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DEPARTAMENTO DE FÍSICA
PROGRAMA DE PÓS-GRADUAÇÃO EM FÍSICA.

**SOBRE ALGUNS ASPECTOS
GEOMÉTRICOS DA GRAVITAÇÃO:
MERGULHOS DO ESPAÇO-TEMPO EM
DIMENSÕES SUPERIORES, A
CONJECTURA DE WHEELER E
VÁRIEDADES DE WEYL**

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João Pessoa - PB

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FEDERAL UNIVERSITY OF PARAÍBA - UFPB
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ON SOME GEOMETRIC ASPECTS OF
GRAVITY: SPACE-TIME EMBEDDINGS,
WHEELER'S THIN-SANDWICH
CONJECTURE AND WEYL MANIFOLDS

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WHEELER'S THIN-SANDWICH
CONJECTURE AND WEYL MANIFOLDS

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ABSTRACT

In this thesis we present several embedding problems related to modern theories about space-time. First we deal with the so called *thin sandwich conjecture* proposed by J. A. Wheeler. We show that the Bartnik-Fodor theorem extends naturally to higher dimensions and that, furthermore, the geometric hypotheses needed for the proofs can always be satisfied on compact manifolds. From this result, we conclude that on any compact n -dimensional manifold, $n \geq 3$, there is an open set in the space of possible initial data where the thin sandwich problem is well-posed. Then, we apply this result to prove that any compact n -dimensional Lorentzian manifold, $n \geq 3$, with metric in the Sobolev space H_{s+3} , $s > \frac{n}{2}$, has an embedding in a $(2n+2)$ -dimensional Ricci-flat semi-Riemannian manifold. This result improves the codimension needed for the embedding quite drastically compared with previously known results. Finally, we study some embedding problems within the context of Weyl's geometry. With respect to this problem, we show that the Campbell-Magaard theorem naturally extends to the Weyl integrable case, but, in the general case we find some no-go results and, as a very particular case, for the embedding of 3-dimensional Weyl structures, there is as an analogue of the Campbell-Magaard theorem. We conclude by analysing the initial value formulation of a particular class of geometric scalar-tensor theories of gravity, where we will show that the vacuum case is well-posed.

Keywords: General Relativity; Cauchy problem; Thin-Sandwich conjecture; Ricci-flat embeddings; Campbell-Magaard theorems; Weyl structures; Geometric scalar-tensor theories.

RESUMO

Nesta tese, abordamos diferentes aspectos geométricos da gravitação: a conjectura do “*thin sandwich*” de Wheeler, o mergulho do espaço-tempo em dimensões superiores e variedades de Weyl. Começando pelo primeiro, lembramos que esta conjectura, apresentada no contexto do modelo do superespaço para o espaço-tempo, propõe que, nos dados iniciais para o problema de Cauchy da relatividade geral, é possível especificar livremente a métrica induzida sobre a hipersuperfície inicial, tanto quanto sua derivada temporal, *i.e.*, g e $\dot{g}$, e resolver as equações de vínculo em termos do *shift* e do *lapse*. Este problema tem sido estudado e é sabido que a conjectura falha em condições genéricas [16]. Apesar disto, R. Bartnik e G. Fodor mostraram que, em um entorno de dados iniciais com condições geométricas particulares, o problema do sanduíche fino está bem colocado e a conjectura é verdadeira [17]. Esta prova, feita para o caso da relatividade geral, ou seja, um espaço-tempo de quatro dimensões, é generalizada neste trabalho para dimensões superiores, e, além disso, também é mostrado que as condições geométricas necessárias para as provas podem ser sempre satisfeitas no caso de variedades compactas. Desta forma, concluímos que em qualquer variedade compacta de n -dimensões, $n \geq 3$, existe um aberto no espaço dos possíveis dados iniciais para o problema do *thin-sandwich* em que a conjectura de Wheeler é verdadeira.

Em seguida, mostramos que o resultado obtido para o problema do *thin-sandwich* tem um valor geométrico próprio, oferecendo uma caracterização do espaço de soluções das equações de vínculo da relatividade geral que nos fornece as ferramentas necessárias para mostrar um resultado de mergulho para variedades Lorentzianas compactas, com métricas no espaço de Sobolev H_{s+3} , $s > \frac{n}{2}$, em espaços Ricci-flat. Este resultado, que se insere no programa iniciado com as extensões dos teoremas de Campbell-Magaard [25],[26],[27],[28], estabelece uma nova estratégia para o estudo do mergulho do espaço-tempo da relatividade geral em espaços de maior dimensão com propriedades geométricas particulares. Também resolve duas das principais limitações dos teoremas de Campbell-Magaard, já que o resultado requer uma regularidade baixa para as métricas envolvidas, e, ademais, o resultado é global. Além disso, comparado com os melhores resultados conhecidos para este tipo de mergulhos, nosso resultado reduz drasticamente a codimensão do espaço ambiente.

Finalmente, apresentamos um conjunto de resultados sobre imersão no contexto da

geometria de Weyl. Nesta direção, primeiramente destacamos a relevância física dessas estruturas, mediante o enfoque axiomático para o espaço-tempo proposto por Elhers, Pirani e Schild em [39]. Assim, apresentamos uma discussão rigorosa sobre a existência do segundo efeito relógio em um espaço-tempo de Weyl e concluímos que, dentro deste enfoque axiomático, a estrutura mais geral que pode modelar o espaço-tempo é uma estrutura do tipo Weyl integrável. Logo depois disto, mostramos que o teorema de Campbell-Magaard tem uma extensão natural no contexto das estruturas Weyl integrável, mas que, no contexto não-integrável, é possível mostrar obstruções para a existência da imersão. Ademais, mostramos um forte teorema que proíbe a existência de imersões isométricas de estruturas de Weyl não-integráveis de dimensão maior que 3, em estruturas de Weyl de qualquer dimensão, com métricas positivas definidas, nas quais o tensor de Ricci tem parte simétrica nula e com qualquer regularidade para a métrica e 1-forma de Weyl (pelo menos C^2). Particularmente, no caso de imersões de estruturas não integráveis de três dimensões em espaços de Weyl de quatro dimensões, estas restrições não são válidas e é possível provar um teorema do tipo Campbell-Magaard.

Concluiremos analisando o problema de valores iniciais para um tipo de teorias escalar-tensor geométricas baseadas em um espaço-tempo modelado em uma estrutura de Weyl integrável, onde mostraremos que tal problema está bem colocado no vácuo.

Palavras chave: Relatividade geral; problema de Cauchy; Conjectura do sanduíche-fino; Mergulhos Ricci-flat; Teoremas de Campbell-Magaard; Estruturas de Weyl; Teorias escalar-tensor geométricas.

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Chapter 1

Introduction

Even though geometry and physics have always been intimately related, it is a fact that since the advent of relativity these two areas have become even more interconnected. This is a consequence of the fact that, as our views about space-time evolved, the geometric structure of space-time became more intimately related with the fundamental physical properties that can be extracted from a physical theory. In particular, general relativity has intertwined both things to the point that gravity itself is now best understood as a manifestation of the geometric properties of space-time. In this context several fundamental problems in general relativity are naturally translated into well-defined problems in differential geometry. In this way, mathematical general relativity has been increasingly incorporating the standard tools of differential geometry and geometric analysis. Furthermore, problems in both areas have been pushing forward the other. One of the most outstanding problems which has connected these areas during the last century is the initial value formulation of general relativity and problems related to it.

According to general relativity (GR), space-time is modelled as a Lorentzian 4-dimensional manifold $(M, \bar{g})$, and the gravitational field is described by the geometry of space-time, which itself interacts with the other physical fields according to the Einstein equations

$$\text{Ric}(\bar{g}) - \frac{1}{2}R(\bar{g})\bar{g} = T(\bar{g}, \bar{\psi}), \quad (1.1)$$

where $\bar{\psi}$ collectively denotes the set of tensor fields which describe the matter fields, and T represents the energy-momentum tensor associated with such fields.

In this setting, the initial value formulation of general relativity becomes quite distinct from the initial value formulation of other physical theories, for which there is a background space-time where the physical fields propagate. In this case, the geometry of space-time, described by a semi-Riemannian structure associated with the space-time Lorentzian metric, is itself dynamical. Thus, the geometry of space-time evolves with the other fields, each one interacting with the other. When translated to a well-defined geometrical problem, the initial value formulation of general relativity becomes an embedding problem where an initial space-like 3-dimensional Riemannian manifold, representing space-time at a given-time, must be embedded in a 4-dimensional space-time satisfying the Einstein equations. In this context, space-time is typically understood as a globally hyperbolic space-time. It has been shown by R. Geroch that a globally hyperbolic $(n+1)$ -dimensional space-time V is diffeomorphic (and thus equivalent) to $M^n \times \mathbb{R}$ [1]. This point again highlights how much the space-time structure is related to physical properties, since a sensible physical requirement is translated into a topological condition for space-time.

The initial value problem for GR is naturally translated into a partial differential equation problem with a clear geometric origin. The aim here is to write the Einstein equations as a hyperbolic system on space-time, posed for the space-time metric together with the other relevant physical fields. This procedure, of course, depends on the matter models used to describe a particular physical situation, and thus each situation must be treated on its own. Nevertheless, there are some features common for several cases of interest. It was through the pioneer work of Y. Choquet-Bruhat that it was shown that the Einstein equations can be splitted into a set of evolution equations and a set of constraint equations, posed for the initial data of the evolution system, and that the evolution equations actually have a well-posed Cauchy problem in cases such as vacuum, electro-vacuum, and several fluid sources, namely, perfect fluids, dust, charged dust, etc [2].¹ In this way we see that, given an initial data set, the well-posedness of the problem depends on whether this data satisfies the following system of constraint equations on the initial space slice:

$$\begin{aligned} R_g - |K|_g^2 + (\text{tr}_g K)^2 &= 2\epsilon, \\ \nabla \text{tr}_g K - \text{div}_g K &= S, \end{aligned} \tag{1.2}$$

¹See [3] for a more modern and updated review.

where (ϵ, S) denote the energy and momentum densities on M induced by the matter fields, respectively, R_g represents the scalar curvature of the induced metric g on M , $|\cdot|_g$ denotes the pointwise-tensor norm in the metric g and $\operatorname{div}_g K$ denotes the divergence of K in the metric g .

The scenario described above triggered plenty of research in this area. A significant amount of effort was dedicated to showing that the constraint equations could be written as an elliptic system. Some of the pioneer work in this direction was due to Lichnerowicz, Choquet-Bruhat and York (see [4],[5],[6]). Their studies led to the conformal method as the standard tool for solving the constraints. This method consists in freely specifying a Riemannian metric on the initial manifold and fixing the mean curvature to a constant. Then, looking for a conformal metric to the one fixed, leads to an elliptic system for the conformal factor and the trace-free part of the extrinsic curvature on the initial slice. The conformal problem has been widely studied both for the compact case and the asymptotically Euclidean case, which define two cases of interest (see [3] and [7] for recent reviews). It should be noted that these problems are intimately related with the problem of fixing scalar curvature on a given manifold, which has received a considerable amount of attention in geometric analysis (see [8] for a thorough review). Also, a great deal of research has been drawn to the problem of relaxing the constant mean curvature condition and looking for low regularity results [7],[9].

It is important to remark that a great deal of the global aspects about space-time is related to properties that can be extracted from the initial value formulation of GR. Some of these problems have been posed as conjectures regarding generic solutions of the Einstein equations, such as the cosmic censorship conjecture (still active object of research) or the positive energy problems, which have been solved by R. Schoen and S. T. Yau [10].² Also, long-time stability problems are an active object of study, with plenty of open problems.

It is a remarkable fact that plenty of the global properties about the solutions of the Einstein equations can be condensed on the constraint equations. Furthermore, some generic properties about the solutions of the Einstein equations can be reduced to properties of the solutions of the constraint equations. For instance, it has recently been shown by R. Schoen, R. Beig and P. T. Chruściel that, for a variety of situations, the set of pos-

²The positive energy problem in arbitrary dimensions has been an open issue until this year. E. Witten provided a partial solution to this problem in [20] and Schoen-Yau proved the general case in [21].

sible initial data evolving into space-times without continuous symmetries forms a dense subset in the space of solutions of the constraint equations [11]. In this line, it had been shown by A. E. Fischer, J. Marsden and V. Moncrieff that around these generic initial data, the space of solutions has a manifold structure [12].

Related to this last problem we have Wheeler's conjectures [13],[14].³ In this context space-time is seen as a history of space-geometries. After shortly reviewing some background material in chapter 2, in chapter 3 we will be centred in studying one of these conjectures, known as the *thin sandwich conjecture*. There, we will define the set of *geometries* on a given manifold M as the superspace related to it, which is denoted by $S(M)$ [15]. The thin sandwich conjecture stated that given a point in superspace, represented by a pair (M, g) , and a tangent vector to it $\dot{g}$, there is a unique space-time development for this initial data satisfying the Einstein equations, describing a curve in $S(M)$. Note that, for this statement to be even precise, we need to show that superspace actually has a manifold structure, at least around those point where the conjecture can be posed. It is clear that this conjecture cannot hold for arbitrary data, as has been shown in [16]. For the 3-dimensional case, G. Fodor and R. Bartnik have shown that in a neighbourhood of solutions of the constraint equations satisfying some specific geometric properties, the thin sandwich conjecture holds [17].⁴ The geometric conditions we are referring to, are a *constraint* on the scalar curvature, explicitly $2\epsilon - R(g) > 0$; a technical condition on the extrinsic curvature, which is used to guarantee the ellipticity of the constraint equations in their thin sandwich formulation and, finally, another technical condition concerning the kernel of the linearization of the differential operator involved in this problem. In chapter 3, we will extend this result to arbitrary dimensions and, furthermore, we will show that those points around which the conjecture holds, are, in some precise sense, a generic property in the compact setting. By this we mean that on any given compact manifold M , there are initial data sets in a neighbourhood of which the thin sandwich problem is well-posed [18]. This is a novel result even for the 3-dimensional case.

An important consequence of the results of chapter 3 is that, around the points where the results described above hold, the thin sandwich problem in vacuum can be consistently pictured as giving as initial data an element $(g, \dot{g}) \in T_g S(M)$ and finding a unique space-

³For a detailed review on these conjectures see [22].

⁴The specific topologies in which such neighbourhoods are considered will be properly defined in chapter 3.

time related to this initial data, represented by a curve in $S(M)$. It is quite interesting that this result gives us the tools for studying embeddings of compact Lorentzian manifolds in Ricci-flat spaces, which will be the content of Chapter 4. In this chapter our main motivation is the well-known Campbell-Magaard theorem together with its extensions [23]-[24],[25],[26],[27],[28]. These results guarantee the existence of local isometric embeddings for semi-Riemannian n -dimensional manifolds with analytic metrics, around any given point, in $(n + 1)$ -dimensional manifolds with specific geometric properties, namely, being Ricci-flat, an Einstein-space or sourced by a scalar field. Such theorems have been widely used in the context of higher-dimensional space-time theories [29],[30],[31],[32],[33]. The applications of such theorems in this context is related with trying to guarantee that the classical space-times resulting as solutions of the Einstein equations, can be embedded in these higher-dimensional structures.

The previous ideas have been quite popular in the search for modified gravitational theories, but, nevertheless, the applicability of the Campbell-Magaard theorems in this context is limited. This is because the analyticity assumption is not well-suited to model realistic physical properties. For instance, the domain of dependence properties, which are physically related to the finite-propagation speed for physical interactions, would not work if analytic initial data were the only type of data allowed. Furthermore, the Campbell-Magaard theorems, which are proven by means of the Cauchy-Kovalevskaya theorem, do not guarantee Cauchy-stability in the higher-dimensional bulk. By this we mean that, considering the space-time metric as initial data for the embedding problem, there is no guarantee that the embedding is stable under small perturbations of the space-time metric, even near the *brane* representing space-time. In contrast, a conjectured C^k theorem proven using hyperbolic theory, would be better suited for preserving these two properties, which are vital for relativistic physical theories. Such properties, for instance, are standard (and very important) properties that can be extracted from the initial value formulation of GR for the 4-dimensional classical space-time. Furthermore, it is important to note that the claim that general solutions of GR can be embedded in these higher-dimensional space-times require global theorems, not local ones, as the Campbell-Magaard theorems guarantee.⁵

The remarks made above give us the starting point of Chapter 4, where we will also

⁵A discussion on these topics can be found in [34]-[35].

motivate the pure mathematical interest in the study of global and non-analytic embedding problems for Lorentzian manifolds in Ricci-flat spaces. In this chapter, using tools developed in chapter 3, we will show a C^k -embedding theorem for compact Lorentzian manifolds in Ricci-flat manifolds. Even though it seems that compact space-times are not best suited to model physical space-times, we have been able to drastically drop the regularity assumptions and address the global problem using a new strategy, which we consider a first step towards sharper results better suited for more realistic situations. Furthermore, the sharpest result known so far for this type of embeddings, due to R. Greene, is a much stronger statement, which guarantees the isometric embedding of any compact n -dimensional pseudo-Riemannian manifold in an $n(n+5)$ Riemann-flat space. Compared with this result, by relaxing the Riemann-flat condition, we have been able to drastically drop the number of codimensions needed, saving, in the case of 4-dimensional Lorentzian manifolds, 26-extra dimensions [36].⁶

Continuing in the line of alternative space-time theories, in chapter 5 we will focus our attention on Weyl manifolds. As it is quite well-known, these type of structures were first proposed by H. Weyl in an attempt of describing the electromagnetic field as a manifestation of geometric properties of space-time [37]. In this direction, by relaxing the compatibility condition between the metric and the connection, he proposed a geometric structure which had more degrees of freedom than Riemannian geometry [38]. On this setting he proposed a new theory, which attracted much attention during the first half of the twentieth century. Despite the elegance of Weyl's theory, Einstein pointed out that a space-time modelled by a Weyl structure would exhibit a *second clock effect*, which he regarded as a fatal flaw. Nevertheless, it is true that Einstein's argument, even though physically well-guided, was never put into rigorous terms. Because of this, we will begin chapter 5 by doing a detailed analysis of whether a general Weyl structure does exhibit such an effect without invoking any additional assumptions. We will do this analysis within the well-known axiomatic framework for space-time proposed by Ehlers, Pirani and Schild in [39], where, interestingly enough, Weyl structures appear quite naturally as a model for space-time, and it is only after adding further axioms that the Weyl structure is reduced to a Lorentzian one.

Finally, after motivating the physical interest on Weyl structures, we will analyse

⁶The precise regularity needed for the Lorentzian metric is dimension-dependent, but smooth metrics are always allowed.

some embedding problems within this broader framework. In this line we will investigate whether the Campbell-Magaard theorem extends to this new scenario. Again, these results are of interest for higher-dimensional theories based on Weyl's geometry. We will see that, even though for the Weyl integrable case, which has attracted the attention of some cosmologists [40],[41],[42], the Campbell-Magaard theorem extends naturally, for the general case we will present some rather unexpected no-go theorems, and a particular result for the 3-dimensional case where the theorem holds.

Following the ideas described in the above paragraphs, we will conclude this thesis by analysing the initial value formulation of particular type of geometric scalar-tensor theories, in which space-time is modelled by a Weyl-integrable structure, where we will conclude that the vacuum case is well-posed.

Chapter 2

Background on differential geometry

This chapter is intended to present some general definitions and background material on differential geometry. We will assume the reader to be familiarized with differentiable manifolds, Riemannian and semi-Riemannian geometry. We will begin by establishing some definitions and notations and then quickly go to the study of submanifolds. After presenting a review on this topic, we will present an overview of Weyl manifolds, which will be used in Chapters 5 and 6. Since these structures, in some way, generalize semi-Riemannian ones, we will present the theory of Weyl-submanifolds and the results we will need for semi-Riemannian submanifolds can be recovered by setting Weyl's 1-form to zero.

For further details in the theory of manifolds and submanifolds we refer the reader to the following standard textbooks [44],[45],[46],[47].

2.1 Differentiable Manifolds

2.1.1 Conventions

Throughout this thesis we will take all manifolds to be Hausdorff, second-countable topological spaces equipped with a complete atlas, and we will only work with smooth manifolds. When dealing with semi-Riemannian manifolds, we will typically write (M, g) , where g is some semi-Riemannian metric defined on M , and we will typically assume g to be smooth. This will not be the case in some parts of chapters 3 and 4, where we will explicitly state the functional spaces where the different tensor fields are supposed to live

in. Also, when working in the context of semi-Riemannian geometry, we will assume we are working with the associated Riemannian connection, usually denoted by ∇ , defined to be the unique torsionless connection *compatible* with the given semi-Riemannian metric. When dealing with Weyl manifolds, we will make explicit the changes in notation.

Finally, we will denote by $T_q^p M \xrightarrow{\pi} M$ the vector bundle over M whose typical fibre over a point $m \in M$ is the vector space of tensors of type (p, q) . We will denote the set of tensor fields of type (p, q) , which are defined to be sections of such tensor bundles, by $\Gamma(T_q^p M)$. Such tensor fields, unless explicitly stated, will be assumed to be smooth.

2.2 Submanifolds

We will now present the basic definitions and properties concerning submanifolds. Since the main content of this thesis is concerned with the study of embeddings and submanifolds in the context of modern gravitational theories, during this section we will go into some details and try to be thorough.

Definition 2.2.0.1. A manifold P is a submanifold of a manifold M if:

- 1) P is a topological subspace of M .
- 2) The inclusion map $i : P \hookrightarrow M$ is smooth and for each $p \in P$ is di_p is 1-1.

Recall that a subset P (with a topology of its own) of a topological space M is a topological subspace if its topology is the induced topology from M . Also note that if P is a submanifold of M and $\phi : M \rightarrow N$ is smooth, then $\phi|_P \doteq \phi \circ i$ is also smooth, since i is smooth by definition. When the context makes it clear, we will sometimes omit the map i , since it is an obvious mapping. In this way, we will use injectivity of di to state that $T_p P$ is a linear subspace of $T_p M$, omitting the inclusion map.¹

Example 2.2.0.1. Any open set of M , with the induced topology, naturally acquires the structure of a submanifold. If $\{(V_\alpha, \mu_\alpha), \alpha \in \mathcal{I}\}$ is an open cover of M by coordinate charts, then, there is a subset of this covering $\mathcal{A} = \{(V_j, \mu_j), j \in \mathcal{J} \subset \mathcal{I}\}$ which covers P . Also, if we give the induced topology to P , since $V'_i \doteq V_i \cap P$ is an open set of P (by definition of the induced topology), then the coordinate systems of M produce coordinate systems for P since $\mu_j \circ \mu_i^{-1}|_P : \mu_i(V'_j \cap V'_i) \subset \mu_i(V_j \cap V_i) \mapsto \mu_j(V'_j \cap V'_i) \subset \mathbb{R}^m$ are

¹The “more precise” thing to say would be that $di_p(T_p P)$ is a subspace of $T_{i(p)} M$.

diffeomorphisms. Thus, by considering the maximal atlas on P constructed from $\mathcal{A}$, any open subset P of M naturally gets the structure of a differentiable manifold of the same dimension than M , whose topology is the induced topology from M . Also, given the inclusion map $i : P \hookrightarrow M$, $p \mapsto p$, we can compute the coordinate expression of such a map using these induced coordinate systems: $\mu_i \circ i \circ \mu_i^{-1} : \mu_i(V'_i) \mapsto \mathbb{R}^m$, $x \mapsto x$. Thus $\mu_i \circ i \circ \mu_i^{-1} = Id$, so we get $di_p = Id : T_p P \mapsto T_p M$, hence P is a submanifold of M .

Example 2.2.0.2. A slightly less trivial example is the unit sphere S^n in $\mathbb{R}^{n+1}$. We can again give to it the induced topology, so as to make it a topological subspace. In this case it is standard how to produce a covering by coordinate systems and a maximal atlas. In particular, around any point, we know there is a coordinate system $\{(x^1, \dots, x^n) : U \subset S^n \mapsto \mathbb{R}^n\}$, such that the inclusion map $i : S^n \mapsto \mathbb{R}^{n+1}$ has the following coordinate expression: $i \circ \mu^{-1}(x) = (x^1, \dots, x^n, \sqrt{1 - (x^1)^2 - \dots - (x^n)^2})$. Thus

$$di_i^\alpha = \frac{\partial(i \circ \mu^{-1})^\alpha}{\partial x^i} = \left[\begin{array}{ccc} Id_{n \times n} & & \\ \frac{-x^1}{\sqrt{1 - (x^1)^2 - \dots - (x^n)^2}} & \dots & \frac{-x^n}{\sqrt{1 - (x^1)^2 - \dots - (x^n)^2}} \end{array} \right]$$

which shows that the inclusion is injective, and thus that the S^n is a submanifold of $\mathbb{R}^{n+1}$.

Example 2.2.0.3. Given a coordinate system $\mu = (x^1, \dots, x^m) : U \subset M \mapsto \mathbb{R}^m$ on some manifold M , consider the subset of $\mu(U)$ given by $\overline{P} \doteq \{x \in \mu(U) / x^{n+1}, \dots, x^m \text{ are constant}\}$. Then take $P \doteq \mu^{-1}(\overline{P}) \subset U \subset M$. On this set consider the induced topology, so as to make it a topological subspace of M . Also, clearly,

$$\begin{aligned} \mu|_P : P &\mapsto \mathbb{R}^m \\ p &\mapsto (x^1, \dots, x^n, c^1, \dots, c^{m-n}) \end{aligned}$$

gives a coordinate system on P , denoted by $\tilde{\mu}$, constructed by considering the projection of $\mu|_P$ onto its first n -coordinates, which actually covers P , and thus we can construct a complete atlas for P by considering all the admissible coordinate systems compatible with this one. We thus give P a manifold structure. Finally, using this coordinate system, it is clear that

$$di_i^\alpha = \frac{\partial(\mu \circ i \circ \tilde{\mu}^{-1})^\alpha}{\partial x^i} = \left[\begin{array}{c} Id_{n \times n} \\ 0_{(m-n) \times n} \end{array} \right]$$

and thus $i : P \mapsto M$ is injective. So P is a submanifold of M .

Submanifolds constructed as in the previous example are called μ -**coordinate slices** of $U \subset M$.

Definition 2.2.0.2. Let P be a subset of M^m . We say that a coordinate system $\mu : U \mapsto \mathbb{R}^m$ is adapted to P if $U \cap P$ is a μ -coordinate slice of U .

Suppose that P is a submanifold of M and that $\mu = (x^1, \dots, x^m) : U \mapsto \mathbb{R}^m$ is a coordinate system for M adapted to P . Without loss of generality, we can assume that the last $(m - n)$ -coordinates are the ones which are constant on $U \cap P$. By hypothesis $U \cap P$ is open in P . Consider the projection $\pi : \mathbb{R}^m \mapsto \mathbb{R}^n$, $(x^1, \dots, x^m) \mapsto (x^1, \dots, x^n)$, which projects onto the first n -coordinates. Then

$$\begin{aligned}\tilde{\mu} &\doteq \pi \circ \mu \circ i : U \cap P \mapsto \pi(\mu(U \cap P)) \subset \mathbb{R}^n \\ p &\mapsto (x_p^1, \dots, x_p^n)\end{aligned}$$

is smooth since it is composition of smooth maps. Similarly, $\tilde{\mu} \circ \mathcal{Y}^{-1}$ is a smooth map from $\mathbb{R}^n$ to $\mathbb{R}^n$, for any coordinate system $\mathcal{Y} = (y^1, \dots, y^n)$ on P , since, again, it is a composition of smooth maps (note that we are using that the inclusion is smooth, which comes from the hypothesis that P is a submanifold). Finally, note that, since $\mu \circ i : U \cap P \mapsto \mathbb{R}^m$ is given by $\mu \circ i(p) = (x^1, \dots, x^n, c^1, \dots, c^{m-n})$, then:

$$\frac{\partial(\mu \circ i \circ \mathcal{Y}^{-1})^\alpha}{\partial y^i} = \begin{bmatrix} \left(\frac{\partial x^i}{\partial y^j}\right)_{n \times n} \\ 0_{(m-n) \times n} \end{bmatrix}$$

Since P is a submanifold, then $di_p : T_p P \mapsto T_p M$ is injective. The previous expression is just a coordinate expression for di , which implies that the previous matrix must have rank n , and thus that

$$\frac{\partial(\pi \circ \mu \circ i \circ \mathcal{Y}^{-1})^j}{\partial y^i} = \left(\frac{\partial x^j}{\partial y^i}\right)_{n \times n}$$

is an isomorphism for each y_p , $p \in U \cap P$. Then, by means of the inverse function theorem, it is a diffeomorphism around each of these points, and thus $\tilde{\mu}$ is an admissible coordinate system for P . So we get that each coordinate system of M adapted to some submanifold P naturally induces a coordinate system on P . In fact we have much more than this. The following proposition is a standard (and very useful) characterization for submanifolds.

Proposition 2.2.0.1. *If P^n is a submanifold of M^m , then around each $p \in P$ there is a coordinate system of M adapted to P .* $\square$

Corollary 2.2.0.1. *Let P^k be a submanifold of M^m . If $\phi : N^n \mapsto M^m$ is a smooth mapping such that $\phi(N) \subset P$, then the induced map $\bar{\phi} : N \mapsto P$ is smooth.*

Proof. What we need to check in order to prove the corollary, is that the coordinate expression of $\bar{\phi}$ is smooth. So, consider a coordinate system $(x^1, \dots, x^m) : V \mapsto \mathbb{R}^m$ on M around $\phi(q)$, for some $q \in N$, adapted to P . In order to compute the coordinate expression of $\bar{\phi}$ using such coordinate system, we need a coordinate system $\mathcal{Y} : U \subset N \mapsto \mathbb{R}^n$ on N around q such that $\phi(U) \subset V \cap P$. Take $\phi^{-1}(V \cap P)$ which is an open neighbourhood of q in N since ϕ is continuous. Then take a coordinate neighbourhood $(U, \mathcal{Y})$ of q and restrict it to $W \doteq U \cap \phi^{-1}(V \cap P)$ which is open in N , i.e., consider $(W, \mathcal{Y})$. Finally, notice that the coordinate expression of $\bar{\phi}$ is given by

$$\begin{aligned}\bar{\phi}^i(y) &= x^i|_P \circ \bar{\phi} \circ \mathcal{Y}^{-1}(y) = x^i \circ i \circ \bar{\phi} \circ \mathcal{Y}^{-1}(y), \\ &= x^i \circ \phi \circ \mathcal{Y}^{-1}(y), \quad i = 1, \dots, k \\ &= \pi^i(x^1 \circ \phi \circ \mathcal{Y}^{-1}(y), \dots, x^m \circ \phi \circ \mathcal{Y}^{-1}(y)),\end{aligned}$$

where π^i is the projection onto the i th-coordinate. Since by hypothesis ϕ is smooth, the previous expression is smooth. $\square$

Corollary 2.2.0.2. *A subset P of a smooth manifold M can be made a submanifold of M in at most one way.*

Proof. In order for P to be a submanifold, it must have the induced topology. Thus, once we have an atlas on P , and thus P is manifold on its own, we do not have any freedom left. Thus, if there are two different ways (resulting in *different* submanifolds) to make P a submanifold, it means that there are two different complete atlases generating the manifolds P_1 and P_2 such that both manifolds are submanifolds of M . In such a case, the inclusions $i_{1,2} : P_{1,2} \mapsto M$ are smooth. Also $i_1(P_1) \subset P_2$. Hence the previous corollary gives us that $\bar{i}_1 : P_1 \mapsto P_2$ is smooth. In fact, this map is the identity between the two sets. Similarly we have that $\bar{i}_2 : P_2 \mapsto P_1$ is smooth. In particular these maps are inverses of each other, and thus P_1 and P_2 are diffeomorphic. $\square$

Notice that this corollary states that it is sensible to ask the question of whether or not a subset of a manifold is a submanifold, since, either way, there is no ambiguity in the answer. A basic criterion for determining this, is given in the following proposition.

Proposition 2.2.0.2. *A subset P of a manifold M^m is an n -dimensional submanifold iff around each point $p \in P$ there is a coordinate system of M adapted to P by an n -coordinate slice.*

Proof. The first part is **Proposition 2.2.0.1**. For the converse, see [44] for the details. The idea is that once we give the induced topology to P , we have to prove that the adapted coordinate systems patch together nicely in order to form an atlas for P , giving a manifold structure to P . Once this is established, the fact that P is a submanifold can be easily proved using such adapted coordinate systems. $\square$

2.2.1 Immersions and Embeddings

Definition 2.2.1.1. An **immersion** $\phi : M \mapsto N$ is a differentiable map such that $d\phi_p$ is 1-1 for all $p \in M$.

The following lemma gives us a useful characterization of the injectivity property required for immersions.

Lemma 2.2.1.1. *Let $\phi : M^m \mapsto N^n$ be smooth and $p \in M$. Then the following are equivalent:*

- 1) $d\phi_p$ is 1-1.
- 2) The Jacobian matrix of $d\phi_p$ has rank m .
- 3) If $(y^1, \dots, y^n)$ is a coordinate system on N in a neighbourhood of $\phi(p)$, then there are integers $1 \leq i_1 \leq \dots \leq i_m \leq n$ such that the functions $y^{i_1} \circ \phi, \dots, y^{i_m} \circ \phi$ are a coordinate system around $p \in M$.

Proof. 1) $\Leftrightarrow$ 2) is something well-known from linear algebra.

2) $\Rightarrow$ 3)

Since the Jacobian matrix of $d\phi_p$ has rank m , we know that it has m -linearly independent rows. We can assume that the first m -rows are actually the independent ones

(if necessary, we can just relabel the coordinates in order to get this). Thus, given a coordinate system $\mu = (x^1, \dots, x^m) : U \subset M \mapsto \mathbb{R}^m$ on M , we get that

$$\left(\frac{\partial y^i \circ \phi \circ \mu^{-1}(x_p)}{\partial x^j} \right)_{m \times m}, \quad i, j = 1, \dots, m$$

is invertible. Thus, applying the inverse function theorem, there is a neighbourhood V of $x_p \in \mathbb{R}^m$ such that $\tilde{\xi} \doteq (y^1 \circ \phi \circ \mu^{-1}, \dots, y^m \circ \phi \circ \mu^{-1}) : V \mapsto \tilde{\xi}(V)$ is a diffeomorphism. This implies that $\xi \doteq (y^1 \circ \phi, \dots, y^m \circ \phi)$ is an admissible coordinate system for M .

3) $\Rightarrow$ 2)

Pick a coordinate system $\mu = (x^1, \dots, x^m) : U \mapsto \mathbb{R}^m$ around $p \in M$. Since $(y^1, \dots, y^n)$ is a coordinate system on N around $\phi(p)$, we can compute $d\phi_p$ using these coordinates. That is:

$$d\phi_j^\alpha = \left(\frac{\partial y^\alpha \circ \phi \circ \mu^{-1}(x)}{\partial x^j} \right)_{n \times m}, \quad \alpha = 1, \dots, n ; \quad j = 1, \dots, m. \quad (2.1)$$

Then, if (3) holds, this matrix must have m -linearly independent rows, since by grouping together those y^{i_k} , $1 \leq k \leq m$, for which (3) holds, then $\frac{\partial y^{i_k} \circ \phi \circ \mu^{-1}}{\partial x^j}$, $k, j = 1, \dots, m$, is the Jacobian matrix of a coordinate change, and thus is invertible, thus all the rows in such matrix must be independent, and all these rows appear in (2.1). $\square$

Definition 2.2.1.2. An **embedding** $\phi : P \mapsto M$ of a manifold P in another manifold M is a 1 – 1 immersion such that the map $\phi : P \mapsto \phi(P)$ is an homeomorphism when $\phi(P)$ is seen with the induced topology.

Proposition 2.2.1.1. *Let P be a submanifold of a differentiable manifold M , then $i : P \mapsto M$ is an embedding.*

Proof. By hypothesis $i : P \mapsto M$ is smooth and di is injective at every point. Thus i is an immersion. Also, by hypothesis P has the induced topology from M , thus the induced map $\bar{i} : P \mapsto i(M)$ is just the identity, where both sets are the same set with the same topology, thus $\bar{i}$ is a homeomorphism. Hence $i : P \mapsto M$ is an embedding. $\square$

Proposition 2.2.1.2. *If $\phi : P \mapsto M$ is an embedding, then $\phi(P)$ is a submanifold of M .*

Proof. First give to $\phi(P)$ the induced topology. With this topology $\bar{\phi} : P \mapsto \phi(P)$ is an homomorphism. In order to give $\phi(P)$ a manifold structure we need to give an atlas.

Consider a covering by coordinate charts of P given by $\{U_\alpha, \mu_\alpha\}$. Let

$$\begin{aligned}\xi_\alpha &: \phi(U_\alpha) \subset \phi(P) \mapsto \mathbb{R}^n \\ \xi_\alpha &\doteq \mu_\alpha \circ \bar{\phi}^{-1}\end{aligned}$$

The claim is that $\{\phi(U_\alpha), \xi_\alpha\}$ gives a differentiable structure to $\phi(P)$. Clearly it gives an open covering, and on the overlaps $\xi_\alpha \circ \xi_\beta^{-1} = \mu_\alpha \circ \bar{\phi}^{-1} \circ \bar{\phi} \circ \mu_\beta^{-1} = \mu_\alpha \circ \mu_\beta^{-1}$, which is smooth by hypothesis. Thus we can consider the maximal atlas compatible with this differentiable structure, making $\phi(P)$ a manifold. Finally, we need to check that $i : \phi(P) \mapsto M$ is smooth and has injective differential. Note that we can write $i = \phi \circ \bar{\phi}^{-1}$, and then $di = d\phi \circ d\bar{\phi}^{-1}$, which is injective, which completes the proof. $\square$

Corollary 2.2.1.1. *If M and N are manifolds and $M \subset N$, then M is a submanifold of N iff $i : M \mapsto N$ is an embedding.* $\square$.

Lemma 2.2.1.2. *Let $\psi : M^m \mapsto N^n$ be smooth and let $p \in M$. Then the following are equivalent:*

- 1) $d\psi_p$ is surjective.
- 2) The Jacobian matrix of $d\psi_p$ has rank n .
- 3) If $y^1, \dots, y^n$ is a coordinate system on N around $\psi(p)$, then there is a coordinate system for M around p of the form $(y^1 \circ \psi, \dots, y^n \circ \psi, x^{n+1}, \dots, x^m)$.

Proof. 1) $\Leftrightarrow$ 2) is something well-known from linear algebra.

2) $\Rightarrow$ 3)

Suppose that $\eta = (x^1, \dots, x^m)$ is a coordinate system around $p \in M$ and $\xi = (y^1, \dots, y^n)$ a coordinate system around $\psi(p) \in N$. By hypothesis $\frac{\partial y^i \circ \psi \circ \eta^{-1}(x_p)}{\partial x^\alpha}$ has rank n . We can suppose that the first n -columns are the independent ones (if necessary we can relabel the x^α coordinates in order to get this). Then consider

$$\mu \doteq (y^1 \circ \psi, \dots, y^n \circ \psi, x^{n+1}, \dots, x^m) : U \subset M \mapsto \mathbb{R}^m$$

where U is a neighbourhood of $p \in M$. Then,

$$\frac{\partial(\mu \circ \eta^{-1})^\alpha(x_p)}{\partial x^\beta} = \left[\begin{array}{c|c} \left(\frac{\partial y^i \circ \psi \circ \eta^{-1}(x_p)}{\partial x^j} \right)_{n \times n} & A \\ \hline 0 & Id_{(m-n) \times (m-n)} \end{array} \right]$$

since $\frac{\partial y^i \circ \psi \circ \eta^{-1}(x_p)}{\partial x^j}$ is invertible, thus $\frac{\partial(\mu \circ \eta^{-1})^\alpha(x_p)}{\partial x^\beta}$ is an isomorphism, and hence the inverse function theorem guarantees that $\mu \circ \eta^{-1}$ is an diffeomorphism around x_p . Thus μ defines a coordinate system around $p \in M$.

3) $\Rightarrow$ 2)

Suppose $\xi = (y^1, \dots, y^n)$ is a coordinate system on N and $\mu = (y^1 \circ \psi, \dots, y^n \circ \psi, x^{n+1}, \dots, x^m) \doteq (z^1, \dots, z^m)$ a coordinate system on M . Then, noticing that $\xi \circ \psi \circ \mu^{-1}(z) = (z^1, \dots, z^n)$, we get

$$\frac{\partial y^j \circ \psi \circ \mu^{-1}(z)}{\partial z^\alpha} = \left[Id_{n \times n} \mid 0_{n \times (m-n)} \right]$$

which has rank n . □

Definition 2.2.1.3. A point $q \in N$ is called a **regular value** of $\psi : M \mapsto N$, with ψ smooth, if $d\psi_p$ is surjective $\forall p \in \psi^{-1}(q)$.

Corollary 2.2.1.2. If $q \in \psi(M)$ is a regular value of $\psi : M^m \mapsto N^n$, then $\psi^{-1}(q)$ is a submanifold of M and $\dim M = \dim N + \dim \psi^{-1}(q)$.

Proof. Using **Lemma 2.2.1.2**, since by hypothesis $d\psi_p$ is surjective for each $p \in \psi^{-1}(q)$, we have that, reordering the coordinates, there is a coordinate system around each such p of the form $(x^{n+1}, \dots, x^m, y^1 \circ \psi, \dots, y^n \circ \psi)$ defined on a neighbourhood $U \subset M$. In particular, $U \cap \psi^{-1}(q)$ is an $(m - n)$ -coordinate slice. Thus, we get that around any point in $\psi^{-1}(q)$ there is a coordinate system of M adapted to $\psi^{-1}(q)$. Then, appealing to **Proposition 2.2.0.2**, we get that $\psi^{-1}(q)$ is an $(m - n)$ -dimensional submanifold. Thus $\dim \psi^{-1}(q) = \dim M - \dim N$. □

Definition 2.2.1.4. A **hypersurface** in a manifold M is a submanifold P whose *codimension* ($\dim M - \dim P$) is 1.

Example 2.2.1.1. Consider $\psi : \mathbb{R}^{n+1} \mapsto \mathbb{R}$, given by $\psi(x) = \sum_{i=1}^{n+1} (x^i)^2$. Let $q = 1 \in \mathbb{R}$. Then $\psi^{-1}(q) = S^n(1)$ is the unit n -sphere. Noticing that the Jacobian matrix of this

mapping is $d\psi_i = 2(x^1, \dots, x^{n+1})_{1 \times (n+1)}$, we see that $d\psi_x : T_x \mathbb{R}^{n+1} \mapsto T_{\psi(x)} \mathbb{R}$ is surjective $\forall x \neq 0$. Thus, in particular, $q = 1$ is a regular value of ψ , and hence S^n is an n -dimensional hypersurface of $\mathbb{R}^{n+1}$.

Example 2.2.1.2. Consider an n -dimensional manifold M , and construct the $(n + 1)$ -dimensional manifold $V \doteq \mathbb{R} \times M$. Consider the mapping

$$\begin{aligned} f : V &\mapsto \mathbb{R} \\ (t, x) &\mapsto t \end{aligned}$$

In this case, using adapted product coordinates, we get $df_i = (1, 0, \dots, 0)_{1 \times (n+1)}$, and we see that df is always surjective, and hence the sets $\{t\} \times M \cong M$ are n -dimensional hypersurfaces in V diffeomorphic to M . This simple structure, results of substantial interest in general relativity.

We will finish this section with the following definition.

Definition 2.2.1.5. A **submersion** $\psi : M \mapsto B$ is a smooth surjective map, such that $d\psi_p$ is surjective $\forall p \in M$.

Notice that in this case every point in B is a regular value of ψ , thus the sets $\psi^{-1}(q)$, $q \in B$, produce a slicing of M by submanifolds.

Examples of such mappings are projections on product manifolds, or even on fibre bundles. Also, the last example given before the above definition gives a submersion. In that example we can foliate a “*space-time*” by diffeomorphic “space-slices”.

2.3 Weyl structures

When working with Riemannian geometry we consider a pair (M, g) where M is a differentiable manifold and g a (semi)-Riemannian metric defined on M . In this context we know that there is unique torsionless linear connection *compatible* with g . We will shortly review what compatibility means in the last sentence. When we endow any differentiable manifold with a linear connection ∇ we have an associated notion of parallel transport. It is well known that parallel transport defines isomorphisms between tangent spaces. The compatibility condition is defined as the requirement that this isomorphism is also an

isometry. This turns out to be equivalent to the following requirement

$$\nabla g = 0.$$

When working with Weyl's geometry we relax the requirement of ∇ being compatible with g , this means that parallel transports are not required to define isometries any more. Instead, we propose as a starting point the following. We endow M not only with a semi-Riemannian metric, but also with a 1-form field ω , so now we consider a triple (M, g, ω) . Weyl's connection is defined by requiring it to be torsionless and that, for any parallel vector field V along any smooth curve γ , the following condition is satisfied:

$$\frac{d}{dt}g(V(t), V(t)) = \omega(\gamma'(t))g(V(t), V(t)). \quad (2.2)$$

Before proving the existence and uniqueness of such a connection we will try to get some insight on what this condition means geometrically.

First of all, note that if V, W are parallel fields along some curve γ , then, since parallel transport is a linear application, $V+W$ also is a parallel field along γ . Also by polarization we get

$$g(V, W) = \frac{1}{2}(g(V+W, V+W) - g(V, V) - g(W, W)).$$

Thus, using (2.2), we get

$$\begin{aligned} \frac{d}{dt}g(V, W) &= \frac{1}{2}(\omega(\gamma')g(V+W, V+W) - \omega(\gamma')g(V, V) - \omega(\gamma')g(W, W)), \\ &= \omega(\gamma')g(V, W). \end{aligned}$$

Hence we have that if ∇ is *Weyl compatible* with (M, g, ω) then for any pair of parallel vectors along any smooth curve γ it holds that

$$\frac{d}{dt}g(V(t), W(t)) = \omega(\gamma'(t))g(V(t), W(t)).$$

We can solve this last expression and get that

$$g(V(t), W(t)) = g(V(0), W(0))e^{\int_0^t \omega(\gamma'(s))ds}. \quad (2.3)$$

In particular, if $V = W$, this last expression gives us precisely how much parallel transport fails to be an isometry:

$$g(V(t), V(t)) = g(V(0), V(0)) e^{\int_0^t \omega(\gamma'(s)) ds}.$$

Also note that (2.3) tells us that if $V(0)$ and $W(0)$ are orthogonal, then they remain orthogonal when parallel transported along some curve (although their respective "norms" may change).

Now we will establish analogue results for a Weyl connection to those known for Riemannian connections.

Proposition 2.3.0.1. *A connection ∇ is compatible with a Weyl structure (M, g, ω) iff for any pair of vector fields V, W along any smooth curve γ in M it holds that*

$$\frac{d}{dt}g(V, W) = g\left(\frac{DV}{dt}, W\right) + g\left(V, \frac{DW}{dt}\right) + \omega(\gamma')g(V, W) \quad (2.4)$$

Proof. $\Leftarrow$) Suppose (2.4) holds for any pair of vector fields along any curve γ in M . In particular it holds if V, W are parallel, and if $V = W$. So in this case, since $\frac{DV}{dt} = 0$, (2.4) gives us

$$\frac{d}{dt}g(V, V) = \omega(\gamma')g(V, V)$$

and hence ∇ is compatible with the Weyl structure.

$\Rightarrow$) Consider an orthogonal basis $\{e_1(p), \dots, e_n(p)\}$ for $T_p M$ and consider any smooth curve γ passing through p such that $\gamma(0) = p$. Let $\{P_1, \dots, P_n\}$ be the set of parallel vector fields along γ such that $P_i(0) = e_i(p)$. We know that they not only remain linearly independent, but also orthogonal at any point of the curve. So, they form an orthogonal basis of the tangent space at that point. This means that we can write

$$V = V^i(t)P_i(t) \quad ; \quad W = W^i(t)P_i(t)$$

and then

$$\frac{DV}{dt} = \frac{dV^i}{dt}P_i \quad ; \quad \frac{DW}{dt} = \frac{dW^i}{dt}P_i.$$

Thus we get that

$$\begin{aligned}
g\left(\frac{DV}{dt}, W\right) + g\left(V, \frac{DW}{dt}\right) &= \frac{dV^i}{dt} W^j g(P_i, P_j) + V^i \frac{dW^j}{dt} g(P_i, P_j) \\
&= \sum_{i=1}^n \left(\frac{dV^i}{dt} W^i + V^i \frac{dW^i}{dt} \right) g(P_i, P_i) \\
&= \frac{d}{dt} \left(\sum_{i=1}^n g(P_i, P_i) V^i W^i \right) - \sum_{i=1}^n \frac{d}{dt} g(P_i, P_i) V^i W^i \\
&= \frac{d}{dt} g(V, W) - \sum_{i=1}^n \omega(\gamma') g(P_i, P_i) V^i W^i.
\end{aligned}$$

So finally we get

$$g\left(\frac{DV}{dt}, W\right) + g\left(V, \frac{DW}{dt}\right) = \frac{d}{dt} g(V, W) - \omega(\gamma') g(V, W).$$

□

Corollary 2.3.0.1. *A linear connection ∇ is compatible with a Weyl structure (M, g, ω) iff $\forall p \in M$ and for every vector fields X, Y, Z on M it holds that*

$$X_p(g(Y, Z)) = g_p(\nabla_{X_p} Y, Z_p) + g_p(Y_p, \nabla_{X_p} Z) + \omega_p(X_p) g_p(Y_p, Z_p) \quad (2.5)$$

Proof. Just consider the previous proposition with $\gamma(0) = p$ and $\gamma'(0) = X_p$ □

In the last proposition we can actually drop the p and write

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) + \omega(X) g(Y, Z)$$

and use this to prove the following.

Proposition 2.3.0.2. *A linear connection ∇ is compatible with a Weyl structure (M, g, ω) iff it satisfies*

$$\nabla g = \omega \otimes g. \quad (2.6)$$

Proof. $\Leftarrow$) This is just a computation

$$\begin{aligned}
X(g(Y, Z)) &= \nabla_X(g(Y, Z)) \\
&= (\nabla_X g)(Y, Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\
&= \omega(X)g(Y, Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z)
\end{aligned}$$

Where in the last line we used our hypothesis, and this last equality, by means of the previous corollary, gives that ∇ is compatible with (M, g, ω) .

$\Rightarrow$) Consider X, Y, Z arbitrary vector fields. Then the previous corollary gives us the following

$$\begin{aligned}
(\nabla g)(X, Y, Z) &\stackrel{\text{def}}{=} (\nabla_X g)(Y, Z) \\
&= X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) \\
&\stackrel{\text{hyp}}{=} \omega(X)g(Y, Z).
\end{aligned}$$

Thus,

$$\nabla g = \omega \otimes g.$$

□

Now the following result is easily established.

Proposition 2.3.0.3. *There is a unique torsionless connection compatible with the Weyl structure (M, g, ω) .*

Proof. Consider a torsionless connection ∇ on M . Then in any coordinate system we have

$$\begin{aligned}
\nabla_a g_{bc} &= \partial_a g_{bc} - \Gamma_{ab}^u g_{uc} - \Gamma_{ac}^u g_{bu}, \\
\nabla_b g_{ca} &= \partial_b g_{ca} - \Gamma_{bc}^u g_{ua} - \Gamma_{ba}^u g_{cu}, \\
\nabla_c g_{ab} &= \partial_c g_{ab} - \Gamma_{ca}^u g_{ub} - \Gamma_{cb}^u g_{au}.
\end{aligned}$$

Then we have that

$$\nabla_a g_{bc} + \nabla_c g_{ab} - \nabla_b g_{ca} = \partial_a g_{bc} + \partial_c g_{ab} - \partial_b g_{ca} - 2\Gamma_{ac}^u g_{bu}.$$

Hence, if ∇ satisfies (2.6), we have that

$$\Gamma_{ac}^u = \frac{1}{2}g^{bu}(\partial_a g_{bc} + \partial_c g_{ab} - \partial_b g_{ca}) + \frac{1}{2}g^{bu}(\omega_b g_{ca} - \omega_a g_{bc} - \omega_c g_{ab}) \quad (2.7)$$

Thus, we see that there is a connection compatible with the Weyl structure, and that in any coordinate system the components of such connection read as in (2.7), which proves uniqueness. $\square$

It is important to note that a Weyl manifold defines an equivalence class of such structures all linked by the following transformations

$$\begin{cases} \bar{g} = e^{-f} g \\ \bar{\omega} = \omega - df \end{cases}$$

where f is a smooth function defined on M . It is easy to check that this transformations define an equivalence relation between Weyl manifolds, and that if ∇ is compatible with (M, g, ω) then it is also compatible with $(M, \bar{g}, \bar{\omega})$. So every member in the class is compatible with the same connection and, because of that, will have the same geodesics, curvature tensor and every other property that depends only on the connection. This is the reason why, when working with Weyl manifolds, it is the whole equivalence class $(M, [g, \omega])$ that is considered of interest and not only a specific element in the class. This issue will play a fundamental role when we discuss Weyl structures as *possible* models for space-time. We finish this section with the following definition.

Definition 2.3.0.1. Given a Weyl structure $(M, [g, \omega])$, if the 1-form field is an exact form, *i.e.* if there exists a (smooth) function on M such that $\omega = df$, then we will say that the Weyl structure is **integrable**, and denote it by $(M, [g, f])$.

2.3.1 Symmetries of the curvature tensor

In this section we intend to show what the Bianchi identities look like in the context of Weyl geometry. These identities will be useful when studying embedding problems. We

will consider all quantities defined on an n -dimensional Weyl manifold. We will begin by adopting the following sign convention for the curvature tensor:

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, \quad (2.8)$$

where X, Y and Z are vector fields on M . When writing coordinate expressions, we will adopt the following notations:

$$R_{kji}^l = dx^l(R(\partial_i, \partial_j)\partial_k). \quad (2.9)$$

We have the associated $(0, 4)$ curvature tensor, defined by

$$R(V, Z, Y, X) = \langle R(X, Y)Z, V \rangle, \quad (2.10)$$

which has coordinates

$$R_{lkji} = \langle R(\partial_i, \partial_j)\partial_k, \partial_l \rangle. \quad (2.11)$$

Before going to the symmetries of the curvature tensor, note that since tensors are pointwise objects, we can look at the symmetries at a given point.

Proposition 2.3.1.1. *If $x, y, z, v, w \in T_p M$, then*

- 1) $R(x, y)z = -R(y, x)z$
- 2) $\langle R(x, y)z, w \rangle + \langle R(x, y)w, z \rangle = 2\langle w, z \rangle F(y, x)$
- 3) $R(x, y)z + R(y, z)x + R(z, x)y = 0$ (*First Bianchi identity*),
- 4) $\langle R(v, w)x, y \rangle - \langle R(x, y)v, w \rangle = F(x, y)g(v, w) + F(w, v)g(x, y) - F(y, v)g(x, w) - F(w, y)g(v, x) - F(x, w)g(v, y) - F(v, x)g(y, w),$

where the tensor $F \doteq d\omega$, i.e., $F_{ij} = \frac{1}{2}(\nabla_i \omega_j - \nabla_j \omega_i) = d\omega_{ij}$.

Proof. (1) is a simple consequence of the definition of the curvature tensor. In order to prove the other identities, consider extensions of the vectors x, y, z, v, w to vector fields X, Y, Z, V, W defined in a neighbourhood of $p \in M$ such that the Lie brackets between these extensions are zero (in order to do this simply extend the vectors to vector fields with constant coefficients with respect to some coordinate system).

In order to prove (2) first notice the following:

$$\begin{aligned}
\langle R(X, Y)V, V \rangle &= \langle \nabla_X \nabla_Y V, V \rangle - \langle \nabla_Y \nabla_X V, V \rangle \\
&= X(\langle \nabla_Y V, V \rangle) - \langle \nabla_Y V, \nabla_X V \rangle - \omega(X)\langle \nabla_Y V, V \rangle \\
&\quad - Y(\langle \nabla_X V, V \rangle) + \langle \nabla_X V, \nabla_Y V \rangle + \omega(Y)\langle \nabla_X V, V \rangle, \\
&= \frac{1}{2}XY(\langle V, V \rangle) - \frac{1}{2}X(\omega(Y)\langle V, V \rangle) - \frac{1}{2}\omega(X)Y(\langle V, V \rangle) + \frac{1}{2}\omega(X)\omega(Y)\langle V, V \rangle \\
&\quad - \frac{1}{2}YX(\langle V, V \rangle) + \frac{1}{2}Y(\omega(X)\langle V, V \rangle) + \frac{1}{2}\omega(Y)X(\langle V, V \rangle) - \frac{1}{2}\omega(Y)\omega(X)\langle V, V \rangle, \\
&= \frac{1}{2}[X, Y](\langle V, V \rangle) + \frac{1}{2}(Y(\omega(X)) - X(\omega(Y)))\langle V, V \rangle.
\end{aligned}$$

In the above expression, the first term vanishes because of the choice of extension we made. Thus, we get

$$\langle R(X, Y)V, V \rangle = F(Y, X)\langle V, V \rangle. \quad (2.12)$$

Now, using polarization, we get

$$\begin{aligned}
\langle R(X, Y)Z, W \rangle + \langle R(X, Y)W, Z \rangle &= \langle R(X, Y)(Z + W), Z + W \rangle - \langle R(X, Y)Z, Z \rangle \\
&\quad - \langle R(X, Y)W, W \rangle, \\
&= F(Y, X)(\langle Z + W, Z + W \rangle - \langle Z, Z \rangle - \langle W, W \rangle), \\
&= 2F(Y, X)\langle Z, W \rangle.
\end{aligned}$$

Now, in order to prove (3), first consider any $\mathbb{R}$ -linear function $F : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \mapsto \mathfrak{X}(M)$ and define the following operation on F :

$$\mathfrak{G}F(X, Y, Z) \doteq F(X, Y, Z) + F(Y, Z, X) + F(Z, X, Y). \quad (2.13)$$

Notice that making a cyclic permutation in $\mathfrak{G}(F(X, Y, Z))$ does not change anything. Then, we have that

$$\begin{aligned}
\mathfrak{G}R(X, Y)Z &= \mathfrak{G}(\nabla_X \nabla_Y Z) - \mathfrak{G}(\nabla_Y \nabla_X Z), \\
&= \mathfrak{G}(\nabla_Y \nabla_Z X) - \mathfrak{G}(\nabla_Y \nabla_X Z).
\end{aligned}$$

Then, using the torsionless condition, together with the fact that the brackets between our fields vanish, we get

$$\begin{aligned}\mathfrak{G}R(X, Y)Z &= \mathfrak{G}(\nabla_Y \nabla_X Z) - \mathfrak{G}(\nabla_X \nabla_Y Z) \\ &= 0,\end{aligned}$$

which is precisely equivalent to (3).

Finally in order to prove (4), just consider the following four identities, coming from (3):

$$\begin{aligned}\langle R(x, v)y + R(v, y)x + R(y, x)v, w \rangle &= 0, \\ \langle R(v, y)w + R(y, w)v + R(w, v)y, x \rangle &= 0, \\ \langle R(y, w)x + R(w, x)y + R(x, y)w, v \rangle &= 0, \\ \langle R(w, x)v + R(x, v)w + R(v, w)x, y \rangle &= 0.\end{aligned}$$

Adding the four equalities, we get twelve terms, which, using the identities (1) and (2) are reduced to twice

$$\begin{aligned}\langle R(v, w)x, y \rangle - \langle R(x, y)v, w \rangle &= F(x, y)g(v, w) + F(w, v)g(x, y) - F(y, v)g(x, w) - \\ &\quad F(w, y)g(v, x) - F(x, w)g(v, y) - F(v, x)g(y, w).\end{aligned}$$

□

Note that the coordinate expression for the identity (2) of the previous Proposition is

$$R_{lkji} + R_{klji} = 2g_{lk}F_{ji}. \quad (2.14)$$

Proposition 2.3.1.2 (Second Bianchi identity). *If $\sigma \in T_p^*M$ and $x, y, z, v \in T_pM$ then*

$$\nabla_z R(\sigma, x, y, v) + \nabla_x R(\sigma, y, z, v) + \nabla_y R(\sigma, z, x, v) = 0. \quad (2.15)$$

Proof. Consider normal coordinates at p and extend the vectors x, y, z to vector fields in a such neighbourhood considering such extensions with constant coefficients. For σ and v consider arbitrary extensions, denoted by θ and V respectively. Now, we can compute

the following

$$\nabla_Z(R(\theta, X, Y, V)) \stackrel{\text{def}}{=} \nabla_Z(\theta(R(X, Y)V)),$$

Now, notice that, using the Leibnitz rule, we have

$$\begin{aligned} \nabla_Z(R(\theta, X, Y, Z)) &= \nabla_Z R(\theta, X, Y, V) + R(\nabla_Z \theta, X, Y, V) + R(\theta, \nabla_Z X, Y, V) + \\ &\quad R(\theta, X, \nabla_Z Y, V) + R(\theta, X, Y, \nabla_Z V). \end{aligned}$$

Because of the way we chose our extension, all nine covariant derivatives related to the vector fields X, Y, Z vanish at p . Thus,

$$\begin{aligned} \{\nabla_Z R(\theta, X, Y, V) + \nabla_Z \theta(R(X, Y)V) + \theta(R(X, Y)\nabla_Z V)\}|_p &= \{\nabla_Z \theta(R(X, Y)V) \\ &\quad + \theta(\nabla_Z(R(X, Y)V))\}|_p, \end{aligned}$$

which gives us

$$\begin{aligned} \nabla_Z R(\theta, X, Y, V)|_p &= \theta\left(\nabla_Z(R(X, Y)V) - R(X, Y)\nabla_Z V\right)|_p \\ &= \theta(\{\nabla_Z[\nabla_X, \nabla_Y] - [\nabla_X, \nabla_Y]\nabla_Z\}V)|_p \\ &= \theta([\nabla_Z, [\nabla_X, \nabla_Y]]V)|_p. \end{aligned}$$

Thus finally we get

$$\begin{aligned} \{\nabla_Z R(\theta, X, Y, V) + \nabla_X R(\theta, Y, Z, V) + \nabla_Y R(\theta, Z, X, V)\}|_p &= \\ \theta(\{[\nabla_Z, [\nabla_X, \nabla_Y]] + [\nabla_X, [\nabla_Y, \nabla_Z]] + [\nabla_Y, [\nabla_Z, \nabla_X]]\}V)|_p. \end{aligned}$$

A straightforward computation gives that the operators $\nabla_X, \nabla_Y, \nabla_Z$ satisfy a Jacobi-type identity, showing that the right-hand side of the previous expression vanishes, finally giving us:

$$\nabla_z R(\sigma, x, y, v) + \nabla_x R(\sigma, y, z, v) + \nabla_y R(\sigma, z, x, v) = 0. \quad (2.16)$$

□

Notice that in coordinates (2.15) gives us:

$$\nabla_i R_{ukj}^l + \nabla_j R_{uik}^l + \nabla_k R_{uji}^l = 0. \quad (2.17)$$

We will now use both Bianchi identities to produce the contracted Bianchi identities in this context. First notice that, unlike in Riemannian geometry, in the context of Weyl geometry the Ricci tensor is no longer symmetric. So we can compute two divergences for the Einstein tensor, which will be called the contracted Bianchi identities for Weyl's geometry.

Start by contracting l and j in (2.17) and then contract the result with g^{uk} to get

$$g^{uk} \nabla_i R_{uk} + g^{uk} \nabla_l R_{uik}^l - g^{uk} \nabla_k R_{ui} = 0. \quad (2.18)$$

Using the symmetries (1)-(2) of **Proposition 2.3.1.1** and Weyl's compatibility condition, we get that

$$g^{uk} \nabla_l R_{uik}^l = -g^{lv} \nabla_l R_{vi} + 2g^{uk} \nabla_u F_{ik}. \quad (2.19)$$

Plugging this in (2.18), we get

$$\nabla_i R + \omega_i R - g^{lu} \nabla_l R_{ui} + 2g^{uk} \nabla_u F_{ik} - g^{uk} \nabla_k R_{ui} = 0$$

Since $\nabla_i R = \nabla_l (g^{lk} (g_{ki} R)) = g^{lk} \nabla_l (g_{ik} R) - \omega_i R$, we finally get

$$-2g^{uk} \nabla_k R_{ui} + g^{lk} \nabla_l (g_{ki} R) + 2g^{uk} \nabla_u F_{ik} = 0,$$

thus

$$g^{uk} \nabla_k G_{ui} = g^{uk} \nabla_u F_{ik}. \quad (2.20)$$

In order to compute the other divergence, first use the first Bianchi identity in each term of the second one, to get

$$\nabla_i R_{kju}^l + \nabla_i R_{juk}^l + \nabla_j R_{iku}^l + \nabla_j R_{kui}^l + \nabla_k R_{jiu}^l + \nabla_k R_{iuj}^l = 0.$$

Now contract l and j and then contract with g^{iu} . Doing this, we get

$$-g^{iu}\nabla_i R_{ku} + g^{iu}\nabla_i R_{luk}^l + g^{iu}\nabla_l R_{iku}^l + g^{iu}\nabla_l R_{kui}^l + g^{iu}\nabla_k R_{liu}^l + g^{iu}\nabla_k R_{iul} = 0$$

Now, again using (2.19) and the following identity (which also comes from the symmetries of the curvature tensor stated in **Proposition 2.3.1.1**)

$$g^{iu}\nabla_i R_{luk}^l = ng^{iu}\nabla_i F_{uk}, \quad (2.21)$$

we get the following identity

$$-g^{iu}\nabla_i R_{ku} + ng^{iu}\nabla_i F_{uk} - g^{lu}\nabla_l R_{uk} + 2g^{ul}\nabla_u F_{kl} + g^{iu}\nabla_k R_{iul} = 0.$$

We can now use (2.20) in the third term in the previous expression. After doing this, we get

$$-g^{iu}\nabla_i R_{ku} + ng^{iu}\nabla_i F_{uk} + g^{lu}\nabla_u F_{kl} - \frac{1}{2}g^{ul}\nabla_l(Rg_{uk}) + \nabla_k R + R\omega_k = 0.$$

Again, using $\nabla_k R = g^{lu}\nabla_l(g_{uk}R) - \omega_k R$ in the fifth term, we finally get

$$g^{lu}\nabla_l G_{ku} = (1 - n)g^{lu}\nabla_l F_{ku}. \quad (2.22)$$

Putting together (2.20) and (2.22), we get the following expression, which will be useful for us when studying embeddings in the context of Weyl geometry:

$$g^{lu}\nabla_l {}^S G_{ku} = \frac{2-n}{2}g^{iu}\nabla_i F_{ku}. \quad (2.23)$$

2.3.2 Splitting of the Ricci tensor

In this section we will present some useful expressions for the Ricci tensor, splitting it into the *associated* Riemannian Ricci tensor and into its symmetric and antisymmetric

parts. These kind of splittings will become useful when studying embedding problems in the context of Weyl's geometry.

We will first split the curvature tensor and then get the corresponding splitting for the Ricci tensor. Notice that under our conventions for the curvature tensor, we have that the its components are given by:

$$R_{kji}^u = \partial_i \Gamma_{jk}^u - \partial_j \Gamma_{ik}^u + \Gamma_{jk}^l \Gamma_{il}^u - \Gamma_{ik}^l \Gamma_{jl}^u. \quad (2.24)$$

It is standard that, after choosing a representative of a Weyl structure $(M, [g, \omega])$, *i.e.*, (M, g, ω) , we can write the components the Weyl connection associated with the Weyl structure as $\Gamma_{jk}^u = \mathbb{T}_{jk}^u + \gamma_{jk}^u$, where $\mathbb{T}_{jk}^u$ denotes the components of the Riemannian connection associated with g . Using this in (2.24), a straightforward computations gives us that

$$R_{kji}^u = {}^\circ R_{kji}^u + \nabla_i \gamma_{jk}^u - \nabla_j \gamma_{ik}^u + \gamma_{jk}^l \gamma_{il}^u - \gamma_{ik}^l \gamma_{jl}^u,$$

where all the objects denoted with a $^\circ$ are constructed with the Riemannian connection $\mathbb{T}_{jk}^u$. We know that $\gamma_{jk}^u = \frac{1}{2}(\omega^u g_{jk} - \delta_j^u \omega_k - \delta_k^u \omega_j)$. Thus, we get that

$$\nabla_i \gamma_{jk}^u - \nabla_j \gamma_{ik}^u = g_{k[j} \nabla_{i]} \omega^u - \delta_{[j}^u \nabla_{i]} \omega_k - \delta_k^u F_{ij}, \quad (2.25)$$

where we have used $[]$ to denote antisymmetrization between a pair of indices, *i.e.*, $T_{[ab]} = \frac{1}{2}(T_{ab} - T_{ba})$. Also, we can compute the following:

$$\gamma_{jk}^l \gamma_{il}^u - \gamma_{ik}^l \gamma_{jl}^u = \frac{1}{2} \omega_k \omega_{[j} \delta_{i]}^u - \frac{1}{2} g_{k[i} \omega_{j]} \omega^u - \frac{1}{2} \omega_l \omega^l g_{k[j} \delta_{i]}^u. \quad (2.26)$$

Putting these results together, we get the following splitting for the curvature tensor

$$R_{kji}^u = {}^\circ R_{kji}^u + g_{k[j} \nabla_{i]} \omega^u - \delta_{[j}^u \nabla_{i]} \omega_k - \delta_k^u F_{ij} + \frac{1}{2} \omega_k \omega_{[j} \delta_{i]}^u - \frac{1}{2} g_{k[i} \omega_{j]} \omega^u - \frac{1}{2} \omega_l \omega^l g_{k[j} \delta_{i]}^u. \quad (2.27)$$

Now, from this expression plus some computations, we get the following splitting for the

Ricci tensor:

$$R_{kj} = {}^\circ R_{kj} + \frac{n-2}{4}(\nabla_j \omega_k + \nabla_k \omega_j) + \frac{n-2}{4}\omega_k \omega_j + \frac{1}{2}g_{kj}(\nabla_u \omega^u - \frac{n-2}{2}\omega_u \omega^u) - \frac{n}{2}F_{kj}. \quad (2.28)$$

This expression not only gives us the splitting of the Ricci tensor into its “Riemannian” and “non-Riemannian” parts, but also into its symmetric and antisymmetric parts.

2.3.3 Weyl Submanifolds

Consider $(\overline{M}, \overline{g}, \overline{\omega})$ a Weyl manifold with its compatible connection $\overline{\nabla}$. Suppose that $M \hookrightarrow \overline{M}$ is a submanifold. As usual we consider the induced connection on M , which we will denote ∇ . This means that if X, Y are vector fields on M and $\overline{X}, \overline{Y}$ are extensions of these vector fields to $\overline{M}$, then $\nabla_X Y \doteq (\overline{\nabla}_{\overline{X}} \overline{Y})^T$. It is easily checked that this definition does not depend on how we extend X, Y . In addition to this induced connection we have the induced metric on M and an induced one form, (g, ω) , which are just the restrictions of $(\overline{g}, \overline{\omega})$ to fields defined on M .

Proposition 2.3.3.1. *Given a Weyl manifold $(\overline{M}, \overline{g}, \overline{\omega})$ and a submanifold $M \hookrightarrow \overline{M}$, the induced connection ∇ on M is the Weyl connection compatible with the induced Weyl structure on (M, g, ω) .*

Proof. We just need to show that the induced connection is compatible with (g, ω) and torsionless, then the uniqueness of such a connection gives our result. To see that ∇ is torsionless, just consider two vector field on M , given by X, Y , and their extensions to $\overline{M}$, given by $\overline{X}$ and $\overline{Y}$, and then compute

$$\begin{aligned} \nabla_X Y - \nabla_Y X &= (\overline{\nabla}_{\overline{X}} \overline{Y} - \overline{\nabla}_{\overline{Y}} \overline{X})^T|_M \\ &= [\overline{X}, \overline{Y}]^T|_M \\ &= [X, Y]. \end{aligned}$$

Hence ∇ is torsionless. Now suppose X, Y and Z are vector fields on M . Then, since on

M we have $\bar{g} = g$, $\bar{X} = X$, $\bar{Y} = Y$ and $\bar{Z} = Z$; for any $p \in M$ we have that

$$\begin{aligned}
X_p(g(Y, Z)) &= \bar{X}_p(\bar{g}(\bar{Y}, \bar{Z})) \\
&= \bar{g}_p((\bar{\nabla}_{\bar{X}}\bar{Y})(p), \bar{Z}_p) + \bar{g}_p(\bar{Y}_p, (\bar{\nabla}_{\bar{X}}\bar{Z})(p)) + \bar{\omega}_p(\bar{X}_p)\bar{g}_p(\bar{Y}_p, \bar{Z}_p) \\
&= \bar{g}_p((\bar{\nabla}_{\bar{X}}\bar{Y})(p), Z_p) + \bar{g}_p(Y_p, (\bar{\nabla}_{\bar{X}}\bar{Z})(p)) + \omega_p(X_p)g_p(Y_p, Z_p) \\
&= \bar{g}_p((\nabla_X Y)(p), Z_p) + \bar{g}_p(Y_p, (\nabla_X Z)(p)) + \omega_p(X_p)g_p(Y_p, Z_p) \\
&= g_p((\nabla_X Y)(p), Z_p) + g_p(Y_p, (\nabla_X Z)(p)) + \omega_p(X_p)g_p(Y_p, Z_p) \quad \forall \quad p \in M.
\end{aligned}$$

So we have that ∇ is compatible with (M, g, ω) . So ∇ is the Weyl connection of the induced Weyl structure. $\square$

Now, as usual, we define the second fundamental form, α , on M in the following way.

$$\begin{aligned}
\alpha : \Gamma(TM) \times \Gamma(TM) &\mapsto \Gamma(TM^\perp) \\
\alpha(X, Y) &\doteq (\bar{\nabla}_{\bar{X}}\bar{Y})^\perp
\end{aligned}$$

One must check that this definition does not depend on how we extend X and Y to $\bar{M}$. This is the case, since, if we give two vector fields X, Y on M and consider extensions to $\bar{M}$ given by $\bar{X}, \bar{Y}$ and consider a coordinate system (U, μ) in $\bar{M}$, then

$$\bar{\nabla}_{\bar{X}}\bar{Y} = \bar{X}(\bar{Y}^i)\partial_i + \bar{Y}^i\bar{\nabla}_{\bar{X}}\partial_i,$$

and for any $p \in M \cap U$ we have $(\bar{X}(\bar{Y}^i))(p) = \bar{X}_p(\bar{Y}^i|_{M \cap U}) = X_p(Y^i|_{M \cap U})$ and $\bar{\nabla}_{\bar{X}}\partial_i|_p = \bar{\nabla}_{X_p}\partial_i$. So

$$(\bar{\nabla}_{\bar{X}}\bar{Y})(p) = X_p(Y^i|_{M \cap U})\partial_i|_p + Y^i(p)\bar{\nabla}_{X_p}\partial_i$$

and we see that given two vector fields X, Y on M and two extensions $\bar{X}, \bar{Y}$ of them, $\bar{\nabla}_{\bar{X}}\bar{Y}$ does not depend on the extensions. Now, suppose that $(\bar{X}_1, \bar{Y}_1), (\bar{X}_2, \bar{Y}_2)$ are two different extensions of these fields to $\bar{M}$. Then, since $T_p\bar{M} = (T_pM)^T \oplus (T_pM)^\perp$, we have that

$$(\bar{\nabla}_{\bar{X}_1}\bar{Y}_1 - \bar{\nabla}_{\bar{X}_2}\bar{Y}_2)^\perp = (\bar{\nabla}_{\bar{X}_1}\bar{Y}_1 - \bar{\nabla}_{\bar{X}_2}\bar{Y}_2) - (\bar{\nabla}_{\bar{X}_1}\bar{Y}_1 - \bar{\nabla}_{\bar{X}_2}\bar{Y}_2)^T$$

The first term on the right hand side vanishes from the above arguments and the second because, as we have stated at the beginning of this section, the tangent part is independent of the extensions. This proves that α does not depend on how we extend X, Y . So, for X, Y fields on M , it is legitimate to write

$$\bar{\nabla}_X Y = \nabla_X Y + \alpha(X, Y).$$

Proposition 2.3.3.2. *The second fundamental form α is symmetric and $C^\infty(M)$ -linear in both arguments.*

Proof. To prove the symmetry statement consider the following:

$$\begin{aligned} \alpha(X, Y) - \alpha(Y, X) &= (\bar{\nabla}_{\bar{X}} \bar{Y} - \bar{\nabla}_{\bar{Y}} \bar{X})^\perp, \\ &= [\bar{X}, \bar{Y}]^\perp. \end{aligned}$$

Since these expressions are defined on points of M we get that

$$\begin{aligned} \alpha(X, Y) - \alpha(Y, X) &= [X, Y]^\perp \\ &= 0. \end{aligned}$$

Now, to check the linearity property consider $f \in C^\infty(M)$. Then,

$$\begin{aligned} \alpha(fX, Y) &= (\bar{\nabla}_{f\bar{X}} \bar{Y})^\perp \\ &= f(\bar{\nabla}_{\bar{X}} \bar{Y})^\perp \end{aligned}$$

and then the previous symmetry property finishes the proof. $\square$

From now on we will just consider hypersurfaces. In this case we can define a unit normal vector field η , which at least locally is unique up to a sign. Then we define the **scalar second fundamental form**, which in the physics literature is usually called **extrinsic curvature**, by

$$\begin{aligned} K : \Gamma(TM) \times \Gamma(TM) &\mapsto C^\infty(M), \\ (X, Y) &\mapsto \bar{g}(\alpha(X, Y), \eta) \end{aligned}$$

where η is a unit normal vector field to M .

Because of the properties we have shown for α , we know that K is symmetric and $C^\infty(M)$ -linear, that is, K is a $(0, 2)$ -tensor on M .

2.3.4 Gauss-Codazzi equations

We will now write the Gauss-Codazzi equations for hypersurfaces in the context of Weyl geometry.

Gauss equation We know that for X and Y vector fields on M we can write

$$\bar{\nabla}_X Y = \nabla_X Y + \alpha(X, Y).$$

Then, for X, Y, Z and W vector fields on M it holds that

$$\begin{aligned}\bar{\nabla}_X \bar{\nabla}_Y Z &= \bar{\nabla}_X \nabla_Y Z + \bar{\nabla}_X \alpha(Y, Z) \\ &= \nabla_X \nabla_Y Z + \alpha(X, \nabla_Y Z) + \bar{\nabla}_X \alpha(Y, Z)\end{aligned}$$

and

$$\bar{\nabla}_Y \bar{\nabla}_X Z = \nabla_Y \nabla_X Z + \alpha(Y, \nabla_X Z) + \bar{\nabla}_Y \alpha(X, Z)$$

and finally

$$\bar{\nabla}_{[X, Y]} Z = \nabla_{[X, Y]} Z + \alpha([X, Y], Z).$$

Then,

$$\begin{aligned}\bar{g}(\bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X, Y]} Z, W) &= \bar{g}(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, W) + \\ &\quad \bar{g}(\alpha(X, \nabla_Y Z) - \alpha(Y, \nabla_X Z) - \alpha([X, Y], Z) + \\ &\quad \bar{\nabla}_X \alpha(Y, Z) - \bar{\nabla}_Y \alpha(X, Z), W)\end{aligned}\tag{2.29}$$

Then, since α gives a field orthogonal to W , we have that

$$\bar{g}(\bar{R}(X, Y)Z, W) = \bar{g}(R(X, Y)Z, W) + \bar{g}(\bar{\nabla}_X \alpha(Y, Z), W) - \bar{g}(\bar{\nabla}_Y \alpha(X, Z), W).\tag{2.30}$$

Now, from Weyl's compatibility condition we get

$$\begin{aligned}
\bar{g}(\bar{\nabla}_X \alpha(Y, Z), W) &= X(\bar{g}(\alpha(Y, Z), W)) - \bar{g}(\alpha(Y, Z), \bar{\nabla}_X W) - \omega(X)\bar{g}(\alpha(Y, Z), W) \\
&= -\bar{g}(\alpha(Y, Z), \bar{\nabla}_X W) \\
&= -\bar{g}(\alpha(Y, Z), \alpha(X, W)),
\end{aligned}$$

and similarly

$$\bar{g}(\bar{\nabla}_Y \alpha(X, Z), W) = -\bar{g}(\alpha(X, Z), \alpha(Y, W)).$$

So we get the Gauss equation

$$\bar{g}(\bar{R}(X, Y)Z, W) = \bar{g}(R(X, Y)Z, W) + \bar{g}(\alpha(X, Z), \alpha(Y, W)) - \bar{g}(\alpha(Y, Z), \alpha(X, W)). \quad (2.31)$$

In terms of the extrinsic curvature K , we get that the Gauss equation reads:

$$\bar{g}(\bar{R}(X, Y)Z, W) = \bar{g}(R(X, Y)Z, W) + \epsilon(K(X, Z)K(Y, W) - K(Y, Z)K(X, W)), \quad (2.32)$$

where $\epsilon = \bar{g}(\eta, \eta) = \pm 1$ depending on whether the orthogonal direction is time-like or space-like.

Codazzi's equation If we consider a unit normal field ξ to M , from (2.29), we get that

$$\begin{aligned}
\bar{g}(\bar{R}(X, Y)Z, \xi) &= \bar{g}(\bar{\nabla}_X \alpha(Y, Z) - \alpha(Y, \nabla_X Z), \xi) - \bar{g}(\bar{\nabla}_Y \alpha(X, Z) - \alpha(X, \nabla_Y Z), \xi) - \\
&\quad \bar{g}(\alpha([X, Y], Z), \xi)
\end{aligned}$$

Since ∇ is torsionless, we get that

$$\bar{g}(\alpha([X, Y], Z), \xi) = \bar{g}(\alpha(\nabla_X Y, Z), \xi) - \bar{g}(\alpha(\nabla_Y X, Z), \xi).$$

Replacing this last expression in the previous one, we get the following form of Codazzi's equation

$$\bar{g}(\bar{R}(X, Y)Z, \xi) = \bar{g}(\bar{\nabla}_X \alpha(Y, Z) - \alpha(Y, \nabla_X Z) - \alpha(\nabla_X Y, Z), \xi) - \quad (2.33)$$

$$\bar{g}(\bar{\nabla}_Y \alpha(X, Z) - \alpha(X, \nabla_Y Z) - \alpha(\nabla_Y X, Z), \xi). \quad (2.34)$$

Notice that $\alpha(X, Y) = \epsilon K(X, Y)\xi$, where $\epsilon = g(\xi, \xi) = \pm 1$, and this sign depends on whether the restriction of g to each $T_p M^\perp$ is positive or negative definite. Using this, we get the following.

$$\begin{aligned} \bar{g}(\bar{\nabla}_X \alpha(Y, Z), \xi) &= \epsilon \bar{g}(X(K(Y, Z))\xi + K(Y, Z)\bar{\nabla}_X \xi, \xi) \\ &= X(K(Y, Z)) + \epsilon K(Y, Z)\bar{g}(\bar{\nabla}_X \xi, \xi), \end{aligned}$$

and similarly

$$\bar{g}(\nabla_Y \alpha(X, Z), \xi) = Y(K(X, Z)) + \epsilon K(X, Z)\bar{g}(\bar{\nabla}_Y \xi, \xi).$$

Replacing these relations in Codazzi's equation we get the following

$$\begin{aligned} \bar{g}(\bar{R}(X, Y)Z, \xi) &= (X(K(Y, Z)) - K(Y, \nabla_X Z) - K(\nabla_X Y, Z)) + \epsilon K(Y, Z)\bar{g}(\bar{\nabla}_X \xi, \xi) - \\ &\quad (Y(K(X, Z)) - K(X, \nabla_Y Z) - K(\nabla_Y X, Z)) - \epsilon K(X, Z)\bar{g}(\bar{\nabla}_Y \xi, \xi) \\ &= (\nabla_X K)(Y, Z) - (\nabla_Y K)(X, Z) + \epsilon K(Y, Z)\bar{g}(\bar{\nabla}_X \xi, \xi) - \epsilon K(X, Z)\bar{g}(\bar{\nabla}_Y \xi, \xi). \end{aligned}$$

Now Weyl's compatibility condition gives us

$$\bar{g}(\bar{\nabla}_X \xi, \xi) = X(\bar{g}(\xi, \xi)) - \bar{g}(\xi, \bar{\nabla}_X \xi) - \omega(X)\bar{g}(\xi, \xi).$$

Thus,

$$\bar{g}(\bar{\nabla}_X \xi, \xi) = -\frac{\epsilon}{2}\omega(X)$$

and so finally we get the following form of Codazzi's equation

$$\bar{g}(\bar{R}(X, Y)Z, \xi) = (\nabla_X K)(Y, Z) - (\nabla_Y K)(X, Z) + \frac{1}{2}(\omega(Y)K(X, Z) - \omega(X)K(Y, Z)). \quad (2.35)$$

From these equations, we get that in the case $\omega = 0$, where we “recover” Riemannian geometry, the Gauss-Codazzi equations take the following form:

$$\bar{g}(\bar{R}(X, Y)Z, W) = \bar{g}(R(X, Y)Z, W) + \epsilon(K(X, Z)K(Y, W) - K(Y, Z)K(X, W)), \quad (2.36)$$

$$\bar{g}(\bar{R}(X, Y)Z, \xi) = (\nabla_X K)(Y, Z) - (\nabla_Y K)(X, Z). \quad (2.37)$$

Chapter 3

Wheeler's thin-sandwich conjecture

3.1 Introduction

The initial value formulation (Cauchy problem) of any physical theory lies at the core of its interpretation, and even more, in some way, it is part of the basic program of most modern physical theories. This is related to the fact that we expect a physical theory to be able to tell us how a physical system will evolve in time, given some initial configuration of the physically relevant data at some given time. For instance, this is a feature shared by classical mechanics, electromagnetism and quantum mechanics. For the case of general relativity (GR), this problem becomes particularly difficult to study and its consequences are also deeply interesting even for the interpretation of space-time itself.

Given that GR is a theory about space-time, which states that the gravitational interaction is actually determined by the geometry of space-time, the Cauchy problem for GR becomes a geometric problem which consists in finding a solution of the Einstein equations in a 4-dimensional Lorentzian manifold, which satisfies some prescribed initial conditions on a 3-dimensional Riemannian hypersurface. This can be understood as studying whether we can propagate some initial space-like hypersurface, such that the resulting space-time satisfies the Einstein equations. This problem has long been studied and there are results which show that general relativity has a well-posed Cauchy problem for initial data satisfying some constraint equations [48]. This implies that we cannot arbitrarily give the initial data set for the Cauchy problem, motivating the study of this set of constraint equations so as to determine under what conditions it has a solution, and what part of this data can in fact be given arbitrarily on the initial manifold. It is

customary to regard this system as a system of partial differential equations for a Riemannian metric and for some $(0, 2)$ symmetric tensor field defined on this Riemannian hypersurface, which in the end will play the role of the extrinsic curvature. We now know that, under some hypotheses on the topology of the space-like manifold, we can specify a conformal metric to the physical one (that is the one which will solve the constraints and hence will have a development in space-time) and the trace of the second fundamental form, and then get a well-posed system for the remaining undetermined quantities [3].

Another way to look at this problem was proposed by Wheeler. His idea was to consider space-time as a curve in what he called Superspace [15]. Intuitively, given a 3-dimensional manifold M , the Superspace $S(M)$ related to it would be the space of *geometries* that can be defined on M . In this way, a point in $S(M)$ is regarded as an equivalence class $(M, [g])$, represented by a pair (M, g) with g a Riemannian metric defined on M , where two Riemannian metrics are considered equivalent if they are related to each other by a diffeomorphism via pullback. A detailed review on this structure can be found in [15]. With this in mind we can think of space-time as a curve in Superspace. In this context Wheeler proposed the Thin Sandwich Problem (TSP) [13], where the idea is to give as initial data a Riemannian metric g and a tangent vector $\partial_t g$ to (M, g) , and then study whether we can solve the constraint equations for these initial data. If we can, then, for many cases of interest, it can be shown that this initial data set has a unique Cauchy development in space-time (see [3] for a review on this matter).¹ This means that there would be a unique curve in Superspace satisfying these initial conditions and compatible with the Einstein equations. This problem has been recently investigated by some authors [17],[16],[19]. In this chapter we will be particularly interested in the results obtained by Bartnik and Fodor, who, for the 3-dimensional case, which is the arena of classical general relativity, have established sufficient conditions for the TSP to be well-posed [17]. More precisely, they have shown that given some *free* data $(g_0, \dot{g}_0, \epsilon_0, S_0)$ satisfying some specific geometric conditions and for which a solution of the constraints in their thin-sandwich formulation exists, there is a neighbourhood of this free data set where the TSP is well-posed.

¹Whether such initial data admits a Cauchy development in a space-time depends on whether the evolution equations coming from the Einstein equations are well-posed. This will depend on the type of physical fields we are considering. For instance, it is known that in vacuum, with or without a cosmological constant, the evolution equations are well-posed. Furthermore, the case of a minimally coupled scalar field for quite generic potentials is also well-posed, as is the electromagnetic case. Some fluid sources are also known to generate well-posed systems (see [3]).

Even though the results presented in [17] mainly rely on elliptic theory and an implicit function argument, which do not generally depend on the dimension, in the proofs they explicitly take advantage of the fact that they are working in 3-dimensions to manipulate expressions in a way which is not practical in arbitrary dimensions. But since, just as for the evolution problem in GR, the setting of the constraint equations in its classical formulation does not strongly depend on the dimension, it would be expected that the results presented in [17] should extend to arbitrary dimensions ($n \geq 3$). We will show that this is actually true, and that there is in fact an n -dimensional analog of the Bartnik-Fodor theorem.

Also, in [17], in order to show that there are reference solutions for the constraint equations where their main theorem applies, they produce an example using the initial data induced by the spatially compactified Friedman-Robertson-Walker cosmological solution with $k = -1$, where all the conditions needed for this theorem are satisfied. Nevertheless, it is not shown that on any compact 3-dimensional manifold a reference solution exists. In this chapter we will show that this last statement actually holds, that is, on any compact n -dimensional manifold, $n \geq 3$, there are reference solutions of the constraint equations satisfying all the hypotheses needed to apply the implicit function argument. In this way we will be concerned with the local well-posedness of the TPS, where by this we mean that we will show that in a neighbourhood of free data with specific properties the TSP has a unique solution [18].

3.2 The Cauchy problem in General Relativity

We will begin by rapidly reviewing the main points about the initial value formulation of GR, focusing on those points which are most relevant for the TSP. The following definition is intended to summarise the scenario presented by GR.

Definition 3.2.0.1. An $(n+1)$ -dimensional **globally hyperbolic space-time** is defined to be an $(n+1)$ -dimensional Lorentzian manifold $(V \doteq M^n \times \mathbb{R}, \bar{g})$ satisfying the Einstein equations:

$$Ric(\bar{g}) - \frac{1}{2}R(\bar{g})\bar{g} = T(\bar{g}, \bar{\psi}) \quad (3.1)$$

where, $Ric(g)$ and $R(g)$ represent the Ricci tensor and Ricci scalar of $\bar{g}$, respectively, and T represents some $(0, 2)$ -tensor field, called the energy-momentum tensor, depending on

$\bar{g}$ and (possibly) on a collection of tensor fields, collectively denoted by $\bar{\psi}$, representing other *physical* fields.²

In this context, the initial value formulation becomes an embedding problem, in which we try to find an **isometric embedding** of an initial space-like hypersurface, representing space-time at some given time, into a globally hyperbolic space-time satisfying the Einstein equations.

The standard approach towards this problem is to start with an initial data set of the form $(M^n, g, K, \psi, \psi_t)$ consisting of an n -dimensional manifold M , a Riemannian metric g , a second rank tensor field K , and, if needed, initial data (ψ, ψ_t) for the non-geometric fields. In this context, the tensor field K , once we get the embedding into space-time, is intended to stand for the extrinsic curvature of the space-like slice. Since K “*behaves like*” a time derivative of the metric, it is used to complete the initial data set of a system of partial differential equations posed on space-time which are equivalent to (3.1).³ In order for the problem to be well-posed, the aim is to show that this system is equivalent to some hyperbolic system. When we have non-geometric fields present, that is, we are not in vacuum, we need to analyse each case on its own. It is known that, for most sources of interest, this system can be cast as a hyperbolic system plus a system of constraints which the initial data must solve for the problem to be well-posed. Two very thorough modern reviews on these results can be found in [3] and [49]. In order to get an intuitive picture of what is going on, a pictorial representation of the procedure we just described, in the vacuum case, is given in Figure 3.1.

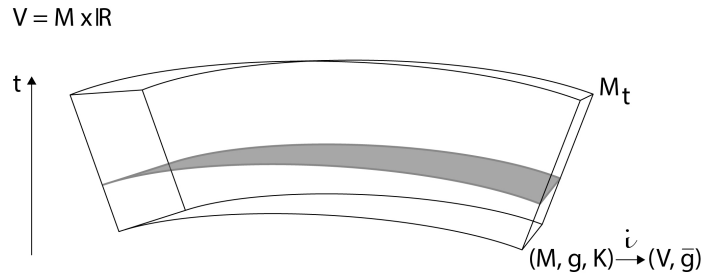


Figure 3.1: Space-time evolution of the geometric initial data set (M, g, K) .

²Note that in the presence of non-geometric fields ψ , the contracted Bianchi identities imply a set of extra field equations given by $\text{div}_{\bar{g}} T = 0$. The full system to be solved is (3.1) plus these equations.

³Once we have our initial data set, in order to pose our problem in space-time, we trivially embed $M \hookrightarrow M \times \mathbb{R}$, and on this manifold we pose the system (3.1) for some Lorentzian metric $\bar{g}$ and the non-geometric fields $\bar{\psi}$.

In the study of the Cauchy problem, after embedding $M \hookrightarrow M \times \mathbb{R}$, it is customary to make an “ $(n + 1)$ -splitting” for the space-time metric $\bar{g}$. This means that we consider local co-frames where we can write the metric $\bar{g}$ in a convenient way, such that we have a “space-time splitting”. We will now give a simple explanation on how this splitting is achieved.

3.2.1 Space-time $(n+1)$ -splitting

Consider $(V = M \times \mathbb{R}, \bar{g})$ a Lorentzian manifold. Define the following function on V .

$$\begin{aligned} f : M \times \mathbb{R} &\mapsto \mathbb{R} \\ (p, t) &\mapsto t \end{aligned} \tag{3.2}$$

We can use an adapted coordinate system to the product structure around any point $q \in V$, with adapted coordinates (x, t) , to compute the differential of this function. In such a coordinate system it is easy to see that $df : T_q V \mapsto T_{f(q)} \mathbb{R}$ is surjective $\forall q \in V$ (and this statement is a coordinate independent statement). Thus $t \in \mathbb{R}$ is always a regular value of f and hence we can make the following two statements:

- $f^{-1}(t) = M \times \{t\}$ are hypersurfaces in V .
- $grad(f)_{(p,t)}$ is orthogonal to $M \times \{t\}$ for each t .

We now want to write $\bar{g}$ in a convenient coframe. We work under the hypothesis that $grad(f)$ is time-like, that is, that $M \times \{t\}$ are space-like. In order to do this, pick adapted coordinates for $M \times \mathbb{R}$, (x, t) , and choose $e_i = \partial_i$, where $\partial_i = \frac{\partial}{\partial x^i}$ are the usual local fields tangent to the coordinate lines in this adapted system. Also, in this coordinates, we can compute that $\partial_t \tilde{f} = 1$ and $\partial_i \tilde{f} = 0$, where $\tilde{f}$ denotes the coordinate expression of the function f . So we can compute the following:

$$\begin{aligned} grad(f) &= \bar{g}^{\alpha\beta} \partial_\beta \tilde{f} \partial_\alpha \\ &= \bar{g}^{00} \partial_t + \bar{g}^{i0} \partial_i \end{aligned} \tag{3.3}$$

Now since $grad(f)$ is orthogonal to each $M \times \{t\}$, we see that, in these coordinates, we must have $\bar{g}^{00} \neq 0$. Now define the following unit vector field, normal to each $M \times \{t\}$,

$n \doteq -\frac{1}{|grad(f)|}grad(f)$. This vector field and the coordinate vector field ∂_t are globally defined fields on $M \times \mathbb{R}$. So we define the following vector field on the product structure:

$$\beta \doteq \partial_t - \frac{1}{|grad(f)|}n. \quad (3.4)$$

Now we want to compute the coordinate expressions for the fields β and $\frac{1}{|grad(f)|}n$ in an adapted coordinate system. In order to do this, first note that, in such a coordinate system, we have the following.

$$\begin{aligned} |grad(f)|_g^2 &\stackrel{def}{=} |g_{\alpha\beta}g^{\alpha\sigma}\partial_\sigma\tilde{f}g^{\beta\gamma}\partial_\gamma\tilde{f}| \\ &= |g^{\sigma\gamma}\partial_\sigma\tilde{f}\partial_\gamma\tilde{f}| \\ &= |g^{00}| \end{aligned}$$

Also, since $grad(f)$ is supposed to be time-like, we have that $g(grad(f), grad(f)) < 0$, so $g^{00} < 0$ in an adapted coordinate system.⁴ Then we have that

$$\begin{aligned} \frac{1}{|grad(f)|}n &= -\frac{g^{\alpha\beta}}{|g^{00}|}\partial_\beta\tilde{f}\partial_\alpha \\ &= -\frac{g^{00}}{|g^{00}|}\partial_t - \frac{g^{i0}}{|g^{00}|}\partial_i \\ &= \partial_t - \frac{g^{i0}}{|g^{00}|}\partial_i. \end{aligned}$$

Hence we see that in any *adapted* coordinate system, we can write the vector field β in the following way

$$\beta = \frac{g^{i0}}{|g^{00}|}\partial_i. \quad (3.5)$$

Note that this expression is valid in any *adapted* coordinate system, not in a *general* coordinate system. In the general case we must find the coordinate expression for β from its definition, which is coordinate independent. Also note that (3.5) shows that β is tangent to each $M \times \{t\}$. In fact, from (3.4) and the fact that, at any point $q \in M \times \{t\}$, the tangent space splits $T_qV = T_pM \oplus T_pM^\perp$, we have a clear interpretation for both β and $\frac{1}{|grad(f)|}n$: they represent the tangent and orthogonal projections of ∂_t with respect to T_pM .

⁴This shows, together with (3.3), that $grad(f)$ is a past oriented normal vector and n represents the future oriented unit normal vector to M_t .

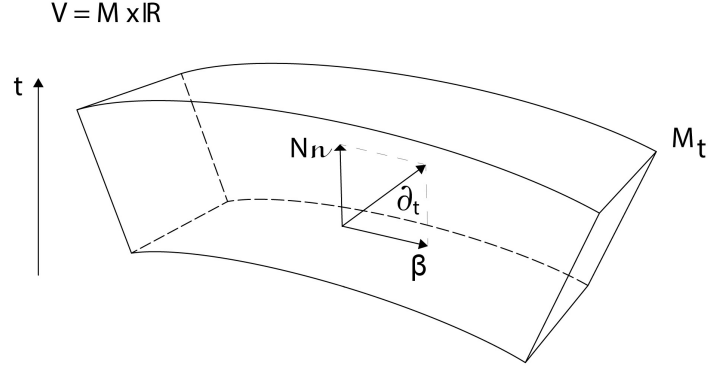


Figure 3.2: Geometric interpretation of the lapse function and shift vector.

Now we choose $e_0 = \frac{1}{|\text{grad}(f)|}n$, so that we have the following local frame $\{e_0, e_i\}$. Note the following facts:

- By definition e_0 is orthogonal to each $M \times \{t\}$ since it is proportional to $\text{grad}(f)$. This means that $\bar{g}(e_0, e_i) = 0 \ \forall \ i = 1, \dots, n$.
- In any adapted coordinate system (x, t) we have that

$$\begin{aligned} e_0 &= \partial_t - \beta, \\ e_i &= \partial_i, \quad i = 1, \dots, n. \end{aligned}$$

We can check that the dual coframe reads

$$\begin{aligned} \theta^0 &= dt, \\ \theta^i &= dx^i + \beta^i dt, \quad i = 1, \dots, n. \end{aligned}$$

Now we will write the metric $\bar{g}$ in this adapted coframe. The metric will read

$$\bar{g} = \bar{g}_{\alpha\beta} \theta^\alpha \otimes \theta^\beta, \tag{3.6}$$

where $\bar{g}_{\alpha\beta} = \bar{g}(e_\alpha, e_\beta)$. Since $\bar{g}(e_0, e_i) = 0 \ \forall \ i = 1, \dots, n$, then we have $\bar{g}_{0i} = 0 \ \forall \ i = 1, \dots, n$. So, in this particular coframe, the metric tensor can be written in the following way:

$$\bar{g} = -N^2 \theta^0 \otimes \theta^0 + \bar{g}_{ij} \theta^i \otimes \theta^j. \tag{3.7}$$

Note that the *lapse* function equals $N^2 = -g(e_0, e_0) = \frac{1}{|grad(f)|^2}$. In this context, the function N is called the **lapse function** and the vector field β the **shift vector**.

When referring to coordinate expressions, we will typically assume them to be written with respect to the above frames. Thus, we will now review what the basic constructions needed for the study of the Cauchy problem look like in such frames.

3.2.2 Adapted extrinsic curvature

In this section we will see how to write the extrinsic curvature in our adapted frames. By definition, we have that in such a frame,

$$K_{ij} = \langle \nabla_{e_i} e_j, n \rangle, \quad (3.8)$$

where we have seen that $e_0 = Nn$. Thus, using that $\nabla_{e_i} e_j = \Gamma_{ij}^\alpha e_\alpha$ defines the components of the connexion with respect to this basis, we get that:

$$K_{ij} = -N\Gamma_{ij}^0. \quad (3.9)$$

In order to get a nicer expression for K_{ij} , we need to compute Γ_{ij}^0 . It is a standard procedure to find the expression of the components of the Riemannian connexion in any basis. The result is the following:

$$\Gamma_{\alpha\gamma}^\sigma = \frac{1}{2}\bar{g}^{\sigma\beta}(e_\alpha(\bar{g}_{\beta\gamma}) + e_\gamma(\bar{g}_{\alpha\beta}) - e_\beta(\bar{g}_{\alpha\gamma})) + \frac{1}{2}g^{\beta\sigma}(C_{\beta\alpha}^\mu \bar{g}_{\gamma\mu} + C_{\beta\gamma}^\mu \bar{g}_{\alpha\mu} - C_{\gamma\alpha}^\mu \bar{g}_{\beta\mu}), \quad (3.10)$$

where $\bar{g}_{\alpha\beta}$ are the components of the metric with respect to the dual basis of $\{e_\alpha\}$, and the coefficients $C_{\beta\gamma}^\alpha$ are given by

$$C_{\beta\gamma}^\alpha \doteq [e_\beta, e_\gamma]^\alpha.$$

From (3.7) and (3.10) we get that

$$\Gamma_{ij}^0 = \frac{1}{2N^2}e_0(\bar{g}_{ij}) - \frac{1}{2N^2}(C_{0i}^k \bar{g}_{kj} + C_{0j}^k \bar{g}_{ki} - C_{ji}^0 \bar{g}_{00}).$$

An easy computation gives that $C_{0i}^k = \partial_i \beta^k$ and $C_{ij}^0 = 0$, and thus we get

$$\begin{aligned}\Gamma_{ij}^0 &= \frac{1}{2N^2}(\partial_t \bar{g}_{ij} - (\beta^k \partial_k \bar{g}_{ij} + \partial_i \beta^k \bar{g}_{kj} + \partial_j \beta^k \bar{g}_{ki})) \\ &= \frac{1}{2N^2}(\partial_t \bar{g}_{ij} - \mathcal{L}_\beta \bar{g}_{ij}).\end{aligned}$$

Thus, from (3.9), we get that in this adapted frame the second fundamental form on each $M \times \{t\}$ takes the form:

$$K_{ij} = -\frac{1}{2N}(\partial_t \bar{g}_{ij} - (\bar{\nabla}_i \beta_j + \bar{\nabla}_j \beta_i)) \quad (3.11)$$

where $\bar{\nabla}$ denotes the induced connection on M compatible with the induced metric $\bar{g}$.⁵

3.2.3 Constraint equations

The origin of the constraint equations is quite simple: If the Einstein equations are to hold on space-time, they must, in particular, hold on the initial hypersurface $M \cong M \times \{0\}$. Thus, the equations

$$G(n, e_i)|_{t=0} = T(n, e_i)|_{t=0}, \quad (3.12)$$

$$G(n, n)|_{t=0} = T(n, n)|_{t=0}, \quad (3.13)$$

must be satisfied. We will see that these equations depend only on the initial data set. This implies that the initial data set cannot be chosen arbitrarily. Only those initial data sets satisfying the constraint equations (3.12)-(3.13) will have a development into a space-time satisfying the Einstein equations.

We note that the right-hand side of the equations (3.12)-(3.13) are interpreted as the induced energy and momentum densities as seen by the family of observers whose worldlines are given by the integral curves of the normal vector field n , and typically denoted by ϵ and S respectively.

⁵Here we taking the opposite sign convention for the extrinsic curvature than in [18]. This convention is related with the choice of past versus future oriented normal field. We have decided to take this convention to have a unified notation throughout the whole thesis.

Note that the Gauss-Codazzi equation (2.36) give us the following:

$$\bar{R}_{ijkl} = R_{ijkl} + K_{jk}K_{il} - K_{ik}K_{jl}, \quad (3.14)$$

$$\bar{R}_{0jkl} = N(\nabla_l K_{kj} - \nabla_k K_{lj}). \quad (3.15)$$

In order to compute the Einstein tensor, it is also necessary to explicitly compute the following components of the Riemann tensor:

$$R_{0jk0} = N(e_0(K_{jk}) + NK_{kl}K_j^l + \nabla_k \nabla_j N). \quad (3.16)$$

With these explicit expressions plus the symmetries of the curvature tensor, we can compute the Einstein tensor in this adapted frame. After a few computations, we get the following

$$\begin{aligned} \bar{G}_{0i} &= \bar{R}_{0i} = N(\nabla_i K_l^l - \nabla_j K_i^j), \\ \bar{G}_{00} &= \frac{N^2}{2}((K_l^l)^2 - K^{ij}K_{ij} + R(g)). \end{aligned}$$

On the other hand, for the energy-momentum tensor, we get

$$\begin{aligned} T_{0i} &= NT(n, e_i) = NS_i, \\ T_{00} &= N^2 T(n, n) = N^2 \epsilon. \end{aligned}$$

Putting these results together, we get the following set of constraint equations, written in a coordinate-free fashion.

$$\begin{aligned} R_g - |K|_g^2 + (\text{tr}_g K)^2 &= 2\epsilon \\ \nabla \text{tr}_g K - \text{div}_g K &= S \end{aligned} \quad (3.17)$$

where (ϵ, S) denote the induced energy and momentum densities on M , respectively, R_g represents the scalar curvature of g , $|\cdot|_g$ denotes the pointwise-tensor norm in the metric g and $\text{div}_g K$ denotes the divergence of K . In coordinates, these equations become:

$$R_g - K^{ij}K_{ij} + (K_i^i)^2 = 2\epsilon \quad (3.18)$$

$$\nabla_i K_j^j - \nabla_j K_i^j = S_i \quad (3.19)$$

These equations are considered on a particular initial hypersurface $M \cong M \times \{t\}$, for example, in the hypersurface defined by $t = 0$. If our initial data set satisfies these constraints, then, it is known that for many cases of practical interest, such as vacuum, Einstein-scalar field, Einstein-Maxwell, some fluid sources (dust, charged dust and perfect fluids for instance), among others, there is an Einsteinian development in a space-time V .⁶

From the previous discussion, we see that a great deal of the Cauchy problem for GR relies on the possibility of solving the constraint equations for the initial data set. This is not an easy problem and is an object of active research in the area of mathematical relativity. This problem has strong physical consequences. For instance, it is clear that the system (3.18)-(3.19) is an under determined system for g and K , which naturally rises the question of which part of the initial data set should be regarded as *free data*, and, consequently, which part is to be determined from this free data by solving the constraints. The most common approach towards this issue is the *conformal method*, in which the free data is given by a Riemannian metric conformal to the *physical* one (that is, to the one which will solve the constraints) and the mean curvature of M , and the system is posed for the remaining undetermined quantities. By means of this conformal technique, the system is cast as an elliptic problem. In this scenario, there are several existence and non-existence results known, which depend, among other things, on the topological assumptions on M (see, for instance, [3],[9],[50],[51] and references therein). Most of the research is devoted mainly to two “opposite” cases: either M compact (“cosmological case”) or M asymptotically Euclidean (which models isolated systems). To construct more sophisticated solutions, gluing techniques have been widely studied (see, for instance, [52],[53]).

3.3 Wheeler’s conjectures

From the discussion of the previous section, we see that one physically relevant part in the study of the initial value formulation of GR, lies in determining which part of the initial data set should be regarded as *free*. In contrast to the conformal problem, J. A. Wheeler proposed two conjectures regarding this issue, both of them are posed in the context of the Superspace picture for space-time. As we have already stated, in this formulation

⁶See [3] for detailed discussions on these existence results.

space-time is seen as a history of space-geometries, each of which is represented by a Riemannian metric g on some manifold M . Thus, the full history of space-time would be represented by a curve in Superspace. This view, in our opinion, seems to be quite sensible within the initial value formulation of GR we just described. A detailed review about the mathematical structure of superspace and the physical properties that can be extracted from such structure can be found in [15] and references therein.

There are two remarks which are in order from the physics perspective. First, this picture of space-time is posed within the program of finding a quantum theory of gravity, since Superspace plays the role of the domain of the *wave function* of the Wheeler-de Witt equation. Secondly, based on the initial value formulation of GR, we end up with a picture where the totality of the information that can be extracted from space-time and the fields defined on it, can actually be condensed on the initial data set.

The first conjecture posed by Wheeler, known as **the sandwich conjecture**, claimed that, given two Riemannian manifolds (M, g_1) and (M, g_2) , representing two points in $S(M)$, there exists a unique space-time connecting these space geometries [13],[14]. Stated in this way this conjecture is quite ambitious and, probably, in order to obtain meaningful results, some conditions should be imposed and more precise statements should be made. The study of this problem is a difficult task, and, as far as we know, there are no significant results concerning this conjecture. In contrast, a simplified version of this problem, also posed by Wheeler, is the **thin sandwich conjecture**. For this last problem some existence results are known [17]-[19], and also some interesting global uniqueness results and non-existence results were found in [16]-[19].

3.3.1 The Thin Sandwich Problem (TSP)

The thin sandwich problem emerges as a simplification of the sandwich conjecture. The idea, to resolve the non-uniqueness issues for instance, is to look at two “infinitesimally close” slices in space-time and, considering an idealized limiting procedure, take as free data on the initial manifold M a Riemannian metric g and the time derivative of this metric on this initial slice $\dot{g}$. In this case, the superspace picture of the TSP is an analogue of the usual dynamical problem for classical particle physics, where, given the initial position and velocities of a system of particles, the dynamical equations determine uniquely the evolution of the whole system as a curve in configuration space. In the TSP,

the conjecture is that given $(g, \dot{g})$, representing a point in superspace and a tangent vector to it, there is a unique curve in superspace representing the space-time development of this initial data set.

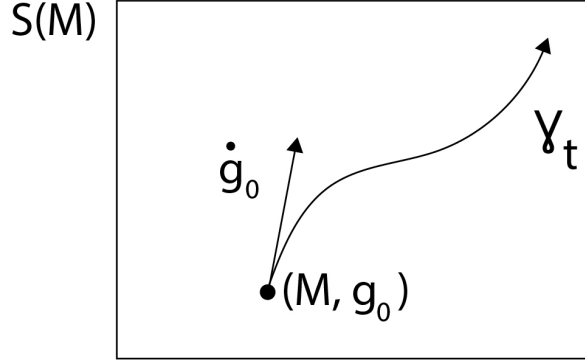


Figure 3.3: Superspace picture of the TSP.

In the case of vacuum, this picture can be treated at the fundamental level. In the case where matter sources are present, the energy and momentum densities (ϵ, S) are typically treated as some phenomenologically given data, completing the freely given data $(g, \dot{g}, \epsilon, S)$, and the constraint equations are posed for the only undetermined quantities: the lapse and shift. This approach seems quite sensible in the sense that in the evolution part of the problem, the initial values of the lapse and shift work as *gauge variables*, which, from the point of view of geometric uniqueness, should not play a physical role. For some cases of interest, such as vacuum, this is actually the content of specific theorems.⁷ Thus, solving the constraints for these variables, and regarding the more physically meaningful quantities $(g, \dot{g})$ as free, seems as a tempting approach. The main problem with this approach is that, in general, the resulting system is not elliptic and existence cannot be guaranteed [19]. This is the reason for assuming (ϵ, S) as given and independently of the the lapse and shift, which is not true in a fundamental model of physical fields, such as a scalar field, eletromagnetic fields, or even fluid sources. Nonetheless, the TSP is still an interesting problem, which has deep theoretical significance.⁸ With the simplifications we have just described, it was shown by Bartnik and Fodor that, for the case of classical GR, where the space slices are 3-dimensional, in a neighbourhood of solutions of the

⁷For further details, see, for instance, [3].

⁸D. Giulini, in [19], worked out a generalised version of this problem where he was able to include realistic models for the matter fields.

constraints which satisfy some particular geometric conditions, the TPS is well-posed.⁹

It is quite interesting that in [17], for their proofs, the fact that they are working with few dimensions is important, since the main results are achieved by manipulation of explicit expressions which would be much harder to handle in arbitrary dimensions. Despite of this fact, their proof outlines the main procedure that should be followed in the general dimensional case. This will be our starting point, since we will show that their result actually holds in any dimension $n \geq 3$. Furthermore, we will show that the geometric hypotheses needed for the proofs can always be satisfied on compact manifolds (without boundary, *i.e.*, closed manifolds), and thus we can always find open subsets in the space of possible initial data for the constraints equations on compact manifolds where the TSP is well-posed [18]. This is a novel result even for the 3-dimensional case.

3.3.2 The reduced equations

As we have noted above, in the context of Wheeler's TSP, equations (3.18)-(3.19) are looked as a system for N and β . In order to do this, we use (3.11) to express (3.18)-(3.19) in terms of the lapse and shift, and then look for solutions with some prescribed data $(g, \dot{g}, \epsilon, S)$. It is worth first noticing the following. Suppose that, given some prescribed data $(g, \dot{g}, \epsilon, S)$, we have a solution (N, β) for the constraint equations. Furthermore, suppose this solution satisfies $2\epsilon - R_g \neq 0$ over all M . Then, introducing (3.11) in (3.18) we can equate the lapse function in terms of the shift vector and the prescribed data. After doing this we obtain

$$N = \sqrt{\frac{(\text{tr}_g \gamma)^2 - |\gamma|_g^2}{2\epsilon - R_g}}, \quad (3.20)$$

where the tensor γ has components

$$\gamma_{ij} = \frac{1}{2}(\dot{g}_{ij} - (\nabla_i \beta_j + \nabla_j \beta_i)). \quad (3.21)$$

It should be noted that in (3.20) we have chosen the positive sign for the square root, since this choice, which corresponds to the choice of positive lapse, is related to the choice of a space-time foliation which *evolves to the future*, whereas the negative sign would represent a foliation *evolving to the past*. Furthermore, a few comments on the individual

⁹The neighbourhoods are taken in the set of possible initial data for the constraints in a topology we will shortly specify.

signs of the numerator and denominator are in order. First of all, note that if M is connected, then the condition $2\epsilon - R_g \neq 0$ all over M implies that $2\epsilon - R_g$ has a definite sign all over M . It should be noted that later on we will impose the condition $2\epsilon - R_g > 0$ for a reference solution of the constraint equations, in a neighbourhood of which we will study the TSP. This condition, which imposes an energy constraint, forces the numerator in (3.20) to be strictly positive, and, furthermore, it forces $\text{tr}_g \gamma \neq 0 \ \forall p \in M$. Using (3.11) and the definition of γ , we see that, if M is connected, this implies that $\tau \doteq \text{tr}_g K$ has a definite sign all over M . That is, if this initial data set has an embedding into a space-time satisfying the Einstein equations, then the hypersurface $M \times \{0\} \cong M$ will be an embedded hypersurface whose mean curvature has a definite sign. This fact carries a clear physical interpretation: the sign of the mean curvature is related to whether to future pointing unit normals are diverging from the hypersurface or converging, which pictures an expanding or contracting space evolving in space-time.

Replacing (3.20) in (3.19) shows that the shift vector satisfies the following equation

$$\nabla_i \left(\sqrt{\frac{2\epsilon - R_g}{(\text{tr}_g \gamma)^2 - |\gamma|_g^2}} (\gamma_j^i - \delta_j^i \text{tr}_g \gamma) \right) = S_j, \quad (3.22)$$

that is,

$$\text{div}_g \left(\sqrt{\frac{2\epsilon - R_g}{(\text{tr}_g \gamma)^2 - |\gamma|_g^2}} (\gamma - \text{tr}_g \gamma g) \right) = S. \quad (3.23)$$

We have a converse procedure to the one just described. That is, if, for a given free initial data set $(g, \dot{g}, \epsilon, S)$, (3.22) is well-posed and β is a solution of (3.22), then taking (3.20) as a definition, the lapse will satisfy (3.18). With this in mind, we will attack the TSP as follows. First, we will assume we have a smooth solution (N_0, β_0) of the constraint equations obtained from some given smooth data $\psi \doteq (g_0, \dot{g}_0, \epsilon_0, S_0)$. Then, we will try to see if for free data sufficiently near to ψ_0 we can find a solution of the constraint equations. If we can, then, in such a neighbourhood of the initial data ψ_0 the TSP will be well-posed.¹⁰

At this point it should be stressed that we do not expect the TSP to be well-posed for arbitrary initial data. For instance, following [16], if we look for solutions of the constraints with given initial data $\psi = (g, \dot{g}, \epsilon, S)$ such that $2\epsilon - R(g) > 0 \ \forall p \in M$, then,

¹⁰When we say that the quantity $\dot{g}$ is a given datum, we mean that some symmetric $(0, 2)$ tensor field on M is given, and that with this tensor field we construct K from (3.11), taking this tensor field as $\partial_t g_{ij}$. Then if we have a solution for the Cauchy problem, this tensor field will coincide with $\partial_t g_{ij}$ on $M \times \{0\}$.

from the Hamiltonian constraint, we see that we must have $\text{tr}_g \gamma \neq 0 \forall p \in M$. But, if M is connected, this condition implies that $\text{tr}_g \gamma$ must have constant sign on M . Furthermore, if M is closed, then $\dot{g}$ cannot be chosen to be L^2 -orthogonal to g , *i.e.*,

$$\int_M \langle g, \dot{g} \rangle_g \mu_g \neq 0.$$

Thus, for instance, choosing in our free data set $\dot{g} = \mathcal{L}_X g$ for any vector field X , gives a free data set for which no solution exists.

In order to address the TSP, we will concentrate in solving (3.23), since we know that this is enough to produce solutions of the constraints. We will start by assuming M to be compact (without boundary) and write this set of non-linear PDE for the shift vector in the following way. Let

$$H_s(T_q^p(M)), \quad s > \frac{n}{2}, \quad s > 2,$$

be the Sobolev space of (p, q) -tensor fields on M with s generalized derivatives in L^2 .¹¹ Define

$$E_1 \doteq H_{s+3}(T_2^0 M) \times H_{s+1}(T_2^0 M) \times H_{s+1}(M) \times H_s(T_1^0 M),$$

which is a Banach space with the norm $\|\cdot\|_{E_1} : E_1 \rightarrow \mathbb{R}$ given by

$$\|(g, \dot{g}, \epsilon, S)\|_{E_1} = \|g\|_{H_{s+3}} + \|\dot{g}\|_{H_{s+1}} + \|\epsilon\|_{H_{s+1}} + \|S\|_{H_s}$$

and let

$$E_2 \doteq H_{s+2}(T_0^1 M) \quad \text{and} \quad F \doteq H_s(T_1^0 M).$$

Now suppose that for given smooth data $\psi_0 \doteq (g_0, \dot{g}_0, \epsilon_0, S_0) \in E_1$ we have a solution $\beta_0 \in E_2$. Then, the continuity of all the maps involved guarantees that (3.22) is well-defined in a neighbourhood $\mathcal{U}$ of (ψ_0, β_0) in $E_1 \times E_2$. With this in mind, we define the map

$$\Phi : \mathcal{U} \subset E_1 \times E_2 \rightarrow F$$

¹¹See Appendix 1.

given by

$$\Phi(\psi, \beta) \doteq \operatorname{div}_g \left(\sqrt{\frac{2\epsilon - R_g}{(\operatorname{tr}_g \gamma)^2 - |\gamma|_g^2}} (\gamma - \operatorname{tr}_g \gamma g) \right) - S \quad (3.24)$$

where we have denoted $\psi = (g, \dot{g}, \epsilon, S)$, and we are using β to denote the shift. Then (3.22) can be written as

$$\Phi(\psi, \beta) = 0. \quad (3.25)$$

Now our problem reduces to the following: we want to see if there are open sets $\mathcal{V} \subset E_1$, $\mathcal{W} \subset E_2$, with $\psi_0 \in \mathcal{V}$ and $\beta_0 \in \mathcal{W}$, and a unique map

$$g : \mathcal{V} \rightarrow \mathcal{W}$$

such that

$$\Phi(\psi, g(\psi)) = 0 \quad \text{for all } \psi \in \mathcal{V}.$$

Notice that, in this case, $\beta = g(\psi) \in \mathcal{W}$ would be the solution to our problem. In order to address this issue, we intend to use the Implicit Function Theorem.¹² Hence, we need to show that

$$L = D_2 \Phi_{(\psi_0, \beta_0)} : E_2 \rightarrow F \quad (3.26)$$

is an isomorphism. This is precisely the procedure followed in [17] in the 3-dimensional case. We will extend their results for arbitrary dimensions ($n \geq 3$). Using (3.24), we compute

$$L\mathcal{Y} = D_2 \Phi_{(\psi, \beta)} \cdot \mathcal{Y} = \operatorname{div}_g \left(\frac{1}{N} \left(\operatorname{div}_g \mathcal{Y} g - {}^S \nabla \mathcal{Y} - \frac{1}{2\epsilon - R_g} \langle \pi, \nabla \mathcal{Y} \rangle \pi \right) \right) \quad (3.27)$$

where π is the tensor given by¹³

$$\pi \doteq \frac{1}{N} (\gamma - \operatorname{tr}_g \gamma g) = K - \operatorname{tr}_g K g \quad (3.28)$$

and

$${}^S \nabla_i \mathcal{Y}_j = \frac{1}{2} (\nabla_i \mathcal{Y}_j + \nabla_j \mathcal{Y}_i).$$

We will study the properties of the linearised operator L . First of all, it is clear that L is

¹²See Appendix 2.

¹³The tensor π , in the Hamiltonian formulation of GR, stands for the conjugate momentum of g .

a linear second order operator. We now have the following proposition.

Proposition 3.3.2.1. *If π is a definite operator all over M , then the linear operator L is elliptic.*

Proof. The first thing we need to do is to compute the symbol of the linear operator L .

$$\begin{aligned}
(\sigma(L)(\xi) \cdot \mathcal{Y})^j &= \frac{1}{N} \left[\bar{g}^{hj} \xi_h \xi_k y^k - \frac{1}{2} (\bar{g}^{hu} \xi_h \xi_u y^j + \bar{g}^{ju} \xi_u \xi_h y^h) - \frac{\pi^{jh} \pi_{ml}}{2\epsilon - \bar{R}} \bar{g}^{mi} \xi_h \xi_i y^l \right] \\
&= \frac{1}{N} \left[\xi^j \xi_k y^k - \frac{1}{2} \xi^u \xi_u y^j - \frac{1}{2} \xi^j \xi_h y^h - \frac{\pi^{jh} \pi_{ml}}{2\epsilon - \bar{R}} \xi_h \xi^m y^l \right] \\
&= \frac{1}{N} \left[\frac{1}{2} \xi^j \xi_k - \frac{1}{2} \xi^u \xi_u \delta_k^j - \frac{\pi^{jh} \pi_{mk}}{2\epsilon - \bar{R}} \xi_h \xi^m \right] y^k
\end{aligned} \tag{3.29}$$

for all $\xi \in \Gamma(T^*M)$ and $\mathcal{Y} \in \Gamma(TM)$. Hence

$$\begin{aligned}
\langle \sigma(L)(\xi) \cdot \mathcal{Y}, \xi \rangle &= (\sigma(L)(\xi) \cdot \mathcal{Y})^j \xi_j = \frac{1}{N} \left(\frac{1}{2} |\xi|_g^2 \langle \xi, \mathcal{Y} \rangle - \frac{1}{2} |\xi|_g^2 \langle \xi, \mathcal{Y} \rangle - \frac{1}{2\epsilon - R_g} \pi(\xi, \xi) \pi(\xi, \mathcal{Y}) \right) \\
&= -\frac{1}{(2\epsilon - R_g)N} \pi(\xi, \xi) \pi(\xi, \mathcal{Y}).
\end{aligned}$$

Consider $\mathcal{Y} \in T_p M$ such that $\sigma(L)(\xi) \cdot \mathcal{Y} = 0$ for some $\xi \neq 0$. Then

$$\pi(\xi, \xi) \pi(\xi, \mathcal{Y}) = 0 \tag{3.30}$$

for some $\xi \neq 0$. Since by assumption π is definite and hence non-degenerate, this implies that $\pi(\xi, \mathcal{Y}) = 0$ for some $\xi \neq 0$. Using this information in (3.29), we get that if $\mathcal{Y}$ is in the null space of $\sigma(L)(\xi)$, then $\mathcal{Y} = \langle \frac{\xi}{|\xi|_g^2}, \mathcal{Y} \rangle \xi$. This condition together with $\pi(\xi, \mathcal{Y}) = 0$, imply

$$\langle \xi, \mathcal{Y} \rangle \pi(\xi, \xi) = 0.$$

Since, by assumption, π is non-degenerate and $\xi \neq 0$, then $\langle \xi, \mathcal{Y} \rangle = 0$, which, together with $\mathcal{Y} = \langle \frac{\xi}{|\xi|_g^2}, \mathcal{Y} \rangle \xi$, implies that $\mathcal{Y} = 0$. Thus, $\sigma(L)$ defines an injective operator form $T_p M \mapsto T_p^* M \forall p \in M$, hence an isomorphism, which shows that L is elliptic. $\square$

It is interesting to note that the condition on π being a definite operator has one particular consequence with a clear physical interpretation. Note that π being definite imposes a condition on $\text{tr}_g K$, since, using (3.28), we get that $\text{tr}_g K = \frac{1}{1-n} \text{tr}_g \pi$. Also, note

that π defines an operator $\pi^\sharp$ on vector fields, given in components by $\pi^\sharp(X)^i \doteq \pi_j^i X^j$. Note that the symmetry of π shows that $\pi^\sharp$ defines a self-adjoint operator (with respect to g) on each tangent space. That is, $\langle v, \pi^\sharp(w) \rangle = \langle \pi^\sharp(v), w \rangle$ for all $v, w \in T_p M$ and $p \in M$. This means that, at each point, there is an g -orthonormal basis diagonalizing π . Using such basis in order to compute $\text{tr}_g \pi$, we see that the trace is the sum of the eigenvalues of π , and thus, that if π is definite, the trace must be either strictly positive or strictly negative. This implies that if π is definite all over M , then, if M is connected, $\text{tr}_g K$ must have constant sign all over M , and it cannot be zero. Now, if a given initial data set (g, K) satisfying this condition on the trace of K has a development into a space-time V , then the embedded hypersurface $M \times \{0\} \cong M$ has mean curvature with a definite sign all over the hypersurface. This, again, can be interpreted as telling us that the whole hypersurface is either expanding or contracting in its space-time evolution (at least for short times).

From now on, we will suppose that π gives a definite operator at every point of M so that the last proposition holds. Having in mind that our aim is to establish sufficient conditions so that L is an isomorphism, the ellipticity condition just established shows that what we need to do is to show the injectivity of both L and its formal adjoint L^* . A straightforward computation, using integration by parts, gives us that L is (formally) self-adjoint. This means that for all smooth vector fields $\mathcal{Y}, \mathcal{Z}$ the following holds

$$\int_M \langle L\mathcal{Y}, \mathcal{Z} \rangle \mu_{g_0} = \int_M \langle \mathcal{Y}, L\mathcal{Z} \rangle \mu_{g_0},$$

where μ_{g_0} is the Riemannian volume form in M induced by the metric g_0 . Thus, if π is a definite operator on M , then L is a (formally) self-adjoint elliptic operator, and what we need to establish is its injectivity, which is the content of the following proposition.

Proposition 3.3.2.2. *Consider a reference solution (ψ, β) for the TSP on a compact n -dimensional manifold M satisfying that: i) π is a definite operator on M ; ii) $2\epsilon - R_g > 0$ on M ; iii) given a function μ , the equation*

$${}^S\nabla \mathcal{Y} = \mu K \tag{3.31}$$

has only the solution $\mathcal{Y} = 0, \mu = 0$. Then L is injective.

Proof. Recall that

$$L\mathcal{Y} = \operatorname{div} \left(\frac{1}{N} \left(\operatorname{div} \mathcal{Y} g - \frac{1}{2} \mathcal{L}_{\mathcal{Y}} g - \frac{1}{2\varepsilon - R_g} \langle \pi, \nabla \mathcal{Y} \rangle \pi \right) \right). \quad (3.32)$$

Let Ω be a relatively compact open subset in M and let $\eta \in C_0^\infty(\Omega)$ with $\eta \equiv 1$ in $\Omega' \subset \Omega$.

Denoting $\mathcal{Z} = \eta \mathcal{Y}$, one obtains

$$\begin{aligned} \langle L\mathcal{Y}, \mathcal{Z} \rangle &= Z^j \nabla_i \left(\frac{1}{N} \left(\operatorname{div} \mathcal{Y} \delta_j^i - \frac{1}{2} (\nabla_j \mathcal{Y}^i + \nabla^i \mathcal{Y}_j) - \frac{1}{2\varepsilon - R_g} \langle \pi, \nabla \mathcal{Y} \rangle \pi_j^i \right) \right) \\ &= \nabla_i \left(\frac{1}{N} \left(\operatorname{div} \mathcal{Y} \delta_j^i - \frac{1}{2} (\nabla_j \mathcal{Y}^i + \nabla^i \mathcal{Y}_j) - \frac{1}{2\varepsilon - R_g} \langle \pi, \nabla \mathcal{Y} \rangle \pi_j^i \right) Z^j \right) \\ &\quad - \frac{1}{N} \left(\operatorname{div} \mathcal{Y} \delta_j^i - \frac{1}{2} (\nabla_j \mathcal{Y}^i + \nabla^i \mathcal{Y}_j) - \frac{1}{2\varepsilon - R_g} \langle \pi, \nabla \mathcal{Y} \rangle \pi_j^i \right) \nabla_i Z^j \end{aligned}$$

which yields

$$\begin{aligned} \langle L\mathcal{Y}, \mathcal{Z} \rangle &= \operatorname{div} \left(\frac{1}{N} \left(\mathcal{Z} \operatorname{div} \mathcal{Y} - \frac{1}{2} \mathcal{L}_{\mathcal{Y}} g(\mathcal{Z}, \cdot) - \frac{1}{2\varepsilon - R_g} \langle \pi, \nabla \mathcal{Y} \rangle \pi(\mathcal{Z}, \cdot) \right) \right) \\ &\quad - \frac{1}{N} \left(\operatorname{div} \mathcal{Y} \operatorname{div} \mathcal{Z} - \langle {}^S \nabla \mathcal{Y}, \nabla \mathcal{Z} \rangle - \frac{1}{2\varepsilon - R_g} \langle \pi, \nabla \mathcal{Y} \rangle \langle \pi, \nabla \mathcal{Z} \rangle \right). \end{aligned}$$

We conclude that

$$\begin{aligned} \int_{\Omega} \langle L\mathcal{Y}, \mathcal{Z} \rangle \, dM &= \int_{\partial\Omega} \frac{1}{N} \left(\langle \mathcal{Z}, \nu \rangle \operatorname{div} \mathcal{Y} - \frac{1}{2} \mathcal{L}_{\mathcal{Y}} g(\mathcal{Z}, \nu) - \frac{1}{2\varepsilon - R_g} \langle \pi, \nabla \mathcal{Y} \rangle \langle \pi(\mathcal{Z}), \nu \rangle \right) \mu_{\partial\Omega} \\ &\quad - \int_{\Omega} \frac{1}{N} \left(\operatorname{div} \mathcal{Y} \operatorname{div} \mathcal{Z} - \langle {}^S \nabla \mathcal{Y}, \nabla \mathcal{Z} \rangle - \frac{1}{2\varepsilon - R_g} \langle \pi, \nabla \mathcal{Y} \rangle \langle \pi, \nabla \mathcal{Z} \rangle \right) \mu_{g_0}. \end{aligned}$$

where ν stands for the outward normal to $\partial\Omega$. Since the integrand of the first term in the right-hand side vanishes at the boundary, it follows that, if $L\mathcal{Y} = 0$, then

$$\int_{\Omega} \frac{1}{N} \left(\operatorname{div} \mathcal{Y} \operatorname{div} \mathcal{Z} - \langle {}^S \nabla \mathcal{Y}, \nabla \mathcal{Z} \rangle - \frac{1}{2\varepsilon - R_g} \langle \pi, \nabla \mathcal{Y} \rangle \langle \pi, \nabla \mathcal{Z} \rangle \right) \mu_{g_0} = 0. \quad (3.33)$$

In particular on $\Omega' \subset \Omega$ we have

$$\int_{\Omega'} \frac{1}{N} \left(\langle {}^S \nabla \mathcal{Y}, \nabla \mathcal{Y} \rangle - (\operatorname{div} \mathcal{Y})^2 + \frac{1}{2\varepsilon - R_g} \langle \pi, \nabla \mathcal{Y} \rangle^2 \right) \mu_{g_0} = 0. \quad (3.34)$$

Now define

$$I \doteq \langle {}^S\nabla\mathcal{Y}, \nabla\mathcal{Y} \rangle - (\operatorname{div}\mathcal{Y})^2 + \frac{1}{2\varepsilon - R_g} \langle \pi, \nabla\mathcal{Y} \rangle^2. \quad (3.35)$$

Since (ψ, β) gives a reference solution for the constraint equations, using (3.18) we know that

$$\frac{(\operatorname{tr}K)^2 - |K|_g^2}{2\varepsilon - R_g} = 1.$$

We note that

$$\langle {}^S\nabla\mathcal{Y}, \nabla\mathcal{Y} \rangle = \langle {}^S\nabla\mathcal{Y}, {}^S\nabla\mathcal{Y} \rangle = |{}^S\nabla\mathcal{Y}|_g^2$$

and

$$\langle \pi, \nabla\mathcal{Y} \rangle = \langle \pi, {}^S\nabla\mathcal{Y} \rangle.$$

We also have

$$\operatorname{div}_g\mathcal{Y} = g^{ij}\nabla_i\mathcal{Y}_j = \langle g, \nabla\mathcal{Y} \rangle.$$

and

$$\langle K, \nabla\mathcal{Y} \rangle = \langle K, {}^S\nabla\mathcal{Y} \rangle.$$

Note that since π is definite, then $|K|_g \neq 0$. Thus we denote

$$\lambda = \frac{1}{|K|_g^2} \langle \nabla\mathcal{Y}, K \rangle$$

and rewrite (3.35) above as

$$\begin{aligned} I &= |{}^S\nabla\mathcal{Y} - \lambda K|_g^2 + 2\lambda \langle {}^S\nabla\mathcal{Y}, K \rangle - \lambda^2 |K|_g^2 - (\operatorname{div}\mathcal{Y})^2 + \frac{1}{2\varepsilon - R} \langle \pi, \nabla\mathcal{Y} \rangle^2 \\ &= |{}^S\nabla\mathcal{Y} - \lambda K|_g^2 + 2\frac{1}{|K|_g^2} \langle {}^S\nabla\mathcal{Y}, K \rangle^2 - \frac{1}{|K|_g^2} \langle {}^S\nabla\mathcal{Y}, K \rangle^2 - (\operatorname{div}\mathcal{Y})^2 + \frac{1}{2\varepsilon - R} \langle \pi, \nabla\mathcal{Y} \rangle^2. \end{aligned}$$

However, since

$$\pi = K - \operatorname{tr}K g$$

and

$$\langle \pi, \nabla\mathcal{Y} \rangle = \langle K, \nabla\mathcal{Y} \rangle - \operatorname{tr}K \langle g, \nabla\mathcal{Y} \rangle = \langle {}^S\nabla\mathcal{Y}, K \rangle - \operatorname{tr}K \operatorname{div}\mathcal{Y},$$

we get that

$$\begin{aligned} I &= |{}^S\nabla\mathcal{Y} - \lambda K|_g^2 + \frac{1}{|K|_g^2} \langle {}^S\nabla\mathcal{Y}, K \rangle^2 - (\operatorname{div}\mathcal{Y})^2 \\ &\quad + \frac{1}{2\varepsilon - R} (\langle {}^S\nabla\mathcal{Y}, K \rangle^2 - 2\operatorname{tr}K \operatorname{div}\mathcal{Y} \langle {}^S\nabla\mathcal{Y}, K \rangle + (\operatorname{tr}K)^2 (\operatorname{div}\mathcal{Y})^2). \end{aligned}$$

Using again the fact that

$$2\varepsilon - R = (\operatorname{tr}K)^2 - |K|_g^2$$

we have

$$\begin{aligned} I &= |{}^S\nabla\mathcal{Y} - \lambda K|_g^2 + \left(\frac{1}{|K|_g^2} + \frac{1}{(\operatorname{tr}K)^2 - |K|_g^2} \right) \langle {}^S\nabla\mathcal{Y}, K \rangle^2 + \left(\frac{(\operatorname{tr}K)^2}{(\operatorname{tr}K)^2 - |K|_g^2} - 1 \right) (\operatorname{div}\mathcal{Y})^2 \\ &\quad - \frac{2}{2\varepsilon - R} \operatorname{tr}K \operatorname{div}\mathcal{Y} \langle {}^S\nabla\mathcal{Y}, K \rangle. \end{aligned}$$

Therefore

$$\begin{aligned} I &= |{}^S\nabla\mathcal{Y} - \lambda K|_g^2 + \frac{(\operatorname{tr}K)^2}{|K|^2((\operatorname{tr}K)^2 - |K|_g^2)} \langle {}^S\nabla\mathcal{Y}, K \rangle^2 + \frac{|K|_g^2}{(\operatorname{tr}K)^2 - |K|_g^2} (\operatorname{div}\mathcal{Y})^2 \\ &\quad - \frac{2}{(\operatorname{tr}K)^2 - |K|_g^2} \operatorname{tr}K \operatorname{div}\mathcal{Y} \langle {}^S\nabla\mathcal{Y}, K \rangle. \end{aligned}$$

Hence we have

$$\begin{aligned} ((\operatorname{tr}K)^2 - |K|_g^2) I &= ((\operatorname{tr}K)^2 - |K|_g^2) |{}^S\nabla\mathcal{Y} - \lambda K|_g^2 + \frac{(\operatorname{tr}K)^2}{|K|_g^2} \langle {}^S\nabla\mathcal{Y}, K \rangle^2 + |K|_g^2 (\operatorname{div}\mathcal{Y})^2 \\ &\quad - 2\operatorname{tr}K \operatorname{div}\mathcal{Y} \langle {}^S\nabla\mathcal{Y}, K \rangle \\ &= ((\operatorname{tr}K)^2 - |K|_g^2) |{}^S\nabla\mathcal{Y} - \lambda K|_g^2 + \frac{(\operatorname{tr}K)^2}{|K|_g^2} \langle {}^S\nabla\mathcal{Y}, K \rangle^2 + |K|_g^2 (\operatorname{div}\mathcal{Y})^2 \\ &\quad - 2\frac{\operatorname{tr}K}{|K|_g} |K|_g \operatorname{div}\mathcal{Y} \langle {}^S\nabla\mathcal{Y}, K \rangle. \end{aligned}$$

Therefore

$$((\operatorname{tr}K)^2 - |K|^2) I = ((\operatorname{tr}K)^2 - |K|^2) \left| {}^S\nabla\mathcal{Y} - \frac{1}{|K|_g^2} \langle \nabla\mathcal{Y}, K \rangle K \right|_g^2 + \left(|K|_g \operatorname{div}\mathcal{Y} - \frac{\operatorname{tr}K}{|K|_g} \langle {}^S\nabla\mathcal{Y}, K \rangle \right)^2.$$

Using this in (3.34), we get the following

$$\int_{\Omega'} \frac{1}{N} |{}^S\nabla \mathcal{Y} - \frac{1}{|K|_g^2} \langle \nabla \mathcal{Y}, K \rangle K|_g^2 \mu_{g_0} + \int_{\Omega'} \frac{1}{(2\epsilon - R_g)N} \left(|K|_g \operatorname{div} \mathcal{Y} - \frac{\operatorname{tr} K}{|K|_g} \langle {}^S\nabla \mathcal{Y}, K \rangle \right)^2 \mu_{g_0} = 0. \quad (3.36)$$

Since, by hypothesis, $2\epsilon - \bar{R} > 0$, then both integrands are non-negative, thus, for the equality to hold, both must equal zero. From the first integral, we get that

$${}^S\nabla \mathcal{Y} = \frac{1}{|K|_g^2} \langle \nabla \mathcal{Y}, K \rangle K \quad (3.37)$$

Taking traces we get

$$\operatorname{div}_g \mathcal{Y} = \frac{1}{|K|_g^2} \langle \nabla \mathcal{Y}, K \rangle \operatorname{tr}_g K$$

and multiplying by $|K|_g$ we obtain

$$|K|_g \operatorname{div}_g \mathcal{Y} = \frac{\operatorname{tr}_g K}{|K|_g} \langle \nabla \mathcal{Y}, K \rangle,$$

which proves that (3.36) holds if and only if (3.37) holds. This shows that if $\mathcal{Y} \in \ker L$ then $\mathcal{Y}$ satisfies (3.37). The converse for this statement is also true. If we consider a field $\mathcal{Y}$ which satisfies (3.37), after some computations, we get that $L\mathcal{Y} = 0$. So if (3.37) has only the trivial solution $\mathcal{Y} = 0$, then L is injective. $\square$

Using these results and appealing to the implicit function theorem, we have shown the following theorem.

Theorem 3.3.2.1. *Suppose $(\psi_0, \beta_0) \in E_1 \times E_2$ satisfies $\Phi(\psi_0, \beta_0) = 0$. Then if π is a definite operator at each point of M , $2\epsilon - R_g > 0$ everywhere on M , and if for a given function μ on M the equation*

$${}^S\nabla \mathcal{Y} = \mu K$$

has only the solution $\mathcal{Y} = 0$, $\mu = 0$, then there are open neighbourhoods $\mathcal{V} \subset E_1$ and $\mathcal{W} \subset E_2$ of ψ_0 and β_0 respectively, and a unique mapping

$$g : \mathcal{V} \rightarrow \mathcal{W}$$

such that $\Phi(\psi, g(\psi)) = 0$ for all $\psi \in \mathcal{V}$.

At this point it is interesting to point out that the curvature condition $2\epsilon - R_g > 0$ by itself does not pose any topological obstruction. This is because, for instance, if ϵ is continuous, then compactness of M gives us that ϵ is bounded, thus any metric on M with scalar curvature which is *more negative* than $2 \min_{p \in M} \epsilon(p)$ satisfies this condition. That this last (stronger) condition can always be satisfied can be seen as consequence of an important result in geometric analysis, which shows that on any compact n -dimensional manifold, $n \geq 3$, we can always choose a smooth metric g' such that $R(g') = -1$ (see [55],[56]). Then we can always find a suitably rescaled metric g satisfying $2\epsilon - R_g > 0$. Later on, using more subtle arguments, we will actually show that can always find such metric within an initial data set satisfying the constraint equations. Note that, under our assumption, the Sobolev embedding theorems imply that we are assuming ϵ at least C^1 , thus, under our regularity hypotheses, this inequality does not impose any *a priori* restriction.

Notice that **Theorem 3.3.2.1** shows that given an initial data set $\psi_0 \in E_1$ for which a solution β_0 of the reduced constraint equations $\Phi(\psi, \beta) = 0$ exists, if the conditions stated in the theorem are satisfied, then for every $\psi \in \mathcal{V} \subset E_1$ there is a unique solution of the reduced constraint equations. Then taking lapse defined as in (3.20), we get a solution of the full constraint equations. This gives an answer to the TSP: In the previously stated conditions, the TSP holds in a neighbourhood of the free data ψ_0 . This means that for each element in such a neighbourhood, the constraint equations, posed for the lapse and shift, admit a solution.

3.3.3 Remark on the embedding problem

The last theorem we proved in the previous section gives an answer to the question of whether we can solve the constraint equations within the TSP scenario. Now, in order to guarantee that the superspace picture presented in the beginning of this section is consistent, we need to be able to guarantee that initial data satisfying the constraints has an embedding in a space-time. For this part, we need to apply the evolution part of the initial value formulation of GR. Here we have to take into account that, while the previous theorem was quite flexible with regard to ϵ and S , in the study of the evolution problem each matter model has to be considered independently, *i.e.*, we need to show that each particular model has a hyperbolic formulation. Thus, given a reference solution (ψ_0, β_0)

as in **Theorem 3.3.2.1** and free initial data $\psi \in \mathcal{V}$ for which a solution of the constraint will exist, in order for (ψ, β) to have an embedding into space-time we need to know what kind of evolution equations in space-time give rise to these constraints (they must be hyperbolic). In this sense **Theorem 3.3.2.1** gives us the following characterization.

Suppose the Einstein field equations admit a hyperbolic formulation for some situation and that we have a solution of these equations $(\bar{g}, \bar{\varphi})$ on a space-time V , and, furthermore, that the non geometric fields do not generate extra constraints. Then, necessarily, the induced data $(g, K, \varphi, \varphi_t)$ satisfies the constraint equations on M . We can use this solution to obtain a reference solution for the TSP. In order to do this, just consider any given smooth N and β , $N > 0$, and take

$$\dot{g}_{ij} \doteq 2NK_{ij} + (\nabla_i \beta_j + \nabla_j \beta_i). \quad (3.38)$$

Then the set $((g, \dot{g}, \epsilon, S), \beta)$, where ϵ and S are the energy and momentum densities induced by the non geometric fields $\bar{\varphi}$, gives a reference solution of the constraint equations. Suppose that this reference solution (ψ, β) satisfies the conditions stated in **Theorem 3.3.2.1**, then for small a perturbation ψ' of ψ , the theorem guarantees a solution β' in a neighbourhood of β . With this solution we can construct the initial data set $(g', K', \varphi', \varphi'_t)$. Here φ' and φ'_t are perturbations of φ and φ_t respectively, which generate small enough perturbations on ϵ and S such that ψ' is sufficiently close to ψ . Then, for instance, posing the same hyperbolic system satisfied by $(\bar{g}, \bar{\varphi})$ for $(\bar{g}', \bar{\varphi}')$ but with this perturbed initial data, the evolution part of the problem guarantees the existence of the embedding into a space-time whose sources are of the same type to those of the original solution. Thus, we see that **Theorem 3.3.2.1** could be used to produce solutions in a neighbourhood of *realistic* solutions of the constraint equations satisfying the hypotheses of the theorem. In particular, in such a case, using Cauchy stability of the evolution equations, we can guarantee that, at least for short times, the perturbed solution of the Einstein equations remains close to the original one. The following example will be of particular interest in the following chapter.

Suppose that we have a solution of the constraint equations in vacuum, *i.e.*, $\epsilon = 0$ and $S = 0$, and that this solution satisfies the hypotheses of **Theorem 3.3.2.1**. For this solution, the corresponding evolution equations are the vacuum Einstein field equations, where the only variable is the space-time metric. As we have already pointed out, this

problem has a unique solution for initial data satisfying the constraints. This implies that if we take initial data $(g', \dot{g}', 0, 0)$ such that $\|(g', \dot{g}', 0, 0) - (g, \dot{g}, 0, 0)\|_{E_1}$ is sufficiently small, then there is a unique solution of the constraints associated to $(g', \dot{g}', 0, 0)$ and consequently a unique vacuum space-time development, *i.e.*, unique curve in superspace representing a vacuum space-time, associated with this free initial data set. It is trivial to see that the same argument applies to solutions of the constraints with a cosmological constant, *i.e.*, $\epsilon = \Lambda g$ and $S = 0$, and that we get corresponding embeddings into Einstein manifolds, representing vacuum cosmological solutions.

Finally, we would like to point out that we can use these ideas to parametrize the space of solution of the Einstein equations in a neighbourhood of solutions where **Theorem 3.3.2.1** applies. In fact, such parametrization is achieved by parametrizing the solutions of the constraints. Then, within the superspace picture for space time, we see that in a neighbourhood of solutions where our theorem applies, the superspace $S(M)$ has the structure of an infinite dimensional manifold modelled on some Sobolev space, and that the thin sandwich picture can be consistently pictured as giving as initial data an element $(g, \dot{g}) \in T_g S(M)$.

3.3.4 Existence of reference solutions

At this point, we would like to show that the conditions stated in the previous theorem are not too restrictive. By this we mean that, generically, there are solutions of the constraint equations satisfying all these conditions. From the discussion presented above, we know that it is enough to show that solutions (g, K) of the constraint equations satisfying the required properties generically exist. Then we can construct sets of free data $(g, \dot{g}, \epsilon, S)$ from this geometric initial data, and pose the TSP in the way previously discussed. A first step in this direction is the following proposition.

Proposition 3.3.4.1. *Suppose (M, g) is a an n -dimensional compact Riemannian manifold. Suppose that (g, K) satisfy the constraint equations (3.18) and (3.19), where K is a $(0, 2)$ tensor field, and suppose that $2\epsilon - R_g > 0$ on M . Then, if the Ricci tensor on M is negative definite, the equation ${}^S\nabla\mathcal{Y} = \mu K$ has only the trivial solution $\mathcal{Y} = 0$ and $\mu = 0$.*

Proof. Suppose $\mathcal{Y}$ and μ satisfy ${}^S\nabla\mathcal{Y} = \mu K$. From the definition of the curvature tensor

we get the following

$$\begin{aligned}
R_{ij}\mathcal{Y}^j &= \nabla_j \nabla_i \mathcal{Y}^j - \nabla_i \nabla_j \mathcal{Y}^j \\
&= 2\nabla_j (g^{jk} {}^S \nabla_i \mathcal{Y}_k) - \nabla_j \nabla^j \mathcal{Y}_i - \nabla_i (\mu K_j^j) \\
&= 2\nabla_j (\mu K_i^j) - \nabla_j \nabla^j \mathcal{Y}_i - \nabla_i (\mu K_j^j).
\end{aligned}$$

Then we have

$$R_{ij}\mathcal{Y}^j \mathcal{Y}^i = 2\mathcal{Y}^i \nabla_j (\mu K_i^j) - \mathcal{Y}^i \nabla_j \nabla^j \mathcal{Y}_i - \mathcal{Y}^i \nabla_i (\mu K_j^j).$$

We can write this last expression in the following way, which is globally defined:

$$\text{Ric}_g(\mathcal{Y}, \mathcal{Y}) = 2 \langle \text{div}_g(\mu K), \mathcal{Y} \rangle - \langle \mathcal{Y}, \Delta \mathcal{Y} \rangle - \langle \nabla \text{tr}_g \mu K, \mathcal{Y} \rangle. \quad (3.39)$$

In this last expression, Δ stands for the connection Laplacian defined by $\text{tr}_g \nabla^2$. From the previous expression we get that

$$\int_M (\text{Ric}_g(\mathcal{Y}, \mathcal{Y}) + \langle \mathcal{Y}, \Delta \mathcal{Y} \rangle + \langle \nabla(\mu \text{tr}_g K), \mathcal{Y} \rangle - 2 \langle \text{div}_g(\mu K), \mathcal{Y} \rangle) \mu_g = 0. \quad (3.40)$$

Applying divergence theorem, we get

$$\int_M (\text{Ric}_g(\mathcal{Y}, \mathcal{Y}) - |\nabla \mathcal{Y}|_g^2 - \mu^2((\text{tr}_g K)^2 - |K|_g^2) + \mu \langle K, \nabla \mathcal{Y} \rangle) \mu_g = 0. \quad (3.41)$$

Now we will analyze the second and fourth terms in the integrand. In order to do this, we rewrite these expressions in the following way:

$$\begin{aligned}
|\nabla \mathcal{Y}|_g^2 - \langle \mu K, \nabla \mathcal{Y} \rangle &= \langle \nabla \mathcal{Y}, \nabla \mathcal{Y} \rangle - \langle {}^S \nabla \mathcal{Y}, \nabla \mathcal{Y} \rangle = \langle \nabla \mathcal{Y} - {}^S \nabla \mathcal{Y}, \nabla \mathcal{Y} \rangle \\
&= \langle {}^A \nabla \mathcal{Y}, \nabla \mathcal{Y} \rangle = \langle {}^A \nabla \mathcal{Y}, {}^A \nabla \mathcal{Y} \rangle \\
&= |{}^A \nabla \mathcal{Y}|_g^2,
\end{aligned}$$

where

$${}^A \nabla_i \mathcal{Y}_j = \nabla_i \mathcal{Y}_j - {}^S \nabla_i \mathcal{Y}_j = \frac{1}{2} (\nabla_i \mathcal{Y}_j - \nabla_j \mathcal{Y}_i).$$

Then, (3.41) can be rewritten as

$$\int_M (\text{Ric}_g(\mathcal{Y}, \mathcal{Y}) - |{}^A\nabla\mathcal{Y}|_g^2 - \mu^2(2\epsilon - R_g)) \mu_{g_0} = 0. \quad (3.42)$$

Thus, if Ric_g is negative definite, the integrand of this last expression is non-positive. Hence, in order for the last equality to hold, each term in the integrand has to equal zero. Thus, finally, since Ric_g is negative definite by hypothesis, this implies that $\mathcal{Y} = 0$, and consequently $\mu = 0$. So the only possible $\mathcal{Y}$ and μ which can satisfy ${}^S\nabla\mathcal{Y} = \mu K$ under these geometric conditions are $\mathcal{Y} = 0$, $\mu = 0$. $\square$

This proposition implies that, given a solution of the constraint equations (g_0, K_0) satisfying $2\epsilon - R > 0$, then, if π is a definite operator on M , and Ric_{g_0} is negative definite, then the linearization $L = D_2\Phi_{(\psi_0, \beta_0)}$ is an isomorphism and **Theorem 3.3.2.1** applies. A 3-dimensional version of the previous proposition was shown in [17]. In their paper, they used this result to show that the Friedman-Robertson-Walker cosmological solution, with negatively curved space slices, when these slices are compactified, give a (meaningful) example of a reference solution of the constraint equations where **Theorem 3.3.2.1** applies.

We will now show a more generic result, which gives us that any compact n -dimensional manifold admits a solution of the constraint equations satisfying all the hypotheses of **Theorem 3.3.2.1**. The first step in this direction is to look for a solution of the constraint equations of the form $(h, \alpha h)$, with h being a Riemannian metric and α a constant. That is, we are considering $K = \alpha h$ from the beginning. We will restrict ourselves to solutions of (3.18)-(3.19) with $S = 0$, *i.e.*, with zero momentum density. With this set up, equation (3.19) is automatically satisfied and we just need to consider equation (3.18), which, under these conditions, becomes the following equation, which is posed for h :

$$R_h = 2\epsilon - \alpha^2 n(n-1). \quad (3.43)$$

In order to guarantee existence of solutions for (3.43), we will appeal to the following well-established theorem:

Theorem 3.3.4.1 (Kazdan-Wagner). *Let M be a C^∞ compact manifold of dimension $n \geq 3$. If $f \in C^\infty(M)$ is negative somewhere, then there is a C^∞ Riemannian metric on*

M with f as its scalar curvature.

This theorem was proved by Kazdan and Warner [54], and its proof can also be found in [8]. Using this theorem, we see that if the right-hand side of (3.43) is negative somewhere, then, for smooth sources ($\epsilon \in C^\infty$), we have that (3.43) always admits a smooth solution. In order to guarantee this last condition, just take $\alpha^2 > \min \frac{2\epsilon}{n(n-1)}$. A solution constructed in this way satisfies two of the three conditions required by **Theorem 3.3.2.1**, that is, it satisfies

- $2\epsilon - R_h > 0$, which comes from (3.43).
- π is definite, since from $K = \alpha h$ we get that $\pi = \alpha(1 - n)h$.

In this context, the last condition of **Theorem 3.3.2.1** becomes the statement that h does not admit conformal Killing fields. We will show that we can always find a solution h of (3.43) with this property. In order to do this, we need to make a remark on how **Theorem 3.3.4.1** is proved¹⁴. The proof of this theorem begins with the statement that we can choose on M a Riemannian metric g' , with $R_{g'} = -1$, which is something known from [55], and then one finds a conformal metric to g' satisfying the theorem. In fact, it is shown that h has the following form:

$$h = (\phi^{-1})^*(u^{\frac{4}{n-2}}g') \quad (3.44)$$

where u is a positive function and ϕ is a suitably chosen diffeomorphism. In this process, we claim that we can choose g' without conformal Killing fields. We support this claim using the results shown by Lohkamp in [56]. There, it is shown that every manifold M of dimension $n \geq 3$ admits a complete metric with negative definite Ricci tensor. As a corollary of this theorem, it is shown that, starting from such a Riemannian metric g on M with negative definite Ricci tensor, we can find a conformal metric $g' = v^{\frac{4}{n-2}}g$, such that $R(g') = -1$. In this way, using this metric as the starting point in the proof of **Theorem 3.3.4.1**, we get that the metric h solving (3.43) has the following form:

$$h = (\phi^{-1})^*(u^{\frac{4}{n-2}}v^{\frac{4}{n-2}}g) \quad (3.45)$$

¹⁴See, for example, T. Aubin, *Some Non-Linear Problems in Riemannian Geometry*, Springer, Berlin, chapter 6 (1998).

where g has negative definite Ricci tensor. We will use this result in a minute, but before we would like to recall the following result.¹⁵

Proposition 3.3.4.2. *Let M and N be smooth manifolds and $\varphi : M \mapsto N$ be a smooth diffeomorphism. Suppose $X \in \Gamma(TM)$ and $Y \in \Gamma(TN)$ are related by $Y = \varphi_* X$ and that $t \in \Gamma(T_s N)$. Then, the Lie derivative satisfies the following equality*

$$\varphi^*(\mathcal{L}_Y t) = \mathcal{L}_X \varphi^* t. \quad (3.46)$$

□

Now we can prove the following proposition.

Proposition 3.3.4.3. *The solution (3.45) obtained for (3.43) does not admit conformal Killing fields.*

Proof. It is a well-known fact that, on compact manifolds, metrics with negative definite Ricci tensor do not admit conformal Killing fields¹⁶, which is the case of the metric g . Now, imagine that h admitted a conformal Killing field $Y \in \Gamma(TM)$, that is $\mathcal{L}_Y h = \lambda h$, for some $\lambda \in C^\infty(M)$, where $\mathcal{L}_Y h$ is the Lie derivative of h with respect to Y . From (3.45) we get that

$$g = (uv)^{\frac{-4}{n-2}} \phi^*(h) \doteq \mu \phi^*(h).$$

Define $X \doteq \phi_*^{-1} Y \in \Gamma(TM)$, so that $Y = \phi_* X$. We claim that under these conditions X is a conformal Killing field of g . To check this we simply have to compute the Lie derivative of g with respect to X , that is

$$\mathcal{L}_X g = X(\mu) \phi^*(h) + \mu \mathcal{L}_X (\phi^* h).$$

Using (3.46), we know that $\mathcal{L}_X (\phi^* h) = \phi^*(\mathcal{L}_Y h)$, then, since Y is a conformal Killing field for h , we get that

$$\mathcal{L}_X g = (X(\log \mu) + \lambda \circ \phi) g,$$

¹⁵For the proof, see J. E. Marsden, T. Ratiu and R. Abraham, *Manifolds, Tensor Analysis, and Applications*, Springer-Verlag Publishing Company, New York (2007), page 368.

¹⁶See, for instance, Yvonne Choquet-Bruhat, *General Relativity and the Einstein equations*, Oxford University Press Inc., New York, chapter 7 (2009).

which shows that X is a conformal Killing field for g . But this contradicts the fact that g has negative definite Ricci tensor. Thus h can not admit conformal Killing fields. $\square$

Using this last proposition we conclude that the solution we have constructed for the constraint equations satisfies all the hypotheses of **Theorem 3.3.2.1** and, therefore, can be used as a reference solution. Then, we can state the following theorem:

Theorem 3.3.4.2. *Any smooth compact n -dimensional manifold M , $n \geq 3$, admits a smooth solution (N, β) of the constraint equations (3.18)-(3.19) with $S = 0$, with prescribed smooth free data $\psi = (g, \dot{g}, \epsilon_\alpha, 0) \in E_1$, such that there is an E_1 -neighbourhood of ψ where the TSP is well-posed.*

With this theorem we conclude this chapter. We would like to remark that, for instance in vacuum, or the case of a cosmological constant, this theorem gives a nice local description of the superspace picture for space-time around the “generic” points where it applies, as an infinite dimensional manifold where the picture proposed in the beginning of this chapter makes sense. Also, in this way, around these elements, we get a way of parametrizing the set of solutions for the constraints in a nice way. It should be remarked that there are some results which characterize the space of solutions of the constraints, around solution that produce space-times without Killing vectors, as a smooth infinite dimensional manifold [12]. Also, it has been shown by R. Beig, P. T. Chruściel and R. Schoen that for a wide variety of physically relevant settings, the set of solutions of the Einstein equations without Killing vectors is dense in the space of solutions [11]. Combining these results we get a nice characterizations of such structures. Also, A. E. Fischer, J. E. Marsden and V. Moncrief have studied the type of singularities of these sets [12].

Chapter 4

Non analytic Ricci-flat embeddings

In this chapter we will present an embedding theorem which states that any compact n -dimensional Lorentzian manifold, $n \geq 3$, can be isometrically embedded in a $2(n+1)$ -dimensional semi-Riemannian Ricci-flat manifold. It is quite interesting that the proof of this result relies on the main theorems we showed in the previous chapter. The relation between these two problems comes about from a standard procedure which consists in appealing to embedding theorems valid in the Riemannian setting, to prove embeddings in semi-Riemannian cases at the cost of increasing the codimension of the embedding quite drastically. In spite of this fact, the results we will present are novel as far as know, and, hopefully, will serve as a starting point towards sharper results.

4.1 A brief review of some important results

The problem of embedding general n -dimensional manifolds in higher-dimensional spaces with particular properties has been an active object of study for more than a century. It is a well known fact that, as mathematicians began to study abstract manifolds, the question of whether these structures were actually more general than the submanifolds of some $\mathbb{R}^N$ naturally arose. In this direction, several very interesting theorems were proved. We will try to review some of them in a chronological order, and show how the increasing interest in higher-dimensional theories of space-time naturally poses new problems in this area which may have a strong significance for modern theoretical physics.

As we said in the above paragraph, the issue of whether the intrinsic definition of a (finite dimensional) manifold gives rise to more general structures than merely the

submanifolds of some Euclidean space is not a trivial one. This issue was settled in 1936 by H. Whitney in a very famous paper where he showed that any n -dimensional manifold can be embedded in $\mathbb{R}^{2n+1}$ [58]. This rather strong result is a topological one, *i.e.*, there is no mention to the geometry of either manifold. In spite of this, this theorem shows that there is no loss of generality in looking at manifolds as subsets of some $\mathbb{R}^N$. Regarding the isometric embedding of Riemannian manifolds in Euclidean spaces, it was conjectured in 1873 by Schlaefli that every n -dimensional smooth Riemannian manifold admits a smooth local isometric embedding in $\mathbb{R}^{\frac{n(n+1)}{2}}$. It was almost 50 years later that this conjecture was proved to be true in the analytic case in what now is known as the Janet-Cartan theorem [61]-[62], which states the following:

Theorem 4.1.0.1 (Janet-Cartan). *Any n -dimensional analytic Riemannian manifold admits a local analytic isometric embedding in $\mathbb{R}^{\frac{n(n+1)}{2}}$.*

It was not until 1954 that global and non-analytic results appeared for this problem. It turned out that the easiest case to prove was the C^1 -embedding problem. The following results were shown by J. Nash in [59] as two separate theorems, which we will condense in one.

Theorem 4.1.0.2 (Nash). *An n -dimensional Riemannian manifold (M, g) admits a C^1 isometric embedding in E^{2n+1} . If M is closed, then the embedding exists in E^{2n} .*

Increasing the regularity of the embedding turned out to be an extremely difficult task, which ended up with Nash proving a new version of the implicit function theorem, now known as the Nash-Moser implicit function theorem, in order to prove the following celebrated result [60].

Theorem 4.1.0.3 (Nash). *A compact n -dimensional Riemannian manifold has a C^k isometric embedding in any small volume of euclidean $\frac{n}{2}(3n+11)$ -space, provided $3 \leq k \leq \infty$.*

For the non-compact case, the same theorem holds, but with a greater dimensional embedding space. In fact the dimension of the embedding space gets of the order of n^3 . These results were extended to semi-Riemannian geometry by R. Greene in 1970 in [63]. The following theorem, which was shown in Greene's paper, will be important for us, since, as far as we know, gives the best known codimension for a smooth embedding of a

compact semi-Riemannian manifolds in a semi-Euclidean space. Not only that, but, to our knowledge, it is the best known codimension for the case of general smooth embeddings for compact Lorentzian manifolds in Ricci-flat semi-Riemannian ones.

Theorem 4.1.0.4 (Greene). *Any smooth compact n -dimensional semi-Riemannian manifold (M, g) admits an isometric embedding in $\mathbb{R}_k^k$, with $k = \frac{n}{2}(n + 5)$.*

In the previous theorem $\mathbb{R}_q^p$ denotes the semi-euclidean space with p space-like and q time-like dimensions. Greene also worked out a generalization of **Theorem 4.1.0.3** to semi-Riemannian geometry, where the dimensionality of the embedding semi-Euclidean space remains of the same order as in Nash's theorem. For the non-compact case Greene gets improved results with respect to Nash, getting isometric embeddings for general semi-Riemannian manifolds in semi-Euclidean spaces of dimension of order n^2 .¹ Furthermore, he gets improved local results, where he can even control, up to some point, the signature of the embedding space. For such local results the embeddings exist in $\mathbb{R}_q^p$, with $p + q = \frac{n}{2}(n + 3)$, and q can be taken as low as the original number of time-like dimensions of g .

Other results worth mentioning, also from 1970, are the ones obtained by C. J. S. Clarke in [64], where it is shown that a semi-Riemannian manifold with metric of signature² " s " can be embedded in a semi-euclidean space of dimension $d = \frac{n}{2}(3n + 12) + 1 - \frac{s}{2}$, for the compact case, and $\frac{n}{6}(2n^2 + 40) + \frac{5}{2}n + 2 - \frac{s}{2}$ for the non-compact case, and, also, that for the case of globally hyperbolic Lorentzian manifolds, the dimension of the Lorentzian embedding space can be shown to be $d = \frac{n}{6}(2n^2 + 37) + \frac{5}{2}n^2 + 2$.

It should be noted that in 1989, in order to prove Nash's theorem, M. Günther applied a simplified method, which consisted in considering an equivalent elliptic problem to which he could apply the contraction mapping theorem [65]. His result gives an improvement in the codimension of the compact case and a rather low dimension for the local case (eventhough it does not improve the one obtained by Greene).

A relevant recent result, shown by O. Müller and M. Sánchez in [66], which speaks about the possibility of embedding realistic physical space-times, states that any globally hyperbolic n -dimensional space-time can be isometrically embedded in a Minkowski space of $d = N_0(n) + 1$ dimensions, where $N_0(n)$ is the optimal dimension for the case of an

¹The precise codimensions can be checked in Greene's original paper [63]. The precise value of this codimension depends on the details of the type of embedding one is looking for.

²The signature of a semi-Riemannian metric is defined as the difference of the number of positive and negative eigenvalues.

embedding of an n -dimensional Riemannian manifold into an N -dimensional Euclidean space (for a discussion on the value of $N_0(n)$ see [63] and [67]).

We should notice how, for instance, the results obtained in [64] and [66] seem to be strongly motivated by realistic physical situations, since both in the original manifolds and in the target spaces physically sensible properties are imposed. Furthermore, there are other embedding problems which are more significant for realistic physical situations. Probably, the most physically relevant of all, is the initial value formulation of general relativity discussed in the beginning of the previous chapter. In this case, the embedding manifold is characterised by a weaker geometric condition compared to the results reviewed above. Specifically, the geometric conditions imposed on the embedding space are different type of conditions on the Ricci tensor. For instance, for the vacuum case this condition is Ricci-flatness or, more generally, an Einstein space. Since we will have particular interest in the vacuum case, we will state the main theorem for this case, whose first versions were shown by Y. Choquet-Bruhat. In order to simplify the statement of the theorem, we will assume that the space-slices are compact without boundary, *i.e.*, closed.³

Theorem 4.1.0.5 (Choquet-Bruhat). *Consider a vacuum initial data set (M, g, K) , where M is a smooth compact n -dimensional manifold. Consider a smooth Riemannian metric e defined on M used to define the Sobolev spaces H_s . Suppose that the initial data set satisfies the following hypotheses: 1) g is a continuous Riemannian metric satisfying $Dg \in H_{s-1}$, with $s - 1 > \frac{n}{2}$; 2) K is a symmetric second rank tensor field on M such that $K \in H_{s-1}$, with $s - 1 > \frac{n}{2}$; 3) The initial data set (M, g, K) satisfies the constraint equations (3.18)-(3.19) in vacuum. Then, this initial data set admits a vacuum Einsteinian development $(V_T, \bar{g}_T)$, $V_T = M \times [0, T)$, such that the space-time metric is a regularly sliced Lorentzian metric in $E_s(T)$.*

In the previous theorem, the functional spaces $E_s(T)$ denote the space of tensor fields of some given type defined by

$$E_s(T) = \cap_{0 \leq k \leq s} C^{s-k}([0, T], H_k), \quad (4.1)$$

³For a review on results on more general setting see [3].

endowed with the norm

$$\|u\|_{E_s} \doteq \sup_{t \in [0, T]} \sup_{0 \leq k \leq s} \|(\partial_t^k u)(\cdot, t)\|_{H_{s-k}}. \quad (4.2)$$

These spaces enjoy the same embedding and multiplication properties as the Sobolev spaces defined in Appendix 1.⁴

We have chosen to present the previous theorem in arbitrary dimensions, since this type of evolution problem is of interest not only for classical 4-dimensional GR, but for any of the current higher-dimensional theories of space-time posing a geometric restriction on the *bulk* manifold as field equations. In particular, Ricci-flatness and Einstein manifolds are quite popular conditions.⁵ Some of these higher-dimensional models can be reviewed in [29],[31],[68],[69],[70],[71]. In this context in which space-time is supposed to have more than 4 dimensions and the ordinary space-time of GR is considered as a submanifold embedded in this *bulk*, natural additional embedding problems arise. For example, in some of these higher-dimensional models, the bulk is supposed to be characterized by some geometric property, such as being Ricci-flat or, more generally, an Einstein space [29],[31],[68]. Thus, it becomes a straightforward question whether or not any solution of the 4-dimensional Einstein field equations can be embedded in such higher-dimensional structures. Showing that this last statement holds would be a rigorous way of showing that the new theory, in some precise sense, *contains* GR solutions, and, in this way, would be successful in reproducing all the results where GR has been successfully tested. This issue triggered some research and it was shown that, for the local and analytic case, the embeddings do exist. These theorems are the content of the Campbell-Magaard theorem and its extensions [23]-[24],[25],[26],[27],[28],[43]. We will state the most well-known version of these theorems:

Theorem 4.1.0.6 (Campbell-Magaard-Dahia-Romero). *Any n -dimensional semi Riemannian manifold (M, g) with analytic metric admits a local isometric embedding, in a neighbourhood of any point, into an $(n + 1)$ -dimensional semi-Riemannian Einstein manifold with arbitrary cosmological constant.*

Applications of some of these theorems to physical situations can be reviewed in [29].

⁴Note that **Theorem 4.1.0.5** can be applied to C^2 metrics and that the solutions will be as regular as the initial data, thus we get a variety of possible embeddings going from C^2 up to C^∞ .

⁵**Theorem 4.1.0.5** can be extended to an Einstein space development without almost any change.

Even though some of these results have been around for a while, there is no reference in the literature, as far as we know, of non-analytic extensions of the Campbell-Magaard theorems. These type of results would be of interest not just as mathematically interesting problems, but also because some properties of vital importance in relativistic theories, namely causality and stability, may demand non-analytic versions of the Campbell-Magaard theorems in order for them to be applicable to realistic physical theories (for a discussion on this matter, see [34]-[35]). Furthermore, if we want to rigorously support the statement that *ordinary* GR solutions can be embedded in these higher-dimensional space-times, then *global* theorems are required. With all this in mind, the aim of this chapter is to present a first result in this direction, proving a new global embedding theorem for compact n -dimensional Lorentzian manifolds, with metric in the Sobolev space H_{s+3} , $s > \frac{n}{2}$ (which requires metrics at least C^3 , and allows in particular smooth metrics, and in general C^k metrics with k depending on n), in Ricci-flat semi-Riemannian spaces, where the dimensionality needed for the embedding manifold is $2n + 2$.

It should be noted that the aforementioned embedding theorems for semi-Riemannian manifolds in semi-Euclidean spaces, set an upper-bound for novel results concerning embeddings of Lorentzian manifolds in Ricci-flat semi-Riemannian spaces. In particular, the sharpest result previously known for the compact case, which will be our object of study, is the one presented by Robert E. Greene in [63], which guarantees the much stronger statement that an n -dimensional compact semi-Riemannian manifold can be embedded in an $n(n + 5)$ -dimensional Riemann-flat space. Thus, the existence of an embedding in an $n(n + 5)$ -dimensional Ricci-flat space is guaranteed. This means that, using the best result known so far, $(n^2 + 3n - 2)$ additional dimensions are needed with respect to our present result. This seems to indicate that the results we will present in this chapter appear as a significant improvement with respect to the results known so far.

4.2 C^k Ricci-flat embeddings

As we have already stated, the aim of this chapter is to prove an embedding theorem for Lorentzian manifolds in Ricci-flat semi-Riemannian ones. Surprisingly enough, there is no much reference in the literature regarding embeddings of general manifolds in Ricci-flat spaces besides the Cauchy problem in GR and the Campbell-Magaard theorem. Conse-

quently it seems natural to try to adapt any of these results for the case we want to study here. Nevertheless, when attempting to do that, we are faced with serious difficulties. On the one hand, the Campbell-Magaard theorem strongly relies on the Cauchy-Kovalesvka theorem, which depends crucially on the analyticity assumptions for the quantities involved. This occurs as a consequence of the fact that the Ricci-flat condition (or any other geometric condition) is translated into a system of partial differential equations. Our task is to guarantee that this problem is well posed with appropriate initial data. For the local and analytic case the Cauchy-Kovaleskya theorem does the trick, but relaxing the analyticity assumptions prohibits the direct application of this theorem. Thus, we see that the approach taken in the Campbell-Magaard theorem does not seem to be well-suited as a starting point for the non-analytic case, which is our present object of study. This type of difficulty is not an odd feature of embedding problems. An analogous situation was presented when trying to generalize the Janet-Cartan embedding theorem, that also relies on the Cauchy-Kovaleskya theorem, which culminated with Nash's theorem.

On the other hand, the Cauchy problem in GR strongly depends on the hyperbolic character of the evolution equations and the elliptic one of the constraint equations, and these properties strongly depend on the signatures of the space-time metric and the induced metric on the space-like slices. Nevertheless, these results require weak regularity assumptions, and we will use some of the results known for the Cauchy problem in GR to show our main theorem.

We will begin by reviewing some auxiliary results that are needed for our proofs. In order to motivate the introduction of these results, we will first outline the idea behind our proof.

Suppose we are given a smooth compact n -dimensional Lorentzian manifold (M, g) and we want to embed it in some Ricci-flat manifold. If we can perturb the metric g in such a way that the perturbed metric g' is now a properly Riemannian metric, and furthermore g' is part of an initial data set solving the vacuum constraint equations (3.18)-(3.19), then, **theorem (4.1.0.5)** could be invoked to guarantee the existence of an isometric embedding of (M, g') into an $(n + 1)$ -dimensional space-time $(V, \bar{g}')$. If we can arrange things in such a way that the perturbation g' lies in a sufficiently small neighbourhood of the reference solution of the constraints used in **Theorem 3.3.4.2**, denoted by g_0 , then we can guarantee the existence of the previous embedding appealing to that theorem, and

we would have embeddings for both g_0 and g' . Roughly speaking, the more or less obvious thing to do now, is to embed (M, g) into the product of these two embedding Lorentzian manifolds, and use the embeddings for g_0 and g' to construct the embedding for (M, g) . With all this in mind, we will begin by reviewing some geometric properties of product manifolds.

4.2.1 Product of semi-Riemannian manifolds

Recall that given two smooth manifolds M and N , we can construct the product manifold, denoted by $M \times N$, and that an atlas for $M \times N$ is given by taking the product of all coordinate systems on M and N . That is, if $\phi : (x^1, \dots, x^m) : U \subset M \mapsto \mathbb{R}^m$ and $\varphi : (y^1, \dots, y^n) : V \subset N \mapsto \mathbb{R}^n$ are coordinate systems on M and N respectively, then

$$\begin{aligned} \phi \times \varphi : U \times V &\mapsto \mathbb{R}^{n+m} \\ (p, q) &\mapsto (x^1(p), \dots, x^m(p), y^1(q), \dots, y^n(q)) \end{aligned}$$

is a coordinate system on $M \times N$. In fact the maximal atlas with which we endow this product manifold is the one compatible with all these charts. The following standard constructions on product manifold will be useful for us.

Let $\pi : M \times N \mapsto M$ and $\sigma : M \times N \mapsto N$ denote the standard projections of $M \times N$ onto its first and second factors. Both of this maps are submersions, and thus the sets $M \times \{q\}$ and $\{p\} \times N$ are submanifolds of $M \times N$. Thus we have that $T_{(p,q)}M \doteq T_{(p,q)}(M \times \{q\})$ and $T_{(p,q)}N \doteq T_{(p,q)}(\{p\} \times N)$ are subspaces of $T_{(p,q)}(M \times N) \forall (p, q) \in M \times N$. Furthermore, an easy computation gives that $\pi|_{M \times \{q\}}$ and $\sigma|_{\{p\} \times N}$ define diffeomorphisms from $M \times \{q\} \mapsto M$ and $\{p\} \times N \mapsto N$ respectively. Actually we have the following:

$$\begin{aligned} d\pi|_{T_{(p,q)}(M \times \{q\})} &= Id \quad ; \quad d\pi|_{T_{(p,q)}(\{p\} \times N)} = 0 \\ d\sigma|_{T_{(p,q)}(M \times \{q\})} &= 0 \quad ; \quad d\sigma|_{T_{(p,q)}(\{p\} \times N)} = Id \end{aligned}$$

which can be used to see that the tangent spaces split: $T_{(p,q)}(M \times N) = T_{(p,q)}M \oplus T_{(p,q)}N$. Taking into account these facts we can omit notational differences between objects liked by diffeomorphisms, and make precise sense to writing $T_{(p,q)}(M \times N) = T_pM \oplus T_qN$ without invoking the horizontal and vertical lifts. Thus, we see that a vector field $X \in$

$\Gamma(T(M \times N))$ can be written $X = X_1 + X_2$ with $X_1 \in T_p M$ and $X_2 \in T_q N$, where again, we are omitting a change in notation between these fields and their horizontal and vertical lifts. In particular, we have that, given X, Y vector fields on the product then $[X, Y] = [X_1, Y_1] + [X_2, Y_2]$, where $X = X_1 + X_2$ and $Y = Y_1 + Y_2$ with X_1, Y_1 are vector fields on M and X_2, Y_2 are vector fields on N . With all this in mind, we will now link the geometry of the product structure to the geometries of its factors. We begin by presenting the following lemma.

Lemma 4.2.1.1. *Let (M, g_M) and (N, g_N) be two semi-Riemannian manifolds. If π and σ denote the projections of $M \times N$ onto M and N respectively, define*

$$g \doteq \pi^* g_M + \sigma^* g_N. \quad (4.3)$$

Then, g is a metric tensor on $M \times N$ making it a semi-Riemannian manifold, whose index is the sum of the indexes of g_M and g_N . $\square$

A proof of this lemma can be found in [44]. When we think of the product of two semi-Riemannian manifolds, we will think about it as a semi-Riemannian manifold endowed the metric defined in the previous lemma.

Having endowed $M \times N$ with a semi-Riemannian structure, we know that we have a Riemannian connection associated with it, which we will denote ∇^{M+N} .

Lemma 4.2.1.2. *Consider two semi-Riemannian manifolds (M, g_M) , (N, g_N) and the semi-Riemannian product manifold $(M \times N, g)$. Take X, Y vector fields on $M \times N$. Then we have that*

$$\nabla_X^{M+N} Y = \nabla_{X_1}^M Y_1 + \nabla_{X_2}^N Y_2, \quad (4.4)$$

where $X_1, Y_1 \in \Gamma(TM)$ and $X_2, Y_2 \in \Gamma(TN)$ are the unique vector fields such that $X = X_1 + X_2$ and $Y = Y_1 + Y_2$.

Proof. Clearly (4.4) defines a linear connection on the product manifold. In order to show that the lemma holds, we need to show that (4.4) is the Riemannian connection associated with g . Thus, we need to show that it is torsionless and compatible with g . First note

that

$$\begin{aligned}
[X, Y] &= [X_1, Y_1] + [X_2, Y_2] \\
&= (\nabla_{X_1}^M Y_1 - \nabla_{Y_1}^M X_1) + (\nabla_{X_2}^N Y_2 - \nabla_{Y_2}^N X_2) \\
&= (\nabla_{X_1}^M Y_1 + \nabla_{X_2}^N Y_2) - (\nabla_{Y_1}^M X_1 + \nabla_{Y_2}^N X_2) \\
&= \nabla_X^{M+N} Y - \nabla_Y^{M+N} X,
\end{aligned}$$

which shows that ∇^{M+N} is torsionless. In order to check the compatibility condition consider X, Y, Z vector fields on $M \times N$, with splittings $X = X_1 + X_2$, $Y = Y_1 + Y_2$ and $Z = Z_1 + Z_2$ where X_1, Y_1, Z_1 are vector fields on M and X_2, Y_2, Z_2 vector fields on N . Then,

$$\begin{aligned}
X(g(Y, Z)) &= X(g_M(Y_1, Z_1)) + X(g_N(Y_2, Z_2)) \\
&= X_1(g_M(Y_1, Z_1)) + X_2(g_N(Y_2, Z_2)) \\
&= g_M(\nabla_{X_1}^M Y_1, Z_1) + g_M(Y_1, \nabla_{X_1}^M Z_1) + g_N(\nabla_{X_2}^N Y_2, Z_2) + g_N(Y_2, \nabla_{X_2}^N Z_2) \\
&= (g_M(\nabla_{X_1}^M Y_1, Z_1) + g_N(\nabla_{X_2}^N Y_2, Z_2)) + (g_M(Y_1, \nabla_{X_1}^M Z_1) + g_N(Y_2, \nabla_{X_2}^N Z_2)) \\
&= g(\nabla_{X_1}^M Y_1 + \nabla_{X_2}^N Y_2, Z_1 + Z_2) + g(Y_1 + Y_2, \nabla_{X_1}^M Z_1 + \nabla_{X_2}^N Z_2) \\
&= g(\nabla_X^{M+N} Y, Z) + g(Y, \nabla_X^{M+N} Z).
\end{aligned}$$

□

Corollary 4.2.1.1. *Consider two semi-Riemannian manifolds (M, g_M) , (N, g_N) and the semi-Riemannian product manifold $(M \times N, g)$. Take X, Y, Z vector fields on $M \times N$. Denote by ${}^M R$, ${}^N R$ and ${}^{M+N} R$ the curvature tensors of ∇^M , ∇^N and ∇^{M+N} respectively. Then we have that*

$${}^{M+N} R(X, Y)Z = {}^M R(X_1, Y_1)Z_1 + {}^N R(X_2, Y_2)Z_2, \quad (4.5)$$

where $X = X_1 + X_2$, $Y = Y_1 + Y_2$ and $Z = Z_1 + Z_2$, with X_1, Y_1, Z_1 vector fields on M and X_2, Y_2, Z_2 vector fields on N .

Proof. By definition we know that

$${}^{M+N} R(X, Y)Z = \nabla_X^{M+N} \nabla_Y^{M+N} Z - \nabla_Y^{M+N} \nabla_X^{M+N} Z - \nabla_{[X, Y]}^{M+N} Z.$$

Using the previous lemma, we have that $\nabla_Y^{M+N} Z = \nabla_{Y_1}^M Z_1 + \nabla_{Y_2}^N Z_2$, and consequently

$$\nabla_X^{M+N} \nabla_Y^{M+N} Z = \nabla_{X_1}^M \nabla_{Y_1}^M Z_1 + \nabla_{X_2}^N \nabla_{Y_2}^N Z_2,$$

and similarly

$$\nabla_Y^{M+N} \nabla_X^{M+N} Z = \nabla_{Y_1}^M \nabla_{X_1}^M Z_1 + \nabla_{Y_2}^N \nabla_{X_2}^N Z_2.$$

Finally, since we have that $[X, Y] = [X_1, Y_1] + [X_2, Y_2]$, we get

$$\nabla_{[X, Y]}^{M+N} Z = \nabla_{[X_1, Y_1]}^M Z_1 + \nabla_{[X_2, Y_2]}^N Z_2.$$

Putting together these computations, we see that the results holds. $\square$

Corollary 4.2.1.2. *Under the same hypotheses of the previous corollary, if (M, g_M) and (N, g_N) are Ricci-flat then $(M \times N, g)$ is Ricci-flat.*

Proof. In order to make a clean computation, consider an orthonormal frame in a neighbourhood of some point in $M \times N$ given by $\{E_\alpha\}_{\alpha=1}^{m+n}$, where $\{E_\alpha\}_{\alpha=1}^m$ is a frame on M and $\{E_\alpha\}_{\alpha=m+1}^{m+n}$ is a frame of N . The Ricci tensor is given by

$$^{M+N}Ric(X, Y) = \sum_{\alpha} \epsilon_{\alpha} g(^{M+N}R(E_{\alpha}, X)Y, E_{\alpha}),$$

where $\epsilon_{\alpha} = g(E_{\alpha}, E_{\alpha}) = \pm 1$. Using the previous corollary, $^{M+N}R(E_{\alpha}, X)Y = ^MR(E_{\alpha_1}, X_1)Y_1 + ^NR(E_{\alpha_2}, X_2)Y_2$, and thus

$$\begin{aligned} ^{M+N}Ric(X, Y) &= \sum_{\alpha=1}^m \epsilon_{\alpha} g_M(^MR(E_{\alpha_1}, X_1)Y_1, E_{\alpha_1}) + \sum_{\alpha=m+1}^{m+n} \epsilon_{\alpha} g_N(^NR(E_{\alpha_2}, X_2)Y_2, E_{\alpha_2}) \\ &= ^MRic(X_1, Y_1) + ^NRic(X_2, Y_2) \\ &= 0. \end{aligned}$$

$\square$

4.2.2 The embedding problem

Before going to the proof of our main theorem we will need two auxiliary results regarding fibre bundles. Recall that given a topological space E , which consists of a base B and a fibre F , where both B and F are topological spaces, and a surjective projection $\pi : E \mapsto B$, such that for each $p \in B$ the sets $\pi^{-1}(p)$ are homeomorphic to F , we call $\pi^{-1}(p)$ the fibre of E over p . Also, if for any $p \in B$ there exists a neighbourhood U and an homeomorphism $\varphi : \pi^{-1}(U) \mapsto U \times F$, which maps fibres to fibres homeomorphically, we say that E is a fibre bundle with typical fibre F .

Usually we require that on the overlapping of two trivializations $\varphi_1 : \pi^{-1}(U_1) \mapsto U_1 \times F$ and $\varphi_2 : \pi^{-1}(U_2) \mapsto U_2 \times F$, the *transition maps* are smooth. These maps are defined as maps of the following form:

$$g_{12} : U_1 \cap U_2 \mapsto \text{Aut}(F),$$

such that

$$\begin{aligned} \varphi_2 \circ \varphi_1^{-1} : (U_1 \cap U_2) \times F &\mapsto (U_1 \cap U_2) \times F \\ \varphi_2 \circ \varphi_1^{-1}(p, f) &= (p, g_{12}(f)), \end{aligned}$$

where $\text{Aut}(F)$ denotes the group of homomorphism of F . In fact, the transition maps, in some way, encode the entire structure of the fibre bundle. It is typical to construct such fibre bundles from local trivial bundles $U_\alpha \times F \mapsto U_\alpha$ plus the transition maps. In this way we define the fibre bundle E as

$$E \doteq \coprod_{\alpha} U_\alpha \times F / \sim,$$

where $(p_{\alpha_1}, f_{\alpha_1}) \sim (p_{\alpha_2}, f_{\alpha_2})$ if $p_{\alpha_1} = p_{\alpha_2} \in U_{\alpha_1} \cap U_{\alpha_2}$ and $f_{\alpha_2} = g_{12}(f_{\alpha_1})$. In this setting the following very general theorem holds.

Theorem 4.2.2.1. *Suppose $E \mapsto B$ is a fibre bundle with base B and fibre F . If both the base and the fibre are Hausdorff compact spaces, then E is a hausdorff compact topological space.* □

The proof of this theorem relies on the characterization of compact sets which states

that a Hausdorff topological space is compact iff every *net* has a convergent *subnet*.⁶ This result is bit too general for our case of interest where the base, fibre and total space are manifolds, that is, they are also second countable. In such a case there is a celebrated result which states the topologies of each manifold are metrizable.⁷ Thus, we can use a simpler characterization of compactness using sequences: A metric space is compact iff every sequence has a convergent subsequence. With this in mind we will present the following lemma.

Lemma 4.2.2.1. *Suppose $E \xrightarrow{\pi} B$ is a fibre bundle with base B and fibre F . If both the base and the fibre are compact manifolds, then E is a compact manifold.*

Proof. Take a sequence $\{x_n\}_{n=1}^\infty \subset E$ and consider the sequence $\{\pi(x_n)\}_{n=1}^\infty \subset B$. Since B is compact, then it has a convergent subsequence $\{\pi(x_{n_j})\}_{j=1}^\infty$. Denote the limit of this sequence $p \in B$. Take a trivialization $\pi^{-1}(U) \mapsto U \times F$ with $p \in U$. From now on, just consider the subsequence of $\{x_n\}$ formed by those elements which lie in $\pi^{-1}(U)$. In particular, consider the subsequence $\{x_{n_j} = (p_j, q_j)\}_{j=1}^\infty \subset \pi^{-1}(U)$. Take the projection on the fibre $\{\sigma(x_{n_j})\} \subset F$, where $\sigma : \pi^{-1}(U) \mapsto F$. Again, by compactness of F , $\{\sigma(x_{n_j})\}$ has a convergent subsequence $\{\sigma(x_{n_{j_k}})\}_{k=1}^\infty$ in F with limit $q \in F$. Thus, the sequence $\{x_{n_{j_k}}\}_{k=1}^\infty \subset \pi^{-1}(U) \subset E$ satisfies that $\{\pi(x_{n_{j_k}})\}_{k=1}^\infty \subset U$ and $\{\sigma(x_{n_{j_k}})\}_{k=1}^\infty \subset F$ are convergent, thus, from the definition of the product topology, $\{x_{n_{j_k}}\}_{k=1}^\infty \subset E$ is convergent. Thus we have a convergent subsequence of $\{x_n\}_{n=1}^\infty$, and hence E is compact. $\square$

We can now go to the the embedding theorem. To make the assumptions explicit, we recall that we will work on compact (without boundary) manifolds and consider a reference solution of the vacuum constraint equations on such manifold denoted by $(\psi_0 \doteq (g_0, \dot{g}_0), N_0, \beta_0)$, which from **Theorem 3.3.4.2** we know exists, and there is an E_1 -neighbourhood of ψ_0 such that for any ψ in such a neighbourhood there is an associated solution of the constraints (3.18)-(3.19), which satisfies the functional hypotheses of **Theorem 4.1.0.5**, and thus has an associated vacuum development. We will now need the following lemma:

Lemma 4.2.2.2. *Given a compact n -dimensional Lorentzian manifold (M^n, g) , with $g \in H_{s+3}$, it is always possible to find a Riemannian metric $\tilde{g}$ on M such that:*

⁶See, for instance, [102].

⁷See [73] for different levels of this result.

i) $\tilde{g}$ is as close of g_0 as we want.

ii) $g = \lambda(\tilde{g} - g_0)$

for some positive constant λ .

Proof. On M we have both g and g_0 defined. Now let

$$TU \doteq \coprod_{p \in M} \{v \in T_p M \mid g_0(v, v) = 1\}$$

be the unit bundle associated to g_0 , and define

$$F : TU \mapsto \mathbb{R}$$

$$v = (x, v_x) \mapsto \frac{g_x(v_x, v_x)}{g_{0x}(v_x, v_x)}$$

Noticing that F defines a continuous function on TU , which, since M is compact, is a compact set, then F attains its minimum. Furthermore, since g is Lorentzian, then this minimum must be a negative number. Thus, we get that $\exists \alpha < 0$ such that

$$\alpha \leq F(v) \quad \forall v \in TU.$$

Then, $\forall \lambda > 0$ satisfying $-\lambda < \alpha$ it holds that:

$$-\lambda \leq \frac{g_x(v_x, v_x)}{g_{0x}(v_x, v_x)} \quad \forall v \in TU,$$

which gives us the following

$$0 \leq \frac{1}{\lambda} g_x(v_x, v_x) + g_{0x}(v_x, v_x) \quad \forall v \in TU. \quad (4.6)$$

Now define $\tilde{g} \doteq \frac{1}{\lambda} g + g_0$. The claim is that this metric is positive-definite. In order to see this, simply note that for arbitrary $m \in M$ and $\forall V \in T_m M$, $V \neq 0$, the following holds:

$$\tilde{g}_m(V, V) = \|V\|_{g_0}^2 \tilde{g}_m\left(\frac{V}{\|V\|_{g_0}}, \frac{V}{\|V\|_{g_0}}\right) > 0$$

since $\frac{V}{\|V\|_{g_0}} \in T_m U$, where $\|V\|_{g_0} = (g_0(V, V))^{\frac{1}{2}}$. This last relation shows that $\tilde{g}$ is a well-defined Riemannian metric on M . Now, in order to show our first statement, we simply note that λ can in fact be taken as large as we want, thus, given $\epsilon > 0$, $\exists \lambda > 0$ satisfying

the previous statements together the following one:

$$\|\tilde{g} - g_0\|_{H_{s+3}} = \frac{1}{\lambda} \|g\|_{H_{s+3}} < \epsilon. \quad (4.7)$$

Having established our first claim, we see that the second one is a trivial consequence of the definition of $\tilde{g}$, and thus the proposition holds. $\square$

We now present the main theorem:

Theorem 4.2.2.2. *Any n -dimensional compact Lorentzian manifold (M, g) , with $n \geq 3$ and $g \in H_{s+3}$, $s > \frac{n}{2}$, admits an embedding in a Ricci-flat $(2n + 2)$ -dimensional semi-Riemannian manifold with index $n + 1$ (that is, with $n + 1$ time-like dimensions).*

Proof. We will start by appealing to **Lemma 4.2.2.2** and writing the Lorentzian metric g in the following way

$$g = \lambda(\tilde{g} - g_0) \quad (4.8)$$

where both g_0 and $\tilde{g}$ are Riemannian metrics, and g_0 is part of a reference solution $(g_0, \dot{g}_0, N_0, \beta_0)$ of the vacuum constraint equation (3.18)-(3.19) on M . As we have already stated above, **Theorem 3.3.4.2** guarantess that under our hypotheses we can always pick such a solution of the constraint equations, with g_0 being actually smooth, and guarantee that for initial data $(g, \dot{g})$ in a small enough neighborhood of $(g_0, \dot{g}_0)$, there is a unique solution (N, β) of the constraint equations (3.18)-(3.19). Thus, picking λ and $\tilde{g}$ so that $\tilde{g}$ lies in a small enough H_{s+3} -neighborhood of g_0 , then, given the initial data $(\tilde{g}, \dot{g}_0)$, we know that there is a solution of the constraint equations, and, thus, that there are isometric embeddings ϕ_1 and ϕ_2 , of $(M, \tilde{g})$ and (M, g_0) , into the Lorentzian manifolds (V_1, h_1) and (V_2, h_2) respectively, with h_1 and h_2 at least C^2 . We now consider the semi-Riemannian manifold $(V_1 \times V_2, h)$, where

$$h \doteq \lambda(\pi^* h_1 - \sigma^* h_2) \quad (4.9)$$

and π and σ denote the projections of $V \times V$ onto its first and second factors respectively. Thus, this manifold is the product semi-Riemannian manifold resulting from $(V_1, \lambda h_1)$

and $(V_2, -\lambda h_2)$. We claim that the following map is an isometric embedding

$$\begin{aligned}\phi : M &\mapsto V \times V \\ m &\mapsto (\phi_1(m), \phi_2(m))\end{aligned}\tag{4.10}$$

The fact that ϕ is an embedding comes from the fact that both ϕ_1 and ϕ_2 are embeddings. To check the isometry condition, given $v, w \in T_m M$, we compute the following:

$$\begin{aligned}\phi^*(h)_m(v, w) &= h(d\phi_m(v), d\phi_m(w)) \\ &= \lambda(h_1(d\pi_{\phi(m)} \circ d\phi_m(v), d\pi_{\phi(m)} \circ d\phi_m(w)) - h_2(d\sigma_{\phi(m)} \circ d\phi_m(v), d\pi_{\phi(m)} \circ d\phi_m(w))) \\ &= \lambda(h_1(d\phi_{1m}(v), d\phi_{1m}(w)) - h_2(d\phi_{2m}(v), d\phi_{2m}(w))) \\ &= \lambda(\phi_1^*(h_1)_m(v, w) - \phi_2^*(h_2)_m(v, w)) \\ &= \lambda(\tilde{g} - g_0)_m(v, w) \\ &= g_m(v, w).\end{aligned}$$

This last equality shows the isometry condition. As a final step, we have to show that $(V_1 \times V_2, h)$ is Ricci-flat. But, appealing to **Lemma 4.2.1.2**, this is a consequence of the Ricci-flatness of both factors. $\square$

This theorem possesses quite some intrinsic geometric value. It provides a general embedding result for compact Lorentzian manifolds with smooth metrics into Ricci-flat spaces, and the codimension needed for the embedding space, even though greater than in the Campbell-Magaard theorem, is much lower than the one needed using the result obtained by R. Greene for the much stronger condition of Riemann-flatness on the embedding space. Explicitly, $(n^2 + 3n - 2)$ -extra dimensions are saved. Furthermore, we can get a few corollaries from the previous theorem which are quite relevant for (possible) physical applications.

Corollary 4.2.2.1. *Any given compact n -dimensional Lorentzian manifold (M, g) , with metric in the Sobolev space H_{s+3} , $s > \frac{n}{2}$, admits an isometric embedding in a $2(n+1)$ -dimensional Ricci-flat semi-Riemannian manifold $(\tilde{M}^{2n+2}, h)$, with index $h = n+1$, where both the embedding and h depend continuously on g .*

Proof. From the proof of the previous theorem we get that the embedding $\phi : M \mapsto V \times V$, depends on g only through the embedding $\phi_1 : (M, \tilde{g}) \mapsto (V = M \times [0, T], h_1)$, $T > 0$.

Now, from basic facts about the Cauchy problem for the vacuum Einstein equations, we know that ϕ is just the inclusion map, and that, under our functional hypotheses, h_1 is continuous with respect to $\tilde{g} = \frac{1}{\lambda(g)}g + g_0$. Now, from (4.7), we see that we any $\lambda(g) > \frac{\|g\|_{H_{s+3}}}{\epsilon}$ makes the procedure work. In particular, we can fix a choice which satisfies this condition and makes λ a continuous function on g , for instance $\lambda_0(g) \doteq \frac{\|g\|_{H_{s+3}}}{\epsilon} + 1$. With this choice, it is clear that $\tilde{g}_0 \doteq \frac{1}{\lambda_0(g)}g + g_0$ depends continuously on g , with respect to the H_{s+3} topology. Thus, using the fact that the Cauchy development of vacuum initial data for the Einstein field equations is continuous with respect to the initial data, then h_1 is continuous with respect to g , which proves our statement. $\square$

The physical interpretation of this corollary is that the embedding is **stable** against small perturbations on the space-time metric g . Thus, if we think of a compact space-time (M, g) as an embedded submanifold in (M^{2n+2}, h) , then, we can guarantee that we can do this embedding in such a way that, near the submanifold representing the 4-dimensional space-time, the embedding is stable against small perturbations on the space-time metric g .

Now, we will present an important corollary which shows that **Theorem 4.2.2.2** can be used to guarantee the embedding of *strips* of globally hyperbolic space-times with compact space-slices.

Corollary 4.2.2.2. *Suppose we have a strip $(M \times [0, T], g)$, $T > 0$, of an n -dimensional globally hyperbolic space-time, where M is an $(n - 1)$ -dimensional closed manifold, and the Lorentzian metric g is smooth. Then, for any $0 < T_1 < T_2 < T$, the closed strip $M \times [T_1, T_2]$ admits an isometric embedding in a $(2n + 2)$ -dimensional Ricci-flat manifold $(\tilde{M}^{2n+2}, h)$. Furthermore, both the embedding and h can be chosen to be continuous with respect to the space-time metric g .*

Proof. Given the T_1 and T_2 satisfying the hypotheses of the corollary, take T_3 and T_4 such that $0 < T_1 < T_2 < T_3 < T_4 < T$. Then, consider two bump functions $f_1, f_2 : [0, T] \mapsto \mathbb{R}^+$. Pick f_1 such that $f_1|_{[0, T_2]} \equiv 1$ and $f_1|_{[T_3, T]} \equiv 0$, and pick f_2 such that $f_2|_{[0, T_2]} \equiv 0$ and $f_2|_{[T_3, T]} \equiv 1$. Then, define on $M \times [T_2, T)$ the following $(0, 2)$ -tensor field

$$g_2(\cdot, t) \doteq f_2(t)g(\cdot, t - (T_4 - T_1)), \quad (4.11)$$

and then extend it for $t \in [0, T_2]$ as zero. In order for the previous expression to be

well-defined, take T_3 and T_4 sufficiently close, and T_3 sufficiently close to T_2 .⁸ Thus, on $M \times (T_2, T)$ g_2 defines a Lorentzian metric. Now, define the following Lorentzian metric on $M \times [T_1, T_4]$:⁹

$$G \doteq f_1 g + g_2. \quad (4.12)$$

Note that $G(\cdot, t) = g(\cdot, t) \forall T_1 \leq t \leq T_2$, and $G(\cdot, t) = g(\cdot, t - (T_4 - T_1))$ for $T_3 \leq t \leq T_4$. In particular, this gives us that:

$$\begin{aligned} \partial_t^k G(\cdot, T_1) &= \partial_t^k G(\cdot, T_4) = \partial_t^k g(\cdot, T_1) \quad \forall \quad k \in \mathbb{N}_0, \\ \partial_a^k G(\cdot, T_1) &= \partial_a^k G(\cdot, T_4) = \partial_a^k g(\cdot, T_1) \quad \forall \quad k \in \mathbb{N}_0, \end{aligned}$$

where ∂_a denote derivatives on the coordinates on M . The above relations give us that the metric G can be glued smoothly when identifying the slices $M \times \{T_1\}$ with $M \times \{T_4\}$. This gluing gives us a quotient manifold $(M \times [T_1, T_4]) / \sim \cong M \times S^1$, with an induced smooth Lorentzian metric G' , such that the embedding $(M \times [T_1, T_2], g) \hookrightarrow (M \times S^1, G')$ is isometric. We can then apply **Theorem 4.2.2.2** to $(M \times S^1, G')$ and get an isometric embedding $(M \times S^1, G') \xrightarrow{\phi} (\tilde{M}^{2n+2}, h)$. The composition of these two embeddings gives us an isometric embedding of $(M \times [T_1, T_2], g)$ in $(\tilde{M}^{2n+2}, h)$.

Finally, the continuous dependence with respect to the metric g is a consequence of **Corollary 4.2.2.1**. □

4.3 Discussion

In **Theorem 4.2.2.2**, we have obtained a non-analytic embedding theorem for compact Lorentzian manifolds in Ricci-flat spaces, where the regularity assumptions are rather weak. In fact, by means of the Sobolev embedding theorems, we get that the minimal regularity needed for the Lorentzian metric g is C^3 , since H_{s+3} embeds in C^3 for $s > \frac{n}{2}$. In general, in order for $g \in H_{s+3}$, we will need g in some C^k space, with k depending on n , and in particular a smooth g always satisfies this condition. As far as we know, this is a new result, and we regard this theorem as a first step in the search of sharper theorems

⁸This choice has to be done such that $T_1 + T_2 > T_4$.

⁹Notice that the metric G is actually well-defined on the whole strip $M \times [0, T)$, but we are restricting it to the closed strip $M \times [T_1, T_4]$.

in this context.

Even though the direct application of this theorem to physics would seem to be limited, since the compactness condition on space-time poses strong consequences, such as the existence of a closed time-like curve¹⁰, we have been able to extract two important corollaries from **Theorem 4.2.2.2**, which give us that any closed strip in an n -dimensional globally hyperbolic space-time, with a compact (without boundary) Cauchy surface, can be isometrically embedded in a $2(n+1)$ -dimensional Ricci-flat semi-Riemannian manifold. Notice that such strip can be taken to be as large as we like as long as it remains finite. Thus, for instance, any cosmological model with compact space-slices, considered from some arbitrary finite time $T_1 > 0$ up to some arbitrary large (but finite) time $T_2 < \infty$ can be isometrically embedded in a $2(n+1)$ -dimensional Ricci-flat manifold. It is quite interesting that, for the case of the ordinary 4-dimensional space-time of general relativity, this result gives us an embedding in a 10-dimensional manifold, which is, at least, a curious coincidence, since these number of space-time dimensions have been considered with more interest by theoreticians working on string theory.

We would also like to point out that, in our opinion, the main theorem presented here has an intrinsic geometric value, offering a substantial improvement in the number of dimensions needed to get the embedding. In fact, compared with the sharpest results we are aware of for this kind of embedding we are saving as much as $n^2 + 3n - 2$ dimension, which, for instance, in the context of physics where $n = 4$, 26 extra dimensions are saved.

It is also worth pointing out that results such as the Campbell-Magaard theorems, which have been invoked in the context of some higher-dimensional space-time theories, are both local and analytic theorems. By local we mean that they guarantee the existence of the embedding in a neighbourhood of an arbitrary point in space-time, and, in fact, the size of such neighbourhood is not controlled. In this sense, we have shown that for globally hyperbolic space-times with compact Cauchy hypersurfaces, we can prove the existence of embeddings which are *global in space* and *local in time*, where by local in time we mean that as long as we chose a finite time interval, the embedding will exist. Moreover, regarding analyticity, we have dramatically lowered the regularity needed for the space-time metric, and this has enabled us to prove the stability of the embedding.

Finally, we would like to remark that it would be desirable to control the number

¹⁰See B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, Academic Press Inc., New York. Chapter 14. (1983).

of time-like dimensions of the embedding manifold, even though this has been known to be a difficult task in analogous contexts (see [63]). Also, we leave as a future research perspective the weakening of the compactness condition.

Chapter 5

Ricci-flat embeddings for Weyl's space-time

The aim of this chapter is to present some results concerning existence and non-existence of local isometric embeddings of Weyl structures in some other Ricci-flat Weyl structure. The spirit of these embedding results is to try to study whether the Campbell-Magaard theorem extends naturally to this scenario. Besides the obvious mathematical motivation for this problem, which merely consists in trying to extend known theorems to more general settings, there is also a possible physical motivation for such a problem. Such motivation comes by putting together two different approaches to alternative theories of gravity. As we discussed in the first section of the previous chapter, one quite popular approach is to consider the 4-dimensional space-time of GR as an embedded submanifold of some higher-dimensional space-time, usually characterized by some geometric property. Another approach which became quite popular immediately after Einstein presented the theory of general relativity, was to try to write down an analogous theory in a more general geometric framework, where there were enough degrees of freedom so as to try to accommodate other fields besides the gravitational one, in particular the electromagnetic field. In this direction several ideas were put forward and discussed by some of the most prominent theoretical physicists of the time, such as Eddington, Weyl, Pauli, Dirac and Einstein himself.¹ In particular we will be interested in Weyl's proposal, since, for instance, his proposal of considering the mathematical model for space-time as, what now

¹For a discussion on such proposals together with some of the original correspondence between these figures, see [75].

is called, a Weyl structure, can be actually motivated from *first* principles [39]. Thus, the combination of these two approaches results in a picture where space-time is endowed with a non-Riemannian geometry and posses extra dimensions. In this setting, naturally, embedding problems will appear. We will deal with some of these problems related with extending the Campbell-Magaard theorems to the context of Weyl's geometry.

We will begin this chapter by making a quick overview of the physical motivations for Weyl's geometry, in particular following the axiomatic approach to space-time put forward by Ehler, Pirani and Schild in [39]. Then, we will make a detailed review of the main objection to this model, which goes back to Einstein, known as the *second clock effect*. We will present a careful analysis of such an objection. After this introduction to Weyl's space-time, we will go to the embedding problems.

5.1 Introduction to Weyl's space-time

Shortly after the theory of general relativity was put forward, Hermann Weyl, in the attempt of unifying electromagnetism and gravitation, proposed a theory based on the “*geometrization*” of the electromagnetic field. In order to achieve this, he postulated that the mathematical model for space-time was that of a Weyl structure. In this scheme, Weyl's 1-form ω plays the role of the electromagnetic potential, and Weyl's *length* curvature F plays the role of the electromagnetic Faraday tensor. In this context Weyl's transformations play the role of the gauge transformations of the electromanetic field. With this in mind Weyl proposed a set of field equations, derived from a Lagrangian quadratic on the curvature scalar and coupled with the electromagnetic field (now represented by the geometric tensor F) in the ordinary fashion. In this context, Weyl obtained a theory that could pass some of the basic observational tests of GR, such us the perihelion precession of Mercury and the deflection of light rays, but that would generally deviate from GR. In such theory the electromagnetic field would appear as a consequence of the geometric properties of space-time, just as much as the gravitational field does in GR. A revision of this theory can be found in [37]-[84].

Even though Weyl's idea got some attention and was discussed by some the most renowned theoretical physicists of the time, there was in particular one objection which yielded the theory as *non-physical*. This objection, put forward originally by Einstein,

stated that the *proper time* as measured by a *standard clock* within Weyl's theory would be path-dependent, and that this would lead to a *second clock effect*, which is something that has not been observed in experience, and going further from Einstein's original critic, there are quite strong arguments which would in principle set strong constraints to such an effect, if it existed.² The core of Einstein's objection was based upon the fact that, in a Weyl structure, parallel transported vectors change their "norms" according to the following expression:

$$g(V(t), V(t)) = g(V_0, V_0) e^{\int_{t_0}^t \omega(\gamma'(s)) ds} \quad (5.1)$$

where γ is the curve used to make the transport of the initial vector V_0 , and ω is Weyl's 1-form field. From the previous expression it is clear that the norm of such parallel transported vector at a particular point depends on the curve chosen. It was then argued that this effect was related with a dependence of the ticking rate of a clock on its history, concluding that a Weyl structure leads to a *second clock effect*. In our opinion, this discussion, even though physically well-guided, is not rigorous, since the association of the tick-tack of a clock with the norm of a parallel transported vector is not trivial, and, even more, norms or lengths are not well-defined concepts in the context of Weyl's space-time. Thus, we believe that a detailed and rigorous proof of this intuitive statement is needed, and for this, a plausible notion for proper time in the context of Weyl's theory seems to be essential.

It is remarkable that from a seemingly unrelated context Weyl structures appear as a natural model for space-time within an axiomatic construction. It will be more interesting for us to present the discussion of the second clock effect within this more general context, since, in such a case, the conclusion we draw will be valid for any theory which uses a Weyl structure as a mathematical model for space-time, and not just for one particular theory (such as Weyl's). Also, it should be noted that the objections for Weyl's theory are not directly related to the particular predictions of his field equations, but are actually related with the mathematical framework on which the theory is constructed. Thus, we regard the seminal paper written by Elhers, Pirani and Schild (EPS) as the most appropriate starting point for our discussion [39].

²See, for instance, R. Geroch. *General Relativity*, 1972 Lecture Notes. Minkowski Institute Press. Chapter 3. (2013); and R. Penrose, *The Road to Reality*, Jonathan Cape, London, Ch. 19 (2004).

5.1.1 Axiomatic approach to space-time

In [39] EPS proposed an axiomatic approach to space-time, which, in a constructive way, enables us to get the more sophisticated mathematical structures used to model space-time from more or less simple (and well-established) properties about the behaviour of physical objects. Their proposal is to consider freely falling particles and light-rays as the primitive concepts, and to derive the mathematical model for space-time from these concepts plus some simple hypotheses about their behaviour. We will not go into the details of their construction, which can be reviewed in their original paper, but only comment on their conclusions.

First of all, using the set of freely falling particles and considering light rays as means of communication between these particles, they succeed in giving space-time the structure of a differentiable manifold. Then, a few assumptions about the behaviour of light, which are related with the possibility of distinguishing between time-like and space-like vectors and between past and future, are used to provide space-time with a *conformal structure*. After that, a generalized *law of inertial*, basically containing the equivalence principle, is used to show that the motion of freely falling particles determines a *projective structure* on space-time.³ At this point, a compatibility condition between these two structures is imposed: The conformal and projective structures are said to be *compatible* if light rays are actually geodesics of the projective structure. This is motivated by the fact that, using their construction, light rays *are shown* to be geodesics of the conformal structure, and since, based on experience, we can “*approximate*” the path of a light ray using the path of a freely falling massive particle “*as close as*” we want⁴, then it seems reasonable to impose that light rays should be geodesics of the projective structure as a compatibility requirement. Surprisingly enough, this requirement shows that the resulting mathematical model for space-time is that of a Weyl structure. It should be noted that in the EPS paper Weyl transformations are not discussed⁵. Since they play a fundamental role in Weyl’s original idea, we will continue to consider a Weyl structure as an equivalence class of Weyl manifolds, sharing the same Weyl connection and liked by the action of Weyl’s

³In [39] a projective structure is defined by an equivalence class of affine connections, where the equivalence relation is given by declaring two connections with the same pregeodesics (unparametrized geodesics) as equivalent.

⁴In EPS words: A photon can be chased by a massive particle as closely as we want. This is nothing more than the more abstract statement that freely falling particles fill the light cone.

⁵See, for instance, the footnote in page 68 of [39].

transformations, as described in Chapter 2. At this point, in order to reduce this Weyl structure to a Riemannian one, EPS proposed two seemingly different possible axioms related with the behaviour of *clocks*. First of all, it is clear that the notion of proper time known from GR does not make much sense in a Weyl structure since it is not invariant under Weyl transformations, and thus could not be sensibly used to describe a physical property in a mathematical model for space-time based on a Weyl structure. Nevertheless, EPS proposed the following definition for proper-time.

Definition 5.1.1.1 (EPS definition of proper time). We will say that a time-like curve γ is parametrized by **proper time** if its tangent vector γ' is *congruent* at each point with a parallel transported vector field V along γ , *i.e.* if $g(\gamma'_p, \gamma'_p) = g(V_p, V_p)$.

We will later see that this notion of proper time is actually well-defined, that is, it is actually invariant under Weyl-transformation. Also, that every time-like curve can be parametrized by such parametrization and it reproduces the definition known from GR when the Weyl structure is reduced to a semi-Riemannian one. At first sight, in our opinion, none of these properties seem self-evident. Furthermore, this definition, unlike the rest of their paper, does not stick to the original framework, where *complex* notions were to be constructed from simple hypotheses about the behaviour of freely falling particles and light rays. We will show that this definition is actually equivalent to a definition proposed by Volker Perlick in [77], which can be motivated within the EPS framework.

The two axioms proposed by EPS as two apparently different hypotheses which could reduce the Weyl structure which results as a consequence of their first axioms, to a semi-Riemannian one, are the following.

On the one hand, they postulate that the “norm” of a parallel transported vector should be independent of the curve chosen to make the transport. Looking at (5.1) it is clear that this implies that ω should be closed, and if we assume space-time to be simply connected, ω should be exact. Thus the Weyl structure is actually integrable, and since in any Weyl integrable structure there is a Riemannian element in the class, that is, there is an element of the form $(M, e^{-\phi}g, 0)$ for which the Weyl connection of the class is its Riemannian connection, then we arrive at the conclusion that space-time has a semi-Riemannian description. Since for EPS Weyl transformations do not play any role, this is enough to conclude that this hypothesis reduces the Weyl structure to a Riemannian one. It seems quite clear that the introduction of this hypothesis is motivated by Einstein’s

objection towards Weyl's proposal, which was related to the existence of a second clock effect. But, in our opinion, the link between equation (5.1) and the ticking rate of a standard clock is not clear, and this makes this hypothesis much weaker than the previous ones. Even more, again, this axiom is not clearly motivated by the motion of freely falling particles or light-rays.

On the other hand, EPS do propose another possible axiom which could replace the previous one and which is in fact well-motivated within their axiomatic framework. This axiom is stated as follows. Given two freely falling, infinitesimally proximate clocks C_1 and C_2 , if we consider a regular sequence of events $(p_1, p_2, \dots)$ in the world line of C_1 determined by the ticking of this clock, and the *Einstein-simultaneous* sequence of events $(q_1, q_2, \dots)$ in the world line of C_2 , then $(q_1, q_2, \dots)$ should also be a regular sequence of events. With the help of the geodesic deviation equation, EPS show that this hypothesis also forces Weyl's one form to be closed, and thus we can draw the same conclusions as above.

In our opinion, none of the above arguments are as natural as the original set of axioms which led to a Weyl structure. Thus we will try to analyse the existence of the second clock effect within this minimal set of axioms, and after showing that such an effect is a characteristic of non-integrable Weyl structures, we will see that the most general model for space-time not exhibiting such an effect, within this framework, is that of a Weyl integrable structure.

5.2 The problem of proper-time and the second clock effect

In this section, the main idea is to introduce a definition of *proper time* which would naturally fit the axiomatic approach proposed by Ehlers, Pirani and Schild. From now on we will disregard the last of the EPS axioms, however retaining that the motion of freely falling particles and light rays determine a Weyl structure as a suitable mathematical model of space-time. In this way we will think of the space-time geometry as given by a structure of the form $(M, [(g, \omega)])$. In this scenario, an axiomatic definition for proper time, guided by the EPS framework, should come as a natural way of distinguishing the proper time parametrization among all the possible parametrizations of a time-like

curve. A natural way of doing this for the case of free falling particles (pregeodesics of the Weyl structure), using the primitive concepts available in our axiomatic framework, is to distinguish the parametrization which makes the free-falling particle satisfy the geodesic equation, which will be clearly defined uniquely up to an affine transformation. This idea has already been suggested in [79]. A particle $\gamma : I \mapsto M$, $u \mapsto \gamma(u)$, is freely falling if it is a pregeodesic, that is, if

$$\frac{D\gamma'(u)}{du} = f(\gamma(u))\gamma'(u). \quad (5.2)$$

Our principle is that the proper time parametrization is a reparametrization of γ that transforms it into a geodesic. Then, if $\gamma(\tau)$ is a time-like curve representing a freely falling particle, it is straightforward to see that γ is parametrized by proper time if, and only if

$$g(\gamma'(\tau), \frac{D\gamma'(\tau)}{d\tau}) = 0. \quad (5.3)$$

This notion of proper time fits naturally into the picture proposed by EPS, since we are using the basic concepts of their construction to motivate it. Our next step should be to generalize this definition for arbitrary time-like curves, which represent arbitrary particles. Obviously, since, in general, particles are not freely falling, (5.2) does not generalize naturally to the general case. However, (5.3) does, and, as we have just seen, (5.2) and (5.3) are equivalent in the case of freely falling particles. This leads us to the following definition, which is precisely the one given by V. Perlick in [77]:

Definition 5.2.0.1. A time-like curve $\gamma : I \mapsto M$, $u \mapsto \gamma(u)$, is called a *standard clock* if $\frac{D\gamma'}{du}$ is orthogonal to $\gamma'(u)$.

In order to check that this definition is mathematically consistent with the fact that we are working with a Weyl structure, we should check that it is independent of the representative member of the class chosen to carry out the computations. Suppose that for some particular $g \in [g]$ we have $g(\gamma', \frac{D\gamma'}{du}) = 0$. Then, since ∇ is independent of the choice of the representative, so is the covariant derivative, and hence $\frac{D\gamma'}{du}$ does not depend on this choice. Also, since any other $\tilde{g} \in [g]$ is related to g by a conformal transformation, then orthogonality of vectors is preserved. Thus, the definition is consistent in the whole

class. Also, in order for this definition to be sensible, we should show that any time-like curve can be parametrized by this kind of parametrization. To see this, consider a time-like curve

$$\begin{aligned}\gamma : I &\mapsto M \\ t &\mapsto \gamma(t)\end{aligned}$$

We are looking for a reparametrization of γ such that the reparametrized curve is a standard clock. This means that we are looking for a diffeomorphism

$$\begin{aligned}\mu : I &\mapsto I' \\ t &\mapsto \tau\end{aligned}$$

such that

$$\begin{aligned}\tilde{\gamma} &= \gamma \circ \mu^{-1} : I' \mapsto M \\ \tau &\mapsto \gamma(\mu^{-1}(\tau))\end{aligned}$$

is a standard clock. With this set-up, we see that $\gamma = \tilde{\gamma} \circ \mu : I \mapsto M$ and

$$\frac{D\gamma'(t)}{dt} = \frac{d^2\mu}{dt^2}\tilde{\gamma}'(\mu(t)) + \left(\frac{d\mu}{dt}\right)^2 \frac{D\tilde{\gamma}'}{d\tau}(\mu(t)). \quad (5.4)$$

Also from (5.4) we get

$$\begin{aligned}g\left(\frac{D\gamma'(t)}{dt}, \gamma'(t)\right) &= \frac{d^2\mu}{dt^2}g(\tilde{\gamma}'(\mu(t)), \gamma'(t)) + \left(\frac{d\mu}{dt}\right)^2 g\left(\frac{D\tilde{\gamma}'}{d\tau}(\mu(t)), \gamma'(t)\right) \\ &= \frac{1}{\frac{d\mu}{dt}} \frac{d^2\mu}{dt^2} g(\gamma'(t), \gamma'(t)) + \left(\frac{d\mu}{dt}\right)^3 g\left(\frac{D\tilde{\gamma}'}{d\tau}(\mu(t)), \tilde{\gamma}'(\mu(t))\right)\end{aligned}$$

Then, the following equation is satisfied:

$$\frac{d^2\mu}{dt^2} - \frac{g(\gamma'(t), \frac{D\gamma'(t)}{dt})}{g(\gamma'(t), \gamma'(t))} \frac{d\mu}{dt} + \left(\frac{d\mu}{dt}\right)^4 \frac{g(\frac{D\tilde{\gamma}'}{d\tau}(\mu(t)), \tilde{\gamma}'(\mu(t)))}{g(\gamma'(t), \gamma'(t))} = 0.$$

From this last equation we see that $\tilde{\gamma}$ is a standard clock if, and only if, the reparametriza-

tion μ satisfies the following differential equation:

$$\frac{d^2\mu}{dt^2} - \frac{g(\gamma'(t), \frac{D\gamma'(t)}{dt})}{g(\gamma'(t), \gamma'(t))} \frac{d\mu}{dt} = 0. \quad (5.5)$$

We then see that if a reparametrization μ^{-1} makes $\gamma \circ \mu^{-1}$ a standard clock, then it satisfies (5.5). Conversely, given a solution of (5.5) with initial conditions such that $\frac{d\mu}{dt}(t_0) \neq 0$, then there is a neighborhood I of t_0 where $\mu : I \mapsto \mu(I) = I'$ is a diffeomorphism, and hence the reparametrization $\gamma \circ \mu^{-1} : I' \mapsto M$ will be a standard clock. Thus, we can state that given a time-like curve $\gamma(t)$, there is a reparametrization which makes it a standard clock if, and only if, the equation (5.5) admits a solution with $\frac{d\mu}{dt}(t_0) \neq 0$. Since this type of differential equation always admits solutions for given initial data $\mu(t_0), \mu'(t_0)$, then a given time-like curve γ can always be reparametrized, in a neighborhood of any point, so as to make it a standard clock. Seeing that this definition makes mathematical sense, we will use it to define *proper time*.

Definition 5.2.0.2 (Perlick's definition of proper time). We will say that a time-like curve γ is parametrized by proper time if the parametrized curve is a standard clock.

Concerning the above definitions, the following remarks should be made:

- 1) Using the same notations as above, the map $\mu : I \mapsto I'$, maps an arbitrary parametrization t to proper time $\tau = \mu(t)$.
- 2) If μ is a solution of (5.5), then $\tilde{\mu}(t) = a\mu(t) + b$ is also a solution, where a, b are arbitrary constants. These constants are determined by the initial conditions, and they just represent the scale (a) and the zero of the clock (b).

It will be important to note that a general solution of equation (5.5) can be obtained. In order to do this, first note that from (2.6) we have

$$\frac{d}{dt} \ln(-g(\gamma'(t), \gamma'(t))) = \frac{1}{g(\gamma'(t), \gamma'(t))} \left\{ 2g(\gamma'(t), \frac{D\gamma'(t)}{dt}) + \omega(\gamma'(t))g(\gamma'(t), \gamma'(t)) \right\}.$$

This leads to

$$\frac{g(\gamma'(t), \frac{D\gamma'(t)}{dt})}{g(\gamma'(t), \gamma'(t))} = \frac{1}{2} \left\{ \frac{d}{dt} \ln(-g(\gamma'(t), \gamma'(t))) - \omega(\gamma'(t)) \right\}. \quad (5.6)$$

Going back to (5.5) and using (5.6) we get

$$\frac{d^2\mu}{dt^2} - \frac{1}{2}\left\{\frac{d}{dt}\ln(-g(\gamma'(t), \gamma'(t))) - \omega(\gamma'(t))\right\}\frac{d\mu}{dt} = 0. \quad (5.7)$$

In order to integrate this equation, first define $\psi \doteq \frac{d\mu}{dt}$, and then (5.7) is reduced to a first order linear ordinary differential equation for ψ , which can easily be integrated, yielding

$$\frac{d\mu(t)}{dt} = \frac{d\mu(t_0)}{dt} \left[\frac{g(\gamma'(t), \gamma'(t))}{g(\gamma'(t_0), \gamma'(t_0))} \right]^{\frac{1}{2}} e^{-\frac{1}{2} \int_{t_0}^t \omega(\gamma'(u)) du}. \quad (5.8)$$

We can now integrate this equation once more and get the general solution for (5.5). It is not difficult to see that doing this we get the following:

$$\mu(t) = \frac{\frac{d\mu(t_0)}{dt}}{(-g(\gamma'(t_0), \gamma'(t_0)))^{\frac{1}{2}}} \int_{t_0}^t e^{-\frac{1}{2} \int_{t_0}^u \omega(\gamma'(s)) ds} (-g(\gamma'(u), \gamma'(u)))^{\frac{1}{2}} du + \mu_0,$$

If $\Delta\tau(t)$ denotes the elapsed proper time between t_0 and t , we can write

$$\Delta\tau(t) = \frac{\frac{d\tau(t_0)}{dt}}{(-g(\gamma'(t_0), \gamma'(t_0)))^{\frac{1}{2}}} \int_{t_0}^t e^{-\frac{1}{2} \int_{t_0}^u \omega(\gamma'(s)) ds} (-g(\gamma'(u), \gamma'(u)))^{\frac{1}{2}} du. \quad (5.9)$$

As a final comment about the mathematical consistency of our present definition of proper time, we remark that the expression (5.9), which is the expression to be used when computing the elapsed proper time measured by an observer between two events, is invariant both under the Weyl transformations and reparametrizations of γ . Therefore, we can say that we have a mathematically consistent definition of proper time in the framework of Weyl geometry. Let us now analyze whether it is physically sensible to adopt this definition. In order to address this question we will consider three points.

i) The Riemannian limit

We claim that the above definition of proper time reduces to the usual definition adopted in general relativity and other metric theories of gravity where the underlying space-time structure is that of a Riemannian manifold. In order to see this, consider this definition when the manifold (M, g) is Riemannian. Let γ be a time-like curve. In this case, the

compatibility of ∇ with g gives

$$g\left(\frac{D\gamma'}{dt}(t), \gamma'(t)\right) = \frac{1}{2} \frac{d}{dt} g(\gamma'(t), \gamma'(t)). \quad (5.10)$$

So $\gamma(t)$ is standard clock if and only if $g(\gamma'(t), \gamma'(t))$ is constant along $\gamma(t)$. This means that γ is parametrized by arc-length, or an affine reparametrization of it. We conclude that our definition is consistent with the Riemannian limit.

ii) The WIST limit

If the 1-form field ω is exact, i.e, if $\omega = d\phi$, where ϕ is some smooth scalar field ϕ on M , then we say that the Weyl structure is *integrable*, and accordingly the resulting space-time is called *Weyl Integrable Space-Time (WIST)*. This kind of geometry has recently attracted the attention of some cosmologists (see, for instance, [40],[41],[42],[79],[80]). A particular interesting recent review on this topic can be found in [81]. In the case of WIST, it is already known that it is possible to define the proper time interval between two events along a curve $\gamma(t)$ in an invariant way as

$$\Delta\tau = \int_{u_1}^{u_2} e^{-\frac{1}{2}\phi(\gamma(u))} \sqrt{-g(\gamma'(u), \gamma'(u))} du. \quad (5.11)$$

It is easy to see that this quantity is invariant under the action of Weyl transformation. We want to show that Perlick's definition reduces to this expression in the case of a WIST model. In order to do this, consider the explicit expression (5.9) and set $\omega = d\phi$. This will lead to

$$\begin{aligned} \Delta\tau &= \frac{\frac{d\tau(u_0)}{du}}{(-g(\gamma'(u_0), \gamma'(u_0))e^{-\phi(\gamma(u_1))})^{\frac{1}{2}}} \int_{u_1}^{u_2} e^{-\frac{1}{2}\phi(\gamma(u'))} (-g(\gamma'(u'), \gamma'(u'))^{\frac{1}{2}} du' \\ &= C \int_{u_1}^{u_2} e^{-\frac{1}{2}\phi(\gamma(u'))} (-g(\gamma'(u'), \gamma'(u'))^{\frac{1}{2}} du' \end{aligned}$$

which, by an appropriate setting of the scale to make $C = 1$, reproduces the WIST definition of proper time.

iii) Additivity

It seems that in any plausible physical definition proper time intervals should be additive. This means that if an observer experiences three events A, B and C in that order, then given two identical clocks, the time interval measured by a single clock from A to C should be the same as the sum of the time intervals measured by the other from A to B , and from B to C . Thus, for the above definition of proper time to be acceptable, it must be the case that if we use (5.9) to make computations in both situations, the results should be the same.

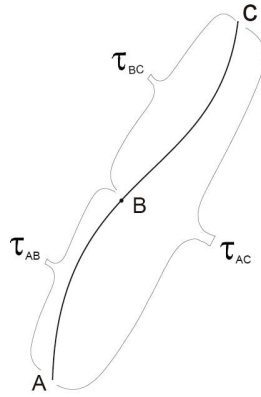


Figure 5.1: Additivity of proper time.

In order to check that the definition that we have presented for proper time is additive, we need to compute the proper time elapsed between A and B , and between B and C , separately by using (5.9), which would represent the readings of two standard clocks used to compute these elapsed times. Then, we have to add these results, and this should equal the result we would get if we were to make a single computation of the proper time elapsed between A and C , which represents the reading of a single clock. Carrying out these calculations we get

$$\tau_{AB} = \frac{\tau'(t_A)}{(-g(\gamma'(t_A), \gamma'(t_A)))^{\frac{1}{2}}} \int_{t_A}^{t_B} e^{-\frac{1}{2} \int_{t_A}^t \omega(\gamma'(s)) ds} (-g(\gamma'(t), \gamma'(t)))^{\frac{1}{2}} dt$$

$$\tau_{BC} = \frac{\tau'(t_B)}{(-g(\gamma'(t_B), \gamma'(t_B)))^{\frac{1}{2}}} \int_{t_B}^{t_C} e^{-\frac{1}{2} \int_{t_B}^t \omega(\gamma'(s)) ds} (-g(\gamma'(t), \gamma'(t)))^{\frac{1}{2}} dt.$$

On the other hand, from (5.8) we know that

$$\tau'(t_B) = \tau'(t_A) \left[\frac{g(\gamma'(t_B), \gamma'(t_B))}{g(\gamma'(t_A), \gamma'(t_A))} \right]^{\frac{1}{2}} e^{-\frac{1}{2} \int_{t_A}^{t_B} \omega(\gamma'(s)) ds}.$$

After substituting the above equation into the expression for τ_{BC} we have that

$$\begin{aligned} \tau_{AB} + \tau_{BC} &= \frac{\tau'(t_A)}{(-g(\gamma'(t_A), \gamma'(t_A)))^{\frac{1}{2}}} \int_{t_A}^{t_C} e^{-\frac{1}{2} \int_{t_A}^t \omega(\gamma'(s)) ds} (-g(\gamma'(t), \gamma'(t)))^{\frac{1}{2}} dt \\ &= \tau_{AC} \end{aligned}$$

which proves the additivity of proper time intervals.

From the above arguments we are led to conclude that Perlick's definition of proper time is both mathematically consistent and physically sensible. We will now show that this definition is, in fact, equivalent to the definition proposed by Ehlers, Pirani and Schild in their original paper [39].

Proposition 5.2.0.1 (Equivalence of Perlick's and EPS proper time). *A time-like curve is parametrized by proper time according to **Definition 5.1.1.1** iff it is parametrized by proper time according to **Definition 5.2.0.2**.*

Proof. Suppose that we have a time-like curve γ parametrized by proper time according to EPS and let τ denote such parametrization. Then, by definition, there exists a parallel vector field V along γ satisfying the following condition:

$$g(\gamma'(\tau), \gamma'(\tau)) = g(V(\tau), V(\tau)).$$

Thus, differentiating the previous identity, using Weyl's compatibility condition and the fact that V is a parallel vector field, we get the following.

$$\begin{aligned} \omega(\gamma'(\tau))g(V(\tau), V(\tau)) &= 2g(\gamma'(\tau), \frac{D\gamma'(\tau)}{d\tau}) + \omega(\gamma'(\tau))g(\gamma'(\tau), \gamma'(\tau)) \\ &= 2g(\gamma'(\tau), \frac{D\gamma'(\tau)}{d\tau}) + \omega(\gamma'(\tau))g(V(\tau), V(\tau)) \Rightarrow \\ 0 &= g(\gamma'(\tau), \frac{D\gamma'(\tau)}{d\tau}) \end{aligned}$$

Hence γ is parametrized by proper time according to **Definition 5.2.0.2**.

In order to prove the converse, suppose that γ is parametrized by proper time according to **Definition 5.2.0.2**. Then, by hypothesis, $\gamma'(\tau)$ and $\frac{D\gamma'(\tau)}{d\tau}$ are orthogonal. Now

consider the following initial value problem:

$$\begin{aligned}\frac{DV(\tau)}{d\tau} &= 0, \\ V(\tau_0) &= \gamma'(\tau_0),\end{aligned}$$

which defines a unique parallel vector field along γ . Since V is a parallel vector, using Weyl's compatibility condition, we see that $g(V(\tau), V(\tau))$ satisfies the following equation

$$\frac{d}{d\tau}g(V(\tau), V(\tau)) = \omega(\gamma'(\tau))g(V(\tau), V(\tau)).$$

Using the compatibility condition and the fact that $\gamma'(\tau)$ and $\frac{D\gamma'(\tau)}{d\tau}$ are orthogonal, we see that $g(\gamma'(\tau), \gamma'(\tau))$ also satisfies the previous equation. Furthermore, both solutions initially agree, that is, $g(V(\tau_0), V(\tau_0)) = g(\gamma'(\tau_0), \gamma'(\tau_0))$. Then, by uniqueness of solutions, we get that $g(V(\tau), V(\tau)) = g(\gamma'(\tau), \gamma'(\tau))$, which means that γ is parametrized by proper time according to EPS. $\square$

With the previous result we have established the equivalence of both definitions. This is an interesting fact, since, at first sight, the two definitions do not seem to be intimately related to each other. It is also worth noting that these two definitions have been widely used in the literature. However, it seems to us that when an author accepts one of them, no reference to the other is made ([39],[77],[78],[79],[82]). Thus, the establishing of their equivalence takes away any possible ambiguity.

5.2.1 Analysis of the Second Clock Effect

In this section, we investigate the question of whether or not a space-time modelled as a Weyl structure, in which proper time is understood according to Perlick's definition, exhibits the so-called *second clock effect*. As is well known, we say that a space-time model exhibits the second clock effect if the clock rate of clocks depends on their histories. The analysis of whether a Weyl structure presents such an effect has been usually done in a more or less *intuitive* way, using the fact that the “norms” of parallel transported vectors depend on the path along which the transport is made, and accepting that such a parallel transported vector along a time-like curve represents the clock rate of a clock. Even though this argument might seem compelling, it is not a rigorous proof, and depends on

some untested hypotheses, such as the fact that the tick tack of a clock can be related with the norm of a parallel transported vector field along the worldline of the clock, let alone the fact that such a discussion should be made using concepts which are well-defined within the context of Weyl's geometry. In this section, we intend to make a detailed analysis of the second clock effect, and settle the question of whether or not a general Weyl structure leads to the second-clock effect.

In order to give an answer to this question consider the following situation. Suppose that we transport two identical standard clocks along a time-like curve from A to B , and then, at B , they separate, following different paths γ_1 and γ_2 until they merge again at event C (see Fig. 5.2), after which they continue their journey together along the same path. Suppose that both clocks were synchronized at A . Thinking of a clock as a device that counts the number of cycles of some periodic process, what we are saying when we refer to synchronization is that identical clocks use the same type of process and that the periods of these cycles were set to be equal at A (both clocks are set with the same scale at A). Now assume that our space-time model does not exhibit a second clock effect. Accepting this hypothesis means that we would expect that the clock rate of a clock, at a given event in space-time, should depend only on *local properties* of the clock, that is, its position, instant velocity, instant acceleration, etc, but not on its history. Therefore, in the particular case we are considering, after we bring back the two identical clocks together at C , and keep them together, we would expect their clock rates to coincide. In other words, we would expect the number of cycles counted by either clock after C to be the same. This also means that the readings of the two clocks would coincide at any subsequent event D ($\tau_{CD} = \bar{\tau}_{CD}$). These considerations give us a way to test a possible existence of the second clock effect: compute the elapsed time for both clocks between C and some subsequent event D and see whether they agree or not. If they do not, then clearly there is a second clock effect.

Let us now, with the help of (5.9), carry out some calculations to make the above discussion more precise. First, let us recall that after the event C both clocks are being transported along the same time-like curve γ . Suppose that the curve γ is parametrized by some arbitrary parameter u . Then, from (5.9) we have

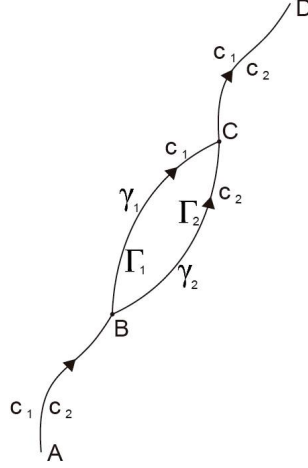


Figure 5.2: Transport of two clocks c_1 and c_2 . The curves γ_1 and γ_2 represent the world lines of c_1 and c_2 respectively, while Γ_1 and Γ_2 denote the portions of these curves between B and C .

$$\tau = \frac{\tau'(u_C)}{(-g(\gamma'(u_C), \gamma'(u_C)))^{\frac{1}{2}}} \int_{u_C}^u e^{-\frac{1}{2} \int_{u_C}^{u'} \omega(\gamma'(s)) ds} (-g(\gamma'(u'), \gamma'(u')))^{\frac{1}{2}} du'$$

$$\bar{\tau} = \frac{\bar{\tau}'(u_C)}{(-g(\gamma'(u_C), \gamma'(u_C)))^{\frac{1}{2}}} \int_{u_C}^u e^{-\frac{1}{2} \int_{u_C}^{u'} \omega(\gamma'(s)) ds} (-g(\gamma'(u'), \gamma'(u')))^{\frac{1}{2}} du'$$

where τ and $\bar{\tau}$ represent the readings of clocks 1 and 2 respectively. We can also compute τ_{AC} and $\bar{\tau}_{AC}$ using (5.9). In order to make this computation, we will consider that the parametrization u used for γ after C is the parametrization used to parametrize the whole time-like path of clock 1. On the other hand, for the path of clock 2 we will use $\bar{u}$ as a parameter. Then, a straightforward calculation shows that

$$\tau'(u_C) = \tau'(u_A) \left(\frac{g(\gamma'_1(u_C), \gamma'_1(u_C))}{g(\gamma'_1(u_A), \gamma'_1(u_A))} \right)^{\frac{1}{2}} e^{-\frac{1}{2} \int_{u_A}^{u_C} \omega(\gamma'_1(s)) ds}$$

$$\bar{\tau}'(\bar{u}_C) = \bar{\tau}'(\bar{u}_A) \left(\frac{g(\gamma'_2(\bar{u}_C), \gamma'_2(\bar{u}_C))}{g(\gamma'_2(\bar{u}_A), \gamma'_2(\bar{u}_A))} \right)^{\frac{1}{2}} e^{-\frac{1}{2} \int_{\bar{u}_A}^{\bar{u}_C} \omega(\gamma'_2(s)) ds}$$

where γ_1 and γ_2 are the curves representing the world lines of each clock. Now, since from A to B and after C both world lines are the same, both curves being equal in these intervals, we could reparametrize γ_2 by changing from $\bar{u}$ to u to obtain

$$\bar{\tau}'(\bar{u}_C) = \bar{\tau}'(u_A) \frac{du(\bar{u}_C)}{d\bar{u}} \left(\frac{g(\gamma'_2(u_C), \gamma'_2(u_C))}{g(\gamma'_2(u_A), \gamma'_2(u_A))} \right)^{\frac{1}{2}} e^{-\frac{1}{2} \int_{u_A}^{\bar{u}_C} \omega(\gamma'_2(s)) ds}.$$

Thus we have

$$\bar{\tau}'(u_C) = \frac{d\bar{u}(u_C)}{du} \bar{\tau}'(\bar{u}_C) = \bar{\tau}'(u_A) \left(\frac{g(\gamma'_2(u_C), \gamma'_2(u_C))}{g(\gamma'_2(u_A), \gamma'_2(u_A))} \right)^{\frac{1}{2}} e^{-\frac{1}{2} \int_{\bar{u}_A}^{\bar{u}_C} \omega(\gamma'_2(s)) ds}.$$

Since $\gamma'_1(u_A) = \gamma'_2(u_A)$ and $\gamma'_1(u_C) = \gamma'_2(u_C) = \gamma'(u_C)$ (recalling that it is the same curve with the same parametrization), then for the reading of clock 2 we get

$$\begin{aligned} \bar{\tau} &= \bar{\tau}'(u_A) \left(\frac{g(\gamma'_1(u_C), \gamma'_1(u_C))}{g(\gamma'_1(u_A), \gamma'_1(u_A))} \right)^{\frac{1}{2}} e^{-\frac{1}{2} \int_{\bar{u}_A}^{\bar{u}_C} \omega(\gamma'_2(s)) ds} \frac{\int_{u_C}^u e^{-\frac{1}{2} \int_{u_C}^{u'} \omega(\gamma'(s)) ds} (-g(\gamma'(u'), \gamma'(u')))^{\frac{1}{2}} du'}{(-g(\gamma'(u_C), \gamma'(u_C)))^{\frac{1}{2}}} \\ &= \frac{\bar{\tau}'(u_A)}{\tau'(u_A)} e^{\frac{1}{2} \int_{u_A}^{u_C} \omega(\gamma'_1(s)) ds - \frac{1}{2} \int_{\bar{u}_A}^{\bar{u}_C} \omega(\gamma'_2(s)) ds} \tau. \end{aligned}$$

Also, since both clocks have the same scale at the event A , that is,

$$\frac{\bar{\tau}'(u_A)}{\tau'(u_A)} = \frac{d\bar{\tau}(\tau_A)}{d\tau} = 1,$$

we finally get

$$\bar{\tau} = e^{\frac{1}{2} \int_{\Gamma_1} \omega - \frac{1}{2} \int_{\Gamma_2} \omega} \tau. \quad (5.12)$$

Thus we are led to conclude that

A Weyl space-time does not exhibit a second clock effect if, and only if, $\int_{\Gamma_1} \omega = \int_{\Gamma_2} \omega$, where Γ_1 and Γ_2 are arbitrary time-like curves joining the same pair of events A and B .

Taking into consideration the previous statement, we are now in position to discuss whether or not a Weyl structure is a suitable model for space-time. In this framework we have an unambiguous, well-defined and physically sensible definition for proper time, and we have arrived at the conclusion that a general Weyl space-time exhibits a second clock effect. We note that up to now this kind of effect has never been measured, and as we have remarked in the introduction, some interesting arguments can be given, which, at least, set strong constraints on the magnitude of this effect. Then, it is natural to ask what would be the most general mathematical structure of space-time if we were to reject space-times exhibiting a second clock effect. If, in the previous statement, we could drop the condition that the curves Γ_1 and Γ_2 are time-like, then it is immediate to see that the non-existence of a second clock effect would imply that the 1-form ω must be closed. If, in turn, we assume that space-time is simply connected, then we would

be led to a Weyl integrable space-time. In what follows, we will show that even if the timelike character of the curves is kept, a Weyl integrable space-time still emerges as a consequence of the non-existence of a second clock effect. Thus, let us suppose that ω is path independent when integrated over time-like curves and see what this implies. First consider an arbitrary event in space-time, $p \in M$, and the set of events I_p that are causally connected with p . Thus I_p is defined as

$$I_p \doteq I_p^+ \cup I_p^- = \{q \in M, \text{ such that there is a time-like curve joining } q \text{ and } p\},$$

which is an open subset of M . On this set define the following function:

$$f(q) \doteq \int_{\gamma} \omega$$

where γ is any time-like curve joining p and q (see figure 5.3) Since, by hypothesis, the integral of ω over any such time-like curve does not depend on the choice of the curve, then the previous function is well-defined.

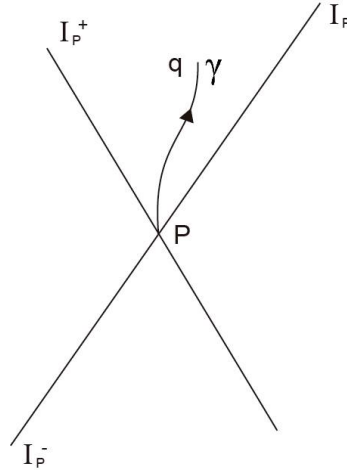


Figure 5.3: Domain of f .

Before going any further, it is important to remark that f is differentiable near p . To see this, we explicitly compute f in a normal coordinate system around p . In such a coordinate system computing f is not difficult since we can choose as the time-like curve γ joining p and q , with q sufficiently near to p , the unique time-like geodesic joining these points inside the normal neighborhood. Then, since the coordinate expression of a geodesic in a normal coordinate system is given by a straight line in the direction of the initial velocity of the geodesic, we can make an easy explicit computation for f . Doing

this, we get the following:

$$f(x) = x^\alpha \int_0^1 \omega_\alpha(tx) dt. \quad (5.13)$$

Assuming ω to be smooth, it is clear that f is differentiable near p . From now on we will restrict f to such a neighborhood of p , so that we know that df exists. Consider now a neighborhood U_q of q such that $U_q \subset I_p$ and a time-like vector $X_q \in T_q M$. Then, we can construct a smooth curve μ defined in a neighborhood J_q of the origin satisfying

$$\begin{aligned} \mu(0) &= q \\ \mu'(0) &= X_q \end{aligned}$$

Also, since $g_q(X(q), X(q)) < 0$, then, by continuity, there is a neighborhood of $0 \in \mathbb{R}$ where μ is time-like. To avoid complications in the notation we will take J_q to be such a neighborhood (if needed, we could shrink the original J_q in order for this to be true). Now we wish to compute

$$df_q(X_q) = \left. \frac{df \circ \mu(t)}{dt} \right|_{t=0}.$$

We can compute $f \circ \mu(t)$ in the following way. First consider a piecewise smooth time-like curve joining p and $\mu(t)$ consisting of a time-like curve β , joining p and $\mu(-\epsilon)$, $-\epsilon \in J_q$, and then the curve μ joining $\mu(-\epsilon)$ and $\mu(t)$. In this set-up we have the following:

$$\begin{aligned} f \circ \mu(t) &= \int_\beta \omega + \int_\mu \omega \\ &= f(\mu(-\epsilon)) + \int_{-\epsilon}^t \omega(\mu'(u)) du. \end{aligned}$$

Therefore

$$\begin{aligned} \left. \frac{df \circ \mu(t)}{dt} \right|_{t=0} &= \omega_{\mu(0)}(\mu'(0)) \\ &= \omega_q(X_q). \end{aligned}$$

Thus we have shown that $df_q(X_q) = \omega_q(X_q)$, for any time-like $X_q \in T_q M$.

To see how df_q acts on an arbitrary vector of $T_q M$ consider an orthonormal basis

$\{e_0, e_i\}$ of $T_q M$, where e_0 is time-like and e_i are space-like. Now if we consider the set of vectors in $T_q M$ given by

$$\begin{aligned}\tilde{e}_0 &\doteq e_0 \\ \tilde{e}_i &\doteq 2e_0 + e_i\end{aligned}$$

then $\{\tilde{e}_0, \tilde{e}_i\}$ gives a basis of time-like vectors for $T_q M$. Then if we pick $V \in T_q M$ arbitrary and we write it in this basis $V = V^\alpha \tilde{e}_\alpha$, we can then compute the action of df_q on this element:

$$\begin{aligned}df_q(V) &= V^\alpha df_q(\tilde{e}_\alpha) \\ &= V^\alpha \omega_q(\tilde{e}_\alpha) \\ &= \omega_q(V) \Rightarrow \\ df_q(V) &= \omega_q(V) \quad \forall V \in T_q M \text{ and } q \in I_p.\end{aligned}$$

Therefore we have shown that, given any point $q \in I_p$, there is a neighborhood of q where ω is exact. Then, if we accept that any event q in space-time lies in I_p for some other event p , we conclude that ω is closed. Finally, assuming that space-time is simply connected, then, to avoid the second clock effect, our space-time model has to be reduced to a Weyl integrable structure $(M, [(g, \phi)])$.

5.3 The Campbell-Magaard theorem for Weyl structures

Taking into account the previous sections as a motivation for considering Weyl structures as possible interesting mathematical models for space-time, we will now study some embedding problems for these structures. Even though we concluded that the strongest original objection towards general Weyl structures modelling space-time actually holds, we will consider this general case since present data would only set bounds on the magnitude of such effect (it cannot be ruled out), and, again, appealing to the axiomatic approach put forward by EPS in [39], these structures appear quite naturally from basic physical phenomena.

In this context, we will investigate possible extensions of the Campbell-Magaard (CM)

theorem (see **Theorem (4.1.0.6)**). It should be noted that the CM theorem caught the attention of some theoretical physicists motivated by the problem of trying to embedded n -dimensional solutions of the Einstein equations in $(n+1)$ -dimensional “*vacuum*” space-times. In this line, a few extensions of this theorem were proved, which aimed to adapt the CM theorem to less restrictive situations where the embedding space could admit sources [25],[26],[27],[28]. It is also important to remark that all these theorems are both local theorems which require analyticity assumptions.⁶ In this chapter we will analyse the existence of local and analytic isometric embeddings for Weyl manifolds, imposing a “Ricci-flatness-type” condition on the embedding manifold. Note that from (2.28), imposing Ricci-flatness already imposes the embedding manifold to be integrable. With this in mind we will first study embeddings for Weyl integrable manifolds into Ricci-flat Weyl manifolds. Then, since we want to study embeddings for general Weyl manifolds, we will impose a weaker condition on the embedding manifold which does not automatically reduce the Weyl structure to an integrable one, this condition being the vanishing of the symmetric-part of the Ricci tensor. Quite surprisingly, in this more general setting, we will find some obstructions for the embedding, which will give a *no go* result. We will also be able to show a particular version of the CM theorem which actually extends to this more general setting. It is remarkable that this extension is highly dependent on the dimension, and will only work for $n = 3$.

First of all, we shall define what we understand by an *isometric embedding* in the context of Weyl geometry.

Definition 5.3.0.1. An *isometric immersion* $\phi : M \mapsto \tilde{M}$ between two Weyl manifolds (M, g, ω) and $(\tilde{M}, \tilde{g}, \tilde{\omega})$ is a smooth mapping satisfying:

- 1) $d\phi_p$ is injective $\forall p \in M$
- 2) $\phi^*(\tilde{g}) = g$
- 3) $\phi^*(\tilde{\omega}) = \omega$.

If, furthermore, ϕ is one-to-one and the induced map $M \mapsto \phi(M)$ is an homeomorphism, where $\phi(M) \subset \tilde{M}$ is seen with the induced topology, then we say that ϕ is an *isometric embedding*. Also, we shall say that ϕ is a *local isometric embedding at* $p \in M$ if there is a

⁶See Chapter 4 for a discussion on this matter.

neighborhood of p where ϕ is an embedding .

As we stated above, at first sight, a natural extension of the Campbell-Magaard theorem [25] in the context of Weyl geometry seems to be to prove the existence of a local analytic isometric embedding of an arbitrary Weyl manifold (M^n, g, ω) in an $(n + 1)$ -dimensional Weyl manifold $(\tilde{M}^{n+1}, \tilde{g}, \tilde{\omega})$ satisfying $\tilde{R}_{\alpha\beta} = 0$ around some arbitrary point $p \in M$. This turns out to be a simple extension which can be treated in complete analogy to [25] after making some considerations. First, note that $\tilde{R}_{\alpha\beta} = 0$ implies that both its symmetric and antisymmetric parts of $\tilde{R}_{\alpha\beta}$ must vanish. However, we already know that for an n -dimensional Weyl manifold we have ${}^A\tilde{R}_{\alpha\beta} = -\frac{n}{2}F_{\alpha\beta}$. Therefore, this condition implies $F_{\alpha\beta} = 0$, which is locally equivalent to setting $\omega = d\phi$, for some function ϕ . In other words, in this case $(\tilde{M}, \tilde{g}, \tilde{\omega})$ gives an integrable Weyl structure. From this, we see that if (M, g, ω) is non-integrable, then it does not exist any isometric embedding of (M, g, ω) in Ricci-flat manifolds, irrespective of the codimension considered. This previous statement, even quite obvious in this context, becomes quite remarkable when compared to standard results in the context of semi-Riemannian geometry, where any semi-Rimannian manifold admits an embedding in a Riemann-flat space. Thus, let us first consider integrable Weyl manifolds and look for analytic isometric embeddings of an integrable Weyl manifold (M^n, g, ϕ) in a Ricci-flat integrable Weyl manifold $(\tilde{M}^{n+1}, \tilde{g}, \tilde{\phi})$. We now proceed to set up the notation that will be used throughout the rest of this chapter.

Henceforth we shall consider $\tilde{M} = M \times \mathbb{R}$, a local chart in M defined in a neighbourhood of p with coordinates $(x^1, \dots, x^n)$, while in the product structure we have a coordinate system around $(p, 0)$ with coordinates $(x^1, \dots, x^n, y)$, where y denotes the coordinate in $\mathbb{R}$. In this coordinate system we write

$$\begin{aligned}\tilde{g} &= \bar{g}_{ik}dx^i \otimes dx^k + \epsilon\bar{\psi}^2 dy \otimes dy, \\ \tilde{\omega} &= \bar{\omega}_i dx^i + \tilde{\omega}_{n+1} dy,\end{aligned}$$

and consider the unit normal field given by

$$\xi = \frac{1}{|\tilde{g}(\partial_{n+1}, \partial_{n+1})|^{\frac{1}{2}}} \partial_{n+1}.$$

From now on, we will typically write a bar over quantities constructed with the induced metric and 1-form, $\bar{g}_{ij}$ and $\bar{\omega}_k$, related to the Weyl structure induced on each slice defined

by $y = cte$.

Now, we will need to write down explicitly the components of the Ricci tensor (recall that the Ricci tensor is no longer symmetric). In order to do this, we will need Gauss' equation, which reads

$$\tilde{R}_{likj} = \bar{R}_{likj} + \epsilon(K_{ji}K_{kl} - K_{ki}K_{jl}), \quad (5.14)$$

with

$$K_{ji} = \epsilon\bar{\psi}\tilde{\Gamma}_{ji}^{n+1} = -\frac{1}{2\bar{\psi}}\frac{\partial}{\partial y}\bar{g}_{ij} + \frac{1}{2\bar{\psi}}\bar{g}_{ij}\tilde{\omega}_{n+1}. \quad (5.15)$$

Also, we will need the following explicit expression for the connection coefficients:

$$\begin{aligned} \tilde{\Gamma}_{ij}^l &= \bar{\Gamma}_{ij}^l, \\ \tilde{\Gamma}_{(n+1)\beta}^{n+1} &= \frac{1}{\bar{\psi}}\partial_\beta\bar{\psi} - \frac{1}{2}\bar{\omega}_\beta, \\ \tilde{\Gamma}_{(n+1)i}^l &= -\bar{\psi}K_i^l, \\ \tilde{\Gamma}_{(n+1)(n+1)}^u &= -\epsilon\bar{\psi}\bar{g}^{ul}\partial_l\bar{\psi} + \frac{\epsilon}{2}\bar{\psi}^2\bar{\omega}^u. \end{aligned} \quad (5.16)$$

Then, from Gauss' equation, we get

$$\tilde{R}_{ij} = \bar{g}^{lk}\bar{R}_{lijk} + \epsilon\bar{g}^{lk}(K_{ik}K_{jl} - K_{ji}K_{lk}) + \tilde{R}^{n+1}_{ij(n+1)}.$$

Using the connection coefficients (5.16), it is straightforward to get

$$\begin{aligned} \tilde{R}^{n+1}_{ij(n+1)} &= \frac{\epsilon}{\bar{\psi}}\frac{\partial K_{ij}}{\partial y} + \epsilon\bar{g}^{lu}K_{iu}K_{lj} + \frac{1}{2}\bar{\nabla}_j\bar{\omega}_i - \frac{1}{4}\bar{\omega}_i\bar{\omega}_j - \frac{1}{\bar{\psi}}\bar{\nabla}_j\bar{\nabla}_i\bar{\psi} + \frac{1}{2\bar{\psi}}(\bar{\nabla}_i\bar{\psi}\bar{\omega}_j \\ &\quad + \bar{\nabla}_j\bar{\psi}\bar{\omega}_i - \epsilon\tilde{\omega}_{n+1}K_{ij}), \end{aligned} \quad (5.17)$$

thus,

$$\begin{aligned} \tilde{R}_{ij} &= \bar{g}^{lk}\bar{R}_{lijk} + \epsilon\bar{g}^{lk}(K_{ik}K_{jl} - K_{ji}K_{lk}) + \frac{\epsilon}{\bar{\psi}}\frac{\partial K_{ij}}{\partial y} + \epsilon\bar{g}^{lu}K_{iu}K_{lj} + \frac{1}{2}\bar{\nabla}_j\bar{\omega}_i - \frac{1}{4}\bar{\omega}_i\bar{\omega}_j - \frac{1}{\bar{\psi}}\bar{\nabla}_j\bar{\nabla}_i\bar{\psi} \\ &\quad + \frac{1}{2\bar{\psi}}(\bar{\nabla}_i\bar{\psi}\bar{\omega}_j + \bar{\nabla}_j\bar{\psi}\bar{\omega}_i - \epsilon\tilde{\omega}_{n+1}K_{ij}). \end{aligned} \quad (5.18)$$

Continuing with the other components, we get that

$$\tilde{R}_{(n+1)i} = \bar{g}^{ul}\tilde{R}_{l(n+1)lu} + \tilde{R}^{n+1}_{(n+1)i(n+1)}. \quad (5.19)$$

From the symmetry properties of the curvature tensor from **Proposition 2.3.1.1**, we get the following.

$$\begin{aligned}\tilde{R}_{l(n+1)iu} &\stackrel{def}{=} \langle R(\partial_u, \partial_i) \partial_{n+1}, \partial_l \rangle \\ &= -\bar{\psi} \langle R(\partial_u, \partial_i) \partial_l, \xi \rangle.\end{aligned}$$

Now, using Codazzi's equation,

$$\tilde{R}_{l(n+1)iu} = -\bar{\psi} \{ \bar{\nabla}_u K_{il} - \bar{\nabla}_i K_{ul} + \frac{1}{2}(\bar{\omega}_i K_{ul} - \omega_u K_{il}) \}. \quad (5.20)$$

Using the expressions (5.16) we can compute that

$$\tilde{R}^{n+1}_{(n+1)i(n+1)} = F_{i(n+1)} = \frac{1}{2}(\partial_i \tilde{\omega}_{n+1} - \partial_{n+1} \bar{\omega}_i).$$

Thus, we get the following expression for (5.19):

$$\tilde{R}_{(n+1)i} = \bar{\psi} \bar{g}^{ul} (\bar{\nabla}_i K_{ul} - \bar{\nabla}_u K_{il}) + \frac{\bar{\psi}}{2} (K_i^u \bar{\omega}_u - K_u^i \bar{\omega}_i) + F_{i(n+1)} \quad (5.21)$$

Going on, using the antisymmetry of the curvature tensor in its last two indices, we get

$$\tilde{R}_{i(n+1)} = \bar{g}^{ul} \tilde{R}_{li(n+1)u}.$$

Using the symmetry properties of the curvature tensor and the special form of the metric $\tilde{g}$, we get the following.

$$\begin{aligned}\tilde{R}_{li(n+1)u} &\stackrel{def}{=} \langle R(\partial_u, \partial_{n+1}) \partial_i, \partial_l \rangle \\ &= \bar{\psi} \langle R(\partial_i, \partial_l) \partial_u, \xi \rangle + \bar{g}_{il} F_{(n+1)u} - \bar{g}_{ui} F_{(n+1)l} + \bar{g}_{ul} F_{(n+1)i},\end{aligned}$$

which gives us that

$$\tilde{R}_{i(n+1)} = \bar{\psi} \bar{g}^{ul} (\bar{\nabla}_i K_{lu} - \bar{\nabla}_l K_{iu}) + \frac{\bar{\psi}}{2} (K_i^l \bar{\omega}_l - K_u^i \bar{\omega}_i) + n F_{(n+1)i}. \quad (5.22)$$

Finally, for the only remaining component, we get that

$$\begin{aligned}
\tilde{R}_{(n+1)(n+1)} &= \bar{g}^{ul} \tilde{R}_{l(n+1)(n+1)u} \\
&= -\bar{g}^{ul} \langle \tilde{R}(\partial_{n+1}, \partial_u) \partial_{n+1}, \partial_l \rangle \\
&= \bar{g}^{ul} \langle \tilde{R}(\partial_{n+1}, \partial_u) \partial_l, \partial_{n+1} \rangle - \bar{g}^{ul} F_{u(n+1)} \tilde{g}_{l(n+1)} \\
&= \bar{g}^{ul} \tilde{R}_{(n+1)lu(n+1)} \\
&= \epsilon \bar{\psi}^2 \bar{g}^{ul} \tilde{R}^{n+1}_{lu(n+1)}.
\end{aligned}$$

Thus, using (5.17) in the previous expression, we finally get

$$\begin{aligned}
\tilde{R}_{(n+1)(n+1)} &= \bar{\psi} \bar{g}^{lu} \frac{\partial K_{lu}}{\partial y} + \bar{\psi}^2 K^{lu} K_{lu} + \frac{\epsilon \bar{\psi}^2}{2} \bar{g}^{ul} \bar{\nabla}_u \bar{\omega}_l - \frac{\epsilon}{4} \bar{\psi}^2 \bar{\omega}_l \bar{\omega}^l - \epsilon \bar{\psi} \Delta_{\bar{g}} \bar{\psi} \\
&\quad + \frac{\epsilon \bar{\psi}}{2} \bar{\omega}^l \bar{\nabla}_l \bar{\psi} - \frac{\bar{\psi}}{2} \tilde{\omega}_{n+1} K_l^l
\end{aligned} \tag{5.23}$$

We will also need the following component of the Einstein tensor, which can easily be computed from the previous expressions for the Ricci tensor.

$$\tilde{G}_{n+1}^{n+1} = -\frac{1}{2} (\bar{R} + \epsilon (K^{lu} K_{lu} - (K_l^l)^2)). \tag{5.24}$$

5.3.1 The integrable case

In this section we will discuss the embedding problem for Weyl integrable manifolds. It is worth noticing that, in this case, there is a stronger analogy with some Riemannian problems already studied in contact with General Relativity. This is because if for one particular member of the class $(\tilde{M}^{n+1}, [\tilde{g}, \tilde{\phi}])$, we split the Ricci tensor into its *Riemannian part* and the *extra terms*, then the Ricci-flat condition becomes equivalent to the Einstein field equations with a scalar field as a source. In the Riemannian framework, embeddings in such structures have been studied by Ponce de Leon, who constructed explicit embeddings of general vacuum solutions of n -dimensional general relativity (with a possible presence of the cosmological constant) into $(n+1)$ -Semi-Riemannian manifolds sourced by a scalar field [83]. We should also mention that embeddings in such structures were also treated by Anderson *et al.*, in which they worked out one of the known extensions of the Campell-Magaard theorem [28]. Even though these results are clearly related to the problem we intend to study here, there are important differences, one of them and maybe

the main one, is that, since in both [28] and [83] the underlying structure is Riemannian, the results presented there would not guarantee the embedding of a whole Weyl integrable structure $(M^n, [g, \phi])$ in a Ricci-flat Weyl integrable structure $(\tilde{M}^{n+1}, [\tilde{g}, \tilde{\phi}])$, as will be shown in this section. Another difference with respect to [83] is that there it is shown that given a solution of the vacuum Einstein field equations in n -dimensions, then it is possible to construct embeddings for a such solution in $(n + 1)$ -dimensional manifolds sourced by scalar fields. In contrast, we will not impose any restriction, besides the regularity assumptions, for the initial data (it does not need to solve any field equations on the original manifold). In this way, we can make an interesting contact with these results known in Riemannian geometry, while having some important differences with them.

In order to start with the discussion of the present embedding problem, note that in the case where $(\tilde{M}, \tilde{g}, \tilde{\omega})$ is integrable, that is, when $\tilde{\omega} = d\tilde{\phi}$, the expressions for the components of the Ricci tensor are simplified. In fact, as we have already seen, the Ricci tensor turns out to be symmetric in this case, and from (2.23), we obtain

$$\tilde{g}^{\alpha\beta} \tilde{\nabla}_\alpha \tilde{G}_{\nu\beta} = 0. \quad (5.25)$$

From the above, we see that the natural approach to this problem is to follow the same procedure adopted in [25], which consists in considering the *evolution* equations $\tilde{R}_{ij} = 0$ in a neighborhood of $0 \in \mathbb{R}^{n+1}$, as well as the *constraint* equations $\tilde{R}_{i(n+1)} = 0$ and $\tilde{G}_{n+1}^{n+1} = 0$ on the hypersurface Σ_0 given by $y = 0$. The justification for this approach is a simple consequence of (5.25). In order to understand how this works, notice that, in this case, from (5.18) we can translate the equations $\tilde{R}_{ij} = 0$ into a second order system for $\bar{g}_{ij}$ of the form

$$\frac{\partial^2 \bar{g}_{ij}}{\partial y^2} = \mathcal{F}_{ij}(\bar{g}_{ab}, \partial_\alpha \bar{g}_{ab}, \partial_{k\alpha} \bar{g}_{ab}) ; \quad a, b, i, j = 1, \dots, n \quad ; \quad \alpha = 1, \dots, n+1,$$

where we are considering both $\tilde{\phi}$ and $\bar{\psi}$ as given analytic functions in a neighbourhood of the origin of $\mathbb{R}^{n+1}$, with the only restriction that $\tilde{\phi}(x, 0) = \phi(x)$. It is a straightforward argument that, if we give analytic initial data defined in a neighbourhood of the origin in $\mathbb{R}^n$, then this system is of the Cauchy-Kovalevskya-type, and thus there will be a unique analytic solution for this system, defined in a neighbourhood of $\mathbb{R}^{n+1}$, taking the prescribed initial data. Clearly, in order to satisfy the isometry condition, the initial data should

be chosen such that $\bar{g}_{ij}(x, 0) = g_{ij}(x)$ and $\frac{\partial \bar{g}_{ij}(x, 0)}{\partial y} = -2\bar{\psi}(x, 0)\Omega_{ij}(x) + g_{ij}(x)\partial_{n+1}\tilde{\phi}(x, 0)$, where Ω_{ij} are analytic functions in a neighbourhood of the origin of $\mathbb{R}^n$, which represent the values of the extrinsic curvature on the slice $y = 0$ (this procedure will be discussed in detail for the non-integrable case). Then, it is also a straightforward argument that for the Weyl integrable manifold $(\tilde{M}^{n+1}, \tilde{g}, \tilde{\phi})$, defined by the solution of this system, (5.25) stands as a linear first order homogeneous system for $\tilde{G}_{\alpha(n+1)}$. This system is again of the Cauchy-Kovaleskya-type, and thus if $\tilde{G}_{\alpha(n+1)}|_{y=0}$ are zero, then, by uniqueness of solutions, these function must be zero in a neighbourhood of $0 \in \mathbb{R}^{n+1}$ (See **Lemma 5.3.3.1** below). Since $\tilde{G}_{\alpha(n+1)} = 0 \Leftrightarrow \tilde{R}_{\alpha(n+1)}$, then we see that the equations $\tilde{R}_{\alpha(n+1)} = 0$ work as constraint equations on the initial data g, Ω_{ij} , which, if the evolution equations $\tilde{R}_{ij} = 0$ are satisfied in a neighbourhood of $0 \in \mathbb{R}^{n+1}$, are propagated by the contracted Bianchi identities (5.25).

It is remarkable that for the analytic case these constraints can be posed entirely for the functions Ω_{ij} and the function g_{ij} are left as **free** initial data. In this procedure, analyticity of the free data combined with the Cauchy-Kovaleskya theorem guarantees the existence of solutions. This procedure is entirely analogous to the one followed in [25]. Thus, we see that given an n -dimensional Weyl manifold (M^n, g, ϕ) , then in a neighbourhood of any given point $p \in M$ there will be a local isometric embedding into an $(n+1)$ -dimensional Ricci-flat Weyl manifold $(\tilde{M}^{n+1}, \tilde{g}, \tilde{\phi})$. Thus we have the following statement:

Theorem 5.3.1.1. *Any analytic integrable n -dimensional Weyl manifold (M^n, g, ϕ) admits a local analytic isometric embedding around any point $p \in M$ in an analytic Ricci-flat integrable Weyl manifold $(\tilde{M}^{n+1}, \tilde{g}, \tilde{\phi})$.*

It is interesting to note that this result guarantees the existence of isometric embeddings for *Weyl manifolds*, not for a *Weyl structure* $(M, [g, \omega])$. Indeed, in order to take into account the whole *Weyl structure* we need to show that for every element of $(M, [g, \omega])$ there is an isometric embedding of this element in an element of some $(n+1)$ -dimensional Weyl structure $(\tilde{M}, [\tilde{g}, \tilde{\omega}])$. Since, as already remarked, when working with Weyl manifolds all the relevant geometric (and physical) quantities are to be defined on the whole class, it is of much more interest to look for an embedding for the whole structure. We claim that we can show this from our previous results. To do this, let us consider the following argument.

Suppose that a particular n -dimensional Weyl manifold (M, g, ω) admits a local analytic isometric embedding into an $(n + 1)$ -dimensional Weyl manifold $(\tilde{M}, \tilde{g}, \tilde{\omega})$, and that this embedding has been constructed following our previous prescription, namely, that the embedding is just the inclusion. On the other hand, any other element of the class $(M, [g, \omega])$ can be written as $(M, e^{-h}g, \omega - dh)$ for some analytic function h . The question is whether there is some analytic function f on $\tilde{M}$ such that, for this element of the class, there is a local analytic isometric embedding into $(\tilde{M}, e^{-f}\tilde{g}, \tilde{\omega} - df)$. Using the same set up we have developed, we define the function $f(x, y)$ in a neighbourhood of the point $p \in \tilde{M}$ (where we know the isometric embedding exists) by:

$$f(x, y) \doteq h(x) + y.$$

We then get

$$\begin{aligned} e^{-f(x,0)}\tilde{g}_{ij}(x, 0) &= e^{-h(x)}g_{ij}(x), \\ \tilde{\omega}_i(x, 0) - \partial_i f(x, 0) &= \omega_i(x) - \partial_i h(x), \end{aligned}$$

which gives us the isometry condition. Also, since the Ricci tensor is an invariant of the class of Weyl manifolds, we have shown the following result.

Theorem 5.3.1.2. *Any analytic n -dimensional integrable Weyl structure $(M^n, [g, \phi])$ admits a local analytic isometric embedding in an $(n + 1)$ -dimensional integrable Weyl structure $(\tilde{M}^{n+1}, [\tilde{g}, \tilde{\phi}])$ with vanishing Ricci tensor.*

5.3.2 The general case

We now turn our attention to the more general problem of embedding of Weyl manifolds which are not necessarily integrable, dropping the condition of Ricci-flatness.

In this section we shall investigate the existence of a local isometric embedding of an arbitrary Weyl manifold (M^n, g, ω) around some point $p \in M$ in a Weyl manifold $(\tilde{M}^{n+1}, \tilde{g}, \tilde{\omega})$ which has ${}^S\tilde{R}_{\alpha\beta} = 0$. This is the same as requiring that ${}^S\tilde{G}_{\alpha\beta} = 0$. From the identity

$$\tilde{g}^{\alpha\beta}\tilde{\nabla}_\alpha{}^S\tilde{G}_{\nu\beta} = \frac{2-n}{2}\tilde{g}^{\alpha\beta}\tilde{\nabla}_\alpha F_{\nu\beta},$$

we see that our requirement on $(\tilde{M}, \tilde{g}, \tilde{\omega})$ imposes the condition

$$\tilde{g}^{\alpha\beta} \tilde{\nabla}_\alpha F_{\nu\beta} = 0, \quad (5.26)$$

which must hold in a neighborhood of p . As we shall see, (5.26) will impose further restrictions on $(\tilde{M}, \tilde{g}, \tilde{\omega})$. To see this we shall need to make use of some geometric identities.

Proposition 5.3.2.1. *Suppose we have a semi-Riemannian manifold M endowed with a torsionless connection ∇ . Then, for any $T \in \Gamma(T_2^0 M)$ the following identity holds:*

$$\nabla_\nu \nabla_\mu T_{\alpha\beta} - \nabla_\mu \nabla_\nu T_{\alpha\beta} = R^\sigma_{\alpha\nu\mu} T_{\sigma\beta} + R^\sigma_{\beta\nu\mu} T_{\alpha\sigma}.$$

Proof. Consider $T \in \Gamma(T_2^0 M)$, which in local coordinates can be written $T = T_{\alpha\beta} dx^\alpha \otimes dx^\beta$. A straightforward computation, using Leibniz rule, gives that

$$\nabla_\nu \nabla_\mu T - \nabla_\mu \nabla_\nu T = T_{\alpha\beta} [\nabla_\nu \nabla_\mu dx^\alpha - \nabla_\mu \nabla_\nu dx^\alpha] \otimes dx^\beta + T_{\alpha\beta} dx^\alpha \otimes [\nabla_\nu \nabla_\mu dx^\beta - \nabla_\mu \nabla_\nu dx^\beta].$$

Also, it is just a computational matter to see that

$$\nabla_\nu \nabla_\mu dx^\alpha - \nabla_\mu \nabla_\nu dx^\alpha = R^\alpha_{\sigma\nu\mu} dx^\sigma,$$

thus,

$$\begin{aligned} \nabla_\nu \nabla_\mu T - \nabla_\mu \nabla_\nu T &= R^\alpha_{\sigma\nu\mu} T_{\alpha\beta} dx^\sigma \otimes dx^\beta + R^\beta_{\sigma\nu\mu} T_{\alpha\beta} dx^\alpha \otimes dx^\sigma \\ &= \{R^\sigma_{\alpha\nu\mu} T_{\sigma\beta} + R^\sigma_{\beta\nu\mu} T_{\alpha\sigma}\} dx^\alpha \otimes dx^\beta. \end{aligned}$$

Which gives us that

$$\nabla_\nu \nabla_\mu T_{\alpha\beta} - \nabla_\mu \nabla_\nu T_{\alpha\beta} = R^\sigma_{\alpha\nu\mu} T_{\sigma\beta} + R^\sigma_{\beta\nu\mu} T_{\alpha\sigma}. \quad (5.27)$$

□

A corollary of this proposition in the context of Weyl geometry is given by the statement below.

Corollary 5.3.2.1. *Suppose we have a Weyl manifold (M, g, ω) , endowed with its Weyl-compatible connection ∇ , and let $F \doteq d\omega$. Then, for any 2-form T on M we have the*

identity

$$g^{\nu\alpha}g^{\mu\beta}\nabla_\nu\nabla_\mu T_{\alpha\beta} = \frac{4-n}{2}F^{\sigma\alpha}T_{\sigma\alpha}$$

Proof. From the previous proposition, we have that

$$g^{\nu\alpha}g^{\mu\beta}\nabla_\nu\nabla_\mu T_{\alpha\beta} - g^{\nu\alpha}g^{\mu\beta}\nabla_\mu\nabla_\nu T_{\alpha\beta} = g^{\nu\alpha}g^{\mu\beta}R^\sigma_{\alpha\nu\mu}T_{\sigma\beta} + g^{\nu\alpha}g^{\mu\beta}R^\sigma_{\beta\nu\mu}T_{\alpha\sigma}.$$

Using the symmetries of the curvature tensor, we have the following

$$\begin{aligned} g^{\nu\alpha}g^{\mu\beta}R^\sigma_{\alpha\nu\mu} &= -g^{\mu\beta}g^{\lambda\sigma}g^{\nu\alpha}R_{\lambda\alpha\mu\nu} \\ &= -g^{\mu\beta}g^{\lambda\sigma}g^{\nu\alpha}(-R_{\alpha\lambda\mu\nu} + 2g_{\lambda\alpha}F_{\mu\nu}) \\ &= g^{\mu\beta}g^{\lambda\sigma}R_{\lambda\mu} - 2g^{\mu\beta}g^{\nu\alpha}\delta^\sigma_\alpha F_{\mu\nu} \\ &= R^{\sigma\beta} - 2F^{\beta\sigma}. \end{aligned}$$

Also, we have that

$$\begin{aligned} g^{\nu\alpha}g^{\mu\beta}g^{\lambda\sigma}R_{\lambda\beta\nu\mu} &= g^{\nu\alpha}g^{\lambda\sigma}g^{\mu\beta}(-R_{\beta\lambda\nu\mu} + 2g_{\lambda\beta}F_{\nu\mu}) \\ &= -g^{\nu\alpha}g^{\lambda\sigma}R_{\lambda\nu} + 2g^{\nu\alpha}g^{\lambda\sigma}\delta^\mu_\lambda F_{\nu\mu} \\ &= -R^{\sigma\alpha} + 2F^{\alpha\sigma}. \end{aligned}$$

Furthermore, the antisymmetry of T , gives us that

$$\begin{aligned} g^{\nu\alpha}g^{\mu\beta}\nabla_\mu\nabla_\nu T_{\alpha\beta} &= -g^{\nu\alpha}g^{\mu\beta}\nabla_\mu\nabla_\nu T_{\beta\alpha} \\ &= -g^{\mu\beta}g^{\nu\alpha}\nabla_\nu\nabla_\mu T_{\alpha\beta}, \end{aligned}$$

thus, finally, we get that

$$\begin{aligned} 2g^{\nu\alpha}g^{\mu\beta}\nabla_\nu\nabla_\mu T_{\alpha\beta} &= R^{\sigma\beta}T_{\sigma\beta} - 2F^{\beta\sigma}T_{\sigma\beta} - R^{\sigma\alpha}T_{\alpha\sigma} + 2F^{\alpha\sigma}T_{\alpha\sigma} \\ &= 2(R^{\sigma\alpha}T_{\sigma\alpha} + 2F^{\sigma\alpha}T_{\sigma\alpha}). \end{aligned}$$

Since, on an n -dimensional manifold M , the antisymmetric part of $R^{\sigma\alpha} = -\frac{n}{2}F^{\sigma\alpha}$, we finally get

$$g^{\nu\alpha}g^{\mu\beta}\nabla_\nu\nabla_\mu T_{\alpha\beta} = \frac{4-n}{2}F^{\sigma\alpha}T_{\sigma\alpha}. \quad (5.28)$$

□

A direct consequence of the above is the following corollary:

Corollary 5.3.2.2. *Let (M, g, ω) be a n -dimensional Weyl manifold whose symmetric part of the Ricci tensor is zero. Then, for $n \neq 4$ we must have*

$$F^{\mu\nu} F_{\mu\nu} = 0. \quad (5.29)$$

Proof. Using Weyl's compatibility condition we get the following:

$$g^{\mu\nu} \nabla_\mu (g^{\alpha\beta} \nabla_\alpha F_{\nu\beta}) = g^{\mu\nu} g^{\alpha\beta} \nabla_\mu \nabla_\alpha F_{\nu\beta} - \omega^\nu g^{\alpha\beta} \nabla_\alpha F_{\nu\beta}$$

We know that under our hypotheses (5.26) is satisfied. Then the second term in the right-hand side of the previous expression vanishes and so does the left-hand side. Also, we know that the previous corollary holds for the 2-form F . This gives us the following:

$$0 = \frac{4-n}{2} F^{\mu\nu} F_{\mu\nu}$$

So we get that if $n \neq 4$ then it must hold that:

$$F^{\mu\nu} F_{\mu\nu} = 0.$$

□

It is worth noticing that the above condition will lead to unexpected and interesting *no go* results. For example, if $\tilde{g}$ is a positive definite metric, then $F^{\mu\nu} F_{\mu\nu} = 0$ implies $F_{\mu\nu} = 0$; hence $(\tilde{M}, \tilde{g}, \tilde{\omega})$ is integrable. Therefore, we have the following result:

Theorem 5.3.2.1. *Let (M, g, ω) be an n -dimensional non-integrable Weyl manifold, with $n \geq 4$. If g is positive definite, then it is not possible to isometrically immerse (M, g, ω) into a Weyl manifold $(\tilde{M}, \tilde{g}, \tilde{\omega})$, with a positive definite metric $\tilde{g}$ and a Ricci tensor whose symmetric part is vanishing, regardless of the codimension of the embedding .*

This result shows that the previous corollary imposes a very strong restriction on the existence of embeddings in the case of Weyl manifolds. For example, we had already said that non-integrable Weyl manifold cannot be embedded in Riemann-flat Weyl manifolds,

but **Theorem 5.3.2.1** implies an even stronger restriction for the case of positive definite metrics. Notice that these obstruction do not suppose any particularly high degree of regularity and they set no-go results irrespective of the codimension.

We shall now treat the very particular 4-dimensional case for which this restriction does not apply. In doing this we will make use of the restriction on the dimensionality of the embedding manifold only when necessary, so that the difficulties implied for the general dimensional case are made explicit.

5.3.3 The 4-dimensional case

The idea is to divide the equations ${}^S\tilde{R}_{\alpha\beta} = 0$ into a set of constraint equations and a set of evolution equations. To do this, we shall impose an additional set of equations coming from the contracted Bianchi identities. Explicitly, we shall impose the equations $\tilde{g}^{\alpha\beta}\tilde{\nabla}_\alpha{}^S\tilde{G}_{\beta\sigma} = 0$, which, as can be seen from (2.23), is equivalent to imposing the following set of additional partial differential equations (PDE):

$$\tilde{g}^{\alpha\beta}\tilde{\nabla}_\alpha F_{\sigma\beta} = 0. \quad (5.30)$$

The above equations will be looked upon as a set of equations imposed on $\tilde{\omega}_\beta$. Thus, our complete system consists of (5.30) together with the following set of equations:

$${}^S\tilde{R}_{ij} = 0 \quad (5.31)$$

$${}^S\tilde{R}_{i(n+1)} = 0 \quad (5.32)$$

$${}^S\tilde{G}_{n+1}^{n+1} = 0 \quad (5.33)$$

As we shall show, by using this scheme we can treat the problem as consisting of a set of evolution equations plus some constraint equations. In what follows we will denote by Σ_0 the hypersurface defined by the condition $y = 0$.

Lemma 5.3.3.1. *Let $\bar{g}_{ik}(x, y), \bar{\psi}(x, y)$ and $\tilde{\omega}_\alpha(x, y)$ be analytic functions at $0 \in \Sigma_0 \subset \mathbb{R}^{n+1}$. Suppose that $\bar{g}_{ik} = \bar{g}_{ki}$, $\det(\bar{g}_{ik}) \neq 0$ and $\bar{\psi} \neq 0$ in a neighborhood of $0 \in \mathbb{R}^{n+1}$, that $\bar{g}_{ik}, \bar{\psi}$ and $\tilde{\omega}_\alpha$ satisfy (5.30) and (5.31) in a neighborhood V of $0 \in \mathbb{R}^{n+1}$ and also (5.32) and (5.33) in a neighborhood of $0 \in \Sigma_0$. Then, $\bar{g}_{ik}, \bar{\psi}$ and $\tilde{\omega}_\alpha$ will satisfy (5.32) and (5.33) in a neighbourhood of $0 \in \mathbb{R}^{n+1}$.*

Proof. Since equation (5.30) is equivalent to $\tilde{g}^{\alpha\beta}\tilde{\nabla}_\alpha^S\tilde{G}_{\beta\sigma}=0$, then by hypothesis we have that:

$$\bar{g}^{ij}\tilde{\nabla}_j^S\tilde{G}_{i\sigma}+\frac{\epsilon}{\bar{\psi}^2}\tilde{\nabla}_{n+1}^S\tilde{G}_{(n+1)\sigma}=0$$

which is equivalent to the following:

$$\frac{\partial^S\tilde{G}_{(n+1)\sigma}}{\partial y}=-\epsilon\bar{\psi}^2\bar{g}^{ij}\partial_j^S\tilde{G}_{i\sigma}+\tilde{\Gamma}_{(n+1)(n+1)}^\gamma{}^S\tilde{G}_{\gamma\sigma}+\tilde{\Gamma}_{(n+1)\sigma}^\gamma{}^S\tilde{G}_{(n+1)\gamma}+\epsilon\bar{\psi}^2\bar{g}^{ij}(\tilde{\Gamma}_{ij}^\gamma{}^S\tilde{G}_{\gamma\sigma}+\tilde{\Gamma}_{j\sigma}^\gamma{}^S\tilde{G}_{i\gamma}) \quad (5.34)$$

To analyse these equations first set $\sigma=k$. Since (5.31) holds in a neighborhood of $0\in\mathbb{R}^{n+1}$, then in such a neighborhood we have that the following holds ${}^S\tilde{G}_{ik}=-\frac{\epsilon}{\bar{\psi}^2}\tilde{g}_{ik}{}^S\tilde{G}_{(n+1)(n+1)}$. Therefore,

$$\begin{aligned} \frac{\partial}{\partial y}{}^S\tilde{G}_{(n+1)k} &= \partial_k^S\tilde{G}_{(n+1)(n+1)}+\bar{\psi}^2\bar{g}^{ij}\partial_j(\frac{\bar{g}_{ik}}{\bar{\psi}^2}){}^S\tilde{G}_{(n+1)(n+1)}-\frac{\epsilon}{\bar{\psi}^2}\tilde{\Gamma}_{(n+1)(n+1)}^j\bar{g}_{jk}{}^S\tilde{G}_{(n+1)(n+1)} \\ &\quad +\tilde{\Gamma}_{(n+1)(n+1)}^{n+1}{}^S\tilde{G}_{(n+1)k}+\tilde{\Gamma}_{(n+1)k}^\gamma{}^S\tilde{G}_{(n+1)\gamma}+\epsilon\bar{\psi}^2\bar{g}^{ij}(\tilde{\Gamma}_{ij}^{n+1}{}^S\tilde{G}_{(n+1)k}+\tilde{\Gamma}_{jk}^{n+1}{}^S\tilde{G}_{i(n+1)} \\ &\quad -\frac{\epsilon}{\bar{\psi}^2}\tilde{\Gamma}_{ij}^l\bar{g}_{lk}{}^S\tilde{G}_{(n+1)(n+1)}-\frac{\epsilon}{\bar{\psi}^2}\tilde{\Gamma}_{jk}^l\bar{g}_{il}{}^S\tilde{G}_{(n+1)(n+1)}) \end{aligned} \quad (5.35)$$

Also setting $\sigma=n+1$ in (5.34) we get:

$$\begin{aligned} \frac{\partial}{\partial y}{}^S\tilde{G}_{(n+1)(n+1)} &= -\epsilon\bar{\psi}^2\bar{g}^{ij}\partial_j^S\tilde{G}_{i(n+1)}+2\tilde{\Gamma}_{(n+1)(n+1)}^\gamma{}^S\tilde{G}_{\gamma(n+1)}+\epsilon\bar{\psi}^2\bar{g}^{ij}(\tilde{\Gamma}_{ij}^\gamma{}^S\tilde{G}_{\gamma(n+1)} \\ &\quad +\tilde{\Gamma}_{j(n+1)}^{n+1}{}^S\tilde{G}_{i(n+1)}-\frac{\epsilon}{\bar{\psi}^2}\tilde{\Gamma}_{j(n+1)}^l\bar{g}_{il}{}^S\tilde{G}_{(n+1)(n+1)}) \end{aligned} \quad (5.36)$$

So we get that (5.34) is equivalent to the system of PDE formed by the equations (5.35) and (5.36), which are linear homogeneous equations on ${}^S\tilde{G}_{(n+1)\sigma}$ which can be written in the following form:

$$\frac{\partial}{\partial y}{}^S\tilde{G}_{(n+1)\sigma}=\mathcal{U}_\sigma(x,y,{}^S\tilde{G}_{(n+1)\beta},\partial_j\tilde{G}_{(n+1)\beta}) \quad (5.37)$$

and under our hypothesis the functions on the right hand side are analytic functions in some neighbourhood of the origin in $\mathbb{R}^{n+1}$. Also under our hypothesis we have that, not

only this set of equations are satisfied, but they also satisfy the following initial data:

$${}^S\tilde{G}_{(n+1)\sigma}(x, 0) = 0 \quad (5.38)$$

Now we know that the Cauchy-Kovalevskaya theorem asserts that this system admits just one set of analytic solutions satisfying these initial data, and since the system is homogeneous, we know that the trivial solution ${}^S\tilde{G}_{(n+1)\sigma} = 0$ is such a solution, then this is the only solution. Hence the functions ${}^S\tilde{G}_{(n+1)\sigma}$ are actually zero in a neighbourhood of the origin in $\mathbb{R}^{n+1}$ and this finishes the proof. $\square$

We shall now show that (5.30) and (5.31) have a solution in a neighbourhood of $0 \in \mathbb{R}^{n+1}$. In order to do this we need to write down these equations explicitly. From (5.18) we find that:

$$\begin{aligned} {}^S\tilde{R}_{ij} = & {}^S\bar{R}_{ij} + \epsilon \bar{g}^{lk} (K_{ik} K_{jl} - K_{ji} K_{lk}) + \frac{\epsilon}{\bar{\psi}} \frac{\partial K_{ij}}{\partial y} + \epsilon \bar{g}^{lu} K_{iu} K_{lj} + \frac{1}{4} (\bar{\nabla}_i \bar{\omega}_j + \bar{\nabla}_j \bar{\omega}_i) - \frac{1}{4} \bar{\omega}_i \bar{\omega}_j \\ & - \frac{1}{\bar{\psi}} \bar{\nabla}_j \bar{\nabla}_i \bar{\psi} + \frac{1}{2\bar{\psi}} (\bar{\nabla}_i \bar{\psi} \bar{\omega}_j + \bar{\nabla}_j \bar{\psi} \bar{\omega}_i - \epsilon \tilde{\omega}_{n+1} K_{ij}). \end{aligned}$$

Using the fact that

$$K_{ij} = -\frac{1}{2\bar{\psi}} \frac{\partial \bar{g}_{ij}}{\partial y} + \frac{1}{2\bar{\psi}} \tilde{\omega}_{n+1} \bar{g}_{ij},$$

we can write (5.31) in the form

$$\begin{aligned} \frac{\epsilon}{2\bar{\psi}^2} \frac{\partial^2 \bar{g}_{ij}}{\partial y^2} = & {}^S\bar{R}_{ij} + \epsilon \bar{g}^{kl} (2K_{ki} K_{jl} - K_{ij} K_{kl}) - \frac{1}{\bar{\psi}} \bar{\nabla}_j \bar{\nabla}_i \bar{\psi} + \frac{1}{4} (\bar{\nabla}_j \bar{\omega}_i + \bar{\nabla}_i \bar{\omega}_j) - \frac{1}{4} \bar{\omega}_i \bar{\omega}_j \\ & + \frac{1}{2\bar{\psi}} (\bar{\omega}_i \partial_j \bar{\psi} + \bar{\omega}_j \partial_i \bar{\psi} - \epsilon \tilde{\omega}_{n+1} K_{ji}) + \frac{\epsilon}{2\bar{\psi}^2} (\bar{g}_{ij} \frac{\partial \tilde{\omega}_{n+1}}{\partial y} + \tilde{\omega}_{n+1} \frac{\partial \bar{g}_{ij}}{\partial y}) \\ & + \frac{\epsilon}{2\bar{\psi}^3} \frac{\partial \bar{\psi}}{\partial y} (\frac{\partial \bar{g}_{ij}}{\partial y} - \tilde{\omega}_{n+1} \bar{g}_{ij}) \end{aligned} \quad (5.39)$$

On the other hand, (5.30) is equivalent to:

$$\tilde{g}^{\mu\lambda} \tilde{\nabla}_\lambda \tilde{\nabla}_\nu \tilde{\omega}_\mu - \tilde{g}^{\mu\lambda} \tilde{\nabla}_\lambda \tilde{\nabla}_\mu \tilde{\omega}_\nu = 0. \quad (5.40)$$

Thus, from the compatibility condition we can rewrite the first term as

$$\tilde{g}^{\mu\lambda} \tilde{\nabla}_\lambda \tilde{\nabla}_\nu \tilde{\omega}_\mu = \tilde{\nabla}_\lambda \tilde{\nabla}_\nu \tilde{\omega}^\lambda - \tilde{\omega}^\mu \tilde{\omega}_\mu \tilde{\omega}_\nu + \tilde{\omega}^\lambda \tilde{\nabla}_\lambda \tilde{\omega}_\nu + \tilde{g}^{\mu\lambda} \tilde{\nabla}_\lambda \tilde{\omega}_\mu \tilde{\omega}_\nu + \tilde{\omega}^\mu \tilde{\nabla}_\nu \tilde{\omega}_\mu.$$

From the definition of the curvature tensor we have

$$\tilde{\nabla}_\lambda \tilde{\nabla}_\nu \tilde{\omega}^\lambda = \tilde{R}^\lambda_{\sigma\nu\lambda} \tilde{\omega}^\sigma + \tilde{\nabla}_\nu \tilde{\nabla}_\lambda \tilde{\omega}^\lambda,$$

that is,

$$\tilde{\nabla}_\lambda \tilde{\nabla}_\nu \tilde{\omega}^\lambda = \tilde{\nabla}_\nu \tilde{\nabla}_\lambda \tilde{\omega}^\lambda + \tilde{R}_{\sigma\nu} \tilde{\omega}^\sigma.$$

In this way, we get

$$\tilde{g}^{\mu\lambda} \tilde{\nabla}_\lambda \tilde{\nabla}_\nu \tilde{\omega}_\mu = \tilde{\nabla}_\nu \tilde{\nabla}_\lambda \tilde{\omega}^\lambda + \tilde{R}_{\sigma\nu} \tilde{\omega}^\sigma + \tilde{\omega}^\lambda \tilde{\nabla}_\lambda \tilde{\omega}_\nu + \tilde{g}^{\mu\lambda} \tilde{\nabla}_\lambda \tilde{\omega}_\mu \tilde{\omega}_\nu + \tilde{\omega}^\mu \tilde{\nabla}_\nu \tilde{\omega}_\mu - \tilde{\omega}^\mu \tilde{\omega}_\mu \tilde{\omega}_\nu.$$

Using this in (5.40) we obtain

$$\tilde{g}^{\mu\lambda} \tilde{\nabla}_\lambda \tilde{\nabla}_\mu \tilde{\omega}_\nu - \tilde{\nabla}_\nu \tilde{\nabla}_\lambda \tilde{\omega}^\lambda - \tilde{\omega}^\lambda \tilde{\nabla}_\lambda \tilde{\omega}_\nu - \tilde{\omega}^\mu \tilde{\nabla}_\nu \tilde{\omega}_\mu - \tilde{g}^{\mu\lambda} \tilde{\nabla}_\lambda \tilde{\omega}_\mu \tilde{\omega}_\nu + \tilde{\omega}^\mu \tilde{\omega}_\mu \tilde{\omega}_\nu - \tilde{R}_{\sigma\nu} \tilde{\omega}^\sigma = 0. \quad (5.41)$$

These equations are equivalent to (5.40). Unfortunately, they cannot be written in a form where we can apply the Cauchy-Kovalevskaya theorem. However, if we consider these equations in the *Lorentz gauge* $\tilde{\nabla}_\lambda \tilde{\omega}^\lambda = 0$, we can show that the resulting set of *reduced equations* can be cast in the form required by this theorem. Now, writing these equations explicitly we get

$$\begin{aligned} \frac{\epsilon}{\tilde{\psi}^2} \frac{\partial^2 \tilde{\omega}_\nu}{\partial y^2} = & -\tilde{g}^{ij} \tilde{\nabla}_i \tilde{\nabla}_j \tilde{\omega}_\nu + \tilde{\omega}^\lambda \tilde{\nabla}_\lambda \tilde{\omega}_\nu + \tilde{\omega}^\mu \tilde{\nabla}_\nu \tilde{\omega}_\mu + \tilde{g}^{\mu\lambda} \tilde{\nabla}_\lambda \tilde{\omega}_\mu \tilde{\omega}_\nu - \tilde{\omega}^\mu \tilde{\omega}_\mu \tilde{\omega}_\nu + \tilde{R}_{\sigma\nu} \tilde{\omega}^\sigma \\ & + \frac{\epsilon}{\tilde{\psi}^2} \left(\frac{\partial \tilde{\Gamma}_{(n+1)\nu}^\sigma}{\partial y} \tilde{\omega}_\sigma + \tilde{\Gamma}_{(n+1)\nu}^\sigma \frac{\partial \tilde{\omega}_\sigma}{\partial y} + \tilde{\Gamma}_{(n+1)\nu}^\beta \frac{\partial \tilde{\omega}_\beta}{\partial y} - \tilde{\Gamma}_{(n+1)\nu}^\beta \tilde{\Gamma}_{(n+1)\beta}^\sigma \tilde{\omega}_\sigma \right. \\ & \left. + \tilde{\Gamma}_{(n+1)(n+1)}^\beta \partial_\beta \tilde{\omega}_\nu - \tilde{\Gamma}_{(n+1)(n+1)}^\beta \tilde{\Gamma}_{\beta\nu}^\sigma \tilde{\omega}_\sigma \right) \end{aligned} \quad (5.42)$$

We shall regard these equations together with (5.39) as a system of PDEs for $(\tilde{g}, \tilde{\omega})$. It is important to remark that (5.42) depends on $\frac{\partial^2 \tilde{g}_{ij}}{\partial y^2}$ through terms such as $\frac{\partial \tilde{\Gamma}_{(n+1)\nu}^\sigma}{\partial y} \tilde{\omega}_\sigma$ or $\tilde{R}_{\sigma\nu} \tilde{\omega}^\sigma$. But, as we are regarding (5.39) and (5.42) as a system, we just replace $\frac{\partial^2 \tilde{g}_{ij}}{\partial y^2}$ in (5.42) using (5.39). Thus, if we consider that $\bar{\psi}$ is a given analytic function in a neighborhood of the origin of $\mathbb{R}^{n+1}$ which satisfies $\bar{\psi} \neq 0$ in this neighborhood, then (5.39) and (5.42) yield a system of the form

$$\begin{aligned}
\frac{\partial^2 \bar{g}_{ij}}{\partial y^2} &= F_{ij}(x, y, \bar{g}_{ij}, \tilde{\omega}_\alpha, \partial_\alpha \bar{g}_{ij}, \partial_\beta \tilde{\omega}_\alpha, \partial_{i\alpha} \bar{g}_{ij}, \partial_{i\beta} \tilde{\omega}_\alpha) \quad 1 \leq i < j \leq n \quad ; \quad \alpha, \beta = 1, \dots, n+1 \\
\frac{\partial^2 \tilde{\omega}_\beta}{\partial y^2} &= \mathcal{U}_\beta(x, y, \bar{g}_{ij}, \tilde{\omega}_\alpha, \partial_\alpha \bar{g}_{ij}, \partial_\beta \tilde{\omega}_\alpha, \partial_{i\alpha} \bar{g}_{ij}, \partial_{ij} \tilde{\omega}_\alpha) \quad 1 \leq i < j \leq n \quad ; \quad \alpha, \beta = 1, \dots, n+1
\end{aligned}
\tag{5.43}$$

Therefore, if we choose a specific order for the $\frac{n(n+1)}{2}$ components of $\bar{g}_{ij}$ and the $n+1$ components of $\tilde{\omega}_\beta$, then (5.43) may be regarded as a system of $\frac{(n+1)(n+2)}{2}$ PDEs for the $\frac{(n+1)(n+2)}{2}$ functions $(\bar{g}_{ij}, \tilde{\omega}_\beta)$. For such a system we give the following initial data

$$\begin{aligned}
\bar{g}_{ik}(x, 0) &= g_{ik}(x) \quad 1 \leq i < k \leq n \\
\tilde{\omega}_\beta(x, 0) &= \omega_\beta(x) \quad \beta = 1, \dots, n+1
\end{aligned}
\tag{5.44}$$

$$\begin{aligned}
\frac{\partial \bar{g}_{ik}}{\partial y}(x, 0) &= -2\bar{\psi}(x, 0)\Omega_{ik}(x) + g_{ik}(x)\omega_{n+1}(x) \doteq g'_{ik}(x) \quad 1 \leq i < k \leq n \\
\frac{\partial \tilde{\omega}_\beta}{\partial y}(x, 0) &= \omega'_\beta(x) \quad \beta = 1, \dots, n+1
\end{aligned}
\tag{5.45}$$

where $\omega_\beta, \omega'_\beta, \Omega_{ik}$ and g_{ik} are all analytic functions at $0 \in \mathbb{R}^n$, and it is required that the initial data g_{ik} also satisfy that the condition $\det(g_{ik})(0) \neq 0$. It is important to note that the right-hand side of (5.43) consists of rational functions of the variables $\bar{g}_{ij}, \tilde{\omega}_\alpha, \partial_\alpha \bar{g}_{ij}, \partial_\beta \tilde{\omega}_\alpha, \partial_{a\alpha} \bar{g}_{ij}, \partial_{a\beta} \tilde{\omega}_\alpha$, and all the denominators are just $\det(\bar{g}_{ik})$. On the other hand, it follows from the initial data that $\det(\bar{g}_{ik})(0, 0) = \det(g_{ik})(0) \neq 0$. Thus, since $\det(\bar{g}_{ik})$ is a polynomial of the functions $\bar{g}_{ik}$, and we know that for $\bar{g}_{ik}^\circ \doteq \bar{g}_{ik}(0, 0)$ this polynomial is different from zero, then there is a neighborhood of $(\bar{g}_{ik}^\circ)$ where this polynomial does not vanish. Using this fact and that the functions F_{ij} and $\mathcal{U}_\beta$ are just these rational functions multiplied by some power of $\bar{\psi}$, which, in turn, is an analytic function in a neighborhood of $0 \in \mathbb{R}^{n+1}$ and $\bar{\psi} \neq 0$ in this neighborhood, we get that F_{ij} and $\mathcal{U}_\beta$ are analytic functions at $P = (0, 0, \bar{g}_{ij}(0, 0), \tilde{\omega}_\alpha(0, 0), \partial_\alpha \bar{g}_{ij}(0, 0), \partial_\beta \tilde{\omega}_\alpha(0, 0), \partial_{a\alpha} \bar{g}_{ij}(0, 0), \partial_{a\beta} \tilde{\omega}_\alpha(0, 0))$. We can thus use the Cauchy-Kovalevskaya theorem to guarantee the existence of solutions of the system (5.43) with the initial data (5.44)-(5.45). It is important to remark that, since the Cauchy-Kovalevskaya theorem guarantees the existence of analytic solutions in

a neighbourhood of $0 \in \mathbb{R}^{n+1}$, and as $\det(\bar{g}_{ij})(0,0) \neq 0$ from the initial data, then by continuity we know that there exists a neighborhood of $0 \in \mathbb{R}^{n+1}$ where $\det(\bar{g}_{ij})(x,y) \neq 0$.

In this way we have constructed a set of analytic solutions of the reduced equations (5.43). In order for these solutions to satisfy the original set of equations (5.30)-(5.31) we need to show that they satisfy the *gauge condition* $\tilde{\nabla}_\lambda \tilde{\omega}^\lambda = 0$. With this in mind we present the following lemma.

Lemma 5.3.3.2. *Consider $n + 1 = 4$ and suppose that $(\tilde{g}, \tilde{\omega})$ is an analytic solution of (5.43) satisfying the initial data (5.44)-(5.45), and also assume that*

$$\tilde{\nabla}_\lambda \tilde{\omega}^\lambda|_{\Sigma_0} = 0 \quad (5.46)$$

$$\frac{\partial \tilde{\nabla}_\lambda \tilde{\omega}^\lambda}{\partial y}|_{\Sigma_0} = 0 \quad (5.47)$$

Then, $(\tilde{g}, \tilde{\omega})$ satisfy the complete system of equations (5.30)-(5.31).

Proof. If $\dim(\tilde{M}) = 4$ then we have seen that the following identity is satisfied on $\tilde{M}$:

$$\tilde{g}^{\alpha\beta} \tilde{\nabla}_\alpha (\tilde{g}^{\mu\nu} \tilde{\nabla}_\mu F_{\beta\nu}) + \tilde{\omega}^\beta \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu F_{\beta\nu} = 0.$$

Also if $(\tilde{g}, \tilde{\omega})$ satisfy the reduced equations, then from (5.41) we get that

$$\tilde{g}^{\mu\nu} \tilde{\nabla}_\mu F_{\beta\nu} = \frac{1}{2} \tilde{\nabla}_\beta \tilde{\nabla}_\lambda \tilde{\omega}^\lambda \quad (5.48)$$

Then the previous identity gives us the following:

$$\tilde{g}^{\alpha\beta} \tilde{\nabla}_\alpha \tilde{\nabla}_\beta (\tilde{\nabla}_\lambda \tilde{\omega}^\lambda) + \tilde{\omega}^\beta \tilde{\nabla}_\beta (\tilde{\nabla}_\lambda \tilde{\omega}^\lambda) = 0.$$

It is not difficult to show that this is a second order linear and homogeneous equation for the function $\tilde{\nabla}_\lambda \tilde{\omega}^\lambda$ which can be rewritten as follows:

$$\frac{\partial^2 (\tilde{\nabla}_\lambda \tilde{\omega}^\lambda)}{\partial y^2} = \mathcal{F}(x, y, \tilde{\nabla}_\lambda \tilde{\omega}^\lambda, \partial_\alpha (\tilde{\nabla}_\lambda \tilde{\omega}^\lambda), \partial_{i\alpha} (\tilde{\nabla}_\lambda \tilde{\omega}^\lambda)) \quad (5.49)$$

where the right-hand side is an analytic function at the origin. Then the Cauchy-Kovalevskaya theorem guarantees the existence of a unique solution for this equation satisfying the initial data (5.46)-(5.47). Since $\tilde{\nabla}_\lambda \tilde{\omega}^\lambda = 0$ satisfies all these requirements,

we get that this is the unique solution. Using this in (5.48) we see that under these conditions $(\tilde{g}, \tilde{\omega})$ satisfies the full system (5.30)-(5.31). $\square$

Using this lemma, which only works in the 4-dimensional case, we see that we should look at (5.46) and (5.47) as additional constraints. Thus, our system of constraint equations consists of the equations (5.32)-(5.33) on the hypersurface Σ_0 , together with the equations (5.46)-(5.47). This system is posed for the second fundamental form of Σ_0 , Ω_{ij} , and the initial data ω'_β , and will be referred as the *Weyl constraints equations*. We shall denote the initial data set by $(\Sigma_0, g, \omega, \Omega, \omega')$, where (Σ_0, g, ω) gives the Weyl structure of the hypersurface Σ_0 . With these notations, we can state the following theorem:

Theorem 5.3.3.1. *Let $(\Sigma_0, g, \omega, \Omega, \omega')$ be an initial data set satisfying the Weyl constraint equations. Then (Σ_0, g, ω) admits a local analytic isometric embedding around $p \in \Sigma_0$ in a Weyl manifold $(\tilde{M}^4, \tilde{g}, \tilde{\omega})$ such that the symmetric part of the Ricci tensor of the embedding manifold vanishes.*

Using this theorem, we see that in order to guarantee the existence of an isometric embedding of (M^3, g, ω) at $p \in M^3$ in a Weyl manifold $(\tilde{M}^4, \tilde{g}, \tilde{\omega})$ having vanishing symmetric part of its Ricci tensor, we just need to show that we can always find an initial data set $(M^3, g, \omega, \Omega, \omega')$ satisfying the Weyl constraint equations in a neighborhood of $p \in M^3$. When dealing with these constraints, we shall make use of the *gauge* freedom we have in Weyl's geometry. We have already seen that if we can construct an embedding for some element $(M, g, \omega) \in (M, [g, \omega])$ in $(\tilde{M}, \tilde{g}, \tilde{\omega}) \in (\tilde{M}, [\tilde{g}, \tilde{\omega}])$ then we can construct an embedding for each element of $(M, [g, \omega])$ in some element of $(\tilde{M}, [\tilde{g}, \tilde{\omega}])$.⁷ Thus, we shall select a particular element of $(M, [g, \omega])$ where (5.46) is satisfied. Let us show that we can always do this. First, consider that $(\bar{g}_{ij}, \bar{\omega}_\beta)$ is a solution of (5.43) in a neighborhood U of $0 \in \mathbb{R}^{n+1}$ satisfying the initial data (5.44)-(5.45). Then, defining

$$\tilde{g} \doteq \bar{g}_{ij} dx^i \otimes dx^j + \epsilon \bar{\psi}^2 dy \otimes dy,$$

we have that $(U, \tilde{g}, \tilde{\omega})$ is a well-defined Weyl manifold. Under these assumptions note that

$$\tilde{\nabla}_\lambda \tilde{\omega}^\lambda = \bar{\nabla}_k \tilde{\omega}^k + \tilde{\nabla}_{n+1} \tilde{\omega}^{n+1},$$

⁷Note that, just as for the full Ricci tensor, the symmetric part of the Ricci tensor is invariant under Weyl transformations. This can easily be seen from (2.28) since ${}^S\tilde{R}_{\alpha\beta} = \tilde{R}_{\alpha\beta} + \frac{n+1}{2}F_{\alpha\beta}$, where both terms on the right-hand side are clearly invariant.

hence

$$\tilde{\nabla}_\lambda \tilde{\omega}^\lambda|_{\Sigma_0} = \nabla_k \omega^k + \tilde{\nabla}_{n+1} \tilde{\omega}^{n+1}(x, 0).$$

This last expression only depends on the initial data (g, ω) and $\tilde{\omega}_{n+1}|_{\Sigma_0}, \partial_{n+1} \tilde{\omega}_{n+1}|_{\Sigma_0}$.

Recall that what we are trying to do is to find appropriate initial data $(g^*, \omega^*) \in [g, \omega]$ such that for our solution $(\tilde{g}_{ij}, \tilde{\omega}_k)$ of (5.43) with initial data (5.44)-(5.45) constructed with (g^*, ω^*) the constraint equation (5.46) is automatically satisfied. Thus, consider the Weyl transformation on the initial data:

$$\begin{aligned} g &\rightarrow e^{-f} g \\ \omega_k &\rightarrow \omega_k - \partial_k f \end{aligned}$$

for some analytic function f . Then, for $(e^{-f} g, \omega - df)$ (5.46) is equivalent to the equation

$$g^{ku} \nabla_k \nabla_u f + g^{ku} (\omega_k - \nabla_k f)(\omega_u - \nabla_u f) + g^{ku} \nabla_k \omega_u + e^{-f} (\partial_y \tilde{\omega}^4 + \tilde{\Gamma}_{4\sigma}^4 \tilde{\omega}^\sigma)|_{y=0} = 0, \quad (5.50)$$

where $\tilde{\Gamma}_{4\sigma}^4 = \frac{1}{\bar{\psi}} \partial_\sigma \bar{\psi} - \frac{1}{2} \tilde{\omega}_\sigma$. Since $\bar{\psi}$ is considered as a given analytic function and both $\tilde{\omega}_4|_{\Sigma_0}$ and $\partial_y \tilde{\omega}_4|_{\Sigma_0}$ are also arbitrary given analytic functions, then (5.50) is a second-order PDE for the function f . Thus, from now on, we shall regard $\bar{\psi}|_{\Sigma_0} = \psi(x)$, $\tilde{\omega}_4|_{\Sigma_0} \doteq \omega_4(x)$ and $\partial_y \tilde{\omega}_4|_{\Sigma_0} \doteq \eta(x)$ as given analytic functions, which will be involved in the initial data of the system (5.43). Then, we can guarantee the existence of an analytic solution for (5.50). To see this, we can use a coordinate system (x^i) on M around p , satisfying that $g_{1k'} = 0$, with $k' = 2, 3$. In this way, (5.50) can be cast in the form

$$\begin{aligned} &g^{11} \nabla_1 \nabla_1 f + g^{k'u'} \nabla_{k'} \nabla_{u'} f + g^{ku} (\omega_k - \nabla_k f)(\omega_u - \nabla_u f) + g^{ku} \nabla_k \omega_u + e^{-f} \left\{ \frac{\epsilon}{\psi^2} \eta \right. \\ &\left. - \frac{2\epsilon}{\psi^3} \frac{\partial \bar{\psi}}{\partial y} \Big|_{\Sigma_0} \omega_4 + \left(\frac{1}{\psi} \partial_k \psi - \frac{1}{2} (\omega_k - \nabla_k f) \right) e^f g^{ku} (\omega_u - \nabla_u f) + \frac{\epsilon}{\psi} \left(\frac{1}{\psi} \frac{\partial}{\partial y} \bar{\psi} \Big|_{\Sigma_0} - \frac{1}{2} \omega_4 \right) \omega_4 \right\} = 0, \end{aligned}$$

where $g^{11} \neq 0$ in a neighborhood of the origin, $k', u' = 2, 3$, and all the known quantities involved are analytic in a neighborhood of $0 \in \mathbb{R}^n$. Therefore, this last equation has the form

$$\frac{\partial^2 f}{\partial (x^1)^2} = \mathcal{U}(x, f, \partial_i f, \partial_{k'i} f),$$

where the right-hand side is analytic at the origin, and hence the Cauchy-Kovalevskaya

theorem guarantees the existence of an analytic solution. We thus have shown that, given a Weyl manifold (M^3, g, ω) we can always find an element in $(M^3, [g, \omega])$ for which (5.46) is satisfied. Hence there is no loss of generality in assuming that (M^3, g, ω) satisfies this condition. In this way, we can reduce the Weyl constraint equations to the following set of equations:

$$\psi g^{kl}(\nabla_k \Omega_{il} - \nabla_i \Omega_{kl}) + \frac{\psi}{2} g^{kl}(\omega_i \Omega_{kl} - \omega_k \Omega_{il}) + \frac{n-1}{4}(\partial_i \eta - \omega'_i) = 0 \quad (5.51)$$

$$g^{ij} g^{kl}(R_{kilj} + \epsilon(\Omega_{ij} \Omega_{kl} - \Omega_{ki} \Omega_{jl})) = 0 \quad (5.52)$$

$$\frac{\partial \tilde{\nabla}_\lambda \tilde{\omega}^\lambda}{\partial y}|_{\Sigma_0} = 0. \quad (5.53)$$

Equations (5.51) and (5.52) are dealt with in the same way it was done in [25], just carrying the extra terms along the same computations. Doing this, from (5.52) we find an explicit expression for Ω_{11} in terms of the other variables and from (5.51) a set first-order PDEs of the Cauchy-Kovalevskaya-type for the functions $\Omega_{1k'}$ and $\Omega_{r'3}$, where $k' = 2, 3$ and r' is fixed, having either the value 2 or 3. In this system, the remaining components of Ω_{ij} are set as arbitrary analytic functions. In the same way we did when dealing with (5.46), in this procedure, a coordinate system on M^3 around p is chosen such that $g_{1k'} = 0$, and the coordinate x^1 is chosen as the variable with respect to which we pose the constraint equations in the Cauchy-Kovalevskaya form. Now, we shall deal with the remaining equation (5.53). First, let us write it down explicitly:

$$\frac{\partial(\tilde{\nabla}_\lambda \tilde{\omega}^\lambda)}{\partial y} = \frac{\partial(\partial_k \tilde{\omega}^k)}{\partial y} + \frac{\partial^2 \tilde{\omega}^4}{\partial y^2} + \frac{\partial \tilde{\Gamma}_{kj}^k}{\partial y} \tilde{\omega}^j + \tilde{\Gamma}_{kj}^k \frac{\partial \tilde{\omega}^j}{\partial y} + \frac{\partial \tilde{\Gamma}_{4\sigma}^4}{\partial y} \tilde{\omega}^\sigma + \tilde{\Gamma}_{4\sigma}^4 \frac{\partial \tilde{\omega}^\sigma}{\partial y}. \quad (5.54)$$

Using the following expressions for the connection components involved in the previous expression

$$\begin{aligned} \tilde{\Gamma}_{4\beta}^4 &= \frac{1}{\psi} \partial_\beta \bar{\psi} - \frac{1}{2} \tilde{\omega}_\beta, \\ \tilde{\Gamma}_{ij}^l &= \bar{\Gamma}_{ij}^l, \end{aligned}$$

together with the definition of the second fundamental form Ω ,

$$\Omega_{ji} = \left\{ -\frac{1}{2\bar{\psi}} \frac{\partial \bar{g}_{ij}}{\partial y} + \frac{1}{2\bar{\psi}} \bar{g}_{ij} \tilde{\omega}_4 \right\} |_{\Sigma_0},$$

we see that when we restrict (5.54) to Σ_0 the last four terms depend on given data $g_{ij}, \omega_k, \eta, \omega'_4, \psi$ and terms up to first-order in Ω_{ij} and the remaining ω'_k . Also, since $(\tilde{g}, \tilde{\omega})$ satisfy the reduced equations (5.43), then:

$$\frac{\partial^2 \tilde{\omega}_4}{\partial y^2} = \mathcal{U}_{n+1}(x, y, \bar{g}_{ij}, \tilde{\omega}_\alpha, \partial_\alpha \bar{g}_{ij}, \partial_\beta \tilde{\omega}_\alpha, \partial_{i\alpha} \bar{g}_{ij}, \partial_{ai} \tilde{\omega}_\alpha), \quad 1 \leq i < j \leq 3; \quad a = 1, 2, 3.$$

It follows that

$$\frac{\partial^2 \tilde{\omega}_4}{\partial y^2} |_{\Sigma_0} = \mathcal{U}'_{n+1}(x, \Omega_{ij}, \omega'_\alpha, \partial_a \Omega_{ij}), \quad 1 \leq i < j \leq 3; \quad a = 1, 2, 3.$$

Thus, constraining (5.54) to Σ_0 and setting the left-hand side equal to zero, we get

$$g^{ku} \partial_k \omega'_u + \partial_k g^{ku} \omega'_u + \partial_{k4} \bar{g}^{ku} |_{\Sigma_0} \omega_u + \frac{\partial \bar{g}^{ku}}{\partial y} |_{\Sigma_0} \partial_k \omega_u + \mathcal{O}(x, \Omega_{ij}, \omega'_k, \partial_k \Omega_{ij}) = 0.$$

Using the same special form of the metric in the coordinate system used to study the constraints, we can rewrite this last equation as

$$\frac{\partial \omega'_1}{\partial x^1} = \mathcal{O}'(x, \Omega_{ij}, \omega'_k, \partial_{k'} \omega'_{j'}, \partial_k \Omega_{ij}) \quad i, j, k = 1, 2, 3; \quad j', k' = 2, 3.$$

Then, we see that the constraint equations (5.51)-(5.53) can be written as a set of first-order PDEs of the following form:

$$\begin{aligned} \frac{\partial \Omega_{1k'}}{\partial x^1} &= \mathcal{H}_{k'}(x, \Omega_{1j'}, \Omega_{3r'}, \omega'_1, \partial_{u'} \Omega_{1j'}, \partial_{u'} \Omega_{3r'}), \quad u', j', k' = 2, 3; \quad r' \text{ fixed with } r' = 2 \text{ or } r' = 3 \\ \frac{\partial \Omega_{3r'}}{\partial x^1} &= \mathcal{H}_{r'}(x, \Omega_{1j'}, \Omega_{3r'}, \omega'_1, \partial_{u'} \Omega_{1j'}, \partial_{u'} \Omega_{3r'}), \quad u', j' = 2, 3; \quad r' \text{ fixed with } r' = 2 \text{ or } r' = 3 \\ \frac{\partial \omega'_1}{\partial x^1} &= \mathcal{O}'(x, \Omega_{1j'}, \Omega_{3r'}, \omega'_1, \partial_{u'} \Omega_{1j'}, \partial_{u'} \Omega_{3r'}), \quad u', j' = 2, 3; \quad r' \text{ fixed with } r' = 2 \text{ or } r' = 3 \end{aligned} \quad (5.55)$$

together with an explicit algebraic expression for Ω_{11} . In this set up the rest of the Ω_{ij} and $\omega'_2, \dots, \omega'_n$ are set as given arbitrary analytic functions. The equations (5.55) are of the Cauchy-Kovalevskaya type and hence we know that this system admits a solution.

We now can state the main result of this section.

Theorem 5.3.3.2. *Any 3-dimensional Weyl structure $(M^3, [g, \omega])$ admits a local analytic isometric embedding at any point $p \in M^3$ in a 4-dimensional Weyl structure $(\tilde{M}^4, [\tilde{g}, \tilde{\omega}])$ having vanishing symmetric part of its Ricci tensor.*

5.4 Final Comments

During this chapter we have extensively discussed the physical motivations for studying Weyl structures in the context of modern space-time theories. We have revisited the notion of proper time in the framework of Weyl's space-time within the axiomatic approach put forward by Elhers, Pirani and Schild. We have shown that the EPS original definition of proper time is equivalent to the definition proposed by V. Perlick, even though the latter seems to be more easily motivated in the context of EPS's axiomatic approach to space-time. After showing that Perlick's notion leads to a well-defined and physically sensible definition of proper time in a general Weyl space-time, without invoking any additional axioms, we proved that this kind of space-time exhibits a second clock effect. We then derived the condition a space-time must obey so that a second clock effect does not appear. We have shown that in this case the geometric space-time structure should be that of a Weyl integrable space-time. Our final conclusion is that, within the slightly modified EPS axiomatic approach, taking into account Perlick's proper time, Weyl integrable space-time appears naturally as the most general model for space-time. However, the question of a possible existence of a second clock effect, which would then widen this scenario leaving us with a general Weyl space-time as a suitable mathematical model, is something that should be settled by experiment. This poses both integrable and non-integrable Weyl structures as interesting mathematical structures within physics, which could actually serve as a model for space-time.

The above considerations led us to the study of embedding problems for Weyl structures, which, within the context of physics are related to the problem of trying to embed a space-time modelled by such structure in a higher dimensional space-time with some specified geometric property. The geometric properties we considered on the embedding manifolds were both Ricci-flatness and "symmetric-Ricci-flatness". Both properties, within the context of physical theories which consider space-time as a Weyl structure,

are related to vacuum conditions (see [40],[41],[42],[79],[80],[81]).⁸ In this direction, we were able to prove that some of the Campbell-Magaard-type theorems can be naturally extended from Riemannian to Weyl's geometry, although some instances appear that are not exactly analogous to their Riemannian counterpart. In particular we discovered some interesting and rather unexpected no-go results, and established an important geometrical identity which seems to be essential for studying embeddings in Weyl spaces, in arbitrary dimensions, in which the symmetric part of its Ricci tensor vanishes. We have worked out the embedding problem in the 3-dimensional case and showed that this solution does not hold in other dimensions. The complete solution of the general problem, still left open, may be regarded as a mathematical motivation for studying other embedding problems in the framework of Weyl's geometry, which may originate from modern theoretical physics.

⁸For theories based on non-integrable Weyl geometry see [37],[84],[86], for instance.

Chapter 6

The Cauchy problem for geometric scalar-tensor theories

In this chapter we will analyse the initial value formulation for a particular type of scalar-tensor theories, which are based on Weyl integrable geometry.

6.1 Weyl Geometric scalar tensor theories (WGSTT)

We will now make a short review of the type of theories we will be analysing during this chapter. Our starting point will be the model proposed in [41]. There, the starting point is an action functional of the following type

$$S(\nabla, g, \phi) = \int_V e^{-\phi} (R(\nabla) + \omega \langle d\phi, d\phi \rangle) \mu_g, \quad (6.1)$$

where V stands for a 4-dimensional space-time, g for a Lorentzian metric, ϕ for a scalar field, ω is a coupling constant and μ_g the volume form associated with g . In this context the Ricci scalar R is associated to a torsionless connection ∇ .

It should be remarked that, if we consider that the background geometry is Riemannian, then (6.1) is equivalent to the Brans-Dicke action. The idea adopted in [41] is to consider a Palatini-type action principle for (6.1), and thus determine the relation between the connection and the other geometric objects (in this context g and ϕ) from the field equations. Interestingly enough, this procedure leads to the Weyl compatibility condition between the connection, the metric and the scalar field. Thus, we are led to a

theory based on a Weyl integrable space-time. Written as in (6.1) this theory is posed in vacuum, but a prescription for coupling other fields is given [41].

It should be noted that, once it is established that the physical theory which comes from (6.1) should be posed on a Weyl integrable manifold, it is straightforward to try pose it on a Weyl integrable structure. This is a consequence of the natural invariance of the connection under Weyl transformations. Furthermore, following for instance the EPS axiomatic approach described in the previous chapter, we have concluded that such structures appear as the most general mathematical models for space-time within the original scope proposed in [39]. In this scenario, freely falling point-like massive particles follow geodesics of the Weyl connection. This is a consequence of taking the equivalence principle as one of the axioms for space-time. Thus, the trajectories of massive particles will be invariant under Weyl transformations. All this ideas point towards adopting the view that that physical properties, in this context, should be invariant under Weyl transformations, and thus that the theory described by (6.1) should be posed on the equivalence class $(V, [g, \phi])$.

With the above considerations in mind, we should stress that the second term in (6.1) is not invariant under Weyl transformations. Thus, we must expect the field equations derived from such action to be *frame*-dependent. This forces us to take a minute to give a sensible interpretation for such theory.

If we are to consider, as it is done in the type of models we will be investigating, that the physical theory described by (6.1) must be invariant under Weyl transformations, then we are forced to concede that the field equations cannot represent by themselves anything physical, but it must be the solutions who carry the physical information. Thus, from (6.1) we will get a set of frame-dependent field equations, which define an equivalence class $(V, [g, \phi])$, and, from the physical point of view, any member in the equivalence class is on equal footing, and thus any physical property must be invariant under Weyl transformation. This point of view has been used to describe some cosmological scenarios in [87],[88].

Finally, we should stress that, as maybe expected, there is a strong connection between the theory described above and Brans-Dicke theory. In fact, another other interesting aspect of this geometric model, is that the frame transformations connected with the so called Einstein and Jordan frames in Brans-Dicke theory, can be understood as Weyl

transformation within this context [89].

The issue of the equivalence of frames in Brans-Dicke theory has been treated (from the classical point of view) in some previous works [90, 91]. In particular, [90] recaptures the arguments of Dicke’s original paper [92], in which the notion of running units is addressed and applied to analyse the equivalence between frames. In this line of argument, it is claimed that any measurement of a physical observable is performed by considering the ratio between such a quantity, derived in a certain frame, and some chosen unit. Being this ratio what is actually observed, it is this quantity which should be invariant under frame transformations. Therefore, although the field equations are not invariant, the measured physical observable would be.

Contrasting with the above interpretation, the kind of models proposed in [41] also consider that different frames should be treated on equal footing, but this idea is translated into the mathematical structure on which the models are based. This is accomplished by considering space-time, and physical phenomena, described by an equivalence class $(V, [g, \phi])$ defining a Weyl integrable structure, and proposing that physical observables should be invariant under Weyl transformations. It is worth noticing that a simple way of defining invariant quantities in this formulation, consists in using the invariant metric $\gamma = e^{-\phi}g$ to perform contractions on invariant tensors.

To conclude this section, we will write down the d -dimensional version of the action functional (6.1) and the field equations we get from it. We will pose the problem on a Weyl manifold from the beginning. Thus, the curvature terms are to be understood as computed with the Weyl connection. For quantities computed with a Riemannian connection we will use the same notations as in the previous chapter, putting the symbol ‘ $\circ$ ’ on the upper left corner of such quantities.

Our d -dimensional action is the following:

$$S(g, \phi) = \int_V e^{-\frac{d-2}{2}\phi} (R + \omega \langle d\phi, d\phi \rangle) \mu_g. \quad (6.2)$$

Straightforward computations show that the Euler-Lagrange equations for such an action are the following:

$$\begin{aligned}
G_{\alpha\beta} &= -\omega \nabla_\alpha \phi \nabla_\beta \phi + \frac{\omega}{2} g_{\alpha\beta} \nabla_\sigma \phi \nabla^\sigma \phi \\
2\omega \nabla_\sigma \nabla^\sigma \phi + \frac{d+2}{2} \omega \nabla_\sigma \phi \nabla^\sigma \phi + \frac{d-2}{2} R &= 0.
\end{aligned} \tag{6.3}$$

These equations can be rewritten in a more convenient way by taking the trace of the first equations and using this information in the second one. After a few computations we get the following.

$$\begin{aligned}
G_{\alpha\beta} &= -\omega \nabla_\alpha \phi \nabla_\beta \phi + \frac{\omega}{2} g_{\alpha\beta} \nabla_\sigma \phi \nabla^\sigma \phi, \\
g^{\sigma\mu} \nabla_\sigma \nabla_\mu \phi &= 0.
\end{aligned} \tag{6.4}$$

Recall that the point of view that we are taking is that a solution of these field equations defines an equivalence class $(V, [g, \phi])$, where each element is related to another by a Weyl transformation, and that there is no preferred element in the class to describe physical properties. Thus, it will be instructive to look at the field equations satisfied by a generic element in the class $(V, [g, \phi])$ defined by a solution of (6.4). Since any element (V, g', ϕ') in the equivalence class of (V, g, ϕ) is of the form $(V, e^{-f}g, \phi - f)$ for some $f \in C^\infty(V)$, then $(g, \phi) = (e^f g', \phi' + f)$. Replacing these expressions in (6.4), we see that (g', ϕ') satisfy the following system:

$$\begin{aligned}
G_{\alpha\beta}(g', \phi') &= -\omega \nabla_\alpha (\phi' + f) \nabla_\beta (\phi' + f) + \frac{\omega}{2} g'_{\alpha\beta} g'^{\mu\nu} \nabla_\mu (\phi' + f) \nabla_\nu (\phi' + f), \\
g'^{\mu\nu} \nabla_\mu \nabla_\nu (\phi' + f) &= 0.
\end{aligned} \tag{6.5}$$

We have used the fact that the Einstein tensor is invariant under Weyl transformations, thus $G_{\alpha\beta}(e^f g', \phi' + f) = G_{\alpha\beta}(g', \phi') \forall f$.

6.2 The initial value formulation for WGSTT

In this section we will analyse the initial value formulation for the theory described in the previous section. As usual, we will consider space-time to be globally hyperbolic. Thus, $V = M \times \mathbb{R}$, where V is an $(n+1)$ -dimensional space-time and M an n -dimensional hypersurface. On this setting we have the same $(n+1)$ -splitting for the space-time metric that we described in chapter 3.

In what follows, our strategy will be to first analyse the well-posedness of the initial value formulation in a frame, that is, to analyse the well-posedness of the set of equations (6.4), and then to show that this is enough to guarantee the well-posedness for the class $(V, [g, \phi])$. We will now be more specific on what we mean by saying that the Cauchy problem is well-posed on the equivalence class.

Since we are regarding that every meaningful quantity, which can in fact be observed, must be invariant under Weyl transformation, then, certainly, the initial value formulation should satisfy this condition if this interpretation is to hold. By this we mean that if we have an initial data set for the evolution problem, which has a development into a space-time (V, g, ϕ) , then any other equivalent space-time (V, g', ϕ') related to (V, g, ϕ) by a Weyl transformation, must also be the product of the evolution of initial data on M , and, in a specific sense, both initial data sets should be equivalent. Similarly, given two *equivalent* initial data sets, if one of them has a development into a space-time satisfying (6.4), then the other must also have a development into an equivalent space-time, thus satisfying (6.5) for some gauge function f . Pictorially, what we intend to show is that the following commutative diagram holds.

$$\begin{array}{ccc}
 (V, g, \phi) & \xleftrightarrow{W} & (V, g', \phi') \\
 \uparrow C & & \uparrow C \\
 (M, \bar{g}, \bar{\phi}, K, \bar{\phi}_t) & \xleftrightarrow{W} & (M, \bar{g}', \bar{\phi}', K', \bar{\phi}'_t)
 \end{array}$$

Where in the above diagram C stands for *Cauchy evolution* and W for Weyl transformation.

We should now make sense of what we mean by a Weyl transformation on the initial data set. This is because, even though it is clear how a Weyl transformation has to act on $\bar{g}$ and $\bar{\phi}$, the objects K and $\bar{\phi}_t$ are extrinsic objects, and the way they should transform comes from looking at the embedded manifold $M \hookrightarrow M \times \mathbb{R}$ and analysing their transformation behaviour. Thus, consider a space-time Weyl transformation:

$$g' = e^{-f} g, \tag{6.6}$$

$$\phi' = \phi - f, \tag{6.7}$$

and using the definition of the extrinsic curvature, we have that

$$K'(X, Y) = g'(\nabla_X Y, n'),$$

where n' stands for the unit normal vector field, in the metric g' , to each slice $M \times \{t\}$. It is clear that using the adapted frames, then $N'^2 = e^{-f} N^2$, where N and N' are the lapse functions in the frames (V, g, ϕ) and (V, g', ϕ') . Then, we have that, by definition of the lapse,

$$n' = \frac{1}{N'} e_0 = e^{\frac{f}{2}} n.$$

Then, clearly, we have that

$$K'(X, Y) = e^{-\frac{f}{2}} K(X, Y), \quad (6.8)$$

that is, under a space-time Weyl transformation the extrinsic curvature to each hypersurface $M \times \{t\}$ transforms as $K' = e^{-\frac{f}{2}} K$. Thus, on the initial slice $M \cong M \times \{0\}$, setting $h(\cdot) \doteq f(\cdot, 0) \in C^\infty(M)$, we have the following induced Weyl transformation on M

$$K' = e^{-\frac{h}{2}} K.$$

Also, with the same line of argument, we have that $\partial_t \phi'(\cdot, 0) = \partial_t \phi(\cdot, 0) - \partial_t f(\cdot, 0)$. Thus, since $\eta(\cdot) \doteq \partial_t f(\cdot, 0)$ represents an arbitrary smooth function on M , we get the following transformations on the initial data set

$$\begin{aligned} \bar{g}' &= e^{-h} \bar{g}, \\ \bar{\phi}' &= \bar{\phi} - h, \\ K' &= e^{-\frac{h}{2}} K, \\ \bar{\phi}'_t &= \bar{\phi}_t - \eta. \end{aligned} \quad (6.9)$$

Thus, we will take as a definition that a Weyl transformation on the initial data set $(\bar{g}, \bar{\phi}, K, \bar{\phi}_t)$, is given by (6.9), where h and η are arbitrary smooth functions on M .

6.2.1 Constraint equations

It should be noted that the same line of argument explained in chapter 3, shows that we will have a set of constraint equations which must be satisfied by the initial data. Before writing these equations, we will first establish some notation. The field equation can be rewritten as

$$G_{\alpha\beta} = T_{\alpha\beta}, \quad (6.10)$$

$$g^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi = 0, \quad (6.11)$$

where

$$T_{\alpha\beta} \doteq -\omega \nabla_\alpha \phi \nabla_\beta \phi + \frac{\omega}{2} g_{\alpha\beta} \nabla_\sigma \phi \nabla^\sigma \phi. \quad (6.12)$$

The constraint equations are given by

$$\begin{aligned} G(e_0, e_0) &= T(e_0, e_0), \\ G(e_0, e_i) &= T(e_0, e_i). \end{aligned} \quad (6.13)$$

In order to compute the components of the Einstein tensor, we can take advantage that we know their form in the Riemannian context, and that within each class $(V, [g, \phi])$ there is a Riemannian element $(V, e^{-\phi}g, 0)$, *i.e.*, for this element the Weyl connection coincides with its Riemannian connection. Thus, since we know that the Einstein tensor is invariant in the equivalence class, using (3.17), we see that for

$$\begin{aligned} g' &= e^{-\phi} g, \\ K' &= e^{-\frac{\phi}{2}} K, \end{aligned}$$

we get that

$$G(e_0, e_0) = \frac{N'^2}{2} ((K'^l_l)^2 - K'^{ij} K'_{ij} + R(\bar{g}')).$$

Using (6.9), we get

$$G(e_0, e_0) = \frac{N^2}{2}((K_l^l)^2 - K^{ij}K_{ij} + R(\bar{g}, \bar{\phi})). \quad (6.14)$$

Also from (3.17) we get that

$$G(e_0, e_i) = N'(\bar{\nabla}_i K_l^l - \bar{\nabla}_j K_i^{lj}).$$

Again, using (6.9) and Weyl's compatibility condition, we get

$$G(e_0, e_i) = N\bar{g}^{uj}(\bar{\nabla}_i K_{uj} - \bar{\nabla}_j K_{ui}) + \frac{N}{2}(\bar{\nabla}_j \phi K_i^j - \bar{\nabla}_i \phi K_l^l). \quad (6.15)$$

Now, in order to write down the constraints we need to compute the same components for the tensor T . Notice that, using an adapted frame, we have

$$T_{g,\phi} = -\omega\{e_\alpha(\phi)e_\beta(\phi) - \frac{g_{\alpha\beta}}{2}g^{\mu\nu}e_\mu(\phi)e_\nu(\phi)\}\theta^\alpha \otimes \theta^\beta.$$

From this expression it is straightforward to see that

$$\begin{aligned} T(e_0, e_0) &= -\frac{N^2\omega}{2}\{\pi^2 + |\bar{\nabla}\phi|_{\bar{g}}^2\} \\ T(e_0, e_i) &= -\omega N\pi\bar{\nabla}_i\phi, \end{aligned} \quad (6.16)$$

where $\pi \doteq \frac{1}{N}e_0(\phi)$, $\bar{g}$ denotes the induced metric on each slice $M \times \{t\}$ and $\bar{\nabla}$ denotes the induced Weyl connection. From these computations we get that the constraint equations are the following

$$\begin{aligned} (K_l^l)^2 - K^{ij}K_{ij} + R &= -\omega\{\pi^2 + |\bar{\nabla}\phi|_{\bar{g}}^2\}, \\ \bar{g}^{uj}(\bar{\nabla}_i K_{uj} - \bar{\nabla}_j K_{ui}) + \frac{1}{2}(\bar{\nabla}_j \phi K_i^j - \bar{\nabla}_i \phi K_l^l) &= -\omega\pi\bar{\nabla}_i\phi. \end{aligned} \quad (6.17)$$

Since these constraints are to be posed for the initial data on the initial hypersurface M , it becomes clear that, just as discussed for K , the way that π transforms under a Weyl transformation on the initial data set does not come from the intrinsic Weyl transformation on M , but is induced from the way it transforms under space-time Weyl transformations. Also note that π is a quantity that we get by analysing the right-hand

side of (6.4), which is frame-dependent. Thus, we should expect π to be frame-dependent. In fact, the above computations applied to the transformed system (6.5), give us the following constraints, valid in each frame:

$$\begin{aligned} (K^{*l})^2 - K^{*ij} K^*_{ij} + R(\bar{g}^*, \bar{\phi}^*) &= -\omega \left\{ \left(\frac{e^{\frac{f(\cdot, 0)}{2}}}{N} e_0(\bar{\phi}^* + f) \right) \Big|_{t=0}^2 + |\bar{\nabla}(\bar{\phi}^* + f(\cdot, 0))|_{\bar{g}^*}^2 \right\}, \\ \bar{g}^{*uj} (\bar{\nabla}_i K^*_{uj} - \bar{\nabla}_j K^*_{ui}) + \frac{1}{2} (\bar{\nabla}_j \bar{\phi}^* K^{*j}_i - \bar{\nabla}_i \bar{\phi}^* K^{*l}_l) &= -\omega \left(\frac{e^{\frac{f(\cdot, 0)}{2}}}{N} e_0(\bar{\phi}^* + f) \right) \Big|_{t=0} \bar{\nabla}_i (\bar{\phi}^* + f(\cdot, 0)). \end{aligned} \quad (6.18)$$

where the notation comes from considering the space-time Weyl transformation

$$\begin{aligned} g^* &= e^{-f} g, \\ \phi^* &= \phi - f. \end{aligned}$$

From the above expression, we finally get that in an arbitrary frame,

$$\begin{aligned} \pi^* &\doteq \frac{e^{\frac{f}{2}}}{N} e_0(\phi^* + f) = \frac{e^{\frac{f}{2}}}{N} e_0(\phi - f + f) \\ &= e^{\frac{f}{2}} \pi. \end{aligned}$$

Thus, we can write the constraints of the transformed system (6.5) as

$$\begin{aligned} (K^{*l})^2 - K^{*ij} K^*_{ij} + R(\bar{g}^*, \bar{\phi}^*) &= -\omega \left\{ \pi^{*2} + |\bar{\nabla}(\bar{\phi}^* + f(\cdot, 0))|_{\bar{g}^*}^2 \right\}, \\ \bar{g}^{*uj} (\bar{\nabla}_i K^*_{uj} - \bar{\nabla}_j K^*_{ui}) + \frac{1}{2} (\bar{\nabla}_j \bar{\phi}^* K^{*j}_i - \bar{\nabla}_i \bar{\phi}^* K^{*l}_l) &= -\omega \pi^* \bar{\nabla}_i (\bar{\phi}^* + f(\cdot, 0)). \end{aligned} \quad (6.19)$$

Now, just as instead of $\partial_t \bar{g}|_{t=0}$, we actually consider K as part of the geometric initial data, we will consider π instead of $\bar{\phi}_t$ as part of the geometric initial data. In this way, when given the initial data set $(\bar{g}, \bar{\phi}, K, \pi)$, we will have two *gauge* choices: one for the lapse-shift and another one for the values of $f, \partial_t f|_{t=0}$. In this way we get that an initial data set for the system (6.4) is a set of the form $(\bar{g}, \bar{\phi}, K, \pi)$, which under a Weyl transformation

on the initial slice M , transforms according to

$$\begin{aligned}\bar{g}' &= e^{-h}\bar{g}, \\ \bar{\phi}' &= \bar{\phi} - h, \\ K' &= e^{-\frac{h}{2}}K, \\ \pi' &= e^{\frac{h}{2}}\pi.\end{aligned}\tag{6.20}$$

Clearly, given a solution of (6.17) and applying these transformations to it, we get that, under the choice $h \doteq f(\cdot, 0)$, the transformed quantities $(\bar{g}^*, \bar{\phi}^*, K^*, \pi^*)$ satisfy the system (6.19), which is the constraint system for the space-time equations (6.5).

6.2.2 The evolution problem

We will now analyse the set of equations (6.4). We will follow the same line of argument typically applied to the vacuum system in GR.¹ With this in mind, we should begin by making the following remarks. First, note that the geometric identities implied by the symmetries of the curvature tensor are vital in this problem. Note that, from (2.23), the usual contracted Bianchi identities are valid in this context. This identity clearly imposes a condition on the right-hand side of (6.4), which is typically referred to as a *conservation equation*. In order for the system (6.4) to be sensible, this geometric condition must be compatible with the equation posed for ϕ . It is a simple matter of computation to show that both conditions are equivalent.

Secondly, we will rewrite (6.4) merely in terms of the Ricci tensor. In order to achieve this, notice that taking the trace of this equation, we get

$$R = -\omega g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi.$$

Using this information, we can rewrite our system in the following way:

$$\begin{aligned}R_{\alpha\beta} &= -\omega \nabla_\alpha \phi \nabla_\beta \phi, \\ g^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi &= 0.\end{aligned}\tag{6.21}$$

¹Thorough reviews on this topic can be found in [3] and [49].

Now, using (2.28), we can write down the Ricci tensor as:

$$R_{\alpha\beta} = {}^\circ R_{\alpha\beta} + \frac{n-1}{2} \nabla_\alpha \nabla_\beta \phi + \frac{n-1}{4} \nabla_\alpha \phi \nabla_\beta \phi + \frac{1}{2} g_{\alpha\beta} (g^{\mu\nu} \nabla_\mu \nabla_\nu \phi - \frac{n-1}{2} \nabla_\mu \phi \nabla^\mu \phi).$$

Furthermore, a straightforward computation gives that

$$g^{\mu\nu} \nabla_\mu \nabla_\nu \phi = g^{\mu\nu} \nabla_\mu \nabla_\nu \phi - \frac{n-1}{2} \nabla_\mu \phi \nabla^\mu \phi.$$

Thus, the system (6.21) is equivalent to the following set of equations:

$${}^\circ R_{\alpha\beta} + \frac{n-1}{2} \nabla_\alpha \nabla_\beta \phi + \frac{n-1}{4} \nabla_\alpha \phi \nabla_\beta \phi = -\omega \nabla_\alpha \phi \nabla_\beta \phi, \quad (6.22)$$

$$g^{\mu\nu} \nabla_\mu \nabla_\nu \phi - \frac{n-1}{2} \nabla_\mu \phi \nabla^\mu \phi = 0. \quad (6.23)$$

The above field equations are posed for (g, ϕ) . Looking at (6.23), it is clear that this is a linear wave equation which does not pose any difficulties from the point of view of the theory of partial differential equations. In contrast, in the set of equations (6.22), the second order term $\nabla_\alpha \nabla_\beta \phi$ brings some difficulties. If this term was not present, the same approach towards the vacuum Einstein equations, which shows that such equations can be written as a system of non-linear wave equations for the metric, could be applied here, and the lower order terms in ϕ would not cause any difficulties. This approach does not work here. It is interesting to remark that the very same problem appears when studying, in the Jordan frame, the Cauchy problem for Brans-Dicke theory and other scalar tensor theories. This problem has been studied before, for instance, in [93],[94],[95],[96]. In [93] and [94], their considerations, even though not always made explicit, would be limited to analytic initial data.² In [95], the *local* well-posedness of the initial value formulation for the smooth case (or even C^k) is addressed. By local we mean that the problem is studied within a coordinate system, which is chosen by using wave-coordinates, and this condition is not invariant under coordinate changes. Thus, the well-posedness is both local in space and in time.³ In [96], more general type of scalar-tensor theories are studied and

²The method used in these papers is similar to the one used in the proof of the Campbell-Magaard theorem in the previous chapter, where the field equations are split into a set of constraints and evolution equations, and the analyticity assumptions are used to propagate the equations in a neighbourhood of the hypersurface.

³The locality in time, meaning that only short-time existence is generically guaranteed, comes from the theory of hyperbolic equations, or, as a particular case, non-linear wave equations, where, generically, only short-time existence is guaranteed. Showing long-time existence for specific systems is an active

they are shown to have a well-posed initial value formulation by rewriting them as a first order hyperbolic system where classical theorems on partial differential equations can be applied. In this approach, they also treat the local in space and time problem, using a generalized wave-gauge condition.

In contrast to the above mentioned approaches, we will address the global in space initial value formulation for the set of equations (6.22)-(6.23), and, furthermore, we will do this in such a way that the invariance of the problem under Weyl transformations becomes transparent. This will be done by following the standard approach to the global in space problem in GR, but adapting our gauge conditions to our new setting. In this direction, let us begin by defining the following properly Rimeannian metric on space-time:

$$\hat{e} \doteq dt \otimes dt + e, \quad (6.24)$$

where e is some smooth Riemannian metric on M . Now define the following vector field on space-time

$$F^\lambda \doteq g^{\alpha\beta}(\Gamma_{\alpha\beta}^\lambda - \hat{\Gamma}_{\alpha\beta}^\lambda), \quad (6.25)$$

where $\Gamma_{\alpha\beta}^\lambda$ are connection components of the Weyl connection associated with the space-time Weyl structure (g, ϕ) , and $\hat{\Gamma}_{\alpha\beta}^\lambda$ are the components of the Riemannian connection associated with the metric $\hat{e}$. The fact that F defines a vector field on space-time, comes from the fact that the difference of two connections defines a tensor field. We can rewrite this vector field in the following way:

$$F^\lambda = {}^\circ F^\lambda + g^{\alpha\beta}(\Gamma_{\alpha\beta}^\lambda - {}^\circ \Gamma_{\alpha\beta}^\lambda),$$

where ${}^\circ F^\lambda \doteq g^{\alpha\beta}({}^\circ \Gamma_{\alpha\beta}^\lambda - \hat{\Gamma}_{\alpha\beta}^\lambda)$. The vector field ${}^\circ F$ has been used to analyse the global in space problem in GR [3]. Now, since $g^{\alpha\beta}(\Gamma_{\alpha\beta}^\lambda - {}^\circ \Gamma_{\alpha\beta}^\lambda) = \frac{n-1}{2} {}^\circ \nabla^\lambda \phi$, we get that

$$F^\lambda = {}^\circ F^\lambda + \frac{n-1}{2} {}^\circ \nabla^\lambda \phi. \quad (6.26)$$

At this point, we will take advantage of some useful expressions known from the Cauchy

object of study in mathematical relativity, which is related to the problem of long-time stability with respect to the initial data.

problem in GR. For instance, it is known that the Ricci tensor ${}^\circ R_{\alpha\beta}$ has the following decomposition:⁴

$$\begin{aligned} {}^\circ R_{\alpha\beta} &= R_{\alpha\beta}^{(\hat{e})} + \frac{1}{2}(g_{\alpha\lambda}D_\beta {}^\circ F^\lambda + g_{\beta\lambda}D_\alpha {}^\circ F^\lambda) \\ R_{\alpha\beta}^{(\hat{e})} &= -\frac{1}{2}g^{\lambda\mu}D_\lambda D_\mu g_{\alpha\beta} + f_{\alpha\beta}(g, Dg), \end{aligned}$$

where D represents the $\hat{e}$ -covariant derivative and $f_{\alpha\beta}$ has smooth dependence on both g and Dg . Using these expressions together with (6.26), we can rewrite

$${}^\circ R_{\alpha\beta} = R_{\alpha\beta}^{(\hat{e})} + \frac{1}{2}(g_{\alpha\lambda}D_\beta F^\lambda + g_{\beta\lambda}D_\alpha F^\lambda) - \frac{n-1}{4}(g_{\alpha\lambda}D_\beta g^{\lambda\mu}D_\mu \phi + g_{\beta\lambda}D_\alpha g^{\lambda\mu}D_\mu \phi) - \frac{n-1}{2}D_\alpha D_\beta \phi. \quad (6.27)$$

Now, the idea is to rewrite (6.22) in terms of the $\hat{e}$ -covariant derivatives, and to use the previous decomposition for the Riemannian Ricci tensor. The interesting thing is that the last term in (6.27) will exactly cancel the problematic term in (6.22). In this way, we get the following system:

$$\begin{aligned} {}^r R_{\alpha\beta} - \rho_{\alpha\beta} + \frac{1}{2}(g_{\alpha\lambda}D_\beta F^\lambda + g_{\beta\lambda}D_\alpha F^\lambda) &= 0, \\ g^{\alpha\beta}D_\alpha D_\beta \phi - {}^\circ F^\sigma D_\sigma \phi - \frac{n-1}{2}D_\sigma \phi D^\sigma \phi &= 0, \end{aligned} \quad (6.28)$$

where

$$\begin{aligned} {}^r R_{\alpha\beta} &\doteq R_{\alpha\beta}^{(\hat{e})} - \frac{n-1}{4}(g_{\alpha\lambda}D_\beta g^{\lambda\mu}D_\mu \phi + g_{\beta\lambda}D_\alpha g^{\lambda\mu}D_\mu \phi) - \frac{n-1}{2}\Delta^\circ \Gamma_{\alpha\beta}^\sigma D_\sigma \phi \\ &\quad + \frac{n-1}{4}D_\alpha \phi D_\beta \phi, \\ \rho_{\alpha\beta} &\doteq -\omega D_\alpha \phi D_\beta \phi, \\ \Delta^\circ \Gamma_{\alpha\beta}^\sigma &\doteq \mathbb{T}_{\alpha\beta}^\sigma - \hat{\Gamma}_{\alpha\beta}^\sigma. \end{aligned} \quad (6.29)$$

Following an approach analogous to the standard approach to the Einstein equations, we will consider the reduced system:

$${}^r R_{\alpha\beta} - \rho_{\alpha\beta} = 0, \quad (6.30)$$

$$g^{\alpha\beta}D_\alpha D_\beta \phi - {}^\circ F^\sigma D_\sigma \phi - \frac{n-1}{2}D_\sigma \phi D^\sigma \phi = 0. \quad (6.31)$$

⁴See Yvonne Choquet-Bruhat, *General Relativity and the Einstein equations*, Oxford University Press Inc., New York (2009), page 163.

Using the expression for $R_{\alpha\beta}^{(\hat{e})}$ in (6.30), we see that (6.30) and (6.31) form a system of hyperquasilinear wave equations for (g, ϕ) , thus, given appropriate initial data, the system has one and only one solution on $M \times [0, T)$, for some $T > 0$.⁵ Now, since $T_{\alpha\beta} = \rho_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}g^{\mu\nu}\rho_{\mu\nu}$, we get that, given a solution (g, ϕ) for (6.30)-(6.31), its Einstein tensor satisfies the following identity:

$$G_{\alpha\beta} - T_{\alpha\beta} = \frac{1}{2}(g_{\alpha\lambda}D_{\beta}F^{\lambda} + g_{\beta\lambda}D_{\alpha}F^{\lambda} - g_{\alpha\beta}D_{\lambda}F^{\lambda}).$$

Now, the contracted Bianchi identities for Weyl's connection together with (6.31), imply that the divergence of the previous expression vanishes. This gives us a geometric identity satisfied by F . Some straightforward computations give us that F satisfies the following homogeneous system of linear wave equations:

$$g^{\alpha\mu}D_{\alpha}D_{\mu}F^{\lambda} + A_{\nu}^{\alpha\lambda}(g, \phi, Dg, D\phi)D_{\alpha}F^{\nu} + g^{\beta\lambda}R_{\mu\beta}(\hat{e})F^{\mu} = 0, \quad (6.32)$$

where $A_{\nu}^{\alpha\lambda}$ is a $(2, 1)$ -tensor field depending on $(g, \phi, Dg, D\phi)$. Uniqueness of solutions for such a system gives us that, if $F(\cdot, 0) = 0$ and $\partial_t F(\cdot, 0) = 0$, then $F \equiv 0$. Thus, if we can guarantee that the initial data for (6.30)-(6.31) satisfy these conditions, then the solution to this system actually satisfies the full system of equations (6.28).

Lemma 6.2.2.1. *Suppose that (g, ϕ) satisfy the reduced system of equations (6.30)-(6.31) and that the initial data satisfies $F(\cdot, 0) = 0$. Then $\partial_t F(\cdot, 0) = 0$ iff the constraint equations (6.17) are satisfied.*

Proof. We have the following decomposition:

$$G_{\alpha\beta} = {}^rG_{\alpha\beta} + \frac{1}{2}(g_{\alpha\lambda}D_{\beta}F^{\lambda} + g_{\beta\lambda}D_{\alpha}F^{\lambda} - g_{\alpha\beta}D_{\lambda}F^{\lambda}),$$

where ${}^rG_{\alpha\beta} = {}^rR_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}(g^{\mu\nu}{}^rR_{\mu\nu})$.

Note that if $F|_{t=0} = 0$, then $D_i F|_{t=0} = 0$. Then we have that:

$$G_{\alpha 0}|_{t=0} = {}^rG_{\alpha 0}|_{t=0} + \frac{1}{2}(g_{\alpha i}D_0 F^i - N^2 D_{\alpha} F^0)|_{t=0}.$$

⁵By appropriate initial data, we mean initial data belonging to some appropriate functional spaces. The regularity of the solution will be directly tied to this choice. In particular, for smooth data, we get smooth solutions. For a detailed review of these properties, and low regularity results, see Yvonne Choquet-Bruhat, *General Relativity and the Einstein equations*, Oxford University Press Inc., New York (2009). Appendix III.

Thus,

$$(G_{\alpha 0} - T_{\alpha 0})|_{t=0} = ({}^r G_{\alpha 0} - T_{\alpha 0})|_{t=0} + \frac{1}{2}(g_{\alpha i} D_0 F^i - N^2 D_\alpha F^0)|_{t=0}.$$

Then, since (g, ϕ) solve the reduced system, we get that

$$(G_{\alpha 0} - T_{\alpha 0})|_{t=0} = \frac{1}{2}(g_{\alpha i} D_0 F^i - N^2 D_\alpha F^0)|_{t=0}.$$

Thus we see that the constraint equations are satisfied iff

$$0 = (g_{\alpha i} D_0 F^i - N^2 D_\alpha F^0)|_{t=0}.$$

Setting $\alpha = 0$, the previous expression gives $\partial_t F^0 = 0$ and setting $\alpha = j$ gives $\partial_t F^j = 0$. $\square$

Using this lemma, we see that if we give an initial data set satisfying the constraint equations plus the *gauge condition* $F|_{t=0} = 0$, then the system (6.30)-(6.31) will have a unique solution, which, in fact, solves the full system (6.22)-(6.23). Let us see that we can always pick initial data satisfying the gauge condition.

Our geometric initial data set for (6.30)-(6.31) is $(M, \bar{g}, \bar{\phi}, K, \pi)$. We need to provide initial data for (g, ϕ) . It is clear that $\bar{g}, \bar{\phi}$ determine the initial data $g_{ij}|_{t=0}, \phi|_{t=0}$. Also that, once we fix the initial data for the lapse and shift, K and π determine the initial data $\partial_t g_{ij}|_{t=0}$ and $\partial_t \phi|_{t=0}$. Then, we freely set $g_{00}|_{t=0} = -1$ and $g_{0i}|_{t=0} = 0$, which amounts to setting $N|_{t=0} = 1$ and $\beta|_{t=0} = 0$. We still need initial the data $\partial_t g_{00}|_{t=0}, \partial_t g_{0i}|_{t=0}$. These initial data will come from the solving the gauge condition. Note that

$$F^\lambda|_{t=0} = \{g^{\alpha\beta} \mathring{\Gamma}_{\alpha\beta}^\lambda + \frac{n-1}{2} \circ \nabla^\lambda \phi - g^{\alpha\beta} \hat{\Gamma}_{\alpha\beta}^\lambda\}|_{t=0}.$$

Then, since we have that

$$\begin{aligned} g^{\alpha\beta} \mathring{\Gamma}_{\alpha\beta}^0|_{t=0} &= -\frac{1}{2} \partial_t g_{00}|_{t=0} - \bar{g}^{ij} K_{ij}, \\ g^{\alpha\beta} \mathring{\Gamma}_{\alpha\beta}^i|_{t=0} &= -\bar{g}^{ij} \partial_t g_{j0}|_{t=0} + \bar{g}^{kl} \bar{\Gamma}_{kl}^i, \end{aligned}$$

we can solve the algebraic equations $F^\lambda|_{t=0} = 0$ for $\partial_t g_{00}|_{t=0}, \partial_t g_{0i}|_{t=0}$. With these choices for $\partial_t g_{00}|_{t=0}, \partial_t g_{0i}|_{t=0}$ we complete the initial data for the reduced system. Then, the

theory of hyperbolic partial differential equations guarantees the existence of a unique solution for this system, which will be as regular as the initial data. Thus, we have the following theorem:

Theorem 6.2.2.1 (Existence in a frame). *Given initial data $(M, \bar{g}, \bar{\phi}, K, \pi)$ satisfying the constraint equations (6.17), there is a Cauchy development of this initial data set into a space-time (V, g, ϕ) satisfying the field equations (6.22)-(6.23).*

Finally, we need to see that the problem is *invariant under Weyl transformations*. By this we mean that if we make a space-time Weyl transformation, then, the space-time described in this other frame results as the evolution of initial data *equivalent* to $(\bar{g}, \bar{\phi}, K, \pi)$. And conversely, given an *equivalent* initial data set to $(\bar{g}, \bar{\phi}, K, \pi)$, then this initial data set admits a development into a space-time *equivalent* to the one generated by $(\bar{g}, \bar{\phi}, K, \pi)$.

Consider that we have an initial data set $(\bar{g}, \bar{\phi}, K, \pi)$ satisfying the constraint equations (6.17) for the system (6.4). Then, using **Theorem 6.2.2.1**, we know that this initial data set admits a development into a space-time (V, g, ϕ) satisfying the field equations (6.4). Then, perform a space-time Weyl transformation of the form

$$g' = e^{-f}g, \quad \phi' = \phi - f \quad (6.33)$$

We know that (g', ϕ') will satisfy the system of equations (6.5). Furthermore, since the gauge condition $F = 0$ is Weyl-invariant, *i.e.*, if it is satisfied in one frame, then it is satisfied in all the equivalence class, then applying the same procedure described above for the system (6.4) to the transformed system (6.5), we see that (6.5) is actually equivalent to the set of reduced equations that result from setting $F(g', \phi') = 0$. Thus, we see that (g', ϕ') actually solve a system of non-linear wave equations in the space-time metric g' analogous to (6.30)-(6.31). Furthermore, this system takes initial data satisfying the constraints (6.19). This shows that making a Weyl transformation on the initial data $(\bar{g}, \bar{\phi}, K, \pi)$ of the form

$$\bar{g}' = e^{-h}\bar{g}, \quad \bar{\phi}' = \bar{\phi} - h, \quad K' = e^{-\frac{h}{2}}K, \quad \pi' = e^{\frac{h}{2}}\pi, \quad (6.34)$$

and setting $h \doteq f(\cdot, 0)$, then (V, g', ϕ') results as the evolution of the initial data set $(\bar{g}', \bar{\phi}', K', \pi')$, which is equivalent to $(\bar{g}, \bar{\phi}, K, \pi)$.

Conversely, if we make a Weyl transformation on the initial data $(\bar{g}, \bar{\phi}, K, \pi)$ of the form (6.34), then the transformed initial data will satisfy the system (6.19) if we consider $f|_{t=0} = h$. Thus, if we pick a function f on space-time satisfying $f|_{t=0} = h$, and then we perform a space-time Weyl transformation on (g, ϕ) of the form (6.33), we see that (g', ϕ') on M_0 satisfy the initial conditions $(\bar{g}', \bar{\phi}', K', \bar{\phi}'_t)$. Thus, any initial data set equivalent to $(\bar{g}, \bar{\phi}, K, \pi)$ has an evolution into a space-time equivalent to (V, g, ϕ) . We summarize this with the following theorem.

Theorem 6.2.2.2. *Given an initial data set $(M, \bar{g}, \bar{\phi}, K, \pi)$ satisfying the constraint equations (6.17), then the Cauchy problem for the WGSTT is well-posed.*

□

6.3 Discussion

We would like to conclude this chapter by making a few remarks about the results presented above. First, we have been able to show that the WGSTT presented in the first section of this chapter, have a well-posed initial value formulation for initial data satisfying a system of constraint equations. Furthermore, it is an interesting fact that the Weyl structure results as the evolution of initial data, where the space-time structure, in each frame, is achieved as a solution of a system of non-linear wave equations in the space-time metric, which shows that in each frame we have that the speed of propagation of the gravitational interaction, now described by the whole Weyl structure, is determined by the null cones of (any) of the space-time metrics in the conformal class, which is an invariant property in Weyl structure.⁶

Using the above results, we see that the evolution problem for the theories described above is well-posed in vacuum. We leave for future research the study of the evolution problem in the presence of matter fields. Furthermore, it would be essential to analyse the well-posedness of the system of constraint equations, and all the related problems to this issue.

⁶When we say that we have a system of wave equations in the space-time metric, we mean that the *principal part* (the top order terms) has the structure of a wave operator in the space-time metric.

Conclusions

In this thesis, we have studied different embedding problems that naturally arise in the context of modern space-time theories. A particularly important problem that we have addressed is the initial value formulation for general relativity. The different ramifications of this problem are among the most studied problems in the overlap between differential geometry, geometric analysis and theoretical physics during the past decades, and the consequences of such studies have had a profound impact on the foundations of modern physical theories. In this line, the characterization of the space of solutions of the constraint equations is a problem with the capability of describing generic properties of the solutions allowed by general relativity. In this direction, we have continued the study of the thin sandwich problem, originally proposed as a conjecture by J. A. Wheeler, and we have been able to conclude that the Bartnik-Fodor theorem, which, within the superspace picture of space-time, deals with the local well-posedness of the thin sandwich equations around points with specific geometric properties, naturally extends to arbitrary dimensions. Just as the 3-dimensional problem is of interest for classical general relativity, this extension is clearly of interest for higher-dimensional theories of gravity. Furthermore, we have been able to show that the type of initial data sets where this result applies are a generic feature on the compact setting. This last result is a novelty even for the 3-dimensional case.

The results mentioned above for the thin sandwich problem give us several insights on the space of solutions of the Einstein equations. For instance, they give us a clear picture of the structure of the superspace around these *generic*⁷ points as an infinite dimensional manifold modelled locally as a Banach space. Furthermore, we have shown

⁷When studying properties of the space of solutions of the constraint equations, the phrase *generic data* is typically used to refer to a class of data which is dense in the space of solution in some specified topology. We do not mean the same. What we mean by generic data is a type of initial data sets which always exist on any compact manifold.

that there are neighbourhoods of such *generic* data where the thin sandwich picture for space-time holds. It is remarkable that this result on its own provides a powerful geometric tool, which we have used to analyse the existence of isometric embedding for compact Lorentzian manifolds into Ricci-flat semi-Riemannian spaces. With this approach, we have been able to obtain a new result, which, with regard to the codimension needed for the embedding, improves previously known results quite drastically. Also, the regularity requirements needed are quite low, which overcomes one of the main limitations of the Campbell-Magaard theorem. Despite this fact, in order for the theorem to be better suited for physical applications, the compactness condition should be dropped and it would be desirable to have control on the number of time-like dimensions of the ambient space. We regard these two problems as perspectives for future research, and we highlight the role that the initial value formulation for general relativity, with different type of asymptotic behaviour, can play in this problem.

Finally, we have addressed some embedding problems outside the context of Riemannian geometry. In particular, we analysed the relevance of Weyl structures for space-time theories, which becomes transparent from the EPS axiomatic formulation. In this context, we provided a detailed and rigorous discussion about the possible existence of a second clock effect, and concluded that if we are to reject space-time models exhibiting such effect, then the most general mathematical structure, within the EPS axiomatic framework, which can serve as a model for space-time, is that of a Weyl integrable structure.

With the above motivations in mind, and taking into account the current interest in both higher-dimensional theories of space-time and the interest in non-Riemannian geometries as a tool for providing new degrees of freedom, we have studied possible extensions of the Campbell-Magaard theorems in the context of both integrable and non-integrable Weyl geometry. We have shown that in the integrable case the theorem extends naturally. In the non-integrable case we found a particular geometric identity which poses obstructions for embeddings into Weyl manifolds with Ricci tensor whose symmetric part is vanishing. This led us to an unexpected no-go theorem when the embedding manifold is endowed with a positive definite metric and the dimension of the initial manifold is greater or equal to 4. Quite interestingly, for the particular case of embedding a 3-dimensional Weyl structure in a 4-dimensional Weyl structure whose Ricci tensor has vanishing symmetric part, the previous geometric obstructions disappear and a Campbell-Magaard-like

theorem can be proved.

Finally, continuing in the context of Weyl structures as possible models of space-time, we have analysed the initial value formulation of a particular type of geometric scalar-tensor theory, which assumes that space-time is modelled as a Weyl integrable structure, for the case of vacuum. We have been able to conclude that this problem is well-posed within the invariant interpretation proposed at the beginning of chapter 6.

As for future perspectives, regarding the thin sandwich problem there is still work to be done. First, it would be interesting to apply some of the ideas used by D. Giulini in [19] in order to incorporate realistic models for the matter fields into the picture. Furthermore, it would be interesting to study the problem outside the compact case. Probably the next case to look at is the asymptotically Euclidean case. Finally, it would be quite interesting to analyse the full sandwich conjecture in more detail, since there are virtually no results concerning this problem.

Concerning the embedding problem for Lorentzian manifolds, as we have stated above, we consider the results presented here, even though they are novel, as a first step in the search for sharper theorems better suited for more realistic physical applications. In this direction, maybe further extensions of the thin sandwich problems on manifolds with different type of asymptotics can translate into embedding theorems of the type presented here.

Related to the initial value formulation of the geometric scalar-tensor theories described in chapter 6, it would be interesting to analyse this problem in the presence of matter fields, and furthermore, to analyse the well-posedness of the constraint system.

Appendix A

Sobolev spaces

In this appendix we will gather the most basic definitions about Sobolev spaces and state the fundamental results that are used in this thesis. The reader who is not familiarized with this results can find further references, for instance, in [8],[102],[103].

Definition A.0.0.1. Let (M, e) be a smooth Riemannian manifold. We denote by ∇ the covariant derivative and by μ_e the volume element in the metric e .

Definition A.0.0.2. The Sobolev space W_s^p is the space of functions, or tensor fields of some given type, on (M, e) with L^p integrable generalized derivatives of order $\leq s$ in the metric e . It is a Banach space with the norm

$$\|f\|_{W_s^p} \doteq \left\{ \sum_{k=0}^s \int |\nabla^k f|^p \right\}^{\frac{1}{p}}, \quad 1 \leq p < \infty \quad (\text{A.1})$$

with $|\cdot|$ the pointwise norm of tensors in the metric e .

It is possible to prove many different density theorems regarding the density of the space $C_0^\infty(M)$ of C^∞ tensor fields of a given type with compact support in the spaces W_s^p . In particular, for the case of M compact, it holds thar $C^\infty(M)$ is always dense in W_s^p . Also for M compact the following embedding theorems hold.

Proposition A.0.0.1. *1. If $s > \frac{n}{p}$ then the space W_s^p is continuously embedded into $\overline{C}^0$ (continuous and bounded fields); it holds that, with $C(M, e)$ a number depending on (M, e)*

$$\sup_M |u| \leq C(M, e) \|u\|_{W_s^p}$$

Also $W_n^1 \subset \overline{C}^0$

2. If $s = \frac{n}{p}$ the space W_s^p is continuously embedded in L^q for all q such that $p \leq q < \infty$

3. If $s < \frac{n}{p}$, the space W_s^p is continuously embedded in L^q for all q such that $p \leq q \leq \frac{np}{n-sp}$

In cases 2 and 3 it holds that

$$\|u\|_{L^q} \leq C(M, e) \|u\|_{W_s^p}$$

Corollary A.0.0.1. *An induction argument shows the continuous embeddings:*

1. $W_{s+m}^p \subset \overline{C}^m$ if $s > \frac{n}{p}$

2. $W_{s+m}^p \subset W_m^q$ if $s < \frac{n}{p}$, $p \leq q \leq \frac{np}{n-sp}$

Proposition A.0.0.2. *The spaces W_s^p admit the continuous multiplication property:*

$$\begin{aligned} W_{s_1}^p \times W_{s_2}^p &\rightarrow W_s^p, \\ (u, v) &\mapsto u \otimes v \end{aligned}$$

$$\|u \otimes v\|_{W_s^p} \leq C(M, e) \|u\|_{W_{s_1}^p} \|v\|_{W_{s_2}^p} \quad (\text{A.2})$$

if $s_1 + s_2 > s + \frac{n}{p}$.

We will have particular interest in the L^2 based Sobolev spaces, which are denoted by $H_s \doteq W_s^2$.

As a final comment, we note that when dealing with differential operators acting on sections of some tensor bundle belonging to some Sobolev space, we usually need to know that the range of such an operator lies in some other, appropriate, Sobolev space. For this purpose the multiplication property becomes extremely useful. For instance, since typically such operators are given in terms of tensor fields, we get that their action is given by contractions of tensor fields, composed with simple operations. Then, using the following pointwise estimates, combined with the multiplication property, we can usually get in a straightforward way the target space of such a mapping.

Lemma A.0.0.1. *Suppose we have a Riemannian manifold (M, e) and $|\cdot|$ denotes the usual pointwise tensor norm in the metric e . Then, for any two tensor fields u and v , the*

following pointwise estimates hold:

$$|u.v| \leq |u||v|, \tag{A.3}$$

$$|u \otimes v| = |u||v|, \tag{A.4}$$

where $u.v$ denotes any contraction between these tensor fields.

Appendix B

Differential calculus in Banach spaces

The idea for this appendix, is to provide the basic notions about calculus in general (infinite dimensional) Banach spaces. These techniques are needed to analyse the linearisation of non-linear maps acting between Banach spaces, which is a typical situation when studying non linear partial differential equations. In particular, we intend to present the implicit function theorem in this context, which is a standard technique for solving non linear problems. The main references for this section are [73] and [97]. Also, the same topics can be found in [102].

Definition B.0.0.1. Let X, Y be normed linear spaces (NLS). We will denote $B(X, Y)$ the set of bounded (continuous) linear maps from X to Y

Definition B.0.0.2. Let $f : U \rightarrow Y$, with $U \subset X$ open and X, Y NLS's. Let $x \in U$. We say that f is Fréchet differentiable at x if $\exists A \in B(X, Y)$ such that

$$\frac{\|f(x+h) - f(x) - Ah\|_Y}{\|h\|_X} \xrightarrow{h \rightarrow 0} 0$$

which is equivalent to $\|f(x+h) - f(x) - Ah\|_Y = o(\|h\|_X)$. When it exists, we call A the Fréchet derivative of f at x : $A = Df(x)$

Proposition B.0.0.1. *If f is Fréchet differentiable at x then $Df(x)$ is unique and f is continuous at x . Moreover if g is Fréchet differentiable and $\alpha, \beta \in \mathbb{F}$ then*

$$D(\alpha f + \beta g)(x) = \alpha Df(x) + \beta Dg(x)$$

Proof. Suppose $A, B \in B(X, Y)$ and that

$$\frac{\|f(x+h) - f(x) - Ah\|_Y}{\|h\|_X} \xrightarrow{h \rightarrow 0} 0$$

$$\frac{\|f(x+h) - f(x) - Bh\|_Y}{\|h\|_X} \xrightarrow{h \rightarrow 0} 0$$

Define

$$R_A(x, h) \doteq f(x+h) - f(x) - Ah$$

$$R_B(x, h) \doteq f(x+h) - f(x) - Bh$$

Then

$$\begin{aligned} \|A - B\|_{B(X, Y)} &= \sup_{\|x\|_X=1} \|(A - B)(x)\|_Y \\ &= \frac{1}{\epsilon} \sup_{\|x\|_X=1} \|(A - B)(\epsilon x)\|_Y \\ &= \frac{1}{\epsilon} \sup_{\|h\|_X=\epsilon} \|(A - B)(h)\|_Y \\ &= \sup_{\|h\|_X=\epsilon} \frac{\|R_B(x, h) - R_A(x, h)\|_Y}{\|h\|_X} \\ &\leq \sup_{\|h\|_X=\epsilon} \frac{\|R_B(x, h)\|_Y}{\|h\|_X} + \sup_{\|h\|_X=\epsilon} \frac{\|R_A(x, h)\|_Y}{\|h\|_X} \quad \forall \epsilon > 0. \end{aligned}$$

By hypothesis, the right-hand side of the last equation goes to zero as $\epsilon \rightarrow 0$, so we have $A = B$.

Continuity of f at x is straightforward since

$$\begin{aligned} \|f(x+h) - f(x)\|_Y &= \|Df(x) \cdot h + R(x, h)\| \\ &\leq \|Df(x)\|_{B(X, Y)} \|h\|_X + \|R(x, h)\| \xrightarrow{h \rightarrow 0} 0 \end{aligned}$$

where we used the fact that, by definition, $Df(x)$ is bounded.

The linearity property follows from straightforward calculations.

□

In fact we have more than just continuity. We state this in the following lemma.

Lemma B.0.0.1. *If $f : U \rightarrow Y$ is differentiable, with $U \subset X$ open, X, Y NLS's and $x \in U$ then, given $\epsilon > 0 \exists \delta(x, \epsilon) > 0$ such that $\forall h \in X$ with $\|h\|_X \leq \delta$:*

$$\|f(x+h) - f(x)\|_Y \leq (\|Df(x)\|_{B(X,Y)} + \epsilon)\|h\|_X$$

Proof.

$$\begin{aligned} \|f(x+h) - f(x)\|_Y &= \|R(x, h) + Df(x) \cdot h\|_Y \\ &\leq \|R(x, h)\|_Y + \|Df(x)\|_{B(X,Y)}\|h\|_X \end{aligned}$$

Since f is differentiable, then $\forall \epsilon > 0 \exists \delta > 0$ such that if $\|h\| < \delta$:

$$\frac{\|R(x, h)\|_Y}{\|h\|_X} < \epsilon$$

then taking $\|h\|_X < \delta(\epsilon, x)$ we get

$$\|f(x+h) - f(x)\|_Y \leq (\|Df(x)\|_{B(X,Y)} + \epsilon)\|h\|_X$$

□

Example B.0.0.1. Let X, Y be NLS's and

$$\begin{aligned} f: X &\rightarrow Y \\ x &\mapsto Ax. \end{aligned}$$

with $A \in B(X, Y)$. Then

$$\begin{aligned} \|f(x+h) - f(x) - Ah\|_Y &= \|Ax + Ah - Ax - Ah\|_Y \\ &= 0 \end{aligned}$$

so $Df(x) = A \in B(X, Y)$.

This example can be used to compute the Fréchet differential of a covariant derivative ∇ , which is a continuous map from H_{s+1} into H_s .

Theorem B.0.0.1. *Let X, Y, Z be NLS's and $U \subset X, V \subset Y$ be open sets; $f : U \rightarrow Y$ and $g : V \rightarrow Z$. Let $x \in U$ and $y = f(x) \in V$. Suppose g is Fréchet differentiable at y*

and f is Fréchet differentiable at x . Then $g \circ f$ is Fréchet differentiable at x and

$$D(g \circ f)(x) = Dg(y)Df(x)$$

Proof. Let

$$R_f(x, h) \doteq f(x + h) - f(x) - Df(x) \cdot h \quad \text{with } x \in U,$$

$$R_g(y, k) \doteq g(y + k) - g(y) - Dg(y) \cdot k \quad \text{with } y \in V.$$

The hypotheses are

$$\frac{\|R_f(x, h)\|_Y}{\|h\|_X} \xrightarrow{h \rightarrow 0} 0,$$

$$\frac{\|R_g(y, k)\|_Z}{\|k\|_Y} \xrightarrow{k \rightarrow 0} 0.$$

Define $u: X \rightarrow Y$ by

$$u(h) \doteq f(x + h) - f(x) = f(x + h) - y.$$

Since f is differentiable at $x \in U$ then f is continuous at x , we have that $u(h) \xrightarrow[h \rightarrow 0]{Y} 0$.

Now consider

$$\begin{aligned} g(f(x + h)) - g(f(x)) &= g(f(x + h)) - g(y) \\ &= g(y + u) - g(y) - Dg(y) \cdot u + Dg(y) \cdot u \\ &= Dg(y) \cdot [f(x + h) - f(x)] + R_g(y, u) \\ &= Dg(y) \cdot (Df(x) \cdot h) + Dg(y) \cdot R_f(x, h) + R_g(y, u). \end{aligned}$$

Looking at the last expression, we need to show that the last two terms go to zero as $h \rightarrow 0$. Consider the second term:

$$\frac{\|Dg(y) \cdot R_f(x, h)\|_Z}{\|h\|_X} \leq \|Dg(y)\|_{B(Y, Z)} \frac{\|R_f(x, h)\|_Y}{\|h\|_X} \xrightarrow{h \rightarrow 0} 0$$

For the last term we need to use the Lipschitz property. If $u = 0$, then clearly $R_g(y, u) = 0$

and then $\frac{\|R_g(y,u)\|_Z}{\|h\|_X} = 0$. If $u \neq 0$, then

$$\frac{\|R_g(y,u)\|_Z}{\|h\|_X} = \frac{\|R_g(y,u)\|_Z}{\|u\|_Y} \frac{\|u\|_Y}{\|h\|_X} \quad (\text{B.1})$$

From our hypotheses, we know that given $\epsilon > 0 \exists \sigma > 0$ such that if $\|u\|_Y < \sigma$ then

$$\frac{\|R_g(y,u)\|_Z}{\|u\|_Y} < \epsilon. \quad (\text{B.2})$$

Also, from the Lipschitz property we know that because f is differentiable at x , then given $\epsilon > 0 \exists \delta(\epsilon, x) > 0$ such that $\forall \|h\|_X < \delta$:

$$\|u\|_Y = \|f(x+h) - f(x)\|_Y \leq (\|Df(x)\|_{B(X,Y)} + \epsilon)\|h\|_X. \quad (\text{B.3})$$

Taking ϵ arbitrarily small and $\epsilon < 1$ and δ sufficiently small so that the last expression holds for this ϵ and also such that $\delta < \frac{\sigma}{\|Df(x)\|_{B(X,Y)} + 1}$, then from (B.3) we see that (B.2) holds. In this case (B.1) reads

$$\frac{\|R_g(y,u)\|_Z}{\|h\|_X} \leq \epsilon(\|Df(x)\|_{B(X,Y)} + 1) \xrightarrow{\epsilon \rightarrow 0} 0$$

which finishes the proof. □

Now consider $F_1, \dots, F_n$ NLS's. Then

$$\begin{aligned} \|\cdot\|: F_1 \times \dots \times F_n &\rightarrow \mathbb{R} \\ (f_1, \dots, f_n) &\mapsto \|f_1\|_{F_1} + \dots + \|f_n\|_{F_n} \end{aligned}$$

defines a norm on the product. It can be seen that if all the F_i 's are Banach, then the product with this norm is also a Banach space. In this context we have the following result.

Proposition B.0.0.2. *Let $f_i: U \subset E \rightarrow F_i$ with E, F_i $i = 1, \dots, n$ Banach spaces, be a collection of differentiable mappings. Then*

$$f = f_1 \times \dots \times f_n: U \subset E \rightarrow F_1 \times \dots \times F_n$$

$$u \mapsto (f_1(u), \dots, f_n(u))$$

is differentiable and $Df = Df_1 \times \dots \times Df_n$

Proof. Consider $u \in U$ and the following linear map:

$$(Df_1 \times \dots \times Df_n)(u): E \rightarrow F_1 \times \dots \times F_n.$$

$$e \mapsto (Df_1(u) \cdot e, \dots, Df_n(u) \cdot e)$$

Also note that

$$\frac{\|(Df_1 \times \dots \times Df_n)(u) \cdot e\|_F}{\|e\|_E} = \frac{\|Df_1(u) \cdot e\|_{F_1}}{\|e\|_E} + \dots + \frac{\|Df_n(u) \cdot e\|_{F_n}}{\|e\|_E}.$$

Thus, we have that the following holds

$$\sup_{\|e\|_E=1} \|(Df_1 \times \dots \times Df_n)(u) \cdot e\| \leq \sup_{\|e\|_E=1} \frac{\|Df_1(u) \cdot e\|_{F_1}}{\|e\|_E} + \dots + \sup_{\|e\|_E=1} \frac{\|Df_n(u) \cdot e\|_{F_n}}{\|e\|_E},$$

and since the right-hand side is bounded by hypothesis, then $Df_1 \times \dots \times Df_n \in B(E, F_1 \times \dots \times F_n)$. Also, if we consider $x \in U$:

$$\frac{\|f(x+h) - f(x) - (Df_1 \times \dots \times Df_n)(x) \cdot h\|_F}{\|h\|_E} =$$

$$\frac{\|f_1(x+h) - f_1(x) - Df_1(x) \cdot h\|_{F_1}}{\|h\|_E} + \dots$$

$$+ \frac{\|f_n(x+h) - f_n(x) - Df_n(x) \cdot h\|_{F_n}}{\|h\|_E}$$

and the right-hand side goes to zero as $h \rightarrow 0$. □

The following proposition is not difficult to prove.

Proposition B.0.0.3. *Let $B: E \times F \rightarrow G$ be a continuous bilinear map between NLS's.*

Then B is differentiable and

$$DB(u, v) \cdot (e, f) = B(u, f) + B(e, v)$$

□

With this result we can state the following theorem.

Theorem B.0.0.2. *Let $f_i : U \subset E \rightarrow F_i$ with $i = 1, 2$ be differentiable maps and $B \in B(F_1, F_2; G)$ a bilinear continuous map. Then $B(f_1, f_2) = B \circ (f_1 \times f_2) : U \rightarrow G$ is differentiable, and*

$$D(B(f_1, f_2))(u) \cdot e = B(Df_1(u) \cdot e, f_2(u)) + B(f_1(u), Df_2(u) \cdot e)$$

Proof. From the last proposition we know how to compute the differential of a continuous bilinear map. So we can apply the chain rule to the map $B \circ (f_1, f_2)$ to get

$$\begin{aligned} D(B \circ (f_1, f_2))(u) \cdot e &= DB(f_1(u), f_2(u)) \cdot (D(f_1 \times f_2)(u) \cdot e) \\ &= B(f_1(u), Df_2(u) \cdot e) + B(Df_1(u) \cdot e, f_2(u)) \end{aligned}$$

where we have used that $D(f_1 \times f_2)(u) \cdot e = (Df_1(u) \cdot e, Df_2(u) \cdot e)$

□

Example B.0.0.2. Suppose $F_1, F_2, G = \mathbb{R}$ and $B(u, v) = u \cdot v \in \mathbb{R}$. Then,

$$D(B(f_1, f_2))(u) \cdot e = f_2(u)Df_1(u) \cdot e + f_1(u)Df_2(u) \cdot e$$

which is the usual product rule.

Example B.0.0.3. A more interesting example for us is the following. Consider the following Sobolev spaces based on some Riemannian manifold: $E = H_s$, $F_1 = H_{s_1}$, $F_2 = H_{s_2}$ and $G = H_{s_3}$, with $s_1 + s_2 > s_3 + \frac{n}{2}$. Consider mappings between these spaces $f_1 : E \mapsto F_1$ and $f_2 : E \mapsto F_2$. Finally, suppose that the bilinear operator B is given by either the tensor product or some contraction between tensor fields. Then, the Sobolev multiplication property gives us that $B : F_1 \times F_2 \mapsto G$ is continuous. Thus, applying the theorem, we easily compute the Fréchet derivatives of either $f_1 \otimes f_2$ or $f_1 \cdot f_2$.

The results we have presented up to now, besides serving as computational tools that are needed in the main body of this thesis, also give an idea of how most of the results

known from ordinary calculus can be extended to this more general setting. We will now omit several very important results, such as the mean value inequality, higher derivatives and their properties, a version of Taylor's theorem in this context just to state a few. We would like to remark that these concepts are useful, not only in pure mathematics, where they are standard tools for analysing partial differential equations and the spaces of solutions, but also in physics, since variational techniques are formulated with this language. We refer the interested reader to [73] for an excellent review on these topics.

We will now jump to the notion of partial derivatives which are necessary to state the implicit function theorem, which is one of our main aims.

Cosider $X_1 \times \cdots \times X_m$ NLS's and $X = X_1 \times \cdots \times X_m$ equipped with the norm we have been using for this type of product space.

Definition B.0.0.3. Let $X = X_1 \times \cdots \times X_m$ as above. Let $U \subset X$ be open and $F : U \rightarrow Y$, Y a NLS. Let $(x_1, \dots, x_m) \in U$ and fix an integer $k \in [1, m]$. For z near x_k in X_k , the point $(x_1, \dots, x_{k-1}, z, x_{k+1}, \dots, x_m)$ lies in U , since U is open. Define

$$f_k(z) \doteq F(X_1, \dots, x_{k-1}, z, x_{k+1}, \dots, x_m)$$

Then f_k maps an open subset of X_k into Y . If f_k has a Fréchet derivative at $z = x_k$, then we say that F has a k th-partial derivative at x and we denote it by $D_k F(x) \doteq Df_k(x_k)$. In this case $D_k F(x) \in B(X_k, Y)$.

The following result is just a generalization of the already known result on $\mathbb{R}^n$.

Proposition B.0.0.4. Let $X = X_1 \times \cdots \times X_m$, $U \subset X$ open, and $F : U \rightarrow Y$, where all this are NLS's. Suppose $D_j F(x)$ exists for $x \in U$ and $1 \leq j \leq m$ and that these linear maps are continuous as functions of x at $x_0 \in U$. Then F is Fréchet differentiable at x_0 and for $h = (h_1, \dots, h_m) \in X$:

$$DF(x_0) \cdot h = \sum_{j=1}^m D_j F(x_0) \cdot h_j$$

□

B.0.1 Implicit function theorem

In order to arrive at the implicit function theorem, one uses the fix point theorem, which is used to prove the inverse function theorem, and then this last theorem allows us to prove the implicit function theorem. A detailed discussion on these proofs can be found in [73] and [97], where several independently interesting results are also shown.

Theorem B.0.1.1 (Implicit Function Theorem). *Let X, Y, Z be Banach and suppose $f : Z \times X \rightarrow Y$ to be a C^1 -map defined at least in a neighbourhood of a point (z_0, x_0) . Denote by $y_0 = f(z_0, x_0)$. Suppose $D_x f(z_0, x_0) \in GL(X, Y)$. Then there are open sets $W \subset Z$, $U \subset X$, $V \subset Y$, with $z_0 \in W$, $x_0 \in U$, $y_0 \in V$ and a unique mapping $g : W \times V \rightarrow U$ such that*

$$f(z, g(z, y)) = y \quad \forall (z, y) \in W \times V. \quad (\text{B.4})$$

Moreover, g is C^1 and, with $x = g(z, y)$,

$$Dg(z, y) \cdot (\eta, \xi) = (D_x f(z, x))^{-1}(\xi - D_z f(z, x) \cdot \eta) \quad \forall (z, y) \in W \times V \text{ and } (\eta, \xi) \in Z \times Y. \quad (\text{B.5})$$

Versions of this theorem with higher degrees for regularity can be found in [73] together with important corollaries of this theorem.

Appendix C

Linear elliptic systems

The idea for this appendix is to present the basic results concerning the theory of linear elliptic systems on compact manifolds which are used in chapters 4 and 5. In this sense, we will merely try to present these results in a manner that can be understood by someone who is not familiarized with the theory of partial differential equations. Thus, we will just present the necessary definitions, which, together with the material of the previous appendix, provide the language to understand claims made in the main theorems. We will not provide the proofs of those theorems, which can be found in the suggested references, since doing that would require going much further into functional analytical tools than what we have done.

The main reference that we will follow, both regarding the content and presentation of the results, is [3]. Also, the interested reader can find a detailed treatment on the topics we will cover in references such as [74],[100],[101] and [104]. Also, the following notes, available on line, provide the main results that are needed for this theses by means of pseudodifferential techniques [98].

C.1 Principal symbol of a differential operator

Let (M, g) be a smooth Riemannian manifold; $E \rightarrow M$, $F \rightarrow M$ be two tensor bundles over M and L a linear differential operator of order- m acting between sections of E and

F , *i.e.*,

$$L = \sum_{k=0}^m a_k \nabla^k \quad (\text{C.1})$$

where each a_k is a linear map from tensor fields to tensor fields, given also in terms of tensor fields over M . That is, if $E = T_s^r M$, $F = T_{s'}^{r'} M$ and $u \in E$, then

$$\begin{aligned} a_k : \Gamma(T_{s+k}^r M) &\rightarrow \Gamma(F) \\ \nabla^k u &\mapsto a_k \cdot \nabla^k u \end{aligned}$$

Notice that, in order to get a linear map from $E_p \mapsto F_p$, we can take the contraction of $a_k(p)$ with k covectors in $T_p^* M$. In particular, consider the top term (C.1), and the mapping we get by replacing ∇^m by $(\otimes \xi)^m$. The resulting mapping, denoted by $\sigma(L)$, is defined as the **principal symbol** of the operator L , and is a well-defined function on the cotangent bundle $T^* M$, taking values in the space of linear maps between E and F , *i.e.*, $\sigma(L) : T^* M \mapsto \text{Hom}(E, F)$, where $\text{Hom}(E, F)$ is the vector bundle over M whose fibre over a point $(x, \xi) \in T^* M$ is the space of linear transformation from E_x to F_x .

Example C.1.0.1. Let (M, g) be a Riemannian manifold and let $\Delta_g : \Gamma(TM) \mapsto \Gamma(TM)$ be the usual Laplacian operator, defined by $\Delta_g \doteq \text{tr}_g(\nabla_g^2)$. Then, in coordinates we have that

$$\Delta_g = g^{ij} \nabla_i \nabla_j.$$

Thus, we get that the principal symbol for the Laplacian is given by $\sigma(\Delta_g)(y, \xi) = |\xi|_g$.

Example C.1.0.2. Let (M, g) be a Riemannian manifold and let $\Delta_{g, \text{conf}} : \Gamma(TM) \mapsto \Gamma(TM)$ be the conformal Laplacian, which is defined to be the contravariant associate of the divergence of the conformal Killing operator $\mathcal{L}_X g - \frac{2}{n} \text{div}_g X g$. That is, in coordinates we have

$$\Delta_{g, \text{conf}} X^j \doteq g^{ju} g^{il} \nabla_l (\nabla_u X_i + \nabla_i X_u - \frac{2}{n} \nabla_k X^k g_{iu}).$$

It is a simple exercise to compute that

$$\Delta_{g, conf} X^j = \left\{ \delta_k^l g^{ju} + g^{ul} \delta_k^j - \frac{2}{n} g^{jl} \delta_k^u \right\} \nabla_l \nabla_u X^k.$$

Thus, we can directly compute the symbol:

$$\sigma(\Delta_{g, conf})_k^j(y, \xi) = \left\{ |\xi|_g^2 \delta_k^j + \frac{n-2}{n} \xi^j \xi_k \right\}.$$

It is worth to note that if $D_1 \in \text{Diff}^{l_1}(E_1, E_2)$ and $D_2 \in \text{Diff}^{l_2}(E_2, E_3)$ are differential operators of order l_1 and l_2 respectively, then $D_2 \circ D_1 \in \text{Diff}^{l_1+l_2}(E_1, E_3)$ and we have that

$$\sigma_{l_1+l_2}(D_2 \circ D_1)(p, \xi) = \sigma_{l_2}(D_2) \circ \sigma_{l_1}(D_1)(p, \xi). \quad (\text{C.2})$$

Also, we have that the symbol *respects adjoints*, i.e, if $D \in \text{Diff}^l(E, F)$, then

$$\sigma_l(D^*) = (\sigma_l(D))^*, \quad (\text{C.3})$$

where the right-hand side is the adjoint taken in $\text{Hom}(E, F)$.

Definition C.1.0.1. The linear operator (C.1) is said to be **elliptic** if its symbol $\sigma(L)(p, \xi)$ is invertible $\forall \xi \neq 0, \forall p \in M$.

Definition C.1.0.2. The ellipticity constant of the operator L is the infimum of the norms of the linear maps $\sigma(\xi)$. If a_m is continuous and bounded on (M, e) this ellipticity constant is a positive number given by:

$$\inf_{x \in M, \xi \in T_x^* M, X_x \in E_x} \frac{|(a_m(x)(\otimes \xi)^m) X_x|_e}{|\xi|_e^m |X_x|_e}$$

Considering the previous examples, it is clear that the Laplacian is elliptic. It is also easy to see that for the conformal Laplacian the following equality holds

$$\sigma(\Delta_{g, conf})_k^j X^k X_j = |\xi|_g |X|_g^2 + \frac{n-2}{2} (\xi_k X^k)^2.$$

Thus, $\forall \xi \neq 0$ and $n \geq 2$, $\sigma(\Delta_{g, conf})(p, \xi_p)$ defines a positive definite operator on $T_p M$ $\forall p \in M$, which shows that it is non-degenerate, thus invertible. Then, we see that the

conformal Laplacian is also elliptic.

In the next section, we will review the main results which are needed for the main content of this thesis. We will present them in the same way they appear in [3] and omit the proofs, which can be found therein. The results we are concerned with speak about the mapping properties of operators of the form of (C.1) acting between H_s Sobolev spaces. Such results (on compact manifolds) can be also reviewed in classical textbooks, such as [100]-[101]. Also, by means of pseudodifferential operator techniques, they can be obtained in an elegant manner [74],[98],[104]. Since we will only be concerned with second order systems, we will present the results for this type of systems, even though they can be presented in a more general setting.

C.2 Second Order Systems

Suppose M is a compact Manifold. Then it can be covered by a finite number of charts with domains being bounded open sets in $\mathbb{R}^n$. We endow it with a Riemannian metric e equivalent to the euclidean one in each coordinate chart.

Theorem C.2.0.1. *Let (M, e) be a smooth compact orientable Riemannian manifold. Let*

$$Lu = \sum_{i=0}^2 a_i \partial^i u$$

be a second order elliptic operator on (M, e) . Suppose the coefficients of L are such that

$$a_2 \in H_s, a_1 \in H_{s-1}, a_0 \in H_{s-2}, s > \frac{n}{2}, s \geq 2$$

Then $\forall 2 \leq \sigma \leq s$ it holds that:

1. *L is a continuous mapping from $H_\sigma \rightarrow H_{\sigma-2}$*

2.

$$\|u\|_{H_\sigma} \leq C_L \{\|Lu\|_{H_{\sigma-2}} + \|u\|_{H_{\sigma-1}}\} \forall u \in H_\sigma$$

with C_L a number depending only on the norms of the a_k 's and the ellipticity constant of a_2 .

After the following definition, we will present the main result we need concerning

elliptic linear systems.

Definition C.2.0.1. Given a linear differential operator L such as (C.1), its **formal adjoint**, denoted by L^* , is defined to be the linear operator $L^* : \Gamma(F) \mapsto \Gamma(E)$, satisfying the following identity

$$\int_M L\psi \cdot \phi \mu_e = \int_M \psi \cdot L^*\phi \mu_e.$$

Theorem C.2.0.2. *Under the hypothesis of theorem (C.2.0.1) it holds that:*

1. *The operator L maps H_s into H_{s-2} with finite dimensional kernel and closed range.*
2. *If L is injective on H_s then $\exists C_L > 0$ such that for each $u \in H_s$ the following inequality holds: $\|u\|_{H_s} \leq C_L \|Lu\|_{H_{s-2}}$*
3. *If the formal adjoint L^* of L satisfies the same hypothesis as L and is injective, then L is surjective from H_s onto H_{s-2} , and hence is an isomorphism if it is also injective.*

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