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Programa Associado de Pós Graduação em Matemática
Doutorado em Matemática

Hardy-Sobolev type inequalities in the upper half-space and their applications

por

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João Pessoa - PB
Setembro de 2019

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sob orientação do

Prof. Dr. Everaldo Souto de Medeiros

Tese apresentada ao Corpo Docente do Programa Associado de Pós Graduação em Matemática UFPB/UFCG como requisito parcial para obtenção do título de Doutor em Matemática.

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"All analysts spend half their time hunting through the literature for inequalities which they want to use and cannot prove."

- G. H. Hardy.

Abstract

In this thesis, we prove two Hardy-Sobolev type inequalities and as a consequence we establish embedding results of a certain Sobolev space defined on the upper half-space into weighted Lebesgue spaces. Furthermore, some Trudinger-Moser type inequalities for functions defined in the upper half-space are obtained. As applications, we also prove existence, nonexistence and multiplicity of solutions for three class of indefinite quasilinear elliptic problems with weights in anisotropic spaces.

Keywords: Hardy-Sobolev inequality, Sobolev space, Weighted Lebesgue space, Quasilinear elliptic problem, Anisotropic space.

Resumo

Nesta tese, provamos duas desigualdades do tipo Hardy-Sobolev e, como consequência, estabelecemos resultados de imersão de um determinado espaço de Sobolev definido no semi espaço superior em espaços de Lebesgue com peso. Além disso, algumas desigualdades do tipo Trudinger-Moser para funções definidas no semi espaço superior são obtidas. Como aplicações, também provamos a existência, não existência e multiplicidade de soluções para três classes de problemas elípticos quasilineares indefinidos com pesos em espaços anisotrópicos.

Palavras-chave: Desigualdade de Hardy-Sobolev, Espaço de Sobolev, Espaço de Lebesgue com peso, Problema elíptico quasilinear, Espaço anisotrópico.

Contents

Notation

We select here some notations used throughout the work.

Spaces

- $L^p(\Omega, a(x)) = \left\{ \varphi : \Omega \to \mathbb{R} : \begin{array}{l} \varphi \text{ is Lebesgue measurable with} \\ \int_\Omega a(x)|\varphi(x)|^p dx < \infty \end{array} \right\}$, $1 \leq p < \infty$;
- $L^\infty(\Omega, a(x)) = \{\varphi : \Omega \to \mathbb{R},\ a(x)\varphi \text{ is bounded and Lebesgue measurable}\}$;
- $\partial\Omega, \overline{\Omega}, \Omega^c$ denote boundary, closure, and complement of the set Ω, respectively.
- $\mathbb{R}^n$ denotes the usual euclidean space with the norm $|x| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$, $x \in \mathbb{R}^n$
- $\mathbb{R}^n_+ := \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}$;
- $W^{1,p}(\mathbb{R}^n_+)$ denotes the usual Sobolev space of p-weak derivatives;
- $\mathcal{E}^{1,p}(\mathbb{R}^n_+) := \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^n_+) : \begin{array}{l} u_{|_{\mathbb{R}^{n-1}}} \in L^p(\mathbb{R}^{n-1}) \quad \text{and} \\ u_{x_i} \in L^p(\mathbb{R}^n_+), \quad \forall\, i = 1, \ldots, n \end{array} \right\}$;
- $C(\Omega)$ denotes the space of continuous real functions in $\Omega \subset \mathbb{R}^n$;
- For an integer $k \geq 1$, $C^k(\Omega)$ denotes the space of k-times continuously differentiable real functions in $\Omega \subset \mathbb{R}^n_+$;
- $C^\infty(\Omega) = \cap_{k\in\mathbb{N}} C^k(\Omega)$;
- $C_0^\infty(\Omega)$ denotes the space of infinitely differentiable real functions whose support is compact in $\Omega \subset \mathbb{R}^n$;
- $\mathcal{D}^{1,p}(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_p = (\int_\Omega |\nabla u|^p dx)^{1/p}$, $1 < p < n$;
- E' denotes the topological dual of the Banach space E;

Norms

- For $1 \leq p < \infty$, the standard norm in $L^p(\mathbb{R}^n_+, a(x))$ is denoted by $\|\cdot\|_{L^p(\mathbb{R}^n_+, a(x))}$;

Other Notation

- $|A|$ denotes the Lebesgue measure of a set $A \subset \mathbb{R}^n$;
- $supp(\varphi)$ denotes the support of function φ;
- C, C_0, C_1, C_2, C_3, ... denote positive constants possibly different;
- $C(s)$ denotes constant which depends of s;
- $o_k(1)$ denotes a sequence which converges to 0 as $k \to \infty$;
- $\rightharpoonup$ denotes weak convergence in a normed space;
- $\to$ denotes strong convergence in a normed space;
- $\hookrightarrow$ denotes continuous embedding;
- $\langle \cdot, \cdot \rangle$ denotes the duality pairing between E and E';
- *Weight functions* are functions measurable and positive almost everywhere (a.e.)

Introduction

The purpose of this thesis is twofold: firstly, we prove the Hardy-Sobolev type inequalities

$$\int_{\mathbb{R}^n_+} \frac{|u|^p}{(1+x_n)^p} dx \le \left(\frac{p}{p-1}\right)^p \left(\int_{\mathbb{R}^n_+} |\nabla u|^p dx + \int_{\mathbb{R}^{n-1}} |u|^p dx'\right), \quad \forall\, u \in C_0^\infty(\mathbb{R}^n), \tag{1}$$

where $n \ge 2$ and $1 < p < \infty$ and

$$\left(\int_{\mathbb{R}^n_+} \frac{|u|^{p_*}}{(1+x_n)^\alpha}\right)^{p/p_*} dx \le C \int_{\mathbb{R}^n_+} |\nabla u|^p dx, \quad \forall\, u \in C_0^1(\mathbb{R}^n), \tag{2}$$

where $n \ge 3$, $\alpha > 1$ and $p_* := p(n-1)/(n-p)$; and secondly, we use these inequalities to study the following class of quasilinear elliptic

$$\left\{ \begin{array}{rcll} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) & = & g(x,u) & \text{in} \quad \mathbb{R}^n_+, \\ |\nabla u|^{p-2}\nabla u \cdot \nu + \kappa |u|^{p-2} u & = & 0 & \text{on} \quad \mathbb{R}^{n-1}, \end{array} \right. \tag{3}$$

where $1 < p \le n$ and g satisfies some suitable growth conditions that will be specified later.

Our interest in the type of inequalities has been mostly motivated by their deep connections with Hardy and Sobolev inequalities. We quote here that the proof of (1) was strongly inspired by the arguments used in [23, Theorem 1.4], where the authors obtain a similar result for functions in $C_0^\infty(\mathbb{R}^n_+)$ and the proof of inequality (2) was inspired by the arguments used in the paper [11, Proposition 3.4].

We also point out that (1) is an extension of the weighted Hardy-type inequality proved in [34, Lemma 1] where the author proved a similar result:

$$\int_\Omega \frac{|u|^p}{(1+|x|)^p} dx \le C_0 \left(\int_\Omega |\nabla u|^p dx + \int_\Gamma \frac{|\nu \cdot x|}{(1+|x|)^p} |u|^p d\Gamma\right), \tag{4}$$

for $1 < p < n$ and some $C_0 > 0$.

As a consequence of (1) we establish embedding results of a certain Sobolev space $\mathcal{E}^{1,p}(\mathbb{R}^n_+)$ defined on the upper half-space and we investigate existence, nonexistence and multiplicity of solutions for a class of indefinite quasilinear elliptic problems with weights in anisotropic spaces.

The problems studied in this thesis have the form

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \lambda a(x)\bar{f}(x,u) - b(x)|u|^{r-2}u & \text{in} \quad \mathbb{R}^n_+, \\ |\nabla u|^{p-2}\nabla u\cdot\nu + \kappa|u|^{p-2}u = 0 & \text{on} \quad \mathbb{R}^{n-1}, \end{cases} \tag{5}$$

where ν denotes the unit outward normal on the boundary $\mathbb{R}^{n-1}$, $\kappa \geq 0$, λ is a real parameter, the nonlinearity $\bar{f}$ can assume polynomial growth or exponential growth in the Trudinger-Moser sense in $\mathbb{R}^n_+$, $n \geq 2$, and the weight functions a, b satisfy some suitable conditions that we will describe later on.

Problems of this type have been investigated by many authors, see for instance [12, 14, 16, 25, 32–35] and references therein. In many of these papers a relevant Sobolev inequality proved by Pflüger in [34] has played an important role in their variational approach. Precisely, let $1 < p < n$ and assume that the weight function $h(x)$ satisfies the hypothesis:

$$1/C(1+|x|)^{p-1} \leq h(x) \leq C/(1+|x|)^{p-1}, \quad \text{a.e. in} \quad \Omega,$$

for some $C > 0$ and denote by $C^\infty_\delta(\Omega)$ the space of $C^\infty_0(\mathbb{R}^n)-$functions restricted to Ω. Defining the weighted Sobolev space E as the completion of $C^\infty_\delta(\Omega)$ in the norm

$$\|u\|_E := \left(\int_\Omega |\nabla u|^p dx + \int_\Omega \frac{|u|^p}{(1+|x|)^p} dx\right)^{1/p},$$

in [34], the author proved that $\|\cdot\|_E$ is an equivalent norm to (see [34, Lemma 2])

$$|||u|||_E := \left(\int_\Omega |\nabla u|^p + \int_{\partial\Omega} h(x)|u|^p dx'\right)^{1/p}.$$

To this, the Hardy-Sobolev type inequality (4) was crucial.

Let us now describe the content of this thesis. The thesis is written in four chapters. Each chapter corresponds to a submitted paper. In this way, each chapter in this thesis is self-contained.

In **Chapter 1**, we consider the Sobolev space defined by

$$\mathcal{E}^{1,p}(\mathbb{R}^n_+) := \left\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^n_+) \ : \quad \begin{array}{l} u_{|_{\mathbb{R}^{n-1}}} \in L^p(\mathbb{R}^{n-1}) \quad \text{and} \\ u_{x_i} \in L^p(\mathbb{R}^n_+), \quad \forall\, i = 1, \ldots, n \end{array} \right\},$$

where u_{x_i} denotes the distributional derivative of u, $u_{|_{\mathbb{R}^{n-1}}}$ is understood in the trace sense and $\mathbb{R}^{n-1}$ denotes the boundary of $\mathbb{R}^n_+$ and $\mathcal{E}^{1,p}(\mathbb{R}^n_+)$, from now on denoted by $\mathcal{E}^{1,p}$, is equipped with the norm

$$\|u\|_{\mathcal{E}^{1,p}} := \left(\int_{\mathbb{R}^n_+} |\nabla u|^p dx + \int_{\mathbb{R}^{n-1}} |u|^p dx'\right)^{1/p}.$$

For $1 < p < n$ we obtain the following embedding result.

Theorem 0.0.1. *Assume $1 < p < n$. Then the weighted Sobolev embedding*

$$\mathcal{E}^{1,p} \hookrightarrow L^q\left(\mathbb{R}^n_+, \frac{1}{(1+x_n)^p}\right), \quad \forall\, p \le q \le p^*,$$

and the Sobolev trace embedding

$$\mathcal{E}^{1,p} \hookrightarrow L^q(\mathbb{R}^{n-1}), \quad \forall\, p \le q \le p_* := \frac{(n-1)p}{n-p}$$

are continuous.

In the borderline case $p = n$, we establish the following result:

Theorem 0.0.2. *Assume $p = n$. Then the weighted Sobolev embedding*

$$\mathcal{E}^{1,n} \hookrightarrow L^q\left(\mathbb{R}^n_+, \frac{1}{(1+x_n)^n}\right), \quad \forall\, n \le q < \infty, \tag{6}$$

and the Sobolev trace embedding

$$\mathcal{E}^{1,n} \hookrightarrow L^q(\mathbb{R}^{n-1}), \quad \forall\, n \le q < \infty \tag{7}$$

are continuous.

The embeddings (6) and (7) are not valid if $q = \infty$, see Remark 1.1.5. Thus, it is natural to study embedding from $\mathcal{E}^{1,n}$ into Orlicz space. To this end, we introduce a new weighted Sobolev space, which plays a central role in the proof of our Trudinger-Moser type inequality.

Definition 0.0.3. *Consider the weight function $b(x) := (1+|x|)^{-n}$. We define the space $\mathcal{E}^{1,n}_b$ as the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm*

$$\|u\|^n_{\mathcal{E}^{1,n}_b} := \int_{\mathbb{R}^n_+} \left(|\nabla u|^n + b(x)|u|^n\right) dx.$$

We establish the following embedding result involving $\mathcal{E}^{1,n}_b$.

Theorem 0.0.4. *For any $n \le q < \infty$, the embedding*

$$\mathcal{E}^{1,n}_b \hookrightarrow L^q\left(\mathbb{R}^n_+, b(x)\right) \tag{8}$$

is continuous. Furthermore, this embedding is false if $q = \infty$.

Considering the Young function defined by

$$\Psi_\alpha(s) = e^{\alpha|s|^{n'}} - \sum_{k=0}^{n-2} \frac{\alpha^k}{k!}|s|^{n'k}, \quad s \in \mathbb{R},$$

where $n' := n/(n-1)$ and according to (8), inspired by the arguments used in [18], we prove the following Trudinger-Moser type inequality in the setting of $\mathcal{E}^{1,n}_b$.

Theorem 0.0.5. *For any $\alpha > 0$ we have that $\Psi_\alpha(u) \in L^1(\mathbb{R}^n_+, b(x))$. Moreover, there exists a constant $\alpha_0 > 0$, independent of $u \in \mathcal{E}_b^{1,n}$, such that*

$$L(\alpha) := \sup_{\{u \in \mathcal{E}_b^{1,n}:\ \|u\|_{\mathcal{E}_b^{1,n}} \leq 1\}} \int_{\mathbb{R}^n_+} b(x)\Psi_\alpha(u)dx < +\infty,$$

for any $0 < \alpha \leq \alpha_0$.

As a consequence of Theorem 0.0.5, the following Trudinger-Moser type inequality in the setting of $\mathcal{E}^{1,n}$ holds.

Corollary 0.0.6. *For any $u \in \mathcal{E}^{1,n}$ and $\alpha \geq 0$, we have that $\Psi_\alpha(u) \in L^1(\mathbb{R}^n_+, b(x))$. Moreover,*

$$l(\alpha) := \sup_{\{u \in \mathcal{E}^{1,n}:\ \|u\|_{\mathcal{E}^{1,n}} \leq 1\}} \int_{\mathbb{R}^n_+} b(x)\Psi_\alpha(u)dx < +\infty,$$

for any $0 < \alpha \leq \alpha_0 / \left(2n/(n-1)\right)^{n'}$.

In the trace sense, we have the following Trudinger-Moser type inequality:

Corollary 0.0.7. *For any $u \in \mathcal{E}^{1,n}$ and $\alpha \geq 0$, we have that $\Psi_\alpha(u(\cdot, 0)) \in L^1(\mathbb{R}^{n-1}, b(x', 0))$. Moreover,*

$$T(\alpha) := \sup_{\{u \in \mathcal{E}^{1,n}:\ \|u\|_{\mathcal{E}^{1,n}} \leq 1\}} \int_{\mathbb{R}^{n-1}} b(x', 0)\Psi_\alpha(u(x', 0))dx' < +\infty,$$

for any $0 < \alpha \leq \left((n-1)/n\right)^2 \alpha_0 / \left(2n/(n-1)\right)^{n'}$.

In the case $p > n$ we obtain a Morrey's type inequality.

Theorem 0.0.8. *Assume $n < p < \infty$ and $a(x) := (1 + x_n)^{-p}$. Then the following weighted Sobolev embedding holds*

$$\mathcal{E}^{1,p} \hookrightarrow L^\infty(\mathbb{R}^n_+, a(x)).$$

Furthermore, for all $u \in \mathcal{E}^{1,p}$ there exists $C_0 = C_0(n, p) > 0$ such that for a.e. $x, y \in \mathbb{R}^n_+$

$$|a(x)u(x) - a(y)u(y)| \leq C_0 |x - y|^\gamma \left(\|\nabla u\|_{L^p(\mathbb{R}^n_+)} + \|u\|_{L^p(\mathbb{R}^n_+ a(x))} \right),$$

where $\gamma = 1 - n/p$.

In **Chapter 2**, we investigate the problem (5) when $\kappa > 0$ and $\bar{f}$ has polynomial growth in $\mathbb{R}^n_+$. Precisely, the problem studied in this chapter has the form

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \lambda a(x)|u|^{q-2}u - b(x)|u|^{r-2}u & \text{in} \quad \mathbb{R}^n_+, \\ |\nabla u|^{p-2}\nabla u \cdot \nu + |u|^{p-2}u = 0 & \text{on} \quad \mathbb{R}^{n-1}. \end{cases} \tag{9}$$

We begin by considering the case $r > q$. To this end, we shall assume the following assumptions:

(H_1) $a : \mathbb{R}^n_+ \to \mathbb{R}$ is a nontrivial measurable function and there are constants $\alpha \geq n$ and $c_1 > 0$ such that

$$0 \leq a(x) \leq \frac{c_1}{(1+x_n)^\alpha}, \quad \text{a.e. in} \quad \mathbb{R}^n_+;$$

(H_2) $b : \mathbb{R}^n_+ \to \mathbb{R}$ is a positive continuous function satisfying

$$\int_{\mathbb{R}^n_+} \frac{a^{\frac{r}{r-q}}}{b^{\frac{q}{r-q}}} dx < \infty.$$

Under this hypotheses, our main result can be stated as follows.

Theorem 0.0.9. *Let $r > q$ and assume the hypotheses $(H_1) - (H_2)$.*

(i) If $1 < p \leq q \leq p^*$, there exists $\lambda^* > 0$ such that problem (9) has only the trivial solution for all $\lambda \in (-\infty, \lambda^*)$;

(ii) If $\max\{2, p\} \leq q < p^*$, there exists $\tilde{\lambda} > 0$ such that problem (9) has at least a nontrivial weak solution for all $\lambda \in (\tilde{\lambda}, \infty)$. Furthermore, if $p < q$ then $(\mathcal{P}_{\tilde{\lambda}})$ has a nontrivial weak solution;

(iii) If $\max\{2, p\} \leq q < p^*$, there exists $\Lambda \geq \tilde{\lambda}$ such that problem (9) has at least two nontrivial weak solutions $u_\lambda \geq \tilde{u}_\lambda$ for all $\lambda \in (\Lambda, \infty)$;

(iv) If $1 < p \leq q < p^*$, for any $m \in \mathbb{N}$ there exists $\Lambda_m > 0$ such that problem (9) has at least m pairs of nontrivial weak solutions for all $\lambda > \Lambda_m$.

Next we deal with the case $r < q$. In order to prove the existence of solutions for problem (9), instead of hypotheses $(H_1) - (H_2)$, we will assume:

$(\widetilde{H}_1)$ $a : \mathbb{R}^n_+ \to \mathbb{R}$ is a nontrivial measurable function and there are $c_2 > 0$ and $\alpha \geq n$ such that

$$0 \leq a(x) \leq \frac{c_2}{(1+|x|)^\alpha}, \quad \text{a.e. in} \quad \mathbb{R}^n_+.$$

$(\widetilde{H}_2)$ $b : \mathbb{R}^n_+ \to \mathbb{R}$ is a measurable positive function.

In this case, our main result is stated as follows.

Theorem 0.0.10. *Let $1 < p \leq r < q < p^*$ and assume the hypotheses $(\widetilde{H}_1) - (\widetilde{H}_2)$. Then*

(i) the problem (9) has no nontrivial weak solution for every $\lambda \in (-\infty, 0]$;

(ii) the problem (9) has an infinite number of nontrivial weak solutions for every $\lambda \in (0, \infty)$.

Chapter 3 contains our study of the problem (5) when $\kappa > 0$ and $\bar{f}$ has exponential growth in the Trudinger-Moser sense. Precisely, we study the following class of quasilinear elliptic problems

$$\begin{cases} -\text{div}(|\nabla u|^{n-2}\nabla u) + b(x)|u|^{r-2}u = \lambda a(x) f(x,u) & \text{in} \quad \mathbb{R}^n_+ \\ |\nabla u|^{n-2}\nabla u \cdot \nu + |u|^{n-2}u = 0 & \text{on} \quad \mathbb{R}^{n-1}, \end{cases} \tag{10}$$

where ν denotes the unit outward normal on the boundary $\mathbb{R}^{n-1}$, λ is a real parameter, $1 < p \leq r < \infty$, the weight functions $a(x)$ and $b(x)$ satisfy some suitable conditions that we will describe later on and we assume that f is a continuous function with subcritical exponential growth in the Trudinger-Moser sense, i.e., for any $\beta > 0$

$$\lim_{|s|\to\infty} \frac{|f(x,s)|}{e^{\beta|s|^{n'}}} = 0, \text{ uniformly in } x \in \mathbb{R}^n_+.$$

Setting $F(x,s) = \int_0^s f(x,t)dt$, we also will assume that f satisfies the following assumptions:

(f_1) $\overline{\lim_{s\to 0^+}} \frac{nF(x,s)}{s^n} < \lambda_1$ uniformly with respect to $x \in \mathbb{R}^n_+$, where

$$\lambda_1 := \inf\left\{ \frac{\int_{\mathbb{R}^n_+} |\nabla u|^n dx + \int_{\partial\mathbb{R}^n_+} |u|^n dx'}{\int_{\mathbb{R}^n_+} a|u|^n dx} : u \in C^1_0(\mathbb{R}^n) \setminus \{0\} \right\};$$

(f_2) there exists $\mu > r$ such that

$$0 < \mu F(x,s) \leq f(x,s)s, \quad \forall\, x \in \mathbb{R}^n_+ \quad \text{and} \quad s \neq 0;$$

(f_3) there exist constants $R_0, M_0 > 0$ such that for all $x \in \mathbb{R}^n_+$ and $s \geq R_0$

$$F(x,s) \leq M_0 f(x,s).$$

We assume the following assumptions on the weighted functions $a(x)$, $b(x)$:

(H_1) $a : \mathbb{R}^n_+ \to \mathbb{R}$ is a nontrivial mensurable function and there exists $c_1 > 0$ such that

$$0 \leq a(x) \leq \frac{c_1}{(1+|x|)^n}, \quad \text{a.e. in} \quad \mathbb{R}^n_+;$$

(H_2) $b : \mathbb{R}^n_+ \to \mathbb{R}$ is a positive continuous function satisfying

$$\int_{\mathbb{R}^n_+} \frac{a^{\frac{r}{r-n}}}{b^{\frac{2}{r-n}}} dx < \infty.$$

Under these hypotheses, our first result concerning problem (10) is as follows.

Theorem 0.0.11. *Assume* $(f_1)-(f_3)$ *and* $(H_1)-(H_2)$. *If* $n \leq r < \infty$ *then*

(i) Problem (10) *has no nontrivial weak solution for every* $\lambda \in (-\infty, 0]$;

(ii) Problem (10) *has at least a nontrivial weak solution for every* $\lambda \in (0, \infty)$.

In order to obtain a multiplicity result, in addition, we will assume the assumption on f:

(f_4) there exist $\nu_0, s_0 > 0$ and $\gamma_0 > r$ such that

$$F(x,s) \geq \nu_0|s|^{\gamma_0}, \text{ uniformly with respect to } x \in \mathbb{R}^n_+, \quad \forall\, |s| \leq s_0.$$

Our multiplicity result is established as follows.

Theorem 0.0.12. *Assume* $(H_2) - (H_2)$ *and that* $f(x,\cdot)$ *is odd and satisfies* $(f_1) - (f_4)$*. If* $n \leq r < \infty$*, then Problem* (10) *has an infinite number of nontrivial weak solutions for every* $\lambda \in (0,\infty)$*.*

Finally, in **Chapter 4**, using inequality (2) we develop our approach to problem (5) with $\kappa = 0$ and $\bar{f}$ having polynomial growth which correspond to Neumann boundary value problem. Precisely, we concerned with the following quasilinear elliptic problem

$$\begin{cases} -\text{div}(|\nabla u|^{p-2}\nabla u) &= \lambda a(x)|u|^{q-2}u - b(x)|u|^{r-2}u \quad \text{in} \quad \mathbb{R}^n_+, \\ |\nabla u|^{p-2}\nabla u \cdot \nu &= 0 \qquad\qquad\qquad\qquad\qquad\quad\;\; \text{on} \quad \mathbb{R}^{n-1}, \end{cases} \tag{11}$$

where $n \geq 3$, ν denotes the unit outward normal on the boundary, λ is a real parameter and the weighted functions $a(x)$ and $b(x)$ satisfy some suitable conditions that we will describe later on. As our interest is to analyze the interplay between the powers q and r, we will consider two cases:

(I) $r > q$ and $\frac{p(n-1)}{n-p} =: p_* < q < p^* := \frac{np}{n-p}$ if $1 < p < n$;

(II) $1 < p_* < r < q < p^*$.

We begin by considering the case $r > q$. To this end, we shall assume the following assumptions:

(H_1) $a : \mathbb{R}^n_+ \to \mathbb{R}$ is a nontrivial measurable function and there are constants $\alpha \geq n$ and $c_1 > 0$ such that
$$0 \leq a(x) \leq \frac{c_1}{(1+x_n)^\alpha}, \quad \text{a.e. in} \quad \mathbb{R}^n_+;$$

(H_2) $b : \mathbb{R}^n_+ \to \mathbb{R}$ is a positive continuous function satisfying
$$\int_{\mathbb{R}^n_+} \frac{a^{\frac{r}{r-q}}}{b^{\frac{q}{r-q}}} dx < \infty.$$

Under this hypotheses, our main result can be stated as follows.

Theorem 0.0.13. *Let* $r > q$ *and assume the hypotheses* $(H_1) - (H_2)$*.*

(i) If $1 < p \leq q \leq p^*$*, there exists* $\lambda^* > 0$ *such that problem* (11) *has only the trivial solution for all* $\lambda \in (-\infty, \lambda^*)$*;*

(ii) If $\max\{2,p\} \le q < p^*$, *there exists* $\tilde{\lambda} > 0$ *such that problem* (11) *has at least a nontrivial weak solution for all* $\lambda \in (\tilde{\lambda}, \infty)$. *Furthermore, if* $p < q$ *then* $(\mathcal{P}_{\tilde{\lambda}})$ *has a nontrivial weak solution;*

(iii) If $\max\{2,p\} \le q < p^*$, *there exists* $\Lambda \ge \tilde{\lambda}$ *such that problem* (11) *has at least two nontrivial weak solutions* $u_\lambda \ge \tilde{u}_\lambda$ *for all* $\lambda \in (\Lambda, \infty)$*;*

(iv) If $1 < p \le q < p^*$, *for any* $m \in \mathbb{N}$ *there exists* $\Lambda_m > 0$ *such that problem* (11) *has at least* m *pairs of nontrivial weak solutions for all* $\lambda > \Lambda_m$.

Next we deal with the case $r < q$. In order to prove the existence of solutions for problem (11), instead of hypotheses $(H_1) - (H_2)$, we will assume:

$(\widetilde{H}_1)$ $a : \mathbb{R}^n_+ \to \mathbb{R}$ is a nontrivial measurable function and there are $c_2 > 0$ and $\alpha \ge n$ such that

$$0 \le a(x) \le \frac{c_2}{(1+|x|)^\alpha}, \quad \text{a.e. in} \quad \mathbb{R}^n_+.$$

$(\widetilde{H}_2)$ $b : \mathbb{R}^n_+ \to \mathbb{R}$ is a measurable positive function.

In this case, our main result is stated as follows.

Theorem 0.0.14. *Let* $1 < p \le r < q < p^*$ *and assume the hypotheses* $(\widetilde{H}_1) - (\widetilde{H}_2)$. *Then*

(i) the problem (11) *has no nontrivial weak solution for every* $\lambda \in (-\infty, 0]$*;*

(ii) the problem (11) *has an infinite number of nontrivial weak solutions for every* $\lambda \in (0, \infty)$.

Motivated by the works of Alama-Tarrantelo [7], Filippucci-Pucci-Radulescu [25], Lyberopoulos [27], Perera [31] and Pflüger [34], we will use the variational method to study the problems (9), (10) and (11).

In order to do not get resorting to Introduction and for the sake of independence of the chapters, we will present again in each chapter the main results and the hypotheses.

Part I

A Hardy-Sobolev type inequality and its applications

Chapter 1

A Hardy-Sobolev type inequality and its consequences

This chapter is devoted to the paper [2], where we prove a new Hardy-Sobolev type inequality and as a consequence we establish embedding results of a Sobolev space $\mathcal{E}^{1,p}(\mathbb{R}^n_+)$ defined on the upper half-space. Precisely, for $1 < p < n$ we obtain an embedding from $\mathcal{E}^{1,p}(\mathbb{R}^n_+)$ into weighted Lebesgue spaces. In the borderline case $p = n$, we derive some Trudinger-Moser type inequalities, and in the case $p > n$ we obtain a Morrey's type inequality.

1.1 Introduction and main results

Let $n \geq 2$ an integer number and denote by $\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}$ the upper half-space. Inspired by the paper [6], for any $1 < p < \infty$, we consider the Sobolev space defined by

$$\mathcal{E}^{1,p}(\mathbb{R}^n_+) := \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^n_+) \ : \ \begin{array}{l} u_{|_{\mathbb{R}^{n-1}}} \in L^p(\mathbb{R}^{n-1}) \quad \text{and} \\ u_{x_i} \in L^p(\mathbb{R}^n_+), \quad \forall\, i = 1, \dots, n \end{array} \right\}, \tag{1.1}$$

where u_{x_i} denotes the distributional derivative of u, $u_{|_{\mathbb{R}^{n-1}}}$ is understood in the trace sense and $\mathbb{R}^{n-1}$ denotes the boundary of $\mathbb{R}^n_+$. We can see that $\mathcal{E}^{1,p}(\mathbb{R}^n_+)$, from now on denoted by $\mathcal{E}^{1,p}$, is a reflexive Banach space when equipped with the norm

$$\|u\|_{\mathcal{E}^{1,p}} := \left(\int_{\mathbb{R}^n_+} |\nabla u|^p dx + \int_{\mathbb{R}^{n-1}} |u|^p dx' \right)^{1/p}.$$

From the classical Sobolev trace embedding $W^{1,p}(\mathbb{R}^n_+) \hookrightarrow L^p(\mathbb{R}^{n-1})$, one can see that the embedding $W^{1,p}(\mathbb{R}^n_+) \hookrightarrow \mathcal{E}^{1,p}$ is continuous, but with strict inclusion. In fact, a straightforward computation shows that, for any $(n-1)/2 < \beta < n/2$, the function

$$u_\beta(x', x_n) = \left(1 + |x'|^2 + x_n^2\right)^{-\beta/p}, \quad x' \in \mathbb{R}^{n-1}, \quad x_n > 0,$$

belongs to the Sobolev space $\mathcal{E}^{1,p}$ but not in $W^{1,p}(\mathbb{R}^n_+)$. Moreover, if for any open set $\Omega \subset \mathbb{R}^n$ $\mathcal{D}^{1,p}(\Omega)$ with $1 < p < n$ denotes the completion of $C_0^\infty(\Omega)$ with respect to the norm

$\|u\|_p = (\int_\Omega |\nabla u|^p dx)^{1/p}$, it is well known that the best constant of the Sobolev trace embedding $\mathcal{D}^{1,p}(\mathbb{R}^n_+) \hookrightarrow L^{p_*}(\mathbb{R}^{n-1})$ with $p_* = p(n-1)/(n-p)$, is achieved (see [20]) by the function

$$u(x', x_n) = c_n(1 + |x'|^2 + x_n^2)^{(p-n)/2(p-1)}, \quad x' \in \mathbb{R}^{n-1}, \quad x_n > 0,$$

for some convenient constant $c_n > 0$, however $u \notin \mathcal{E}^{1,p}$. Therefore, we have the continuous embeddings with strict inclusions

$$W^{1,p}(\mathbb{R}^n_+) \hookrightarrow \mathcal{E}^{1,p} \hookrightarrow \mathcal{D}^{1,p}(\mathbb{R}^n_+).$$

In this chapter we focus our attention on embedding results of $\mathcal{E}^{1,p}$. To this end, we start by proving the following weighted Hardy-Sobolev type inequality:

Theorem 1.1.1. *Let $n \geq 2$ and $1 < p < \infty$. Then the following inequality holds*

$$\int_{\mathbb{R}^n_+} \frac{|u|^p}{(1+x_n)^p} dx \leq \left(\frac{p}{p-1}\right)^p \left(\int_{\mathbb{R}^n_+} |\nabla u|^p dx + \int_{\mathbb{R}^{n-1}} |u|^p dx'\right), \quad \forall\, u \in C_0^\infty(\mathbb{R}^n).$$

Our interest in this type of inequality has been mostly motivated by their deep connections with Hardy-type inequalities. As it is well known, the Hardy inequality in the upper half space for function $u \in C_0^\infty(\mathbb{R}^n_+)$ has been extensively investigated by many authors, see for instance [23,24,28,34,42] and references therein. We quote here that the proof of Theorem 1.1.1 is strongly inspired by the arguments used in [23, Theorem 1.4], where the authors obtain a similar result for function in $C_0^\infty(\mathbb{R}^n_+)$. We also point out that Theorem 1.1.1 is an extension of the weighted Hardy-type inequality proved in [34, Lemma 1] where the author proved a similar result with $1 < p < n$ (see also [48] for a related results).

For future applications, let us introduce the Banach space $E^{1,p}$ defined as the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{E^{1,p}} := \left(\int_{\mathbb{R}^n_+} |\nabla u|^p dx + \int_{\mathbb{R}^n_+} \frac{|u|^p}{(1+x_n)^p} dx\right)^{1/p}.$$

An useful consequence of Theorem 1.1.1 is the following corollary which extends [34, Lemma 2] to the case $p \geq n$:

Corollary 1.1.2. *Assume $n \geq 2$ and $1 < p < \infty$. Then the norms $\|\cdot\|_{\mathcal{E}^{1,p}}$ and $\|\cdot\|_{E^{1,p}}$ are equivalent in $E^{1,p}$.*

In order to put our results into perspective, we recall some well known results concerning Hardy inequalities in the upper half-space. As pointed in the paper [42], in the well known book Sobolev Spaces by Maz'ya [28], the following inequality is obtained

$$\frac{1}{16}\int_{\mathbb{R}^n_+} \frac{|u|^2}{(x_{n-1}^2 + x_n^2)^{1/2}} dx \leq \int_{\mathbb{R}^n_+} \frac{|\nabla u|^2}{x_n} dx, \quad \forall\, u \in C_0^\infty(\mathbb{R}^n). \tag{1.2}$$

Taking $u(x) = |x_n|^{-1/2}v(x)$ into (1.2) we get

$$\frac{1}{16}\int_{\mathbb{R}^n_+} \frac{|v|^2}{(x_{n-1}^2 + x_n^2)^{1/2}|x_n|}dx + \frac{1}{4}\int_{\mathbb{R}^n_+} \frac{|v|^2}{x_n^2}dx \leq \int_{\mathbb{R}^n_+} |\nabla v|^2 dx, \quad \forall\, v \in C_0^\infty(\mathbb{R}^n_+). \tag{1.3}$$

In fact, this inequality was improved in [42, Lemma 3.1]. It is an open problem, formulated by Maz'ya, whether the following generalization of the above inequality holds or not:

$$\alpha(p,\tau)\int_{\mathbb{R}^n_+} \frac{|u|^p}{x_n^{p-\tau}(x_{n-1}^2 + x_n^2)^{\tau/2}}dx + \left(\frac{p-1}{p}\right)^p \int_{\mathbb{R}^n_+} \frac{|u|^p}{x_n^p}dx \leq \int_{\mathbb{R}^n_+} |\nabla u|^p dx,$$

where $p > 1$, $\tau > 0$, $\alpha(p,\tau)$ is a positive constant and $u \in C_0^\infty(\mathbb{R}^n_+)$. It is worth pointing out that inequality (1.2) and their variants were studied by a great number of authors, most of them focused on the context of $C_0^\infty(\mathbb{R}^n_+)$, which is motivated mainly on the study of Dirichlet boundary value problems. However, motivated by study of elliptic problems involving nonlinear boundary conditions, it is quite natural to ask if similar results can by forwarded to inequality (1.2) and its variants in the setting of $C_0^\infty(\mathbb{R}^n)$, which is used in many papers, see for instance [16, 32] and references therein. In the last section of this chapter, we comment some applications of our embedding results for the study of some nonlinear elliptic problems involving nonlinear boundary conditions.

With the aid Theorem 1.1.1, we can now prove the embedding of $\mathcal{E}^{1,p}$ into weighted Lebesgue spaces, as it is showed in the next theorem.

Theorem 1.1.3. *Assume $1 < p < n$. Then the weighted Sobolev embedding*

$$\mathcal{E}^{1,p} \hookrightarrow L^q\left(\mathbb{R}^n_+, \frac{1}{(1+x_n)^p}\right), \quad \forall\, p \leq q \leq p^* := \frac{np}{n-p}, \tag{1.4}$$

and the Sobolev trace embedding

$$\mathcal{E}^{1,p} \hookrightarrow L^q(\mathbb{R}^{n-1}), \quad \forall\, p \leq q \leq p_* := \frac{(n-1)p}{n-p} \tag{1.5}$$

are continuous.

In the borderline case $p = n$, we establish the following result:

Theorem 1.1.4. *Assume $p = n$. Then the weighted Sobolev embedding*

$$\mathcal{E}^{1,n} \hookrightarrow L^q\left(\mathbb{R}^n_+, \frac{1}{(1+x_n)^n}\right), \quad \forall\, n \leq q < \infty, \tag{1.6}$$

and the Sobolev trace embedding

$$\mathcal{E}^{1,n} \hookrightarrow L^q(\mathbb{R}^{n-1}), \quad \forall\, n \leq q < \infty \tag{1.7}$$

are continuous.

Remark 1.1.5. *The embeddings* (1.6) *and* (1.7) *are not valid if* $q = \infty$. *In fact, considering the function* $u(x', x_n) := (1 + x_n)^n \ln(1 - \ln|x|)$ *if* $(x', x_n) \in B_1^+$ *and zero otherwise, where* $B_1^+ := \{x = (x', x_n) \in \mathbb{R}^n_+;\ |x| < 1\}$, *one can see that* $u \in \mathcal{E}^{1,n}$ *but* $u \notin L^\infty\left(\mathbb{R}^n_+, (1 + x_n)^{-n}\right)$ *as well as* $u \notin L^\infty(\mathbb{R}^{n-1})$.

Remark 1.1.6. *Since* $(1 + x_n)^n \le (1 + |x|)^n$, *by Theorem 1.1.1 with* $p = n$, *one has*

$$\int_{\mathbb{R}^n_+} \frac{|u|^n}{(1+|x|)^n} dx \le \left(\frac{n}{n-1}\right)^n \left(\int_{\mathbb{R}^n_+} |\nabla u|^n dx + \int_{\mathbb{R}^{n-1}} |u|^n dx'\right), \quad \forall\, u \in C_0^\infty(\mathbb{R}^n).$$

Furthermore, from the embedding (1.6) *we have the continuous embedding*

$$\mathcal{E}^{1,n} \hookrightarrow L^q\left(\mathbb{R}^n_+, \frac{1}{(1+|x|)^n}\right), \quad \forall\, n \le q < \infty. \tag{1.8}$$

which is not valid if $q = \infty$. *In fact, considering the function* $u(x', x_n) := (1+|x|)^n \ln(1 - \ln|x|)$ *if* $(x', x_n) \in B_1^+$ *and zero otherwise, one can see that* $u \in \mathcal{E}^{1,n}$ *but,* $u \notin L^\infty\left(\mathbb{R}^n_+, (1 + |x|)^{-n}\right)$.

In view of Remarks 1.1.5 and 1.1.6, it is natural to study embedding from $\mathcal{E}^{1,n}$ into Orlicz space. To this end, we introduce a new weighted Sobolev space, which plays a central role in the proof of our Trudinger-Moser type inequality.

Definition 1.1.7. *Consider the weight function* $b(x) := (1 + |x|)^{-n}$. *We define the space* $\mathcal{E}_b^{1,n}$ *as the completion of* $C_0^\infty(\mathbb{R}^n)$ *with respect to the norm*

$$\|u\|^n_{\mathcal{E}_b^{1,n}} := \int_{\mathbb{R}^n_+} \left(|\nabla u|^n + b(x)|u|^n\right) dx.$$

We establish the following embedding result involving $\mathcal{E}_b^{1,n}$:

Theorem 1.1.8. *For any* $n \le q < \infty$, *the embedding*

$$\mathcal{E}_b^{1,n} \hookrightarrow L^q\left(\mathbb{R}^n_+, b(x)\right) \tag{1.9}$$

is continuous. Furthermore, the same example in Remark 1.1.6 shows that this embedding is false if $q = \infty$.

Considering the Young function defined by

$$\Psi_\alpha(s) = e^{\alpha|s|^{n'}} - \sum_{k=0}^{n-2} \frac{\alpha^k}{k!}|s|^{n'k}, \quad s \in \mathbb{R}, \tag{1.10}$$

where $n' := n/(n-1)$ and according to (1.9), the following Trudinger-Moser type inequality in the setting of $\mathcal{E}_b^{1,n}$ is natural.

Theorem 1.1.9. *For any $\alpha > 0$ we have that $\Psi_\alpha(u) \in L^1(\mathbb{R}^n_+, b(x))$. Moreover, there exists a constant $\alpha_0 > 0$, independent of $u \in \mathcal{E}_b^{1,n}$, such that*

$$L(\alpha) := \sup_{\{u\in\mathcal{E}_b^{1,n}:\ \|u\|_{\mathcal{E}_b^{1,n}}\le 1\}} \int_{\mathbb{R}^n_+} b(x)\Psi_\alpha(u)dx < +\infty, \tag{1.11}$$

for any $0 < \alpha \le \alpha_0$.

As a consequence of Theorem 1.1.9, the following Trudinger-Moser type inequality in the setting of $\mathcal{E}^{1,n}$ holds.

Corollary 1.1.10. *For any $u \in \mathcal{E}^{1,n}$ and $\alpha \ge 0$, we have that $\Psi_\alpha(u) \in L^1(\mathbb{R}^n_+, b(x))$. Moreover,*

$$l(\alpha) := \sup_{\{u\in\mathcal{E}^{1,n}:\ \|u\|_{\mathcal{E}^{1,n}}\le 1\}} \int_{\mathbb{R}^n_+} b(x)\Psi_\alpha(u)dx < +\infty,$$

for any $0 < \alpha \le \alpha_0/\left(2n/(n-1)\right)^{n'}$.

In the trace sense, we have the following Trudinger-Moser type inequality:

Corollary 1.1.11. *For any $u \in \mathcal{E}^{1,n}$ and $\alpha \ge 0$, we have that $\Psi_\alpha(u(\cdot,0)) \in L^1(\mathbb{R}^{n-1}, b(x',0))$. Moreover,*

$$T(\alpha) := \sup_{\{u\in\mathcal{E}^{1,n}:\ \|u\|_{\mathcal{E}^{1,n}}\le 1\}} \int_{\mathbb{R}^{n-1}} b(x',0)\Psi_\alpha(u(x',0))dx' < +\infty,$$

for any $0 < \alpha \le \left((n-1)/n\right)^2 \alpha_0/\left(2n/(n-1)\right)^{n'}$.

Remark 1.1.12. *It is worthwhile to mention here, that we believe that the natural weight function to consider in the Trudinger-Moser inequalities above must be $a(x) = (1+x_n)^{-n}$. However, we were not able to consider this situation in our approach. As usual, we can not apply Schwarz symmetrization arguments as considered in many papers (see for instance* [39] *and references therein).*

Finally, we consider the case $p > n$ that corresponds to the Morrey's case.

Theorem 1.1.13. *Assume $n < p < \infty$ and $a(x) := (1+x_n)^{-p}$. Then the following weighted Sobolev embedding holds*

$$\mathcal{E}^{1,p} \hookrightarrow L^\infty(\mathbb{R}^n_+, a(x)). \tag{1.12}$$

Furthermore, for all $u \in \mathcal{E}^{1,p}$ there exists $C_0 = C_0(n,p) > 0$ such that for a.e. $x, y \in \mathbb{R}^n_+$

$$|a(x)u(x) - a(y)u(y)| \le C_0|x-y|^\gamma \left(\|\nabla u\|_{L^p(\mathbb{R}^n_+)} + \|u\|_{L^p(\mathbb{R}^n_+ a(x))}\right), \tag{1.13}$$

where $\gamma = 1 - n/p$.

This chapter is organized as follows. In Section 1.2, we prove Theorem 1.1.1 and Corollary 1.1.2. The Sobolev embedding, for $1 < p \le n$, into Lebesgue spaces are proved in Sections 1.3 and 1.4. In Section 1.5, we prove the Trudinger-Moser inequalities established in Theorem 1.1.9 and Corollaries 1.1.10 and 1.1.11. Finally, in Section 1.6 we prove Theorem 1.1.13.

1.2 A Hardy-Sobolev type inequality

This section is devoted to the proof of Theorem 1.1.1 which is the main step in the proof of our embedding results. The proof is inspired in the paper [23], where the authors obtain a similar result for function in $C_0^\infty(\mathbb{R}^n_+)$. We also included the proof of Corollary 1.1.2.

Proof of Theorem 1.1.1. Let $v \in C_0^1(\mathbb{R}^n)$ and $\sigma \in \mathbb{R}$ with $\sigma \neq -1$. Using integration by parts, we obtain

$$\begin{aligned}(\sigma+1)\int_{\mathbb{R}^n_+}(1+x_n)^\sigma |v| dx &= \int_{\mathbb{R}^n_+}\frac{\partial}{\partial x_n}((1+x_n)^{\sigma+1})|v|dx \\ &= -\int_{\mathbb{R}^n_+}(1+x_n)^{\sigma+1}(|v|)_{x_n}dx - \int_{\mathbb{R}^{n-1}}|v|dx',\end{aligned}$$

where above we used that $\eta = (0', -1)$ is the outwards normal to $\mathbb{R}^{n-1}$. Thus, we get

$$|\sigma+1|\int_{\mathbb{R}^n_+}(1+x_n)^\sigma |v| dx \le \int_{\mathbb{R}^n_+}(1+x_n)^{\sigma+1}|\nabla v|dx + \int_{\mathbb{R}^{n-1}}|v|dx'.$$

Applying this inequality with $v = |u|^p$, $p > 1$ and $u \in C_0^\infty(\mathbb{R}^n)$ we infer that

$$|\sigma+1|\int_{\mathbb{R}^n_+}(1+x_n)^\sigma |u|^p dx \le \int_{\mathbb{R}^n_+}(1+x_n)^{\sigma+1}p|u|^{p-1}|\nabla u|dx + \int_{\mathbb{R}^{n-1}}|u|^p dx'. \tag{1.14}$$

Now, for any $\varepsilon > 0$ and $a, b \ge 0$ we can use the elementary inequality

$$ab = \left(\varepsilon^{\frac{p-1}{p}}a\right)\left(\frac{b}{\varepsilon^{\frac{p-1}{p}}}\right) \le \frac{(p-1)}{p}\left(\varepsilon^{\frac{p-1}{p}}a\right)^{p/(p-1)} + \frac{1}{p}\left(\frac{b}{\varepsilon^{\frac{p-1}{p}}}\right)^p,$$

to derive the inequality

$$p\int_{\mathbb{R}^n_+}(1+x_n)^{\sigma+1}|u|^{p-1}|\nabla u|dx \;\; \le (p-1)\varepsilon\int_{\mathbb{R}^n_+}(1+x_n)^{\frac{(\sigma+1)p}{p-1}}|u|^p dx + \frac{1}{\varepsilon^{(p-1)}}\int_{\mathbb{R}^n_+}|\nabla u|^p dx. \tag{1.15}$$

Choosing $\sigma = p(\sigma+1)/(p-1)$, that is, $\sigma = -p$ and combining inequalities (1.14) and (1.15), one has

$$\int_{\mathbb{R}^n_+}\frac{|u|^p}{(1+x_n)^p}dx \le \frac{1}{(p-1)(\varepsilon^{p-1}-\varepsilon^p)}\left(\int_{\mathbb{R}^n_+}|\nabla u|^p dx + \int_{\mathbb{R}^{n-1}}|u|^p dx'\right),$$

for any $0 < \varepsilon < 1$. Taking into account that the function $g(\varepsilon) = 1/(\varepsilon^{p-1} - \varepsilon^p)$ with $0 < \varepsilon < 1$ achieves its minimum at $\varepsilon_0 = (p-1)/p$ and $g(\varepsilon_0) = p^p/(p-1)^{p-1}$ we conclude that

$$\int_{\mathbb{R}^n_+}\frac{|u|^p}{(1+x_n)^p}dx \le \left(\frac{p}{p-1}\right)^p\left(\int_{\mathbb{R}^n_+}|\nabla u|^p dx + \int_{\mathbb{R}^{n-1}}|u|^p dx'\right),$$

which completes the proof. □

Proof of Corollary 1.1.2. Let $u \in C_0^\infty(\mathbb{R}^n)$. By Theorem 1.1.1 we have $\|u\|_{E^{1,p}} \leq C_1 \|u\|_{\mathcal{E}^{1,p}}$. To prove the reverse inequality, using the Young inequality with the conjugate exponents p and $p/(p-1)$, we observe that

$$\begin{aligned}
|u(x',0)|^p &= -\int_0^{+\infty} \frac{\partial}{\partial x_n}\left(\frac{|u|^p}{(1+x_n)^p}\right) dx_n \\
&\leq p\int_0^{+\infty} \frac{|u|^{p-1}|\nabla u|}{(1+x_n)^p} dx_n + p\int_0^{+\infty} \frac{|u|^p}{(1+x_n)^{p+1}} dx_n \\
&\leq \int_0^{+\infty} |\nabla u|^p dx_n + (2p-1)\int_0^{+\infty} \frac{|u|^p}{(1+x_n)^p} dx_n,
\end{aligned}$$

where above we used that $1/(1+x_n)^{p+1} \leq 1/(1+x_n)^p$. Integrating this inequality we obtain

$$\int_{\mathbb{R}^{n-1}} |u(x',0)|^p dx' \leq \int_{\mathbb{R}^n_+} |\nabla u|^p dx + (2p-1)\int_{\mathbb{R}^n_+} \frac{|u|^p}{(1+x_n)^p} dx.$$

Thus,

$$\begin{aligned}
\|u\|^p_{\mathcal{E}^{1,p}} &= \int_{\mathbb{R}^n_+} |\nabla u|^p dx + \int_{\mathbb{R}^{n-1}} |u(x',0)|^p dx' \\
&\leq 2\int_{\mathbb{R}^n_+} |\nabla u|^p dx + (2p-1)\int_{\mathbb{R}^n_+} \frac{|u|^p}{(1+x_n)^p} dx \\
&\leq \max\{2, 2p-1\}\|u\|^p_{E^{1,p}},
\end{aligned}$$

and the proof is complete. □

1.3 Embedding into Lebesgue spaces $(1 < p < n)$

In this section, we prove Theorem 1.1.3. To this end, we first establish a density result in the context of the Sobolev space $\mathcal{E}^{1,p}$, which is a consequence of Theorem 1.1.1. Hereafter in this chapter, B_R denotes the ball of center zero and radius $R > 0$ in $\mathbb{R}^n$, $B_R^+ := B_R \cap \mathbb{R}^n_+$, $(B_R)^c$ denotes $\mathbb{R}^n \setminus B_R$, the complement of the set $B_R \subset \mathbb{R}^n$, and $(B_R^+)^c$ denotes $\mathbb{R}^n_+ \setminus B_R^+$ the complement of the set $B_R^+ \subset \mathbb{R}^n_+$.

Lemma 1.3.1. *Let $n \geq 2$. Then the set of restrictions to $\mathbb{R}^n_+$ of functions in $C_0^\infty(\mathbb{R}^n)$ is dense in $\mathcal{E}^{1,p}$.*

Proof. We start by proving that the Sobolev space $W^{1,p}(\mathbb{R}^n_+)$ is dense in $\mathcal{E}^{1,p}$. In fact, let $R > 0$ and consider a smooth function $\varphi_R : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\varphi_R(x', x_n) = \begin{cases} 1, & |(x', x_n)| \leq R \\ 0, & |(x', x_n)| \geq R+1, \end{cases}$$

$0 \leq \varphi_R \leq 1$ and $|\nabla \varphi_R| \leq 1/R$. Given $u \in \mathcal{E}^{1,p}$ we can see that $u_R = u\varphi_R \in W^{1,p}(\mathbb{R}^n_+)$ and

$$\begin{aligned}\|u-u_R\|^p_{\mathcal{E}^{1,p}} &= \int_{(B_R^+)^c} |\nabla u - \nabla u_R|^p dx + \int_{|x'|>R} |u(1-\varphi_R)|^p dx' \\ &= o_R(1) + 2^p \int_{(B_R^+)^c} |\nabla u_R|^p dx + o_R(1),\end{aligned}$$

where $o_R(1) \to 0$ denotes a quantity that goes to zero as $R \to \infty$. Using straightforward calculations we obtain

$$\begin{aligned}\int_{(B_R^+)^c} |\nabla u_R|^p dx &\leq C\left(\int_{(B_R^+)^c} |\nabla u|^p \varphi_R^p dx + \int_{(B_R^+)^c} |u|^p |\nabla \varphi_R|^p dx\right) \\ &\leq C\left(\int_{(B_R^+)^c} |\nabla u|^p dx + \frac{1}{R^p}\int_{A^+_{R,R+1}} |u|^p dx\right) \\ &\leq o_R(1) + \frac{C}{R^p}\int_{A^+_{R,R+1}} |u|^p dx,\end{aligned}$$

where $A^+_{R,R+1} := \{(x', x_n) \in \mathbb{R}^n_+ :\ R \leq |(x', x_n)| \leq R+1\}$. We claim that

$$\frac{1}{R^p}\int_{A^+_{R,R+1}} |u|^p dx = o_R(1). \tag{1.16}$$

Indeed, by the Friedrichs inequality there exists $C_1 > 0$ satisfying the inequality

$$\int_{A^+_{R,R+1}} |v|^p dx \leq C_1 \left(\int_{A^+_{R,R+1}} |\nabla v|^p dx + \int_{\Gamma_{R,R+1}} |v|^p dx\right),$$

where $\Gamma_{R,R+1} = \{(x', 0); R \leq |(x', 0)| \leq R+1\}$. Choosing $v(x) = u(Rx)$ in this inequality and performing a change of variable we obtain

$$\frac{1}{R^p}\int_{A^+_{R,R+1}} |u|^p dx \leq C_1 \left(\int_{A^+_{R,R+1}} |\nabla u|^p dx + \frac{1}{R^{p-1}}\int_{\mathbb{R}^{n-1}} |v|^p dx'\right) = o_R(1)$$

as claimed in (1.16). Now fixed $u \in \mathcal{E}^{1,p}$ and $\varepsilon > 0$, by the first step there exists $u_1 \in W^{1,p}(\mathbb{R}^n_+)$ such that

$$\|u - u_1\|_{\mathcal{E}^{1,p}} \leq \varepsilon. \tag{1.17}$$

On the other hand, taking into account that $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{1,p}(\mathbb{R}^n_+)$ (see [5, Theorem 3.18]), there exists $v \in C_0^\infty(\mathbb{R}^n)$ such that $\|u_1 - v\|_{W^{1,p}(\mathbb{R}^n_+)} \leq \varepsilon$. Since $u_1 - v \in W^{1,p}(\mathbb{R}^n_+) \hookrightarrow \mathcal{E}^{1,p}$ we get $\|u_1 - v\|_{\mathcal{E}^{1,p}} \leq C\|u_1 - v\|_{W^{1,p}(\mathbb{R}^n_+)} \leq C\varepsilon$, which in combination with (1.17) imply $\|u - v\|_{\mathcal{E}^{1,p}} \leq \|u - u_1\|_{\mathcal{E}^{1,p}} + \|u_1 - v\|_{\mathcal{E}^{1,p}} \leq \varepsilon + C\varepsilon$, and this completes the proof. □

As an immediate consequence of Lemma 1.3.1, we have the following result.

Corollary 1.3.2. *Define $\widetilde{\mathcal{E}}^{1,p}$ as the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm*

$$\|u\| := \left(\int_{\mathbb{R}^n_+} |\nabla u|^p dx + \int_{\mathbb{R}^{n-1}} |u|^p dx' \right)^{1/p}.$$

Then, $\mathcal{E}^{1,p} = \widetilde{\mathcal{E}}^{1,p}$.

Now we are ready to prove Theorem 1.1.3.

Proof of Theorem 1.1.3. To prove the embedding (1.4), we first recall that for $1 \le p < n$ it follows from the Gagliardo-Nirenberg-Sobolev inequality and a suitable reflexion argument (see [43, Lemma 2.10]) that there exists $C_0 = C_0(n,p)$ such that

$$\left(\int_{\mathbb{R}^n_+} |u|^{p^*} dx \right)^{(n-p)/n} \le C_0 \int_{\mathbb{R}^n_+} |\nabla u|^p dx, \quad \forall\, u \in C_0^1(\mathbb{R}^n). \tag{1.18}$$

This in combination with the fact that $(1+x_n)^{-p} \le 1$ and Lemma 1.3.1 imply that $\mathcal{E}^{1,p} \hookrightarrow L^{p^*}(\mathbb{R}^n_+, (1+x_n)^{-p})$. This together with Theorem 1.1.1 and an interpolation argument imply that $\mathcal{E}^{1,p} \hookrightarrow L^q(\mathbb{R}^n_+, (1+x_n)^{-p})$ for all $q \in [p, p^*]$, as stated in (1.4). Now we will prove the embedding (1.5). From the trace inequality (see [20])

$$\left(\int_{\mathbb{R}^{n-1}} |u|^{p_*} dx' \right)^{(n-p)/(n-1)} \le C_0 \int_{\mathbb{R}^n_+} |\nabla u|^p dx, \quad \forall\, u \in C_0^1(\mathbb{R}^n), \tag{1.19}$$

we see that $\mathcal{E}^{1,p} \hookrightarrow L^{p_*}(\mathbb{R}^{n-1})$. On the other hand, by definition we have $\mathcal{E}^{1,p} \hookrightarrow L^p(\mathbb{R}^{n-1})$. Thus, by an interpolation argument we obtain that $\mathcal{E}^{1,p} \hookrightarrow L^q(\mathbb{R}^{n-1})$ for any $q \in [p, p_*]$ and this completes the proof of Theorem 1.1.3. □

1.4 Embedding into Lebesgue spaces $(p = n)$

In this section we present the proof of Theorems 1.1.4 and 1.1.8.

Proof of Theorem 1.1.4. First we prove the embedding (1.6). For that, from estimate (1.18) with $p = 1$ we have

$$\left(\int_{\mathbb{R}^n_+} |v|^{\frac{n}{n-1}} dx \right)^{(n-1)/n} \le C_0 \int_{\mathbb{R}^n_+} |\nabla v| dx, \quad \forall\, v \in C_0^1(\mathbb{R}^n). \tag{1.20}$$

Applying (1.20) with $v = (1+x_n)^\alpha |u|^n$ for any $u \in C_0^\infty(\mathbb{R}^n)$ we infer that

$$\left(\int_{\mathbb{R}^n_+} |(1+x_n)^\alpha |u|^n|^{\frac{n}{n-1}} dx\right)^{(n-1)/n} \leq C_0 \int_{\mathbb{R}^n_+} |\alpha|(1+x_n)^{\alpha-1}|u|^n dx + C_0 n \int_{\mathbb{R}^n_+} (1+x_n)^\alpha |u|^{n-1}|\nabla u| dx.$$

Choosing $\alpha = -(n-1)$ and using the Young inequality with the conjugate exponents n and $n/(n-1)$, ones has

$$\left(\int_{\mathbb{R}^n_+} \frac{|u|^{\frac{n^2}{n-1}}}{(1+x_n)^n} dx\right)^{(n-1)/n} \leq C_1 \left(\int_{\mathbb{R}^n_+} \frac{|u|^n}{(1+x_n)^n} dx + \int_{\mathbb{R}^n_+} |\nabla u|^n dx\right),$$

where C_1 depends only on n. This in combination with Theorem 1.1.1 and Lemma 1.3.1 imply that $\mathcal{E}^{1,n} \hookrightarrow L^{\frac{n^2}{n-1}}(\mathbb{R}^n_+, (1+x_n)^{-n})$. If $n \leq q \leq n^2/(n-1)$, by an interpolation argument, there exists $0 < \theta < 1$ such that

$$\|u\|_{L^q(\mathbb{R}^n_+, \frac{1}{(1+x_n)^n})} \leq \|u\|^\theta_{L^n(\mathbb{R}^n_+, \frac{1}{(1+x_n)^n})} \|u\|^{1-\theta}_{L^{\frac{n^2}{(n-1)}}(\mathbb{R}^n_+, \frac{1}{(1+x_n)^n})} \leq C\|u\|_{\mathcal{E}^{1,n}}.$$

In particular, using that $n < n+1 < n^2/(n-1)$, one has $\mathcal{E}^{1,n} \hookrightarrow L^{n+1}\left(\mathbb{R}^n_+, (1+x_n)^{-n}\right)$. On the other hand, applying again (1.20) with $v = (1+x_n)^{-(n-1)}|u|^{n+1}$ and using the Young inequality with the conjugate exponents n and $n/(n-1)$ we get

$$\begin{aligned}\left(\int_{\mathbb{R}^n_+} \frac{|u|^{\frac{n(n+1)}{n-1}}}{(1+x_n)^n} dx\right)^{(n-1)/n} &\leq (n-1)C \int_{\mathbb{R}^n_+} \frac{|u|^{n+1}}{(1+x_n)^n} dx + (n+1)C \int_{\mathbb{R}^n_+} \frac{|u|^n |\nabla u|}{(1+x_n)^{(n-1)}} dx \\ &\leq (n-1)C \int_{\mathbb{R}^n_+} \frac{|u|^{n+1}}{(1+x_n)^n} dx + (n+1)C \left(\int_{\mathbb{R}^n_+} \frac{|u|^{\frac{n^2}{n-1}}}{(1+x_n)^n} dx + \int_{\mathbb{R}^n_+} |\nabla u|^n dx\right),\end{aligned}$$

which implies that $\mathcal{E}^{1,n} \hookrightarrow L^{\frac{n(n+1)}{(n-1)}}(\mathbb{R}^n_+, (1+x_n)^{-n})$ and by using an interpolation argument we get $\mathcal{E}^{1,n} \hookrightarrow L^q(\mathbb{R}^n_+, (1+x_n)^{-n})$ for any $n \leq q \leq n(n+1)/(n-1)$. Reiterating this argument with $k = n+2, n+3, \ldots$, one has $\mathcal{E}^{1,n} \hookrightarrow L^{\frac{nk}{n-1}}(\mathbb{R}^n_+, (1+x_n)^{-n})$. Now, given $q \in [n, \infty)$, one can choose $k \geq n$ such that $n \leq q \leq nk/(n-1)$ and once again using an interpolation argument we get

$$\|u\|_{L^q(\mathbb{R}^n_+, (1+x_n)^{-n})} \leq \|u\|^\theta_{L^n(\mathbb{R}^n_+, (1+x_n)^{-n})} \|u\|^{1-\theta}_{L^{\frac{nk}{n-1}}(\mathbb{R}^n_+, (1+x_n)^{-n})} \leq C\|u\|_{\mathcal{E}^{1,n}},$$

which proves the embedding (1.6). Now we will prove the trace embedding (1.7). For that, by

Lemma 1.3.1 we may assume that $u \in C_0^\infty(\mathbb{R}^n)$. Fixed $q \geq n$ we have

$$|u(x',0)|^q = -\int_0^{+\infty} \frac{\partial}{\partial x_n}\left(\frac{|u|^q}{(1+x_n)^n}\right) dx_n$$
$$\leq q\int_0^{+\infty} \frac{|u|^{q-1}|\nabla u|}{(1+x_n)^n} dx_n + n\int_0^{+\infty} \frac{|u|^q}{(1+x_n)^{n+1}} dx_n.$$

Integrating this inequality and using the Hölder inequality together with the fact that $(1+x_n)^{-1} < 1$ we infer that

$$\int_{\mathbb{R}^{n-1}} |u(x',0)|^q dx' \leq q\left(\int_{\mathbb{R}^n_+} \frac{|u|^{(q-1)\frac{n}{n-1}}}{(1+x_n)^n} dx\right)^{(n-1)/n}\left(\int_{\mathbb{R}^n_+} |\nabla u|^n dx\right)^{1/n} + n\int_{\mathbb{R}^n_+} \frac{|u|^q}{(1+x_n)^n} dx.$$

Since $n(q-1)/(n-1) \geq n$, by the embedding (1.6) we get $\|u\|^q_{L^q(\mathbb{R}^{n-1})} \leq C_1\|u\|^{q-1}_{\mathcal{E}^{1,n}}\|u\|_{\mathcal{E}^{1,n}} + C_2\|u\|^q_{\mathcal{E}^{1,n}}$, which completes the proof of Theorem 1.1.4. □

Now we present the proof of Theorem 1.1.8.

Proof of Theorem 1.1.8. Applying (1.20) with $v = (1+|x|)^\alpha |u|^n$, we get

$$\left(\int_{\mathbb{R}^n_+} |(1+|x|)^\alpha |u|^n|^{\frac{n}{n-1}} dx\right)^{(n-1)/n} \leq C_0\int_{\mathbb{R}^n_+} |\alpha|(1+|x|)^{\alpha-1}\frac{x}{|x|}|u|^n dx$$
$$+ C_0 n\int_{\mathbb{R}^n_+} (1+|x|)^\alpha |u|^{n-1}|\nabla u| dx.$$

Choosing $\alpha = -(n-1)$ and using the Young inequality with the conjugate exponents n and $n/(n-1)$ we obtain

$$\left(\int_{\mathbb{R}^n_+} b(x)|u|^{\frac{n^2}{n-1}} dx\right)^{(n-1)/n} \leq C_1\left(\int_{\mathbb{R}^n_+} b(x)|u|^n dx + \int_{\mathbb{R}^n_+} |\nabla u|^n dx\right),$$

where C_1 depends only on n. This implies that $\mathcal{E}^{1,n}_b \hookrightarrow L^q(\mathbb{R}^n_+, b(x))$ for any $n \leq q \leq n^2/(n-1)$. In particular, one has $\mathcal{E}^{1,n}_b \hookrightarrow L^{n+1}\left(\mathbb{R}^n_+, b(x)\right)$. Applying again (1.20) with $v = (1+|x|)^{-(n-1)}|u|^{n+1}$ and using the Young inequality with the conjugate exponents n and $n/(n-1)$ we get

$$\left(\int_{\mathbb{R}^n_+} b(x)|u|^{\frac{n(n+1)}{n-1}} dx\right)^{(n-1)/n} \leq |(n-1)|C\int_{\mathbb{R}^n_+} b(x)|u|^{n+1}\frac{x}{|x|} dx + (n+1)C\int_{\mathbb{R}^n_+} \frac{|u|^n|\nabla u|}{(1+|x|)^{n-1}} dx$$
$$\leq C_2\left(\int_{\mathbb{R}^n_+} b(x)|u|^{n+1} dx + \int_{\mathbb{R}^n_+} b(x)|u|^{\frac{n^2}{n-1}} dx + \int_{\mathbb{R}^n_+} |\nabla u|^n dx\right).$$

where C_2 depends only on n. Hence by an interpolation argument $\mathcal{E}^{1,n}_b \hookrightarrow L^q(\mathbb{R}^n_+, b(x))$, for any $n \leq q \leq n(n+1)/(n-1)$. Reiterating this argument with $k = n+2, n+3, \ldots$, one has $\mathcal{E}^{1,n}_b \hookrightarrow L^{\frac{nk}{n-1}}(\mathbb{R}^n_+, b(x))$. Now, given $q \in [n, \infty)$, one can choose $k \geq n$ such that $n \leq q \leq nk/(n-1)$

and once again by an interpolation argument we get $\mathcal{E}_b^{1,n} \hookrightarrow L^{\frac{nk}{n-1}}(\mathbb{R}^n_+, b(x))$ which proves the embedding (1.9) and this completes the proof. $\square$

1.5 Trudinger-Moser inequalities

In this section, we present the proof of Theorem 1.1.9 and Corollary 1.1.10. To prove Theorem 1.1.9 we will combine the ideas of Kufner-Opic [30] and Yang-Zhu [47]. First we recall a basic fact.

Lemma 1.5.1 ([47]). *Let $\alpha_n := n\omega_{n-1}^{\frac{1}{n-1}}$, where ω_{n-1} is the measure of the unit sphere in $\mathbb{R}^n$. For any $R > 0$, there exists a constant $C_0 = C_0(n) > 0$ such that for any $y \in \mathbb{R}^n$ and $v \in W_0^{1,n}(B_R(y))$ with $\|\nabla v\|_{L^n(B_R(y))} \leq 1$ we have*

$$\int_{B_R(y)} \Psi_{\alpha_n}(v)dx \leq C_0 R^n.$$

Our strategy to prove Theorem 1.1.9 is consider for $u \in \mathcal{E}_b^{1,n}$ its extension to the whole space $\mathbb{R}^n$

$$\bar{u}(x, x_n) = \begin{cases} u(x, x_n), & x_n > 0 \\ u(x, -x_n), & x_n < 0. \end{cases} \tag{1.21}$$

For any $R > 0$ we can split the integral in (1.11) as follows

$$2\int_{\mathbb{R}^n_+} b(x)\Psi_\alpha(u)dx = \int_{B_R} b(x)\Psi_\alpha(\bar{u})dx + \int_{(B_R)^c} b(x)\Psi_\alpha(\bar{u})dx. \tag{1.22}$$

Now we will estimate the first integral on the right hand side of (1.22).

Lemma 1.5.2. *Let $u \in \mathcal{E}_b^{1,n}$ be such that $\|u\|_{\mathcal{E}_b^{1,n}} \leq 1$ and $R > 1$. Then there are $\alpha_1 > 0$ and $C_0 = C_0(R) > 0$ such that*

$$\int_{B_R} b(x)\Psi_\alpha(\bar{u})dx \leq C_0,$$

for any $0 < \alpha \leq \alpha_1$.

Proof. Consider a cut-off function $\varphi \in C_0^\infty(B_{2R})$ such that $0 \leq \varphi \leq 1$ and

$$\varphi \equiv 1 \text{ in } B_R \quad \text{and} \quad |\nabla\varphi| \leq \frac{C}{R} \text{ in } B_{2R}$$

for some $C > 0$. Note that $\varphi\bar{u} \in W_0^{1,n}(B_{2R})$ and by straightforward calculation we check that

$$\begin{aligned}\int_{B_{2R}} |\nabla(\varphi\bar{u})|^n dx &\le 2^{n-1}\left(\int_{B_{2R}} |\varphi|^n|\nabla\bar{u}|^n dx + \int_{B_{2R}} |\nabla\varphi|^n|\bar{u}|^n dx\right)\\ &\le 2^{n-1}\left(\int_{B_{2R}} |\nabla\bar{u}|^n dx + \frac{C^n}{R^n}\int_{B_{2R}} |\bar{u}|^n dx\right)\\ &\le 2^{n-1}\left(\int_{B_{2R}} |\nabla\bar{u}|^n dx + C^n\frac{(1+2R)^n}{R^n}\int_{B_{2R}} b(x)|\bar{u}|^n dx\right),\end{aligned}$$

and hence,

$$\int_{B_{2R}} |\nabla(\varphi\bar{u})|^n dx \le C_1\int_{B_{2R}} \left(|\nabla\bar{u}|^n + b(x)|\bar{u}|^n\right) dx,$$

where $C_1 := 2^{n-1}\max\{1,(3C)^n\}$. Note that $v := \varphi\bar{u}/\sqrt[n]{2C_1} \in W_0^{1,n}(B_{2R})$ and

$$\|\nabla v\|^n_{L^n(B_{2R})} = \frac{\|\nabla(\varphi\bar{u})\|^n_{L^n(B_{2R})}}{2C_1} \le \frac{1}{2}\int_{\mathbb{R}^n} \left(|\nabla\bar{u}|^n + b(x)|\bar{u}|^n\right) dx \le 1.$$

Since $b(x) \le 1$, in view of Lemma 1.5.1 and the fact that $\Psi_\alpha(cs) = \Psi_{\alpha|c|^{n'}}(s)$, for all $c \in \mathbb{R}$, we conclude that

$$\int_{B_R} b(x)\Psi_\alpha(\bar{u})dx \le \int_{B_R} \Psi_\alpha(\varphi\bar{u})dx \le \int_{B_{2R}} \Psi_{\alpha(2C_1)^{\frac{1}{n-1}}}(v)dx \le C_0(2R)^n,$$

if $0 < \alpha \le \alpha_1 := \alpha_n/(2C_1)^{\frac{1}{n-1}}$ and this completes the proof of Lemma 1.5.2. □

Now we proceed to estimate the second integral on the right hand side of (1.22).

Lemma 1.5.3. *Let $u \in \mathcal{E}_b^{1,n}$ be such that $\|u\|_{\mathcal{E}_b^{1,n}} \le 1$. Then there are $\alpha_2 > 0$ and $C_2 > 0$ independent of $u \in \mathcal{E}_b^{1,n}$ such that*

$$\int_{B_{3r}^c} b(x)\Psi_\alpha(\bar{u})dx \le C_2,$$

for any $r > 1$ and $0 < \alpha \le \alpha_2$.

Proof. Given $r \ge 1$ and $\sigma > r$ we define the annuli

$$A_r^\sigma := \{x \in B_r^c : |x| < \sigma\} = \{x \in \mathbb{R}^n : r < |x| < \sigma\}.$$

A trick adaption of Besicovitch covering lemma [26] (see [18, estimate (4.8)]) shows that there exist a sequence of points $\{x_k\}_k \in A_1^\sigma$ and a universal constant $\theta > 0$ such that

$$A_1^\sigma \subseteq \bigcup_k U_k^{1/2} \quad \text{and} \quad \sum_k \chi_{U_k}(x) \le \theta, \quad \forall\, x \in \mathbb{R}^n,$$

where $U_k^{1/2} := B\left(x_k, \frac{|x_k|}{6}\right)$ and χ_{U_k} denotes the function characteristic of $U_k := B\left(x_k, \frac{|x_k|}{3}\right)$. Let $u \in \mathcal{E}_b^{1,n}$ be such that $\|u\|_{\mathcal{E}_b^{1,n}} \le 1$. In order to estimate the integral of $\bar{u}$ in A_{3r}^σ, we fix $1 < r < \sigma$

and we follow as in [30] introducing the set of indices

$$K_{r,\sigma} := \left\{ k \in \mathbb{N} : \; U_k^{1/2} \cap B_{3r}^c \neq \varnothing \right\}.$$

It is easy to see that, if $U_k \cap B_{3r}^c \neq \varnothing$, then $U_k \subset B_r^c$. Moreover, since $1 < r < 3r$, we have that $A_{3r}^\sigma \subset A_1^\sigma$. Now using and the definition of $K_{r,\sigma}$ we get

$$A_{3r}^\sigma \subseteq \bigcup_{k \in K_{r,\sigma}} U_k^{1/2} \subseteq \bigcup_{k \in K_{r,\sigma}} U_k \subseteq B_r^c \subseteq B_1^c \tag{1.23}$$

and hence

$$\int_{A_{3r}^\sigma} b(x)\Psi_\alpha(\bar{u})dx \leq \sum_{k \in K_{r,\sigma}} \int_{U_k^{1/2}} b(x)\Psi_\alpha(\bar{u})dx. \tag{1.24}$$

Next, we estimate the integral on the right hand side of (1.24). Since

$$\frac{2}{3}|x_k| \leq |x| \leq \frac{4}{3}|x_k|, \quad \forall\, x \in U_k,$$

we have

$$\frac{1}{(1+(4/3)|x_k|)^n} \leq b(x) \leq \frac{1}{(1+(2/3)|x_k|)^n}, \quad \forall\, x \in U_k. \tag{1.25}$$

For any $k \in K_{r,\sigma}$ fixed, in view of (1.25) we get

$$\int_{U_k^{1/2}} b(x)\Psi_\alpha(\bar{u})dx \leq \frac{1}{(1+(2/3)|x_k|)^n} \int_{U_k^{1/2}} \Psi_\alpha(\bar{u})dx. \tag{1.26}$$

Now, consider a cut-off function $\varphi_k \in C_0^\infty(U_k)$ such that $0 \leq \varphi_k \leq 1$ in U_k, $\varphi_k \equiv 1$ in $U_k^{1/2}$ and $|\nabla \varphi_k| \leq C/|x_k|$ in U_k for some constant $C > 0$. Then we see that $\varphi_k \bar{u} \in W_0^{1,n}(U_k)$ and by straightforward computation we have

$$\begin{aligned}
\int_{U_k} |\nabla(\varphi_k \bar{u})|^n dx &\leq 2^{n-1} \left(\int_{U_k} |\varphi_k|^n |\nabla \bar{u}|^n dx + \int_{U_k} |\nabla \varphi_k|^n |\bar{u}|^n dx \right) \\
&\leq 2^{n-1} \left(\int_{U_k} |\nabla \bar{u}|^n dx + \frac{C^n}{|x_k|^n} \int_{U_k} |\bar{u}|^n dx \right) \\
&\leq 2^{n-1} \left(\int_{U_k} |\nabla \bar{u}|^n dx + C^n \frac{(1+(4/3)|x_k|)^n}{|x_k|^n} \int_{U_k} b(x)|\bar{u}|^n dx \right).
\end{aligned}$$

Recalling that $k \in K_{r,\sigma}$, in view of (1.23), we have that $x_k \in B_r^c$ and consequently $|x_k| \geq r > 1$. This and the above estimate imply that

$$\int_{U_k} |\nabla(\varphi_k \bar{u})|^n dx \leq C_3 \int_{U_k} \left(|\nabla \bar{u}|^n + b(x)|\bar{u}|^n \right) dx,$$

where $C_3 := 2^{n-1} \max\{1, (7C/3)^n\}$. Thus, the function $v_k := \varphi_k \bar{u} / \sqrt[n]{2C_3} \in W_0^{1,n}(U_k)$ and

$$\|\nabla v_k\|_{L^n(U_k)}^n = \frac{\|\nabla \varphi_k \bar{u}\|_{L^n(U_k)}^n}{2C_3} \leq \frac{1}{2} \int_{U_k} \left(|\nabla \bar{u}|^n + b(x)|\bar{u}|^n \right) dx \leq 1.$$

Applying Lemma 1.5.1 with $B_R(y) = U_k$, $v = v_k$ and using the fact that $\Psi_\alpha(cs) = \Psi_{\alpha|c|^{n'}}(s)$, for all $c \in \mathbb{R}$, we obtain

$$\int_{U_k^{1/2}} \Psi_\alpha(\varphi_k \bar{u})dx \leq \int_{U_k} \Psi_{\alpha(2C_3)^{\frac{1}{n-1}}}(v_k)dx \leq C_0 \left(\frac{|x_k|}{3}\right)^n \int_{U_k} |\nabla v_k|^n dx,$$

for any $0 < \alpha \leq \alpha_2 := \alpha_n/(2C_3)^{\frac{1}{n-1}}$ and hence

$$\int_{U_k^{1/2}} \Psi_\alpha(\bar{u})dx \leq \frac{C_0|x_k|^n}{3^n 2} \int_{U_k} (|\nabla \bar{u}|^n + b(x)|\bar{u}|^n)\, dx.$$

This together with estimates (1.24), (1.26) and the fact that $s^n/(1+cs)^n \leq 1/c^n$ for any $c, s > 0$ imply that

$$\begin{aligned}\int_{A_{3r}^\sigma} b(x)\Psi_\alpha(\bar{u})dx &\leq \frac{C_0}{3^n 2} \sum_{k \in K_{r,\sigma}} \frac{|x_k|^n}{(1 + (2/3)|x_k|)^n} \int_{U_k} (|\nabla \bar{u}|^n + b(x)|\bar{u}|^n)\, dx \\ &\leq \frac{C_0}{2^{n+1}} \sum_{k \in K_{r,\sigma}} \int_{B_r^c} (|\nabla \bar{u}|^n + b(x)|\bar{u}|^n)\, \chi_{U_k} dx,\end{aligned}$$

where the last inequality we used (1.23). In view of the Besicovitch covering lemma we obtain

$$\int_{A_{3r}^\sigma} b(x)\Psi_\alpha(\bar{u})dx \leq \frac{C_0\theta}{2^{n+1}} \int_{B_r^c} (|\nabla \bar{u}|^n + b(x)|\bar{u}|^n)\, dx.$$

Taking the limit as $\sigma \to +\infty$ we get

$$\int_{B_{3r}^c} b(x)\Psi_\alpha(\bar{u})dx \leq C\theta \int_{B_r^c} (|\nabla \bar{u}|^n + b(x)|\bar{u}|^n)\, dx,$$

for any $0 < \alpha \leq \alpha_2 := \alpha_n/(2C_3)^{\frac{1}{n-1}}$ and this completes the proof of Lemma 1.5.3. □

Finalizing the proof of Theorem 1.1.9. The proof follows directly from (1.22), Lemmas 1.5.2 and Lemma 1.5.3 by choosing $R = 3r$ and $\alpha_0 = \min\{\alpha_1, \alpha_2\}$. □

Proof of Corollary 1.1.10. By Remark 1.1.6 we have that $\|u\|_{\mathcal{E}_b^{1,n}} \leq 2n/(n-1)\|u\|_{\mathcal{E}^{1,n}}$, for all $u \in \mathcal{E}^{1,n}$. This together with the fact that $\Psi_\alpha(cs) = \Psi_{\alpha|c|^{n'}}(s)$, for all $c \in \mathbb{R}$, imply that

$$\int_{\mathbb{R}^n_+} b(x)\Psi_\alpha(u)dx \leq \int_{\mathbb{R}^n_+} b(x)\Psi_{(2n/(n-1))^{n'}\alpha}\left(\frac{u}{\|u\|_{\mathcal{E}_b^{1,n}}}\right) dx,$$

for all $u \in \mathcal{E}^{1,n}$ with $\|u\|_{\mathcal{E}^{1,n}} \leq 1$. Thus, the result follows from Theorem 1.1.9. □

To prove Corollary 1.1.11, we need establish some auxiliary results. First we observe that we can write the function $\Psi_\alpha(s)$ defined in (1.10) as

$$\Psi_\alpha(s) = \Phi_\alpha(s) + \frac{\alpha^{n-1}}{(n-1)!}|s|^n,$$

where

$$\Phi_\alpha(s) = e^{\alpha|s|^{n'}} - \sum_{k=0}^{n-1} \frac{\alpha^k}{k!}|s|^{n'k}, \quad s \in \mathbb{R}.$$

The crucial point for the proof of Corollary 1.1.11 is the following result.

Lemma 1.5.4. *For any $\alpha > 0$ and $u \in \mathcal{E}^{1,n}$, we have*

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} b(x',0)\Phi_\alpha(u(x',0))dx' \leq \alpha n' &\left(\int_{\mathbb{R}^n_+} b(x)|u|^{\frac{n}{(n-1)^2}}\Psi_{n'\alpha}(u)dx\right)^{(n-1)/n} \\ &\times \left(\int_{\mathbb{R}^n_+} |\nabla u|^n dx\right)^{1/n} + n\int_{\mathbb{R}^n_+} b(x)\Psi_\alpha(u)dx.\end{aligned} \tag{1.27}$$

Proof. By Lemma 1.3.1 we may assume that $u \in C_0^\infty(\mathbb{R}^n)$. Straightforward computation shows that $\Phi'_\alpha(s) = \alpha n'|s|^{n'-1}\Psi_\alpha(s)$. Thus,

$$\begin{aligned} b(x',0)\Phi_\alpha(u(x',0)) &= -\int_0^{+\infty} \frac{\partial}{\partial x_n}\left(b(x)\Phi_\alpha(u)\right)dx_n \\ = -\alpha n' &\int_0^{+\infty} b(x)|u|^{n'-1}\Psi_\alpha(u)u_{x_n}dx_n + n\int_0^{+\infty}\frac{\Phi_\alpha(u)x_n}{(1+|x|)^{n+1}|x|}dx_n \\ \leq \alpha n' &\int_0^{+\infty} b(x)|u|^{n'-1}\Psi_\alpha(u)|\nabla u|dx_n + n\int_0^{+\infty} b(x)\Psi_\alpha(u)dx_n. \end{aligned} \tag{1.28}$$

Integrating this inequality on $\mathbb{R}^{n-1}$ and using Hölder's estimates we deduce

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} b(x',0)\Phi_\alpha(u(x',0))dx' \leq \alpha n' &\left(\int_{\mathbb{R}^n_+} (b(x))^{n'}|u|^{\frac{n}{(n-1)^2}}(\Psi_\alpha(u))^{n'}dx\right)^{(n-1)/n} \\ &\times \left(\int_{\mathbb{R}^n_+} |\nabla u|^n dx\right)^{1/n} + n\int_{\mathbb{R}^n_+} b(x)\Psi_\alpha(u)dx.\end{aligned}$$

Now using that $(b(x))^{n'} \leq b(x)$ and $(\Psi_\alpha(s))^{n'} \leq \Psi_{n'\alpha}(s)$ (see [45, Lemma 2.1]) we finish the proof. □

Proof of Corollary 1.1.11. Since

$$\int_{\mathbb{R}^{n-1}} b(x',0)\Psi_\alpha(u(x',0))dx' = \int_{\mathbb{R}^{n-1}} b(x',0)\Phi_\alpha(u(x',0))dx + \frac{\alpha^{n-1}}{(n-1)!}\int_{\mathbb{R}^{n-1}} b(x',0)|u(x',0)|^n dx,$$

by Lemma 1.5.4 and the trace embedding (1.7) it is sufficient to prove that

$$\sup_{\{u\in\mathcal{E}^{1,n}:\|u\|_{\mathcal{E}^{1,n}}\leq 1\}} \int_{\mathbb{R}^n_+} b(x)|u|^{\frac{n}{(n-1)^2}}\Psi_{n'\alpha}(u)dx < +\infty, \tag{1.29}$$

for any $0 < \alpha \leq ((n-1)/n)^2\alpha_0/(2n/(n-1))^{n'}$ where $\alpha_0 > 0$ is the constant obtained in

Theorem 1.1.9. To this end, by Hölder's estimates we see that

$$\int_{\mathbb{R}^n_+} b(x)|u|^{\frac{n}{(n-1)^2}}\Psi_{n'\alpha}(u)dx \le \left(\int_{\mathbb{R}^n_+} b(x)|u|^{\frac{n^2}{(n-1)^2}}dx\right)^{1/n}\left(\int_{\mathbb{R}^n_+} b(x)(\Psi_{n'\alpha}(u))^{n'}dx\right)^{1/n'}.$$

Now using that $(\Psi_{n'\alpha}(s))^{n'} \le \Psi_{(n')^2\alpha}(s)$ (see [45, Lemma 2.1]) by Proposition 1.1.8 we get

$$\int_{\mathbb{R}^n_+} b(x)|u|^{\frac{n}{(n-1)^2}}\Psi_{n'\alpha}(u)dx \le C\|u\|_{\mathcal{E}^{1,n}}\left(\int_{\mathbb{R}^n_+} b(x)\Psi_{(n')^2\alpha}(u)dx\right)^{1/n'}.$$

This together with Corollary 1.1.10 imply that (1.29) holds. The proof is complete. □

1.6 Morrey-Sobolev type embedding $(p > n)$

In this section we prove Theorem 1.1.13. To this end, we fix $z = (0', z_0) \in \mathbb{R}^n_+$ and let $Q \subset \mathbb{R}^n$ be an open cube centered at the origin 0 containing z whose sides-of length r-are parallel to the coordinate axes. Setting

$$\bar{u} = \frac{1}{|Q^+|}\int_{Q^+} a(x)u(x)dx,$$

where $Q^+ := Q \cap \mathbb{R}^n_+$ and $a(x) := (1 + x_n)^{-p}$, we have the following result.

Lemma 1.6.1. *There exists $C_0 = C_0(p) > 0$ such that*

$$|\bar{u} - a(x)u(x)| \le C_0 r^{1-\frac{n}{p}}\left(\|\nabla u\|_{L^p(Q^+)} + \|u\|_{L^p(Q^+,a(y))}\right), \quad \forall\, x \in Q^+.$$

Proof. Note that for any $x \in Q^+$ we have

$$a(x)u(x) - a(z)u(z) = \int_0^1 \frac{d}{dt}[a(tx + (1-t)z)u(tx + (1-t)z)]dt.$$

Since

$$\frac{d}{dt}[a(tx + (1-t)z)u(tx + (1-t)z)] = -p\frac{u(tx + (1-t)z)(x_n - z_0)}{(1 + (tx_n + (1-t)z_0))^{p+1}} + \frac{\nabla u(tx + (1-t)z)\cdot(x - z)}{(1 + (tx_n + (1-t)z_0))^{p}},$$

and taking into account that $a(x) \le 1$, for $x \in Q^+$, it follows that

$$|a(x)u(x) - a(z)u(z)| \le pr\int_0^1 a(tx + (1-t)z)|u(tx + (1-t)z)|dt + r\int_0^1 \sum_{i=1}^n |u_{x_i}(tx + (1-t)z)|dt.$$

Integrating this inequality on Q^+ with respect to the variable x we obtain

$$\begin{aligned}|\bar{u} - a(z)u(z)| &\leq \frac{pr}{|Q^+|}\int_{Q^+}\int_0^1 a(tx+(1-t)z)|u(tx+(1-t)z|dtdx \\ &+ \frac{r}{|Q^+|}\int_{Q^+}\int_0^1 \sum_{i=1}^n |u_{x_i}(tx+(1-t)z)|dtdx \\ &\leq \frac{pC_1}{r^{n-1}}\int_0^1\int_{tQ^+}\frac{|u(y)|}{(1+y_n)^p}\frac{dy}{t^n}dt + \frac{C_1}{r^{n-1}}\int_0^1\int_{tQ^+}\sum_{i=1}^n |u_{x_i}(y)|\frac{dy}{t^n}dt.\end{aligned}$$

Since $tQ^+ \subset Q^+$ for $t \in (0,1)$, from Hölders estimate we get

$$\int_{tQ^+}\frac{|u(y)|}{(1+y_n)^p}dy \leq \|u\|_{L^p(Q^+,a(y))}|tQ^+|^{\frac{1}{p'}} \leq C_2\|u\|_{L^p(Q^+,a(y))}(rt)^{\frac{n}{p'}}$$

$$\int_{tQ^+}|u_{x_i}(y)|dy \leq \left(\int_{Q^+}|u_{x_i}(y)|^p dy\right)^{\frac{1}{p}}|tQ^+|^{\frac{1}{p'}} \leq C_2\|\nabla u\|_{L^p(Q^+)}(rt)^{\frac{n}{p'}}.$$

This immediately implies that for all $p > n$,

$$|\bar{u} - a(z)u(z)| \leq C_0 r^{1-\frac{n}{p}}\left(\|\nabla u\|_{L^p(Q^+)} + \|u\|_{L^p(Q^+,a(y))}\right).$$

By translation, this inequality remains true for all cubes Q whose sides-of length r-are parallel to the coordinate axes and this completes the proof. □

Finalizing the proof of Theorem 1.1.13. First we shall prove (1.12). Let $u \in C_0^1(\mathbb{R}^n)$, $x \in \mathbb{R}^n_+$ and $Q \subset \mathbb{R}^n$ be an open cube centered at the 0 containing x. In view of Lemma 1.6.1 and Theorem 1.1.1 we obtain

$$\begin{aligned}|a(x)u(x)| &\leq |\bar{u}| + C_0 r^{1-\frac{n}{p}}\left(\|\nabla u\|_{L^p(Q^+)} + \|u\|_{L^p(Q^+,a(y))}\right) \\ &\leq C_3\int_{Q^+} a(x)|u|dx + C_0 r^{1-\frac{n}{p}}\left(\|\nabla u\|_{L^p(Q^+)} + \|u\|_{L^p(Q^+,a(y))}\right) \\ &\leq C_3\|u\|_{L^p(Q^+,a(x))} + C_0 r^{1-\frac{n}{p}}\left(\|\nabla u\|_{L^p(Q^+)} + \|u\|_{L^p(Q^+,a(y))}\right) \leq C\|u\|_{\mathcal{E}^{1,p}(\mathbb{R}^n_+)}.\end{aligned}$$

Therefore, $\|u\|_{L^\infty(\mathbb{R}^n_+,a(x))} \leq C\|u\|_{\mathcal{E}^{1,p}(\mathbb{R}^n_+)}$, as stated in (1.12). Next we will prove the estimate (1.13). To do this, we observe that given any two points $x, y \in \mathbb{R}^n_+$ there exists a cube Q with side $r = 2|x-y|$ such that $x, y \in Q$. Since $x, y \in Q^+$ form Lemma 1.6.1 we infer that

$$\begin{aligned}|a(x)u(x) - a(y)u(y)| &\leq |a(x)u(x) - \bar{u}| + |\bar{u} - a(y)u(y)| \\ &\leq C_0|x-y|^{1-\frac{n}{p}}\left(\|\nabla u\|_{L^p(\mathbb{R}^n_+)} + \|u\|_{L^p(\mathbb{R}^n_+,a(y))}\right),\end{aligned}$$

for all $u \in C_0^1(\mathbb{R}^n)$, as stated in (1.13) and this finishes the proof. □

1.7 Final remarks and comments

As it is well known, Sobolev embeddings turn out to be efficient tools for study nonlinear boundary value problems. In the sequel, we comment a few further examples for which our main theorem can be applied.

A remarkable class of nonlinear equations appears in the study of the best constant of certain Sobolev trace embeddings in bounded domain is the limit problem (see for instance [1] and references therein):

$$\begin{cases} -\Delta u &= 0, & \text{in} & \mathbb{R}^n_+, \\ \frac{\partial u}{\partial \nu} + u &= |u|^{q-2}u, & \text{on} & \partial\mathbb{R}^n_+, \end{cases} \tag{$\mathcal{P}_1$}$$

where ν denotes the unit outer normal to the boundary $\mathbb{R}^n_+$, $2 < q \leq 2_* := 2(n-1)/(n-2)$ if $n \geq 3$ and $q > 2$ if $n = 2$.

Another illustrative example we bring up here concerns the weighted eigenvalue problem:

$$\begin{cases} -\Delta u &= \lambda a(x) u, & \text{in} & \mathbb{R}^n_+, \\ \frac{\partial u}{\partial \nu} + u &= 0, & \text{on} & \partial\mathbb{R}^n_+. \end{cases} \tag{$\mathcal{P}_2$}$$

where $n \geq 2$, λ is a real parameter and $a(x) \leq (1 + x_n)^{-2}$. Let u be an eigenfunction with corresponding eigenvalue λ. As a consequence of Theorem 1.1.1, we get

$$\begin{aligned} \int_{\mathbb{R}^n_+} |\nabla u|^2 dx + \int_{\mathbb{R}^{n-1}} |u|^2 dx' &= \lambda \int_{\mathbb{R}^n_+} a(x) u^2 \\ &\leq \lambda \int_{\mathbb{R}^n_+} \frac{u^2}{(1+x_n)^2} \leq \lambda 4 \left(\int_{\mathbb{R}^n_+} |\nabla u|^2 dx + \int_{\mathbb{R}^{n-1}} |u|^2 dx' \right) \end{aligned}$$

which implies $\lambda \geq 1/4$. Therefore, all eigenvalues of the problem $(\mathcal{P}_2)$ are greater than or equal to $1/4$. This eigenvalue problem has been studied in the paper [6] with more general elliptic operator perturbed by a potential, and with Robin boundary conditions. We also quote here that this eigenvalue problem is related with the Sobolev trace inequality see [10]. We also mention the works [13, 36, 37] where the authors studied the eigenvalue problem with Robin boundary conditions in a bounded domain.

Finally, we mention that based on the theorems proved in the present chapter one can study a wide class of quasilinear elliptic problems. Precisely, if $1 < p \leq n$ and $a(x) = (1 + x_n)^{-\alpha}$ with $\alpha \geq p$ then with the aid of Theorems 1.1.3 and 1.1.4 we can study the problem:

$$\begin{cases} -\Delta_p u &= a(x) f(u), & \text{in} & \mathbb{R}^n_+, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} + |u|^{p-2}u &= g(u), & \text{on} & \partial\mathbb{R}^n_+, \end{cases} \tag{$\mathcal{P}_3$}$$

when $f(u) \approx |u|^r$ with $p < r \leq p^* := np/(n-p)$ $(p < r < \infty$ if $p = n)$ and $g(u) \approx |u|^q$ with $p < r \leq p_* := (n-1)p/(n-p)$ $(p < q < \infty$ if $p = n$). Furthermore, in the limit case $p = n = 2$

(to simplify), we can study the problem:

$$\left\{ \begin{array}{rcllr} -\Delta u & = & a(x)f(u), & \text{in} & \mathbb{R}^2_+, \\ \frac{\partial u}{\partial \nu} + u & = & b(x')g(u), & \text{on} & \partial\mathbb{R}^2_+, \end{array} \right. \tag{$\mathcal{P}_4$}$$

when f and g have growth in the Trudinger-Moser sense, that is, $f(u) \approx e^{\alpha_0 \pi u^2}$ and $g(u) \approx e^{\beta_0 u^2}$, for some $\alpha_0, \beta_0 > 0$, at infinity and $a(x), b(x')$ satisfying suitable conditions.

Part II

An indefinite quasilinear elliptic problem with weights in anisotropic spaces (Chapter 2)

A quasilinear elliptic equation with exponential growth and weights in anisotropic spaces (Chapter 3)

Chapter 2

An indefinite quasilinear elliptic problem with weights in anisotropic spaces

In this chapter we present the results of the paper [3], where we investigate existence, nonexistence and multiplicity of solutions for a class of indefinite quasilinear elliptic problems in the upper half-space involving weights in anisotropic Lebesgue spaces. One of our basic tools consists in a Hardy type inequality proved in [3] that allows us to establish Sobolev embeddings into Lebesgue spaces with weights in anisotropic Lebesgue spaces.

2.1 Introduction and main results

Consider the Euclidean upper half-space $\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}$ with $n \geq 2$ and denote by $\mathbb{R}^{n-1}$ its boundary. This chapter is concerned with the existence, nonexistence and multiplicity of solutions for the indefinite quasilinear elliptic problem:

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \lambda a(x)|u|^{q-2}u - b(x)|u|^{r-2}u & \text{in } \mathbb{R}^n_+, \\ |\nabla u|^{p-2}\nabla u \cdot \nu + |u|^{p-2}u = 0 & \text{on } \mathbb{R}^{n-1}, \end{cases} \tag{$\mathcal{P}_\lambda$}$$

where ν denotes the unit outward normal on the boundary, $1 < p \leq n$, λ is a real parameter and the weight functions $a(x)$ and $b(x)$ satisfy some suitable conditions that we will describe later on. Our main interest is to analyze the interplay between the powers q and r. Thus, we will consider two cases:

Case I: $r > q$ and $1 < p \leq q \leq p^* := \frac{np}{n-p}$ if $1 < p < n$ ($p^* := \infty$ if $p = n$);

Case II: $1 < p \leq r \leq q \leq p^*$.

The model problem $(\mathcal{P}_\lambda)$ arises in the study of nonlinear diffusion equations, in particular, in the mathematical modeling of non-Newtonian fluids, see [34]. For a Physics background related to this problem, we refer the reader to [19] and references therein.

The existence, nonexistence and multiplicity of solutions for quasilinear elliptic problems of the form

$$\begin{cases} -\text{div}(|\nabla u|^{p-2}\nabla u) = f(x,u), & \text{in} \quad \Omega, \\ |\nabla u|^{p-2}\nabla u \cdot \nu + h(x)|u|^{p-2}u = 0, & \text{on} \quad \Gamma, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is an unbounded domain, Γ denotes the boundary of Ω and the functions h, f satisfy some growth conditions, have been investigated by many authors, see for instance [12, 14, 16, 25, 32–35] and references therein. In many of these papers, a relevant Sobolev type inequality proved by Pflüger in [34] has played an important role in their variational approach. Precisely, let $1 < p < n$ and assume that the weight function $h(x)$ satisfies the hypothesis

$$1/C(1+|x|)^{p-1} \leq h(x) \leq C/(1+|x|)^{p-1}, \quad \text{a.e.} \quad x \in \Omega,$$

for some $C > 0$ and denote by $C_\delta^\infty(\Omega)$ the space of $C_0^\infty(\mathbb{R}^n)-$functions restricted to Ω. Defining the weighted Sobolev space E as the completion of $C_\delta^\infty(\Omega)$ in the norm

$$\|u\|_E := \left(\int_\Omega |\nabla u|^p dx + \int_\Omega \frac{|u|^p}{(1+|x|)^p} dx\right)^{1/p},$$

in [34], the author proved that $\|\cdot\|_E$ is an equivalent norm to (see [34, Lemma 2])

$$|||u|||_E := \left(\int_\Omega |\nabla u|^p + \int_{\partial\Omega} h(x)|u|^p dx'\right)^{1/p}.$$

To this, the following Hardy type inequality was crucial:

$$\int_\Omega \frac{|u|^p}{(1+|x|)^p} dx \leq C_0 \left(\int_\Omega |\nabla u|^p dx + \int_\Gamma \frac{|\nu \cdot x|}{(1+|x|)^p} |u|^p d\Gamma\right), \tag{2.1}$$

for some $C_0 > 0$. As pointed in [27], in contrast to the classical Hardy and Sobolev inequalities in $\mathbb{R}^n$, the exact values of the constant C_0 in inequality (2.1) is not known and their determinations seem to be a challenging problem even when the domain Ω has special geometry such as, for example, the half-space $\mathbb{R}^n_+$.

Here, we will prove a version of the Hardy type inequality (2.1) that includes $p = n$ and this inequality will allows us to consider weights like $a(x) = (1+x_n)^{-\alpha}$ for some $\alpha \geq n$, which belongs to an anisotropic Lebesgue space (for the definition of anisotropic Lebesgue spaces, see [5, 9]). Then, we set up some new Sobolev embeddings into weighted Lebesgue spaces with where the weight belongs is in anisotropic Lebesgue spaces. We also quote that recent developments on Hardy type inequalities in the half-space were addressed in the context of $C_0^\infty(\mathbb{R}^n_+)$ in [23, 24, 42].

Motivated by the works of Alama-Tarrantelo [7], Filippucci-Pucci-Radulescu [25], Lyberopoulos [27], Perera [31] and Pflüger [34], our main purpose in the present chapter is to use variational techniques to investigate the existence, nonexistence and multiplicity of nontrivial weak solutions for the problem $(\mathcal{P}_\lambda)$. We want to remark that the main features of this class of problems is that we are facing an indefinite nonlinearity and the weight function $a(x)$ is allowed to be in anisotropic Lebesgue spaces.

We begin by considering the case $r > q$. To this end, we shall assume the following assumptions:

(H_1) $a : \mathbb{R}^n_+ \to \mathbb{R}$ is a nontrivial measurable function and there are constants $\alpha \geq n$ and $c_1 > 0$ such that

$$0 \leq a(x) \leq \frac{c_1}{(1+x_n)^\alpha}, \quad \text{a.e.} \quad x \in \mathbb{R}^n_+;$$

(H_2) $b : \mathbb{R}^n_+ \to \mathbb{R}$ is a positive continuous function satisfying

$$\int_{\mathbb{R}^n_+} \frac{a^{\frac{r}{r-q}}}{b^{\frac{q}{r-q}}} dx < \infty.$$

It is worthwhile mentioning that the hypothesis (H_2) appears in the paper [7].

Remark 2.1.1. *Note that if $a(x)$ satisfies (H_1) then the function $b(x) = (1+|x|)^\theta/(1+x_n)^{\frac{\alpha r}{q}}$ with $\theta > n(r-q)/q$ satisfies the assumption (H_2). In fact, if $\theta > n(r-q)/q$ we have*

$$\int_{\mathbb{R}^n_+} \frac{a^{\frac{r}{r-q}}}{b^{\frac{q}{r-q}}} dx \leq \int_{\mathbb{R}^n_+} \frac{c_1}{(1+x_n)^{\frac{\alpha r}{r-q}}} \frac{(1+x_n)^{\frac{\alpha r}{r-q}}}{(1+|x|)^{\frac{\theta q}{r-q}}} dx = \int_{\mathbb{R}^n_+} \frac{c_1}{(1+|x|)^{\frac{\theta q}{r-q}}} dx < \infty.$$

Under these hypotheses, our main result can be stated as follows.

Theorem 2.1.2. *Let $r > q$ and assume the hypotheses $(H_1) - (H_2)$.*

(i) *If $1 < p \leq q \leq p^*$, there exists $\lambda^* > 0$ such that problem $(\mathcal{P}_\lambda)$ has only the trivial solution for all $\lambda \in (-\infty, \lambda^*)$;*

(ii) *If $\max\{2,p\} < q < p^*$, there exists $\tilde{\lambda} > 0$ such that problem $(\mathcal{P}_\lambda)$ has at least a nontrivial weak solution for all $\lambda \in (\tilde{\lambda}, \infty)$. Furthermore, if $p < q$ then $(\mathcal{P}_{\tilde{\lambda}})$ has a nontrivial weak solution;*

(iii) *If $\max\{2,p\} < q < p^*$, there exists $\Lambda \geq \tilde{\lambda}$ such that problem $(\mathcal{P}_\lambda)$ has at least two nontrivial weak solutions $u_\lambda \geq \tilde{u}_\lambda$ for all $\lambda \in (\Lambda, \infty)$;*

(iv) *If $1 < p \leq q < p^*$, for any $m \in \mathbb{N}$ there exists $\Lambda_m > 0$ such that problem $(\mathcal{P}_\lambda)$ has at least m pairs of nontrivial weak solutions for all $\lambda > \Lambda_m$.*

The proof of the existence in Theorem 2.1.2 is based on minimization techniques. To obtain the second solution we will follow a truncation argument. The multiplicity result is obtained by applying the symmetric mountain pass theorem.

Next, we deal with the case $r < q$. In order to prove the existence of solutions for problem $(\mathcal{P}_\lambda)$, instead of hypotheses $(H_1) - (H_2)$, we will assume:

$(\widetilde{H}_1)$ $a : \mathbb{R}^n_+ \to \mathbb{R}$ is a nontrivial measurable function and there are $c_2 > 0$ and $\alpha \geq n$ such that

$$0 \leq a(x) \leq \frac{c_2}{(1+|x|)^\alpha}, \quad \text{a.e. in} \quad x \in \mathbb{R}^n_+;$$

$(\widetilde{H}_2)$ $b : \mathbb{R}^n_+ \to \mathbb{R}$ is a measurable positive function.

In this case, our main result is stated as follows.

Theorem 2.1.3. *Let $1 < p \leq r < q < p^*$ and assume the hypotheses $(\widetilde{H}_1) - (\widetilde{H}_2)$. Then*

(i) *the problem $(\mathcal{P}_\lambda)$ has no nontrivial weak solution for every $\lambda \in (-\infty, 0]$;*

(ii) *the problem $(\mathcal{P}_\lambda)$ has an infinite number of nontrivial weak solutions for every $\lambda \in (0, \infty)$.*

The proof of Theorem 2.1.3 is obtained by performing a variational approach based on the symmetric mountain pass theorem.

Hereafter in this chapter, B_R denotes the ball of center zero and radius $R > 0$ in $\mathbb{R}^n$, $B_R^+ := B_R \cap \mathbb{R}^n_+$, $(B_R)^c$ denotes $\mathbb{R}^n \setminus B_R$, the complement of the set $B_R \subset \mathbb{R}^n$, and $(B_R^+)^c$ denotes $\mathbb{R}^n_+ \setminus B_R^+$ the complement of the set $B_R^+ \subset \mathbb{R}^n_+$.

This chapter is organized as follows. Section 2.2 contains the necessary preliminary results on the weighted Sobolev embeddings needed in the sequel. In Section 2.3, we present the proof of Theorem 2.1.2. Finally, in Section 2.4, we discuss the proof of Theorem 2.1.3.

2.2 Variational framework

In this section, in order to perform a variational approach we introduce our functional space and its embeddings into weighted Lebesgue spaces. To this, denote by $C^\infty_\delta(\mathbb{R}^n_+)$ the space of $C^\infty_0(\mathbb{R}^n)-$functions restricted to $\mathbb{R}^n_+$. We define the weighted Sobolev space E as the completion of $C^\infty_\delta(\mathbb{R}^n_+)$ with respect to the norm

$$\|u\| := \left[\int_{\mathbb{R}^n_+} \left(|\nabla u|^p + \frac{|u|^p}{(1+x_n)^p} \right) dx \right]^{1/p}.$$

We have the following embedding result.

Lemma 2.2.1. *Assume $1 < p < n$. Then the weighted Sobolev embedding*

$$E \hookrightarrow L^q\left(\mathbb{R}^n_+, \frac{1}{(1+x_n)^p}\right), \quad \forall\, p \leq q \leq p^*, \tag{2.2}$$

and the trace embedding

$$E \hookrightarrow L^q(\mathbb{R}^{n-1}), \quad \forall\, p \leq q \leq p_*, \tag{2.3}$$

are continuous.

Proof. We first recall that for any $1 \leq p < n$ it follows from the Gagliardo-Nirenberg-Sobolev inequality that there exists $C_0 = C_0(n,p) > 0$ such that

$$\left(\int_{\mathbb{R}^n_+} |u|^{p^*} dx\right)^{(n-p)/n} \leq C_0 \int_{\mathbb{R}^n_+} |\nabla u|^p dx, \quad \forall\, u \in C_0^1(\mathbb{R}^n). \tag{2.4}$$

This in combination with the fact that $(1+x_n)^{-p} \leq 1$ imply that $E \hookrightarrow L^{p^*}\left(\mathbb{R}^n_+, (1+x_n)^{-p}\right)$. By interpolation we see that $E \hookrightarrow L^q\left(\mathbb{R}^n_+, (1+x_n)^{-p}\right)$ for all $q \in [p, p^*]$, as stated in (2.2). Now we will prove the embedding (2.3). For that, observe that for all $u \in C_0^1(\mathbb{R}^n)$ we have

$$|u(x',0)|^p = -\int_0^{+\infty} \frac{\partial}{\partial x_n}\left(\frac{|u|^p}{(1+x_n)^p}\right) dx_n \leq p\int_0^{+\infty} \frac{|u|^{p-1}|\nabla u|}{(1+x_n)^p} dx_n + p\int_0^{+\infty} \frac{|u|^p}{(1+x_n)^{p+1}} dx_n.$$

Integrating this inequality and using the Hölder inequality together with the fact that $(1+x_n)^{-1} < 1$ we obtain

$$\int_{\mathbb{R}^{n-1}} |u(x',0)|^p dx' \leq p\left(\int_{\mathbb{R}^n_+} \frac{|u|^p}{(1+x_n)^p} dx\right)^{(p-1)/p} \left(\int_{\mathbb{R}^n_+} |\nabla u|^p dx\right)^{1/p} + p\int_{\mathbb{R}^n_+} \frac{|u|^p}{(1+x_n)^p} dx.$$

Using the embedding (2.2) we get

$$\|u\|^p_{L^p(\mathbb{R}^{n-1})} \leq C_1 \|u\|^{p-1}\|u\| + C_2\|u\|^p,$$

which implies that $E \hookrightarrow L^p(\mathbb{R}^{n-1})$. On the other hand, from the trace inequality (see [20, 29]), for all $1 < p < n$ we have

$$\left(\int_{\mathbb{R}^{n-1}} |u|^{p_*} dx\right)^{(n-p)/(n-1)} \leq C_0 \int_{\mathbb{R}^n_+} |\nabla u|^p dx, \quad \forall\, u \in C_0^1(\mathbb{R}^n),$$

and hence $E \hookrightarrow L^{p_*}(\mathbb{R}^{n-1})$. Thus, by interpolation we obtain that $E \hookrightarrow L^q(\mathbb{R}^{n-1})$ for any $q \in [p, p_*]$ and this completes the proof of Lemma 2.2.1. □

Next we consider the borderline case $p = n$.

Lemma 2.2.2. *Assume $p = n$. For any $n \leq q < \infty$ the weighted Sobolev embedding*

$$E \hookrightarrow L^q\left(\mathbb{R}^n_+, \frac{1}{(1+x_n)^n}\right) \tag{2.5}$$

and the Sobolev trace embedding

$$E \hookrightarrow L^q(\mathbb{R}^{n-1}) \tag{2.6}$$

are continuous.

Proof. We first prove the embedding (2.5). To this, using inequality (2.4) with $p = 1$ we get

$$\left(\int_{\mathbb{R}^n_+} |v|^{\frac{n}{n-1}} dx\right)^{(n-1)/n} \leq C \int_{\mathbb{R}^n_+} |\nabla v| dx, \quad \forall\, v \in C^1_0(\mathbb{R}^n). \tag{2.7}$$

Applying (2.7) with $v = (1 + x_n)^\alpha |u|^n$ and $u \in C^1_0(\mathbb{R}^n)$ we obtain

$$\begin{aligned}\left(\int_{\mathbb{R}^n_+} |(1 + x_n)^\alpha |u|^n|^{\frac{n}{n-1}} dx\right)^{(n-1)/n} &\leq C_0 \int_{\mathbb{R}^n_+} |\alpha|(1 + x_n)^{\alpha-1}|u|^n dx \\ &+ C_0 n \int_{\mathbb{R}^n_+} (1 + x_n)^\alpha |u|^{n-1}|\nabla u| dx.\end{aligned}$$

Choosing $\alpha = -(n - 1)$ and using the Young inequality we obtain

$$\left(\int_{\mathbb{R}^n_+} \frac{|u|^{n^2/(n-1)}}{(1 + x_n)^n} dx\right)^{(n-1)/n} \leq C_1 \int_{\mathbb{R}^n_+} \left(|\nabla u|^n + \frac{|u|^n}{(1 + x_n)^n}\right) dx,$$

where C_1 depends only on n and hence we conclude that

$$E \hookrightarrow L^{\frac{n^2}{n-1}} \left(\mathbb{R}^n_+, \frac{1}{(1 + x_n)^n}\right).$$

If $n < q < n^2/(n - 1)$, by interpolation, there exists $0 < \theta < 1$ such that

$$\|u\|_{L^q\left(\mathbb{R}^n_+,(1+x_n)^{-n}\right)} \leq \|u\|^\theta_{L^n\left(\mathbb{R}^n_+,(1+x_n)^{-n}\right)} \|u\|^{1-\theta}_{L^{\frac{n^2}{(n-1)}}\left(\mathbb{R}^n_+,(1+x_n)^{-n}\right)} \leq C\|u\|.$$

In particular, using that $n < n + 1 < n^2/(n - 1)$, one has $E \hookrightarrow L^{n+1} \left(\mathbb{R}^n_+, (1 + x_n)^{-n}\right)$. On the other hand, applying again (2.7) with $v = (1 + x_n)^{-(n-1)}|u|^{n+1}$ and using the Young inequality we get

$$\begin{aligned}\left(\int_{\mathbb{R}^n_+} \frac{|u|^{\frac{n(n+1)}{n-1}}}{(1 + x_n)^n} dx\right)^{(n-1)/n} &\leq C_1(n) \int_{\mathbb{R}^n_+} \frac{|u|^{n+1}}{(1 + x_n)^n} dx \\ &+ C_2(n) \left(\int_{\mathbb{R}^n_+} \frac{|u|^{\frac{n^2}{n-1}}}{(1 + x_n)^n} dx + \int_{\mathbb{R}^n_+} |\nabla u|^n dx\right),\end{aligned}$$

which implies that $E \hookrightarrow L^{\frac{n(n+1)}{(n-1)}}(\mathbb{R}^n_+, (1 + x_n)^{-n})$. For any $n \leq q \leq n(n + 1)/(n - 1)$, by interpolation, there exists $0 \leq \theta \leq 1$ such that

$$\|u\|_{L^q(\mathbb{R}^n_+,(1+x_n)^{-n})} \leq \|u\|^\theta_{L^n(\mathbb{R}^n_+,(1+x_n)^{-n})} \|u\|^{1-\theta}_{L^{\frac{n(n+1)}{(n-1)}}(\mathbb{R}^n_+,(1+x_n)^{-n})} \leq C\|u\|.$$

Reiterating this argument with $k = n + 2, n + 3, \ldots$, one has $E \hookrightarrow L^{\frac{nk}{n-1}}(\mathbb{R}^n_+, (1 + x_n)^{-n})$. Now, given $q \in [n, \infty)$, one can choose $k \geq n$ such that $n < q < nk/(n - 1)$ and once again using interpolation we get

$$\|u\|_{L^q(\mathbb{R}^n_+,(1+x_n)^{-n})} \le \|u\|^{\theta}_{L^n(\mathbb{R}^n_+,(1+x_n)^{-n})} \|u\|^{1-\theta}_{L^{\frac{nk}{n-1}}(\mathbb{R}^n_+,(1+x_n)^{-n})} \le C\|u\|,$$

which proves the embedding (2.5). Now we will prove the trace embedding (2.6). For that, fixed $q \ge n$, we have

$$|u(x',0)|^q = -\int_0^{+\infty} \frac{\partial}{\partial x_n}\left(\frac{|u|^q}{(1+x_n)^n}\right) dx_n \le q\int_0^{+\infty} \frac{|u|^{q-1}|\nabla u|}{(1+x_n)^n} dx_n + n\int_0^{+\infty} \frac{|u|^q}{(1+x_n)^{n+1}} dx_n.$$

Integrating this inequality and using the Hölder inequality together with the fact that $(1+x_n)^{-1} < 1$ we obtain

$$\int_{\mathbb{R}^{n-1}} |u(x',0)|^q dx' \le q\left(\int_{\mathbb{R}^n_+} \frac{|u|^{(q-1)\frac{n}{n-1}}}{(1+x_n)^n} dx\right)^{(n-1)/n} \left(\int_{\mathbb{R}^n_+} |\nabla u|^n dx\right)^{1/n} + n\int_{\mathbb{R}^n_+} \frac{|u|^q}{(1+x_n)^n} dx.$$

Since $(q-1)n/(n-1) \ge n$, by the embedding (2.5) we get

$$\|u\|^q_{L^q(\mathbb{R}^{n-1})} \le C_1\|u\|^{q-1}\|u\| + C_2\|u\|^q,$$

which completes the proof of Lemma 2.2.2. □

Next we prove a weighted Hardy-type inequality which is in some way a version of [34, Lemma 2].

Lemma 2.2.3. *Let $1 < p \le n$. Then the following inequality holds*

$$\int_{\mathbb{R}^n_+} \frac{|u|^p}{(1+x_n)^p} dx \le \left(\frac{p}{p-1}\right)^p \left(\int_{\mathbb{R}^n_+} |\nabla u|^p dx + \int_{\mathbb{R}^{n-1}} |u|^p dx'\right), \quad \forall\, u \in C^1_0(\mathbb{R}^n).$$

Proof. Let $v \in C^1_0(\mathbb{R}^n)$ and $\sigma \in \mathbb{R}$ with $\sigma \ne -1$. Using the divergence theorem we obtain

$$\sigma\int_{\mathbb{R}^n_+} \frac{v}{(1+x_n)^{\sigma+1}} dx = \int_{\mathbb{R}^n_+} \frac{v_{x_n}}{(1+x_n)^{\sigma}} dx + \int_{\mathbb{R}^{n-1}} v dx',$$

where we are using that the normal unit vector pointing out of $\mathbb{R}^{n-1}$ is $\eta = (0', -1)$. Applying this equality with $v = |u|^p$, we get

$$|\sigma|\int_{\mathbb{R}^n_+} \frac{|u|^p}{(1+x_n)^{\sigma+1}} dx \le \int_{\mathbb{R}^n_+} \frac{p|u|^{p-1}|\nabla u|}{(1+x_n)^{\sigma}} dx + \int_{\mathbb{R}^{n-1}} |u|^p dx'. \tag{2.8}$$

Now using the Young inequality with $0 < \varepsilon < 1$ we obtain

$$p\int_{\mathbb{R}^n_+} \frac{|u|^{p-1}|\nabla u|}{(1+x_n)^{\sigma}} dx \le (p-1)\varepsilon\int_{\mathbb{R}^n_+} \frac{|u|^p}{(1+x_n)^{\frac{\sigma p}{p-1}}} dx + \frac{1}{\varepsilon^{(p-1)}}\int_{\mathbb{R}^n_+} |\nabla u|^p dx. \tag{2.9}$$

Choosing $\sigma p/(p-1) = \sigma + 1$, that is, $\sigma = p-1$ and combining inequalities (2.8) and (2.9), one has

$$\int_{\mathbb{R}^n_+} \frac{|u|^p}{(1+x_n)^p} dx \leq \frac{1}{(p-1)(\varepsilon^{p-1}-\varepsilon^p)} \left(\int_{\mathbb{R}^n_+} |\nabla u|^p dx + \int_{\mathbb{R}^{n-1}} |u|^p dx' \right).$$

Using that the function $g(\varepsilon) = 1/(\varepsilon^{p-1} - \varepsilon^p)$ for $\varepsilon > 0$ achieves its minimum at $\varepsilon = (p-1)/p$ we conclude that

$$\int_{\mathbb{R}^n_+} \frac{|u|^p}{(1+x_n)^p} dx \leq \left(\frac{p}{p-1}\right)^p \left(\int_{\mathbb{R}^n_+} |\nabla u|^p dx + \int_{\mathbb{R}^{n-1}} |u|^p dx' \right),$$

which is the desired result. □

As a consequence of Lemma 2.2.2 and Lemma 2.2.3 we have

Corollary 2.2.4. *The quantity*

$$\|u\|_{E^{1,p}} := \left(\int_{\mathbb{R}^n_+} |\nabla u|^p dx + \int_{\mathbb{R}^{n-1}} |u|^p dx' \right)^{1/p}, \quad 1 < p \leq n,$$

define an equivalent norm on E.

Proof. By Lemma 2.2.3 we have $\|u\| \leq C_1 \|u\|_{E^{1,p}}$. On the other hand, using Lemmas 2.2.2 and 2.2.3 we obtain

$$\|u\|^p_{E^{1,p}} = \int_{\mathbb{R}^n_+} |\nabla u|^p dx + \int_{\mathbb{R}^{n-1}} |u(x',0)|^p dx' \leq \int_{\mathbb{R}^n_+} |\nabla u|^p dx + C_1 \|u\|^p,$$

which implies the desired result and this completes the proof. □

In view of Corollary 2.2.4, from now on we consider the space E equipped with the norm $\|\cdot\|_{E^{1,p}}$, and we denote by $E^{1,p}$.

Remark 2.2.5. *Suppose that the weight function $a(x)$ satisfies hypotheses (H_1) or $(\widetilde{H}_1)$. By Lemma 2.2.1 and Lemma 2.2.2, respectively, the weighted Sobolev embeddings*

$$E^{1,p} \hookrightarrow L^q\left(\mathbb{R}^n_+, a(x)\right), \quad \forall\, p \leq q \leq p^* \quad \textit{if} \quad 1 < p < n, \tag{2.10}$$

and

$$E^{1,n} \hookrightarrow L^q\left(\mathbb{R}^n_+, a(x)\right), \quad \forall\, n \leq q < \infty, \tag{2.11}$$

are continuous.

Now we are ready to define our variational approach. Since the weight function $b(x)$ does not belong to any Lebesgue space we need to consider the subspace of $E^{1,p}$ defined by

$$E^{r,p} = \left\{ u \in E^{1,p} : \int_{\mathbb{R}^n_+} b(x)|u|^r dx < \infty \right\},$$

equipped with the norm

$$\|u\|_{E^{r,p}} := \left(\|u\|^p_{E^{1,p}} + \|u\|^p_{L^r(\mathbb{R}^n_+,b(x))}\right)^{1/p}.$$

The next two compactness results play a crucial role in the proof of Theorem 2.1.2 and Theorem 2.1.3, respectively.

Lemma 2.2.6. *Assume hypotheses $(H_1)-(H_2)$. Then the embedding $E^{r,p} \hookrightarrow L^q\left(\mathbb{R}^n_+, a(x)\right)$ is compact:*

(i) *For all $p \le q \le p^*$ if $1 < p < n$;*

(ii) *For all $n \le q < \infty$ if $p = n$.*

Proof. We will show that $u_k \to 0$ in $L^q(\mathbb{R}^n_+, a(x))$ whenever $u_k \rightharpoonup 0$ in $E^{r,p}$. Indeed, let $C > 0$ be such that $\|u_k\|_{E^{r,p}} \le C$ and $R > 0$ to be chosen during the proof independently of u. We have

$$\int_{\mathbb{R}^n_+} a|u_k|^q dx = \int_{B^+_R} a|u_k|^q dx + \int_{\mathbb{R}^n_+\setminus B^+_R} a|u_k|^q dx. \tag{2.12}$$

Since the restriction operator $u \mapsto u_{|_{B^+_R}}$ is continuous from $E^{r,p}$ into $E^{r,p}(B^+_R) := \left\{v_{|_{B^+_R}} : v \in E^{r,p}\right\}$ and the embedding $E^{r,p}(B^+_R) \hookrightarrow L^q(B^+_R, a(x))$ is compact, in case that, $p \le q < p^*$ if $1 < p < n$ and $n \le q$ if $p = n$, there exists $k_1 \in \mathbb{N}$ such that

$$\int_{B^+_R} a|u_k|^q dx < \frac{\varepsilon}{2}, \quad \forall\, k \ge k_1 \tag{2.13}$$

for any $p \le q < p^*$ if $1 < p < n$ and $n \le q$ if $p = n$. On the other hand, by assumption (H_2), the Hölder inequality and choosing $R > 0$ sufficiently large, we get

$$\begin{aligned}\int_{\mathbb{R}^n_+\setminus B^+_R} a|u_k|^q dx &\le \left(\int_{\mathbb{R}^n_+\setminus B^+_R} \frac{a^{\frac{r}{r-q}}}{b^{\frac{q}{r-q}}} dx\right)^{(r-q)/r} \left(\int_{\mathbb{R}^n_+\setminus B^+_R} b|u_k|^r dx\right)^{q/r} \\ &\le C_1 \left(\int_{\mathbb{R}^n_+\setminus B^+_R} \frac{a^{\frac{r}{r-q}}}{b^{\frac{q}{r-q}}} dx\right)^{(r-q)/r} \le \frac{\varepsilon}{2}.\end{aligned}$$

This combined with (2.12) and (2.13) imply the desired result. □

Lemma 2.2.7. *Assume hypothesis $(\widetilde{H}_1)$. If $\alpha > n$ then the weighted Sobolev embeddings* (2.10) *and* (2.11) *are compact.*

Proof. Since $E^{1,p} \hookrightarrow L^q\left(\mathbb{R}^n_+, (1+|x|)^{-\alpha}\right) \hookrightarrow L^q\left(\mathbb{R}^n_+, a(x)\right)$, is sufficient to show that $u_k \to 0$ in $L^q(\mathbb{R}^n_+, (1+|x|)^{-\alpha})$ whenever $u_k \rightharpoonup 0$ in $E^{1,p}$. To this end, let $C > 0$ be such that $\|u_k\|_{E^{1,p}} \le C$ and $R > 0$ to be chosen later on. Note that

$$\int_{\mathbb{R}^n_+} \frac{|u_k|^q}{(1+|x|)^\alpha} dx = \int_{B^+_R} \frac{|u_k|^q}{(1+|x|)^\alpha} dx + \int_{\mathbb{R}^n_+\setminus B^+_R} \frac{|u_k|^q}{(1+|x|)^\alpha} dx.$$

Arguing as in the proof of Lemma 2.2.6 we obtain $k_1 \in \mathbb{N}$ such that

$$\int_{B_R^+} \frac{|u_k|^q}{(1+|x|)^\alpha} dx < \frac{\varepsilon}{2}, \quad \forall\, k \geq k_1.$$

On the other hand, using that $\alpha > p$ we see that $(1+x_n)^p/(1+|x|)^\alpha \to 0$ as $|x| \to \infty$. Thus, we can choose $R > 0$ sufficiently large such that $(1+x_n)^p/(1+|x|)^\alpha \leq \varepsilon/2C$. Hence, there exists $k_2 \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^n_+ \setminus B_R^+} \frac{|u_k|^q}{(1+|x|)^\alpha} dx = \int_{\mathbb{R}^n_+ \setminus B_R^+} \frac{|u_k|^q}{(1+x_n)^p} \frac{(1+x_n)^p}{(1+|x|)^\alpha} dx < \frac{\varepsilon}{2}, \quad \forall\, k \geq k_2,$$

which implies the desired result. □

In this chapter we seek for weak solutions of problem $(\mathcal{P}_\lambda)$, which means a function $u \in E^{r,p}$ verifying

$$\int_{\mathbb{R}^n_+} |\nabla u|^{p-2} \nabla u \nabla \varphi dx + \int_{\mathbb{R}^{n-1}} |u|^{p-2} u \varphi dx' = \lambda \int_{\mathbb{R}^n_+} a|u|^{q-2} u \varphi dx - \int_{\mathbb{R}^n_+} b|u|^{r-2} u \varphi dx, \tag{2.14}$$

for all $\varphi \in E^{r,p}$. In view of assumption (H_1) and using Lemmas 2.2.1 and 2.2.2 the energy functional associated to problem $(\mathcal{P}_\lambda)$, namely $I_\lambda : E^{r,p} \to \mathbb{R}$ defined by

$$I_\lambda(u) = \frac{1}{p} \int_{\mathbb{R}^n_+} |\nabla u|^p dx + \frac{1}{p} \int_{\mathbb{R}^{n-1}} |u|^p dx' + \frac{1}{r} \int_{\mathbb{R}^n_+} b|u|^r dx - \frac{\lambda}{q} \int_{\mathbb{R}^n_+} a|u|^q dx,$$

is well defined. Furthermore, standard arguments show that $u \in E^{r,p}$ is a critical point of I_λ if, and only if, it is a weak solution of problem $(\mathcal{P}_\lambda)$.

2.3 Proof of Theorem 2.1.2

In this section, we present the proof of Theorem 2.1.2. We split the proof into three parts.

2.3.1 Nonexistence

First, we present the proof of the nonexistence statement **(i)** in Theorem 2.1.2. Suppose that $u \in E^{r,p}$ is a nontrivial weak solution of $(\mathcal{P}_\lambda)$. If $\lambda \leq 0$ the result is immediate. Thus, we assume that $\lambda > 0$ and taking $\varphi = u$ as a test function in (2.14) we obtain

$$\int_{\mathbb{R}^n_+} |\nabla u|^p dx + \int_{\mathbb{R}^{n-1}} |u|^p dx' = \lambda \int_{\mathbb{R}^n_+} a|u|^q dx - \int_{\mathbb{R}^n_+} b|u|^r dx. \tag{2.15}$$

Using the Young inequality we get

$$\lambda \int_{\mathbb{R}^n_+} a|u|^q dx = \int_{\mathbb{R}^n_+} \frac{\lambda a}{b^{\frac{q}{r}}} \left(b^{\frac{q}{r}} |u|^q \right) dx \leq \frac{r-q}{r} \lambda^{\frac{r}{r-q}} \int_{\mathbb{R}^n_+} \frac{a^{\frac{r}{r-q}}}{b^{\frac{q}{r-q}}} dx + \frac{q}{r} \int_{\mathbb{R}^n_+} b|u|^r dx. \tag{2.16}$$

This together with (2.15) and the fact that $q < r$ imply

$$\|u\|^p_{E^{1,p}} \leq \frac{r-q}{r}\lambda^{\frac{r}{r-q}} \int_{\mathbb{R}^n_+} \frac{a^{\frac{r}{r-q}}}{b^{\frac{q}{r-q}}} dx + \frac{q-r}{r}\int_{\mathbb{R}^n_+} b|u|^r dx \leq \frac{r-q}{r}\lambda^{\frac{r}{r-q}} \int_{\mathbb{R}^n_+} \frac{a^{\frac{r}{r-q}}}{b^{\frac{q}{r-q}}} dx. \tag{2.17}$$

Since $p \leq q$, combining (2.2) with (2.15) and the fact that $b > 0$ we get

$$\bar{C}\left(\int_{\mathbb{R}^n_+} a|u|^q dx\right)^{p/q} \leq \|u\|^p_{E^{1,p}} \leq \lambda \int_{\mathbb{R}^n_+} a|u|^q dx, \tag{2.18}$$

for some constant $\bar{C} > 0$. If $p = q$ we obtain $\lambda \geq \bar{C}$. In case that $q > p$ we have

$$\left(\bar{C}\lambda^{-1}\right)^{\frac{q}{q-p}} \leq \int_{\mathbb{R}^n_+} a|u|^q dx.$$

Using the first inequality in (2.18) we obtain $\bar{C}\left(\bar{C}\lambda^{-1}\right)^{\frac{p}{q-p}} \leq \|u\|^p_{E^{1,p}}$. This together with (2.17) imply that

$$\lambda \geq \bar{\lambda} := \left(\bar{C}^{\frac{q}{q-p}} \frac{r}{r-q}\left(\int_{\mathbb{R}^n_+} \frac{a^{\frac{r}{r-q}}}{b^{\frac{q}{r-q}}} dx\right)^{-1}\right)^{(r-q)(q-p)/q(r-p)}.$$

To conclude, we define

$$\lambda^* = \sup\left\{\lambda > 0 : (\mathcal{P}_\mu) \text{ does not admits any nontivial weak solution for all } \mu < \lambda\right\}.$$

Therefore, $\lambda^* \geq \bar{C} > 0$ if $p = q$ and $\lambda^* \geq \bar{\lambda} > 0$ if $q > p$, and hence item **(i)** in Theorem 2.1.2 holds for all $\lambda < \lambda^*$.

2.3.2 The first solution

In this subsection, by using minimization argument we will prove item **(ii)** in Theorem 2.1.2. We first recall a basic estimate (see [7]).

Remark 2.3.1. *Let $0 \leq \beta < \gamma$ and $k, l \in (0, \infty)$. Then there exists a constant $C = C(\beta,\gamma) > 0$ such that*

$$k|s|^\beta - l|s|^\gamma \leq C(\beta,\gamma)k\left(\frac{k}{l}\right)^{\frac{\beta}{\gamma-\beta}}, \quad \forall s \in \mathbb{R}.$$

In order to use the direct methods of the calculus of variations we need the following result.

Lemma 2.3.2. *Let $\max\{2,p\} < q < p^*$, $r > q$ and assume $(H_1) - (H_2)$. Then, for all $\lambda > 0$, the functional $J_\lambda : E^{r,p} \to \mathbb{R}$ defined by*

$$J_\lambda(u) := \int_{\mathbb{R}^n_+} F(x,u),$$

where $F_\lambda(x,s) := \lambda a(x)|s|^q/q - b(x)|s|^r/r$, is weakly lower semicontinuous. As a consequence the functional I_λ is lower semicontinuous in $E^{r,p}$.

Proof. Assume that $u_k \rightharpoonup u_0$ in $E^{r,p}$. Taking into account that

$$F_s(x,s) = a(x)|s|^{q-2}s - b(x)|s|^{r-2}s, \quad F_{ss}(x,s) = (q-1)a(x)|s|^{q-2} - (r-1)b(x)|s|^{r-2} \quad s \in \mathbb{R}\backslash\{0\},$$

we get

$$F(x,u_k) - F(x,u_0) = \int_0^1 F_s(x, u_0 + t(u_k - u_0))(u_k - u_0)dt$$

and

$$F_s(x, u_0 + t(u_k - u_0)) - F_s(x,u_0) = \int_0^t F_{ss}(x, u_0 + s(u_k - u_0))(u_k - u_0)ds.$$

Consequently,

$$\begin{aligned} F(x,u_k) - F(x,u_0) &= \int_0^1 \left[\int_0^t F_{uu}(x, u_0 + s(u_k - u_0))(u_k - u_0)ds + F_u(x,u_0) \right] (u_k - u_0)dt \\ &= \int_0^1 \int_0^t F_{uu}(x, u_0 + s(u_k - u_0))(u_k - u_0)^2 ds dt + \frac{1}{2} F_u(x,u_0)(u_k - u_0). \end{aligned}$$

Thus, using Remark 2.3.1 we get

$$|F(x,u_k) - F(x,u_0)| \leq C_2 \frac{a^{\frac{r-2}{r-q}}}{b^{\frac{q-2}{r-q}}} (u_k - u_0)^2 + |F_u(x,u_0)(u_k - u_0)|,$$

where $C_2 = C_1(q,r)\lambda^{\frac{r-2}{r-q}}$. Applying the Hölder inequality and using Lemma 2.2.6 we obtain

$$\int_{\mathbb{R}^n_+} (u_k - u_0)^2 \frac{a^{\frac{r-2}{r-q}}}{b^{\frac{q-2}{r-q}}} \leq \left(\int_{\mathbb{R}^n_+} \frac{a^{\frac{r}{r-q}}}{b^{\frac{q}{r-q}}} dx \right)^{(q-2)/q} \left(\int_{\mathbb{R}^n_+} a|u_k - u_0|^q dx \right)^{2/q} \to 0.$$

On the other hand, considering the linear functional $\Phi_0 : E^{r,p} \to \mathbb{R}$ defined by

$$\Phi_0(v) = \int_{\mathbb{R}^n_+} F_u(x,u_0) v dx,$$

we see that

$$\begin{aligned} |\Phi_0(v)| &\leq \lambda \int_{\mathbb{R}^n_+} a|u_0|^{q-1}|v|dx + \int_{\mathbb{R}^n_+} b|u_0|^{r-1}|v|dx \\ &\leq \|u_0\|^{q-1}_{L^q(\mathbb{R}^n_+, a(x))} \|v\|_{L^q(\mathbb{R}^n_+, a(x))} + \|u_0\|^{r-1}_{L^r(\mathbb{R}^n_+, b(x))} \|v\|_{L^r(\mathbb{R}^n_+, b(x))} \leq C\|v\|_{E^{r,p}}, \end{aligned}$$

and hence Φ_0 is continuous. Therefore, if $u_k \rightharpoonup u_0$ in $E^{r,p}$ we have

$$\lim_{k\to\infty} \int_{\mathbb{R}^n_+} F_u(x,u_0)(u_k - u_0)dx = 0,$$

which implies the desired result. □

Now we establish some geometric properties of the energy functional I_λ.

Lemma 2.3.3. *Let $1 < p \leq q < p^*$, $r > q$ and assume $(H_1) - (H_2)$. For all $\lambda > 0$ the functional I_λ is coercive.*

Proof. Since $\lambda, a, b > 0$ and $q < r$, by Remark 2.3.1 we obtain

$$\int_{\mathbb{R}^n_+} \left(\frac{\lambda a}{q} |u|^q - \frac{b}{2r} |u|^r \right) \leq C_{r,q} \frac{1}{q r^{\frac{q}{r-q}}} \int_{\mathbb{R}^n_+} \lambda a \left(\frac{\lambda a}{b} \right)^{\frac{q}{r-q}} = C_{r,q} \lambda^{\frac{r}{r-q}} \int_{\mathbb{R}^n_+} \left(\frac{a^{\frac{r}{r-q}}}{b^{\frac{q}{r-q}}} \right) < \infty.$$

Thus, we get

$$\begin{aligned} I_\lambda(u) &= \frac{1}{p} \int_{\mathbb{R}^n_+} |\nabla u|^p dx + \frac{1}{p} \int_{\mathbb{R}^{n-1}} |u|^p dx' + \frac{1}{2r} \int_{\mathbb{R}^n_+} b |u|^r dx - \int_{\mathbb{R}^n_+} \left(\frac{\lambda a}{q} |u|^q - \frac{b}{2r} |u|^r \right) dx \\ &\geq \frac{1}{p} \|u\|^p_{E^{1,p}} + \frac{1}{2r} \int_{\mathbb{R}^n_+} b |u|^r dx - C_1, \end{aligned}$$

which implies that I_λ is coercive and the proof is completed. ☐

Lemma 2.3.4. *Let $\max\{2, p\} < q < p^*$, $r > q$ and assume $(H_1) - (H_2)$. Then there exists $\Lambda > 0$ such that*

$$-\infty < \inf_{u \in E^{r,p}} I_\lambda(u) < 0, \quad \forall \lambda > \Lambda. \tag{2.19}$$

Proof. Let

$$\Lambda := \inf_{u \in E^{r,p}} \left\{ \frac{q}{p} \|u\|^p_{E^{1,p}} + \frac{q}{r} \int_{\mathbb{R}^n_+} b |u|^r dx : \int_{\mathbb{R}^n_+} a |u|^q = 1 \right\}.$$

We claim that $\Lambda > 0$. Otherwise, there exists a sequence $(u_k) \subset E^{r,p}$ such that

$$\frac{q}{p} \|u_k\|^p_{E^{1,p}} + \frac{q}{r} \int_{\mathbb{R}^n_+} b |u_k|^r dx = o_k(1) \quad \text{and} \quad \int_{\mathbb{R}^n_+} a |u_k|^q = 1.$$

Thus, by using the Hölder inequality we have

$$1 = \int_{\mathbb{R}^n_+} a |u_k|^q \leq \left(\int_{\mathbb{R}^n_+} \frac{a^{\frac{r}{r-q}}}{b^{\frac{q}{r-q}}} dx \right)^{(r-q)/r} \left(\int_{\mathbb{R}^n_+} b |u_k|^r dx \right)^{r/q} \to 0,$$

which is a contradiction. Now if $\lambda > \Lambda$, by the definition of Λ there exists $u_\lambda \in E^{r,p}$ with $\int_{\mathbb{R}^n_+} a |u_\lambda|^q = 1$ such that

$$\lambda > \frac{q}{p} \|u_\lambda\|^p_{E^{1,p}} + \frac{q}{r} \int_{\mathbb{R}^n_+} b |u_\lambda|^r dx.$$

Consequently,

$$I_\lambda(u_\lambda) = \frac{1}{p} \|u_\lambda\|^p_{E^{1,p}} + \frac{1}{r} \int_{\mathbb{R}^n_+} b |u_\lambda|^r dx - \frac{\lambda}{q} \int_{\mathbb{R}^n_+} a |u_\lambda|^q < 0.$$

Therefore, (2.19) holds. ☐

Lemma 2.3.5. *Let $\max\{2, p\} < q < p^*$, $r > q$ and assume $(H_1) - (H_2)$. For all $\lambda > \Lambda$ problem $(\mathcal{P}_\lambda)$ has a nontrivial weak solution.*

Proof. Using the direct method of the calculus of variations, from Lemmas 2.3.2, 2.3.3 and 2.3.4, for all $\lambda > \Lambda$ there exists $u_\lambda \in E^{r,p} \setminus \{0\}$ such that

$$-\infty < \inf_{u \in E^{r,p}} I_\lambda(u) = I_\lambda(u_\lambda) < 0.$$

Therefore, problem $(\mathcal{P}_\lambda)$ has a nontrivial weak solution u_λ with $I_\lambda(u_\lambda) < 0$ for all $\lambda > \Lambda$. Since $I_\lambda(u_\lambda) = I_\lambda(|u_\lambda|)$ we may assume that $u_\lambda \geq 0$. □

Setting

$$\tilde{\lambda} := \inf\{\lambda > 0 : (\mathcal{P}_\mu) \text{ has a nontrivial weak solution for all } \mu > \lambda\},$$

we clearly have that $\lambda^* \leq \tilde{\lambda} \leq \Lambda$.

Next we will prove that problem $(\mathcal{P}_{\tilde{\lambda}})$ has a nontrivial weak solution when $p < q$. To this end, we need the following result.

Lemma 2.3.6. *Let $1 < p < q < p^*$, $r > q$ and assume $(H_1) - (H_2)$. If $\lambda > 0$ and $u \in E^{r,p}$ is a nontrivial weak solution of problem $(\mathcal{P}_\lambda)$ then*

$$\|u\|^p_{E^{1,p}} + \frac{r-q}{r} \int_{\mathbb{R}^n_+} b|u|^r dx \leq \frac{r-q}{r} \lambda^{\frac{r}{r-q}} \int_{\mathbb{R}^n_+} \frac{a^{\frac{r}{r-q}}}{b^{\frac{q}{r-q}}} dx. \tag{2.20}$$

Furthermore, there exists a constant $K > 0$ independent of u such that

$$\|u\|_{E^{1,p}} \geq K \lambda^{\frac{-1}{q-p}}. \tag{2.21}$$

Proof. If $u \in E^{r,p}$ is a nontrivial weak solution of problem $(\mathcal{P}_\lambda)$, proceeding as in (2.16), we get

$$\|u\|^p_{E^{1,p}} + \int_{\mathbb{R}^n_+} b|u|^r dx = \lambda \int_{\mathbb{R}^n_+} a|u|^q dx \leq \frac{r-q}{r} \lambda^{\frac{r}{r-q}} \int_{\mathbb{R}^n_+} \frac{a^{\frac{r}{r-q}}}{b^{\frac{q}{r-q}}} dx + \frac{q}{r} \int_{\mathbb{R}^n_+} b|u|^r dx,$$

which gives estimate (2.20). Now we will prove (2.21). Using again that u is a nontrivial weak solution of problem $(\mathcal{P}_\lambda)$ we see that

$$\frac{1}{\lambda} \|u\|^p_{E^{1,p}} \leq \|u\|^q_{L^q(\mathbb{R}^n_+, a(x))}.$$

This combined with Lemmas 2.2.1 and 2.2.2 show that

$$C_q \|u\|^q_{E^{1,p}} \geq \|u\|^q_{L^q(\mathbb{R}^n_+, a(x))} \geq \frac{1}{\lambda} \|u\|^p_{E^{1,p}}, \quad \forall\, u \in E^{1,p},$$

for some constant $C_q > 0$. Thus, using that $p < q$ and $u \neq 0$ we get

$$\|u\|_{E^{1,p}} \geq C_q^{\frac{-1}{q-p}} \lambda^{\frac{-1}{q-p}},$$

which implies that (2.21) holds by choosing $K = C_q^{\frac{-1}{q-p}}$. □

Lemma 2.3.7. *The problem* $(\mathcal{P}_{\tilde\lambda})$ *has a nontrivial weak solution.*

Proof. Consider a sequence $\lambda_k \to \tilde\lambda$ with $\lambda_k > \tilde\lambda$. By the definition of $\tilde\lambda$, for each k the problem $(\mathcal{P}_{\lambda_k})$ has a nontrivial weak solution u_k. Furthermore, the sequence (u_k) is bounded in $E^{1,p}$ in view of Lemma 2.3.6. Thus, we may assume that $u_k \rightharpoonup u_{\tilde\lambda}$ in $E^{1,p}$ and, by Lemma 2.2.6, $u_k \to u_{\tilde\lambda}$ in $L^q(\mathbb{R}^n_+, a(x))$. Consequently, $u_{\tilde\lambda}$ is a nontrivial weak solution of $(\mathcal{P}_{\tilde\lambda})$. We claim that $u_{\tilde\lambda}$ is not trivial. Indeed, since u_k and $u_{\tilde\lambda}$ are weak solutions of $(\mathcal{P}_{\lambda_k})$ and $(\mathcal{P}_{\tilde\lambda})$, respectively, we have

$$\begin{aligned} o_k(1) = \langle I'_{\lambda_k}(u_k) - I'_{\tilde\lambda}(u_{\tilde\lambda}), u_k - u_{\tilde\lambda}\rangle &= \int_{\mathbb{R}^n_+} \left(|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_{\tilde\lambda}|^{p-2}\nabla u_{\tilde\lambda}\right)\left(\nabla u_k - \nabla u_{\tilde\lambda}\right) dx \\ &+ \int_{\mathbb{R}^{n-1}} \left(|u_k|^{p-2}u_k - |u_{\tilde\lambda}|^{p-2}u_{\tilde\lambda}\right)\left(u_k - u_{\tilde\lambda}\right) dx' + \int_{\mathbb{R}^n_+} b\left(|u_k|^{r-2}u_k - |u_{\tilde\lambda}|^{r-2}u_{\tilde\lambda}\right)\left(u_k - u_{\tilde\lambda}\right) dx \\ &- (J_{1,k} + J_{2,k}), \end{aligned}$$

where

$$J_{1,k} = \lambda_k \int_{\mathbb{R}^n_+} a\left(|u_k|^{q-2}u_k - |u_{\tilde\lambda}|^{q-2}u_{\tilde\lambda}\right)\left(u_k - u_{\tilde\lambda}\right) dx$$

and

$$J_{2,k} = (\lambda_k - \tilde\lambda) \int_{\mathbb{R}^n_+} a|u_{\tilde\lambda}|^{q-2}u_{\tilde\lambda}\left(u_k - u_{\tilde\lambda}\right) dx.$$

Using the Höder inequality together with the fact that (λ_k) is bounded we get

$$|J_{1,k}| \le C\left[\left(\int_{\mathbb{R}^n_+} a|u_k|^q dx\right)^{(q-1)/q} + \left(\int_{\mathbb{R}^n_+} a|u_{\tilde\lambda}|^q dx\right)^{(q-1)/q}\right]\left(\int_{\mathbb{R}^n_+} a|u_k - u_{\tilde\lambda}|^q dx\right)^{1/q}.$$

Consequently, by Lemma 2.2.6 we obtain $J_{1,k} = o_k(1)$. Similarly, we have $J_{2,k} = o_k(1)$. Therefore, we conclude that

$$\begin{aligned} \Bigg(&\int_{\mathbb{R}^n_+} \left(|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_{\tilde\lambda}|^{p-2}\nabla u_{\tilde\lambda}\right)\left(\nabla u_k - \nabla u_{\tilde\lambda}\right) dx \\ &+ \int_{\mathbb{R}^{n-1}} \left(|u_k|^{p-2}u_k - |u_{\tilde\lambda}|^{p-2}u_{\tilde\lambda}\right)\left(u_k - u_{\tilde\lambda}\right) dx' \\ &+ \int_{\mathbb{R}^n_+} b\left(|u_k|^{r-2}u_k - |u_{\tilde\lambda}|^{r-2}u_{\tilde\lambda}\right)\left(u_k - u_{\tilde\lambda}\right) dx\Bigg) = o_k(1). \end{aligned} \tag{2.22}$$

Now we recall that for all $\xi, \zeta \in \mathbb{R}^n$, we know that there exists a constant $C = C(p) > 0$ (see inequality (2.2) in [40]) such that

$$(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)(\xi - \zeta) \ge C\begin{cases} |\xi - \zeta|^p, & \text{if} \quad p \ge 2, \\ |\xi - \zeta|^2(|\xi| + |\zeta|)^{p-2}, & \text{if} \quad 1 < p \le 2. \end{cases} \tag{2.23}$$

If $p \geq 2$, using the fact that $b > 0$ together with (2.22) we obtain

$$\|u_k - u_{\tilde{\lambda}}\|^p_{E^{1,p}} \leq C \left(\int_{\mathbb{R}^n_+} \left(|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_{\tilde{\lambda}}|^{p-2}\nabla u_{\tilde{\lambda}}\right)\left(\nabla u_k - \nabla u_{\tilde{\lambda}}\right) dx \right.$$
$$\left. + \int_{\mathbb{R}^{n-1}} \left(|u_k|^{p-2}u_k - |u_{\tilde{\lambda}}|^{p-2}u_{\tilde{\lambda}}\right)\left(u_k - u_{\tilde{\lambda}}\right) dx' \right) = o_k(1).$$

On the other hand, if $1 < p < 2$ we can use the inequality (2.23) again to obtain

$$\int_{\mathbb{R}^n_+} (|\nabla u_k - \nabla u_{\tilde{\lambda}}|^2)^{\frac{p}{2}} dx$$
$$\leq \int_{\mathbb{R}^n_+} \left[\left(|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_{\tilde{\lambda}}|^{p-2}\nabla u_{\tilde{\lambda}}\right)\left(\nabla u_k - \nabla u_{\tilde{\lambda}}\right)\right]^{\frac{p}{2}} \left((|\nabla u_k| + |\nabla u_{\tilde{\lambda}}|)^p\right)^{\frac{(2-p)}{2}} dx.$$

This together with the Höder inequality, (2.22) and the fact that (u_k) is bounded imply that

$$\tilde{C}_p \int_{\mathbb{R}^n_+} |\nabla u_k - \nabla u_{\tilde{\lambda}}|^p dx$$
$$\leq \left(\int_{\mathbb{R}^n_+} \left(|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_{\tilde{\lambda}}|^{p-2}\nabla u_{\tilde{\lambda}}\right)\left(\nabla u_k - \nabla u_{\tilde{\lambda}}\right) dx \right)^{p/2} \left(\int_{\mathbb{R}^n_+} (|\nabla u_k|^p + |\nabla u_{\tilde{\lambda}}|^p) dx \right)^{(2-p)/2}$$
$$= o_k(1).$$

Similarly, we obtain

$$\int_{\mathbb{R}^{n-1}} |u_k - u_{\tilde{\lambda}}|^p dx' = o_k(1).$$

Hence, $u_k \to u_{\tilde{\lambda}}$ in $E^{1,p}$. Since u_k is a nontrivial weak solution of problem $(\mathcal{P}_{\lambda_k})$, by Lemma 2.3.6 there exists $K = K(p, q)$ such that

$$\|u_k\|_{E^{1,p}} \geq K \lambda_k^{-\frac{1}{q-p}}, \quad \forall\, k \in \mathbb{N}.$$

Since $\|u_k\|_{E^{1,p}} \to \|u_{\tilde{\lambda}}\|_{E^{1,p}}$ and $\lambda_k \to \tilde{\lambda} > 0$ we get

$$\|u_{\tilde{\lambda}}\|_{E^{1,p}} \geq K(\tilde{\lambda})^{-\frac{1}{q-p}} > 0,$$

and hence $u_{\tilde{\lambda}}$ is nontrivial. Since $I_{\tilde{\lambda}}(u_{\tilde{\lambda}}) = I_{\tilde{\lambda}}(|u_{\tilde{\lambda}}|)$ we may assume that $u_{\tilde{\lambda}} \geq 0$ a.e. in $\mathbb{R}^n_+$. □

2.3.3 The second solution

In what follows we will prove item **(iii)** in Theorem 2.1.2. This will be done by using a truncation argument. Let $\lambda > \Lambda$ be fixed and consider the truncated Carathéodory function defined by

$$g_\lambda(x,t) = \begin{cases} 0, & \text{if } \ t < 0, \\ \lambda a(x) t^{q-1} - b(x) t^{r-1}, & \text{if } \ 0 \leq t \leq u_\lambda(x), \\ \lambda a(x) u_\lambda^{q-1} - b(x) u_\lambda^{r-1}, & \text{if } \ t > u_\lambda(x), \end{cases}$$

where $u_\lambda \in E^{r,p}$ is the weak solution of problem $(\mathcal{P}_\lambda)$ with $I_\lambda(u_\lambda) < 0$ obtained in Lemma 2.3.5. Setting $G_\lambda(x,t) = \int_0^t g_\lambda(x,s)ds$, we define the functional $\tilde{I}_\lambda : E^{1,p} \to \mathbb{R}$ by

$$\tilde{I}_\lambda(u) = \frac{1}{p}\|u\|_{E^{1,p}}^p - \int_{\mathbb{R}^n_+} G_\lambda(x,u)dx.$$

Notice that for all $v, \varphi \in E^{1,p}$ it holds

$$\tilde{I}_\lambda'(v)\varphi = \int_{\mathbb{R}^n_+} |\nabla v|^{p-2}\nabla v \nabla\varphi dx + \int_{\mathbb{R}^{n-1}} |v|^{p-2}v\varphi dx' \\ - \int_{\{0\le v\le u_\lambda\}} [\lambda a v^{q-1} - b v^{r-1}]\varphi dx - \int_{\{v > u_\lambda\}} [\lambda a u_\lambda^{q-1} - b u_\lambda^{r-1}]\varphi dx.$$

Furthermore, by choosing $\varphi = v^- := -\min\{v, 0\}$ we see that critical points of $\tilde{I}_\lambda$ are nonnegative.

Next, to prove that critical point of $\tilde{I}_\lambda$ is a critical point of I_λ, inspired in [31, Lemma 2.1] (see also [35]) we have the following *a priori estimate.*

Lemma 2.3.8. *Let* $\max\{2,p\} < q < p^*$, $r > q$ *and assume* $(H_1) - (H_2)$*. If* u_λ *is the solution obtained in item **(ii)** of Theorem 2.1.2 and* $\tilde{u}_\lambda$ *is a critical point of* $\tilde{I}_\lambda$ *then* $\tilde{u}_\lambda \le u_\lambda$*.*

Proof. For a function $v \in E^{1,p}$ let us denote by $v^+(x) = \max\{v(x), 0\}$. Since u_λ is a critical point of I_λ and $\tilde{u}_\lambda$ is a critical point of $\tilde{I}_\lambda$ we get

$$0 = \langle \tilde{I}_\lambda'(\tilde{u}_\lambda) - I_\lambda'(u_\lambda), (\tilde{u}_\lambda - u_\lambda)^+\rangle = \int_{\{\tilde{u}_\lambda > u_\lambda\}} \left(|\nabla\tilde{u}_\lambda|^{p-2}\nabla\tilde{u}_\lambda - |\nabla u_\lambda|^{p-2}\nabla u_\lambda\right)(\nabla\tilde{u}_\lambda - \nabla u_\lambda)\, dx \\ + \int_{\{\tilde{u}_\lambda > u_\lambda\}\cap\mathbb{R}^{n-1}} \left(|\tilde{u}_\lambda|^{p-2}\tilde{u}_\lambda - |u_\lambda|^{p-2}u_\lambda\right)(\tilde{u}_\lambda - u_\lambda)\, dx' \ge 0.$$

This combined with inequality (2.23) imply that $|\{x \in \mathbb{R}^n_+ : \tilde{u}_\lambda(x) > u_\lambda(x)\}| = 0$. Thus, $(\tilde{u}_\lambda - u_\lambda)^+ = 0$ a.e. in $\mathbb{R}^n_+$. Therefore, $\tilde{u}_\lambda \le u_\lambda$ and the proof is complete. □

Lemma 2.3.9. *Let* $\max\{2,p\} < q < p^*$, $r > q$ *and assume* $(H_1) - (H_2)$*. Then there exist* $\rho \in (0, \|u_\lambda\|_{E^{r,p}})$ *and* $\alpha > 0$ *such that* $\tilde{I}_\lambda(u) \ge \alpha > 0$ *if* $\|u\|_{E^{1,p}} = \rho$*;*

Proof. Notice that for all $u \in E^{1,p}$ we can write

$$\int_{\mathbb{R}^n_+} G_\lambda(x,u)dx = \int_{\{0\le u\le u_\lambda\}} G_\lambda(x,u)dx + \int_{\{u > u_\lambda\}} G_\lambda(x,u)dx.$$

Now observing that

$$\int_{\{0\le u\le u_\lambda\}} G_\lambda(x,u)dx = \int_{\{0\le u\le u_\lambda\}} \left[\frac{\lambda a}{q}u^q - \frac{b}{r}u^r\right] dx \le \frac{\lambda}{q}\int_{\{0\le u\le u_\lambda\}} a u^q dx$$

and

$$\begin{aligned}\int_{\{u>u_\lambda\}} G_\lambda(x,u)dx &= \int_{\{u>u_\lambda\}} \left[\int_0^{u_\lambda} g_\lambda(x,t)dt + \int_{u_\lambda}^{u} g_\lambda(x,t)dt\right] dx \\ &= \int_{\{u>u_\lambda\}} \left[\frac{\lambda a u_\lambda^q}{q} - \frac{b u_\lambda^r}{r} + \left(\lambda a u_\lambda^{q-1} - b u_\lambda^{r-1}\right)(u-u_\lambda)\right] dx \\ &\leq \int_{\{u>u_\lambda\}} \left[\frac{\lambda a u_\lambda^q}{q} + \lambda a u_\lambda^{q-1} u\right],\end{aligned}$$

we get

$$\tilde{I}_\lambda(u) \geq \frac{1}{p}\|u\|_{E^{1,p}}^p - \frac{\lambda}{q}\int_{\{0\leq u\leq u_\lambda\}} a u^q dx - \lambda \int_{\{u>u_\lambda\}} a\left[\frac{u_\lambda^q}{q} + u_\lambda^{q-1}u\right]. \tag{2.24}$$

This combined with Remark 2.2.5 imply that there exists $C_1 > 0$ such that

$$\tilde{I}_\lambda(u) \geq \frac{1}{p}\|u\|_{E^{1,p}}^p - \frac{\lambda}{q}C_1\|u\|_{E^{1,p}}^q = \left(\frac{1}{p} - \lambda C_1 \|u\|_{E^{1,p}}^{q-p}\right)\|u\|_{E^{1,p}}^p.$$

Since $q > p$ we obtain the desired result and the proof is completed. □

By Lemma 2.3.9 we have that

$$\inf_{\|u\|_{E^{1,p}}=\rho} \tilde{I}_\lambda(u) > 0 \geq \tilde{I}_\lambda(u_\lambda), \quad \forall\, \lambda > \Lambda.$$

Thus, the minimax level

$$c_\lambda = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \tilde{I}_\lambda(\gamma(t)) > 0, \quad \forall\, \lambda > \Lambda,$$

where $\Gamma := \{\gamma \in C([0,1], E^{1,p}) : \gamma(0) = 0 \text{ and } \gamma(1) = u_\lambda\}$. Applying the mountain pass theorem without the Palais-Smale condition (see [44, Theorem 1.15])), or (PS) for short, we find a sequence $(u_k) \subset E^{1,p}$ at the minimax level c_λ, that is

$$\tilde{I}_\lambda(u_k) \to c_\lambda \quad \text{and} \quad \tilde{I}'_\lambda(u_k) \to 0. \tag{2.25}$$

Lemma 2.3.10. *Let* $\max\{2,p\} < q < p^*$, $r > q$ *and assume* $(H_1) - (H_2)$. *Then, the sequence* (u_k) *in* (2.25) *has a convergent subsequence.*

Proof. From estimate (2.24), there exists $C_1 > 0$ such that

$$\tilde{I}_\lambda(u) \geq \frac{1}{p}\|u\|_{E^{1,p}}^p - \frac{\lambda}{q}\int_{\mathbb{R}_+^n} a u_\lambda^q dx - \lambda C_1 \|u_\lambda\|_{L^q(\mathbb{R}_+^n, a(x))}^{q-1}\|u\|_{E^{1,p}},$$

from where we obtain that $\tilde{I}_\lambda$ is coercive and consequently (u_k) is bounded in $E^{1,p}$. By Lemma 2.2.7, up to a subsequence, we can assume that

$$\begin{cases} u_k \rightharpoonup \tilde{u}_\lambda & \text{in } E^{1,p} \\ u_k(x) \to \tilde{u}_\lambda(x) & \text{a.e. in } \mathbb{R}_+^n \\ u_k \to \tilde{u}_\lambda & \text{in } L^q(\mathbb{R}_+^n, a(x)). \end{cases}$$

Arguing as in proof of Lemma 2.3.2 we can see that $\tilde{I}'_\lambda(\tilde{u}_\lambda) = 0$ and hence $0 \leq \tilde{u}_\lambda \leq u_\lambda$ in $\mathbb{R}_+^n$

by Lemma 2.3.8. Thus, we get

$$o_k(1) = \langle \tilde{I}'_\lambda(u_k) - \tilde{I}'_\lambda(\tilde{u}_\lambda), u_k - \tilde{u}_\lambda \rangle = A_k - B_k + C_k, \tag{2.26}$$

where $o_k(1)$ denotes a quantity that goes to zero as $k \to +\infty$ and

$$\begin{aligned}
A_k &= \int_{\mathbb{R}^n_+} \left(|\nabla u_k|^{p-2}\nabla u_k - |\nabla \tilde{u}_\lambda|^{p-2}\nabla \tilde{u}_\lambda\right)(\nabla u_k - \nabla \tilde{u}_\lambda)\, dx \\
&\quad + \int_{\mathbb{R}^{n-1}} \left(|u_k|^{p-2}u_k - |\tilde{u}_\lambda|^{p-2}\tilde{u}_\lambda\right)(u_k - \tilde{u}_\lambda)\, dx' \\
B_k &= \int_{\{0\le u_k \le u_\lambda\}} [\lambda a u_k^{q-1} - b u_k^{r-1}]\,(u_k - \tilde{u}_\lambda)\, dx + \int_{\{u_k > u_\lambda\}} [\lambda a u_\lambda^{q-1} - b u_\lambda^{r-1}]\,(u_k - \tilde{u}_\lambda)\, dx \\
C_k &= \int_{\{0\le \tilde{u}_\lambda \le u_\lambda\}} [\lambda a \tilde{u}_\lambda^{q-1} - b \tilde{u}_\lambda^{r-1}]\,(u_k - \tilde{u}_\lambda)\, dx + \int_{\{\tilde{u}_\lambda > u_\lambda\}} [\lambda a \tilde{u}_\lambda^{q-1} - b \tilde{u}_\lambda^{r-1}]\,(u_k - \tilde{u}_\lambda)\, dx.
\end{aligned}$$

Therefore,

$$A_k = o_k(1) + \int_{\{0\le u_k \le u_\lambda\}} [\lambda a u_k^{q-1} - b u_k^{r-1}]\,(u_k - \tilde{u}_\lambda)\, dx - \int_{\{0\le \tilde{u}_\lambda \le u_\lambda\}} [\lambda a \tilde{u}_\lambda^{q-1} - b \tilde{u}_\lambda^{r-1}]\,(u_k - \tilde{u}_\lambda)\, dx.$$

Now, proceeding as in the proof of Lemma 2.3.2 we see that

$$\int_{\{0\le u_k \le u_\lambda\}} [\lambda a u_k^{q-1} - b u_k^{r-1}]\,(u_k - \tilde{u}_\lambda)\, dx = o_k(1)$$

and

$$\int_{\{0\le \tilde{u}_\lambda \le u_\lambda\}} [\lambda a \tilde{u}_\lambda^{q-1} - b \tilde{u}_\lambda^{r-1}]\,(u_k - \tilde{u}_\lambda)\, dx = o_k(1).$$

Thus, we conclude that $A_k = o_k(1)$. If $2 \le p \le q < r$, using inequality (2.23), we get $\|u_k - \tilde{u}_\lambda\|^p_{E^{1,p}} = o_k(1)$. Furthermore, if $1 < p < 2$, arguing as in the proof of Lemma 2.3.7 we obtain $\|u_k - \tilde{u}_\lambda\|^p_{E^{1,p}} = o_k(1)$. This completes the proof of Lemma 2.3.10. □

Finalizing the proof of item ***(iii)*** *in Theorem 2.1.2.* By Lemma 2.3.10, and standard arguments we conclude that $\tilde{u}_\lambda$ is a critical point of I_λ. To conclude, by Lemma 2.3.8, we have $0 \le \tilde{u}_\lambda \le u_\lambda$. Thus,

$$g(x, \tilde{u}_\lambda) = \lambda a(x)\tilde{u}_\lambda^{q-1} - b(x)\tilde{u}_\lambda^{r-1} \quad \text{and} \quad G(x, \tilde{u}_\lambda) = \frac{\lambda a(x)\tilde{u}_\lambda^q}{q} - \frac{b(x)\tilde{u}_\lambda^r}{r}$$

so that

$$\tilde{I}_\lambda(\tilde{u}_\lambda) = I_\lambda(\tilde{u}_\lambda) \quad \text{and} \quad \tilde{I}'_\lambda(\tilde{u}_\lambda) = I'_\lambda(\tilde{u}_\lambda).$$

More precisely, we find

$$I_\lambda(\tilde{u}_\lambda) > 0 \ge I_\lambda(u_\lambda) \quad \text{and} \quad I'_\lambda(\tilde{u}_\lambda) = 0.$$

Therefore, $\tilde{u}_\lambda$ is a nontrivial weak solution of problem $(\mathcal{P}_\lambda)$ such that $0 \le \tilde{u}_\lambda \le u_\lambda$, $\tilde{u}_\lambda \ne 0$ and $\tilde{u}_\lambda \ne u_\lambda$. □

2.3.4 Multiplicity

Finally, in this subsection we will complete the proof of Theorem 2.1.2 by proving statement **(iv)**. It consists in applying the symmetric mountain pass theorem due to Ambrosetti-Rabinowitz [8] and Clark [17]. To this, we need to recall some notations. Let E be a Banach space and denotes by $\mathcal{E}$ the class of all subsets of $E \setminus \{0\}$ closed and symmetric with respect to the origin :

$$\mathcal{E} := \{A \subset E \setminus \{0\} : A \text{ is closed and } A = -A\}.$$

For $A \in \mathcal{E} \setminus \{\varnothing\}$ the genus $\gamma(A)$ is define by

$$\gamma(A) := \min\{m \in \mathbb{N} : \exists\, \varphi \in C(A, \mathbb{R}^m \setminus \{0\}) \text{ such that } \varphi(x) = -\varphi(-x)\}.$$

If the minimum does not exist, we define $\gamma(A) = \infty$ and $\gamma(\varnothing) = 0$. Let $\mathcal{E}_m = \{A \in \mathcal{E} : \gamma(A) \geq m\}$. The main properties of the genus can be found in [38, 41].

Now, we recall the following classical multiplicity result (see for instance [8, 17]).

Theorem 2.3.11. *Let E be an infinite dimensional Banach space and $I \in C^1(E, \mathbb{R})$ satisfying*

(A_1) *$I(u)$ is even, bounded from below, $I(0) = 0$ and $I(u)$ satisfies the Palais-Smale condition (PS);*

(A_2) *For each $m \in \mathbb{N}$, there exists an $A_m \in \mathcal{E}_m$ such that $\sup_{u \in A_m} I(u) < 0$.*

Defining

$$c_m = \inf_{A \in \mathcal{E}_m} \sup_{u \in A} I_\lambda(u),$$

then each c_k is a critical value of $I(u)$, $c_m \leq c_{m+1} < 0$ for $m \in \mathbb{N}$ and (c_m) converges to zero. Moreover, if $c = c_m = c_{m+1} = \cdots = c_{m+j} < \infty$, then $\gamma(K_c) \geq j + 1$. Here, K_c is defined by

$$K_c = \{u \in E^{r,p} : I_\lambda(u) = c \text{ and } I'_\lambda(u) = 0\}.$$

To prove item **(iv)** in Theorem 2.1.2, it is sufficient to show that I_λ satisfies the conditions (A_1) and (A_2) above. Arguing as in the proof of Lemma 2.3.10 one can see that I_λ satisfies condition (A_1). In order to verify condition (A_2), we consider $\Omega_0 = \{x \in \mathbb{R}^n_+ : a(x) = 0\}$ and $\Omega_0^c = \mathbb{R}^n_+ \setminus \Omega_0$. Denote

$$E_0 = \{u \in E^{r,p} : u(x) = 0 \quad \text{a.e.} \quad x \in \Omega_0\}.$$

If $\Omega_0 = \varnothing$, i.e., $a(x) > 0$ in $\mathbb{R}^n_+$ then we let $E_0 = E^{r,p}$. Obviously, E_0 is an infinitely dimensional linear subspace of $E^{r,p}$. A seminorm $[\cdot]_q$ on $E^{r,p}$ is defined by

$$[u]_q = \left(\int_{\mathbb{R}^n_+} a(x)|u|^q dx \right)^{1/q}.$$

Lemma 2.3.12. *The seminorm $[\cdot]_q$ is a norm in E_0.*

Proof. It is sufficient to show that $u \in E_0$, $[u]_q = 0$ implies that $u = 0$, a.e. in $\mathbb{R}^n_+$. Indeed,

$$0 = [u]_q^q = \int_{\mathbb{R}^n_+} a(x)|u|^q dx = \int_{\Omega_0^c} a(x)|u|^q dx.$$

This together with fact $a(x) > 0$ in Ω_0^c imply that $u(x) = 0$, a.e. in Ω_0^c. Since $u \in E_0$, $u(x) = 0$, a.e. in Ω_0. Therefore, $u(x) = 0$, a.e. in $\mathbb{R}^n_+$ and this completes the proof. □

Lemma 2.3.13. *Let $1 < p \leq q < p^*$, $r > q$ and assume (H_1). Then for each $m \in \mathbb{N}$, there exist an $A_m \in \mathcal{E}_m$ and λ_m such that*

$$\sup_{u \in A_m} I_\lambda(u) < 0, \quad \forall\, \lambda > \lambda_m.$$

Proof. Let E_m be a m-dimensional subspace of E_0. Since all norms on the finite dimension space E_m are equivalent, there exists $b_m > 0$ such that

$$I_\lambda(u) \leq \frac{1}{p}\|u\|^p_{E^{r,p}} + \frac{1}{r}\|u\|^r_{E^{r,p}} - \frac{\lambda b_m}{q}\|u\|^q_{E^{r,p}} \leq \frac{2}{p} - \frac{\lambda b_m}{q}$$

for all $u \in E_m$ with $\|u\|_{E^{r,p}} = 1$. Thus, for $\lambda_m = 4q/pb_m$, $I_\lambda(u) < -2/p$ if $\|u\|_{E^{r,p}} = 1$, for all $\lambda > \lambda_m$. Let $A_m = S^m(1)$ be a sphere with radius 1 in E_m. Then

$$\sup_{u \in A_m} I_\lambda(u) < 0, \quad \forall\, \lambda > \lambda_m$$

and by properties of genus $A_m \in \mathcal{E}_m$. □

*Finalizing the proof of item **(iv)** in Theorem 2.1.2.* It follows directly from Theorem 2.3.11. □

2.4 Proof of Theorem 2.1.3

This section is devoted to the proof of Theorem 2.1.3. In order to prove our multiplicity result we recall the original statement of the symmetric mountain pass theorem (see [8]).

Theorem 2.4.1. *Let E be a real infinite-dimensional Banach space and $I \in C^1(E, R)$ an even functional satisfying the (PS) condition and the following hypotheses:*

(I_1) *$I(0) = 0$ and there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho(0)} \geq \alpha$;*

(I_2) *for any finite dimensional $\widetilde{E} \subset E$, $\widetilde{E} \cap \{u \in E : I(u) \geq 0\}$ is bounded.*

Then I has an unbounded sequence of critical values.

Now, we establish some properties of the energy functional I_λ.

Lemma 2.4.2. *Let $1 < p \leq r < q < p^*$ and assume $(\widetilde{H}_1) - (\widetilde{H}_2)$. Then for each $\lambda > 0$ there exist $\rho, \alpha_0 > 0$ such that $I_\lambda(u) \geq \alpha_0 > 0$ if $\|u\|_{E^{r,p}} = \rho$.*

Proof. First we observe that

$$\|u\|^r_{E^{r,p}} \leq \left(\|u\|^p_{E^{1,p}} + \|u\|^p_{L^r(\mathbb{R}^n_+,b(x))}\right)^{r/p} \leq 2^{\frac{r}{p}} \left(\|u\|^r_{E^{1,p}} + \|u\|^r_{L^r(\mathbb{R}^n_+,b(x))}\right). \tag{2.27}$$

Without loss of generality we may assume that $\|u\|^p_{E^{1,p}} + \|u\|^p_{L^r(\mathbb{R}^n_+,b(x))} = \|u\|^p_{E^{r,p}} = \rho^p \leq 1$ and using that $p \leq r$ we see that $\|u\|^p_{E^{1,p}} \geq \|u\|^r_{E^{1,p}}$. Thus, we conclude that

$$I_\lambda(u) \geq \frac{1}{p}\|u\|^r_{E^{1,p}} + \frac{1}{r}\|u\|^r_{L^r(\mathbb{R}^n_+,b(x))} - \frac{\lambda}{q}\|u\|^q_{L^q(\mathbb{R}^n_+,a(x))}.$$

This together with (2.27), Lemmas 2.2.1 and 2.2.2 and the fact that $r < q$ imply

$$I_\lambda(u) \geq \frac{1}{r2^{\frac{r}{p}}}\|u\|^r_{E^{r,p}} - \frac{\lambda}{q}C_1\|u\|^q_{E^{r,p}} = \left(\frac{1}{r2^{\frac{r}{p}}} - \frac{\lambda}{q}C_1\rho^{q-r}\right)\rho^r,$$

which implies (I_1) by choosing ρ sufficiently small. □

Next, let us ensure that any (PS) sequence associated to I_λ has a convergent subsequence. This is done in the next lemma.

Lemma 2.4.3. *Let $1 < p \leq r < q < p^*$ and assume $(\widetilde{H}_1)-(\widetilde{H}_2)$. Then any sequence $(u_k) \subset E^{r,p}$ such that*

$$I_\lambda(u_k) \to c \quad \textit{and} \quad I'_\lambda(u_k) \to 0, \tag{2.28}$$

has a convergent subsequence.

Proof. First, we observe that

$$\left(\frac{1}{p} - \frac{1}{q}\right)\|u_k\|^p_{E^{1,p}} + \left(\frac{1}{r} - \frac{1}{q}\right)\int_{\mathbb{R}^n_+} b|u_k|^r dx = I_\lambda(u_k) - \frac{1}{q}\langle I'_\lambda(u_k), u_k\rangle \leq c_\lambda + o_k(\|u_k\|_{E^{r,p}}). \tag{2.29}$$

We claim that $(u_k) \subset E^{r,p}$ is bounded. Arguing by contradiction, let us suppose that $\|u_k\|_{E^{r,p}} \to \infty$. Since $1 < p \leq r < q$, in view of (2.29) we get

$$\frac{\|u_k\|^p_{E^{1,p}}}{\|u_k\|_{E^{r,p}}} = o_k(1) \quad \text{and} \quad \frac{\|u_k\|^r_{L^r(\mathbb{R}^n_+,b(x))}}{\|u_k\|_{E^{r,p}}} = o_k(1). \tag{2.30}$$

This in combination with the fact that

$$\frac{\|u_k\|^p_{E^{1,p}}}{\|u_k\|_{E^{r,p}}} + \frac{\|u_k\|^p_{L^r(\mathbb{R}^n_+,b(x))}}{\|u_k\|_{E^{r,p}}} = \|u_k\|^{p-1}_{E^{r,p}} \to \infty, \quad \text{as} \quad k \to \infty$$

imply that

$$\frac{\|u_k\|^p_{L^r(\mathbb{R}^n_+,b(x))}}{\|u_k\|_{E^{r,p}}} \to \infty, \quad \text{as} \quad k \to \infty. \tag{2.31}$$

If $p = r$, combining (2.30) and (2.31) we obtain a contradiction. In case that $p < r$, using again (2.31) we conclude that $\|u_k\|^p_{L^r(\mathbb{R}^n_+,b(x))} \to \infty$ as $k \to \infty$ and hence $\|u_k\|^{p-r}_{L^r(\mathbb{R}^n_+,b(x))} \leq C$. On the other hand,

$$\frac{\|u_k\|^p_{L^r(\mathbb{R}^n_+,b(x))}}{\|u_k\|_{E^{r,p}}} = \|u_k\|^{p-r}_{L^r(\mathbb{R}^n_+,b(x))} \frac{\|u_k\|^r_{L^r(\mathbb{R}^n_+,b(x))}}{\|u_k\|_{E^{r,p}}} \to 0, \quad \text{as} \quad k \to \infty,$$

which contradicts (2.31) and hence (u_k) is bounded. By Lemma 2.2.7 we may assume that

$$\begin{cases} u_k \rightharpoonup u_0 & \text{in } E^{r,p} \\ u_k(x) \to u_0(x) & \text{a.e. in } \mathbb{R}^n_+ \\ u_k \to u_0 & \text{in } L^q(\mathbb{R}^n_+, a(x)) \end{cases}$$

as $k \to \infty$. From (2.28), it follows that

$$o_k(1) = \langle I'_\lambda(u_k) - I'_\lambda(u_0), u_k - u_0 \rangle = A_k - \int_{\mathbb{R}^n_+} \lambda a \left(|u_k|^{q-2}u_k - |u_0|^{q-2}u_0\right)(u_k - u_0)\, dx, \quad (2.32)$$

where

$$\begin{aligned} A_k = &\int_{\mathbb{R}^n_+} \left(|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_0|^{p-2}\nabla u_0\right)(\nabla u_k - \nabla u_0)\, dx \\ &+ \int_{\mathbb{R}^{n-1}} \left(|u_k|^{p-2}u_k - |u_0|^{p-2}u_0\right)(u_k - u_0)\, dx' + \int_{\mathbb{R}^n_+} b\left(|u_k|^{r-2}u_k - |u_0|^{r-2}u_0\right)(u_k - u_0)\, dx. \end{aligned}$$

By the Hölder inequality and Lemma 2.2.7, we obtain

$$\int_{\mathbb{R}^n_+} \lambda a \left(|u_k|^{q-2}u_k - |u_0|^{q-2}u_0\right)(u_k - u_0)\, dx = o_k(1).$$

Thus, from (2.32) we conclude that $A_k = o_k(1)$. If $2 \le p \le r < q$, we can use the inequality (2.23) and the fact that $b \ge 0$ to get

$$\begin{aligned} &\int_{\mathbb{R}^n_+} \left(|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_0|^{p-2}\nabla u_0\right)(\nabla u_k - \nabla u_0)\, dx = o_k(1) \\ &\int_{\mathbb{R}^{n-1}} \left(|u_k|^{p-2}u_k - |u_0|^{p-2}u_0\right)(u_k - u_0)\, dx' = o_k(1) \\ &\int_{\mathbb{R}^n_+} b\left(|u_k|^{r-2}u_k - |u_0|^{r-2}u_0\right)(u_k - u_0)\, dx = o_k(1). \end{aligned} \quad (2.33)$$

Using once again inequality (2.23), we get

$$\|u_k - u_0\|^p_{E^{r,p}} = \|u_k - u_0\|^p_{E^{1,p}} + \|u_k - u_0\|^p_{L^r(\mathbb{R}^n_+,b(x))} = o_k(1),$$

which implies that $u_k \to u_0$ in $E^{r,p}$. Now, if $1 < p < 2$ we have two cases to consider, $r \ge 2$ and $p \le r < 2$. If $r \ge 2$, by inequality (2.23) and (2.33) we obtain

$$\|u_k - u_0\|^r_{L^r(\mathbb{R}^n_+,b(x))} \le \int_{\mathbb{R}^n_+} b\left(|u_k|^{r-2}u_k - |u_0|^{r-2}u_0\right)(u_k - u_0)\, dx = o_k(1). \quad (2.34)$$

Now, if $p \leq r < 2$, by inequality (2.23) and the Höder inequality we get

$$\|u_k - u_0\|^r_{L^r(\mathbb{R}^n_+,b(x))} \leq \int_{\mathbb{R}^n_+} b\left(\left(|u_k|^{r-2}u_k - |u_0|^{r-2}u_0\right)(u_k - u_0)\right)^{\frac{r}{2}} \left((|u_k| + |u_0|)^r\right)^{\frac{(2-r)}{2}} dx$$
$$\leq \left(\int_{\mathbb{R}^n_+} b\left(|u_k|^{r-2}u_k - |u_0|^{r-2}u_0\right)(u_k - u_0)\, dx\right)^{r/2} \left(\int_{\mathbb{R}^n_+} b\,(|u_k| + |u_0|)^r\, dx\right)^{(2-r)/2}.$$

This combined with inequalities (2.33) and (2.34) and the fact that (u_k) is bounded imply that $\|u_k - u_0\|^r_{L^r(\mathbb{R}^n_+,b(x))} = o_k(1)$. Now, if $1 < p < 2$, arguing as in the proof of Lemma 2.3.7 we obtain $\|u_k - u_0\|^p_{E^{1,p}} = o_k(1)$. Therefore, $\|u_k - u_0\|^p_{E^{r,p}} = \|u_k - u_0\|^p_{E^{1,p}} + \|u_k - u_0\|^p_{L^r(\mathbb{R}^n_+,b(x))} = o_k(1)$, and this completes the proof. □

Finalizing the proof of Theorem 2.1.3. If u is a weak solution of problem $(\mathcal{P}_\lambda)$, choosing $\varphi = u$ in (2.14) we get $\|u\|^p_{E^{1,p}} + \|u\|^r_{L^r(\mathbb{R}^n_+,b(x))} = \lambda\|u\|^q_{L^r(\mathbb{R}^n_+,a(x))}$, which implies that $u = 0$ if $\lambda \leq 0$ and item **(i)** in Theorem 2.1.3 is proved. Now we will use Theorem 2.4.1 to prove item **(ii)** in Theorem 2.1.3. By Lemma 2.4.2, for any $\lambda > 0$ the functional I_λ satisfies condition (I_1). Now we prove item (I_2). Suppose by contradiction that (I_2) is false. Then, there exist a finite dimensional $\widetilde{E} \subset E^{r,p}$ and a sequence $(u_k) \subset \widetilde{E}$ satisfying

$$I_\lambda(u_k) > 0, \quad k \in \mathbb{N} \quad \text{and} \quad \|u_k\|_{E^{r,p}} \to \infty \quad \text{as} \quad k \to \infty. \tag{2.35}$$

Using the fact that all the norms in $\tilde{E}$ are equivalent, there exists $\tilde{c} > 0$ such that

$$0 < I_\lambda(u_k) \leq \frac{1}{p}\|u_k\|^p_{E^{r,p}} + \frac{1}{r}\|u_k\|^r_{E^{r,p}} - \frac{\lambda\tilde{c}}{q}\|u_k\|^q_{E^{r,p}}, \quad \forall\, k \in \mathbb{N}.$$

Thus,

$$\frac{\lambda\tilde{c}}{q}\|u_k\|^q_{E^{r,p}} < \frac{1}{p}\|u_k\|^p_{E^{r,p}} + \frac{1}{r}\|u_k\|^r_{E^{r,p}}, \quad \forall\, k \in \mathbb{N},$$

which contradicts (2.35), since $p \leq r < q$, and item (I_2) is proved. In view of Lemma 2.4.3, for each $\lambda > 0$ we can apply Theorem 2.4.1 to obtain an unbounded sequence of critical values of I_λ to which we can associate at least two critical points because the functional I_λ is even. This completes the proof. □

Chapter 3

A quasilinear elliptic equation with exponential growth and weights in anisotropic spaces

In this chapter we establish embedding results of a certain Sobolev space into weighted Lebesgue spaces and we derive some Trudinger-Moser type inequalities. As an application we prove existence, nonexistence and multiplicity of solutions for a class of quasilinear elliptic problems with nonlinear boundary condition and involving exponential nonlinearities and weights in anisotropic Lebesgue spaces. This chapter is also in article format submitted in [21].

3.1 Introduction and main results

Here, we are concerning with the existence, nonexistence and multiplicity of solutions for the following nonlinear eigenvalue problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{n-2}\nabla u) + h(x)|u|^{r-2}u = \lambda a(x) f(x,u) & \text{in} \quad \mathbb{R}^n_+ \\ |\nabla u|^{n-2}\nabla u \cdot \eta + |u|^{n-2}u = 0 & \text{on} \quad \partial\mathbb{R}^n_+, \end{cases} \qquad (\mathcal{P}_\lambda)$$

where $n \geq 2$ is an integer, $\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}$ denotes the upper half-space, η is the unit outward normal vector on the boundary $\partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$, $n \leq r < \infty$, and a, h and f satisfy some suitable conditions.

We assume that f is a continuous function with subcritical exponential growth in the Trudinger-Moser sense, i.e., for any $\alpha > 0$

$$\lim_{|s|\to\infty} \frac{|f(x,s)|}{e^{\alpha|s|^{n'}}} = 0, \text{ uniformly in } x \in \mathbb{R}^n_+. \tag{3.1}$$

Setting $F(x,s) = \int_0^s f(x,t)dt$, we assume that f is continuous and satisfies the following assumptions:

(f_1) $\overline{\lim\limits_{s\to 0^+}} \frac{nF(x,s)}{s^n} < \lambda_1$ uniformly with respect to $x \in \mathbb{R}^n_+$, where

$$\lambda_1 := \inf\left\{ \frac{\int_{\mathbb{R}^n_+} |\nabla u|^n dx + \int_{\partial\mathbb{R}^n_+} |u|^n dx'}{\int_{\mathbb{R}^n_+} a|u|^n dx} : u \in C^1_0(\mathbb{R}^n) \setminus \{0\} \right\};$$

(f_2) there exists $\mu > r$ such that

$$0 < \mu F(x,s) \leq f(x,s)s, \quad \forall\, x \in \mathbb{R}^n_+ \quad \text{and} \quad s \neq 0;$$

(f_3) there exist $R_0, M_0 > 0$ such that

$$F(x,s) \leq M_0 f(x,s), \quad \forall\, x \in \mathbb{R}^n_+ \quad \text{and} \quad s \geq R_0.$$

We assume the following assumptions on the weight functions a, h:

(H_1) $a : \mathbb{R}^n_+ \to \mathbb{R}$ is a nontrivial mensurable function and there are $c_1 > 0$ and $\beta \geq n$ such that

$$0 \leq a(x) \leq \frac{c_1}{(1+|x|)^\beta}, \quad \text{for a.e.} \quad x \in \mathbb{R}^n_+.$$

(H_2) $h : \mathbb{R}^n_+ \to \mathbb{R}$ is a positive continuous function.

Under these hypotheses, our first result concerning problem $(\mathcal{P}_\lambda)$ can be stated as follows:

Theorem 3.1.1. *Assume* $(H_1) - (H_2)$ *and* $(f_1) - (f_3)$. *If* $n \leq r < \infty$ *then*

(i) *Problem* $(\mathcal{P}_\lambda)$ *has no nonzero weak solution for every* $\lambda \in (-\infty, 0]$*;*

(ii) *Problem* $(\mathcal{P}_\lambda)$ *has at least a nonzero weak solution for every* $\lambda \in (0, \infty)$.

In order to obtain a multiplicity result, in addition, we assume that

(f_4) there exist $\nu_0, s_0 > 0$ and $\gamma_0 > r$ such that

$$F(x,s) \geq \nu_0 |s|^{\gamma_0}, \text{ uniformly with respect to } x \in \mathbb{R}^n_+, \quad \forall\, |s| \leq s_0.$$

Our multiplicity result is stated below:

Theorem 3.1.2. *Assume* $(H_1) - (H_2)$ *and that* $f(x,\cdot)$ *is odd and satisfies* $(f_1) - (f_4)$. *If* $n \leq r < \infty$, *then problem* $(\mathcal{P}_\lambda)$ *has an infinite number of nonzero weak solutions for every* $\lambda \in (0, \infty)$.

3.2 Variational Framework

In this section, we set-up under which space we shall work in the present chapter. Firstly we collect a few definitions and embeddings results. Denote by $C^\infty_\delta(\mathbb{R}^n_+)$ the space of $C^\infty_0(\mathbb{R}^n)-$functions restricted to $\mathbb{R}^n_+$. We define the weighted Sobolev space E as the completion of $C^\infty_\delta(\mathbb{R}^n_+)$ with respect to the norm

$$\|u\| := \left[\int_{\mathbb{R}^n_+}\left(|\nabla u|^n + \frac{|u|^n}{(1+x_n)^n}\right)dx\right]^{1/n}.$$

Hereafter in this chapter, B_R denotes the ball of center zero and radius $R > 0$ in $\mathbb{R}^n$, $B_R^+ := B_R \cap \mathbb{R}^n_+$, $(B_R)^c$ denotes $\mathbb{R}^n \setminus B_R$, the complement of the set $B_R \subset \mathbb{R}^n$, and $(B_R^+)^c$ denotes $\mathbb{R}^n_+ \setminus B_R^+$ the complement of the set $B_R^+ \subset \mathbb{R}^n_+$.

3.2.1 Sobolev embedding

In this subsection, we establish some embedding results from E into weighted Lebesgue spaces. We start with following:

Lemma 3.2.1. *For any $n \le q < \infty$ the weighted Sobolev embedding*

$$E \hookrightarrow L^q\left(\mathbb{R}^n_+, \frac{1}{(1+x_n)^n}\right), \tag{3.2}$$

and the Sobolev trace embedding

$$E \hookrightarrow L^q(\mathbb{R}^{n-1}), \tag{3.3}$$

are continuous.

Proof. Recall that, for any $1 \le p < n$, by the Gagliardo-Nirenberg-Sobolev inequality and a suitable reflexion argument (see [43, Lemma 2.10]) that there exists $C = C(n,p)$ such that

$$\left(\int_{\mathbb{R}^n_+} |v|^{p^*}dx\right)^{(n-p)/np} \le C\int_{\mathbb{R}^n_+}|\nabla v|^p dx, \quad \forall\, v \in C_0^\infty(\mathbb{R}^n).$$

In particular, if $p = 1$ we have

$$\left(\int_{\mathbb{R}^n_+} |v|^{\frac{n}{n-1}}dx\right)^{(n-1)/n} \le C_0\int_{\mathbb{R}^n_+}|\nabla v| dx, \quad \forall\, v \in C_0^\infty(\mathbb{R}^n). \tag{3.4}$$

Applying (3.4) with $v = (1+x_n)^\sigma |u|^n$, $\sigma \in \mathbb{R}$ to be chosen later on, we obtain

$$\left(\int_{\mathbb{R}^n_+} |(1+x_n)^\sigma |u|^n|^{\frac{n}{n-1}}dx\right)^{\frac{n-1}{n}} \le C_0\int_{\mathbb{R}^n_+}|\sigma|(1+x_n)^{\sigma-1}|u|^n dx + C_0 n\int_{\mathbb{R}^n_+}(1+x_n)^\sigma|u|^{n-1}|\nabla u| dx.$$

Choosing $\sigma = -(n-1)$ and using the Young inequality we obtain

$$\left(\int_{\mathbb{R}^n_+} \frac{|u|^{\frac{n^2}{n-1}}}{(1+x_n)^n}dx\right)^{(n-1)/n} \le C_1\int_{\mathbb{R}^n_+}\left(|\nabla u|^n + \frac{|u|^n}{(1+x_n)^n}\right)dx, \tag{3.5}$$

where C_1 depends only on n. Hence, we conclude that

$$E \hookrightarrow L^{\frac{n^2}{n-1}}\left(\mathbb{R}^n_+, \frac{1}{(1+x_n)^n}\right).$$

If $n < q < n^2/(n-1)$, by interpolation, there exists $0 < \theta < 1$ such that

$$\|u\|_{L^q\left(\mathbb{R}^n_+,(1+x_n)^{-n}\right)} \leq \|u\|^{\theta}_{L^n\left(\mathbb{R}^n_+,(1+x_n)^{-n}\right)}\|u\|^{1-\theta}_{L^{\frac{n^2}{n-1}}\left(\mathbb{R}^n_+,(1+x_n)^{-n}\right)} \leq C\|u\|.$$

In particular, using that $n < n+1 < n^2/(n-1)$, one has $E \hookrightarrow L^{n+1}\left(\mathbb{R}^n_+, (1+x_n)^{-n}\right)$. On the other hand, applying again (3.4) with $v = (1+x_n)^{-(n-1)}|u|^{n+1}$ and using the Young inequality we get

$$\begin{aligned}\left(\int_{\mathbb{R}^n_+} \frac{|u|^{\frac{n(n+1)}{n-1}}}{(1+x_n)^n}dx\right)^{(n-1)/n} &\leq (n-1)C\int_{\mathbb{R}^n_+} \frac{|u|^{n+1}}{(1+x_n)^n}dx + (n+1)C\int_{\mathbb{R}^n_+} \frac{|u|^n|\nabla u|}{(1+x_n)^{(n-1)}}dx \\ &\leq (n-1)C\int_{\mathbb{R}^n_+} \frac{|u|^{n+1}}{(1+x_n)^n}dx + (n+1)C_2\left(\int_{\mathbb{R}^n_+} \frac{|u|^{\frac{n^2}{n-1}}}{(1+x_n)^n}dx + \int_{\mathbb{R}^n_+}|\nabla u|^n dx\right).\end{aligned}$$

This together with (3.5) imply that $E \hookrightarrow L^{\frac{n(n+1)}{(n-1)}}(\mathbb{R}^n_+, (1+x_n)^{-n})$. If $n \leq q \leq n(n+1)/(n-1)$, by interpolation, there exists $0 \leq \theta \leq 1$ such that

$$\|u\|_{L^q(\mathbb{R}^n_+,(1+x_n)^{-n})} \leq \|u\|^{\theta}_{L^n(\mathbb{R}^n_+,(1+x_n)^{-n})}\|u\|^{1-\theta}_{L^{\frac{n(n+1)}{(n-1)}}(\mathbb{R}^n_+,(1+x_n)^{-n})} \leq C\|u\|.$$

Iterating this process with $k = n+2, n+3, \ldots$, one has $E \hookrightarrow L^{\frac{nk}{n-1}}(\mathbb{R}^n_+, (1+x_n)^{-n})$. Now, given $q \in [0, \infty)$, one can choose $k \geq n$ such that $n < q < nk/(n-1)$ and once again use interpolation to get

$$\|u\|_{L^q(\mathbb{R}^n_+,(1+x_n)^{-n})} \leq \|u\|^{\theta}_{L^n(\mathbb{R}^n_+,(1+x_n)^{-n})}\|u\|^{1-\theta}_{L^{\frac{nk}{n-1}}(\mathbb{R}^n_+,(1+x_n)^{-n})} \leq C\|u\|,$$

which proves the embedding (3.2).

Now we will prove the trace embedding (3.3). For that, we fix $q \geq n$ and compute

$$|u(x',0)|^q = -\int_0^{+\infty} \frac{\partial}{\partial x_n}\left(\frac{|u|^q}{(1+x_n)^n}\right)dx_n \leq q\int_0^{+\infty} \frac{|u|^{q-1}|\nabla u|}{(1+x_n)^n}dx_n + n\int_0^{+\infty} \frac{|u|^q}{(1+x_n)^{n+1}}dx_n.$$

Integrating and using Hölder's inequality together with the fact that $(1+x_n)^{-1} < 1$ we obtain

$$\int_{\mathbb{R}^{n-1}} |u(x',0)|^q dx' \leq q\left(\int_{\mathbb{R}^n_+} \frac{|u|^{(q-1)\frac{n}{n-1}}}{(1+x_n)^n}dx\right)^{(n-1)/n}\left(\int_{\mathbb{R}^n_+}|\nabla u|^n dx\right)^{1/n} + n\int_{\mathbb{R}^n_+} \frac{|u|^q}{(1+x_n)^n}dx.$$

Since $(q-1)n/(n-1) \geq n$, we obtain from the embedding (3.2) that

$$\|u\|^q_{L^q(\mathbb{R}^{n-1})} \leq C_1\|u\|^{q-1}\|u\| + C_2\|u\|^q,$$

which completes the proof of Lemma 3.2.1. □

A fundamental result in the context of this chapter regards on a weighted Hardy-type inequality. This is the subject of the next lemma (see for instance [33], for a similar result in dimension $n \geq 3$).

Lemma 3.2.2. *Let $n \geq 2$. Then the following inequality holds*

$$\int_{\mathbb{R}^n_+} \frac{|u|^n}{(1+x_n)^n} dx \leq \left(\frac{n}{n-1}\right)^n \left(\int_{\mathbb{R}^n_+} |\nabla u|^n dx + \int_{\mathbb{R}^{n-1}} |u|^n dx'\right), \quad \forall\, u \in C_0^\infty(\mathbb{R}^n).$$

Proof. Let $v \in C_0^\infty(\mathbb{R}^n)$ and $\sigma \in \mathbb{R}$ with $\sigma \neq -1$. Using the Divergence Theorem we obtain

$$\sigma \int_{\mathbb{R}^n_+} \frac{v}{(1+x_n)^{\sigma+1}} dx = \int_{\mathbb{R}^n_+} \frac{v_{x_n}}{(1+x_n)^\sigma} dx + \int_{\mathbb{R}^{n-1}} v dx',$$

where we used that the normal unit vector pointing out of $\mathbb{R}^{n-1}$ is $\eta = (0', -1)$. Applying this equality with $v = |u|^n$, we get

$$|\sigma| \int_{\mathbb{R}^n_+} \frac{|u|^n}{(1+x_n)^{\sigma+1}} dx \leq \int_{\mathbb{R}^n_+} \frac{n|u|^{n-1}|\nabla u|}{(1+x_n)^\sigma} dx + \int_{\mathbb{R}^{n-1}} |u|^n dx'. \tag{3.6}$$

Now using the Young inequality with $0 < \varepsilon < 1$ we obtain

$$\begin{aligned} n\int_{\mathbb{R}^n_+} \frac{|u|^{n-1}|\nabla u|}{(1+x_n)^\sigma} dx &= n\int_{\mathbb{R}^n_+} \frac{\sqrt[n']{\varepsilon}|u|^{n-1}}{(1+x_n)^\sigma} \frac{|\nabla u|}{\sqrt[n']{\varepsilon}} dx \\ &\leq (n-1)\varepsilon \int_{\mathbb{R}^n_+} \frac{|u|^n}{(1+x_n)^{\frac{\sigma n}{n-1}}} dx + \frac{1}{\varepsilon^{(n-1)}} \int_{\mathbb{R}^n_+} |\nabla u|^n dx. \end{aligned} \tag{3.7}$$

Choosing $\frac{\sigma n}{n-1} = \sigma + 1$, that is, $\sigma = n-1$ and combining inequalities (3.6) and (3.7), one has

$$\int_{\mathbb{R}^n_+} \frac{|u|^n}{(1+x_n)^n} dx \leq \frac{1}{(n-1)(\varepsilon^{n-1} - \varepsilon^n)} \left(\int_{\mathbb{R}^n_+} |\nabla u|^n dx + \int_{\mathbb{R}^{n-1}} |u|^n dx'\right).$$

Using that the function $g(\varepsilon) = 1/(\varepsilon^{n-1} - \varepsilon^n)$ for $0 < \varepsilon < 1$ achieves its minimum at $\varepsilon = (n-1)/n$ we conclude that

$$\int_{\mathbb{R}^n_+} \frac{|u|^n}{(1+x_n)^n} dx \leq \left(\frac{n}{n-1}\right)^n \left(\int_{\mathbb{R}^n_+} |\nabla u|^n dx + \int_{\mathbb{R}^{n-1}} |u|^n dx'\right),$$

which is the desired result. □

As a corollary of Lemma 3.2.1 and Lemma 3.2.2 we have

Corollary 3.2.3. *The quantity*

$$\|u\|_E := \left(\int_{\mathbb{R}^n_+} |\nabla u|^n dx + \int_{\mathbb{R}^{n-1}} |u|^n dx'\right)^{1/n}$$

define an equivalent norm on E.

Proof. Let $u \in C_0^\infty(\mathbb{R}^n)$. By Lemma 3.2.2 we have $\|u\| \leq C(n)\|u\|_E$. Now we observe that

$$\begin{aligned} |u(x',0)|^n &= -\int_0^{+\infty} \frac{\partial}{\partial x_n}\left(\frac{|u|^n}{(1+x_n)^n}\right) dx_n \\ &\leq n\int_0^{+\infty} \frac{|u|^{n-1}|\nabla u|}{(1+x_n)^n} dx_n + n\int_0^{+\infty} \frac{|u|^n}{(1+x_n)^{n+1}} dx_n \\ &\leq \int_0^{+\infty} |\nabla u|^n dx_n + (2n-1)\int_0^{+\infty} \frac{|u|^n}{(1+x_n)^n} dx_n, \end{aligned}$$

where above we used the Young inequality and the fact that $(1+x_n)^{-(n+1)} \leq (1+x_n)^{-n} \leq 1$. Integrating we obtain

$$\int_{\mathbb{R}^{n-1}} |u(x',0)|^n dx' \leq \int_{\mathbb{R}^n_+} |\nabla u|^n dx + (2n-1)\int_{\mathbb{R}^n_+} \frac{|u|^n}{(1+x_n)^n} dx.$$

Therefore,

$$\begin{aligned} \|u\|_E^n &= \int_{\mathbb{R}^n_+} |\nabla u|^n dx + \int_{\mathbb{R}^{n-1}} |u(x',0)|^n dx' \\ &\leq 2\int_{\mathbb{R}^n_+} |\nabla u|^n dx + (2n-1)\int_{\mathbb{R}^n_+} \frac{|u|^n}{(1+x_n)^n} dx \leq (2n-1)\|u\|^n, \end{aligned}$$

and this completes the proof. □

From now on, the space E is equipped with the norm $\|\cdot\|_E$.

Remark 3.2.4. *The embeddings* (3.2) *and* (3.3) *are not valid if $q = \infty$. In fact, considering the function $u(x', x_n) := (1+x_n)^n \ln(1 - \ln|x|)$ if $(x', x_n) \in B_1^+$ and zero otherwise, one can see that $u \in E$ but $u \notin L^\infty\left(\mathbb{R}^n_+, (1+x_n)^{-n}\right)$.*

Remark 3.2.5. *Using that $(1+|x|)^{-\beta} \leq (1+x_n)^{-n}$ for any $\beta \geq n$ and assumptions (H_1), in view of Lemma 3.2.1, the embedding*

$$E \hookrightarrow L^q\left(\mathbb{R}^n_+, a(x)\right), \quad \forall\, n \leq q < \infty, \tag{3.8}$$

is continuous, which also is not valid if $q = \infty$. In fact, considering the function

$$u(x) := \frac{\ln(1 - \ln|x|)}{a(x)}$$

if $x \in B_1^+$ and zero otherwise, one can see that $u \in E$, but $u \notin L^\infty\left(\mathbb{R}^n_+, a(x)\right)$.

The next two compactness results play a crucial role in the proof of Theorem 3.1.1.

Lemma 3.2.6. *Assume hypothesis (H_1). If $\beta > n$ then the weighted Sobolev embedding* (3.8) *is compact.*

Proof. Since $E \hookrightarrow L^q\left(\mathbb{R}^n_+, (1+|x|)^{-\beta}\right) \hookrightarrow L^q\left(\mathbb{R}^n_+, a(x)\right)$, is sufficient to show that $u_k \to 0$ in $L^q(\mathbb{R}^n_+, (1+|x|)^{-\beta})$ whenever $u_k \rightharpoonup 0$ in E. To this end, let $C > 0$ be such that $\|u_k\|_E \leq C$ and $R > 0$ to be chosen later on. Note that

$$\int_{\mathbb{R}^n_+} \frac{|u_k|^q}{(1+|x|)^\beta} dx = \int_{B_R^+} \frac{|u_k|^q}{(1+|x|)^\beta} dx + \int_{\mathbb{R}^n_+ \setminus B_R^+} \frac{|u_k|^q}{(1+|x|)^\beta} dx. \tag{3.9}$$

Since the restriction operator $u \mapsto u_{|_{B_R^+}}$ is continuous from E into $E(B_R^+) := \left\{ v_{|_{B_R^+}} : v \in E \right\}$ and the embedding $E(B_R^+) \hookrightarrow L^q(B_R^+, (1+|x|)^{-\beta})$ is compact for any $q \geq n$, there exists $k_1 \in \mathbb{N}$ such that

$$\int_{B_R^+} \frac{|u_k|^q}{(1+|x|)^\beta} dx < \frac{\varepsilon}{2}, \quad \forall\, k \geq k_1. \tag{3.10}$$

On the other hand, using that $\beta > n$ we see that $(1+x_n)^n/(1+|x|)^\beta \to 0$ as $|x| \to \infty$. Thus, we can choose $R > 0$, large enough, such that $(1+x_n)^n/(1+|x|)^\beta \leq \varepsilon/2C$. Hence, there exists $k_2 \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^n_+ \setminus B_R^+} \frac{|u_k|^q}{(1+|x|)^\beta} dx = \int_{\mathbb{R}^n_+ \setminus B_R^+} \frac{|u_k|^q}{(1+x_n)^n} \frac{(1+x_n)^n}{(1+|x|)^\beta} dx < \frac{\varepsilon}{2}, \quad \forall\, k \geq k_2. \tag{3.11}$$

Since $\varepsilon > 0$ is arbitrary, the result follows from (3.9), (3.10) and (3.11). □

3.2.2 Trudinger-Moser type inequalities

In view of Remarks 3.2.4 and 3.2.5, it is natural to study embedding from E into Orlicz spaces. In all this section we consider the weight function $b(x) := 1/(1+|x|)^n \leq 1/(1+x_n)^n$. In view of Lemma 3.2.1, for any $n \leq q < \infty$ the embedding

$$E \hookrightarrow L^q\left(\mathbb{R}^n_+, b(x)\right) \tag{3.12}$$

is continuous. Furthermore, the same example in Remark 3.2.5 shows that this embedding is false if $q = \infty$.

Now, considering the Young function defined by

$$\Psi_\alpha(s) = e^{\alpha|s|^{n'}} - \sum_{k=0}^{n-2} \frac{\alpha^k}{k!} |s|^{n'k}, \quad s \in \mathbb{R},$$

where $n' := n/(n-1)$ and according to embedding 3.12, the following Trudinger-Moser type inequality in the setting of E is natural.

Proposition 3.2.7. *For any $\alpha > 0$ we have that $\Psi_\alpha(u) \in L^1(\mathbb{R}^n_+, b(x))$. Moreover, there exists a constant $\alpha_0 > 0$, independent of $u \in E$, such that*

$$L(\alpha) := \sup_{\{u \in E:\ \|u\|_E \leq 1\}} \int_{\mathbb{R}^n_+} b(x) \Psi_\alpha(u) dx < +\infty, \tag{3.13}$$

for any $0 < \alpha \leq \alpha_0$.

Before to present the proof of Proposition 3.2.7, we establish the following version Trudinger-Moser type inequality which will be used in the proof of Theorem 3.1.1.

Corollary 3.2.8. *Assume* (H_1) *and let* $\alpha_0 > 0$ *be given by Proposition 3.2.7. Then, for any* $u \in E$ *and* $\alpha > 0$, *we have that* $\Psi_\alpha(u) \in L^1(\mathbb{R}^n_+, a(x))$. *Moreover,*

$$l(\alpha) := \sup_{\{u \in E\colon \|u\|_E \leq 1\}} \int_{\mathbb{R}^n_+} a(x)\Psi_\alpha(u)dx < +\infty,$$

for any $0 < \alpha \leq \alpha_0$.

Proof. By assumption (H_1) we get

$$\int_{\mathbb{R}^n_+} a(x)\Psi_\alpha(u)dx \leq \int_{\mathbb{R}^n_+} b(x)\Psi_\alpha(u)dx,$$

for all $u \in E$ with $\|u\|_E \leq 1$. Thus, the result follows from Proposition 3.2.7. □

Now we will prove (3.13). Since

$$\|u\|_b := \left(\int_{\mathbb{R}^n_+} (|\nabla u|^n + b(x)|u|^n)dx\right)^{1/n} \leq C(n)\|u\|_E, \quad \forall\, u \in E, \tag{3.14}$$

it is sufficient to prove that for some $\alpha_1 > 0$ we have

$$\sup_{\{u \in E\colon \|u\|_b \leq 1\}} \int_{\mathbb{R}^n_+} b(x)\Psi_\alpha(u)dx < +\infty, \quad \forall\, 0 < \alpha \leq \alpha_1. \tag{3.15}$$

To prove (3.15) we will combine the ideas of Kufner-Opic [30] and Yang-Zhu [47] and this will be fulfilled in some lemmatas. First we recall a local estimate concerning the Trudinger-Moser inequality.

Lemma 3.2.9 ([47])**.** *For any* $R > 0$, *there exists a constant* $C_0 = C_0(n) > 0$ *such that for any* $y \in \mathbb{R}^n$ *and* $v \in W_0^{1,n}(B_R(y))$ *with* $\|\nabla v\|_{L^n(B_R(y))} \leq 1$ *we have*

$$\int_{B_R(y)} \Psi_{\alpha_n}(v)dx \leq C_0 R^n \int_{B_R(y)} |\nabla v|^n dx.$$

Proof. For the proof, we refer the reader to [46, Lemma 4.1] or [47, Lemma 1]. □

Our strategy to prove Proposition 3.2.7 consists in to consider for any $u \in E$ their extensions to the whole space $\mathbb{R}^n$ defined by:

$$\bar{u}(x', x_n) := \begin{cases} u(x', x_n), & x_n > 0 \\ u(x', x_n), & x_n < 0. \end{cases} \tag{3.16}$$

For any $R > 0$ we can split the integral (3.13) as follows

$$2\int_{\mathbb{R}^n_+} b(x)\Psi_\alpha(u)dx = \int_{B_R} b(x)\Psi_\alpha(\bar{u})dx + \int_{B_R^c} b(x)\Psi_\alpha(\bar{u})dx. \tag{3.17}$$

Now we will estimate the first integral on the right hand side of (3.17).

Lemma 3.2.10. *Let $u \in E$ be such that $\|u\|_E \leq 1$ and $R > 1$. Then there are $\alpha_2 > 0$ and $C_0 = C_0(R) > 0$ such that*

$$\int_{B_R} b(x)\Psi_\alpha(\bar{u})dx \leq C_0,$$

for any $0 < \alpha \leq \alpha_2$.

Proof. Consider a cut-off function $\varphi \in C_0^\infty(B_{2R})$ such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ in B_R and $|\nabla\varphi| \leq C/R$ in B_{2R} for some $C > 0$. Note that $\varphi\bar{u} \in W_0^{1,n}(B_{2R})$ and by straightforward calculation we check that

$$\begin{aligned}\int_{B_{2R}} |\nabla(\varphi\bar{u})|^n dx &\leq 2^{n-1}\left(\int_{B_{2R}} |\varphi|^n|\nabla\bar{u}|^n dx + \int_{B_{2R}} |\nabla\varphi|^n|\bar{u}|^n dx\right)\\ &\leq 2^{n-1}\left(\int_{B_{2R}} |\nabla\bar{u}|^n dx + \frac{C^n}{R^n}\int_{B_{2R}} |\bar{u}|^n dx\right)\\ &\leq 2^{n-1}\left(\int_{B_{2R}} |\nabla\bar{u}|^n dx + C^n\frac{(1+2R)^n}{R^n}\int_{B_{2R}} b(x)|\bar{u}|^n dx\right),\end{aligned}$$

and hence,

$$\int_{B_{2R}} |\nabla(\varphi\bar{u})|^n dx \leq C_1\int_{B_{2R}} \left(|\nabla\bar{u}|^n + b(x)|\bar{u}|^n\right) dx,$$

where $C_1 := 2^{n-1}\max\{1, (3C)^n\}$. Note that $v := \varphi\bar{u}/\sqrt[n]{2C_1C(n)} \in W_0^{1,n}(B_{2R})$ and

$$\|\nabla v\|^n_{L^n(B_{2R})} = \frac{\|\nabla\varphi\bar{u}\|^n_{L^n(B_{2R})}}{2C_1C(n)} \leq \frac{1}{2C(n)}\int_{\mathbb{R}^n} \left(|\nabla\bar{u}|^n + b(x)|\bar{u}|^n\right) dx \leq \|u\|^n_E \leq 1,$$

where we have used $b(x) \leq (1 + x_n)^{-n}$, if $x_n \geq 0$, and (3.14). Since $b(x) \leq 1$, in view of Lemma 3.2.9 and the fact that $\Psi_\alpha(cs) = \Psi_{\alpha|c|^{n'}}(s)$, for all $c \in \mathbb{R}$, we conclude that

$$\int_{B_R} b(x)\Psi_\alpha(\bar{u})dx \leq \int_{B_R} \Psi_\alpha(\varphi\bar{u})dx \leq \int_{B_{2R}} \Psi_{\alpha(2C_1C(n))^{\frac{1}{n-1}}}(v)dx \leq C_0(2R)^n,$$

if $0 < \alpha \leq \alpha_2 := \alpha_n/(2C_1C(n))^{\frac{1}{n-1}}$ and this completes the proof of Lemma 3.2.10. □

Now we proceed to estimate the second integral on the right hand side of (3.17).

Lemma 3.2.11. *Let $u \in E$ be such that $\|u\|_E \leq 1$. Then there exist $\alpha_3 > 0$ and $C_2 > 0$ independent of $u \in E$ such that*

$$\int_{B_{3r}^c} b(x)\Psi_\alpha(\bar{u})dx \leq C_2,$$

for any $r > 1$ and $0 < \alpha \leq \alpha_3$.

Proof. Given $r \geq 1$ and $\sigma > r$ we define the annuli

$$A_r^\sigma := \{x \in B_r^c : |x| < \sigma\} = \{x \in \mathbb{R}^n : r < |x| < \sigma\}.$$

A trick adaption of Besicovitch covering lemma [26] (see [18, estimate (4.8)]) shows that there exist a sequence of points $\{x_k\}_k \in A_1^\sigma$ and a universal constant $\theta > 0$ such that

$$A_1^\sigma \subseteq \bigcup_k U_k^{1/2} \quad \text{and} \quad \sum_k \chi_{U_k}(x) \leq \theta, \quad \forall\, x \in \mathbb{R}^n,$$

where $U_k^{1/2} := B\left(x_k, \frac{|x_k|}{6}\right)$ and χ_{U_k} denotes the function characteristic of $U_k := B\left(x_k, \frac{|x_k|}{3}\right)$. Let $u \in E$ be such that $\|u\|_E \leq 1$. In order to estimate the integral of $\bar{u}$ in A_{3r}^σ, we fix $1 < r < \sigma$ and we follow as in [30] introducing the set of indices

$$K_{r,\sigma} := \left\{k \in \mathbb{N} : \; U_k^{1/2} \cap B_{3r}^c \neq \varnothing\right\}.$$

It is easy to see that, if $U_k \cap B_{3r}^c \neq \varnothing$, then $U_k \subset B_r^c$. Moreover, since $1 < r < 3r$, we have that $A_{3r}^\sigma \subset A_1^\sigma$. Now using and the definition of $K_{r,\sigma}$ we get

$$A_{3r}^\sigma \subseteq \bigcup_{k \in K_{r,\sigma}} U_k^{1/2} \subseteq \bigcup_{k \in K_{r,\sigma}} U_k \subseteq B_r^c \subseteq B_1^c \tag{3.18}$$

and hence

$$\int_{A_{3r}^\sigma} b(x)\Psi_\alpha(\bar{u})dx \leq \sum_{k \in K_{r,\sigma}} \int_{U_k^{1/2}} b(x)\Psi_\alpha(\bar{u})dx. \tag{3.19}$$

Next, we estimate the integral on the right hand side of (3.19). Since

$$\frac{2}{3}|x_k| \leq |x| \leq \frac{4}{3}|x_k|, \quad \forall\, x \in U_k,$$

we have

$$\frac{1}{(1 + (4/3)|x_k|)^n} \leq b(x) \leq \frac{1}{(1 + (2/3)|x_k|)^n}, \quad \forall\, x \in U_k. \tag{3.20}$$

For any $k \in K_{r,\sigma}$ fixed, in view of (3.20) we get

$$\int_{U_k^{1/2}} b(x)\Psi_\alpha(\bar{u})dx \leq \frac{1}{(1 + (2/3)|x_k|)^n} \int_{U_k^{1/2}} \Psi_\alpha(\bar{u})dx. \tag{3.21}$$

Now, consider a cut-off function $\varphi_k \in C_0^\infty(U_k)$ such that $0 \leq \varphi_k \leq 1$ in U_k, $\varphi_k \equiv 1$ in $U_k^{1/2}$ and $|\nabla \varphi_k| \leq C/|x_k|$ in U_k for some constant $C > 0$. Then we see that $\varphi_k \bar{u} \in W_0^{1,n}(U_k)$ and by straightforward computation we have

$$\begin{aligned}\int_{U_k} |\nabla(\varphi_k \bar{u})|^n dx &\leq 2^{n-1}\left(\int_{U_k} |\varphi_k|^n |\nabla \bar{u}|^n dx + \int_{U_k} |\nabla \varphi_k|^n |\bar{u}|^n dx\right) \\ &\leq 2^{n-1}\left(\int_{U_k} |\nabla \bar{u}|^n dx + \frac{C^n}{|x_k|^n}\int_{U_k} |\bar{u}|^n dx\right) \\ &\leq 2^{n-1}\left(\int_{U_k} |\nabla \bar{u}|^n dx + C^n \frac{(1+(4/3)|x_k|)^n}{|x_k|^n}\int_{U_k} b(x)|\bar{u}|^n dx\right).\end{aligned}$$

Recalling that $k \in K_{r,\sigma}$, in view of (3.18), we have that $x_k \in B_r^c$ and consequently $|x_k| \geq r > 1$. This and the above estimate imply that

$$\int_{U_k} |\nabla(\varphi_k \bar{u})|^n dx \leq C_3 \int_{U_k} (|\nabla \bar{u}|^n + b(x)|\bar{u}|^n)\, dx,$$

where $C_3 := 2^{n-1}\max\{1, (7C/3)^n\}$. Thus, the function $v_k := \varphi_k \bar{u}/\sqrt[n]{2C_3 C(n)} \in W_0^{1,n}(U_k)$ and

$$\|\nabla v_k\|^n_{L^n(U_k)} = \frac{\|\nabla \varphi_k \bar{u}\|^n_{L^n(U_k)}}{2C_3 C(n)} \leq \frac{1}{2C(n)} \int_{U_k} (|\nabla \bar{u}|^n + b(x)|\bar{u}|^n)\, dx \leq 1.$$

Applying Lemma 3.2.9 with $B_R(y) = U_k$, $v = v_k$ and using the fact that $\Psi_\alpha(cs) = \Psi_{\alpha|c|^{n'}}(s)$, for all $c \in \mathbb{R}$, we get

$$\int_{U_k^{1/2}} \Psi_\alpha(\varphi_k \bar{u}) dx \leq \int_{U_k} \Psi_{\alpha(2C_3C(n))^{\frac{1}{n-1}}}(v_k) dx \leq C_0 \left(\frac{|x_k|}{3}\right)^n \int_{U_k} |\nabla v_k|^n dx,$$

for any $0 < \alpha \leq \alpha_3 := \alpha_n/(2C_3 C(n))^{\frac{1}{n-1}}$ and hence

$$\int_{U_k^{1/2}} \Psi_\alpha(\bar{u}) dx \leq \frac{C_0 |x_k|^n}{3^n 2C(n)} \int_{U_k} (|\nabla \bar{u}|^n + b(x)|\bar{u}|^n)\, dx.$$

This together with estimates (3.19), (3.21) and the fact that $s^n/(1+cs)^n \leq 1/c^n$ for any $c, s > 0$ imply that

$$\begin{aligned}\int_{A_{3r}^\sigma} b(x)\Psi_\alpha(\bar{u}) dx &\leq \frac{C_0}{3^n 2C(n)} \sum_{k \in K_{r,\sigma}} \frac{|x_k|^n}{(1+(2/3)|x_k|)^n} \int_{U_k} (|\nabla \bar{u}|^n + b(x)|\bar{u}|^n)\, dx \\ &\leq \frac{C_0}{2^{n+1}C(n)} \sum_{k \in K_{r,\sigma}} \int_{B_r^c} (|\nabla \bar{u}|^n + b(x)|\bar{u}|^n)\, \chi_{U_k} dx,\end{aligned}$$

where the last inequality we used (3.18). In view of the Besicovitch covering lemma we obtain

$$\int_{A_{3r}^\sigma} b(x)\Psi_\alpha(\bar{u}) dx \leq \frac{C_0 \theta}{2^{n+1}C(n)} \int_{B_r^c} (|\nabla \bar{u}|^n + b(x)|\bar{u}|^n)\, dx.$$

Taking the limit as $\sigma \to +\infty$ we get

$$\int_{B_{3r}^c} b(x)\Psi_\alpha(\bar{u})dx \le C\theta \int_{B_r^c} \left(|\nabla \bar{u}|^n + b(x)|\bar{u}|^n\right) dx,$$

for any $0 < \alpha \le \alpha_3 := \alpha_n/(2C_3C(n))^{\frac{1}{n-1}}$ and this completes the proof of Lemma 3.2.11. □

Finalizing the proof of (3.15). The proof follows directly from (3.17), Lemmas 3.2.10 and 3.2.11 by choosing $R = 3r$ and $\alpha_1 = \min\{\alpha_2, \alpha_3\}$. □

Finalizing the proof of Proposition 3.2.7. By (3.14) and the fact that $\Psi_\alpha(cs) = \Psi_{\alpha|c|^{n'}}(s)$, for all $c \in \mathbb{R}$, we get

$$\int_{\mathbb{R}_+^2} b(x)\Psi_\alpha(u)dx = \int_{\mathbb{R}_+^2} b(x)\Psi_{\alpha\|u\|_b^{n'}}\left(\frac{u}{\|u\|_b}\right) dx \le \int_{\mathbb{R}_+^2} b(x)\Psi_{\alpha(C(n))^{n'}}\left(\frac{u}{\|u\|_b}\right) dx$$

for all $u \in E$ with $\|u\|_E \le 1$. This together with (3.14) imply that

$$\begin{aligned}\sup_{\{u\in E:\ \|u\|_E\le 1\}} \int_{\mathbb{R}_+^2} b(x)\Psi_\alpha(u)dx &\le \sup_{\{u\in E:\ \|u\|_E\le 1\}} \int_{\mathbb{R}_+^2} b(x)\Psi_{\alpha(C(n))^{n'}}\left(\frac{u}{\|u\|_b}\right) dx \\ &\le \sup_{\{u\in E:\ \|u\|_b\le 1\}} \int_{\mathbb{R}_+^2} b(x)\Psi_{\alpha(C(n))^{n'}}\left(\frac{u}{\|u\|_b}\right) dx\end{aligned}$$

and the result follows from (3.15) by choosing $\alpha_0 = \alpha_1/(C(n))^{n'}$. □

3.3 Proof of the main results

In this section, we prove Theorems 3.1.1 and 3.1.2. Since h does not belong to any Lebesgue space we will consider the subspace of E defined by

$$E^r = \left\{u \in E : \int_{\mathbb{R}_+^n} h|u|^r dx < \infty\right\},$$

equipped with the norm

$$\|u\|_{E^r} := \left(\|u\|_E^n + \left(\int_{\mathbb{R}_+^n} h|u|^r dx\right)^{n/r}\right)^{1/n}.$$

Here we seek for weak solutions of problem $(\mathcal{P}_\lambda)$, which means a function $u \in E^r$ verifying

$$\int_{\mathbb{R}_+^n} |\nabla u|^{n-2}\nabla u \nabla\varphi dx + \int_{\mathbb{R}^{n-1}} |u|^{n-2}u\varphi dx' + \int_{\mathbb{R}_+^n} h|u|^{r-2}u\varphi dx = \lambda \int_{\mathbb{R}_+^n} af(x,u)\varphi dx, \qquad (3.22)$$

for all $\varphi \in E^r$. In view of assumption (H_1), Lemma 3.2.1 and Theorem 3.2.8 the energy functional associated to problem $(\mathcal{P}_\lambda)$, namely $I_\lambda : E^r \to \mathbb{R}$ defined by

$$I_\lambda(u) = \frac{1}{n}\int_{\mathbb{R}^n_+} |\nabla u|^n dx + \frac{1}{n}\int_{\mathbb{R}^{n-1}} |u|^n dx' + \frac{1}{r}\int_{\mathbb{R}^n_+} h|u|^r dx - \lambda \int_{\mathbb{R}^n_+} aF(x,u)dx,$$

is well defined. Furthermore, standard arguments show that $u \in E^r$ is a critical points of I_λ if, and only if, it is a weak solution of problem $(\mathcal{P}_\lambda)$.

Now, we will prove that I_λ satisfies the Mountain Pass geometry.

Lemma 3.3.1. *Assume $n \leq r < \infty$, (f_1), (f_2) and $(H_1)-(H_2)$. Then*

(i) *there exist $\rho > 0$ and $c_0 > 0$ such that $I_\lambda(u) \geq c_0 > 0$ if $\|u\|_{E^r} = \rho$;*

(ii) *there exists $v_0 \in E^r$ with $\|v_0\|_{E^r} > \rho$ such that $I_\lambda(v_0) < 0$.*

Proof. By (f_1), given $\tau > 0$ there exists $\delta > 0$ such that

$$F(x,s) \leq \frac{(\lambda_1 - \tau)}{n}|s|^n, \quad \forall\, (x,s) \in \mathbb{R}^n_+ \times (0,\delta).$$

This together with (3.1) imply that there exists $C_0 > 0$ such that

$$F(x,s) \leq \frac{(\lambda_1 - \tau)}{n}|s|^n + C_0 \Psi_{\alpha_0}(s)|s|^{r+1}, \quad \forall\, (x,s) \in \mathbb{R}^n_+ \times \mathbb{R}, \tag{3.23}$$

where $\alpha_0 > 0$ is given by Corollary 3.2.8. By Hölder inequality with conjugate exponents $1/r_1 + 1/r_2 = 1$, we get

$$\int_{\mathbb{R}^n_+} a|u|^{r+1}\Psi_{\alpha_0}(u)dx \leq \left(\int_{\mathbb{R}^n_+} a|u|^{(r+1)r_1}\right)^{1/r_1} \left(\int_{\mathbb{R}^n_+} a\left[\Psi_{\alpha_0}(u)\right]^{r_2} dx\right)^{1/r_2}.$$

Using that $\Psi_\alpha(cs) = \Psi_{\alpha|c|^{n'}}(s)$, for all $c \in \mathbb{R}$, $(\Psi_\alpha(s))^{r_2} \leq \Psi_{r_2\alpha}(s)$ (see [45, Lemma 2.1]), $\|u\|_E \leq \|u\|_{E^r}$, for all $u \in E^r$, and Corollary 3.2.8 we get

$$\int_{\mathbb{R}^n_+} a|u|^{r+1}\Psi_{\alpha_0}(u)dx \leq \left(\int_{\mathbb{R}^n_+} a|u|^{(r+1)r_1}\right)^{1/r_1} \left(\int_{\mathbb{R}^n_+} a\Psi_{r_2\alpha_0\|u\|^{n'}_{E^r}}\left(\frac{u}{\|u\|^{n'}_{E^r}}\right)dx\right)^{1/r_2} \leq C_1\|u\|^{r+1}_{E^r},$$

if $r_2 > 1$ sufficiently close to 1 and $\|u\|_{E^r}$ sufficiently small such that $r_2\|u\|^{n'}_{E^r} \leq 1$. This together with (3.23) and the definition of λ_1 imply that

$$\int_{\mathbb{R}^n_+} aF(x,u)dx \leq \frac{(\lambda_1 - \tau)}{n}\|u\|^n_{L^n(\mathbb{R}^n_+,a)} + C_3\|u\|^{r+1}_{E^r} \leq \frac{(\lambda_1 - \tau)}{n\lambda_1}\|u\|^n_E + C_3\|u\|^{r+1}_{E^r}.$$

Thus,

$$\begin{aligned} I_\lambda(u) &= \frac{1}{n}\|u\|_E^n + \frac{1}{r}\|u\|_{L^r(\mathbb{R}^n_+,h)}^r - \lambda\int_{\mathbb{R}^n_+} aF(x,u)dx \\ &\geq \frac{1}{n}\|u\|_E^n + \frac{1}{r}\|u\|_{L^r(\mathbb{R}^n_+,h)}^r - \lambda\left(\frac{(\lambda_1-\tau)}{n\lambda_1}\|u\|_E^n + C_3\|u\|_{E^r}^{r+1}\right) \\ &= \left(\frac{1}{n} - \frac{\lambda(\lambda_1-\tau)}{n\lambda_1}\right)\|u\|_E^n + \frac{1}{r}\|u\|_{L^r(\mathbb{R}^n_+,h)}^r - C_3\|u\|_{E^r}^{r+1}. \end{aligned} \tag{3.24}$$

Since we can choose $0 < \tau < \lambda_1$ sufficiently close to λ_1 such that $\lambda(\lambda_1-\tau)/n\lambda_1 < 1/2n$ and assume without loss of generality that $\|u\|_E \leq 1$, from (3.24), we get

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{2n}\|u\|_E^r + \frac{1}{r}\|u\|_{L^r(\mathbb{R}^n_+,h)}^r - C_3\|u\|_{E^r}^{r+1} \\ &\geq \min\left\{\frac{1}{2n}, \frac{1}{r}\right\}\frac{1}{2^{\frac{r}{n}}}\|u\|_{E^r}^r - C_3\|u\|_{E^r}^{r+1}, \end{aligned}$$

where in the last inequality we used that $\|u\|_{E^r}^r \leq 2^{\frac{r}{n}}(\|u\|_E^r + \|u\|_{L^r(\mathbb{R}^n_+,h)}^r)$, and the item (i) is proved. Now we prove item (ii). By (f_2) for each $M > 0$ there exists $s_0 > 0$ such that

$$F(x,s) \geq Ms^\mu, \quad \forall\,(x,s) \in \mathbb{R}^n_+ \times (s_0,\infty). \tag{3.25}$$

Assume $\varphi \neq 0$ is supported in a bounded domain $\Omega \subset \mathbb{R}^n_+$. Using that F is continuous, we conclude that F is bounded in the compact $\overline{\Omega} \times [0, s_0]$. This together with (3.25) imply that there exist $c_1, c_2 > 0$ such that

$$F(x,s) \geq c_1 s^\mu - c_2, \quad \forall\,(x,s) \in \mathbb{R}^n_+ \times \mathbb{R}^+.$$

It follows that

$$\begin{aligned} I_\lambda(t\varphi) &= \frac{1}{n}\|t\varphi\|_E^n + \frac{1}{r}\int_\Omega h|t\varphi|^r dx - \lambda\int_\Omega aF(x,t\varphi)dx \\ &\leq \frac{t^n}{n}\|\varphi\|_E^n + \frac{t^r}{r}\int_\Omega h|\varphi|^r dx - c_1\lambda t^\mu\int_\Omega a|\varphi|^\mu dx + c_2|\Omega|. \end{aligned}$$

Since $\mu > r$, I_λ verifies (ii) by choosing $v_0 = t_0\varphi$ with t sufficiently large. □

In view of Lemma 3.3.1, it is well defined the minimax level

$$c_\lambda = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\lambda(\gamma(t)) > 0, \quad \forall\,\lambda > 0,$$

where $\Gamma := \{\gamma \in C([0,1], E^r) : \gamma(0) = 0 \text{ and } \gamma(1) = v_0\}$. Applying the mountain pass theorem without the (PS) condition (see [44, Theorem 1.15])) we find a sequence $(u_k) \subset E^r$ at the minimax level c_λ, that is

$$I_\lambda(u_k) \to c_\lambda \quad \text{and} \quad I'_\lambda(u_k) \to 0.$$

The next result prove that I_λ satisfies the Palais-Smale condition.

Lemma 3.3.2. *Assume $n < r < \infty$ and (H_1). Then the functional I_λ satisfies (PS)-condition.*

Proof. Let $(u_k) \subset E^r$ be a (PS)-sequence associated to I_λ, i.e.,

$$I_\lambda(u_k) \to c \quad \text{and} \quad I'_\lambda(u_k) \to 0.$$

By assumption (f_2) we have

$$\begin{aligned} I_\lambda(u_k) - \frac{1}{\mu}\langle I'_\lambda(u_k), u_k\rangle &= \left(\frac{1}{n} - \frac{1}{\mu}\right)\|u_k\|_E^n + \left(\frac{1}{r} - \frac{1}{\mu}\right)\int_{\mathbb{R}^n_+} h|u_k|^r dx \\ &+ \lambda \int_{\mathbb{R}^n_+} a\left(\frac{1}{\mu}f(x,u)u - F(x,u)\right) dx \\ &\geq \left(\frac{1}{n} - \frac{1}{\mu}\right)\|u_k\|_E^n + \left(\frac{1}{r} - \frac{1}{\mu}\right)\int_{\mathbb{R}^n_+} h|u_k|^r dx. \end{aligned}$$

Thus, we get

$$\left(\frac{1}{n} - \frac{1}{\mu}\right)\|u_k\|_E^n + \left(\frac{1}{r} - \frac{1}{\mu}\right)\int_{\mathbb{R}^n_+} h|u_k|^r dx \leq c + o_k(\|u_k\|_{E^r}), \quad \forall\, k \in \mathbb{N}. \tag{3.26}$$

We claim that (u_k) is bounded. Indeed, suppose by contradiction that $\|u_k\|_{E^r} \to \infty$ as $k \to \infty$. Since $n \leq r < \mu$, in view of (3.26) we get

$$\frac{\|u_k\|_E^n}{\|u_k\|_{E^r}} \to 0 \quad \text{and} \quad \frac{\|u_k\|^r_{L^r(\mathbb{R}^n_+,h)}}{\|u_k\|_{E^r}} \to 0, \quad \text{as} \quad k \to \infty. \tag{3.27}$$

This combined with the fact that

$$\frac{\|u_k\|_E^n}{\|u_k\|_{E^r}} + \frac{\|u_k\|^n_{L^r(\mathbb{R}^n_+,h)}}{\|u_k\|_{E^r}} = \|u_k\|_{E^r}^{n-1} \to \infty, \quad \text{as} \quad k \to \infty,$$

imply that

$$\frac{\|u_k\|^n_{L^r(\mathbb{R}^n_+,h)}}{\|u_k\|_{E^r}} \to \infty, \quad \text{as} \quad k \to \infty.$$

Consequently,

$$\|u_k\|^n_{L^r(\mathbb{R}^n_+,h)} \to \infty, \quad \text{as} \quad k \to \infty.$$

This together with (3.27) and the fact that $n < r$ imply

$$\frac{\|u_k\|^n_{L^r(\mathbb{R}^n_+,h)}}{\|u_k\|_{E^r}} = \|u_k\|^{n-r}_{L^r(\mathbb{R}^n_+,h)}\frac{\|u_k\|^r_{L^r(\mathbb{R}^n_+,h)}}{\|u_k\|_{E^r}} \to 0, \quad \text{as} \quad k \to \infty,$$

which is a contradiction and hence (u_k) is bounded. Hence, we can use Lemma 3.2.6 to assume that

$$\begin{cases} u_k \rightharpoonup u \text{ in } E^r \\ u_k(x) \to u(x) \quad \text{a.e. in } \mathbb{R}^n_+ \\ u_k \to u \text{ in } L^q(\mathbb{R}^n_+, a) \end{cases}$$

and $\langle I'_\lambda(u_k) - I'_\lambda(u), u_k - u\rangle = o_k(1)$. Now we observe that

$$o_k(1) = \langle I'_\lambda(u_k) - I'_\lambda(u), u_k - u\rangle = A(k) - \lambda B(k), \tag{3.28}$$

where

$$\begin{aligned} A(k) = &\int_{\mathbb{R}^n_+} \left(|\nabla u_k|^{n-2}\nabla u_k - |\nabla u|^{n-2}\nabla u\right)(\nabla u_k - \nabla u)\, dx \\ &+ \int_{\mathbb{R}^{n-1}} \left(|u_k|^{n-2}u_k - |u|^{n-2}u\right)(u_k - u)\, dx' + \int_{\mathbb{R}^n_+} h\left(|u_k|^{r-2}u_k - |u|^{r-2}u\right)(u_k - u)\, dx \end{aligned}$$

and

$$B(k) = \int_{\mathbb{R}^n_+} a\left(f(x,u_k) - f(x,u)\right)(u_k - u)\, dx.$$

We claim that $B(k) = o_k(1)$. In fact, we have

$$|B(k)| \le \int_{\mathbb{R}^n_+} a\left(|f(x,u_k)| + |f(x,u)|\right)|u_k - u| dx \le B_1(k) + B_2(k),$$

where

$$B_1(k) := \int_{\mathbb{R}^n_+} a|f(x,u_k)||u_k - u| dx \quad \text{and} \quad B_2(k) := \int_{\mathbb{R}^n_+} a|f(x,u)||u_k - u| dx.$$

On the other hand, it follows from (3.1) and (f_1) that there are constants $C_1, C_2 > 0$ and $\alpha > 0$ to be chosen later such that

$$|f(x,s)| \le C_1|s|^{n-1} + C_2\Psi_\alpha(s), \quad \forall\, (x,s) \in \mathbb{R}^n_+ \times \mathbb{R}.$$

Thus,

$$B_1(k) \le C_1 \int_{\mathbb{R}^n_+} a|u_k|^{n-1}|u_k - u| dx + C_2 \int_{\mathbb{R}^n_+} a\Psi_\alpha(u_k)|u_k - u| dx.$$

Now we observe that using the Hölder inequality with conjugate exponents $1/n + 1/n' = 1$ and the fact that $(\Psi_\alpha(s))^{n'} \le \Psi_{n'\alpha}(s)$ (see [45, Lemma 2.1]) we get

$$B_1(k) \le C_1\|u_k\|^{n-1}_{L^n(\mathbb{R}^n_+,a(x))}\|u_k - u\|^n_{L^n(\mathbb{R}^n_+,a(x))} + C_2\left(\int_{\mathbb{R}^n_+} a\Psi_{n'\alpha}(u_k) dx\right)^{1/n'} \|u_k - u\|_{L^n(\mathbb{R}^n_+,a(x))}$$

Since $u_k \to u$ in $L^n(\mathbb{R}^n_+, a(x))$ we get

$$B_1(k) \le o_k(1) + \left(\int_{\mathbb{R}^n_+} a\Psi_{n'\alpha}(u_k) dx\right)^{1/n'} o_k(1).$$

Using that (u_k) is bounded in E, by choosing $\alpha > 0$ such that $n'\alpha\|u_k\|^{n'}_E \le \alpha_0$, and applying the

Corollary 3.2.8 we obtain $C > 0$ that does not depend on k such that

$$\int_{\mathbb{R}^n_+} a\Psi_{n'\alpha}(u_k)dx \leq \int_{\mathbb{R}^n_+} a\Psi_{n'\alpha\|u_k\|_E^{n'}}\left(\frac{u_k}{\|u_k\|_E}\right)dx \leq C$$

Hence $B_1(k) = o_k(1)$, and similarly $B_2(k) = o_k(1)$. Thus, from (3.28) we concluded that $A(k) = o_k(1)$. For all $\xi, \zeta \in \mathbb{R}^n$, we know that there exists a constant $C_1 = C_1(n) > 0$ (see inequality (2.2) in [40]) such that

$$C_1|\xi - \zeta|^n \leq (|\xi|^{n-2}\xi - |\zeta|^{n-2}\zeta)(\xi - \zeta), \quad \text{if} \quad n \geq 2. \tag{3.29}$$

This inequality together with the facts that $A(k) = o_k(1)$ and $h \geq 0$ imply that

$$C(k) := \int_{\mathbb{R}^n_+} \left(|\nabla u_k|^{n-2}\nabla u_k - |\nabla u_0|^{n-2}\nabla u_0\right)(\nabla u_k - \nabla u_0)\,dx = o_k(1)$$
$$D(k) := \int_{\mathbb{R}^{n-1}} \left(|u_k|^{n-2}u_k - |u_0|^{n-2}u_0\right)(u_k - u_0)\,dx' = o_k(1)$$
$$E(k) := \int_{\mathbb{R}^n_+} h\left(|u_k|^{r-2}u_k - |u_0|^{r-2}u_0\right)(u_k - u_0)\,dx = o_k(1).$$

Using once again inequality (3.29), we get $C_1 = C_1(n) > 0$ such that

$$\|u_k - u_0\|^n_{E^r} = \|u_k - u_0\|^n_E + \|u_k - u_0\|^n_{L^r(\mathbb{R}^n_+,h)} \leq C_1\left(C(k) + D(k) + (E(k))^{\frac{n}{r}}\right) = o_k(1),$$

which implies that $u_k \to u_0$ in E^r and this completes the proof. □

Finalizing the proof of Theorem 3.1.1. Taking $\varphi = u$ in (3.22), we have that any weak solution u of problem $(\mathcal{P}_\lambda)$ satisfies the equality

$$\|u\|^n_E + \|u\|^r_{L^r(\mathbb{R}^n_+,h)} = \lambda\|u\|^q_{L^r(\mathbb{R}^n_+,a)}$$

so that problem $(\mathcal{P}_\lambda)$ does not have any nontrivial solution whenever $\lambda \leq 0$ and hence the item (i) in Theorem 3.1.1 is provided. Finally the item (ii) in Theorem 3.1.1 follows by Lemmas 3.3.1 and 3.3.2 and the Mountain Pass theorem. □

In order to prove our multiplicity result we shall use the following version of the symmetric mountain pass theorem (see [8]).

Theorem 3.3.3. *Let E be a real infinite-dimensional Banach space and $I \in C^1(E, R)$ an even functional satisfying the PS condition and the following hypotheses:*

(I_1) *$I(0) = 0$ and there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho(0)} \geq \alpha$;*

(I_2) *for any finite dimensional $\tilde{E} \subset E$, $\tilde{E} \cap \{u \in E : I(u) \geq 0\}$ is bounded.*

Then I has an unbounded sequence of critical values.

Finalizing the proof of Theorem 3.1.2. The proof of item (I_1) follows as in the proof of Lemma 3.3.1. Now we prove item (I_2). Suppose by contradiction that (I_2) does not hold. Then, there exists a finite dimensional $\tilde{E} \subset E^r$ and a sequence $(u_k) \subset \tilde{E}$ satisfying

$$I_\lambda(u_k) > 0, \quad k \in \mathbb{N} \quad \text{and} \quad \|u_k\|_{E^r} \to \infty \text{ as } k \to \infty. \tag{3.30}$$

By using the local condition (f_4) and (f_3) we can obtain $\nu > 0$ such that

$$F(x,s) \geq \nu |s|^{\gamma_0}, \quad \text{uniformly with respect to } x \in \mathbb{R}^n_+, \quad \forall\, s \in \mathbb{R}.$$

This inequality together with the fact that all the norms in $\tilde{E}$ are equivalent, there exist $\tilde{b} > 0$ such that

$$\begin{aligned} 0 < I_\lambda(u_k) &= \frac{1}{n}\|u_k\|^n_E + \frac{1}{r}\|u_k\|^r_{L^r(\mathbb{R}^n_+,h)} - \lambda \int_{\mathbb{R}^n_+} aF(x,u_k)dx \\ &\leq \frac{1}{n}\|u_k\|^n_E + \frac{1}{r}\|u_k\|^r_{L^r(\mathbb{R}^n_+,h)} - \lambda\nu\|u_k\|^{\gamma_0}_{L^{\gamma_0}(\mathbb{R}^n_+,a)} \\ &\leq \frac{1}{n}\|u_k\|^n_{E^r} + \frac{c}{r}\|u_k\|^r_{E^r} - \lambda\nu\tilde{b}\|u\|^{\gamma_0}_{E^r}, \quad \forall\, k \in \mathbb{N}. \end{aligned}$$

Thus,

$$\lambda\nu\tilde{b}\|u\|^{\gamma_0}_{E^r} < \frac{1}{p}\|u_k\|^n_{E^r} + \frac{1}{r}\|u_k\|^r_{E^r}, \quad \forall\, k \in \mathbb{N},$$

which contradicts (3.30), since $n < r < \gamma_0$, and item (I_2) is proved. In view of Lemma 3.3.2, for each $\lambda > 0$ we can apply Theorem 3.3.3 to obtain an unbounded sequence of critical values of I_λ to which we can associate at least two critical points because the functional I_λ is even. Therefore, the proof of Theorem 3.1.1 is complete. □

Part III

A Hardy-Sobolev type inequality without the trace and its applications

Chapter 4

A Hardy-Sobolev type inequality and its applications

In this chapter we present a new Hardy-Sobolev type inequality without the the trace and as an application we prove existence, nonexistence and multiplicity of solutions for an indefinite quasilinear elliptic equation with Neumann boundary condition and weights in anisotropic spaces. This results is part of the submitted paper [22].

This chapter is organized as follows. Section 4.1 contains the necessary preliminary results on the weighted Sobolev embeddings needed in the sequel. Section 4.2 presents the indefinite quasilinear elliptic problem which will be studied in this chapter. In Section 4.3, we present the proof of Theorem 4.2.2. Finally, in Section 4.4, we discuss the proof of Theorem 4.2.3.

4.1 A Hardy-Sobolev type inequality and its consequences

In order to study quasilinear elliptic problems in the upper half-space with Neumann boundary conditions we will establish another Hardy-Sobolev type inequality. The next weighed Hardy-Sobolev inequality will be fundamental in our approach.

Proposition 4.1.1. *Let $1 < p < n$ and $\alpha > 1$. Then there is $C = C(n, \alpha, p) > 0$ such that*

$$\left(\int_{\mathbb{R}^n_+} \frac{|u|^{p_*}}{(1+x_n)^\alpha}\right)^{p/p_*} dx \leq C \int_{\mathbb{R}^n_+} |\nabla u|^p dx, \quad \forall\, u \in C_0^1(\mathbb{R}^n), \quad \textit{where} \quad p_* := \frac{p(n-1)}{n-p}.$$

Proof. Let $v \in C_0^1(\mathbb{R}^n)$ and $\sigma \in \mathbb{R}$ with $\sigma \neq -1$. Using integration by parts, we obtain

$$\begin{aligned}(\sigma+1)\int_{\mathbb{R}^n_+}(1+x_n)^\sigma |v| dx &= \int_{\mathbb{R}^n_+} \partial_{x_n}((1+x_n)^{\sigma+1})|v| dx \\ &= -\int_{\mathbb{R}^n_+}(1+x_n)^{\sigma+1}(|v|)_{x_n} dx - \int_{\mathbb{R}^{n-1}} |v| dx',\end{aligned}$$

where we are using that the normal unit vector pointing out of $\mathbb{R}^{n-1}$ is $\eta = (0', -1)$. Thus,

$$|\sigma + 1| \int_{\mathbb{R}^n_+} (1 + x_n)^{\sigma} |v| dx \leq \int_{\mathbb{R}^n_+} (1 + x_n)^{\sigma+1} |\nabla v| dx + \int_{\mathbb{R}^{n-1}} |v| dx'.$$

Applying this inequality with $v = |u|^q$, $q > 1$, and $\sigma + 1 < 0$ we get

$$\begin{aligned} |\sigma + 1| \int_{\mathbb{R}^n_+} (1 + x_n)^{\sigma} |u|^q dx &\leq \int_{\mathbb{R}^n_+} (1 + x_n)^{\sigma+1} q |u|^{q-1} |\nabla u| dx + \int_{\mathbb{R}^{n-1}} |u|^q dx' \\ &\leq q \int_{\mathbb{R}^n_+} (1 + x_n)^{\sigma+1} |u|^{q-1} |\nabla u| dx + \int_{\mathbb{R}^{n-1}} |u|^q dx'. \end{aligned} \tag{4.1}$$

From the trace inequality, for $u \in C^1_0(\mathbb{R}^n)$ and $q = p_*$, there exists $C_1 = C_1(n) > 0$ such that

$$\int_{\mathbb{R}^{n-1}} |u|^{p_*} dx' \leq C_1 \left(\int_{\mathbb{R}^n_+} |\nabla u|^p dx \right)^{(n-1)/(n-p)}. \tag{4.2}$$

On the other hand, by the Hölder inequality and the embedding $\mathcal{D}^{1,p}(\mathbb{R}^n_+) \hookrightarrow L^{p^*}(\mathbb{R}^n_+)$ we see that

$$\begin{aligned} p_* \int_{\mathbb{R}^n_+} (1 + x_n)^{\sigma+1} |u|^{p_*-1} |\nabla u| dx &\leq p_* \left(\int_{\mathbb{R}^n_+} |u|^{p^*} dx \right)^{(p-1)/p} \left(\int_{\mathbb{R}^n_+} |\nabla u|^p dx \right)^{1/p} \\ &\leq p_* C_2 \left(\int_{\mathbb{R}^n_+} |\nabla u|^p dx \right)^{(n-1)/(n-p)}. \end{aligned} \tag{4.3}$$

Combining inequalities (4.1), (4.2) and (4.3) we get

$$\begin{aligned} |\sigma + 1| \int_{\mathbb{R}^n_+} \frac{|u|^{p_*}}{(1 + x_n)^{-\sigma}} dx &\leq p_* C_2 \left(\int_{\mathbb{R}^n_+} |\nabla u|^p dx \right)^{(n-1)/(n-p)} + C_1 \left(\int_{\mathbb{R}^n_+} |\nabla u|^p dx \right)^{(n-1)/(n-p)} \\ &= (p_* C_2 + C_1) \left(\int_{\mathbb{R}^n_+} |\nabla u|^p dx \right)^{(n-1)/(n-p)}. \end{aligned}$$

Thus, considering $\alpha = -\sigma$, we obtain

$$\left(\int_{\mathbb{R}^n_+} \frac{|u|^{p_*}}{(1 + x_n)^{\alpha}} dx \right)^{p/p_*} \leq \left(\frac{p_* C_2 + C_1}{|-\alpha + 1|} \right)^{p/p_*} \int_{\mathbb{R}^n_+} |\nabla u|^p dx,$$

which is the desired result. □

As a consequence of Proposition 4.1.1 we have the following inequality.

Lemma 4.1.2. *Let $1 < p < n$ and $\alpha > 1$. Then there is $C = C(n, \alpha, p, q) > 0$ such that*

$$\left(\int_{\mathbb{R}^n_+} \frac{|u|^q}{(1 + x_n)^{\alpha}} \right)^{p/q} dx \leq C \int_{\mathbb{R}^n_+} |\nabla u|^p dx, \quad \forall\, u \in C^\infty_0(\mathbb{R}^n), \quad \forall\, p_* \leq q \leq p^* := \frac{np}{n-p}. \tag{4.4}$$

Furthermore, the condition $q \geq p_*$ is necessary.

Proof. The proof follows of Proposition 4.1.1 and an interpolation argument. To see that the condition $q \geq p_*$ is necessary, we will argue as in [11, Proposition 3.5]. Let $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $\phi(x) = 1$ for $|x| \leq 1$ and $\phi(x) = 0$ for $|x| \geq 2$. We define, for any $t > 0$, $\phi_t(x) = \phi(x/t)$ for $x \in \mathbb{R}^n$. A straightforward calculation shows that there exist $C_1, C_2 > 0$ independent of t such that

$$\int_{\mathbb{R}^n_+} |\nabla \phi_t|^p dx = C_1 t^{n-p} \tag{4.5}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n_+} \frac{|\phi_t(x)|^q}{(1+x_n)^\alpha} dx &\geq \int_0^{t/\sqrt{2}} \int_{|x'|\leq t/\sqrt{2}} \frac{1}{(1+x_n)^\alpha} dx' dx_n \\ &= \left[\frac{1}{(\alpha-1)} - \frac{1}{(\alpha-1)(1+t/\sqrt{2})^{\alpha-1}}\right] C_2 t^{n-1}.\end{aligned}$$

Assume by contradiction that there exists a constant $C_3 > 0$ such that for all $u \in C_0^\infty(\mathbb{R}^n)$ we have

$$0 < C_3 \leq \frac{\displaystyle\int_{\mathbb{R}^n_+} |\nabla u|^p dx}{\left(\displaystyle\int_{\mathbb{R}^n_+} \frac{|u|^q}{(1+x_n)^\alpha} dx\right)^{p/q}} \leq \frac{\displaystyle\int_{\mathbb{R}^n_+} |\nabla \phi_t|^p dx}{\left(\displaystyle\int_{\mathbb{R}^n_+} \frac{|\phi_t|^q}{(1+x_n)^\alpha} dx\right)^{p/q}} \leq C_4 t^{n-p-\frac{p(n-1)}{q}} \tag{4.6}$$

for some $C_4 > 0$ and t large. If $q < p_*$ we obtain a contradiction letting $t \to \infty$ and this finishes the proof. □

In order to perform a variational approach we introduce our functional space and its embeddings into weighted Lebesgue spaces. To this, denote by $C_\delta^\infty(\mathbb{R}^n_+)$ the space of $C_0^\infty(\mathbb{R}^n)-$functions restricted to $\mathbb{R}^n_+$. We define the weighted Sobolev space E as the completion of $C_\delta^\infty(\mathbb{R}^n_+)$ with respect to the norm

$$\|u\|_E := \left(\int_{\mathbb{R}^n_+} |\nabla u|^p dx\right)^{1/p}.$$

As a consequence of Lemma 4.1.2, we have the following embedding result.

Lemma 4.1.3. *Assume $1 < p < n$ and $\alpha > 1$. Then the weighted Sobolev embedding*

$$E \hookrightarrow L^q\left(\mathbb{R}^n_+, \frac{1}{(1+x_n)^\alpha}\right), \quad \forall\, p_* \leq q \leq p^*, \tag{4.7}$$

is continuous.

4.2 Applications

In this section, we present the indefinite quasilinear elliptic problem which will be studied existence, nonexistence and multiplicity of nontrivial solutions, as a consequence of Hardy-

Sobolev inequality obtained in Proposition (4.1.1), to namely:

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) &= \lambda a(x)|u|^{q-2}u - b(x)|u|^{r-2}u \quad \text{in} \quad \mathbb{R}^n_+, \\ |\nabla u|^{p-2}\nabla u\cdot\nu &= 0 \quad \text{on} \quad \mathbb{R}^{n-1}, \end{cases} \tag{$\mathcal{P}_\lambda$}$$

where $1 < p < n$, ν denotes the unit outward normal on the boundary, λ is a real parameter and the weighted functions $a(x)$ and $b(x)$ satisfy some suitable conditions that we will describe later on. Our interest is to analyze the interplay between the powers q and r. Thus, we will consider two cases:

(I) $r > q$ and $p_* \leq q \leq p^*$;

(II) $1 < p \leq r < q$ and $p_* \leq q \leq p^*$.

Motivated by the works of Alama-Tarrantelo [7], Filippucci-Pucci-Radulescu [25], Lyberopoulos [27], Perera [31] and Pflüger [34], our main purpose in the present paper is to use variational techniques to investigate the existence, nonexistence and multiplicity of nontrivial weak solutions for the problem $(\mathcal{P}_\lambda)$. We want to remark that the main features of this class of problems is that we are facing an indefinite nonlinearity and the weight function $a(x)$ is allowed to be in anisotropic Lebesgue spaces.

We begin by considering the case $r > q$. To this end, we shall assume the following assumptions:

(H_1) $a : \mathbb{R}^n_+ \to \mathbb{R}$ is a nontrivial measurable function and there are constants $\alpha > 1$ and $c_1 > 0$ such that

$$0 \leq a(x) \leq \frac{c_1}{(1+x_n)^\alpha}, \quad \text{a.e. in} \quad \mathbb{R}^n_+;$$

(H_2) $b : \mathbb{R}^n_+ \to \mathbb{R}$ is a positive continuous function satisfying

$$\int_{\mathbb{R}^n_+} \frac{a^{\frac{r}{r-q}}}{b^{\frac{q}{r-q}}} dx < \infty.$$

It is worthwhile mentioning that the hypothesis (H_2) appears in the paper [7].

Remark 4.2.1. *Note that if $a(x)$ satisfies (H_1) then the function $b(x) = (1+|x|)^\theta/(1+x_n)^{\frac{\alpha r}{q}}$ with $\theta > n(r-q)/q$ satisfies the assumption (H_2). In fact, if $\theta > n(r-q)/q$ we have*

$$\int_{\mathbb{R}^n_+} \frac{a^{\frac{r}{r-q}}}{b^{\frac{q}{r-q}}} dx \leq \int_{\mathbb{R}^n_+} \frac{1}{(1+x_n)^{\frac{\alpha r}{r-q}}} \frac{(1+x_n)^{\frac{\alpha r}{r-q}}}{(1+|x|)^{\frac{\theta q}{r-q}}} dx = \int_{\mathbb{R}^n_+} \frac{1}{(1+|x|)^{\frac{\theta q}{r-q}}} dx < \infty.$$

Under this hypotheses, our main result can be stated as follows.

Theorem 4.2.2. *Let $r > q$ and assume the hypotheses $(H_1) - (H_2)$.*

(i) *If $p_* \leq q \leq p^*$, there exists $\lambda^* > 0$ such that problem $(\mathcal{P}_\lambda)$ has only the trivial solution for all $\lambda \in (-\infty, \lambda^*)$;*

(ii) *If $p_* \leq q < p^*$, there exists $\tilde{\lambda} > 0$ such that problem $(\mathcal{P}_\lambda)$ has at least a nontrivial weak solution for all $\lambda \in [\tilde{\lambda}, \infty)$;*

(iii) *If $p_* \leq q < p^*$, there exists $\Lambda \geq \tilde{\lambda}$ such that problem $(\mathcal{P}_\lambda)$ has at least two nontrivial weak solutions $u_\lambda \geq \tilde{u}_\lambda$ for all $\lambda \in (\Lambda, \infty)$;*

(iv) *If $p_* \leq q < p^*$, for any $m \in \mathbb{N}$ there exists $\Lambda_m > 0$ such that problem $(\mathcal{P}_\lambda)$ has at least m pairs of nontrivial weak solutions for all $\lambda > \Lambda_m$.*

The proof of the existence in Theorem 4.2.2 is based on minimization techniques. To obtain the second solution we will follow a truncation argument. The multiplicity result is obtained by applying the symmetric mountain pass theorem.

Next we deal with the case $r < q$. In order to prove the existence of solutions for problem $(\mathcal{P}_\lambda)$, instead of hypotheses $(H_1) - (H_2)$, we will assume:

$(\widetilde{H}_1)$ $a : \mathbb{R}^n_+ \to \mathbb{R}$ is a nontrivial measurable function and there are $c_2 > 0$ and $\alpha > 1$ such that

$$0 \leq a(x) \leq \frac{c_2}{(1+|x|)^\alpha}, \quad \text{a.e. in} \quad \mathbb{R}^n_+.$$

$(\widetilde{H}_2)$ $b : \mathbb{R}^n_+ \to \mathbb{R}$ is a measurable positive function.

In this case, our main result is stated as follows.

Theorem 4.2.3. *Let $p_* \leq r < q < p^*$ and assume the hypotheses $(\widetilde{H}_1) - (\widetilde{H}_2)$. Then*

(i) *the problem $(\mathcal{P}_\lambda)$ has no nontrivial weak solution for every $\lambda \in (-\infty, 0]$;*

(ii) *the problem $(\mathcal{P}_\lambda)$ has an infinite number of nontrivial weak solutions for every $\lambda \in (0, \infty)$.*

The proof of Theorem 4.2.3 is obtained by performing a variational approach based on the symmetric mountain pass theorem.

Hereafter in this chapter, B_R denotes the ball of center zero and radius $R > 0$ in $\mathbb{R}^n$, $B_R^+ := B_R \cap \mathbb{R}^n_+$, $(B_R)^c$ denotes $\mathbb{R}^n \setminus B_R$, the complement of the set $B_R \subset \mathbb{R}^n$, and $(B_R^+)^c$ denotes $\mathbb{R}^n_+ \setminus B_R^+$ the complement of the set $B_R^+ \subset \mathbb{R}^n_+$.

4.3 Proof of Theorem 4.2.2

In this section, we present the proof of Theorem 4.2.2. We will split the proof into three subsections.

First, we will define our variational approach. Since the weighted function $b(x)$ does not belongs to any Lebesgue space we need consider the subspace of E defined by

$$E^{r,p} = \left\{ u \in E : \int_{\mathbb{R}^n_+} b|u|^r dx < \infty \right\},$$

equipped with the norm

$$\|u\|_{E^{r,p}} := \left(\|u\|_E^p + \|u\|_{L^r(\mathbb{R}^n_+, b(x))}^p\right)^{1/p}.$$

Remark 4.3.1. *Suppose that the weight function $a(x)$ satisfies hypotheses (H_1) or $(\widetilde{H}_1)$. By Lemma 4.1.3, the weighted Sobolev embeddings*

$$E \hookrightarrow L^q\left(\mathbb{R}^n_+, a(x)\right) \tag{4.8}$$

and

$$E^{r,p} \hookrightarrow L^q\left(\mathbb{R}^n_+, a(x)\right) \tag{4.9}$$

are continuous if $p_* \leq q \leq p^*$.

The next two compactness results play a crucial role in the proof of Theorem 4.2.2 and Theorem 4.2.3, respectively.

Lemma 4.3.2. *Assume $1 < p < n$ and $(H_1) - (H_2)$. Then the weighted Sobolev embedding (4.9) is compact if $p_* \leq q < p^*$.*

Proof. We will show that $u_k \to 0$ in $L^q(\mathbb{R}^n_+, a(x))$ whenever $u_k \rightharpoonup 0$ in $E^{r,p}$. Indeed, let $C > 0$ be such that $\|u_k\|_{E^{r,p}} \leq C$ and $R > 0$ to be chosen during the proof independently of u. We have

$$\int_{\mathbb{R}^n_+} a|u_k|^q dx = \int_{B_R^+} a|u_k|^q dx + \int_{\mathbb{R}^n_+ \setminus B_R^+} a|u_k|^q dx. \tag{4.10}$$

Since the restriction operator $u \mapsto u_{|_{B_R^+}}$ is continuous from $E^{r,p}$ into $E^{r,p}(B_R^+) := \left\{ v_{|_{B_R^+}} : v \in E^{r,p} \right\}$ and the embedding $E^{r,p}(B_R^+) \hookrightarrow L^q(B_R^+, a(x))$ is compact, there exists $k_1 \in \mathbb{N}$ such that

$$\int_{B_R^+} a|u_k|^q dx < \frac{\varepsilon}{2}, \quad \forall\, k \geq k_1, \tag{4.11}$$

for any $p_* \leq q < p^*$. On the other hand, by assumption (H_2), the Hölder inequality and choosing $R > 0$ sufficiently large, we get

$$\begin{aligned}\int_{\mathbb{R}^n_+ \setminus B_R^+} a|u_k|^q dx &\leq \left(\int_{\mathbb{R}^n_+ \setminus B_R^+} \frac{a^{\frac{r}{r-q}}}{b^{\frac{q}{r-q}}} dx\right)^{(r-q)/r} \left(\int_{\mathbb{R}^n_+ \setminus B_R^+} b|u_k|^r dx\right)^{q/r} \\ &\leq C \left(\int_{\mathbb{R}^n_+ \setminus B_R^+} \frac{a^{\frac{r}{r-q}}}{b^{\frac{q}{r-q}}} dx\right)^{(r-q)/r} \leq \frac{\varepsilon}{2}.\end{aligned}$$

This combined with (4.10) and (4.11) imply the desired result. □

Lemma 4.3.3. *Assume $1 < p < n$ and $(\widetilde{H}_1)$. If $\alpha > n$ then the weighted Sobolev embedding (4.8) is compact if $p_* \leq q < p^*$.*

Proof. Since $E \hookrightarrow L^q\left(\mathbb{R}^n_+, (1 + |x|)^{-\alpha}\right) \hookrightarrow L^q\left(\mathbb{R}^n_+, a(x)\right)$, is sufficient to show that $u_k \to 0$ in $L^q(\mathbb{R}^n_+, (1 + |x|)^{-\alpha})$ whenever $u_k \rightharpoonup 0$ in E. To this end, let $C > 0$ be such that $\|u_k\|_E \leq C$ and

$R > 0$ to be chosen during the proof independently of u. We have

$$\int_{\mathbb{R}^n_+} \frac{|u_k|^q}{(1+|x|)^\alpha} dx = \int_{B_R^+} \frac{|u_k|^q}{(1+|x|)^\alpha} dx + \int_{\mathbb{R}^n_+ \setminus B_R^+} \frac{|u_k|^q}{(1+|x|)^\alpha} dx.$$

Arguing as in the proof of Lemma 4.3.2, we obtain $k_1 \in \mathbb{N}$ such that

$$\int_{B_R^+} \frac{|u_k|^q}{(1+|x|)^\alpha} dx < \frac{\varepsilon}{2}, \quad \forall\, k \geq k_1,$$

for any $p_* \leq q < p^*$. On the other hand, choosing $1 < \beta < \alpha$ we see that $(1+x_n)^\beta/(1+|x|)^\alpha \to 0$ as $|x| \to \infty$. Thus, we can choose $R > 0$, large enough, such that $(1+x_n)^\beta/(1+|x|)^\alpha \leq \varepsilon/2C$. Hence, there exists $k_2 \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^n_+ \setminus B_R^+} \frac{|u_k|^q}{(1+|x|)^\alpha} dx = \int_{\mathbb{R}^n_+ \setminus B_R^+} \frac{|u_k|^q}{(1+x_n)^\beta} \frac{(1+x_n)^\beta}{(1+|x|)^\alpha} dx < \frac{\varepsilon}{2}, \quad \forall\, k \geq k_2,$$

which implies the desired result. □

Here, by a weak solution of problem $(\mathcal{P}_\lambda)$, we mean a nontrivial function $u \in E^{r,p}$ verifying

$$\int_{\mathbb{R}^n_+} |\nabla u|^{p-2} \nabla u \nabla \varphi dx = \lambda \int_{\mathbb{R}^n_+} a|u|^{q-2} u \varphi dx - \int_{\mathbb{R}^n_+} b|u|^{r-2} u \varphi dx, \quad \forall\, \varphi \in E^{r,p}. \tag{4.12}$$

In view of assumption (H_1), Lemma 4.1.3 the energy functional associated to problem $(\mathcal{P}_\lambda)$ $I_\lambda : E^{r,p} \to \mathbb{R}$ defined by

$$I_\lambda(u) = \frac{1}{p} \int_{\mathbb{R}^n_+} |\nabla u|^p dx + \frac{1}{r} \int_{\mathbb{R}^n_+} b|u|^r dx - \frac{\lambda}{q} \int_{\mathbb{R}^n_+} a|u|^q dx,$$

is well defined. Furthermore, standard arguments show that $u \in E^{r,p}$ is a critical point of I_λ if and only if is a weak solution of problem $(\mathcal{P}_\lambda)$.

4.3.1 Nonexistence

In this section we present the proof of item (i) in Theorem 4.2.2. Suppose that $u \in E^{r,p}$ is a nontrivial weak solution of $(\mathcal{P}_\lambda)$. If $\lambda \leq 0$ the result is immediate. Thus, we assume that $\lambda > 0$ and taking $\varphi = u$ as a test function in (4.12) we obtain

$$\int_{\mathbb{R}^n_+} |\nabla u|^p dx = \lambda \int_{\mathbb{R}^n_+} a|u|^q dx - \int_{\mathbb{R}^n_+} b|u|^r dx. \tag{4.13}$$

Using the Young inequality we get

$$\lambda \int_{\mathbb{R}^n_+} a|u|^q dx = \int_{\mathbb{R}^n_+} \frac{\lambda a}{b^{\frac{q}{r}}} \left(b^{\frac{q}{r}} |u|^q \right) dx \leq \frac{r-q}{r} \lambda^{\frac{r}{r-q}} \int_{\mathbb{R}^n_+} \frac{a^{\frac{r}{r-q}}}{b^{\frac{q}{r-q}}} dx + \frac{q}{r} \int_{\mathbb{R}^n_+} b|u|^r dx.$$

This together with (4.13) and the fact that $q < r$ imply

$$\|u\|_E^p \leq \frac{r-q}{r}\lambda^{\frac{r}{r-q}} \int_{\mathbb{R}^n_+} \frac{a^{\frac{r}{r-q}}}{b^{\frac{q}{r-q}}} dx + \frac{q-r}{r} \int_{\mathbb{R}^n_+} b|u|^r dx \leq \frac{r-q}{r}\lambda^{\frac{r}{r-q}} \int_{\mathbb{R}^n_+} \frac{a^{\frac{r}{r-q}}}{b^{\frac{q}{r-q}}} dx. \tag{4.14}$$

Since $p < p_* \leq q$, combining (4.13) with Lemma 4.1.3 and the fact that $b > 0$ we get

$$\bar{C} \left(\int_{\mathbb{R}^n_+} a|u|^q dx \right)^{p/q} \leq \|u\|_E^p \leq \lambda \int_{\mathbb{R}^n_+} a|u|^q dx \tag{4.15}$$

for some constant $\bar{C} > 0$. Thus,

$$\left(\bar{C}\lambda^{-1}\right)^{\frac{q}{q-p}} \leq \int_{\mathbb{R}^n_+} a|u|^q dx.$$

Using the first inequality in (4.15) we obtain $\bar{C}\left(\bar{C}\lambda^{-1}\right)^{\frac{p}{q-p}} \leq \|u\|_E^p$. This together with (4.14) imply that

$$\lambda \geq \bar{\lambda} := \left(\bar{C}^{\frac{q}{q-p}} \frac{r}{r-q} \left(\int_{\mathbb{R}^n_+} \frac{a^{\frac{r}{r-q}}}{b^{\frac{q}{r-q}}} dx \right)^{-1} \right)^{(r-q)(q-p)/q(r-p)}.$$

To conclude, we define

$$\lambda^* = \sup \left\{ \lambda > 0 : (\mathcal{P}_\mu) \text{ does not admits any nontivial weak solution for all } \mu < \lambda \right\}.$$

Therefore, $\lambda^* \geq \bar{\lambda} > 0$ and Theorem 4.2.2 holds true for all $\lambda < \lambda^*$.

4.3.2 The first solution

In this subsection, by using minimization argument we will prove item **(ii)** in Theorem 4.1.3. We first recall a basic estimate (see [7]).

Remark 4.3.4. *Let $0 \leq \beta < \gamma$ and $k, l \in (0, \infty)$. Then there exists a constant $C = C(\beta, \gamma) > 0$ such that*

$$k|s|^\beta - l|s|^\gamma \leq C(\beta, \gamma) k \left(\frac{k}{l}\right)^{\frac{\beta}{\gamma-\beta}}, \quad \forall\, s \in \mathbb{R}.$$

In order to use the direct methods of the calculus of variations we need the following result.

Lemma 4.3.5. *Let $p_* \leq q < p^*$, $r > q$ and assume $(H_1) - (H_2)$. Then, for all $\lambda > 0$, the functional $J_\lambda : E^{r,p} \to \mathbb{R}$ defined by*

$$J_\lambda(u) := \int_{\mathbb{R}^n_+} F(x, u),$$

where $F_\lambda(x, s) := \lambda a(x)|s|^q/q - b(x)|s|^r/r$ is weakly lower semicontinuous. As a consequence the functional I_λ is lower semicontinuous in $E^{r,p}$.

Proof. Assume that $u_k \rightharpoonup u_0$ in $E^{r,p}$. Taking into account that

$$F_s(x,s) = a(x)|s|^{q-2}s - b(x)|s|^{r-2}s, \quad F_{ss}(x,s) = (q-1)a(x)|s|^{q-2} - (r-1)b(x)|s|^{r-2} \quad s \in \mathbb{R}\backslash\{0\},$$

we get

$$F(x,u_k) - F(x,u_0) = \int_0^1 F_s(x, u_0 + t(u_k - u_0))(u_k - u_0)dt$$

and

$$F_s(x, u_0 + t(u_k - u_0)) - F_s(x, u_0) = \int_0^t F_{ss}(x, u_0 + s(u_k - u_0))(u_k - u_0)ds.$$

Consequently,

$$\begin{aligned} F(x,u_k) - F(x,u_0) &= \int_0^1 \left[\int_0^t F_{uu}(x, u_0 + s(u_k - u_0))(u_k - u_0)ds + F_u(x,u_0)\right](u_k - u_0)dt \\ &= \int_0^1 \int_0^t F_{uu}(x, u_0 + s(u_k - u_0))(u_k - u_0)^2 ds dt + F_u(x,u_0)(u_k - u_0). \end{aligned}$$

Thus, using Remark 4.3.4 we get

$$|F(x,u_k) - F(x,u_0)| \le C_2 \frac{a^{\frac{r-2}{r-q}}}{b^{\frac{q-2}{r-q}}}(u_k - u_0)^2 + |F_u(x,u_0)(u_k - u_0)|,$$

where $C_2 = C_1(q,r)\lambda^{\frac{r-2}{r-q}}$. Applying the Hölder inequality and using Lemma 4.3.2 we obtain

$$\int_{\mathbb{R}^n_+} (u_k - u_0)^2 \frac{a^{\frac{r-2}{r-q}}}{b^{\frac{q-2}{r-q}}} \le \left(\int_{\mathbb{R}^n_+} \frac{a^{\frac{r}{r-q}}}{b^{\frac{q}{r-q}}} dx\right)^{(q-2)/q} \left(\int_{\mathbb{R}^n_+} a|u_k - u_0|^q dx\right)^{2/q} \to 0.$$

On the other hand, considering the linear functional $\Phi_0 : E^{r,p} \to \mathbb{R}$ defined by

$$\Phi_0(v) = \int_{\mathbb{R}^n_+} F_u(x,u_0) v dx,$$

we see that

$$\begin{aligned} |\Phi_0(v)| &\le \lambda \int_{\mathbb{R}^n_+} a|u_0|^{q-1}|v|dx + \int_{\mathbb{R}^n_+} b|u_0|^{r-1}|v|dx \\ &\le \|u_0\|^{q-1}_{L^q(\mathbb{R}^n_+, a(x))}\|v\|_{L^q(\mathbb{R}^n_+, a(x))} + \|u_0\|^{r-1}_{L^r(\mathbb{R}^n_+, b(x))}\|v\|_{L^r(\mathbb{R}^n_+, b(x))} \le C\|u\|_{E^{r,p}}, \end{aligned}$$

and hence Φ_0 is continuous. Therefore, if $u_k \rightharpoonup u_0$ in $E^{r,p}$ we have

$$\lim_{k\to\infty} \int_{\mathbb{R}^n_+} F_u(x,u_0)(u_k - u_0)dx = 0,$$

which implies the desired result. □

Now we establish some geometric properties of the energy functional I_λ.

Lemma 4.3.6. *Let $p_* \leq q < p^*$, $r > q$ and assume $(H_1) - (H_2)$. For all $\lambda > 0$ the functional I_λ is coercive.*

Proof. Since $\lambda, a, b > 0$ and $q < r$, by Remark 4.3.4 we obtain

$$\int_{\mathbb{R}^n_+} \left(\frac{\lambda a}{q} |u|^q - \frac{b}{2r} |u|^r \right) \leq C_{r,q} \frac{1}{q r^{\frac{q}{r-q}}} \int_{\mathbb{R}^n_+} \lambda a \left(\frac{\lambda a}{b} \right)^{\frac{q}{r-q}} = C_{r,q} \lambda^{\frac{r}{r-q}} \int_{\mathbb{R}^n_+} \left(\frac{a^{\frac{r}{r-q}}}{b^{\frac{q}{r-q}}} \right) < \infty.$$

Thus, we get

$$\begin{aligned} I_\lambda(u) &= \frac{1}{p} \int_{\mathbb{R}^n_+} |\nabla u|^p dx + \frac{1}{2r} \int_{\mathbb{R}^n_+} b|u|^r dx - \int_{\mathbb{R}^n_+} \left(\frac{\lambda a}{q} |u|^q - \frac{b}{2r} |u|^r \right) dx \\ &\geq \frac{1}{p} \|u\|_E^p + \frac{1}{2r} \int_{\mathbb{R}^n_+} b|u|^r dx - C_1, \end{aligned}$$

which implies that I_λ is coercive and the proof is completed. □

Lemma 4.3.7. *Let $p_* \leq q < p^*$, $r > q$ and assume $(H_1) - (H_2)$. Then there exists $\Lambda > 0$ such that*

$$-\infty < \inf_{u \in E^{r,p}} I_\lambda(u) < 0, \quad \forall \lambda > \Lambda. \tag{4.16}$$

Proof. Let

$$\Lambda := \inf_{u \in E^{r,p}} \left\{ \frac{q}{p} \|u\|_E^p + \frac{q}{r} \int_{\mathbb{R}^n_+} b|u|^r dx : \int_{\mathbb{R}^n_+} a|u|^q = 1 \right\}.$$

We claim that $\Lambda > 0$. Otherwise, there exists a sequence $(u_k) \subset E^{r,p}$ such that

$$\frac{q}{p} \|u_k\|_E^p + \frac{q}{r} \int_{\mathbb{R}^n_+} b|u_k|^r dx = o_k(1) \quad \text{and} \quad \int_{\mathbb{R}^n_+} a|u_k|^q = 1.$$

Thus, by using the Hölder inequality we have

$$1 = \int_{\mathbb{R}^n_+} a|u_k|^q \leq \left(\int_{\mathbb{R}^n_+} \frac{a^{\frac{r}{r-q}}}{b^{\frac{q}{r-q}}} dx \right)^{(r-q)/r} \left(\int_{\mathbb{R}^n_+} b|u_k|^r dx \right)^{r/q} \to 0, \tag{4.17}$$

which is a contradiction. Now if $\lambda > \Lambda$, by the definition of Λ there exists $u_\lambda \in E^{r,p}$ with $\int_{\mathbb{R}^n_+} a|u_\lambda|^q = 1$ such that

$$\lambda > \frac{q}{p} \|u_\lambda\|_E^p + \frac{q}{r} \int_{\mathbb{R}^n_+} b|u_\lambda|^r dx.$$

Consequently,

$$I_\lambda(u_\lambda) = \frac{1}{p} \|u_\lambda\|_E^p + \frac{1}{r} \int_{\mathbb{R}^n_+} b|u_\lambda|^r dx - \frac{\lambda}{q} \int_{\mathbb{R}^n_+} a|u_\lambda|^q < 0.$$

Therefore, (4.16) holds. □

Lemma 4.3.8. *Let $p_* \leq q < p^*$, $r > q$ and assume $(H_1) - (H_2)$. For all $\lambda > \Lambda$ problem $(\mathcal{P}_\lambda)$ has a nontrivial weak solution.*

Proof. Using the direct method of the calculus of variations, from Lemmas 4.3.5, 4.3.6 and 4.3.7, for all $\lambda > \Lambda$ there exists $u_\lambda \in E^{r,p} \setminus \{0\}$ such that

$$-\infty < \inf_{u \in E^{r,p}} I_\lambda(u) = I_\lambda(u_\lambda) < 0.$$

Therefore, problem $(\mathcal{P}_\lambda)$ has a nontrivial weak solution u_λ with $I_\lambda(u_\lambda) < 0$ for all $\lambda > \Lambda$. Since $I_\lambda(u_\lambda) = I_\lambda(|u_\lambda|)$ we may assume that $u_\lambda \geq 0$. □

Setting

$$\tilde{\lambda} := \inf\{\lambda > 0 : (\mathcal{P}_\mu) \text{ has a nontrivial weak solution for all } \mu > \lambda\},$$

we clearly have that $\lambda^* \leq \tilde{\lambda} \leq \Lambda$.

Next we will prove that problem $(\mathcal{P}_{\tilde{\lambda}})$ has a nontrivial weak solution when $p < q$. To this end, we need the following result.

Lemma 4.3.9. *Let $p_* \leq q < p^*$, $r > q$ and assume $(H_1) - (H_2)$. If $\lambda > 0$ and $u \in E^{r,p}$ is a nontrivial weak solution of problem $(\mathcal{P}_\lambda)$ then*

$$\|u\|_E^p + \frac{r-q}{r} \int_{\mathbb{R}^n_+} b|u|^r dx \leq \frac{r-q}{r} \lambda^{\frac{r}{r-q}} \int_{\mathbb{R}^n_+} \frac{a^{\frac{r}{r-q}}}{b^{\frac{q}{r-q}}} dx. \tag{4.18}$$

Furthermore, there exists a constant $K > 0$ independent of u such that

$$\|u\|_E \geq K \lambda^{\frac{-1}{q-p}}. \tag{4.19}$$

Proof. If $u \in E$ is a weak solution of problem $(\mathcal{P}_\lambda)$, proceeding as in (4.17), we get

$$\|u\|_E^p + \int_{\mathbb{R}^n_+} b|u|^r dx = \lambda \int_{\mathbb{R}^n_+} a|u|^q dx \leq \frac{r-q}{r} \lambda^{\frac{r}{r-q}} \int_{\mathbb{R}^n_+} \frac{a^{\frac{r}{r-q}}}{b^{\frac{q}{r-q}}} dx + \frac{q}{r} \int_{\mathbb{R}^n_+} b|u|^r dx$$

which gives estimate (4.18). Now we will prove (4.19). Using again that u is a weak solution of problem $(\mathcal{P}_\lambda)$ we see that

$$\frac{1}{\lambda} \|u\|_E^p \leq \|u\|^q_{L^q(\mathbb{R}^n_+, a(x))}.$$

This combined with Lemma 4.1.3 show that

$$C_q \|u\|_E^q \geq \|u\|^q_{L^q(\mathbb{R}^n_+, a(x))} \geq \frac{1}{\lambda} \|u\|_E^p, \quad \forall\, u \in E,$$

for some constant $C_q > 0$. Thus, using that $p < q$ and $u \neq 0$ we get

$$\|u\|_E \geq C_q^{\frac{-1}{q-p}} \lambda^{\frac{-1}{q-p}},$$

which implies that (4.19) holds by choosing $K = C_q^{\frac{-1}{q-p}}$. □

Lemma 4.3.10. *The problem $(\mathcal{P}_{\tilde{\lambda}})$ has a nontrivial weak solution.*

Proof. Consider a sequence $\lambda_k \to \tilde{\lambda}$ with $\lambda_k > \tilde{\lambda}$. By the definition of $\tilde{\lambda}$, for each k the problem $(\mathcal{P}_{\lambda_k})$ has a nontrivial weak solution u_k. Furthermore, the sequence (u_k) is bounded in E in view of Lemma 4.3.9. Thus, we may assume that $u_k \rightharpoonup u_{\tilde{\lambda}}$ in E and, by Lemma 4.3.3, $u_k \to u_{\tilde{\lambda}}$ in $L^q(\mathbb{R}^n_+, a(x))$. Consequently, $u_{\tilde{\lambda}}$ is a nontrivial weak solution of $(\mathcal{P}_{\tilde{\lambda}})$. We claim that $u_{\tilde{\lambda}}$ is not trivial. Indeed, since u_k and $u_{\tilde{\lambda}}$ are weak solutions of $(\mathcal{P}_{\lambda_k})$ and $(\mathcal{P}_{\tilde{\lambda}})$, respectively, we have

$$o_k(1) = \langle I'_{\lambda_k}(u_k) - I'_{\tilde{\lambda}}(u_{\tilde{\lambda}}), u_k - u_{\tilde{\lambda}} \rangle = \int_{\mathbb{R}^n_+} \left(|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_{\tilde{\lambda}}|^{p-2}\nabla u_{\tilde{\lambda}} \right)(\nabla u_k - \nabla u_{\tilde{\lambda}})\, dx$$
$$+ \int_{\mathbb{R}^n_+} b \left(|u_k|^{r-2}u_k - |u_{\tilde{\lambda}}|^{r-2}u_{\tilde{\lambda}} \right)(u_k - u_{\tilde{\lambda}})\, dx - (J_{1,k} + J_{2,k}),$$

where

$$J_{1,k} = \lambda_k \int_{\mathbb{R}^n_+} a \left(|u_k|^{q-2}u_k - |u_{\tilde{\lambda}}|^{q-2}u_{\tilde{\lambda}} \right)(u_k - u_{\tilde{\lambda}})\, dx$$

and

$$J_{2,k} = (\lambda_k - \tilde{\lambda}) \int_{\mathbb{R}^n_+} a |u_{\tilde{\lambda}}|^{q-2}u_{\tilde{\lambda}}\,(u_k - u_{\tilde{\lambda}})\, dx.$$

Using the Höder inequality together with the fact that (λ_k) is bounded we get

$$|J_{1,k}| \le C \left[\left(\int_{\mathbb{R}^n_+} a|u_k|^q dx \right)^{(q-1)/q} + \left(\int_{\mathbb{R}^n_+} a|u_{\tilde{\lambda}}|^q dx \right)^{(q-1)/q} \right] \left(\int_{\mathbb{R}^n_+} a|u_k - u_{\tilde{\lambda}}|^q dx \right)^{1/q}.$$

Consequently, by Lemma 4.3.3 we obtain $J_{1,k} = o_k(1)$. Similarly, we have $J_{2,k} = o_k(1)$. Therefore, we conclude that

$$\left(\int_{\mathbb{R}^n_+} \left(|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_{\tilde{\lambda}}|^{p-2}\nabla u_{\tilde{\lambda}} \right)(\nabla u_k - \nabla u_{\tilde{\lambda}})\, dx \right.$$
$$\left. + \int_{\mathbb{R}^n_+} b \left(|u_k|^{r-2}u_k - |u_{\tilde{\lambda}}|^{r-2}u_{\tilde{\lambda}} \right)(u_k - u_{\tilde{\lambda}})\, dx \right) = o_k(1). \tag{4.20}$$

Now we recall that for all $\xi, \zeta \in \mathbb{R}^n$, we know that there exists a constant $C = C(p) > 0$ (see inequality (2.2) in [40]) such that

$$(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)(\xi - \zeta) \ge C \begin{cases} |\xi - \zeta|^p, & \text{if} \quad p \ge 2, \\ |\xi - \zeta|^2(|\xi| + |\zeta|)^{p-2}, & \text{if} \quad 1 < p \le 2. \end{cases} \tag{4.21}$$

If $p \ge 2$, using the fact that $b > 0$ together with (4.20) we obtain

$$\|u_k - u_{\tilde{\lambda}}\|_E^p \le C \left(\int_{\mathbb{R}^n_+} \left(|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_{\tilde{\lambda}}|^{p-2}\nabla u_{\tilde{\lambda}} \right)(\nabla u_k - \nabla u_{\tilde{\lambda}})\, dx \right) = o_k(1).$$

On the other hand, if $1 < p < 2$ we can use the inequality (4.21) again to obtain

$$\int_{\mathbb{R}^n_+} (|\nabla u_k - \nabla u_{\tilde{\lambda}}|^2)^{\frac{p}{2}} dx$$
$$\leq \int_{\mathbb{R}^n_+} \left[\left(|\nabla u_k|^{p-2} \nabla u_k - |\nabla u_{\tilde{\lambda}}|^{p-2} \nabla u_{\tilde{\lambda}} \right) (\nabla u_k - \nabla u_{\tilde{\lambda}}) \right]^{\frac{p}{2}} \left((|\nabla u_k| + |\nabla u_{\tilde{\lambda}}|)^p \right)^{\frac{2-p}{2}} dx.$$

This together with the Höder inequality, (4.20) and the fact that (u_k) is bounded imply that

$$\tilde{C}_p \int_{\mathbb{R}^n_+} |\nabla u_k - \nabla u_{\tilde{\lambda}}|^p dx \leq \left(\int_{\mathbb{R}^n_+} \left(|\nabla u_k|^{p-2} \nabla u_k - |\nabla u_{\tilde{\lambda}}|^{p-2} \nabla u_{\tilde{\lambda}} \right) (\nabla u_k - \nabla u_{\tilde{\lambda}}) \, dx \right)^{p/2}$$
$$\times \left(\int_{\mathbb{R}^n_+} (|\nabla u_k|^p + |\nabla u_{\tilde{\lambda}}|^p) dx \right)^{(2-p)/2} = o_k(1).$$

Hence, $u_k \to u_{\tilde{\lambda}}$ in E. Since u_k is a nontrivial weak solution of problem $(\mathcal{P}_{\lambda_k})$, by Lemma 4.3.9 there exists $K = K(p, q)$ such that

$$\|u_k\|_E \geq K \lambda_k^{-\frac{1}{q-p}}, \quad \forall \, k \in \mathbb{N}.$$

Since $\|u_k\|_E \to \|u_{\tilde{\lambda}}\|_E$ and $\lambda_k \to \tilde{\lambda} > 0$ we get

$$\|u_{\tilde{\lambda}}\|_E \geq K (\tilde{\lambda})^{-\frac{1}{q-p}} > 0,$$

and hence $u_{\tilde{\lambda}}$ is nontrivial. Since $I_{\tilde{\lambda}}(u_{\tilde{\lambda}}) = I_{\tilde{\lambda}}(|u_{\tilde{\lambda}}|)$ we may assume that $u_{\tilde{\lambda}} \geq 0$ a.e. in $\mathbb{R}^n_+$. □

4.3.3 The second solution

In what follows we will prove item **(iii)** in Theorem 4.1.3. This will be done by using a truncation argument. Let $\lambda > \Lambda$ be fixed and consider the truncated Carathéodory function defined by

$$g_\lambda(x, t) = \begin{cases} 0, & \text{if } \; t < 0, \\ \lambda a(x) t^{q-1} - b(x) t^{r-1}, & \text{if } \; 0 \leq t \leq u_\lambda(x), \\ \lambda a(x) u_\lambda^{q-1} - b(x) u_\lambda^{r-1}, & \text{if } \; t > u_\lambda(x), \end{cases}$$

where $u_\lambda \in E^{r,p}$ is the weak solution of problem $(\mathcal{P}_\lambda)$ with $I_\lambda(u_\lambda) < 0$ obtained in Lemma 4.3.8. Setting $G_\lambda(x, t) = \int_0^t g_\lambda(x, s) ds$, we define the functional $\tilde{I}_\lambda : E \to \mathbb{R}$ by

$$\tilde{I}_\lambda(u) = \frac{1}{p} \|u\|_E^p - \int_{\mathbb{R}^n_+} G_\lambda(x, u) dx.$$

Notice that for all $v, \varphi \in E$ it holds

$$\tilde{I}'_\lambda(v)\varphi = \int_{\mathbb{R}^n_+} |\nabla v|^{p-2} \nabla v \nabla \varphi dx - \int_{\{0 \leq v \leq u_\lambda\}} [\lambda a v^{q-1} - b v^{r-1}] \varphi dx - \int_{\{v > u_\lambda\}} [\lambda a u_\lambda^{q-1} - b u_\lambda^{r-1}] \varphi dx.$$

Furthermore, by choosing $\varphi = v^- := -\min\{v, 0\}$ we see that critical points of $\tilde{I}_\lambda$ are nonnegative.

Next, to prove that critical point of $\tilde{I}_\lambda$ is a critical point of I_λ, inspired in [31, Lemma 2.1] (see also [35]) we have the following *a priori estimate*.

Lemma 4.3.11. *Let $p_* \leq q < p^*$, $r > q$ and assume $(H_1) - (H_2)$. If u_λ is the solution obtained in item **(ii)** of Theorem 4.1.3 and $\tilde{u}_\lambda$ is a critical point of $\tilde{I}_\lambda$ then $0 \leq \tilde{u}_\lambda \leq u_\lambda$ in $\mathbb{R}^n_+$.*

Proof. For any $v \in E$ let us denote by $v^+(x) = \max\{v(x), 0\}$. If $\tilde{u}_\lambda$ is a critical point of $\tilde{I}_\lambda$ we get

$$0 = \langle \tilde{I}'_\lambda(\tilde{u}_\lambda) - I'_\lambda(u_\lambda), (\tilde{u}_\lambda - u_\lambda)^+\rangle = \int_{\{u>u_\lambda\}} \left(|\nabla \tilde{u}_\lambda|^{p-2}\nabla \tilde{u}_\lambda - |\nabla u_\lambda|^{p-2}\nabla u_\lambda\right)(\nabla \tilde{u}_\lambda - \nabla u_\lambda)\, dx.$$

This combined with inequality (4.21) imply that $|\{x \in \mathbb{R}^n_+ : \tilde{u}_\lambda(x) > u_\lambda(x)\}| = 0$. Thus, $(\tilde{u}_\lambda - u_\lambda)^+ = 0$ a.e. in $\mathbb{R}^n_+$. Therefore, $\tilde{u}_\lambda \leq u_\lambda$ and the proof is complete. □

Lemma 4.3.12. *Let $p_* \leq q < p^*$, $r > q$ and assume $(H_1) - (H_2)$. Then there exist $\rho \in (0, \|u_\lambda\|_{E^{r,p}})$ and $\alpha > 0$ such that $\tilde{I}_\lambda(v) \geq \alpha > 0$ if $\|v\|_E = \rho$.*

Proof. Notice that for all $v \in E$ we can write

$$\int_{\mathbb{R}^n_+} G_\lambda(x, v)dx = \int_{\{0\leq v\leq u_\lambda\}} G_\lambda(x, v)dx + \int_{\{v>u_\lambda\}} G_\lambda(x, v)dx.$$

Now observing that

$$\int_{\{0\leq v\leq u_\lambda\}} G_\lambda(x, v)dx = \int_{\{0\leq v\leq u_\lambda\}} \left[\frac{\lambda a}{q}v^q - \frac{b}{r}v^r\right] dx \leq \frac{\lambda}{q}\int_{\{0\leq v\leq u_\lambda\}} av^q dx$$

and

$$\begin{aligned}\int_{\{v>u_\lambda\}} G_\lambda(x, v)dx &= \int_{\{v>u_\lambda\}} \left[\int_0^{u_\lambda} g_\lambda(x, t)dt + \int_{u_\lambda}^{v} g_\lambda(x, t)dt\right] dx \\ &= \int_{\{v>u_\lambda\}} \left[\frac{\lambda a u_\lambda^q}{q} - \frac{b u_\lambda^r}{r} + \left(\lambda a u_\lambda^{q-1} - b u_\lambda^{r-1}\right)(v - u_\lambda)\right] dx \\ &\leq \int_{\{v>u_\lambda\}} \left[\frac{\lambda a u_\lambda^q}{q} + \lambda a u_\lambda^{q-1} v\right],\end{aligned}$$

we get

$$\tilde{I}_\lambda(v) \geq \frac{1}{p}\|v\|_E^p - \frac{\lambda}{q}\int_{\{0\leq v\leq u_\lambda\}} av^q dx - \lambda\int_{\{v>u_\lambda\}} a\left[\frac{u_\lambda^q}{q} + u_\lambda^{q-1}v\right]. \tag{4.22}$$

This combined with Remark 4.3.1 imply that there exists $C_1 > 0$ such that

$$\tilde{I}_\lambda(v) \geq \frac{1}{p}\|v\|_E^p - \frac{\lambda}{q}C_1\|v\|_E^q = \left(\frac{1}{p} - \lambda C_1\|v\|_E^{q-p}\right)\|v\|_E^p.$$

Since $q > p$ we obtain the desired result and the proof is completed. □

By Lemma 4.3.12 we have that

$$\inf_{\|v\|_E=\rho} \tilde{I}_\lambda(v) > 0 \geq \tilde{I}_\lambda(u_\lambda), \quad \forall\, \lambda > \Lambda.$$

Thus, the minimax level

$$c_\lambda = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \tilde{I}_\lambda(\gamma(t)) > 0, \quad \forall\,\lambda > \Lambda,$$

where $\Gamma := \{\gamma \in C([0,1], E) : \gamma(0) = 0 \text{ and } \gamma(1) = u_\lambda\}$. Applying the mountain pass theorem without the (PS) (see [44, Theorem 1.15])) we find a sequence $(u_k) \subset E$ at the minimax level c_λ, that is

$$\tilde{I}_\lambda(u_k) \to c_\lambda \quad \text{and} \quad \tilde{I}'_\lambda(u_k) \to 0. \tag{4.23}$$

Lemma 4.3.13. *Let $p_* \le q < p^*$, $r > q$ and assume $(H_1) - (H_2)$. Then, the sequence (u_k) in (4.23) has a convergent subsequence.*

Proof. From estimate (4.22), there exists $C_1 > 0$ such that

$$\tilde{I}_\lambda(u) \ge \frac{1}{p}\|u\|_E^p - \frac{\lambda}{q}\int_{\mathbb{R}^n_+} a u_\lambda^q dx - \lambda C_1 \|u_\lambda\|^{q-1}_{L^q(\mathbb{R}^n_+, a(x))}\|u\|_E,$$

from where we obtain that $\tilde{I}_\lambda$ is coercive and consequently (u_k) is bounded in E. By Lemma 4.3.3, up to a subsequence, we can assume that

$$\begin{cases} u_k \rightharpoonup \tilde{u}_\lambda & \text{in } E \\ u_k(x) \to \tilde{u}_\lambda(x) & \text{a.e. in } \mathbb{R}^n_+ \\ u_k \to \tilde{u}_\lambda & \text{in } L^q(\mathbb{R}^n_+, a(x)). \end{cases}$$

Arguing as in proof of Lemma 4.3.5 we can see that $\tilde{I}'_\lambda(\tilde{u}_\lambda) = 0$ and hence $0 \le \tilde{u}_\lambda \le u_\lambda$ in $\mathbb{R}^n_+$ by Lemma 4.3.11. Thus, we get

$$o_k(1) = \langle \tilde{I}'_\lambda(u_k) - \tilde{I}'_\lambda(\tilde{u}_\lambda), u_k - \tilde{u}_\lambda\rangle = A_k - B_k + C_k, \tag{4.24}$$

where $o_k(1)$ denotes a quantity that goes to zero as $k \to +\infty$ and

$$\begin{aligned} A_k &= \int_{\mathbb{R}^n_+} \left(|\nabla u_k|^{p-2}\nabla u_k - |\nabla \tilde{u}_\lambda|^{p-2}\nabla \tilde{u}_\lambda\right)(\nabla u_k - \nabla \tilde{u}_\lambda)\,dx \\ B_k &= \int_{\{0\le u_k\le u_\lambda\}} [\lambda a u_k^{q-1} - b u_k^{r-1}]\,(u_k - \tilde{u}_\lambda)\,dx + \int_{\{u_k > u_\lambda\}} [\lambda a u_\lambda^{q-1} - b u_\lambda^{r-1}]\,(u_k - \tilde{u}_\lambda)\,dx \\ C_k &= \int_{\{0\le \tilde{u}_\lambda\le u_\lambda\}} [\lambda a \tilde{u}_\lambda^{q-1} - b \tilde{u}_\lambda^{r-1}]\,(u_k - \tilde{u}_\lambda)\,dx + \int_{\{\tilde{u}_\lambda > u_\lambda\}} [\lambda a \tilde{u}_\lambda^{q-1} - b \tilde{u}_\lambda^{r-1}]\,(u_k - \tilde{u}_\lambda)\,dx. \end{aligned}$$

Therefore,

$$A_k = o_k(1) + \int_{\{0\le u_k\le u_\lambda\}} [\lambda a u_k^{q-1} - b u_k^{r-1}]\,(u_k - \tilde{u}_\lambda)\,dx - \int_{\{0\le \tilde{u}_\lambda\le u_\lambda\}} [\lambda a \tilde{u}_\lambda^{q-1} - b \tilde{u}_\lambda^{r-1}]\,(u_k - \tilde{u}_\lambda)\,dx.$$

Now, proceeding as in the proof of Lemma 4.3.5 we see that

$$\int_{\{0\le u_k\le u_\lambda\}} [\lambda a u_k^{q-1} - b u_k^{r-1}]\,(u_k - \tilde{u}_\lambda)\,dx = o_k(1)$$

and

$$\int_{\{0\le\tilde u_\lambda\le u_\lambda\}}[\lambda a\tilde u_\lambda^{q-1}-b\tilde u_\lambda^{r-1}]\,(u_k-\tilde u_\lambda)\,dx=o_k(1).$$

Thus, we conclude that $A_k = o_k(1)$. If $2 \le p \le q < r$, using inequality (4.21), we get $\|u_k-\tilde u_\lambda\|_E^p = o_k(1)$. Furthermore, if $1 < p < 2$, arguing as in the proof of Lemma 4.3.10 we obtain $\|u_k-\tilde u_\lambda\|_E^p = o_k(1)$. This completes the proof of Lemma 4.3.13. □

Finalizing the proof of item ***(iii)*** *in Theorem 4.2.2.* By Lemma 4.3.13, and standard arguments we conclude that $\tilde u_\lambda$ is a critical point of I_λ. To conclude, by Lemma 4.3.11, we have $0\le\tilde u_\lambda\le u_\lambda$. Thus,

$$g(x,\tilde u_\lambda)=\lambda a(x)\tilde u_\lambda^{q-1}-b(x)\tilde u_\lambda^{r-1}\quad\text{and}\quad G(x,\tilde u_\lambda)=\frac{\lambda a(x)\tilde u_\lambda^{q}}{q}-\frac{b(x)\tilde u_\lambda^{r}}{r}$$

so that

$$\tilde I_\lambda(\tilde u_\lambda)=I_\lambda(\tilde u_\lambda)\quad\text{and}\quad\tilde I'_\lambda(\tilde u_\lambda)=I'_\lambda(\tilde u_\lambda).$$

More precisely, we find

$$I_\lambda(\tilde u_\lambda)>0\ge I_\lambda(u_\lambda)\quad\text{and}\quad I'_\lambda(\tilde u_\lambda)=0.$$

Therefore, $\tilde u_\lambda$ is a nontrivial weak solution of problem $(\mathcal{P}_\lambda)$ such that $0\le\tilde u_\lambda\le u_\lambda$, $\tilde u_\lambda\ne 0$ and $\tilde u_\lambda\ne u_\lambda$. □

4.3.4 Multiplicity

Finally, in this subsection we will complete the proof of Theorem 4.1.3 by proving statement **(iv)**. It consists in applying the symmetric mountain pass theorem due to Ambrosetti-Rabinowitz [8] and Clark [17]. To this, we need to recall some notations. Let E be a Banach space and denotes by $\mathcal{E}$ the class of all subsets of $E\setminus\{0\}$ closed and symmetric with respect to the origin :

$$\mathcal{E}:=\{A\subset E\setminus\{0\}: A\text{ is closed and }A=-A\}.$$

For $A\in\mathcal{E}\setminus\{\varnothing\}$ the genus $\gamma(A)$ is define by

$$\gamma(A):=\min\{m\in\mathbb{N}:\exists\,\varphi\in C(A,\mathbb{R}^m\setminus\{0\})\text{ such that }\varphi(x)=-\varphi(-x)\}.$$

If the minimum does not exist, we define $\gamma(A)=\infty$ and $\gamma(\varnothing)=0$. Let $\mathcal{E}_m=\{A\in\mathcal{E}:\gamma(A)\ge m\}$. The main properties of the genus can be found in [38, 41].

Now, we recall the following classical multiplicity result (see for instance [8, 17]).

Theorem 4.3.14. *Let E be an infinite dimensional Banach space and $I\in C^1(E,\mathbb{R})$ satisfying*

(A_1) *$I(u)$ is even, bounded from below, $I(0)=0$ and $I(u)$ satisfies the Palais-Smale condition (PS);*

(A_2) *For each $m\in\mathbb{N}$, there exists an $A_m\in\mathcal{E}_m$ such that $\sup_{u\in A_m}I(u)<0$.*

Defining

$$c_m = \inf_{A \in \mathcal{E}_m} \sup_{u \in A} I_\lambda(u),$$

then each c_k *is a critical value of* $I(u)$, $c_m \leq c_{m+1} < 0$ *for* $m \in \mathbb{N}$ *and* (c_m) *converges to zero. Moreover, if* $c = c_m = c_{m+1} = \cdots = c_{m+j} < \infty$, *then* $\gamma(K_c) \geq j+1$. *Here,* K_c *is defined by*

$$K_c = \{u \in E^{r,p} : I_\lambda(u) = c \text{ and } I'_\lambda(u) = 0\}.$$

To prove item **(iv)** in Theorem 4.1.3, it is sufficient to show that I_λ satisfies the conditions (A_1) and (A_2) above. Arguing as in the proof of Lemma 4.3.13 one can see that I_λ satisfies condition (A_1). In order to verify condition (A_2), we consider $\Omega_0 = \{x \in \mathbb{R}^n_+ : a(x) = 0\}$ and $\Omega_0^c = \mathbb{R}^n_+ \setminus \Omega_0$. Denote

$$E_0 = \{u \in E^{r,p} : u(x) = 0 \quad \text{a.e.} \quad x \in \Omega_0\}.$$

If $\Omega_0 = \varnothing$, i.e., $a(x) > 0$ in $\mathbb{R}^n_+$ then we let $E_0 = E^{r,p}$. Obviously, E_0 is an infinitely dimensional linear subspace of $E^{r,p}$. A seminorm $[\cdot]_q$ on $E^{r,p}$ is defined by

$$[u]_q = \left(\int_{\mathbb{R}^n_+} a(x)|u|^q dx \right)^{1/q}.$$

Lemma 4.3.15. *The seminorm* $[\cdot]_q$ *is a norm in* E_0.

Proof. It is sufficient to show that $u \in E_0$, $[u]_q = 0$ implies that $u = 0$, a.e. in $\mathbb{R}^n_+$. Indeed,

$$0 = [u]_q^q = \int_{\mathbb{R}^n_+} a(x)|u|^q dx = \int_{\Omega_0^c} a(x)|u|^q dx.$$

This together with fact $a(x) > 0$ in Ω_0^c imply that $u(x) = 0$, a.e. in Ω_0^c. Since $u \in E_0$, $u(x) = 0$, a.e. in Ω_0. Therefore, $u(x) = 0$, a.e. in $\mathbb{R}^n_+$ and this completes the proof. □

Lemma 4.3.16. *Let* $1 < p \leq q < p^*$, $r > q$ *and assume* (H_1). *Then for each* $m \in \mathbb{N}$, *there exist an* $A_m \in \mathcal{E}_m$ *and* λ_m *such that*

$$\sup_{u \in A_m} I_\lambda(u) < 0, \quad \forall \lambda > \lambda_m.$$

Proof. Let E_m be a m-dimensional subspace of E_0. Since all norms on the finite dimension space E_m are equivalent, there exists $b_m > 0$ such that

$$I_\lambda(u) \leq \frac{1}{p}\|u\|^p_{E^{r,p}} + \frac{1}{r}\|u\|^r_{E^{r,p}} - \frac{\lambda b_m}{q}\|u\|^q_{E^{r,p}} \leq \frac{2}{p} - \frac{\lambda b_m}{q}$$

for all $u \in E_m$ with $\|u\|_{E^{r,p}} = 1$. Thus, for $\lambda_m = 4q/pb_m$, $I_\lambda(u) < -2/p$ if $\|u\|_{E^{r,p}} = 1$, for all

$\lambda > \lambda_m$. Let $A_m = S^m(1)$ be a sphere with radius 1 in E_m. Then

$$\sup_{u \in A_m} I_\lambda(u) < 0, \quad \forall\, \lambda > \lambda_m$$

and by properties of genus $A_m \in \mathcal{E}_m$. □

Finalizing the proof of item ***(iv)*** *in Theorem 4.1.3.* It follows directly from Theorem 4.3.14. □

4.4 Proof of Theorem 4.2.3

This section is devoted to the proof of Theorem 4.2.3. In order to prove our multiplicity result we recall the original statement of the symmetric mountain pass theorem (see [8]).

Theorem 4.4.1. *Let E be a real infinite-dimensional Banach space and $I \in C^1(E, R)$ an even functional satisfying the (PS) condition and the following hypotheses:*

(I_1) *$I(0) = 0$ and there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho(0)} \geq \alpha$;*

(I_2) *for any finite dimensional $\widetilde{E} \subset E$, $\widetilde{E} \cap \{u \in E : I(u) \geq 0\}$ is bounded.*

Then I has an unbounded sequence of critical values.

Now, we establish some properties of the energy functional I_λ.

Lemma 4.4.2. *Let $1 < p \leq r < q < p^*$ and assume $(\widetilde{H}_1) - (\widetilde{H}_2)$. Then for each $\lambda > 0$ there exist $\rho, \alpha_0 > 0$ such that $I_\lambda(u) \geq \alpha_0 > 0$ if $\|u\|_{E^{r,p}} = \rho$.*

Proof. First we observe that

$$\|u\|^r_{E^{r,p}} \leq \left(\|u\|^p_E + \|u\|^p_{L^r(\mathbb{R}^n_+, b(x))}\right)^{r/p} \leq 2^{\frac{r}{p}} \left(\|u\|^r_E + \|u\|^r_{L^r(\mathbb{R}^n_+, b(x))}\right). \tag{4.25}$$

Without loss of generality we may assume that $\|u\|^p_E + \|u\|^p_{L^r(\mathbb{R}^n_+, b(x))} = \|u\|^p_{E^{r,p}} = \rho^p \leq 1$ and using that $p \leq r$ we see that $\|u\|^p_E \geq \|u\|^r_E$. Thus, we conclude that

$$I_\lambda(u) \geq \frac{1}{p}\|u\|^r_E + \frac{1}{r}\|u\|^r_{L^r(\mathbb{R}^n_+, b(x))} - \frac{\lambda}{q}\|u\|^q_{L^q(\mathbb{R}^n_+, a(x))}.$$

This together with (4.25), Lemmas 4.1.3 and the fact that $r < q$ imply

$$I_\lambda(u) \geq \frac{1}{r 2^{\frac{r}{p}}}\|u\|^r_{E^{r,p}} - \frac{\lambda}{q} C_1 \|u\|^q_{E^{r,p}} = \left(\frac{1}{r 2^{\frac{r}{p}}} - \frac{\lambda}{q} C_1 \rho^{q-r}\right)\rho^r,$$

which implies (I_1) by choosing ρ sufficiently small. □

Next, let us ensure that any (PS) sequence associated to I_λ has a convergent subsequence. This is done in the next lemma.

Lemma 4.4.3. *Let* $1 < p \leq r < q < p^*$ *and assume* $(\widetilde{H}_1)-(\widetilde{H}_2)$*. Then any sequence* $(u_k) \subset E^{r,p}$ *such that*

$$I_\lambda(u_k) \to c \quad \text{and} \quad \|I'_\lambda(u_k)\|_{(E^{r,p})^*} \to 0, \quad \text{as} \quad k \to \infty, \tag{4.26}$$

has a convergent subsequence.

Proof. First, we observe that

$$\left(\frac{1}{p} - \frac{1}{q}\right) \|u_k\|_E^p + \left(\frac{1}{r} - \frac{1}{q}\right) \int_{\mathbb{R}^n_+} b|u_k|^r dx = I_\lambda(u_k) - \frac{1}{q}\langle I'_\lambda(u_k), u_k\rangle \leq c_\lambda + o_k(\|u_k\|_{E^{r,p}}). \tag{4.27}$$

We claim that $(u_k) \subset E^{r,p}$ is bounded. Arguing by contradiction, let us suppose that $\|u_k\|_{E^{r,p}} \to \infty$. Since $1 < p \leq r < q$, in view of (4.27) we get

$$\frac{\|u_k\|_E^p}{\|u_k\|_{E^{r,p}}} = o_k(1) \quad \text{and} \quad \frac{\|u_k\|^r_{L^r(\mathbb{R}^n_+, b(x))}}{\|u_k\|_{E^{r,p}}} = o_k(1). \tag{4.28}$$

This in combination with the fact that

$$\frac{\|u_k\|_E^p}{\|u_k\|_{E^{r,p}}} + \frac{\|u_k\|^p_{L^r(\mathbb{R}^n_+, b(x))}}{\|u_k\|_{E^{r,p}}} = \|u_k\|^{p-1}_{E^{r,p}} \to \infty, \quad \text{as} \quad k \to \infty$$

imply that

$$\frac{\|u_k\|^p_{L^r(\mathbb{R}^n_+, b(x))}}{\|u_k\|_{E^{r,p}}} \to \infty, \quad \text{as} \quad k \to \infty. \tag{4.29}$$

If $p = r$, combining (4.28) and (4.29) we obtain a contradiction. In case that $p < r$, using again (4.29) we conclude that $\|u_k\|^p_{L^r(\mathbb{R}^n_+, b(x))} \to \infty$ as $k \to \infty$ and hence $\|u_k\|^{p-r}_{L^r(\mathbb{R}^n_+, b(x))} \leq C$. On the other hand,

$$\frac{\|u_k\|^p_{L^r(\mathbb{R}^n_+, b(x))}}{\|u_k\|_{E^{r,p}}} = \|u_k\|^{p-r}_{L^r(\mathbb{R}^n_+, b(x))} \frac{\|u_k\|^r_{L^r(\mathbb{R}^n_+, b(x))}}{\|u_k\|_{E^{r,p}}} \to 0, \quad \text{as} \quad k \to \infty,$$

which contradicts (4.29) and hence (u_k) is bounded in $E^{r,p}$. By Lemma 4.3.2 we may assume that

$$\begin{cases} u_k \rightharpoonup u_0 & \text{in } E^{r,p} \\ u_k(x) \to u_0(x) & \text{a.e. in } \mathbb{R}^n_+ \\ u_k \to u_0 & \text{in } L^q(\mathbb{R}^n_+, a(x)) \end{cases}$$

as $k \to \infty$. From (4.26), it follows that

$$o_k(1) = \langle I'_\lambda(u_k) - I'_\lambda(u_0), u_k - u_0\rangle = A_k - \int_{\mathbb{R}^n_+} \lambda a \left(|u_k|^{q-2}u_k - |u_0|^{q-2}u_0\right)(u_k - u_0)\, dx, \tag{4.30}$$

where

$$A_k = \int_{\mathbb{R}^n_+} \left(|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_0|^{p-2}\nabla u_0\right)(\nabla u_k - \nabla u_0)\,dx$$
$$+ \int_{\mathbb{R}^n_+} b\left(|u_k|^{r-2}u_k - |u_0|^{r-2}u_0\right)(u_k - u_0)\,dx.$$

By the Hölder inequality and Lemma 4.3.3, we obtain

$$\int_{\mathbb{R}^n_+} \lambda a\left(|u_k|^{q-2}u_k - |u_0|^{q-2}u_0\right)(u_k - u_0)\,dx = o_k(1).$$

Thus, from (4.30) we conclude that $A_k = o_k(1)$. If $2 \leq p \leq r < q$, we can use the inequality (4.21) and the fact that $b \geq 0$ to get

$$\begin{aligned}&\int_{\mathbb{R}^n_+} \left(|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_0|^{p-2}\nabla u_0\right)(\nabla u_k - \nabla u_0)\,dx = o_k(1)\\ &\int_{\mathbb{R}^n_+} b\left(|u_k|^{r-2}u_k - |u_0|^{r-2}u_0\right)(u_k - u_0)\,dx = o_k(1).\end{aligned} \tag{4.31}$$

Using once again inequality (4.21), we get

$$\|u_k - u_0\|^p_{E^{r,p}} = \|u_k - u_0\|^p_E + \|u_k - u_0\|^p_{L^r(\mathbb{R}^n_+,b(x))} = o_k(1),$$

which implies that $u_k \to u_0$ in $E^{r,p}$. Now, if $1 < p < 2$ we have two cases to consider, $r \geq 2$ and $p \leq r < 2$. If $r \geq 2$, by inequality (4.21) and (4.31) we obtain

$$\|u_k - u_0\|^r_{L^r(\mathbb{R}^n_+,b(x))} \leq \int_{\mathbb{R}^n_+} b\left(|u_k|^{r-2}u_k - |u_0|^{r-2}u_0\right)(u_k - u_0)\,dx = o_k(1). \tag{4.32}$$

Now, if $p \leq r < 2$, by inequality (4.21) and the Höder inequality we get

$$\begin{aligned}\|u_k - u_0\|^r_{L^r(\mathbb{R}^n_+,b(x))} &\leq \int_{\mathbb{R}^n_+} b\left(\left(|u_k|^{r-2}u_k - |u_0|^{r-2}u_0\right)(u_k - u_0)\right)^{\frac{r}{2}}\left((|u_k| + |u_0|)^r\right)^{\frac{(2-r)}{2}}dx\\ &\leq \left(\int_{\mathbb{R}^n_+} b\left(|u_k|^{r-2}u_k - |u_0|^{r-2}u_0\right)(u_k - u_0)\,dx\right)^{r/2}\left(\int_{\mathbb{R}^n_+} b\left(|u_k| + |u_0|\right)^r dx\right)^{(2-r)/2}.\end{aligned}$$

This combined with (4.31), (4.32) and the fact (u_k) is bounded imply that $\|u_k - u_0\|^r_{L^r(\mathbb{R}^n_+,b(x))} = o_k(1)$. Now, if $1 < p < 2$, arguing as in the proof of Lemma 4.3.10 we obtain $\|u_k - u_0\|^p_E = o_k(1)$. Therefore, $\|u_k - u_0\|^p_{E^{r,p}} = \|u_k - u_0\|^p_E + \|u_k - u_0\|^p_{L^r(\mathbb{R}^n_+,b(x))} = o_k(1)$, and this completes the proof. □

Finalizing the proof of Theorem 4.2.3. If u is a weak solution of problem $(\mathcal{P}_\lambda)$, choosing $\varphi = u$ in (4.12) we get $\|u\|^p_E + \|u\|^r_{L^r(\mathbb{R}^n_+,b(x))} = \lambda\|u\|^q_{L^r(\mathbb{R}^n_+,a(x))}$, which implies that $u = 0$ if $\lambda \leq 0$ and item **(i)** in Theorem 4.2.3 is proved. Now we will use Theorem 4.4.1 to prove item **(ii)** in Theorem 4.2.3. By Lemma 4.4.2, for any $\lambda > 0$ the functional I_λ satisfies condition (I_1). Now we prove item (I_2). Suppose by contradiction that (I_2) is false. Then, there exist a finite dimensional $\widetilde{E} \subset E^{r,p}$ and a sequence $(u_k) \subset \widetilde{E}$ satisfying

$$I_\lambda(u_k) > 0, \quad k \in \mathbb{N} \quad \text{and} \quad \|u_k\|_{E^{r,p}} \to \infty \quad \text{as} \quad k \to \infty. \tag{4.33}$$

Using the fact that all the norms in $\tilde{E}$ are equivalent, there exists $\tilde{c} > 0$ such that

$$0 < I_\lambda(u_k) \leq \frac{1}{p}\|u_k\|^p_{E^{r,p}} + \frac{1}{r}\|u_k\|^r_{E^{r,p}} - \frac{\lambda\tilde{c}}{q}\|u_k\|^q_{E^{r,p}}, \quad \forall\, k \in \mathbb{N}.$$

Thus,

$$\frac{\lambda\tilde{c}}{q}\|u_k\|^q_{E^{r,p}} < \frac{1}{p}\|u_k\|^p_{E^{r,p}} + \frac{1}{r}\|u_k\|^r_{E^{r,p}}, \quad \forall\, k \in \mathbb{N},$$

which contradicts (4.33), since $p \leq r < q$, and item (I_2) is proved. In view of Lemma 4.4.3, for each $\lambda > 0$ we can apply Theorem 4.4.1 to obtain an unbounded sequence of critical values of I_λ to which we can associate at least two critical points because the functional I_λ is even. This completes the proof. □

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