{\partial l}{\partial \theta_J} = \frac{\partial l}{\partial \phi_i} \frac{d\phi_i}{d\eta_{2i}} \frac{\partial \eta_{2i}}{\partial \theta_J}, \quad J = 1, ..., q. \end{aligned}$$

Logo,

$$U_j(\beta, \theta) = \frac{\partial l}{\partial \beta_j} = \sum_{i=1}^{k} \left[\frac{1}{\phi_i} \left[-r + \sum_{s=1}^{r} \exp \left(\frac{y_{(s,n_i)} - \mu_i}{\phi_i} \right) + (n_i - r) \exp \left(\frac{y_{(r,n_i)} - \mu_i}{\phi_i} \right) \right] \right] \frac{d\mu_i}{d\eta_{1i}} \frac{\partial \eta_{1i}}{\partial \beta_j}$$

com $j = 1, ..., p$.

e

$$U_J(\beta, \theta) = \frac{\partial l}{\partial \theta_J} = \sum_{i=1}^{k} \left[\frac{1}{\phi_i} \left[-r + \sum_{i=1}^{r} \left[-\left(\frac{y_{(s,n_i)} - \mu_i}{\phi_i} \right) + \exp \left(\frac{y_{(s,n_i)} - \mu_i}{\phi_i} \right) \left(\frac{y_{(s,n_i)} - \mu_i}{\phi_i} \right) \right] \right. \right.$$
$$\left. \left. + (n_i - r) \exp \left(\frac{y_{(r,n_i)} - \mu_i}{\phi_i} \right) \left(\frac{y_{(r,n_i)} - \mu_i}{\phi_i} \right) \right] \right] \frac{d\phi_i}{d\eta_{2i}} \frac{\partial \eta_{2i}}{\partial \theta_J},$$

com $J = 1, ..., q$. Para mais detalhes ver o Apêndice(C).

Em notação matricial, temos que

$$U_j(\beta, \theta) = \frac{\partial l}{\partial \beta_j} = \widetilde{X}^T M_1 u_{s1} + \widetilde{X}^T M_1 u_{r1},$$

e

$$U_J(\beta, \theta) = \frac{\partial l}{\partial \beta_J} = \widetilde{S}^T M_2 v_{s1} + \widetilde{S}^T M_2 v_{r1},$$

onde os vetores de ordem $k \times 1$,

$$u_{s1}^T = \left(-\sum_{s=1}^{r} \frac{1}{\phi_1} + \sum_{s=1}^{r} \frac{1}{\phi_1} \exp \left(\tfrac{y_{(s,n_1)} - \mu_1}{\phi_1} \right), ..., -\sum_{s=1}^{r} \frac{1}{\phi_k} + \sum_{s=1}^{r} \frac{1}{\phi_k} \exp \left(\tfrac{y_{(s,n_k)} - \mu_k}{\phi_k} \right) \right),$$

$$u_{r1}^T = \left(\sum_{s=1}^{r} \frac{(n_1 - r)}{\phi_1} \exp\left(\frac{y_{(r,n_1)} - \mu_1}{\phi_1} \right), ..., \sum_{s=1}^{r} \frac{(n_k - r)}{\phi_k} \exp\left(\frac{y_{(r,n_k)} - \mu_k}{\phi_k} \right) \right),$$

$$v_{s1}^T = \left(-\sum_{s=1}^{r} \frac{1}{\phi_1} - \sum_{s=1}^{r} \left(\frac{y_{(s,n_1)} - \mu_1}{\phi_1^2} \right) + \sum_{s=1}^{r} \exp\left(\frac{y_{(s,n_1)} - \mu_1}{\phi_1^2} \right), ..., -\sum_{s=1}^{r} \frac{1}{\phi_k} - \sum_{s=1}^{r} \left(\frac{y_{(s,n_k)} - \mu_k}{\phi_k^2} \right) + \sum_{s=1}^{r} \exp\left(\frac{y_{(s,n_k)} - \mu_k}{\phi_k^2} \right) \right),$$

e

$$v_{r1}^T = \left(\sum_{s=1}^{r} \frac{(n_1 - r)}{\phi_1} \exp\left(\frac{y_{(r,n_1)} - \mu_1}{\phi_1} \right) \left(\frac{y_{(r,n_1)} - \mu_1}{\phi_1} \right), ..., \sum_{s=1}^{r} \frac{(n_k - r)}{\phi_k} \exp\left(\frac{y_{(r,n_k)} - \mu_k}{\phi_k} \right) \left(\frac{y_{(r,n_k)} - \mu_k}{\phi_k} \right) \right),$$

e as matrizes $\widetilde{X} = \left(\frac{\partial \eta_{1i}}{\partial \beta_j} \right)_{i,j}$ de ordem $k \times p$, $\widetilde{S} = \left(\frac{\partial \eta_{2i}}{\partial \beta_J} \right)_{i,J}$ de ordem $k \times q$, e as matrizes de ordem $k \times k$, $M_1 = \text{diag}\left(\frac{d\mu_i}{d\eta_{1i}} \right)$ e $M_2 = \text{diag}\left(\frac{d\phi_i}{d\eta_{2i}} \right)$.

Os estimadores de máxima verossimilhança para os parâmetros β e θ são obtidos resolvendo o sistema não linear $U = 0$ e não há uma forma fechada para tal solução; Portanto, utilizamos um algoritmo de optimização não linear, como o algoritmo de Newton ou quase-Newton, para encontrar estimadores de máxima verossimilhança.

A matriz informação de Fisher é dada por

$$K'' = K''(\beta, \theta) = \begin{pmatrix} K_{\beta\beta}^2 & K_{\beta\theta}^2 \\ K_{\theta\beta}^2 & K_{\theta\theta}^2 \end{pmatrix} = \begin{pmatrix} \widetilde{X}^T W_{\beta\beta}^2 \widetilde{X} & \widetilde{X}^T W_{\beta\theta}^2 \widetilde{S} \\ \widetilde{S}^T W_{\theta\beta}^2 \widetilde{X} & \widetilde{S}^T W_{\theta\theta}^2 \widetilde{S} \end{pmatrix}.$$

onde,

$$W_{\beta\beta}^2 = \text{diag}\left(\frac{1}{\phi_i^2} \left[\sum_{s=1}^{r} q_{3si} + (n_i - r) q_{4ri} \right] \left(\frac{d\mu_i}{d\eta_{1i}} \right)^2 \right),$$

$$W_{\theta\theta}^2 = \text{diag}\left(\frac{1}{\phi_i^2} \left[-q_1 - \sum_{s=1}^{r} (2q_{2si} - q_{7si} - 2q_{5si}) + (n_i - r)(q_{8ri} + 2q_{6ri}) \right] \left(\frac{d\phi_i}{d\eta_{2i}} \right)^2 \right),$$

e

$$W_{\beta\theta}^2 = \text{diag}\left(\frac{1}{\phi_i^2} \left[-q_1 + \sum_{s=1}^{r} (q_{3si} + q_{5si}) + (n_i - r)(q_{6ri} + q_{4ri}) \right] \frac{d\mu_i}{d\eta_{1i}} \frac{d\phi_i}{d\eta_{2i}} \right)$$

A inversa da matriz de Fisher é dada por

$$(K'')^{-1} = (K'')^{-1}(\beta, \theta) = \begin{pmatrix} K_{\beta\beta}^2 & K_{\beta\theta}^2 \\ K_{\theta\beta}^2 & K_{\theta\theta}^2 \end{pmatrix}^{-1}.$$

Definimos as matrizes $\mathbb{X}$ e $\widetilde{W}_2$ com dimensões $2k \times (p+q)$ e $2k \times 2k$ respectivamente, com

$$\mathbb{X} = \begin{pmatrix} \widetilde{X} & 0 \\ 0 & \widetilde{S} \end{pmatrix} \quad \text{e} \quad \widetilde{W}_2 = \begin{pmatrix} W^2_{\beta\beta} & W^2_{\beta\theta} \\ W^2_{\theta\beta} & W^2_{\theta\theta} \end{pmatrix}.$$

Então podemos escrever a matriz de Fisher como

$$K'' = \mathbb{X}^T \widetilde{W}_2 \mathbb{X}.$$

4.1 Correção Do Viés De Um EMVs

Nesta seção, obtemos uma expressão para os vieses de segunda ordem dos EMVs dos parâmetros do modelo geral de regressão do valor extremo usando Cox e Snell (1968). As derivadas parciais da log-verossimilhança com respeito as componentes dos vetores desconhecidos β e θ são indicados pelos indices $\{j, l, ...\}$ e $\{J, L, ...\}$, respectivamente. Assim, definimos $U_j = \partial l/\partial\beta_j$, $U_J = \partial l/\partial\theta_J$, $U_{jL} = \partial^2 l/\partial\beta_j\partial\theta_L$, $U_{jlM} = \partial^3 l/\partial\beta_j\partial\beta_l\partial\theta_M$ e etc. Para denotar os cumulantes das derivadas parciais acima, usamos a notação introduzida por Lawley (1956): $k_{jl} = E(U_{jl})$, $k_{j,l} = E(U_j U_l)$, $k_{jl,M} = E(U_{jl}U_M)$ e etc, em geral k's são da ordem de $O(n)$. As derivadas destes cumulantes são denotadas da seguinte forma $k_{jl}^{(m)} = \partial k_{jl}/\partial\beta_m$, $k_{jl}^{(M)} = \partial k_{jl}/\partial\theta_M$ e etc. Nem todos os k's são necessariamente independentes. Seja $k^{j,l} = -k^{jl}$ e $k^{J,L} = -k^{JL}$ os elementos de seus respectivos inversos $K_{\beta\beta}$ e $K_{\theta\theta}$ que são $O(n^{-1})$.

A fórmula de Cox e Snell (1968) pode ser usada para obter o viés segunda ordem do EMV para a a-ésima componente do vetor paramétrico $\widehat{\tau} =$

$(\widehat{\tau}_1, ..., \widehat{\tau}_{p+q}) = (\widehat{\beta}^T, \widehat{\theta}^T)$, a qual é dada por

$$\begin{aligned} B(\widehat{\tau}_a) &= \sum_{j,l,m} k^{aj} k^{lm} \left\{ k_{jl}^{(m)} - \tfrac{1}{2} k_{jlm} \right\} + \sum_{J,l,m} k^{aJ} k^{lm} \left\{ k_{Jl}^{(m)} - \tfrac{1}{2} k_{Jlm} \right\} \\ &+ \sum_{j,L,m} k^{aj} k^{Lm} \left\{ k_{jL}^{(m)} - \tfrac{1}{2} k_{jLm} \right\} + \sum_{j,l,M} k^{aj} k^{lM} \left\{ k_{jl}^{(M)} - \tfrac{1}{2} k_{jlM} \right\} \\ &+ \sum_{J,L,m} k^{aJ} k^{Lm} \left\{ k_{JL}^{(m)} - \tfrac{1}{2} k_{JLm} \right\} + \sum_{J,l,M} k^{aJ} k^{lM} \left\{ k_{Jl}^{(M)} - \tfrac{1}{2} k_{JlM} \right\} \\ &+ \sum_{j,L,M} k^{aj} k^{LM} \left\{ k_{jL}^{(M)} - \tfrac{1}{2} k_{jLM} \right\} + \sum_{J,L,M} k^{aJ} k^{LM} \left\{ k_{JL}^{(M)} - \tfrac{1}{2} k_{JLM} \right\} \end{aligned} \quad (4.2)$$

Os parâmetros β e θ não são ortogonais. Assim as entradas da matriz $W_{\beta\theta}$ não são todas nulas. Por isso, todos os termos em (4.2) devem ser considerados o que faz derivação ser mais forte.

4.2 Correção do viés dos MLEs para o modelo de Gumbel com censura tipo I

Usando a expressão (4.2) e utilizando algumas manipulações algébricas, que pode ser encontrada no Apêndice (C), podemos obter uma expressão para o viés de segunda ordem de $\widehat{\beta}$ e $\widehat{\theta}$ na forma matricial:

$$\begin{aligned} B_2(\widehat{\beta}) &= K_2^{\beta\beta} \widetilde{X}^T \left[W_2'' Z_{\beta d}'' + W_2'' D_\beta'' + (W_3'' + W_5'') Z_{\beta\theta d}'' + W_4'' D_\theta'' + W_7'' Z_{\theta d}'' \right] 1_{n\times 1} \\ &+ K_2^{\beta\theta} \widetilde{S}^T \left[W_3'' Z_{\beta d}'' + W_4'' D_\beta'' + (W_6'' + W_7'') Z_{\beta\theta d}'' + W_8'' Z_{\theta d}'' + W_9'' D_\theta'' \right] 1_{n\times 1}, \end{aligned} \quad (4.3)$$

e

$$\begin{aligned} B_2(\widehat{\theta}) &= K_2^{\theta\beta} \widetilde{X}^T \left[W_1'' Z_{\beta d}'' + W_2'' D_\beta'' + (W_3'' + W_5'') Z_{\beta\theta d}'' + W_4'' D_\theta'' + W_7'' Z_{\theta d}'' \right] 1_{n\times 1} \\ &+ K_2^{\theta\theta} \widetilde{S}^T \left[W_3'' Z_{\beta d}'' + W_4'' D_\beta'' + (W_6'' + W_7'') Z_{\beta\theta d}'' + W_8'' Z_{\theta d}'' + W_9'' D_\theta'' \right] 1_{n\times 1}, \end{aligned} \quad (4.4)$$

onde $W_k'' = diag\{w_{k1}'', ..., w_{kn}''\}$ para $i = 1, ..., n$ e $k = 1, ..., 9$, $1_{n\times 1}$ denota um vetor com n entradas igual a 1, $Z_{\beta d}'' = diag(\widetilde{X} K_2^{\beta\beta} \widetilde{X}^T)$, $Z_{\beta\theta d}'' = diag(\widetilde{X} K_2^{\beta\beta} \widetilde{S}^T)$,

$Z''_{\theta d} = diag(\widetilde{S}K_1^{\theta\theta}\widetilde{S}^T)$, $D''_\beta = diag(d''_{1\beta}, ..., d''_{n\beta})$ e $D''_\theta = diag(d''_{1\theta}, ..., d''_{n\theta})$ com $d''_{i\beta} = tr(\widetilde{\widetilde{X}}K_2^{\beta\beta})$, $d''_{i\theta} = tr(\widetilde{\widetilde{S}}K_2^{\theta\theta})$, $\widetilde{\widetilde{X}}_i = (\partial^2\eta_1 i/\partial\beta_j\beta_l)_{j,l}$ e $\widetilde{\widetilde{S}}_i = (\partial^2\eta_2 i/\partial\theta_J\theta_L)_{j,l}$ para $i = 1, ..., n$.

Considere os vetores de ordem $(2n \times 1)$, δ''_1 e δ''_2 como

$$\delta''_1 = \begin{pmatrix} [W''_1 Z''_{\beta d} + (W''_3 + W''_5)Z''_{\beta\theta d} + W''_7 Z''_{\theta d}]1_{n\times 1} \\ [W''_3 Z''_{\beta d} + (W''_6 + W''_7)Z''_{\beta\theta d} + W''_8 Z''_{\theta d}]1_{n\times 1} \end{pmatrix} \tag{4.5}$$

e

$$\delta''_2 = \begin{pmatrix} [W''_2 D''_\beta + W''_4 D''_\theta]1_{n\times 1} \\ [W''_4 D''_\beta + W''_9 D''_\theta]1_{n\times 1} \end{pmatrix} \tag{4.6}$$

os blocos inferiores de ordem $p \times (p+q)$ e superior de ordem $q \times (p+q)$ da matriz $K_1(\tau)^{-1}$ por $K_2^{\beta *} = (k_2^{\beta\beta} k_2^{\beta\theta})$ e $K_2^{\theta *} = (k_2^{\theta\beta} k_2^{\theta\theta})$, respectivamente. Com estas expressões, podemos escrever o viés de segunda ordem de $\widehat{\beta}$ e $\widehat{\theta}$ como

$$B_2(\widehat{\beta}) = K_2^{\beta *}\mathbb{X}^T(\delta''_1 + \delta''_2) \quad \text{e} \quad B_2(\widehat{\theta}) = K_2^{\theta *}\mathbb{X}^T(\delta''_1 + \delta''_2), \tag{4.7}$$

respectivamente. Então, por (3.8) concluímos que o viés de segunda ordem do EMV do vetor conjunto $\widehat{\tau} = (\widehat{\beta}^T, \widehat{\theta}^T)$ possui a forma

$$B_2(\widehat{\tau}) = K_2^{-1}\widetilde{X}^T(\delta''_1 + \delta''_2) = (\mathbb{X}^T\widetilde{W}_2\mathbb{X})^{-1}\mathbb{X}^T(\delta''_1 + \delta''_2).$$

Definindo $\xi''_1 = \widetilde{W}_2^{-1}\delta''_1$ e $\xi''_2 = \widetilde{W}_2^{-1}\delta''_2$, assim

$$B_2(\widehat{\tau}) = (\mathbb{X}^T\widetilde{W}_2\mathbb{X})^{-1}\mathbb{X}^T\widetilde{W}_2(\xi''_1 + \xi''_2). \tag{4.8}$$

A fórmula (4.8) mostra que o viés de segunda ordem de $\widehat{\tau}$ e facilmente obtida com os vetores dos coeficientes de regressão na forma de regressão linear de ξ''_1 e ξ''_2 nas colunas de $\mathbb{X}$ com $\widetilde{W}_2$ sendo a matriz peso. Podemos expressar (4.8) como

$$B_2(\widehat{\tau}) = B''_1(\widehat{\tau}) + B''_2(\widehat{\tau}),$$

com $B_1''(\widehat{\tau}) = (\mathbb{X}^T\widetilde{\widetilde{W}}_2\mathbb{X})^{-1}\mathbb{X}^T\widetilde{\widetilde{W}}_1\xi_1''$ e $B_2''(\widehat{\tau}) = (\mathbb{X}^T\widetilde{\widetilde{W}}_2\mathbb{X})^{-1}\mathbb{X}^T\widetilde{\widetilde{W}}_2\xi_2''$.

Se $\xi_2'' = 0$, a fórmula (4.8) dá o viés de segunda ordem para Modelos Lineares de Valores Extremo de Regressão com covariavéis de dispersão linear. Portanto, $B_1''(\widehat{\tau})$ e $B_2''(\widehat{\tau})$ podem ser considerados respectivamente, como a linearidade e não-linearidade em termos do viés total.

4.3 Correção Do Viés De Um MLEs De μ e ϕ

Primeiramente, expandimos as funções $\widehat{\eta_{1i}} = f_1(x_i^T, \widehat{\beta})$ e $\widehat{\eta_{2i}} = f_2(x_i^T, \widehat{\theta})$ dado em (3.1) em série de Taylor até a segunda ordem em torno dos pontos β e θ, respectivamente, obtemos

$$\widehat{\eta_{1i}} - \eta_{1i} = \widetilde{X}_i^T(\widehat{\beta} - \beta) + \frac{1}{2}(\widehat{\beta} - \beta)^T\widetilde{\widetilde{X}}_i(\widehat{\beta} - \beta) + o_p(\|(\widehat{\beta} - \beta)\|^2)$$

e

$$\widehat{\eta_{2i}} - \eta_{2i} = \widetilde{S}_i^T(\widehat{\theta} - \theta) + \frac{1}{2}(\widehat{\theta} - \theta)^T\widetilde{\widetilde{S}}_i(\widehat{\theta} - \theta) + o_p(\|(\widehat{\theta} - \theta)\|^2)$$

onde $\widetilde{X}_i$ e $\widetilde{S}_i$ são a i-ésima linha das matrizes $\widetilde{X}$ e $\widetilde{S}$ respectivamente. Assim, os vieses de segunda ordem de $\widehat{\eta_{1i}}$ e $\widehat{\eta_{2i}}$ na notação matricial são dados por

$$B(\widehat{\eta_{1i}}) = \widetilde{X}B(\widetilde{\beta}) + \frac{1}{2}D_\beta 1_{n\times 1} \text{ e } B(\widehat{\eta_{2i}}) = \widetilde{S}B(\widetilde{\theta}) + \frac{1}{2}D_\theta 1_{n\times 1}$$

Vamos agora expandir as funções $\widehat{\mu}_{1i} = g_1^{-1}(\widetilde{\eta}_{1i})$ e $\widehat{\phi}_{1i} = g_2^{-1}(\widetilde{\eta}_{2i})$ em séries de taylor até a segunda ordem, em torno dos pontos η_{1i} e η_{2i} respectivamente. Com isto, segue que

$$\widehat{\mu}_i - \mu_i = \frac{d\mu_i}{d\eta_{1i}}(\widehat{\eta_{1i}} - \eta_{1i}) + \frac{1}{2}\frac{d^2\mu_i}{d\eta_{1i}^2}(\widehat{\eta_{1i}} - \eta_{1i})^2 + o_p((\widehat{\eta_{1i}} - \eta_{1i})^2)$$

e

$$\widehat{\phi}_i - \phi_i = \frac{d\phi_i}{d\eta_{2i}}(\widehat{\eta_{2i}} - \eta_{2i}) + \frac{1}{2}\frac{d^2\phi_i}{d\eta_{2i}^2}(\widehat{\eta_{2i}} - \eta_{2i})^2 + o_p((\widehat{\eta_{2i}} - \eta_{2i})^2)$$

Assim, obtemos os vieses de segunda ordem de $\widehat{\mu}_i$ e $\widehat{\phi}_i$

$$B(\widehat{\mu}_i) = B(\widehat{\eta_{1i}})\frac{d\mu_i}{d\eta_{1i}} + \frac{1}{2}Var(\widehat{\eta_{1i}})\frac{d^2\mu_i}{d\eta_{1i}^2} \text{ e } B(\widehat{\phi}_i) = B(\widehat{\eta_{2i}})\frac{d\phi_i}{d\eta_{2i}} + \frac{1}{2}Var(\widehat{\eta_{2i}})\frac{d^2\phi_i}{d\eta_{2i}^2} \quad (4.9)$$

A fórmula anterior irá nos fornecer uma expressão para os vies de segunda ordem do EMVs de μ e ϕ, em notação matricial, fica como segue

$$B_2(\widehat{\mu}_i) = \frac{1}{2}\left\{M_1[2\widetilde{X}B_2(\widehat{\beta}) + D''_\beta 1_{n\times 1}] + Z''_{\beta d}T_1 1_{n\times 1}\right\}$$

e

$$B_2(\widehat{\phi}_i) = \frac{1}{2}\left\{M_2[2\widetilde{S}B_2(\widehat{\theta}) + D''_\theta 1_{n\times 1}] + Z''_{\theta d}T_2 1_{n\times 1}\right\}$$

Para o modelo de regressão dos valores extremos, usando (4.7) temos,

$$B_2(\widehat{\mu}_i) = \frac{1}{2}\left\{M_1[2\widetilde{X}K_2^{\beta *}\widetilde{X}^T(\delta''_1 + \delta''_2) + D''_\beta 1_{n\times 1}] + Z''_{\beta d}T_1 1_{n\times 1}\right\}$$

e

$$B_2(\widehat{\phi}_i) = \frac{1}{2}\left\{M_2[2\widetilde{S}K_2^{\theta *}\widetilde{X}^T(\delta''_1 + \delta''_2) + D''_\theta 1_{n\times 1}] + Z''_{\theta d}T_2 1_{n\times 1}\right\}$$

Definimos as matrizes diagonais $T_1 = diag\left\{d^2\mu_i/d\eta_{1i}^2\right\}$ e $T_2 = diag\left\{d^2\phi_i/d\eta_{2i}^2\right\}$ de ordem n.

Os estimadores corrigidos $\widetilde{\mu} = \widehat{\mu} - \widehat{B}_2(\widehat{\mu})$ e $\widetilde{\phi} = \widehat{\phi} - \widehat{B}_2(\widehat{\phi})$ de μ e ϕ respectivamente, tem viéses de ordem $O(n^{-2})$, onde $\widehat{B}_2(\cdot)$ denota o EMV de $B_2(\cdot)$, isto é, os parâmetros desconhecidos são substituídos por seus EMVs.

5 ANALISE NUMÉRICA

Neste capítulo apresentamos os resultados numéricos o qual usamos a simulação de Monte carlo com 5000 replicas, e faremos também uma aplicação do nosso modelo a dados reais.

5.1 Simulação

Nesta seção, analisaremos uma simulação, a qual utilizamos a simulação Monte carlo. O objetivo desta simulação, foi comparar os desempenhos dos EMVs corrigido com os não corrigidos dos parâmetros. Usamos um modelo não linear de regressão de Gumbel, com covariavéis de dispersão linear, nas tabelas 5.1, 5.2 e 5.3, podemos observar que os EMVs corrigido apresentam menor viés e menor erro quadrático médio que os EMVs não corrigido. Para essa simulação utilizado o software R (www.r-project.org) na sua versão 3.2.2. Para gerarmos as tabelas das simulações, usamos 5000 replicas Monte carlo, simulamos o modelo de regressão com $\beta_1 = 2$, $\beta_2 = 1$, $\theta_1 = 1$ e $\theta_2 = 2$.

s= 13,3%	β_1	β_2	θ_1	θ_2
EMVs	1.9841894	0.9385983	0.9020137	2.0035233
EMVs Corrigido	2.0010960	0.9943084	0.9843714	2.0028233
EQM EMVs	0.02312897	0.22287039	0.09794647	0.450711522
EQM EMVs Corrigido	0.02107961	0.18025302	0.08862047	0.430311321

Tabela 5.1: Simulação 1.

s= 20%	β_1	β_2	θ_1	θ_2
EMVs	1.9884107	0.9951531	0.9481695	2.0312621
EMVs Corrigido	1.9987930	0.9995286	0.9938284	2.0058405
EQM EMVs	0.006291786	0.004690152	0.0193308113	0.030571569
EQM EMVs Corrigido	0.006013692	0.004556922	0.0166785860	0.028874163

Tabela 5.2: Simulação 2.

s= 33.3%	β_1	β_2	θ_1	θ_2
EMVs	1.9812884	0.9911473	0.9370526	2.0296384
EMVs Corrigido	1.9976929	0.9983948	0.9964153	1.9988143
EQM EMVs	0.008038732	0.008278457	0.025232690	0.042788843
EQM EMVs Corrigido	0.007481709	0.007983603	0.021294664	0.040556555

Tabela 5.3: Simulação 3.

5.2 Aplicação

Neste Seção, faremos uma aplicação a dados reais, para isto foi utilizado o software R (www.r-project.org) na sua vers ao 3.2.2.

O estudo foi baseado no banco de dados de tempo de falha apresentado no trabalho de McCool (1980). Ele nos deu os tempos de falha para amostras de aço endurecido num ensaio de fadiga constante de um rolamento, 10 observações independentes foram tomadas em cada um dos 4 valores de tensão de contato ilustrado na tabela 5.5. A Engenharia sugere que pelo nível de tensão s , o tempo de falha deve ter aproximadamente uma distribuição Weibull com um parâmetro de escala α relacionado com s pela lei de potência dada por $\alpha = cs^p$. Reparametrizando obtemos que nosso problema pode ser tratado como um problema que segue aproximadamente o Modelo de Regressão de Valor Extremo onde $\mu(x) = log(\alpha)$, $x = log(s)$, $v_0 = log(c)$ e $v_1 = p$, com um parâmetro ϕ independente de s.

Tensão ($psi^2 \div 10^6$)	Tempo de falha (ordenados)									
0.87	1.67	2.20	2.51	3.00	3.90	4.70	7.53	14.70	27.80	37.40
0.99	0.80	1.00	1.37	2.25	2.95	3.70	6.07	6.65	7.05	7.37
1.09	0.0012	0.18	0.20	0.24	0.26	0.32	0.32	0.42	0.44	0.08
1.18	0.073	0.098	0.117	0.135	0.175	0.262	0.270	0.350	0.386	0.456

Tabela 5.4: Dados Reais.

Utilizando os dados reais acima e consideramos função de ligação a função identidade, pois esta satisfaz as nossa hipotese; Tomanos o nosso problema, com $33,3\%$ dados amostrais censurados.

Obtemos assim, o estimador de máxima verossimilhança e o estimador de máxima verossimilhança corrigido

s= 33.3%	β_1	β_2	θ_1
EMVs	0.8188506	-12.9572220	0.9376049
EMVs Corrigido	0.8499224	-12.9915189	0.9800770

Tabela 5.5: Dados Reais.

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Apêndice A

- Função geradora de momentos, esperança e variância.

Iremos achar primeiramente, a função geradora de momentos de (2.1), em seguida, calcularemos a esperança é a variância da mesma. Estamos tratando da função de distribuição do valor extremo, ou seja,

$$g(y;\mu,\phi)=\frac{1}{\phi}\exp\left(\frac{y-\mu}{\phi}\right)\exp\left(-\exp\left(\frac{y-\mu}{\phi}\right)\right);\ y\in\mathbb{R}.$$

onde $Y \sim EV(\mu,\phi)$, com $\mu \in \mathbb{R}$ e $\phi > 0$.

Seja Y uma variável aleatória com distribuição do valor extremo. Note que se consideramos $\mu = 0$ e $\phi = 1$, e tomando $X = \frac{Y-\mu}{\phi}$, obtemos a forma padrão

$$g(x) = \exp(x - \exp(x)).$$

Seja $Z = \exp\left(\frac{Y-\mu}{\phi}\right) = \exp(X)$, iremos mostrar agora que a função de densidade de probabilidade de Z e da forma exponencial, ou seja,

$$g(z) = \exp(-z)\,;\ z > 0 \tag{A.1}$$

De fato,

$$G_Z(z) = P(Z \leq z) = P(\exp(X) \leq z) = P(X \leq \log(z)) = G_X(\log(z)).$$

Derivando ambos os lados da expressão acima com relação a z, obtemos

$$\begin{aligned}
g(z) = G'_Z(z) &= \frac{d}{dz}G_X(\log(z)) \\
&= G'_X(\log(z))\cdot\frac{1}{z} \\
&= g_X(\log(z))\cdot\frac{1}{z} \\
&= \exp(\log(z) - \exp(\log(z)))\cdot\frac{1}{z} \\
&= \exp(\log(z))\cdot\exp(-\exp(\log(z)))\cdot\frac{1}{z}
\end{aligned}$$

$$
\begin{aligned}
&= z \cdot \exp(-z) \cdot \frac{1}{z} \\
&= \exp(-z);\ \ z > 0.
\end{aligned}
$$

Agora, iremos calcular a função geradora de momentos de Y. Utilizando a expressão (A.1), iremos obter

$$
\begin{aligned}
E(\exp(tX)) &= E(\exp(Xt)) \\
&= E(Z^t) \\
&= \int_0^\infty z^t\, g(z)\, dz \\
&= \int_0^\infty z^t \exp^{-z}\ dz \\
&= \Gamma(1+t),\ t > -1.
\end{aligned}
$$

Portanto a função geradora de momentos e dada por

$$
\begin{aligned}
\mu_Y(t) = E(\exp(tY)) &= E(\exp(t(X\phi + \mu))) \\
&= E(\exp(tX\phi + t\mu)) \\
&= E(\exp(tX\phi) \cdot \exp(t\mu)) \\
&= \exp(t\mu)\, E(\exp(tX\phi)) \\
&= \exp(t\mu)\, \Gamma(1+\phi t),\ \ t > -\phi^{-1}.
\end{aligned}
$$

Agora, para encontrarmos a esperança e a variância da função de distribuição do valor extremo, iremos primeiramente encontrar a função geradora acumulada.

$$
k(t) = \log(M_Y(t)) \ \ = \ \ t\mu + \log(\Gamma(1+\phi t)), \tag{A.2}
$$

com isso, o valor esperado de Y é obtido, derivando a expressão (A.2), e em seguida tomando $t = 0$, assim

$$
\begin{aligned}
k_1(0) = E(Y) &= \mu + \frac{1}{\Gamma(1)} \cdot \Gamma'(1)\phi \\
&= \mu + \psi(1)\phi
\end{aligned}
$$

$$= \mu - (-\psi(1)\phi)$$
$$= \mu - \gamma\phi,$$

o valor da variância de Y é obtido, derivando duas vezes a expressão (A.2), e em seguida tomando $t = 0$, iremos obter

$$k_2(0) = V(Y) = \phi^2 \cdot \psi'(1) = \frac{\pi^2}{6}\phi^2,$$

onde a função digama é a derivada do logaritmo da função gama, ou seja,

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

Os valores de $\psi(1)$ e $\psi'(1)$, podem ser encontrados em Escobar and Meeker (1998)

- Calculando a função acumulada de (2.1).

$$G(y) = \int_{-\infty}^{y} g(y)\, dy = \frac{1}{\phi} \int_{-\infty}^{y} \exp\left(\frac{y-\mu}{\phi}\right) \exp\left(-\exp\left(\frac{y-\mu}{\phi}\right)\right) dy,$$

Usando o método da substituição, considere $u = \exp\left(\frac{y-\mu}{\phi}\right)$ então $du = \exp\left(\frac{y-\mu}{\phi}\right) \frac{1}{\phi} = u\frac{1}{\phi}\, dy$. Assim

$$\begin{aligned} G(y) &= \frac{1}{\phi} \int_{0}^{u} u \exp(-u) \phi \frac{1}{u}\, du = \int_{0}^{u} \exp(-u)\, du = -\exp(-u)\Big|_{0}^{u} \\ &= 1 - \exp(-u) = 1 - \exp\left(-\exp\left(\frac{y-\mu}{\phi}\right)\right); \; y \in \mathbb{R}. \end{aligned}$$

Apêndice B

- Calculando as esperanças que serão necessárias para o modelo com censura tipo I.

Utilizamos a seguinte expressão para calcular as esperanças envolvidas nesta dissertação

$$E[h(\delta_i, Y_i)] = E_1(h(1, Y_i)) + E_2(h(0, Y_i)),$$

onde

$$E_1(h(Y_i)) = \int_{-\infty}^{T_i} h\left(\frac{y_i - \mu_i}{\phi_i}\right) g\left(y_i; \mu_i, \phi_i\right) \, dy_i$$

e

$$E_2[h(Y_i)] = h\left(\frac{T_i - \mu_i}{\phi_i}\right) \cdot P(Y_i > T_i).$$

$*$ Calculando a esperança $h_{1i} = E[\delta_i]$.

$$\begin{aligned} h_{1i} &= E[\delta_i] = E_1(1) + E_2(0) \\ &= \frac{1}{\phi_i} \int_{-\infty}^{T_i} \exp\left[\tfrac{y_i - \mu_i}{\phi_i} - \exp\left(\tfrac{y_i - \mu_i}{\phi_i}\right)\right] \, dy_i \end{aligned}$$

Usando o método de substituição, tomando $x_i = \exp\left(\frac{y_i - \mu_i}{\phi_i}\right)$ então $dx_i = \frac{x_i}{\phi_i} \, dy_i$. Assim

$$\begin{aligned} h_{1i} = E[\delta_i] &= \int_{0}^{\exp\left(\frac{T_i - \mu_i}{\phi_i}\right)} \exp(-x_i) \, dx_i = -\exp\left(-x_i\right) \Big|_{0}^{\exp\left(\frac{T_i - \mu_i}{\phi_i}\right)} \\ &= -\exp\left(-\exp\left(\tfrac{T_i - \mu_i}{\phi_i}\right)\right) + 1 \end{aligned}$$

$*$ Calculando a esperança $h_{2i} = E\left[\exp\left(\frac{Y_i - \mu_i}{\phi_i}\right)\right]$.

$$\begin{aligned} h_{2i} &= E\left[\exp\left(\tfrac{Y_i - \mu_i}{\phi_i}\right)\right] = E_1\left(\exp\left(\tfrac{Y_i - \mu_i}{\phi_i}\right)\right) + E_2\left(\exp\left(\tfrac{Y_i - \mu_i}{\phi_i}\right)\right) \\ &= \frac{1}{\phi_i} \int_{-\infty}^{T_i} exp\left(\tfrac{y_i - \mu_i}{\phi_i}\right) \exp\left[\tfrac{y_i - \mu_i}{\phi_i} - \exp\left(\tfrac{y_i - \mu_i}{\phi_i}\right)\right] \, dy_i + \exp\left(\tfrac{T_i - \mu_i}{\phi_i}\right) \exp\left(-\exp\left(\tfrac{T_i - \mu_i}{\phi_i}\right)\right) \end{aligned}$$

Tomando a mesma mudança de variável anterior, temos

$$\begin{aligned} h_{2i} &= E\left[\exp\left(\tfrac{Y_i-\mu_i}{\phi_i}\right)\right] \\ &= \int_0^{\exp\left(\frac{T_i-\mu_i}{\phi_i}\right)} x_i\, exp\left(-x_i\right)\, dx_i + \exp\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\exp\left(-\exp\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\right) \end{aligned}$$

Usando o método de integral por partes, tome $u_i = x_i$ então $du_i = dx_i$ e $dv_i = \exp(-x_i)\, dx_i$ então $v_i = -\exp(-x_i)$, assim

$$\begin{aligned} h_{2i} &= E\left[\exp\left(\tfrac{Y_i-\mu_i}{\phi_i}\right)\right] \\ &= -x_i\exp(-x_i)\Big|_0^{\exp\left(\frac{T_i-\mu_i}{\phi_i}\right)} + \int_0^{\exp\left(\frac{T_i-\mu_i}{\phi_i}\right)} exp\left(-x_i\right)\, dx_i + \exp\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\exp\left(-\exp\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\right) \\ &= -\exp\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\exp\left(-\exp\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\right) + 0 - \exp\left(-\exp\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\right) + 1 \\ &+ exp\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\exp\left(-\exp\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\right) \\ &= 1 - \exp\left(-\exp\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\right). \end{aligned}$$

As demais esperanças, utilizaremos a função geradora de momentos, do seguinte modo

$$\begin{aligned} E\left[\exp\left(t\left(\tfrac{Y_i-\mu_i}{\phi_i}\right)\right)\right] &= E_1\left[\exp\left(t\left(\tfrac{Y_i-\mu_i}{\phi_i}\right)\right)\right] + E_2\left[\exp\left(t\left(\tfrac{Y_i-\mu_i}{\phi_i}\right)\right)\right] \\ &= \frac{1}{\phi_i}\int_{-\infty}^{T_i} \exp\left(t\left(\tfrac{y_i-\mu_i}{\phi_i}\right)\right)\exp\left[\tfrac{y_i-\mu_i}{\phi_i} - \exp\left(\tfrac{y_i-\mu_i}{\phi_i}\right)\right]\, dy_i \\ &+ \exp\left(t\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\right)\exp\left(-\exp\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\right) \end{aligned}$$

Tomando a seguinte mudança de variável, $x_i = \dfrac{y_i-\mu_i}{\phi_i}$ então $dx_i = \dfrac{1}{\phi_i}\, dy_i$. Daí

$$\begin{aligned} E\left[\exp\left(tx_i\right)\right] &= \int_{-\infty}^{\frac{T_i-\mu_i}{\phi_i}} \exp\left(tx_i\right)\exp\left[x_i - \exp\left(x_i\right)\right]\, dx_i \\ &+ \exp\left(t\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\right)\exp\left(-\exp\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\right) \end{aligned}$$

Tomando a seguinte mudança de variável, $u_i = \exp(x_i)$ então $du_i = u_i\, dx_i$. Daí

$$\begin{aligned} E\left[\exp\left(tx_i\right)\right] &= \int_0^{\exp\left(\frac{T_i-\mu_i}{\phi_i}\right)} u_i^t \exp\left(-u_i\right)\, du_i \\ &+ \exp\left(t\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\right)\exp\left(-\exp\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\right) \end{aligned}$$

Lembremos que a função gama incompleta é obtida pela mesma integral que a função gama, porém com uma integral indefinida no lugar da integral definida:

$$\Gamma(a,x) = \int_x^\infty t^{a-1}\, e^{-t}\, dt.$$

$$\gamma(a,x) = \int_0^x t^{a-1}\, e^{-t}\, dt.$$

Considere

$$\varphi(t) = \gamma\left(t+1, \exp\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\right) + \exp\left(t\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\right)\exp\left(-\exp\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\right)$$

Observe que

$$p(t) = E\left[\delta_i \exp\left(\tfrac{t(Y_i-\mu_i)}{\phi_i}\right)\right] = E_1\left[\exp\left(\tfrac{t(y_i-\mu_i)}{\phi_i}\right)\right] + 0 = \gamma\left(t+1, \exp\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\right)$$

e

$$p^{'}(t) = \frac{\partial}{\partial t}\gamma\left(t+1, \exp\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\right)$$

Então

$$h_{3i} = E\left[\delta_i\left(\tfrac{Y_i-\mu_i}{\phi_i}\right)\right] = p^{'}(0) = \frac{\partial}{\partial t}\gamma\left(1, \exp\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\right)$$

Note que,

$$\varphi^{'}(t) = \frac{\partial}{\partial t}\gamma\left(t+1, \exp\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\right) + \left(\tfrac{T_i-\mu_i}{\phi_i}\right)\exp\left(t\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\right)\exp\left(-\exp\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\right)$$

$$\varphi^{''}(t) = \frac{\partial^2}{\partial t^2}\gamma\left(t+1, \exp\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\right) + \left(\tfrac{T_i-\mu_i}{\phi_i}\right)^2\exp\left(t\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\right)\exp\left(-\exp\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\right)$$

$$\varphi^{'''}(t) = \frac{\partial^3}{\partial t^3}\gamma\left(t+1, \exp\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\right) + \left(\tfrac{T_i-\mu_i}{\phi_i}\right)^3\exp\left(t\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\right)\exp\left(-\exp\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\right)$$

Portanto

$$\begin{aligned} h_{4i} &= E\left[\left(\tfrac{Y_i-\mu_i}{\phi_i}\right)\exp\left(\tfrac{Y_i-\mu_i}{\phi_i}\right)\right] = \varphi^{'}(1) \\ &= \frac{\partial}{\partial t}\gamma\left(2, \exp\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\right) + \left(\tfrac{T_i-\mu_i}{\phi_i}\right)\exp\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\exp\left(-\exp\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\right) \end{aligned}$$

$$
\begin{aligned}
h_{5i} &= E\left[\left(\tfrac{Y_i-\mu_i}{\phi_i}\right)^2 \exp\left(\tfrac{Y_i-\mu_i}{\phi_i}\right)\right] = \varphi^{''}(1)\\
&= \frac{\partial^2}{\partial t^2}\gamma\left(2, \exp\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\right) + \left(\tfrac{T_i-\mu_i}{\phi_i}\right)^2 \exp\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\exp\left(-\exp\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\right)\\
h_{6i} &= E\left[\left(\tfrac{Y_i-\mu_i}{\phi_i}\right)^3 \exp\left(\tfrac{Y_i-\mu_i}{\phi_i}\right)\right] = \varphi^{'''}(1)\\
&= \frac{\partial^3}{\partial t^3}\gamma\left(2, \exp\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\right) + \left(\tfrac{T_i-\mu_i}{\phi_i}\right)^3 \exp\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\exp\left(-\exp\left(\tfrac{T_i-\mu_i}{\phi_i}\right)\right)
\end{aligned}
$$

- Calculando as seguintes derivadas primeiras

$$U_j(\beta,\theta) = \frac{\partial l}{\partial \mu_i}\frac{d\mu_i}{d\eta_{1i}}\frac{\partial \eta_{1i}}{\partial \beta_j},\ \ U_J(\beta,\theta) = \frac{\partial l}{\partial \phi_i}\frac{d\phi_i}{d\eta_{2i}}\frac{\partial \eta_{2i}}{\partial \theta_J}.$$

$*$ Calculando $\dfrac{\partial l}{\partial \mu_i}$.

$$\frac{\partial l}{\partial \mu_i} = \delta_i\left(\frac{-1\phi_i}{\phi_i^2}\right) - \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{-1\phi_i}{\phi_i^2}\right) = \frac{1}{\phi_i}\left[-\delta_i + \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right].$$

Iremos escrever $\dfrac{\partial l}{\partial \mu_i}$ usando μ_i° e y_i°.

$$
\begin{aligned}
\left[-\delta_1 + \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\frac{1}{\phi_i} &= \frac{1}{\phi_i}\left[-1 + \exp\left(\frac{y_i}{\phi_i}\right)\exp\left(-\frac{\mu_i}{\phi_i}\right)\right]\\
&= \frac{1}{\phi_i}\left[\frac{-\exp\left(\frac{\mu_i}{\phi_i}\right)}{\exp\left(\frac{\mu_i}{\phi_i}\right)} + \frac{\exp\left(\frac{y_i}{\phi_i}\right)}{\exp\left(\frac{\mu_i}{\phi_i}\right)}\right]\\
&= \frac{1}{\mu_i^{\circ}\phi_i}\left(y_i^{\circ} - \mu_i^{\circ}\right).
\end{aligned}
$$

Portanto, $\dfrac{\partial l}{\partial \beta_j} = \displaystyle\sum_{i=1}^{n}\left[\frac{1}{\mu_i^{\circ}\phi_i}\left(y_i^{\circ} - \delta_i\mu_i^{\circ}\right)\right]\frac{d\mu_i}{d\eta_{1i}}\frac{\partial \eta_{1i}}{\partial \beta_j},\ j = 1, ..., p.$

$*$ Calculando $\dfrac{\partial l}{\partial \phi_i}$.

$$
\begin{aligned}
\frac{\partial l}{\partial \phi_i} =& -\frac{\delta_i}{\phi_i} + \delta_i\left(\frac{-(y_i-\mu_i)}{\phi_i^2}\right) - \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{-(y_i-\mu_i)}{\phi_i^2}\right)\\
=&\frac{1}{\phi_i}\left[\delta_i\left[-1-\left(\frac{y_i-\mu_i}{\phi_i}\right)\right] + \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\\
=&v_i\frac{1}{\phi_i}.
\end{aligned}
$$

Portanto, $\dfrac{\partial l}{\partial \theta_J} = \displaystyle\sum_{i=1}^{n}\left[v_i\frac{1}{\phi_i}\right]\frac{d\phi_i}{d\eta_{2i}}\frac{\partial \eta_{2i}}{\partial \theta_J},\ \ J=1,...,q.$

- Agora iremos calcular as seguintes derivadas

$$
U_{jl} = \frac{\partial^2 l}{\partial \beta_j \partial \beta_l}, U_{jL} = \frac{\partial^2 l}{\partial \theta_J \partial \theta_L} \text{ e } U_{JL} = \frac{\partial^2 l}{\partial \beta_j \partial \theta_L}.
$$

Para obtermos as derivadas segundas acima, iremos necessitar do valor das seguintes expressões:

$*$ Iremos calcular agora $\dfrac{\partial^2 l}{\partial \mu_i^2}$.

$$
\begin{aligned}
\frac{\partial^2 l}{\partial \mu_i^2} =& \frac{\partial}{\partial \mu_i}\left[\frac{\partial l}{\partial \mu_i}\right] = \frac{\partial}{\partial \mu_i}\left[\frac{1}{\phi_i^2}\left[-\delta_i + \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\right]\\
=&\frac{1}{\phi_i^2}\left[0+\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{-1\phi_i}{\phi_i^2}\right)\right]\\
=&-\frac{1}{\phi_i^2}\exp\left(\frac{y_i-\mu_i}{\phi_i}\right).
\end{aligned}
$$

$*$ Iremos calcular agora $\dfrac{\partial^2 l}{\partial \mu_i \partial \phi_i}$.

$$
\begin{aligned}
\frac{\partial^2 l}{\partial \mu_i \partial \phi_i} =& \frac{\partial}{\partial \mu_i}\left[\frac{\partial l}{\partial \phi_i}\right] = \frac{\partial}{\partial \mu_i}\left[\frac{-\delta_i}{\phi_i} - \delta_i\left[\frac{y_i-\mu_i}{\phi_i}\right] + \exp\left[\frac{y_i-\mu_i}{\phi_i}\right]\left[\frac{y_i-\mu_i}{\phi_i^2}\right]\right]\\
=&-\delta_i\left(\frac{-1}{\phi_i^2}\right) + \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{-1}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i^2}\right) + \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{-1}{\phi_i^2}\right)\\
=&\frac{1}{\phi_i^2}\left[\delta_i - \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right) - \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right].
\end{aligned}
$$

$*$ Iremos calcular agora $\dfrac{\partial^2 l}{\partial \phi_i^2}$.

$$
\begin{aligned}
\frac{\partial^2 l}{\partial \phi_i^2} =& \frac{\partial}{\partial \phi_i}\left[\frac{\partial l}{\partial \phi_i}\right] = \frac{\partial}{\partial \phi_i}\left[\frac{-\delta_i}{\phi_i} - \delta_i\left[\frac{y_i-\mu_i}{\phi_i}\right] + \exp\left[\frac{y_i-\mu_i}{\phi_i}\right]\left[\frac{y_i-\mu_i}{\phi_i^2}\right]\right] \\
=& -\delta_i\left(\frac{-1}{\phi_i^2}\right) - \delta_i\left(\frac{-(y_i-\mu_i)2\phi_i}{\phi_i^4}\right) + \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{-(y_i-\mu_i)}{\phi_i^2}\right)\left(\frac{y_i-\mu_i}{\phi_i^2}\right) \\
&+ \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{-(y_i-\mu_i)2\phi_i}{\phi_i^4}\right) \\
=& \frac{\delta_i}{\phi_i^2} + 2\delta_i\left(\frac{y_i-\mu_i}{\phi_i^3}\right) - \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i^2}\right)^2 - 2\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i^3}\right) \\
=& \frac{1}{\phi_i^2}\left[\delta_i\left[1+2\left(\frac{y_i-\mu_i}{\phi_i}\right)\right] - \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)^2 - 2\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)\right].
\end{aligned}
$$

Portanto, obtemos as seguintes expressões para as derivadas segundas

$$
\begin{aligned}
U_{jl} =& \frac{\partial^2 l}{\partial \beta_j \partial \beta_l} = \frac{\partial}{\partial \beta_j}\left[\frac{\partial l}{\partial \beta_l}\right] = -\sum_{i=1}^{n}\frac{1}{\phi_i^2}\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2\frac{\partial \eta_{1i}}{\partial \beta_j}\frac{\partial \eta_{1i}}{\partial \beta_l} \\
&+ \sum_{i=1}^{n}\frac{1}{\phi_i}\left[-\delta_i + \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\left[\frac{d^2\mu_i}{d\eta_{1i}^2}\frac{\partial \eta_{1i}}{\partial \beta_j}\frac{\partial \eta_{1i}}{\partial \beta_l} + \frac{d\mu_i}{d\eta_{1i}}\frac{\partial^2 \eta_{1i}}{\partial \beta_j \partial \beta_l}\right].
\end{aligned}
$$

$$
\begin{aligned}
U_{jL} =& \frac{\partial^2 l}{\partial \theta_J \partial \theta_L} = \frac{\partial}{\partial \theta_J}\left[\frac{\partial l}{\partial \theta_L}\right] = \sum_{i=1}^{n}\frac{1}{\phi_i^2}\left[\delta_i - \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)\right. \\
&\left. - \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\frac{d\mu_i}{d\eta_{1i}}\frac{d\phi_i}{d\eta_{2i}}\frac{\partial \eta_{1i}}{\partial \beta_j}\frac{\partial \eta_{2i}}{\partial \theta_J}.
\end{aligned}
$$

$$
\begin{aligned}
U_{JL} =& \frac{\partial^2 l}{\partial \beta_j \partial \theta_L} = \frac{\partial}{\partial \beta_j}\left[\frac{\partial l}{\partial \theta_L}\right] = \sum_{i=1}^{n}\frac{1}{\phi_i^2}\left[\delta_i\left[1+2\left(\frac{y_i-\mu_i}{\phi_i}\right)\right] - \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)^2\right. \\
&\left. -2\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\left(\frac{d\phi_i}{d\eta_{2i}}\right)^2\frac{\partial \eta_{2i}}{\partial \theta_L}\frac{\partial \eta_{2i}}{\partial \theta_J} + \sum_{i=1}^{n}\left[\frac{-\delta_i}{\phi_i} - \delta_i\left[\frac{y_i-\mu_i}{\phi_i}\right]\right. \\
&\left. + \exp\left[\frac{y_i-\mu_i}{\phi_i}\right]\left[\frac{y_i-\mu_i}{\phi_i^2}\right]\right]\left[\frac{d^2\phi_i}{d\eta_{2i}^2}\frac{\partial \eta_{2i}}{\partial \theta_J}\frac{\partial \eta_{2i}}{\partial \theta_L} + \frac{d\phi_i}{d\eta_{2i}}\frac{\partial^2 \eta_{1i}}{\partial \theta_J \partial \theta_L}\right].
\end{aligned}
$$

- Calculando os cumulantes de segunda ordem

$$
\begin{aligned}
k'_{jl} =& E\left[-\sum_{i=1}^{n}\frac{1}{\phi_i^2}\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2\frac{\partial\eta_{1i}}{\partial\beta_j}\frac{\partial\eta_{1i}}{\partial\beta_l} + E\left[\sum_{i=1}^{n}\frac{1}{\phi_i}\left(-\delta_i+\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right)\right] \\
&\times\left[\frac{d^2\mu_i}{d\eta_{1i}^2}\frac{\partial\eta_{1i}}{\partial\beta_j}\frac{\partial\eta_{1i}}{\partial\beta_l}+\frac{d\mu_i}{d\eta_{1i}}\frac{\partial^2\eta_{1i}}{\partial\beta_j\partial\beta_l}\right] \\
=& -\sum_{i=1}^{n}\frac{1}{\phi_i^2}E\left[\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2\frac{\partial\eta_{1i}}{\partial\beta_j}\frac{\partial\eta_{1i}}{\partial\beta_l}+\sum_{i=1}^{n}\frac{1}{\phi_i}\left[E[-\delta_i]+E\left[\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\right] \\
&\times\left[\frac{d^2\mu_i}{d\eta_{1i}^2}\frac{\partial\eta_{1i}}{\partial\beta_j}\frac{\partial\eta_{1i}}{\partial\beta_l}+\frac{d\mu_i}{d\eta_{1i}}\frac{\partial^2\eta_{1i}}{\partial\beta_j\partial\beta_l}\right] \\
=& -\sum_{i=1}^{n}\frac{h_{2i}}{\phi_i^2}\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2\frac{\partial\eta_{1i}}{\partial\beta_j}\frac{\partial\eta_{1i}}{\partial\beta_l}.
\end{aligned}
$$

$$
\begin{aligned}
k'_{jL} =& E\left[\sum_{i=1}^{n}\frac{1}{\phi_i^2}\left[\delta_i-\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)-\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\right]\frac{d\mu_i}{d\eta_{1i}}\frac{d\phi_i}{d\eta_{2i}}\frac{\partial\eta_{1i}}{\partial\beta_j}\frac{\partial\eta_{2i}}{\partial\theta_J} \\
=& \sum_{i=1}^{n}\frac{1}{\phi_i^2}\left[E[\delta_i]-E\left[\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]-E\left[\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\right]\frac{d\mu_i}{d\eta_{1i}}\frac{d\phi_i}{d\eta_{2i}}\frac{\partial\eta_{1i}}{\partial\beta_j}\frac{\partial\eta_{2i}}{\partial\theta_J} \\
=& \sum_{i=1}^{n}\frac{h_{1i}-h_{4i}-h_{2i}}{\phi_i^2}\frac{d\mu_i}{d\eta_{1i}}\frac{d\phi_i}{d\eta_{2i}}\frac{\partial\eta_{1i}}{\partial\beta_j}\frac{\partial\eta_{2i}}{\partial\theta_L}.
\end{aligned}
$$

$$
\begin{aligned}
k'_{JL} =& E\left[\sum_{i=1}^{n}\frac{1}{\phi_i^2}\left[\delta_i\left[1+2\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]-\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)^2\right.\right. \\
&\left.\left.-2\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\right]\left(\frac{d\phi_i}{d\eta_{2i}}\right)^2\frac{\partial\eta_{2i}}{\partial\theta_L}\frac{\partial\eta_{2i}}{\partial\theta_J}+E\left[\sum_{i=1}^{n}\frac{1}{\phi_i}\left[\delta_i\left(-1-\left(\frac{y_i-m_i}{\phi_i}\right)\right)\right.\right. \\
&\left.\left.+\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\right]\left[\frac{d^2\phi_i}{d\eta_{2i}^2}\frac{\partial\eta_{2i}}{\partial\theta_J}\frac{\partial\eta_{2i}}{\partial\theta_L}+\frac{d\phi_i}{d\eta_{2i}}\frac{\partial^2\eta_{1i}}{\partial\theta_J\partial\theta_L}\right] \\
=& \sum_{i=1}^{n}\frac{1}{\phi_i^2}\left[E[\delta_i]+2E\left[\delta_i\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]-E\left[\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)^2\right]\right. \\
&\left.-2E\left[\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\right]\left(\frac{d\phi_i}{d\eta_{2i}}\right)^2\frac{\partial\eta_{2i}}{\partial\theta_L}\frac{\partial\eta_{2i}}{\partial\theta_J}+\sum_{i=1}^{n}\frac{1}{\phi_i}\left[E[-\delta_i]-E\left[\delta_i\left(\frac{y_i-m_i}{\phi_i}\right)\right]\right. \\
&\left.+E\left[\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\right]\left[\frac{d^2\phi_i}{d\eta_{2i}^2}\frac{\partial\eta_{2i}}{\partial\theta_J}\frac{\partial\eta_{2i}}{\partial\theta_L}+\frac{d\phi_i}{d\eta_{2i}}\frac{\partial^2\eta_{1i}}{\partial\theta_J\partial\theta_L}\right] \\
=& \sum_{i=1}^{n}\frac{h_{1i}+2h_{3i}-h_{5i}-2h_{4i}}{\phi_i^2}\left(\frac{d\phi_i}{d\eta_{2i}}\right)^2\frac{\partial\eta_{2i}}{\partial\theta_L}\frac{\partial\eta_{2i}}{\partial\theta_J}.
\end{aligned}
$$

- Iremos agora, calcular as seguintes derivadas

$$\frac{\partial^3 l}{\partial \mu_i^3}, \frac{\partial^3 l}{\partial \phi_i^3}, \frac{\partial^3 l}{\partial \mu_i^2 \partial \phi_i}, \text{ e } \frac{\partial^3 l}{\partial \phi_i^2 \partial \mu_i},$$

para podemos calcular as derivadas terceiras e os cumulantes de terceira ordem.

$*$ Calculando $\frac{\partial^3 l}{\partial \mu_i^3}$.

$$\frac{\partial^3 l}{\partial \mu_i^3} = \frac{\partial}{\partial \mu_i}\left[\frac{\partial^2}{\partial \mu_i^2}\right] = \frac{\partial}{\partial \mu_i}\left[-\frac{1}{\phi_i^2}\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right] = -\frac{1}{\phi_i^2}\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{-1\phi_i}{\phi_i^2}\right) = \frac{1}{\phi_i^3}\exp\left(\frac{y_i-\mu_i}{\phi_i}\right).$$

$*$ Calculando $\frac{\partial^3 l}{\partial \phi_i^3}$.

$$\begin{aligned}
\frac{\partial^3 l}{\partial \phi^3} =& \frac{\partial}{\partial \phi}\left[\frac{\partial^2 l}{\partial \phi^2}\right] = \frac{\partial}{\partial \phi}\left[\frac{1}{\phi_i^2}\left[\delta_i\left[1+2\left(\frac{y_i-\mu_i}{\phi_i}\right)\right] - \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)^2\right.\right.\\
&\left.\left.-2\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\right]\\
=&\delta_i\left[\frac{-2\phi_i}{\phi_i^3} + 2\left(\frac{-(y_i-\mu_i)3\phi_i^2}{\phi_i^6}\right)\right] + \left[-\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{-(y_i-\mu_i)}{\phi_i^2}\right)\left(\frac{y_i-\mu_i}{\phi_i^2}\right)^2\right.\\
&-2\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i^2}\right)\left(\frac{-(y_i-\mu_i)2\phi_i}{\phi_i^4}\right) - 2\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{-(y_i-\mu_i)3\phi_i^2}{\phi_i^6}\right)\\
&\left.-2\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{-(y_i-\mu_i)}{\phi_i^2}\right)\left(\frac{y_i-\mu_i}{\phi_i^3}\right)\right]\\
=&\frac{\delta_i}{\phi_i^3}\left[-2-6\left(\frac{y_i-\mu_i}{\phi_i}\right)\right] + \frac{1}{\phi_i^3}\left[\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)^3 + 6\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)\right.\\
&\left.+4\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)^2 + 2\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)^2\right]\\
=&\frac{1}{\phi_i^3}\left[\delta_i\left[-2-6\left(\frac{y_i-\mu_i}{\phi_i}\right)\right] + \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)^3 + 6\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)\right.\\
&\left.+6\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)^2\right].
\end{aligned}$$

$*$ Calculando $\frac{\partial^3 l}{\partial \mu_i^2 \partial \phi_i}$.

$$
\begin{aligned}
\frac{\partial^3 l}{\partial\mu_i^2\partial\phi_i} =& \frac{\partial}{\partial\mu_i}\left[\frac{\partial^2 l}{\partial\mu_i\partial\phi_i}\right] = \frac{\partial}{\partial\mu_i}\left[\frac{1}{\phi_i^2}\left[\delta_i - \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right) - \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\right]\\
=& \frac{1}{\phi_i^2}\left[-\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{-1}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right) - \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{-1}{\phi_i}\right) - \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{-1}{\phi_i}\right)\right]\\
=& \frac{1}{\phi_i^3}\left[\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right) + \exp\left(\frac{y_i-\mu_i}{\phi_i}\right) + \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\\
=& \frac{1}{\phi_i^3}\left[\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right) + 2\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right].
\end{aligned}
$$

$*$ Calculando $\dfrac{\partial^3 l}{\partial\phi_i^2\partial\mu_i}$.

$$
\begin{aligned}
\frac{\partial^3 l}{\partial\phi_i^2\partial\mu_i} =& \frac{\partial l}{\partial\phi_i}\left[\frac{\partial^2 l}{\partial\phi_i\partial\mu_i}\right] = \frac{\partial l}{\partial\phi_i}\left[\frac{1}{\phi_i^2}\left[\delta_i - \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right) - \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\right]\\
=& \left[\delta_i\left(\frac{-2}{\phi_i^3}\right) - \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{-(y_i-\mu_i)}{\phi_i^2}\right)\left(\frac{y_i-\mu_i}{\phi_i^3}\right) - \left(\frac{-2}{\phi_i^3}\right)\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right.\\
&\left. - \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{-(y_i-\mu_i)3\phi_i^2}{\phi_i^6}\right) - \frac{1}{\phi_i^2}\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{-(y_i-\mu_i)}{\phi_i^2}\right)\right]\\
=& \frac{1}{\phi_i^3}\left[-2\delta_i + \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)^2 + 3\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right) + 2\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right.\\
&\left. + \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\\
=& \frac{1}{\phi_i^3}\left[-2\delta_i + \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)^2 + 4\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right) + 2\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right].
\end{aligned}
$$

- Temos as seguintes expressões para as derivadas terceiras

$$
\begin{aligned}
U_{jlm} =& \sum_{i=1}^{n}\frac{1}{\phi_i^3}\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{d\mu_i}{d\eta_{1i}}\right)^3\frac{\partial\eta_{1i}}{\partial\beta_m}\frac{\partial\eta_{1i}}{\partial\beta_l}\frac{\partial\eta_{1i}}{\partial\beta_j} - \sum_{i=1}^{n}\frac{1}{\phi_i^2}\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\\
&\times\left[3\frac{d\mu_i}{d\eta_{1i}}\frac{d^2\mu_i}{d\eta_{1i}^2}\frac{\partial\eta_{1i}}{\partial\beta_m}\frac{\partial\eta_{1i}}{\partial\beta_l}\frac{\partial\eta_{1i}}{\partial\beta_j} + \left(\frac{d\mu_i}{d\eta_{1i}}\right)^2\left[\frac{\partial^2\eta_{1i}}{\partial\beta_m\partial\beta_l}\frac{\partial\eta_{1i}}{\partial\beta_j} + \frac{\partial^2\eta_{1i}}{\partial\beta_m\partial\beta_j}\frac{\partial\eta_{1i}}{\partial\beta_l} + \frac{\partial^2\eta_{1i}}{\partial\beta_l\partial\beta_j}\frac{\partial\eta_{1i}}{\partial\beta_m}\right]\right]\\
&+\sum_{i=1}^{n}\frac{1}{\phi_i}\left[-\delta_i + \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\left[\frac{d^3\mu_i}{d\eta_{1i}^3}\frac{\partial\eta_{1i}}{\partial\beta_m}\frac{\partial\eta_{1i}}{\partial\beta_l}\frac{\partial\eta_{1i}}{\partial\beta_j} + \frac{d^2\mu_i}{d\eta_{1i}^2}\left[\frac{\partial^2\eta_{1i}}{\partial\beta_m\partial\beta_l}\frac{\partial\eta_{1i}}{\partial\beta_l}\right.\right.\\
&\left.\left.+\frac{\partial^2\eta_{1i}}{\partial\beta_m\partial\beta_l}\frac{\partial\eta_{1i}}{\partial\beta_j} + \frac{\partial^2\eta_{1i}}{\partial\beta_l\partial\beta_j}\frac{\partial\eta_{1i}}{\partial\beta_m}\right] + \frac{d\mu_i}{d\eta_{1i}}\frac{\partial^3\eta_{1i}}{\partial\beta_m\partial\beta_l\partial\beta_j}\right].
\end{aligned}
$$

$$
\begin{aligned}
U_{jlM} =& \sum_{i=1}^{n}\frac{1}{\phi_i^3}\left[\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right) + 2\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2\frac{\partial\eta_{1i}}{\partial\beta_j}\frac{\partial\eta_{1i}}{\partial\beta_l}\frac{d\phi_i}{d\eta_{2i}}\frac{\partial\eta_{2i}}{\partial\theta_M}\\
&+\sum_{i=1}^{n}\frac{1}{\phi_i^2}\left[\delta_i - \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right) - \exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]
\end{aligned}
$$

$$\times \left[\frac{d^2\mu_i}{d\eta_{1i}^2}\frac{\partial\eta_{1i}}{\partial\beta_l}\frac{d\phi_i}{d\eta_{2i}}\frac{\partial\eta_{2i}}{\partial\theta_M}\frac{\partial\eta_{1i}}{\partial\beta_j}+\frac{d\mu_i}{d\eta_{1i}}\frac{\partial^2\eta_{1i}}{\partial\beta_j\partial\beta_l}\frac{d\phi_i}{d\eta_{2i}}\frac{\partial\eta_{2i}}{\partial\theta_M}\right].$$

$$\begin{aligned}U_{JLm} =&\sum_{i=1}^{n}\frac{1}{\phi_i^3}\left(-2\delta_i+\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)^2+4\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)\right.\\
&\left.+2\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right)\left(\frac{d\phi_i}{d\eta_{2i}}\right)^2\frac{d\mu_i}{d\eta_{1i}}\frac{\partial\eta_{2i}}{\partial\theta_J}\frac{\partial\eta_{2i}}{\partial\beta_m}\frac{\partial\eta_{2i}}{\partial\theta_L}+\sum_{i=1}^{n}\frac{1}{\phi_i^2}\left[\delta_i-\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right.\\
&\left.-\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\left[\left(\frac{d\mu_i}{d\eta_{1i}}\right)\left[\frac{d^2\phi_i}{d\eta_{2i}^2}\frac{\partial\eta_{1i}}{\partial\beta_m}\frac{\partial\eta_{2i}}{\partial\theta_J}\frac{\partial\eta_{2i}}{\partial\beta_L}+\frac{d\phi_i}{d\eta_{2i}}\frac{\partial\eta_{1i}}{\partial\beta_m}\frac{\partial^2\eta_{2i}}{\partial\theta_J\partial\theta_L}\right]\right].\end{aligned}$$

$$\begin{aligned}U_{JLM} =&\sum_{i=1}^{n}\frac{\delta_i}{\phi_i^3}\left[-2-6\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]+\sum_{i=1}^{n}\frac{1}{\phi_i^3}\left[\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)^3\right.\\
&\left.+6\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i^2}\right)^2+6\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\left(\frac{d\phi_i}{d\eta_{2i}}\right)^3\frac{\partial\eta_{2i}}{\partial\theta_M}\frac{\partial\eta_{2i}}{\partial\theta_J}\frac{\partial\eta_{2i}}{\partial\theta_L}\\
&+\sum_{i=1}^{n}\frac{\delta_i}{\phi_i^2}\left[1+2\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]+\sum_{i=1}^{n}\frac{1}{\phi_i^2}\left[-\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)^2\right.\\
&\left.-2\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\left[3\cdot\frac{d\phi_i}{d\eta_{2i}}\frac{d^2\phi_i}{d\eta_{2i}^2}\frac{\partial\eta_{2i}}{\partial\theta_M}\frac{\partial\eta_{2i}}{\partial\theta_J}\frac{\partial\eta_{2i}}{\partial\theta_L}+\left(\frac{d\phi_i}{d\eta_{2i}}\right)^2\left[\frac{\partial\eta_{2i}}{\partial\theta_M}\frac{\partial^2\eta_{2i}}{\partial\theta_L\partial\theta_J}\right.\right.\\
&\left.\left.+\frac{\partial^2\eta_{2i}}{\partial\theta_M\partial\theta_J}\frac{\partial\eta_{2i}}{\partial\theta_M}+\frac{\partial^2\eta_{2i}}{\partial\theta_M\partial\theta_L}\frac{\partial\eta_{2i}}{\partial\theta_J}\right]\right]+\sum_{i=1}^{n}\frac{\delta_i}{\phi_i}\left[-1-\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\\
&+\sum_{i=1}^{n}\frac{1}{\phi_i}\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)\left[\frac{d^3\phi_i}{d\eta_{2i}}\frac{\partial\eta_{2i}}{\partial\theta_M}\frac{\partial\eta_{2i}}{\partial\theta_L}\frac{\partial\eta_{2i}}{\partial\theta_J}+\left(\frac{d^2\phi_i}{d\eta_{2i}^2}\right)\left[\frac{\partial^2\eta_{2i}}{\partial\theta_M\partial\theta_L}\frac{\partial\eta_{2i}}{\partial\theta_J}\right.\right.\\
&\left.\left.+\frac{\partial^2\eta_{2i}}{\partial\theta_M\partial\theta_J}\frac{\partial\eta_{2i}}{\partial\theta_L}+\frac{\partial\eta_{2i}}{\partial\theta_M}\frac{\partial^2\eta_{2i}}{\partial\theta_L\partial\theta_J}\right]+\frac{d\phi_i}{d\eta_{2i}}\frac{\partial^3\eta_{2i}}{\partial\theta_M\partial\theta_L\partial\theta_J}\right].\end{aligned}$$

- Agora, iremos calcular os cumulantes de terceira ordem.

$$\begin{aligned}k'_{jlM} =&E\left[\sum_{i=1}^{n}\frac{1}{\phi_i^3}\left[\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)+2\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\right]\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2\frac{\partial\eta_{1i}}{\partial\beta_j}\frac{\partial\eta_{1i}}{\partial\beta_l}\frac{d\phi_i}{d\eta_{2i}}\frac{\partial\eta_{2i}}{\partial\theta_M}\\
&+E\left[\sum_{i=1}^{n}\frac{1}{\phi_i^2}\left[\delta_i-\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)-\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\right]\\
&\times\left[\frac{d^2\mu_i}{d\eta_{1i}^2}\frac{\partial\eta_{1i}}{\partial\beta_l}\frac{d\phi_i}{d\eta_{2i}}\frac{\partial\eta_{2i}}{\partial\theta_M}\frac{\partial\eta_{1i}}{\partial\beta_j}+\frac{d\mu_i}{d\eta_{1i}}\frac{\partial^2\eta_{1i}}{\partial\beta_j\partial\beta_l}\frac{d\phi_i}{d\eta_{2i}}\frac{\partial\eta_{2i}}{\partial\theta_M}\right]\\
=&\sum_{i=1}^{n}\frac{h_{4i}+2h_{2i}}{\phi_i^3}\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2\frac{d\phi_i}{d\eta_{2i}}(j,l,M)_i+\sum_{i=1}^{n}\frac{h_{1i}-h_{4i}-h_{2i}}{\phi_i^2}\left[\frac{d^2\mu_i}{d\eta_{1i}^2}\frac{d\phi_i}{d\eta_{2i}}(j,l,M)_i\right.\end{aligned}$$

$$
\begin{aligned}
&+\frac{d\mu_i}{d\eta_{1i}}\frac{d\phi_i}{d\eta_{2i}}(jl,M)_i\Bigg] \\
&=\sum_{i=1}^{n}\left[\frac{h_{4i}+2h_{2i}}{\phi_i^3}\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2+\frac{h_{1i}-h_{4i}-h_{2i}}{\phi_i^2}\frac{d^2\mu_i}{d\eta_{1i}^2}\right]\frac{d\phi_i}{d\eta_{2i}}(j,l,M)_i+\sum_{i=1}^{n}\frac{h_{1i}-h_{4i}-h_{2i}}{\phi_i^2} \\
&\times\frac{d\mu_i}{d\eta_{1i}}\frac{d\phi_i}{d\eta_{2i}}(jl,M)_i.
\end{aligned}
$$

$$
\begin{aligned}
k'_{jlm} =&E\left[\sum_{i=1}^{n}\frac{1}{\phi_i^3}\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\left(\frac{d\mu_i}{d\eta_{1i}}\right)^3\frac{\partial\eta_{1i}}{\partial\beta_m}\frac{\partial\eta_{1i}}{\partial\beta_l}\frac{\partial\eta_{1i}}{\partial\beta_j}+E\left[-\sum_{i=1}^{n}\frac{1}{\phi_i^2}\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right] \\
&\times\left[3\frac{d\mu_i}{d\eta_{1i}}\frac{d^2\mu_i}{d\eta_{1i}^2}\frac{\partial\eta_{1i}}{\partial\beta_m}\frac{\partial\eta_{1i}}{\partial\beta_l}\frac{\partial\eta_{1i}}{\partial\beta_j}+\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2\left[\frac{\partial^2\eta_{1i}}{\partial\beta_m\partial\beta_l}\frac{\partial\eta_{1i}}{\partial\beta_j}+\frac{\partial^2\eta_{1i}}{\partial\beta_m\partial\beta_j}\frac{\partial\eta_{1i}}{\partial\beta_l}+\frac{\partial^2\eta_{1i}}{\partial\beta_l\partial\beta_j}\frac{\partial\eta_{1i}}{\partial\beta_m}\right]\right] \\
&+E\left[\sum_{i=1}^{n}\frac{1}{\phi_i}\left[-\delta_i+\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\right]\left[\frac{d^3\mu_i}{d\eta_{1i}^3}\frac{\partial\eta_{1i}}{\partial\beta_m}\frac{\partial\eta_{1i}}{\partial\beta_l}\frac{\partial\eta_{1i}}{\partial\beta_j}+\frac{d^2\mu_i}{d\eta_{1i}^2}\left[\frac{\partial^2\eta_{1i}}{\partial\beta_m\partial\beta_l}\frac{\partial\eta_{1i}}{\partial\beta_l}\right.\right. \\
&\left.\left.+\frac{\partial^2\eta_{1i}}{\partial\beta_m\partial\beta_l}\frac{\partial\eta_{1i}}{\partial\beta_j}+\frac{\partial^2\eta_{1i}}{\partial\beta_l\partial\beta_j}\frac{\partial\eta_{1i}}{\partial\beta_m}\right]+\frac{d\mu_i}{d\eta_{1i}}\frac{\partial^3\eta_{1i}}{\partial\beta_m\partial\beta_l\partial\beta_j}\right] \\
=&\sum_{i=1}^{n}\frac{h_{2i}}{\phi_i^3}\left(\frac{d\mu_i}{d\eta_{1i}}\right)^3(j,l,m)_i-\sum_{i=1}^{n}\frac{h_{2i}}{\phi_i^2}\left[3\frac{d\mu_i}{d\eta_{1i}}\frac{d^2\mu_i}{d\eta_{1i}^2}(j,l,m)_i+\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2[(lm,j)_i+(jm,l)_i\right. \\
&\left.+(jl,m)_i]\right] \\
=&\sum_{i=1}^{n}\left[\frac{h_{2i}}{\phi_i^3}\left(\frac{d\mu_i}{d\eta_{1i}}\right)^3-\frac{3h_{2i}}{\phi_i^2}\frac{d\mu_i}{d\eta_{1i}}\frac{d^2\mu_i}{d\eta_{1i}^2}\right](j,l,m)_i+\sum_{i=1}^{n}\left[\frac{-h_{2i}}{\phi_i^2}\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2\right][(lm,j)_i+(jm,l)_i \\
&+(jl,m)_i]
\end{aligned}
$$

$$
\begin{aligned}
k'_{JLm} =&E\left[\sum_{i=1}^{n}\frac{1}{\phi_i^3}\left(-2\delta_i+\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)^2+4\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)\right.\right. \\
&\left.\left.+2\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right)\right]\left(\frac{d\phi_i}{d\eta_{2i}}\right)^2\frac{d\mu_i}{d\eta_{1i}}\frac{\partial\eta_{2i}}{\partial\theta_J}\frac{\partial\eta_{2i}}{\partial\beta_m}\frac{\partial\eta_{2i}}{\partial\theta_L}+E\left[\sum_{i=1}^{n}\frac{1}{\phi_i^2}\left[\delta_i-\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\right.\right. \\
&\left.\left.-\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\right]\left[\left(\frac{d\mu_i}{d\eta_{1i}}\right)\left[\frac{d^2\phi_i}{d\eta_{2i}^2}\frac{\partial\eta_{1i}}{\partial\beta_m}\frac{\partial\eta_{2i}}{\partial\theta_J}\frac{\partial\eta_{2i}}{\partial\beta_L}+\frac{d\phi_i}{d\eta_{2i}}\frac{\partial\eta_{1i}}{\partial\beta_m}\frac{\partial^2\eta_{2i}}{\partial\theta_J\partial\theta_L}\right]\right] \\
=&\sum_{i=1}^{n}\frac{-2h_{1i}+h_{5i}+4h_{4i}+2h_{2i}}{\phi_i^3}\left(\frac{d\phi_i}{d\eta_{2i}}\right)^2\frac{d\mu_i}{d\eta_{1i}}(J,L,m)_i+\sum_{i=1}^{n}\frac{h_{1i}-h_{4i}-h_{2i}}{\phi_i^2} \\
&\times\left[\left(\frac{d\mu_i}{d\eta_{1i}}\right)\left[\frac{d^2\phi_i}{d\eta_{2i}^2}(J,L,m)_i+\frac{d\phi_i}{d\eta_{2i}}(JL,m)_i\right]\right] \\
=&\sum_{i=1}^{n}\left[\frac{-2h_{1i}+h_{5i}+4h_{4i}+2h_{2i}}{\phi_i^3}\left(\frac{d\phi_i}{d\eta_{2i}}\right)^2+\frac{h_{1i}-h_{4i}-h_{2i}}{\phi_i^2}\frac{d^2\phi_i}{d\eta_{2i}^2}\right]\frac{d\mu_i}{d\eta_{1i}}(J,L,m)_i \\
&+\sum_{i=1}^{n}\left[\frac{h_{1i}-h_{4i}-h_{2i}}{\phi_i^2}\right]\frac{d\mu_i}{d\eta_{1i}}\frac{d\phi_i}{d\eta_{2i}}(JL,m)_i.
\end{aligned}
$$

$$
\begin{aligned}
k'_{JLM} =& E\left[\sum_{i=1}^{n} \frac{\delta_i}{\phi_i^3}\left[-2-6\left(\frac{y_i-\mu_i}{\phi_i}\right)\right] + \sum_{i=1}^{n}\frac{1}{\phi_i^3}\left[\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)^3\right.\right.\\
&\left.\left.+6\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i^2}\right)^2 + 6\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\right]\left(\frac{d\phi_i}{d\eta_{2i}}\right)^3\frac{\partial\eta_{2i}}{\partial\theta_M}\frac{\partial\eta_{2i}}{\partial\theta_J}\frac{\partial\eta_{2i}}{\partial\theta_L}\\
&+E\left[\sum_{i=1}^{n}\frac{\delta_i}{\phi_i^2}\left[1+2\left(\frac{y_i-\mu_i}{\phi_i}\right)\right] + \sum_{i=1}^{n}\frac{1}{\phi_i^2}\left[-\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)^2\right.\right.\\
&\left.\left.-2\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\right]\left[3\cdot\frac{d\phi_i}{d\eta_{2i}}\frac{d^2\phi_i}{d\eta_{2i}^2}\frac{\partial\eta_{2i}}{\partial\theta_M}\frac{\partial\eta_{2i}}{\partial\theta_J}\frac{\partial\eta_{2i}}{\partial\theta_L} + \left(\frac{d\phi_i}{d\eta_{2i}}\right)^2\left[\frac{\partial\eta_{2i}}{\partial\theta_M}\frac{\partial^2\eta_{2i}}{\partial\theta_L\partial\theta_J}\right.\right.\\
&\left.\left.+\frac{\partial^2\eta_{2i}}{\partial\theta_M\partial\theta_J}\frac{\partial\eta_{2i}}{\partial\theta_M}+\frac{\partial^2\eta_{2i}}{\partial\theta_M\partial\theta_L}\frac{\partial\eta_{2i}}{\partial\theta_J}\right]\right] + E\left[\sum_{i=1}^{n}\frac{\delta_i}{\phi_i}\left[-1-\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\right.\\
&\left.+\sum_{i=1}^{n}\frac{1}{\phi_i}\exp\left(\frac{y_i-\mu_i}{\phi_i}\right)\left(\frac{y_i-\mu_i}{\phi_i}\right)\right]\left[\frac{d^3\phi_i}{d\eta_{2i}}\frac{\partial\eta_{2i}}{\partial\theta_M}\frac{\partial\eta_{2i}}{\partial\theta_L}\frac{\partial\eta_{2i}}{\partial\theta_J} + \left(\frac{d^2\phi_i}{d\eta_{2i}^2}\right)\left[\frac{\partial^2\eta_{2i}}{\partial\theta_M\partial\theta_L}\frac{\partial\eta_{2i}}{\partial\theta_J}\right.\right.\\
&\left.\left.+\frac{\partial^2\eta_{2i}}{\partial\theta_M\partial\theta_J}\frac{\partial\eta_{2i}}{\partial\theta_L}+\frac{\partial\eta_{2i}}{\partial\theta_M}\frac{\partial^2\eta_{2i}}{\partial\theta_L\partial\theta_J}\right]\frac{d\phi_i}{d\eta_{2i}}\frac{\partial\eta_{2i}}{\partial\theta_J}+\frac{\partial^3\eta_{2i}}{\partial\theta_M\partial\theta_L\partial\theta_J}\right]\\
=&\sum_{i=1}^{n}\frac{-2h_{1i}-6h_{3i}+h_{6i}+6h_{5i}+6h_{4i}}{\phi_i^3}\left(\frac{d\phi_i}{d\eta_{2i}}\right)^3(J,L,M)_i + \sum_{i=1}^{n}\frac{h_{1i}+2h_{3i}-h_{5i}-2h_{4i}}{\phi_i^2}\\
&\times\left[3\cdot\frac{d\phi_i}{d\eta_{2i}}\frac{d^2\phi_i}{d\eta_{2i}^2}(J,L,M)_i + \left(\frac{d\phi_i}{d\eta_{2i}}\right)^2\left[(JL,M)_i+(JM,L)_i+(LM,J)_i\right]\right]\\
=&\sum_{i=1}^{n}\left[\frac{-2h_{1i}-6h_{3i}+h_{6i}+6h_{5i}+6h_{4i}}{\phi_i^3}\left(\frac{d\phi_i}{d\eta_{2i}}\right)^3 + \frac{3\left[h_{1i}+2h_{3i}-h_{5i}-2h_{4i}\right]}{\phi_i^2}\frac{d\phi_i}{d\eta_{2i}}\frac{d^2\phi_i}{d\eta_{2i}^2}\right]\\
&\times(J,L,M)_i + \sum_{i=1}^{n}\frac{h_{1i}+2h_{3i}-h_{5i}-2h_{4i}}{\phi_i^2}\left(\frac{d\phi_i}{d\eta_{2i}}\right)^2\left[(LM,J)_i+(JM,L)_i+(JL,M)_i\right].
\end{aligned}
$$

Como todas as esperanças envolvidas nos cumulantes de segunda e terceira ordem são finitas, podemos fazer a correção do viés.

- As derivadas dos elementos da Matriz de Informação de Fisher, são

$$
\begin{aligned}
k_{jl}^{'(m)} = -\sum_{i=1}^{n}\frac{1}{\phi_i^2}\frac{\partial h_{2i}}{\partial\mu_i}\left(\frac{d\mu_i}{d\eta_{1i}}\right)^3(j,l,m)_i - 2\sum_{i=1}^{n}\frac{h_{2i}}{\phi_i^2}\frac{d\mu_i}{d\eta_{1i}}\frac{d^2\mu_i}{d\eta_{1i}^2}(j,l,m)_i\\
-\sum_{i=1}^{n}\frac{h_{2i}}{\phi_i^2}\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2\left[(jm,l)_i+(lm,j)_i\right],
\end{aligned}
$$

$$
k_{jl}^{'(M)} = -\sum_{i=1}^{n}\left[\frac{\partial}{\partial\phi_i}\left(\frac{h_{2i}}{\phi_i^2}\right)\right]\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2\frac{d\phi_i}{d\eta_{2i}}(j,l,M)_i,
$$

$$k_{JL}^{'(m)} = \sum_{i=1}^{n} \left[\frac{\partial}{\partial \mu_i} \left(\frac{h_{1i} + 2h_{3i} - h_{5i} - 2h_{4i}}{\phi_i^2} \right) \right] \left(\frac{d\phi_i}{d\eta_{2i}} \right)^2 \frac{d\mu_i}{d\eta_{1i}} (J, L, m)_i,$$

$$k_{JL}^{'(M)} = \sum_{i=1}^{n} \left[\frac{\partial}{\partial \phi_i} \left(\frac{h_{1i} + 2h_{3i} - h_{5i} - 2h_{4i}}{\phi_i^2} \right) \left(\frac{d\phi_i}{d\eta_{2i}} \right)^3 + \frac{2(h_{1i} + 2h_{3i} - h_{5i} - 2h_{4i})}{\phi_i^2} \frac{d\phi_i}{d\eta_{2i}} \frac{d^2\phi_i}{d\eta_{2i}^2} \right]$$
$$\times (J, L, M)_i + \sum_{i=1}^{n} \frac{h_{1i} + 2h_{3i} - h_{5i} - 2h_{4i}}{\phi_i^2} \left(\frac{d\phi_i}{d\eta_{2i}} \right)^2 [(LM, J)_i + (JM, L)_i],$$

$$k_{jL}^{'(m)} = \sum_{i=1}^{n} \left[\frac{\partial}{\partial \mu_i} \left(\frac{-h_{4i}}{\phi_i^2} \right) \left(\frac{d\mu_i}{d\eta_{1i}} \right)^2 + \frac{-h_{4i}}{\phi_i^2} \frac{d^2\mu_i}{d\eta_{1i}^2} \right] \frac{d\phi_i}{d\eta_{2i}} (j, L, m)_i + \sum_{i=1}^{n} \frac{-h_{4i}}{\phi_i^2} \frac{d\phi_i}{d\eta_{2i}} \frac{d\mu_i}{d\eta_{1i}} (jm, L)_i,$$

e

$$k_{jL}^{'(M)} = \sum_{i=1}^{n} \left[\frac{\partial}{\partial \phi_i} \left(\frac{-h_{4i}}{\phi_i^2} \right) \left(\frac{d\phi_i}{d\eta_{2i}} \right)^2 + \frac{-h_{4i}}{\phi_i^2} \frac{d^2\phi_i}{d\eta_{2i}^2} \right] \frac{d\mu_i}{d\eta_{1i}} (j, L, M) + \sum_{i=1}^{n} \frac{-h_{4i}}{\phi_i^2} \frac{d\mu_i}{d\eta_{1i}} \frac{d\phi_i}{d\eta_{2i}} (LM, j)_i.$$

- Definiremos agora algumas expressões

$$w_{1i}^{'} = \left[\left(\frac{-1}{\phi_i^2} \frac{\partial h_{2i}}{\partial \mu_i} - \frac{h_{2i}}{2\phi_i^3} \right) \left(\frac{d\mu_i}{d\eta_{1i}} \right)^3 - \frac{h_{2i}}{2\phi_i^2} \frac{d\mu_i}{d\eta_{1i}} \frac{d^2\mu_i}{d\eta_{1i}^2} \right],$$

$$w_{2i}^{'} = \frac{1}{2} \left[\frac{-h_{2i}}{\phi_i^2} \left(\frac{d\mu_i}{d\eta_{1i}} \right)^2 \right],$$

$$w_{3i}^{'} = \left[\frac{-h_{4i}}{2\phi_i^2} \frac{d^2\mu_i}{d\eta_{1i}^2} + \left(\frac{\partial}{\partial \mu_i} \left(\frac{-h_{4i}}{\phi_i^2} \right) - \frac{h_{4i} + 2h_{2i}}{2\phi_i^3} \right) \left(\frac{d\mu_i}{d\eta_{1i}} \right)^2 \right] \frac{d\phi_i}{d\eta_{2i}},$$

$$w_{4i}^{'} = \frac{1}{2} \left[\frac{-h_{4i}}{\phi_i^2} \frac{d\mu_i}{d\eta_{1i}} \frac{d\phi_i}{d\eta_{2i}} \right],$$

$$w'_{5i} = \left[\left(\frac{\partial}{\partial\phi_i}\left(\frac{-h_{2i}}{\phi_i^2}\right) - \frac{2h_{2i}+h_{4i}}{2\phi_i^3}\right)\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2 + \frac{h_{4i}}{2\phi_i^2}\frac{d^2\mu_i}{d\eta_{1i}^2}\right]\frac{d\phi_i}{d\eta_{2i}},$$

$$w'_{6i} = \left[\left(\frac{\partial}{\partial\mu_i}\left(\frac{(h_{1i}+2h_{3i}-h_{5i}-2h_{4i})}{\phi_i^2}\right) - \frac{(h_{5i}+4h_{4i})}{2\phi_i^3}\right)\left(\frac{d\phi_i}{d\eta_{2i}}\right)^2 + \frac{(h_{4i})}{2\phi_i^2}\frac{d^2\phi_i}{d\eta_{2i}^2}\right]\frac{d\mu_i}{d\eta_{1i}},$$

$$w'_{7i} = \left[\left(\frac{\partial}{\partial\phi_i}\left(\frac{-h_{4i}}{\phi_i^2}\right) - \frac{(h_{5i}+4h_{4i})}{2\phi_i^3}\right)\left(\frac{d\phi_i}{d\eta_{2i}}\right)^2 + \frac{(-h_{4i})}{2\phi_i^2}\frac{d^2\phi_i}{d\eta_{2i}^2}\right]\frac{d\mu_i}{d\eta_{1i}},$$

$$w'_{8i} = \left[\left(\frac{\partial}{\partial\phi_i}\left(\frac{-2h_{4i}+2h_{3i}-h_{5i}+h_{1i}}{\phi_i^2}\right) - \frac{(-2h_{1i}-6h_{3i}+h_{6i}+6h_{5i}+6h_{4i})}{2\phi_i^3}\right)\left(\frac{d\phi_i}{d\eta_{2i}}\right)^3 + \frac{(-2h_{4i}+2h_{3i}-h_{5i}+h_{1i})}{2\phi_i^2}\frac{d^2\phi_i}{d\eta_{2i}^2}\frac{d\phi_i}{d\eta_{2i}}\right],$$

e

$$w'_{9i} = \frac{1}{2}\left[\frac{(h_{1i}+2h_{3i}-h_{5i}-2h_{4i})}{\phi_i^2}\left(\frac{d\phi_i}{d\eta_{2i}}\right)^2\right].$$

- Calcularemos agora, algumas expressões que serão necessárias para obtermos a correção do viés.

$$k_{jl}^{'(m)} - \frac{1}{2}k'_{jlm} = \sum_{i=1}^{n} w'_{1i}(j,l,m)_i + \sum_{i=1}^{n} w'_{2i}[(jm,l)_i + (lm,j)_i - (jl,m)_i],$$

$$k_{Jl}^{'(m)} - \frac{1}{2}k'_{Jlm} = \sum_{i=1}^{n} w'_{3i}(J,l,m)_i + \sum_{i=1}^{n} w'_{4i}(lm,J)_i,$$

$$k_{jL}^{'(m)} - \frac{1}{2}k'_{jLm} = \sum_{i=1}^{n} w'_{3i}(j,L,m)_i + \sum_{i=1}^{n} w'_{4i}(jm,L)_i,$$

$$k_{jl}^{\prime(M)} - \frac{1}{2}k_{jlM}^{\prime} = \sum_{i=1}^{n} w_{5i}^{\prime}(j,l,M)_i - \sum_{i=1}^{n} w_{4i}^{\prime}(jl,M)_i,$$

$$k_{JL}^{\prime(m)} - \frac{1}{2}k_{JLm}^{\prime} = \sum_{i=1}^{n} w_{6i}^{\prime}(J,L,m)_i - \sum_{i=1}^{n} w_{4i}^{\prime}(JL,m)_i,$$

$$k_{Jl}^{\prime(M)} - \frac{1}{2}k_{JlM}^{\prime} = \sum_{i=1}^{n} w_{7i}^{\prime}(J,l,M)_i + \sum_{i=1}^{n} w_{4i}^{\prime}(JM,l)_i,$$

$$k_{jL}^{\prime(M)} - \frac{1}{2}k_{jLM}^{\prime} = \sum_{i=1}^{n} w_{7i}^{\prime}(j,L,M)_i + \sum_{i=1}^{n} w_{4i}^{\prime}(LM,j)_i,$$

e

$$k_{JL}^{\prime(M)} - \frac{1}{2}k_{JLM}^{\prime} = \sum_{i=1}^{n} w_{8i}^{\prime}(J,L,M)_i + \sum_{i=1}^{n} w_{9i}^{\prime}[(LM,J)_i + (JM,L)_i - (JL,M)_i].$$

- Calculando as expressões da correções do viés para $B_1(\widehat{\beta})$ e $B_1(\widehat{\theta})$ e colocando na forma matricial.

$*$ Para $B_1(\widehat{\beta})$

$$\begin{aligned}
\sum_{j,l,m} k_1^{aj}k_1^{lm}\left\{k_{jl}^{\prime(m)} - \frac{1}{2}k_{jlm}^{\prime}\right\} &= \sum_{j,l,m} k_1^{aj}k_1^{lm}\left\{\sum_{i=1}^{n} w_{1i}^{\prime}(j,l,m)_i + \sum_{i=1}^{n} w_{2i}^{\prime}[(jm,l)_i + (lm,j)_i - (jl,m)_i]\right\} \\
&= \sum_{i=1}^{n} w_{1i}^{\prime}\sum_{j} k_1^{aj}(j)_i \sum_{l,m} k_1^{lm}(l,m)_i + \sum_{i=1}^{n} w_{2i}^{\prime}\sum_{l,m} k_1^{lm}(l)_i \sum_{j} k_1^{aj}(jm)_i \\
&+ \sum_{i=1}^{n} w_{2i}^{\prime}\sum_{j} k_1^{aj}(j)_i \sum_{l,m} k_1^{lm}(lm)_i - \sum_{i=1}^{n} w_{2i}^{\prime}\sum_{l,m} k_1^{lm}(m)_i \sum_{j} k_1^{aj}(jl)_i \\
&= \sum_{i=1}^{n} w_{1i}^{\prime}\sum_{j} k_1^{aj}(j)_i \sum_{l,m} k_1^{lm}(l,m)_i + \sum_{i=1}^{n} w_{2i}^{\prime}\sum_{j} k_1^{aj}(j)_i \sum_{l,m} k_1^{lm}(lm)_i \\
&= e_a^T K_1^{\beta\beta}\widetilde{X}^T W_1^{\prime} Z_{\beta d}^{\prime} 1_{n\times 1} + e_a^T K_1^{\beta\beta}\widetilde{X}^T W_2^{\prime} D_{\beta}^{\prime} 1_{n\times 1},
\end{aligned}$$

$$\sum_{J,l,m} k_1^{aJ} k_1^{lm} \left\{ k_{Jl}^{'(m)} - \frac{1}{2} k_{Jlm}^{'} \right\} = \sum_{J,l,m} k_1^{aJ} k_1^{lm} \left\{ \sum_{i=1}^{n} w_{3i}^{'}(J,l,m)_i + \sum_{i=1}^{n} w_{4i}^{'}(lm,J)_i \right\}$$

$$= \sum_{i=1}^{n} w_{3i}^{'} \sum_{J} k_1^{aJ}(J)_i \sum_{l,m} k_1^{lm}(l,m)_i + \sum_{i=1}^{n} w_{4i}^{'} \sum_{J} k_1^{aJ}(J)_i \sum_{l,m} k_1^{lm}(lm)_i$$

$$= e_a^T K_1^{\beta\theta} \widetilde{S}^T W_3^{'} Z_{\beta d}^{'} 1_{n\times 1} + e_a^T K_1^{\beta\theta} \widetilde{S}^T W_4^{'} D_{\beta}^{'} 1_{n\times 1},$$

$$\sum_{j,L,m} k_1^{aj} k_1^{Lm} \left\{ k_{jL}^{'(m)} - \frac{1}{2} k_{jLm}^{'} \right\} = \sum_{j,L,m} k_1^{aj} k_1^{Lm} \left\{ \sum_{i=1}^{n} w_{3i}^{'}(j,L,m)_i + \sum_{i=1}^{n} w_{4i}^{'}(jm,L)_i \right\}$$

$$= \sum_{i=1}^{n} w_{3i}^{'} \sum_{j} k_1^{aj}(j)_i \sum_{L,m} k_1^{Lm}(L,m)_i + \sum_{i=1}^{n} w_{4i}^{'} \sum_{L,m} k_1^{Lm}(L)_i \sum_{j} k_1^{aj}(jm)_i$$

$$= e_a^T K_1^{\beta\beta} \widetilde{X}^T W_3^{'} Z_{\beta\theta d}^{'} 1_{n\times 1} + L_{1,a}^{'},$$

$$\sum_{j,l,M} k_1^{aj} k_1^{lM} \left\{ k_{jl}^{'(M)} - \frac{1}{2} k_{jlM}^{'} \right\} = \sum_{j,l,M} k_1^{aj} k_1^{lM} \left\{ \sum_{i=1}^{n} w_{5i}^{'}(j,l,M)_i - \sum_{i=1}^{n} w_{4i}^{'}(jl,M)_i \right\}$$

$$= \sum_{i=1}^{n} w_{5i}^{'} \sum_{j} k_1^{aj}(j)_i \sum_{l,M} k_1^{lM}(l,M)_i - \sum_{i=1}^{n} w_{4i}^{'} \sum_{l,M} k_1^{lM}(M)_i \sum_{j} k_1^{aj}(jl)_i$$

$$= e_a^T K_1^{\beta\beta} \widetilde{X}^T W_5^{'} Z_{\beta\theta d}^{'} 1_{n\times 1} - L_{1,a}^{'},$$

$$\sum_{J,L,m} k_1^{aJ} k_1^{Lm} \left\{ k_{JL}^{'(m)} - \frac{1}{2} k_{JLm}^{'} \right\} = \sum_{J,L,m} k_1^{aJ} k_1^{Lm} \left\{ \sum_{i=1}^{n} w_{6i}^{'}(J,L,m)_i - \sum_{i=1}^{n} w_{4i}^{'}(JL,m)_i \right\}$$

$$= \sum_{i=1}^{n} w_{6i}^{'} \sum_{J} k_1^{aJ}(J)_i \sum_{L,m} k_1^{Lm}(L,m)_i - \sum_{i=1}^{n} w_{4i}^{'} \sum_{L,m} k_1^{Lm}(m)_i \sum_{J} k_1^{aJ}(JL)_i$$

$$= e_a^T K_1^{\beta\theta} \widetilde{S}^T W_6^{'} Z_{\beta\theta d}^{'} 1_{n\times 1} - L_{2,a}^{'},$$

$$\sum_{J,l,M} k_1^{aJ} k_1^{lM} \left\{ k_{Jl}^{'(M)} - \frac{1}{2} k_{JlM}^{'} \right\} = \sum_{J,l,M} k_1^{aJ} k_1^{lM} \left\{ \sum_{i=1}^{n} w_{7i}^{'}(J,l,M)_i + \sum_{i=1}^{n} w_{4i}^{'}(JM,l)_i \right\}$$

$$= \sum_{i=1}^{n} w_{7i}^{'} \sum_{J} k_1^{aJ}(J)_i \sum_{l,M} k_1^{lM}(l,M)_i + \sum_{i=1}^{n} w_{4i}^{'} \sum_{l,M} k_1^{lM}(l)_i \sum_{J} k_1^{aJ}(JM)_i$$

$$= e_a^T K_1^{\beta\theta} \widetilde{S}^T W_7^{'} Z_{\beta\theta d}^{'} 1_{n\times 1} + L_{2,a}^{'},$$

$$\sum_{j,L,M} k_1^{aj} k_1^{LM} \left\{ k_{jL}^{'(M)} - \frac{1}{2} k_{jLM}^{'} \right\} = \sum_{j,L,M} k_1^{aj} k_1^{LM} \left\{ \sum_{i=1}^{n} w_{7i}^{'}(j,L,M)_i + \sum_{i=1}^{n} w_{4i}^{'}(LM,j)_i \right\}$$

$$= \sum_{i=1}^{n} w_{7i}^{'} \sum_{j} k_1^{aj}(j)_i \sum_{L,M} k_1^{LM}(L,M)_i + \sum_{i=1}^{n} w_{4i}^{'} \sum_{j} k_1^{aj}(j)_i \sum_{L,M} k_1^{LM}(LM)_i$$

$$= e_a^T K_1^{\beta\beta} \widetilde{X}^T W_7^{'} Z_{\theta d}^{'} 1_{n\times 1} + e_a^T K_1^{\beta\beta} \widetilde{X}^T W_4^{'} D_{\theta}^{'} 1_{n\times 1},$$

e

$$\sum_{J,L,M} k_1^{aJ}k_1^{LM}\left\{k_{JL}^{'(M)}-\frac{1}{2}k_{JLM}^{'}\right\} = \sum_{J,L,M} k_1^{aJ}k_1^{LM}\left\{\sum_{i=1}^{n} w_{8i}^{'}(J,L,M)_i+\sum_{i=1}^{n} w_{9i}^{'}[(JM,L)_i+(LM,J)_i-(JL,M)_i]\right\}$$
$$=\sum_{i=1}^{n} w_{8i}^{'}\sum_{J}k_1^{aJ}(J)_i\sum_{L,M}k_1^{LM}(L,M)_i+\sum_{i=1}^{n} w_{9i}^{'}\sum_{J}k_1^{aJ}(J)_i\sum_{L,M}k_1^{LM}(LM)_i$$
$$+\sum_{i=1}^{n} w_{9i}^{'}\sum_{L,M}k_1^{LM}(L)_i\sum_{J}k_1^{aJ}(JM)_i-\sum_{i=1}^{n} w_{9i}^{'}\sum_{L,M}k_1^{LM}(M)_i\sum_{J}k_1^{aJ}(JL)_i$$
$$=\sum_{i=1}^{n} w_{8i}^{'}\sum_{J}k_1^{aJ}(J)_i\sum_{L,M}k_1^{LM}(L,M)_i+\sum_{i=1}^{n} w_{9i}^{'}\sum_{J}k_1^{aJ}(J)_i\sum_{L,M}k_1^{LM}(LM)_i$$
$$=e_a^TK_1^{\beta\theta}\widetilde{S}^TW_8^{'}Z_{\theta d}^{'}1_{n\times 1}+e_a^TK_1^{\beta\theta}\widetilde{S}^TW_9^{'}D_{\theta}^{'}1_{n\times 1}.$$

$*$ Para $B_1(\widehat{\theta})$

$$\sum_{j,l,m} k_1^{Aj}k_1^{lm}\left\{k_{jl}^{'(m)}-\frac{1}{2}k_{jlm}^{'}\right\} = \sum_{j,l,m} k_1^{Aj}k_1^{lm}\left\{\sum_{i=1}^{n} w_{1i}^{'}(j,l,m)_i+\sum_{i=1}^{n} w_{2i}^{'}[(jm,l)_i+(lm,j)_i-(jl,m)_i]\right\}$$
$$=\sum_{i=1}^{n} w_{1i}^{'}\sum_{j}k_1^{Aj}(j)_i\sum_{l,m}k_1^{lm}(l,m)_i+\sum_{i=1}^{n} w_{2i}^{'}\sum_{l,m}k_1^{lm}(l)_i\sum_{j}k_1^{Aj}(jm)_i$$
$$+\sum_{i=1}^{n} w_{2i}^{'}\sum_{j}k_1^{Aj}(j)_i\sum_{l,m}k_1^{lm}(lm)_i-\sum_{i=1}^{n} w_{2i}^{'}\sum_{l,m}k_1^{lm}(m)_i\sum_{j}k_1^{Aj}(jl)_i$$
$$=\sum_{i=1}^{n} w_{1i}^{'}\sum_{j}k_1^{Aj}(j)_i\sum_{l,m}k_1^{lm}(l,m)_i+\sum_{i=1}^{n} w_{2i}^{'}\sum_{j}k_1^{Aj}(j)_i\sum_{l,m}k_1^{lm}(lm)_i$$
$$=e_A^TK_1^{\theta\beta}\widetilde{X}^TW_1^{'}Z_{\beta d}^{'}1_{n\times 1}+e_A^TK_1^{\theta\beta}\widetilde{X}^TW_2^{'}D_{\beta}^{'}1_{n\times 1},$$

$$\sum_{J,l,m} k_1^{AJ}k_1^{lm}\left\{k_{Jl}^{'(m)}-\frac{1}{2}k_{Jlm}^{'}\right\} = \sum_{J,l,m} k_1^{AJ}k_1^{lm}\left\{\sum_{i=1}^{n} w_{3i}^{'}(J,l,m)_i+\sum_{i=1}^{n} w_{4i}^{'}(lm,J)_i\right\}$$
$$=\sum_{i=1}^{n} w_{3i}^{'}\sum_{J}k_1^{AJ}(J)_i\sum_{l,m}k_1^{lm}(l,m)_i+\sum_{i=1}^{n} w_{4i}^{'}\sum_{J}k_1^{AJ}(J)_i\sum_{l,m}k_1^{lm}(lm)_i$$
$$=e_A^TK_1^{\theta\theta}\widetilde{S}^TW_3^{'}Z_{\beta d}^{'}1_{n\times 1}+e_A^TK_1^{\theta\theta}\widetilde{S}^TW_4^{'}D_{\beta}^{'}1_{n\times 1},$$

$$\sum_{j,L,m} k_1^{Aj}k_1^{Lm}\left\{k_{jL}^{'(m)}-\frac{1}{2}k_{jLm}^{'}\right\} = \sum_{j,L,m} k_1^{aj}k_1^{Lm}\left\{\sum_{i=1}^{n} w_{3i}^{'}(j,L,m)_i+\sum_{i=1}^{n} w_{4i}^{'}(jm,L)_i\right\}$$
$$=\sum_{i=1}^{n} w_{3i}^{'}\sum_{j}k_1^{Aj}(j)_i\sum_{L,m}k_1^{Lm}(L,m)_i+\sum_{i=1}^{n} w_{4i}^{'}\sum_{L,m}k_1^{Lm}(L)_i\sum_{j}k_1^{Aj}(jm)_i$$
$$=e_A^TK_1^{\theta\beta}\widetilde{X}^TW_3^{'}Z_{\beta\theta d}^{'}1_{n\times 1}+L_{1,A}^{'},$$

$$\sum_{j,l,M} k_1^{Aj} k_1^{lM} \left\{ k_{jl}^{'(M)} - \frac{1}{2} k_{jlM}^{'} \right\} = \sum_{j,l,M} k_1^{Aj} k_1^{lM} \left\{ \sum_{i=1}^{n} w_{5i}^{'} (j,l,M)_i - \sum_{i=1}^{n} w_{4i}^{'} (jl,M)_i \right\}$$

$$= \sum_{i=1}^{n} w_{5i}^{'} \sum_{j} k_1^{Aj} (j)_i \sum_{l,M} k_1^{lM} (l,M)_i - \sum_{i=1}^{n} w_{4i}^{'} \sum_{l,M} k_1^{lM} (M)_i \sum_{j} k_1^{Aj} (jl)_i$$

$$= e_A^T K_1^{\theta\beta} \widetilde{X}^T W_5^{'} Z_{\beta\theta d}^{'} 1_{n\times 1} - L_{1,A}^{'},$$

$$\sum_{J,L,m} k_1^{AJ} k_1^{Lm} \left\{ k_{JL}^{'(m)} - \frac{1}{2} k_{JLm}^{'} \right\} = \sum_{J,L,m} k_1^{AJ} k_1^{Lm} \left\{ \sum_{i=1}^{n} w_{6i}^{'} (J,L,m)_i - \sum_{i=1}^{n} w_{4i}^{'} (JL,m)_i \right\}$$

$$= \sum_{i=1}^{n} w_{6i}^{'} \sum_{J} k_1^{AJ} (J)_i \sum_{L,m} k_1^{Lm} (L,m)_i - \sum_{i=1}^{n} w_{4i}^{'} \sum_{L,m} k_1^{Lm} (m)_i \sum_{J} k_1^{AJ} (JL)_i$$

$$= e_A^T K_1^{\theta\theta} \widetilde{S}^T W_6^{'} Z_{\beta\theta d}^{'} 1_{n\times 1} - L_{2,A}^{'},$$

$$\sum_{J,l,M} k_1^{AJ} k_1^{lM} \left\{ k_{Jl}^{'(M)} - \frac{1}{2} k_{JlM}^{'} \right\} = \sum_{J,l,M} k_1^{AJ} k_1^{lM} \left\{ \sum_{i=1}^{n} w_{7i}^{'} (J,l,M)_i + \sum_{i=1}^{n} w_{4i}^{'} (JM,l)_i \right\}$$

$$= \sum_{i=1}^{n} w_{7i}^{'} \sum_{J} k_1^{AJ} (J)_i \sum_{l,M} k_1^{lM} (l,M)_i + \sum_{i=1}^{n} w_{4i}^{'} \sum_{l,M} k_1^{lM} (l)_i \sum_{J} k_1^{AJ} (JM)_i$$

$$= e_A^T K_1^{\theta\theta} \widetilde{S}^T W_7^{'} Z_{\beta\theta d}^{'} 1_{n\times 1} + L_{2,A}^{'},$$

$$\sum_{j,L,M} k_1^{Aj} k_1^{LM} \left\{ k_{jL}^{'(M)} - \frac{1}{2} k_{jLM}^{'} \right\} = \sum_{j,L,M} k_1^{Aj} k_1^{LM} \left\{ \sum_{i=1}^{n} w_{7i}^{'} (j,L,M)_i + \sum_{i=1}^{n} w_{4i}^{'} (LM,j)_i \right\}$$

$$= \sum_{i=1}^{n} w_{7i}^{'} \sum_{j} k_1^{Aj} (j)_i \sum_{L,M} k_1^{LM} (L,M)_i + \sum_{i=1}^{n} w_{4i}^{'} \sum_{j} k_1^{Aj} (j)_i \sum_{L,M} k_1^{LM} (LM)_i$$

$$= e_A^T K_1^{\theta\beta} \widetilde{X}^T W_7^{'} Z_{\theta d}^{'} 1_{n\times 1} + e_A^T K_1^{\theta\beta} \widetilde{X}^T W_4^{'} D_{\theta}^{'} 1_{n\times 1},$$

e

$$\sum_{J,L,M} k_1^{AJ} k_1^{LM} \left\{ k_{JL}^{'(M)} - \frac{1}{2} k_{JLM}^{'} \right\} = \sum_{J,L,M} k_1^{AJ} k_1^{LM} \left\{ \sum_{i=1}^{n} w_{8i}^{'} (J,L,M)_i + \sum_{i=1}^{n} w_{9i}^{'} [(JM,L)_i + (LM,J)_i - (JL,M)_i] \right\}$$

$$= \sum_{i=1}^{n} w_{8i}^{'} \sum_{J} k_1^{AJ} (J)_i \sum_{L,M} k_1^{LM} (L,M)_i + \sum_{i=1}^{n} w_{9i}^{'} \sum_{J} k_1^{AJ} (J)_i \sum_{L,M} k_1^{LM} (LM)_i$$

$$+ \sum_{i=1}^{n} w_{9i}^{'} \sum_{L,M} k_1^{LM} (L)_i \sum_{J} k_1^{AJ} (JM)_i - \sum_{i=1}^{n} w_{9i}^{'} \sum_{L,M} k_1^{LM} (M)_i \sum_{J} k_1^{AJ} (JL)_i$$

$$= \sum_{i=1}^{n} w_{8i}^{'} \sum_{J} k_1^{AJ} (J)_i \sum_{L,M} k_1^{LM} (L,M)_i + \sum_{i=1}^{n} w_{9i}^{'} \sum_{J} k_1^{AJ} (J)_i \sum_{L,M} k_1^{LM} (LM)_i$$

$$= e_A^T K_1^{\theta\theta} \widetilde{S}^T W_8^{'} Z_{\theta d}^{'} 1_{n\times 1} + e_A^T K_1^{\theta\theta} \widetilde{S}^T W_9^{'} D_{\theta}^{'} 1_{n\times 1}.$$

Apêndice C

- Calculando as esperanças que serão usadas para o modelo de Gumbel com censura tipo II.

Para fazermos a correção do viés da distribuição de Gumbel com censura tipo II, nos baseamos na estatística de ordem, onde mostraremos abaixo as expressões para calcular as esperanças envolvidas. Mais antes disto, como estamos tratando da estatística de ordem, sabemos por David (1981), que a forma de calcular a densidade de uma estatística de ordem e dada por:

$$f_{s:n} = \frac{n!}{(s-1)!(n-s)!} \left[F(x)\right]^{s-1} \left[1 - F(x)\right]^{n-s} f(x), \text{ com } -\infty < x < \infty,$$

onde $f(x)$ e $F(x)$ denotam respectivamente, a função de densidade e a função acumulada.

Utilizando a expressão acima, iremos obter a seguinte função de densidade

$$\begin{aligned}
f_{s:n} =& \frac{n!}{(s-1)!(n-s)!} \left[1 - \exp(-\exp(x))\right]^{s-1} \left[\exp(-\exp(x))\right]^{n-s} f(x) \\
=& \frac{n!}{(s-1)!(n-s)!} \exp(-(n-s)\exp(x)) \sum_{a=0}^{s-1} \binom{s-1}{a} (-1)^a \exp(-a\exp(x)) f(x) \\
=& \frac{n!}{(s-1)!(n-s)!} \sum_{a=0}^{s-1} \binom{s-1}{a} (-1)^a \exp(-(n-s+a)\exp(x)) f(x) \\
=& \frac{n!}{(s-1)!(n-s)!} \sum_{a=0}^{s-1} \binom{s-1}{a} (-1)^a \exp(x - (n-s+1+a)\exp(x)).
\end{aligned}$$

Considere $Z_{s:n_i} = \left(\frac{y_{(s,n_i)} - \mu_i}{\phi_i} \right)$, com $s = 1, ..., n_i$ e $i = 1, ..., k$.

Utilizando esta notação, temos

$$E\left[Z_{s:n}\right] = \frac{n!}{(s-1)!(n-s)!} \sum_{a=0}^{s-1} (-1)^{a+1} \binom{s-1}{a} \frac{\gamma + \log(n-s+1+a)}{n-s+1+a},$$

está expressão pode ser encontrada em Balakrishnan and Chan (1992).

Nosso objetivo agora e calcular a esperança da seguinte expressão $E[Z_{s:n}^d \exp(Z_{s:n})]$, onde $d = 0, 1, 2, 3$.

Baseado no artigo de Lieblein (1953), temos

$$
\begin{aligned}
E[Z_{s:n}^d \exp(Z_{s:n})] &= \int_{-\infty}^{\infty} x^d \exp(x) f_{s:n}\, dx \\
&= \frac{n!}{(s-1)!(n-s)!} \sum_{a=0}^{s-1} (-1)^a \binom{s-1}{a} \int_{-\infty}^{\infty} x^d \exp(x) \exp(x - (n-s+1+a)\exp(x))\, dx \\
&= \frac{n!}{(s-1)!(n-s)!} \sum_{a=0}^{s-1} (-1)^a \binom{s-1}{a} \int_{-\infty}^{\infty} x^d \exp(2x - (n-s+1+a)\exp(x))\, dx \\
&= \frac{n!}{(s-1)!(n-s)!} \sum_{a=0}^{s-1} (-1)^a \binom{s-1}{a} g_d(n-s+1+a),
\end{aligned}
$$

onde $g_d(n-s+1+a) = \int_{-\infty}^{\infty} x^d \exp(2x - (n-s+1+a)\exp(x)))\, dx$.

Usando o método de substituição na função $g_d(c)$, tomando $v = \exp(x)$ então $dv = \exp(x)\, dx$. Assim

$$g_d(c) = \int_0^{\infty} v\, (\log(v))^d \exp(-cv)\, dv.$$

Para k inteiros não negativos, temos que

$$g_d(c) = \frac{d^d}{dt^d} \int_0^{\infty} v^t \exp(-cv)\, dv \Big|_{t=1} = \frac{d^d}{dt^d} \left[\Gamma(t+1) c^{-(t+1)} \right] \Big|_{t=1},$$

onde $\Gamma(\cdot)$ denota a função gamma completa, que e dada por

$$\Gamma(t) = \int_0^{\infty} x^{t-1}\, e^{-x}\, dx.$$

Agora, iremos calcular $g_0(c), g_1(c), g_2(c), g_3(c)$, que serão necessários para obtenção da esperanças envolvidas nesta dissertação.

* $g_0(c)$

$$g_0(c) = \int_0^{\infty} v \exp(-cv)\, dv.$$

Usando o método de integral por partes, tome $u = v$ então $du = dv$ e $dv = \exp(-cv)\, dv$ então $v = \frac{-1}{c}\exp(-cv)$, assim

$$
\begin{aligned}
g_0(c) &= \frac{-v}{c} \exp(-cv) \Big|_0^{\infty} + \frac{1}{c} \int_0^{\infty} \exp(-cv)\, dv \\
&= \frac{1}{c} \left[\frac{-1}{c} \exp(-cv) \Big|_0^{\infty} \right] = \frac{1}{c^2}.
\end{aligned}
$$

$*$ $g_1(c)$

$$\begin{aligned} g_1(c) =& \frac{d}{dt}\left[\frac{\Gamma(t+1)}{c^{(t+1)}}\right]\Big|_{t=1} \\ =& \frac{\Gamma^{(1)}(t+1)c^{(t+1)} - \Gamma(t+1)c^{(t+1)}\log(c)}{(c^{(t+1)})^2}\Big|_{t=1} \\ =& \frac{\Gamma^{(1)}(2) - \Gamma(2)\log(c)}{c^{(2)}} \\ =& \frac{-\gamma + 1 - \log(c)}{c^2}. \end{aligned}$$

$*$ $g_2(c)$

$$\begin{aligned} g_2(c) =& \frac{d}{dt}\left[\frac{\Gamma^{(1)}(t+1) - \Gamma(t+1)\log(c)}{c^{(t+1)}}\right]\Big|_{t=1} \\ =& (1/(6c^2)(6\gamma^2 + 12\log(c)\gamma + \pi^2 + 6\log(c)^2 - 12\gamma - 12\log(c))). \end{aligned}$$

$*$ $g_3(c)$

$$\begin{aligned} g_3(c) =& \frac{d}{dt}\left[\frac{(\Gamma^{(2)}(t+1) - \Gamma^{(1)}(t+1)\log(c)) - (\Gamma^{(1)}(t+1) - \Gamma(t+1)\log(c))\log(c)}{c^{(t+1)}}\right]\Big|_{t=1} \\ =& -1/(2c^2)(2\gamma^3 + 6\log(c)\gamma^2 + \pi^2\gamma + 6\log(c)^2\gamma + \log(c)\pi^2 + 2\log(c)^3 - 6\gamma^2 \\ & - 12\log(c)\gamma - \pi^2 - 6\log(c)^2 + 4\xi(3)), \end{aligned}$$

usamos, $\Gamma(2) = 1$, $\Gamma^{(2)}(2) = \frac{\pi^2}{6} - \gamma(2-\gamma)$, $\Gamma^{(3)}(2) = (1-\gamma)\frac{\pi^2}{2} + \gamma^2(3-\gamma) - 2\xi(3)$, onde $\xi(3) \approx 1.202$.

Dessa forma, obtemos

$$q_1 = E[r] = r,$$

$$q_{2si} = E[Z_{s:n_i}] = \frac{n_i!}{(s-1)!(n_i-s)!}\sum_{a=0}^{s-1}(-1)^{a+1}\binom{s-1}{a}\frac{\gamma + \log(n-s+1+a)}{n-s+1+a},$$

$$q_{3si} = E[\exp(Z_{s:n_i})] = \frac{n_i!}{(s-1)!(n_i-s)!}\sum_{a=0}^{s-1}(-1)^{a}\binom{s-1}{a}g_0(n-s+1+a),$$

$$q_{4ri} = E[\exp(Z_{r:n_i})] = \frac{n_i!}{(r-1)!(n_i-r)!}\sum_{a=0}^{r-1}(-1)^{a}\binom{r-1}{a}g_0(n-r+1+a),$$

$$q_{5si} = E[Z^1_{s:n_i}\exp(Z_{s:n_i})] = \frac{n_i!}{(s-1)!(n_i-s)!}\sum_{a=0}^{s-1}(-1)^{a}\binom{s-1}{a}g_1(n-s+1+a),$$

$$q_{6ri} = E[Z^1_{r:n_i}\exp(Z_{r:n_i})] = \frac{n_i!}{(r-1)!(n_i-r)!}\sum_{a=0}^{r-1}(-1)^{a}\binom{r-1}{a}g_1(n-r+1+a),$$

$$q_{7si} = E[Z_{s:n_i}^2 \exp(Z_{s:n_i})] = \frac{n_i!}{(s-1)!(n_i-s)!}\sum_{a=0}^{s-1}(-1)^a\binom{s-1}{a}g_2(n-s+1+a),$$

$$q_{8ri} = E[Z_{r:n_i}^2 \exp(Z_{r:n_i})] = \frac{n_i!}{(r-1)!(n_i-r)!}\sum_{a=0}^{r-1}(-1)^a\binom{r-1}{a}g_2(n-r+1+a),$$

$$q_{9si} = E[Z_{s:n_i}^3 \exp(Z_{s:n_i})] = \frac{n_i!}{(s-1)!(n_i-s)!}\sum_{a=0}^{s-1}(-1)^a\binom{s-1}{a}g_3(n-s+1+a)$$

e

$$q_{10ri} = E[Z_{r:n_i}^3 \exp(Z_{r:n_i})] = \frac{n_i!}{(r-1)!(n_i-r)!}\sum_{a=0}^{r-1}(-1)^a\binom{r-1}{a}g_3(n-r+1+a).$$

- Calculando as seguintes derivadas primeiras

$$U_j(\beta,\theta) = \frac{\partial l}{\partial \mu_i}\frac{d\mu_i}{d\eta_{1i}}\frac{\partial \eta_{1i}}{\partial \beta_j}, \;\; U_J(\beta,\theta) = \frac{\partial l}{\partial \phi_i}\frac{d\phi_i}{d\eta_{2i}}\frac{\partial \eta_{2i}}{\partial \theta_J}.$$

$*$ Calculando $\frac{\partial l}{\partial \mu_i}$.

$$\begin{aligned}\frac{\partial l}{\partial \mu_i} =& r\left(\frac{-1\phi_i}{\phi_i^2}\right) - \sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{-1\phi_i}{\phi_i^2}\right) - (n_i-r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{-1\phi_i}{\phi_i}\right)\\ =& \frac{-r}{\phi_i} + \sum_{s=1}^{r}\frac{1}{\phi_i}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right) + (n_i-r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\frac{1}{\phi_i}\\ =& \frac{1}{\phi_i}\left[-r + \sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right) + (n_i-r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right].\end{aligned}$$

Portanto,

$$\frac{\partial l}{\partial \beta_j} = \sum_{i=1}^{k}\left[\frac{1}{\phi_i}\left[-r + \sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right) + (n_i-r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\frac{d\mu_i}{d\eta_{1i}}\frac{\partial \eta_{1i}}{\partial \beta_j},$$

com $j = 1, ..., p$.

$*$ Calculando $\frac{\partial l}{\partial \phi_i}$.

$$
\begin{aligned}
\frac{\partial l}{\partial \phi_i} =& -\frac{r}{\phi_i} + \sum_{i=1}^{r}\left(\frac{-(y_{(s,n_i)}-\mu_i)}{\phi_i^2}\right) - \sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{-(y_{(s,n_i)}-\mu_i)}{\phi_i^2}\right)\\
&+ (n_i - r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i^2}\right)\\
=& -\frac{r}{\phi_i} - \sum_{i=1}^{r}\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i^2}\right) + \sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i^2}\right)\\
&+ (n_i - r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i^2}\right)\\
=&\frac{1}{\phi_i}\left[-r + \sum_{i=1}^{r}\left[-\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right) + \exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]\right.\\
&\left.+(n_i - r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right].
\end{aligned}
$$

Portanto,

$$
\begin{aligned}
\frac{\partial l}{\partial \theta_J} =& \sum_{i=1}^{k}\left[\frac{1}{\phi_i}\left[-r + \sum_{i=1}^{r}\left[-\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right) + \exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]\right.\right.\\
&\left.\left.+(n_i - r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\frac{d\phi_i}{d\eta_{2i}}\frac{\partial \eta_{2i}}{\partial \theta_J},
\end{aligned}
$$

com $J = 1, ..., q$.

- Agora iremos calcular as seguintes derivadas

$$
U_{jl} = \frac{\partial^2 l}{\partial \beta_j \partial \beta_l}\,, U_{jL} = \frac{\partial^2 l}{\partial \theta_J \partial \theta_L} \text{ e } U_{JL} = \frac{\partial^2 l}{\partial \beta_j \partial \theta_L}.
$$

Para obtermos as derivadas segundas acima, iremos necessitar do valor das seguintes expressões:

∗ Iremos calcular agora $\dfrac{\partial^2 l}{\partial \mu_i^2}$.

$$
\begin{aligned}
\frac{\partial^2 l}{\partial \mu_i^2} =& \frac{\partial}{\partial \mu_i}\left[\frac{\partial l}{\partial \mu_i}\right] = \frac{\partial}{\partial \mu_i}\left[\frac{1}{\phi_i}\left[-r + \sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right) + (n_i - r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\\
=&\frac{1}{\phi_i}\left[0 + \sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{-1\phi_i}{\phi_i^2}\right) + (n_i - r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{-1\phi_i}{\phi_i^2}\right)\right]\\
=&\frac{1}{\phi_i^2}\left[-\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right) - (n_i - r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right].
\end{aligned}
$$

$*$ Iremos calcular agora $\dfrac{\partial^2 l}{\partial \mu_i \partial \phi_i}$.

$$
\begin{aligned}
\frac{\partial^2 l}{\partial \mu_i \partial \phi_i} =& \frac{\partial}{\partial \mu_i}\left[\frac{\partial l}{\partial \phi_i}\right] = \frac{\partial}{\partial \mu_i}\left[\frac{1}{\phi_i}\left[-r - \sum_{s=1}^{r}\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right) + \sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right.\right.\\
&\left.\left.+(n_i-r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\\
=&\frac{r}{\phi_i^2} - \sum_{s=1}^{r}\left(\frac{1}{\phi_i^2}\right)\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right) - \sum_{s=1}^{r}\frac{1}{\phi_i^2}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{-(y_{(s,n_i)}-\mu_i)}{\phi_i^2}\right)\\
&-\frac{(n_i-r)}{\phi_i^2}\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right) - \frac{(n_i-r)}{\phi_i^2}\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\\
=&\frac{1}{\phi_i^2}\left[r - \sum_{s=1}^{r}\left[\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right) + \exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]\right.\\
&\left.-(n_i-r)\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i}-\mu_i)}{\phi_i}\right) + \exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right].
\end{aligned}
$$

$*$ Iremos calcular agora $\dfrac{\partial^2 l}{\partial \phi_i^2}$.

$$
\begin{aligned}
\frac{\partial^2 l}{\partial \phi_i^2} =& \frac{\partial}{\partial \phi_i}\left[\frac{\partial l}{\partial \phi_i}\right] = \frac{\partial}{\partial \phi_i}\left[\frac{1}{\phi_i}\left[-r - \sum_{s=1}^{r}\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right) + \sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right.\right.\\
&\left.\left.+(n_i-r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\\
=&\frac{r}{\phi_i^2} + 2\sum_{s=1}^{r}\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right) - \sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)^2\frac{1}{\phi_i^2}\\
&-\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\frac{2}{\phi_i^2} - \frac{(n_i-r)}{\phi_i^2}\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)^2\\
&-2\frac{(n_i-r)}{\phi_i^2}\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\\
=&\frac{1}{\phi_i^2}\left[r + \sum_{s=1}^{r}\left[-2\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right) - \exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)^2\right.\right.\\
&\left.+2\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right] - (n_i-r)\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)^2\right.\\
&\left.\left.+2\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right].
\end{aligned}
$$

Portanto, obtemos as seguintes expressões para as derivadas segundas

$$
\begin{aligned}
U_{jl} =& \frac{\partial^2 l}{\partial\beta_j \partial\beta_l} = \frac{\partial}{\partial\beta_j}\left[\frac{\partial l}{\partial\beta_l}\right]\\
=& \sum_{i=1}^{k}\left[\frac{1}{\phi_i^2}\left[-\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)-(n_i-r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2\frac{\partial\eta_{1i}}{\partial\beta_j}\frac{\partial\eta_{1i}}{\partial\beta_l}\\
&+\sum_{i=1}^{k}\left[\frac{1}{\phi_i}\left[-r+\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)+(n_i-r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\\
&\times\left[\frac{d^2\mu_i}{d\eta_{1i}^2}\frac{\partial\eta_{1i}}{\partial\beta_j}\frac{\partial\eta_{1i}}{\partial\beta_l}+\frac{d\mu_i}{d\eta_{1i}}\frac{\partial^2\eta_{1i}}{\partial\beta_j\partial\beta_l}\right].
\end{aligned}
$$

$$
\begin{aligned}
U_{jL} =& \frac{\partial^2 l}{\partial\theta_J \partial\theta_L} = \frac{\partial}{\partial\theta_J}\left[\frac{\partial l}{\partial\theta_L}\right]\\
=& \sum_{i=1}^{k}\left[\frac{1}{\phi_i^2}\left[r-\sum_{s=1}^{r}\left[\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)+\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]\right.\right.\\
&\left.\left.-(n_i-r)\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i}-\mu_i)}{\phi_i}\right)+\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\right]\frac{d\mu_i}{d\eta_{1i}}\frac{d\phi_i}{d\eta_{2i}}\frac{\partial\eta_{1i}}{\partial\beta_j}\frac{\partial\eta_{2i}}{\partial\theta_J}.
\end{aligned}
$$

$$
\begin{aligned}
U_{JL} =& \frac{\partial^2 l}{\partial\beta_j \partial\theta_L} = \frac{\partial}{\partial\beta_j}\left[\frac{\partial l}{\partial\theta_L}\right]\\
=& \sum_{i=1}^{k}\left[\frac{1}{\phi_i^2}\left[r+\sum_{s=1}^{r}\left[-2\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)-\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)^2\right.\right.\right.\\
&\left.\left.\left.+2\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]-(n_i-r)\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)^2+2\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\right]\\
&\times\left(\frac{d\phi_i}{d\eta_{2i}}\right)^2\frac{\partial\eta_{2i}}{\partial\theta_L}\frac{\partial\eta_{2i}}{\partial\theta_J}+\sum_{i=1}^{k}\left[\frac{1}{\phi_i}\left[-r+\sum_{i=1}^{r}\left[-\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)+\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]\right.\right.\\
&\left.\left.+(n_i-r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\left[\frac{d^2\phi_i}{d\eta_{2i}^2}\frac{\partial\eta_{2i}}{\partial\theta_J}\frac{\partial\eta_{2i}}{\partial\theta_L}+\frac{d\phi_i}{d\eta_{2i}}\frac{\partial^2\eta_{1i}}{\partial\theta_J\partial\theta_L}\right].
\end{aligned}
$$

- Calculando os cumulantes de segunda ordem

$$
\begin{aligned}
k_{jl}^{''} =&E\left[\sum_{i=1}^{k}\left[\frac{1}{\phi_i^2}\left[-\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)-(n_i-r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\right]\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2\frac{\partial\eta_{1i}}{\partial\beta_j}\frac{\partial\eta_{1i}}{\partial\beta_l}\\
&+E\left[\sum_{i=1}^{k}\left[\frac{1}{\phi_i}\left[-r+\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)+(n_i-r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\right]\\
&\times\left[\frac{d^2\mu_i}{d\eta_{1i}^2}\frac{\partial\eta_{1i}}{\partial\beta_j}\frac{\partial\eta_{1i}}{\partial\beta_l}+\frac{d\mu_i}{d\eta_{1i}}\frac{\partial^2\eta_{1i}}{\partial\beta_j\partial\beta_l}\right]\\
=&\sum_{i=1}^{k}\left[\frac{1}{\phi_i^2}\left[E\left[-\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]-(n_i-r)E\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\right]\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2\frac{\partial\eta_{1i}}{\partial\beta_j}\frac{\partial\eta_{1i}}{\partial\beta_l}\\
&+\sum_{i=1}^{k}\left[\frac{1}{\phi_i}\left[E[-r]+\sum_{s=1}^{r}E\left[\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]+(n_i-r)E\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\right]\\
&\times\left[\frac{d^2\mu_i}{d\eta_{1i}^2}\frac{\partial\eta_{1i}}{\partial\beta_j}\frac{\partial\eta_{1i}}{\partial\beta_l}+\frac{d\mu_i}{d\eta_{1i}}\frac{\partial^2\eta_{1i}}{\partial\beta_j\partial\beta_l}\right]\\
=&\sum_{i=1}^{k}\frac{1}{\phi_i^2}\left[-\sum_{s=1}^{r}q_{3si}-(n_i-r)q_{4ri}\right]\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2\frac{\partial\eta_{1i}}{\partial\beta_j}\frac{\partial\eta_{1i}}{\partial\beta_l}.
\end{aligned}
$$

$$
\begin{aligned}
k_{jL}^{''} =&E\left[\sum_{i=1}^{k}\left[\frac{1}{\phi_i^2}\left[r-\sum_{s=1}^{r}\left[\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)+\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]\right.\right.\right.\\
&\left.\left.\left.-(n_i-r)\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i}-\mu_i)}{\phi_i}\right)+\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\right]\right]\\
&\times\frac{d\mu_i}{d\eta_{1i}}\frac{d\phi_i}{d\eta_{2i}}\frac{\partial\eta_{1i}}{\partial\beta_j}\frac{\partial\eta_{2i}}{\partial\theta_L}\\
=&\sum_{i=1}^{k}\left[\frac{1}{\phi_i^2}\left[E[r]-\sum_{s=1}^{r}\left[E\left[\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]+E\left[\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\right.\right.\\
&\left.\left.-(n_i-r)\left[E\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i}-\mu_i)}{\phi_i}\right)\right]+E\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\right]\right]\\
&\times\frac{d\mu_i}{d\eta_{1i}}\frac{d\phi_i}{d\eta_{2i}}\frac{\partial\eta_{1i}}{\partial\beta_j}\frac{\partial\eta_{2i}}{\partial\theta_L}\\
=&\sum_{i=1}^{k}\frac{1}{\phi_i^2}\left[q_1-\sum_{s=1}^{r}(q_{3si}+q_{5si})-(n_i-r)(q_{6ri}+q_{4ri})\right]\frac{d\mu_i}{d\eta_{1i}}\frac{d\phi_i}{d\eta_{2i}}\frac{\partial\eta_{1i}}{\partial\beta_j}\frac{\partial\eta_{2i}}{\partial\theta_L}.
\end{aligned}
$$

$$
\begin{aligned}
k_{JL}^{''} =&E\left[\sum_{i=1}^{k}\left[\frac{1}{\phi_i^2}\left[r+2\sum_{s=1}^{r}\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)-\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)^2\right.\right.\right.\\
&-2\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)-(n_i-r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)^2\\
&\left.\left.\left.-2(n_i-r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\left(\frac{d\phi_i}{d\eta_{2i}}\right)^2\frac{\partial\eta_{2i}}{\partial\theta_L}\frac{\partial\eta_{2i}}{\partial\theta_J}+\sum_{i=1}^{k}\left[\frac{1}{\phi_i}\left[-r-\sum_{i=1}^{r}\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right.\right.\right.\\
&\left.\left.+\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)+(n_i-r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]
\end{aligned}
$$

$$
\begin{aligned}
&\times\left[\frac{d^2\phi_i}{d\eta_{2i}^2}\frac{\partial\eta_{2i}}{\partial\theta_J}\frac{\partial\eta_{2i}}{\partial\theta_L}+\frac{d\phi_i}{d\eta_{2i}}\frac{\partial^2\eta_{1i}}{\partial\theta_J\partial\theta_L}\right]\\
=&\sum_{i=1}^{k}\left[\frac{1}{\phi_i^2}\left[E[r]+2\sum_{s=1}^{r}E\left[\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]-\sum_{s=1}^{r}E\left[\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)^2\right]\right.\right.\\
&-2\sum_{s=1}^{r}E\left[\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]-(n_i-r)E\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)^2\right]\\
&\left.\left.-2(n_i-r)E\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\right]\left(\frac{d\phi_i}{d\eta_{2i}}\right)^2\frac{\partial\eta_{2i}}{\partial\theta_L}\frac{\partial\eta_{2i}}{\partial\theta_J}\\
&+\sum_{i=1}^{k}\left[\frac{1}{\phi_i}\left[E[-r]-\sum_{i=1}^{r}E\left[\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]+\sum_{s=1}^{r}E\left[\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]\right.\right.\\
&\left.\left.+(n_i-r)E\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\right]\left[\frac{d^2\phi_i}{d\eta_{2i}^2}\frac{\partial\eta_{2i}}{\partial\theta_J}\frac{\partial\eta_{2i}}{\partial\theta_L}+\frac{d\phi_i}{d\eta_{2i}}\frac{\partial^2\eta_{1i}}{\partial\theta_J\partial\theta_L}\right]\\
=&\sum_{i=1}^{k}\frac{1}{\phi_i^2}\left[q_1+\sum_{s=1}^{r}(2q_{2si}-q_{7si}-2q_{5si})-(n_i-r)(q_{8ri}+2q_{6ri})\right]\left(\frac{d\phi_i}{d\eta_{2i}}\right)^2\frac{\partial\eta_{2i}}{\partial\theta_L}\frac{\partial\eta_{2i}}{\partial\theta_J}.
\end{aligned}
$$

- Iremos agora, calcular as seguintes derivadas

$$\frac{\partial^3 l}{\partial\mu_i^3},\ \frac{\partial^3 l}{\partial\phi_i^3},\ \frac{\partial^3 l}{\partial\mu_i^2\partial\phi_i},\ \text{e}\ \frac{\partial^3 l}{\partial\phi_i^2\partial\mu_i},$$

para podemos calcular as derivadas terceiras e os cumulantes de terceira ordem.

$*$ Calculando $\frac{\partial^3 l}{\partial\mu_i^3}$.

$$
\begin{aligned}
\frac{\partial^3 l}{\partial\mu_i^3}=&\frac{\partial}{\partial\mu_i}\left[\frac{\partial^2 l}{\partial\mu_i^2}\right]=\frac{\partial}{\partial\mu_i}\left[\frac{1}{\phi_i^2}\left[-\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)-(n_i-r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\\
=&\frac{1}{\phi_i^2}\left[-\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{-1\phi_i}{\phi_i^2}\right)-(n_i-r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{-1\phi_i}{\phi_i^2}\right)\right]\\
=&\frac{1}{\phi_i^3}\left[\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)+(n_i-r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right].
\end{aligned}
$$

$*$ Calculando $\frac{\partial^3 l}{\partial\phi_i^3}$.

$$
\begin{aligned}
\frac{\partial^3 l}{\partial\phi^3}=&\frac{\partial}{\partial\phi}\left[\frac{\partial^2 l}{\partial\phi^2}\right]=\frac{\partial}{\partial\phi}\left[\frac{1}{\phi_i^2}\left[r+2\sum_{s=1}^{r}\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)-\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)^2\right.\right.\\
&-2\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)-(n_i-r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)^2\\
&\left.\left.-2(n_i-r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{-2r}{\phi_i^3}-6\sum_{s=1}^{r}\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i^4}\right)+6\frac{(n_i-r)}{\phi^3}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i^2}\right)\\
&+4\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i^2}\right)^2\frac{1}{\phi^3}+2\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i^2}\right)^2\frac{1}{\phi^3}\\
&+6\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i^2}\right)\frac{1}{\phi^3}+\frac{(n_i-r)}{\phi^3}\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i^2}\right)^3\\
&+6\frac{(n_i-r)}{\phi^3}\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i^2}\right)^2+\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i^2}\right)^3\\
=&\frac{1}{\phi_i^3}\left[-2r+\sum_{s=1}^{r}\left[-6\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)+6\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right.\right.\\
&\left.+6\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)^2+\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)^3\right]\\
&+(n_i-r)\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)^3+6\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)^2\right.\\
&\left.\left.+6\exp\left(\frac{y_{(r,n)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right].
\end{aligned}
$$

$*$ Calculando $\dfrac{\partial^3 l}{\partial\mu_i^2\partial\phi_i}$.

$$
\begin{aligned}
\frac{\partial^3 l}{\partial\mu_i^2\partial\phi_i}=&\frac{\partial}{\partial\mu_i}\left[\frac{\partial^2 l}{\partial\mu_i\partial\phi_i}\right]=\frac{\partial}{\partial\mu_i}\left[\frac{1}{\phi_i^2}\left[r-\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)-\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right.\right.\\
&\left.\left.-(n_i-r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i}-\mu_i)}{\phi_i}\right)-(n_i-r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\\
=&\frac{1}{\phi_i^2}\left[0-\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{-(y_{(s,n_i)}-\mu_i)}{\phi_i^2}\right)-\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{-1}{\phi_i}\right)\right.\\
&-\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{-1}{\phi_i}\right)+\frac{(n_i-r)}{\phi_i}\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)+\frac{(n_i-r)}{\phi_i}\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\\
&\left.+\frac{(n_i-r)}{\phi_i}\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i}-\mu_i)}{\phi_i}\right)\right]\\
=&\frac{1}{\phi_i^3}\left[\sum_{s=1}^{r}\left[2\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)+\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]\right.\\
&\left.+(n_i-r)\left[2\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)+\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i}-\mu_i)}{\phi_i}\right)\right]\right].
\end{aligned}
$$

$*$ Calculando $\dfrac{\partial^3 l}{\partial\phi_i^2\partial\mu_i}$.

$$
\begin{aligned}
\frac{\partial^3 l}{\partial\phi_i^2\partial\mu_i} =& \frac{\partial l}{\partial\phi_i}\left[\frac{\partial^2 l}{\partial\phi_i\partial\mu_i}\right] = \frac{\partial l}{\partial\phi_i}\left[\frac{1}{\phi_i^2}\left[r-\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)-\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right.\right.\\
&\left.\left.-(n_i-r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i}-\mu_i)}{\phi_i}\right)-(n_i-r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\\
=&\sum_{s=1}^{r}\frac{-2}{\phi_i^3}+\sum_{s=1}^{r}\frac{1}{\phi_i^3}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)^2+\frac{2}{\phi_i^3}\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\\
&+\frac{2}{\phi_i^3}\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)+2\frac{(n_i-r)}{\phi_i^3}\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\\
&+\frac{(n_i-r)}{\phi_i^3}\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i}-\mu_i)}{\phi_i}\right)^2+\frac{2}{\phi_i^3}\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\\
&+2\frac{(n_i-r)}{\phi_i^3}\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i}-\mu_i)}{\phi_i}\right)+2\frac{(n_i-r)}{\phi_i^3}\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\\
=&\frac{1}{\phi_i^3}\left[\sum_{s=1}^{r}\left[-2+\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)^2+4\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right.\right.\\
&\left.+2\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]+(n_i-r)\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)^2+2\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right.\\
&\left.\left.+4\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i}-\mu_i)}{\phi_i}\right)\right]\right].
\end{aligned}
$$

- Temos as seguintes expressões para as derivadas terceiras

$$
\begin{aligned}
U_{jlm} =& \sum_{i=1}^{k}\left[\frac{1}{\phi_i^3}\left[\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)+(n_i-r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\left(\frac{d\mu_i}{d\eta_{1i}}\right)^3\frac{\partial\eta_{1i}}{\partial\beta_m}\frac{\partial\eta_{1i}}{\partial\beta_l}\frac{\partial\eta_{1i}}{\partial\beta_j}\\
&+\sum_{i=1}^{k}\left[\frac{1}{\phi_i^2}\left[-\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)-(n_i-r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\\
&\times\left[3\frac{d\mu_i}{d\eta_{1i}}\frac{d^2\mu_i}{d\eta_{1i}^2}\frac{\partial\eta_{1i}}{\partial\beta_m}\frac{\partial\eta_{1i}}{\partial\beta_l}\frac{\partial\eta_{1i}}{\partial\beta_j}+\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2\left[\frac{\partial^2\eta_{1i}}{\partial\beta_m\partial\beta_l}\frac{\partial\eta_{1i}}{\partial\beta_j}+\frac{\partial^2\eta_{1i}}{\partial\beta_m\partial\beta_j}\frac{\partial\eta_{1i}}{\partial\beta_l}+\frac{\partial^2\eta_{1i}}{\partial\beta_l\partial\beta_j}\frac{\partial\eta_{1i}}{\partial\beta_m}\right]\right]\\
&+\sum_{i=1}^{k}\left[\frac{1}{\phi_i}\left[-r+\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)+(n_i-r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\left[\frac{d^3\mu_i}{d\eta_{1i}^3}\frac{\partial\eta_{1i}}{\partial\beta_m}\frac{\partial\eta_{1i}}{\partial\beta_l}\frac{\partial\eta_{1i}}{\partial\beta_j}\right.\\
&\left.+\frac{d^2\mu_i}{d\eta_{1i}^2}\left[\frac{\partial^2\eta_{1i}}{\partial\beta_m\partial\beta_l}\frac{\partial\eta_{1i}}{\partial\beta_l}+\frac{\partial^2\eta_{1i}}{\partial\beta_m\partial\beta_l}\frac{\partial\eta_{1i}}{\partial\beta_j}+\frac{\partial^2\eta_{1i}}{\partial\beta_l\partial\beta_j}\frac{\partial\eta_{1i}}{\partial\beta_m}\right]+\frac{d\mu_i}{d\eta_{1i}}\frac{\partial^3\eta_{1i}}{\partial\beta_m\partial\beta_l\partial\beta_j}\right].
\end{aligned}
$$

$$
\begin{aligned}
U_{jlM} =& \sum_{i=1}^{k}\left[\frac{1}{\phi_i^3}\left[\sum_{s=1}^{r}\left[2\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)+\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]+(n_i-r)\left[2\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right.\right.\right.\\
&\left.\left.\left.+\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i}-\mu_i)}{\phi_i}\right)\right]\right]\right]\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2\frac{\partial\eta 1i}{\partial\beta_j}\frac{\partial\eta_{1i}}{\partial\beta_l}\frac{d\phi_i}{d\eta_{2i}}\frac{\partial\eta_{2i}}{\partial\theta_M}+\sum_{i=1}^{k}\left[\frac{1}{\phi_i^2}\right.[r\\
&-\sum_{s=1}^{r}\left[\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)+\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]-(n_i-r)\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right.\\
&\left.\left.\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i}-\mu_i)}{\phi_i}\right)\right]\right]\right]\left[\frac{d^2\mu_i}{d\eta_{1i}^2}\frac{\partial\eta_{1i}}{\partial\beta_l}\frac{d\phi_i}{d\eta_{2i}}\frac{\partial\eta 2i}{\partial\theta_M}\frac{\partial\eta 1i}{\partial\beta_j}+\frac{d\mu_i}{d\eta_{1i}}\frac{\partial^2\eta_{1i}}{\partial\beta_j\partial\beta_l}\frac{d\phi_i}{d\eta_{2i}}\frac{\partial\eta_{2i}}{\partial\theta_M}\right].
\end{aligned}
$$

$$
\begin{aligned}
U_{JLm} =& \sum_{i=1}^{k}\left[\frac{1}{\phi_i^3}\left[\sum_{s=1}^{r}\left[-2+\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)^2+4\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right.\right.\right.\\
&+2\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\Bigg]+(n_i-r)\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)^2+2\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right.\\
&\left.\left.\left.+4\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i}-\mu_i)}{\phi_i}\right)\right]\right]\right]\left(\frac{d\phi_i}{d\eta_{2i}}\right)^2\frac{d\mu_i}{d\eta_{1i}}\frac{\partial\eta_{2i}}{\partial\theta_J}\frac{\partial\eta_{2i}}{\partial\beta_m}\frac{\partial\eta_{2i}}{\partial\theta_L}\\
&+\sum_{i=1}^{k}\left[\frac{1}{\phi_i^2}\left[r-\sum_{s=1}^{r}\left[\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)+\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]\right.\right.\\
&\left.\left.-(n_i-r)\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i}-\mu_i)}{\phi_i}\right)+\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\right]\\
&\times\left[\left(\frac{d\mu_i}{d\eta_{1i}}\right)\left[\frac{d^2\phi_i}{d\eta_{2i}^2}\frac{\partial\eta_{1i}}{\partial\beta_m}\frac{\partial\eta_{2i}}{\partial\theta_J}\frac{\partial\eta_{2i}}{\partial\beta_L}+\frac{d\phi_i}{d\eta_{2i}}\frac{\partial\eta_{1i}}{\partial\beta_m}\frac{\partial^2\eta_{2i}}{\partial\theta_J\partial\theta_L}\right]\right]
\end{aligned}
$$

$$
\begin{aligned}
U_{JLM} =& \sum_{i=1}^{k}\left[\frac{1}{\phi_i^3}\left[-2r+\sum_{s=1}^{r}\left[-6\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)+6\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i^2}\right)\right.\right.\right.\\
&\left.+6\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)^2+\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)^3\right]\\
&+(n_i-r)\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i^2}\right)^3+6\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)^2\right.\\
&\left.+6\exp\left(\frac{y_{(r,n)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i^2}\right)\right]\left(\frac{d\phi_i}{d\eta_{2i}}\right)^3\frac{\partial\eta_{2i}}{\partial\theta_M}\frac{\partial\eta_{2i}}{\partial\theta_J}\frac{\partial\eta_{2i}}{\partial\theta_L}+\sum_{i=1}^{k}\left[\frac{1}{\phi_i^2}\left[r+\sum_{s=1}^{r}\left[2\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right.\right.\right.\\
&\left.-2\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)-\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)^2\right]\\
&\left.\left.-(n_i-r)\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)^2+2\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\right]\\
&\times\left[3\cdot\frac{d\phi_i}{d\eta_{2i}}\frac{d^2\phi_i}{d\eta_{2i}^2}\frac{\partial\eta_{2i}}{\partial\theta_M}\frac{\partial\eta_{2i}}{\partial\theta_J}\frac{\partial\eta_{2i}}{\partial\theta_L}+\left(\frac{d\phi_i}{d\eta_{2i}}\right)^2\left[\frac{\partial\eta_{2i}}{\partial\theta_M}\frac{\partial^2\eta_{2i}}{\partial\theta_L\partial\theta_J}+\frac{\partial^2\eta_{2i}}{\partial\theta_M\partial\theta_J}\frac{\partial\eta_{2i}}{\partial\theta_M}+\frac{\partial^2\eta_{2i}}{\partial\theta_M\partial\theta_L}\frac{\partial\eta_{2i}}{\partial\theta_J}\right]\right]\\
&+\sum_{i=1}^{k}\frac{1}{\phi_i}\left[-r+\sum_{i=1}^{r}\left[-\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)+\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]\right.\\
&\left.+(n_i-r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\left[\frac{d^3\phi_i}{d\eta_{2i}}\frac{\partial\eta_{2i}}{\partial\theta_M}\frac{\partial\eta_{2i}}{\partial\theta_L}\frac{\partial\eta_{2i}}{\partial\theta_J}+\left(\frac{d^2\phi_i}{d\eta_{2i}^2}\right)\left[\frac{\partial^2\eta_{2i}}{\partial\theta_M\partial\theta_L}\frac{\partial\eta_{2i}}{\partial\theta_J}\right.\right.\\
&\left.\left.+\frac{\partial^2\eta_{2i}}{\partial\theta_M\partial\theta_J}\frac{\partial\eta_{2i}}{\partial\theta_L}+\frac{\partial\eta_{2i}}{\partial\theta_M}\frac{\partial^2\eta_{2i}}{\partial\theta_L\partial\theta_J}\right]+\frac{d\phi_i}{d\eta_{2i}}\frac{\partial^3\eta_{2i}}{\partial\theta_M\partial\theta_L\partial\theta_J}\right].
\end{aligned}
$$

- Agora, iremos calcular os cumulantes de terceira ordem.

$$
\begin{aligned}
k''_{jlm} =& E\left[\sum_{i=1}^{k}\left[\frac{1}{\phi_i^3}\left[\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)+(n_i-r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\right]\left(\frac{d\mu_i}{d\eta_{1i}}\right)^3\frac{\partial\eta_{1i}}{\partial\beta_m}\frac{\partial\eta_{1i}}{\partial\beta_l}\frac{\partial\eta_{1i}}{\partial\beta_j}\\
&+E\left[\sum_{i=1}^{k}\left[\frac{1}{\phi_i^2}\left[-\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)-(n_i-r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\right]
\end{aligned}
$$

$$
\begin{aligned}
&\times\left[3\frac{d\mu_i}{d\eta_{1i}}\frac{d^2\mu_i}{d\eta_{1i}^2}\frac{\partial\eta_{1i}}{\partial\beta_m}\frac{\partial\eta_{1i}}{\partial\beta_l}\frac{\partial\eta_{1i}}{\partial\beta_j}+\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2\left[\frac{\partial^2\eta_{1i}}{\partial\beta_m\partial\beta_l}\frac{\partial\eta_{1i}}{\partial\beta_j}+\frac{\partial^2\eta_{1i}}{\partial\beta_m\partial\beta_j}\frac{\partial\eta_{1i}}{\partial\beta_l}+\frac{\partial^2\eta_{1i}}{\partial\beta_l\partial\beta_j}\frac{\partial\eta_{1i}}{\partial\beta_m}\right]\right]\\
&+E\left[\sum_{i=1}^{k}\left[\frac{1}{\phi_i}\left[-r+\sum_{s=1}^{r}\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)+(n_i-r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\right]\left[\frac{d^3\mu_i}{d\eta_{1i}^3}\frac{\partial\eta_{1i}}{\partial\beta_m}\frac{\partial\eta_{1i}}{\partial\beta_l}\frac{\partial\eta_{1i}}{\partial\beta_j}\right.\\
&\left.+\frac{d^2\mu_i}{d\eta_{1i}^2}\left[\frac{\partial^2\eta_{1i}}{\partial\beta_m\partial\beta_l}\frac{\partial\eta_{1i}}{\partial\beta_l}+\frac{\partial^2\eta_{1i}}{\partial\beta_m\partial\beta_l}\frac{\partial\eta_{1i}}{\partial\beta_j}+\frac{\partial^2\eta_{1i}}{\partial\beta_l\partial\beta_j}\frac{\partial\eta_{1i}}{\partial\beta_m}\right]+\frac{d\mu_i}{d\eta_{1i}}\frac{\partial^3\eta_{1i}}{\partial\beta_m\partial\beta_l\partial\beta_j}\right]\\
=&\sum_{i=1}^{k}\left[\frac{1}{\phi_i^3}\left[\sum_{s=1}^{r}E\left[\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]+(n_i-r)E\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\right]\left(\frac{d\mu_i}{d\eta_{1i}}\right)^3(j,l,m)_i\\
&+\sum_{i=1}^{k}\left[\frac{1}{\phi_i^2}\left[-\sum_{s=1}^{r}E\left[\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]-(n_i-r)E\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\right]\left[3\frac{d\mu_i}{d\eta_{1i}}\frac{d^2\mu_i}{d\eta_{1i}^2}(j,l,m)_i\right.\\
&\left.+\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2[(lm,j)_i+(jm,l)_i+(jl,m)_i]\right]+\sum_{i=1}^{k}\left[\frac{1}{\phi_i}\left[E[-r]+\sum_{s=1}^{r}E\left[\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]\right.\right.\\
&\left.\left.+(n_i-r)E\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\right]\left[\frac{d^3\mu_i}{d\eta_{1i}^3}(j,l,m)_i+\frac{d^2\mu_i}{d\eta_{1i}^2}[(jm,l)_i+(lm,j)_i+(lj.m)_i]+\frac{d\mu_i}{d\eta_{1i}}(jlm)_i\right]\\
=&\sum_{i=1}^{k}\frac{1}{\phi_i^3}\left[\sum_{s=1}^{r}q_{3si}+(n_i-r)q_{4ri}\right]\left(\frac{d\mu_i}{d\eta_{1i}}\right)^3(j,l,m)_i+\sum_{i=1}^{k}\frac{1}{\phi_i^2}\left[-\sum_{s=1}^{r}q_{3si}-(n_i-r)q_{4ri}\right]\\
&\times\left[3\frac{d\mu_i}{d\eta_{1i}}\frac{d^2\mu_i}{d\eta_{1i}^2}(j,l,m)_i+\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2[(lm,j)_i+(jm,l)_i+(jl,m)_i]\right]\\
=&\sum_{i=1}^{k}\left[\frac{1}{\phi_i^3}\left[\sum_{s=1}^{r}q_{3si}+(n_i-r)q_{4ri}\right]\left(\frac{d\mu_i}{d\eta_{1i}}\right)^3+\frac{3}{\phi_i^2}\left[-\sum_{s=1}^{r}q_{3si}-(n_i-r)q_{4ri}\right]\frac{d^2\mu_i}{d\eta_{1i}^2}\frac{d\mu_i}{d\eta_{1i}}\right](j,l,m)_i\\
&+\sum_{i=1}^{k}\frac{1}{\phi_i^2}\left[-\sum_{s=1}^{r}q_{3si}-(n_i-r)q_{4ri}\right]\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2[(lm,j)_i+(jm,l)_i+(jl,m)_i].
\end{aligned}
$$

$$
\begin{aligned}
k''_{jlM}=&E\left[\sum_{i=1}^{k}\left[\frac{1}{\phi_i^3}\left[\sum_{s=1}^{r}\left[2\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]+\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]+(n_i-r)\left[2\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right.\right.\right.\\
&\left.\left.\left.+\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i}-\mu_i)}{\phi_i}\right)\right]\right]\right]\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2\frac{\partial\eta_{1i}}{\partial\beta_j}\frac{\partial\eta_{1i}}{\partial\beta_l}\frac{d\phi_i}{d\eta_{2i}}\frac{\partial\eta_{2i}}{\partial\theta_M}+E\left[\sum_{i=1}^{k}\left[\frac{1}{\phi_i^2}\left[r-\sum_{s=1}^{r}\left[\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right.\right.\right.\right.\\
&\left.\left.\left.\left.+\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]-(n_i-r)\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i}-\mu_i)}{\phi_i}\right)+\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\right]\right]\\
&\times\left[\frac{d^2\mu_i}{d\eta_{1i}^2}\frac{\partial\eta_{1i}}{\partial\beta_l}\frac{d\phi_i}{d\eta_{2i}}\frac{\partial\eta 2i}{\partial\theta_M}\frac{\partial\eta 1i}{\partial\beta_j}+\frac{d\mu_i}{d\eta_{1i}}\frac{\partial^2\eta_{1i}}{\partial\beta_j\partial\beta_l}\frac{d\phi_i}{d\eta_{2i}}\frac{\partial\eta_{2i}}{\partial\theta_M}\right]\\
=&\sum_{i=1}^{k}\left[\frac{1}{\phi_i^3}\left[\sum_{s=1}^{r}\left[2E\left[\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]+E\left[\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\right]\right.\\
&\left.+(n_i-r)\left[2E\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]+E\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i}-\mu_i)}{\phi_i}\right)\right]\right]\right]\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2\frac{d\phi_i}{d\eta_{2i}}(j,l,M)_i\\
&+\sum_{i=1}^{k}\left[\frac{1}{\phi_i^2}\left[E[r]-\sum_{s=1}^{r}\left[E\left[\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]+E\left[\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\right]\right.\\
&\left.-(n_i-r)\left[E\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i}-\mu_i)}{\phi_i}\right)\right]+E\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\right]\left[\frac{d^2\mu_i}{d\eta_{1i}^2}\frac{d\phi_i}{d\eta_{2i}}(j,l,M)_i\right.\\
&\left.+\frac{d\mu_i}{d\eta_{1i}}\frac{d\phi_i}{d\eta_{2i}}(lj,M)_i\right]\\
=&\sum_{i=1}^{k}\frac{1}{\phi_i^3}\left[\sum_{s=1}^{r}(2q_{3si}+q_{5si})+(n_i-r)(2q_{4ri}+q_{6ri})\right]\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2\frac{d\phi_i}{d\eta_{2i}}(j,l,M)_i+\sum_{i=1}^{k}\frac{1}{\phi_i^2}\left[q_1-\sum_{s=1}^{r}(q_{3si}+q_{5si})\right.\\
&\left.-(n_i-r)(q_{6ri}+q_{4ri})\right]\left[\frac{d^2\mu_i}{d\eta_{1i}^2}\frac{d\phi_i}{d\eta_{2i}}(j,l,M)_i+\frac{d\mu_i}{d\eta_{1i}}\frac{d\phi_i}{d\eta_{2i}}(lj,M)_i\right]
\end{aligned}
$$

$$
\begin{aligned}
=&\sum_{i=1}^{k}\left[\frac{1}{\phi_i^3}\left[\sum_{s=1}^{r}(2q_{3si}+q_{5si})+(n_i-r)(2q_{4ri}+q_{6ri})\right]\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2+\frac{1}{\phi_i^2}\left[q_1-\sum_{s=1}^{r}(q_{3si}+q_{5si})\right.\right.\\
&\left.\left.-(n_i-r)(q_{6ri}+q_{4ri})\right]\frac{d^2\mu_i}{d\eta_{1i}^2}\right]\frac{d\phi_i}{d\eta_{2i}}(j,l,M)_i+\sum_{i=1}^{k}\frac{1}{\phi_i^2}\left[q_1-\sum_{s=1}^{r}(q_{3si}+q_{5si})-(n_i-r)(q_{6ri}+q_{4ri})\right]\\
&\times\frac{d\mu_i}{d\eta_{1i}}\frac{d\phi_i}{d\eta_{2i}}(lj,M)_i.
\end{aligned}
$$

$$
\begin{aligned}
k''_{JLm}=&E\left[\sum_{i=1}^{k}\left[\frac{1}{\phi_i^3}\left[\sum_{s=1}^{r}\left[-2+\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)^2+4\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right.\right.\right.\right.\\
&\left.+2\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]+(n_i-r)\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)^2+2\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right.\\
&\left.\left.\left.\left.+4\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i}-\mu_i)}{\phi_i}\right)\right]\right]\right]\right]\left(\frac{d\phi_i}{d\eta_{2i}}\right)^2\frac{d\mu_i}{d\eta_{1i}}\frac{\partial\eta_{2i}}{\partial\theta_J}\frac{\partial\eta_{2i}}{\partial\beta_m}\frac{\partial\eta_{2i}}{\partial\theta_L}\\
&+E\left[\sum_{i=1}^{k}\left[\frac{1}{\phi_i^2}\left[r-\sum_{s=1}^{r}\left[\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)+\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]\right.\right.\right.\\
&\left.\left.\left.-(n_i-r)\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i}-\mu_i)}{\phi_i}\right)+\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\right]\right]\\
&\times\left[\left(\frac{d\mu_i}{d\eta_{1i}}\right)\left[\frac{d^2\phi_i}{d\eta_{2i}^2}\frac{\partial\eta_{1i}}{\partial\beta_m}\frac{\partial\eta_{2i}}{\partial\theta_J}\frac{\partial\eta_{2i}}{\partial\beta_L}+\frac{d\phi_i}{d\eta_{2i}}\frac{\partial\eta_{1i}}{\partial\beta_m}\frac{\partial^2\eta_{2i}}{\partial\theta_J\partial\theta_L}\right]\right]\\
=&\sum_{i=1}^{k}\left[\frac{1}{\phi_i^3}\left[\sum_{s=1}^{r}\left[E[-2]+E\left[\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)^2\right]+2E\left[\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]\right.\right.\right.\\
&\left.+4E\left[\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]\right]+(n_i-r)\left[E\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)^2\right]\right.\\
&\left.\left.\left.+2E\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]+4E\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i}-\mu_i)}{\phi_i}\right)\right]\right]\right]\right]\left(\frac{d\phi_i}{d\eta_{2i}}\right)^2\frac{d\mu_i}{d\eta_{1i}}(J,L,m)_i\\
&+\sum_{i=1}^{k}\left[\frac{1}{\phi_i^2}\left[E[r]-\sum_{s=1}^{r}\left[E\left[\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]+E\left[\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\right]\right.\\
&\left.\left.-(n_i-r)\left[E\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i}-\mu_i)}{\phi_i}\right)\right]+E\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\right]\right]\\
&\times\left[\left(\frac{d\mu_i}{d\eta_{1i}}\right)\left[\frac{d^2\phi_i}{d\eta_{2i}^2}(J,L,m)_i+\frac{d\phi_i}{d\eta_{2i}}(JL,m)_i\right]\right]\\
=&\sum_{i=1}^{k}\frac{1}{\phi_i^3}\left[\sum_{s=1}^{r}(-2+q_{7si}+4q_{5si}+2q_{3si})+(n_i-r)(q_{8ri}+4q_{6ri}+2q_{4ri})\right]\left(\frac{d\phi_i}{d\eta_{2i}}\right)^2\frac{d\mu_i}{d\eta_{1i}}(J,L,m)_i\\
&+\sum_{i=1}^{k}\frac{1}{\phi_i^2}\left[q_1-\sum_{s=1}^{r}(q_{3si}+q_{5si})-(n_i-r)(q_{6ri}+q_{4ri})\right]\left[\left(\frac{d\mu_i}{d\eta_{1i}}\right)\left[\frac{d^2\phi_i}{d\eta_{2i}^2}(J,L,m)_i+\frac{d\phi_i}{d\eta_{2i}}(JL,m)_i\right]\right]\\
=&\sum_{i=1}^{k}\left[\frac{1}{\phi_i^3}\left[\sum_{s=1}^{r}(-2+q_{7si}+4q_{5si}+2q_{3si})+(n_i-r)(q_{8ri}+4q_{6ri}+2q_{4ri})\right]\left(\frac{d\phi_i}{d\eta_{2i}}\right)^2+\frac{1}{\phi_i^2}\left[q_1-\sum_{s=1}^{r}(q_{3si}+q_{5si})\right.\right.\\
&\left.\left.-(n_i-r)(q_{6ri}+q_{4ri})\right]\frac{d^2\phi_i}{d\eta_{2i}^2}\right]\frac{d\mu_i}{d\eta_{1i}}(J,L,m)_i+\sum_{i=1}^{k}\frac{1}{\phi_i^2}\left[q_1-\sum_{s=1}^{r}(q_{3si}+q_{5si})-(n_i-r)(q_{6ri}+q_{4ri})\right]\\
&\times\left(\frac{d\mu_i}{d\eta_{1i}}\right)\frac{d\phi_i}{d\eta_{2i}}(JL,m)_i.
\end{aligned}
$$

$$
k''_{JLM}=E\left[\sum_{i=1}^{k}\left[\frac{1}{\phi_i^3}\left[-2r+\sum_{s=1}^{r}\left[-6\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)+6\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i^2}\right)\right.\right.\right.\right.
$$

$$
\begin{aligned}
&+6\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)^2+\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)^3\Bigg]\\
&+(n_i-r)\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i^2}\right)^3+6\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)^2\right.\\
&+6\exp\left(\frac{y_{(r,n)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i^2}\right)\Bigg]\Bigg]\left(\frac{d\phi_i}{d\eta_{2i}}\right)^3\frac{\partial\eta_{2i}}{\partial\theta_M}\frac{\partial\eta_{2i}}{\partial\theta_J}\frac{\partial\eta_{2i}}{\partial\theta_L}+E\left[\sum_{i=1}^{k}\left[\frac{1}{\phi_i^2}\right[r\right.\\
&+\sum_{s=1}^{r}\left[2\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)-2\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)-\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)^2\right]\\
&-(n_i-r)\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)^2+2\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\Bigg]\Bigg]\Bigg]\\
&\times\left[3\cdot\frac{d\phi_i}{d\eta_{2i}}\frac{d^2\phi_i}{d\eta_{2i}^2}\frac{\partial\eta_{2i}}{\partial\theta_M}\frac{\partial\eta_{2i}}{\partial\theta_J}\frac{\partial\eta_{2i}}{\partial\theta_L}+\left(\frac{d\phi_i}{d\eta_{2i}}\right)^2\left[\frac{\partial\eta_{2i}}{\partial\theta_M}\frac{\partial^2\eta_{2i}}{\partial\theta_L\partial\theta_J}+\frac{\partial^2\eta_{2i}}{\partial\theta_M\partial\theta_J}\frac{\partial\eta_{2i}}{\partial\theta_L}+\frac{\partial^2\eta_{2i}}{\partial\theta_M\partial\theta_L}\frac{\partial\eta_{2i}}{\partial\theta_J}\right]\right]\\
&+E\left[\sum_{i=1}^{k}\frac{1}{\phi_i}\left[-r+\sum_{i=1}^{r}\left[-\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)+\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]\right.\right.\\
&+(n_i-r)\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\Bigg]\Bigg]\left[\frac{d^3\phi_i}{d\eta_{2i}}\frac{\partial\eta_{2i}}{\partial\theta_M}\frac{\partial\eta_{2i}}{\partial\theta_L}\frac{\partial\eta_{2i}}{\partial\theta_J}+\left(\frac{d^2\phi_i}{d\eta_{2i}^2}\right)\left[\frac{\partial^2\eta_{2i}}{\partial\theta_M\partial\theta_L}\frac{\partial\eta_{2i}}{\partial\theta_J}\right.\right.\\
&+\frac{\partial^2\eta_{2i}}{\partial\theta_M\partial\theta_J}\frac{\partial\eta_{2i}}{\partial\theta_L}+\frac{\partial\eta_{2i}}{\partial\theta_M}\frac{\partial^2\eta_{2i}}{\partial\theta_L\partial\theta_J}\Bigg]+\frac{d\phi_i}{d\eta_{2i}}\frac{\partial^3\eta_{2i}}{\partial\theta_M\partial\theta_L\partial\theta_J}\Bigg]\\
=&\sum_{i=1}^{k}\left[\frac{1}{\phi_i^3}\left[E[-2r]+\sum_{s=1}^{r}\left[-6E\left[\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]+6E\left[\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]\right.\right.\right.\\
&+6E\left[\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)^2\right]+E\left[\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)^3\right]\Bigg]\\
&+(n_i-r)\left[E\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i^2}\right)^3\right]+6E\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)^2\right]\right.\\
&+6E\left[\exp\left(\frac{y_{(r,n)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\Bigg]\Bigg]\Bigg]\left(\frac{d\phi_i}{d\eta_{2i}}\right)^3(J,L,M)_i+\sum_{i=1}^{k}\left[\frac{1}{\phi_i^2}\left[E[r]+\sum_{s=1}^{r}\left[2E\left[\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]\right.\right.\right.\\
&-2E\left[\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]-E\left[\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)^2\right]\Bigg]\\
&-(n_i-r)\left[E\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)^2\right]+2E\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\Bigg]\Bigg]\\
&\left[3\frac{d\phi_i}{d\eta_{2i}}\frac{d^2\phi_i}{d\eta_{2i}^2}(J,L,M)_i+\left(\frac{d\phi_i}{d\eta_{2i}}\right)^2[(JL,M)_i+(JM,L)_i+(LM,J)_i]\right]+\sum_{i=1}^{k}\frac{1}{\phi_i}\left[E[-r]\right.\\
&+\sum_{i=1}^{r}\left[-E\left[\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]+E\left[\exp\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(s,n_i)}-\mu_i}{\phi_i}\right)\right]\right]\\
&+(n_i-r)E\left[\exp\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\left(\frac{y_{(r,n_i)}-\mu_i}{\phi_i}\right)\right]\Bigg]\times\left[\frac{d^3\phi_i}{d\eta_{2i}}(J,L,M)_i+\left(\frac{d^2\phi_i}{d\eta_{2i}^2}\right)[(LM,J)_i+(JM,L)_i\right.\\
&+(JL,M)_i]+\frac{d\phi_i}{d\eta_{2i}}(JLM)_i\Bigg].\\
=&\sum_{i=1}^{k}\frac{1}{\phi_i^3}\left[-2q_1+\sum_{s=1}^{r}(-6q_{2si}+q_{9si}+6q_{7si}+6q_{5si})+(n_i-r)(q_{10si}+6q_{8si}+6q_{6si})\right]\left(\frac{d\phi_i}{d\eta_{2i}}\right)^3(J,L,M)_i\\
&+\sum_{i=1}^{k}\frac{1}{\phi_i^2}\left[q_1+\sum_{s=1}^{r}(2q_{2si}-q_{7si}-2q_{5si})-(n_i-r)(q_{8ri}+2q_{6ri})\right]\left[3\frac{d\phi_i}{d\eta_{2i}}\frac{d^2\phi_i}{d\eta_{2i}^2}(J,L,M)_i+\left(\frac{d\phi_i}{d\eta_{2i}}\right)^2[(JL,M)_i\right.\\
&+(JM,L)_i+(LM,J)_i]\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
=&\sum_{i=1}^{k}\left[\frac{1}{\phi_i^3}\left[-2q_1+\sum_{s=1}^{r}(-6q_{2si}+q_{9si}+6q_{7si}+6q_{5si})+(n_i-r)(q_{10si}+6q_{8si}+6q_{6si})\right]\left(\frac{d\phi_i}{d\eta_{2i}}\right)^3\right.\\
&+\left.\frac{3}{\phi_i^2}\left[q_1+\sum_{s=1}^{r}(2q_{2si}-q_{7si}-2q_{5si})-(n_i-r)(q_{8ri}+2q_{6ri})\right]\frac{d\phi_i}{d\eta_{2i}}\frac{d^2\phi_i}{d\eta_{2i}^2}\right](J,L,M)_i+\sum_{i=1}^{k}\frac{1}{\phi_i^2}\left[q_1\right.\\
&+\left.\sum_{s=1}^{r}(2q_{2si}-q_{7si}-2q_{5si})-(n_i-r)(q_{8ri}+2q_{6ri})\right]\left(\frac{d\phi_i}{d\eta_{2i}}\right)^2\left[(JL,M)_i+(JM,L)_i+(LM,J)_i\right].
\end{aligned}
$$

Como todas as esperanças envolvidas nos cumulantes de segunda e terceira ordem são finitas, podemos fazer a correção do viés.

- As derivadas dos elementos da Matriz de Informação de Fisher, são

$$
\begin{aligned}
k_{jl}^{''(m)}=&\sum_{i=1}^{k}\left[\frac{2}{\phi_i^2}\left(-\sum_{s=1}^{r}q_{3si}-(n_i-r)q_{4ri}\right)\frac{d\mu_i}{d\eta_{1i}}\frac{d^2\mu_i}{d\eta_{1i}^2}\right]\times(j,l,m)_i+\\
&\sum_{i=1}^{k}\frac{1}{\phi_i^2}\left[-\sum_{s=1}^{r}q_{3si}-(n_i-r)q_{4ri}\right]\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2\left[(jm,l)_i+(lm,j)_i\right],
\end{aligned}
$$

$$
k_{jl}^{''(M)}=\sum_{i=1}^{k}\frac{2}{\phi_i^3}\left[\sum_{s=1}^{r}q_{3si}+(n_i-r)q_{4ri}\right]\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2\frac{d\phi_i}{d\eta_{2i}}(j,l,M)_i,
$$

$$
k_{JL}^{''(m)}=0,
$$

$$
\begin{aligned}
k_{JL}^{''(M)}=&\sum_{i=1}^{k}\left[\frac{2}{\phi_i^3}\left[-q_1-\sum_{s=1}^{r}(2q_{2si}-q_{7si}-2q_{5si})+(n_i-r)(q_{8ri}+2q_{6ri})\right]\left(\frac{d\phi_i}{d\eta_{2i}}\right)^3+\frac{2}{\phi_i^2}\left[q_1\right.\right.\\
&+\left.\left.\sum_{s=1}^{r}(2q_{2si}-q_{7si}-2q_{5si})-(n_i-r)(q_{8ri}+2q_{6ri})\right]\frac{d\phi_i}{d\eta_{2i}}\frac{d^2\phi_i}{d\eta_{2i}^2}\right](J,L,M)_i+\sum_{i=1}^{k}\frac{1}{\phi_i^2}\left[q_1\right.\\
&+\left.\sum_{s=1}^{r}(2q_{2si}-q_{7si}-2q_{5si})-(n_i-r)(q_{8ri}+2q_{6ri})\right]\left(\frac{d\phi_i}{d\eta_{2i}}\right)^2\left[(LM,J)_i+(JM,L)_i\right],
\end{aligned}
$$

$$
\begin{aligned}
k_{jL}^{''(m)}=&\sum_{i=1}^{k}\frac{1}{\phi_i^2}\left[q_1-\sum_{s=1}^{r}(q_{3si}+q_{5si})-(n_i-r)(q_{6ri}+q_{4ri})\right]\frac{d^2\mu_i}{d\eta_{1i}^2}\frac{d\phi_i}{d\eta_{2i}}(j,L,m)_i+\sum_{i=1}^{k}\frac{1}{\phi_i^2}\left[q_1\right.\\
&-\left.\sum_{s=1}^{r}(q_{3si}+q_{5si})-(n_i-r)(q_{6ri}+q_{4ri})\right]\frac{d\phi_i}{d\eta_{2i}}\frac{d\mu_i}{d\eta_{1i}}(jm,L)_i,
\end{aligned}
$$

e

$$
\begin{aligned}
k''^{(M)}_{jL} = & \sum_{i=1}^{n}\left[\frac{2}{\phi_i^3}\left[-q_1+\sum_{s=1}^{r}(q_{3si}+q_{5si})+(n_i-r)(q_{6ri}+q_{4ri})\right]\left(\frac{d\phi_i}{d\eta_{2i}}\right)^2\right.\\
& \left.+\frac{1}{\phi_i^2}\left[q_1-\sum_{s=1}^{r}(q_{3si}+q_{5si})-(n_i-r)(q_{6ri}+q_{4ri})\right]\frac{d^2\phi_i}{d\eta_{2i}^2}\right]\frac{d\mu_i}{d\eta_{1i}}(j,L,M)\\
& +\sum_{i=1}^{k}\frac{1}{\phi_i^2}\left[q_1-\sum_{s=1}^{r}(q_{3si}+q_{5si})-(n_i-r)(q_{6ri}+q_{4ri})\right]\frac{d\mu_i}{d\eta_{1i}}\frac{d\phi_i}{d\eta_{2i}}(LM,j)_i.
\end{aligned}
$$

- Definiremos agora algumas expressões

$$
w''_{1i}=\left[\frac{1}{2\phi_i^3}\left(-\sum_{s=1}^{r}q_{3si}-(n_i-r)q_{4ri}\right)\left(\frac{d\mu_i}{d\eta_{1i}}\right)^3+\frac{1}{2\phi_i^2}\left(-\sum_{s=1}^{r}q_{3si}-(n_i-r)q_{4ri}\right)\frac{d\mu_i}{d\eta_{1i}}\frac{d^2\mu_i}{d\eta_{1i}^2}\right],
$$

$$
w''_{2i}=\frac{1}{2\phi_i^2}\left[-\sum_{s=1}^{r}q_{3si}-(n_i-r)q_{4ri}\right]\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2,
$$

$$
\begin{aligned}
w''_{3i}= & \left[\frac{1}{2\phi_i^3}\left[-\sum_{s=1}^{r}(2q_{3si}+q_{5si})-(n_i-r)(2q_{4ri}+q_{6ri})\right]\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2\right.\\
& \left.+\frac{1}{2\phi_i^2}\left[q_1-\sum_{s=1}^{r}(q_{3si}+q_{5si})-(n_i-r)(q_{6ri}+q_{4ri})\right]\frac{d^2\mu_i}{d\eta_{1i}^2}\right]\frac{d\phi_i}{d\eta_{2i}},
\end{aligned}
$$

$$
w''_{4i}=\frac{1}{2\phi_i^2}\left[q_1-\sum_{s=1}^{r}(q_{3si}+q_{5si})-(n_i-r)(q_{6ri}+q_{4ri})\right]\frac{d\mu_i}{d\eta_{1i}}\frac{d\phi_i}{d\eta_{2i}},
$$

$$
\begin{aligned}
w''_{5i}= & \left[\frac{1}{2\phi_i^3}\left[\sum_{s=1}^{r}(2q_{3si}-q_{5si})+(n_i-r)(2q_{4ri}-q_{6ri})\right]\left(\frac{d\mu_i}{d\eta_{1i}}\right)^2\right.\\
& \left.-\frac{1}{2\phi_i^2}\left[q_1-\sum_{s=1}^{r}(q_{3si}+q_{5si})-(n_i-r)(q_{6ri}+q_{4ri})\right]\frac{d^2\mu_i}{d\eta_{1i}^2}\right]\frac{d\phi_i}{d\eta_{2i}},
\end{aligned}
$$

$$
\begin{aligned}
w''_{6i}= & \left[-\frac{1}{2\phi_i^3}\left[\sum_{s=1}^{r}(-2+q_{7si}+4q_{5si}+2q_{3si})+(n_i-r)(q_{8ri}+4q_{6ri}+2q_{4ri})\right]\left(\frac{d\phi_i}{d\eta_{2i}}\right)^2\right.\\
& \left.-\frac{1}{2\phi_i^2}\left[q_1-\sum_{s=1}^{r}(q_{3si}+q_{5si})-(n_i-r)(q_{6ri}+q_{4ri})\right]\frac{d^2\phi_i}{d\eta_{2i}^2}\right]\frac{d\mu_i}{d\eta_{1i}},
\end{aligned}
$$

$$w^{''}_{7i} = \left[\frac{1}{2\phi_i^3}\left[-2q_1 + \sum_{s=1}^{r}(-q_{7si} + 2q_{3si}) + (n_i - r)(-q_{8ri} + 2q_{4ri})\right]\left(\frac{d\phi_i}{d\eta_{2i}}\right)^2 + \frac{1}{2\phi_i^2}\left[q_1 - \sum_{s=1}^{r}(q_{3si} + q_{5si}) - (n_i - r)(q_{6ri} + q_{4ri})\right]\frac{d^2\phi_i}{d\eta_{2i}^2}\right]\frac{d\mu_i}{d\eta_{1i}},$$

$$w^{''}_{8i} = \left[\frac{1}{2\phi_i^3}\left[-2q_1 + \sum_{s=1}^{r}(-2q_{2si} - q_{7si} - q_{9si} + 2q_{5si}) + (n_i - r)(-q_{10ri} - 2q_{8ri} + 2q_{6ri})\right]\left(\frac{d\phi_i}{d\eta_{2i}}\right)^3 + \frac{1}{2\phi_i^2}\left[q_1 + \sum_{s=1}^{r}(2q_{2si} - q_{7si} - 2q_{5si}) - (n_i - r)(q_{8ri} + 2q_{6ri})\right]\frac{d^2\phi_i}{d\eta_{2i}^2}\frac{d\phi_i}{d\eta_{2i}}\right],$$

e

$$w^{''}_{9i} = \frac{1}{2\phi_i^2}\left[q_1 + \sum_{s=1}^{r}(2q_{2si} - q_{7si} - 2q_{5si}) - (n_i - r)(q_{8ri} + 2q_{6ri})\right]\left(\frac{d\phi_i}{d\eta_{2i}}\right)^2.$$

- Calcularemos agora, algumas expressões que serão necessárias para obtermos a correção do viés.

$$k^{''(m)}_{jl} - \frac{1}{2}k^{''}_{jlm} = \sum_{i=1}^{n} w^{''}_{1i}(j,l,m)_i + \sum_{i=1}^{n} w^{''}_{2i}[(jm,l)_i + (lm,j)_i - (jl,m)_i],$$

$$k^{''(m)}_{Jl} - \frac{1}{2}k^{''}_{Jlm} = \sum_{i=1}^{n} w^{''}_{3i}(J,l,m)_i + \sum_{i=1}^{n} w^{''}_{4i}(lm,J)_i,$$

$$k^{''(m)}_{jL} - \frac{1}{2}k^{''}_{jLm} = \sum_{i=1}^{n} w^{''}_{3i}(j,L,m)_i + \sum_{i=1}^{n} w^{''}_{4i}(jm,L)_i,$$

$$k^{''(M)}_{jl} - \frac{1}{2}k^{''}_{jlM} = \sum_{i=1}^{n} w^{''}_{5i}(j,l,M)_i - \sum_{i=1}^{n} w^{''}_{4i}(jl,M)_i,$$

$$k^{''(m)}_{JL} - \frac{1}{2}k^{''}_{JLm} = \sum_{i=1}^{n} w^{''}_{6i}(J,L,m)_i - \sum_{i=1}^{n} w^{''}_{4i}(JL,m)_i,$$

$$k_{Jl}^{''(M)}-\frac{1}{2}k_{JlM}^{''}=\sum_{i=1}^{n}w_{7i}^{''}(J,l,M)_i+\sum_{i=1}^{n}w_{4i}^{''}(JM,l)_i,$$

$$k_{jL}^{''(M)}-\frac{1}{2}k_{jLM}^{''}=\sum_{i=1}^{n}w_{7i}^{''}(j,L,M)_i+\sum_{i=1}^{n}w_{4i}^{''}(LM,j)_i,$$

e

$$k_{JL}^{''(M)}-\frac{1}{2}k_{JLM}^{''}=\sum_{i=1}^{n}w_{8i}^{''}(J,L,M)_i+\sum_{i=1}^{n}w_{9i}^{''}[(LM,J)_i+(JM,L)_i-(JL,M)_i].$$

- Calculando as expressões da correções do viés para $B_1(\widehat{\beta})$ e $B_1(\widehat{\theta})$ e colocando na forma matricial.

∗ Para $B_2(\widehat{\beta})$

$$\begin{aligned}\sum_{j,l,m}k_2^{aj}k_2^{lm}\left\{k_{jl}^{''(m)}-\frac{1}{2}k_{jlm}^{''}\right\}&=\sum_{j,l,m}k_2^{aj}k_2^{lm}\left\{\sum_{i=1}^{n}w_{1i}^{''}(j,l,m)_i+\sum_{i=1}^{n}w_{2i}^{''}[(jm,l)_i+(lm,j)_i-(jl,m)_i]\right\}\\&=\sum_{i=1}^{n}w_{1i}^{''}\sum_{j}k_2^{aj}(j)_i\sum_{l,m}k_2^{lm}(l,m)_i+\sum_{i=1}^{n}w_{2i}^{''}\sum_{l,m}k_2^{lm}(l)_i\sum_{j}k_2^{aj}(jm)_i\\&+\sum_{i=1}^{n}w_{2i}^{''}\sum_{j}k_2^{aj}(j)_i\sum_{l,m}k_2^{lm}(lm)_i-\sum_{i=1}^{n}w_{2i}^{''}\sum_{l,m}k_2^{lm}(m)_i\sum_{j}k_2^{aj}(jl)_i\\&=\sum_{i=1}^{n}w_{1i}^{''}\sum_{j}k_2^{aj}(j)_i\sum_{l,m}k_2^{lm}(l,m)_i+\sum_{i=1}^{n}w_{2i}^{''}\sum_{j}k_2^{aj}(j)_i\sum_{l,m}k_2^{lm}(lm)_i\\&=e_a^TK_2^{\beta\beta}\widetilde{X}^TW_1^{''}Z_{\beta d}^{''}1_{n\times 1}+e_a^TK_2^{\beta\beta}\widetilde{X}^TW_2^{''}D_{\beta}^{''}1_{n\times 1},\end{aligned}$$

$$\begin{aligned}\sum_{J,l,m}k_2^{aJ}k_2^{lm}\left\{k_{Jl}^{''(m)}-\frac{1}{2}k_{Jlm}^{''}\right\}&=\sum_{J,l,m}k_2^{aJ}k_2^{lm}\left\{\sum_{i=1}^{n}w_{3i}^{''}(J,l,m)_i+\sum_{i=1}^{n}w_{4i}^{''}(lm,J)_i\right\}\\&=\sum_{i=1}^{n}w_{3i}^{''}\sum_{J}k_2^{aJ}(J)_i\sum_{l,m}k_2^{lm}(l,m)_i+\sum_{i=1}^{n}w_{4i}^{''}\sum_{J}k_2^{aJ}(J)_i\sum_{l,m}k_2^{lm}(lm)_i\\&=e_a^TK_2^{\beta\theta}\widetilde{S}^TW_3^{''}Z_{\beta d}^{''}1_{n\times 1}+e_a^TK_2^{\beta\theta}\widetilde{S}^TW_4^{''}D_{\beta}^{''}1_{n\times 1},\end{aligned}$$

$$\sum_{j,L,m} k_2^{aj} k_2^{Lm} \left\{ k_{jL}^{''(m)} - \frac{1}{2} k_{jLm}^{''} \right\} = \sum_{j,L,m} k_2^{aj} k_2^{Lm} \left\{ \sum_{i=1}^{n} w_{3i}^{''}(j,L,m)_i + \sum_{i=1}^{n} w_{4i}^{''}(jm,L)_i \right\}$$

$$= \sum_{i=1}^{n} w_{3i}^{''} \sum_{j} k_2^{aj}(j)_i \sum_{L,m} k_2^{Lm}(L,m)_i + \sum_{i=1}^{n} w_{4i}^{''} \sum_{L,m} k_2^{Lm}(L)_i \sum_{j} k_2^{aj}(jm)_i$$

$$= e_a^T K_2^{\beta\beta} \widetilde{X}^T W_3^{''} Z_{\beta\theta d}^{''} 1_{n\times 1} + L_{1,a}^{''},$$

$$\sum_{j,l,M} k_2^{aj} k_2^{lM} \left\{ k_{jl}^{''(M)} - \frac{1}{2} k_{jlM}^{''} \right\} = \sum_{j,l,M} k_2^{aj} k_2^{lM} \left\{ \sum_{i=1}^{n} w_{5i}^{''}(j,l,M)_i - \sum_{i=1}^{n} w_{4i}^{''}(jl,M)_i \right\}$$

$$= \sum_{i=1}^{n} w_{5i}^{''} \sum_{j} k_2^{aj}(j)_i \sum_{l,M} k_2^{lM}(l,M)_i - \sum_{i=1}^{n} w_{4i}^{''} \sum_{l,M} k_2^{lM}(M)_i \sum_{j} k_2^{aj}(jl)_i$$

$$= e_a^T K_2^{\beta\beta} \widetilde{X}^T W_5^{''} Z_{\beta\theta d}^{''} 1_{n\times 1} - L_{1,a}^{''},$$

$$\sum_{J,L,m} k_2^{aJ} k_2^{Lm} \left\{ k_{JL}^{''(m)} - \frac{1}{2} k_{JLm}^{''} \right\} = \sum_{J,L,m} k_2^{aJ} k_2^{Lm} \left\{ \sum_{i=1}^{n} w_{6i}^{''}(J,L,m)_i - \sum_{i=1}^{n} w_{4i}^{''}(JL,m)_i \right\}$$

$$= \sum_{i=1}^{n} w_{6i}^{''} \sum_{J} k_2^{aJ}(J)_i \sum_{L,m} k_2^{Lm}(L,m)_i - \sum_{i=1}^{n} w_{4i}^{''} \sum_{L,m} k_2^{Lm}(m)_i \sum_{J} k_2^{aJ}(JL)_i$$

$$= e_a^T K_2^{\beta\theta} \widetilde{S}^T W_6^{''} Z_{\beta\theta d}^{''} 1_{n\times 1} - L_{2,a}^{''},$$

$$\sum_{J,l,M} k_2^{aJ} k_2^{lM} \left\{ k_{Jl}^{''(M)} - \frac{1}{2} k_{JlM}^{''} \right\} = \sum_{J,l,M} k_2^{aJ} k_2^{lM} \left\{ \sum_{i=1}^{n} w_{7i}^{''}(J,l,M)_i + \sum_{i=1}^{n} w_{4i}^{''}(JM,l)_i \right\}$$

$$= \sum_{i=1}^{n} w_{7i}^{''} \sum_{J} k_2^{aJ}(J)_i \sum_{l,M} k_2^{lM}(l,M)_i + \sum_{i=1}^{n} w_{4i}^{''} \sum_{l,M} k_2^{lM}(l)_i \sum_{J} k_2^{aJ}(JM)_i$$

$$= e_a^T K_2^{\beta\theta} \widetilde{S}^T W_7^{''} Z_{\beta\theta d}^{''} 1_{n\times 1} + L_{2,a}^{''},$$

$$\sum_{j,L,M} k_2^{aj} k_2^{LM} \left\{ k_{jL}^{''(M)} - \frac{1}{2} k_{jLM}^{''} \right\} = \sum_{j,L,M} k_2^{aj} k_2^{LM} \left\{ \sum_{i=1}^{n} w_{7i}^{''}(j,L,M)_i + \sum_{i=1}^{n} w_{4i}^{''}(LM,j)_i \right\}$$

$$= \sum_{i=1}^{n} w_{7i}^{''} \sum_{j} k_2^{aj}(j)_i \sum_{L,M} k_2^{LM}(L,M)_i + \sum_{i=1}^{n} w_{4i}^{''} \sum_{j} k_2^{aj}(j)_i \sum_{L,M} k_2^{LM}(LM)_i$$

$$= e_a^T K_2^{\beta\beta} \widetilde{X}^T W_7^{''} Z_{\theta d}^{''} 1_{n\times 1} + e_a^T K_2^{\beta\beta} \widetilde{X}^T W_4^{''} D_\theta^{''} 1_{n\times 1},$$

e

$$\sum_{J,L,M} k_2^{aJ} k_2^{LM} \left\{ k_{JL}^{''(M)} - \frac{1}{2} k_{JLM}^{''} \right\} = \sum_{J,L,M} k_2^{aJ} k_2^{LM} \left\{ \sum_{i=1}^{n} w_{8i}^{''}(J,L,M)_i + \sum_{i=1}^{n} w_{9i}^{''}[(JM,L)_i + (LM,J)_i - (JL,M)_i] \right\}$$

$$= \sum_{i=1}^{n} w_{8i}^{''} \sum_{J} k_2^{aJ}(J)_i \sum_{L,M} k_2^{LM}(L,M)_i + \sum_{i=1}^{n} w_{9i}^{''} \sum_{J} k_2^{aJ}(J)_i \sum_{L,M} k_2^{LM}(LM)_i$$

$$+ \sum_{i=1}^{n} w_{9i}^{''} \sum_{L,M} k_2^{LM}(L)_i \sum_{J} k_2^{aJ}(JM)_i - \sum_{i=1}^{n} w_{9i}^{''} \sum_{L,M} k_2^{LM}(M)_i \sum_{J} k_2^{aJ}(JL)_i$$

$$= \sum_{i=1}^{n} w_{8i}^{''} \sum_{J} k_2^{aJ}(J)_i \sum_{L,M} k_2^{LM}(L,M)_i + \sum_{i=1}^{n} w_{9i}^{''} \sum_{J} k_2^{aJ}(J)_i \sum_{L,M} k_2^{LM}(LM)_i$$

$$= e_a^T K_2^{\beta\theta} \widetilde{S}^T W_8^{''} Z_{\theta d}^{''} 1_{n\times 1} + e_a^T K_2^{\beta\theta} \widetilde{S}^T W_9^{''} D_\theta^{''} 1_{n\times 1}.$$

$*$ Para $B_2(\widehat{\theta})$

$$\begin{aligned}\sum_{j,l,m} k_2^{Aj} k_2^{lm} \left\{ k_{jl}^{''(m)} - \frac{1}{2} k_{jlm}^{''} \right\} &= \sum_{j,l,m} k_2^{Aj} k_2^{lm} \left\{ \sum_{i=1}^{n} w_{1i}^{''} (j,l,m)_i + \sum_{i=1}^{n} w_{2i}^{''} [(jm,l)_i + (lm,j)_i - (jl,m)_i] \right\} \\ &= \sum_{i=1}^{n} w_{1i}^{''} \sum_{j} k_2^{Aj}(j)_i \sum_{l,m} k_2^{lm}(l,m)_i + \sum_{i=1}^{n} w_{2i}^{''} \sum_{l,m} k_2^{lm}(l)_i \sum_{j} k_2^{Aj}(jm)_i \\ &+ \sum_{i=1}^{n} w_{2i}^{''} \sum_{j} k_2^{Aj}(j)_i \sum_{l,m} k_2^{lm}(lm)_i - \sum_{i=1}^{n} w_{2i}^{''} \sum_{l,m} k_2^{lm}(m)_i \sum_{j} k_2^{Aj}(jl)_i \\ &= \sum_{i=1}^{n} w_{1i}^{''} \sum_{j} k_2^{Aj}(j)_i \sum_{l,m} k_2^{lm}(l,m)_i + \sum_{i=1}^{n} w_{2i}^{''} \sum_{j} k_2^{Aj}(j)_i \sum_{l,m} k_2^{lm}(lm)_i \\ &= e_A^T K_2^{\theta\beta} \widetilde{X}^T W_1^{''} Z_{\beta d}^{''} 1_{n\times 1} + e_A^T K_2^{\theta\beta} \widetilde{X}^T W_2^{''} D_{\beta}^{''} 1_{n\times 1},\end{aligned}$$

$$\begin{aligned}\sum_{J,l,m} k_2^{AJ} k_2^{lm} \left\{ k_{Jl}^{''(m)} - \frac{1}{2} k_{Jlm}^{''} \right\} &= \sum_{J,l,m} k_2^{AJ} k_2^{lm} \left\{ \sum_{i=1}^{n} w_{3i}^{''} (J,l,m)_i + \sum_{i=1}^{n} w_{4i}^{''} (lm,J)_i \right\} \\ &= \sum_{i=1}^{n} w_{3i}^{''} \sum_{J} k_2^{AJ}(J)_i \sum_{l,m} k_2^{lm}(l,m)_i + \sum_{i=1}^{n} w_{4i}^{''} \sum_{J} k_2^{AJ}(J)_i \sum_{l,m} k_2^{lm}(lm)_i \\ &= e_A^T K_2^{\theta\theta} \widetilde{S}^T W_3^{''} Z_{\beta d}^{''} 1_{n\times 1} + e_A^T K_2^{\theta\theta} \widetilde{S}^T W_4^{''} D_{\beta}^{''} 1_{n\times 1},\end{aligned}$$

$$\begin{aligned}\sum_{j,L,m} k_2^{Aj} k_2^{Lm} \left\{ k_{jL}^{''(m)} - \frac{1}{2} k_{jLm}^{''} \right\} &= \sum_{j,L,m} k_2^{aj} k_2^{Lm} \left\{ \sum_{i=1}^{n} w_{3i}^{''} (j,L,m)_i + \sum_{i=1}^{n} w_{4i}^{''} (jm,L)_i \right\} \\ &= \sum_{i=1}^{n} w_{3i}^{''} \sum_{j} k_2^{Aj}(j)_i \sum_{L,m} k_2^{Lm}(L,m)_i + \sum_{i=1}^{n} w_{4i}^{''} \sum_{L,m} k_2^{Lm}(L)_i \sum_{j} k_2^{Aj}(jm)_i \\ &= e_A^T K_2^{\theta\beta} \widetilde{X}^T W_3^{''} Z_{\beta\theta d}^{''} 1_{n\times 1} + L_{1,A}^{''},\end{aligned}$$

$$\begin{aligned}\sum_{j,l,M} k_2^{Aj} k_2^{lM} \left\{ k_{jl}^{''(M)} - \frac{1}{2} k_{jlM}^{''} \right\} &= \sum_{j,l,M} k_2^{Aj} k_2^{lM} \left\{ \sum_{i=1}^{n} w_{5i}^{''} (j,l,M)_i - \sum_{i=1}^{n} w_{4i}^{''} (jl,M)_i \right\} \\ &= \sum_{i=1}^{n} w_{5i}^{''} \sum_{j} k_2^{Aj}(j)_i \sum_{l,M} k_2^{lM}(l,M)_i - \sum_{i=1}^{n} w_{4i}^{''} \sum_{l,M} k_2^{lM}(M)_i \sum_{j} k_2^{Aj}(jl)_i \\ &= e_A^T K_2^{\theta\beta} \widetilde{X}^T W_5^{''} Z_{\beta\theta d}^{''} 1_{n\times 1} - L_{1,A}^{''},\end{aligned}$$

$$\begin{aligned}\sum_{J,L,m} k_2^{AJ} k_2^{Lm} \left\{ k_{JL}^{''(m)} - \frac{1}{2} k_{JLm}^{''} \right\} &= \sum_{J,L,m} k_2^{AJ} k_2^{Lm} \left\{ \sum_{i=1}^{n} w_{6i}^{''} (J,L,m)_i - \sum_{i=1}^{n} w_{4i}^{''} (JL,m)_i \right\} \\ &= \sum_{i=1}^{n} w_{6i}^{''} \sum_{J} k_2^{AJ}(J)_i \sum_{L,m} k_2^{Lm}(L,m)_i - \sum_{i=1}^{n} w_{4i}^{''} \sum_{L,m} k_2^{Lm}(m)_i \sum_{J} k_2^{AJ}(JL)_i \\ &= e_A^T K_2^{\theta\theta} \widetilde{S}^T W_6^{''} Z_{\beta\theta d}^{''} 1_{n\times 1} - L_{2,A}^{''},\end{aligned}$$

$$\sum_{J,l,M} k_2^{AJ} k_2^{lM} \left\{ k_{Jl}^{''(M)} - \frac{1}{2} k_{JlM}^{''} \right\} = \sum_{J,l,M} k_2^{AJ} k_2^{lM} \left\{ \sum_{i=1}^{n} w_{7i}^{''}(J,l,M)_i + \sum_{i=1}^{n} w_{4i}^{''}(JM,l)_i \right\}$$
$$= \sum_{i=1}^{n} w_{7i}^{''} \sum_{J} k_2^{AJ}(J)_i \sum_{l,M} k_2^{lM}(l,M)_i + \sum_{i=1}^{n} w_{4i}^{''} \sum_{l,M} k_2^{lM}(l)_i \sum_{J} k_2^{AJ}(JM)_i$$
$$= e_A^T K_2^{\theta\theta} \widetilde{S}^T W_7^{''} Z_{\beta\theta d}^{''} 1_{n\times 1} + L_{2,A}^{''},$$

$$\sum_{j,L,M} k_2^{Aj} k_2^{LM} \left\{ k_{jL}^{''(M)} - \frac{1}{2} k_{jLM}^{''} \right\} = \sum_{j,L,M} k_2^{Aj} k_2^{LM} \left\{ \sum_{i=1}^{n} w_{7i}^{''}(j,L,M)_i + \sum_{i=1}^{n} w_{4i}^{''}(LM,j)_i \right\}$$
$$= \sum_{i=1}^{n} w_{7i}^{''} \sum_{j} k_2^{Aj}(j)_i \sum_{L,M} k_2^{LM}(L,M)_i + \sum_{i=1}^{n} w_{4i}^{''} \sum_{j} k_2^{Aj}(j)_i \sum_{L,M} k_2^{LM}(LM)_i$$
$$= e_A^T K_2^{\theta\beta} \widetilde{X}^T W_7^{''} Z_{\theta d}^{''} 1_{n\times 1} + e_A^T K_2^{\theta\beta} \widetilde{X}^T W_4^{''} D_\theta^{''} 1_{n\times 1},$$

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$$\sum_{J,L,M} k_2^{AJ} k_2^{LM} \left\{ k_{JL}^{''(M)} - \frac{1}{2} k_{JLM}^{''} \right\} = \sum_{J,L,M} k_2^{AJ} k_2^{LM} \left\{ \sum_{i=1}^{n} w_{8i}^{''}(J,L,M)_i + \sum_{i=1}^{n} w_{9i}^{''}[(JM,L)_i + (LM,J)_i - (JL,M)_i] \right\}$$
$$= \sum_{i=1}^{n} w_{8i}^{''} \sum_{J} k_2^{AJ}(J)_i \sum_{L,M} k_2^{LM}(L,M)_i + \sum_{i=1}^{n} w_{9i}^{''} \sum_{J} k_2^{AJ}(J)_i \sum_{L,M} k_2^{LM}(LM)_i$$
$$+ \sum_{i=1}^{n} w_{9i}^{''} \sum_{L,M} k_2^{LM}(L)_i \sum_{J} k_2^{AJ}(JM)_i - \sum_{i=1}^{n} w_{9i}^{''} \sum_{L,M} k_2^{LM}(M)_i \sum_{J} k_2^{AJ}(JL)_i$$
$$= \sum_{i=1}^{n} w_{8i}^{''} \sum_{J} k_2^{AJ}(J)_i \sum_{L,M} k_2^{LM}(L,M)_i + \sum_{i=1}^{n} w_{9i}^{''} \sum_{J} k_2^{AJ}(J)_i \sum_{L,M} k_2^{LM}(LM)_i$$
$$= e_A^T K_2^{\theta\theta} \widetilde{S}^T W_8^{''} Z_{\theta d}^{''} 1_{n\times 1} + e_A^T K_2^{\theta\theta} \widetilde{S}^T W_9^{''} D_\theta^{''} 1_{n\times 1}.$$