Universidade Federal da Paraíba
Centro de Ciências Exatas e da Natureza
Programa de Pós-Graduação em Matemática
Doutorado em Matemática

Asymptotic behavior of solutions for Klein-Gordon and thermoelastic plate systems

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João Pessoa - PB
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Tese apresentada ao Corpo Docente do Programa de Pós-Graduação em Matemática - UFPB, como requisito parcial para obtenção do título de Doutor em Matemática.

João Pessoa - PB

November, 2019

[†]Este trabalho contou com apoio financeiro da UEPB

Catalogação na publicação
Seção de Catalogação e Classificação

S586a Silva, Cládio Odair Pereira da.
Asymptotic behavior of solutions for Klein-Gordon and thermoelastic plate systems / Cládio Odair Pereira da Silva. - João Pessoa, 2019.
155 f. : il.

Orientação: Flank David Morais Bezerra.
Coorientação: Aldo Trajano Louredo, Manuel Antolino Milla Miranda.
Tese (Doutorado) - UFPB/CCEN.

1. global attractor. 2. thermoelastic plate systems. 3. concentrated term in the boundary. 4. Klein-Gordon system. 5. asymptotic behavior. 6. energy functional. I. Bezerra, Flank David Morais. II. Louredo, Aldo Trajano. III. Miranda, Manuel Antolino Milla. IV. Título.

UFPB/BC

Universidade Federal da Paraíba

Programa de Pós-Graduação em Matemática

Doutorado em Matemtica

Área de Concentração: Análise

Aprovada em: 29/11/2019

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November, 2019

Aos meus pais, à minha esposa Cláudia e
ao meu filho Pedro, com amor.

Acknowledgement (in Portuguese)

Eu gostaria de agradecer primeiramente a Deus, por minha vida, família e amigos e por ter me dado saúde e forças para superar as dificuldades.

Ao Departamento de Matemática da Universidade Federal da Paraíba e a Unidade Acadêmica de Matemática e Estatística da Universidade Federal de Campina Grande pelo apoio e pela minha formação.

À Universidade Estadual da Paraíba pelo apoio financeiro, em especial ao Centro de Ciências Humanas e Exatas pela confiança depositada em mim.

Aos professores doutores Flank David Morais Bezerra, Manuel Milla Miranda e Aldo Trajano Louredo, pela orientação e co-orientação e pela amizade, paciência, confiança e dedicação na construção deste trabalho.

Aos professores doutores Antônio Luiz Pereira, Gleiciane Silva Aragão e Jacson Simsen pela grande colaboração neste trabalho e pela participação na banca.

À minha esposa Cláudia e meu filho Pedro, pelo amor, companheirismo e compreensão do tempo ausente.

Aos meus pais, Agenildo e Damiana e meus irmãos, pelo carinho e apoio que apesar das dificuldades da vida, sempre me incentivaram na busca do conhecimento.

Às minhas avós, *in memorian*, Ana Alexandrina e Maria Nita e ao meu primo Francisco e seus pais, Miguel Jorge e Maria José que contribuíram bastante para que eu pudesse chegar até aqui.

Finalmente a todos que direta e indiretamente fizeram parte da minha formação, o meu muito obrigado!

"O dinheiro faz homens ricos, o conhecimento faz homens sábios e a humildade faz grandes homens."

Mahatma Gandhi

Abstract

In this work we study a Klein-Gordon system with mixed boundary conditions and a thermoelastic plate system with Neumann boundary conditions. In the first system we analyze the existence and uniqueness of global solution. Moreover, we show the exponential decay of energy associated to solution. In the second system we show the existence, uniform boundedness, and continuity of the global attractors when some reaction terms are concentrated in a neighborhood of the boundary and this neighborhood shrinks to boundary as a parameter goes to zero.

Keywords: global attractor; thermoelastic plate systems; concentrated term in the boundary; Klein-Gordon system; asymptotic behavior; energy functional.

Resumo

Neste trabalho estudamos um sistema de Klein-Gordon com condições de fronteira mista e um sistema termoelástico da placa com condições de fronteira de Neumann. No primeiro sistema, analisamos a existência e unicidade de solução global. Além disso, mostramos o decaimento exponencial da energia associada a solução. No segundo sistema mostramos a existência, limitação uniforme, e continuidade dos atratores globais quando alguns termos de reação estão concentrado em uma vizinhança da fronteira e essa vizinhança comprime para a fronteira quando um parâmetro vai para zero.

Palavras-chave: atrator global; sistema termoelástico da placa; termo concentrado na fronteira; sistema de Klein-Gordon; comportamento assintótico; funcional energia.

Contents

Introduction

This thesis is divided in two parts. In the first part we analyze the asymptotic behavior of solution for a Klein-Gordon system with mixed boundary conditions and with linear damping acting on part of the boundary, which is represented by

$$\begin{cases} \partial_t^2 u - \Delta u + |u|^\rho |v|^\rho v = 0 & \text{in} \quad \Omega \times (0,\infty) \\ \partial_t^2 v - \Delta v + |u|^\rho u |v|^\rho = 0 & \text{in} \quad \Omega \times (0,\infty) \\ u = v = 0 & \text{on} \quad \Gamma_0 \times (0,\infty) \\ \dfrac{\partial u}{\partial \vec{n}} + \delta(\cdot)\partial_t u = 0 & \text{on} \quad \Gamma_1 \times (0,\infty) \\ \dfrac{\partial v}{\partial \vec{n}} + \delta(\cdot)\partial_t v = 0 & \text{on} \quad \Gamma_1 \times (0,\infty) \\ u(0) = u^0, \; v(0) = v^0, \partial_t u(0) = u^1, \partial_t v(0) = v^1, \end{cases} \tag{1}$$

where Ω is an open bounded set of $\mathbb{R}^n$ with boundary $\Gamma = \partial\Omega$ of class C^2, Γ is constituted by two parts Γ_0 and Γ_1, both with positive measure and $\overline{\Gamma}_0 \cap \overline{\Gamma}_1$ empty, see Figure 1

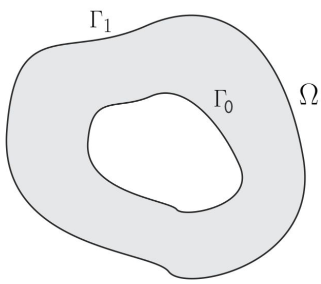

Figure 1: The set Ω.

and $\vec{n}(x)$ is represented the unit outward normal at $x \in \Gamma_1$, δ is a real function belong to $W^{1,\infty}(\Gamma_1)$ such that $\delta(x) \geqslant \delta_0 > 0$ on Γ_1 and ρ is a positive real number which depend of the spatial dimension of the space $\mathbb{R}^n$.

The system (1) is a generalization of the model proposed by Segal [39]

$$\begin{cases} \partial_t^2 u - \Delta u + \sigma^2 u + \phi v^2 u = 0, \\ \partial_t^2 v - \Delta v + \varrho^2 v + \tau u^2 v = 0, \end{cases}$$

which describes the interaction of two electromagnetic fields u and v with masses σ and ϱ, respectively, and with interaction constants $\phi > 0$ and $\tau > 0$.

Further generalizations of these problems are given in Medeiros and Milla Miranda [33], Medeiros and Milla Miranda [35], in this papers the authors have analyzed the existence and uniqueness of weak solutions of the mixed problem for a class of systems of nonlinear Klein-Gordon equations by using Galerkin methods and potential well method, respectively. The existence of solutions and decay of the energy of the problem for the coupled system of Klein-Gordon equations by using Galerkin methods is analyzed by Lourêdo and Milla Miranda [28].

Motivated by these papers we study the problem (1) using the ideas of Milla Miranda, Lourêdo and Medeiros [32] and Milla Miranda, Lourêdo and H. Clark [34]. More precisely, we analyze global existence, uniqueness and decay of solutions of the problem (1). Our approach in this problem be through of Faedo-Galerkin method, which consists of three steps. The first setp is the construction of the approximate problem in a finite dimensional space, the second step is to obtain a priori estimates to prolong the solutions of approximate problem, and finally, the third is the passage to the limit in the approximate solutions.

In order to establish the existence of global solution of the system (1), firstly we note that its energy which will be defined later, does not definite sign. Therefore the energy method to obtain global solution of (1) does not work. To overcome this serious difficulty we use a method introduced by Milla Miranda, Lourêdo and Medeiros [32], which was inspirated in one idea of Tartar [40]. This method simplifies the potential well one. We complement our approach by using the Faedo-Galerkin method with a special basis, due to the dissipative boundary conditions, and compactness argument. With this considerations, we obtain a global weak solution of (1) with restrictions on the norm of initial data and $\rho > 0$ which depends of the dimension of $\mathbb{R}^n$.

The uniqueness of solutions is derived by using the energy method. Thus if $\rho > 1$, we consider $n = 1, 2$ and if $\rho = 1$ we consider $n \leqslant 3$. This restriction on n is due to the fact that we need to differentiate with respect to t the difference of the nonlinear parts in order to apply the mean value theorem.

To obtain the decay of the energy of problem (1), we consider the same restrictions of the uniqueness of solutions and make $\delta(x) = m(x) \cdot \vec{n}(x)$, where $m(x) = x - x^0$, $x, x^0 \in \mathbb{R}^n$. In this conditions, by using the multiplier method and the ideas contained in Komornik and Zuazua [25] and Komornik [24], we obtain the exponential decay of the energy.

In the second part of this thesis, we analyze the asymptotic behavior of an autonomous thermoelastic plate systems with Neumann boundary conditions when some reaction terms are concentrated in a neighborhood of the boundary, and this neighborhood shrinks to boundary as a parameter ε goes to zero, which is represented by

$$\begin{cases} \partial_t^2 u^\varepsilon + \Delta^2 u^\varepsilon + u^\varepsilon + \Delta\theta^\varepsilon - \theta^\varepsilon = f(u^\varepsilon) + \dfrac{1}{\varepsilon}\chi_{\omega_\varepsilon} g(u^\varepsilon) & \text{in} \quad \Omega \times (0,\infty), \\ \partial_t\theta^\varepsilon - \Delta\theta^\varepsilon + \theta^\varepsilon - \Delta\partial_t u^\varepsilon + \partial_t u^\varepsilon = 0 & \text{in} \quad \Omega \times (0,\infty), \\ \dfrac{\partial u^\varepsilon}{\partial \vec{n}} = 0, \quad \dfrac{\partial(\Delta u^\varepsilon)}{\partial \vec{n}} = 0, \quad \dfrac{\partial \theta^\varepsilon}{\partial \vec{n}} = 0 & \text{on} \quad \Gamma \times (0,\infty), \\ u^\varepsilon(0) = u_0 \in H^2(\Omega), \quad \partial_t u^\varepsilon(0) = v_0 \in L^2(\Omega), \quad \theta^\varepsilon(0) = \theta_0 \in L^2(\Omega), \end{cases} \tag{2}$$

where Ω is a bounded and smooth open set of $\mathbb{R}^n$, $n \geqslant 2$, with boundary $\Gamma = \partial\Omega$ smooth, ω_ε, $0 < \varepsilon \leqslant \varepsilon_0$, is a neighborhood of Γ, see Figure 2,

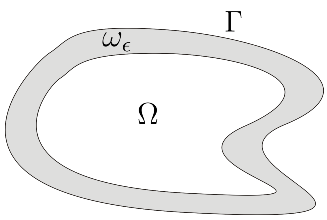

Figure 2: The set $\omega_\varepsilon \subset \overline{\Omega}$.

$\chi_{\omega_\varepsilon}$ is the characteristic function of set ω_ε, $0 < \varepsilon \leqslant \varepsilon_0$, and $f, g : \mathbb{R} \to \mathbb{R}$ are nonlinearities under suitable growth conditions.

The above system represents a certain plate subject to small vibrations in which $u(x,t)$ denotes the displacement of wave at point x at time t and $\theta(x,t)$ denotes the temperature at point x at time t, which the plate is subjected.

The first work to consider this technique of concentrating terms in a neighborhood of boundary was done by Arrieta, Jiménez-Casas and Rodríguez-Bernal [10]. In this work they analyzed the limit of the solutions of an elliptic problem when some reaction and potential terms are concentrated in a neighborhood of certain partition of the boundary and this neighborhood shrinks to this partition as a parameter ε goes to zero. They proved that these solutions converge, in certain Sobolev spaces, to the solution of an elliptic problem, where the reaction term and the concentrating potential are transformed into a flux condition and a potential in the partition. This convergence result can be seen as a tool to transfer information from interior of the domain to its boundary. After this same technique was used by Jiménez-Casas and Rodríguez-Bernal [22], [23] for parabolic problems. They have analyzed the asymptotic behavior of the attractors of a parabolic problem, more precisely, they have proved the existence of a family of attractors and that this family is upper semicontinuous at $\varepsilon = 0$. With this same technique of concentrating terms in the boundary we can still mention some papers, for instance, Aragão and Oliva [6], [7], Aragão and Pereira [8], [9] and Jiménez-Casas and Rodríguez-Bernal [21]. In Aragão and Bezerra [2] was analyzed the asymptotic behavior of the pullback attractors of a non-autonomous damped wave equation with terms concentrating on the boundary, that is, has been proved a regularity result of the pullback attractors and that the family of these attractors is upper semicontinuous at $\varepsilon = 0$ and in Aragão and Bezerra [3] was shown the continuity of the set of equilibria of the same equation considered in Aragão and Bezerra [2].

Motivated by Aragão and Bezerra [2], [3] and using results of Arrieta, Jiménez-Casas and Rodríguez-Bernal [10] and Jiménez-Casas and Rodríguez-Bernal [23] we study the asymptotic behavior of the problem (2) in the sense of global attractors.

Note that in (2) the nonlinear term $g(u^\varepsilon)$ is only effective on the region ω_ε which collapses to Γ as $\varepsilon \to 0$, then we show that the limit problem for the autonomous thermoelastic plate system (2) is given by

$$\begin{cases} \partial_t^2 u + \Delta^2 u + u + \Delta\theta - \theta = f(u) & \text{in} \quad \Omega \times (\tau, \infty), \\ \partial_t \theta - \Delta\theta + \theta - \Delta\partial_t u + \partial_t u = 0 & \text{in} \quad \Omega \times (0, \infty), \\ \dfrac{\partial u}{\partial \vec{n}} = 0, \quad \dfrac{\partial(\Delta u)}{\partial \vec{n}} = -g(u), \quad \dfrac{\partial \theta}{\partial \vec{n}} = 0 & \text{on} \quad \Gamma \times (0, \infty), \\ u(0) = u_0 \in H^2(\Omega), \quad \partial_t u(0) = v_0 \in L^2(\Omega), \quad \theta(0) = \theta_0 \in L^2(\Omega). \end{cases} \tag{3}$$

In other words, we prove that the nonlinear semigroup associated to (2) converges to the nonlinear semigroup associated to (3). Moreover, we show the existence, uniform boundedness and continuity of the global attractors at $\varepsilon = 0$ associated to these semigroups.

Firstly we write the problems (2) and (3) in the abstract forms and to analyze the local and global well-posedness of this problems, we use strongly continuous semigroup theory; namely, we rewrite the problems as abstract Cauchy problems and we show that the linear part of the problem is a parabolic problem to according Henry [20]. We also analyze the behavior of nonlinearity F_ε, related to Lipschitz and differentiable conditions, that allows us to use the classic results of the theory of ordinary differential equations in Banach spaces to ensure the local well-posedness of the parabolic problems associated to (2) and (3). After we make use of the functional energy associated to (2) and (3) with the same results above, for show that the parabolic problems are global well-posedness and that the semigroups associated the this parabolic problems, which are given by a variation of constants formula, are dissipative. On the other hand, to ensure the existence of attractor of the abstract problems associated to (2) and (3), first we observe that, due to the functional energy, the systems are gradient and asymptotically smooth, then using a result of Hale [19, Theorem 3.8.5, p.51], we show that the problems has attractors and that this attractors are characterized by manifold unstable of the set of equilibria of nonlinear semigroup generated by parabolic problems.

Finally, we show the continuity of the attractors at $\varepsilon = 0$; first, we prove upper semicontinuity, that follows as a consequence of its uniform bounds and of the convergence result of the nonlinear semigrups. After, we prove lower semicontinuity, in this case was need to show the continuity of the set of equilibria associated to abstract problems associated to (2) and (3) and also we have to show the continuity of local unstable manifolds around these equilibria. With this and using the results due to Henry [20, Chapter 6] we obtain the lower semicontinuity of these attractors at $\varepsilon = 0$.

This thesis is composed of three chapters and three appendices. In the Chapter 1 we analyze global existence, uniqueness and decay of solutions of problem (1). This chapter resulted in the following paper:

• C. O. P. Da Silva, A. T. Louredo and M. Milla Miranda, Existence and asymptotic behavior of solutions for a Klein-Gordon system, see [17].

In the Chapter 2 we show existence of global attractors for nonlinear semigroups associated to the problems (2) and (3). Moreover, we show the continuity of these attractors at $\varepsilon = 0$, in the sense of Hausdorff distance. This chapter resulted in two papers:

• G. S. Aragão, F. D. M. Bezerra and C. O. P. Da Silva, Dynamics of thermoelastic plate system with terms concentrated in the boundary, Differential Equations and Applications, **11**, 3 (2019), 379 – 407, see [4].

• G. S. Aragão, F. D. M. Bezerra and C. O. P. Da Silva, Dynamics of thermoelastic plate system with terms concentrated in the boundary: the lower semicontinuity of the global attractors, see [5].

In the Chapter 3 we present final considerations and conclusions on the Chapter 1 and Chapter 2.

Finally, in the Appendix A we present concepts and results related to the theory of partial differential equations. In the Appendices B and C we present concepts and results related to the theory of linear and nonlinear semigroups and global attractors.

Notations

General

- $|\Omega|$ measure of Lesbegue of $\Omega \subset \mathbb{R}^n$;
- $p' = \dfrac{p}{p-1}$ exponent conjugate of p;
- $\operatorname{supp}(u) = \overline{\{x \in \Omega \,;\, u(x) \neq 0\}}$;
- $\hookrightarrow$ embedding continuous;
- $\overset{c}{\hookrightarrow}$ embedding compact;
- $\{e^{\mathbb{A}t} : t \geqslant 0\}$ linear semigroup generated by operator $\mathbb{A}$;
- $\{S(t) : t \geqslant 0\}$ nonlinear semigroup;
- $\rho(A)$ resolvent set of operator A;
- $\sigma(A)$ spectrum of operator A.

Spaces of functions

- $C(\Omega) = \{u : \Omega \to \mathbb{R} \,;\; u \text{ is continuous}\}$;
- $C^k(\Omega) = \{u : \Omega \to \mathbb{R} \,;\; u \text{ is } k\text{-times continuously differentiable}\}$;
- $C^\infty(\Omega) = \{u : \Omega \to \mathbb{R} \,;\; u \text{ is infinitely differentiable}\}$;
- $C_0^k(\Omega) = \{u \in C^k(\Omega) \,;\; \operatorname{supp} u \subset \Omega \text{ is compact}\}, \quad k \in \mathbb{N} \text{ or } k = \infty$;
- $C^{k,\alpha}(\Omega) = \{u \in C^k(\Omega) \,;\; D^\alpha u \text{ is } \alpha - \text{Hölder continuous}\}$;

- $\mathscr{D}(\Omega)$ space of test functions;

- $L^p_{loc}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ mensurable; } u \in L^p(K),\ \forall\ K \subset \Omega \text{ compact}\}$;

- $W^{m,p}(\Omega) = \{u \in L^p(\Omega)\ ;\ D^\alpha u \in L^p(\Omega),\ 0 \leqslant |\alpha| \leqslant m\}$;

- $W^{m,p}_0(\Omega) = \overline{C^\infty_0(\Omega)}^{W^{m,p}(\Omega)}$;

- $C([0,T]; X) = \{u : [0,T] \to X\ ;\ u(t) \text{ is continuous}\}$;

- $C^k([0,T]; X) = \{u : [0,T] \to X\ ;\ u(t) \text{ is } k\text{-times continuously differentiable}\}$;

- $L^p(0,T;X) = \{u : (0,T) \to X \text{ mensurable ; } \int_0^T \|u(t)\|^p_X dt < \infty\}$;

- $L^\infty(0,T;X) = \{u : (0,T) \to X \text{ mensurable ; } ess\sup_{t\in(0,T)} \|u(t)\|_X < \infty\}$;

- $L^p_{loc}(0,T;X) = \{u : (0,T) \to X \text{ mensurable ; } \|u(s)\|_X \in L^p(I), I \subset (0,T) \text{ compact}\}$;

- $\mathscr{L}(X,Y) = \{T : X \to Y\ ;\ T \text{ is linear continuous}\}$;

- $X' = \mathscr{L}(X,\mathbb{R})$ dual space of X;

- $\mathscr{D}'(\Omega) = \mathscr{L}(\mathscr{D}(\Omega),\mathbb{R})$;

- $\mathscr{D}'(0,T;X) = \mathscr{L}(\mathscr{D}(0,T);X)$;

- $H^{s,p}(\Omega)$, $s \in \mathbb{R}$ and $1 \leqslant p < \infty$, Bessel Potentials spaces.

Norms

- $\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|^p_X dt\right)^{\frac{1}{p}}$;

- $\|u\|_{L^\infty(0,T;X)} = ess \sup\limits_{t\in(0,T)} \|u(t)\|_X$;

- $\|u\|_{C^k([0,T];X)} = \sum_{i=0}^k \max\limits_{t\in[0,T]} \left\|\dfrac{d^i u(t)}{dt^i}\right\|_X$;

- $\|f\|_{X'} = \sup\limits_{x\in X, \|x\|_X \leqslant 1} |\langle f, x\rangle|$;

- $\|T\|_{\mathscr{L}(X,Y)} = \sup\limits_{x\in X, x\neq 0} \dfrac{\|Tx\|_Y}{\|x\|_X} = \sup\limits_{x\in X, \|x\|_X = 1} \|Tx\|_Y$.

Convergences

- $\to$ convergence strong;
- $\rightharpoonup$ convergence weak;
- $\overset{*}{\rightharpoonup}$ convergence weak star.

Chapter 1

Asymptotic behavior of solutions for a Klein-Gordon system

In this chapter we present results on existence and uniqueness of global solution for the system (1) and we analyze the asymptotic behavior of this solution. In the first section we treat the preliminary part. In the second section we analyze the existence of a global solution presenting some results according to the values of the real number $\rho > 0$ and spatial dimension n. In the third section concerns the uniqueness of solution. We show the uniqueness when $\rho = 1$ and $n \leqslant 3$, $\rho > 1$ and $n = 1, 2$. Finally in the fourth section we analyze asymptotic behavior with the same restrictions on ρ and n, in the case of uniqueness.

1.1 Preliminary

In this section we introduce some notations and also show results related to separability, density and trace theory that will be important throughout this chapter.

1.1.1 Separability

In this chapter the inner product and norm of $L^2(\Omega)$ are represented, respectively, by $(\cdot, \cdot)$ and $\|\cdot\|_{L^2(\Omega)}$. Denote by V the Hilbert space

$$V = \{u \in H^1(\Omega) \ ; \ u = 0 \quad \text{on} \quad \Gamma_0\}$$

equipped with the inner product and norm, respectively,

$$((u,v)) = \sum_{i=1}^{n} \int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx \quad \text{and} \quad \|u\|_V^2 = \sum_{i=1}^{n} \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^2 dx.$$

Let θ be a real number with $1 \leqslant \theta < 2$ such that $\dfrac{1}{\theta} + \dfrac{1}{\theta'} = 1$. We consider the following Banach spaces equipped with respective norms

$$W^{1,\theta'}(\Omega), \quad \|u\|_{W^{1,\theta'}(\Omega)} = \left(\int_{\Omega} |u(x)|^{\theta'} dx + \sum_{i=1}^{n} \int_{\Omega} \left| \frac{\partial u(x)}{\partial x_i} \right|^{\theta'} dx \right)^{\frac{1}{\theta'}}$$

and

$$W_{\Gamma_0}^{1,\theta'}(\Omega) = \{u \in W^{1,\theta'}(\Omega); u = 0 \text{ on } \Gamma_0\}; \quad \|u\|_{W_{\Gamma_0}^{1,\theta'}(\Omega)} = \left(\sum_{i=1}^{n} \int_{\Omega} \left| \frac{\partial u(x)}{\partial x_i} \right|^{\theta'} dx \right)^{\frac{1}{\theta'}}.$$

We also consider the Banach space

$$\mathscr{X} = \{u \in V; \Delta u \in L^{\theta}(\Omega)\}$$

with the norm

$$\|u\|_{\mathscr{X}} = \|u\|_V + \|\Delta u\|_{L^{\theta}(\Omega)}.$$

Now consider X, Y and W be Banach spaces such that $W \hookrightarrow X$ and $W \hookrightarrow Y$. Let Z be a topological vector space that separates points, such that $X \hookrightarrow Z$ and $Y \hookrightarrow Z$. Then the space $E = X \cap Y$ provided with the norm

$$\|u\|_E = \|u\|_X + \|u\|_Y$$

is a Banach space.

Proposition 1.1.1 *If W is dense in X and dense in Y then W is dense in E.*

Proof. Consider $T \in E'$ such that

$$\langle T, w \rangle_{E' \times E} = 0, \quad \forall w \in W.$$

Note that W has the same topology considered as a subspace of $X \cap Y$ or as a subspace of $X \times Y$. So T is continuous on W with the topology of $X \cap Y$. Then by the Hahn-Banach Theorem there exist $R \in X'$ and $S \in Y'$ such that

$$\langle T, w \rangle_{E' \times E} = \langle R, w \rangle_{X' \times X} + \langle S, w \rangle_{Y' \times Y}, \quad \forall w \in W. \tag{1.1}$$

Observe that $X' \hookrightarrow W'$ and $Y' \hookrightarrow W'$. Then

$$\begin{aligned} \langle R, w\rangle_{X'\times X} &= \langle R, w\rangle_{W'\times W} \\ \langle S, w\rangle_{Y'\times Y} &= \langle S, w\rangle_{W'\times W}. \end{aligned} \tag{1.2}$$

Thanks to (1.1) and (1.2), we obtain

$$\langle R + S, w\rangle_{W'\times W} = \langle T, w\rangle_{E'\times E} = 0, \quad \forall w \in W. \tag{1.3}$$

By the density of W in X and in Y, using (1.3) and Brezis [11, Corollary 1.8, p. 8] implies

$$R + S = 0 \text{ on } X \quad \text{and} \quad R + S = 0 \text{ on } Y.$$

Therefore $T = R + S = 0 \quad \text{on} \quad E$. Again using Brezis [11, Corollary 1.8, p. 8], we conclude that W is dense in E. ■

Proposition 1.1.2 *Assume the hypotheses of Proposition 1.1.1. Then if W is separable we have that E is separable.*

Proof. Let $\{w_1, w_2, \ldots\}$ be a basis of W. Consider $u \in E$. Then by Proposition 1.1.1, there exists $\varphi \in W$ such that

$$\|u - \varphi\|_E < \frac{\epsilon}{2}.$$

Also there exists $\sum_{j=1}^{n} \alpha_j w_j$ such that

$$\left\| \varphi - \sum_{j=1}^{n} \alpha_j w_j \right\|_W < \frac{\epsilon}{2c}.$$

Thus

$$\left\| u - \sum_{j=1}^{n} \alpha_j w_j \right\|_E \leqslant \|u-\varphi\|_E + \left\| \varphi - \sum_{j=1}^{n} \alpha_j w_j \right\|_E \leqslant \|u-\varphi\|_E + c\left\| \varphi - \sum_{j=1}^{n} \alpha_j w_j \right\|_W < \epsilon.$$

■

1.1.2 Density of $\mathscr{D}(\overline{\Omega})$ in $\mathscr{X}$

In what follows we show that $\mathscr{D}(\overline{\Omega})$ is dense in $\mathscr{X}$. For this, let $\mathscr{O}$ be a star-shaped subset of $\mathbb{R}^n$ with respect to $0 \in \mathbb{R}^n$. Consider the linear homotetic transformation $\sigma_\eta(x) = \eta x,\ \eta > 0$. Note that for $\eta > 1$,

$$\mathscr{O} \subset \overline{\mathscr{O}} \subset \sigma_\eta(\mathscr{O}). \tag{1.4}$$

Consider a function $w : \mathscr{O} \to \mathbb{R}$ defined in $\mathscr{O}$. For $\eta > 0$ introduce the function

$$\sigma_\eta \circ w : \sigma_\eta(\mathscr{O}) \to \mathbb{R}, \quad y \longmapsto (\sigma_\eta \circ w)(y) = w(\sigma_{\frac{1}{\eta}}(y)).$$

Note that when $\eta > 1$, the domain of the function $\sigma_\eta \circ w$ contain the domain of w (see (1.4))

Proposition 1.1.3 *Let $S \in \mathscr{D}'(\mathscr{O})$. Then*

1) *$\sigma_\eta \circ S$ defined by*

$$\langle \sigma_\eta \circ S, \xi \rangle = \frac{1}{\eta^n} \langle S, \sigma_\eta \circ \xi \rangle, \quad \xi \in \mathscr{D}(\sigma_\eta(\mathscr{O})),$$

belongs to $\mathscr{D}'(\sigma_\eta(\mathscr{O}))$, $(\eta > 0)$.

2) $\dfrac{\partial}{\partial y_i}(\sigma_\eta \circ S) = \eta \sigma_\eta \circ \left(\dfrac{\partial}{\partial y_i} S \right)$, $(\eta > 0)$.

3) *If $\eta > 1$, $\eta \to 1$, the restriction to $\mathscr{O}$ of $\sigma_\eta \circ S$ convergs to S in the distribution sense.*

4) *If $v \in L^p(\mathscr{O})$, $(1 \leqslant p < \infty)$, $\sigma_\eta \circ v \in L^p(\sigma_\eta(\mathscr{O}))$, $(\eta > 0)$. For $\eta > 1$, $\eta \to 1$, the restriction to $\mathscr{O}$ of $\sigma_\eta \circ v$ convergs to v in $L^p((\mathscr{O}))$.*

Proof. The proof can be found in Temam [42, Lemma 1.1, p. 7]. ∎

We have the following results:

Theorem 1.1.4 *The space $\mathscr{D}(\overline{\Omega})$ is dense in $\mathscr{X}$.*

Proof. Let U be an open set of $\mathbb{R}^n$ with boundary ∂U of class C^2. Introduce the Banach space

$$\mathscr{X}(U) = \{u \in V(U); \Delta u \in L^\theta(U)\}$$

equipped with the norm

$$\|u\|_{\mathscr{X}(U)} = \|u\|_{V(U)} + \|\Delta u\|_{L^\theta(U)}.$$

We divide the proof in four parts.

First part. By truncation and regularization we prove that $\mathscr{D}(\mathbb{R}^n)$ is dense in $\mathscr{X}(\mathbb{R}^n)$. For more details see Medeiros and Milla Miranda [30, Theorem 1.1, p. 8].

Second part. Let $(U_k)_{1\leqslant k\leqslant m}$ be an open covering of Γ_0 and Γ_1 with $U_k^+ = \Omega \cap U_k$ star-shaped with respect to one of its points, $k = 1, \ldots, m$. Let $(\varphi_k)_{0\leqslant k\leqslant m}$ be a $C^\infty-$partition of unity subordinate to the open covering $\Omega, (U_k)_{1\leqslant k\leqslant m}$ of $\overline{\Omega}$. Thus

$$\varphi_0(x) + \sum_{k=1}^{m} \varphi_k(x) = 1, \quad \forall x \in \overline{\Omega},\ \varphi_0 \in \mathscr{D}(\Omega),\ \varphi_k \in \mathscr{D}(U_k), k = 1, \ldots, m.$$

Consider $u \in \mathscr{X}$. Then

$$u = \varphi_0(x)u + \sum_{k=1}^{m} \varphi_k(x)u. \tag{1.5}$$

We use the notations

$$v_k = \varphi_k(x)u, \quad k = 0, 1, \ldots, m.$$

We analyze v_0. Represent by U_0 an open set of $\mathbb{R}^n$ such that $(\operatorname{supp}\varphi_0)\cap\Omega$ is contained in U_0. After translation, we can choose U_0 such that U_0 is star-shaped with respect to $0 \in \mathbb{R}^n$. Define $\sigma_\eta \circ v_0,\ \eta > 1$. Then by (1.4) and Proposition 1.1.3, first part, we have that $\sigma_\eta \circ v_0$ is defined in $\sigma_\eta(U_0)$. Consider

$$\psi \in \mathscr{D}(\sigma_\eta(U_0)) \text{ such that } \psi \equiv 1 \text{ on } U_0, \quad \text{and} \quad w_{0\eta} = \psi[\sigma_\eta \circ v_0],\ \eta > 1.$$

Then $\operatorname{supp}(w_{0\eta})$ is contained in $\sigma_\eta(U_0)$. By Proposition 1.1.3, item 2), we obtain

$$\frac{\partial w_{0\eta}}{\partial x_i} = \frac{\partial \psi}{\partial x_i}[\sigma_\eta \circ v_0] + \eta\psi\left(\sigma_\eta \circ \frac{\partial v_0}{\partial x_i}\right), \tag{1.6}$$

$$\Delta w_{0\eta} = \eta^2\psi[\sigma_\eta \circ \Delta v_0] + \Delta\psi[\sigma_\eta \circ \Delta v_0] + 2\eta\sum_{i=n}^{n} \frac{\partial \psi}{\partial x_i}\left[\sigma_\eta \circ \frac{\partial v_0}{\partial x_i}\right]. \tag{1.7}$$

By the preceding equalities, we obtain that $w_{0\eta} \in \mathscr{X}(\sigma_\eta(U_0))$. Consider $\tilde{w}_{0\eta}$ the extension of $w_{0\eta}$, that is,

$$\tilde{w}_{0\eta} = \begin{cases} w_{0\eta} & \text{in} \quad \sigma_\eta(U_0); \\ 0 & \text{in} \quad \mathbb{R}^n/\sigma_\eta(U_0). \end{cases}$$

Then $\tilde{w}_{0\eta} \in \mathscr{X}(\mathbb{R}^n)$. By the first part we have $\tilde{w}_{0\eta}$ can be approximated in $\mathscr{X}(\mathbb{R}^n)$ by functions of $\mathscr{D}(\mathbb{R}^n)$. Consequently

$$w_{0\eta} \text{ can be approximated in } \mathscr{X}(\sigma_\eta(U_0)) \text{ by functions of } \mathscr{D}(\overline{\sigma_\eta(U_0)}). \tag{1.8}$$

By (1.6) and (1.7) we have

$$w_{0\eta}|_{U_0} = [\sigma_\eta \circ v_0]|_{U_0}, \quad \frac{\partial w_{0\eta}}{\partial x_i}\Big|_{U_0} = \eta\left[\sigma_\eta \circ \frac{\partial v_0}{\partial x_i}\right]\Big|_{U_0}, \quad \Delta w_{0\eta}|_{U_0} = \eta^2[\sigma_\eta \circ \Delta v_0]|_{U_0}.$$

Then by Proposition 1.1.3, items 3) and 4), we obtain

$$
\begin{aligned}
& w_{0\eta}|_{U_0} \to v_0 \quad \text{in} \quad L^2(U_0) \quad \text{as} \quad \eta \to 1; \\
& \frac{\partial w_{0\eta}}{\partial x_i}\Big|_{U_0} \to \frac{\partial v_0}{\partial x_i} \quad \text{in} \quad L^2(U_0) \quad \text{as} \quad \eta \to 1; \\
& \Delta w_{0\eta}|_{U_0} \to \Delta v_0 \quad \text{in} \quad L^\theta(U_0) \quad \text{as} \quad \eta \to 1.
\end{aligned}
$$

By (1.8) and the last three convergences we conclude that v_0 can be approximated in $\mathscr{X}(U_0)$ by functions of $\mathscr{D}(\overline{U}_0)$.

Third part. Analyze $v_k,\ k = 1, \ldots, m$. In this case we apply similar arguments to those used in the case v_0. Thus we take U_k^+ instead U_0. We can assume that U_k^+ is star-shaped with respect to $0 \in \mathbb{R}^n$. Consider $\sigma_\eta(U_k^+)$ instead $\sigma_\eta(U_0)$. Introduce

$$\psi \in \mathscr{D}(\sigma_\eta(U_k^+)) \quad \text{with} \quad \psi \equiv 1 \quad \text{on} \quad U_k^+.$$

Consider $w_{k\eta} = \psi[\sigma_\eta \circ v_k],\ \eta > 1$. Then

$$w_{k\eta} \in \mathscr{X}(\sigma_\eta(U_k^+)); \quad \text{supp}(w_{k\eta}) \subset \mathscr{X}(\sigma_\eta(U_k^+)); \quad \tilde{w}_{k\eta} \in \mathscr{X}(\mathbb{R}^n)$$

and

$$w_{k\eta}|_{U_k^+} \to v_k \quad \text{in} \quad \mathscr{X}(\sigma_\eta(U_k^+)) \quad \text{as} \quad \eta \to 1.$$

Thus v_k can be approximated in $\mathscr{X}(U_k^+)$ by functions of $\mathscr{D}(\overline{U_k^+})$.

By (1.5) and the above results we conclude that $u \in \mathscr{X}$ can be approximated in $\mathscr{X}(U)$ by functions of $\mathscr{D}(\overline{U})$.

Fourth part. The theorem follow since that $\mathscr{X}(U)$ and $\mathscr{X}$ has equivalent norms in $\mathscr{X}$. ■

1.1.3 A trace theorem

It is known by trace theorem that there exists a linear continuous and sobrejective map

$$\gamma_0 : W^{1,\theta'}(\Omega) \to W^{\frac{1}{\theta},\theta'}(\Gamma), \quad \gamma_0 u = u|_\Gamma$$

and has inverse continuous,

$$W^{\frac{1}{\theta},\theta'}(\Gamma) \to W^{1,\theta'}(\Omega), \quad \xi \longmapsto u$$

In particular, we have

$$\gamma_0 : W^{1,\theta'}_{\Gamma_0}(\Omega) \to W^{\frac{1}{\theta},\theta'}(\Gamma_1), \quad \gamma_0 u = u|_{\Gamma_1}$$

and

$$W^{\frac{1}{\theta},\theta'}(\Gamma_1) \to W^{1,\theta'}_{\Gamma_0}(\Omega), \quad \xi \longmapsto u,$$

are continuous. For more details see Nečas [37, Theorem 5.5, p. 95].

We want to prove a similar result for the functions in $\mathscr{X}$. We have the following trace theorem, which means that we can define $\dfrac{\partial u}{\partial \vec{n}}$ on Γ_1 when $u \in \mathscr{X}$.

Theorem 1.1.5 *There exists a linear continuous map*

$$\mathscr{X} \to W^{-\frac{1}{\theta},\theta}(\Gamma_1), \quad u \longmapsto \gamma_1 u = \frac{\partial u}{\partial \vec{n}}$$

such that

$$\langle \gamma_1 u, \gamma_0 z \rangle_{W^{-\frac{1}{\theta},\theta}(\Gamma_1) \times W^{\frac{1}{\theta},\theta'}(\Gamma_1)} = \langle \Delta u, z \rangle_{L^\theta(\Omega) \times L^{\theta'}(\Omega)} + \sum_{i=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial z}{\partial x_i} dx, \tag{1.9}$$

for all $z \in W^{1,\theta'}_{\Gamma_0}(\Omega)$.

Proof. Note that (1.9) is well defined and hold for $u \in \mathscr{D}(\overline{\Omega})$. In fact, let $u \in \mathscr{D}(\overline{\Omega})$, using $W^{1,\theta'}_{\Gamma_0}(\Omega) \hookrightarrow V$ and $W^{1,\theta'}_{\Gamma_0}(\Omega) \hookrightarrow L^{\theta'}(\Omega)$, we have

$$\begin{aligned}
\left| \int_{\Gamma_1} (\gamma_1 u)(\gamma_0 z) d\Gamma \right| &\leqslant \|u\|_V \|z\|_V + \|\Delta u\|_{L^\theta(\Omega)} \|z\|_{L^{\theta'}(\Omega)} \\
&\leqslant c\|u\|_V \|z\|_{W^{1,\theta'}_{\Gamma_0}(\Omega)} + c\|\Delta u\|_{L^\theta(\Omega)} \|z\|_{W^{1,\theta'}_{\Gamma_0}(\Omega)} \\
&\leqslant c\|u\|_{\mathscr{X}} \|z\|_{W^{1,\theta'}_{\Gamma_0}(\Omega)},
\end{aligned}$$

for some positive constant c.

Let $\xi \in W^{\frac{1}{\theta},\theta'}(\Gamma_1)$. Then by trace theorem there exists $z \in W^{1,\theta'}_{\Gamma_0}(\Omega)$ such that $\xi = \gamma_0 z$ and

$$\|z\|_{W^{1,\theta'}_{\Gamma_0}(\Omega)} \leqslant c\|\xi\|_{W^{\frac{1}{\theta},\theta'}(\Gamma_1)}.$$

Thus

$$\left| \int_{\Gamma_1} (\gamma_1 u) \xi d\Gamma \right| \leqslant c\|u\|_{\mathscr{X}} \|\xi\|_{W^{\frac{1}{\theta},\theta'}(\Gamma_1)},$$

that is,

$$\gamma_1 u \in (W^{\frac{1}{\theta},\theta'}(\Gamma_1))' = W^{-\frac{1}{\theta},\theta}(\Gamma_1)$$

and

$$\|\gamma_1 u\|_{W^{-\frac{1}{\theta},\theta}(\Gamma_1)} \leqslant c\|u\|_{\mathscr{X}}, \quad \forall u \in \mathscr{D}(\overline{\Omega}).$$

Now, the results follows by density. ∎

1.2 Existence of global solutions

This section concerns the existence of global solution for the problem (1) with linear damping at the boundary.

Introduce the hypothesis

$$\rho > 0 \text{ and } \theta > 1 \text{ with } 4\rho\theta \geqslant 1, \quad \text{if} \quad n = 1, 2; \tag{1.10}$$

$$\frac{n+2}{8n} \leqslant \rho \leqslant \frac{n+2}{4(n-2)}, \quad \text{if} \quad 3 \leqslant n \leqslant 6. \tag{1.11}$$

Remark 1.2.1 (i) *Note that for* $n \geqslant 3$ *we have* $0 < \rho < \dfrac{2}{n-2}$, *then* $1 < 2(\rho+1) \leqslant q$, *therefore the following embeddings of Sobolev*

$$V \hookrightarrow L^q(\Omega) \hookrightarrow L^{2(\rho+1)}(\Omega),$$

are holds, where $q = \dfrac{2n}{n-2}$. *In particular, for* $\rho = 1$ *we have* $V \hookrightarrow L^4(\Omega)$. *Thus, there exist positive constants* c_0 *and* c_1 *such that*

$$\|v\|_{L^{2(\rho+1)}(\Omega)} \leqslant c_0 \|v\|_V, \quad \textit{and} \quad \|v\|_{L^4(\Omega)} \leqslant c_1 \|v\|_V, \ \forall \ v \in V. \tag{1.12}$$

(ii) *Under the restrictions* (1.11) *on* ρ *and* n *we obtain*

$$V \hookrightarrow L^q(\Omega) \hookrightarrow L^{\frac{8n\rho}{n+2}}(\Omega) \quad \textit{and} \quad V \hookrightarrow L^q(\Omega) \hookrightarrow L^{\frac{4n}{n+2}}(\Omega).$$

Introduce the following restrictions on the initial data and some constants:

$$\|u^0\|_V, \ \|v^0\|_V < \lambda^* \quad \text{and} \quad L < \frac{1}{4}(\lambda^*)^2, \tag{1.13}$$

where

$$\lambda^* = \left(\frac{1}{4N}\right)^{\frac{1}{2\rho}}; \tag{1.14}$$

$$L = \frac{1}{2}\left[\|u^1\|^2_{L^2(\Omega)} + \|v^1\|^2_{L^2(\Omega)}\right] + \frac{1}{2}\left[\|u^0\|^2_V + \|v^0\|^2_V\right] + N\left[\|u^0\|^{2(\rho+1)}_V + \|v^0\|^{2(\rho+1)}_V\right]; \tag{1.15}$$

$$N = \frac{c_0^{2(\rho+1)}}{2(\rho+1)}; \tag{1.16}$$

$$\delta \in W^{1,\infty}(\Gamma_1) \quad \text{such that} \quad \delta(x) \geqslant \delta_0 > 0 \quad \text{on} \quad \Gamma_1. \tag{1.17}$$

Theorem 1.2.2 *Assume hypotheses* (1.11) *and* (1.13)–(1.17). *Consider* $u^0, v^0 \in V$ *and* $u^1, v^1 \in L^2(\Omega)$. *Then there exists functions* u, v *in the class*

$$\begin{cases} u, v \in L^\infty(0,\infty; V), \\ \partial_t u, \partial_t v \in L^\infty(0,\infty; L^2(\Omega)), \\ \partial_t u, \partial_t v \in L^\infty(0,\infty; L^2(\Gamma_1)), \end{cases}$$

such that u *and* v *satisfies the equations*

$$\begin{cases} \partial_t^2 u - \Delta u + |u|^\rho |v|^\rho v = 0 & in \quad H^{-1}_{loc}(0,\infty; L^{q'}(\Omega)) \\ \partial_t^2 v - \Delta v + |u|^\rho u |v|^\rho = 0 & in \quad H^{-1}_{loc}(0,\infty; L^{q'}(\Omega)) \\ \frac{\partial u}{\partial \vec{n}} + \delta(\cdot)\partial_t u = 0 & in \quad L^2_{loc}(0,\infty; L^2(\Gamma_1)) \\ \frac{\partial v}{\partial \vec{n}} + \delta(\cdot)\partial_t v = 0 & in \quad L^2_{loc}(0,\infty; L^2(\Gamma_1)). \end{cases} \tag{1.18}$$

and the initial conditions

$$u(0) = u^0, v(0) = v^0, \quad \partial_t u(0) = u^1, \ \partial_t v(0) = v^1.$$

Proof. To following, we use Faedo-Galerkin Method with compactness arguments and ideas used by MiIla Miranda, Lourêdo and Medeiros [32].

Approximate problem. Let $(w_i)_{i\in\mathbb{N}}$ be a basis of the separable Banach space V, that is, the vectors $(w_i)_{i\in\mathbb{N}}$ are linearly independent and the finite linear combinations of vectors of $(w_i)_{i\in\mathbb{N}}$ are denses in V. Let $V_m = [w_1, \cdots, w_m]$ be the subspace generated by the m first vectores $w_1, w_2, \cdots, w_m$. Consider

$$u_m(t) = \sum_{j=1}^m g_{jm}(t) w_j, \quad \text{and} \quad v_m(t) = \sum_{\ell=1}^m h_{\ell m}(t) w_\ell$$

such that u_m and v_m are approximate solutions of the problem (1); that is,

$$\begin{cases} (\partial_t^2 u_m(t), w_j) + ((u_m(t), w_j)) + \int_{\Gamma_1} \delta \partial_t u_m(t) w_j d\Gamma + \int_\Omega |u_m(t)|^\rho |v_m(t)|^\rho v_m(t) w_j dx = 0, \\ (\partial_t^2 v_m(t), w_\ell) + ((v_m(t), w_\ell)) + \int_{\Gamma_1} \delta \partial_t v_m(t) w_\ell d\Gamma + \int_\Omega |u_m(t)|^\rho u_m(t) |v_m(t)|^\rho w_\ell dx = 0, \\ u_m(0) = u_{0m} \to u^0 \text{ in } V \quad \text{and} \quad \partial_t u_m(0) = u_{1m} \to u^1 \text{ in } L^2(\Omega), \\ v_m(0) = v_{0m} \to v^0 \text{ in } V \quad \text{and} \quad \partial_t v_m(0) = v_{1m} \to v^1 \text{ in } L^2(\Omega), \end{cases} \tag{1.19}$$

for all $j = 1, 2, \ldots, m$ and for all $\ell = 1, 2, \ldots, m$.

The above finite-dimensional system has solutions $\{u_m(t), v_m(t)\}$ defined on $[0, t_m)$. The following estimate allows us to extend this solution to the interval $[0, \infty)$.

Remark 1.2.3 *We prove initially that the integral*

$$\int_\Omega |u_m(t)|^\rho |v_m(t)|^\rho v_m(t) w_j dx \tag{1.20}$$

makes sense. Indeed, firstly we note that $w_j \in L^q(\Omega)$, q *and* ρ *as in the Remark 1.2.1. If* $3 \leqslant n \leqslant 6$ *and we use the item (ii) of Remark 1.2.1. We obtain, noting that* $q' = \dfrac{2n}{n+2}$

$$\begin{aligned}
&\int_\Omega |u_m(t)|^{\rho q'} |v_m(t)|^{\rho q'} |v_m(t)|^{q'} dx = \int_\Omega |u_m(t)|^{\frac{2n\rho}{n+2}} |v_m(t)|^{\frac{2n\rho}{n+2}} |v_m(t)|^{\frac{2n}{n+2}} dx \\
&\leqslant \left(\int_\Omega |u_m(t)|^{\frac{8n\rho}{n+2}} dx\right)^{\frac{1}{4}} \left(\int_\Omega |v_m(t)|^{\frac{8n\rho}{n+2}} dx\right)^{\frac{1}{4}} \left(\int_\Omega |v_m(t)|^{\frac{4n}{n+2}} dx\right)^{\frac{1}{2}} \\
&= \|u_m(t)\|^{\frac{2n\rho}{n+2}}_{L^{\frac{8n\rho}{n+2}}(\Omega)} \|v_m(t)\|^{\frac{2n\rho}{n+2}}_{L^{\frac{8n\rho}{n+2}}(\Omega)} \|v_m(t)\|^{\frac{2n}{n+2}}_{L^{\frac{4n}{n+2}}(\Omega)} \\
&\leqslant C \|u_m(t)\|_V^{\frac{2n\rho}{n+2}} \|v_m(t)\|_V^{\frac{2n\rho}{n+2}} \|v_m(t)\|_V^{\frac{2n}{n+2}}.
\end{aligned}$$

Therefore the above integral (1.20) *makes sense. Similar considerations for the integral*

$$\int_\Omega |u_m(t)|^\rho u_m(t) |v_m(t)|^\rho w_\ell dx.$$

A priori estimates. Multiplying both of sides of $(1.19)_1$ by $g'_{jm}(t)$ and adding from $j = 1$ to $j = m$. We obtain

$$\begin{aligned}
&(\partial_t^2 u_m(t), \partial_t u_m(t)) + ((u_m(t), \partial_t u_m(t))) + \int_{\Gamma_1} \delta[\partial_t u_m(t)]^2 d\Gamma \\
&+ \int_\Omega (|u_m(t)|^\rho \partial_t u_m(t))(|v_m(t)|^\rho v_m(t)) dx = 0.
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_t u_m(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}\|u_m(t)\|_V^2 + \int_{\Gamma_1} \delta[\partial_t u_m(t)]^2 d\Gamma \\
&+ \int_\Omega (|u_m(t)|^\rho \partial_t u_m(t))(|v_m(t)|^\rho v_m(t)) dx = 0.
\end{aligned} \tag{1.21}$$

We observe that

$$\frac{d}{dt}(|u_m(t)|^\rho u_m(t)) = (\rho + 1)|u_m(t)|^\rho \partial_t u_m(t). \tag{1.22}$$

Taking into account (1.22) in (1.21), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_t u_m(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}\|u_m(t)\|_V^2 + \int_{\Gamma_1} \delta[\partial_t u_m(t)]^2 d\Gamma \\
&+ \frac{1}{\rho + 1}\int_\Omega \frac{d}{dt}(|u_m(t)|^\rho u_m(t))(|v_m(t)|^\rho v_m(t)) dx = 0.
\end{aligned} \tag{1.23}$$

Similarly multiplying both of sides of $(1.19)_2$ by $h'_{jm}(t)$ we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\partial_t v_m(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}\|v_m(t)\|^2_V + \int_{\Gamma_1} \delta[\partial_t v_m(t)]^2 d\Gamma \\ &+ \frac{1}{\rho+1}\int_\Omega (|u_m(t)|^\rho u_m(t))\frac{d}{dt}(|v_m(t)|^\rho v_m(t))dx = 0. \end{aligned} \tag{1.24}$$

Adding (1.23) and (1.24), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big\{\|\partial_t u_m(t)\|^2_{L^2(\Omega)} + \|u_m(t)\|^2_V + \|\partial_t v_m(t)\|^2_{L^2(\Omega)} + \|v_m(t)\|^2_V\Big\} + \int_{\Gamma_1} \delta[\partial_t u_m(t)]^2 d\Gamma \\ &+ \int_{\Gamma_1} \delta[\partial_t v_m(t)]^2 d\Gamma + \frac{1}{\rho+1}\frac{d}{dt}\int_\Omega (|u_m(t)|^\rho u_m(t))(|v_m(t)|^\rho v_m(t))dx = 0. \end{aligned}$$

Integrating the above expression from 0 to t with $t < t_m$, and using the hypothesis on δ, we obtain

$$\begin{aligned} &\frac{1}{2}\Big\{\|\partial_t u_m(t)\|^2_{L^2(\Omega)} + \|u_m(t)\|^2_V + \|\partial_t v_m(t)\|^2_{L^2(\Omega)} + \|v_m(t)\|^2_V\Big\} + \delta_0\int_0^t\int_{\Gamma_1} [\partial_t u_m(s)]^2 d\Gamma ds \\ &+ \delta_0\int_0^t\int_{\Gamma_1} [\partial_t v_m(s)]^2 d\Gamma ds + \frac{1}{\rho+1}\int_\Omega (|u_m(t)|^\rho u_m(t))(|v_m(t)|^\rho v_m(t))dx \\ &\leqslant \frac{1}{2}\Big\{\|u_{1m}\|^2_{L^2(\Omega)} + \|u_{0m}\|^2_V + \|v_{1m}\|^2_{L^2(\Omega)} + \|v_{0m}\|^2_V\Big\} \\ &+ \frac{1}{\rho+1}\int_\Omega (|u_{0m}(x)|^\rho u_{0m}(x))(|v_{0m}(x)|^\rho v_{0m}(x))dx. \end{aligned} \tag{1.25}$$

By Young inequality, we get

$$\begin{aligned} &\left|\frac{1}{\rho+1}\int_\Omega (|u_m(t)|^\rho u_m(t))(|v_m(t)|^\rho v_m(t))dx\right| \leqslant \frac{1}{\rho+1}\int_\Omega |u_m(t)|^{\rho+1}|v_m(t)|^{\rho+1}dx \\ &\leqslant \frac{1}{2(\rho+1)}\left\{\|u_m(t)\|^{2(\rho+1)}_{L^{2(\rho+1)}(\Omega)} + \|v_m(t)\|^{2(\rho+1)}_{L^{2(\rho+1)}(\Omega)}\right\}. \end{aligned}$$

Now using the fact $V \hookrightarrow L^{2(\rho+1)}(\Omega)$, (see (1.12)), we obtain

$$\left|\frac{1}{\rho+1}\int_\Omega (|u_m(t)|^\rho u_m(t))(|v_m(t)|^\rho v_m(t))dx\right| \leqslant N\Big[\|u_m(t)\|^{2(\rho+1)}_V + \|v_m(t)\|^{2(\rho+1)}_V\Big], \tag{1.26}$$

where N was defined in (1.16). Analogously, we obtain

$$\left|\frac{1}{\rho+1}\int_\Omega (|u_{0m}|^\rho u_{0m})(|v_{0m}|^\rho v_{0m})dx\right| \leqslant N\Big[\|u_{0m}\|^{2(\rho+1)}_V + \|v_{0m}\|^{2(\rho+1)}_V\Big]. \tag{1.27}$$

Substituting (1.26) and (1.27) in (1.25), we obtain

$$\begin{aligned}
&\frac{1}{2}\Big\{\|u_m'(t)\|^2_{L^2(\Omega)} + \|u_m(t)\|^2_V + \|\partial_t v_m(t)\|^2_{L^2(\Omega)} + \|v_m(t)\|^2_V\Big\}\\
&- N\Big\{\|u_m(t)\|_V^{2(\rho+1)} + \|v_m(t)\|_V^{2(\rho+1)}\Big\} + \delta_0\int_0^t\int_{\Gamma_1}[\partial_t u_m(s)]^2 d\Gamma ds\\
&+ \delta_0\int_0^t\int_{\Gamma_1}[\partial_t v_m(s)]^2 d\Gamma ds\\
&\leqslant \frac{1}{2}\Big\{\|u_{1m}\|^2_{L^2(\Omega)} + \|u_{0m}\|^2_V + \|v_{1m}\|^2_{L^2(\Omega)} + \|v_{0m}\|^2_V\Big\} + N\Big\{\|u_{0m}\|_V^{2(\rho+1)} + \|v_{0m}\|_V^{2(\rho+1)}\Big\}.
\end{aligned} \tag{1.28}$$

By hypotheses and convergences (1.19), for small $\eta > 0$, there exists $m_0 \in \mathbb{N}$ such that

$$\|u_{0m}\|_V < \|u^0\|_V + \eta < \lambda^*, \quad \|v_{0m}\|_V < \|v^0\|_V + \eta < \lambda^*, \quad \forall m \geqslant m_0 \tag{1.29}$$

and

$$\begin{aligned}
L_m &= \frac{1}{2}[\|u_{1m}\|^2_{L^2(\Omega)} + \|v_{1m}\|^2_{L^2(\Omega)}] + \frac{1}{2}[\|u_{0m}\|^2_V + \|v_{0m}\|^2_V] + N[\|u_{0m}\|_V^{2(\rho+1)} + \|v_{0m}\|_V^{2(\rho+1)}]\\
&< L + \eta < \frac{1}{2}(\lambda^*)^2, \quad \forall m \geqslant m_0,
\end{aligned} \tag{1.30}$$

where L was introduced in (1.15). Therefore from (1.28) and (1.30), we have for small $\eta > 0$,

$$\begin{aligned}
&\frac{1}{2}\Big\{\|\partial_t u_m(t)\|^2_{L^2(\Omega)} + \|u_m(t)\|^2_V + \|\partial_t v_m(t)\|^2_{L^2(\Omega)} + \|v_m(t)\|^2_V\Big\}\\
&- N\Big\{\|u_m(t)\|_V^{2(\rho+1)} + \|v_m(t)\|_V^{2(\rho+1)}\Big\} + \delta_0\int_0^t\int_{\Gamma_1}[\partial_t u_m(s)]^2 d\Gamma ds\\
&+ \delta_0\int_0^t\int_{\Gamma_1}[\partial_t v_m(s)]^2 d\Gamma ds < L + \eta < \frac{1}{4}(\lambda^*)^2, \ \forall m \geqslant m_0.
\end{aligned} \tag{1.31}$$

Motivated by (1.31), we set the function

$$J(\lambda) = \frac{1}{4}\lambda^2 - N\lambda^{2(\rho+1)}, \quad \lambda \geqslant 0. \tag{1.32}$$

Then (1.31) provides

$$\begin{aligned}
&\frac{1}{2}\|\partial_t u_m(t)\|^2_{L^2(\Omega)} + \frac{1}{4}\|u_m(t)\|^2_V + J(\|u_m(t)\|_V) + \frac{1}{2}\|\partial_t v_m(t)\|^2_{L^2(\Omega)} + \frac{1}{4}\|v_m(t)\|^2_V\\
&+ J(\|v_m(t)\|_V) + \delta_0\int_0^t\int_{\Gamma_1}[\partial_t u_m(s)]^2 d\Gamma ds + \delta_0\int_0^t\int_{\Gamma_1}[\partial_t v_m(s)]^2 d\Gamma ds\\
&< L + \eta < \frac{1}{4}(\lambda^*)^2, \quad \forall m \geqslant m_0.
\end{aligned} \tag{1.33}$$

In order to obtain a priori estimates for the approximate solutions $\{u_m(t), v_m(t)\}$, we need that the left side of (1.33) would be non-negative. It is possible if $J(\|u_m(t)\|_V)$ and $J(\|v_m(t)\|_V)$ are non-negative. In the next result, we prove that if the hypothesis (1.13) is satisfied then

$$J(\|u_m(t)\|_V) \geqslant 0, \quad J(\|v_m(t)\|_V) \geqslant 0, \quad \forall\, t \in [0, \infty).$$

Remark 1.2.4 *We have*

$$J(\lambda) = \frac{1}{4}\lambda^2 - N\lambda^{2(\rho+1)} \geqslant 0, \quad \forall\; 0 \leqslant \lambda \leqslant \lambda^*.$$

This fact is consequence of

$$J(\lambda) = \lambda^2 \left(\frac{1}{4} - N\lambda^{2\rho}\right), \quad \lambda \geqslant 0.$$

Lemma 1.2.5 *Consider $u^0, v^0 \in V$ and $u^1, v^1 \in L^2(\Omega)$ such that*

$$\|u^0\|_V,\ \|v^0\|_V < \lambda^*$$

and

$$L < \frac{1}{4}(\lambda^*)^2$$

where λ^* and L were defined, respectively, in (1.14) *and* (1.15)*. Then*

$$\|u_m(t)\|_V < \lambda^* \quad \textit{and} \quad \|v_m(t)\|_V < \lambda^*, \quad \forall t \in [0, \infty) \quad \textit{and} \quad \forall m \geqslant m_0.$$

Proof. We fix $m \geqslant m_0$. We show the lemma by contradiction argument. Thus assume that there exists $t_1 \in (0, t_m)$ or $t_2 \in (0, t_m)$ such that

$$\|u_m(t_1)\|_V \geqslant \lambda^* \quad \text{or} \quad \|v_m(t_2)\|_V \geqslant \lambda^*.$$

There are two possibilities, which are

$$\begin{aligned} &1) \quad \|u_m(t_1)\|_V \geqslant \lambda^* \quad \text{and} \quad \|v_m(t_2)\|_V \geqslant \lambda^*, \\ &2) \quad \|u_m(t_1)\|_V \geqslant \lambda^* \quad \text{and} \quad \|v_m(t)\|_V < \lambda^*, \quad \forall t \in [0, \infty). \end{aligned} \tag{1.34}$$

Assume that occurs possibility 1). Note that

$$\|u_m(t_1)\|_V \geqslant \lambda^* > \|u_m(0)\|_V \quad \text{and} \quad \|v_m(t_2)\|_V \geqslant \lambda^* > \|v_m(0)\|_V.$$

Then by intermediate value theorem there exists $\tau_1 \in (0, t_m)$ and $\tau_2 \in (0, t_m)$ such that

$$\|u_m(\tau_1)\|_V = \lambda^* \quad \text{and} \quad \|v_m(\tau_2)\|_V = \lambda^*.$$

Set

$$t_1^* = \inf\{\tau \in (0, t_m); \|u_m(\tau)\|_V = \lambda^*\}$$

$$t_2^* = \inf\{\tau \in (0, t_m); \|v_m(\tau)\|_V = \lambda^*\}.$$

By continuity of $\|u_m(t)\|_V$ and $\|v_m(t)\|_V$, we obtain

$$\|u_m(t_1^*)\|_V = \lambda^* \quad \text{and} \quad \|v_m(t_2^*)\|_V = \lambda^*.$$

From $(1.29)_1$ it follows that $t_1^* > 0$ and $t_2^* > 0$. Thus

$$\|u_m(t)\|_V < \lambda^* \quad \text{for} \quad 0 \leqslant t < t_1^*$$

$$\|v_m(t)\|_V < \lambda^* \quad \text{for} \quad 0 \leqslant t < t_2^*.$$

Therefore by Remark 1.2.4, we get

$$J(\|u_m(t)\|_V) \geqslant 0 \quad \text{for} \quad 0 \leqslant t < t_1^*$$

$$J(\|v_m(t)\|_V) \geqslant 0 \quad \text{for} \quad 0 \leqslant t < t_2^*.$$

Assume $t_1^* \leqslant t_2^*$. Similar arguments if $t_2^* \leqslant t_1^*$. Return to expression (1.33). Then

$$\frac{1}{4}\|u_m(t)\|_V^2 + J(\|u_m(t)\|_V) + \frac{1}{4}\|v_m(t)\|_V^2 + J(\|v_m(t)\|_V) \leqslant L + \eta < \frac{1}{4}(\lambda^*)^2, \quad 0 \leqslant t < t_1^*.$$

So

$$\frac{1}{4}\|u_m(t)\|_V^2 \leqslant L + \eta < \frac{1}{4}(\lambda^*)^2, \quad 0 \leqslant t < t_1^*, \ \forall m \geqslant m_0.$$

Taking the limit as $t \to t_1^*$, $0 < t < t_1^*$, in this inequality we obtain a contradiction. This prove the part 1) of (1.34).

The proof of possibility 2) of (1.34) follows by applying the arguments used in part 1) to $\|u_m(t_1)\|_V \geqslant \lambda^*$ and this conclude the proof of the lemma. ■

By Lemma 1.2.5 we have

$$\|u_m(t)\|_V < \lambda^* \quad \text{and} \quad \|v_m(t)\|_V < \lambda^*, \quad \forall\ 0 \leqslant t < \infty \quad \text{and} \quad \forall\ m \geqslant m_0.$$

Consequently

$$J(\|u_m(t)\|_V) \geqslant 0 \quad \text{and} \quad J(\|v_m(t)\|_V) \geqslant 0, \quad \forall\ t \in [0, \infty).$$

Therefore, from (1.33)

$$\begin{aligned}&\frac{1}{2}\|\partial_t u_m(t)\|^2_{L^2(\Omega)}+\frac{1}{4}\|u_m(t)\|^2_V+\frac{1}{2}\|\partial_t v_m(t)\|^2_{L^2(\Omega)}+\frac{1}{4}\|v_m(t)\|^2_V\\&+\delta_0\int_0^t\int_{\Gamma_1}[\partial_t u_m(s)]^2 d\Gamma ds+\delta_0\int_0^t\int_{\Gamma_1}[\partial_t v_m(s)]^2 d\Gamma ds\leqslant L+\eta<\frac{1}{4}(\lambda^*)^2,\end{aligned}\tag{1.35}$$

for all $t\in[0,\infty)$ and for all $m\geqslant m_0$. By (1.35) we obtain

$$\begin{cases}(u_m),(v_m) \text{ are bounded in } \ L^\infty(0,\infty;V),\ \forall m\geqslant m_0;\\ (\partial_t u_m),(\partial_t v_m) \text{ are bounded in } \ L^\infty(0,\infty;L^2(\Omega)),\ \forall m\geqslant m_0;\\ (\partial_t u_m),(\partial_t v_m) \text{ are bounded in } L^2(0,\infty;L^2(\Gamma_1)),\ \forall m\geqslant m_0.\end{cases}\tag{1.36}$$

With similar arguments used in the item (ii) of Remark 1.2.3 we obtain

$$\||u_m(t)|^\rho|v_m(t)|^\rho v_m(t)\|_{L^{q'}(\Omega)}\leqslant C,\quad \forall\ m\geqslant m_0.$$

where the constant $C>0$ is independent of t and m. It follows that

$$(|u_m|^\rho|v_m|^\rho v_m) \text{ is bounded in } \ L^\infty(0,\infty;L^{q'}(\Omega)),\quad \forall\ m\geqslant m_0.\tag{1.37}$$

In similar way, we find

$$(|u_m|^\rho u_m|v_m|^\rho) \text{ is bounded in } \ L^\infty(0,\infty;L^{q'}(\Omega)),\quad \forall\ m\geqslant m_0.\tag{1.38}$$

Passage to the limit. Estimates (1.36), (1.37) and (1.38) allow us, by induction and diagonal process, to obtain a subsequences of (u_m) and (v_m), still denoted by (u_m) and (v_m), and functions $u,v:\Omega\times[0,\infty)\to\mathbb{R}$, such that

$$\begin{cases}u_m\overset{*}{\rightharpoonup}u \quad\text{and}\quad v_m\overset{*}{\rightharpoonup}v & \text{in}\quad L^\infty(0,\infty;V),\\ \partial_t u_m\overset{*}{\rightharpoonup}\partial_t u \quad\text{and}\quad \partial_t v_m\overset{*}{\rightharpoonup}\partial_t v & \text{in}\quad L^\infty(0,\infty;L^2(\Omega)),\\ \partial_t u_m\rightharpoonup\partial_t u \quad\text{and}\quad \partial_t v_m\rightharpoonup\partial_t v & \text{in}\quad L^2(0,\infty;L^2(\Gamma_1)),\\ |u_m|^\rho|v_m|^\rho v_m\overset{*}{\rightharpoonup}\xi & \text{in}\quad L^\infty(0,\infty;L^{q'}(\Omega)),\\ |u_m|^\rho u_m|v_m|^\rho\overset{*}{\rightharpoonup}\zeta & \text{in}\quad L^\infty(0,\infty;L^{q'}(\Omega)).\end{cases}\tag{1.39}$$

We must show that $\xi=|u|^\rho|v|^\rho v$ and $\zeta=|u|^\rho u|v|^\rho$.

Consider $T>0$ fixed but arbitrary. By convergences $(1.39)_1$ and $(1.39)_2$ and noting that $V\overset{c}{\hookrightarrow}L^2(\Omega)$, we obtain by Aubin-Lions Theorem, see Lions [26, Theorem

5.1, p. 58], that there are subsequences of (u_m) and (v_m), which we still denoted by (u_m) and (v_m), respectively, such that

$$\begin{aligned} u_m &\to u \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)), \\ v_m &\to v \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)). \end{aligned} \tag{1.40}$$

By (1.40) there are subsequences of (u_m) and (v_m) such that

$$\begin{aligned} u_m &\to u \quad \text{a.e. in} \quad \Omega \times (0,T), \\ v_m &\to v \quad \text{a.e. in} \quad \Omega \times (0,T). \end{aligned} \tag{1.41}$$

By (1.41) we have that

$$\begin{aligned} &|u_m|^\rho \to |u|^\rho \quad \text{a.e. in} \quad \Omega \times (0,T), \\ &|v_m|^\rho v_m \to |v|^\rho v \quad \text{a.e. in} \quad \Omega \times (0,T). \end{aligned}$$

Therefore

$$|u_m|^\rho |v_m|^\rho v_m \to |u|^\rho |v|^\rho v \quad \text{a.e. in} \quad \Omega \times (0,T). \tag{1.42}$$

From (1.37), (1.42) and of Lions' Lemma, see [26, Lemma 1.3, p. 12], we obtain

$$|u_m|^\rho |v_m|^\rho v_m \rightharpoonup |u|^\rho |v|^\rho v \quad \text{in} \quad L^2(0,T;L^{q'}(\Omega)).$$

In similar way, we find

$$|u_m|^\rho u_m |v_m|^\rho \rightharpoonup |u|^\rho u |v|^\rho \quad \text{in} \quad L^2(0,T;L^{q'}(\Omega)).$$

By a diagonal process we obtain

$$|u_m|^\rho |v_m|^\rho v_m \overset{*}{\rightharpoonup} |u|^\rho |v|^\rho v \quad \text{in} \quad L^\infty_{loc}(0,\infty;L^{q'}(\Omega)). \tag{1.43}$$

In similar way, we find

$$|u_m|^\rho u_m |v_m|^\rho \overset{*}{\rightharpoonup} |u|^\rho u |v|^\rho \quad \text{in} \quad L^\infty_{loc}(0,\infty;L^{q'}(\Omega)). \tag{1.44}$$

By (1.39), (1.43) and (1.44) we have $\xi = |u|^\rho |v|^\rho v$ and $\zeta = |u|^\rho u |v|^\rho$.

Multiplying both sides of the approximate equation $(1.19)_1$ by $\varphi \in \mathscr{D}(0,\infty)$, integrating in $[0,\infty)$, using the convergences (1.39) and noting that V_m is dense in V, we obtain

$$\begin{aligned} &\int_0^\infty (\partial_t^2 u(t), w)\varphi(t)dt + \int_0^\infty ((u(t),w))\varphi(t)dt + \int_0^\infty \int_{\Gamma_1} \delta \partial_t u(t) w \varphi(t) d\Gamma dt \\ &+ \int_0^\infty (|u(t)|^\rho |v(t)|^\rho v(t), w)\varphi(t)dt = 0, \quad \forall w \in V,\ \forall \varphi \in \mathscr{D}(0,\infty). \end{aligned} \tag{1.45}$$

Since V is dense in $L^2(\Omega)$ it follows that (1.45) is true for all $w \in L^2(\Omega)$.

In similar way, we find

$$\int_0^\infty (\partial_t^2 v(t), z)\varphi(t)dt + \int_0^\infty ((v(t), z))\varphi(t)dt + \int_0^\infty \int_{\Gamma_1} \delta\partial_t u(t) z\varphi(t) d\Gamma dt$$
$$+ \int_0^\infty (|u(t)|^\rho u(t)|v(t)|^\rho, z)\varphi(t)dt = 0, \quad \forall z \in V,\ \forall \varphi \in \mathscr{D}(0,\infty).$$

Taking in (1.45) $w \in \mathscr{D}(\Omega) \subset V$, it follows that

$$\partial_t^2 u - \Delta u + |u|^\rho |v|^\rho v = 0 \quad \text{in} \quad \mathscr{D}'(\Omega \times (0,\infty)). \tag{1.46}$$

In similar way

$$\partial_t^2 v - \Delta v + |u|^\rho u |v|^\rho = 0 \quad \text{in} \quad \mathscr{D}'(\Omega \times (0,\infty)).$$

Let $T > 0$ fix. Note that $\partial_t u \in L^2(0,T;L^2(\Omega)) \hookrightarrow L^2(0,T;L^{q'}(\Omega))$ then $\partial_t^2 u \in H^{-1}(0,T;L^{q'}(\Omega))$ see Proposition A.16. Since

$$|u|^\rho |v|^\rho v \in L^\infty(0,\infty;L^{q'}(\Omega)) \hookrightarrow L^2(0,T;L^{q'}(\Omega)) \hookrightarrow H^{-1}(0,T;L^{q'}(\Omega))$$

then by (1.46) we have $-\Delta u \in H^{-1}(0,T;L^{q'}(\Omega))$. Therefore

$$\partial_t^2 u - \Delta u + |u|^\rho |v|^\rho v = 0 \quad \text{in} \quad H^{-1}_{loc}(0,\infty;L^{q'}(\Omega)). \tag{1.47}$$

In similar way

$$\partial_t^2 v - \Delta v + |u|^\rho u |v|^\rho = 0 \quad \text{in} \quad H^{-1}_{loc}(0,\infty;L^{q'}(\Omega)).$$

As $u \in L^2(0,\infty;V)$ and $\Delta u \in H^{-1}(0,T;L^{q'}(\Omega))$ then by Theorem 1.1.5 with $\theta = q'$, we obtain

$$\frac{\partial u}{\partial \vec{n}} \in H^{-1}(0,T;W^{-\frac{1}{q'},q'}(\Gamma_1)) \tag{1.48}$$

Multiplying both sides of (1.46) by $w\varphi$ with $w \in V$ and $\varphi \in \mathscr{D}(0,\infty)$, integrating on $\Omega \times (0,\infty)$ and using (1.48) and Green's Formula of the Theorem 1.1.5,

$$\int_0^\infty (\partial_t^2 u(t), w)\varphi(t)dt + \int_0^\infty ((u(t), w))\varphi(t)dt - \int_0^\infty \left\langle \frac{\partial u(t)}{\partial \vec{n}}, w \right\rangle \varphi(t)dt$$
$$+ \int_0^\infty (|u(t)|^\rho u(t)|v(t)|^\rho, w)\varphi(t)dt = 0, \quad \forall w \in V,\ \forall \varphi \in \mathscr{D}(0,\infty),$$

where $\langle \cdot, \cdot \rangle$ denotes the duality paring between $W^{-\frac{1}{q'},q'}(\Gamma_1)$ and $W^{\frac{1}{q'},q}(\Gamma_1)$. Comparing this lest equation with (1.45), we obtain

$$\int_0^\infty \left\langle \frac{\partial u(t)}{\partial \vec{n}} + \delta\partial_t u(t), w \right\rangle \varphi(t)dt = 0,\ \forall w \in V,\ \forall \varphi \in \mathscr{D}(0,\infty).$$

Therefore

$$\frac{\partial u}{\partial \vec{n}} + \delta \partial_t u = 0 \quad \text{in} \quad W^{-\frac{1}{q'},q'}(\Gamma_1).$$

In similar way

$$\frac{\partial v}{\partial \vec{n}} + \delta \partial_t v = 0 \quad \text{in} \quad W^{-\frac{1}{q'},q'}(\Gamma_1).$$

Since $\delta \partial_t u \in L^2(0,\infty, L^2(\Gamma_1))$, then

$$\frac{\partial u}{\partial \vec{n}} + \delta \partial_t u = 0 \quad \text{in} \quad L^2_{loc}(0,\infty; L^2(\Gamma_1)).$$

In similar way

$$\frac{\partial v}{\partial \vec{n}} + \delta \partial_t v = 0 \quad \text{in} \quad L^2_{loc}(0,\infty; L^2(\Gamma_1)).$$

Initial conditions. We see that $u \in L^\infty(0,\infty;V)$, $\partial_t u \in L^\infty(0,\infty;L^2(\Omega))$ and $\partial_t^2 u \in H^{-1}(0,\infty;L^{q'}(\Omega))$, then $u \in C([0,\infty);L^2(\Omega))$ and $\partial_t u \in C([0,\infty);L^{q'}(\Omega))$. So it makes sense to calculate $u(0)$ and $\partial_t u(0)$.

We show that $u(0) = u^0$. In fact, let $\varphi \in C^1([0,T];\mathbb{R})$ such that $\varphi(0) = 1$ and $\varphi(T) = 0$. By $(1.39)_2$ we have

$$\int_0^T (\partial_t u_m(t), w)\varphi(t)dt \to \int_0^T (\partial_t u(t), w)\varphi(t)dt. \tag{1.49}$$

Integrating by parts (1.49) we obtain

$$-(u_m(0), w) - \int_0^T (u_m(t), w)\varphi'(t)dt \to -(u(0), w) - \int_0^T (u(t), w)\varphi'(t)dt. \tag{1.50}$$

By $(1.39)_1$ we have $u_m \overset{*}{\rightharpoonup} u$ in $L^\infty(0,T;L^2(\Omega))$ and

$$\int_0^T (u_m(t), w)\varphi'(t)dt \to \int_0^T (u(t), w)\varphi'(t)dt. \tag{1.51}$$

Adding (1.50) and (1.51) we obtain

$$(u_m(0), w) \to (u(0), w), \quad \forall\, w \in L^2(\Omega). \tag{1.52}$$

On the other hand, by $(1.19)_3$ we have $u_m(0) \rightharpoonup u_0$ in $L^2(\Omega)$ and thus

$$(u_m(0), w) \to (u_0, w), \quad \forall\, w \in L^2(\Omega). \tag{1.53}$$

By (1.52), (1.53) and uniquenesses of limit $u(0) = u^0$.

Now we show that $\partial_t u(0) = u^1$. In fact, let $\varphi \in C^1([0,T];\mathbb{R})$ such that $\varphi(0) = 1$ and $\varphi(T) = 0$. Multiplying both of sides of $(1.19)_1$ by φ we have

$$\int_0^T (\partial_t^2 u_m(t), w)\varphi(t)dt + \int_0^T ((u_m(t), w))\varphi(t)dt + \int_0^T \int_{\Gamma_1} \delta\partial_t u_m(t) w\varphi(t) d\Gamma dt$$
$$+ \int_0^T (|u_m(t)|^\rho |v_m(t)|^\rho v_m(t), w)\varphi(t)dt = 0, \ \forall w \in V.$$

Integrating by parts the expression above we get

$$-(\partial_t u_m(0), w) - \int_0^T (\partial_t u_m(t), w)\varphi'(t)dt + \int_0^T ((u_m(t), w))\varphi(t)dt$$
$$+ \int_0^T \int_{\Gamma_1} \delta\partial_t u_m(t) w\varphi(t) d\Gamma dt + \int_0^T (|u_m(t)|^\rho |v_m(t)|^\rho v_m(t), w)\varphi(t)dt = 0, \ \forall w \in V.$$

Taking the limit in the expression above we obtain

$$\begin{aligned} &-(u^1, w) - \int_0^T (\partial_t u(t), w)\varphi'(t)dt + \int_0^T ((u(t), w))\varphi(t)dt \\ &+ \int_0^T \int_{\Gamma_1} \delta\partial_t u(t) w\varphi(t) d\Gamma dt + \int_0^T (|u(t)|^\rho |v(t)|^\rho v(t), w)\varphi(t)dt = 0, \ \forall w \in V. \end{aligned} \tag{1.54}$$

Let $T > 0$. Introduce the notation $Y = L^1(0,T;X)$. Then $Y' = L^\infty(0,T;X')$. Consider $\varphi \in C^1([0,T])$ with $\varphi(0) = 1$, $\varphi(T) = 0$ and $w \in X$. Then $\varphi w \in Y$. By (1.47) we obtain

$$\langle \partial_t^2 u, \varphi w\rangle_{Y'\times Y} + \langle -\Delta u, \varphi w\rangle_{Y'\times Y} + \langle |u|^\rho |v|^\rho v, \varphi w\rangle_{Y'\times Y} = 0. \tag{1.55}$$

Noting that $u' \in C^0([0,T];X')$, we find

$$\begin{aligned} \langle \partial_t^2 u, \varphi w\rangle_{Y'\times Y} &= \int_0^T \langle \partial_t^2 u(t), w\rangle_{X'\times X}\varphi(t)dt = \int_0^T \frac{d}{dt}\langle \partial_t u(t), w\rangle_{X'\times X}\varphi(t)dt \\ &= -\int_0^T \langle \partial_t u(t), w\rangle_{X'\times X}\varphi(t)'dt - \langle \partial_t u(0), w\rangle_{X'\times X}. \end{aligned}$$

We also that

$$\langle -\Delta u, \varphi w\rangle_{Y'\times Y} = \int_0^T ((u(t), w))\varphi(t)dt + \int_0^T \int_{\Gamma_1} \delta\partial_t u(t) w\varphi(t) d\Gamma dt,$$

$$\langle |u|^\rho |v|^\rho v, \varphi w\rangle_{Y'\times Y} = \int_0^T \langle |u(t)|^\rho |v(t)|^\rho v(t), w\rangle_{X'\times X}\varphi(t)dt.$$

The last three equalities and (1.55) provide

$$-\langle \partial_t u(0), w\rangle_{X'\times X} - \int_0^T \langle \partial_t u(t), w\rangle_{X'\times X}\varphi(t)'dt + \int_0^T ((u(t), w))\varphi(t)dt$$
$$+ \int_0^T \int_{\Gamma_1} \delta\partial_t u(t) w\varphi(t) d\Gamma dt + \int_0^T \langle |u(t)|^\rho |v(t)|^\rho v(t), w\rangle_{X'\times X}\varphi(t)dt = 0.$$

Combining this expression with (1.54), we get $\partial_t u(0) = u^1$.

In similar way, we obtain

$$v(0) = v^0 \quad \text{and} \quad \partial_t v(0) = v^1.$$

Therefore, we conclude the proof of the Theorem 1.2.2. ■

Corollary 1.2.6 *We obtain similar results to the Theorem 1.2.2 for the case $\rho > 0$ and $n = 1, 2$.*

Now we consider the following hypothesis

$$\rho > 0 \quad \text{and} \quad \theta > 1 \quad \text{with} \quad 4\rho\theta \geqslant 1, \quad \text{if} \quad n = 1, 2; \tag{1.56}$$

$$\rho = \frac{2}{n-2} \quad \text{and} \quad \theta = \frac{n}{n-2}, \quad \text{if} \quad 7 \leqslant n \leqslant 11. \tag{1.57}$$

Remark 1.2.7 *Under the restrictions* (1.57) *on ρ and n we have:*

$$V \hookrightarrow L^q(\Omega) \hookrightarrow L^{4\rho\theta}(\Omega), \quad \text{and} \quad V \hookrightarrow L^q(\Omega) \hookrightarrow L^{2\theta}(\Omega).$$

Theorem 1.2.8 *Consider $u^0, v^0 \in V \cap L^{\theta'}(\Omega)$ and $u^1, v^1 \in L^2(\Omega)$. Then under hypotheses* (1.13)–(1.17) *and* (1.57)*, we have that there exist functions u, v in the class*

$$\begin{cases} u, v \in L^\infty(0, \infty; V); \\ \partial_t u, \partial_t v \in L^\infty(0, \infty; L^2(\Omega)); \\ \partial_t u, \partial_t v \in L^\infty(0, \infty; L^2(\Gamma_1)), \end{cases}$$

such that u and v satisfies the equations

$$\begin{cases} \partial_t^2 u - \Delta u + |u|^\rho |v|^\rho v = 0 & in \quad H^{-1}_{loc}(0, \infty; L^\theta(\Omega)) \\ \partial_t^2 v - \Delta v + |u|^\rho u |v|^\rho = 0 & in \quad H^{-1}_{loc}(0, \infty; L^\theta(\Omega)) \\ \dfrac{\partial u}{\partial \vec{n}} + \delta(\cdot)\partial_t u = 0 & in \quad L^2_{loc}(0, \infty; L^2(\Gamma_1)) \\ \dfrac{\partial v}{\partial \vec{n}} + \delta(\cdot)\partial_t v = 0 & in \quad L^2_{loc}(0, \infty; L^2(\Gamma_1)) \end{cases} \tag{1.58}$$

and the initial conditions

$$u(0) = u^0, v(0) = v^0, \quad \partial_t u(0) = u^1, \ \partial_t v(0) = v^1.$$

Proof. Since the separable space $W^{1,\theta'}_{\Gamma_0}(\Omega)$ is dense in V and dense in $L^{\theta'}(\Omega)$ and $W^{1,\theta'}_{\Gamma_0}(\Omega) \hookrightarrow V \cap L^{\theta'}(\Omega)$ by Proposition 1.1.1 and Proposition 1.1.2 we have that $V \cap L^{\theta'}(\Omega)$ is a separable Banach space. Thus, taking a basis $(w_\ell)_{\ell \in \mathbb{N}}$ in $V \cap L^{\theta'}(\Omega)$, where $\dfrac{1}{\theta} + \dfrac{1}{\theta'} = 1$, and using similar arguments to those used in the Theorem 1.2.2 we show the Theorem 1.2.8. ■

Corollary 1.2.9 *Under the hypothesis* (1.56), *we obtain similar results to Theorem 1.2.8.*

In order to obtain results on the uniqueness and decay of solutions of problem (1), we prove the following theorem on existence of solutions for $\rho = 1$ and $n = 1, 2, 3$.

Remark 1.2.10 *We observe that for* $0 < \rho \leqslant \dfrac{1}{n-2}$ *and from trace theorem, we have*

$$V \hookrightarrow H^{\frac{1}{2}}(\Gamma_1) \hookrightarrow L^{q_1}(\Gamma_1) \hookrightarrow L^{2(\rho+1)}(\Gamma_1),$$

where $q_1 = \frac{2(n-1)}{n-2}$ *for* $n \geqslant 3$. *In particular for* $\rho = 1$ *or* $n = 3$, *we obtain*

$$V \hookrightarrow H^{\frac{1}{2}}(\Gamma_1) \hookrightarrow L^4(\Gamma_1) \hookrightarrow L^2(\Gamma_1).$$

Thus there exists positive constants c_2 *and* c_3 *such that*

$$\|w\|_{L^4(\Gamma_1)} \leqslant c_2 \|w\|_V \quad \textit{and} \quad \|w\|_{L^2(\Gamma_1)} \leqslant c_3 \|w\|_V, \ \forall w \in V. \tag{1.59}$$

We also consider the following restrictions on the initial data and some constants:

$$\|u_0\|_V, \ \|v_0\|_V < \lambda_1^* \quad \text{and} \quad L_1 < \frac{1}{4}(\lambda_1^*)^2, \tag{1.60}$$

where

$$\lambda_1^* = \left(\frac{1}{4N_1}\right)^{\frac{1}{2}}; \tag{1.61}$$

$$L_1 = \frac{1}{2}\left[\|u_1\|^2_{L^2(\Omega)} + \|v_1\|^2_{L^2(\Omega)}\right] + \frac{1}{2}\left[\|u_0\|_V^2 + \|v_0\|_V^2\right] + N_1\left[\|u_0\|_V^4 + \|v_0\|_V^4\right]; \tag{1.62}$$

$$N_1 = \frac{c_1^4}{2}\left[n + \frac{1}{4}\right] + \frac{Rc_2^4}{2} + c_1^4(n-1). \tag{1.63}$$

To show the next theorem we need of the following propositions.

Proposition 1.2.11 *Let us consider* $f \in L^2(\Omega)$ *and* $g \in H^{\frac{1}{2}}(\Gamma_1)$. *Then, the solution* u *of the boundary value problem:*

$$\begin{cases} -\Delta w = f & \textit{in} \quad \Omega, \\ w = 0 & \textit{on} \quad \Gamma_0, \\ \dfrac{\partial w}{\partial \vec{n}} = g & \textit{on} \quad \Gamma_1, \end{cases}$$

belongs to $V \cap H^2(\Omega)$ *and*

$$\|w\|_{H^2(\Omega)} \leqslant \left(|f| + \|g\|_{H^{\frac{1}{2}}(\Gamma_1)}\right).$$

Proof. See Milla Miranda and Medeiros [36, Proposition 1, p. 49]. ■

Proposition 1.2.12 *Suppose* $u^0 \in V \cap H^2(\Omega)$, $u^1 \in V$ *and*

$$\frac{\partial u^0}{\partial \vec{n}} + \delta(\cdot) u^1 = 0 \quad on \quad \Gamma_1.$$

Then, for each ε*, there exists* w *and* z *in* $V \cap H^2(\Omega)$ *such that:*

$$\|w - u^0\|_{V\cap H^2(\Omega)} < \varepsilon, \quad \|z - u^1\|_V < \varepsilon \quad and \quad \frac{\partial w}{\partial \vec{n}} + \delta(\cdot) z = 0 \quad on \quad \Gamma_1.$$

Proof. See Milla Miranda and Medeiros [36, Proposition 3, p. 50]. ■

Theorem 1.2.13 *Let* $\rho = 1$ *and* $n = 3$. *Consider* (1.60)–(1.63) *and that* $u^0, v^0 \in V \cap H^2(\Omega)$ *and* $u^1, v^1 \in V$ *satisfying*

$$\frac{\partial u^0}{\partial \vec{n}} + \delta(\cdot) u^1 = 0 \quad on \quad \Gamma_1$$
$$\frac{\partial v^0}{\partial \vec{n}} + \delta(\cdot) v^1 = 0 \quad on \quad \Gamma_1.$$

Then there exists functions u, v *in the class*

$$\begin{cases} u, v \in L^\infty(0,\infty; V \cap H^2(\Omega)), \ \partial_t u, \partial_t v \in L^\infty_{loc}(0,\infty; V) \\ \partial_t^2 u, \partial_t^2 v \in L^\infty_{loc}(0,\infty; L^2(\Omega)), \\ \partial_t u, \partial_t v \in L^\infty(0,\infty; L^2(\Gamma_1)), \ \partial_t^2 u, \partial_t^2 v \in L^\infty_{loc}(0,\infty; L^2(\Gamma_1)), \end{cases} \tag{1.64}$$

such that u *and* v *satisfies the equations*

$$\begin{cases} \partial_t^2 u - \Delta u + |u||v|v = 0 & in \quad L^\infty_{loc}(0,\infty; L^2(\Omega)) \\ \partial_t^2 v - \Delta v + |u|u|v| = 0 & in \quad L^\infty_{loc}(0,\infty; L^2(\Omega)) \\ \dfrac{\partial u}{\partial \vec{n}} + \delta(\cdot)\partial_t u = 0 & in \quad L^\infty_{loc}(0,\infty; H^{\frac{1}{2}}(\Gamma_1)) \\ \dfrac{\partial v}{\partial \vec{n}} + \delta(\cdot)\partial_t v = 0 & in \quad L^\infty_{loc}(0,\infty; H^{\frac{1}{2}}(\Gamma_1)). \end{cases} \tag{1.65}$$

and the initial conditions

$$u(0) = u^0, v(0) = v^0, \quad \partial_t u(0) = u^1, \ \partial_t v(0) = v^1.$$

Proof. The proof of Theorem 1.2.13 be done by applying the Faedo-Galerkin Method with a special basis of $V \cap H^2(\Omega)$.

Approximate problem. From Proposition 1.2.12, we obtain sequences $(u_k^0), (v_k^0)$ and $(u_k^1), (v_k^1)$ of vectors of $V \cap H^2(\Omega)$ such that

$$\begin{aligned} &u_k^0 \to u^0 \quad \text{in} \quad \text{and} \quad v_k^0 \to v^0 \quad \text{in} \quad V \cap H^2(\Omega) \\ &u_k^1 \to u^1 \quad \text{in} \quad \text{and} \quad v_k^1 \to v^1 \quad \text{in} \quad V, \end{aligned} \tag{1.66}$$

and

$$\frac{\partial u_k^0}{\partial \vec{n}} + \delta u_k^1 = 0 \quad \text{and} \quad \frac{\partial v_k^0}{\partial \vec{n}} + \delta v_k^1 = 0 \quad \text{on} \quad \Gamma_1 \quad \text{for all} \quad k \in \mathbb{N}.$$

Now we fix $k \in \mathbb{N}$ and consider the basis $\{w_1^k, w_2^k, w_3^k, w_4^k, \ldots\}$ of $V \cap H^2(\Omega)$ such that u_k^0, v_k^0, u_k^1 and v_k^1 belong to the subspace $[w_1^k, w_2^k, w_3^k, w_4^k]$ spanned by w_1^k, w_2^k, w_3^k and w_4^k. For each $m \in \mathbb{N}$ we built the subspace $V_m^k = [w_1^k, w_2^k, \ldots, w_m^k]$. Consider

$$u_{km}(t) = \sum_{j=1}^{m} g_{jkm}(t) w_j^k, \quad \text{and} \quad v_{km}(t) = \sum_{j=1}^{m} h_{jkm}(t) w_j^k$$

such that u_{km} and v_{km} are approximate solutions of the problem (1); that is,

$$\begin{cases} (\partial_t^2 u_{km}(t), w) + ((u_{km}(t), w)) + \displaystyle\int_{\Gamma_1} \delta \partial_t u_{km}(t) z d\Gamma + \int_\Omega |u_{km}(t)||v_{km}(t)| v_{km}(t) w dx = 0, \\ (\partial_t^2 v_{km}(t), z) + ((v_{km}(t), z)) + \displaystyle\int_{\Gamma_1} \delta \partial_t v_{km}(t) z d\Gamma + \int_\Omega |u_{km}(t)| u_{km}(t) |v_{km}(t)| z dx = 0, \\ u_{km}(0) = u_k^0, \quad \partial_t u_{km}(0) = u_k^1, \\ v_{km}(0) = v_k^0, \quad \partial_t v_{km}(0) = v_k^1, \end{cases} \tag{1.67}$$

for all $w, z \in V_m^k$. The above finite-dimensional system has solutions $\{u_{km}(t), v_{km}(t)\}$ defined on $[0, t_{km})$. The following estimate allows us to extend this solution to the interval $[0, \infty)$.

Remark 1.2.14 *We prove initially that the integral*

$$\int_\Omega |u_{km}(t)||v_{km}(t)| v_{km}(t) w dx \tag{1.68}$$

makes sense. Indeed, firstly we note that $w \in L^2(\Omega)$*,*

$$\begin{aligned} &\int_\Omega [|u_{km}(t)||v_{km}(t)| v_{km}(t)]^2 dx \leqslant \int_\Omega |u_{km}(t)|^2 |v_{km}(t)|^2 |v_{km}(t)|^2 dx \\ &\leqslant \left(\int_\Omega |u_{km}(t)|^6 dx \right)^{\frac{1}{3}} \left(\int_\Omega |u_{km}(t)|^6 dx \right)^{\frac{1}{3}} \left(\int_\Omega |v_{km}(t)|^6 dx \right)^{\frac{1}{3}} \\ &= \|u_{km}(t)\|_{L^6(\Omega)}^2 \|v_{km}(t)\|_{L^6(\Omega)}^2 \|v_{km}(t)\|_{L^6(\Omega)}^2 \\ &\leqslant C \|u_{km}(t)\|_V^2 \|v_{km}(t)\|_V^2 \|v_{km}(t)\|_V^2. \end{aligned}$$

Therefore the above integral (1.68) *makes sense. Similar considerations for the integral*

$$\int_\Omega |u_{km}(t)| u_{km}(t) |v_{km}(t)| z dx.$$

First estimate. To obtain the first estimate we apply similar arguments used in Theorem 1.2.2 with $\rho = 1$. In this case, we replace the function J by

$$J_1(\lambda) = \frac{1}{4}\lambda^2 - N_1\lambda^4, \tag{1.69}$$

where N_1 was defined in (1.63). We also obtain the following lemma

Lemma 1.2.15 *Consider $u^0, v^0 \in V \cap H^2(\Omega)$ and $u^1, v^1 \in V$ such that*

$$\|u^0\|_V,\ \|v^0\|_V < \lambda_1^*$$

and

$$L_1 < \frac{1}{4}(\lambda_1^*)^2,$$

where λ_1^* and L_1 were defined, respectively, in (1.61) *and* (1.62). *Then*

$$\|u_{km}(t)\|_V < \lambda_1^* \quad \text{and} \quad \|u_{km}(t)\|_V < \lambda_1^*, \quad \forall t \in [0,\infty),\ \forall k \geqslant k_0,\ \forall m.$$

Therefore, we get

$$\begin{cases} (u_{km}), (v_{km}) \text{ are bounded in } L^\infty(0,\infty;V), & \forall k \geqslant k_0,\ \forall m \in \mathbb{N}, \\ (\partial_t u_{km}), (\partial_t v_{km}) \text{ are bounded in } L^\infty(0,\infty;L^2(\Omega)), & \forall k \geqslant k_0,\ \forall m \in \mathbb{N}, \\ (\partial_t u_{km}), (\partial_t v_{km}) \text{ are bounded in } L^2(0,\infty;L^2(\Gamma_1)), & \forall k \geqslant k_0,\ \forall m \in \mathbb{N}, \\ (|u_{km}||v_{km}|v_{km}) \text{ is bounded in } L^\infty(0,\infty;L^2(\Omega)), & \forall k \geqslant k_0,\ \forall m \in \mathbb{N}, \\ (|u_{km}|u_{km}|v_{km}|) \text{ is bounded in } L^\infty(0,\infty;L^2(\Omega)), & \forall k \geqslant k_0,\ \forall m \in \mathbb{N}. \end{cases} \tag{1.70}$$

Second estimate Deriving $(1.67)_1$ with respect to t, as the function $F(\lambda) = |\lambda|, \lambda \in \mathbb{R}$ is Lipschitz continuous, $F(0) = 0$, using Brezis and Cazenave [12, Theorem A.3.12, p. 35], we obtain

$$\begin{aligned} &(\partial_t^3 u_{km}(t), w) + ((\partial_t u_{km}(t), w)) + \int_{\Gamma_1} \delta \partial_t^2 u_{km}(t) w d\Gamma \\ &\leqslant \int_\Omega |\partial_t u_{km}(t)||v_{km}(t)|^2 |w| dx + 2\int_\Omega |u_{km}(t)||v_{km}(t)||\partial_t v_{km}(t)||w| dx \end{aligned}$$

Making $w = \partial_t^2 u_{km}(t)$ in the inequality above we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\partial_t^2 u_{km}(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}\|\partial_t u_{km}(t)\|_V^2 + \int_{\Gamma_1} \delta[\partial_t^2 u_{km}(t)]^2 d\Gamma \\ &\leqslant \int_\Omega |\partial_t u_{km}(t)||\partial_t^2 u_{km}(t)||v_{km}(t)|^2 dx + 2\int_\Omega |u_{km}(t)||\partial_t^2 u_{km}(t)||v_{km}(t)||\partial_t v_{km}(t)| dx. \end{aligned} \tag{1.71}$$

Using the Hölder inequality, the Sobolev embedding $V \hookrightarrow L^6(\Omega)$ and (1.70) we have

$$\begin{aligned}\int_\Omega |\partial_t u_{km}(t)||\partial_t^2 u_{km}(t)||v_{km}(t)|^2 dx &\leqslant \|\partial_t u_{km}(t)\|_{L^6(\Omega)}\|v_{km}(t)\|^2_{L^6(\Omega)}\|\partial_t^2 u_{km}(t)\|_{L^2(\Omega)}\\ &\leqslant C\|\partial_t u_{km}(t)\|_V\|\partial_t^2 u_{km}(t)\|_{L^2(\Omega)}\\ &\leqslant C(\|\partial_t u_{km}(t)\|_V^2+\|\partial_t^2 u_{km}(t)\|^2_{L^2(\Omega)}).\end{aligned} \tag{1.72}$$

Analogously, we obtain

$$\int_\Omega |u_{km}(t)||\partial_t^2 u_{km}(t)||v_{km}(t)||\partial_t v_{km}(t)|dx \leqslant C(\|\partial_t v_{km}(t)\|_V^2+\|\partial_t^2 u_{km}(t)\|^2_{L^2(\Omega)}), \tag{1.73}$$

where C denote the several constant independent of k and m.

Combining (1.72) and (1.73) with (1.71) and using the fact that $\delta(x) \geqslant \delta_0 > 0$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial_t^2 u_{km}(t)\|^2_{L^2(\Omega)}+\frac{1}{2}\frac{d}{dt}\|\partial_t u_{km}(t)\|_V^2+\delta_0\int_{\Gamma_1}[\partial_t^2 u_{km}(t)]^2 d\Gamma\\ &\leqslant C(\|\partial_t u_{km}(t)\|_V^2+\|\partial_t v_{km}(t)\|_V^2+2\|\partial_t^2 u_{km}(t)\|^2_{L^2(\Omega)}).\end{aligned}$$

In similar way

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial_t^2 v_{km}(t)\|^2_{L^2(\Omega)}+\frac{1}{2}\frac{d}{dt}\|\partial_t v_{km}(t)\|_V^2+\delta_0\int_{\Gamma_1}[\partial_t^2 v_{km}(t)]^2 d\Gamma\\ &\leqslant C(\|\partial_t u_{km}(t)\|_V^2+\|\partial_t v_{km}(t)\|_V^2+2\|\partial_t^2 v_{km}(t)\|^2_{L^2(\Omega)}).\end{aligned}$$

Adding the last inequalities above and integrating on $[0,t]$, we get

$$\begin{aligned}&\frac{1}{2}(\|\partial_t^2 u_{km}(t)\|^2_{L^2(\Omega)}+\|\partial_t^2 v_{km}(t)\|^2_{L^2(\Omega)}+\|\partial_t u_{km}(t)\|_V^2+\|\partial_t v_{km}(t)\|_V^2)\\ &+\delta_0\int_0^t\int_{\Gamma_1}[\partial_t^2 u_{km}(s)]^2 d\Gamma ds+\delta_0\int_0^t\int_{\Gamma_1}[\partial_t^2 v_{km}(s)]^2 d\Gamma ds\\ &\leqslant \frac{1}{2}(\|\partial_t^2 u_{km}(0)\|^2_{L^2(\Omega)}+\|\partial_t^2 v_{km}(0)\|^2_{L^2(\Omega)}+\|u_k^1\|_V^2+\|v_k^1\|_V^2)\\ &+\int_0^t C(\|\partial_t^2 u_{km}(s)\|^2_{L^2(\Omega)}+\|\partial_t^2 v_{km}(s)\|^2_{L^2(\Omega)}+\|\partial_t u_{km}(s)\|_V^2+\|\partial_t v_{km}(s)\|_V^2)ds.\end{aligned} \tag{1.74}$$

We need to bound $\|\partial_t^2 u_{km}(0)\|^2_{L^2(\Omega)}$ and $\|\partial_t^2 v_{km}(0)\|^2_{L^2(\Omega)}$ by a constant independent of k and m. This is one of the key points of the proof. These bounds are obtained thanks to the choice of the special basis of $V \cap H^2(\Omega)$. In fact, taking $t = 0$ in $(1.67)_1$ we obtain

$$(\partial_t^2 u_{km}(0), w)+((u_{km}(0), w))+\int_{\Gamma_1}\delta(x)\partial_t u_{km}(0) w d\Gamma+\int_\Omega |u_{km}(0)||v_{km}(0)|v_{km}(0) w dx = 0. \tag{1.75}$$

We have $u_{km}(0) = u_k^0$, $\partial_t u_{km}(0) = u_k^1$ and $\dfrac{\partial u_k^0}{\partial \vec{n}} = -\delta(\cdot)u_k^1$ on Γ_1. Applying Green's formula to (1.75), we see that

$$(\partial_t^2 u_{km}(0), w) = (\Delta u_k^0, w) + \int_\Omega |u_k^0||v_k^0|v_k^0 w dx, \quad \forall w \in V_m^k.$$

Taking $w = \partial_t^2 u_{km}(0)$ in this equality, using Hölder inequality and observing the convergences (1.66), we have

$$\|\partial_t^2 u_{km}(0)\|_{L^2(\Omega)} \leqslant \|\Delta u_k^0\|_{L^2(\Omega)} + \|u_k^0\|_{L^6(\Omega)}\|v_k^0\|^2_{L^6(\Omega)} \leqslant C, \quad \forall k, m.$$

Thus $(\partial_t^2 u_{km}(0))$ is bounded in $L^2(\Omega)$, for all k, m. In similar way $(\partial_t^2 v_{km}(0))$ is bounded in $L^2(\Omega)$, for all k, m.

From (1.74), observing the fact $(\partial_t^2 u_{km}(0)), (\partial_t^2 v_{km}(0))$ are bounded in $L^2(\Omega)$ and the convergences (1.66) we have

$$\begin{aligned}
&\frac{1}{2}(\|\partial_t^2 u_{km}(t)\|^2_{L^2(\Omega)} + \|\partial_t^2 v_{km}(t)\|^2_{L^2(\Omega)} + \|\partial_t u_{km}(t)\|^2_V + \|\partial_t v_{km}(t)\|^2_V) \\
&+ \delta_0 \int_0^t \int_{\Gamma_1} [\partial_t^2 u_{km}(s)]^2 d\Gamma ds + \delta_0 \int_0^t \int_{\Gamma_1} [\partial_t^2 v_{km}(s)]^2 d\Gamma ds \\
&\leqslant C + \int_0^t C(\|\partial_t^2 u_{km}(s)\|^2_{L^2(\Omega)} + \|\partial_t^2 v_{km}(s)\|^2_{L^2(\Omega)} + \|\partial_t u_{km}(s)\|^2_V + \|\partial_t v_{km}(s)\|^2_V) ds.
\end{aligned}$$

Therefore by Gronwall's inequality there exists $C(t),\ t > 0$, such that

$$\begin{aligned}
&\|\partial_t^2 u_{km}(t)\|^2_{L^2(\Omega)} + \|\partial_t^2 v_{km}(t)\|^2_{L^2(\Omega)} + \|\partial_t u_{km}(t)\|^2_V + \|\partial_t v_{km}(t)\|^2_V \\
&+ \int_0^t \int_{\Gamma_1} [\partial_t^2 u_{km}(s)]^2 d\Gamma ds + \int_0^t \int_{\Gamma_1} [\partial_t^2 v_{km}(s)]^2 d\Gamma ds \leqslant C(t),
\end{aligned}$$

it follows that

$$\begin{cases}
(\partial_t u_{km}), (\partial_t v_{km}) \text{ are bounded in } L^\infty_{loc}(0, \infty; V); \\
(\partial_t^2 u_{km}), (\partial_t^2 v_{km}) \text{ are bounded in } L^\infty_{loc}(0, \infty; L^2(\Omega)); \\
(\partial_t^2 u_{km}), (\partial_t^2 v_{km}) \text{ are bounded in } L^2_{loc}(0, \infty; L^2(\Gamma_1)).
\end{cases} \tag{1.76}$$

Passage to the limit in m. Estimates (1.70), (1.76) and using similar arguments to

Theorem 1.2.2 with $\rho = 1$ allow us

$$\begin{cases} u_{km} \overset{*}{\rightharpoonup} u_k \quad \text{and} \quad v_{km} \overset{*}{\rightharpoonup} v_k & \text{in} \quad L^\infty(0,\infty;V), \\ \partial_t u_{km} \overset{*}{\rightharpoonup} \partial_t u_k \quad \text{and} \quad \partial_t v_{km} \overset{*}{\rightharpoonup} \partial_t v_k & \text{in} \quad L^\infty_{loc}(0,\infty;V), \\ \partial_t^2 u_{km} \overset{*}{\rightharpoonup} \partial_t^2 u_k \quad \text{and} \quad \partial_t^2 v_{km} \overset{*}{\rightharpoonup} \partial_t^2 v_k & \text{in} \quad L^\infty_{loc}(0,\infty;L^2(\Omega)), \\ \partial_t u_{km} \rightharpoonup \partial_t u_k \quad \text{and} \quad \partial_t v_{km} \rightharpoonup \partial_t v_k & \text{in} \quad L^2(0,\infty;L^2(\Gamma_1)), \\ \partial_t^2 u_{km} \rightharpoonup \partial_t^2 u_k \quad \text{and} \quad \partial_t^2 v_{km} \rightharpoonup \partial_t^2 v_k & \text{in} \quad L^2_{loc}(0,\infty;L^2(\Gamma_1)), \\ |u_{km}||v_{km}|v_{km} \overset{*}{\rightharpoonup} |u_k||v_k|v_k & \text{in} \quad L^\infty(0,\infty;L^2(\Omega)), \\ |u_{km}|u_{km}|v_{km}| \overset{*}{\rightharpoonup} |u_k|u_k|v_k| & \text{in} \quad L^\infty(0,\infty;L^2(\Omega)). \end{cases} \tag{1.77}$$

From $(1.76)_2$ and trace theorem we obtain

$$(\partial_t u_{km}), (\partial_t v_{km}) \text{ are bounded in } \ L^\infty_{loc}(0,\infty;H^{\frac{1}{2}}(\Gamma_1)),$$

and thus

$$\begin{aligned} \partial_t u_{km} &\overset{*}{\rightharpoonup} \partial_t u_k \quad \text{in} \quad L^\infty_{loc}(0,\infty;H^{\frac{1}{2}}(\Gamma_1)); \\ \partial_t v_{km} &\overset{*}{\rightharpoonup} \partial_t v_k \quad \text{in} \quad L^\infty_{loc}(0,\infty;H^{\frac{1}{2}}(\Gamma_1)). \end{aligned} \tag{1.78}$$

Multiplying both sides of the approximate equation $(1.67)_1$ by $\varphi \in \mathscr{D}(0,\infty)$, integrating in $[0,\infty)$, using the convergences $(1.77)_{1,3,6}$ and $(1.78)_1$, we obtain

$$\begin{aligned} &\int_0^\infty (\partial_t^2 u_k(t), w)\varphi(t)dt + \int_0^\infty ((u_k(t), w))\varphi(t)dt + \int_0^\infty \int_{\Gamma_1} \delta\partial_t u_k(t) w\varphi(t) d\Gamma dt \\ &+ \int_0^\infty (|u_k(t)||v_k(t)|v_k(t), w)\varphi(t)dt = 0, \quad \forall w \in V_m^k, \ \forall \varphi \in \mathscr{D}(0,\infty). \end{aligned} \tag{1.79}$$

Since V_m^k is dense in $V \cap H^2(\Omega)$ it follows that (1.79) is true for all $w \in V \cap H^2(\Omega)$. In similar way, we find

$$\begin{aligned} &\int_0^\infty (\partial_t^2 v_k(t), w)\varphi(t)dt + \int_0^\infty ((v_k(t), w))\varphi(t)dt + \int_0^\infty \int_{\Gamma_1} \delta\partial_t u_k(t) w\varphi(t) d\Gamma dt \\ &+ \int_0^\infty (|u_k(t)|u_k(t)|v_k(t)|, w)\varphi(t)dt = 0, \quad \forall w \in V \cap H^2(\Omega), \ \forall \varphi \in \mathscr{D}(0,\infty). \end{aligned}$$

We can see that the estimates (1.70) and (1.76) are also independent of k. Therefore by the same argument used to obtain (1.77) and (1.78) we get a diagonal sequence

$(u_{k_k}), (v_{k_k})$, still denoted by $(u_k), (v_k)$, and functions $u, v : \Omega \times [0, \infty) \to \mathbb{R}$ such that

$$\begin{cases} u_k \overset{*}{\rightharpoonup} u \quad \text{and} \quad v_k \overset{*}{\rightharpoonup} v & \text{in} \quad L^\infty(0,\infty;V); \\ \partial_t u_k \overset{*}{\rightharpoonup} \partial_t u \quad \text{and} \quad \partial_t v_k \overset{*}{\rightharpoonup} \partial_t v & \text{in} \quad L^\infty_{loc}(0,\infty;V); \\ \partial_t^2 u_k \overset{*}{\rightharpoonup} \partial_t^2 u \quad \text{and} \quad \partial_t^2 v_k \overset{*}{\rightharpoonup} \partial_t^2 v & \text{in} \quad L^\infty_{loc}(0,\infty;L^2(\Omega)); \\ \partial_t u_k \rightharpoonup \partial_t u \quad \text{and} \quad \partial_t v_k \rightharpoonup \partial_t v & \text{in} \quad L^2(0,\infty;L^2(\Gamma_1)); \\ \partial_t^2 u_k \rightharpoonup \partial_t^2 u \quad \text{and} \quad \partial_t^2 v_k \rightharpoonup \partial_t^2 v & \text{in} \quad L^2_{loc}(0,\infty;L^2(\Gamma_1)); \\ |u_k||v_k|v_k \overset{*}{\rightharpoonup} |u||v|v & \text{in} \quad L^\infty(0,\infty;L^2(\Omega)); \\ |u_k|u_k|v_k| \overset{*}{\rightharpoonup} |u|u|v| & \text{in} \quad L^\infty(0,\infty;L^2(\Omega)); \\ \partial_t u_k \overset{*}{\rightharpoonup} \partial_t u \quad \text{and} \quad \partial_t v_k \overset{*}{\rightharpoonup} \partial_t v & \text{in} \quad L^\infty_{loc}(0,\infty;H^{\frac{1}{2}}(\Gamma_1)). \end{cases} \tag{1.80}$$

Taking the limit in (1.79), using convergences $(1.80)_{1,3,6,8}$ and observing that $V \cap H^2(\Omega)$ is dense in V, we obtain

$$\begin{aligned} &\int_0^\infty (\partial_t^2 u(t), w)\varphi(t)dt + \int_0^\infty ((u(t), w))\varphi(t)dt + \int_0^\infty \int_{\Gamma_1} \delta \partial_t u(t) w \varphi(t) d\Gamma dt \\ &+ \int_0^\infty (|u(t)||v(t)|v(t), w)\varphi(t)dt = 0, \ \forall w \in V, \ \forall \varphi \in \mathscr{D}(0,\infty). \end{aligned} \tag{1.81}$$

In similar way, we find

$$\begin{aligned} &\int_0^\infty (\partial_t^2 v(t), z)\varphi(t)dt + \int_0^\infty ((v(t), z))\varphi(t)dt + \int_0^\infty \int_{\Gamma_1} \delta \partial_t u(t) z \varphi(t) d\Gamma dt \\ &+ \int_0^\infty (|u(t)|u(t)|v(t)|, z)\varphi(t)dt = 0, \ \forall z \in V, \ \forall \varphi \in \mathscr{D}(0,\infty). \end{aligned}$$

Taking in (1.81) $w \in \mathscr{D}(\Omega) \subset V$, it follows that

$$\partial_t^2 u - \Delta u + |u||v|v = 0 \quad \text{in} \quad \mathscr{D}'(\Omega \times (0,T)).$$

In similar way

$$\partial_t^2 v - \Delta v + |u|u|v| = 0 \quad \text{in} \quad \mathscr{D}'(\Omega \times (0,T)).$$

Therefore, by $(1.80)_{3,6}$, we get

$$\begin{aligned} &\partial_t^2 u - \Delta u + |u||v|v = 0 \quad \text{in} \quad L^\infty_{loc}(0,\infty, L^2(\Omega)), \\ &\partial_t^2 v - \Delta v + |u|u|v| = 0 \quad \text{in} \quad L^\infty_{loc}(0,\infty, L^2(\Omega)). \end{aligned} \tag{1.82}$$

Since $u \in L^\infty(0,\infty;V)$ and by $(1.82)_1$, $\Delta u \in L^\infty_{loc}(0,T;L^2(\Omega))$ then, by Milla Miranda [31] we obtain

$$\frac{\partial u}{\partial \vec{n}} \in L^\infty(0,\infty;H^{-\frac{1}{2}}(\Gamma_1)). \tag{1.83}$$

Multiplying both sides of (1.82) by $w\varphi$ with $w \in V$ and $\varphi \in \mathscr{D}(0,\infty)$, integrating on $\Omega \times (0,T)$ and using (1.83) and Green's formula

$$\int_0^\infty (\partial_t^2 u(t), w)\varphi(t)dt + \int_0^\infty ((u(t),w))\varphi(t)dt - \int_0^\infty \left\langle \frac{\partial u(t)}{\partial \vec{n}}, w \right\rangle \varphi(t)dt$$
$$+ \int_0^\infty (|u(t)|u(t)|v(t)|, w)\varphi(t)dt = 0,\ \forall w \in V,\ \forall \varphi \in \mathscr{D}(0,\infty),$$

where $\langle \cdot,\cdot \rangle$ denotes the duality paring between $H^{-\frac{1}{2}}(\Gamma_1)$ and $H^{\frac{1}{2}}(\Gamma_1)$. Comparing this last equation with (1.81), we obtain

$$\int_0^\infty \left\langle \frac{\partial u(t)}{\partial \vec{n}} + \delta\partial_t u(t), w \right\rangle \varphi(t)d\Gamma dt = 0,\ \forall w \in V,\ \forall \varphi \in \mathscr{D}(0,\infty).$$

Therefore

$$\frac{\partial u}{\partial \vec{n}} + \delta\partial_t u = 0 \quad \text{in} \quad H^{-\frac{1}{2}}(\Gamma_1).$$

In similar way

$$\frac{\partial v}{\partial \vec{n}} + \delta\partial_t v = 0 \quad \text{in} \quad H^{-\frac{1}{2}}(\Gamma_1).$$

Since $\delta\partial_t u \in L^\infty_{loc}(0,\infty,H^{\frac{1}{2}}(\Gamma_1))$, then

$$\frac{\partial u}{\partial \vec{n}} + \delta\partial_t u = 0 \quad \text{in} \quad L^\infty_{loc}(0,\infty,H^{\frac{1}{2}}(\Gamma_1)). \tag{1.84}$$

In similar way

$$\frac{\partial v}{\partial \vec{n}} + \delta\partial_t v = 0 \quad \text{in} \quad L^\infty_{loc}(0,\infty,H^{\frac{1}{2}}(\Gamma_1)). \tag{1.85}$$

To complete the proof of the Theorem 1.2.13, we shall show that $u \in L^\infty_{loc}(0,\infty,H^2(\Omega))$. In fact, note that $u \in L^\infty(0,\infty;V)$. With this, using $(1.82)_1$, (1.84) we see that u is the solution of the following boundary value problem:

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega \times [0,T], \\ u = 0 & \text{on} \quad \Gamma_0 \times [0,T], \\ \dfrac{\partial u}{\partial \vec{n}} = g & \text{on} \quad \Gamma_1 \times [0,T], \end{cases}$$

for all real number $T > 0$. Since

$$f = -\partial_t^2 u - |u||v|v \in L^\infty_{loc}(0,\infty;L^2(\Omega)) \quad \text{and} \quad g = -\delta\partial_t u \in L^\infty_{loc}(0,\infty;H^{\frac{1}{2}}(\Gamma_1)),$$

it follows by the Proposition 1.2.11 that

$$u \in L^\infty_{loc}(0,\infty; V \cap H^2(\Omega)). \tag{1.86}$$

In similar way

$$v \in L^\infty_{loc}(0,\infty; V \cap H^2(\Omega)). \tag{1.87}$$

The verification of the initial conditions follows by similar arguments used in the Theorem 1.2.2. ■

Remark 1.2.16 *From* (1.77)*, we obtain* u_k *and* v_k *in the class* (1.64)*. From* (1.79) *and using the same arguments for obtain* (1.82)*,* (1.84) *and* (1.85)*, we get*

$$\begin{cases} \partial_t^2 u_k - \Delta u_k + |u_k||v_k|v_k = 0 & in \quad L^\infty_{loc}(0,\infty; L^2(\Omega)) \\ \partial_t^2 v_k - \Delta v_k + |u_k|u_k|v_k| = 0 & in \quad L^\infty_{loc}(0,\infty; L^2(\Omega)) \\ \dfrac{\partial u_k}{\partial \nu} + \delta(\cdot)\partial_t u_k = 0 & in \quad L^\infty_{loc}(0,\infty; H^{\frac{1}{2}}(\Gamma_1)) \\ \dfrac{\partial v_k}{\partial \nu} + \delta(\cdot)\partial_t v_k = 0 & in \quad L^\infty_{loc}(0,\infty; H^{\frac{1}{2}}(\Gamma_1)). \end{cases} \tag{1.88}$$

Also using the same arguments for obtain the regularities (1.86) *and* (1.87)*, we get*

$$u_k, v_k \in L^\infty_{loc}(0,\infty; V \cap H^2(\Omega)). \tag{1.89}$$

Corollary 1.2.17 *We obtain similar results to Theorem 1.2.13 for the case* $\rho > 1$ *and* $n = 1, 2$*.*

1.3 Uniqueness of solutions

In this section we show that the solution in the case of the Theorem 1.2.13 and Corollary 1.2.17 is unique. For this we use the energy method.

Theorem 1.3.1 *The solution* $\{u, v\}$ *obtained in the Theorem 1.2.13 or Corollary 1.2.17 is unique.*

Proof. First we show the case $\rho = 1$ for $n = 3$. Let $\{u, v\}$ and $\{w, z\}$ solutions of system (1) satisfying the Theorem 1.2.13. We define

$$U = u - w \quad \text{and} \quad Z = v - z.$$

Then U and Z satisfies the following problem

$$\begin{cases} \partial_t^2 U - \Delta U + |u||v|v - |w||z|z = 0; & \\ \partial_t^2 Z - \Delta Z + |u|u|v| - |w|w|z| = 0; & \\ U = Z = 0 & \text{on } \Gamma_0; \\ \dfrac{\partial U}{\partial \vec{n}} + \delta(\cdot)\partial_t U = 0 & \text{on } \Gamma_1; \\ \dfrac{\partial Z}{\partial \vec{n}} + \delta(\cdot)\partial_t Z = 0 & \text{on } \Gamma_1, \end{cases} \tag{1.90}$$

with initial conditions $U(0) = 0, \partial_t U(0) = 0, Z(0) = 0$ and $\partial_t Z(0) = 0$.

Remark 1.3.2 *We note that*

$$\partial_t^2 U(t) \in L^2(\Omega) \quad \textit{and} \quad \partial_t U(t) \in L^2(\Omega).$$

Therefore, make sense to calculate the duality $\langle \partial_t U(t), \partial_t U(t)\rangle$. Thus, the uniqueness results from the energy method.

Taking the scalar product of $(1.90)_1$ and $(1.90)_2$ with $\partial_t U$ and $\partial_t Z$, respectively and integrating on $[0, T]$ we have

$$\begin{aligned} &\frac{1}{2}\|\partial_t U(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|U(t)\|^2_V + \delta_0 \int_0^t \|\partial_t U(s)\|^2_{L^2(\Gamma_1)} dt \\ &+ \int_0^t (|u(s)||v(s)|v(s) - |w(s)||z(s)|z(s), \partial_t U(s)) dt \leqslant 0. \end{aligned} \tag{1.91}$$

$$\begin{aligned} &\frac{1}{2}\|\partial_t Z(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|Z(t)\|^2_V + \delta_0 \int_0^t \|\partial_t Z(s)\|^2_{L^2(\Gamma_1)} dt \\ &+ \int_0^t (|u(s)|u(s)|v(s)| - |w(s)|w(s)|z(s)|, \partial_t Z(s)) dt \leqslant 0. \end{aligned} \tag{1.92}$$

Adding (1.91) and (1.92) and denoting by M and N the left hand side of (1.91) and (1.92), respectively, we have

$$\frac{1}{2}\|\partial_t U(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\partial_t Z(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|U(t)\|^2_V + \frac{1}{2}\|Z(t)\|^2_V \leqslant M + N. \tag{1.93}$$

We write M and N respectively by

$$\begin{aligned} M = &\int_0^t ([|u(s)| - |w(s)|]|v(s)|v(s), \partial_t U(s)) dt \\ &+ \int_0^t ([|v(s)|v(s) - |z(s)|z(s)]|w(s)|, \partial_t U(s)) dt \end{aligned} \tag{1.94}$$

and

$$N = \int_0^t ([|v(s)| - |z(s)|]|u(s)|u(s), \partial_t Z(s))dt \\ + \int_0^t ([|u(s)|u(s) - |w(s)|w(s)]|z(s)|, \partial_t Z(s))dt. \tag{1.95}$$

We examine each of the integrals above.

- Analysis of $I_1 = ([|u(s)| - |w(s)|]|v(s)|v(s), \partial_t U(s))$:

  Using the fact that the function $F(\lambda) = |\lambda|$ is Lipschitz continuous and $F(0) = 0$, it follows by Brezis and Cazenave [12, Theorem A.3.12, p. 35] that

$$\begin{aligned} I_1 &= ([|u(s)| - |w(s)|]|v(s)|v(s), \partial_t U(s)) \\ &\leqslant \int_\Omega ||u(s)| - |w(s)|||v(s)|^2|\partial_t U(s)|dx \\ &\leqslant \int_\Omega |v(s)|^2|U(s)||\partial_t U(s)|dx \\ &\leqslant \|v(t)\|^2_{L^6(\Omega)}\|\partial_t U(s)\|_{L^2(\Omega)}\|U(s)\|_{L^6(\Omega)} \\ &\leqslant C\|\partial_t U(s)\|_{L^2(\Omega)}\|U(s)\|_V, \end{aligned} \tag{1.96}$$

  for some positive constant C.

- Analysis of $I_2 = ([|v(s)|v(s) - |z(s)|z(s)]|w(s)|, \partial_t U(s))$:

  Again by Brezis and Cazenave [12, Theorem A.3.12, p. 35] and using the mean value theorem we can conclude that there exists $C > 0$ such that

$$\begin{aligned} I_2 &= ([|v(s)|v(s) - |z(s)|z(s)]|w(s)|, \partial_t U(s)) \\ &\leqslant \int_\Omega ||v(s)|v(s) - z(s)z(s)||w(s)||\partial_t U(s)|dx \\ &\leqslant C\int_\Omega [|v(s)| + |z(s)|]|w(s)||Z(s)||\partial_t U(s)|dx \\ &\leqslant C(\|v(s)\|_{L^6(\Omega)} + \|z(s)\|_{L^6(\Omega)})\|w(s)\|_{L^6(\Omega)}\|\partial_t U(s)\|_{L^2(\Omega)}\|Z(s)\|_{L^6(\Omega)} \\ &\leqslant C\|\partial_t U(s)\|_{L^2(\Omega)}\|Z(s)\|_V. \end{aligned} \tag{1.97}$$

  Therefore from (1.94), (1.96) and (1.97) we obtain

$$M \leqslant C\int_0^t (\|U(s)\|_V + \|Z(s)\|_V)\|\partial_t U(s)\|_{L^2(\Omega)}dt.$$

By a similar argument, with (1.95) we also find

$$N \leqslant C \int_0^t (\|U(s)\|_V + \|Z(t)\|_V)\|\partial_t Z(s)\|_{L^2(\Omega)} dt. \tag{1.98}$$

It follows that

$$|M| \leqslant \frac{C}{2}\int_0^t \|U(s)\|_V^2 dt + \frac{C}{2}\int_0^t \|Z(s)\|_V^2 dt + C\int_0^t \|\partial_t U(s)\|^2_{L^2(\Omega)} dt, \tag{1.99}$$

for some constant $C > 0$. Similarly from (1.98) we have

$$|N| \leqslant \frac{C}{2}\int_0^t \|U(s)\|_V^2 dt + \frac{C}{2}\int_0^t \|Z(s)\|_V^2 dt + C\int_0^t \|\partial_t Z(s)\|^2_{L^2(\Omega)} dt. \tag{1.100}$$

Combining (1.99) and (1.100) with (1.93), we get

$$\begin{aligned} &\|\partial_t U(s)\|^2_{L^2(\Omega)} + \|\partial_t Z(s)\|^2_{L^2(\Omega)} + \|U(t)\|_V^2 + \|Z(t)\|_V^2 \\ &\leqslant 2C\int_0^t (\|\partial_t U(s)\|^2_{L^2(\Omega)} + \|\partial_t Z(s)\|^2_{L^2(\Omega)} + \|U(s)\|_V^2 + \|Z(s)\|_V^2) dt \end{aligned} \tag{1.101}$$

Thus from (1.101) and Gronwall's inequality we get $U(t) = Z(t) = 0$ for all $0 \leqslant t \leqslant T$.

For the cases $n = 1, 2$, we have $V \hookrightarrow L^q(\Omega),\ 1 \leqslant q < \infty$. Then the estimates (1.99) and (1.100) are hold. The proof in this case follows by similar arguments used for $n = 3$. ■

1.4 Asymptotic behavior: energy estimates

Next we state the result on the decay of solutions of the problem (1) in the cases of the Theorem 1.2.13 and Corollary 1.2.17. To this we assume that there exists a point $x^0 \in \mathbb{R}^n$, such that

$$\Gamma_0 = \{x \in \Gamma : m(x)\cdot \vec{n}(x) \leqslant 0\} \quad \text{and} \quad \Gamma_1 = \{x \in \Gamma : m(x)\cdot \vec{n}(x) > 0\},$$

where $m(x) = x - x^0,\ x \in \mathbb{R}^n$, see Figure 1.1.

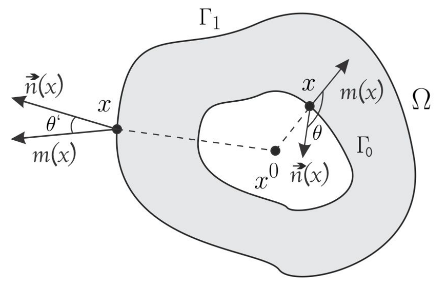

Figure 1.1: The sets Γ_0 and Γ_1.

In this section we consider $\delta(x) = m(x)\cdot\vec{n}(x)$ and $R = \max_{x\in\overline{\Omega}} \|m(x)\|_{\mathbb{R}^n}$. The energy of system (1) with $\rho = 1$ is defined by

$$E(t) = \frac{1}{2}\Big(\|\partial_t u(t)\|^2_{L^2(\Omega)} + \|\partial_t v(t)\|^2_{L^2(\Omega)} + \|u(t)\|^2_V + \|v(t)\|^2_V\Big) + \frac{1}{2}\int_\Omega |u(t)|u(t)|v(t)|v(t)dx.$$

We have the following result:

Theorem 1.4.1 *Let $\{u, v\}$ be the solution obtained in Theorem 1.2.13. Then*

$$E(t) \leqslant 3E(0)e^{-\frac{\tau}{3}t}, \quad \forall\, t \in [0,\infty), \tag{1.102}$$

where

$$\begin{aligned} &\tau = \min\left\{\frac{1}{2P}, \frac{m_0}{D}\right\} > 0;\\ &P = 4\left(2R + \frac{n-1}{2} + \frac{n-1}{2\lambda_1}\right);\\ &D = R^3 + R + R^2(n-1)^2c_3^2;\\ &m_0 = \min\{m(x)\cdot\vec{n}(x); x \in \Gamma_1\} > 0. \end{aligned} \tag{1.103}$$

Proof. To prove the Theorem 1.4.1 we show that the energy

$$\begin{aligned} E_k(t) = &\frac{1}{2}\Big(\|\partial_t u_k(t)\|^2_{L^2(\Omega)} + \|\partial_t v_k(t)\|^2_{L^2(\Omega)} + \|u_k(t)\|^2_V + \|v_k(t)\|^2_V\Big)\\ &+ \frac{1}{2}\int_\Omega |u_k(t)|u_k(t)|v_k(t)|v_k(t)dx \end{aligned} \tag{1.104}$$

associated with the solution $\{u_k(t), v_k(t)\}$ of the equations in (1.88) satisfies the inequality (1.102). Thus, the exponential decay of $E(t)$ be obtained by taking the lim inf of $E_k(t)$ as $k \to \infty$.

Now, we introduce the function

$$\begin{aligned} \psi_k(t) = &\,2(\partial_t u_k(t), m\cdot\nabla u_k(t)) + (n-1)(\partial_t u_k(t), u_k(t))\\ &+ 2(\partial_t v_k(t), m\cdot\nabla v_k(t)) + (n-1)(\partial_t v_k(t), v_k(t)). \end{aligned} \tag{1.105}$$

For $\varepsilon > 0$, we introduce the perturbed energy

$$E_{k\varepsilon}(t) := E_k(t) + \varepsilon\psi_k(t).$$

First we prove that $E_{k\varepsilon}(t)$ and $E_k(t)$ are equivalent. Then we show that

$$E'_{k\varepsilon}(t) \leqslant -\varepsilon E_k(t). \tag{1.106}$$

Equivalence between $E_{k\varepsilon}(t)$ and $E_k(t)$. First of all, we note that

$$A_k(t) := \frac{1}{4}(\|u_k(t)\|_V^2 + \|v_k(t)\|_V^2) + \frac{1}{2}\int_\Omega (|u_k(t)|u_k(t))(|v_k(t)|v_k(t))dx \geqslant 0,\ \forall t \in [0,\infty). \tag{1.107}$$

In fact,

$$\frac{1}{2}\left|\int_\Omega (|u_k(t)|u_k(t))(|v_k(t)|v_k(t))dx\right| \leqslant \frac{1}{4}c_1^4(\|u_k(t)\|_V^4 + \|v_k(t)\|_V^4).$$

Then

$$A_k(t) \geqslant \frac{1}{4}\|u_k(t)\|_V^2 - \frac{1}{4}c_1^4\|u_k(t)\|_V^4 + \frac{1}{4}\|v_k(t)\|_V^2 - \frac{1}{4}c_1^4\|v_k(t)\|_V^4. \tag{1.108}$$

As $-\frac{1}{4}c_1^4 > -N_1$, we obtain

$$\frac{1}{4}\|u_k(t)\|_V^2 - \frac{1}{4}c_1^4\|u_k(t)\|_V^4 \geqslant \frac{1}{4}\|u_k(t)\|_V^2 - N_1\|u_k(t)\|_V^4.$$

If we take the limit $m \to \infty$ in Lemma 1.2.15, we find

$$J_1(\|u_k(t)\|_V) = \frac{1}{4}\|u_k(t)\|_V^2 - N_1\|u_k(t)\|_V^4 \geqslant 0, \quad \forall t \in [0,\infty). \tag{1.109}$$

In similar way

$$\frac{1}{4}\|v_k(t)\|_V^2 - N_1\|v_k(t)\|_V^4 \geqslant 0, \quad \forall t \in [0,\infty). \tag{1.110}$$

Taking into account (1.109) and (1.110) in (1.108), we derive (1.107).

Observe that

$$E_k(t) \geqslant \frac{1}{4}(\|\partial_t u_k(t)\|_{L^2(\Omega)}^2 + \|\partial_t v_k(t)\|_{L^2(\Omega)}^2) + \frac{1}{4}(\|u_k(t)\|_V^2 + \|v_k(t)\|_V^2) + A_k(t).$$

Then by (1.107)

$$E_k(t) \geqslant \frac{1}{4}(\|\partial_t u_k(t)\|_{L^2(\Omega)}^2 + \|\partial_t v_k(t)\|_{L^2(\Omega)}^2) + \frac{1}{4}(\|u_k(t)\|_V^2 + \|v_k(t)\|_V^2), \quad \forall t \in [0,\infty). \tag{1.111}$$

On the other side, we have

$$|\psi_k(t)| \leqslant \left(R + \frac{n-1}{2}\right)(\|\partial_t u_k(t)\|^2_{L^2(\Omega)} + \|\partial_t v_k(t)\|^2_{L^2(\Omega)})$$
$$+ \left(R + \frac{n-1}{2\lambda_1}\right)(\|u_k(t)\|^2_V + \|v_k(t)\|^2_V),$$

where λ_1 is the first eigenvalue of the spectral problem $((u,v)) = \lambda(u,v),\ u,v \in V$. Thus

$$|\psi_k(t)| \leqslant \frac{P}{4}(\|\partial_t u_k(t)\|^2_{L^2(\Omega)} + \|\partial_t v_k(t)\|^2_{L^2(\Omega)} + \|u_k(t)\|^2_V + \|v_k(t)\|^2_V), \tag{1.112}$$

where P was defined in (1.103).

From (1.111) and (1.112) it follows that

$$|\psi_k(t)| \leqslant PE_k(t), \quad \forall t \in [0,\infty).$$

Since that

$$|E_{k\varepsilon}(t) - E_k(t)| = |\varepsilon\psi_k(t)| \leqslant \varepsilon P E_k(t)$$

we have

$$E_k(t)(1-\varepsilon P) \leqslant E_{k\varepsilon}(t) \leqslant (1+\varepsilon P)E_k(t).$$

Then

$$\frac{1}{2}E_k(t) \leqslant E_{k\varepsilon_1}(t) \leqslant \frac{3}{2}E_k(t), \quad 0 < \varepsilon_1 \leqslant \frac{1}{2P}. \tag{1.113}$$

From now on, to simplify the notation we will do not write the variable t.

Proof of (1.106). Multiplying $(1.88)_1$ and $(1.88)_2$ by $\partial_t u_k$ and $\partial_t v_k$, respectively, using the fact $\delta(x) = m(x)\cdot\vec{n}(x)$ the hypothesis (1.103), we get

$$E'_k \leqslant -m_0(\|\partial_t u_k\|^2_{L^2(\Gamma_1)} + \|\partial_t v_k\|^2_{L^2(\Gamma_1)}). \tag{1.114}$$

The idea to prove (1.106) is to find that

$$\psi'_k \leqslant -E_k - \left[\frac{1}{4}(\|u_k\|^2_V + \|v_k\|^2_V) - N_1(\|u_k\|^4_V + \|v_k\|^4_V)\right]$$
$$+ D(\|\partial_t u_k\|^2_{L^2(\Gamma_1)} + \|\partial_t v_k\|^2_{L^2(\Gamma_1)}),$$

where N_1 and D are positive constants independent of k.

Then to prove an existence theorem of solutions which permits us to say

$$\frac{1}{4}(\|u_k\|^2_V + \|v_k\|^2_V) - N_1(\|u_k\|^4_V + \|v_k\|^4_V) \geqslant 0, \quad \forall t \in [0,\infty).$$

Thus

$$E'_{k\varepsilon} = E'_k + \varepsilon\psi'_k \leqslant -\varepsilon E_k - (m_0 - \varepsilon D)(\|\partial_t u_k\|^2_{L^2(\Gamma_1)} + \|\partial_t v_k\|^2_{L^2(\Gamma_1)}).$$

For small $\varepsilon > 0$, we obtain (1.106).

Differentiating the function ψ_k, we obtain

$$\psi'_k = 2(\partial_t^2 u_k, m\cdot\nabla u_k) + 2(\partial_t u_k, m\cdot\nabla\partial_t u_k) + (n-1)(\partial_t^2 u_k, u_k) + (n-1)\|\partial_t u_k\|^2_{L^2(\Omega)}$$
$$+ 2(\partial_t^2 v_k, m\cdot\nabla v_k) + 2(\partial_t v_k, m\cdot\nabla\partial_t v_k) + (n-1)(\partial_t^2 v_k, v_k) + (n-1)\|\partial_t v_k\|^2_{L^2(\Omega)}.$$

From $(1)_1$ and $(1)_2$, we find

$$\begin{aligned}\psi'_k &= 2(\Delta u_k, m\cdot\nabla u_k) - 2(|u_k||v_k|v_k, m\cdot\nabla u_k) + 2(\partial_t u_k, m\cdot\nabla\partial_t u_k)\\ &\quad + (n-1)(\Delta u_k, u_k) - (n-1)(|u_k||v_k|v_k, u_k) + (n-1)\|\partial_t u_k\|^2_{L^2(\Omega)}\\ &\quad + 2(\Delta v_k, m\cdot\nabla v_k) - 2(|u_k|u_k|v_k|, m\cdot\nabla v_k) + 2(\partial_t v_k, m\cdot\nabla v_k)\\ &\quad + (n-1)(\Delta v_k, v_k) - (n-1)(|u_k|u_k|v_k|, v_k) + (n-1)\|\partial_t v_k\|^2_{L^2(\Omega)}\\ &=: I_1 + \cdots + I_{12}.\end{aligned} \tag{1.115}$$

respectively.

Our goal is to derive a bound above for each terms on the right hand side of (1.115).

- The regularity (1.89) allows us to obtain Rellich's identity for u_k, see Komornik and Zuazua [25, Remark 2.3, p. 41]; that is,

  $$I_1 = 2(\Delta u_k, m\cdot\nabla u_k) = (n-2)\|u_k\|^2_V + 2\int_\Gamma \frac{\partial u_k}{\partial\vec{n}}(m\cdot\nabla u_k)d\Gamma - \int_\Gamma (m\cdot\vec{n})|\nabla u_k|^2 d\Gamma.$$

  Since $|\nabla u_k|^2 = \left(\frac{\partial u_k}{\partial\vec{n}}\right)^2$ and $m\cdot\nabla u_k = (m\cdot\vec{n})\frac{\partial u_k}{\partial\vec{n}}$ on Γ_0, then

  $$\int_\Gamma (m\cdot\vec{n})|\nabla u_k|^2 d\Gamma = \int_{\Gamma_0}(m\cdot\vec{n})\left(\frac{\partial u_k}{\partial\vec{n}}\right)^2 d\Gamma + \int_{\Gamma_1}(m\cdot\vec{n})|\nabla u_k|^2 d\Gamma$$

  and

  $$\int_\Gamma \frac{\partial u_k}{\partial\vec{n}}(m\cdot\nabla u_k)d\Gamma = \int_{\Gamma_0}(m\cdot\vec{n})\left(\frac{\partial u_k}{\partial\vec{n}}\right)^2 d\Gamma + \int_{\Gamma_1}\frac{\partial u_k}{\partial\vec{n}}(m\cdot\nabla u_k)d\Gamma.$$

  Thus

  $$\begin{aligned}I_1 &= (n-2)\|u_k\|^2_V + 2\int_{\Gamma_0}(m\cdot\vec{n})\left(\frac{\partial u_k}{\partial\vec{n}}\right)^2 d\Gamma + 2\int_{\Gamma_1}\frac{\partial u_k}{\partial\vec{n}}(m\cdot\nabla u_k)d\Gamma\\ &\quad - \int_{\Gamma_0}(m\cdot\vec{n})\left(\frac{\partial u_k}{\partial\vec{n}}\right)^2 d\Gamma - \int_{\Gamma_1}(m\cdot\vec{n})|\nabla u_k|^2 d\Gamma.\end{aligned} \tag{1.116}$$

Since $\dfrac{\partial u_k}{\partial \vec{n}} + (m\cdot\vec{n})\partial_t u_k = 0$ on Γ_1 we have

$$\begin{aligned}\left|2\int_{\Gamma_1}\frac{\partial u_k}{\partial \vec{n}}(m\cdot\nabla u_k)d\Gamma\right| &= \left|2\int_{\Gamma_1}(m\cdot\vec{n})\partial_t u_k(m\cdot\nabla u_k)d\Gamma\right| \\ &\leqslant R^3\int_{\Gamma_1}|\partial_t u_k|^2 d\Gamma + \int_{\Gamma_1}(m\cdot\vec{n})|\nabla u_k|^2.\end{aligned} \tag{1.117}$$

Substituting (1.117) in (1.116), making the reduction of similar terms and observing that $m\cdot\vec{n}\leqslant 0$ on Γ_0, we get

$$I_1 \leqslant (n-2)\|u_k\|_V^2 + R^3\|\partial_t u_k\|^2_{L^2(\Gamma_1)}. \tag{1.118}$$

In similar way

$$I_7 \leqslant (n-2)\|v_k\|_V^2 + R^3\|\partial_t v_k\|^2_{L^2(\Gamma_1)}. \tag{1.119}$$

- Note that

$$\begin{aligned} I_2 &= -2(|u_k||v_k|v_k, m\cdot\nabla u_k) \\ &= -2\int_\Omega |u_k||v_k|v_k(m\cdot\nabla u_k)dx \\ &= -2\sum_{i=1}^n\int_\Omega |u_k||v_k|v_k m_i\frac{\partial u_k}{\partial x_i}dx \\ &= -\sum_{i=1}^n\int_\Omega m_i\left[\frac{\partial}{\partial x_i}(|u_k|u_k)\right](|v_k|v_k)dx.\end{aligned}$$

Similarly we find

$$I_8 = -2(|u_k|u_k|v_k|, m\cdot\nabla v_k) = -\sum_{i=1}^n\int_\Omega m_i(|u_k|u_k)\left[\frac{\partial}{\partial x_i}(|v_k|v_k)\right]dx.$$

From Green's Theorem, using the fact $\dfrac{\partial m_i}{\partial x_i}=1$ and $u_k=v_k=0$ on Γ_0, we have

$$I_2+I_8 = n\int_\Omega(|u_k|u_k)(|v_k|v_k)dx - \int_{\Gamma_1}(m\cdot\vec{n})(|u_k|u_k)(|v_k|v_k)d\Gamma.$$

Using (1.59), we have

$$\left|-\int_{\Gamma_1}(m\cdot\vec{n})(|u_k|u_k)(|v_k|v_k)d\Gamma\right| \leqslant \frac{Rc_2^4}{2}\left(\|u_k\|_V^4+\|v_k\|_V^4\right).$$

Therefore

$$I_2+I_8 \leqslant n\int_\Omega(|u_k|u_k)(|v_k|v_k)dx + \frac{Rc_2^4}{2}\left(\|u_k\|_V^4+\|v_k\|_V^4\right). \tag{1.120}$$

- From Green's Theorem and since $\dfrac{\partial m_i}{\partial x_i} = 1$ and $\partial_t u_k = 0$ on Γ_0, we obtain

$$\begin{aligned} I_3 = 2(\partial_t u_k, m\cdot\nabla\partial_t u_k) &= 2\sum_{i=1}^{n}\int_\Omega \partial_t u_k m_i \frac{\partial}{\partial x_i}\partial_t u_k dx = \sum_{i=1}^{n}\int_\Omega m_i \frac{\partial}{\partial x_i}(\partial_t u_k)^2 dx \\ &= -\sum_{i=1}^{n}\int_\Omega \frac{\partial m_i}{\partial x_i}(\partial_t u_k)^2 dx + \sum_{i=1}^{n}\int_{\Gamma_1}(m_i\cdot\vec{n}_i)(\partial_t u_k)^2 d\Gamma, \end{aligned}$$

it follows that

$$I_3 \leqslant -n\|\partial_t u_k\|^2_{L^2(\Omega)} + R\|\partial_t u_k\|^2_{L^2(\Gamma_1)}.$$

In similar way

$$I_9 \leqslant -n\|\partial_t v_k\|^2_{L^2(\Omega)} + R\|\partial_t v_k\|^2_{L^2(\Gamma_1)}.$$

- From the boundary conditions $\dfrac{\partial u_k}{\partial \vec{n}} + (m\cdot\vec{n})\partial_t u_k = 0$ on Γ_1, we find

$$(\Delta u_k, u_k) = -\|u_k\|_V^2 - \int_{\Gamma_1}(m\cdot\vec{n})\partial_t u_k u_k d\Gamma.$$

Note that, using (1.59) we obtain

$$\begin{aligned} \left|\int_{\Gamma_1}(m\cdot\vec{n})\partial_t u_k u_k d\Gamma\right| &\leqslant R\int_{\Gamma_1}|\partial_t u_k||u_k| d\Gamma \\ &\leqslant \frac{1}{2}R^2 c_3^2(n-1)\|\partial_t u_k\|^2_{L^2(\Gamma_1)} + \frac{1}{2(n-1)c_3^2}\|u_k\|^2_{L^2(\Gamma_1)} \\ &\leqslant \frac{1}{2}R^2(n-1)c_3^2\|\partial_t u_k\|^2_{L^2(\Gamma_1)} + \frac{1}{2(n-1)}\|u_k\|^2_V. \end{aligned}$$

Thus

$$I_4 = (n-1)(\Delta u_k, u_k) \leqslant -(n-1)\|u_k\|_V^2 + \frac{1}{2}R^2(n-1)^2c_3^2\|\partial_t u_k\|^2_{L^2(\Gamma_1)} + \frac{1}{2}\|u_k\|_V^2.$$

In similar way

$$I_{10} \leqslant -(n-1)\|v_k\|_V^2 + \frac{1}{2}R^2(n-1)^2c_3^2\|\partial_t v_k\|^2_{L^2(\Gamma_1)} + \frac{1}{2}\|v_k\|_V^2.$$

- From (1.12) we have

$$(|u_k||v_k|v_k, u_k) \leqslant \int_\Omega |u_k|^2|v_k|^2 dx \leqslant \frac{c_1^4}{2}(\|u_k\|_V^4 + \|v_k\|_V^4).$$

Therefore

$$I_5 = -(n-1)(|u_k||v_k|v_k, u_k) \leqslant \frac{c_1^4}{2}(n-1)(\|u_k\|_V^4 + \|v_k\|_V^4).$$

In similar way

$$I_{11} \leqslant \frac{c_1^4}{2}(n-1)(\|u_k\|_V^4 + \|v_k\|_V^4). \tag{1.121}$$

Taking into account (1.118)–(1.121) in (1.115) and reducing similar terms, we obtain

$$\begin{aligned} I_1 + \cdots + I_{12} &\leqslant -(\|\partial u_k\|^2_{L^2(\Omega)} + \|\partial v_k\|^2_{L^2(\Omega)}) - \frac{1}{2}(\|u_k\|^2_V + \|v_k\|^2_V) \\ &\quad + n\int_\Omega (|u_k|u_k)(|v_k|v_k)dx + \left[\frac{Rc_2^4}{2} + c_1^4(n-1)\right](\|u_k\|^4_V + \|v_k\|^4_V) \\ &\quad + D(\|u_k'\|^2_{L^2(\Gamma_1)} + \|v_k'\|^2_{L^2(\Gamma_1)}), \end{aligned}$$

where D was defined in (1.103). Thus,

$$\begin{aligned} I_1 + \cdots + I_{12} &\leqslant -\frac{1}{2}(\|\partial u_k\|^2_{L^2(\Omega)} + \|\partial v_k\|^2_{L^2(\Omega)}) - \frac{1}{4}(\|u_k\|^2_V + \|v_k\|^2_V) \\ &\quad - \frac{1}{4}\int_\Omega (|u_k|u_k)(|v_k|v_k)dx - \frac{1}{4}(\|u_k\|^2_V + \|v_k\|^2_V) \\ &\quad + \left[n + \frac{1}{4}\right]\int_\Omega (|u_k|u_k)(|v_k|v_k)dx + \left[\frac{Rc_2^4}{2} + c_1^4(n-1)\right](\|u_k\|^4_V + \|v_k\|^4_V) \\ &\quad + D(\|u_k'\|^2_{L^2(\Gamma_1)} + \|v_k'\|^2_{L^2(\Gamma_1)}) \end{aligned} \tag{1.122}$$

Now using (1.12), we have

$$\left|\int_\Omega (|u_k|v_k)(|v_k|v_k)dx\right| \leqslant \frac{c_1^4}{2}(\|u_k\|^4_V + \|v_k\|^4_V). \tag{1.123}$$

Combining (1.123) with (1.122) we get

$$\begin{aligned} I_1 + \cdots + I_{12} &\leqslant -\frac{1}{2}E_k - \left(\frac{1}{4}(\|u_k\|^2_V + \|v_k\|^2_V) - N_1(\|u_k\|^4_V + \|v_k\|^4_V)\right) \\ &\quad + D(\|u_k'\|^2_{L^2(\Gamma_1)} + \|v_k'\|^2_{L^2(\Gamma_1)}), \end{aligned}$$

where N_1 was defined in (1.63). From (1.109) and (1.110) we obtain

$$\frac{1}{4}(\|u_k\|^2_V + \|v_k\|^2_V) - N_1(\|u_k\|^4_V + \|v_k\|^4_V) \geqslant 0.$$

Therefore

$$\psi_k' \leqslant -\frac{1}{2}E_k + D(\|\partial_t u_k\|^2_{L^2(\Gamma_1)} + \|\partial_t v_k\|^2_{L^2(\Gamma_1)}). \tag{1.124}$$

Thanks to (1.114), (1.124) and $E'_{k\varepsilon} = E'_k + \varepsilon\psi'_k$ we get

$$E'_{k\varepsilon} \leqslant -\frac{\varepsilon}{2}E_k - (m_0 - D\varepsilon)(\|\partial_t u_k\|^2_{L^2(\Gamma_1)} + \|\partial_t v_k\|^2_{L^2(\Gamma_1)}).$$

Therefore

$$E'_{k\varepsilon_2} \leqslant -\frac{\varepsilon_2}{2}E_k, \quad \text{for all} \quad 0 \leqslant \varepsilon_2 \leqslant \frac{m_0}{D}. \tag{1.125}$$

The choice τ given in (1.103) implies that (1.113) and (1.125) hold simultaneously for this τ. Thus, from (1.113) we have

$$-\frac{\tau}{2}E_k \leqslant -\frac{\tau}{3}E_{k\tau}.$$

Consequently, using the above inequality in (1.125), we obtain

$$E'_{k\tau} \leqslant -\frac{\tau}{3}E_{k\tau}.$$

This give us that

$$E_{k\tau}(t) \leqslant e^{-\frac{\tau}{3}t}E_{k\tau}(0).$$

From this inequality and (1.113) we have

$$E_k(t) \leqslant 3E_k(0)e^{-\frac{\tau}{3}t}, \quad \text{for all} \quad t \in [0, \infty).$$

With this we conclude the proof of the Theorem 1.4.1. ■

Chapter 2

Asymptotic behavior of solutions for a thermoelastic plate system

In this chapter we aim to make a study of the asymptotic behavior, in the sense of global attractors, of the solutions of an autonomous thermoelastic plate system with $n \geqslant 2$ and Neumann boundary conditions when some reaction terms are concentrated in a neighborhood of the boundary and this neighborhood shrinks to boundary as a parameter ε goes to zero. More precisely, we show the existence, uniform bound of the global attractors of the problems (2) and (3) and that the semigroup associated to (2) converges for the semigroup associated to (3). Moreover we show the continuity of these attractors at $\varepsilon = 0$.

2.1 Preliminary

We begin this section with some notations and we present hypotheses and a dissipative condition on nonlinearities. After we write (2) and (3) in abstract problems. We finished the section with a result, that ensure us the sectoriality of operator and with an exponential estimate for the linear semigroup.

2.1.1 Abstract setting

To better describe the problem we introduce some terminology, let Ω be an open bounded smooth set in $\mathbb{R}^n$, $n \geqslant 2$ with a smooth boundary $\Gamma = \partial\Omega$. We define the strip of width ε and base $\partial\Omega$ as

$$\omega_\varepsilon = \{x - \sigma \vec{n}\,(x) : \ x \in \Gamma \ \text{ and } \ \sigma \in [0, \varepsilon)\},$$

for sufficiently small ε, say $0 < \varepsilon \leqslant \varepsilon_0$, where $\vec{n}(x)$ denotes the outward normal vector at $x \in \Gamma$. We note that the set ω_ε has Lebesgue measure $|\omega_\varepsilon| = O(\varepsilon)$ with $|\omega_\varepsilon| \leqslant k\,|\Gamma|\,\varepsilon$, for some $k > 0$ independent of ε, and that for small ε, the set ω_ε is a neighborhood of Γ in $\overline{\Omega}$, that collapses to the boundary when the parameter ε goes to zero.

We are interested in the behavior, for small ε, of the solutions of the autonomous thermoelastic plate systems with concentrated terms given in (2)

We take $j : \mathbb{R} \to \mathbb{R}$ to be $\mathscr{C}^2$ and assume that it satisfies the growth estimates

$$|j(s)| + |j'(s)| + |j''(s)| \leqslant K, \quad \forall s \in \mathbb{R}, \tag{2.1}$$

for some constant $K > 0$, we also assume the standard dissipative assumption given by

$$\limsup_{|s| \to +\infty} \frac{j(s)}{s} \leqslant 0, \tag{2.2}$$

with $j = f$ or $j = g$. We note that (2.2) is equivalent to saying that for any $\gamma > 0$ there exists $c_\gamma > 0$ such that

$$sj(s) \leqslant \gamma s^2 + c_\gamma, \quad \forall s \in \mathbb{R}. \tag{2.3}$$

Let us consider the Hilbert space $Y := L^2(\Omega)$ and the unbounded linear operator $\Lambda : D(\Lambda) \subset Y \to Y$ defined by

$$\Lambda u = (-\Delta)^2 u, \quad u \in D(\Lambda),$$

with domain

$$D(\Lambda) := \Big\{ u \in H^4(\Omega) \ : \ \frac{\partial u}{\partial \vec{n}} = \frac{\partial(\Delta u)}{\partial \vec{n}} = 0 \text{ on } \Gamma \Big\}.$$

The operator Λ has a discrete spectrum formed of eigenvalues satisfying

$$0 = \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_n \leqslant \cdots, \quad \lim_{n \to \infty} \mu_n = \infty.$$

Since this operator turns out to be sectorial in Y in the sense of Henry [20, Definition 1.3.1, p.18] and Cholewa and Dłotko [15, Example 1.3.9, p.42], associated to it there is a scale of Banach spaces (the fractional power spaces) Y^α, $\alpha \in \mathbb{R}$, denoting the domain of the fractional power operators associated with $\mathit{\Lambda}$, that is, $Y^\alpha := D(\mathit{\Lambda}^\alpha)$, $\alpha \geqslant 0$. Let us consider Y^α endowed with the norm $\|(\cdot)\|_{Y^\alpha} = \|\mathit{\Lambda}^\alpha(\cdot)\|_Y + \|(\cdot)\|_Y, \alpha \geqslant 0$. The fractional power spaces are related to the Bessel Potentials spaces $H^s(\Omega)$, $s \in \mathbb{R}$, and it is well known that

$$Y^\alpha \hookrightarrow H^{2\alpha}(\Omega), \quad Y^{-\alpha} = (Y^\alpha)', \quad \alpha \geqslant 0, \tag{2.4}$$

with

$$Y^{\frac{1}{2}} = \Big\{ u \in H^2(\Omega) : \frac{\partial u}{\partial \vec{n}} = 0 \text{ on } \Gamma \Big\}.$$

We also have

$$Y^{-\frac{1}{2}} = (Y^{\frac{1}{2}})', \quad Y = Y^0 = L^2(\Omega) \quad \text{and} \quad Y^1 = D(\Lambda).$$

Since the problem (3) has a nonlinear term on boundary, choosing $\frac{1}{2} < s \leqslant 1$ and using the standard trace theory results that for any function $v \in H^s(\Omega)$, the trace of v is well defined and lies in $L^2(\Gamma)$. Moreover, the scale of negative exponents $Y^{-\alpha}$, for $\alpha > 0$, it is necessary to introduce the nonlinear term of (3) in the abstract equation, since we are using the operator $\mathit{\Lambda}$ with homogeneous boundary conditions. If we consider the realizations of $\mathit{\Lambda}$ in this scale, then the operator $\mathit{\Lambda}_{-\frac{1}{2}} \in \mathscr{L}(Y^{\frac{1}{2}}, Y^{-\frac{1}{2}})$ is given by

$$\langle \mathit{\Lambda}_{-\frac{1}{2}} u, v \rangle_Y = \int_\Omega \Delta u \Delta v dx + \int_\Omega uv dx, \qquad u, v \in Y^{\frac{1}{2}}.$$

With some abuse of notation we identify all different realizations of this operator and we write them all as Λ.

We also consider the operator $\mathit{\Lambda} + I : D(\mathit{\Lambda} + I) \subset Y \to Y$, it is a positive defined and sectorial operator in Y in the sense of Henry [20, Definition 1.3.1, p.18] and Cholewa and Dłotko [15, Example 1.3.9, p.42], associated to it there is a scale of Banach spaces (the fractional power spaces) $D((\mathit{\Lambda} + I)^\alpha)$, $\alpha \geqslant 0$, domain of the operator $(\mathit{\Lambda} + I)^\alpha$. Let us consider $D((\mathit{\Lambda} + I)^\alpha)$ endowed with the graph norm

$$\|(\cdot)\|_{D((\mathit{\Lambda}+I)^\alpha)} = \|(\mathit{\Lambda} + I)^\alpha(\cdot)\|_Y, \quad \alpha \geqslant 0 \ (0 \in \rho((\mathit{\Lambda} + I)^\alpha)).$$

Consequentely, by Cholewa and Dłotko [15, Corollary 1.3.5] and $D(\mathit{\Lambda}+I) = D(\mathit{\Lambda})$, we also have that

$$Y^\alpha = [Y, D(\mathit{\Lambda})]_\alpha = [Y, D(\mathit{\Lambda} + I)]_\alpha = D((\mathit{\Lambda} + I)^\alpha), \quad 0 < \alpha < 1,$$

endowed with equivalent norms.

The operator $\varLambda + I$ has a discrete spectrum formed of eigenvalues satisfying

$$1 = \mu_1^I \leqslant \mu_2^I \leqslant \cdots \leqslant \mu_n^I \leqslant \cdots, \quad \lim_{n\to\infty} \mu_n^I = \infty.$$

Also, let us consider the following Hilbert space

$$X = X^0 = Y^{\frac{1}{2}} \times Y \times Y,$$

equipped with the inner product

$$\left\langle \begin{pmatrix} u_1 \\ v_1 \\ \theta_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \\ \theta_2 \end{pmatrix} \right\rangle_X = \langle u_1, u_2 \rangle_{Y^{\frac{1}{2}}} + \langle v_1, v_2 \rangle_Y + \langle \theta_1, \theta_2 \rangle_Y,$$

where $\langle \cdot, \cdot \rangle_Y$ is the usual inner product in $L^2(\Omega)$ and

$$\mathscr{H} = H^2(\Omega) \times H^{-s}(\Omega) \times L^2(\Omega)$$

equipped with the usual inner product with $\frac{1}{2} < s \leqslant 1$.

To better explain the results in the chapter, initially, we define the abstract problems associated to (2) and (3) respectively. For this we define the unbounded linear operator $\mathbb{A} : D(\mathbb{A}) \subset X \to X$ by

$$\mathbb{A}\begin{pmatrix} u \\ v \\ \theta \end{pmatrix} = \begin{pmatrix} 0 & I & 0 \\ -\varLambda - I & 0 & \varLambda^{\frac{1}{2}} + I \\ 0 & -\varLambda^{\frac{1}{2}} - I & -\varLambda^{\frac{1}{2}} - I \end{pmatrix} \begin{pmatrix} u \\ v \\ \theta \end{pmatrix} = \begin{pmatrix} v \\ -\varLambda u - u + \varLambda^{\frac{1}{2}}\theta + \theta \\ -\varLambda^{\frac{1}{2}} v - v - \varLambda^{\frac{1}{2}}\theta - \theta \end{pmatrix}, \tag{2.5}$$

for all $\begin{pmatrix} u \\ v \\ \theta \end{pmatrix} \in D(\mathbb{A})$, with domain

$$D(\mathbb{A}) = Y^1 \times Y^{\frac{1}{2}} \times Y^{\frac{1}{2}}. \tag{2.6}$$

For each $\varepsilon \in (0, \varepsilon_0]$, we write (2) in the abstract form as

$$\begin{cases} \dfrac{dw^\varepsilon}{dt} = \mathbb{A}w^\varepsilon + F_\varepsilon(w^\varepsilon), \quad t > 0, \\ w^\varepsilon(0) = w_0, \end{cases} \tag{2.7}$$

with

$$w^\varepsilon = \begin{pmatrix} u^\varepsilon \\ \partial_t u^\varepsilon \\ \theta^\varepsilon \end{pmatrix}, \quad w_0 = \begin{pmatrix} u_0 \\ v_0 \\ \theta_0 \end{pmatrix} \in X$$

and nonlinear map $F_\varepsilon : X \to \mathscr{H}$, with $\frac{1}{2} < s \leqslant 1$, defined by

$$F_\varepsilon(w) = \begin{pmatrix} 0 \\ f_\Omega(u) + \dfrac{1}{\varepsilon}\chi_{\omega_\varepsilon} g_\Omega(u) \\ 0 \end{pmatrix}, \quad w = \left(\begin{smallmatrix} u \\ v \\ \theta \end{smallmatrix}\right) \in X,$$

where $f_\Omega, \dfrac{1}{\varepsilon}\chi_{\omega_\varepsilon} g_\Omega : H^2(\Omega) \to H^{-s}(\Omega)$ are the operators, respectively, given by

$$\langle f_\Omega(u), \varphi \rangle = \int_\Omega f(u)\varphi dx, \quad u \in H^2(\Omega) \text{ and } \varphi \in H^s(\Omega) \tag{2.8}$$

and

$$\left\langle \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} g_\Omega(u), \varphi \right\rangle = \frac{1}{\varepsilon}\int_{\omega_\varepsilon} g(u)\varphi dx, \quad u \in H^2(\Omega) \text{ and } \varphi \in H^s(\Omega). \tag{2.9}$$

While the problem (3) can be written in the abstract form as

$$\begin{cases} \dfrac{dw}{dt} = \mathbb{A}w + F_0(w), \quad t > 0, \\ w(0) = w_0, \end{cases} \tag{2.10}$$

with

$$w = \left(\begin{smallmatrix} u \\ \partial_t u \\ \theta \end{smallmatrix}\right)$$

and nonlinear map $F_0 : X \to \mathscr{H}$, with $\frac{1}{2} < s \leqslant 1$, defined by

$$F_0(w) = \begin{pmatrix} 0 \\ f_\Omega(u) + g_\Gamma(u) \\ 0 \end{pmatrix}, \quad w = \left(\begin{smallmatrix} u \\ v \\ \theta \end{smallmatrix}\right) \in X,$$

where f_Ω is defined in (2.8) and $g_\Gamma : H^2(\Omega) \to H^{-s}(\Omega)$ is the operator given by

$$\langle g_\Gamma(u), \varphi \rangle = \int_\Gamma \gamma(g(u))\gamma(\varphi) dS, \quad u \in H^2(\Omega) \text{ and } \varphi \in H^s(\Omega), \tag{2.11}$$

where $\gamma : H^s(\Omega) \to L^2(\Gamma)$ is the trace operator, to according with Triebel [43].

2.1.2 Sectoriality

In this section we prove that the unbounded linear operator $\mathbb{A}$ generates a analytic semigroup, which we denote $\{e^{\mathbb{A}t} : t \geqslant 0\}$, more precisely, we show that unbounded linear operator $-\mathbb{A}$ is sectorial.

On analyticity of a C_0-semigroup of contractions on a Hilbert space, we have following result.

Theorem 2.1.1 *Let $\{S(t) : t \geqslant 0\}$ be a C_0-semigroup of contractions of linear operators in a Hilbert space with infinitesimal generator $\mathscr{B}$. Suppose that $i\mathbb{R} \subset \rho(\mathscr{B})$. Then, $\{S(t) : t \geqslant 0\}$ is analytic if and only if* $\limsup_{|\beta|\to+\infty} \|\beta(i\beta I - \mathscr{B})^{-1}\| < \infty$.

Proof. For the proof, see Liu and Zheng [27, Theorem 1.3.3, p.5]. ■

In the following two results we verify that the unbounded linear operator $\mathbb{A}$ is dissipative, closed and densely defined.

Lemma 2.1.2 *The unbounded linear operator $\mathbb{A} : D(\mathbb{A}) \subset X \to X$ defined in* (2.5)-(2.6) *satisfy the following equality*

$$Re\left\langle \mathbb{A}\begin{pmatrix} u \\ v \\ \theta \end{pmatrix}, \begin{pmatrix} u \\ v \\ \theta \end{pmatrix}\right\rangle_X = -\|\Lambda^{\frac{1}{4}}\theta\|_Y^2 - \|\theta\|_Y^2 \leqslant 0, \quad \forall \begin{pmatrix} u \\ v \\ \theta \end{pmatrix} \in D(\mathbb{A}). \tag{2.12}$$

Proof. Note that

$$\begin{aligned}
\left\langle \mathbb{A}\begin{pmatrix} u \\ v \\ \theta \end{pmatrix}, \begin{pmatrix} u \\ v \\ \theta \end{pmatrix}\right\rangle_X &= \left\langle \begin{pmatrix} v \\ -\Lambda u - u + \Lambda^{\frac{1}{2}}\theta + \theta \\ -\Lambda^{\frac{1}{2}}v - v - \Lambda^{\frac{1}{2}}\theta - \theta \end{pmatrix}, \begin{pmatrix} u \\ v \\ \theta \end{pmatrix}\right\rangle_{Y^{\frac{1}{2}}\times Y\times Y} \\
&= \langle v, u\rangle_{Y^{\frac{1}{2}}} - \langle \Lambda u - \Lambda^{\frac{1}{2}}\theta, v\rangle_Y - \langle u - \theta, v\rangle_Y - \langle \Lambda^{\frac{1}{2}}v + \Lambda^{\frac{1}{2}}\theta, \theta\rangle_Y - \langle v + \theta, \theta\rangle_Y \\
&= \overline{\langle \Lambda^{\frac{1}{2}}u, \Lambda^{\frac{1}{2}}v\rangle_Y} - \langle \Lambda^{\frac{1}{2}}u, \Lambda^{\frac{1}{2}}v\rangle_Y + \overline{\langle u, v\rangle_Y} - \langle u, v\rangle_Y + \langle \Lambda^{\frac{1}{2}}\theta, v\rangle_Y - \overline{\langle \Lambda^{\frac{1}{2}}\theta, v\rangle_Y} \\
&+ \overline{\langle v, \theta\rangle_Y} - \langle v, \theta\rangle_Y - \|\Lambda^{\frac{1}{4}}\theta\|_Y^2 - \|\theta\|_Y^2.
\end{aligned}$$

Finally, from this we get (2.12). ■

Theorem 2.1.3 *The unbounded linear operator $\mathbb{A} : D(\mathbb{A}) \subset X \to X$ defined in* (2.5)-(2.6) *is closed and densely defined.*

Proof. Let $w_n = [u_n \ \ v_n \ \ \theta_n]^T \in D(\mathbb{A})$ with $w_n \to [u \ \ v \ \ \theta]^T$ in X as $n \to \infty$, and $\mathbb{A}w_n \to \varphi = [\varphi_1 \ \ \varphi_2 \ \ \varphi_3]^T$ in X as $n \to \infty$, or equivalently

$$\begin{cases} v_n \to \varphi_1 & \text{in } Y^{\frac{1}{2}} \text{ as } n \to \infty; \\ -\Lambda u_n - u_n + \Lambda^{\frac{1}{2}}\theta_n + \theta_n \to \varphi_2 & \text{in } Y \text{ as } n \to \infty; \\ -\Lambda^{\frac{1}{2}}v_n - v_n - \Lambda^{\frac{1}{2}}\theta_n - \theta_n \to \varphi_3 & \text{in } Y \text{ as } n \to \infty, \end{cases}$$

then $v = \varphi_1 \in Y^{\frac{1}{2}}$. Since

$$-(\Lambda^{\frac{1}{2}} + I)\theta_n = [-(\Lambda^{\frac{1}{2}} + I)v_n - (\Lambda^{\frac{1}{2}} + I)\theta_n] + (\Lambda^{\frac{1}{2}} + I)v_n \to \varphi_3 + \Lambda^{\frac{1}{2}}\varphi_1 + \varphi_1$$

in Y as $n \to \infty$, we have

$$\theta \in D(\Lambda^{\frac{1}{2}} + I) = Y^{\frac{1}{2}} \quad \text{and} \quad -(\Lambda^{\frac{1}{2}} + I)\theta = \varphi_3 + \Lambda^{\frac{1}{2}}\varphi_1 + \varphi_1.$$

Finally, since

$$-(\varLambda+I)u_n = [-(\varLambda+I)u_n + (\varLambda^{\frac{1}{2}}+I)\theta_n] - (\varLambda^{\frac{1}{2}}+I)\theta_n \to \varphi_2+\varphi_3+\varLambda^{\frac{1}{2}}\varphi_1+\varphi_1$$

in Y as $n\to\infty$, we conclude

$$u \in D(\varLambda+I) = Y^1 \quad \text{and} \quad -(\varLambda+I)u = \varphi_2+\varphi_3+\varLambda^{\frac{1}{2}}\varphi_1+\varphi_1,$$

that is, $[u \;\; v \;\; \theta]^T \in D(\mathbb{A})$ and

$$[\varphi_1 \;\; \varphi_2 \;\; \varphi_3]^T = [v \quad -(\varLambda+I)u+(\varLambda^{\frac{1}{2}}+I)\theta \quad -(\varLambda^{\frac{1}{2}}+I)v-(\varLambda^{\frac{1}{2}}+I)\theta]^T = \mathbb{A}[u \;\; v \;\; \theta]^T.$$

Clearly $\mathbb{A}$ is densely defined. ■

Remark 2.1.4 *Note that zero is in the resolvent set of* $\mathbb{A}$ *and*

$$\mathbb{A}^{-1} = \begin{pmatrix} -(\varLambda+I)^{-1}(\varLambda^{\frac{1}{2}}+I) & -(\varLambda+I)^{-1} & -(\varLambda+I)^{-1} \\ I & 0 & 0 \\ -I & 0 & -(\varLambda^{\frac{1}{2}}+I)^{-1} \end{pmatrix}.$$

Since $\mathbb{A}$ is dissipative, closed, densely defined and zero is in the resolvent set of $\mathbb{A}$, by Lumer-Phillips theorem, $\mathbb{A}$ is generator of a C_0-semigroup of contractions.

The next theorem shows that the operator $\mathbb{A}$ generates an analytic semigroup, that is, $-\mathbb{A}$ is a sectorial operator, for this we use the Theorem 2.1.1.

Theorem 2.1.5 *The unbounded linear operator* $-\mathbb{A}$ *such that* $\mathbb{A} : D(\mathbb{A}) \subset X \to X$ *is defined in* (2.5)-(2.6) *is sectorial with* $Re\sigma(-\mathbb{A}) > 0$. *Thus the semigroup of contractions* $\{e^{\mathbb{A}t} : t \geqslant 0\}$ *is analytic.*

Proof. First, we show that $i\mathbb{R} \subset \rho(\mathbb{A})$. We show this result by a contradiction argument. That is, let us suppose that there exists $0 \neq \beta \in \mathbb{R}$, such that $i\beta$ is in the spectrum of $\mathbb{A}$. Then $i\beta$ must be an eigenvalue of $\mathbb{A}$, because $\mathbb{A}^{-1}$ is compact. Thus there is a vector function $w = [u \;\; v \;\; \theta]^T \in D(\mathbb{A})$, $\|w\|_X = 1$, such that

$$(i\beta I - \mathbb{A})w = 0 \text{ in } X$$

or equivalenty

$$\begin{cases} i\beta u - v = 0, \\ i\beta v + (\varLambda+I)u - (\varLambda^{\frac{1}{2}}+I)\theta = 0, \\ i\beta\theta + (\varLambda^{\frac{1}{2}}+I)v + (\varLambda^{\frac{1}{2}}+I)\theta = 0, \end{cases} \tag{2.13}$$

and so

$$Re\langle \mathbb{A}w, w\rangle_X = -\|\varLambda^{\frac{1}{4}}\theta\|_Y^2 - \|\theta\|_Y^2 = 0.$$

Thus $\theta = 0$ and by (2.13), $u = v = 0$. Thus, we have a contradition. Therefore, $i\mathbb{R} \subset \rho(\mathbb{A})$.

Finally, we show that there exists a positive constant C such that

$$|\beta| \left\| \begin{pmatrix} u \\ v \\ \theta \end{pmatrix} \right\|_X \leqslant C\|\mathscr{F}\|_X, \text{ for all } \mathscr{F} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \in X,\ \beta \in \mathbb{R},$$

where $w = [u \quad v \quad \theta]^T = (i\beta I - \mathbb{A})^{-1}\mathscr{F} \in D(\mathbb{A})$. In fact, multiplying equation

$$(i\beta I - \mathbb{A})w = \mathscr{F} \text{ in } X \tag{2.14}$$

with $w = [u \quad v \quad \theta]^T$; that is, in terms of its components yields

$$\begin{cases} i\beta u - v = f_1, \\ i\beta v + (\varLambda + I)u - (\varLambda^{\frac{1}{2}} + I)\theta = f_2, \\ i\beta\theta + (\varLambda^{\frac{1}{2}} + I)v + (\varLambda^{\frac{1}{2}} + I)\theta = f_3, \end{cases} \tag{2.15}$$

we get

$$i\beta\|w\|_X^2 - \langle \mathbb{A}w, w\rangle_X = \langle \mathscr{F}, w\rangle_X. \tag{2.16}$$

Taking the real part in (2.16) it follows that

$$|Re\langle \mathbb{A}w, w\rangle_X| = \|\varLambda^{\frac{1}{4}}\theta\|_Y^2 + \|\theta\|_Y^2 \leqslant \|\mathscr{F}\|_X\|w\|_X, \tag{2.17}$$

and taking the imaginary parts in (2.16), and using (2.17) and Young's inequality we have that

$$\begin{aligned} |\beta|\|w\|_X^2 &\leqslant 2|\langle \varLambda^{\frac{1}{2}}u, \varLambda^{\frac{1}{2}}v\rangle_Y| + 2|\langle u, v\rangle_Y| + 2|\langle \varLambda^{\frac{1}{2}}v, \theta\rangle_Y| + 2|\langle v, \theta\rangle_Y| + 2\|\mathscr{F}\|_X\|w\|_X \\ &= 2|\langle \varLambda^{\frac{3}{4}}u, \varLambda^{\frac{1}{4}}v\rangle_Y| + 2|\langle u, v\rangle_Y| + 2|\langle \varLambda^{\frac{1}{4}}v, \varLambda^{\frac{1}{4}}\theta\rangle_Y| + 2|\langle v, \theta\rangle_Y| + 2\|\mathscr{F}\|_X\|w\|_X \\ &\leqslant \|\varLambda^{\frac{3}{4}}u\|_Y^2 + \|u\|_Y^2 + 2\|\varLambda^{\frac{1}{4}}v\|_Y^2 + 2\|v\|_Y^2 + \|\varLambda^{\frac{1}{4}}\theta\|_Y^2 + \|\theta\|_Y^2 + 2\|\mathscr{F}\|_X\|w\|_X \\ &\leqslant \|\varLambda^{\frac{3}{4}}u\|_Y^2 + \|u\|_Y^2 + 2\|\varLambda^{\frac{1}{4}}v\|_Y^2 + 2\|v\|_Y^2 + \|\varLambda^{\frac{1}{4}}\theta\|_Y^2 + \|\theta\|_Y^2 + 2\|\mathscr{F}\|_X\|w\|_X \end{aligned} \tag{2.18}$$

Thanks to (2.17) and (2.18) we obtain that

$$|\beta|\|w\|_X^2 \leqslant \|\varLambda^{\frac{3}{4}}u\|_Y^2 + \|u\|_Y^2 + 2(\|\varLambda^{\frac{1}{4}}v\|_Y^2 + \|v\|_Y^2) + 3\|\mathscr{F}\|_X\|w\|_X. \tag{2.19}$$

Multiplying (2.14) by $[0 \;\; 0 \;\; v]^T$, in the sense of X, using the second equation of the (2.15) and the Young's inequality we have that

$$
\begin{aligned}
&2(\|\Lambda^{\frac{1}{4}} v\|_Y^2 + \|v\|_Y^2) \\
&\leqslant 2(\|f_2\|_Y \|\theta\|_Y + \|f_3\|_Y \|v\|_Y) + \|\Lambda^{\frac{3}{4}} u\|_Y^2 + \|u\|_Y^2 + \left(1 + \frac{1}{\gamma_0}\right)(\|\Lambda^{\frac{1}{4}}\theta\|_Y^2 + \|\theta\|_Y^2) \\
&+ \gamma_0(\|\Lambda^{\frac{1}{4}} v\|_Y^2 + \|v\|_Y^2),
\end{aligned}
$$

for some constant $\gamma_0 > 0$ to be choose later.

Thus

$$
\begin{aligned}
(2-\gamma_0)(\|\Lambda^{\frac{1}{4}} v\|_Y^2 + \|v\|_Y^2) \leqslant{}& 2(\|f_2\|_Y \|\theta\|_Y + \|f_3\|_Y \|v\|_Y) + \|\Lambda^{\frac{3}{4}} u\|_Y^2 + \|u\|_Y^2 \\
&+ \left(1 + \frac{1}{\gamma_0}\right)(\|\Lambda^{\frac{1}{4}}\theta\|_Y^2 + \|\theta\|_Y^2),
\end{aligned}
$$

for some constant $\gamma_0 > 0$ to be choose later.

With this, by (2.17) and choosing $0 < \gamma_0 < 2$ we get

$$
(2-\gamma_0)(\|\Lambda^{\frac{1}{4}} v\|_Y^2 + \|v\|_Y^2) \leqslant C_1 \|\mathscr{F}\|_X \|w\|_X + \|\Lambda^{\frac{3}{4}} u\|_Y^2 + \|u\|_Y^2, \tag{2.20}
$$

for some constant $C_1 > 0$.

Now, multiplying (2.14) by $[0 \;\; \Lambda^{\frac{1}{2}} u + u \;\; 0]^T$, in the sense of X, we have

$$
\langle i\beta v + (\Lambda + I)u - (\Lambda^{\frac{1}{2}} + I)\theta, \Lambda^{\frac{1}{2}} u + u\rangle_Y = \langle f_2, \Lambda^{\frac{1}{2}} u + u\rangle_Y,
$$

that is, using the first equation of the (2.15) in the above equation,

$$
\begin{aligned}
&\|\Lambda^{\frac{3}{4}} u\|_Y^2 + \|\Lambda^{\frac{1}{2}} u\|_Y^2 + \|\Lambda^{\frac{1}{4}} u\|_Y^2 + \|u\|_Y^2 \\
&\leqslant (\|\Lambda^{\frac{1}{2}} f_1\|_Y + \|f_1\|_Y)\|v\|_Y + \|f_2\|_Y(\|\Lambda^{\frac{1}{2}} u\|_Y + \|u\|_Y) + \frac{2+\gamma_1}{2\gamma_1}\|\Lambda^{\frac{1}{4}}\theta\|_Y^2 + \frac{1}{2}\|\Lambda^{\frac{3}{4}} u\|_Y^2 \\
&+ \gamma_1\|\Lambda^{\frac{1}{4}} u\|_Y^2 + \frac{1}{2}\|\theta\|_Y^2 + \frac{1}{2}\|u\|_Y^2 - \|\Lambda^{\frac{1}{4}} v\|_Y^2 - \|v\|_Y^2 \\
&\leqslant (\|\Lambda^{\frac{1}{2}} f_1\|_Y + \|f_1\|_Y)\|v\|_Y + \|f_2\|_Y(\|\Lambda^{\frac{1}{2}} u\|_Y + \|u\|_Y) + \frac{2+\gamma_1}{2\gamma_1}(\|\Lambda^{\frac{1}{4}}\theta\|_Y^2 + \|\theta\|_Y^2) \\
&+ \frac{1}{2}\|\Lambda^{\frac{3}{4}} u\|_Y^2 + \gamma_1\|\Lambda^{\frac{1}{4}} u\|_Y^2 + \frac{1}{2}\|u\|_Y^2,
\end{aligned}
$$

for some constant $\gamma_1 > 0$ to be choose later.

Thus

$$
\begin{aligned}
&\frac{1}{2}(\|\Lambda^{\frac{3}{4}} u\|_Y^2 + \|u\|_Y^2) + (1-\gamma_1)\|\Lambda^{\frac{1}{4}} u\|_Y^2 \\
&\leqslant (\|\Lambda^{\frac{1}{2}} f_1\|_Y + \|f_1\|_Y)\|v\|_Y + \|f_2\|_Y(\|\Lambda^{\frac{1}{2}} u\|_Y + \|u\|_Y) + \frac{2+\gamma_1}{2\gamma_1}(\|\Lambda^{\frac{1}{4}}\theta\|_Y^2 + \|\theta\|_Y^2),
\end{aligned}
$$

for some constant $\gamma_1 > 0$ to be choose later.

Now, take $0 < \gamma_1 < 1$ and see that by (2.17) we get

$$\|\Lambda^{\frac{3}{4}}u\|_Y^2 + \|u\|_Y^2 \leqslant C_2\|\mathscr{F}\|_X\|w\|_X,$$

for some constant $C_2 > 0$.

Thanks to (2.20) we have

$$\|\Lambda^{\frac{1}{4}}v\|_Y^2 + \|v\|_Y^2 \leqslant C_3\|\mathscr{F}\|_X\|w\|_X,$$

for some constant $C_3 > 0$.

Finally, from (2.19) we obtain

$$|\beta|\|w\|_X^2 \leqslant c_0\|\mathscr{F}\|_X\|w\|_X,$$

for some constant $c_0 > 0$, and we conclude by Theorem 2.1.1 that $\mathbb{A}$ generates an analytic semigroup and therefore $-\mathbb{A}$ is a sectorial operator. ■

Theorem 2.1.6 *The following conditions hold:*
(i) $-\mathbb{A}$ is maximal accretive or, equivalently, $\mathbb{A}$ is maximal dissipative;
(ii) $\mathbb{A}$ has compact resolvent;
(iii) $\mathbb{A}$ has imaginary powers are bounded and

$$\|\mathbb{A}^{it}\|_{\mathscr{L}(X)} \leqslant e^{\frac{\pi}{2}|t|}, \quad t \in \mathbb{R};$$

(iv) The semigroup $\{e^{\mathbb{A}t} : t \geqslant 0\}$ is compact.

Proof. A part of item (i) follows of Lemma 2.1.2. To complete part (i) it suffices to note that the equation

$$(I_d - \mathbb{A})\begin{pmatrix} u \\ v \\ \theta \end{pmatrix} = \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{\theta} \end{pmatrix}$$

possesses, for each $\begin{pmatrix} \tilde{u} & \tilde{v} & \tilde{\theta} \end{pmatrix}^T \in X$, a unique solution

$$\begin{pmatrix} u \\ v \\ \theta \end{pmatrix} =$$

$$\begin{pmatrix} (\Lambda^{\frac{3}{2}} + 3\Lambda + 4\Lambda^{\frac{1}{2}} + 5I)^{-1}[(\Lambda + 3\Lambda^{\frac{1}{2}} + 3I)\tilde{u} - (\Lambda^{\frac{3}{2}} + 2\Lambda + \Lambda^{\frac{1}{2}} + 2I)\tilde{v}] \\ (\Lambda^{\frac{3}{2}} + 3\Lambda + 4\Lambda^{\frac{1}{2}} + 5I)^{-1}[-(\Lambda^{\frac{3}{2}} + 2\Lambda + \Lambda^{\frac{1}{2}} + 2I)\tilde{u} + (\Lambda^{\frac{1}{2}} + 2I)\tilde{v} - (\Lambda + 2\Lambda^{\frac{1}{2}} + I)\tilde{\theta}] \\ (\Lambda^{\frac{3}{2}} + 3\Lambda + 4\Lambda^{\frac{1}{2}} + 5I)^{-1}[-(\Lambda + 2\Lambda^{\frac{1}{2}} + I)\tilde{v} + (\Lambda^{\frac{3}{2}} + 2\Lambda + 2\Lambda^{\frac{1}{2}} + 4I)\tilde{\theta}] \end{pmatrix}$$

belong to X^1. The item (ii) follows from Remark 2.1.4 and compactness of the Sobolev inclusions between Y^α spaces resulting from compactness of the resolvent of $\Lambda + I$ and $\Lambda^{\frac{1}{2}} + I$. The item (iii) follows from the observations concerning powers of accretive operators reported in [1, Example 4.7.3 (b), p. 164]. The item (iv) is a consequence of (ii). ■

2.1.3 Partial description of the fractional power scale

Connecting the properties of listed above with the results of Amann [1, Chapter v] we obtain a partial description of the fractional power scale associated to $\mathbb{A}$. Before we can proceed we need the following general interpolation result:

Proposition 2.1.7 *Let $\mathscr{W}_i, \mathscr{Z}_i,\ i = 1, 2, 3$ be the Banach spaces such that*

$$\mathscr{W}_i \subset \mathscr{Z}_i,$$

topologically and algebraically. Then,

$$[\mathscr{Z}_1 \times \mathscr{Z}_2 \times \mathscr{Z}_3, \mathscr{W}_1 \times \mathscr{W}_2 \times \mathscr{W}_3]_\alpha = [\mathscr{Z}_1, \mathscr{W}_1]_\alpha \times [\mathscr{Z}_2, \mathscr{W}_2]_\alpha \times [\mathscr{Z}_3, \mathscr{W}_3]_\alpha, \quad \alpha \in (0, 1). \tag{2.21}$$

Proof. The proof is an immediate consequence of the definition of complex interpolation spaces in Triebel [43, Section 1.9.2]. ■

Based on Proposition 2.1.7 it is now easy to get characterizations of the fractional power spaces $X^\alpha,\ \alpha \in (0, 1)$.

Proposition 2.1.8 *For $\alpha \in [0, 1]$ we have:*

$$X^\alpha = D(\mathbb{A}^\alpha) = Y^{\frac{\alpha+1}{2}} \times Y^{\frac{\alpha}{2}} \times Y^{\frac{\alpha}{2}}$$

Proof. Recall that $X^0 = Y^{\frac{1}{2}} \times Y^0 \times Y^0,\ X^1 = Y^1 \times Y^{\frac{1}{2}} \times Y^{\frac{1}{2}}$ and from Theorem 2.1.5,

$$X^\alpha = [X^0, X^1]_\alpha, \quad \alpha \in [0, 1]. \tag{2.22}$$

Combining (2.22) and (2.21) we obtain

$$X^\alpha = [Y^{\frac{1}{2}}, Y^1]_\alpha \times [Y^0, Y^{\frac{1}{2}}]_\alpha \times [Y^0, Y^{\frac{1}{2}}]_\alpha.$$

Next, by our assumptions on Λ, we have the equalities:

$$[Y^{\frac{1}{2}}, Y^1]_\alpha = Y^{\frac{\alpha+1}{2}}, \quad [Y^0, Y^{\frac{1}{2}}]_\alpha = Y^{\frac{\alpha}{2}}, \quad [Y^0, Y^{\frac{1}{2}}]_\alpha = Y^{\frac{\alpha}{2}},$$

which justify the relation for $D(\mathbb{A})$. ■

Remark 2.1.9 *Denote by X_{-1} the extrapolation space of $X = Y^{\frac{1}{2}} \times Y \times Y$ generated by the operator $\mathbb{A}^{-1}$. The following equality holds*

$$X_{-1} = Y \times Y^{-\frac{1}{2}} \times Y^{-\frac{1}{2}}.$$

In fact, recall first that X_{-1} is the completion of the normed space $(X, \|\mathbb{A}^{-1} \cdot \|)$. Now, note that

$$\begin{aligned}\left\|\mathbb{A}^{-1}\left(\begin{smallmatrix} u \\ v \\ \theta \end{smallmatrix}\right)\right\|_X &= \left\|\begin{pmatrix} -(\Lambda+I)^{-1}(\Lambda^{\frac{1}{2}}+I)u - (\Lambda+I)^{-1}v - (\Lambda+I)^{-1}\theta \\ u \\ -u - (\Lambda^{\frac{1}{2}}+I)^{-1}\theta \end{pmatrix}\right\|_X \\ &\leqslant \|(\Lambda+I)^{-1}(\Lambda^{\frac{1}{2}}+I)u\|_{Y^{\frac{1}{2}}} + \|(\Lambda+I)^{-1}v\|_{Y^{\frac{1}{2}}} + \|(\Lambda+I)^{-1}\theta\|_{Y^{\frac{1}{2}}} \\ &\quad + 2\|u\|_Y + \|(\Lambda^{\frac{1}{2}}+I)^{-1}\theta\|_Y \\ &\leqslant 3\|u\|_Y + \|v\|_{Y^{-\frac{1}{2}}} + 2\|\theta\|_{Y^{-\frac{1}{2}}} \\ &\leqslant C_1 \left\|\left(\begin{smallmatrix} u \\ v \\ \theta \end{smallmatrix}\right)\right\|_{X_{-1}},\end{aligned}$$

for any $\left(\begin{smallmatrix} u \\ v \\ \theta \end{smallmatrix}\right) \in X_{-1}$ and for some constant $C_1 > 0$. We also have that

$$\begin{aligned}\left\|\left(\begin{smallmatrix} u \\ v \\ \theta \end{smallmatrix}\right)\right\|_{X_{-1}} &= \|u\|_Y + \|v\|_{Y^{-\frac{1}{2}}} + \|\theta\|_{Y^{-\frac{1}{2}}} \\ &= \|u\|_Y + \|(\Lambda+I)^{-\frac{1}{2}}v\|_Y + \|(\Lambda^{\frac{1}{2}}+I)^{-1}\theta\|_Y.\end{aligned} \tag{2.23}$$

The last two parcels of (2.23) *can be estimated as follows*

$$\begin{aligned}\|(\Lambda+I)^{-\frac{1}{2}}v\|_Y &\leqslant \|(\Lambda+I)^{-\frac{1}{2}}v + (\Lambda+I)^{-\frac{1}{2}}\theta + (\Lambda+I)^{-\frac{1}{2}}(\Lambda^{\frac{1}{2}}+I)u\|_Y \\ &\quad + \|(\Lambda+I)^{-\frac{1}{2}}(\Lambda^{\frac{1}{2}}+I)u\|_Y + \|(\Lambda+I)^{-\frac{1}{2}}\theta\|_Y \\ &= \|(\Lambda+I)^{\frac{1}{2}}[(\Lambda+I)^{-1}v + (\Lambda+I)^{-1}\theta + (\Lambda+I)^{-1}(\Lambda^{\frac{1}{2}}+I)u]\|_Y \\ &\quad + \|(\Lambda+I)^{-\frac{1}{2}}(\Lambda^{\frac{1}{2}}+I)u\|_Y + \|(\Lambda+I)^{-\frac{1}{2}}\theta\|_Y \\ &= \|(\Lambda+I)^{-1}v + (\Lambda+I)^{-1}\theta + (\Lambda+I)^{-1}(\Lambda^{\frac{1}{2}}+I)u\|_{Y^{\frac{1}{2}}} \\ &\quad + 2\|u\|_Y + \|u + (\Lambda^{\frac{1}{2}}+I)^{-1}\|_Y\end{aligned} \tag{2.24}$$

and

$$\|(\Lambda^{\frac{1}{2}}+I)^{-1}\|_Y \leqslant \|u + (\Lambda^{\frac{1}{2}}+I)^{-1}\|_Y + \|u\|_Y. \tag{2.25}$$

Then, combining (2.23) *with* (2.24) *and* (2.25), *we obtain that for some constant* $C_2 > 0$,

$$\left\| \begin{pmatrix} u \\ v \\ \theta \end{pmatrix} \right\|_{X_1} \leqslant \|(\Lambda + I)^{-1} v + (\Lambda + I)^{-1}\theta + (\Lambda + I)^{-1}(\Lambda^{\frac{1}{2}} + I)u\|_{Y^{\frac{1}{2}}}$$
$$+ 4\|u\|_Y + 2\|u + (\Lambda^{\frac{1}{2}} + I)^{-1}\|_Y$$
$$\leqslant C_2 \left\| \mathbb{A}^{-1} \begin{pmatrix} u \\ v \\ \theta \end{pmatrix} \right\|_X .$$

So we conclude that the completion of $(X, \|\mathbb{A}^{-1}\cdot\|_X)$ *and* $(X, \|\cdot\|_{X_{-1}})$ *coincide.*

Remark 2.1.10 *The operator* $\mathbb{A}$ *can be extended to its closed* $X_{-1}-$*realization (see Amann [1]), which we still denote by the same symbol so that* $\mathbb{A}$ *considered in* X_{-1} *is then sectorial positive operator. Our next concern be to obtain embedding of the spaces from the fractional powers scale* $X_{\alpha-1}$, $\alpha \geqslant 0$, *generated by* $(\mathbb{A}, X_{-1})$.

Below we have a partial description of the fractional power spaces scale for $\mathbb{A}$: *for convenience we denote* X *by* X_0, *then*

$$X_0 \hookrightarrow X_{\alpha-1} \hookrightarrow X_{-1}, \text{ for all } 0 < \alpha < 1,$$

where

$$X_{\alpha-1} = [X_{-1}, X_0]_\alpha = Y^{\frac{\alpha}{2}} \times Y^{\frac{\alpha-1}{2}} \times Y^{\frac{\alpha-1}{2}}$$

and $[\cdot,\cdot]_\alpha$ *denotes the complex interpolation functor (see Triebel [43]). The first equality follows from Theorem 2.1.5 (since* $0 \in \rho(\mathbb{A})$*) see Amann [1, Example 4.7.3 (b)] and the second equality follows from Carvalho and Cholewa [13, Proposition 2].*

Remark 2.1.11 *The operator* $\mathbb{A}$ *or, more precisely, a suitable realization of it, generates an analytic semigroup,* $\{e^{\mathbb{A}t} : t \geqslant 0\}$, *in* X_{-1}, *this semigroup is order preserving and satisfies the smoothing estimates. Thanks to Henry [20, Theorem 1.4.3, p. 26] we have*

$$\left\|e^{\mathbb{A}t}v\right\|_X \leqslant Me^{-\omega t}t^{-1}\|v\|_{X_{-1}},$$

for any $t > 0$, $v \in X_{-1}$, *for some constants* $M > 0$ *and* $\omega > 0$.

Finally, thanks to (2.4) *we have* $Y^{\frac{1}{2}} \hookrightarrow H^s(\Omega)$, $s \leqslant 2$ *and consequently,* $\mathscr{H} \hookrightarrow X_{-1}$ *and*

$$\left\|e^{\mathbb{A}t}v\right\|_X \leqslant Me^{-\omega t}t^{-1}\|v\|_{\mathscr{H}}, \tag{2.26}$$

for any $t > 0$, $v \in \mathscr{H}$, *for some constants* $M > 0$ *and* $\omega > 0$.

2.2 Existence and uniqueness of local solutions and differentiability

Since the operator $-\mathbb{A}$ is sectorial, we prove local existence and uniqueness of the solutions of the abstract problems (2.7) and (2.10) and that the solutions are continuously differentiable with respect to initial conditions.

2.2.1 Existence and uniqueness of local solutions

We are interested in obtaining the local well-posedness of the parabolic problems (2.7) and (2.10) (or (2) and (3)), for this it is necessary to study the behavior of nonlinearity F_ε, $\varepsilon \in [0, \varepsilon_0]$.

The next lemmas are crucial results in our analysis.

Lemma 2.2.1 *Assume that $v \in H^{s,p}(\Omega)$ with $\frac{1}{p} < s \leqslant 2$ and $s - \frac{N}{p} \geqslant -\frac{N-1}{q}$, or $v \in H^{1,1}(\Omega)$, i.e., $s = 1 = p$ and $q = 1$ below. Then for sufficiently small ε_0, we have*

(i) The map

$$[0, \varepsilon_0] \ni \sigma \mapsto \int_{\Gamma_\sigma} |v|^q dS$$

is continuous, where for sufficiently small $\sigma \geqslant 0$, $\Gamma_\sigma = \{x - \sigma \vec{n}(x) : x \in \Gamma\}$ is the "parallel" interior boundary.

(ii) There exists $C > 0$ independent of ε and v such that for any $0 < \varepsilon \leqslant \varepsilon_0$, we have

$$\sup_{\sigma \in [0,\varepsilon)} \|v\|_{L^q(\Gamma_\sigma)} \leqslant C\|v\|_{H^{s,p}(\Omega)},$$

$$\int_{\omega_\varepsilon} |v|^q dx = \int_0^\varepsilon \Big(\int_{\Gamma_\sigma} |v|^q dS \Big) d\sigma,$$

with the same equality, without the absolute value, if $q = 1$.

In particular

$$\frac{1}{\varepsilon} \int_{\omega_\varepsilon} |v|^q dx \leqslant C\|v\|^q_{H^{s,p}(\Omega)},$$

and

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\omega_\varepsilon} |v|^q dx = \int_\Gamma |v|^q dS.$$

Proof. See Arrieta, Jiménez-Casas and Rodríguez-Bernal [10, Lemma 2.1]. ■

Now, we consider a family of functions $g^0_\varepsilon : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ for $0 \leqslant \varepsilon \leqslant \varepsilon_0$, satisfying the following conditions:

(i) $\{g^0_\varepsilon(x, u)\}_\varepsilon$ is uniformly bounded in $\overline{\Omega}$ on bounded sets of $\mathbb{R}$, that is, for any $R > 0$ there exists a positive constant $C(R)$ independent of ε such that

$$\big|g^0_\varepsilon(x, u)\big| \leqslant C(R), \quad \text{for all } x \in \overline{\Omega}, \text{ and } |u| \leqslant R. \tag{2.27}$$

(ii) $\{g_\varepsilon^0(x,u)\}_\varepsilon$ is uniformly bounded in $\overline{\Omega}$ on bounded sets of $\mathbb{R}$ and also uniformly Lipschitz on bounded sets of $\mathbb{R}$, that is, for any $R>0$ there exists a positive constant $L(R)$ independent of ε such that

$$\left|g_\varepsilon^0(x,u)-g_\varepsilon^0(x,v)\right| \leqslant L(R)|u-v|, \quad \text{for all } x\in\overline{\Omega}, \text{ and } |u|\leqslant R,\ |v|\leqslant R. \tag{2.28}$$

(iii) $g_\varepsilon^0(x,u)$ converges to $g_0^0(x,u)$ uniformly on Γ and on bounded sets of $\mathbb{R}$, that is, for any $R>0$

$$g_\varepsilon^0(x,u)\to g_0^0(x,u) \quad \text{as } \varepsilon\to 0, \text{ uniformly on } x\in\Gamma \text{ and } |u|\leqslant R. \tag{2.29}$$

Then we have the following result.

Lemma 2.2.2 *Consider a family of functions*

$$g_\varepsilon^0:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$$

for $0\leqslant\varepsilon\leqslant\varepsilon_0$. Also, consider a family of functions, C, in Ω such that, for some $1<p<\infty$ and $R>0$

$$\|v\|_{H^{1,p}(\Omega)\cap L^\infty(\Omega)}\leqslant R, \quad \textit{for all } v\in C.$$

(i) If $\{g_\varepsilon^0\}_\varepsilon$ satisfies (2.27), then there exists a positive constant, $M(R)$, independent of ε such that for every $1<q<\infty$ and any $\varphi\in H^{s,q'}(\Omega)$ with $s>\frac{1}{q'}$ and every $v\in C$ we have

$$\left|\frac{1}{\varepsilon}\int_{\omega_\varepsilon} g_\varepsilon^0(\cdot,v)\varphi\right|\leqslant M(R)\|\varphi\|_{H^{s,q'}(\Omega)}.$$

In particular

$$\sup_{v\in\mathscr{C}}\left\|\frac{1}{\varepsilon}\chi_{\omega_\varepsilon}g_\varepsilon^0(\cdot,v)\right\|_{H^{-s,q}(\Omega)}\leqslant M(R).$$

(ii) If $\{g_\varepsilon^0\}_\varepsilon$ satisfies (2.27), (2.28) and (2.29), then there exists $M(\varepsilon,R)$ with $M(\varepsilon,R)\to 0$ as $\varepsilon\to 0$ such that for every $\varphi\in H^{1,q'}(\Omega)$ and $v\in C$

$$\left|\frac{1}{\varepsilon}\int_{\omega_\varepsilon} g_\varepsilon^0(\cdot,v)\varphi-\int_\Gamma g_0^0(\cdot,v)\varphi\right|\leqslant M(\varepsilon,R)\|\varphi\|_{H^{1,q'}(\Omega)},$$

provided

$$p\geqslant\frac{q(N-1)}{N}.$$

In particular

$$\frac{1}{\varepsilon}\chi_{\omega_\varepsilon}g_\varepsilon^0(\cdot,v)\to g_0^0(\cdot,v), \quad \textit{in } H^{-1,q}(\Omega), \textit{ uniformly in } v\in C.$$

Proof. See Jiménez-Casas and Rodríguez-Bernal [23, Lemma 5.2]. ■

Lemma 2.2.3 *Suppose that f and g satisfy the growth estimate (2.1) and $\frac{1}{2} < s \leqslant 1$. Then:*

(i) There exists $C > 0$, independent of ε, such that

$$\|F_\varepsilon(w)\|_{\mathscr{H}} \leqslant C, \quad \text{for all} \quad w = \begin{pmatrix} u \\ v \\ \theta \end{pmatrix} \in X \quad \text{and} \quad 0 \leqslant \varepsilon \leqslant \varepsilon_0.$$

(ii) For each $0 \leqslant \varepsilon \leqslant \varepsilon_0$, the map $F_\varepsilon : X \to \mathscr{H}$ is globally Lipschitz, uniformly in ε.

(iii) For each $w = \begin{pmatrix} u \\ v \\ \theta \end{pmatrix} \in X$, we have

$$\|F_\varepsilon(w) - F_0(w)\|_{\mathscr{H}} \to 0, \quad \text{as } \varepsilon \to 0.$$

Furthermore, this limit is uniform for $w \in X$ such that $\|w\|_X \leqslant R$, for some $R > 0$.

(iv) If $w_\varepsilon \to w$ in X, as $\varepsilon \to 0$, then

$$\|F_\varepsilon(w_\varepsilon) - F_0(w)\|_{\mathscr{H}} \to 0, \quad \text{as } \varepsilon \to 0.$$

Proof. (i) Initially note that

$$\|F_\varepsilon(w)\|_{\mathscr{H}} = \left\| f_\Omega(u) + \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} g_\Omega(u) \right\|_{H^{-s}(\Omega)}, \quad \varepsilon \in (0, \varepsilon_0],$$

$$\|F_0(w)\|_{\mathscr{H}} = \|f_\Omega(u) + g_\Gamma(u)\|_{H^{-s}(\Omega)},$$

with f_Ω, $\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} g_\Omega$ and g_Γ defined, respectively, by (2.8), (2.9) and (2.11).

Using (2.1), Cauchy-Schwarz inequality and Sobolev embedding $H^s(\Omega) \hookrightarrow L^2(\Omega)$ with $\frac{1}{2} < s \leqslant 1$, we have

$$\begin{aligned} |\langle f_\Omega(u), \varphi\rangle| &\leqslant \int_\Omega |f(u(x))||\varphi(x)|dx \leqslant \int_\Omega K|\varphi(x)|dx \\ &\leqslant cK \|\varphi\|_{L^2(\Omega)} \leqslant k_1 \|\varphi\|_{H^s(\Omega)}, \quad \forall \varphi \in H^s(\Omega). \end{aligned}$$

Thus,

$$\|f_\Omega(u)\|_{H^{-s}(\Omega)} \leqslant k_1. \tag{2.30}$$

Using (2.1), Cauchy-Schwarz inequality, $|\omega_\varepsilon| \leqslant k\,|\Gamma|\,\varepsilon$ for some $k > 0$ independent of ε, and Lemma 2.2.1, we have

$$\begin{aligned} \left|\left\langle \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} g_\Omega(u), \varphi \right\rangle\right| &\leqslant \frac{1}{\varepsilon}\int_{\omega_\varepsilon} |g(u(x))||\varphi(x)|dx \leqslant \frac{K}{\varepsilon}\int_{\omega_\varepsilon} |\varphi(x)|dx \\ &\leqslant K\left[\frac{1}{\varepsilon}\int_{\omega_\varepsilon} 1dx\right]^{\frac{1}{2}}\left[\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |\varphi(x)|^2dx\right]^{\frac{1}{2}} \leqslant k_2 \|\varphi\|_{H^s(\Omega)}, \quad \forall \varphi \in H^s(\Omega), \end{aligned}$$

with $k_2 > 0$ independent of ε. Thus,

$$\left\| \frac{1}{\varepsilon} \chi_{\omega_\varepsilon} g_\Omega(u) \right\|_{H^{-s}(\Omega)} \leqslant k_2. \tag{2.31}$$

Now, using (2.1), Cauchy-Schwarz inequality and the continuity of the trace operator $\gamma : H^s(\Omega) \to L^2(\Gamma)$ with $\frac{1}{2} < s \leqslant 1$, we have

$$\begin{aligned} |\langle g_\Gamma(u), \varphi \rangle| &\leqslant \int_\Gamma |\gamma(g(u(x)))||\gamma(\varphi(x))| d\sigma \leqslant K \int_\Gamma |\gamma(\varphi(x))| d\sigma \\ &\leqslant cK \left[\int_\Gamma |\gamma(\varphi(x))|^2 d\sigma \right]^{\frac{1}{2}} = cK \|\gamma(\varphi)\|_{L^2(\Gamma)} \leqslant k_3 \|\varphi\|_{H^s(\Omega)}, \quad \forall \varphi \in H^s(\Omega). \end{aligned}$$

Thus,

$$\|g_\Gamma(u)\|_{H^{-s}(\Omega)} \leqslant k_3. \tag{2.32}$$

Now, the item (i) follows in a straightforward from (2.30), (2.31) and (2.32).

(ii) Initially, note that

$$\begin{aligned} &\|F_\varepsilon(w_1) - F_\varepsilon(w_2)\|_{\mathscr{H}} \\ &= \left\| [f_\Omega(u_1) - f_\Omega(u_2)] + \frac{1}{\varepsilon} \chi_{\omega_\varepsilon} [g_\Omega(u_1) - g_\Omega(u_2)] \right\|_{H^{-s}(\Omega)}, \quad \varepsilon \in (0, \varepsilon_0], \end{aligned}$$

and

$$\|F_0(w_1) - F_0(w_2)\|_{\mathscr{H}} = \|[f_\Omega(u_1) - f_\Omega(u_2)] + [g_\Gamma(u_1) - g_\Gamma(u_2)]\|_{H^{-s}(\Omega)},$$

with f_Ω, $\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} g_\Omega$ and g_Γ defined, respectively, by (2.8), (2.9) and (2.11).

Using (2.1), Cauchy-Schwarz inequality and Sobolev embeddings $H^2(\Omega) \hookrightarrow L^2(\Omega)$ and $H^s(\Omega) \hookrightarrow L^2(\Omega)$ with $\frac{1}{2} < s \leqslant 1$, we have

$$\begin{aligned} |\langle f_\Omega(u_1) - f_\Omega(u_2), \varphi \rangle| &\leqslant \int_\Omega |f(u_1(x)) - f(u_2(x))||\varphi(x)| dx \\ &\leqslant \int_\Omega |f'(\sigma(x) u_1(x) + (1 - \sigma(x)) u_2(x))||u_1(x) - u_2(x)||\varphi(x)| dx \\ &\leqslant K \left[\int_\Omega |u_1(x) - u_2(x)|^2 dx \right]^{\frac{1}{2}} \left[\int_\Omega |\varphi(x)|^2 dx \right]^{\frac{1}{2}} \\ &= K \|u_1 - u_2\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)} \\ &\leqslant c_1 \|u_1 - u_2\|_{H^2(\Omega)} \|\varphi\|_{H^s(\Omega)}, \quad \forall \varphi \in H^s(\Omega), \end{aligned}$$

for some $0 \leqslant \sigma(x) \leqslant 1, x \in \overline{\Omega}$. Thus,

$$\|f_\Omega(u_1) - f_\Omega(u_2)\|_{H^{-s}(\Omega)} \leqslant c_1 \|u_1 - u_2\|_{H^2(\Omega)}. \tag{2.33}$$

Using (2.1), Cauchy-Schwarz inequality and Lemma 2.2.1, we have

$$\left|\left\langle \frac{1}{\varepsilon}\chi_{\omega_\varepsilon}[g_\Omega(u_1)-g_\Omega(u_2)],\varphi\right\rangle\right| \leqslant \frac{1}{\varepsilon}\int_{\omega_\varepsilon}|g(u_1(x))-g(u_2(x))||\varphi(x)|dx$$
$$\leqslant \frac{1}{\varepsilon}\int_{\omega_\varepsilon}|g'(\sigma(x)u_1(x)+(1-\sigma(x))u_2(x))||u_1(x)-u_2(x)||\varphi(x)|dx$$
$$\leqslant K\left[\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|u_1(x)-u_2(x)|^2dx\right]^{\frac{1}{2}}\left[\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|\varphi(x)|^2dx\right]^{\frac{1}{2}}$$
$$\leqslant c_2\left\|u_1-u_2\right\|_{H^2(\Omega)}\left\|\varphi\right\|_{H^s(\Omega)}, \quad \forall\varphi\in H^s(\Omega),$$

with $c_2>0$ independent of ε and for some $0\leqslant\sigma(x)\leqslant 1, x\in\overline{\Omega}$. Thus,

$$\left\|\frac{1}{\varepsilon}\chi_{\omega_\varepsilon}[g_\Omega(u_1)-g_\Omega(u_2)]\right\|_{H^{-s}(\Omega)}\leqslant c_2\left\|u_1-u_2\right\|_{H^2(\Omega)}. \tag{2.34}$$

Now, using (2.1), Cauchy-Schwarz inequality and the continuity of the trace operators $\gamma: H^2(\Omega)\to L^2(\Gamma)$ and $\gamma: H^s(\Omega)\to L^2(\Gamma)$ with $\frac{1}{2}<s\leqslant 1$, we have

$$|\langle g_\Gamma(u_1)-g_\Gamma(u_2),\varphi\rangle|\leqslant\int_\Gamma|\gamma(g(u_1(x))-g(u_2(x)))||\gamma(\varphi(x))|d\sigma$$
$$\leqslant\int_\Gamma|\gamma(g'(\sigma(x)u_1(x)+(1-\sigma(x))u_2(x)))||\gamma(u_1(x)-u_2(x))||\gamma(\varphi(x))|d\sigma$$
$$\leqslant K\left[\int_\Gamma|\gamma(u_1(x)-u_2(x))|^2d\sigma\right]^{\frac{1}{2}}\left[\int_\Gamma|\gamma(\varphi(x))|^2d\sigma\right]^{\frac{1}{2}}$$
$$\leqslant c_3\left\|u_1-u_2\right\|_{H^2(\Omega)}\left\|\varphi\right\|_{H^s(\Omega)}, \quad \forall\varphi\in H^s(\Omega),$$

for some $0\leqslant\sigma(x)\leqslant 1, x\in\Gamma$. Thus,

$$\left\|g_\Gamma(u_1)-g_\Gamma(u_2)\right\|_{H^{-s}(\Omega)}\leqslant c_3\left\|u_1-u_2\right\|_{H^2(\Omega)}. \tag{2.35}$$

Now, the item (ii) follows in a straightforward from (2.33), (2.34) and (2.35).

(iii) Notice that

$$\left\|F_\varepsilon(w)-F_0(w)\right\|_{\mathscr{H}}=\left\|\frac{1}{\varepsilon}\chi_{\omega_\varepsilon}g_\Omega(u)-g_\Gamma(u)\right\|_{H^{-s}(\Omega)}.$$

As in Lemma 2.2.2 we can prove that there exists $M(\varepsilon,R)$ with $M(\varepsilon,R)\to 0$ as $\varepsilon\to 0$ such that

$$\left|\left\langle\frac{1}{\varepsilon}\chi_{\omega_\varepsilon}g_\Omega(u)-g_\Gamma(u),\varphi\right\rangle\right|=\left|\frac{1}{\varepsilon}\int_{\omega_\varepsilon}g(u(x))\varphi(x)dx-\int_\Gamma\gamma(g(u(x)))\gamma(\varphi(x))dS\right|$$
$$\leqslant M(\varepsilon,R)\left\|\varphi\right\|_{H^1(\Omega)}, \quad \forall\varphi\in H^1(\Omega).$$

Thus,

$$\left\| \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} g_\Omega(u) - g_\Gamma(u) \right\|_{H^{-1}(\Omega)} \to 0, \quad \text{as } \varepsilon \to 0, \tag{2.36}$$

uniformly for $u \in H^2(\Omega)$ such that $\|u\|_{H^2(\Omega)} \leqslant R$.

Now, fix $\frac{1}{2} < s_0 < 1$. Then for any s such that $-1 < -s < -s_0 < -\frac{1}{2}$, using interpolation we have

$$\left\| \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} g_\Omega(u) - g_\Gamma(u) \right\|_{H^{-s}(\Omega)} \leqslant \left\| \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} g_\Omega(u) - g_\Gamma(u) \right\|_{H^{-s_0}(\Omega)}^{\theta} \left\| \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} g_\Omega(u) - g_\Gamma(u) \right\|_{H^{-1}(\Omega)}^{1-\theta},$$

for some $0 < \theta < 1$. By (2.31) and (2.32), the first term in the right hand side above is uniformly bounded while, by (2.36), the second goes to zero, both uniformly for $u \in H^2(\Omega)$ such that $\|u\|_{H^2(\Omega)} \leqslant R$.

(iv) This item follows from *(ii)* and *(iii)*, adding and subtracting $F_\varepsilon(w)$. In fact

$$\begin{aligned} \|F_\varepsilon(w_\varepsilon) - F_0(w)\|_{\mathscr{H}} &\leqslant \|F_\varepsilon(w_\varepsilon) - F_\varepsilon(w)\|_{\mathscr{H}} + \|F_\varepsilon(w) - F_0(w)\|_{\mathscr{H}} \\ &\leqslant L\|w_\varepsilon - w\|_X + \|F_\varepsilon(w) - F_0(w)\|_{\mathscr{H}} \to 0, \quad \text{as } \varepsilon \to 0, \end{aligned}$$

where $L > 0$ is the constant of Lipschitz, and we conclude the proof of Lemma 2.2.3.

■

From Lemma 2.2.3 follows that the map $F_\varepsilon : X \to \mathscr{H}$ is bounded, uniformly in ε, in bounded set of X, and it is locally Lipschitz, uniformly in ε. Thus, it follows from the classic results of the theory of ordinary differential equations in Banach spaces that, given $w_0 \in X$, there is an unique local solution $w^\varepsilon(t, w_0)$ of (2.7), with $\varepsilon \in (0, \varepsilon_0]$, defined on a maximal interval of existence $[0, t^\varepsilon_{max}(w_0))$, and there is an unique local solution $w(t, w_0)$ of (2.10) defined on a maximal interval of existence $[0, t_{max}(w_0))$. Moreover, these solutions depend continuously on the initial data.

2.2.2 The differentiability

We prove that the solutions of (2.7) and (2.10) are continuously differentiable with respect to initial conditions, for this it is necessary to prove the Fréchet differentiability of $F_\varepsilon : X \to \mathscr{H}$, $\varepsilon \in [0, \varepsilon_0]$. It is enough to prove the Fréchet differentiability of $f_\Omega, \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} g_\Omega, g_\Gamma : H^2(\Omega) \to H^{-s}(\Omega)$.

We define the maps $Df_\Omega, \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega, Dg_\Gamma : H^2(\Omega) \to \mathscr{L}(H^2(\Omega), H^{-s}(\Omega))$, with

$\frac{1}{2} < s \leqslant 1$, respectively by

$$\langle Df_\Omega(u) \cdot h, \varphi \rangle = \int_\Omega f'(u) h\varphi dx, \quad \forall u, h \in H^2(\Omega) \text{ and } \forall \varphi \in H^s(\Omega), \tag{2.37}$$

$$\left\langle \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(u) \cdot h, \varphi \right\rangle = \frac{1}{\varepsilon}\int_{\omega_\varepsilon} g'(u) h\varphi dx, \quad \forall u, h \in H^2(\Omega) \text{ and } \forall \varphi \in H^s(\Omega) \tag{2.38}$$

and

$$\langle Dg_\Gamma(u) \cdot h, \varphi \rangle = \int_\Gamma \gamma(g'(u)h)\gamma(\varphi) dS, \quad \forall u, h \in H^2(\Omega) \text{ and } \forall \varphi \in H^s(\Omega), \tag{2.39}$$

where $\gamma : H^s(\Omega) \to L^2(\Gamma)$ is the trace operator.

Lemma 2.2.4 *Suppose that f and g satisfy the growth estimates (2.1) and $\frac{1}{2} < s \leqslant 1$. Then, f_Ω, $\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} g_\Omega$,$g_\Gamma : H^2(\Omega) \to H^{-s}(\Omega)$ are Fréchet differentiable, uniformly in ε, and your Fréchet differentials are respectively given by (2.37), (2.38) and (2.39). Consequently, for each $\varepsilon \in [0, \varepsilon_0]$, $F_\varepsilon : X \to \mathscr{H}$ is also Fréchet differentiable, uniformly in ε.*

Proof. First we check that (2.37), (2.38) and (2.39) are well defined. In fact, for $h \in H^2(\Omega)$, using (2.1), Cauchy-Schwarz inequality and Sobolev embeddings, we get

$$\begin{aligned}
|\langle Df_\Omega(u) \cdot h, \varphi \rangle| &\leqslant \int_\Omega |f'(u)h||\varphi| dx \leqslant K \int_\Omega |h||\varphi| dx \\
&\leqslant K \left[\int_\Omega |h|^2 dx\right]^{\frac{1}{2}} \left[\int_\Omega |\varphi|^2 dx\right]^{\frac{1}{2}} \\
&= K\|h\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)} \leqslant k_1\|h\|_{H^2(\Omega)}\|\varphi\|_{H^s(\Omega)}, \quad \forall \varphi \in H^s(\Omega).
\end{aligned}$$

Thus,

$$\|Df_\Omega(u) \cdot h\|_{H^{-s}(\Omega)} \leqslant k_1\|h\|_{H^2(\Omega)}, \quad \forall h \in H^2(\Omega), \tag{2.40}$$

and $Df_\Omega(u) \in \mathscr{L}(H^2(\Omega), H^{-s}(\Omega))$.

Using (2.1), Cauchy-Schwarz inequality and Lemma 2.2.1, we have

$$\begin{aligned}
\left|\left\langle \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(u) \cdot h, \varphi \right\rangle\right| &\leqslant \frac{1}{\varepsilon}\int_{\omega_\varepsilon} |g'(u)h||\varphi| dx \leqslant \frac{K}{\varepsilon}\int_{\omega_\varepsilon} |h||\varphi| dx \\
&\leqslant K \left[\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |h|^2 dx\right]^{\frac{1}{2}} \left[\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |\varphi|^2 dx\right]^{\frac{1}{2}} \\
&\leqslant k_2\|h\|_{H^2(\Omega)}\|\varphi\|_{H^s(\Omega)}, \quad \forall \varphi \in H^s(\Omega),
\end{aligned}$$

where the positive constant k_2 is independent of ε. Thus,

$$\left\|\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(u) \cdot h\right\|_{H^{-s}(\Omega)} \leqslant k_2\|h\|_{H^2(\Omega)}, \quad \forall h \in H^2(\Omega), \tag{2.41}$$

and $\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(u) \in \mathscr{L}(H^2(\Omega), H^{-s}(\Omega))$.

Now, using (2.1), Cauchy-Schwarz inequality and trace theorem, we get

$$\begin{aligned}
|\langle Dg_\Gamma(u)\cdot h, \varphi\rangle| &\leqslant \int_\Gamma |\gamma(g'(u)h)||\gamma(\varphi)|d\sigma \leqslant K\int_\Gamma |\gamma(h)||\gamma(\varphi)|d\sigma \\
&\leqslant K\left[\int_\Gamma |\gamma(h)|^2 d\sigma\right]^{\frac{1}{2}}\left[\int_\Gamma |\gamma(\varphi)|^2 d\sigma\right]^{\frac{1}{2}} \\
&\leqslant k_3\|h\|_{H^2(\Omega)}\|\varphi\|_{H^s(\Omega)}, \quad \forall \varphi \in H^s(\Omega).
\end{aligned}$$

Thus,

$$\|Dg_\Gamma(u)\cdot h\|_{H^{-s}(\Omega)} \leqslant k_3\|h\|_{H^2(\Omega)}, \quad \forall h \in H^2(\Omega), \tag{2.42}$$

and $Dg_\Gamma(u) \in \mathscr{L}(H^2(\Omega), H^{-s}(\Omega))$.

Now, let $u, h \in H^2(\Omega)$ and using (2.1), Cauchy-Schwarz inequality and Sobolev embeddings, we have

$$\begin{aligned}
|\langle f_\Omega(u+h) - f_\Omega(u) - Df_\Omega(u)\cdot h, \varphi\rangle| &\leqslant \int_\Omega |f(u+h) - f(u) - f'(u)h||\varphi|dx \\
&= \int_\Omega |f'(u+\sigma h) - f'(u)||h||\varphi|dx \\
&= \int_\Omega |f''(\theta(u+\sigma h) + (1-\theta)u)||\sigma h||h||\varphi|dx \\
&\leqslant K\int_\Omega |h|^2|\varphi|dx \\
&\leqslant K\|h\|^2_{L^4(\Omega)}\|\varphi\|_{L^2(\Omega)} \\
&\leqslant c_1\|h\|^2_{H^2(\Omega)}\|\varphi\|_{H^s(\Omega)}, \quad \forall \varphi \in H^s(\Omega),
\end{aligned}$$

where $\sigma = \sigma(x) \in [0,1]$ and $\theta = \theta(x) \in [0,1]$, $x \in \bar{\Omega}$. Thus,

$$\|f_\Omega(u+h) - f_\Omega(u) - Df_\Omega(u)\cdot h\|_{H^{-s}(\Omega)} \leqslant c_1\|h\|^2_{H^2(\Omega)}.$$

This proves that f_Ω is Fréchet diferentiable and your Fréchet diferential is given by (2.37).

Let $u, h \in H^2(\Omega)$ and using (2.1), Cauchy-Schwarz and Lemma 2.2.1, we have

$$
\begin{aligned}
&\left|\left\langle \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} g_\Omega(u+h) - \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} g_\Omega(u) - \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(u)\cdot h, \varphi \right\rangle\right| \\
&\leqslant \frac{1}{\varepsilon}\int_{\omega_\varepsilon} |g(u+h) - g(u) - g'(u)h||\varphi| dx \\
&= \frac{1}{\varepsilon}\int_{\omega_\varepsilon} |g'(u+\sigma h) - g'(u)||h||\varphi| dx \\
&= \frac{1}{\varepsilon}\int_{\omega_\varepsilon} |g''(\theta(u+\sigma h) + (1-\theta)u)||\sigma h||h||\varphi| dx \\
&\leqslant K\left[\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |h|^4 dx\right]^{\frac{1}{2}} \left[\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |\varphi|^2 dx\right]^{\frac{1}{2}} \\
&\leqslant c_2\|h\|^2_{H^2(\Omega)}\|\varphi\|_{H^s(\Omega)}, \quad \forall \varphi \in H^s(\Omega),
\end{aligned}
$$

where $\sigma = \sigma(x) \in [0,1]$ and $\theta = \theta(x) \in [0,1]$, $x \in \bar{\Omega}$, and with $c_2 > 0$ independent of ε. Thus,

$$
\left\|\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} g_\Omega(u+h) - \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} g_\Omega(u) - \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(u)\cdot h\right\|_{H^{-s}(\Omega)} \leqslant c_2\|h\|^2_{H^2(\Omega)}.
$$

This proves that $\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} g_\Omega$ is Fréchet differentiable, uniformly in ε, and your Fréchet differential is given by (2.38).

Now, let $u, h \in H^2(\Omega)$ and using (2.1), Cauchy-Schwarz and trace theorem, we have

$$
\begin{aligned}
|\langle g_\Gamma(u+h) - g_\Gamma(u) - Dg_\Gamma(u)\cdot h, \varphi\rangle| &\leqslant \int_\Gamma |\gamma(g(u+h)) - \gamma(g(u)) - \gamma(g'(u)h)||\gamma(\varphi)| d\sigma \\
&= \int_\Gamma |\gamma(g''(\theta(u+\sigma h) + (1-\theta)u))||\gamma(h)|^2|\gamma(\varphi)| d\sigma \\
&\leqslant K\|\gamma(h)\|^2_{L^4(\Gamma)}\|\gamma(\varphi)\|_{L^2(\Gamma)} \\
&\leqslant c_3\|h\|^2_{H^2(\Omega)}\|\varphi\|_{H^s(\Omega)}, \quad \forall \varphi \in H^s(\Omega),
\end{aligned}
$$

where $\sigma = \sigma(x) \in [0,1]$ and $\theta = \theta(x) \in [0,1]$, $x \in \Gamma$. Thus,

$$
\|g_\Gamma(u+h) - g_\Gamma(u) - Dg_\Gamma(u)\cdot h\|_{H^{-s}(\Omega)} \leqslant c_3\|h\|^2_{H^2(\Omega)}.
$$

This proves that g_Γ is Fréchet differentiable and your Fréchet differential is given by (2.39).

The Fréchet differentiability of F_ε, uniformly in ε, follows immediately. ■

Lemma 2.2.5 *Suppose that f and g satisfy the growth estimates (2.1). Then, Df_Ω, $\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega$, $Dg_\Gamma : H^2(\Omega) \to \mathscr{L}(H^2(\Omega), H^{-s}(\Omega))$ are globally Lipschitz, uniformly in ε. Consequently, for $\varepsilon \in [0, \varepsilon_0]$, $DF_\varepsilon : X \to \mathscr{L}(X, \mathscr{H})$ is also globally Lipschitz, uniformly in ε.*

Proof. Let $u, v \in H^2(\Omega)$ and using (2.1), Hölder's inequality and Sobolev embeddings, we have

$$\begin{aligned}
|\langle Df_\Omega(u)\cdot h - Df_\Omega(v)\cdot h, \varphi\rangle| &\leqslant \int_\Omega |f'(u)h - f'(v)h||\varphi|dx \\
&= \int_\Omega |f''(u+\sigma v)||u-v||h||\varphi|dx \\
&\leqslant K\int_\Omega |u-v||h||\varphi|dx \\
&\leqslant K\|u-v\|_{L^6(\Omega)}\|h\|_{L^3(\Omega)}\|\varphi\|_{L^2(\Omega)} \\
&\leqslant k_1\|u-v\|_{H^2(\Omega)}\|h\|_{H^2(\Omega)}\|\varphi\|_{H^s(\Omega)},
\end{aligned}$$

for any $h \in H^2(\Omega)$ and $\varphi \in H^s(\Omega)$, where $\sigma = \sigma(x) \in [0,1]$, $x \in \overline{\Omega}$. Thus,

$$\|Df_\Omega(u) - Df_\Omega(v)\|_{\mathscr{L}(H^2(\Omega),H^{-s}(\Omega))} \leqslant k_1\|u-v\|_{H^2(\Omega)}.$$

Let $u, v \in H^2(\Omega)$ and using (2.1), Hölder's inequality and Lemma 2.2.1, we have

$$\begin{aligned}
&\Big|\Big\langle \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(u)\cdot h - \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(v)\cdot h, \varphi\Big\rangle\Big| \leqslant \frac{1}{\varepsilon}\int_{\omega_\varepsilon} |g'(u)h - g'(v)h||\varphi|dx \\
&= \frac{1}{\varepsilon}\int_{\omega_\varepsilon} |g''(u+\sigma v)||u-v||h||\varphi|dx \\
&\leqslant K\left[\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|u-v|^4dx\right]^{\frac{1}{4}}\left[\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|h|^4dx\right]^{\frac{1}{4}}\left[\frac{1}{\varepsilon}\int_{\omega_\varepsilon}|\varphi|^2dx\right]^{\frac{1}{2}} \\
&\leqslant k_2\|u-v\|_{H^2(\Omega)}\|h\|_{H^2(\Omega)}\|\varphi\|_{H^s(\Omega)},
\end{aligned}$$

for any $h \in H^2(\Omega)$ and $\varphi \in H^s(\Omega)$, where $k_2 > 0$ is independent of ε and $\sigma = \sigma(x) \in [0,1]$, $x \in \overline{\Omega}$. Thus,

$$\Big\|\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(u) - \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(v)\Big\|_{\mathscr{L}(H^2(\Omega),H^{-s}(\Omega))} \leqslant k_2\|u-v\|_{H^2(\Omega)}.$$

Now, let $u, v \in H^2(\Omega)$ and using (2.1), Hölder's inequality and trace theorems, we have

$$\begin{aligned}
\Big|\langle Dg_\Gamma(u)\cdot h - Dg_\Gamma(v)\cdot h, \varphi\rangle\Big| &\leqslant \int_\Gamma |\gamma(g'(u)h) - \gamma(g'(v)h)||\gamma(\varphi)|d\sigma \\
&= \int_\Gamma |\gamma(g''(u+\sigma v))||\gamma(u-v)||\gamma(h)||\gamma(\varphi)|d\sigma \\
&\leqslant K\|\gamma(u-v)\|_{L^4(\Gamma)}\|\gamma(h)\|_{L^4(\Gamma)}\|\gamma(\varphi)\|_{L^2(\Gamma)} \\
&\leqslant k_3\|u-v\|_{H^2(\Omega)}\|h\|_{H^2(\Omega)}\|\varphi\|_{H^s(\Omega)},
\end{aligned}$$

for any $h \in H^2(\Omega)$ and $\varphi \in H^s(\Omega)$, where $\sigma = \sigma(x) \in [0,1]$, $x \in \Gamma$. Thus,

$$\|Dg_\Gamma(u) - Dg_\Gamma(v)\|_{\mathscr{L}(H^2(\Omega),H^{-s}(\Omega))} \leqslant k_3\|u-v\|_{H^2(\Omega)}.$$

Consequently, it is immediate that for each $\varepsilon \in [0, \varepsilon_0]$, DF_ε is globally Lipschitz, uniformly in ε. ■

Under the assumptions of Lemma 2.2.4 and Lemma 2.2.5, we have that the map F_ε is continuously Fréchet differentiable. Now, from the classic results of the theory of ordinary differential equations in Banach spaces the solutions of (2.7) and (2.10) are continuously differentiable with respect to initial conditions.

2.3 Existence and uniqueness of global solutions and dissipativity

In this section we wish to prove that the solutions $w^\varepsilon(t, w_0)$, $\varepsilon \in (0, \varepsilon_0]$, and $w(t, w_0)$ of the problems (2.7) and (2.10), respectively, are globally defined, that is, that for each $w_0 \in X$, $t^\varepsilon_{max}(w_0) = \infty$ and $t_{max}(w_0) = \infty$. Moreover, we show that the semigroups associated to solutions are strongly bounded dissipativite. To prove this, we assume the previous hypotheses and additional dissipativity assumption (2.2)(which is equivalent to (2.3)) and we consider continuous functionals on X which are bounded in bounded subsets of X and non-increasing along solutions of these problems.

2.3.1 Perturbed problems

Let $V_\varepsilon : X \to \mathbb{R}$ be the continuous functional defined by

$$\begin{aligned} V_\varepsilon\left(\begin{smallmatrix} u \\ v \\ \theta \end{smallmatrix}\right) &= \frac{1}{2}\Big[\|\Delta u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} + \|\theta\|^2_{L^2(\Omega)}\Big] - \int_\Omega \int_0^u f(s)dsdx \\ &\quad - \frac{1}{\varepsilon}\int_{\omega_\varepsilon}\int_0^u g(s)dsdx, \quad \varepsilon \in (0, \varepsilon_0]. \end{aligned} \tag{2.43}$$

It follows from (2.2) that for any $\gamma_1 > 0$ and $\gamma_2 > 0$, there exists $k_1 = k_1(\gamma_1) > 0$ and $k_2 = k_2(\gamma_2) > 0$ such that

$$\int_0^u f(s)ds \leqslant \int_0^u \Big[\frac{\gamma_1 s}{2} + k_1\Big]ds \leqslant \frac{\gamma_1 u^2}{4} + k_1 u \leqslant \gamma_1 u^2 + c_1 \tag{2.44}$$

and

$$\int_0^u g(s)ds \leqslant \int_0^u \Big[\frac{\gamma_2 s}{2} + k_2\Big]ds \leqslant \frac{\gamma_2 u^2}{4} + k_2 u \leqslant \gamma_2 u^2 + c_2, \tag{2.45}$$

where $c_1 = c_1(\gamma_1) > 0$ and $c_2 = c_2(\gamma_2) > 0$ are independent of ε.

Using (2.44) and (2.45), it follows that

$$\begin{aligned}
&\frac{1}{2}\|\Delta u\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{L^2(\Omega)} + \frac{1}{2}\|v\|^2_{L^2(\Omega)} + \frac{1}{2}\|\theta\|^2_{L^2(\Omega)} \\
&= V_\varepsilon\begin{pmatrix} u \\ v \\ \theta \end{pmatrix} + \int_\Omega \int_0^u f(s)dsdx + \frac{1}{\varepsilon}\int_{\omega_\varepsilon}\int_0^u g(s)dsdx \\
&\leqslant V_\varepsilon\begin{pmatrix} u \\ v \\ \theta \end{pmatrix} + \int_\Omega (\gamma_1|u|^2 + c_1)dx + \frac{1}{\varepsilon}\int_{\omega_\varepsilon}(\gamma_2|u|^2 + c_2)dx.
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{1}{2}\|\Delta u\|^2_{L^2(\Omega)} + \Big(\frac{1}{2} - \gamma_1\Big)\|u\|^2_{L^2(\Omega)} + \frac{1}{2}\|v\|^2_{L^2(\Omega)} + \frac{1}{2}\|\theta\|^2_{L^2(\Omega)} \\
&\leqslant V_\varepsilon\begin{pmatrix} u \\ v \\ \theta \end{pmatrix} + \frac{\gamma_2}{\varepsilon}\int_{\omega_\varepsilon}|u|^2dx + c_2k|\Gamma| + c_1|\Omega|,
\end{aligned}$$

and from Lemma 2.2.1 there exists $C > 0$ independent of ε such that

$$\frac{\gamma_2}{\varepsilon}\int_{\omega_\varepsilon}|u|^2dx \leqslant \gamma_2 C\|u\|^2_{H^2(\Omega)}, \tag{2.46}$$

and this implies that

$$\begin{aligned}
&\frac{1}{2}\|\Delta u\|^2_{L^2(\Omega)} + \Big(\frac{1}{2} - \gamma_1\Big)\|u\|^2_{L^2(\Omega)} - \gamma_2 C\|u\|^2_{H^2(\Omega)} + \frac{1}{2}\|v\|^2_{L^2(\Omega)} + \frac{1}{2}\|\theta\|^2_{L^2(\Omega)} \\
&\leqslant V_\varepsilon\begin{pmatrix} u \\ v \\ \theta \end{pmatrix} + c_2k|\Gamma| + c_1|\Omega|.
\end{aligned}$$

Consequently, for $w^\varepsilon(t) = \begin{pmatrix} u^\varepsilon \\ v^\varepsilon \\ \theta^\varepsilon \end{pmatrix}(t)$ being the solution of the problem (2) we have that

$$\begin{aligned}
&\frac{1}{2}\|\Delta u^\varepsilon\|^2_{L^2(\Omega)} + \Big(\frac{1}{2} - \gamma_1\Big)\|u^\varepsilon\|^2_{L^2(\Omega)} - \gamma_2 C\|u^\varepsilon\|^2_{H^2(\Omega)} + \frac{1}{2}\|v^\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{2}\|\theta^\varepsilon\|^2_{L^2(\Omega)} \\
&\leqslant V_\varepsilon\begin{pmatrix} u^\varepsilon \\ v^\varepsilon \\ \theta^\varepsilon \end{pmatrix} + c_2k|\Gamma| + c_1|\Omega|.
\end{aligned}$$

For $0 < \gamma_1 < \frac{1}{2}$ and choosing γ_2 sufficientely small in the inequality above, we obtain

$$\|w^\varepsilon(t)\|^2_X \leqslant C_1 V_\varepsilon(w^\varepsilon(t)) + C_2, \tag{2.47}$$

for some $C_1, C_2 > 0$ independent of ε.

We note that by subsection 2.2.2 we obtain that a map $t \mapsto w^\varepsilon(t, w_0)$ is differentiable.

It is clear that for $w^\varepsilon(t) = \begin{pmatrix} u^\varepsilon \\ v^\varepsilon \\ \theta^\varepsilon \end{pmatrix}(t)$ being the solution of the problem (2) we have that $[0, t_{max}(w_0)) \ni t \mapsto V_\varepsilon(w^\varepsilon(t, w_0)) \in \mathbb{R}$ is non-increasing because

$$\frac{dV_\varepsilon}{dt}(t) = -\|\nabla\theta^\varepsilon(t)\|^2_{L^2(\Omega)} - \|\theta^\varepsilon(t)\|^2_{L^2(\Omega)} \leqslant 0,$$

for $V_\varepsilon(t) = V_\varepsilon(w^\varepsilon(t, w_0))$, $t \in [0, t_{max}(w_0))$.

Using Lemma 2.2.1 we can prove that V_ε is continous and uniformly bounded in uniformly bounded subsets of X. From (2.47) we have that given $r > 0$, there is a constant $C(r) > 0$ independent of ε such that

$$\sup\{\|w^\varepsilon(t, w_0)\|_X : \|w_0\|_X \leqslant r,\ t \in [0, t^\varepsilon_{max}(w_0))\} \leqslant C. \tag{2.48}$$

From (2.48) we have that for each $w_0 \in X$, the solution of (2.7) is defined for all $t \geqslant 0$, that is, $t^\varepsilon_{max}(w_0) = \infty$. Consequently, for each $\varepsilon \in [0, \varepsilon_0)$, we can to define a nonlinear semigroup $\{S_\varepsilon(t) :\ t \geqslant 0\}$ in X by

$$S_\varepsilon(t)w_0 = w^\varepsilon(t, w_0), \quad t \geqslant 0.$$

This also implies that each uniformly bounded subset of X has orbit and global orbit uniformly bounded in ε.

Note that the nonlinear semigroups are given by the variation of constants formula

$$S_\varepsilon(t)w_0 = e^{\mathbb{A}t}w_0 + \int_0^t e^{\mathbb{A}(t-s)}F_\varepsilon(S_\varepsilon(s)w_0)ds, \quad t \geqslant 0,$$

see Henry [20, Chapter 3] for details.

Remark 2.3.1 *Note that* (2.43) *is a Lyapunov function with the properties of Definition C.17 and thus* $\{S_\varepsilon(t) : t \geqslant 0\},\ \varepsilon \in (0, \varepsilon_0]$ *is a gradient system.*

2.3.2 Limit problem

Let $V_0 : X \to \mathbb{R}$ be the continuous functional defined by

$$\begin{aligned} V_0\left(\begin{smallmatrix} u \\ v \\ \theta \end{smallmatrix}\right) &= \frac{1}{2}\Big[\|\Delta u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} + \|\theta\|^2_{L^2(\Omega)}\Big] - \int_\Omega\int_0^u f(s)dsdx \\ &\quad - \int_\Gamma\int_0^u g(s)dsdx. \end{aligned} \tag{2.49}$$

Using (2.44) and (2.45), it follows that

$$\begin{aligned} &\frac{1}{2}\|\Delta u\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{L^2(\Omega)} + \frac{1}{2}\|v\|^2_{L^2(\Omega)} + \frac{1}{2}\|\theta\|^2_{L^2(\Omega)} \\ &= V_0\left(\begin{smallmatrix} u \\ v \\ \theta \end{smallmatrix}\right) + \int_\Omega\int_0^u f(s)dsdx + \int_\Gamma\int_0^u g(s)dsd\sigma \\ &\leqslant V_0\left(\begin{smallmatrix} u \\ v \\ \theta \end{smallmatrix}\right) + \int_\Omega(\gamma_1|u|^2 + c_1)dx + \int_\Gamma(\gamma_2|\gamma(u)|^2 + c_2)d\sigma. \end{aligned}$$

Thus

$$\frac{1}{2}\|\Delta u\|^2_{L^2(\Omega)} + \Big(\frac{1}{2} - \gamma_1\Big)\|u\|^2_{L^2(\Omega)} + \frac{1}{2}\|v\|^2_{L^2(\Omega)} + \frac{1}{2}\|\theta\|^2_{L^2(\Omega)}$$
$$\leqslant V_0\left(\begin{smallmatrix} u \\ v \\ \theta \end{smallmatrix}\right) + \gamma_2 \int_\Gamma |\gamma(u)|^2 dS + c_2|\Gamma| + c_1|\Omega|,$$

and from trace theorem there exist $C > 0$ such that

$$\gamma_2 \int_\Gamma |\gamma(u)|^2 dS \leqslant \gamma_2 C \|u\|^2_{H^2(\Omega)},$$

and this implies that

$$\frac{1}{2}\|\Delta u\|^2_{L^2(\Omega)} + \Big(\frac{1}{2} - \gamma_1\Big)\|u\|^2_{L^2(\Omega)} - \gamma_2 C\|u\|^2_{H^2(\Omega)} + \frac{1}{2}\|v\|^2_{L^2(\Omega)} + \frac{1}{2}\|\theta\|^2_{L^2(\Omega)}$$
$$\leqslant V_0\left(\begin{smallmatrix} u \\ v \\ \theta \end{smallmatrix}\right) + c_2|\Gamma| + c_1|\Omega|.$$

Consequently, for $w(t) = \left(\begin{smallmatrix} u \\ v \\ \theta \end{smallmatrix}\right)(t)$ being the solution of the problem (3) we have that

$$\frac{1}{2}\|\Delta u\|^2_{L^2(\Omega)} + \Big(\frac{1}{2} - \gamma_1\Big)\|u\|^2_{L^2(\Omega)} - \gamma_2 C\|u\|^2_{H^2(\Omega)} + \frac{1}{2}\|v\|^2_{L^2(\Omega)} + \frac{1}{2}\|\theta\|^2_{L^2(\Omega)}$$
$$\leqslant V_0\left(\begin{smallmatrix} u \\ v \\ \theta \end{smallmatrix}\right) + c_2|\Gamma| + c_1|\Omega|.$$

For $0 < \gamma_1 < \frac{1}{2}$ and choosing γ_2 sufficiently small in the inequality above, we have that

$$\|w(t)\|^2_X \leqslant C_1 V(w(t)) + C_2, \tag{2.50}$$

for some $C_1,\ C_2 > 0$.

Again in the subsection 2.2.2 we obtain that a map $t \mapsto w(t, w_0)$ is differentiable.

It is clear that for $w(t) = \left(\begin{smallmatrix} u \\ v \\ \theta \end{smallmatrix}\right)(t)$ being the solution of the problem (3) we have that $[0, t_{max}(w_0)) \ni t \mapsto V_0(w(t, w_0)) \in \mathbb{R}$ is non-increasing because

$$\frac{dV_0}{dt}(w(t)) = -\|\nabla\theta(t)\|^2_{L^2(\Omega)} - \|\theta(t)\|^2_{L^2(\Omega)} \leqslant 0,$$

for $V_0(t) = V_0(w(t, w_0))$ and $t \in [0, t_{max}(w_0))$.

Using trace theorem we can prove that V is continuous and uniformly bounded in uniformly bounded subsets of X. From (2.50) we have that given $r > 0$, there is a constant $C(r) > 0$ such that

$$\sup\{\|w(t, w_0)\|_X : \|w_0\|_X \leqslant r,\ t \in [0, t_{max}(w_0))\} \leqslant C. \tag{2.51}$$

From (2.51) we have that for each $w_0 \in X$, the solution of (2.10) is defined for all $t \geqslant 0$, that is $t_{max}(w_0) = \infty$. Consequently, we can to define a nonlinear semigroup $\{S_0(t) :\ t \geqslant 0\}$ in X by

$$S_0(t)w_0 = w(t, w_0), \quad t \geqslant 0.$$

This also implies that each uniformly bounded subset of X has orbit and global orbit uniformly bounded.

Note that the nonlinear semigroup is given by the variation of constants formula

$$S_0(t)w_0 = e^{\mathbb{A}t}w_0 + \int_0^t e^{\mathbb{A}(t-s)}F_0(S_0(s)w_0)ds, \quad t \geqslant 0,$$

see Henry [20, Chapter 3] for details.

Remark 2.3.2 *Note that* (2.49) *is a Lyapunov function with the properties of Definition C.17 and thus* $\{S_0(t) : t \geqslant 0\}$ *is a gradient system.*

2.4 Existence and upper semicontinuity of global attractors

From this section onwards we be assuming all the previous hypotheses. The results obtained in the previous sections and smoothing effect of the equations assure us that the nonlinear semigroups generated by our problems (2.7) and (2.10) have global compact attractors $\mathscr{A}_\varepsilon$ for $0 \leqslant \varepsilon \leqslant \varepsilon_0$. Moreover, we get a result of boundedness uniform in ε of the attractores, the convergence of the nonlinear semigroups and upper semicontinuity of the global attractors.

2.4.1 Existence of the global attractors

In this subsection, we establish the existence and characterization of the global compact attractors for the nonlinear semigroups generated by our problems (2.7) and (2.10) using the results of Hale [19, Theorem 3.8.5, p. 51]. Moreover, we obtain uniform boundedness of the attractors.

Theorem 2.4.1 *For sufficiently small* $\varepsilon \geqslant 0$*, the parabolic problems* (2.7) *and* (2.10) *have a global compact attractor* $\mathscr{A}_\varepsilon$ *and* $\mathscr{A}_\varepsilon = W^u(\mathscr{E}_\varepsilon)$*, where*

$$W^u(\mathscr{E}_\varepsilon) = \Big\{w \in X : S_\varepsilon(-t)w \text{ is defined for } t \geqslant 0 \text{ and } \lim_{t\to+\infty} dist(S_\varepsilon(-t)w, \mathscr{E}_\varepsilon) = 0\},$$

and $\mathscr{E}_\varepsilon$ *denotes the set of equilibria of the nonlinear semigroup* $\{S_\varepsilon(t) : t \geqslant 0\}$ *generated by our problems* (2.7) *and* (2.10)*. Moreover,* $\mathscr{A}_\varepsilon$ *is connected.*

Proof. Using the functionals V_ε and V_0 defined in (2.43) and (2.49), respectively, for $\varepsilon \geqslant 0$ enough small, from the smoothing effect of the systems and the Theorem C.18

we get that the problems (2.7) and (2.10) have global attractor $\mathscr{A}_\varepsilon$ in X with the characterization $\mathscr{A}_\varepsilon = W^u(\mathscr{E}_\varepsilon)$, for $0 \leqslant \varepsilon \leqslant \varepsilon_0$. Moreover, $\mathscr{A}_\varepsilon$ is connected because X is a Hilbert space. ■

Here, we present a result on the uniform bounds of the attractors that we use to show the upper semicontinuity at $\varepsilon = 0$ of the attractors.

Theorem 2.4.2 *For sufficiently small $\varepsilon \geqslant 0$, the union of the global attractors $\bigcup_{\varepsilon\in[0,\varepsilon_0]} \mathscr{A}_\varepsilon$ is a bounded set in X.*

Proof. For sufficiently small $\varepsilon \geqslant 0$, it is important to note that for global bounded solutions of (2.7) in (2.47), we can estimate $V_\varepsilon(w^\varepsilon(t))$ by a constant independent of ε thanks to (2.46), as well as, the constant $C_2 > 0$ in (2.47) is independent of ε. Hence, this boundedness uniform in ε jointly with (2.48), (2.51), and the invariance of the attractors by the semigroups, allows to conclude that the union of the global attractors $\bigcup_{\varepsilon\in[0,\varepsilon_0]} \mathscr{A}_\varepsilon$ is a bounded set in X. ■

2.4.2 Convergence of the nonlinear semigroups

From now on we show the convergence of the nonlinear semigroups as $\varepsilon \to 0$. With this convergence result we concluded that the limit problems for the autonomous thermoelastic plate system (2) is given by (3). Initially, we estimate the linear semigroup.

We use the Remark 2.1.11 to show that the nonlinear semigroups behave continuously at $\varepsilon \to 0$.

Proposition 2.4.3 *Under the above hypothesis, let $\frac{1}{2} < s \leqslant 1$ and some fixed $\tau > 0$. Then, there exists a function $C(\varepsilon) \geqslant 0$ with $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that for $w_0 \in B$, where $B \subset X$ is a bounded set, we have*

$$\Big\| S_\varepsilon(t)w_0 - S_0(t)w_0 \Big\|_X \leqslant M(\tau, B)C(\varepsilon), \quad \forall\, t \in [0, \tau], \tag{2.52}$$

for some constant $M(\tau, B) > 0$.

Proof. Let $B \subset X$ be a bounded set, and let $w_0 \in B$. Fixed $\tau > 0$, we consider the nonlinear semigroups given by the variation of constant formula

$$S_\varepsilon(t)w_0 = e^{\mathbb{A}t}w_0 + \int_0^t e^{\mathbb{A}(t-\xi)} F_\varepsilon(S_\varepsilon(\xi)w_0)d\xi,\ \varepsilon \in [0, \varepsilon_0] \tag{2.53}$$

associated with (2.7) and (2.10).

Note that from (2.53), for $t \in (0,\tau]$ we have

$$\Big\|S_\varepsilon(t)w_0 - S_0(t)w_0\Big\|_X \leqslant \int_0^t \Big\|e^{\mathbb{A}(t-\xi)}\Big\|_{\mathscr{L}(\mathscr{H},X)}\Big\|F_\varepsilon(S_\varepsilon(\xi)w_0) - F_0(S_0(\xi)w_0)\Big\|_{\mathscr{H}} d\xi. \tag{2.54}$$

Adding and subtracting the term $F_\varepsilon(S_0(\xi)w_0)$ in the second norm on right side of (2.54), from (2.26) we can to write the inequality above of the following form

$$\begin{aligned}\Big\|S_\varepsilon(t)w_0 - S_0(t)w_0\Big\|_X &\leqslant \int_0^t \Big\|e^{\mathbb{A}(t-\xi)}\Big\|_{\mathscr{L}(\mathscr{H},X)}\Big\|F_\varepsilon(S_\varepsilon(\xi)w_0) - F_\varepsilon(S_0(\xi)w_0)\Big\|_{\mathscr{H}} d\xi \\ &+ \int_0^t \Big\|e^{\mathbb{A}(t-\xi)}\Big\|_{\mathscr{L}(\mathscr{H},X)}\Big\|F_\varepsilon(S_0(\xi)w_0) - F_0(S_0(\xi)w_0)\Big\|_{\mathscr{H}} d\xi \\ &\leqslant M_\omega \int_0^t (t-\xi)^{-1}e^{-\omega(t-\xi)}\Big\|F_\varepsilon(S_\varepsilon(\xi)w_0) - F_\varepsilon(S_0(\xi)w_0)\Big\|_{\mathscr{H}} d\xi \\ &+ M_\omega \int_0^t (t-\xi)^{-1}e^{-\omega(t-s)}\Big\|F_\varepsilon(S_0(\xi)w_0) - F_0(S_0(\xi)w_0)\Big\|_{\mathscr{H}} d\xi.\end{aligned} \tag{2.55}$$

We analyze each term on right side of (2.55) separately. From (2.48) and (2.51) we have that there exists $C = C(w_0) > 0$ independent of ε, such that

$$\|S_\varepsilon(\xi)w_0\|_X \leqslant C, \quad \forall\, \varepsilon \in [0,\varepsilon_0] \quad \text{and} \quad \forall\, \xi \in [0,\tau].$$

Now, from item (ii) of Lemma 2.2.3, F_ε is globally Lipschitz, uniformly in ε, thus there exists $L > 0$ independent of ε, such that

$$\begin{aligned}&\int_0^t (t-\xi)^{-1}e^{-\omega(t-\xi)}\Big\|F_\varepsilon(S_\varepsilon(\xi)w_0) - F_\varepsilon(S_0(\xi)w_0)\Big\|_{\mathscr{H}} d\xi \\ &\leqslant L\int_0^t (t-\xi)^{-1}e^{-\omega(t-\xi)}\Big\|S_\varepsilon(\xi)w_0 - S_0(\xi)w_0\Big\|_X d\xi.\end{aligned} \tag{2.56}$$

Since $\{S_0(s)w_0 : s \in [0,\tau]\}$ is bounded set contained in X. Thanks to item (iii) of Lemma 2.2.3, there exists a function $C(\varepsilon) \geqslant 0$ with $C(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$\begin{aligned}&\int_0^t (t-\xi)^{-1}e^{-\omega(t-\xi)}\Big\|F_\varepsilon(S_0(\xi)w_0) - F_0(S_0(\xi)w_0)\Big\|_{\mathscr{H}} d\xi \\ &\leqslant M(\tau,w_0)C(\varepsilon)\int_0^t (t-\xi)^{-1}e^{-\omega(t-\xi)} d\xi \\ &\leqslant M(\tau,w_0)C(\varepsilon)\int_0^{+\infty} z^{-1}e^{-z}dz \\ &= M(\tau,w_0)C(\varepsilon)\Gamma(0),\ \ (\Gamma(0)=1),\end{aligned} \tag{2.57}$$

where $M(\tau, w_0) > 0$ and $\Gamma(x) = \int_0^\infty z^{x-1}e^{-z}dz$ is the gamma function.

Combining (2.55) with (2.56) and (2.57), we get for all $t \in (0, \tau]$,

$$\begin{aligned}&\Big\|S_\varepsilon(t)w_0 - S_0(t)w_0\Big\|_X\\ &\leqslant C(\varepsilon)M(\tau, w_0)M + LM_\omega \int_0^t (t-\xi)^{-1}e^{-\omega(t-\xi)}\Big\|S_\varepsilon(\xi)w_0 - S_0(\xi)w_0\Big\|_X d\xi,\end{aligned}$$

where $C(\varepsilon) \geqslant 0$ with $C(\varepsilon) \to 0$ as $\varepsilon \to 0$.

From Gronwall's inequality, Henry [20, Lemma 7.1.1, p.188] it follows that

$$\Big\|S_\varepsilon(t)w_0 - S_0(t)w_0\Big\|_X \leqslant M(\tau, \omega, L, B)C(\varepsilon)e^{-\omega t},$$

and consequently we conclude that (2.52) holds. ■

Similarly, we can prove the following result.

Proposition 2.4.4 *Under the above hypothesis, let $\frac{1}{2} < s \leqslant 1$ and some fixed $\tau > 0$. Then, there exists a function $C(\varepsilon) \geqslant 0$ with $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that for $w^\varepsilon \in \mathscr{A}_\varepsilon$, $\varepsilon \in (0, \varepsilon_0]$, we have*

$$\Big\|S_\varepsilon(t)w^\varepsilon - S_0(t)w^\varepsilon\Big\|_X \leqslant M(\tau)C(\varepsilon), \quad \forall\, t \in [0, \tau], \tag{2.58}$$

for some constant $M(\tau) > 0$.

2.4.3 Upper semicontinuity of the global attractors

Finally, in this subsection we show the upper semicontinuity of global attractors at $\varepsilon = 0$, in the sense of Hausdorff semidistance in X.

Theorem 2.4.5 *The family of attractors $\mathscr{A}_\varepsilon$ is upper semicontinuous at $\varepsilon = 0$; that is,*

$$\mathrm{dist}_H(\mathscr{A}_\varepsilon, \mathscr{A}_0) \to 0, \quad \text{as} \quad \varepsilon \to 0,$$

where

$$\mathrm{dist}_H(\mathscr{A}_\varepsilon, \mathscr{A}_0) := \sup_{w^\varepsilon \in \mathscr{A}_\varepsilon} \mathrm{dist}(w^\varepsilon, \mathscr{A}_0) = \sup_{w^\varepsilon \in \mathscr{A}_\varepsilon} \inf_{w^0 \in \mathscr{A}_0} \{\|w^\varepsilon - w^0\|_X\}.$$

Proof. Thanks to Theorem 2.4.2, there exists $B_0 \subset X$ a bounded set such that $B_0 \supset \bigcup_{\varepsilon \in [0,\varepsilon_0]} \mathscr{A}_\varepsilon$ for some $\varepsilon_0 > 0$. Hence, $\mathscr{A}_0$ attracts $\bigcup_{\varepsilon \in [0,\varepsilon_0]} \mathscr{A}_\varepsilon \supset \mathscr{A}_\varepsilon$ under the nonlinear semigroup $S_0(\cdot)$. Thus, given $\delta > 0$, there exists $\tau = \tau(\delta) > 0$ such that

$$\mathrm{dist}(S_0(\tau)w^\varepsilon, \mathscr{A}_0) < \frac{\delta}{2}, \quad \forall w^\varepsilon \in \mathscr{A}_\varepsilon. \tag{2.59}$$

Since $\mathscr{A}_\varepsilon$ is invariant then given $\varphi^\varepsilon \in \mathscr{A}_\varepsilon$ there exists $\vartheta^\varepsilon \in \mathscr{A}_\varepsilon$ such that $\varphi^\varepsilon = S_\varepsilon(\tau)\vartheta^\varepsilon$. Thus,

$$\begin{aligned}\operatorname{dist}(\varphi^\varepsilon, \mathscr{A}_0) = \inf_{w^0 \in \mathscr{A}_0} \|\varphi^\varepsilon - w^0\|_X &\leqslant \inf_{w^0 \in \mathscr{A}_0} \{\|\varphi^\varepsilon - S_0(\tau)\vartheta^\varepsilon\|_X + \|S_0(\tau)\vartheta^\varepsilon - w^0\|_X\} \\ &= \|S_\varepsilon(\tau)\vartheta^\varepsilon - S_0(\tau)\vartheta^\varepsilon\|_X + \operatorname{dist}(S_0(\tau)\vartheta^\varepsilon, \mathscr{A}_0).\end{aligned}$$

From Proposition 2.4.4, for ε enough small, we get

$$\|S_\varepsilon(\tau)\vartheta^\varepsilon - S_0(\tau)\vartheta^\varepsilon\|_X \leqslant \frac{\delta}{2}. \tag{2.60}$$

Using (2.59) and (2.60), for ε enough small, we have

$$\operatorname{dist}(\varphi^\varepsilon, \mathscr{A}_0) < \delta, \quad \forall\ \varphi^\varepsilon \in \mathscr{A}_\varepsilon,$$

and thus we conclude the upper semicontinuity of the family of attractors at $\varepsilon = 0$. ■

2.5 Lower semicontinuity of global attractors

In this section we finished the analysis on the continuity of the global attractors of the nonlinear semigroups generated by the abstract problems (2.7) and (2.10), showing the lower semicontinuity of these attractors, since in the previous sections was showed the existence and upper semicontinuity. But for this end we need to show the continuity of the set of equilibria associated to abstract problems (2.7) and (2.10) and also we have to show the continuity of local unstable manifolds around these equilibria. With this and using the results of Henry [20, Chapter 6] we obtain the lower semicontinuity of these attractors.

2.5.1 Continuity of the set of equilibria

Firstly, we prove a result of uniform boundedness and convergence of the Fréchet differential of the nonlinearity F_ε, that we need for show some results that we utilize in the proof of the lower semicontinuity of the set of equilibria at $\varepsilon = 0$.

Lemma 2.5.1 *Suppose that f and g satisfy the growth estimates (2.1) and $\frac{1}{2} < s \leqslant 1$. Then*

(i) There exists $k > 0$ independent of ε such that

$$\|DF_\varepsilon(w)\|_{\mathscr{L}(X,\mathscr{H})} \leqslant k, \quad w \in X \quad \textit{and} \quad \varepsilon \in [0, \varepsilon_0].$$

(ii) *For each $w \in X$, we have*

$$\|DF_\varepsilon(w) - DF_0(w)\|_{\mathscr{L}(X,\mathscr{H})} \to 0 \quad as \quad \varepsilon \to 0,$$

and this limit is uniform for $w \in X$ such that $\|w\|_X \leqslant R$, for some $R > 0$.

(iii) *If $w^\varepsilon \to w^0$ in X as $\varepsilon \to 0$, then*

$$\|DF_\varepsilon(w^\varepsilon) - DF_0(w^0)\|_{\mathscr{L}(X,\mathscr{H})} \to 0 \quad as \quad \varepsilon \to 0.$$

(iv) *If $w^\varepsilon \to w^0$ in X as $\varepsilon \to 0$, and $\mathbf{h}^\varepsilon \to \mathbf{h}^0$ in X as $\varepsilon \to 0$, then*

$$\|DF_\varepsilon(w^\varepsilon)\mathbf{h}^\varepsilon - DF_0(w^0)\mathbf{h}^0\|_{\mathscr{H}} \to 0 \quad as \quad \varepsilon \to 0.$$

Proof. *(i)* Let $w \in X$, $\varepsilon \in [0, \varepsilon_0]$ we have

$$\|DF_\varepsilon(w)\|_{\mathscr{L}(X,\mathscr{H})} = \sup_{\substack{\mathbf{h} \in X \\ \|\mathbf{h}\|_X = 1}} \|DF_\varepsilon(w)\mathbf{h}\|_{\mathscr{H}}.$$

Note that, for each $\mathbf{h} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} \in X$,

$$\|DF_\varepsilon(w)\mathbf{h}\|_{\mathscr{H}} = \left\| Df_\Omega(u)h_1 + \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(u)h_1 \right\|_{H^{-s}(\Omega)}, \quad \varepsilon \in (0, \varepsilon_0],$$

$$\|DF_0(w)\mathbf{h}\|_{\mathscr{H}} = \|Df_\Omega(u)h_1 + Dg_\Gamma(u)h_1\|_{H^{-s}(\Omega)},$$

where the maps Df_Ω, $\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega$ and Dg_Γ are given respectively by (2.37), (2.38) and (2.39). From (2.40), (2.41) and (2.42) we conclude (i).

(ii) For each $w \in X$, notice that

$$\|DF_\varepsilon(w) - DF_0(w)\|_{\mathscr{L}(X,\mathscr{H})} = \left\| \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(u) - Dg_\Gamma(u) \right\|_{\mathscr{L}(H^2(\Omega), H^{-s}(\Omega))}.$$

As in Lemma 2.2.2 we can prove that there exists $M(\varepsilon, R)$ with $M(\varepsilon, R) \to 0$ as $\varepsilon \to 0$ such that

$$\left| \left\langle \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(u)h_1 - Dg_\Gamma(u)h_1, \varphi \right\rangle \right| = \left| \frac{1}{\varepsilon}\int_{\omega_\varepsilon} g'(u)h_1\varphi dx - \int_\Gamma \gamma(g'(u)h_1)\gamma(\varphi)dS \right|$$
$$\leqslant M(\varepsilon, R)\, \|h_1\|_{H^2(\Omega)} \|\varphi\|_{H^1(\Omega)}, \quad \forall h_1 \in H^2(\Omega) \text{ and } \forall \varphi \in H^1(\Omega).$$

Thus,

$$\left\| \frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(u) - Dg_\Gamma(u) \right\|_{\mathscr{L}(H^2(\Omega), H^{-1}(\Omega))} \to 0, \quad \text{as } \varepsilon \to 0, \tag{2.61}$$

uniformly for $u \in H^2(\Omega)$ such that $\|u\|_{H^2(\Omega)} \leqslant R$.

Now, fix $\frac{1}{2} < s_0 < 1$. Then for any s such that $-1 < -s < -s_0 < -\frac{1}{2}$, using interpolation, (2.41) and (2.42) we have

$$\begin{aligned}
&\left\|\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(u)h_1 - Dg_\Gamma(u)h_1\right\|_{H^{-s}(\Omega)} \\
&\leqslant \left\|\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(u)h_1 - Dg_\Gamma(u)h_1\right\|_{H^{-s_0}(\Omega)}^{\theta} \left\|\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(u)h_1 - Dg_\Gamma(u)h_1\right\|_{H^{-1}(\Omega)}^{1-\theta} \\
&\leqslant (k_2+k_3)^{\theta} \left\|\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(u) - Dg_\Gamma(u)\right\|_{\mathscr{L}(H^2(\Omega),H^{-1}(\Omega))}^{1-\theta} \|h_1\|_{H^2(\Omega)}, \quad \forall h_1 \in H^2(\Omega),
\end{aligned}$$

for some $0 < \theta < 1$. Thus using (2.61), we obtain

$$\left\|\frac{1}{\varepsilon}\chi_{\omega_\varepsilon} Dg_\Omega(u) - Dg_\Gamma(u)\right\|_{\mathscr{L}(H^2(\Omega),H^{-s}(\Omega))} \to 0, \quad \text{as } \varepsilon \to 0,$$

uniformly for $u \in H^2(\Omega)$ such that $\|u\|_{H^2(\Omega)} \leqslant R$.

(iii) From Lemma 2.2.5, we have that there exists $L > 0$ independent of ε such that

$$\begin{aligned}
&\|DF_\varepsilon(w^\varepsilon) - DF_0(w^0)\|_{\mathscr{L}(X,\mathscr{H})} \\
&\leqslant \|DF_\varepsilon(w^\varepsilon) - DF_\varepsilon(w^0)\|_{\mathscr{L}(X,\mathscr{H})} + \|DF_\varepsilon(w^0) - DF_0(w^0)\|_{\mathscr{L}(X,\mathscr{H})} \\
&\leqslant L\|w^\varepsilon - w^0\|_X + \|DF_\varepsilon(w^0) - DF_0(w^0)\|_{\mathscr{L}(X,\mathscr{H})} \to 0, \quad \text{as } \varepsilon \to 0,
\end{aligned}$$

where we also use the item (ii) and $w^\varepsilon \to w^0$ in X, as $\varepsilon \to 0$.

(iv) We take $w^\varepsilon \to w^0$ in X, as $\varepsilon \to 0$, and $\mathbf{h}^\varepsilon \to \mathbf{h}^0$ in X, as $\varepsilon \to 0$. Using the items *(i)* and *(iii)*, we get

$$\begin{aligned}
&\|DF_\varepsilon(w^\varepsilon)\mathbf{h}^\varepsilon - DF_0(w^0)\mathbf{h}^0\|_{\mathscr{H}} \\
&\leqslant \|DF_\varepsilon(w^\varepsilon)\mathbf{h}^\varepsilon - DF_\varepsilon(w^\varepsilon)\mathbf{h}^0\|_{\mathscr{H}} + \|DF_\varepsilon(w^\varepsilon)\mathbf{h}^0 - DF_0(w^0)\mathbf{h}^0\|_{\mathscr{H}} \\
&\leqslant \|DF_\varepsilon(w^\varepsilon)\|_{\mathscr{L}(X,\mathscr{H})}\|\mathbf{h}^\varepsilon - \mathbf{h}^0\|_X + \|DF_\varepsilon(w^\varepsilon) - DF_0(w^0)\|_{\mathscr{L}(X,\mathscr{H})}\|\mathbf{h}^0\|_X \\
&\leqslant k\|\mathbf{h}^\varepsilon - \mathbf{h}^0\|_X + \|DF_\varepsilon(w^\varepsilon) - DF_0(w^0)\|_{\mathscr{L}(X,\mathscr{H})}\|\mathbf{h}\|_X \to 0,
\end{aligned}$$

as $\varepsilon \to 0$. ■

In order to obtain the lower semicontinuity of global attractors at $\varepsilon = 0$ we need to obtain the continuity of the set of equilibria and then study the continuity of the linearization around each equilibrium. In this section we prove that the family $\{\mathscr{E}_\varepsilon : \varepsilon \in [0,\varepsilon_0]\}$ of (2) and (3) is continuous at $\varepsilon = 0$.

Definition 2.5.2 *The equilibrium solutions of* (2) *and* (3) *are those which are independent of time. In other words, the equilibrium solutions of* (2) *are those which are solutions of the elliptic problems*

$$\begin{cases} \Delta^2 u^\varepsilon + u^\varepsilon = f(u^\varepsilon) + \dfrac{1}{\varepsilon}\chi_{\omega_\varepsilon} g(u^\varepsilon) & in \quad \Omega, \\ \dfrac{\partial u^\varepsilon}{\partial \vec{n}} = \dfrac{\partial(\Delta u^\varepsilon)}{\partial \vec{n}} = 0 & on \quad \Gamma,\ 0 < \varepsilon \leqslant \varepsilon_0, \end{cases} \tag{2.62}$$

and

$$\begin{cases} \Delta\theta^\varepsilon - \theta^\varepsilon = 0 & in \quad \Omega, \\ \dfrac{\partial \theta^\varepsilon}{\partial \vec{n}} = 0 & on \quad \Gamma,\ 0 < \varepsilon \leqslant \varepsilon_0, \end{cases} \tag{2.63}$$

that is, θ^ε is identity null in Ω. The equilibrium solutions of (3) *are those which are solutions of the elliptic problems*

$$\begin{cases} \Delta^2 u + u = f(u) & in \quad \Omega, \\ \dfrac{\partial u}{\partial \vec{n}} = \dfrac{\partial(\Delta u)}{\partial \vec{n}} = -g(u) & on \quad \Gamma, \end{cases} \tag{2.64}$$

and

$$\begin{cases} \Delta\theta - \theta = 0 & in \quad \Omega, \\ \dfrac{\partial \theta}{\partial \vec{n}} = 0 & on \quad \Gamma, \end{cases} \tag{2.65}$$

that is, θ is identity null in Ω.

Remark 2.5.3 *Equivalently, for each $\varepsilon \in (0, \varepsilon_0]$ the equilibrium solutions of* (2) *are those which are solutions of the semilinear problems*

$$\mathbb{A} w^\varepsilon + F_\varepsilon(w^\varepsilon) = 0, \quad w^\varepsilon = \begin{pmatrix} u^\varepsilon \\ 0 \\ 0 \end{pmatrix}. \tag{2.66}$$

As well as, the equilibrium solutions of (3) *are those which are solutions of the semilinear problem*

$$\mathbb{A} w + F_0(w) = 0, \quad w = \begin{pmatrix} u \\ 0 \\ 0 \end{pmatrix}. \tag{2.67}$$

Thus, the set of equilibria $\mathscr{E}_\varepsilon$ of (2) and (3), or equivalently, the set of solutions of (2.66) and (2.67) with $\varepsilon \in [0, \varepsilon_0]$, is given by

$$\mathscr{E}_\varepsilon = \left\{ w_*^\varepsilon = \begin{pmatrix} u_*^\varepsilon \\ 0 \\ 0 \end{pmatrix} \in X;\ u_*^\varepsilon \text{ is solution of (2.62)} \right\}, \quad \varepsilon \in (0, \varepsilon_0],$$

and

$$\mathscr{E}_0 = \left\{ w_*^0 = \begin{pmatrix} u_*^0 \\ 0 \\ 0 \end{pmatrix} \in X;\ u_*^0 \text{ is solution of (2.64)} \right\}.$$

We see that each set $\mathscr{E}_\varepsilon$ is not empty and it is compact, but for this, we need of following result

Theorem 2.5.4 *Let X, Y, Z be normed linear spaces, and suppose $T \in \mathscr{L}(X,Y)$, $S \in \mathscr{L}(Y,Z)$. Then ST is compact, whenever S or T is compact.*

Proof. See Taylor and Lay [41, Theorem 7.2, p. 298] ■

Lemma 2.5.5 *For each $\varepsilon \in [0,\varepsilon_0]$ fixed, the set $\mathscr{E}_\varepsilon$ is not empty. Moreover, $\mathscr{E}_\varepsilon$ is compact in X.*

Proof. The bounded linear operator $(\Lambda+I)^{-1} : H^{-s}(\Omega) \to H^2(\Omega)$ is compact, because the linear operator $(\Lambda+I)^{-1} : H^{-s}(\Omega) \to H^{4-s}(\Omega)$ is bounded and we have the compact embedding $H^{4-s}(\Omega) \hookrightarrow H^2(\Omega)$ for $4-s > 2$. Moreover, we have the compact embedding $H^4(\Omega) \hookrightarrow H^2(\Omega)$ and therefore the bounded linear operator $(\Lambda+I)^{-1} : L^2(\Omega) \to H^2(\Omega)$ is compact. We also have the compact embedding $H^2(\Omega) \hookrightarrow L^2(\Omega)$ and therefore the bounded linear operator $(\Lambda^{\frac{1}{2}} + I)^{-1} : L^2(\Omega) \to L^2(\Omega)$ is compact.

Finally, the linear operator $(\Lambda + I)^{-1}(\Lambda^{\frac{1}{2}} + I) : H^2(\Omega) \to H^2(\Omega)$ is compact, because the linear operator $(\Lambda + I)^{-1}(\Lambda^{\frac{1}{2}} + I) : H^2(\Omega) \to H^4(\Omega)$ is bounded and we have the compact embedding $H^4(\Omega) \hookrightarrow H^2(\Omega)$. Therefore the linear operator $\mathbb{A}^{-1} : \mathscr{H} \to X$ is compact and consequently $\mathbb{A}^{-1}F_\varepsilon : X \to X$ is compact.

Now, we show that for each $\varepsilon \in [0,\varepsilon_0]$ fixed, the set $\mathscr{E}_\varepsilon$ is not empty, it is equivalent to show that the compact operator $\mathbb{A}^{-1}F_\varepsilon : X \to X$ has at least one fixed point. From Lemma 2.2.3, we have that there exists $k > 0$ independent of ε such that

$$\|F_\varepsilon(w)\|_{\mathscr{H}} \leqslant k, \quad \forall w \in X \quad \text{and} \quad \varepsilon \in [0,\varepsilon_0].$$

We consider the closed ball $\overline{B}_r(0)$ in X, where $r = k\|\mathbb{A}^{-1}\|_{\mathscr{L}(\mathscr{H},X)}$. For each $w \in X$, we have

$$\|\mathbb{A}^{-1}F_\varepsilon(w)\|_X \leqslant \|\mathbb{A}^{-1}\|_{\mathscr{L}(\mathscr{H},X)}\|F_\varepsilon(w)\|_{\mathscr{H}} \leqslant r. \tag{2.68}$$

Therefore, the compact operator $\mathbb{A}^{-1}F_\varepsilon : X \to X$ takes X in the ball $\overline{B}_r(0)$, in particular, $\mathbb{A}^{-1}F_\varepsilon$ takes $\overline{B}_r(0)$ into itself. From Schauder Fixed Point Theorem, we obtain that $\mathbb{A}^{-1}F_\varepsilon$ has at least one fixed point in X.

Now, for each $\varepsilon \in [0,\varepsilon_0]$ fixed, we prove that $\mathscr{E}_\varepsilon$ is compact in X. For each $\varepsilon \in [0,\varepsilon_0]$ fixed, let $\{w^\varepsilon_{*,n}\}_{n\in\mathbb{N}}$ be a sequence in $\mathscr{E}_\varepsilon$, then $w^\varepsilon_{*,n} = -\mathbb{A}^{-1}F_\varepsilon(w^\varepsilon_{*,n})$, for all $n \in \mathbb{N}$. Similarly to (2.68), we get that $\{w^\varepsilon_{*,n}\}_{n\in\mathbb{N}}$ is a bounded sequence in X. Thus, $\{-\mathbb{A}^{-1}F_\varepsilon(w^\varepsilon_{*,n})\}_{n\in\mathbb{N}}$ has a convergent subsequence, that we denote by $\{-\mathbb{A}^{-1}F_\varepsilon(w^\varepsilon_{*,n_k})\}_{k\in\mathbb{N}}$, with limit $w \in X$, that is,

$$-\mathbb{A}^{-1}F_\varepsilon(w^\varepsilon_{*,n_k}) \to w \quad \text{in} \quad X, \quad \text{as} \quad k \to \infty.$$

Hence, $w^\varepsilon_{*,n_k} \to w$ in X, as $k \to \infty$.

By continuity of operator $\mathbb{A}^{-1}F_\varepsilon : X \to X$, we get

$$-\mathbb{A}^{-1}F_\varepsilon(w^\varepsilon_{*,n_k}) \to -\mathbb{A}^{-1}F_\varepsilon(w) \quad \text{in} \quad X, \quad \text{as } k \to \infty.$$

By the uniqueness of the limit, $w = -\mathbb{A}^{-1}F_\varepsilon(w)$. Thus, $\mathbb{A}w + F_\varepsilon(w) = 0$ and $w \in \mathscr{E}_\varepsilon$. Therefore, $\mathscr{E}_\varepsilon$ is a compact set in X. ■

The upper semicontinuity of the family $\{\mathscr{E}_\varepsilon : \varepsilon \in [0, \varepsilon_0]\}$ at $\varepsilon = 0$ is a consequence of the upper semicontinuity of attractors at $\varepsilon = 0$.

Theorem 2.5.6 *The family $\{\mathscr{E}_\varepsilon : \varepsilon \in [0, \varepsilon_0]\}$ is upper semicontinuous at $\varepsilon = 0$.*

Proof. Initially, we observe that $\mathscr{E}_\varepsilon \subset \mathscr{A}_\varepsilon$ for any $\varepsilon \in [0, \varepsilon_0]$, and therefore, $\mathscr{E}_\varepsilon$ is bounded in X. We prove that for any sequence of $\varepsilon_n \to 0$ and for any $w^{\varepsilon_n}_* \in \mathscr{E}_{\varepsilon_n}$ we can extract a subsequence which converges to an element of $\mathscr{E}_0$. From the upper semicontinuity of the attractors and using that $w^\varepsilon_* \in \mathscr{E}_\varepsilon \subset \mathscr{A}_\varepsilon$, we can extract a subsequence $w^{\varepsilon_k}_* \in \mathscr{E}_{\varepsilon_k}$ with $\varepsilon_k \to 0$, as $k \to \infty$, and we obtain the existence of a $w^0 \in \mathscr{A}_0$ such that

$$\|w^{\varepsilon_k}_* - w^0\|_X \to 0, \quad \text{as } k \to \infty.$$

We need to prove that $w^0 \in \mathscr{E}_0$; that is, $S_0(t)w^0 = w^0$, for any $t \geqslant 0$.

We first observe that for any $t > 0$,

$$\|w^{\varepsilon_k}_* - S_0(t)w^0\|_X \leqslant \|w^{\varepsilon_k}_* - w^0\|_X + \|w^0 - S_0(t)w^0\|_X \to \|w^0 - S_0(t)w^0\|_X, \text{ as } k \to \infty.$$

Moreover, for a fixed $\tau > 0$ and for any $t \in (0, \tau)$, we obtain

$$\begin{aligned} \|w^{\varepsilon_k}_* - S_0(t)w^0\|_X &= \|S_{\varepsilon_k}(t)w^{\varepsilon_k}_* - S_0(t)w^0\|_X \\ &\leqslant \|S_{\varepsilon_k}(t)w^{\varepsilon_k}_* - S_0(t)w^{\varepsilon_k}_*\|_X + \|S_0(t)w^{\varepsilon_k}_* - S_0(t)w^0\|_X \to 0, \text{ as } k \to \infty, \end{aligned}$$

where we have used the continuity of semigroups given by Proposition 2.4.4. In particular, we have that for each $t \geqslant 0$, $S_0(t)w^0 = w^0$, which implies that $w^0 \in \mathscr{E}_0$. ■

The proof of lower semicontinuity requires additional assumptions. We need to assume that the equilibrium solutions of (2.67) are stable under perturbation, this stability under perturbation be given by the hyperbolicity.

Definition 2.5.7 *We say that the solution w_*^0 of* (2.67) *is hyperbolic if the spectrum $\sigma(\mathbb{A} + DF_0(w_*^0))$ of $\mathbb{A} + DF_0(w_*^0)$ is disjoint from the imaginary axis.*

Theorem 2.5.8 *If all solutions of* (2.67) *are isolated, there are only a finite number of them. Any hyperbolic solution of* (2.67) *is isolated.*

Proof. Since $\mathscr{E}_0$ is compact we only need to prove that hyperbolic solution is isolated. We note that $w_*^0 \in \mathscr{E}_0$ is a solution of (2.67) if and only if w_*^0 is a fixed point of

$$T(\xi) := -(\mathbb{A} + DF_0(w_*^0))^{-1}(F_0(\xi) - DF_0(w_*^0)\xi).$$

It is not difficult to see that there is $\delta > 0$ such that T is a contraction map from closed ball centered at w_*^0 and of radius δ in X, $\overline{B}_\delta(w_*^0)$, into itself. Thus we obtain that w_*^0 is the only element in $\mathscr{E}_0$ in the ball $\overline{B}_\delta(w_*^0)$. ■

Lemma 2.5.9 *Let $w^* \in X$. Then, for each $\varepsilon \in [0, \varepsilon_0]$ fixed, the operator $\mathbb{A}^{-1}DF_\varepsilon(w^*) : X \to X$ is compact. For any bounded family $\{w^\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ in X, the family $\{\mathbb{A}^{-1}DF_\varepsilon(w^*)w^\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ is relatively compact in X. Moreover, if $w^\varepsilon \to w^0$ in X, as $\varepsilon \to 0$, then*

$$\mathbb{A}^{-1}DF_\varepsilon(w^*)w^\varepsilon \to \mathbb{A}^{-1}DF_0(w^*)w^0 \quad in \quad X, \quad as\ \varepsilon \to 0.$$

Proof. For each $\varepsilon \in [0, \varepsilon_0]$ fixed, the compactness of linear operator $\mathbb{A}^{-1}DF_\varepsilon(w^*) : X \to X$ follows from item *(i)* of Lemma 2.5.1 and of compactness of linear operator $\mathbb{A}^{-1} : \mathscr{H} \to X$.

Let $\{w^\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ be a bounded family in X. Since

$$\|DF_\varepsilon(w^*)w^\varepsilon\|_{\mathscr{H}} \leqslant \|DF_\varepsilon(w^*)\|_{\mathscr{L}(X,\mathscr{H})} \|w^\varepsilon\|_X, \ \forall\, \varepsilon \in (0, \varepsilon_0],$$

and from item *(i)* of Lemma 2.5.1, $\{DF_\varepsilon(w^*)\}_{\varepsilon\in(0,\varepsilon_0]}$ is a bounded family in $\mathscr{L}(X, \mathscr{H})$, uniformly in ε, then $\{DF_\varepsilon(w^*)w^\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ is a bounded family in $\mathscr{H}$. By compactness of the linear operator $\mathbb{A}^{-1} : \mathscr{H} \to X$, we have that $\{\mathbb{A}^{-1}DF_\varepsilon(w^*)w^\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ has a convergent subsequence in X. Therefore, the family $\{\mathbb{A}^{-1}DF_\varepsilon(w^*)w^\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ is relatively compact.

Now, let us take $w^\varepsilon \to w^0$ in X, as $\varepsilon \to 0$. Thus, from item *(iv)* of Lemma 2.5.1,

$$DF_\varepsilon(w^*)w^\varepsilon \to DF_0(w^*)w^0 \quad \text{in} \quad \mathscr{H}, \quad \text{as } \varepsilon \to 0.$$

By continuity of the linear operator $\mathbb{A}^{-1} : \mathscr{H} \to X$, we conclude that

$$\mathbb{A}^{-1}DF_\varepsilon(w^*)w^\varepsilon \to \mathbb{A}^{-1}DF_\varepsilon(w^*)w^0 \quad \text{in} \quad X, \quad \text{as } \varepsilon \to 0.$$

■

Lemma 2.5.10 *Let $w^* \in X$ such that $0 \notin \sigma(\mathbb{A} + DF_0(w^*))$. Then, there exists ε_0 and $C > 0$ independent of ε such that $0 \notin \sigma(\mathbb{A} + DF_\varepsilon(w^*))$ and*

$$\|(\mathbb{A} + DF_\varepsilon(w^*))^{-1}\|_{\mathscr{L}(\mathscr{H},X)} \leqslant C, \quad \forall \varepsilon \in [0, \varepsilon_0]. \tag{2.69}$$

Furthermore, for each $\varepsilon \in [0, \varepsilon_0]$ fixed, the operator

$$(\mathbb{A} + DF_\varepsilon(w^*))^{-1} : \mathscr{H} \to X$$

is compact. For any bounded family $\{w^\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ in $\mathscr{H}$, we have that the family $\{(\mathbb{A} + DF_\varepsilon(w^*))^{-1}w^\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ is relatively compact in X. Moreover, if $w^\varepsilon \to w^0$ in $\mathscr{H}$, as $\varepsilon \to 0$, then

$$(\mathbb{A} + DF_\varepsilon(w^*))^{-1}w^\varepsilon \to (\mathbb{A} + DF_0(w^*))^{-1}w^0 \quad \text{in} \quad X, \quad \text{as} \quad \varepsilon \to 0.$$

Proof. First, for each $\varepsilon \in [0, \varepsilon_0]$, we note that

$$(\mathbb{A} + DF_\varepsilon(w^*))^{-1} = [\mathbb{A}(I + \mathbb{A}^{-1}DF_\varepsilon(w^*))]^{-1} = (I + \mathbb{A}^{-1}DF_\varepsilon(w^*))^{-1}\mathbb{A}^{-1}.$$

Then, prove that $0 \notin \sigma(\mathbb{A}+DF_\varepsilon(w^*))$ it is equivalent to prove that $1 \in \rho(\mathbb{A}^{-1}DF_\varepsilon(w^*))$. Moreover, to prove that there exists ε_0 and $C > 0$ independent of ε such that (2.69) holds, it is enough to prove that there exist ε_0 and $K > 0$ independent of ε such that

$$\|(I + \mathbb{A}^{-1}DF_\varepsilon(w^*))^{-1}\|_{\mathscr{L}(X)} \leqslant K, \quad \forall \varepsilon \in [0, \varepsilon_0]. \tag{2.70}$$

Indeed, we note that

$$\begin{aligned}\|(\mathbb{A} + DF_\varepsilon(w^*))^{-1}\|_{\mathscr{L}(X,\mathscr{H})} &\leqslant \|(I + \mathbb{A}^{-1}DF_\varepsilon(w^*))^{-1}\|_{\mathscr{L}(X)}\|\mathbb{A}^{-1}\|_{\mathscr{L}(\mathscr{H},X)} \\ &= K\|\mathbb{A}^{-1}\|_{\mathscr{L}(\mathscr{H},X)} = C, \quad \varepsilon \in [0, \varepsilon_0].\end{aligned}$$

Then we show (2.70). From hypothesis $0 \notin \sigma(\mathbb{A} + DF_0(w^*))$ then $1 \in \rho(\mathbb{A}^{-1}DF_0(w^*))$. Thus, there exists the inverse

$$(I + \mathbb{A}^{-1}DF_0(w^*))^{-1} : X \to X$$

and, particular we have $N(I + \mathbb{A}^{-1}DF_0(w^*)) = \{0\}$.

For simplicity of notation, let $J_\varepsilon = \mathbb{A}^{-1}DF_\varepsilon(w^*)$, for all $\varepsilon \in [0, \varepsilon_0]$. From Lemma 2.5.9 we have that, for each $\varepsilon \in [0, \varepsilon_0]$ fixed, the operator $J_\varepsilon : X \to X$ is compact. Using the compactness of J_ε we show that (2.70) hold, if and only if,

$$\|(I + J_\varepsilon)w^\varepsilon\|_X \geqslant \frac{1}{K}, \quad \forall \varepsilon \in [0, \varepsilon_0] \quad \text{and} \quad \|w^\varepsilon\|_X = 1. \tag{2.71}$$

Indeed, suppose that (2.70) holds, then there exists the inverse $(I+J_\varepsilon)^{-1} : X \to X$ and it is continuous. Moreover,

$$\|(I + J_\varepsilon)^{-1}y^\varepsilon\|_X \leqslant K\|y^\varepsilon\|_X, \ \forall \varepsilon \in [0, \varepsilon_0] \quad \text{and} \quad \forall y^\varepsilon \in X.$$

Now if $w^\varepsilon \in X$ is such that $\|w^\varepsilon\|_X = 1$ and taking $y^\varepsilon = (I + J_\varepsilon)w^\varepsilon$, we have

$$\|(I + J_\varepsilon)^{-1}(I + J_\varepsilon)w^\varepsilon\|_X \leqslant K\|(I + J_\varepsilon)w^\varepsilon\|_X$$

and

$$1 = \|w^\varepsilon\|_X \leqslant K\|(I + J_\varepsilon)w^\varepsilon\|_X,$$

in other words,

$$\|(I + J_\varepsilon)w^\varepsilon\|_X \geqslant \frac{1}{K}.$$

On the other hand, suppose that (2.71) holds. We show that there exists the inverse $(I + J_\varepsilon)^{-1} : X \to X$, it is continuous and satisfies (2.70). From (2.71), we obtain the following estimative

$$\|(I + J_\varepsilon)w^\varepsilon\|_X \geqslant \frac{1}{K}\|w^\varepsilon\|_X, \quad \forall \varepsilon \in [0, \varepsilon_0] \quad \text{and} \quad \forall w^\varepsilon \in X. \tag{2.72}$$

Now, let $w^\varepsilon \in X$ such that $(I + J_\varepsilon)w^\varepsilon = 0$. From (2.72) follows $w^\varepsilon = 0$. Thus, for each $\varepsilon \in [0, \varepsilon_0]$, $N(I + J_\varepsilon) = \{0\}$ and the operator $I + J_\varepsilon$ is injective. Since there exists the inverse $(I + J_\varepsilon)^{-1} : R(I + J_\varepsilon) \to X$ and J_ε is compact, then by Fredhlom Alternative Theorem, we have

$$N(I + J_\varepsilon) = \{0\} \Leftrightarrow R(I + J_\varepsilon) = X.$$

Then $I + J_\varepsilon$ is bijective, thus there exists the inverse $(I + J_\varepsilon)^{-1}; X \to X$.

Now, taking $y^\varepsilon \in X$ there exists $w^\varepsilon \in X$ such that $y^\varepsilon = (I + J_\varepsilon)w^\varepsilon$ and $w^\varepsilon = (I + J_\varepsilon)^{-1}y^\varepsilon$. From (2.72) we have

$$\|(I + J_\varepsilon)^{-1}y^\varepsilon\|_X = \|w^\varepsilon\|_X \leqslant K\|(I + J_\varepsilon)w^\varepsilon\|_X = K\|y^\varepsilon\|_X$$

and

$$\|(I+J_\varepsilon)^{-1}\|_{\mathscr{L}(X)} \leqslant K, \quad \forall \varepsilon \in [0,\varepsilon_0],$$

and thus (2.70) holds.

Therefore (2.70) and (2.71) are equivalents, then we show (2.71). Suppose that (2.71) is not true, that is, there exists a sequence $\{w_n\}_{n\in\mathbb{N}}$ in X, with $\|w_n\|_X = 1$ and $\varepsilon_n \to 0$, as $n \to \infty$, such that

$$\|(I+J_{\varepsilon_n})w_n\|_X \to 0, \quad \text{as} \quad n\to\infty.$$

From Lemma 2.5.9 we get that $\{J_{\varepsilon_n}w_n\}_{n\in\mathbb{N}}$ is relatively compact. Thus, $\{J_{\varepsilon_n}w_n\}_{n\in\mathbb{N}}$ has a convergent subsequence, which still we denote by $\{J_{\varepsilon_n}w_n\}_{n\in\mathbb{N}}$, with limit $w \in X$, that is,

$$J_{\varepsilon_n}w_n \to w \quad \text{in} \quad X, \quad \text{in} \quad n\to\infty$$

Since $w_n + J_{\varepsilon_n}w_n \to 0$ in X, as $n\to\infty$, then $w_n \to -w$ in X, as $n\to\infty$ and thus $\|w\|_X = 1$. Moreover, using the Lemma 2.5.9 we get $J_{\varepsilon_n}w_n \to -J_0 w$ as $n\to\infty$. Then,

$$w_n + J_{\varepsilon_n}w_n \to -(w+J_0 w) \quad \text{in} \quad X, \quad \text{as} \quad n\to\infty.$$

By uniqueness of the limit, $(I+J_0)w = 0$, with $w \neq 0$, contradicting the fact of the operator $I+J_0$ be injective, because $0 \notin \sigma(\mathbb{A}+DF_0(w^*))$. Showing that (2.71) holds.

With this we conclude that there exists $\varepsilon_0 > 0$ and $C > 0$ independent of ε such that (2.69) holds.

Now, for each $\varepsilon \in [0,\varepsilon_0]$, the operator $(\mathbb{A}+DF_\varepsilon(w^*))^{-1}$ is compact and the prove of this compactness follows similarly to account below. Let $\{w^\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ be a bounded family in $\mathscr{H}$. For each $\varepsilon \in (0,\varepsilon_0]$, let $\vartheta^\varepsilon = (\mathbb{A}+DF_\varepsilon(w^*))^{-1}w^\varepsilon$. From (2.69) we have

$$\begin{aligned}\|\vartheta^\varepsilon\|_X &\leqslant \|(\mathbb{A}+DF_\varepsilon(w^*))^{-1}w^\varepsilon\|_X \leqslant \|(\mathbb{A}+DF_\varepsilon(w^*))^{-1}\|_{\mathscr{L}(\mathscr{H},X)}\,\|w^\varepsilon\|_{\mathscr{H}}\\ &\leqslant C\,\|w^\varepsilon\|_{\mathscr{H}}.\end{aligned}$$

Hence, $\{\vartheta^\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ is a bounded family in X. Moreover,

$$\vartheta^\varepsilon = (\mathbb{A}+DF_\varepsilon(w^*))^{-1}w^\varepsilon = (I+\mathbb{A}^{-1}DF_\varepsilon(w^*))^{-1}\mathbb{A}^{-1}w^\varepsilon$$

in other words,

$$(I+\mathbb{A}^{-1}DF_\varepsilon(w^*))\vartheta^\varepsilon = \mathbb{A}^{-1}w^\varepsilon,$$

and equivalently,

$$\vartheta^\varepsilon = \mathbb{A}^{-1} DF_\varepsilon(w^*)\vartheta^\varepsilon + \mathbb{A}^{-1} w^\varepsilon.$$

By compactness of $\mathbb{A}^{-1} : \mathscr{H} \to X$, we get that $\{\mathbb{A}^{-1} w^\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ has a convergent subsequence in X. Moreover, using the Lemma 2.5.9, we have that $\{\mathbb{A}^{-1} DF_\varepsilon(w^*)\vartheta^\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ is relatively compact in X, then $\{\mathbb{A}^{-1} DF_\varepsilon(w^*)\vartheta^\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ has a convergent subsequence in X. Therefore, $\{\vartheta^\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ has a convergent subsequence in X, that is, the family $\{(\mathbb{A} + DF_\varepsilon(w^*))^{-1} w^\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ has a convergent subsequence in X, thus it is relatively compact in X.

Now, we take $w^\varepsilon \to w^0$ in $\mathscr{H}$, as $\varepsilon \to 0$. By continuity of operator $\mathbb{A}^{-1} : \mathscr{H} \to X$, we have

$$\mathbb{A}^{-1} w^\varepsilon \to \mathbb{A}^{-1} w^0 \quad \text{in} \quad X, \quad \text{as } \varepsilon \to 0.$$

Moreover, $\{w^\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ is bounded in $\mathscr{H}$, for some $\varepsilon_0 > 0$ enough small, and we have that from the above that $\{\vartheta^\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$, with $\varepsilon_0 > 0$ enough small, has a convergent subsequence, which we again denote by $\{\vartheta^\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$, with limit $\vartheta^0 \in X$, that is,

$$\vartheta^\varepsilon \to \vartheta^0 \quad \text{in} \quad X, \quad \text{as } \varepsilon \to 0.$$

From Lemma 2.5.9 we get

$$\mathbb{A}^{-1} DF_\varepsilon(w^*)\vartheta^\varepsilon \to \mathbb{A}^{-1} DF_0(w^*)\vartheta^0 \quad \text{in} \quad X, \quad \text{as } \varepsilon \to 0.$$

Thus, ϑ^0 satisfies $\vartheta^0 = \mathbb{A}^{-1} DF_0(w^*)\vartheta^0 + \mathbb{A}^{-1} w^0$, and so $\vartheta^0 = (\mathbb{A} + DF_0(w^*))^{-1} w^0$. Therefore,

$$(\mathbb{A} + DF_\varepsilon(w^*))^{-1} w^\varepsilon \to (\mathbb{A} + DF_0(w^*))^{-1} w^0 \quad \text{in} \quad X, \quad \text{as } \varepsilon \to 0.$$

The limit above is independent of the subsequence, thus whole family $\{(\mathbb{A} + DF_\varepsilon(w^*))^{-1} w^\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ converges to $(\mathbb{A} + DF_0(w^*))^{-1} w^0$ in X, as $\varepsilon \to 0$. ■

Theorem 2.5.11 *Suppose that w_*^0 is a solution for* (2.67) *and that* $0 \notin \sigma(\mathbb{A} + DF_0(w_*^0))$. *Then there are $\varepsilon_0 > 0$ and $\delta > 0$ such that the problem* (2.66) *has exactly one solution, w_*^ε, in the closed ball centered at w_*^0 and radius δ, $\{\xi \in X : \|\xi - w_*^0\|_X \leqslant \delta\}$, for any $\varepsilon \in (0, \varepsilon_0]$. Futhermore,*

$$\|w_*^\varepsilon - w_*^0\|_X \to 0, \quad \textit{as } \varepsilon \to 0.$$

Proof. Initially, note that from Lemma 2.5.10 there exists $\varepsilon_0 > 0$ and $C > 0$, independent of ε_0, such that

$$\|(\mathbb{A} + DF_\varepsilon(w_*^0))^{-1}\|_{\mathscr{L}(\mathscr{H},X)} \leqslant C, \quad \forall \varepsilon \in (0, \varepsilon_0]. \tag{2.73}$$

We note that if w_*^ε, $\varepsilon \in (0, \varepsilon_0]$, is a solution of (2.66), then

$$0 = (\mathbb{A} + DF_\varepsilon(w_*^0))[w_*^\varepsilon + (\mathbb{A} + DF_\varepsilon(w_*^0))^{-1}(F_\varepsilon(w_*^\varepsilon) - DF_\varepsilon(w_*^0)w_*^\varepsilon)].$$

Since $(\mathbb{A} + DF_\varepsilon(w_*^0))$ is invertible, then w_*^ε is a solution of (2.66) if, and only if, w_*^ε is a fixed point of the map $T_\varepsilon : X \to X$ defined by

$$T_\varepsilon(w_*^\varepsilon) = -(\mathbb{A} + DF_\varepsilon(w_*^0))^{-1}(F_\varepsilon(w_*^\varepsilon) - DF_\varepsilon(w_*^0)w_*^\varepsilon).$$

Note that

$$T_\varepsilon(w_*^0) \to w_*^0 \quad \text{in} \quad X \quad \text{as} \quad \varepsilon \to 0. \tag{2.74}$$

In fact, using (2.73), item (iii) of Lemma 2.2.3, item (iv) of Lemma 2.5.1 and Lemma 2.5.10, for $\varepsilon \in (0, \varepsilon_0]$, we have

$$\begin{aligned}
&\|T_\varepsilon(w_*^0) - w_*^0\|_X = \|T_\varepsilon(w_*^0) - T(w_*^0)\|_X \\
&\leqslant \| - (\mathbb{A} + DF_\varepsilon(w_*^0))^{-1}[F_\varepsilon(w_*^0) - DF_\varepsilon(w_*^0)w_*^0) - (F_0(w_*^0) - DF_0(w_*^0)w_*^0)\|_X \\
&+ \|[(\mathbb{A} + DF_\varepsilon(w_*^0))^{-1} - (\mathbb{A} + DF_0(w_*^0))^{-1}](DF_0(w_*^0)w_*^0 - F_0(w_*^0))\|_X \\
&\leqslant C(\|F_\varepsilon(w_*^0) - F_0(w_*^0)\|_{\mathscr{H}} + \|DF_\varepsilon(w_*^0)w_*^0 - DF_0(w_*^0)w_*^0\|_{\mathscr{H}}) \\
&+ \|[(\mathbb{A} + DF_\varepsilon(w_*^0))^{-1} - (\mathbb{A} + DF_0(w_*^0))^{-1}](DF_0(w_*^0)w_*^0 - F_0(w_*^0))\|_X \to 0,
\end{aligned}$$

as $\varepsilon \to 0$.

Next, we prove that there exists $\delta > 0$ and that for $\varepsilon \in (0, \varepsilon_0]$, the map T_ε is contraction from

$$\overline{B}_\delta(w_*^0) = \{\xi \in X : \|\xi - w_*^0\|_X \leqslant \delta\}$$

into itself, uniformly in ε. First note that from Lemma 2.2.4 there exist $\tilde{\delta} = \tilde{\delta}(C) > 0$ independent of ε such that

$$C\|F_\varepsilon(w_*^\varepsilon) - F_\varepsilon(z_*^\varepsilon) - DF_\varepsilon(w_*^0)(w_*^\varepsilon - z_*^\varepsilon)\|_{\mathscr{H}} \leqslant \frac{1}{2}\|w_*^\varepsilon - z_*^\varepsilon\|_X, \quad \forall \varepsilon \in (0, \varepsilon_0], \tag{2.75}$$

for $\|w_*^\varepsilon - z_*^\varepsilon\|_X \leqslant \tilde{\delta}$.

We take $\delta = \dfrac{\tilde{\delta}}{2}$ and let $w_*^\varepsilon,\ z_*^\varepsilon \in \overline{B}_\delta(w_*^0)$ and using (2.73) and (2.75), for $\varepsilon \in (0, \varepsilon_0]$ we have

$$\begin{aligned}
\|T_\varepsilon(w_*^\varepsilon) - T_\varepsilon(z_*^\varepsilon)\|_X &= \| - (\mathbb{A} + DF_\varepsilon(w_*^0))^{-1}(F_\varepsilon(w_*^\varepsilon) - F_\varepsilon(z_*^\varepsilon) - DF_\varepsilon(w_*^0)(w_*^\varepsilon - z_*^\varepsilon)\|_X \\
&\leqslant C\|F_\varepsilon(w_*^\varepsilon) - F_\varepsilon(z_*^\varepsilon) - DF_\varepsilon(w_*^0)(w_*^\varepsilon - z_*^\varepsilon)\|_{\mathscr{H}} \\
&\leqslant \frac{1}{2}\|w_*^\varepsilon - z_*^\varepsilon\|_X.
\end{aligned}$$

To show that $T_\varepsilon(\overline{B}_\delta(w_*^0)) \subset \overline{B}_\delta(w_*^0)$, we observe that if $w_*^\varepsilon \in \overline{B}_\delta(w_*^0)$ and from (2.74) there is $\bar{\varepsilon}$ such $\|T_\varepsilon(w_*^0) - w_*^0\|_X \leqslant \frac{\delta}{2}$, then

$$\begin{aligned}
\|T_\varepsilon(w_*^\varepsilon) - w_*^0\|_X &\leqslant \|T_\varepsilon(w_*^\varepsilon) - T_\varepsilon(w_*^0)|_X + \|T_\varepsilon(w_*^0) - w_*^0\|_X \\
&\leqslant \frac{1}{2}\|w_*^\varepsilon - w_*^0\|_X + \|T_\varepsilon(w_*^0) - w_*^0\|_X \\
&\leqslant \frac{\delta}{2} + \frac{\delta}{2} = \delta.
\end{aligned}$$

Therefore, $T_\varepsilon : \overline{B}_\delta(w_*^0) \to \overline{B}_\delta(w_*^0)$ is a contraction, for all $\varepsilon \in (0, \bar{\varepsilon}]$, and then by Contraction Theorem there is only one point fixed of T_ε in $\overline{B}_\delta(w_*^0)$.

Now we show that $w_*^\varepsilon \to w_*^0$ in X as $\varepsilon \to 0$. In fact,

$$\begin{aligned}
\|w_*^\varepsilon - w_*^0\|_X = \|T_\varepsilon(w_*^\varepsilon) - w_*^0\|_X &\leqslant \|T_\varepsilon(w_*^\varepsilon) - T_\varepsilon(w_*^0)\|_X + \|T_\varepsilon(w_*^0) - w_*^0\|_X \\
&\leqslant \frac{1}{2}\|w_*^\varepsilon - w_*^0\|_X + \|T_\varepsilon(w_*^0) - w_*^0\|_X.
\end{aligned}$$

Thus, using again (2.74) we have

$$\|w_*^\varepsilon - w_*^0\|_X \leqslant 2\|T_\varepsilon(w_*^0) - w_*^0\|_X \to 0, \quad \text{as} \quad \varepsilon \to 0.$$

■

Remark 2.5.12 *The Theorem 2.5.6 and the Theorem 2.5.11 show the continuity of the set of equilibria $\mathscr{E}_\varepsilon$, $\varepsilon \in [0, \varepsilon_0]$ at $\varepsilon = 0$; namely, the Theorem 2.5.11 shows the lower semicontinuity of the set of equilibria. Moreover, the Theorem 2.5.11 shows that if w_*^0 is a solution of the problem* (2.67)*, which satisfies $0 \notin \sigma(\mathbb{A} + DF_0(w_*^0))$, then, for each $0 < \varepsilon \leqslant \varepsilon_0$, with ε_0 suficiently small, there exists a unique solution w_*^ε of problem* (2.66) *in a neighborhood of w_*^0.*

Therefore we conclude the continuity of the set of equilibria $\{\mathscr{E}_\varepsilon : \varepsilon \in [0, \varepsilon_0)\}$ at $\varepsilon = 0$.

Remark 2.5.13 *Now that we have obtained a unique solution w_*^ε for* (2.62)-(2.63) *in a small neighborhood of the hyperbolic solution w_*^0 for* (2.64)-(2.65)*, we can consider the* *linearization $\mathbb{A}+DF_\varepsilon(w_*^\varepsilon)$ and from the convergence of w_*^ε to w_*^0 in X it is easy to obtain that $(\mathbb{A}+DF_\varepsilon(w_*^\varepsilon))^{-1}w^\varepsilon$ converges to $(\mathbb{A}+DF_0(w_*^0))^{-1}w^0$ in X, whenever $w^\varepsilon \to w^0$ in $\mathscr{H}$, as $\varepsilon \to 0$. Consequently, the hyperbolicity of w_*^0 implies the hyperbolicity of w_*^ε, for suitably small ε.*

Theorem 2.5.14 *If all solutions w_*^0 of (2.67) satisfy $0 \notin \sigma(\mathbb{A}+DF_0(w_*^0))$, then (2.67) has a finite number k of solutions, $w_*^{0,1}, ..., w_*^{0,k}$, and there exists $\varepsilon_0 > 0$ such that, for each $\varepsilon \in (0, \varepsilon_0]$, the equation (2.66) has exactly k solutions, $w_*^{\varepsilon,1}, ..., w_*^{\varepsilon,k}$. Moreover, for all $i = 1, ..., k$,*

$$w_*^{\varepsilon,i} \to w_*^{0,i} \quad \text{in} \quad X, \quad \text{as } \varepsilon \to 0.$$

Proof. The proof follows of Theorem 2.5.11. ■

2.5.2 Continuity of local unstable manifolds

Next, we show that the local unstable manifolds of $w_*^{\varepsilon,i}$ fixed, are continuous in X as $\varepsilon \to 0$. This fact and the continuity of the set of equilibria enable us to prove the lower semicontinuity of the attractors at $\varepsilon = 0$. For this we use the convergence results of the previous sections and the convergence of the linearized semigroups proved next.

The main aim of this section is the proof of existence unstable local manifolds as a graph of a Lipschitz function, its convergence and exponential attraction. Let us consider $w_*^{\varepsilon,i}$ be an equilibrium solution for (2.7), thus $\mathbb{A}w_*^{\varepsilon,i} + F_\varepsilon(w_*^{\varepsilon,i}) = 0$. To deal with a neighborhood of the equilibrium solution $w_*^{\varepsilon,i}$, we rewrite the problem (2.7) as

$$\begin{cases} \dfrac{d\mathbf{w}^\varepsilon}{dt} = \mathbf{A}_\varepsilon \mathbf{w}^\varepsilon + F_\varepsilon(\mathbf{w}^\varepsilon + w_*^{\varepsilon,i}) - F_\varepsilon(w_*^{\varepsilon,i}) - DF_\varepsilon(w_*^{\varepsilon,i})\mathbf{w}^\varepsilon, \quad t > 0, \\ \mathbf{w}^\varepsilon(0) = w_0 - w_*^{\varepsilon,i} \end{cases} \tag{2.76}$$

where $\mathbf{w}^\varepsilon = w^\varepsilon - w_*^{\varepsilon,i}$ and $\mathbf{A}_\varepsilon = \mathbb{A} + DF_\varepsilon(w_*^{\varepsilon,i})$. With this, one can look for the previous sections with the unbounded linear operator $\mathbf{A}_\varepsilon$ instead of the unbounded linear operator $\mathbb{A}$.

Let γ be a smooth, closed, simple, rectifiable curve in $\{z \in \mathbb{C} : \mathrm{Re} z > 0\}$, oriented counterclockwise and such that the bounded connected component of $\mathbb{C}\backslash\{\gamma\}$; here, $\{\gamma\}$ denotes the trace of γ, contains $\{z \in \sigma(\mathbf{A}_0) : \mathrm{Re} z > 0\}$. Let $\{\gamma\} \subset \rho(\mathbf{A}_\varepsilon)$, for all $\varepsilon \in [0, \varepsilon_1]$, for some $\varepsilon_1 > 0$. We define $\mathbf{Q}_\varepsilon$ by

$$\mathbf{Q}_\varepsilon = \frac{1}{2\pi i}\int_\gamma (\lambda - \mathbf{A}_\varepsilon)^{-1} d\lambda,$$

for any $\varepsilon \in [0, \varepsilon_1]$.

There exist $\beta > 0$ and $C \geq 1$ such that

$$\|e^{\mathbf{A}_\varepsilon t}\mathbf{Q}_\varepsilon\|_{\mathscr{L}(\mathscr{H},X)} \leq Ce^{\beta t}, \tag{2.77}$$

for any $t \leqslant 0$ and

$$\left\|e^{\mathbf{A}_\varepsilon t}(I - \mathbf{Q}_\varepsilon)\right\|_{\mathscr{L}(\mathscr{H},X)} \leqslant Ce^{-\beta t}t^{-1} \tag{2.78}$$

$$\|e^{\mathbf{A}_\varepsilon t}(I - \mathbf{Q}_\varepsilon)\|_{\mathscr{L}(\mathscr{H},X)} \leqslant Ce^{-\beta t} \tag{2.79}$$

for any $t > 0$ and $\varepsilon \in [0, \varepsilon_1]$.

Using the decomposition $X = \mathbf{Q}_\varepsilon X \oplus (I - \mathbf{Q}_\varepsilon)X$ the solution $\mathbf{w}^\varepsilon$ of (2.76) can be decomposed as $\mathbf{w}^\varepsilon = \omega^\varepsilon + \vartheta^\varepsilon$, where $\omega^\varepsilon = \mathbf{Q}_\varepsilon \mathbf{w}^\varepsilon$ and $\vartheta^\varepsilon = (I - \mathbf{Q}_\varepsilon)\mathbf{w}^\varepsilon$. As $\mathbf{Q}_\varepsilon$ and $I - \mathbf{Q}_\varepsilon$ commutate with $\mathbf{A}_\varepsilon$, we rewrite (2.76) as following

$$\begin{cases} \partial_t \omega^\varepsilon = \mathbf{A}_\varepsilon \omega^\varepsilon + H_\varepsilon(\omega^\varepsilon, \vartheta^\varepsilon), \\ \partial_t \vartheta^\varepsilon = \mathbf{A}_\varepsilon \vartheta^\varepsilon + G_\varepsilon(\omega^\varepsilon, \vartheta^\varepsilon), \end{cases} \tag{2.80}$$

where $H_\varepsilon, G_\varepsilon : X \to \mathscr{H}$ are given by

$$H_\varepsilon(\omega^\varepsilon, \vartheta^\varepsilon) := \mathbf{Q}_\varepsilon[F_\varepsilon(\omega^\varepsilon + \vartheta^\varepsilon + w_*^{\varepsilon,i}) - F_\varepsilon(w_*^{\varepsilon,i}) - F_\varepsilon'(w_*^{\varepsilon,i})(\omega^\varepsilon + \vartheta^\varepsilon)]$$

and

$$G_\varepsilon(\omega^\varepsilon, \vartheta^\varepsilon) := (I - \mathbf{Q}_\varepsilon)[F_\varepsilon(\omega^\varepsilon + \vartheta^\varepsilon + w_*^{\varepsilon,i}) - F_\varepsilon(w_*^{\varepsilon,i}) - F_\varepsilon'(w_*^{\varepsilon,i})(\omega^\varepsilon + \vartheta^\varepsilon)],$$

respectively. Thus implies that $H_\varepsilon(0,0) = G_\varepsilon(0,0) = 0$. Moreover the maps H_ε and G_ε are continuously differentiable with $H_\varepsilon'(0,0) = G_\varepsilon'(0,0) = 0$. Hence, given $\rho > 0$, there exists $\varepsilon_1 > 0$ and $r > 0$ such that if $\|\omega^\varepsilon\|_{\mathbf{Q}_\varepsilon X} + \|\vartheta^\varepsilon\|_{(I-\mathbf{Q}_\varepsilon)X} < r$ and $\varepsilon \in [0, \varepsilon_1]$, then

$$\|H_\varepsilon(\omega^\varepsilon, \vartheta^\varepsilon)\|_{\mathscr{H}} \leq \rho \;\text{ and }\; \|G_\varepsilon(\omega^\varepsilon, \vartheta^\varepsilon)\|_{\mathscr{H}} \leq \rho, \tag{2.81}$$

$$\|H_\varepsilon(\omega^\varepsilon, \vartheta^\varepsilon) - H_\varepsilon(\bar{\omega}^\varepsilon, \bar{\vartheta}^\varepsilon)\|_{\mathscr{H}} \leq \rho(\|\omega^\varepsilon - \bar{\omega}^\varepsilon\|_{\mathbf{Q}_\varepsilon X} + \|\vartheta^\varepsilon - \bar{\vartheta}^\varepsilon\|_{(I-\mathbf{Q}_\varepsilon)X}) \tag{2.82}$$

and

$$\|G_\varepsilon(\omega^\varepsilon, \vartheta^\varepsilon) - G_\varepsilon(\bar{\omega}^\varepsilon, \bar{\vartheta}^\varepsilon)\|_{\mathscr{H}} \leq \rho(\|\omega^\varepsilon - \bar{\omega}^\varepsilon\|_{\mathbf{Q}_\varepsilon X} + \|\vartheta^\varepsilon - \bar{\vartheta}^\varepsilon\|_{(I-\mathbf{Q}_\varepsilon)X}). \tag{2.83}$$

Remark 2.5.15 *It is possible to extend H_ε and G_ε outside a ball $B_X(w_*^\varepsilon, \delta)$ in such a way that the conditions* (2.81), (2.82) *and* (2.83) *holds for all $\omega^\varepsilon \in \mathbf{Q}_\varepsilon X$ and $\vartheta^\varepsilon \in (I - \mathbf{Q})_\varepsilon X$. In fact, define $\tilde{H}_\varepsilon : X \to \mathscr{H}$ by*

$$\tilde{H}_\varepsilon(\omega^\varepsilon,\vartheta^\varepsilon)=\begin{cases} H_\varepsilon(\omega^\varepsilon,\vartheta^\varepsilon), & \|\mathbf{w}^\varepsilon\|_X\leqslant\delta,\\ H_\varepsilon\left(\delta\frac{\omega^\varepsilon}{\|\mathbf{w}^\varepsilon\|_X},\delta\frac{\vartheta^\varepsilon}{\|\mathbf{w}^\varepsilon\|_X}\right), & \|\mathbf{w}^\varepsilon\|_X\geqslant\delta,\end{cases}$$

The extension $\tilde{H}_\varepsilon$ becomes globally Lipschitz and its Lipschitz constant is that of H_ε restricted to the ball $B_X(w_*^\varepsilon,\delta)$. In similar way, we have G_ε.

Given $\varepsilon>0$, we denote by $\Sigma_{D,L}$ the metric space of map $S:\mathbf{Q}_\varepsilon X\to(I-\mathbf{Q}_\varepsilon)X$, bounded and globally Lipschitz continuous, that is,

$$\Sigma_{D,L}=\{S:\mathbf{Q}_\varepsilon X\to(I-\mathbf{Q}_\varepsilon)X;\ \sup_{x\in\mathbf{Q}_\varepsilon X}\|S(x)\|_X\leqslant D \text{ e } \|S(x)-S(\widetilde{x})\|_X\leq L\|x-\widetilde{x}\|_{\mathbf{Q}_\varepsilon X}\}.$$

In $\Sigma_{D,L}$ we define the following metric

$$|\!|\!|S-\tilde{S}|\!|\!|:=\sup_{x\in\mathbf{Q}_\varepsilon X}\|S(x)-\tilde{S}(x)\|_X$$

We have that $(\Sigma_{D,L},|\!|\!|\cdot|\!|\!|)$ is a complete metric space.

Considering the coupled system (2.80), we can show an unstable manifold theorem using similar arguments used in the results in Henry [20, Chapter 6]. For this, we consider the following theorem.

Theorem 2.5.16 *Let w_*^0 be an equilibrium hyperbolic of problem* (3)*. Then from Theorem 2.5.11, the problem* (2) *has a unique equilibrium solution, w_*^ε next of w_*^0. Given $D>0$, $L>0$ and $0<\kappa<1$, let $\rho_0>0$ such that, for all $0<\rho\leqslant\rho_0$ and the following estimates are holds*

$$\begin{aligned}&\rho C\leqslant D;\quad \rho C^2(1+L)\leqslant L;\quad \rho C+\rho^2C^2(1+L)\beta^{-1}\leqslant\kappa<1\\ &\rho C+\rho^2C^2\beta^{-1}(1+L)\leqslant\frac{1}{2};\quad \beta-\left[\rho C+\frac{\rho^2C^2(1+L)}{2\beta-\rho C(1+L)}(1+C)\right]>0\\ &2\beta-\rho(1+L)<0.\end{aligned}\tag{2.84}$$

For the choice of ρ above, suppose H_ε and G_ε satisfying the above conditions for all $(\omega,\vartheta)\in\mathbf{Q}_\varepsilon X\times(I-\mathbf{Q}_\varepsilon)X$. Then there exists a map $S_*^\varepsilon:\mathbf{Q}_\varepsilon X\to(I-\mathbf{Q}_\varepsilon)X$ such that the unstable manifold of w_*^ε is given by

$$W^u(w_*^\varepsilon)=\{(\omega,\vartheta)\in X;\ \vartheta=S_*^\varepsilon(\omega),\ \omega\in\mathbf{Q}_\varepsilon X\}.$$

The map S_*^ε satisfies

$$|\!|\!|S_*^\varepsilon|\!|\!|:=\sup_{\omega\in\mathbf{Q}_\varepsilon X}\|S_*^\varepsilon(\omega)\|_X\leqslant D,\quad \|S_*^\varepsilon(\omega)-S_*^\varepsilon(\widetilde{\omega})\|_X\leqslant L\|\omega-\widetilde{\omega}\|_{\mathbf{Q}_\varepsilon X},$$

where $D>0$ is constant independent of ε, and

$$|\!|\!|S_*^\varepsilon-S_*^0|\!|\!|\to 0,$$

as $\varepsilon \to 0$.

Furthermore, there exists $\rho_1 > 0$, $k > 0$, *independents of* ε, *and* $t_0 > 0$ *such that, for any solution* $(\omega^\varepsilon(t), \vartheta^\varepsilon(t)) \in X$ $(t \in [t_0, \infty))$ *of* (2.80), *we have*

$$\|\vartheta^\varepsilon(t) - S_*^\varepsilon(\omega^\varepsilon(t))\|_X \leqslant k e^{-\rho_1(t-t_0)} \|\vartheta^\varepsilon(t_0) - S_*^\varepsilon(\omega^\varepsilon(t_0))\|_X, \quad t \geqslant t_0.$$

Proof. First we show the existence of the unstable manifold, for this, we use the Banach fixed point theorem for contraction.

Let $S^\varepsilon \in \Sigma_{D,L}$ and $\omega^\varepsilon(t) = \psi(t, \tau, \eta, S^\varepsilon)$ a solution of

$$\begin{cases} \partial_t \omega^\varepsilon = \mathbf{A}_\varepsilon \omega^\varepsilon + H_\varepsilon(\omega^\varepsilon, S^\varepsilon(\omega^\varepsilon)), \quad t < \tau \\ \omega^\varepsilon(\tau) = \eta, \end{cases}$$

that is,

$$\omega^\varepsilon(t) = e^{\mathbf{A}_\varepsilon(t-\tau)}\eta + \int_\tau^t e^{\mathbf{A}_\varepsilon(t-s)} H_\varepsilon(\omega^\varepsilon(s), S^\varepsilon(\omega^\varepsilon(s)))ds.$$

We consider the map $\Phi : \Sigma_{D,L} \to \Sigma_{D,L}$ defined by

$$S^\varepsilon \to \Phi(S^\varepsilon) := \int_{-\infty}^\tau e^{\mathbf{A}_\varepsilon(\tau-s)} G_\varepsilon(\omega^\varepsilon(s), S^\varepsilon(\omega^\varepsilon(s)))ds.$$

Note that, by (2.78) and (2.81) we have

$$\begin{aligned} \|\Phi(S^\varepsilon)(\eta)\|_X &\leqslant \int_{-\infty}^\tau \|e^{\mathbf{A}_\varepsilon(\tau-s)} G_\varepsilon(\omega^\varepsilon(s), S^\varepsilon(\omega^\varepsilon(s)))\|_X ds \\ &\leqslant C \int_{-\infty}^\tau (\tau - s)^{-1} e^{-\beta(\tau-s)} \|G_\varepsilon(\omega^\varepsilon(s), S^\varepsilon(\omega^\varepsilon(s)))\|_{\mathscr{H}} ds \\ &\leqslant \rho C \int_{-\infty}^\tau (\tau - s)^{-1} e^{-\beta(\tau-s)} ds = \rho C \Gamma(0). \end{aligned}$$

From (2.84) we obtain

$$\|\Phi(S^\varepsilon)(\eta)\|_X \leqslant D. \tag{2.85}$$

Now we consider $\eta, \bar{\eta} \in \mathbf{Q}_\varepsilon X$ and $S^\varepsilon, \bar{S}^\varepsilon \in \Sigma_{D,L}$. Denote $\omega^\varepsilon(t) = \psi(t, \tau, \eta, S^\varepsilon)$ and $\bar{\omega}^\varepsilon(t) = \psi(t, \tau, \bar{\eta}, \bar{S}^\varepsilon)$. Then

$$\omega^\varepsilon(t) - \bar{\omega}^\varepsilon(t) = e^{\mathbf{A}_\varepsilon(t-\tau)}(\eta - \bar{\eta}) + \int_\tau^t e^{\mathbf{A}_\varepsilon(t-s)} [H_\varepsilon(\omega^\varepsilon(s), S^\varepsilon(\omega^\varepsilon(s))) - H_\varepsilon(\bar{\omega}^\varepsilon(s), \bar{S}^\varepsilon(\bar{\omega}^\varepsilon(s)))]ds.$$

Using (2.77) and (2.82), it follows that

$$
\begin{aligned}
&\|\omega^\varepsilon(t)-\bar{\omega}^\varepsilon(t)\|_{\mathbf{Q}_\varepsilon X} \leqslant \|e^{\mathbf{A}_\varepsilon(t-\tau)}(\eta-\bar{\eta})\|_X \\
&+\int_t^\tau \|e^{\mathbf{A}_\varepsilon(t-s)}[H_\varepsilon(\omega^\varepsilon(s),S^\varepsilon(\omega^\varepsilon(s)))-H_\varepsilon(\bar{\omega}^\varepsilon(s),\bar{S}^\varepsilon(\bar{\omega}^\varepsilon(s)))]ds\|_X \\
&\leqslant Ce^{\beta(t-\tau)}\|\eta-\bar{\eta}\|_{\mathbf{Q}_\varepsilon X} \\
&+C\int_t^\tau e^{\beta(t-s)}\|[H_\varepsilon(\omega^\varepsilon(s),S^\varepsilon(\omega^\varepsilon(s)))-H_\varepsilon(\bar{\omega}^\varepsilon(s),\bar{S}^\varepsilon(\bar{\omega}^\varepsilon(s)))]\|_{\mathscr{H}}ds \\
&\leqslant Ce^{\beta(t-\tau)}\|\eta-\bar{\eta}\|_{\mathbf{Q}_\varepsilon X} \\
&+\rho C\int_t^\tau e^{\beta(t-s)}((1+L)\|\omega^\varepsilon(s)-\omega^\varepsilon(s)\|_{\mathbf{Q}_\varepsilon X}+\|S^\varepsilon(\omega^\varepsilon(s))-\bar{S}^\varepsilon(\bar{\omega}^\varepsilon(s))\|_{(I-\mathbf{Q}_\varepsilon)X})ds \\
&\leqslant Ce^{\beta(t-\tau)}\|\eta-\bar{\eta}\|_{\mathbf{Q}_\varepsilon X} \\
&+\rho C|\!|\!|S^\varepsilon-\bar{S}^\varepsilon|\!|\!|\int_t^\tau e^{\beta(t-s)}ds+\rho C(1+L)\int_t^\tau e^{\beta(t-s)}\|\omega^\varepsilon(s)-\omega^\varepsilon(s)\|_{\mathbf{Q}_\varepsilon X}ds.
\end{aligned}
$$

Making $\phi(t)=e^{-\beta(t-\tau)}\|\omega^\varepsilon(t)-\omega^\varepsilon(t)\|_{\mathbf{Q}_\varepsilon X}$, we obtain

$$
\phi(t)\leqslant C\|\eta-\bar{\eta}\|_{\mathbf{Q}_\varepsilon X}+\rho C|\!|\!|S^\varepsilon-\bar{S}^\varepsilon|\!|\!|\int_t^\tau e^{\beta(\tau-s)}ds+\rho C(1+L)\int_t^\tau \phi(s)ds,
$$

and by Gronwall lemma we conclude

$$
\begin{aligned}
&\|\omega^\varepsilon(t)-\omega^\varepsilon(t)\|_{\mathbf{Q}_\varepsilon X} \\
&\leqslant \left[Ce^{\beta(t-\tau)}\|\eta-\bar{\eta}\|_{\mathbf{Q}_\varepsilon X}+\rho C|\!|\!|S^\varepsilon-\bar{S}^\varepsilon|\!|\!|\int_t^\tau e^{\beta(t-s)}ds\right]e^{\rho C(1+L)(\tau-t)} \\
&\leqslant \left[C\|\eta-\bar{\eta}\|_{\mathbf{Q}_\varepsilon X}+\frac{\rho C}{\beta}|\!|\!|S^\varepsilon-\bar{S}^\varepsilon|\!|\!|\right]e^{\rho C(1+L)(\tau-t)}.
\end{aligned}
\tag{2.86}
$$

Now using again (2.78) and (2.83), it follows that

$$
\begin{aligned}
&\|\Phi(S^\varepsilon)(\eta)-\Phi(\bar{S}^\varepsilon)(\bar{\eta})\|_X \\
&\leqslant \int_{-\infty}^\tau \|e^{\mathbf{A}_\varepsilon(\tau-s)}[G_\varepsilon(\omega^\varepsilon(s),S^\varepsilon(\omega^\varepsilon(s)))-G_\varepsilon(\bar{\omega}^\varepsilon(s),\bar{S}^\varepsilon(\bar{\omega}^\varepsilon(s)))]\|_X ds \\
&\leqslant C\int_{-\infty}^\tau (\tau-s)^{-1}e^{-\beta(\tau-s)}\|G_\varepsilon(\omega^\varepsilon(s),S^\varepsilon(\omega^\varepsilon(s)))-G_\varepsilon(\bar{\omega}^\varepsilon(s),\bar{S}^\varepsilon(\bar{\omega}^\varepsilon(s)))\|_{\mathscr{H}}ds \\
&\leqslant \rho C\int_{-\infty}^\tau (\tau-s)^{-1}e^{-\beta(\tau-s)}[(1+L)\|\omega^\varepsilon(s)-\omega^\varepsilon(s)\|_{\mathbf{Q}_\varepsilon X}+|\!|\!|S^\varepsilon-\bar{S}^\varepsilon|\!|\!|]ds.
\end{aligned}
$$

Substituting (2.86) in the above inequality, we get

$$
\begin{aligned}
&\|\Phi(S^\varepsilon)(\eta) - \Phi(\bar{S}^\varepsilon)(\bar{\eta})\|_X \\
&\leqslant \rho C^2(1+L)\int_{-\infty}^{\tau}(\tau-s)^{-1}e^{-[\beta-\rho C(1+L)](\tau-s)}ds\|\eta-\bar{\eta}\|_{\mathbf{Q}_\varepsilon X} \\
&+\frac{\rho^2C^2}{\beta}(1+L)\int_{-\infty}^{\tau}(\tau-s)^{-1}e^{-[\beta-\rho C(1+L)](\tau-s)}ds\|\!|S^\varepsilon-\bar{S}^\varepsilon\|\!| \\
&+\rho C\int_{-\infty}^{\tau}(\tau-s)^{-1}e^{-\beta(\tau-s)}ds\|\!|S^\varepsilon-\bar{S}^\varepsilon\|\!| \\
&=\rho C^2(1+L)\Gamma(0)\|\eta-\bar{\eta}\|_{\mathbf{Q}_\varepsilon X}+\left[\rho C\Gamma(0)+\rho^2C^2(1+L)\beta^{-1}\Gamma(0)\right]\|\!|S^\varepsilon-\bar{S}^\varepsilon\|\!|,
\end{aligned}
$$

And by (2.84) we conclude

$$
\|\Phi(S^\varepsilon)(\eta) - \Phi(\bar{S}^\varepsilon)(\bar{\eta})\|_X \leqslant L\|\eta-\bar{\eta}\|_{\mathbf{Q}_\varepsilon X}+\kappa\|\!|S^\varepsilon-\bar{S}^\varepsilon\|\!|. \tag{2.87}
$$

The inequality (2.87) with $S^\varepsilon = \bar{S}^\varepsilon$ and (2.85) imply that Φ takes $\Sigma_{D,L}$ into $\Sigma_{D,L}$. And (2.87) with $\eta = \bar{\eta}$, follows from (2.84) that $\Phi : \Sigma_{D,L} \to \Sigma_{D,L}$ is a contraction. Therefore, there exists a unique fixed point $S^\varepsilon_* = \Phi(S^\varepsilon_*)$ in $\Sigma_{D,L}$.

Now we prove that

$$
W^u(w^\varepsilon_*) = \{(\omega^\varepsilon, S^\varepsilon_*(\omega^\varepsilon)) : \omega^\varepsilon \in \mathbf{Q}_\varepsilon X\}
$$

is an invariant manifold for (2.80).

Let $(\omega^\varepsilon_0, \vartheta^\varepsilon_0) \in W^u(w^\varepsilon_*)$, $\vartheta^\varepsilon_0 = S^\varepsilon_*(\omega^\varepsilon_0)$. Denote by ω^ε_* the solution of the initial value problem

$$
\begin{cases}
\omega^\varepsilon_t = \mathbf{A}_\varepsilon\omega^\varepsilon + H_\varepsilon(\omega^\varepsilon, S^\varepsilon_*(\omega^\varepsilon)), \quad t<\tau \\
\omega^\varepsilon(\tau) = \omega^\varepsilon_0.
\end{cases}
$$

Thus $\{(\omega^\varepsilon_*(t), S^\varepsilon_*(\omega^\varepsilon_*(t)))\}_{t\in\mathbb{R}}$ defines a curve on $W^u(w^\varepsilon_*)$. However the unique solution of equation

$$
\vartheta^\varepsilon_t = \mathbf{A}_\varepsilon\vartheta^\varepsilon + G_\varepsilon(\omega^\varepsilon_*(t), S^\varepsilon_*(\omega^\varepsilon_*(t)))
$$

which remains bounded when $t \to -\infty$ is given by

$$
\vartheta^\varepsilon_*(t) = \int_{-\infty}^{t} e^{\mathbf{A}_\varepsilon(t-s)}G_\varepsilon(\omega^\varepsilon_*(s), S^\varepsilon_*(\omega^\varepsilon(s)))ds = S^\varepsilon_*(\omega^\varepsilon_*(t)).
$$

Therefore $(\omega^\varepsilon_*(t), S^\varepsilon_*(\omega^\varepsilon_*(t)))$ is a solution of (2.80) through $(\omega^\varepsilon_0, \vartheta^\varepsilon_0)$, and thus $W^u(w^\varepsilon_*)$ is an invariant manifold.

Next, we show that $\vartheta(t) = S_*^\varepsilon(\omega^\varepsilon(t))$ for all $t \in \mathbb{R}$; that is, there exists $\rho_1 > 0$, independent of ε such that

$$\|\vartheta^\varepsilon(t) - S_*^\varepsilon(\omega^\varepsilon(t))\|_X \leqslant k e^{-\rho_1(t-t_0)} \|\vartheta^\varepsilon(t_0) - S_*^\varepsilon(\omega^\varepsilon(t_0))\|_X, \quad \forall t \geqslant t_0.$$

When $t_0 \to -\infty$, we obtain $\vartheta(t) = S_*^\varepsilon(\omega^\varepsilon(t))$ for all $t \in \mathbb{R}$.

Let

$$\xi^\varepsilon(t) = \vartheta^\varepsilon(t) - S_*^\varepsilon(\omega^\varepsilon(t)) \tag{2.88}$$

and let $y^\varepsilon = y^\varepsilon(s,t)$ be the solution of the initial value problem

$$\begin{cases} \partial_t y^\varepsilon = \mathbf{A}_\varepsilon y^\varepsilon + H_\varepsilon(y^\varepsilon, S_*^\varepsilon(y^\varepsilon)), \quad s \leqslant t \\ y^\varepsilon(t,t) = \omega^\varepsilon(t), \end{cases}$$

that is,

$$y^\varepsilon(s,t) = e^{\mathbf{A}_\varepsilon(s-t)}\omega^\varepsilon(t) + \int_t^s e^{\mathbf{A}_\varepsilon(s-\theta)} H_\varepsilon(y^\varepsilon(\theta,t), S_*^\varepsilon(y^\varepsilon(\theta,t)))d\theta.$$

Thus, by (2.77) we have

$$\begin{aligned} \|y(s,t) - \omega^\varepsilon(s)\|_{\mathbf{Q}_\varepsilon X} &\leqslant \int_s^t \left\| e^{\mathbf{A}_\varepsilon(s-\theta)} H_\varepsilon(y^\varepsilon(\theta,t), S_*^\varepsilon(y^\varepsilon(\theta,t))) - H_\varepsilon(\omega^\varepsilon(\theta), \vartheta^\varepsilon(\theta)) \right\|_X d\theta \\ &\leqslant C \int_s^t e^{\beta(s-\theta)} \left\| H_\varepsilon(y^\varepsilon(\theta,t), S_*^\varepsilon(y^\varepsilon(\theta,t))) - H_\varepsilon(\omega^\varepsilon(\theta), \vartheta^\varepsilon(\theta)) \right\|_{\mathscr{H}} d\theta, \end{aligned}$$

and thanks to (2.82) and (2.88) we get

$$\|y(s,t) - \omega^\varepsilon(s)\|_{\mathbf{Q}_\varepsilon X} \leqslant \rho C \int_s^t e^{\beta(s-\theta)}[(1+L)\|y^\varepsilon(\theta,t) - \omega^\varepsilon(\theta)\|_{\mathbf{Q}_\varepsilon X} + \|\xi^\varepsilon(\theta)\|_X]d\theta.$$

If we denote $\phi^\varepsilon(s) = e^{-\beta s}\|y^\varepsilon(s,t) - \omega^\varepsilon(s)\|_{\mathbf{Q}_\varepsilon X}$ then

$$\phi^\varepsilon(s) \leqslant \rho C \int_s^t e^{-\beta\theta}\|\xi^\varepsilon(\theta)\|_X d\theta + \rho C(1+L)\int_s^t \phi^\varepsilon(\theta)d\theta.$$

Using Gronwall's lemma we obtain

$$\|y^\varepsilon(s,t) - \omega^\varepsilon(s)\|_{\mathbf{Q}_\varepsilon X} \leqslant \rho C \int_s^t e^{-(\beta - \rho C(1+L))(\theta - s)}\|\xi^\varepsilon(\theta)\|_X d\theta. \tag{2.89}$$

Now, using (2.77), for any $s \leqslant t_0 \leqslant t$ we have

$$\begin{aligned} &\|y^\varepsilon(s,t) - y^\varepsilon(s,t_0)\|_{\mathbf{Q}_\varepsilon X} \leqslant \|e^{\mathbf{A}_\varepsilon(s-t_0)}[y^\varepsilon(t_0,t) - \omega^\varepsilon(t_0)]\|_X \\ &+ \left\| \int_{t_0}^s e^{\mathbf{A}_\varepsilon(s-\theta)}[H_\varepsilon(y^\varepsilon(\theta,t), S_*^\varepsilon(y^\varepsilon(\theta,t))) - H_\varepsilon(y^\varepsilon(\theta,t_0), S_*^\varepsilon(y^\varepsilon(\theta,t_0)))]d\theta \right\|_X \\ &\leqslant Ce^{\beta(s-t_0)}\|y^\varepsilon(t_0,t) - \omega^\varepsilon(t_0)\|_{\mathbf{Q}_\varepsilon X} \\ &+ C\int_s^{t_0} e^{\beta(s-\theta)} \left\| H_\varepsilon(y^\varepsilon(\theta,t), S_*^\varepsilon(y^\varepsilon(\theta,t))) - H_\varepsilon(y^\varepsilon(\theta,t_0), S_*^\varepsilon(y^\varepsilon(\theta,t_0))) \right\|_{\mathscr{H}} d\theta. \end{aligned}$$

and thanks to (2.82) we get

$$
\begin{aligned}
&\|y^\varepsilon(s,t)-y^\varepsilon(s,t_0)\|_{\mathbf{Q}_\varepsilon X} \leqslant Ce^{\beta(s-t_0)}\|y^\varepsilon(t_0,t)-\omega^\varepsilon(t_0)\|_{\mathbf{Q}_\varepsilon X}\\
&+\rho C\int_s^{t_0} e^{\beta(s-\theta)}[\|y^\varepsilon(\theta,t)-y^\varepsilon(\theta,t_0)\|_{\mathbf{Q}_\varepsilon X}+\|S_*^\varepsilon(y^\varepsilon(\theta,t)))-S_*^\varepsilon(y^\varepsilon(\theta,t_0)))\|_X]d\theta\\
&\leqslant Ce^{\beta(s-t_0)}\|y^\varepsilon(t_0,t)-\omega^\varepsilon(t_0)\|_{\mathbf{Q}_\varepsilon X}\\
&+\rho C\int_s^{t_0} e^{\beta(s-\theta)}(1+L)\|y^\varepsilon(\theta,t)-y^\varepsilon(\theta,t_0)\|_{\mathbf{Q}_\varepsilon X}d\theta.
\end{aligned}
$$

Using (2.89), in the above inequality

$$
\begin{aligned}
\|y^\varepsilon(s,t)-y^\varepsilon(s,t_0)\|_{\mathbf{Q}_\varepsilon X} &\leqslant \rho C^2 e^{\beta(s-t_0)}\int_{t_0}^{t} e^{-(\beta-\rho C(1+L))(\theta-t_0)}\|\xi^\varepsilon(\theta)\|_X d\theta\\
&+\rho C(1+L)\int_s^{t_0} e^{\beta(s-\theta)}\|y^\varepsilon(\theta,t)-y^\varepsilon(\theta,t_0)\|_{\mathbf{Q}_\varepsilon X}d\theta.
\end{aligned}
$$

Again by Gronwall's lemma follows that, for $s\leqslant t_0\leqslant t$,

$$
\|y^\varepsilon(s,t)-y^\varepsilon(s,t_0)\|_{\mathbf{Q}_\varepsilon X}\leqslant \rho C^2\int_{t_0}^{t} e^{-(\beta-\rho C(1+L))(\theta-s)}\|\xi^\varepsilon(\theta)\|_X d\theta. \tag{2.90}
$$

We use the last bound to estimate $\xi(t)$, we have

$$
\begin{aligned}
&\xi^\varepsilon(t)-e^{\mathbf{A}_\varepsilon(t-t_0)}\xi^\varepsilon(t_0)=\vartheta^\varepsilon(t)-S_*^\varepsilon(\omega^\varepsilon(t))-e^{\mathbf{A}_\varepsilon(t-t_0)}[\vartheta^\varepsilon(t_0)-S_*^\varepsilon(\omega^\varepsilon(t_0))]\\
&=\int_{t_0}^{t} e^{\mathbf{A}_\varepsilon(t-s)}G_\varepsilon(\omega^\varepsilon(s),\vartheta^\varepsilon(s))ds-\int_{-\infty}^{t} e^{\mathbf{A}_\varepsilon(t-s)}G_\varepsilon(y^\varepsilon(s,t),S_*^\varepsilon(y^\varepsilon(s,t)))ds\\
&+e^{\mathbf{A}_\varepsilon(t-t_0)}\int_{-\infty}^{t_0} e^{\mathbf{A}_\varepsilon(t_0-s)}G_\varepsilon(y^\varepsilon(s,t_0),S_*^\varepsilon(y^\varepsilon(s,t_0)))ds\\
&=\int_{t_0}^{t} e^{\mathbf{A}_\varepsilon(t-s)}[G_\varepsilon(\omega^\varepsilon(s),\vartheta^\varepsilon(s))-G_\varepsilon(y^\varepsilon(s,t),S_*^\varepsilon(y^\varepsilon(s,t)))]ds\\
&-\int_{-\infty}^{t_0} e^{\mathbf{A}_\varepsilon(t-s)}[G_\varepsilon(y^\varepsilon(s,t),S_*^\varepsilon(y^\varepsilon(s,t)))-G_\varepsilon(y^\varepsilon(s,t_0),S_*^\varepsilon(y^\varepsilon(s,t_0)))]ds.
\end{aligned}
$$

Taking the norm and using (2.79) and (2.83), we get

$$
\begin{aligned}
&\left\|\xi^\varepsilon(t)-e^{\mathbf{A}_\varepsilon(t-t_0)}\xi^\varepsilon(t_0)\right\|_X\\
&\leqslant\int_{t_0}^{t}\left\|e^{\mathbf{A}_\varepsilon(t-s)}[G_\varepsilon(\omega^\varepsilon(s),\vartheta^\varepsilon(s))-G_\varepsilon(y^\varepsilon(s,t),S_*^\varepsilon(y^\varepsilon(s,t)))]\right\|_X ds\\
&+\int_{-\infty}^{t_0}\left\|e^{\mathbf{A}_\varepsilon(t-s)}[G_\varepsilon(y^\varepsilon(s,t),S_*^\varepsilon(y^\varepsilon(s,t)))-G_\varepsilon(y^\varepsilon(s,t_0),S_*^\varepsilon(y^\varepsilon(s,t_0)))]\right\|_X ds\\
&\leqslant\rho C\int_{t_0}^{t} e^{-\beta(t-s)}[(1+L)\|\omega^\varepsilon(s)-y^\varepsilon(s,t)\|_{\mathbf{Q}_\varepsilon X}+\|\vartheta^\varepsilon(s))-S_*^\varepsilon(y^\varepsilon(s,t)))]\|_X ds\\
&+\rho C(1+L)\int_{-\infty}^{t_0} e^{-\beta(t-s)}\|y^\varepsilon(s,t)-y^\varepsilon(s,t_0)\|_{\mathbf{Q}_\varepsilon X}ds.
\end{aligned}
$$

Thus, using (2.89) and (2.90), we obtain

$$\begin{aligned}
&\left\|\xi^\varepsilon(t) - e^{\mathbf{A}_\varepsilon(t-t_0)}\xi^\varepsilon(t_0)\right\|_X \leqslant \rho C \int_{t_0}^t e^{-\beta(t-s)}\|\xi^\varepsilon(s)\|_X ds \\
&+ \rho^2 C^2(1+L)\int_{t_0}^t e^{-\beta(t-s)}\int_s^t e^{-(\beta-\rho C(1+L))(\theta-s)}\|\xi^\varepsilon(\theta)\|_X d\theta ds \\
&+ \rho^2 C^3(1+L)\int_{-\infty}^{t_0} e^{-\beta(t-s)}\int_{t_0}^t e^{-(\beta-\rho C(1+L))(\theta-s)}\|\xi^\varepsilon(\theta)\|_X d\theta ds \\
&= \rho C\int_{t_0}^t e^{-\beta(t-s)}\|\xi^\varepsilon(s)\|_X ds \\
&+ \rho^2 C^2(1+L)e^{-\beta t}\int_{t_0}^t e^{-(\beta-\rho C(1+L))\theta}\|\xi^\varepsilon(\theta)\|_X\left[\int_{t_0}^\theta e^{(2\beta-\rho C(1+L))s}ds\right]d\theta \\
&+ \rho^2 C^3(1+L)e^{-\beta t}\int_{t_0}^t e^{-(\beta-\rho C(1+L))\theta}\|\xi^\varepsilon(\theta)\|_X\left[\int_{-\infty}^{t_0} e^{(2\beta-\rho C(1+L))s}ds\right]d\theta.
\end{aligned}$$

So we write,

$$\begin{aligned}
&\left\|\xi^\varepsilon(t) - e^{\mathbf{A}_\varepsilon(t-t_0)}\xi^\varepsilon(t_0)\right\|_X \leqslant \left[\rho C + \frac{\rho^2 C^2(1+L)}{2\beta-\rho C(1+L)}\right]\int_{t_0}^t e^{-\beta(t-s)}\|\xi^\varepsilon(s)\|_X ds \\
&+ \frac{\rho^2 C^3(1+L)}{2\beta-\rho C(1+L)}e^{-\beta(t-t_0)}\int_{t_0}^t e^{-(\beta-\rho C(1+L))(\theta-t_0)}\|\xi^\varepsilon(\theta)\|_X d\theta,
\end{aligned}$$

it follows that,

$$\begin{aligned}
&e^{\beta(t-t_0)}\|\xi^\varepsilon(t)\|_X \leqslant C\|\xi^\varepsilon(t_0)\|_X + \left[\rho C + \frac{\rho^2 C^2(1+L)}{2\beta-\rho C(1+L)}\right]\int_{t_0}^t e^{\beta(s-t_0)}\|\xi^\varepsilon(s)\|_X ds \\
&+ \frac{\rho^2 C^3(1+L)}{2\beta-\rho C(1+L)}\int_{t_0}^t e^{-(2\beta-\rho C(1+L))(s-t_0)}e^{\beta(s-t_0)}\|\xi^\varepsilon(s)\|_X ds \\
&\leqslant C\|\xi^\varepsilon(t_0)\|_X + \left[\rho C + \frac{\rho^2 C^2(1+L)}{2\beta-\rho C(1+L)}(1+C)\right]\int_{t_0}^t e^{\beta(s-t_0)}\|\xi^\varepsilon(s)\|_X ds.
\end{aligned}$$

From Gronwall's lemma we obtain

$$\|\xi^\varepsilon(t)\|_X \leqslant C\|\xi^\varepsilon(t_0)\|_X e^{-\rho_1(t-t_0)}, \quad \forall t \geqslant t_0$$

where

$$\rho_1 = \beta - \left[\rho C + \frac{\rho^2 C^2(1+L)}{2\beta-\rho C(1+L)}(1+C)\right] > 0,$$

with ρ_1 independent of ε, once β, C, ρ and L are independents of ε for $0<\varepsilon<\bar{\varepsilon}$. Thus,

$$\|\vartheta\varepsilon(t) - S_*^\varepsilon(\omega^\varepsilon(t))\|_X \leqslant Ce^{-\rho_1(t-t_0)}\|\vartheta\varepsilon(t_0) - S_*^\varepsilon(\omega^\varepsilon(t_0))\|_X, \quad \forall t \geqslant t_0.$$

Therefore, letting $t_0 \to -\infty$, we have $\vartheta^\varepsilon(t) = S_*^\varepsilon(\omega^\varepsilon(t))$, for all $t \in \mathbb{R}$.

Finally we show that the fixed points S_*^ε continuously depend in ε, that is, if $0<\varepsilon<\bar{\varepsilon}$ its such that the unstable manifold is given by the graph of S_*^ε, we want to prove that

$$|||S_*^\varepsilon - S_*^0||| := \sup_{\eta\in \mathbf{Q}_\varepsilon X} \|S_*^\varepsilon(\eta) - S_*^0(\eta)\|_X \to 0,$$

as $\varepsilon\to 0$. To this end, let us first observe that as the map $\mathbf{Q}_\varepsilon X \ni \omega^0 \to G_0(\omega^0, S_*^0(\omega^0))$ is continuous and takes bounded subsets of $\mathbf{Q}_\varepsilon X$ into subsets relatively compacts of X. Therefore, adding and subtracting terms, we get

$$\begin{aligned}
&\|S_*^\varepsilon(\eta) - S_*^0(\eta)\|_X\\
&\leqslant \int_{-\infty}^{\tau} \left\|e^{\mathbf{A}_\varepsilon(\tau-s)}G_\varepsilon(\omega^\varepsilon(s), S_*^\varepsilon(\omega^\varepsilon(s))) - e^{\mathbf{A}_0(\tau-s)}G_0(\omega^0(s), S_*^0(\omega^0(s)))\right\|_X ds\\
&\leqslant \int_{-\infty}^{\tau} \left\|e^{\mathbf{A}_\varepsilon(\tau-s)}[G_\varepsilon(\omega^\varepsilon(s), S_*^\varepsilon(\omega^\varepsilon(s))) - G_\varepsilon(\omega^0(s), S_*^0(\omega^0(s)))]\right\|_X ds\\
&+ \int_{-\infty}^{\tau} \left\|e^{\mathbf{A}_\varepsilon(\tau-s)}[G_\varepsilon(\omega^0(s), S_*^0(\omega^0(s))) - G_0(\omega^0(s), S_*^0(\omega^0(s)))]\right\|_X ds\\
&+ \int_{-\infty}^{\tau} \left\|[e^{\mathbf{A}_\varepsilon(\tau-s)} - e^{\mathbf{A}_0(\tau-s)}]G_0(\omega^0(s), S_*^0(\omega^0(s)))\right\|_X ds\\
&:= I_1 + I_2 + I_3,
\end{aligned}$$

respectively.

Thus, using (2.78) and (2.83), we get

$$\begin{aligned}
I_1 &\leqslant C\int_{-\infty}^{\tau} (\tau-s)^{-1}e^{-\beta(\tau-s)} \left\|G_\varepsilon(\omega^\varepsilon(s), S_*^\varepsilon(\omega^\varepsilon(s))) - G_\varepsilon(\omega^0(s), S_*^0(\omega^0(s)))]\right\|_{\mathscr{H}} ds\\
&\leqslant \rho C\int_{-\infty}^{\tau} (\tau-s)^{-1}e^{-\beta(\tau-s)}[(1+L)\|\omega^\varepsilon(s)-\omega^0(s)\|_{\mathbf{Q}_\varepsilon X} + |||S_*^\varepsilon - S_*^0|||]ds\\
&= \rho C\Gamma(0)|||S_*^\varepsilon - S_*^0||| + \rho C(1+L)\int_{-\infty}^{\tau} (\tau-s)^{-1}e^{-\beta(\tau-s)}\|\omega^\varepsilon(s)-\omega^0(s)\|_{\mathbf{Q}_\varepsilon X} ds.
\end{aligned}$$

Since G_ε converges to G_0 pointwise on compacts as $\varepsilon\to 0$, we have that $G_\varepsilon \to G_0$, as $\varepsilon\to 0$, uniformlly, see item (iii) of Lemma 2.2.3 and therefore I_2 is $o(1)$, where $o(1)$ denote the quantity which goes to zero as $\varepsilon\to 0$. Note that $G_0(\omega^0, S_*^0(\omega^0))$ is in a compact set of X, and by Lemma 2.5.1, item (iii) we obtain $\mathbf{A}_\varepsilon^{-1} \to \mathbf{A}_0^{-1}$. Then by results due to Trotter-Kato, see [38], we get

$$e^{\mathbf{A}_\varepsilon t}u_0 \to e^{\mathbf{A}_0 t}u_0, \quad \forall t \geqslant 0 \quad \text{and} \quad u_0\in X.$$

This we ensure that I_3 its also $o(1)$ as $\varepsilon\to 0$. Thus

$$\begin{aligned}
&\|S_*^\varepsilon(\eta) - S_*^0(\eta)\|_X\\
&\leqslant \rho C\Gamma(0)|||S_*^\varepsilon - S_*^0||| + \rho C(1+L)\int_{-\infty}^{\tau} (\tau-s)^{-1}e^{-\beta(\tau-s)}\|\omega^\varepsilon(s)-\omega^0(s)\|_{\mathbf{Q}_\varepsilon X} ds + o(1).
\end{aligned}$$

Then, it is sufficient to estimate the term $\|\omega^\varepsilon(s)-\omega^0(s)\|_{\mathbf{Q}_\varepsilon X}$. Once $\|\eta\|_{\mathbf{Q}_\varepsilon X}\leqslant R$, proceeding as above we have

$$\begin{aligned}
&\|\omega^\varepsilon(t)-\omega^0(t)\|_{\mathbf{Q}_\varepsilon X}\leqslant \|e^{\mathbf{A}_\varepsilon(t-\tau)}\eta - e^{\mathbf{A}_0(t-\tau)}\eta\|_X\\
&+\int_t^\tau \big\|e^{\mathbf{A}_\varepsilon(t-s)}[H_\varepsilon(\omega^\varepsilon(s),S_*^\varepsilon(\omega^\varepsilon(s)))-H_\varepsilon(\omega^0(s),S_*^0(\omega^0(s)))]\big\|_X\,ds\\
&+\int_t^\tau \big\|e^{\mathbf{A}_\varepsilon(t-s)}[H_\varepsilon(\omega^0(s),S_*^0(\omega^0(s)))-H_0(\omega^0(s),S_*^0(\omega^0(s)))]\big\|_X\,ds\\
&+\int_t^\tau \big\|[e^{\mathbf{A}_\varepsilon(t-s)}-e^{\mathbf{A}_0(t-s)}]H_0(\omega^0(s),S_*^0(\omega^0(s)))\big\|_X\,ds\\
&\leqslant \rho C|\!|\!|S_*^\varepsilon-S_*^0|\!|\!|\int_t^\tau e^{\beta(t-s)}ds+\rho C(1+L)\int_t^\tau e^{\beta(t-s)}\|\omega^\varepsilon(s)-\omega^0(s)\|_{\mathbf{Q}_\varepsilon X}ds+o(1).
\end{aligned}$$

If $\phi(t)=e^{\beta(\tau-t)}\|\omega^\varepsilon(t)-\omega^0(t)\|_{\mathbf{Q}_\varepsilon X}$ we have

$$\phi(t)\leqslant \rho C|\!|\!|S_*^\varepsilon-S_*^0|\!|\!|\int_t^\tau e^{\beta(\tau-s)}ds+\rho C(1+L)\int_t^\tau \phi(s)ds+o(1).$$

From Gronwall's lemma we obtain

$$\|\omega^\varepsilon(t)-\omega^0(t)\|_{\mathbf{Q}_\varepsilon X}\leqslant [o(1)+\rho C\beta^{-1}|\!|\!|S_*^\varepsilon-S_*^0|\!|\!|]e^{(\rho C(1+L)-\beta)(\tau-t)}.$$

Then from (2.84), it follows that

$$\begin{aligned}
&\|S_*^\varepsilon(\eta)-S_*^0(\eta)\|_X\leqslant o(1)+\rho C\Gamma(0)|\!|\!|S_*^\varepsilon-S_*^0|\!|\!|\\
&+\rho C(1+L)[o(1)+\rho C\beta^{-1}|\!|\!|S_*^\varepsilon-S_*^0|\!|\!|]\int_{-\infty}^\tau(\tau-s)^{-1}e^{(-2\beta+\rho C(1+L))(\tau-s)}ds\\
&=o(1)+\big[\rho C\Gamma(0)+\rho^2C^2\beta^{-1}(1+L)\Gamma(0)\big]\,|\!|\!|S_*^\varepsilon-S_*^0|\!|\!|\\
&\leqslant o(1)+\frac{1}{2}|\!|\!|S_*^\varepsilon-S_*^0|\!|\!|.
\end{aligned}$$

Therefore,

$$|\!|\!|S_*^\varepsilon-S_*^0|\!|\!|=\sup_{\eta\in\mathbf{Q}_\varepsilon X}\|S_*^\varepsilon(\eta)-S_*^0(\eta)\|_X\leqslant o(1)+\frac{1}{2}|\!|\!|S_*^\varepsilon-S_*^0|\!|\!|,$$

and thus,

$$|\!|\!|S_*^\varepsilon-S_*^0|\!|\!|\to 0\quad\text{as}\quad\varepsilon\to 0.$$

■

Theorem 2.5.17 *The family of attractors $\{\mathscr{A}_\varepsilon:\varepsilon\in(0,\varepsilon_0]\}$ is lower semicontinuous at $\varepsilon=0$; that is,*

$$\mathrm{dist}_H(\mathscr{A}_0,\mathscr{A}_\varepsilon)\to 0,\quad \textit{as}\quad \varepsilon\to 0,$$

where

$$\mathrm{dist}_H(\mathscr{A}_0,\mathscr{A}_\varepsilon):=\sup_{w^0\in\mathscr{A}_0}\inf_{w^\varepsilon\in\mathscr{A}_\varepsilon}\{\|w^\varepsilon-w^0\|_X\}.$$

Proof. Thanks to the results of previous sections, the proof follows using arguments already known in the literature, see Carvalho, Langa and Robinson [14, Chapter 3], that is, using the item (ii) of the Lemma C.20. Let $w^0 \in \mathscr{A}_0$. Since $\{S_0(t) : t \geqslant 0\}$ is a gradient system, we have that

$$\mathscr{A}_0 = \bigcup_{w_*^0 \in \mathscr{E}_0} W^u(w_*^0)$$

and then $w^0 \in W^u(w_*^0)$, for some $w_*^0 \in \mathscr{E}_0$. Let $\tau \in \mathbb{R}$ and $\varphi^0 \in W^u_{loc}(w_*^0)$ be such that $S_0(\tau)\varphi^0 = w^0$. Let w_*^ε be such that $w_*^\varepsilon \to w_*^0$ as $\varepsilon \to 0$. From the convergence of unstable manifolds there is a sequence $\{\varphi^\varepsilon\}_{\varepsilon \in [0,\varepsilon_0]}$, $\varphi^\varepsilon \in W^u_{loc}(w_*^\varepsilon)$ such that $\varphi^\varepsilon \to \varphi^0$ as $\varepsilon \to 0$. Finally, from Proposition 2.4.4, we obtain that $S_\varepsilon(\tau)\varphi^\varepsilon \to S_0(\tau)\varphi^0 = w^0$. To conclude, we observe that if $w^\varepsilon = S_\varepsilon(\tau)\varphi^\varepsilon$, then $w^\varepsilon \in \mathscr{A}_\varepsilon$, since

$$\varphi^\varepsilon \in \bigcup_{w_*^\varepsilon \in \mathscr{E}_\varepsilon} W^u(w_*^\varepsilon) = \mathscr{A}_\varepsilon$$

and $\mathscr{A}_\varepsilon$ is invariant. ■

Corollary 2.5.18 *The family of global attractors* $\{\mathscr{A}_\varepsilon : \varepsilon \in (0, \varepsilon_0]\}$ *is continuous at* $\varepsilon = 0$.

Chapter 3

Final considerations and conclusions

In this short chapter we will present possible improvements of the results obtained in previous chapters, under the best of our knowledge.

(i) On the Klein-Gordon system, in order to obtain results on the uniqueness and decay of solutions of problem (1), we prove the Theorem 1.2.13 on existence of solutions for the case $\rho = 1$ and $n = 1, 2, 3$. For the case $\rho > 1$ and $n > 3$, the regularity of the solution u obtained is an open problem;

(ii) A new paper is in phase of conclusion, we will consider the system (1) with nonlinearities which are strongly monotone, acting as damping on a part of the boundary;

(iii) We want study a semigroup approach, in the sense of Pazy [38], to the problem associated with the Klein-Gordon system as in Chapter 1. In this way we will search results of regularity of solutions and behavior asymptotic of solutions;

(iv) On the thermoelastic plate systems, the X^1-regularity of the global attractors for the problems (2.7) and (2.10) are open problems, probably this result can be obtained, for example, by using the same arguments of Carvalho, Langa and Robinson [14, Chapter 15, Section 15.6];

(v) Thanks to parabolic structure of the problems (2.7) and (2.10), in the sense of Henry [20], we want study the the behavior asymptotic of solutions, in the sense of global attractors, in the fractional power space X^α for some $0 < \alpha < 1$;

(vi) We also want study evolution systems as in the Chapter 1 and Chapter 2 with reaction terms concentrated in a neighborhood of only part of the boundary and this

neighborhood shrinks to boundary as a parameter goes to zero. Thus, we want to use the analysis done in Chapter 1 and Chapter 2 in one unique research project;
(vii) Finally, we also want study the non-autonomous dynamical systems, in the sense of Carvalho, Langa and Robinson [14], associated with non-autonomous formulations of the systems in Chapter 1 and Chapter 2. In this case, initially, we can consider non-autonomous damped for these systems. More precisely, we will analyze the asymptotic behavior of a non-autonomous thermoelastic plate systems with Neumann boundary conditions when some reaction terms are concentrated in a neighborhood of the boundary, and this neighborhood shrinks to boundary as a parameter ε goes to zero, which is represented by

$$\begin{cases} \partial_t^2 u^\varepsilon + \Delta^2 u^\varepsilon + u^\varepsilon + a(t)\Delta\theta^\varepsilon - a(t)\theta^\varepsilon = f(t,u^\varepsilon) + \dfrac{1}{\varepsilon}\chi_{\omega_\varepsilon} g(u^\varepsilon) & \text{in} \quad \Omega\times(0,\infty),\\ \partial_t\theta^\varepsilon - \Delta\theta^\varepsilon + \theta^\varepsilon - a(t)\Delta\partial_t u^\varepsilon + a(t)\partial_t u^\varepsilon = 0 & \text{in} \quad \Omega\times(0,\infty),\\ \dfrac{\partial u^\varepsilon}{\partial\vec{n}} = 0, \quad \dfrac{\partial(\Delta u^\varepsilon)}{\partial\vec{n}} = 0, \quad \dfrac{\partial\theta^\varepsilon}{\partial\vec{n}} = 0 & \text{on} \quad \Gamma\times(0,\infty),\\ u^\varepsilon(0) = u_0 \in H^2(\Omega), \quad \partial_t u^\varepsilon(0) = v_0 \in L^2(\Omega), \quad \theta^\varepsilon(0) = \theta_0 \in L^2(\Omega), \end{cases} \tag{3.1}$$

where Ω is a bounded and smooth open set of $\mathbb{R}^n$, $n \geqslant 2$, with boundary $\Gamma = \partial\Omega$ smooth, ω_ε, $0 < \varepsilon \leqslant \varepsilon_0$ is a neighborhood of Γ, $\chi_{\omega_\varepsilon}$ is the characteristic function of set ω_ε, $0 < \varepsilon \leqslant \varepsilon_0$, $a \in L^\infty(\mathbb{R})$ is Hölder continuous, and $f, g : \mathbb{R} \to \mathbb{R}$ are nonlinearities under suitable growth conditions.

We want study the asymptotic behavior of the problem (3.1) in the sense of pullback attractors. We also want show that the limit problem for the autonomous thermoelastic plate system (3.1) is given by

$$\begin{cases} \partial_t^2 u + \Delta^2 u + u + a(t)\Delta\theta - a(t)\theta = f(t,u) & \text{in} \quad \Omega\times(\tau,\infty),\\ \partial_t\theta - \Delta\theta + \theta - a(t)\Delta\partial_t u + a(t)\partial_t u = 0 & \text{in} \quad \Omega\times(0,\infty),\\ \dfrac{\partial u}{\partial\vec{n}} = 0, \quad \dfrac{\partial(\Delta u)}{\partial\vec{n}} = -g(u), \quad \dfrac{\partial\theta}{\partial\vec{n}} = 0 & \text{on} \quad \Gamma\times(0,\infty),\\ u(0) = u_0 \in H^2(\Omega), \quad \partial_t u(0) = v_0 \in L^2(\Omega), \quad \theta(0) = \theta_0 \in L^2(\Omega). \end{cases} \tag{3.2}$$

In other words, we prove that the nonlinear evolution process associated to (3.1) converges to the nonlinear evolution process associated to (3.2). Moreover, we show the existence, uniform boundedness, and continuity of the pullback attractors at $\varepsilon = 0$

associated to these process.

Appendix

Appendix A

Preliminary results

In this appendix we will present some definitions and basic concepts related to functional analysis, measure theory and distributions theory that was necessary for the development of our work. As our goal and to establish a theoretical base for our work, we will not worry about formal proofs for the theorems that will be presented here but for more details we recommend the following references, Lions [26], Brezis [11], Medeiros and Milla Miranda [30], Medeiros [29] and Evans [18].

A.1 Functional spaces and basic results

Given a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$. We define the derivative operator of order $|\alpha|$, by

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}} \quad if \quad \alpha \neq (0, 0, \ldots, 0).$$

with $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$.

Remark A.1 *If $\alpha = (0, 0, \ldots, 0)$ we define $D^\alpha u = u$ for all function u. When the multi-index is $\alpha = (0, \ldots, 0, i, 0, \ldots, 0) \in \mathbb{N}^n$ the derivative operator can to be represented by derivative partial $D^i = \frac{\partial}{\partial x_i}$, $i = 1, 2, \ldots, n$.*

For $k = 1, 2, \ldots$, denote by $C^k(\Omega)$ the Banach space of all the functions $u : \Omega \to \mathbb{R}$ k-times differentiable, equipped with the norm

$$\|u\|_{C^k(\Omega)} = \sum_{|\alpha|=0}^{k} \sup_{x \in \Omega} |D^\alpha u(x)|.$$

In particular $C^0(\Omega)$ is the space of the functions continuous on Ω and $C^\infty(\Omega)$ is the space of the functions infinitely differentiable. Also we denote by $C_0^\infty(\Omega)$ the subspace of $C^\infty(\Omega)$ which is constituted of all functions with support compact on Ω, that is, $\text{supp}(u)$ is a compact subset of Ω.

Definition A.2 *Let Ω be an open set of $\mathbb{R}^n$. A sequence $(\varphi_\nu)_{\nu\in\mathbb{N}}$ in $C_0^\infty(\Omega)$* **converge** *for φ in $C_0^\infty(\Omega)$, when the following conditions are satisfied*

(i) There exists a compact $K \subset \Omega$ such that $\text{supp}(\varphi_\nu) \subset K$, $\forall \nu \in \mathbb{N}$.

(ii) For all multi-index α, $D^\alpha \varphi_\nu \longrightarrow D^\alpha \varphi$ uniformly in K.

The vectorial space $C_0^\infty(\Omega)$ equipped of notion of convergence above is denoted by $\mathscr{D}(\Omega)$ and is called of **space of the test functions**.

Let Ω be an open and bounded set of $\mathbb{R}^n$ we denote by $L^p(\Omega)$, $1 \leqslant p < \infty$ the Banach space of (classes of equivalence) measurable functions u in Ω such that $|u|^p$ is an integrable function on Ω, that is,

$$L^p(\Omega) = \left\{ u : \Omega \to \mathbb{R};\ u \text{ is mensurable and } \int_\Omega |u(x)|^p dx < \infty \right\}$$

equipped with the norm

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p dx \right)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty.$$

For $p = \infty$ we denote by $L^\infty(\Omega)$ the Banach space of (classes of equivalence) mensurable functions u in Ω and that are essentialy bounded in Ω, that is,

$$L^\infty(\Omega) = \left\{ u : \Omega \to \mathbb{R};\ u \text{ is mensurable and } ess \sup_{x\in\Omega} |u(x)| < \infty \right\}$$

equipped with the norm

$$\|u\|_{L^\infty(\Omega)} = ess \sup_{x\in\Omega} |u(x)|.$$

In particular the space $L^2(\Omega)$ equipped with norm $\|u\|^2_{L^2(\Omega)} = (u, v)_{L^2(\Omega)}$ where,

$$(u, v)_{L^2(\Omega)} = \int_\Omega u(x)v(x)dx$$

is usual inner product of $L^2(\Omega)$, is a Hilbert space.

We also define by $L^p_{loc}(\Omega)$ as the space of the mensurable functions u in Ω such that $|u|^p$ is a locally integrable function on Ω, that is, there exists $K \subset \Omega$ such that

$$\int_K |u(x)|^p dx < \infty.$$

Definition A.3 *A* **distribution** *is a linear functional $T : \mathscr{D}(\Omega) \to \mathbb{R}$ such that T is continuous, that is, if φ_ν converge for φ in $\mathscr{D}(\Omega)$ then $\langle T, \varphi_\nu \rangle$ converge for $\langle T, \varphi \rangle$ in $\mathbb{R}$.*

We denote by $\mathscr{D}'(\Omega)$ or $\mathscr{L}(\mathscr{D}(\Omega), \mathbb{R})$ the space of all the distributions on Ω.

In what follows we have an example of distribution.

Example A.4 *Let $u \in L^1_{loc}(\Omega)$ the functional $T_u : \mathscr{D}(\Omega) \to \mathbb{R}$, defined by*

$$\langle T_u, \varphi \rangle = \int_\Omega u(x)\varphi(x)dx,$$

is a distribuiton.

Theorem A.5 (Du Bois Reymond) *Let $u \in L^1_{loc}(\Omega)$. Then $T_u = 0$ if, and only if, $u = 0$ a.e. in Ω.*

Proof. See Medeiros and Milla Miranda [30, Proposition 1.4, p. 11] ∎

Remark A.6 *From Du Bois Raymond's Lemma it follows that if $u, v \in L^1_{loc}(\Omega)$ then $T_u = T_v$ in $\mathscr{D}'(\Omega)$, if and only if, $u = v$ a.e. in Ω. For this reason, u is identified with the distribution T_u.*

Definition A.7 *Let T be a distribution on Ω and $\alpha \in \mathbb{N}^n$ a multi-index. The* **derivative** *of order $|\alpha|$ of T is the functional $D^\alpha T : \mathscr{D}(\Omega) \to \mathbb{R}$ defined by*

$$\langle D^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \varphi \rangle.$$

Moreover $D^\alpha T$ is a distribution on Ω called **derivative in the sense of distribuitions**.

Remark A.8 *It follows from the definition that a distribution has derivative of all orders.*

Let Ω be an open and bounded set of $\mathbb{R}^n$. If $u \in L^p(\Omega)$ with $1 \leqslant p \leqslant \infty$, from definition of derivative distributional, we know that u has derivatives of all orders in the sense of distributions, but it is not true in general that $D^\alpha u$ is defined by a function of $L^p(\Omega)$. When $D^\alpha u$ is defined by a function of $L^p(\Omega)$, we can define the Sobolev space.

Give an integer number $m > 0$, we represent by $W^{m,p}(\Omega)$, $1 \leqslant p \leqslant \infty$, the vector space of all the functions u belongs to $L^p(\Omega)$, such that for all multi-index $|\alpha| \leqslant m$, the derivative of u in the sense of distributions $D^\alpha u$ belongs to $L^p(\Omega)$, that is,

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega);\ D^\alpha u \in L^p(\Omega),\ 0 \leqslant |\alpha| \leqslant m\}.$$

For each $u \in W^{m,p}(\Omega)$, we define the norm

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha| \leqslant m} \int_\Omega |D^\alpha u(x)|^p dx \right)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty,$$

and

$$\|u\|_{W^{m,\infty}(\Omega)} = \sum_{|\alpha| \leqslant m} ess \sup_{x \in \Omega} |D^\alpha u(x)|.$$

With this norm it follows that the **Sobolev space** $W^{m,p}(\Omega)$ is a Banach space. In particular, for $p = 2$, we denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ and $W^{0,p}(\Omega) = L^p(\Omega)$.

Remark A.9 *The space $H^m(\Omega)$ equipped with inner product*

$$((u, v))_{H^m(\Omega)} = \sum_{|\alpha| \leqslant m} (D^\alpha u, D^\alpha v)_{L^2(\Omega)}$$

is a Hilbert space.

We also define the space $H_0^m(\Omega)$ as being the closure of $\mathscr{D}(\Omega)$ in $H^m(\Omega)$ and by $H^{-1}(\Omega)$ the dual topological of $H_0^m(\Omega)$.

Next we list some classic results of the Sobolev spaces theory.

Theorem A.10 *The Banach space $W^{1,p}(\Omega)$ is reflexive for $1 < p < \infty$, and it is separable for $1 \leqslant p < \infty$ and $H^1(\Omega)$ is a separable Hilbert space.*

Proof. See Brezis [11, Proposition 9.1, p. 264] ■

Theorem A.11 (Green's formula) *Let Ω be an open and bounded set of $\mathbb{R}^n$ with boundary Γ smooth. If $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$ then*

$$\int_\Omega v \Delta u dx + \int_\Omega \nabla u \nabla v dx = \int_\Gamma \frac{\partial u}{\partial \vec{n}} v dS$$

where $\vec{n}$ denotes the outward normal vector on Γ and $\dfrac{\partial u}{\partial \vec{n}} = \nabla u \cdot \vec{n}$ the derivative normal of u.

Proof. See Evans [18, Appendix C, p. 628] ■

Theorem A.12 (Poincare's inequality) *Let Ω be a bounded, open subset of $\mathbb{R}^n$. Suppose $u \in W_0^{1,p}(\Omega)$ for some $1 \leqslant p < n$. Then there exists a constant $C = C(p, q, n, \Omega) > 0$ such that*

$$\|u\|_{L^q(\Omega)} \leqslant C \|\nabla u\|_{L^p(\Omega)},$$

for each $q \in [1, p^*]$, where $p^* = \dfrac{np}{n-p}$.

Proof. See Evans [18, Theorem 3, p. 265] ■

Theorem A.13 (Sobolev embedding) *Let Ω be an open and bounded set of $\mathbb{R}^n$ with boundary $\partial\Omega$ of class C^m. Then the following embedding are holds*

(i) $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega), \quad 1 \leqslant q \leqslant q^* = \dfrac{np}{n-mp} \quad$ *if* $\quad mp < n$

(ii) $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega), \quad 1 \leqslant q < \infty \quad$ *if* $\quad mp = n$

(iii) $W^{m,p}(\Omega) \hookrightarrow C^{k,\lambda}(\overline{\Omega}),\ k < m - \dfrac{n}{p} \leqslant k+1$ *if* $mp > n$, *where* k *is an integer non-negative and* λ *a real satisfying* $0 < \lambda \leqslant m - k - \dfrac{n}{p} = \lambda_0$ *if* $\lambda_0 < 1$ *and* $0 < \lambda < 1$ *if* $\lambda_0 = 1$.

Proof. See Medeiros and Milla Miranda [30, Theorem 2.15, p. 74] ■

Theorem A.14 (Rellich-Kondrachov) *Let Ω be an open and bounded set of $\mathbb{R}^n$ with boundary $\partial\Omega$ of class C^m. Then the following embedding are holds*

(i) $W^{m,p}(\Omega) \overset{c}{\hookrightarrow} L^q(\Omega), \quad 1 \leqslant q < q^* = \dfrac{np}{n-mp} \quad$ *if* $\quad mp < n$

(ii) $W^{m,p}(\Omega) \overset{c}{\hookrightarrow} L^q(\Omega), \quad 1 \leqslant q < \infty \quad$ *if* $\quad mp = n$

(iii) $W^{m,p}(\Omega) \overset{c}{\hookrightarrow} C^{k}(\overline{\Omega}), \quad k < m - \dfrac{n}{p} \leqslant k+1 \quad$ *if* $\quad mp > n$, *where* k *is an integer non-negative.*

Proof. See Medeiros and Milla Miranda [30, Theorem 2.20, p. 83] ■

Now let us consider the open interval $(0,T)$, of the real line $\mathbb{R}$ and a real Banach space X equipped with norm $\|\cdot\|_X$. We represent by $C([0,T],X)$ the Banach space of the applications u defined in $(0,T)$ with values in X, whose norm is given by

$$\|u\|_\infty = \sup_{t\in[0,T]} \|u(t)\|_X.$$

For $1 \leqslant p < \infty$ we denote by $L^p(0,T;X)$ the vectorial space of the applications $u:(0,T)\to X$ such that, for each $t\in(0,T)$, the vector $u(t)\in X$ is measurable on $(0,T)$ and $\|u(t)\|_X$ belongs to $L^p(0,T)$, that is,

$$L^p(0,T;X) = \left\{ u:(0,T)\to X \text{ mensurable; } \int_0^T \|u(t)\|_X^p dt < \infty \right\}, \quad 1 \leqslant p < \infty$$

and

$$L^\infty(0,T;X) = \left\{ u:(0,T)\to X \text{ mensurable; } ess \sup_{t\in(0,T)} \|u(t)\|_X < \infty \right\}.$$

In $L^p(0,T;X)$ we define the norm

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p dt \right)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty$$

and

$$\|u\|_{L^\infty(0,T;X)} = ess \sup_{t \in (0,T)} \|u(t)\|_X.$$

With this norm it follows that $L^p(0,T;X)$ is a Banach space. In particular, if X is a Hilbert space then $L^2(0,T;X)$ is a Hilbert space equipped with inner product

$$(u,v)_{L^2(0,T;X)} = \int_0^T (u(t),v(t))_X dt.$$

We also define the following space

$$L^p_{loc}(0,T;X) = \{u : (0,T) \to X \quad \text{mensurable}; \|u(s)\|_X \in L^p(I),\ \forall I \subset (0,T)\},$$

where I is a compact set of $\mathbb{R}$.

Finally, we denote by $H_0^1(0,T;X)$ the Hilbert space

$$H_0^1(0,T;X) = \{u \in L^2(0,T;X); u' \in L^2(0,T;X), u(0) = u(T) = 0\},$$

equipped with inner product

$$((u,v))_{H_0^1(0,T;X)} = \int_0^T (u(t),v(t))_X dt + \int_0^T (u'(t),v'(t))_X dt.$$

If X is a reflexive and separable space, then $L^p(0,T;X)$ is a reflexive and separable space, for $1 < p < \infty$, whose topological dual is identified to space $L^{p'}(0,T;X')$, where p and p' are conjugate exponentes, that is, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. More precisely, we show that for each $u \in L^p(0,T;X)$, there exists $\tilde{u} \in L^{p'}(0,T;X')$ such that

$$\langle u, \varphi \rangle_{(L^p(0,T;X))' \times L^p(0,T;X)} = \int_0^T \langle \tilde{u}(t), \varphi(t) \rangle_{X' \times X} dt$$

In particular for $p = 1$, we identify $[L^1(0,T;X)]' = L^\infty(0,T;X')$.

Remark A.15 *If Ω is an open and bounded set of $\mathbb{R}^n$, $T > 0$ and $Q = \Omega \times (0,T)$ the cylinder in $\mathbb{R}^{n+1}$ then*

$$L^p(0,T;L^p(\Omega)) = L^p(Q), \quad 1 \leqslant p < \infty.$$

In general the space $\mathscr{L}(\mathscr{D}(0,T),X)$ is called the space of vector distributions on $(0,T)$ with value in X and is denoted by $\mathscr{D}'(0,T;X)$.

Identifying $L^2(0,T;X)$ with its dual $(L^2(0,T;X))'$ we obtain the following embedding

$$\mathscr{D}(0,T;X) \hookrightarrow H_0^1(0,T;X) \hookrightarrow L^2(0,T;X) \hookrightarrow H^{-1}(0,T;X) \hookrightarrow \mathscr{D}'(0,T;X),$$

where $H^{-1}(0,T;X) = (H_0^1(0,T;X))'$.

Proposition A.16 *Let $u \in L^2(0,T;X)$. There exists an unique $f \in H^{-1}(0,T;X)$ such that*

$$\langle f, \varphi\xi\rangle = (\langle u', \varphi\rangle, \xi), \quad \forall \varphi \in \mathscr{D}(0,T),\ \forall \xi \in X.$$

Proof. See Milla Miranda [31, Proposition 1, p. 175]. ■

From above proposition we can identify u' with f. Therefore, if $u \in L^2(0,T;X)$ then $u' \in H^{-1}(0,T;X)$.

Corollary A.17 *The map*

$$u \in L^2(0,T;X) \longmapsto u' \in H^{-1}(0,T;X)$$

is linear and continuous.

Proof. See Milla Miranda [31, Corollary 1, p. 176] ■

Now we will see the concept of vector distribution and some of its properties. We denote by $C_0^\infty(0,T)$ space of infinitely differentiable functions on $(0,T)$, with compact support on $(0,T)$.

Definition A.18 *We say that a sequence $(\varphi_\nu)_{\nu\in\mathbb{N}}$* **converge** *for φ in $C_0^\infty(0,T)$, when the following conditions are satisfied*

(i) *There exists a compact K of $(0,T)$ such that* $\operatorname{supp}(\varphi_\nu) \subset K,\ \forall \nu \in \mathbb{N}$.

(ii) *The sequence $(\varphi_\nu)_{\nu\in\mathbb{N}}$ converge for φ uniformly in K, together with its derivative of all orders.*

The vectorial space $C_0^\infty(0,T)$, equipped of the notion of convergence above will be represented by $\mathscr{D}(0,T)$.

Let $u \in L^p(0,T;X)$ and $\varphi \in \mathscr{D}(0,T)$, we define the map $T_u : \mathscr{D}(0,T) \to X$ by

$$\langle T_u, \varphi\rangle = \int_0^T u(t)\varphi(t)dt$$

with the integral calculated in X. The mapping T_u, above defined, is linear and continuous on $\mathscr{D}(0,T)$. In this case we say that T_u is a distribution on $(0,T)$ with value in X, called **vector distribution**, defined by u of $L^p(0,T;X)$. Then $T_u \in \mathscr{D}'(0,T;X)$.

Lemma A.19 *If $u \in L^1(0,T;X)$ and*

$$\int_0^T u(t)\varphi(t)dt = 0$$

for all $\varphi \in D(0,T)$, then $u(t) = 0$ a.e. in $(0,T)$.

Proof. See Medeiros [29, Lema 1. p. 4]. ■

From Lemma A.19, it follows that T_u is unically defined by $u \in L^p(0,T;X)$. Then we can identify the vector $u \in L^p(0,T;X)$, with $T_u \in \mathscr{D}'(0,T;X)$ and we say that u is a distribution defined on $(0,T)$ with values on X. We write $L^p(0,T;X) \subset \mathscr{D}'(0,T;X)$. Thus each $u \in L^p(0,T;X)$ is derivable in the sense of distributions, that is,

$$\left\langle \frac{du}{dt}, \varphi \right\rangle = -\left\langle u, \frac{d\varphi}{dt} \right\rangle, \quad \forall\, \varphi \in \mathscr{D}(0,T)$$

In general, we have

$$\left\langle \frac{d^n u}{dt^n}, \varphi \right\rangle = (-1)^n \left\langle u, \frac{d^n\varphi}{dt^n} \right\rangle, \quad \forall\, \varphi \in \mathscr{D}(0,T)$$

Next we list some results that are used in the proof of the result in the chapter 1.

Theorem A.20 *Let X, Y be a Hilbert spaces such that $X \hookrightarrow Y$. If $u \in L^p(0,T;X)$ and $u_t \in L^p(0,T;Y)$, $1 \leqslant p < \infty$, then $u \in C^0([0,T];Y)$.*

Proof. See Medeiros [29, Corollary 1, p. 9]. ■

Lemma A.21 (Lions Lemma) *Let $\mathscr{O}$ be an open and connected set $\mathbb{R}^{n+1}$ and $g_m, g \in L^q(\mathscr{O})$, $1 < q < \infty$ such that*

$$\|g_m\|_{L^q(\mathscr{O})} \leqslant C \quad \text{and} \quad g_m \to g \quad \text{a.e.} \quad \text{in} \quad \mathscr{O}.$$

Then $g_m \rightharpoonup g$ in $L^q(\mathscr{O})$.

Proof. See Lions [26, Lemma 1.3, p. 12]. ■

Theorem A.22 (Aubin-Lions Theorem) *Let X_0, X, X_1 be a Banach spaces such that $X_0 \overset{c}{\hookrightarrow} X \hookrightarrow X_1$ with X_0 and X_1 reflexives. Moreover for any p_0, p_1 with $1 \leqslant p_0, p_1 < \infty$ we consider the space*

$$W = \{u \mid u \in L^{p_0}(0,T,X_0),\ u' \in L^{p_1}(0,T,X_1)\},$$

endowed with the norm $\|u\|_W = \|u\|_{L^{p_0}(0,T,X_0)} + \|u'\|_{L^{p_1}(0,T,X_1)}$. Then

$$W \overset{c}{\hookrightarrow} L^{p_0}(0,T,X).$$

Proof. See Lions [26, Theorem 5.1, p. 58]. ■

As consequence of the above result we have, if $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence in $L^2(0,T,X_0)$ and $(u_k')_{k\in\mathbb{N}}$ is a bounded sequence in $L^2(0,T,X_1)$, then $(u_k)_{k\in\mathbb{N}}$ is bounded in W. It follows that, there exists a subsequence of $(u_k)_{k\in\mathbb{N}}$ still denoted by $(u_k)_{k\in\mathbb{N}}$, such that $u_k \to u$ in $L^2(0,T,X)$.

Theorem A.23 (Compactness weak) *Let X be a reflexive Banach space. If B is a bounded subset of X, then B is compact in weak topology $\sigma(X,X')$, that is, for any sequence $\{x_n\}$ bounded in X there exists a subsequence $\{x_{n_k}\}$ convergent in X in weak topology $\sigma(X,X')$.*

Proof. See Brezis [11, Theorem 3.18, p. 69] ■

Theorem A.24 (Compactness weak star) *Let X be a separable Banach space. If F is a bounded subset of X', then F is compact in weak star topology $\sigma(X',X)$, that is, for any sequence $\{f_n\}$ bounded in X' there exists a subsequence $\{f_{n_k}\}$ convergent in X' in the weak star topology $\sigma(X',X)$.*

Proof. See Brezis [11, Corollary 3.30, p. 76] ■

Theorem A.25 (Banach Fixed Point Theorem) *Let X be a non-empty complete metric space and let $S : X \to X$ be a strict contraction; that is,*

$$d(Sv_1, Sv_2) \leqslant kd(v_1,v_2), \quad \forall v_1, v_2 \in X \quad \text{with} \quad 0 < k < 1.$$

Then S has an unique fixed point, $u = Su$.

Proof. See Brezis [11, Theorem 5.7, p. 138] ■

Theorem A.26 (Schauder Fixed Point Theorem) *Let X be a Banach space and suppose $K \subset E$ is compact and convex, and assume also $T : K \to K$ is continuous. Then T has a fixed point in K.*

Proof. See Evans [18, Theorem 3, p. 502] ■

A.2 Essential results

In this section we will present the Carathéodory theorem which will be used to ensure the existence of solution to a Cauchy problem in the interval $[0, t_m]$ for every $m \in \mathbb{N}$.

We consider the following Cauchy problem

$$\begin{cases} \dfrac{dY}{dt} = f(t, Y(t)), t > t_0, \\ Y(t_0) = Y_0. \end{cases} \tag{A.1}$$

In the case that f is a mensurable function we ensure that there is a solution to (A.1) through of the Carathéodory theorem.

Definition A.1 *We say that the function $f : [0, T] \times \Omega \to \mathbb{R}^n$ satisfies the conditions of Carathéodory on $Q = [0, T] \times \Omega$ if:*

(i) $f(t, x)$ is mensurable in t for each x fixed;

(ii) $f(t, x)$ is continuous in x for each t fixed;

(iii) For each $K \subset \Omega$ compact set, there exists an integrable real function $m_K(t)$, such that

$$\|f(t, x)\|_{\mathbb{R}^n} \leqslant m_K(t), \quad \textit{for all} \quad (t, x) \in K.$$

Theorem A.2 (Caratheodory Theorem) *Suppose that $f : [0, T] \times \Omega \to \mathbb{R}^n$ satisfies the conditions of Carathéodory on $Q = [0, T] \times \Omega$. Then there exists a solution $Y(t)$ of* (A.1) *on some interval $|t - t_0| \leqslant \beta$, where β is a positive constant.*

Proof. See Coddington-Levinson [16] ■

Theorem A.3 (Prolongation Theorem) *Let $\Omega = [0, T] \times B$ with $T > 0$ and $B = \{x \in \mathbb{R}^n; \|x\|_{\mathbb{R}^n} \leqslant b\}$, where b is a positive constant and $\|\cdot\|_{\mathbb{R}^n}$ the norm euclidian of the $\mathbb{R}^n$. Suppose that f is a function that satisfies* (i), (ii) *and that there exists a function $m \in L^1(0, T)$ such that*

$$|f(t, x)| \leqslant m(t), \quad \textit{for all} \quad (t, x) \in \Omega.$$

Let $Y(t)$ a solution of (A.1) *and suppose that $Y(t)$ is defined in I, satisfying $|Y(t)| \leqslant M$ with M independent of I and $M < b$ for all $t \in I$. Then $Y(t)$ can to be prolonged in all interval $[0, T]$.*

Proof. See Coddington-Levinson [16] ■

Now we present inequalities frequently used in our work.

Lemma A.4 (Gronwall's inequality, integral form) *Let $X \in L^1(0,T;\mathbb{R}^+)$ satisfy*

$$X(t) \leqslant a(t) + \int_0^t b(s)X(s)ds, \ \ a.e.\ t \in (0,T)$$

where $a, b \in L^\infty(0,T)$ and $a(\cdot)$ is increasing. Then,

$$X(t) \leqslant a(t)e^{\int_0^t b(s)ds}, \quad \text{for all } t \in [0,\infty].$$

Proof. See Carvalho, Langa and Robinson [14, Lemma 6.23, p. 167]. ■

Lemma A.5 (Gronwall's inequality, differential form) *Let $J(\cdot)$ be a non-negative, absolutely continuous function on $[0,T]$, which satisfies for a.e. t the differential inequality*

$$J'(t) \leqslant \alpha(t)J(t) + \beta(t),$$

where $\alpha(t), \beta(t)$ are non-negative, integrable functions in $[0,T]$. Then,

$$J(t) \leqslant J(0)e^{\int_0^t \alpha(s)ds} + \int_0^t \beta(\tau)e^{\int_\tau^t \alpha(s)ds}d\tau, \quad \text{for all } t \in [0,T].$$

Proof. See Evans [18, Appendix B, p. 624]. ■

Lemma A.6 (Young's inequality) *Let $1 < p, q < \infty$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then,*

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}, \ \forall\ a, b \geqslant 0.$$

Proof. See Evans [18, Appendix B, p. 622]. ■

Theorem A.7 (Hölder's inequality) *Let $\Omega \subset \mathbb{R}^n$ be a bounded open and $1 \leqslant p, q < \infty$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$. If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$ then $uv \in L^1(\Omega)$ and*

$$\|uv\|_{L^1(\Omega)} \leqslant \|u\|_{L^p(\Omega)}\|v\|_{L^q(\Omega)}.$$

Proof. See Evans [18, Appendix B, p. 622]. ■

Theorem A.8 (Minkowski's inequality) *If $u, v \in L^p(\Omega)$ with $1 \leqslant p < \infty$ then $u + v \in L^p(\Omega)$ and*

$$\|u + v\|_{L^p(\Omega)} \leqslant \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}.$$

Proof. See Evans [18, Appendix B, p. 623]. ■

Theorem A.9 (Integration by parts formula) *Let $u, v \in C^1(\overline{\Omega})$. Then*

$$\int_\Omega u_{x_i} v dx + \int_\Omega u v_{x_i} = \int_{\partial\Omega} uv\nu^i dS, \quad (i = 1, \ldots, n).$$

Proof. See Evans [18, Appendix C, p. 628]. ■

Appendix B

Linear semigroups

In this chapter we recall a few from theory semigroup of bounded linear operators but with the main objective of presenting the theory of strongly continuous semigroups and analytic semigroups. We present definitions and results of this theory that we use throughout this work. The proof of the results we do not make here, for more details we recommender Carvalho, Langa and Robinson [14], Henry [20] and Pazy [38].

B.1 Definitions and basic concepts

In what follows let X and Y be Banach space over a field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) and we denote by $\mathscr{L}(X, Y)$ the space of bounded linear operators from X into Y with the usual norm, that is, for $T \in \mathscr{L}(X, Y)$,

$$\|T\|_{\mathscr{L}(X,Y)} = \sup_{x \in X,\ x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X}.$$

If $X = Y$ we write $\mathscr{L}(X)$ to denote $\mathscr{L}(X, Y)$. Let X' be the topological dual of X, that is, $X' = \mathscr{L}(X, \mathbb{K})$ with the norm defined above.

Definition B.1 *A* **semigroup strongly continuous** *(or a* C_0-**semigroup***) of bounded linear operators is a family of maps* $\{S(t) : t \geqslant 0\} \subset \mathscr{L}(X)$ *such that*

(i) $S(0) = I_X$*;*

(ii) $S(t + s) = S(t)S(s)$*, for any* $t, s \geqslant 0$*;*

(iii) $\lim_{t \to 0^+} \|S(t)x - x\|_X = 0$ *or* ($\lim_{t \to 0^+} S(t)x = x$) *for all* $x \in X$*.*

In general in the space of operators the composition of operators does not commute, however if $\{S(t) : t \geqslant 0\} \subset \mathscr{L}(X)$ is a semigroup we have

$$S(t)S(s) = S(s)S(t), \quad \text{for all} \quad t, s \geqslant 0.$$

The study of semigroups of linear operators is associated with the study of linear Cauchy problems of the form

$$\begin{cases} \dfrac{du(t)}{dt} + Au(t) = 0, \ t > 0, \\ u(0) = u_0. \end{cases} \tag{B.1}$$

where $-A : D(A) \subset X \to X$ is linear operator (in general unbounded). The semigroup $\{S(t) : t \geqslant 0\}$ is the solution operator associated to (B.1); that is, for each $u_0 \in X$, the function $[0, \infty) \ni t \mapsto S(t)u_0 \in X$ is the solution (in some sense) of (B.1).

On the other hand given any semigroup of linear operators we can associate it to a differential equation through the following definition.

Definition B.2 *Let $\{S(t) : t \geqslant 0\} \subset \mathscr{L}(X)$ be a C_0-semigroup its infinitesimal generator is the linear operator defined by $A : D(A) \subset X \to X$, where*

$$D(A) = \left\{ x \in X : \lim_{t \to 0^+} \frac{S(t)x - x}{t} \quad \text{exists} \right\}$$

and

$$Ax = \lim_{t \to 0^+} \frac{S(t)x - x}{t}, \quad \text{for all} \quad x \in D(A).$$

The next result show that all C_0-semigroup of bounded linear operator has an exponential bound.

Theorem B.3 *Let $\{S(t) : t \geqslant 0\} \subset \mathscr{L}(X)$ be a C_0-semigroup. There exists constants $M \geqslant 1$ and $\beta \in \mathbb{R}$ such that*

$$\|S(t)\|_{\mathscr{L}(X)} \leqslant M e^{\beta t}, \quad \forall t \geqslant 0.$$

Proof. See Pazy [38, Theorem 2.2, p. 4]. ∎

In above theorem if $\beta < 0$ we tell that the semigroup has decay exponential or is exponentially stable. If $\beta = 0$, that is, $\|S(t)\|_{\mathscr{L}(X)} \leqslant M$ the semigroup is uniformly bounded, moreover if $M = 1$ it is called a C_0-semigroup of contractions.

Now we present some properties of the strongly continuous semigroup which will be the main point in the applications in this work.

Theorem B.4 *Let* $\{S(t) : t \geqslant 0\}$ *be a* C_0*-semigroup and* A *its infinitesimal generator. The following statements are holds.*

(i) For $x \in X$

$$[0, \infty) \ni t \mapsto S(t)x \in X$$

is a continuous map;

(ii) The map

$$[0, \infty) \ni t \mapsto \|S(t)\|_{\mathscr{L}(X)}$$

is lower semicontinuous, and therefore mensurable;

(iii) The operator A *is closed and densely defined. For each* $x \in D(A)$, $S(t)x \in D(A)$ *for all* $t \geqslant 0$, *the map*

$$(0, \infty) \ni t \mapsto S(t)x \in X$$

is continuously differentiable and

$$\frac{d^+}{dt}S(t)x = AS(t)x = S(t)Ax, \quad \forall t > 0;$$

(iv) We have that $\bigcap_{m=1}^{\infty} D(A^m)$ *is dense subspace of* X*;*

(v) (Representation of the resolvent operators of A *through of Laplace transform of the semigroup) If* $\lambda \in \mathbb{C}$ *is such that Re* $\lambda > \beta$, *where* β *is given by Theorem B.3, then* $\lambda \in \rho(A)$ *and*

$$(\lambda - A)^{-1}x = \int_0^{\infty} e^{-\lambda t} S(t)x dt, \quad \text{for all} \quad x \in X.$$

Proof. See Pazy [38, Theorem 2.4, Corollary 2.5 and Theorem 2.7]. ■

Theorem B.5 *Let* $\{S(t) : t \geqslant 0\} \subset \mathscr{L}(X)$ *and* $\{T(t) : t \geqslant 0\} \subset \mathscr{L}(X)$ *be a* C_0*-semigroup with infinitesimal generator* A *and* B *respectively. If* $A = B$ *then* $S(t) = T(t), \quad t \geqslant 0$.

Proof. See Pazy [38, Theorem 2.6, p. 6]. ■

We define the **resolvent set** of a closed linear operator $A : D(A) \subset X \to X$ as

$$\rho(A) = \{\lambda \in \mathbb{C} : \lambda - A \quad \text{is continuous, injective and surjective}\}.$$

The set $\sigma(A) = \mathbb{C} \setminus \rho(A)$ is called **spectral set** or **spectrum** of A.

It is easy see by closed graph theorem that, if $\lambda - A$ is continuous injective and surjective then $(\lambda - A)^{-1} \subset \mathscr{L}(X)$, which is called **resolvent operator** associated with A.

Remark B.6 *The resolvent set* $\rho(A)$ *is an open set; that is, the spectrum* $\sigma(A) = \mathbb{C} \setminus \rho(A)$ *is a closed set.*

Remark B.7 *We consider the Cauchy problem* (B.1) *such that it is known that* $-A$ *is the infinitesimal generator of a* C_0*-semigroup* $\{S(t) : t \geqslant 0\} \subset \mathscr{L}(X)$*, a direct consequence of Theorem B.4 is the fact that* $u : [0,\infty) \to X$ *given by*

$$u(t, u_0) = S(t)u_0, \quad t \geqslant 0$$

is a unique solution of (B.1) *(in some sense) such that*

$$u(\cdot, u_0) = S(\cdot)u_0 \in C([0,\infty); X) \cap C^1([0,\infty); D(A)).$$

Now we will dedicate the characterization of the infinitesimal generator of a C_0-semigroup. We can characterize an infinitesimal generator of a C_0-semigroup through the theorems of Hille-Yosida and Lumer-Phillips.

Theorem B.8 *(Hille-Yosida) Let* $A : D(A) \subset X \to X$ *a linear operator. Then the following statements are equivalent*

(i) A *is the infinitesimal generator of a* C_0*-semigroup* $\{S(t) : t \geqslant 0\} \subset \mathscr{L}(X)$ *such that*

$$\|S(t)\|_{\mathscr{L}(X)} \leqslant e^{\beta t}, \quad \text{for all} \quad t \geqslant 0.$$

(ii) A *is a closed, densely defined linear operator such that* $\rho(A) \supset (\beta, \infty)$ *and*

$$\|(\lambda - A)^{-1}\|_{\mathscr{L}(X)} \leqslant \frac{1}{\lambda - \beta}, \quad \text{for all} \quad \lambda > \beta.$$

Proof. See Pazy [38, Theorem 3.1 and Corollary 3.8, p. 8 and p.12]. ■

Let X^* be the dual space of the Banach space X. We denote the value of $x^* \in X^*$ at $x \in X$ by $\langle x^*, x\rangle$ or $\langle x, x^*\rangle$.

Definition B.9 *For every* $x \in X$ *we define the map duality* $J : X \to 2^{X^*}$ *by*

$$J(x) = \{x^* \in X^* : Re\langle x, x^*\rangle = \|x\|_X^2, \ \|x^*\|_{X^*} = \|x\|_X\}.$$

Definition B.10 *A linear operator* $A : D(A) \subset X \to X$ *is* **dissipative** *if for every* $x \in D(A)$ *there exists* $x^* \in J(x)$ *tal que* $Re\langle Ax, x^*\rangle \leqslant 0$.

The following result give a characterization of dissipative operators.

Theorem B.11 *A linear operator* A *is dissipative if and only if*

$$\|(\lambda - A)x\| \geqslant \lambda\|x\| \quad \forall x \in D(A) \quad \text{and} \quad \lambda > 0.$$

Proof. See Pazy [38, Theorem 4.2, p. 14]. ■

Theorem B.12 *(Lumer-Phillips) Let $A : D(A) \subset X \to X$ be a densely defined linear operator. Then*

(i) If A is the infinitesimal generator of a C_0-semigroup of contractions on X then A is dissipative and $R(\lambda - A) = X$ for all $\lambda > 0$.

(ii) If A is dissipative and $R(\lambda_0 - A) = X$ for some $\lambda_0 > 0$, then A is the infinitesimal generator of a C_0-semigroup of contractions on X.

Proof. See Pazy [38, Theorem 4.3, p. 14]. ■

A direct consequence of the above theorem, and that is used in the applications is given by corollary below.

Corollary B.13 *Let A be a linear operator with dense domain $D(A)$ in a Hilbert space H. If A is dissipative and $0 \in \rho(A)$, then A is the infinitesimal generator of a C_0-semigoup of contractions on H.*

Proof. See Liu and Zheng [27, Theorem 1.2.4, p. 3]. ■

Definition B.14 *Let $A : D(A) \subset X \to X$ be a linear operator with $\overline{D(A)} = X$. The operator $A^* : D(A^*) \subset X^* \to X^*$ defined by*

$$D(A^*) = \{x^* \in X^* : \exists\, y^* \in X^* \text{ with } \langle x^*, Ax\rangle = \langle y^*, x\rangle,\ \forall x \in D(A)\}$$

and

$$A^*x^* = y^*, \quad \forall x^* \in D(A^*),$$

is called the **adjoint operator** *of A.*

The fact $\overline{D(A)} = X$ ensures that there is unique $y^* \in X^*$ with the property above for some $x^* \in X^*$, that is, $D(A^*) \neq \emptyset$.

Remark B.15 *When X is Hilbert space and we identified its topological dual X^* we have the following*

(i) If $\langle Ax, y\rangle = \langle x, Ay\rangle$, for all $x, y \in D(A)$ holds, we tell that A is symmetric and we denote by $A \subset A^*$;

(ii) If $A = A^*$ we tell that A is self-adjoint;

(iii) If $A = -A^*$ we tell that A is skew-adjoint.

Corollary B.16 *Let A be a closed and densely defined linear operator. If both A and A^* are dissipative, then A is the infinitesimal generator of a C_0-semigroup of contractions on X.*

Proof. See Pazy [38, Corollary 4.4, p. 15]. ■

B.2 Sectorial operators and analytic semigroups

In this section we will define the sectorial operators and we present an important class of C_0-semigroups which are the analytic semigroups and we present a result that show that the semigroup generated by this kind of operator is analytic semigroups.

Definition B.1 *We say that the closed densely defined linear operator $-A : D(A) \subset X \to X$ is* **sectorial** *if, for some $a \in \mathbb{R}$ and $\varphi \in (\frac{\pi}{2}, \pi)$,*

$$\Sigma_{\varphi,a} = \{\lambda \in \mathbb{C} : |\arg(\lambda - a)| \leqslant \varphi; \lambda \neq a\} \subset \rho(A)$$

and, for some $M > 0$,

$$\|(\lambda - A)^{-1}\|_{\mathscr{L}(X)} \leqslant \frac{M}{|\lambda - a|}, \quad \text{for all} \quad \lambda \in \Sigma_{a,\varphi}.$$

Now we consider the sector

$$\Delta = \{z \in \mathbb{C} : \phi_1 < \arg z < \phi_2 \quad \text{with} \quad \phi_1 < 0 < \phi_2\}$$

and for each $z \in \Delta$, let $S(z)$ be a bounded linear operator.

Definition B.2 *We say that a C_0-semigroup $\{S(z) : z \in \Delta\} \subset \mathscr{L}(X)$ is an* **analytic semigroup** *on Δ, if $z \mapsto S(z)$ is analytic in Δ.*

Theorem B.3 *If $A : D(A) \subset X \to X$ is a sectorial operator, then $-A$ is the infinitesimal generator of an analytic semigroup $\{S(t) : t \geqslant 0\} \subset \mathscr{L}(X)$. Moreover,*

$$S(t) = e^{-At} = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t}(\lambda I + A)^{-1} d\lambda$$

where Γ is a contour in $\rho(-A)$ with $\arg \lambda \to +\theta$ as $|\lambda| \to \infty$ for some $\frac{\pi}{2} < \theta < \pi$.

Proof. See Henry [20, Theorem 1.3.4, p. 21]. ■

Remark B.4 *The converse is also true, that is, if $-A$ generates an analytic semigroup, then A is sectorial.*

Definition B.5 *Let A be a sectorial operator and Re $\sigma(A) > 0$; then for any $\alpha > 0$ we define*

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-At} dt.$$

The operator A^α is called fractional power operator associated with operator A. We define A^α as being the inverse of the $A^{-\alpha}$ for $\alpha > 0$, with $D(A^\alpha) = R(A^{-\alpha})$ and A^0 as being the identity in X.

If $A : D(A) \subset X \to X$ is a sectorial operator and $\alpha \geqslant 0$, we define the fractional power space X^α associated with A as being $X^\alpha = D(A_1^\alpha)$, equipped with norm of graph $\|x\|_\alpha = \|A_1^\alpha x\|_X$, $x \in X^\alpha$, where $A_1 = A + aI$ satisfies $Re\sigma(A_1) > 0$. In the case $Re\sigma(A) > 0$ then we can take $X^\alpha = D(A^\alpha)$.

Theorem B.6 *If A is a sectorial operator in X with Re $\sigma(A) > 0$, then for any $\alpha > 0$, $A^{-\alpha}$ is a bounded linear operator on X which is one-one and satisfies $A^{-\alpha}A^{-\beta} = A^{-(\alpha+\beta)}$ whenever $\alpha > 0$, $\beta > 0$. Also, for $0 < \alpha < 1$,*

$$A^{-\alpha} = \frac{sen(\pi\alpha)}{\pi} \int_0^\infty \lambda^{-\alpha} (\lambda + A)^{-1} d\lambda.$$

Proof. See Henry [20, Theorem 1.4.2, p. 25]. ■

Theorem B.7 *If $A : D(A) \subset X \to X$ is a sectorial operator with Re $\sigma(A) > 0$, then X^α is a Banach space with the norm $\|\cdot\|_\alpha$, for $\alpha \geqslant 0$, $X^0 = X$ and for $\alpha \geqslant \beta \geqslant 0$, we have $X^\alpha \hookrightarrow X^\beta$. Moreover, if A has resolvent compact, then the embedding $X^\alpha \hookrightarrow X^\beta$ is compact for $\alpha > \beta \geqslant 0$.*

Proof. See Henry [20, Theorem 1.4.8, p. 29]. ■

Theorem B.8 *Suppose that A is sectorial and Re $\sigma(A) > \delta > 0$. For all $\alpha \geqslant 0$ there exixts $C_\alpha < \infty$ such that*

$$\|A^\alpha e^{-At}\|_{\mathscr{L}(X)} \leqslant C_\alpha t^{-\alpha} e^{-\delta t} \quad for \quad t > 0.$$

Proof. See Henry [20, Theorem 1.4.3, p. 26]. ■

We consider the following Cauchy problem nonlinear

$$\begin{cases} \dfrac{du}{dt} + Au = f(t,u), \ t > t_0, \\ u(t_0) = u_0, \end{cases} \tag{B.2}$$

where $A : D(A) \subset X \to X$ is a **positive sectorial operator**, such that the fractional power A^α are well defined and the spaces $X^\alpha = D(A^\alpha)$ with the norma of graph $\|x\|_\alpha = \|A^\alpha x\|_X$ are defined for $\alpha \geqslant 0$ and $f : \mathbb{R} \times X^\alpha \to X$.

Here are some definitions of the type of solutions to the abstract problem given above.

Definition B.9 *A* **classic solution** *of Cauchy problem* (B.2) *on* $[t_0, t_1)$ *is a continuous function* $u : [t_0, t_1) \to X$*, differentiable in* (t_0, t_1)*, with* $u(t_0) = u_0$*, such that* $f(\cdot, u(\cdot)) : [t_0, t_1) \to X$ *is continuous,* $u(t) \in D(A)$*, for* $t \in (t_0, t_1)$ *and that* u *satisfies* (B.2)*.*

Definition B.10 *A* **mild solution** *of Cauchy problem* (B.2) *is a continuous function* $u : [t_0, t_1) \to X$*, satisfying the integral equation*

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)} f(s, u(s))ds, \quad t_0 \leqslant t < t_1,$$

for which $u_0 \in X$ *and* $f(\cdot, u(\cdot)) : [t_0, t_1) \to X$ *is continuous and* A *is a sectorial operator.*

The following theorem ensures us the existence and uniqueness of local solutions for the problem (B.2).

Theorem B.11 *Let* A *be a sectorial operator, with* $0 \leqslant \alpha < 1$*. Suppose that* $f : \mathbb{R} \times X^\alpha \to X$ *is* **Hölder continuous** *in the variable* t *and* **locally Lipschitz continuous** *in the variable* x*, that is,* f *is continuous and, for any bounded set* B *in* $\mathbb{R} \times X^\alpha$*, there is a constant* L_B *such that*

$$\|f(t, u) - f(s, v)\|_X \leqslant L_B(|t - s|^\theta + \|u - v\|_\alpha), \quad (t, u), (s, v) \in B,$$

where θ, L_B are positive constants. Then, there is $\tau = \tau_{(t_0,u_0)} > 0$, such that, the problem (B.2) has a unique solution u defined in $(t_0, \tau + t_0)$. **Proof.** See Hale [19, Theorem 4.2.1, p. 73]. ■

To existence of global solution, dependence continuous and differentiable on initial data, we have the following theorem.

Theorem B.12 *Suppose that the hypothesis on* A, f *as in the Theorem B.11 are holds and that, for all bounded set* $B \subset \mathbb{R} \times X^\alpha$*,* $f(B)$ *be bounded in* X*. If* u *is a solution of* (B.2) *in the interval maximal* (t_0, t_1)*, such that* $t_1 < \infty$*, then there is a sequence* $t_n \to t_1^-$ *such that* $\|u(t_n)\|_X \to \infty$*. Moreover, if* f *is a* C^r*-function in* u*, then the solution* $u(t)$ *is a* C^r*-function in the domain of definition of the function.*

Proof. See Hale [19, Theorem 4.2.1, p. 73]. ■

Appendix C

Nonlinear semigroups

In this chapter we present some concepts of the theory of semigroup of continuous operators that are of fundamental importance for the understanding the solution techniques of autonomous semilinear parabolic and hyperbolic problems. We begin with a review of the concept of the semi-distance of Hausdorff and some properties. We also do a summary of the theory of global attractors which will be very important throughout this work. For more details we recommender Carvalho, Langa and Robinson [14] and Hale [19].

Throughout of this chapter (M, d) denote a complete metric space equipped with metric $d(\cdot,\cdot)$. We also denote by $\mathscr{C}(M)$ the set of all continuous maps defined on M into self equipped with the uniform convergence metric.

C.1 Nonlinear semigroups

In this section we present some definitions related to nonlinear semigroups theory.

Definition C.1 *A family of maps $\{S(t); t \geqslant 0\}$ in $\mathscr{C}(M)$ is a* **nonlinear semigroup** *if satisfies*

(i) $S(0) = I_M$,

(ii) $S(t+s) = S(t)S(s)$ *for all* $t, s \geqslant 0$,

(iii) The map $(t, x) \mapsto S(t)x \in M$ *is continuous, from* $[0,\infty) \times M$ *to* M.

Definition C.2 *Let $\{S(t); t \geqslant 0\}$ be a semigroup in a metric space M. A point $x^* \in M$ is called an* **equilibrium point** *of $\{S(t); t \geqslant 0\}$ if, $S(t)x^* = x^*$ for any $t \geqslant 0$. In this case, the map $\phi : \mathbb{R} \to M$ defined by $\phi(t) = x^*$ for any $t \geqslant 0$ is called a* **equilibrium solution** *or* **stationary solution** *of $\{S(t); t \geqslant 0\}$. We denote by $\mathscr{E}$ the set of equilibrium points for $\{S(t); t \geqslant 0\}$.*

Next we define the Hausdorff semi-distance between two bounded subsets A and B of M. This notion relationship between sets will be extremely useful for us to understand the concept of global attractor.

Definition C.3 *Let A and B be bounded subsets of M. The* **Hausdorff semi-distance** *of A from B is defined by*

$$\operatorname{dist}_H(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b).$$

We admit $\operatorname{dist}_H(\emptyset, B) = 0$ for every $B \subset M$, and $\operatorname{dist}_H(A, \emptyset) = \infty$ if $A \neq \emptyset$.

The Hausdorff semi-distance of A from B, $\operatorname{dist}_H(A, B)$, allows us to examine how the set A is contained in the set B it is what tells us the following result.

Proposition C.4 *Let A and B be bounded subsets of M. Then $\operatorname{dist}_H(A, B) = 0$ if and only if A is a subset of $\overline{B}$.*

Proof. Firstly, we recall that for $x \in M$, $C \subset M$, $d(x, C) = \inf_{c \in C} d(x, c) = 0$ is equivalent to say that for each $\varepsilon > 0$, there exists $c_\varepsilon \in C$ such that $d(x, c_\varepsilon) < \varepsilon$; that is,

$$d(x, C) = 0 \Leftrightarrow x \in \overline{C}.$$

Thus

$$\operatorname{dist}_H(A, B) = 0 \Leftrightarrow \forall a \in A,\ a \in \overline{B} \Leftrightarrow A \subset \overline{B}.$$

■

Definition C.5 *Let $\{S(t); t \geqslant 0\}$ be a semigroup in a metric space M. An equilibrium point $x^* \in M$ of $\{S(t); t \geqslant 0\}$ is said to be* **asymptotically stable** *if, some neighborhood B of x^* is attracted by x^*, i.e.,*

$$\lim_{t \to \infty} \operatorname{dist}_H(S(t)B, x^*) = 0.$$

Definition C.6 *Let $\{S(t); t \geqslant 0\}$ be a semigroup in a metric space M. Let B be a subset of M. The* **positive orbit** *through B is defined by*

$$\gamma^+(B) = \{S(t)x; t \geqslant 0, x \in B\}.$$

The positive orbit through B in the instant $s \geqslant 0$ is defined by

$$\gamma_s^+(B) = \{S(t)x; t \geqslant s, x \in B\} \quad (\gamma_0^+(B) = \gamma^+(B)).$$

In particular, for any $x \in M$ the positive orbit through x is defined by

$$\gamma^+(x) = \{S(t)x; t \geqslant 0\}.$$

The positive orbit through x in the instant $s \geqslant 0$ is defined by

$$\gamma_s^+(x) = \{S(t)x; t \geqslant s\} \quad (\gamma_0^+(x) = \gamma^+(x)).$$

Definition C.7 *Let $\{S(t); t \geqslant 0\}$ be a semigroup in a metric space M. We say that $\{S(t); t \geqslant 0\}$ is a* **bounded semigroup** *if,*

$$\gamma^+(B) = \{S(t)x; t \geqslant 0, x \in B\}$$

is bounded for every bounded $B \subset M$.

Definition C.8 *Let $x \in M$. A* **global solution** *for the semigroup $\{S(t); t \geq 0\}$ through x is a map $\phi : \mathbb{R} \to M$ such that $\phi(0) = x$, and for $t \in \mathbb{R}$,*

$$\forall s \geqslant 0, \ S(s)\phi(t) = \phi(s+t).$$

that is,

$$\phi(t) = \begin{cases} S(t)x, \ \text{if } t > 0, \\ S(s)\phi(t) = \phi(t+s), \ \text{for } 0 \leqslant s \leqslant t, \ \text{if } t < 0, \end{cases}$$

Given $x \in M$ global solutions for the semigroup $\{S(t); t \geqslant 0\}$ through x, not necessarily exist, and if there exists them, can not to be unique.

Definition C.9 *Let $\{S(t); t \geqslant 0\}$ be a semigroup in a metric space M. Let B be a subset of M. The* **negative orbit** *through B is defined by*

$$\gamma^-(B) = \bigcup_{x \in B} \gamma^-(x),$$

where for each $x \in B$,

$$\gamma^-(x) = \bigcup_{t \geqslant 0} H(t, x),$$

the set $\gamma^-(x)$ *is called* **negative orbit** *through* x*, and for each* $t \geqslant 0, x \in B$*,*

$H(t,x) = \{y \in M;$ *there exists a global solution* $\phi : \mathbb{R} \to M$ *through* x *such that* $y = \phi(-t)\}$.

The negative orbit through B *in the instant* $s \geqslant 0$ *is defined by*

$$\gamma_s^-(B) = \bigcup_{x \in B} \gamma_s^-(x),$$

where

$$\gamma_s^-(x) = \bigcup_{t \geqslant s} H(t,x).$$

Note that negative orbit through of a subset B *of* M *not necessarily exists.*

Definition C.10 *Let* $\{S(t); t \geqslant 0\}$ *be a nonlinear semigroup in a metric space* M*. Let* B *be a subset of* M*. The* **complete orbit** *of* B *is defined by*

$$\gamma(B) = \gamma^+(B) \cup \gamma^-(B).$$

In particular, for $x \in M$ *the complete orbit of the point* x *is defined by*

$$\gamma(x) = \gamma^+(x) \cup \gamma^-(x).$$

Note that the complete orbit of a subset B *of* M *not necessarily exists.*

Definition C.11 *Let* $\{S(t); t \geqslant 0\}$ *be a nonlinear semigroup in a metric space* M*. Let* B *a subset of* M*. The* $\omega-$**limit** *and* $\alpha-$**limit sets** *of* B *are defined, respectively by*

$$\omega(B) = \bigcap_{s \geqslant 0} \overline{\gamma_s^+(B)} \quad \textit{and} \quad \alpha(B) = \bigcap_{s \geqslant 0} \overline{\gamma_s^-(B)}.$$

Next we have a characterization of the $\omega-$limit set, and it will follows from these characterization that $\omega-$limit sets are properties of the orbit of a set, and not of a set. More precisely, we will see that all subsets of a same orbit have the same $\omega-$limit set.

Proposition C.12 *Let* $\{S(t); t \geqslant 0\}$ *be a semigroup in a metric space* M*. Let* B *a subset of* M*. The* $\omega-$*limit set of* B *can be characterized by following set,*

$$\omega(B) = \{y \in M;\ \textit{there exist sequences } \{t_n\} \textit{ with } t_n \geqslant 0,\ \lim_{n\to\infty} t_n = \infty \textit{ and } \{x_n\} \subset B$$
$$\textit{such that } y = \lim_{n\to\infty} S(t_n)x_n\}.$$

Proof. Let $y \in \omega(B)$, then for each $s \geqslant 0$, there exist sequences $\{t_n^s\}$ with $t_n^s \geqslant s$, $\lim_{n\to\infty} t_n^s = \infty$, and $\{x_n^s\} \subset B$ such that $y = \lim_{n\to\infty} S(t_n^s)x_n^s$. In particular, for each $k \in \mathbb{N}$, there exist sequences $\{t_n^k\}$ with $t_n^k \geqslant k$, $\lim_{n\to\infty} t_n^k = \infty$, and $\{x_n^k\} \subset B$ such that

$$y = \lim_{n\to\infty} S(t_n^k)x_n^k.$$

Thus, there exits $N_k \in \mathbb{N}$ such that

$$\forall n \geqslant N_k, \; d(y, S(t_n^k)x_n^k) < \frac{1}{k}.$$

In particular, for each $k \in \mathbb{N}$, we have

$$d(y, S(t_{N_k}^k)x_{N_k}^k) < \frac{1}{k}.$$

Reciprocally, if $y = \lim_{n\to\infty} S(t_n)x_n$, where $t_n \geqslant 0$, $\lim_{n\to\infty} t_n = \infty$, and $\{x_n\} \subset B$, then for each $s \geqslant 0$, there exists $N_s \in \mathbb{N}$ such that $t_n \geqslant s$ for all $n \geqslant N_s$, hence $y = \lim_{n\to\infty,\, n\geqslant N_s} S(t_n)x_n$ and $y \in \overline{\gamma_s^+(B)}$. ■

Definition C.13 *Let $\{S(t); t \geqslant 0\}$ be a semigroup in a metric space M. The subset B of M is called* **invariant** *under the semigroup $\{S(t); t \geqslant 0\}$, if $S(t)B = B$ for all $t \geqslant 0$. The subset B of M is called* **positively invariant** *(***negatively invariant***) under the semigroup $\{S(t); t \geqslant 0\}$, if $S(t)B \subset B$ ($S(t)B \supset B$) for all $t \geqslant 0$.*

Proposition C.14 *Let $\{S(t); t \geqslant 0\}$ be a semigroup in a metric space M. Let B a subset of M. The set B is invariant under the semigroup $\{S(t); t \geqslant 0\}$ if and only if it consists of a collection of complete orbits of points of B; that is,*

$$B = \bigcup_{b\in B} \gamma(b).$$

Proof. See Carvalho, Langa and Robinson [14, Lemma 1.4, p. 6] ■

C.2 Global attractors for semigroups

The study of the longtime dynamics of semigroups acting in infinite dimensional spaces can often be reduced to the study of the dynamics on the global attractor. In this section we will present the concept of global attractor for a nonlinear semigroup, for more details see Carvalho, Langa and Robinson [14].

Definition C.1 *Let $\{S(t); t \geqslant 0\}$ be a semigroup in $\mathscr{C}(M)$. Let A and B subsets of M, we say that A* **attracts** *B under the semigroup $\{S(t); t \geqslant 0\}$, if*

$$\lim_{t\to\infty} \mathrm{dist}_H(S(t)B, A) = 0.$$

Definition C.2 *Let $\{S(t); t \geqslant 0\}$ be a semigroup in $\mathscr{C}(M)$. Let A and B subsets of M, we say that A* **absorbs** *B under the semigroup $\{S(t); t \geqslant 0\}$, if there exists $t_0 \geq 0$ such that $S(t)B \subset A$ for all $t \geq t_0$.*

Remark C.3 *If A absorbs B then A attracts B under a semigroup $\{S(t); t \geqslant 0\}$. The converse may not be true.*

Definition C.4 *A subset of M is called an* **attracting set***, if it is attracts all the bounded subsets of M.*

When there is a bounded attracting set by semigroup $\{S(t); t \geqslant 0\}$, we say that the semigroup is **bounded dissipative**.

Definition C.5 *A subset of M is called an* **absorbing set***, if it is absorbs all the bounded subsets of M.*

Now we define a global attractor for a semigroup.

Definition C.6 *A subset $\mathscr{A}$ of M is a* **global attractor** *for the semigroup $\{S(t); t \geqslant 0\}$ if $\mathscr{A}$ is compact, invariant and it is an attracting set for the semigroup.*

Theorem C.7 (Uniqueness of the attractor) *The global attractor for a semigroup $\{S(t); t \geqslant 0\}$, if it exists, is unique.*

Proof. Suppose that $\mathscr{A}_1$ and $\mathscr{A}_2$ are two global attractors. Then, since $\mathscr{A}_2$ is bounded, it is attracted by $\mathscr{A}_1$,

$$\lim_{t\to\infty} \operatorname{dist}_H(S(t)\mathscr{A}_2, \mathscr{A}_1) = 0.$$

But $\mathscr{A}_2$ is invariant, $S(t)\mathscr{A}_2 = \mathscr{A}_2$, and so $\operatorname{dist}_H(\mathscr{A}_2, \mathscr{A}_1) = 0$; by Proposition C.4, $\mathscr{A}_2 \subset \mathscr{A}_1$ (since $\mathscr{A}_1$ is closed). In similar way we have $\mathscr{A}_1 \subset \mathscr{A}_2$, from which it follows that $\mathscr{A}_1 = \mathscr{A}_2$. ■

Two alternative characterizations of the global attractor it follow from a similar argument to prove of Theorem C.7.

(*i*) The global attractor for a semigroup $\{S(t); t \geqslant 0\} \subset \mathscr{C}(M)$ if it exists, is the minimal (with respect to the inclusion relation in M) compact set that attracts each bounded subset of M: In fact, let $\mathscr{A}_*$ be a compact set that attracts all bounded subsets of M. In particular, $\mathscr{A}_*$ attracts $\mathscr{A}$, and so, since $\mathscr{A} = S(t)\mathscr{A}$ for any $t \geqslant 0$,

$$\operatorname{dist}_H(\mathscr{A}, \mathscr{A}_*) = \lim_{t\to\infty} \operatorname{dist}_H(S(t)\mathscr{A}, \mathscr{A}_*) = 0, \quad \text{and} \quad \mathscr{A} \subset \mathscr{A}_*.$$

(*ii*) The global attractor for a semigroup $\{S(t); t \geqslant 0\} \subset \mathscr{C}(M)$ if it exists, is the maximal (with respect to the inclusion relation in M) closed and bounded invariant set: In fact, if $\mathscr{A}_*$ is closed, bounded, and invariant then $\mathscr{A}$ attracts $\mathscr{A}_*$, and so

$$\text{dist}_H(\mathscr{A}_*, \mathscr{A}) = \lim_{t\to\infty} \text{dist}_H(S(t)\mathscr{A}_*, \mathscr{A}) = 0, \quad \text{and} \quad \mathscr{A}_* \subset \mathscr{A}.$$

Remark C.8 *The equilibria set of a semigroup is always a closed and invariant set, and from (ii) above, if equilibria set is bounded then equilibria set is always contained at global attractor of the semigroup. Indeed, it is sufficient to prove that the equilibria set of a semigroup is closed: if $x^* \in M$ is such that $x^* = \lim x_n$ with $\{x_n\} \subset \mathscr{E}$, then for each $t \geqslant 0$, we have*

$$d(S(t)x^*, S(t)x_n) \to 0,$$

as $n \to \infty$, and

$$d(S(t)x^*, x^*) \leqslant d(S(t)x^*, S(t)x_n) + d(x_n, x^*) \to 0,$$

as $n \to \infty$, and therefore $x^* \in \mathscr{E}$.

In addition, the global attractor can be characterised as the collection of all globally defined bounded solutions.

Theorem C.9 *If the smigroup $\{S(t); t \geqslant 0\}$ has a global attractor $\mathscr{A}$, then*

$$\mathscr{A} = \{y \in X; \text{ there exists a bounded global solution } \phi : \mathbb{R} \to X \text{ with } y = \phi(0)\}.$$

Proof. See Carvalho, Langa and Robinson [14, Theorem 1.7, p. 8] ■

Now we will present some existence results for global attractors for the semigroup. We do not do the proof here for more details we refer to Carvalho, Langa and Robinson [14], Hale [19].

Proposition C.10 *Suppose that there exists a compact attracting set K. Then, for any bounded set B, the $\omega-$limit set $\omega(B)$ is a non-empty compact subset of K that is invariant and attracts B.*

Proof. See Carvalho, Langa and Robinson [14, Corrolary 2.6, p. 26]. ■

Proposition C.11 *Let $\{S(t); t \geqslant 0\}$ be a semigroup in a metric space M. If $\{S(t); t \geqslant 0\}$ is an asymptotically compact semigroup and B is a non-empty bounded subset of M, then* *(i) The omega limit set $\omega(B)$ is non-empty, compact, and invariant and attracts B;*

(ii) The omega limit set $\omega(B)$ is the minimal closed set that attracts B.

Proof. See Carvalho, Langa and Robinson [14, Corollary 2.11, p. 28] ■

In many cases we can show something stronger than the existence of a compact attracting set, namely the existence of a compact absorbing set. Clearly the existence of a compact absorbing set implies the existence of a compact attracting set, which we know implies the existence of a global attractor.

Definition C.12 *Let $\{S(t); t \geqslant 0\}$ be a semigroup in a metric space M. We say that the semigroup is* **asymptotically compact** *if, for every sequence $\{t_n\}$ with $t_n \geqslant 0$ and $\lim t_n = \infty$ and $\{x_n\} \subset B$, with $B \subset M$ bounded, $\{S(t_n)x_n\}$ has a convergent subsequence.*

The definition above is equivalent to say that the semigroup $\{S(t); t \geqslant 0\}$ is **asymptotically compact** (or **asymptotically smooth**) if, and only if, for any non-empty, closed, bounded set $B \subset M$ for which $S(t)B \subset B$ for all $t \geqslant 0$, there exists $K \subset B$ compact set such that K attracts B.

Theorem C.13 *Let $\{S(t); t \geqslant 0\}$ be a bounded semigroup defined in M such that for each $t \geqslant 0$, we can write*

$$S(t) = T(t) + U(t)$$

where

(i) For every bounded set B and each $t > 0$ there exists $t_{(B,t)} \geqslant 0$ and compact set $K(B,t)$, such that $U(s)B \subset K(B,t)$, always that $t \geqslant s \geqslant t_{(B,t)}$, ($U$ is strongly compact);

(ii) There exists a function $g : [0,\infty) \times [0,\infty) \to \mathbb{R}$ with $g(\cdot, r)$ non-increasing for each $r > 0$, $\lim_{s\to\infty} g(s,r) = 0$ and for all $x \in M$ with $\|x\| \leqslant r$,

$$\|T(t)x\|_M \leqslant g(t,r).$$

Then the semigroup $\{S(t); t \geqslant 0\}$ is asymptotically compact.

Proof. See Carvalho, Langa and Robinson [14, Theorem 2.37, p. 41] ■

Any finite-dimensional semigroup with a bounded absorbing set is asymptotically compact; in an infinite-dimensional system this is much weaker than the existence of a compact absorbing set.

Definition C.14 *Let* $\{S(t); t \geqslant 0\}$ *be a semigroup system in a metric space* M*. We say that the semigroup is* **eventually compact** *if, it is bounded and there exists a* $t_0 > 0$ *such that* $\overline{S(t_0)B}$ *is compact for each bounded subset* B *of* X*.*

Proposition C.15 *Let* $\{S(t); t \geqslant 0\}$ *be a semigroup in a metric space* M*. If* $\{S(t); t \geqslant 0\}$ *is* **eventually compact***, then* $\{S(t); t \geqslant 0\}$ *is* **asymptotically compact***.*

Proof. See Carvalho, Langa and Robinson [14, Corollary 2.18, p. 33] ■

Theorem C.16 *Let* $\{S(t); t \geqslant 0\} \subset \mathscr{C}(M)$ *be a bounded semigroup. There exists a* **global attractor** $\mathscr{A}$ *if and only if there exists a* **bounded attracting set** *(bounded dissipative) and the semigroup is* **asymptotically compact***, in which case*

$$\mathscr{A} = \bigcup_{B \subset M, B \neq \emptyset \text{ Bounded}} \omega(B).$$

Proof. See Carvalho, Langa and Robinson [14, Corollary 2.21, p.34] ■

Definition C.17 *A semigroup* $\{S(t); t \geqslant 0\}$ *is said to be a* **gradient system** *if there is a continuous function* $V : M \to \mathbb{R}$*, a* **Lyapunov function***, with the following properties:*

(i) $t \mapsto V(S(t)x)$ *is non-increasing for each* $x \in M$*; and*

(ii) if x *is such that* $V(S(t)x) = V(x)$ *for all* $t \geqslant 0$*, then* $x \in \mathscr{E}$*.*

We define the unstable manifold of $x \in \mathscr{E}$ as being the set

$$W^u(x) = \{y \in M : S(-t)y \quad \text{is defined for all} \quad t \geqslant 0 \text{ and } S(-t)y \to x \text{ as } t \to \infty\}.$$

Now we present a result that ensures us the existence of global attractors for gradient systems.

Theorem C.18 *If* $\{S(t); t \geqslant 0\}$*, is a gradient system, asymptotically smooth, and* $\mathscr{E}$ *is bounded, then there is a global attractor* $\mathscr{A}$ *for* $\{S(t); t \geqslant 0\}$ *and*

$$\mathscr{A} = W^u(\mathscr{E}) = \{y \in M : S(-t)y \text{ is defined for } t \geqslant 0 \text{ and } S(-t)y \to \mathscr{E} \text{ as } t \to \infty\}.$$

If M *is a Banach space, then* $\mathscr{A}$ *is connected. If, in addition, each element of* $\mathscr{E}$ *is hyperbolic, then* $\mathscr{E}$ *is a finite set and*

$$\mathscr{A} = \bigcup_{x \in \mathscr{E}} W^u(x).$$

Proof. See Hale [19, Theorem 3.8.5, p. 51] ■

In order we define and we present a characterization to upper and lower semicontinuity to a family subsets of M. In particular we have the upper and lower semicontinuity of global attractors.

Definition C.19 *Let $\mathscr{T}$ be topology space and $\{\mathbf{A}_\lambda\}_{\lambda\in\mathscr{T}}$ a family of subsets of M. We say that the family $\mathbf{A}_\lambda$ is* **upper semicontinuous** *at $\lambda_0 \in \mathscr{T}$ if*

$$\lim_{\lambda\to\lambda_0} \operatorname{dist}_H(\mathbf{A}_\lambda, \mathbf{A}_{\lambda_0}) = 0.$$

We say that $\mathbf{A}_\lambda$ is **lower semicontinuous** *at $\lambda_0 \in \mathscr{T}$ if*

$$\lim_{\lambda\to\lambda_0} \operatorname{dist}_H(\mathbf{A}_{\lambda_0}, \mathbf{A}_\lambda) = 0.$$

We say that $\mathbf{A}_\lambda$ is continuous at $\lambda_0 \in \mathscr{T}$ if it is both **upper** *and* **lower semicontinuous** *as $\lambda \to \lambda_0$.*

The following result show that (semi)continuity with respect to $\lambda \in \mathscr{T}$ at λ_0 is completely characterized by the behavior of sequences $\{\mathbf{A}_{\lambda_n}\}$ where $\lambda \to \lambda_0$.

Lemma C.20 *Let $\mathscr{T}$ be topology space and let $\{\mathbf{A}_\lambda\}_{\lambda\in\mathscr{T}}$ be a family of compact subsets of M. Then*

(i) $\{\mathbf{A}_\lambda\}_{\lambda\in\mathscr{T}}$ *is* **upper semicontinuous** *at $\lambda_0 \in \mathscr{T}$ if and only if, whenever $\lambda_n \to \lambda_0$ as $n \to \infty$, any sequence $\{x_{\lambda_n}\}$ with $x_{\lambda_n} \in \mathbf{A}_{\lambda_n}$ has a convergent subsequence whose limit belongs to $\mathbf{A}_{\lambda_0}$;*

(ii) $\{\mathbf{A}_\lambda\}_{\lambda\in\mathscr{T}}$ *is* **lower semicontinuous** *at $\lambda_0 \in \mathscr{T}$ if and only if, $\mathbf{A}_{\lambda_0}$ is compact and for any $x_0 \in \mathbf{A}_{\lambda_0}$ and $\lambda_n \to \lambda_0$ there is a sequence $x_{\lambda_n} \in \mathbf{A}_{\lambda_n}$ such that $x_{\lambda_n} \to x_0$ as $n \to \infty$.*

Proof. (i) If any sequence $\{x_{\lambda_n}\}$ with $x_{\lambda_n} \in \mathbf{A}_{\lambda_n}$ and $\lambda_n \to \lambda_0$, has a convergent subsequence with limit belonging to $\mathbf{A}_{\lambda_0}$, and $\{\mathbf{A}_\lambda\}_{\lambda\in\mathscr{T}}$ is not upper semicontinuous at $\lambda_0 \in \mathscr{T}$ then, there are $\varepsilon > 0$ and sequence $\{\lambda_n\}$ with $\lambda_n \to \lambda_0$ such that

$$\operatorname{dist}_M(\mathbf{A}_{\lambda_n}, \mathbf{A}_{\lambda_0}) = \sup_{x\in\mathbf{A}_{\lambda_n}} d(x, \mathbf{A}_{\lambda_0}) > \varepsilon, \quad \forall n \in \mathbb{N}.$$

Thus, for some $x_{\lambda_n} \in \mathbf{A}_{\lambda_n}$, we have that $d(x_{\lambda_n}, \mathbf{A}_{\lambda_0}) > \varepsilon, n \in \mathbb{N}$. But this contradicts the fact that x_{λ_n} has a subsequence which converges to an element of $\mathbf{A}_{\lambda_0}$. Conversely, suppose that $\{\mathbf{A}_\lambda\}_{\lambda\in\mathscr{T}}$ is upper semicontinuous at $\lambda_0 \in \mathscr{T}$. If $x_{\lambda_n} \in \mathbf{A}_{\lambda_n}$, where $\lambda_n \to \lambda_0$, then

$$0 \leqslant d(x_{\lambda_n}, \mathbf{A}_{\lambda_0}) \leqslant \operatorname{dist}_H(\mathbf{A}_{\lambda_n}, \mathbf{A}_{\lambda_0}).$$

Thus $d(x_{\lambda_n}, \mathbf{A}_{\lambda_0}) \to 0$, and $\mathbf{A}_{\lambda_0}$ is compact.

(ii) Suppose that for any $x_0 \in \mathbf{A}_{\lambda_0}$ and any sequence $\lambda_n \to \lambda_0$ there is a sequence $\{x_{\lambda_n}\}$ with $x_{\lambda_n} \in \mathbf{A}_{\lambda_n}$ which converges to x_0. If $\{\mathbf{A}_\lambda\}_{\lambda\in\mathscr{T}}$ is not lower semicontinuous at λ_0, then there are $\varepsilon > 0$ and sequence $\{\lambda_n\}$ with $\lambda_n \to \lambda_0$ such that

$$\mathrm{dist}_H(\mathbf{A}_{\lambda_0}, \mathbf{A}_{\lambda_n}) = \sup_{x\in\mathbf{A}_{\lambda_0}} d(x, \mathbf{A}_{\lambda_n}) > \varepsilon, \quad \forall n \in \mathbb{N}$$

Thus for each $n \in \mathbb{N}$ there exists $y_{\lambda_n} \in \mathbf{A}_{\lambda_0}$, such that $d(y_{\lambda_n}, \mathbf{A}_{\lambda_n}) > \varepsilon$. Since $\mathbf{A}_{\lambda_0}$ is compact, we may assume that $y_{\lambda_n} \to x_0 \in \mathbf{A}_{\lambda_0}$ and that

$$d(x_0, \mathbf{A}_{\lambda_n}) > \varepsilon, \quad \forall\ n \in \mathbb{N}.$$

But this contradicts the fact that there must be a sequence $x_{\lambda_n} \in \mathbf{A}_{\lambda_n}$ that converges to x_0. Conversely, suppose that $\{\mathbf{A}_\lambda\}_{\lambda\in\mathscr{T}}$ is lower semicontinuous at $\lambda_0 \in \mathscr{T}$. If $\lambda_n \to \lambda_0$ and $x_0 \in \mathbf{A}_{\lambda_0}$, then there exists $x_{\lambda_n} \in \mathbf{A}_{\lambda_n}$ such that

$$d(x_0, x_{\lambda_n}) \leqslant d(x_0, \mathbf{A}_{\lambda_n}) \leqslant \mathrm{dist}_H(\mathbf{A}_{\lambda_0}, \mathbf{A}_{\lambda_n}).$$

which converges to zero as $n \to \infty$. ■

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