Universidade Federal da Paraíba
Programa de Pós-Graduação em Matemática
Doutorado em Matemática

On r-Trapped Immersions in Lorentzian Spacetimes and A Weighted Inequality for Tensors

por

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Novembro/2020

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sob orientação do

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Tese apresentada ao Corpo Docente do Programa de Pós-Graduação em Matemática da Universidade Federal da Paraíba- UFPB, como requisito parcial para obtenção do título de Doutor em Matemática.

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† Este trabalho contou com apoio financeiro da CAPES

Catalogação na publicação
Seção de Catalogação e Classificação

C957o Cruz Júnior, Francisco Calvi da.
On r-trapped immersions in Lorentzian spacetimes and a weighted inequality for tensors / Francisco Calvi da Cruz Júnior. - João Pessoa, 2020.
103 f.

Orientação: Eraldo Almeida Lima Junior.
Coorientação: Allan George de Carvalho Freitas.
Tese (Doutorado) - UFPB/CCEN.

1. Matemática. 2. Rigidity. 3. r-trapped submanifolds. 4. GRW spacetime. 5. SSST. 6. De Lellis-Topping Inequality. 7. Weighted manifolds. 8. Bakry-Émery-Ricci tensor. 9. Drifting Laplacian. I. Lima Júnior, Eraldo Almeida. II. Freitas, Allan George de Carvalho. III. Título.

UFPB/BC CDU 51(043)

Elaborado por MARILIA RIANNY PEREIRA COSMOS - CRB-15/862

Universidade Federal da Paraíba

Programa de Pós-Graduação em Matemática

Doutorado em Matemática

Área de Concentração: Geometria Diferencial

Aprovada em: 10/11/2020

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Novembro/2020

Abstract

This work was divided into two moments: at first, we set out to study spacelike submanifolds Σ^n immersed in Lorentz spacetimes $\overline{M}^{n+p+1}$. So, we introduce the notion of r-trapped submanifolds as a generalization of the trapped submanifolds introduced by Penrose. In the case where the ambient space is a GRW $-I \times_\rho M^{n+p}$, considering some properties such as parabolicity and stochastic completeness we prove rigidity and nonexistence results for r-trapped in some configurations of GRW spacetimes and, lastly, we provide examples of r-trapped submanifolds, some of them are also simultaneously trapped, but we provided examples proving that the notion of r-trapped submanifolds are different accordingly to the number r. On the other hand, in the case where the ambient space is an standard static spacetime (SSST) $M^{n+p} \times_\rho \mathbb{R}_1$, we calculate the differential operators L_r and $L_{r,\phi}$ applied to the height function $h = \pi_{\mathbb{R}} \circ \psi$ of the immersion $\psi : \Sigma^n \to M^{n+p} \times_\rho \mathbb{R}_1$ and we consider some properties on Σ^n such as parabolicity and maximum principles. In this setting, we prove rigidity and nonexistence results for r-trapped spacelike submanifolds. After, we obtain some De Lellis-Topping type inequalities for general tensors under constraints in the Bakry-Émery Ricci tensor. In particular, we provide new results on manifolds with convex boundary, improving some known results given on manifolds with totally geodesic boundary. Furthemore, we apply our results in a class of locally conserved tensors.

Keywords: Rigidity, r-trapped submanifolds, GRW spacetime, SSST, De Lellis-Topping Inequality, weighted manifolds, Bakry-Émery-Ricci tensor, drifting Laplacian.

Resumo

Este trabalho foi dividido em dois momentos: no primeiro, nos dedicamos ao estudo de subvariedades tipo-espaço Σ^n imersas em espaços-tempo Lorentzianos $\overline{M}^{n+p+1}$. Assim, introduzimos a noção de subvariedades r-trapped como generalização das subvariedades trapped introduzidas por Penrose. No caso em que o espaço ambiente é um GRW $-I \times_\rho M^{n+p}$, considerando algumas propriedades como parabolicidade e completude estocástica, fornecemos resultados de rigidez e de não existência para subvariedades r-trapped em algumas configurações de espaços-tempo GRW e, por último, fornecemos exemplos de subvariedades r-trapped, onde algumas delas são trapped e outras não, comprovando que a noção de subvariedades r-trapped são diferentes de acordo com o número r. Por outro lado, no caso em que o espaço ambiente é um standard static spacetime (SSST) $M^{n+p} \times_\rho \mathbb{R}_1$, calculamos os operadores diferenciais L_r e $L_{r,\phi}$ aplicados à função altura $h = \pi_{\mathbb{R}} \circ \psi$ da imersão $\psi : \Sigma^n \to M^{n+p} \times_\rho \mathbb{R}_1$ e consideramos algumas propriedades em Σ^n como parabolicidade e princípios de máximo. Neste cenário, fornecemos resultados de rigidez e de não existência para subvariedades r-trapped. Depois, obtemos algumas desigualdades do tipo De Lellis-Topping para tensores gerais sob restrições no tensor Bakry-Émery Ricci. Em particular, fornecemos novos resultados em variedades com bordo convexo, melhorando alguns resultados conhecidos em variedades com bordo totalmente geodésico. Além disso, aplicamos nossos resultados em uma classe de tensores localmente conservativos.

Palavras-chave: Rigidez, subvariedades r-trapped, espaço-tempo GRW, SSST, Desigualdade De Lellis-Topping, variedades ponderadas, tensor Bakry-Émery-Ricci, Laplaciano ponderado.

Acknowledgements

First of all, I would like to thank my beloved wife, Bárbara Karolina Santana Arraes, for always being by my side during this entire journey, supporting me and helping me to overcome all the difficult moments (and there were many). I confess that without it - my safe haven - everything would be more difficult and for that reason, i will be forever grateful.

To my Mother, Maria Ilma Alves Evangelista, for her enormous dedication to educating me, for always demanding from me the best and, most important of all, for always believing in my success regardless of the journey I was on.

To my sister, Marta Alves Ribeiro, for always encouraging me to go beyond what i believed i could. It is also worth mentioning the complicity, affection and friendship that underlies our relationship.

To my brothers, Abraão Evangelista Cruz and Otávio Evangelista Cruz, for the countless happy and unforgettable moments we spent together. I have a lot of admiration, pride and respect for each one of them.

To my advisors, Prof. Dr. Eraldo Almeida Lima Júnior, Prof. Dr. Allan George de Carvalho Freitas and Prof. Dr. Márcio Silva Santos. Prof. Dr. Eraldo Lima, advisor, who contributed a lot with my academic formation with his valuable teachings and orientations, always very attentive and patient with my limitations. Prof. Dr. Allan George, co-advisor, who actively followed my first steps in the world of academic research, continuing to assist me throughout the journey, contributing with his guidance and his important advice and referrals. Prof. Dr. Márcio Santos, who, despite not having his official orientation listed in the legal procedures of UFPB, always accompanied all my research work and contributed effectively with his knowledge and experiences. In addition, it is worth highlighting the feeling of friendship and respect that prevailed throughout my walk on the course. I will be forever grateful to these three teachers and will take them with me as great friends.

To the members of the examining board for this work, Prof. Dr. Alfonso Romero Sarabia (UGR), Prof. Dr. Henrique Fernandes de Lima (UFCG), Prof. Dr. Jorge Herbert Soares de Lira (UFC), Prof. Dr. Márcio Henrique Batista da Silva (UFAL) and Rondinelle Marcolino Batista (UFPI), for dedicating a valuable part of their time to read, evaluate, correct and, based on their vast knowledge and experience, indicate valuable and important suggestions to be implemented in this work and some to be explored in future projects. To my course teachers, Prof. Dr. Adriano Medeiros, Prof. Dr. Ricardo Burity, Prof. Dr. Nacib Albuquerque and Prof. Dr. Alexandre Simas for the valuable teachings and for all the encouragement and support during the initial disciplines of the course.

To my great friend and longtime companion, Leon Dénis da Silva, for the countless advices, for the various productive and inspiring conversations and, of course, for the moments of relaxation that, in a way, helped to recharge my forces to continue the work . To my friend, Juscelino Pereira Silva, who in addition to encouraging me a lot, was instrumental in my removal process to pursue a doctorate. I have him as a great friend and as an inspiration.

To my friend, Flávio França Cruz, who always supported and guided me, always pointing a north when it seemed not to exist. To my friends of the master degree, and also of battle, José Deibson, Filipe Mendonça and Tiago Veras, for sharing good times and, together, overcoming all difficulties.

To the companions of the 2017.1 summer course, André Santana Dosea and Ginaldo de Santana Sá, for all the support and help that, in turn, were fundamental in the approval of the selection process for the course in question.

To fellow students Antonival Lopes do Nascimento Filho, Daniel Tomaz de Araujo, Dayane Santos de Lira, Diogo Santana Germano, Fernanda Roing, João Henrique Santos de Andrade, Jonison Lucas dos Santos Carvalho, Kleber Soares Camara, Laise Dias Alves Araújo, Marcos Aurélio Guimarães Monteiro, Oslenne Nogueira de Araújo, Renato Bezerra Silvestre and Thiago Fiel da Costa Cabral, for the intense moments of study and for all the support in difficult times.

To my friends, Lucas de Carvalho Nascimento and José Sérgio Abrantes Furtado, for the many moments of relaxation, for the enlightening conversations/discussions, for the healthy and respectful disputes and for the sincere and true friendship.

To the work colleagues Allana Kellen Lima Santos Pereira, André Wesley Barbosa Rodrigues, Ary Ferreira da Silva, Clarice Dias Albuquerque, Maria Silvana Alcantara Costa, Placido Francisco de Assis Andrade, Vicente Helano Feitosa Batista Sobrinho.

"Even when God seemed to have abandoned me, He was watching. Even when He seemed indifferent to my suffering, He was watching. And when I was beyond all hope of saving, He gave me rest. Then He gave me a sign to continue my journey"

Yann Martel

Dedicatory

To my wife, Bárbara Arraes.

To my mother, Maria Ilma.

To my siblings, Marta, Otávio and Abraão.

Contents

Introduction

The objective here is to give an overview of the results that motivated our work. At first, we dedicate to the study of the r-th mean curvature of spacelike submanifolds immersed in spactimes (which, in this case, were the generalized Robertson-Walker spacetime and the standard static spacetime). After that, we turn our attention to De Lellis - Topping type inequalities for general tensors under constraints in the Bakry-Émery Ricci tensor on weighted Riemannian manifolds with convex boundary.

As a starting point, we dedicate ourselves to the study of the spacelike submanifolds immersed in Lorentz spacetimes. In this context, de Lima, Santos and Velásquez [43] obtained rigidity for trapped submanifolds in Lorentzian spaces forms, they considered assumptions such as parallel mean curvature and pseudo-umbilicity. Later, Alías, Cánovas and Colares [1], considered codimension two trapped submanifolds Σ^n immersed in generalized Robertson-Walker spacetimes $-I\times_\rho M^{n+1}$ and obtained nonexistence and rigidity results. In this work, they used the Laplace-Beltrami operator of Σ^n and obtained the following equation

$$\Delta\sigma(h) = n\big(-\rho'(h) + \rho(h)\langle\vec{H},\partial_t\rangle\big),$$

where h is the height function of Σ^n in $-I\times_\rho M^{n+1}$ and σ is a primitive of ρ. It is important to note that a causal orientation of the mean curvature vector field $\vec{H}$ plays an important role in the study of Laplacian of $\sigma(h)$ and, therefore, constitutes a valuable tool for the work in question. On the other hand, working in a similar context, Alías, Impera and Rigoli [4], analyzed the problem of uniqueness for spacelike hypersurfaces Σ^n with constant mean order curvature immersed in generalized Robertson-Walker spacetimes $-I\times_\rho M^n$. However, in this case, they turned their attention to the differential operator L_r and obtained the following equation

$$L_r\sigma(h) = -k(r)\left(\rho'(h)H_r + \rho(h)H_{r+1}\langle N, \partial_t\rangle\right),$$

where $k(r) = (n-r)\binom{n}{r}$ is a constant and N is the unique unitary timelike normal vector field globally defined on Σ^n with the same orientation as ∂_t. Following the same line, but in a different ambient space, Freitas *et al* [22] obtained

$$\Delta h = -2\langle \nabla \ln \rho, \nabla h\rangle + \frac{1}{\rho^2}\langle \vec{H}, \partial_t\rangle$$

and, with this, studied trapped submanifolds immersed in standard static spacetimes and established sufficient conditions to guarantee that such a spacelike submanifold must be a hypersurface of the Riemannian base of the ambient spacetime, particularly, they showed that there do not exist n-dimensional compact (without boundary) trapped submanifolds immersed in an $(n+2)$-dimensional standard static spacetime which is a classical result due to Mars and Senovilla [31] (see also [45]).

For the r-th mean curvature of spacelike submanifolds immersed in Lorentz spacetimes, which will be generalized Robertson-Walker spacetime and standard static spacetime (see chapters 3 and 4, respectively), we introduce the concept of r-trapped submanifolds which generalizes the definition of trapped submanifold in the sense that 0-trapped coincides with trapped submanifold (see section 2.2). Then, studying the behavior of spacelike submanifolds Σ^n immersed in a generalized Robertson-Walker $-I \times_\rho M^{n+p}$ (respec., standard static spacetimes $M^{n+p} \times_\rho \mathbb{R}_1$) using the causal orientation of their $(r+1)$-th mean curvature $\vec{H}_{r+1}$ wih $0 \leq r < n$ even, we calculate the differential operator L_r applied to the height function h of Σ^n in $-I \times_\rho M^{n+p}$ (respec., $M^{n+p} \times_\rho \mathbb{R}_1$) and we got

$$L_r(h) = -(\rho'(h)/\rho(h))k(r)H_r - (\rho'(h)/\rho(h))\, T_r\left(\nabla h, \nabla h\right) + k(r)\langle \vec{H}_{r+1}, \partial_t\rangle,$$

and

$$L_r\sigma(h) = k(r)\left(-\rho'(h)H_r + \rho(h)\langle \vec{H}_{r+1}, \partial_t\rangle\right).$$

Therefore, with these tools, we obtain rigidity and nonexistence results. Since L_r is a differential operator, the above equations allow an analysis of the spacelike submanifold Σ^n through the causal orientation of the vector field $\vec{H}_{r+1}$, when $0 \leq r < n$ is even. For example, when Σ^n is closed, we get results like:

Theorem A *Let $-I \times_\rho M^{n+p}$ be a GRW spacetime and $0 \leq r < n$ even.*

(i) If $\mathcal{H}(t) \geq 0$, there exist no closed weakly past $r-$trapped submanifold in $-I \times_\rho M^{n+p}$ such that $T_r \geq 0$ and $H_r > 0$.

(ii) If $\mathcal{H}(t) \leq 0$, there exist no closed weakly future $r-$trapped submanifold in $-I \times_\rho M^{n+p}$ such that $T_r \geq 0$ and $H_r > 0$.

However, when Σ^n is noncompact, we can evoke the concepts of parabolicity, stochastic completeness and maximum principles. For instance, we have the following

Theorem B *Let $-I \times_\rho M^{n+p}$ be a GRW spacetime and $0 \leq r < n$ even.*

(i) Let $t^* \in I$ and assume that $\mathcal{H}(t) > 0$ for $t \leq t^*$. Then there exist no weakly past r-trapped complete, non-compact spacelike submanifold bounded away from the future infinity at height t^* immersed into $-I \times_\rho M^{n+p}$ satisfying the condition (3.5) *and such that $T_r \geq 0$, $\sup_\Sigma trT_r < +\infty$ and $H_r > a > 0$, for some constant a.*

(ii) Let $t_* \in I$ and assume that $\mathcal{H}(t) < 0$ for $t \geq t_*$. Then there exist no weakly future r-trapped complete, non-compact spacelike submanifold bounded away from the past infinity at height t_* immersed into $-I \times_\rho M^{n+p}$ satisfying the condition (3.5) *and such that $T_r \geq 0$, $\sup_\Sigma trT_r < +\infty$ and $H_r > a > 0$, for some constant a.*

On the other hand, when the ambient space is a standard static spacetime, we get

$$L_r(h) = -2T_r(\nabla \ln(\rho), \nabla h) + \frac{1}{\rho^2}k(r)\langle \vec{H}_{r+1}, K\rangle.$$

Moreover, using the divergent operator $L_{r,\phi}(\cdot) := \mathrm{div}_\phi\Big(T_r\big(\nabla(\cdot)\big)\Big)$ with $\phi = -2\ln\rho$, we obtain

$$L_{r,\phi}(h) = \frac{1}{\rho^2}k(r)\langle \vec{H}_{r+1}, K\rangle.$$

By the concepts of the principles of maximum and parabolicity for operators L_r and $L_{r,\phi}$, we obtain nonexistence and some rigidity results. In this context we emphasize the following results:

Theorem C *Let $\overline{M}_c^{n+p+1} = M^{n+p} \times_\rho \mathbb{R}_1$ be a standard static spacetime with constant sectional curvature c, $0 \leq r < n$ even and consider $\phi = -2\ln\rho$. Then* *(i) There do not exist n-dimensional spacelike, $L_{r,\phi}$-parabolic, future (or past) r-trapped and bounded away from the future (or past) infinity submanifolds immersed in $\overline{M}_c^{n+p+1}$;*

(ii) There do not exist n-dimensional spacelike, $L_{r,\phi}$-parabolic, marginally future (or past) r-trapped and bounded away from the future (or past) infinity submanifolds immersed in $\overline{M}_c^{n+p+1}$.

(iii) The n-dimensional spacelike $L_{r,\phi}$-parabolic, weakly future (or past) r-trapped and bounded away from the future (or past) infinity submanifolds immersed in $\overline{M}_c^{n+p+1}$ are r-minimal.

and,

Theorem D *Let $\overline{M}^{n+p+1} = M^{n+p} \times_\rho \mathbb{R}_1$ be a standard static spacetime such that ρ and $\nabla\rho$ are bounded and let $\psi : \Sigma^n \longrightarrow \overline{M}^{n+p+1}$ be a complete, non-compact spacelike submanifold with bounded second fundamental form and whose radial sectional curvature satisfies the condition* (1.11)*. Moreover assume that Σ^n is bounded away from the future infinity and, for some $0 \leq r \leq n-1$ even, suppose that $\sup_\Sigma trT_r < +\infty$, $T_r \geq 0$ and $H_r > 0$. Then Σ^n cannot be past r-trapped nor marginally past r-trapped. Particularly, if Σ^n is weakly past r-trapped then Σ^n must be r-minimal.*

It is important to highlight that, in this more general context, we encompass many of the results cited above (as well as some works that were opportunely cited in the course of chapters 3 and 4), considering that

$$\begin{cases} \vec{H}_{r+1} \text{ coincides with the mean curvature } \vec{H} \text{ of the submanifold when } r = 0; \\ \text{The concept of } r\text{-trapped coincides with that of trapped submanifolds when } r = 0; \\ L_r \text{ coincides with the Laplace-Beltrami operator when } r = 0; \\ \text{The codimension of the submanifold is given by } p+1 \text{ with } p \text{ a non-negative integer.} \end{cases}$$

In addition, to emphasize the importance of this new concept of r-trapped submanifold, we provide examples that demonstrate its independence from the definition of trapped submanifold that already exists in the literature (see example 3.5.1).

In a second part, we study almost-Schur type results on weighted manifold. Schur's lemma states that every Einstein manifold of dimension $n \geq 3$ has constant scalar curvature. With that in mind, De Lellis and Peter Topping [20] asked to what extent the scalar curvature is constant if the traceless Ricci tensor is assumed to be small rather than identically zero and, with this, they obtained the following result in the context of closed Riemannian manifolds:

"*Let* (Σ^n, g) *be a closed Riemannian manifold of dimension* $n \geq 3$*, with nonnegative Ricci curvature. Then*

$$\int_{\Sigma}(R-\overline{R})^2 dv_g \leq \frac{4n(n-1)}{(n-2)^2}\int_{\Sigma}|Ric-(R/n)g|^2\, dv_g,$$

where $\overline{R} = \frac{1}{Vol(\Sigma)}\int_{\Sigma} R dv_g$ *is the average value of* R *over* Σ^n*. Furthermore, the equality occurs if and only if* (Σ^n, g) *is an Einstein manifold.*"

For closed Riemannian manifolds, Cheng generalized the work of De Lellis and Topping in two ways: first replacing the hypothesis of non-negativity of the Ricci curvature with the more general condition $Ric \geq -(n-1)K$ (see [16]), for some positive constant K, and then she obtained a De Lellis-Topping type inequality for a symmetric tensor T that satisfies condition $\operatorname{div} T = c\nabla B$, where c is a constant and $B = trT$ (see [15]). Turning her attention to compact Riemannian manifolds with totally geodesic boundary (i.e., a Riemannian manifold M with umbilical boundary ∂M and whose mean curvature H of the immersion $\partial M \hookrightarrow M$ is zero), Ho [28] obtained the same inequality as De Lellis and Topping under the hypothesis of non-negative Ricci curvature. Finally, in a more general context, we can mention the works of Chen [14], Huang and Zeng [29], Meng and Zhang [32] and Wu [49] that address De Lellis-Topping type inequalities for weighted manifolds under a new condition of limitation for the Bakry-Émery Ricci tensor.

With this in mind, we set out to study De Lellis - Topping type inequalities for symmetric tensors T that satisfy second Bianchi's type identity on weighted manifolds $(M, g, e^{-f}dv)$ with convex boundary.

Theorem E *Let* $(\Sigma^n, g, e^{-f}dv)$ *be a compact* n*-dimensional weighted manifold with* $n \geq 3$*, convex boundary* $\partial\Sigma$ *and* $f : \Sigma^n \longrightarrow \mathbb{R}$ *a smooth function such that* $(\partial f/\partial\nu) \equiv 0$ *on* $\partial\Sigma$*, where* ν *is the exterior unit normal vector field along* $\partial\Sigma$*. Let* T *be a symmetric* $(0,2)$*-tensor field such that* $T(\nu,\cdot) \geq 0$ *along the boundary and* $\operatorname{div} T = c\nabla B$*, where* $c \in \mathbb{R}$ *is a constant and* $B = tr_g T$ *denotes the trace of* T *with respect to* g*. If* $Ric_f \geq -(n-1)K_1 g$*, where* $K_1 \geq 0$ *is a constant, and* $K_2 := \max_{x\in M}\Delta f(x)$*, then*

$$(nc-1)^2\int_{\Sigma}\left(B-\overline{B}\right)^2 e^{-f}dv \leq n^2\left(\frac{(n-1)K_1+K_2}{\lambda_1}+\frac{n-1}{n}\right)\int_{\Sigma}|\overset{\circ}{T}|^2 e^{-f}dv,$$

where $\overline{B} = \left(\int_{\Sigma} Be^{-f}dv\right)/\left(\int_{\Sigma} e^{-f}dv\right)$ *is the weighted average value of the* B *over* Σ^n*,* λ_1 *is the first nonzero eigenvalue for weighted Laplacian with Neumann boundary condition and* $\overset{\circ}{T} = T - \left(tr_g T/n\right)g$ *denotes traceless part of the tensor field* T*. Moreover, assuming positivity of Ricci curvature, the equality holds if and only if* f *is constant and* $\overset{\circ}{T} = 0$*.*

On the other hand we obtain a De Lellis-Topping type inequality with weighted objects, that is, we have:

Theorem F *Let $(\Sigma^n, g, e^{-f}dv)$ be a compact smooth metric measure space with $n \geq 3$, convex boundary $\partial\Sigma$ and $f : \Sigma^n \longrightarrow \mathbb{R}$ smooth and such that $(\partial f/\partial\nu) \equiv 0$ on $\partial\Sigma$. Let T be a symmetric $(0,2)$-tensor field such that $\operatorname{div} T = c\nabla B$ and $T(\nu, \cdot) \geq 0$ along of the boundary, where $c \geq 0$ is a constant and $B = tr_g T$. If $Ric_f \geq (\Delta f - (n-1)K)g$, where $K \geq 0$ is a constant, then*

$$\begin{aligned}\int_\Sigma \left(B_f - \overline{B}_f\right)^2 e^{-f} dv \;\; &\leq \;\; \frac{n^2}{(nc-1)^2}\left(\frac{(n-1)K}{\lambda_1} + \frac{n-1}{n}\right)\int_\Sigma |\overset{\circ}{T}_f - \nabla^2 f|^2 e^{-f} dv \\ &\quad + \int_\Sigma (\Delta f)^2 e^{-f} dv,\end{aligned}$$

where $T_f = T + \nabla^2 f$ and $B_f = tr_g T_f$. Moreover, the equality holds if and only if f is constant and $\overset{\circ}{T} = 0$.

With this in mind, we divided this work into 5 chapters. In chapter 1, we provide some concepts and some important results for the development of our research. In addition to the differential operators, Schur's Lemma and the extension of Reilly's formula to weighted manifolds, we approach concepts of the maximum principles and parabolicity in the classic versions for more general operators such as L_r. In chapter 2, we provide the definition of r-th mean curvature of spacelike submanifold Σ^n immersed in spacetime $\overline{M}^{n+m}$ of the Newton transformations T_r and of the differential operator L_r. With this, we introduce the new concepts of r-trapped submanifolds according to the causal orientation of the vector field $\vec{H}_{r+1}$ for $0 \leq r < n$ even. In chapter 3, we explored the n-dimensional r-trapped submanifolds contained in slices from the ambient space $-I \times_\rho M^{n+1}$. With this, we provide a condition for whether or not such a submanifold is r-trapped (see equation 3.3). Right after that, we will calculate the action of the operator L_r in the height function $h = \pi_I \circ \psi$ and in a primitive function σ of the warping function ρ in the search for a tool to help our results. With this tool in mind, we address some results of non-existence and rigidity. Finally, we provide examples of r-trapped submanifolds. It is important to note that the concepts of trapped and r-trapped submanifolds are independent and that the second generalizes the first, since 0-trapped submanifolds coincides with the trapped in the classic sense (see example 3.5.1). For chapter 4, the idea is to study the spacelike submanifolds immersed in a standard static spacetime. In this way, we restrict ourselves to the spacelike submanifolds $\psi : \Sigma \to M^{n+p} \times_\rho \mathbb{R}_1$ and obtain results of rigidity and nonexistence under the hypothesis of causal orientation for the $(r+1)$-th mean curvature $\vec{H}_{r+1}$, with $0 \leq r < n$ even. At first, we calculate $L_r(h)$ and, in addition, we provide a result that guarantees, under some hypotheses, the Omori-Yau maximum principle for the Laplacian (see Lemma 4.1.2). In section 4.2, we discuss some results of nonexistence and rigidity for r-trapped, as well marginally and weakly r-trapped, submanifolds immersed in a standard static spacetime $M^{n+p} \times_\rho \mathbb{R}_1$. In the next section, we explore the definition of weighted divergence (or, more precisely, of ϕ-divergent for some smooth function ϕ on Σ^n) and, under the hypothesis of constant sectional curvature of the ambient space $M^{n+p} \times_\rho \mathbb{R}_1$, we use the differential operator $L_{r,\phi}$ (see equation (4.4) and continue to obtain results of non-existence and rigidity. In section 4.4, we follow the same idea as in the previous section, but this time making use of the principle of maximum for both Laplacian and differential operator L_r. We dedicate the section 4.5 to study the particular case of some of the results in chapters 3 and 4 when the warping function satisfies $\rho \equiv 1$, i.e., we turn our attention to spacelike submanifolds immersed in the product manifold $-I \times M^{n+p}$ (which, in turn, is both a GRW and an SSST). And, likewise, we end the chapter by providing examples of r-trapped submanifolds. Lastly, in chapter 5 we propose study these type inequalities on weighted manifolds with constraints in the Bakry-Émery Ricci tensor. In section 5.1 we enunciate and demonstrate the main results of this chapter (see Theorems 5.1.1 and 5.1.4) and, in addition, we obtain direct corollaries. Finally, in section 5.2, we provide some applications of the main results.

Summary of Basic Notation

1. Let (M, g) be an n-dimensional Riemannian manifold with metric g, $u, f : M \to \mathbb{R}$ are smooth functions, X, Y, Z and W vector fields on M and T a tensor field on M.

   ∇: Levi-Civita connection of M

   ∇T: covariant derivative of T

   $B = tr_g T$: trace with respect metric g of T

   $\overline{B}$: overage value of B over M

   $\overset{\circ}{T}$: traceless part of T

   ∂M: boundary of M

   $\mathfrak{X}(M)$: space of vector fields on M

   $C^\infty(M)$: space of C^∞ functions

   ∇u: gradient of u

   du: differential of u

   $\nabla^2 u$: hessian of u

   Δu: laplacian of u

   $\operatorname{div} X$: divergence of X

   $R(X, Y)Z$: curvature endomorphism

   $R(X, Y, Z, W)$: curvature tensor

   $[X, Y]$: Lie brackets of X and Y

   Ric: Ricci curvature

   R: scalar curvature

   $\overline{R}$: average value of R over M

   dv_g: volume element of M

   $d\mu_g$: volume element of ∂M

   $\Delta_f u$: weighted laplacian of u

   $\mathcal{L}_X$: Lie derivative with respect to X

   $\operatorname{div}_f X$: f-divergence of X

   $\operatorname{div}_f T$: f-divergence of T

   Ric_f: Bakry-Émery Ricci tensor

   K_M^{rad}: radial sectional curvature of M

2. Looking at M as a submanifold immersed in a $(n+m)$-dimensional semi-Riemannian manifold $\overline{M}$, we have

   $\overline{\nabla}$: Levi-Civita connection of $\overline{M}$

   $\alpha(X, Y)$: second fundamental form of M

   H: mean curvature of M

   H_r: r-th mean curvature of M

   K_G^M: Gauss-Kronecker curvature of M

   T_r: r-th Newton transformation of α

Chapter 1

Fundamentals of Geometric Analysis

The purpose of this chapter is to establish notations and provide some tools that, in large part, will be focused on our results in chapter 5. The section 1.1, because it deals with differential operators in Riemannian manifolds, will also support some of our results from chapters 5 and 6, since they are focused on the study of spacelike submanifolds immersed in spacetimes. In what follows, section 1.2 provides Bianchi's identities. The results contained in chapter 5 are, in a way, a generalization of Schur's Lemma and, for this reason, such identities are necessary. Sections 1.3 and 1.4 are linked, since the first serves as a supposition for the second. More precisely, we introduced in section 4 the concept of weighted manifolds and extended the formulas of Böchner and Reilly in the section for this ambient. It is worth mentioning that such formulas are essential tools in the demonstration of Theorems 5.1.1 and 5.1.4. In addition, we have introduced some weighted differential operators that will be used in chapters 3, 4 and 5.

In order to provide support for results in chapters 3 and 4 that address non-compact manifolds, we introduce the concepts of Omori-Yau maximum principle and parabolicity. In addition, we provide a version of the maximum principle for the operator L defined in (1.10). In a natural way, we approach results that, under certain hypotheses, imply these principles. Then, in the next section, we define the concept of parabolicity. Thus, taking advantage of the definition of weighted divergent introduced in chapter 1 (see section 1.4), we define the operator L_ϕ and, with it, the concept of L_ϕ-parabolicity. Finally, we present conditions for which a given manifold to be L_ϕ-parabolic.

1.1 Differential Operators

Let Σ^n be a smooth manifold equipped with a Riemannian metric $g : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to C^\infty(\Sigma)$ (which we will sometimes denote by $\langle \cdot, \cdot \rangle$) and Riemannian connection ∇. In addition, to fix the notations, denote by $\{e_1, e_2, ..., e_n\}$ a local orthonormal frame on Σ^n. Given a smooth function $u : \Sigma^n \to \mathbb{R}$, we define the *gradient* of u as the vector field ∇u given by

$$\langle \nabla u, X \rangle = X(u), \quad \forall X \in \mathfrak{X}(\Sigma).$$

However, note that

$$\begin{array}{rcl} X(u) = du(X) : M & \to & \mathbb{R} \\ p & \mapsto & \big(du(X)\big)(p) := du_p\big(X_{|_p}\big) \end{array}$$

and, with this,

$$\langle \nabla u, X \rangle = du(X), \ \forall X \in \mathfrak{X}(\Sigma).$$

Therefore, using musical isomorphism $(\cdot)^\flat : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)^*$, we can identify the field ∇u with the $(0,1)$-tensor du as follows

$$\big(\nabla u\big)^\flat = du.$$

Remark 1.1.1 It is natural to omit musical isomorphism " $\flat$ " and use ∇f to denote both the vector field ∇u and the differential du.

According to the definition, if $u, v : \Sigma^n \to \mathbb{R}$ and $\xi : \mathbb{R} \to \mathbb{R}$ are smooth functions, then

$$\begin{cases} \nabla\big(u \pm v\big) = \nabla u \pm \nabla v; \\ \nabla\big(uv\big) = u\nabla v + v\nabla u; \\ \nabla\big(\xi \circ u\big) = \xi'\big(u\big)\nabla u. \end{cases} \tag{1.1}$$

Given a vector field $\mathfrak{X}(\Sigma)$, we can use the connection and define an operator $\nabla X : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ given by

$$Y \in \mathfrak{X}(\Sigma) \mapsto \nabla_Y X.$$

In this context, we define the *hessian* of a smooth function $u : \Sigma^n \to \mathbb{R}$ as being the $(0,2)$-tensor $\nabla^2 u : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to C^\infty(\Sigma)$ whose metrically associated $(1,1)$-tensor is precisely the operator ∇u, that is,

$$(X, Y) \in \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \mapsto \langle \nabla_X \nabla u, Y \rangle.$$

Since, for all X and Y in $\mathfrak{X}(\Sigma)$,

$$\begin{aligned}
\langle \nabla_X \nabla u, Y \rangle &= X\langle \nabla u, Y \rangle - \langle \nabla u, \nabla_X Y \rangle \\
&= X\big(Y(u)\big) - \langle \nabla u, \nabla_Y X + [X, Y] \rangle \\
&= Y\big(X(u)\big) + [X, Y](u) - \langle \nabla u, \nabla_Y X + [X, Y] \rangle \\
&= Y\big(X(u)\big) - \langle \nabla u, \nabla_Y X \rangle \\
&= \langle \nabla_Y \nabla u, X \rangle,
\end{aligned}$$

we have that $\nabla^2 u$ is a symmetric tensor.

From the definition of Hessian given above, we define the *laplacian* of a smooth function $u : \Sigma^n \to \mathbb{R}$ as the function $\Delta u \in C^\infty(\Sigma)$ given by

$$\Delta u := tr\big(\nabla^2 u\big).$$

So, for smooth functions $u, v : \Sigma^n \to \mathbb{R}$ and $\xi : \mathbb{R} \to \mathbb{R}$, it follows that

$$\begin{cases} \Delta\big(uv\big) = u\Delta v + v\Delta u + 2\langle \nabla u, \nabla v \rangle; \\ \Delta\big(\xi \circ u\big) = \big(\xi''(u)\big)|\nabla u|^2 + \big(\xi'(u)\big)\Delta u. \end{cases} \tag{1.2}$$

The next differential operator to be defined is the divergent of a vector field and, for this, we will make use of the musical isomorphism $(\cdot)^\flat : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)^*$ (which, as already mentioned, will be omitted). Thus, given a field of vectors $X \in \mathfrak{X}(\Sigma)$, we define its *divergent* as the function $\operatorname{div}\big(X\big) \in C^\infty(\Sigma)$ given by

$$\begin{aligned}
\operatorname{div}\big(X\big) &= tr\big(\nabla X(\cdot)\big) \\
&= \sum_{i=1}^{n} \langle \nabla_{e_i} X, e_i \rangle.
\end{aligned}$$

Immediately, from the previous definition, it follows that

$$\Delta u = \operatorname{div}\big(\nabla u\big).$$

Moreover, if $u : \Sigma^n \to \mathbb{R}$ is a smooth function and $X, Y \in \mathfrak{X}(\Sigma)$, then

$$\begin{cases} \operatorname{div}\big(X \pm Y\big) = \operatorname{div}\big(X\big) \pm \operatorname{div}\big(Y\big); \\ \operatorname{div}\big(uX\big) = u \operatorname{div}\big(X\big) + \langle \nabla u, X\rangle. \end{cases} \tag{1.3}$$

Inspired by the previous definition, consider a symmetric $(0,2)$-tensor $T : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to C^\infty(\Sigma)$ and, also denoting by T its associated $(1,1)$-tensor, note that

$$\begin{aligned} \operatorname{div}\big(T(X)\big) &= \sum_{i=1}^{n} \langle \nabla_{e_i} T(X), e_i\rangle \\ &= \sum_{i=1}^{n} \Big\langle \Big(\nabla_{e_i} T\Big)(X), e_i\Big\rangle + \sum_{i=1}^{n} \langle T(\nabla_{e_i} X), e_i\rangle \\ &= tr\big(\nabla T(\cdot, X)\big) + \sum_{i=1}^{n} \langle \nabla_{e_i} X, T(e_i)\rangle \\ &= tr\big(\nabla T(\cdot, X)\big) + \langle \nabla X, T\rangle. \end{aligned}$$

So, we define the *divergent* of T as being the $(0,1)$-tensor given by

$$X \in \mathfrak{X}(\Sigma) \mapsto \operatorname{div}\big(T\big)(X) = tr\big(T(\cdot, X)\big).$$

In coordinates, we have

$$\begin{aligned} \operatorname{div}\big(T\big)_i &= \sum_{j,k=1}^{n} g^{jk} \nabla_k T_{ij} \\ &= \sum_{j=1}^{n} \nabla_j T_{ij}, \end{aligned}$$

where $\nabla_k T_{ij} = \big(\nabla_{e_k} T\big)(e_i, e_j)$.

With this definition, we will say that a symmetric $(0,2)$-tensor T is *locally conserved* if it is divergence free, i.e., $\operatorname{div}\big(T\big) = 0$.

1.2 Bianchi Identities and the Schur's Lemma

With the definitions of curvature endomorphism $R : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ and curvature tensor $R : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to C^\infty(\Sigma)$ (which we already admit to be known to the reader), we can highlight the following properties

$$R_{ijk} + R_{jki} + R_{kij} = 0$$

and

$$\nabla_i R_{jkl} + \nabla_j R_{kil} + \nabla_k R_{ijl} = 0,$$

where $R_{ijk} = R(e_i, e_j)e_k$ and $\nabla_i R_{jkl} = (\nabla_{e_i} R)(e_j, e_k, e_l)$. Such properties are known as *Bianchi's first and second identity*, respectively.

Remember that, from the curvature tensor $R : \mathfrak{X}(\Sigma)^4 \to C^\infty(\Sigma)$, we can introduce the it Ricci curvature tensor by

$$\begin{aligned}(X, X) \in \mathfrak{X}(\Sigma)^2 \longmapsto Ric(X, X) &= tr\big(R(\cdot, X, X, \cdot)\big) \\ &= \sum_{i=1}^{n} R(e_i, X, X, e_i) \\ &= \sum_{i=1}^{n} \langle R(e_i, X)X, e_i \rangle .\end{aligned}$$

From there, we define the scalar curvature as being the following function

$$\begin{aligned}p \in \Sigma^n \longmapsto R(p) &= tr\big(Ric(\cdot, \cdot)\big) \\ &= \sum_{j=1}^{n} Ric(e_j, e_j).\end{aligned}$$

In view of the punctual character of the tensors, we can ask that the referential $\{e_1, e_2, ..., e_n\}$ be geodesic at a point $p \in \Sigma^n$ and, at this point, obtain

$$\begin{aligned}\nabla R(e_k) &:= e_k(R) \\ &= e_k\left(\sum_{i=1}^{n} Ric(e_i, e_i)\right) \\ &= \sum_{i,j=1}^{n} e_k \langle R_{jii}, e_j \rangle \\ &= \sum_{i,j=1}^{n} \langle \nabla_k R_{jii}, e_j \rangle .\end{aligned}$$

Using the anti-symmetry property of the curvature tensor and Bianchi's second identity, it follows that

$$
\begin{aligned}
\nabla R(e_k) &= -\sum_{i,j=1}^{n} \langle \nabla_k R_{iji}, e_j \rangle \\
&= \sum_{i,j=1}^{n} \langle \nabla_i R_{jki}, e_j \rangle + \sum_{i,j} \langle \nabla_j R_{kii}, e_j \rangle \\
&= \sum_{i,j=1}^{n} \langle \nabla_i R_{jki}, e_j \rangle + \sum_{i,j} \langle \nabla_j R_{ikj}, e_i \rangle .
\end{aligned}
$$

On the other hand, still at point p,

$$
\begin{aligned}
\operatorname{div}\big(Ric\big)(e_k) &= \sum_{i=1}^{n} \langle (\nabla_i Ric)\, e_k, e_i \rangle \\
&= \sum_{i,j=1}^{n} \langle \nabla_i R_{jki}, e_j \rangle .
\end{aligned}
$$

Therefore, we obtain the *second contracted Bianchi identity* given by

$$
\nabla R = 2 \operatorname{div}\big(Ric\big). \tag{1.4}
$$

With these tools, we are already in a position to glimpse Schur's Lemma. However, before that, we need to introduce the concept of Einstein manifold. In this context, we will say that the Riemannian manifold (Σ^n, g) is *Einstein* if it is Ricci tensor satisfies

$$
Ric = \frac{R}{n} g. \tag{1.5}
$$

So, for Einstein manifolds, we have

$$
\begin{aligned}
Ric = \frac{R}{n} g \quad &\Rightarrow \quad \operatorname{div}\big(Ric\big) = \frac{1}{n} \nabla R \\
&\Rightarrow \quad \frac{1}{2} \nabla R = \frac{1}{n} \nabla R.
\end{aligned}
$$

With this, we have the following

Lemma 1.2.1 (Schur's Lemma) *If (Σ^n, g) is Eintein with $n \geq 3$, then M^n has constant scalar curvature.*

1.3 Böchner Formula and Reilly Formula

This section is justified by the fact that, in the next section, we use the classical case to obtain extended versions of the Böchner and Reilly formulas for weighted manifold.

Lemma 1.3.1 (Böchner formula) *Let (Σ^n, g) be a Riemannian manifold and consider $u \in C^\infty(\Sigma)$. Then*

$$\frac{1}{2}\Delta|\nabla u|^2 = Ric(\nabla u, \nabla u) + \langle \nabla u, \nabla(\Delta u)\rangle + |\nabla^2 u|^2.$$

Proof. Take a point $p \in \Sigma^n$ and consider a local orthonormal frame $\{e_1, ..., e_n\}$ that is geodetic in p. So, in p, we have to

$$\begin{aligned}
\frac{1}{2}\Delta|\nabla u|^2 &= \frac{1}{2}\sum_{i=1}^{n}\big(\nabla^2|\nabla u|^2\big)(e_i, e_i) \\
&= \sum_{i=1}^{n} e_i\big(e_i\,\langle \nabla u, \nabla u\rangle\,\big) \\
&= \sum_{i=1}^{n} e_i\,\langle \nabla_{e_i}\nabla u, \nabla u\rangle \\
&= \sum_{i=1}^{n}\langle \nabla_{e_i}\nabla_{e_i}\nabla u, \nabla u\rangle + \sum_{i=1}^{n}|\nabla_i\nabla u|^2 \\
&= \sum_{i=1}^{n}\langle \nabla_{e_i}\nabla_{e_i}\nabla u, \nabla u\rangle + |\nabla^2 u|^2.
\end{aligned}$$

Now, for $X \in \mathfrak{X}(\Sigma)$, we have to

$$\begin{aligned}
Ric(X, \nabla u) &= \sum_{i=1}^{n}\langle R(e_i, X)\nabla u, e_i\rangle \\
&= \sum_{i=1}^{n}\big\langle \nabla_{e_i}\nabla_X\nabla u - \nabla_X\nabla_{e_i}\nabla u - \nabla_{[e_i,X]}\nabla f, e_i\big\rangle\,.
\end{aligned}$$

However, at point p,

$$\begin{aligned}
\sum_{i=1}^{n}\langle \nabla_X\nabla_{e_i}\nabla u, e_i\rangle &= \sum_{i=1}^{n} X\,\langle \nabla_{e_i}\nabla u, e_i\rangle \\
&= \sum_{i=1}^{n} X\,\langle \nabla\nabla u(e_i), e_i\rangle \\
&= X(\Delta u) \\
&= \langle X, \nabla(\Delta u)\rangle
\end{aligned}$$

and, in addition,

$$\begin{aligned}
\left\langle \nabla_{e_i}\nabla_X\nabla u - \nabla_{[e_i,X]}\nabla u, e_i \right\rangle &= e_i \left\langle \nabla_X \nabla u, e_i \right\rangle - \left\langle \nabla\nabla u([e_i,X]), e_i \right\rangle \\
&= e_i \left\langle \nabla\nabla u(X), e_i \right\rangle - \left\langle \nabla\nabla u(e_i), [e_i,X] \right\rangle \\
&= e_i \left\langle \nabla\nabla f(e_i), X \right\rangle - \left\langle \nabla\nabla f(e_i), [e_i,X] \right\rangle \\
&= e_i \left\langle \nabla_{e_i}\nabla u, X \right\rangle - \left\langle \nabla_{e_i}\nabla u, \nabla_{e_i}X - \nabla_X e_i \right\rangle \\
&= \left\langle \nabla_{e_i}\nabla_{e_i}\nabla u, X \right\rangle .
\end{aligned}$$

Therefore,

$$Ric(X,\nabla u) = \sum_{i=1}^{n} \left\langle \nabla_{e_i}\nabla_{e_i}\nabla u, X \right\rangle - \left\langle X, \nabla(\Delta u) \right\rangle$$

and the result follows in the case where $X = \nabla u$. ■

Remember that, under the assumption that Σ^n is a compact Riemannian manifold with boundary, the boundary of Σ^n is a $(n-1)$-dimensional Riemannian manifold $\partial\Sigma^{n-1}$. Looking at the inclusion map $i : \partial\Sigma^{n-1} \hookrightarrow \Sigma^n$, we denote by ν the unitary normal outside Σ^n along $\partial\Sigma^{n-1}$, $A_{\partial\Sigma}$ the second fundamental form of immersion i and H the mean curvature of $\partial\Sigma^{n-1}$ with respect to ν. With that in mind, we can integrate on Σ^n both sides of the Böchner formula and get the following

Lemma 1.3.2 (Reilly's formula) *In the notations above, if $u \in C^\infty(\Sigma)$, then*

$$\begin{aligned}
&\int_\Sigma \Big(Ric(\nabla u, \nabla u) - \big(\Delta u\big)^2 + |\nabla^2 u|^2 \Big) dv_g \\
&= - \int_{\partial\Sigma} \big(-2u_\nu \Delta u + nHu_\nu^2 + A_{\partial\Sigma}(\nabla u, \nabla u) \big)\, d\mu_g,
\end{aligned}$$

where $u_\nu = \langle \nabla u, \nu \rangle$.

1.4 Smooth Metric Measure Space

A *smooth metric measure space* (or *weighted manifold*) is a triple $(\Sigma^n, g, e^{-f}dv)$, where Σ^n is a n-dimensional Riemannian manifold with metric g, dv is volume element of Σ^n with respect the metric g and $f : \Sigma^n \longrightarrow \mathbb{R}$ is a smooth function. Weighted manifolds arise naturally in the study of self-shrinkers, Ricci solitons, harmonic heat flows, warped products and many other subjects. Its natural relevance in modern mathematics can be viewed, for example, because the Ricci solitons play a very important tool in the theory of Ricci flow and warped product Einstein metrics have considerable interest in the General Relativity (see [34]).

Given $(\Sigma^n, g, e^{-f}dv)$ a compact weighted manifold, we can take classical differential operators in the same weighted sense. For example, we define the *weighted Laplacian* (or “drift" Laplacian) of a smooth function $u : \Sigma^n \to \mathbb{R}$ by

$$\Delta_f u = \Delta u - \langle \nabla f, \nabla u \rangle.$$

Furthermore, we introduce the $(0, 2)$-tensor

$$\widetilde{\nabla}^2 u = \nabla^2 u - \frac{\nabla f \otimes \nabla u + \nabla u \otimes \nabla f}{2}.$$

With this, we have

$$tr_g(\widetilde{\nabla}^2 u) = \Delta_f u \quad \text{and} \quad e^f \mathcal{L}_X g = 2\widetilde{\nabla}^2 u,$$

where $\mathcal{L}_X g$ denotes the Lie derivative of the vector field $X = e^{-f}\nabla u$.

On the other hand, the $f-$*divergence* of a vector field $X \in \mathfrak{X}(\Sigma)$ is defined by

$$\operatorname{div}_f X = e^f \operatorname{div}(e^{-f} X) = \operatorname{div} X - \langle \nabla f, X \rangle.$$

So, it's easy to see that

$$\Delta_f u = \operatorname{div}_f \big(\nabla u\big), \;\; \forall u \in C^\infty(\Sigma).$$

From a tensor point of view, we have the $f-$*divergence* of a $(0, 2)$-tensor T given by the following $(0, 1)$-tensor

$$\operatorname{div}_f T = e^f \operatorname{div}(e^{-f} T) = \operatorname{div} T - T(\nabla f, \cdot),$$

where div is the usual divergence for tensors.

Using the properties of the classical differential operators introduced in section 1.1 and applying the divergence theorem, we obtain

$$\int_\Sigma u \Delta_f v e^{-f} dv_g = -\int_\Sigma \langle \nabla u, \nabla v \rangle \, e^{-f} dv_g + \int_{\partial\Sigma} u \frac{\partial v}{\partial \nu} e^{-f} d\mu_g,$$

where $u, v : \Sigma^n \to \mathbb{R}$ are smooth functions.

Moreover, we know that for every $(0,2)$-tensor field T, every function $u \in C^\infty(\Sigma)$ and every vector field $X \in (\Sigma)$,

$$\operatorname{div}\big(T(uX)\big) = u\,\langle \operatorname{div} T, X\rangle + u\,\langle \nabla X, T\rangle + T(\nabla u, X).$$

Thus,

$$\langle \operatorname{div} T, X\rangle\, e^{-f} = -\,\langle T, \nabla X\rangle\, e^{-f} + T\big(\nabla f, X\big)e^{-f} + \operatorname{div}\big(T(e^{-f}X)\big)$$

and, consequently,

$$\begin{aligned}\int_\Sigma \langle \operatorname{div} T, X\rangle\, e^{-f} dv_g \;&=\; -\int_\Sigma \langle T, \nabla X\rangle\, e^{-f} dv_g + \int_\Sigma T\big(\nabla f, X\big)e^{-f} dv_g \\ &\quad + \int_{\partial\Sigma} T(X,\nu)e^{-f} d\mu_g.\end{aligned}$$

Let $\{e^1, ..., e^n\}$ be the coframe of $\{e_1, ..., e_n\}$ on Σ^n and note that

$$\begin{aligned}\langle T, X\otimes Y\rangle \;&=\; T_{ij}\left\langle e^i\otimes e^j, X\otimes Y\right\rangle \\ &=\; T_{ij}\left\langle e^i, X\right\rangle\left\langle e^j, Y\right\rangle \\ &=\; T(X,Y),\end{aligned}$$

for every $X, Y \in \mathfrak{X}(\Sigma)$, where the second equality follows of the universal property of tensor product. Hence, making $X = \nabla h$, we get

$$T(\nabla f, \nabla u) \;=\; \left\langle T, \frac{\nabla f\otimes\nabla u + \nabla u\otimes\nabla f}{2}\right\rangle$$

and, with this,

$$\int_\Sigma \langle \operatorname{div} T, \nabla u\rangle\, e^{-f} dv_g = -\int_\Sigma \left\langle T, \tilde{\nabla}^2 h\right\rangle e^{-f} dv_g + \int_{\partial\Sigma} T(\nabla u,\nu)e^{-f} d\mu_g. \qquad (1.6)$$

As with operators, it is natural to extend some “objects” already existing in Riemannian manifolds to the weighted context. Thus, we define the *Bakry-Émery Ricci tensor* Ric_f by

$$Ric_f := Ric + \nabla^2 f.$$

Or, more generally, we define the f-tensor T_f of a given $(0,2)$-tensor T by the following tensor

$$T_f := T + \nabla^2 f.$$

With these new differential operators and by analogy to classic cases, there are Böchner and Reilly type formulas to such manifolds. For example, using weighted Laplacian Δ_f, we obtain the following generalization of Bochner's formula for a function $u \in C^\infty(\Sigma)$

$$\frac{1}{2}\Delta_f|\nabla u|^2 = |\nabla^2 u|^2 + Ric_f(\nabla u, \nabla u) + \langle \nabla u, \nabla \Delta_f u\rangle. \tag{1.7}$$

To see this, initially consider an arbitrary vector field $X \in \mathfrak{X}(\Sigma)$ and note that

$$\begin{aligned}
\frac{1}{2}\langle X, \nabla|\nabla u|^2\rangle &= \langle \nabla_X \nabla u, \nabla u\rangle \\
&= \langle \nabla_{\nabla u} X + [X, \nabla u], \nabla u\rangle \\
&= \nabla u\langle X, \nabla u\rangle - \langle X, \nabla_{\nabla u}\nabla u\rangle + \langle [X, \nabla u], \nabla u\rangle \\
&= [\nabla u, X]u + X\nabla u(u) - \langle X, \nabla_{\nabla u}\nabla u\rangle + \langle [X, \nabla u], \nabla u\rangle \\
&= \langle X, \nabla|\nabla u|^2\rangle - \langle X, \nabla_{\nabla u}\nabla u\rangle.
\end{aligned}$$

Consequently,

$$\frac{1}{2}\nabla|\nabla u|^2 = \nabla_{\nabla u}\nabla u$$

and, with this

$$\begin{aligned}
\nabla^2 f(\nabla u, \nabla u) &= \langle \nabla_{\nabla u}\nabla f, \nabla u\rangle \\
&= \nabla u\big(\langle \nabla f, \nabla u\rangle\big) - \langle \nabla f, \nabla_{\nabla u}\nabla u\rangle \\
&= \langle \nabla u, \nabla\langle \nabla f, \nabla u\rangle\rangle - \frac{1}{2}\langle \nabla f, \nabla|\nabla u|^2\rangle.
\end{aligned}$$

Therefore, using the weighted Laplacian definition, it follows that

$$\begin{aligned}
\frac{1}{2}\Delta_f|\nabla u|^2 &= \frac{1}{2}\Delta|\nabla u|^2 - \frac{1}{2}\langle \nabla f, |\nabla u|^2\rangle \\
&= Ric(\nabla u, \nabla u) + \langle \nabla u, \nabla(\Delta u)\rangle + |\nabla^2 f|^2 \\
&\quad +\nabla^2 f(\nabla u, \nabla u) - \langle \nabla u, \nabla\langle \nabla f, \nabla u\rangle\rangle \\
&= |\nabla^2 u|^2 + Ric_f(\nabla u, \nabla u) + \langle \nabla u, \nabla \Delta_f u\rangle.
\end{aligned}$$

Now, in possession of the Böchner type formula given above, we obtain the following

Lemma 1.4.1 (Reilly's formula for weighted manifolds, [30]) *Let $(M^n, g, e^{-f}dv)$ be a compact weighted manifold, possibly with nonempty boundary ∂M, and $u \in C^3$ a* *function. Then*

$$\int_M \left(Ric_f(\nabla u, \nabla u) - (\Delta_f u)^2 + |\nabla^2 u|^2 \right) e^{-f} dv \tag{1.8}$$
$$= - \int_{\partial M} \left[\left(\Delta_f u + H_f \frac{\partial u}{\partial \nu} \right) \frac{\partial u}{\partial \nu} - \left\langle \nabla u, \nabla \frac{\partial u}{\partial \nu} \right\rangle + A_{\partial M}(\nabla u, \nabla u) \right] e^{-f} d\mu,$$

where $H_f = H - \langle \nabla f, \nu \rangle$ and $A_{\partial M}$ are the f-mean curvature and second fundamental form of ∂M in M with respect to ν, the exterior unit normal vector, respectively.

1.5 Omori-Yau maximum principle

We started this section by evoking the Omori-Yau maximum principle for the Laplacian. More precisely, the *Omori-Yau maximum principle* is said to hold on a Riemannian manifold (Σ^n, g) (not necessarily complete) if for any function $u \in C^2(\Sigma)$ with $u^* = \sup_\Sigma u < +\infty$, there exists a sequence $\{p_j\}_{j\in\mathbb{N}} \subset \Sigma^n$ with the properties

$$\text{(i) } u(p_j) > u^* - \frac{1}{j}, \text{ (ii) } |\nabla u(p_j)| < \frac{1}{j} \text{ and (iii) } \Delta u(p_j) < \frac{1}{j},$$

for every $j \in \mathbb{N}$. Equivalently, for any function $u \in C^2(\Sigma)$ with $u_* = \inf_\Sigma > -\infty$, there exists a sequence $\{p_j\}_{j\in\mathbb{N}} \subset \Sigma^n$ with the properties

$$\text{(i) } u(p_j) < u_* + \frac{1}{j}, \text{ (ii) } |\nabla u(p_j)| < \frac{1}{j} \text{ and (iii) } \Delta u(p_j) > -\frac{1}{j},$$

for every $j \in \mathbb{N}$. In the case where the stronger statement

$$\text{(iii)}' \ \nabla^2 u(p_j) < \frac{1}{j} g$$

concerning the Hessian is satisfied, we say that the Omori-Yau maximum principle for the Hessian holds on Σ^n.

With this terminology, the results given by Omori [33] and Yau [50] can be stated as the following.

Lemma 1.5.1 (Omori [33] and Yau [50]) *(i) The Omori-Yau maximum principle for the Hessian holds on every complete Riemannian manifold with sectional curvature bounded from below.*

(ii) The Omori-Yau maximum principle for the Laplacian holds on every complete Riemannian manifold with Ricci curvature bounded from below.

There is a weaker version of the Omori-Yau maximum principle. Similarly, the *Omori-Yau maximum principle* holds for the Laplacian Δ on Σ^n if for any function $u \in C^2(\Sigma)$ with $u^* = \sup_\Sigma u < +\infty$, there exists a sequence $\{p_j\}_{j\in\mathbb{N}} \subset \Sigma^n$ with the properties

$$\text{(i) } u(p_j) > u^* - \frac{1}{j} \quad \text{and} \quad \text{(ii) } \Delta u(p_j) < \frac{1}{j},$$

for every $j \in \mathbb{N}$. Equivalently, for any function $u \in C^2(\Sigma)$ with $u_* = \inf_\Sigma > -\infty$, there exists a sequence $\{p_j\}_{j\in\mathbb{N}} \subset \Sigma^n$ with the properties

$$\text{(i) } u(p_j) < u_* + \frac{1}{j} \quad \text{and} \quad \text{(ii) } \Delta u(p_j) > -\frac{1}{j},$$

for every $j \in \mathbb{N}$.

According to [6], a Riemannian manifold Σ^n is said to be *stochastically complete* if for some (and hence, for any) $(x,t) \in \Sigma \times (0,+\infty)$, the heat kernel $p(x,y,t)$ of the Laplace-Beltrami operator Δ satisfies the conservation property

$$\int_\Sigma p(x,y,t)dy = 1. \tag{1.9}$$

From the probabilistic viewpoint, stochastic completeness is the property for a stochastic process to have infinite (intrinsic) lifetime. For the Brownian motion on a manifold, the conservation property (1.9) means that the total probability of the particle being found in the state space is constantly equal to 1.

Pigola, Rigoli and Setti [36] showed that Σ^n is stochastically complete if, and only if, the *weak maximum principle* holds for the Laplacian Δ on Σ^n.

As the reader can see in our results from chapters 4 and 5, we also explore an operator which, in a way, generalizes the Laplace-Beltrami operator. More precisely, for an Riemannian manifold (Σ^n, g) and a positive semi-definite symmetric tensor $T : \mathfrak{X}(\Sigma^n) \times \mathfrak{X}(\Sigma^n) \to \mathbb{R}$, consider the operator

$$L(\cdot) = tr_g(T \circ \nabla^2(\cdot)) = g\big(T, \nabla^2(\cdot)\big). \tag{1.10}$$

Note that L is elliptic if, and only if, T is positive definite and, in the particular case where $T = g$, L is the Laplace-Beltrami operator Δ on (Σ^n, g).

With this, following the notation in [5], given a positive semi-definite symmetric tensor T in Σ^n satisfying $\sup_\Sigma tr_g T < +\infty$, the *Omori-Yau maximum principle* is said to hold on Σ^n for the operator $L(\cdot) = tr(T \circ \nabla^2(\cdot))$ if, for any function $u \in C^2(\Sigma)$ with $u^* = \sup_\Sigma < +\infty$, there exists a sequence $\{p_j\}_{j\in\mathbb{N}} \subset \Sigma^n$ with the properties

$$\text{(i) } u(p_j) > u^* - \frac{1}{j}, \text{ (ii) } |\nabla u(p_j)| < \frac{1}{j} \text{ and (iii) } Lu(p_j) < \frac{1}{j},$$

for every $j \in \mathbb{N}$. Equivalently, for any function $u \in C^2(\Sigma)$ with $u_* = \inf_\Sigma > -\infty$, there exists a sequence $\{p_j\}_{j\in\mathbb{N}} \subset \Sigma^n$ with the properties

$$\text{(i) } u(p_j) < u_* + \frac{1}{j}, \text{ (ii) } |\nabla u(p_j)| < \frac{1}{j} \text{ and (iii) } Lu(p_j) > -\frac{1}{j},$$

for every $j \in \mathbb{N}$.

We define the radial sectional curvature K_Σ^{rad} of Σ^n as being the infimum of the sectional curvature of the 2-planes containing ∇d, where $d := d(o, \cdot) : \Sigma^n \longrightarrow \mathbb{R}$ denotes the distance function from a fixed reference point $o \in \Sigma^n$. We notice this definition is given only away from the cut locus of $\Sigma^n \setminus \{o\}$. Furthermore, let G be a smooth function on $[0, +\infty)$ even at the origin, i.e., $G^{(2k+1)}(0) = 0$ for each $k = 0, 1, ...$, and satisfying the following conditions

(i) $G(0) > 0$;

(ii) $G(t)^{-1/2} \notin L^1(+\infty)$;

(iii) $G'(t) \geq 0$ on $[0, +\infty)$;

(iv) $\limsup_{t\to\infty} \frac{tG(t^{1/2})}{G(t)} < +\infty$.

An example of this type of functions is given by

$$G(t) = t^2 \prod_{j=1}^{N} \left(log^{(j)}(t)\right)^2, \ t \gg 1,$$

where $log^{(j)}$ stands for the j-th iterated logarithm (see Remark 1.12 in [37]).

In this context, Alías, Impera and Rigoli showed the following:

Lemma 1.5.2 (Alías, Impera and Rigoli [5]) *Let (Σ^n, g) be a complete, non-compact Riemannian manifold whose radial sectional curvature satisfies*

$$K_\Sigma^{rad} \geq -G(d). \tag{1.11}$$

Then the Omori-Yau maximum principle holds on Σ^n for any semi-elliptic operator $L(\cdot) = tr_g(T \circ \nabla^2(\cdot)) = g\big(T, \nabla^2(\cdot)\big)$ with $\sup_\Sigma tr_g T < +\infty$.

1.6 Parabolicity

We also studied the case where Σ^n is complete and non-compact. Thus, for some results, we explored the concept of parabolicity. A Riemannian manifold Σ^n is said to be *parabolic* if every subharmonic function on Σ^n which is bounded from above is constant, that is, $\Delta u \geq 0$ and $u^* = \sup_\Sigma u < +\infty$ on Σ^n implies that $u \equiv const.$, for every $u \in C^2(\Sigma)$.

Following the notation in [3], we can make use of ϕ-divergence of vector fields and consider on Σ^n the following operator

$$L_\phi(u) := \operatorname{div}_\phi \Big(|\nabla u|^{-1} \varphi(x, |\nabla u|) T(\nabla u, \cdot)^\sharp \Big), \tag{1.12}$$

where $\sharp$ denotes the musical isomorphism, $\phi \in C^\infty(\Sigma)$, T is a positive definite symmetric $(0,2)$-tensor field on Σ^n and $\varphi : \Sigma \times \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ satisfies $\varphi(\cdot, t) \in C^0(\Sigma)$, for every $t \in \mathbb{R}_0^+$, and $\varphi(p, \cdot) \in C^0(\mathbb{R}_0^+) \cap C^1(\mathbb{R}^+)$, for every $p \in \Sigma^n$. Similarly to the definition of parabolicity already existing in the literature, we say that the Σ^n is L_ϕ*-parabolic* if the only solutions $u : \Sigma^n \to \mathbb{R}$ of the inequality $L_\phi(u) \geq 0$ which are bounded from above are constant.

In [3], Alías, Lira and Rigoli studied conditions that guarantee the L_ϕ-parabolicity for the operator L_ϕ defined in (1.12). More precisely, they assumed that, for some continuous functions ξ_- and ξ_+ defined on $\mathbb{R}_0^+ = [0, +\infty)$, the tensor T is positive definite and satisfies the following bounds

$$0 < \xi_-(d) \leq T(X, X) \leq \xi_+(d), \tag{1.13}$$

for every $X \in T_p\Sigma$, with $|X| = 1$, and for every $p \in \partial B_R$, where $d = \operatorname{dist}_\Sigma(p, o)$ is the geodesic distance in Σ^n from some fixed origin $o \in \Sigma^n$ and $B_R = B_R(o)$ is the geodesic ball centered at o with radius R. In addition, they also assumed that φ satisfies the following structure conditions:

$$\begin{aligned} &\text{(i) } \varphi(p, 0) = 0 \ \text{ for every } \ p \in \Sigma^n; \\ &\text{(ii) } \varphi(p, t) > 0 \ \text{ on } \ \Sigma \times \mathbb{R}^+; \\ &\text{(iii) } \varphi(p, t) \leq A(p) t^\delta \ \text{ on } \ \Sigma \times \mathbb{R}^+, \end{aligned} \tag{1.14}$$

for some $\delta > 0$ and $A(p) \in C^0(\Sigma)$, with $A(p) > 0$. Furthermore, we must also have

$$\inf_\Sigma \frac{\xi_-(d(p))}{\xi_+(d(p))} \frac{1}{A(p)^{1/\delta}} = \frac{1}{C_0^{1/\delta}}, \tag{1.15}$$

for some $C_0 > 0$.

Remark 1.6.1 *Note that condition* (ii) *in* (1.14) *is just an ellipticity condition for the operator* L*.*

Now we highlight the result below which will be used to prove our next results.

Lemma 1.6.2 (Alías, Lira and Rigoli [3]) *Let* Σ^n *a complete manifold,* $o \in \Sigma^n$ *a fixed origin and* $d(p) = \mathrm{dist}_\Sigma(p, o)$*. Let* L_ϕ *be the operator defined in* (1.12) *with* T *and* φ *satisfying the assumptions* (1.13)*,* (1.14) *and* (1.15) *above. Let* $\xi_+(d)$ *be defined in* (1.13)*. If*

$$\frac{1}{\left(\int_{\partial B_t} \xi_+(d) e^{-\varphi}\right)^{1/\delta}} \notin L^1(+\infty),$$

then Σ^n *is* L_ϕ*-parabolic.*

Chapter 2

Spacelike submanifolds immersed in spacetimes

The behavior of spacelike submanifolds immersed in Lorentzian manifolds is an important object of study which has aroused a lot of interest in recent years, from both the physical and mathematical points of view. Into this branch, the trapped submanifolds appear as an important particular case. The concept of trapped submanifolds, originally formulated by Penrose [35], is related to the causal orientation of the mean curvature vector field of the submanifold, that is, a spacelike submanifold of a spacetime is said to be trapped if its mean curvature vector field is timelike. On the other hand, according to [39], it is possible to define a notion of r-mean curvature for submanifolds immersed in spacetimes. In this sense, we generalized the concept of trapped submanifolds to a wider class, considering the r-th mean curvature, since $\vec{H}_{r+1}$ is a vector field when r is even. This new generalization justifies for many reasons, but we emphasize two: we can obtain interesting mathematical uniqueness, generalizing the already existing results for trapped submanifolds and, on the other hand we recall Penrose's paper on trapped surfaces [35] and its importance in Physics describing the region around a singularity in spacetime, indeed his concept has been generalized for higher dimensions of the ambient and the submanifold such as in the aforementioned works. This is crucial since there are models for the universe with more dimensions, moreover there are spacetimes modeling many other problems with multi-variables (dimensions) in Chemistry, Quantum Physics, Biology, Economics, populations behavior and others. Motivated by this, we introduced the different concepts of r-trapped submanifolds immersed in spacetimes. To do this, we begin section 2.1 by exploring the concept of causal orientation of a vector field in a Lorentzian manifold to define a spacetime and, in view of the ambients studied in chapters 3 and 4, we provide two important examples. In section 2.2, we define the r-th mean curvatures of spacelike submanifolds immersed in spacetimes and, with this in mind, we introduce the concept of r-trapped submanifolds. In addition, we define the differential operator L.

2.1 Spacetimes

In General Relativity, a model for the events space is given by a Lorentzian manifold, which is a smooth manifold M^m equipped with a metric $\langle \cdot, \cdot \rangle$ of index 1 and dimension $m \geq 2$. As an example of the Lorentz manifold we can mention the Minkowski space $\mathbb{R}_1^m$ given by the Euclidean space $\mathbb{R}^m$ equipped with the metric

$$\langle v, w \rangle_{\mathbb{R}_1^m} = -v_1 w_1 + \sum_{i=2}^{m} v_i w_i,$$

for all vectors $v = \{v_1, ..., v_m\}$ and $w = \{w_1, ..., w_m\}$ tangent to $\mathbb{R}_1^m$. It is important to note that for each point p in an arbitrary Lorentz manifold $(M^m, \langle \cdot, \cdot \rangle)$, we can consider an orthonormal basis $\{e_i|_p, ..., e_m|_p\}$ for T_pM and obtain

$$\langle x, y \rangle = -x_1 y_1 + \sum_{i=2}^{m} x_i y_i,$$

for every $x = \sum_{i=1}^{m} x_i e_i$ and $y = \sum_{i=1}^{m} y_i e_i$ in T_pM. Thus, each tangent space of a Lorentzian manifold is isometric to Minkowski space. Hence, one may say that Lorentzian manifolds are locally modeled by Minkowski space, just as Riemannian manifolds are locally modeled by Euclidean space.

Let $(M^m, \langle \cdot, \cdot \rangle)$ be a Lorentz manifold and consider a vector field X in $\mathfrak{X}(M)$. We will say that X is

$$\begin{cases} \textit{timelike} & if \quad \langle X, X \rangle < 0; \\ \textit{lightlike (or null)} & if \quad \langle X, X \rangle = 0 \text{ with } X(p) \neq 0 \ \forall p \in M^m; \\ \textit{spacelike} & if \quad \langle X, X \rangle > 0, \end{cases}$$

where $\langle X, X\rangle$ is taken at each point $p \in M^m$. Now, for each $p \in M^m$, consider the set $\mathfrak{T}_pM$ of all timelike vectors T_pM. For $u \in \mathfrak{T}_pM$,

$$\mathcal{TC}(u) = \{v \in T_pM;\ \langle u, v\rangle < 0\}$$

is the timecone of T_pM containing u. On the other hand, consider a function $\tau : M \to \mathfrak{X}(M)$ that assigns to each point $p \in M^m$ a timecone $\tau(p)$ in T_pM. If this function is smooth, i.e., for each $p \in M^m$ there is a (smooth) vector field X on some neighborhood $\mathcal{U}$ of p such that $X(q) \in \tau(q)$ for each $q \in \mathcal{U}$, then τ is called a *time-orientation* of M. With that in mind, we will say that a Lorentz manifold $(M, \langle\ ,\ \rangle)$ is *time-orientable* if there is a timelike vector field $X \in \mathfrak{X}(M)$. In this way, a *spacetime* is a time-oriented Lorentz manifold.

With the definition of spacetimes and taking into account the results of chapter 4, we will highlight two examples of spacetimes:

Example 2.1.1 (Generalized Robertson-Walker - GRW) The *generalized Robertson-Walker* GRW spacetime given by $-I \times_\rho M^{n+p}$ as a Lorentzian manifold, that is, the product manifold time-oriented $I \times_\rho M^{n+p}$, where $n \geq 2$ is a natural number, p is a non-negative integer, I is an open interval of $\mathbb{R}$, endowed with the Lorentzian warped metric

$$\langle\ ,\ \rangle = -\pi_I^*\left(dt^2\right) + (\rho \circ \pi_I)^2 \pi_M^*\left(\langle\ ,\ \rangle_M\right),$$

where π_M and π_I denote the canonical projections from $I \times M$ onto each factor, $\langle\ ,\ \rangle_M$ is the induced Riemannian metric on the fiber M^{n+p} and the positive smooth warping function $\rho : I \longrightarrow (0, +\infty)$. Furthermore, we will choose on $-I \times_\rho M^{n+p}$ the time-orientation given by the globally defined timelike unit vector field

$$\partial_t = (\partial/\partial t)_{|(t,x)},\ (t, x) \in -I \times_\rho M^{n+p}.$$

For every $t \in I$, the slice $M_t = \{t\} \times M \subset -I \times_\rho M^{n+p}$ is an embedded spacelike hypersurface, in the sense that the metric induced on M_t is Riemannian. The restriction of ∂_t to M_t gives its future-directed Gauss map. So, it is easy to see that M_t is a totally umbilical hypersurface in $-I \times_\rho M^{n+p}$ with shape operator (with respect to the future-directed Gauss map ∂_t) given by

$$A_t v = \overline{\nabla}_v \partial_t = \frac{\rho'(t)}{\rho(t)} v$$

for every tangent vector $v \in T_{(t,x)}M_t$, where $\overline{\nabla}$ denote the Levi-Civita connection of $-I \times_\rho M^{n+p}$.

Therefore, $t \in I \longrightarrow M_t \subset -I \times_\rho M^{n+p}$ determines a foliation of $-I \times_\rho M^{n+p}$ by totally umbilical spacelike hypersurfaces with future constant mean curvature given by

$$\mathcal{H}(t) := -\frac{1}{n+p} tr\left(A_t\right) = -\frac{\rho'(t)}{\rho(t)}. \tag{2.1}$$

Example 2.1.2 (Standard Static Spacetime - SSST) Let $\overline{M}^{n+p+1}$ be an $(n+p+1)$-dimensional Lorentzian manifold endowed with a timelike Killing vector field K, where $n \geq 2$ is a natural number and p is a non-negative integer. Suppose that the distribution orthogonal to K, D, is of constant rank and integrable. We denote by $\Psi : M^{n+1} \times I \longrightarrow \overline{M}^{n+p+1}$ the flow generated by K, where M^{n+1} is an arbitrarily fixed spacelike integral leaf of D labeled as $t = 0$, which we will assume to be connected, and I is the maximal interval of definition. In what follows, we will consider $I = \mathbb{R}$.

In this setting, $\overline{M}^{n+p+1}$ can be regard as a standard static spacetime $M^{n+p} \times_\rho \mathbb{R}_1$, that is, the product manifold $M^{n+p} \times \mathbb{R}$ endowed with the warping metric

$$\langle \cdot, \cdot \rangle = \pi_M^*\left(\langle \cdot, \cdot \rangle_M\right) - (\rho \circ \pi_M)^2 \pi_{\mathbb{R}}^*\left(dt^2\right), \tag{2.2}$$

where π_M and $\pi_{\mathbb{R}}$ denote the canonical projections from $M \times \mathbb{R}$ onto each factor, $\langle \cdot, \cdot \rangle_M$ is the induced Riemannian metric on the base M^{n+p}, $\mathbb{R}_1$ is the manifold $\mathbb{R}$ endowed with the metric $-dt^2$ and the warping function $\rho \in C^\infty$ given by

$$\rho = |K| = \sqrt{-\langle K, K \rangle},$$

where $|\cdot|$ denotes the norm of a timelike vector field on M^{n+p}.

2.2 The r-trapped submanifolds

Let $\psi : \Sigma^n \longrightarrow \overline{M}^{n+m}$ be a connected and oriented spacelike submanifold immersed in a spacetime $\overline{M}^{n+m}$, that is, the metric induced on Σ^n via ψ is a Riemannian metric. As usual, we also denote by $\langle \cdot, \cdot \rangle$ the metric on Σ^n induced via ψ. In this context, let $\overline{\nabla}$ and ∇ denote the Levi-Civita connections in $\overline{M}^{n+m}$ and Σ^n, respectively.

The Gauss formula of Σ^n in $\overline{M}^{n+m}$ is given by

$$\overline{\nabla}_X Y = \nabla_X Y - \alpha(X, Y)$$

for every tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$. Here $\alpha : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \longrightarrow \mathfrak{X}(\Sigma)^\perp$, given by $\alpha(X, Y) := -(\overline{\nabla}_X Y)^\perp$, denotes the *vector valued second fundamental form* of Σ^n. With this, the *mean curvature vector field* $\vec{H}$ of the Σ^n is defined by

$$\vec{H} = \frac{1}{n} tr(\alpha).$$

On the other hand, the Weingarten formula is given by

$$A_\xi X = \nabla^\perp_X \xi - \overline{\nabla}_X \xi, \ \forall X \in \mathfrak{X}(\Sigma) \ \text{ and } \ \xi \in \mathfrak{X}^\perp(\Sigma),$$

where $A_\xi : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ denotes the *shape operator*, with respect to the normal vector field $\xi \in \mathfrak{X}(\Sigma)$, defined by

$$\langle A_\xi X, Y\rangle = \langle \alpha(X,Y), \xi\rangle \,, \ X, Y \in \mathfrak{X}(\Sigma).$$

Furthermore, let $\{e_1, ..., e_n\}$ be a local orthonormal frame on Σ^n, $r \in \{1, ..., n\}$ and denoting $\alpha(e_i, e_j)$ by α_{ij}, we define the r*-th mean curvature* by

$$H_r = \begin{pmatrix} n \\ r \end{pmatrix}^{-1} \frac{1}{r!} \sum_{\substack{i_1,...,i_r \\ j_1,...,j_r}} \delta^{i_1 ... i_r}_{j_1 ... j_r} \langle \alpha_{i_1 j_1}, \alpha_{i_2 j_2}\rangle \cdots \left\langle \alpha_{i_{r-1} j_{r-1}}, \alpha_{i_r j_r}\right\rangle,$$

for r even, and

$$\vec{H}_r = \begin{pmatrix} n \\ r \end{pmatrix}^{-1} \frac{1}{r!} \sum_{\substack{i_1,...,i_r \\ j_1,...,j_r}} \delta^{i_1 ... i_r}_{j_1 ... j_r} \langle \alpha_{i_1 j_1}, \alpha_{i_2 j_2}\rangle \cdots \left\langle \alpha_{i_{r-2} j_{r-2}}, \alpha_{i_{r-1} j_{r-1}}\right\rangle \alpha_{i_r j_r},$$

for r odd, where

$$\delta^{i_1 ... i_r}_{j_1 ... j_r} = \begin{cases} 0, \ \text{ if } \ i_k = i_l \ \text{ for some } \ k \neq l \ \text{ or if } \ \{i_1, ..., i_r\} \neq \{j_1, ..., j_r\} \ \text{ as sets;} \\ \text{sign of the permutation } \ (i_1, ..., i_r) \longmapsto (j_1, ..., j_r). \end{cases}$$

By convention, we put $H_0 = 1$. Moreover, it is easy to show that

$$\begin{aligned} \vec{H}_1 &= \frac{1}{n} \sum_{i=1}^n \alpha_{ii} \\ &= \vec{H} \end{aligned}$$

and, consequently, the definition of the r-th mean curvature generalizes the definition of mean curvature vector field.

Remark 2.2.1 *We have that* $\delta^{i_1 ... i_r}_{j_1 ... j_r}$ *are also known as generalized Kronecker symbols and given by*

$$\delta^{i_1 ... i_r}_{j_1 ... j_r} = det \begin{pmatrix} \delta^{i_1}_{j_1} & \delta^{i_1}_{j_2} & \cdots & \delta^{i_1}_{j_r} \\ \delta^{i_2}_{j_1} & \delta^{i_2}_{j_2} & \cdots & \delta^{i_2}_{j_r} \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{i_r}_{j_1} & \delta^{i_r}_{j_2} & \cdots & \delta^{i_r}_{j_r} \end{pmatrix}.$$

If the codimension of Σ^n is 1, is convenient to work with the real valued second fundamental form b given by $b(X,Y) = -\langle \alpha(X,Y), N\rangle$ and, consequently, the r-th mean curvatures of odd order can be defined as a real valued (we replace the vector field $\vec{H}_r$ by the scalar $-\langle \vec{H}_r, N\rangle$). In this case, choosing orthonormal frame $\{e_1, ..., e_n\}$ such that $b(e_i, e_j) = k_i\delta_{ij}$, we will have

$$H_r = (-1)^r \begin{pmatrix} n \\ r \end{pmatrix}^{-1} \sum_{i_1<...<i_r} k_{i_1} \cdots k_{i_r},$$

for every $r \in \{1, ..., n\}$.

Let $\{e^1, ..., e^n\}$ be the dual coframe of $\{e_1, ..., e_n\}$, we define the *r-th Newton transformation* T_r of α by

$$T_r = \frac{1}{r!} \sum_{i,j} \sum_{\substack{i_1,...,i_r \\ j_1,...,j_r}} \delta^{ii_1...i_r}_{jj_1...j_r} \langle \alpha_{i_1j_1}, \alpha_{i_2j_2}\rangle \cdots \left\langle \alpha_{i_{r-1}j_{r-1}}, \alpha_{i_rj_r}\right\rangle e^i \otimes e^j,$$

for r even, and

$$T_r = \frac{1}{r!} \sum_{i,j} \sum_{\substack{i_1,...,i_r \\ j_1,...,j_r}} \delta^{ii_1...i_r}_{jj_1...j_r} \langle \alpha_{i_1j_1}, \alpha_{i_2j_2}\rangle \cdots \left\langle \alpha_{i_{r-2}j_{r-2}}, \alpha_{i_{r-1}j_{r-1}}\right\rangle \alpha_{i_rj_r} \otimes e^i \otimes e^j,$$

for r odd. By convention $T_0 = \langle \cdot, \cdot \rangle$.

Lemma 2.2.2 *For $0 \leq r < n$ even, we get that:*

(i) $tr(T_r) = k(r)H_r$;

(ii) $\sum_{ij} T_r(e_i, e_j)\alpha(e_i, e_j) = k(r)\vec{H}_{r+1}$,

where $k(r) = (n-r)\begin{pmatrix} n \\ r \end{pmatrix}$ is a constant.

Proof. It follows easily from the definitions of T_r and H_r, so we will omit it. ■

Some of our results are on the case where the ambient space has constant sectional curvature. Thus, the following Lemma is necessary.

Lemma 2.2.3 *If $\overline{M}^{n+m}$ has constant sectional curvature, then* $\operatorname{div}(T_r) = 0$.

Proof. For the proof of this result, we will follow reasoning analogous to proof of the Lemma 2.1 in [26]. Since the ambient space has constant sectional curvature, it follows from Codazzi's equation and some straighforward calculation that

$$\operatorname{div}(T_r) = -\operatorname{div}(T_r).$$

Therefore $\operatorname{div}(T_r) = 0$. ■

With these mathematical objects, T_r and H_r, we define the different types of r-trapped submanifolds. Given $0 \leq r < n$ even, we will say that a spacelike submanifold $\psi : \Sigma^n \longrightarrow \overline{M}^{n+m}$ is *future* (*past*) *r-trapped submanifold* if $\vec{H}_{r+1}$ is timelike and it is future (past) pointing. If $\vec{H}_{r+1}$ is lightlike and it is future (past) pointing everywhere on Σ^n then the submanifold is said to be *marginally future* (*past*) *r-trapped.* If $\vec{H}_{r+1}$ is causal or zero, such that it is future (past) pointing when it is causal, the submanifold is said to be *weakly future* (*past*) *r-trapped.* Finally, the submanifold is said to be *r-minimal* when $\vec{H}_{r+1}$ is identically zero.

We are interested in working with an operator which, in a way, generalizes the Laplace-Beltrami operator. More precisely, for an arbitrary Riemannian manifold (N^n, g_N) and a positive semi-definite symmetric tensor T in N^n, consider the operator $L(\cdot) = tr_{g_N}(T \circ D^2(\cdot)) = g_N\big(T, D^2(\cdot)\big)$, where D^2 denotes the Hessian in (N^n, g_N). Note that L is elliptic if and only if T is positive definite and, in the particular case where $T = g_N$, L is the Laplace-Beltrami operator Δ_{g_N} on (N^n, g_N). In this sense, turning our attention to the spacelike submanifold Σ^n, associated to each globally defined Newton tensor $T_r : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ with $0 \leq r \leq n$ even, we may consider the second order differential operator $L_r : C^\infty(\Sigma) \longrightarrow C^\infty(\Sigma)$ given by

$$\begin{aligned} L_r u &:= \langle \nabla^2 u, T_r \rangle \\ &= \operatorname{div}(T_r(\nabla u)) - \langle \operatorname{div} T_r, \nabla u \rangle. \end{aligned}$$

Thus, L_r generalizes the Laplace-Beltrame operator Δ on $(\Sigma^n, \langle \cdot, \cdot \rangle)$ in the sense that $L_0(\cdot) = \Delta(\cdot)$, since $T_0 = \langle \cdot, \cdot \rangle$. If, in addition, we assume that $\overline{M}^{n+p+1}$ has constant sectional curvature, lemma 2.2.3 provides

$$L_r u = \operatorname{div}(T_r(\nabla u)), \quad \text{for every} \quad u \in C^\infty(\Sigma). \tag{2.3}$$

Remark 2.2.4 *If the codimension of Σ^n is 1, it is not necessary for $0 \leq r \leq n$ to be even in the definition of the operator L_r given above.*

Chapter 3

The Generalized Robertson-Walker case

Recently, de Lima, Santos and Velásquez [43] obtained rigidity for trapped submanifolds in Lorentzian spaces forms, they considered assumptions such as parallel mean curvature and pseudo-umbilicity. Later, Alías, Cánovas and Colares [1], considered codimension two trapped submanifolds immersed in generalized Robertson-Walker spacetimes and obtained results of nonexistence and rigidity. Moreover, working in a similar context, Alías, Impera and Rigoli [4], analyzed the problem of uniqueness for space-like hypersurfaces with constant mean order curvature in generalized Robertson-Walker spacetimes. Motivated by these works, we dedicate this chapter to the study of spacelike submanifolds immersed in generalized Robertson-Walker (GRW) spacetimes. More precisely, we obtained results of rigidity and non-existence for spacelike submanifolds $\psi : \Sigma^n \to -I \times_\rho M^{n+p}$ based on a causal orientation of the $(r+1)$-th mean curvature $\vec{H}_{r+1}$, with $0 \leq r < n$ even. We started by exploring the n-dimensional r-trapped submanifolds contained in slices from the ambient space $-I \times_\rho M^{n+1}$. With this, we provide a condition for whether or not such a submanifold is r-trapped (see equation 3.3). Right after that, we will calculate the action of the operator L_r in the function height $h = \pi_I \circ \psi$ and in a primitive function σ of the warping function ρ in the search for a tool to help our results. With this tool in mind, we address some results of non-existence and rigidity. Finally, we provide examples of r-trapped submanifolds.

It is important to note that the concepts of trapped and r-trapped submanifolds are independent and that the second generalizes the first, since 0-trapped submanifolds coincides with the trapped in the classic sense (see example 3.5.1)

3.1 The r-trapped spacelike submanifolds contained in the slices

Following the notation in [1], let $(M^{n+1}, \langle\cdot,\cdot\rangle_M)$ be a Riemannian manifold and consider a hypersurface $\phi : \Sigma^n \longrightarrow M^{n+1}$ with induced metric

$$\langle\cdot,\cdot\rangle_\Sigma = \phi^*\big(\langle\cdot,\cdot\rangle_M\big).$$

Now, for $t_0 \in I$ fixed, consider the immersion $\phi_{t_0} : \Sigma^n \longrightarrow -I \times_\rho M^{n+1}$ given by

$$\phi_{t_0}(p) = \big(t_0, \phi(p)\big),\ p \in \Sigma^n.$$

Note that ϕ_{t_0} is a spacelike immersion of Σ^n into $-I \times_f M^{n+1}$ which is contained in the slice $M_{t_0} = \{t_0\} \times M^{n+1}$ and induced metric

$$\begin{aligned}\langle\cdot,\cdot\rangle_{t_0} &= \phi^*\big(\langle\cdot,\cdot\rangle\big) \\ &= \rho(t_0)^2\,\langle\cdot,\cdot\rangle_\Sigma\,,\end{aligned} \tag{3.1}$$

where $\langle\cdot,\cdot\rangle$ is the Lorentzian metric of $-I \times_\rho M^{n+1}$.

Conversely, let us consider $\psi : \Sigma^n \to -I \times_\rho M^{n+1}$ a spacelike immersion which is contained in a slice $M_{t_0} = \{t_0\} \times M^{n+1}$. Then, it is not difficult to see that the projection $\phi : \Sigma^n \to M^{n+1}$ given by the relation $\psi(p) = (t_0, \phi(p)) = \phi_{t_0}(p)$ yields an immersed hypersurface, for every $p \in \Sigma^n$.

Let N denote a (locally defined) unit normal vector field of the hypersurface $\phi : \Sigma^n \longrightarrow M^{n+1}$ and note that

$$\begin{aligned}\langle N, N\rangle &= \rho(t_0)^2\,\langle N, N\rangle_\Sigma \\ &= \rho(t_0)^2.\end{aligned}$$

then

$$\eta_{t_0}(p) = \frac{1}{\rho(t_0)} N(p) \ \text{ and } \ \xi_{t_0}(p) = \partial_{t_{|(t_0,\phi(p))}},\ p \in \Sigma^n,$$

define a local orthonormal frame of vector fields normal along the immersion ϕ_{t_0}, with

$$\langle \eta_{t_0}, \eta_{t_0} \rangle = 1,\ \langle \eta_{t_0}, \xi_{t_0} \rangle = 0 \ \text{ and } \ \langle \xi_{t_0}, \xi_{t_0} \rangle = -1.$$

So, it is easy to see that the second fundamental form α_{t_0} of the immersion ϕ_{t_0} can be written as

$$\alpha_{t_0}(X,Y) = \frac{1}{\rho(t_0)^2} \langle AX, Y \rangle_{t_0} N - \frac{\rho'(t_0)}{\rho(t_0)} \langle X, Y \rangle_{t_0} \xi_{t_0}, \quad \forall\, X, Y \in \mathfrak{X}(\Sigma),$$

where $A : \mathfrak{X}(\Sigma) \longrightarrow \mathfrak{X}(\Sigma)$ is the shape operator of $\phi : \Sigma^n \longrightarrow M^{n+1}$ with respect to N.

For a local orthonormal frame $\{E_1, E_2, ..., E_n\}$ on $(\Sigma^n, \langle\cdot,\cdot\rangle_\Sigma)$ that diagonalizes A, i.e.,

$$AE_i = k_i E_i, \quad \text{for} \ \ i = 1, 2, ..., n,$$

we define the r-mean curvature of the immersion $\phi : \Sigma^n \to M^{n+1}$ by

$$\mathcal{H}_r = \begin{pmatrix} n \\ r \end{pmatrix}^{-1} \sum_{i_1 < \cdots < i_r} k_{i_1} \cdots k_{i_r},$$

for $r \in \{1, 2, ..., n\}$, and $\mathcal{H}_0 = 1$. Furthermore, by the relation (3.1), we have that $\{e_1, e_2, ..., e_n\}$, with $e_i = \big(1/\rho(t_0)\big) E_i$ for $i = 1, 2, ..., n$, is a local orthonormal frame on $(\Sigma^n, \langle\cdot,\cdot\rangle_{t_0})$ and, with this,

$$\begin{aligned}
\alpha_{t_0}(e_i, e_j) &= \frac{1}{\rho(t_0)^2} \langle Ae_i, e_j \rangle_{t_0} N - \frac{\rho'(t_0)}{\rho(t_0)} \langle e_i, e_j \rangle_{t_0} \xi_{t_0} \\
&= \frac{1}{f(\tau)^3} \langle AE_i, e_j \rangle_\tau N - \frac{f'(\tau)}{f(\tau)} \delta_{ij} \xi_\tau \\
&= \frac{1}{f(\tau)^3} \langle k_i E_i, e_j \rangle_\tau N - \frac{f'(\tau)}{f(\tau)} \delta_{ij} \xi_\tau \qquad (3.2)\\
&= \frac{1}{\rho(t_0)^2} k_i \delta_{ij} N - \frac{\rho'(t_0)}{\rho(t_0)} \delta_{ij} \xi_{t_0}.
\end{aligned}$$

Consequently, the mean curvature vector field $\vec{H}_{t_0}$ of the immersion ϕ_{t_0} is given by

$$\vec{\mathrm{H}}_{t_0} = \frac{\mathcal{H}}{\rho(t_0)^2} N - \frac{\rho'(t_0)}{\rho(t_0)} \xi_{t_0},$$

where $\mathcal{H} = \mathcal{H}_1$ is the mean curvature of the immersion ϕ. Further,

$$\langle \vec{\mathrm{H}}_{t_0}, \vec{\mathrm{H}}_{t_0} \rangle = \frac{\mathcal{H}^2 - \rho'(t_0)^2}{\rho(t_0)^2} \qquad (3.3)$$

and, for $0 < r \leq n$ odd,

$$\vec{H}_r = \frac{1}{\rho(t_0)^r} \sum_{s=0}^{r} c(n,r,s)\rho'(t_0)^{r-s}\mathcal{H}_r \upsilon_s,$$

where $c(n,r,s)$ is a constant that depends only on n, r and s, and

$$\upsilon_s = \begin{cases} \eta_{t_0}, & \text{if } s \text{ is even;} \\ \xi_{t_0}, & \text{if } s \text{ is odd.} \end{cases}$$

3.2 Key Lemma

Let $\overline{M}^{n+p+1} = -I \times_\rho M^{n+p}$ be a generalized Robertson-Walker spacetime and consider a spacelike submanifold $\psi : \Sigma^n \to \overline{M}^{n+p+1}$. We define the *height function* and the *angle functions* of Σ^n in $\overline{M}^{n+p+1}$ by $h := \pi_{\mathbb{R}} \circ \psi : \Sigma^n \to \mathbb{R}$ and $\theta_l := \langle N_l, \partial_t \rangle$, where N_l, $l = 1, \ldots, p+1$ denotes unit normal vector fields on Σ^n with $N_1 = N$ timelike. On the other hand, from a simple calculation, we obtain

$$\begin{aligned} \overline{\nabla}\pi_{\mathbb{R}} &= -\langle \overline{\nabla}\pi_{\mathbb{R}}, \partial_t \rangle \partial_t \\ &= -\partial_t. \end{aligned}$$

So, from the decomposition $\partial_t = (\partial_t)^\top + \sum_{l=1}^{p+1} \epsilon_l \Theta_l N_l$, where $\epsilon_1 = -1$ and $\epsilon_l = 1$ in other cases, it is easy to see that

$$\nabla h = -(\partial_t)^\top .$$

Consequently, we have the following

Lemma 3.2.1 *Let Σ^n be a spacelike submanifold immersed in a GRW spacetime $-I \times_\rho M^{n+p}$. If $0 \leq r < n$ is even, then*

(i) $L_r(h) = -(\rho'(h)/\rho(h))k(r)H_r - (\rho'(h)/\rho(h))\, T_r(\nabla h, \nabla h) + k(r)\langle \vec{H}_{r+1}, \partial_t \rangle$;

(ii) $L_r\sigma(h) = k(r)\left(-\rho'(h)H_r + \rho(h)\langle \vec{H}_{r+1}, \partial_t\rangle\right)$, *where σ is a primitive of ρ.*

Proof. We already know that $\nabla h = -(\partial_t)^\top$ and $\partial_t = (\partial_t)^\top + \sum_{l=1}^{p+1} \epsilon_l \Theta_l N_l$. So,

$$\begin{aligned}
\nabla^2 h(e_i, e_j) &= \left\langle \overline{\nabla}_{e_i} \nabla h, e_j \right\rangle \\
&= \left\langle \overline{\nabla}_{e_i} (-\partial_t^\top), e_j \right\rangle \\
&= -\left\langle \overline{\nabla}_{e_i} \left(\partial_t - \sum_{l=1}^{p+1} \epsilon_l \Theta_l N_l \right), e_j \right\rangle \\
&= -\left\langle \overline{\nabla}_{e_i} \partial_t, e_j \right\rangle + \left\langle \overline{\nabla}_{e_i} \left(\sum_{l=1}^{p+1} \epsilon_l \Theta_l N_l \right), e_j \right\rangle \\
&= -\left\langle \overline{\nabla}_{e_i} \partial_t, e_j \right\rangle + \sum_{l=1}^{p+1} \left\langle \overline{\nabla}_{e_i} \left(\epsilon_l \Theta_l N_l \right), e_j \right\rangle \\
&= -\left\langle \overline{\nabla}_{e_i} \partial_t, e_j \right\rangle + \sum_{l=1}^{p+1} \epsilon_l \Theta_l \left\langle \overline{\nabla}_{e_i} N_l, e_j \right\rangle \\
&= -\left\langle \overline{\nabla}_{e_i} \partial_t, e_j \right\rangle - \sum_{l=1}^{p+1} \epsilon_l \Theta_l \left\langle N_l, \overline{\nabla}_{e_i} e_j \right\rangle \\
&= -\left\langle \overline{\nabla}_{e_i} \partial_t, e_j \right\rangle + \sum_{l=1}^{p+1} \left\langle \alpha(e_i, e_j), \epsilon_l \Theta_l N_l \right\rangle .
\end{aligned}$$

Since

$$\left\langle \overline{\nabla}_{e_i} \partial_t, e_j \right\rangle = \frac{\rho'(h)}{\rho(h)} \langle e_i, e_j \rangle + \frac{\rho'(h)}{\rho(h)} \langle e_i, \partial_t \rangle \langle e_j, \partial_t \rangle ,$$

it follows that

$$\begin{aligned}
\sum_{ij} T_r(e_i, e_j) \nabla^2 h(e_i, e_j) &= -\sum_{ij} T_r(e_i, e_j) \left(\rho'(h) \langle e_i, e_j \rangle + \frac{\rho'(h)}{\rho(h)} \langle e_i, \partial_t \rangle \langle e_j, \partial_t \rangle \right) \\
&\quad + k(r) \langle \vec{H}_{r+1}, \partial_t \rangle \\
&= -\frac{\rho'(h)}{\rho(h)} k(r) H_r - \frac{\rho'(h)}{\rho(h)} T_r \left(\sum_i \langle e_i, \partial_t \rangle e_i, \sum_j \langle e_j, \partial_t \rangle e_j \right) \\
&\quad + k(r) \langle \vec{H}_{r+1}, \partial_t \rangle \\
&= -\frac{\rho'(h)}{\rho(h)} k(r) H_r - \frac{\rho'(h)}{\rho(h)} T_r \left(\nabla h, \nabla h \right) \\
&\quad + k(r) \langle \vec{H}_{r+1}, \partial_t \rangle .
\end{aligned}$$

Finally, to show item (ii), note that

$$\nabla^2 (f \circ h)(X, Y) = f''(h) \langle \nabla h, X \rangle \langle \nabla h, Y \rangle + f'(h) \nabla^2 h(X, Y),$$

for any smooth function $f : \mathbb{R} \longrightarrow \mathbb{R}$, where X is a vector field along Σ^n. Therefore,

$$\begin{aligned}
L_r(f \circ h) &= \langle \nabla^2 (f \circ h), T_r \rangle \\
&= \sum_{ij} T_r(e_i, e_j) f''(h) \langle \nabla h, e_i \rangle \langle \nabla h, e_j \rangle + \sum_{ij} f'(h) \nabla^2 h(e_i, e_j) T_r(e_i, e_j) \\
&= f''(h) T_r(\nabla h, \nabla h) + f'(h) L_r(h)
\end{aligned}$$

The result appears replacing $L_r h$ found in item (i) in the above equation and making $f = \sigma$. ■

3.3 Some nonexistence results

This section is devoted to establish nonexistence results concerning spacelike submanifolds in a GRW spacetime.

Closed Case

Initially, take a GRW spacetime $\overline{M}^{n+p+1} = -I \times_\rho M^{n+p}$ with constant sectional curvature and consider a closed spacelike submanifold $\psi : \Sigma^n \to -I \times_\rho M^{n+p}$. Note that, for item *(ii)* in Lemma 3.2.1,

$$\operatorname{div}(T_r(\nabla \sigma(h)) = L_r \sigma(h) = k(r) \left(-\rho'(h) H_r + \rho(h) \langle \vec{H}_{r+1}, \partial_t \rangle \right).$$

Integrating both sides of the previous equation on Σ^n and applying Stokes theorem, we obtain the following integral identity

$$\int_\Sigma \rho'(h) H_r d\Sigma = \int_\Sigma \rho(h) \langle \vec{H}_{r+1}, \partial_t \rangle d\Sigma, \tag{3.4}$$

where $d\Sigma$ is the volume element of Σ^n. With this we are able to enunciate and prove our first nonexistence result as it follows.

Proposition 3.3.1 *For $0 \leq r < n$ even, there exist no closed submanifold r-minimal immersed in a GRW spacetime with constant seccional curvature such that $\rho'(h) > 0$ and $H_r > 0$.*

Lemma 3.2.1 is a key part of our results. We will use items (i) and (ii) of it to obtain auxiliary Lemmas that will be used in the proof of our main results.

Lemma 3.3.2 *Let $-I \times_\rho M^{n+p}$ be a GRW spacetime and let $\psi : \Sigma^n \to -I \times_\rho M^{n+p}$ be a closed spacelike submanifold such that $T_r \geq 0$ and $H_r > 0$, for some $0 \leq r < n$ even. Then*

(i) $\min\limits_{\Sigma} \left\langle \dfrac{\vec{H}_{r+1}}{H_r}, \partial_t \right\rangle \leq -\mathcal{H}(h^*)$ *where* $h^* = \max_\Sigma h$;

(ii) $\max\limits_{\Sigma} \left\langle \dfrac{\vec{H}_{r+1}}{H_r}, \partial_t \right\rangle \geq -\mathcal{H}(h_*)$ *where* $h_* = \min_\Sigma h$.

Proof. Since $\sigma' = \rho > 0$, the function $\sigma(h)$ is increasing and attains the maximum at the same point of h. So, the item (ii) of Lemma 3.2.1 and the fact that $T_r \geq 0$ provide

$$0 \geq \left\langle T_r, \nabla^2 \sigma(h^*) \right\rangle = k(r) \left(-\rho'(h^*) H_r + \rho(h^*) \langle \vec{H}_{r+1}, \partial_t \rangle|_{h^*} \right)$$

at the maximum point. Since $H_r > 0$, it follows that

$$\min_{\Sigma} \left\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \right\rangle \leq \left\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \right\rangle \Big|_{h^*} \leq \frac{\rho'}{\rho}(h^*) = -\mathcal{H}(h^*).$$

The proof of (ii) is similar, working at the minimum point. ■

By items (i) and (ii) of Lemma 3.3.2 above, we have that the sign of $\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \rangle$ is related to the sign of $-\mathcal{H}$ in $h^*, h_* \in I$ and, within our proposal, this can be translated into the results below.

Theorem 3.3.3 *Let $-I \times_\rho M^{n+p}$ be a GRW spacetime and $0 \leq r < n$ even.*

(i) *If $\mathcal{H}(t) \geq 0$, there exist no closed weakly past $r-$trapped submanifold in $-I \times_\rho M^{n+p}$ such that $T_r \geq 0$ and $H_r > 0$.*

(ii) *If $\mathcal{H}(t) \leq 0$, there exist no closed weakly future $r-$trapped submanifold in $-I \times_\rho M^{n+p}$ such that $T_r \geq 0$ and $H_r > 0$.*

Proof. For item (i), just note that for any weakly past r-trapped submanifold Σ in $-I \times_\rho M^{n+p}$ such that $T_r \geq 0$ and $H_r > 0$

$$\mathcal{H}(h^*) \leq -\min_{\Sigma} \left\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \right\rangle < 0.$$

Item (ii) follows similarly. ■

Noncompact case

A spacelike submanifold $\psi : \Sigma \longrightarrow -I \times_\rho M^{n+p}$ is called *bounded away from the future infinity at height* $t^* \in I$ if

$$\psi(\Sigma) \subset \left\{(t,x) \in -I \times_\rho M^{n+p};\ t \leq t^*\right\}.$$

Similarly, we say that a spacelike submanifold $\psi : \Sigma \longrightarrow -I \times_\rho M^{n+p}$ is *bounded away from the past infinity at height* $t_* \in I$ if

$$\psi(\Sigma) \subset \left\{(t,x) \in -I \times_\rho M^{n+p};\ t \geq t_*\right\}.$$

In this sense, Σ^n is said to be *bounded away from the infinity* of $-I \times_\rho M^{n+p}$ if it is bounded away from the past and future infinity.

Lemma 3.3.4 *Let $-I \times_\rho M^{n+p}$ be a GRW spacetime and let $\psi : \Sigma^n \to -I \times_\rho M^{n+p}$ be a stochastically complete spacelike submanifold bounded away from the future infinity. Moreover, for some $0 \leq r < n$ even, suppose that $L_r \leq \beta\Delta$ and $H_r > a > 0$, where β and a are positive constants. Then*

$$\inf_\Sigma \left\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \right\rangle \leq -\mathcal{H}(h^*)$$

where $h^* = \sup_\Sigma h$.

Proof. Applying the weak maximum principle to the function $u = \sigma(h)$, which satisfies $u^* = \sup_\Sigma u = \sigma(h^*)$. By hypothesis, we have that $u^* < +\infty$ because Σ^n is bounded away from the future infinity. So, there exists a sequence of points $\{p_j\}_{j\in\mathbb{N}}$ in Σ^n such that

$$\text{(i) } u(p_j) > u^* - \frac{1}{j} \ \text{ and } \ \text{(ii) } \Delta u(p_j) < \frac{1}{j}.$$

By Lemma 3.2.1 item (ii), follows that

$$\begin{aligned}
\frac{\beta}{k(r)j} &> \frac{\beta}{k(r)}\Delta u(p_j) \\
&\geq \frac{1}{k(r)} L_r u(p_j) \\
&= -\rho'(h(p_j))H_r(p_j) + \rho(h(p_j))\langle \vec{H}_{r+1}, \partial_t\rangle(p_j).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\inf_\Sigma \left\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \right\rangle &\leq \left\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \right\rangle (p_j) \\
&< \frac{1}{\rho(h(p_j))}\left(\rho'(h(p_j)) + \frac{\beta}{H_r(p_j)k(r)j}\right).
\end{aligned}$$

Making $j \to +\infty$, we have get

$$\inf_{\Sigma} \left\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \right\rangle \leq -\mathcal{H}(h^*).$$

Here, we use the fact that $\lim_{j \to +\infty} h(p_j) = h^*$ because σ is strictly increasing. ■

We recall that a GRW spacetime $\overline{M} = -I \times_\rho M^{n+p}$ is called spatially expanding if $\rho'(t) > 0$. Analogously, $\overline{M} = -I \times_\rho M^{n+p}$ is called spatially contracting if $\rho'(t) < 0$. Analogously to the case of closed submanifolds, we have the following nonexistence result for stochastically complete submanifolds.

Theorem 3.3.5 *Let $\overline{M} = -I \times_\rho M^{n+p}$ be a spatially expanding GRW spacetime and $0 \leq r < n$ even. Then there exist no stochastically complete weakly past $r-$trapped submanifold in $-I \times_\rho M^{n+p}$ bounded away from future infinity such that $L_r \leq \beta\Delta$ and $H_r > a > 0$, where β and a are positive constants.*

Proof. Just note that for any weakly past r-trapped submanifold Σ^n in $-I \times_\rho M^{n+p}$ such that $H_r > 0$

$$\mathcal{H}(h^*) \leq -\inf_{\Sigma} \left\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \right\rangle \leq 0.$$

■

In view of Lemma 1.5.2, we have that the condition (1.11) implies the principle of maximum for the operator $L(\cdot) = tr_g(T \circ \nabla^2(\cdot)) = g\big(T, \nabla^2(\cdot)\big)$, where T is a positive semi-definite symmetric tensor with $\sup_\Sigma tr_g T < +\infty$.

Lemma 3.3.6 *Let $-I \times_\rho M^{n+p}$ be a GRW spacetime and let $\psi : \Sigma^n \to -I \times_\rho M^{n+p}$ be a complete, non-compact spacelike submanifold whose radial sectional curvature satisfies*

$$K_\Sigma^{rad} \geq -G(d). \tag{3.5}$$

Moreover, for some $0 \leq r < n$ even, suppose that $\sup_\Sigma trT_r < +\infty$, $T_r \geq 0$ and $H_r > a > 0$, for some constant a.

(i) Assume that Σ^n is bounded away from the future infinity. Then

$$\inf_{\Sigma} \left\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \right\rangle \leq -\mathcal{H}(h^*)$$

where $h^* = \sup_\Sigma h$;

(ii) Assume that Σ^n is bounded away from the past infinity. Then

$$\sup_{\Sigma}\left\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \right\rangle \geq -\mathcal{H}(h_*)$$

where $h_* = \inf_\Sigma h$.

Proof. Initially, by the Corollary 3.3 in [5], we have that the Omori-Yau maximum principle holds on Σ^n for positive semi-definite operator $L_r(\cdot) = \langle T_r, \nabla^2(\cdot)\rangle$. We start by applying the maximum principle to the function $u = \sigma(h)$, which satisfies $u^* = \sup_\Sigma u = \sigma(h^*)$ and $u_* = \inf_\Sigma u = \sigma(h_*)$, since $\sigma' = \rho > 0$ (i.e., σ is strictly increasing). For item (i), we have that $u^* < +\infty$. So, there exists a sequence of points $\{p_j\}_{j\in\mathbb{N}}$ in Σ^n such that

$$\text{(i) } u(p_j) > u^* - \frac{1}{j}, \text{ (ii) } |\nabla u(p_j)| < \frac{1}{j}, \text{ (iii) } L_r u(p_j) < \frac{1}{j}.$$

By Lemma 3.2.1 item (ii), it follows

$$\begin{aligned}
\frac{1}{k(r)j} &> \frac{1}{k(r)} L_r u(p_j) \\
&= -\rho'(h(p_j))H_r(p_j) + \rho(h(p_j))\langle \vec{H}_{r+1}, \partial_t\rangle(p_j).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\inf_{\Sigma}\left\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \right\rangle &\leq \left\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \right\rangle (p_j) \\
&< \frac{1}{\rho(h(p_j))}\left(\rho'(h(p_j)) + \frac{1}{H_r(p_j)}k(r)j\right).
\end{aligned}$$

Making $j \to +\infty$, we have

$$\inf_{\Sigma}\left\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \right\rangle \leq -\mathcal{H}(h^*).$$

Here, we use the fact that $\lim_{j\to+\infty} h(p_j) = h^*$ because σ is strictly increasing.

The proof of (ii) is similar. ■

Using Lemma 3.3.6 above, we obtain the nonexistence of weakly future r-trapped complete bounded away from the infinity past in a spatially expanding GRW spacetime whenever the radial curvature has a control from below.

Theorem 3.3.7 *Let $-I \times_\rho M^{n+p}$ be a GRW spacetime and $0 \leq r < n$ even.*

(i) Let $t^* \in I$ and assume that $\mathcal{H}(t) > 0$ for $t \leq t^*$. Then there exist no weakly past r-trapped complete, non-compact spacelike submanifold bounded away from the future infinity at height t^* immersed into $-I \times_\rho M^{n+p}$ satisfying the condition (3.5) *and such that $T_r \geq 0$, $\sup_\Sigma trT_r < +\infty$ and $H_r > a > 0$, for some constant a.*

(ii) Let $t_* \in I$ and assume that $\mathcal{H}(t) < 0$ for $t \geq t_*$. Then there exist no weakly future r-trapped complete, non-compact spacelike submanifold bounded away from the past infinity at height t_* immersed into $-I \times_\rho M^{n+p}$ satisfying the condition (3.5) *and such that $T_r \geq 0$, $\sup_\Sigma trT_r < +\infty$ and $H_r > a > 0$, for some constant a.*

Proof. Just note that, for any weakly past r-trapped submanifold Σ^n,

$$\inf_\Sigma \langle \vec{H}_{r+1}, \partial_t \rangle \geq 0$$

and, consequently, the item (i) of Lemma 3.3.6 provides

$$\mathcal{H}(h^*) \leq -\inf_\Sigma \left\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \right\rangle \leq 0.$$

■

3.4 Some Rigidity Results

In this section we impose conditions for a spacelike submanifold to be contained in a t_0-slice of GRW spacetime.

Closed case

Theorem 3.4.1 *Let $-I \times_\rho M^{n+p}$ be a GRW spacetime such that $(\log \rho)'' \leq 0$, and let $\psi : \Sigma^n \to -I \times_\rho M^{n+p}$ be a closed spacelike submanifold. If for some $0 \leq r < n$ even, $T_r \geq 0$ and $H_r > 0$, then*

$$\min_\Sigma \left\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \right\rangle \leq -\mathcal{H}(h^*) \leq -\mathcal{H}(h_*) \leq \max_\Sigma \left\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \right\rangle. \tag{3.6}$$

Consequently, if $\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \rangle$ is constant and $T_r > 0$, then $\psi(\Sigma)$ is contained in a slice $\{t_0\} \times M^{n+p}$, for some $t_0 \in I$.

Proof. From Lemma 3.3.2

$$\min_\Sigma \left\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \right\rangle \leq -\mathcal{H}(h^*)$$

and

$$-\mathcal{H}(h_*) \leq \max_{\Sigma} \left\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \right\rangle.$$

The inequality (3.6) follows, since $-\mathcal{H}(h^*) \leq -\mathcal{H}(h_*)$ because $(\log \rho)'' \leq 0$.

Now, assuming that $\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \rangle$ is constant, we have

$$\min_{\Sigma} \left\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \right\rangle = \left\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \right\rangle = \max_{\Sigma} \left\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \right\rangle$$

and, with this,

$$\left\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \right\rangle \leq -\mathcal{H}(h^*) \leq -\mathcal{H}(h_*) \leq \left\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \right\rangle.$$

Thus, $-\mathcal{H}(h) = \rho'(h)/\rho(h)$ is constant and

$$\frac{\rho'(h)}{\rho(h)} = \left\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \right\rangle.$$

Consequently,

$$L_r \sigma(h) = k(r)(-\rho'(h) H_r + \rho(h) \langle \vec{H}_{r+1}, \partial_t \rangle) = 0 \text{ on } \Sigma^n.$$

Since T_r is positive defined we have that L_r is an elliptic operator defined in the closed Riemannian Σ^n, hence $\sigma(h)$ is constant on Σ^n, and since σ is an increasing function this means that h is itself constant on Σ^n. Hence, $\psi(\Sigma)$ is contained in a slice $\{t_0\} \times M$, for some $t_0 \in I$. ∎

We point out that the above result is an extension of Theorem 5.1 of [2] for submanifolds, when r is even.

Paying our attention to the component of the vector field $\vec{H}_{r+1}$ which is orthogonal to ∂_t, we obtain the following version of Theorem 3.4.1.

Theorem 3.4.2 *Let $-I \times_\rho M^{n+p}$ be a GRW spacetime such that $(\log \rho)'' \leq 0$ and, for some $0 \leq r < n$ even, let $\phi : \Sigma^n \to -I \times_\rho M^{n+p}$ be a closed marginally future r-trapped submanifold. If $T_r \geq 0$ and $H_r > 0$, then*

$$\min_{\Sigma} \frac{|\vec{H}^0_{r+1}|}{H_r} \leq \mathcal{H}(h_*) \leq \mathcal{H}(h^*) \leq \max_{\Sigma} \frac{|\vec{H}^0_{r+1}|}{H_r} \tag{3.7}$$

where $\vec{H}^0_{r+1}$ stands for the spacelike component of the lightlike vector field $\vec{H}_{r+1}$ which is orthogonal to ∂_t. Consequently, if $\frac{|\vec{H}^0_{r+1}|}{H_r}$ is constant and $T_r > 0$, then $\psi(\Sigma)$ is contained in a slice $\{t_0\} \times M^{n+p}$.

Proof. By hypothesis, we can decompose $\vec{H}_{r+1}$ as

$$\vec{H}_{r+1} = \vec{H}^0_{r+1} - \langle \vec{H}_{r+1}, \partial_t \rangle \partial_t$$

and, with this, we get

$$\left\langle \frac{\vec{H}_{r+1}}{H_r}, \frac{\vec{H}_{r+1}}{H_r} \right\rangle = \frac{1}{H_r^2}|\vec{H}^0_{r+1}|^2 - \frac{1}{H_r^2}\langle \vec{H}_{r+1}, \partial_t \rangle^2.$$

Consequently,

$$0 > \left\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \right\rangle = -\frac{|\vec{H}^0_{r+1}|}{H_r}$$

because Σ^n is marginally future r-trapped. So, it follows from Lemma 3.3.2 that

$$\min_{\Sigma} \frac{|\vec{H}^0_{r+1}|}{H_r} = -\max_{\Sigma} \left\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \right\rangle \leq \mathcal{H}(h_*),$$

and

$$\mathcal{H}(h^*) \leq -\min_{\Sigma} \left\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \right\rangle = \max_{\Sigma} \frac{|\vec{H}^0_{r+1}|}{H_r}.$$

Inequality (3.7) follows from the fact that $(\log \rho)'' \leq 0$, that is, $\mathcal{H}(t)$ is non-decreasing and $\mathcal{H}(h_*) \leq \mathcal{H}(h^*)$.

Now, suppose that $\frac{|\vec{H}^0_{r+1}|}{H_r}$ is constant. Thus, since $\mathcal{H}(t)$ is non-decreasing, $\mathcal{H}(t) = \frac{|\vec{H}^0_{r+1}|}{H_r} = const.$ on $[h_*, h^*]$, i.e.,

$$-\frac{\rho'(h)}{\rho(h)} = \mathcal{H}(h) = \frac{|\vec{H}^0_{r+1}|}{H_r} = -\left\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \right\rangle \text{ on } \Sigma^n.$$

Therefore,

$$L_r\sigma(h) = k(r)(-\rho'(h)H_r + \rho(h)\langle \vec{H}_{r+1}, \partial_t \rangle) = 0 \text{ on } \Sigma^n.$$

The result follows since L_r is an elliptic operator and σ is an increasing function because $T_r > 0$ and $\sigma' = \rho > 0$. ■

We emphasize the previous result generalizes Theorem 5.5 of [1] for r-mean curvature and that $p \geq 1$.

Corollary 3.4.3 *Let $-I \times_\rho M^{n+p}$ be a GRW spacetime such that $(\log \rho)'' \leq 0$, and let $\psi : \Sigma^n \to -I \times_\rho M^{n+p}$ be a closed marginally r-trapped spacelike submanifold, for some $0 \leq r < n$ even. Moreover, suppose that $T_r \geq 0$ and $H_r > 0$.*

(i) If $\mathcal{H}(t) \geq 0$, then Σ^n is marginally future r-trapped and

$$\min_{\Sigma} \frac{|\vec{H}^0_{r+1}|}{H_r} \leq \mathcal{H}(h_*) \leq \mathcal{H}(h^*) \leq \max_{\Sigma} \frac{|\vec{H}^0_{r+1}|}{H_r}.$$

(ii) If $\mathcal{H}(t) \leq 0$, then Σ^n is marginally past r-trapped and

$$\min_{\Sigma} \frac{|\vec{H}^0_{r+1}|}{H_r} \leq -\mathcal{H}(h^*) \leq -\mathcal{H}(h_*) \leq \max_{\Sigma} \frac{|\vec{H}^0_{r+1}|}{H_r}.$$

Proof. Assuming $\mathcal{H}(t) \geq 0$ and using Theorem 3.3.3 we obtain that then Σ^n is necessarily marginally future r-trapped. To conclude the demonstration of item (i), just apply the Theorem 3.4.2 and obtain

$$\min_{\Sigma} \frac{|\vec{H}^0_{r+1}|}{H_r} \leq \mathcal{H}(h_*) \leq \mathcal{H}(h^*) \leq \max_{\Sigma} \frac{|\vec{H}^0_{r+1}|}{H_r}.$$

■

Noncompact case

We will need the warping function of the ambient space to satisfy some additional restriction. More precisely, let us suppose that $-I \times_\rho M^{n+p}$ is a proper GRW spacetime, which is when the warping function ρ satisfies $\rho'(t) = 0$ only at isolated points of I.

Theorem 3.4.4 *Let $-I \times_\rho M^{n+p}$ be a GRW spacetime with $(\log \rho)'' \leq 0$ and let $\psi : \Sigma^n \to -I \times_\rho M^{n+p}$ be a complete, non-compact spacelike submanifold whose radial sectional curvature satisfies the condition (3.5). Moreover, for some $0 \leq r < n$ even, suppose that Σ^n is marginally future r-trapped, bounded away from the infinity, $\sup_\Sigma trT_r < +\infty$, $T_r \geq 0$ and $H_r > a > 0$, for some constant a. Then*

$$\inf_{\Sigma} \left\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \right\rangle \leq -\mathcal{H}(h^*) \leq -\mathcal{H}(h_*) \leq \sup_{\Sigma} \left\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \right\rangle. \tag{3.8}$$

Consequently, if $\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \rangle$ is constant and $(\log \rho)'' = 0$ only at isolated points, then $\psi(\Sigma)$ is contained in a slice $\{t_0\} \times M^{n+p}$, for some $t_0 \in I$.

Proof. The inequality (3.8) follows from Lemma 3.3.6 because $(\log \rho)'' \leq 0$, that is, $\mathcal{H}$ is non-decreasing. Now, assuming that $\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \rangle$ is constant, we have

$$\mathcal{H}(h_*) = \mathcal{H}(h^*) = \left\langle \frac{\vec{H}_{r+1}}{H_r}, \partial_t \right\rangle = const.$$

Since that $(\log \rho)''(t) = 0$ holds only at isolated points of I it implies that $\mathcal{H}(t)$ is strictly increasing on I, henceforth we have $h_* = h^*$ and h is constant on Σ^n. ■

Remark 3.4.5 *We can also obtain an analogous of the above theorem for the marginally past r-trapped case. On the other hand, since $|\langle \vec{H}_{r+1}, \partial_t \rangle| = |\vec{H}^0_{r+1}|$ for the marginally r-trapped case, this theorem holds if we replace $\langle \vec{H}_{r+1}, \partial_t \rangle$ for length of spacelike component $-|\vec{H}^0_{r+1}|$.*

As in Corollary 3.4.3, we can use the Theorem 3.3.7 and previous remark to get the following result.

Corollary 3.4.6 *Let $-I \times_\rho M^{n+p}$ be a GRW spacetime with $(\log \rho)'' \leq 0$ and let $\psi : \Sigma^n \to -I \times_\rho M^{n+p}$ be a complete, non-compact spacelike submanifold whose radial sectional curvature satisfies the condition (3.5). Moreover, for some $0 \leq r < n$ even, suppose that Σ^n is marginally r-trapped, bounded away from the infinity, $\sup_\Sigma trT_r < +\infty$, $T_r \geq 0$ and $H_r > a > 0$, for some constant a.*

(i) If $\mathcal{H}(t) > 0$, then Σ^n is marginally future r-trapped and

$$\inf_\Sigma \frac{|\vec{H}^0_{r+1}|}{H_r} \leq \mathcal{H}(h_*) \leq \mathcal{H}(h^*) \leq \sup_\Sigma \frac{|\vec{H}^0_{r+1}|}{H_r}.$$

(ii) If $\mathcal{H}(t) < 0$, then Σ^n is marginally past r-trapped and

$$\inf_\Sigma \frac{|\vec{H}^0_{r+1}|}{H_r} \leq -\mathcal{H}(h^*) \leq -\mathcal{H}(h_*) \leq \sup_\Sigma \frac{|\vec{H}^0_{r+1}|}{H_r}.$$

Now, we recall that a Riemannian manifold Σ^n is said to be *parabolic* if every subharmonic function on Σ^n which is bounded from above is constant, that is, $\Delta u \geq 0$ and $u^* = \sup_\Sigma u < +\infty$ on Σ^n implies that $u \equiv$ *constant*.

Theorem 3.4.7 *Let $-I \times_\rho M^{n+p}$ be a GRW spacetime and let $\psi : \Sigma^n \to -I \times_\rho M^{n+p}$ be a complete parabolic spacelike submanifold which bounded away from the future infinity. Suppose that, for some $0 \leq r < n$ even, Σ^n is weakly past r-trapped, $H_r > 0$ and $L_r \leq \beta\Delta$, for some positive constant β. If*

$$\frac{\rho'(h)}{\rho(h)} H_r \leq |\vec{H}_{r+1}|, \tag{3.9}$$

where $|\ \ | = \sqrt{|\langle\ ,\ \rangle|}$, then $\psi(\Sigma)$ is contained in the slice $\{t_0\} \times M^n$, for some $t_0 \in I$.

Proof. Initially, note that for any weakly past r-trapped submanifold Σ^n one has $\langle \vec{H}_{r+1}, \partial_t \rangle > 0$ on Σ^n. On the other hand, the Lemma 3.2.1 item (ii) provides

$$\begin{aligned} \Delta\sigma(h) \geq \frac{L_r(\sigma(h))}{\beta} &= \frac{k(r)\rho(h)}{\beta}\left(-\frac{\rho'(h)}{\rho(h)} H_r + \langle \vec{H}_{r+1}, \partial_t \rangle\right) \\ &\geq k(r)\rho(h)\left(-\frac{\rho'(h)}{\rho(h)} H_r + |\vec{H}_{r+1}|\right) \\ &\geq 0. \end{aligned}$$

Therefore the function $\sigma(h)$ is subharmonic and, since the Σ^n is parabolic and $\sigma(h)$ is bounded from above, it should be constant, that is, $\psi(\Sigma)$ is contained in a slice. ■

Remark 3.4.8 *From equation* (2.1), *we have that the condition* (3.9) *above establishes a comparison between the mean curvature of the slice at height h and the mean curvatures of high order H_r and $\vec{H}_{r+1}$.*

In the following, we enunciate an auxiliary Lemma due to Caminha [11].

Lemma 3.4.9 (Caminha, A. [11]) *Let X be a vector field on Σ^n, such that $\operatorname{div} X$ does not change sign on Σ^n. If $|X| \in \mathcal{L}^1(\Sigma)$, then $\operatorname{div} X$ vanishes identically on Σ^n.*

Theorem 3.4.10 *Let $-I \times_\rho M^{n+p}$ be a proper GRW spacetime warped product and constant sectional curvature. Let $\psi : \Sigma^n \to -I \times_\rho M^{n+p}$ be a spacelike submanifold bounded away from the infinity and, for some $0 \le r < n$ even, suppose that Σ^n is weakly past r-trapped, the second fundamental form α is bounded. Suppose that $|\nabla h| \in \mathcal{L}^1(\Sigma)$ and*

$$\frac{\rho'(h)}{\rho(h)} H_r \le |\vec{H}_{r+1}|,$$

where $|\ \ | = \sqrt{|\langle\ ,\ \rangle|}$. Then either $\psi(\Sigma)$ is contained in a slice $\{t_0\} \times M^{n+p}$, for some $t_0 \in I$, or $H_r = 0$ and Σ^n is r-minimal.

Proof. First note that if $\frac{\rho'(h)}{\rho(h)} H_r \le |\vec{H}_{r+1}|$, we have

$$\begin{aligned}
\operatorname{div}(T_r \nabla \sigma(h)) &= L_r \sigma(h) \\
&= k(r)\rho(h)\left(-\frac{\rho'(h)}{\rho(h)} H_r + \langle \vec{H}_{r+1}, \partial_t\rangle\right) \\
&\ge k(r)\rho(h)\left(-\frac{\rho'(h)}{\rho(h)} H_r + |\vec{H}_{r+1}|\right) \\
&\ge 0.
\end{aligned}$$

On the other hand, since Σ^n is bounded away from the infinity of $-I \times_\rho M^{n+p}$ and α is bounded, there exists a positive constant C such that

$$|T_r(\nabla \sigma(h))| \le C |\nabla h|.$$

Consequently, the hypothesis $|\nabla h| \in \mathcal{L}^1(\Sigma)$ implies that $|T_r(\nabla \sigma(h))| \in \mathcal{L}^1(\Sigma)$.

So, we can apply Lemma 3.4.9 to assure that $L_r \sigma(h) = \operatorname{div}(T_r \nabla \sigma(h)) = 0$ on Σ^n. With this,

$$\frac{\rho'(h)}{\rho(h)} H_r = \langle \vec{H}_{r+1}, \partial_t\rangle,\ \frac{\rho'(h)}{\rho(h)} H_r = |\vec{H}_{r+1}| \text{ and } \vec{H}_{r+1} = -|\vec{H}_{r+1}|\partial_t.$$

Therefore, if $\vec{H}_{r+1}$ is nonzero, it follows that

$$\vec{H}_{r+1} = -\frac{\rho'(h)}{\rho(h)} H_r \partial_t$$

and ∂_t is orthogonal to Σ^n.

However, if $\vec{H}_{r+1}$ is identically zero, we have that $\big(\rho'(h)/\rho(h)\big)H_r = 0$. Thus, if $\rho'(h) = 0$, we have that the height function h is constant, because $-I \times_\rho M^{n+p}$ is a proper GRW spacetime. In both cases above, the submanifold Σ^n is contained in a slice $\{t_0\} \times M^{n+p}$. On the other hand, if $\rho'(h) \neq 0$, then $H_r = 0$. ■

Remark 3.4.11 *Since* $|\langle \vec{H}_{r+1}, \partial_t\rangle| \geq |\vec{H}^0_{r+1}|$ *for the weakly* r*-trapped case, that is, the timelike component of* H_{r+1} *is not less than the spacelike component in that case. We have that Theorems* 3.4.7 *and* 3.4.10 *hold if we replace* $|\vec{H}_{r+1}|$ *for* $|\vec{H}^0_{r+1}|$.

3.5 Examples of r-trapped submanifolds

Example 3.5.1 Let Σ^3 be an immersed hypersurface into a Riemannian manifold M^4 and, for each $t \in I$ fixed, consider the inclusion $\phi_t : \Sigma^3 \to -I \times_\rho M^4$. It follows from equation (3.2) that

$$\begin{aligned}
\vec{H}_3 &= \begin{pmatrix} 3 \\ 3 \end{pmatrix}^{-1} \frac{1}{3!} \sum_{\substack{i_1,i_2,i_3 \\ j_1,j_2,j_3}} \delta^{i_1 i_2 i_3}_{j_1 j_2 j_3} \langle \alpha_t(e_{i_1}, e_{j_1}), \alpha_t(e_{i_2}, e_{j_2})\rangle\, \alpha_t(e_{i_3}, e_{j_3}) \\
&= \frac{2}{3!}\Bigg(\langle \alpha_t(e_1,e_1), \alpha_t(e_2,e_2)\rangle\, \alpha_t(e_3,e_3) + \langle \alpha_t(e_1,e_1), \alpha_t(e_3,e_3)\rangle\, \alpha_t(e_2,e_2) \\
&\quad + \langle \alpha_t(e_2,e_2), \alpha_t(e_3,e_3)\rangle\, \alpha_t(e_1,e_1)\Bigg) \\
&= \frac{1}{3}\Bigg(\left[\frac{1}{\rho(t)^2} k_1 k_2 - \left(\frac{\rho'(t)}{\rho(t)}\right)^2\right] \left(\frac{1}{\rho(t)^2} k_3 N - \frac{\rho'(t)}{\rho(t)}\xi_t\right) \\
&\quad + \left[\frac{1}{\rho(t)^2} k_1 k_3 - \left(\frac{\rho'(t)}{\rho(t)}\right)^2\right] \left(\frac{1}{\rho(t)^2} k_2 N - \frac{\rho'(t)}{\rho(t)}\xi_t\right) \\
&\quad + \left[\frac{1}{\rho(t)^2} k_2 k_3 - \left(\frac{\rho'(t)}{\rho(t)}\right)^2\right] \left(\frac{1}{\rho(t)^2} k_1 N - \frac{\rho'(t)}{\rho(t)}\xi_t\right)\Bigg) \\
&= \frac{1}{3}\Bigg(\frac{3}{\rho(t)^4} k_1 k_2 k_3 N - \frac{\rho'(t)}{\rho(t)} \left(k_1 k_2 + k_1 k_3 + k_2 k_3\right) \xi_t \\
&\quad - \left(\frac{\rho'(t)}{\rho(t)^2}\right)^2 \left(k_3 + k_2 + k_1\right) N + 3\left(\frac{\rho'(t)}{\rho(t)}\right)^3 \xi_t\Bigg) \\
&= \frac{1}{\rho(t)^4}\mathcal{H}_3 N - \frac{\rho'(t)}{\rho(t)^3}\mathcal{H}_2 \xi_t - \left(\frac{\rho'(t)}{\rho(t)^2}\right)^2 \mathcal{H}_1 N + \left(\frac{\rho'(t)}{\rho(t)}\right)^3 \xi_t \\
&= \frac{1}{\rho(t)^3}\mathcal{H}_3 \eta_t - \frac{\rho'(t)}{\rho(t)^3}\mathcal{H}_2 \xi_t - \frac{\rho'(t)^2}{\rho(t)^3}\mathcal{H}\eta_t + \left(\frac{\rho'(t)}{\rho(t)}\right)^3 \xi_t.
\end{aligned}$$

Note that we can rewrite $\vec{H}_3$ as

$$\vec{H}_3 = \left(\frac{\mathcal{H}_3 - \rho'(t)^2\mathcal{H}}{\rho(t)^3}\right)\eta_t - \left(\frac{\rho'(t)\mathcal{H}_2 - \rho'(t)^3}{\rho(t)^3}\right)\xi_t.$$

Consequently,

$$\left\langle \vec{H}_3, \vec{H}_3 \right\rangle_t = \left(\frac{\mathcal{H}_3 - \rho'(t)^2\mathcal{H}}{\rho(t)^3}\right)^2 - \left(\frac{\rho'(t)\mathcal{H}_2 - \rho'(t)^3}{\rho(t)^3}\right)^2$$

Now, assuming that $\Sigma^3 = \mathbb{S}^2 \times \mathbb{R}$ and $M^4 = \mathbb{R}^4$ endowed with standard metric, we obtain that $k_1 = k_2 = 1$ and $k_3 = 0$. Therefore

$$\begin{aligned}
\langle \vec{H}_3, \vec{H}_3 \rangle_t &= \left(-\frac{2}{3}\frac{f'(\tau)^2}{f(\tau)^3}\right)^2 - \left(\frac{1}{3}\frac{f'(\tau)}{f(\tau)^3} - \frac{f'(\tau)^3}{f(\tau)^3}\right)^2 \\
&= \frac{\rho'(t)^2}{9\rho(t)^6}\Big(- 9\rho'(t)^4 + 10\rho'(t)^2 - 1\Big),
\end{aligned}$$

because

$$\mathcal{H}_3 = 0, \ \mathcal{H}_2 = \frac{1}{3} \ \text{ and } \ \mathcal{H}_1 = \frac{2}{3}.$$

Here, we have that $\phi_t : \Sigma^3 \to -I \times_\rho \mathbb{R}^4$ is 2-trapped if and only if $0 < \rho'(t)^2 < 1/9$ or $\rho'(t)^2 > 1$. On the other hand, by the equation (3.3), we have that ϕ_t is trapped if, and only if, $|\rho'(t)| > 2/3$. More precisely,

$$\begin{cases}
\text{if } \ 0 < |\rho'(t)| < 1/3, \ \phi_t \text{ is 2-trapped but not trapped;} \\
\text{if } \ 1/3 < |\rho'(t)| < 2/3, \ \phi_t \text{ is neither 2-trapped nor trapped;} \\
\text{if } \ 2/3 < |\rho'(t)| < 1, \ \phi_t \text{ is trapped but not 2-trapped;} \\
\text{if } \ |\rho'(t)| > 1, \ \phi_t \text{ is 2-trapped and trapped.}
\end{cases}$$

With an analogous analysis when $|\rho'(t)| \in \{0, 1/3, 2/3, 1\}$ we can also observe that the concepts of marginally 2-trapped and marginally trapped are distinct.

Example 3.5.2 Let M^{n+1} be a Riemannian manifold and consider a totally geodesic hypersurface $\Sigma^n \longrightarrow M^{n+1}$. With this, for each $t \in \mathbb{R}$ fixed, we have that $\Sigma_t = \{t\} \times \Sigma$ is a spacelike submanifold of codimension two immersed into the GRW spacetime $-\mathbb{R} \times_\rho M^{n+1}$. Then, we have that the second fundamental form α_t of Σ_t is given by

$$\alpha_t(X, Y) = -\frac{\rho'(t)}{\rho(t)} \langle X, Y \rangle_t \partial_t. \tag{3.10}$$

Therefore, for $0 \leq r < n$ even,

$$\vec{H}_{r+1} = -C\left(\frac{\rho'(t)}{\rho(t)}\right)^{r+1} \partial_t,$$

where C is a positive constant. Immediately, we have that Σ^n is r-trapped whenever $\rho'(t) \neq 0$.

As a particular case of the general situation described above, we will consider the de Sitter spacetime given by $-\mathbb{R} \times_{\cosh t} \mathbb{S}^n$. Let Σ be the totally geodesic equator of $\mathbb{S}^n$ and $\Sigma_t = \{t\} \times \Sigma$. From (3.10), the shape operator of Σ_t is given by

$$\alpha(X,Y) = -\text{tgh}\ \ t\langle X,Y\rangle_t \partial_t.$$

Then Σ_t is r-trapped for any $t \neq 0$ and $0 \leq r < n$ even.

Example 3.5.3 In this work we introduce the notion of r-trapped submanifolds. However, we only deal with cases in which the ambient space is a GRW spacetime. In this sense, let us provide an example that use this concept to other ambient spaces.

In the same way as in [34], consider the Schwarzschild-Tangherlini spacetime, see [47], the space $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{S}^n$ with the metric

$$-\zeta(s)dt^2 + \zeta(s)^{-1}ds^2 + s^2 g_{\mathbb{S}^n},$$

where $\zeta(s) = (1 - \mu/s^{n-1})$ is the Schwarzschild-Tangherlini function, μ is a positive constant given by

$$\mu = \frac{16\pi M}{n\omega_n},$$

where $\omega_n = |\mathbb{S}^n|$ is the measure of the unit n-sphere. In this case M is representing the mass of the model and s is the coordinate of $\mathbb{R}_+$. This metric degenerates when $s^{n-1} = \mu$. However, we separate the cases $s^{n-1} < \mu$ as inside the Black Hole and $s^{n-1} > \mu$ as outside the Black Hole or the outer Schwarzschild-Tangherlini space which is asymptotic to the Lorentz-Minkowski $\mathbb{L}^{n+2}$.

The spheres with $t \equiv t_0$ and $s \equiv cte$ appear in two types: the inner spheres if $s^{n-1} < \mu$ and outer spheres if $s^{n-1} > \mu$. They are compact submanifolds with codimension two and they are also totally umbilical with the second fundamental form given by

$$\alpha(V,W) = -\frac{\mu(n-1)}{s^n}\langle V,W\rangle \partial_s.$$

This fact comes from a direct computation considering Proposition 7.35 of [34]. Since the vector ∂_s is timelike inside the black hole, the inner spheres are past r-trapped for any $0 \leq r < n$ even.

Chapter 4

The Standard Static Spacetime case

In [22] the authors studied trapped spacelike submanifolds immersed in standard static spacetimes and established sufficient conditions to guarantee that such a spacelike submanifold must be a hypersurface of the Riemannian base of the ambient spacetime and, particularly, they showed that there do not exist n-dimensional compact (without boundary) trapped submanifolds immersed in an $(n+2)$-dimensional standard static spacetime which is a classical result due to Mars and Senovilla [31] (see also [45]). In this context, the idea here is to study the spacelike submanifolds immersed in a standard static spacetime. In this way, we restrict ourselves to the spacelike submanifolds $\psi : \Sigma \to M^{n+p} \times_\rho \mathbb{R}_1$ and obtain results of rigidity and nonexistence under the hypothesis of causal orientation for the $(r+1)$-th mean curvature $\vec{H}_{r+1}$, with $0 \leq r < n$ even. At first, we calculate $L_r(h)$ and, in addition, we provide a result that guarantees, under some hypotheses, the Omori-Yau maximum principle for the Laplacian (see Lemma 4.1.2). In section 4.2, we discuss some results of nonexistence and rigidity for r-trapped, as well marginally and weakly r-trapped, submanifolds immersed in a standard static spacetime $M^{n+p} \times_\rho \mathbb{R}_1$. In the next section, we explore the definition of weighted divergence (or, more precisely, of ϕ-divergence for some smooth function ϕ on Σ^n) and, under the hypothesis of constant sectional curvature of the ambient space $M^{n+p} \times_\rho \mathbb{R}_1$, we use the differential operator $L_{r,\phi}$ (see equation (4.4) and continue to obtain results of non-existence and rigidity. In section 4.4, we follow the same idea as in the previous section, but this time making use of the principle of maximum for both

Laplacian and differential operator L_r. And, likewise, we end the chapter by providing examples of r-trapped submanifolds.

4.1 Key Lemmas

Along this chapter, we will consider a connected and oriented spacelike submanifold $\psi : \Sigma^n \longrightarrow \overline{M}^{n+p+1}$ immersed in a standard static spacetime $\overline{M}^{n+p+1} = M^{n+p} \times_\rho \mathbb{R}_1$, that is, the metric induced on Σ^n via ψ is a Riemannian metric. As usual, we also denote by $\langle \cdot, \cdot \rangle$ the metric on Σ^n induced via ψ. Since K is a globally defined timelike vector field on $\overline{M}^{n+p+1}$, it follows that there exists a unitary timelike normal vector field N globally defined on Σ^n which is in the same time-orientation of K (one can define N as the unitary direction of K minus its projection on Σ^n). We will also consider smooth functions on $\psi : \Sigma^n \longrightarrow \overline{M}^{n+p+1}$, namely, the (vertical) height function $h = \pi_{\mathbb{R}} \circ \psi$ and the angle functions $\Theta_l = \langle N_l, K \rangle$, where N_l, $l = 1, \ldots, p+1$ denotes unit normal vector fields on Σ^n with $N_1 = N$.

From the decomposition $K = K^\top + \sum_{l=1}^{p+1} \epsilon_l \Theta_l N_l$, it is easy to see that

$$\nabla h = -\frac{1}{\rho^2} K^\top \quad \text{and} \quad |\nabla h|^2 = \frac{\sum_{l=1}^{p+1} \epsilon_l \Theta_l^2 - \rho^2}{\rho^4}, \tag{4.1}$$

where $\epsilon_1 = -1$ and $\epsilon_l = 1$ in other cases.

Lemma 4.1.1 *Let Σ^n be a spacelike submanifold immersed in a manifold $\overline{M}^{n+p+1} = \mathbb{R} \times_\rho M^{n+p}$. If $0 \leq r < n$ is even, then*

$$L_r(h) = -2T_r(\nabla \ln(\rho), \nabla h) + \frac{1}{\rho^2} k(r) \langle \vec{H}_{r+1}, K \rangle. \tag{4.2}$$

Proof. Once $\rho = \sqrt{-\langle K, K \rangle}$, we have that $\nabla \rho^2 = 2\left(\overline{\nabla}_K K\right)^\top$. Next, for a local ortonormal frame $\{e_1, ..., e_n\}$ on Σ^n,

$$\begin{aligned}
\nabla^2 h(e_i, e_j) &= \left\langle \overline{\nabla}_{e_i} \nabla h, e_j \right\rangle \\
&= \left\langle \overline{\nabla}_{e_i} \left(-\frac{1}{\rho^2} K^\top \right), e_j \right\rangle \\
&= -e_i \left(\frac{1}{\rho^2} \right) \left\langle K^\top, e_j \right\rangle - \left(\frac{1}{\rho^2} \right) \left\langle \overline{\nabla}_{e_i} K^\top, e_j \right\rangle .
\end{aligned}$$

On the other hand,

$$e_i \left(\frac{1}{\rho^2} \right) = -\frac{1}{\rho^4} \left\langle \overline{\nabla}_K K, e_i \right\rangle ,$$

and

$$
\begin{aligned}
\left\langle \overline{\nabla}_{e_i} K^\top, e_j \right\rangle &= \left\langle \overline{\nabla}_{e_i} \left(K - \sum_{l=1}^{p+1} \epsilon_l \Theta_l N_l \right), e_j \right\rangle \\
&= \left\langle \overline{\nabla}_{e_i} K, e_j \right\rangle - \left\langle \overline{\nabla}_{e_i} \left(\sum_{l=1}^{p+1} \epsilon_l \Theta_l N_l \right), e_j \right\rangle \\
&= \left\langle \overline{\nabla}_{e_i} K, e_j \right\rangle - \sum_{l=1}^{p+1} \left\langle \overline{\nabla}_{e_i} \left(\epsilon_l \Theta_l N_l \right), e_j \right\rangle \\
&= \left\langle \overline{\nabla}_{e_i} K, e_j \right\rangle - \sum_{l=1}^{p+1} \epsilon_l \Theta_l \left\langle \overline{\nabla}_{e_i} N_l, e_j \right\rangle \\
&= \left\langle \overline{\nabla}_{e_i} K, e_j \right\rangle + \sum_{l=1}^{p+1} \epsilon_l \Theta_l \left\langle N_l, \overline{\nabla}_{e_i} e_j \right\rangle \\
&= \left\langle \overline{\nabla}_{e_i} K, e_j \right\rangle - \sum_{l=1}^{p+1} \left\langle \alpha(e_i, e_j), \epsilon_l \Theta_l N_l \right\rangle .
\end{aligned}
$$

Since

$$
\begin{aligned}
\sum_{ij} T_r(e_i, e_j) \left\langle \overline{\nabla}_K K, e_i \right\rangle \langle K, e_j \rangle &= T_r \left(\sum_i \left\langle \overline{\nabla}_K K, e_i \right\rangle e_i, \sum_j \langle K, e_j \rangle e_j \right) \\
&= T_r \left((\overline{\nabla}_K K)^\top, K^\top \right) \\
&= -\frac{1}{2} T_r(\nabla \rho^2, \rho^2 \nabla h) \\
&= -\rho^3 T_r(\nabla \rho, \nabla h),
\end{aligned}
$$

we get

$$
\begin{aligned}
\sum_{ij} T_r(e_i, e_j) \nabla^2 h(e_i, e_j) &= -\frac{2}{\rho} T_r(\nabla \rho, \nabla h) + \frac{1}{\rho} \left(\sum_{l=1}^{p+1} k(r) \langle H_{r+1}, \Theta_l N_l \rangle \right) \\
&= -2 T_r(\nabla \ln(\rho), \nabla h) + \frac{1}{\rho^2} k(r) \langle H_{r+1}, K \rangle .
\end{aligned}
$$

Here we use the fact that

$$
\sum_{ij} T_r(e_i, e_j) \left\langle \overline{\nabla}_{e_i} K, e_j \right\rangle = 0,
$$

since T_r is a symmetric tensor and K is Killing vector field. ■

Now, taking into account the local orthonormal frame $\{N_1, N_2, ..., N_{p+1}\}$ for the $\mathfrak{X}^\perp(\Sigma)$ with $N_{p+1} = N$, we will denote by A and, for $i = 1, 2, ..., p$, $A^{(i)}$ the components of the second fundamental form α of Σ^n with respect to N and N_i, respectively. Thus, we can rewrite the second fundamental form α as the following

$$\alpha(X,Y) = \sum_{i=1}^{p} \langle A^{(i)}X, Y\rangle N_i + \langle AX, Y\rangle N \tag{4.3}$$

and obtain the decomposition below for the mean curvature vector field

$$\vec{H} = \sum_{i=1}^{p} H^{(i)} N_i - H^N N,$$

where $H^{(i)} = \langle \vec{H}, N_i\rangle$ and $H^N = \langle \vec{H}, N\rangle$. With this we can generalize lemma 2 in [22] for higher codimension, as we see in the following

Lemma 4.1.2 *Let $\overline{M}^{n+p+1} = M^{n+p} \times_\rho \mathbb{R}_1$ be a standard static spacetime whose Riemannian base M^{n+p} has nonnegative curvature K_M and convex warping function ρ. Let $\psi : \Sigma^n \to \overline{M}^{n+p+1}$ be a spacelike submanifold. Then*

$$Ric(X,X) \geq \left| AX + \frac{nH^N}{2}X \right|^2 - \sum_{k=1}^{p} \left| A^{(k)}X - \frac{nH^{(k)}}{2}X \right|^2 + \varepsilon \frac{n^2|\vec{H}|^2}{4}|X|^2,$$

where ε stands for the sign of $\langle \vec{H}, \vec{H}\rangle$.

Proof. For every vector field Y tangent to $\overline{M}^{n+p+1}$, we can write

$$Y = Y^* + Y^\perp,$$

where Y^* and $Y^\perp$ are the orthogonal projection of Y onto TM and $T\mathbb{R}_1$, respectively. Consequently,

$$Y^\perp = \frac{\langle Y, K\rangle}{\langle K, K\rangle} K = -\frac{\langle Y, K\rangle}{\rho^2} K.$$

Thus, we can take vector fields U, V and W tangent to $\overline{M}^{n+p+1}$ and, with a straightforward computation, we get

$$\begin{aligned}
\overline{R}(U,V)W &= \overline{R}\left(U^* - \frac{\langle U, K\rangle}{\rho^2}K, V^* - \frac{\langle V, K\rangle}{\rho^2}K\right)\left(W^* - \frac{\langle W, K\rangle}{\rho^2}K\right) \\
&= \overline{R}(U^*, V^*)W^* - \frac{\langle W, K\rangle}{\rho^2}\overline{R}(U^*, V^*)K - \frac{\langle V, K\rangle}{\rho^2}\overline{R}(U^*, K)W^* \\
&\quad + \frac{\langle V, K\rangle}{\rho^2}\langle W, K\rangle \overline{R}(U^*, K)K - \frac{\langle U, K\rangle}{\rho^2}\overline{R}(K, V^*)W* \\
&\quad + \frac{\langle U, K\rangle\langle W, K\rangle}{\rho^2}\overline{R}(K, V^*)K \\
&= R_M(U^*, V^*)W^* - \frac{\langle W, K\rangle}{\rho^2}\overline{R}(U^*, V^*)K - \frac{\langle V, K\rangle}{\rho^2}\overline{R}(U^*, K)W^* \\
&\quad - \frac{\langle U, K\rangle}{\rho^2}\overline{R}(K, V^*)W^* + \frac{\langle W, K\rangle}{\rho^4}\langle V, K\rangle \overline{R}(U^*, K)K \\
&\quad - \frac{\langle W, K\rangle}{\rho^4}\langle U, K\rangle \overline{R}(V^*, K)K.
\end{aligned}$$

By Proposition 7.42 and Lemma 7.34 of [34], follows that

$$\overline{R}(U^*,V^*)K=0,$$

$$\overline{R}(U^*,V^*)W^*=-\overline{R}(K,U^*)W^*=-\frac{D^2\rho(U^*,W^*)}{\rho}K,$$

$$\overline{R}(K,V^*)W^*=\frac{D^2\rho(V^*,W^*)}{\rho}K,$$

$$\overline{R}(U^*,K)K=\frac{\langle K,K\rangle}{\rho}\overline{\nabla}_{U^*}\overline{\nabla}(\rho\circ\pi_M)=\frac{\langle K,K\rangle}{\rho}\nabla_{U^*}\nabla\rho,$$

$$\overline{R}(V^*,K)K=\frac{\langle K,K\rangle}{\rho}\overline{\nabla}_{V^*}\overline{\nabla}(\rho\circ\pi_M)=\frac{\langle K,K\rangle}{\rho}\nabla_{V^*}\nabla\rho$$

and

$$\begin{aligned}\overline{R}(U,V)W &= R_M(U^*,V^*)W^*+\frac{\langle V,K\rangle}{\rho^3}D^2\rho(U^*,W^*)K\\ &\quad-\frac{\langle U,K\rangle}{\rho^3}D^2\rho(V^*,W^*)K+\frac{\langle W,K\rangle}{\rho^5}\langle V,K\rangle\langle K,K\rangle\nabla_{U^*}\nabla\rho\\ &\quad-\frac{\langle W,K\rangle}{\rho^5}\langle U,K\rangle\langle K,K\rangle\nabla_{V^*}\nabla\rho\\ &= R_M(U^*,V^*)W^*+\frac{\langle V,K\rangle}{\rho^3}D^2\rho(U^*,W^*)K\\ &\quad-\frac{\langle U,K\rangle}{\rho^3}D^2\rho(V^*,W^*)K-\frac{\langle W,K\rangle}{\rho^3}\langle V,K\rangle\nabla_{U^*}\nabla\rho\\ &\quad+\frac{\langle W,K\rangle}{\rho^3}\langle U,K\rangle\nabla_{V^*}\nabla\rho,\end{aligned}$$

where D^2 denotes the Hessian on M^{n+p}. In particular, for a local orthonormal frame $\{e_1,...,e_n\}$ on Σ^n and X a vector field tangent to Σ^n, we can take $U=W=X$ and $V=e_i$, with $i=1,...,n$, in the last equation to obtain

$$\begin{aligned}\overline{R}(X,e_i)X &= R_M(X^*,e_i^*)X^*+\frac{\langle e_i,K\rangle}{\rho^3}D^2\rho(X^*,X^*)K\\ &\quad-\frac{\langle X,K\rangle}{\rho^3}D^2\rho(e_i^*,X^*)K-\frac{\langle X,K\rangle}{\rho^3}\langle e_i,K\rangle\nabla_{X^*}\nabla\rho\\ &\quad+\frac{\langle X,K\rangle^2}{\rho^3}\nabla_{e_i^*}\nabla\rho\end{aligned}$$

and, consequently,

$$
\begin{aligned}
\langle \overline{R}(X,e_i)X,e_i\rangle &= \langle R_M(X^*,e_i^*)X^*,e_i^*\rangle + \frac{\langle e_i,K\rangle^2}{\rho^3}D^2\rho(X^*,X^*)\\
&\quad -\frac{\langle X,K\rangle}{\rho^3}\langle e_i,K\rangle D^2\rho(E_i^*,X^*) - \frac{\langle X,K\rangle}{\rho^3}\langle e_i,K\rangle\langle\nabla_{X^*}\nabla\rho,e_i\rangle\\
&\quad +\frac{\langle X,K\rangle^2}{\rho^3}\langle\nabla_{e_i^*}\nabla\rho,e_i^*\rangle\\
&= \langle R_M(X^*,e_i^*)X^*,e_i^*\rangle + \frac{\langle e_i,K\rangle^2}{\rho^3}D^2\rho(X^*,X^*)\\
&\quad -\frac{\langle X,K\rangle}{\rho^3}\langle e_i,K\rangle D^2\rho(e_i^*,X^*) - \frac{\langle X,K\rangle}{\rho^3}\langle e_i,K\rangle D^2\rho(X^*,e_i)\\
&\quad +\frac{\langle X,K\rangle^2}{\rho^3}D^2\rho(e_i^*,e_i^*)\\
&= K_M(X^*,e_i^*)(\langle X^*,X^*\rangle\langle e_i^*,e_i^*\rangle - \langle X^*,e_i^*\rangle^2)\\
&\quad +\frac{\langle e_i,K\rangle^2}{\rho^3}D^2\rho(X^*,X^*) - 2\frac{\langle X,K\rangle}{\rho^3}\langle e_i,K\rangle D^2\rho(X^*,e_i)\\
&\quad +\frac{\langle X,K\rangle^2}{\rho^3}D^2\rho(e_i^*,e_i^*).
\end{aligned}
$$

Thus, for $\tilde{X}_i^* = \frac{\langle e_i,K\rangle}{\rho}X^*$ and $\tilde{e}_i^* = \frac{\langle X,K\rangle}{\rho}e_i^*$, we have

$$
\begin{aligned}
\langle \overline{R}(X,e_i)X,e_i\rangle &= K_M(X^*,e_i^*)(\langle X^*,X^*\rangle\langle e_i^*,e_i^*\rangle - \langle X^*,e_i^*\rangle^2)\\
&\quad +\frac{1}{\rho}D^2\rho(\tilde{X}_i^*,\tilde{X}_i^*) - \frac{2}{\rho}D^2\rho(\tilde{e}_i^*,\tilde{X}_i^*)\\
&\quad +\frac{1}{\rho}D^2\rho(\tilde{e}_i^*,\tilde{e}_i^*)\\
&= K_M(X^*,e_i^*)(\langle X^*,X^*\rangle\langle e_i^*,e_i^*\rangle - \langle X^*,e_i^*\rangle^2)\\
&\quad +\frac{1}{\rho}D^2\rho(\tilde{X}_i^*-\tilde{e}_i^*,\tilde{X}_i^*-\tilde{e}_i^*).
\end{aligned}
$$

Hence, we obtain that

$$
\begin{aligned}
\sum_{i=1}^{n}\langle \overline{R}(X,e_i)X,e_i\rangle &= \sum_{i=1}^{n} K_M(X^*,e_i^*)(\langle X^*,X^*\rangle\langle e_i^*,e_i^*\rangle - \langle X^*,e_i^*\rangle^2)\\
&\quad +\sum_{i=1}^{n}\frac{1}{\rho}D^2\rho(\tilde{X}_i^*-\tilde{e}_i^*,\tilde{X}_i^*-\tilde{e}_i^*).
\end{aligned}
$$

The Gauss equation allows to rewrite Ricci curvature of Σ^n as the following

$$
\begin{aligned}
Ric(X,X) &= \sum_{i=1}^{n}\langle \overline{R}(X,e_i)X,e_i\rangle + \sum_{i=1}^{n}\langle\alpha(X,X),\alpha(e_i,e_i)\rangle\\
&\quad -\sum_{i=1}^{n}\langle\alpha(X,e_i),\alpha(X,e_i)\rangle,
\end{aligned}
$$

for every $X \in \mathfrak{X}(\Sigma^n)$. On the other hand, remembering the decomposition (4.3), we have

$$\begin{aligned}\sum_{i=1}^{n}\langle\alpha(X,X),\alpha(e_i,e_i)\rangle &= \sum_{k=1}^{p}\langle A^{(k)}X,X\rangle\sum_{i=1}^{n}\langle A^{(k)}e_i,e_i\rangle-\langle AX,X\rangle\sum_{i=1}^{n}\langle Ae_i,e_i\rangle\\ &= \sum_{k=1}^{p}\langle S_kX,X\rangle tr(S_k)-\langle S_NX,X\rangle tr(S_N)\\ &= n\{\sum_{k=1}^{p}\langle A^{(k)}X,X\rangle H^{(k)}+\langle AX,X\rangle H^N\}\end{aligned}$$

and

$$\begin{aligned}-\sum_{i=1}^{n}\langle\alpha(X,e_i),\alpha(X,e_i)\rangle &= -\sum_{k=1}^{p}\sum_{i=1}^{n}\langle A^{(k)}X,e_i\rangle^2+\sum_{i=1}^{n}\langle AX,e_i\rangle^2\\ &= -\sum_{k=1}^{p}|A^{(k)}X|^2+|AX|^2.\end{aligned}$$

Therefore,

$$\begin{aligned}Ric(X,X) &= \sum_{i=1}^{n}\langle\alpha(X,X),\alpha(e_i,e_i)\rangle-\sum_{i=1}^{n}\langle\alpha(X,e_i),\alpha(X,e_i)\rangle\\ &= \sum_{i=1}^{n}\langle\overline{R}(X,e_i)X,e_i\rangle+\left|AX+\frac{nH^N}{2}X\right|^2-\sum_{k=1}^{p}\left|A^{(k)}X-\frac{nH^{(k)}}{2}X\right|^2\\ &\quad+\varepsilon\frac{n^2|\vec{H}|^2}{4}|X|^2.\end{aligned}$$

Since $K_M \geq 0$ and ρ is convex (i.e, $D^2\rho \geq 0$), we have $\sum_{i=1}^{n}\langle\overline{R}(X,e_i)X,e_i\rangle \geq 0$ and, with this,

$$Ric(X,X) \geq \left|AX+\frac{nH^N}{2}X\right|^2-\sum_{k=1}^{p}\left|A^{(k)}X-\frac{nH^{(k)}}{2}X\right|^2+\varepsilon\frac{n^2|\vec{H}|^2}{4}|X|^2$$

■

4.2 Main Results

We dedicate this section to some results of nonexistence and rigidity for r-trapped, as well marginally and weakly r-trapped, submanifolds immersed in a standard static spacetime $M^{n+p}\times_\rho\mathbb{R}_1$.

Proposition 4.2.1 *Let $\overline{M}^{n+p+1} = M^{n+p}\times_\rho\mathbb{R}_1$ be a standard static spacetime and $0 \leq r < n$ even. Then* *(i) There do not exist n-dimensional spacelike future (or past) r-trapped submanifolds contained in a slice of $\overline{M}^{n+p+1}$;*

(ii) There do not exist n-dimensional spacelike marginally future (or past) r-trapped submanifolds contained in a slice of $\overline{M}^{n+p+1}$.

Proof. Just note that, for every submanifold Σ^n contained in a slice of $\overline{M}^{n+p+1}$, we have that

$$h = const. \Rightarrow \nabla h = 0$$

and, consequently,

$$0 = L_r(h) = \frac{1}{\rho^2}k(r)\langle \vec{H}_{r+1}, K\rangle \Rightarrow \langle \vec{H}_{r+1}, K\rangle = 0.$$

■

Accordingly to the definition of weakly r-trapped submanifold, we have that $\vec{H}_{r+1}$ is causal (timelike or lightlike) or zero and, with this, we get the following

Proposition 4.2.2 *Let $\overline{M}^{n+p+1} = M^{n+p} \times_\rho \mathbb{R}_1$ be a standard static spacetime and, for $0 \leq r \leq n-1$ even, let $\psi : \Sigma^n \longrightarrow \overline{M}^{n+p+1}$ be a spacelike weakly r-trapped submanifold. If Σ^n is contained in a slice of $\overline{M}^{n+p+1}$, then Σ^n is r-minimal.*

Proof. Similarly to the demonstration of Proposition 1 above, we have $\langle \vec{H}_{r+1}, K\rangle = 0$ because, by hypothesis, Σ^n is contained in a slice of $\overline{M}^{n+p+1}$. Since $\vec{H}_{r+1}$ is causal or zero, it must be zero and, consequently, Σ^n is r-minimal. ■

We also studied the case where Σ^n is complete and non-compact. Our goal now is to study the behavior of the height function h in the case where the submanifold Σ^n is parabolic and, for this, the following definition is necessary: A spacelike submanifold $\psi : \Sigma \longrightarrow \overline{M}^{n+p+1} = M^{n+p} \times_\rho \mathbb{R}_1$ is called *bounded away from the future infinity at height $t^* \in I$* if

$$\psi(\Sigma) \subset \left\{(t, x) \in \overline{M}^{n+p+1};\ t \leq t^*\right\}.$$

Similarly, we say that a spacelike submanifold $\psi : \Sigma \longrightarrow \overline{M}^{n+p+1}$ is *bounded away from the past infinity at height $t_* \in I$* if

$$\psi(\Sigma) \subset \left\{(t, x) \in \overline{M}^{n+p+1};\ t \geq t_*\right\}.$$

Lastly, Σ^n is said to be *bounded away from the infinity* of $\overline{M}^{n+p+1}$ if it is bounded away from the past and future infinity.

Considering $r = 0$ in equation (4.2), follows that

$$\Delta h = -2\langle \nabla \ln(\rho), \nabla h\rangle + \frac{1}{\rho^2} n \langle \vec{H}, K\rangle.$$

So, we can consider the conformal change $\langle \cdot, \cdot \rangle_\rho = \rho^{4/(n-2)} \langle \cdot, \cdot \rangle$ and obtain

$$\widehat{\Delta} h = n\rho^{-2n/(n-2)} \langle \vec{H}, K\rangle,$$

where $\widehat{\Delta} h$ denotes the Laplacian of h with respect the conformal metric $\langle \cdot, \cdot \rangle_\rho$. Thus, the conformal metric change $\langle \cdot, \cdot \rangle_\rho = \rho^{4/(n-2)} \langle \cdot, \cdot \rangle$ is made with the intention of cancelling $\langle \nabla \rho, \nabla h\rangle$. In this context, Aledo, Rubio and Salamanca (see Theorems 15 and 16 in [8]) obtained results of non-existence and rigidity of spacelike submanifold in standard static spacetime such that $\langle \vec{H}, K\rangle \geq 0$ (or ≤ 0). However, the general case in which $r \geq 0$ presents greater difficulties even under a conformal metric change. Firstly, because the parabolicity of the submanifold in question must be transferred to the conformal metric, which is possible under some conditions on r due to a characterization result made by Troyanov (see Proposition 4.1 in [48] with $p = 2$). Secondly, because it is the L_r operator, we should ask for some estimate involving the Laplace-Beltrami operator Δ, that is,

$$L_r \leq \beta \Delta.$$

But, we should obtain, from there, an equivalent estimate involving the operators in the conformal metric. In this case, the difficulty is due to the presence of $T_r(\nabla \ln(\rho), \nabla h)$ in the equation (4.2).

On the other hand, he particular case in which the warping function ρ is constant deserves special mention, since in this case

$$L_r(h) = \frac{1}{\rho^2} k(r) \langle \vec{H}_{r+1}, K\rangle.$$

Consequently, by asking for the estimate $L_r \leq \beta\Delta$, we obtain the following

Proposition 4.2.3 *Let $\overline{M}^{n+p+1} = M^{n+p} \times \mathbb{R}_1$ be a product spacetime and consider $0 \leq r < n$ even.*

(i) There does not exist an n-dimensional parabolic spacelike past r-trapped submanifold immersed in $\overline{M}^{n+p+1}$ bounded away from the future infinity and such that $L_r \leq \beta\Delta$, for some positive constant β;

(ii) There does not exist an n-dimensional parabolic spacelike marginally past r-trapped submanifold immersed in $\overline{M}^{n+p+1}$ bounded away from the future infinity and such that $L_r \leq \beta\Delta$, for some positive constant β;

(iii) Every n-dimensional parabolic spacelike weakly past r-trapped submanifold immersed in $\overline{M}^{n+p+1}$ bounded away from the future infinity and such that $L_r \leq \beta\Delta$, for some positive constant β, must be r-minimal.

As the reader can see in example 4.6.4 in section 4.6, the hypothesis of parabolicity in the Proposition 4.2.3 above cannot be removed.

4.3 Results in Standard Static Spaceforms

For a smooth function $\phi : \Sigma^n \to \mathbb{R}$, remember that the ϕ-divergent of a vector field $X \in \mathfrak{X}(\Sigma)$ is defined by

$$\operatorname{div}_\phi X = e^\phi \operatorname{div}\left(e^{-\phi} X\right) = \operatorname{div} X - \langle \nabla\phi, X\rangle .$$

With this, take $0 \leq r \leq n$ even and consider the operator $L_{r,\phi} : C^\infty(\Sigma) \to C^\infty(\Sigma)$ given by

$$\begin{aligned} L_{r,\phi}(u) &= \operatorname{div}_\phi \big(T_r(\nabla u)\big) \\ &= \operatorname{div}\big(T_r(\nabla u)\big) - T_r(\nabla\phi, \nabla u) \\ &= L_r(u) + \langle \operatorname{div} T_r, \nabla u\rangle - T_r(\nabla\phi, \nabla u). \end{aligned}$$

Thus, if $\overline{M}^{n+p+1}$ has constant sectional curvature, it follows that

$$L_{r,\phi}(u) = L_r(u) - T_r(\nabla\phi, \nabla u), \ \forall\ u \in C^\infty(\Sigma).$$

Consequently, from (4.2),

$$L_{r,\phi}(h) = \frac{1}{\rho^2} k(r) \langle \vec{H}_{r+1}, K\rangle, \tag{4.4}$$

where $\phi = -2\ln\rho$.

Using the definition of L_ϕ-parabolicity introduced in section 1.6 of the chapter 1, we say that the Σ^n is *$L_{r,\phi}$-parabolic* if it is L_ϕ-parabolic for $\varphi(p,t) = t$ and $T = T_r$ with $T_r \geq 0$ and $0 \leq r \leq n$ even, because in this case

$$L_\phi(u) = \operatorname{div}_\phi(T_r(\nabla u)) = L_{r,\phi}(u),$$

for every $u \in C^\infty(\Sigma)$.

Theorem 4.3.1 *Let* $\overline{M}_c^{n+p+1} = M^{n+p} \times_\rho \mathbb{R}_1$ *be a standard static spacetime with constant sectional curvature* c*,* $0 \le r < n$ *even and consider* $\phi = -2\ln\rho$*. Then*

(i) There do not exist n*-dimensional spacelike,* $L_{r,\phi}$*-parabolic, future (or past)* r*-trapped and bounded away from the future (or past) infinity submanifolds immersed in* $\overline{M}_c^{n+p+1}$*;*

(ii) There do not exist n*-dimensional spacelike,* $L_{r,\phi}$*-parabolic, marginally future (or past)* r*-trapped and bounded away from the future (or past) infinity submanifolds immersed in* $\overline{M}_c^{n+p+1}$*.*

(iii) The n*-dimensional spacelike* $L_{r,\phi}$*-parabolic, weakly future (or past)* r*-trapped and bounded away from the future (or past) infinity submanifolds immersed in* $\overline{M}_c^{n+p+1}$ *are* r*-minimal.*

Proof. Firstly we prove item (i) and item (ii) follows in an analogous way. Suppose, by contradiction, that Σ^n is such a submanifold. Once Σ^n is future (or past) r-trapped, it follows from equation (4.4) that

$$L_{r,\phi}(h) = \frac{1}{\rho^2} k(r) \langle H_{r+1}, K\rangle < 0 \text{ (or } > 0).$$

Thus, by the definition of $L_{r,\phi}$-parabolicity given above, we obtain that the height function h of Σ^n in $\overline{M}^{n+p+1}$ must be constant. For (iii) we have that

$$L_{r,\phi}(h) = \frac{1}{\rho^2} k(r) \langle H_{r+1}, K\rangle \le 0 \text{ (or } \ge 0).$$

As previously we get the desired result using $L_{r,\phi}$-parabolicity.

■

As a consequence of Lemma 1.6.2, we replaced the $L_{r,\phi}$-parabolicity hypothesis and obtained the following

Theorem 4.3.2 *Let* $\overline{M}_c^{n+p+1} = M^{n+p} \times_\rho \mathbb{R}_1$ *a standard static spacetimes with constant sectional curvature* c *and consider a complete spacelike submanifold* Σ^n *immersed in* $\overline{M}_c^{n+p+1}$ *bounded away from the future (or past) infinity and such that, for* $1 \le r < n$ *even,* $\langle \vec{H}_{r+1}, K\rangle \ge 0$ *(or* ≤ 0*). Moreover, assuming that* T_r *is positive definite and satisfying the assumptions* (1.13) *and* (1.15) *above. Let* $\xi_+(d)$ *be defined in* (1.13)*. If*

$$\frac{1}{\left(\int_{\partial B_t} \xi_+(d) e^{-t}\right)^{1/\delta}} \notin L^1(+\infty),$$

where $d = d(x, o)$ *is the geodesic distance in* Σ^n *from some fixed origin* $o \in \Sigma^n$ *and* $B_R = B_R(o)$ *is the geodesic ball centered at* o *with radius* R*. Then* Σ^n *is contained in a slice* $\{t_0\} \times M^{n+p}$*, for some* $t_0 \in \mathbb{R}$*.*

Proof. The idea is to use the $L_{r,\phi}$-parabolicity defined above and, for this, we have to $\varphi(t,p) = t$. Consequently, condition (1.14) is naturally satisfied. On the other hand, considering the hypotheses on Σ^n, the equation (4.4) provides

$$L_{r,\phi}(h) \geq 0 \text{ (or } \leq 0).$$

Accordingly to Lemma 1.6.2, we have Σ^n is $L_{r,\phi}$-parabolic and, with this, the height function h of Σ^n in $\overline{M}_c^{n+p+1}$ must be constant. ■

Remark 4.3.3 *Under the assumptions of Theorem 4.3.2, the spacelike submanifold Σ^n immersed in $\overline{M}_c^{n+p+1} = M^{n+p} \times_\rho \mathbb{R}_1$ can be neither r-trapped nor marginally r-trapped. If, in addition, we assume that the respective submanifold is weakly r-trapped, we obtain that Σ^n is r-minimal.*

Now, since the operator $L_{r,\phi}$ is divergent, it is natural to consider the closed case and apply the divergence theorem. More precisely, we have to

$$\Sigma^n \text{ closed } \Rightarrow \Sigma^n \ L_{r,\phi}\text{-parabolic}.$$

Therefore, since equation (4.4) holds when the ambient space has constant sectional curvature, we obtain the following

Corollary 4.3.4 *Let $\overline{M}_c^{n+p+1} = M^{n+p} \times_\rho \mathbb{R}_1$ be a standard static spacetime with constant sectional curvature and $0 \leq r < n$ even.*

(i) There do not exist n-dimensional closed spacelike future (or past) r-trapped submanifolds immersed in $\overline{M}_c^{n+p+1}$;

(ii) There do not exist n-dimensional closed spacelike marginally future (or past) r-trapped submanifolds immersed in $\overline{M}_c^{n+p+1}$;

(iii) Every n-dimensional closed spacelike weakly future (or past) r-trapped submanifold immersed in $\overline{M}_c^{n+p+1}$ is r-minimal.

4.4 An Omori-Yau Approach for the non-parabolic case

With a combination of Lemmas 1.5.1 and 4.1.2, we obtain the following

Theorem 4.4.1-A *Let $\overline{M}^{n+p+1} = M^{n+p} \times_\rho \mathbb{R}_1$ be a standard static spacetime whose Riemannian base M^{n+p} has nonnegative sectional curvature and, for $0 \leq r < n$ even, consider a complete weakly past r-trapped submanifold $\Sigma^n \to M^{n+p} \times_\rho \mathbb{R}_1$ with ρ convex and away from zero with $\nabla\rho$ and ρ bounded, such that $L_r \leq \beta\Delta$ for some a positive constant β, the second fundamental form is bounded and that Σ^n is bounded away from the future infinity. Then $|\vec{H}_{r+1}|$ cannot be away from zero. Particularly, if $\vec{H}_{r+1}$ is parallel then Σ^n is r-minimal.*

Proof. Let us suppose initially $\vec{H}_{r+1}$ is timelike. By Lemma 4.1.1 we have that

$$\begin{aligned} \beta\Delta h \geq L_r(h) &= -2T_r(\nabla \ln(\rho), \nabla h) + \frac{1}{\rho^2} k(r) \langle \vec{H}_{r+1}, K\rangle \\ &\geq -\frac{2C}{\rho}|\nabla\rho||\nabla h| + \frac{k(r)}{\rho}|\vec{H}_{r+1}|, \end{aligned} \tag{4.5}$$

Here, we use the fact that the limitation of the second fundamental form implies $T_r(\nabla\rho, \nabla h) \leq C|\nabla\rho||\nabla h|$ for some constant $C > 0$. Therefore multiplying (4.5) by ρ we obtain

$$\rho\beta\Delta h \geq -2C|\nabla\rho||\nabla h| + k(r)|\vec{H}_{r+1}|.$$

From Lemma 4.1.2 we can use Omori-Yau Maximum principle and, therefore, on an Omori-Yau sequence we obtain $|\vec{H}_{r+1}| \to 0$.

Note that, if $\vec{H}_{r+1}$ is lightlike, we have that

$$\vec{H}_{r+1} = \vec{H}^0_{r+1} - \frac{|\vec{H}^0_{r+1}|}{\rho} K,$$

where $\vec{H}^0_{r+1}$ is spacelike and orthogonal to K.

Therefore

$$\rho\beta\Delta h \geq L_r(h) \geq -2C|\nabla\rho||\nabla h| + k(r)|\vec{H}^0_{r+1}|$$

and on an Omori-Yau sequence we obtain $|\vec{H}^0_{r+1}| \to 0$. That means that $|\vec{H}_{r+1}|$ is not away from zero. ■

In view of lemma 2.2.3, we can impose a condition on the operator T_r with $0 \leq r < n$ even and obtain, under the hypothesis of the theorem **4.4.1-A** above, a result of rigidity for a submanifold immersed in a standard static spacetime $\overline{M}^{n+p+1}_c$ with constant sectional curvature.

Corollary 4.4.2 *Let $\overline{M}^{n+p+1} = M^{n+p} \times_\rho \mathbb{R}_1$ be a standard static spacetime with sectional curvature constant c and whose Riemannian base M^{n+p} has nonnegative sectional curvature and, for $0 \leq r < n$ even, consider a weakly past r-trapped submanifold $\Sigma^n \to M^{n+p} \times_\rho \mathbb{R}_1$ with ρ convex and away from zero with $\nabla\rho$ and ρ bounded, such that $L_r \leq \beta\Delta$, for some a positive constant β, with the second fundamental form is bounded and that Σ^n is bounded away from the future infinity. If $T_{r+1} = \frac{n-(r+1)}{n}\vec{H}_{r+1}I$, then Σ^n is r-minimal.*

Proof. Note that the traceless part $\overset{\circ}{T}_{r+1}$ of T_{r+1} is such that

$$\overset{\circ}{T}_{r+1} = T_{r+1} - \frac{n-(r+1)}{n}\vec{H}_{r+1}I = 0$$

and, from the lemma 2.2.3, $\operatorname{div}(T_{r+1}) = 0$. Therefore,

$$\begin{aligned} 0 &= \operatorname{div}\left(\overset{\circ}{T}_{r+1}\right) \\ &= -\frac{n-(r+1)}{n}\nabla\vec{H}_{r+1} \end{aligned}$$

and, with this, the result follows from theorem **4.4.1-A**. ■

Another important application of theorem **4.4.1-A** is in the particular case in which the ambient space is an m-dimensional vacuum spacetime with cosmological constant Λ, that is, a Lorentzian manifold (N^m, g) satisfying the Einstein equation $Ric = \Lambda g$. Within our configuration, we will have $N = M^{n+p} \times_\rho \mathbb{R}_1$ with $g = -\rho^2 dt^2 + g_M$, where (M^{n+p}, g_M) is an $(n+p)$-dimensional connected Riemannian manifold, that we will take to be orientable. On the other hand, a complete and connected Riemannian manifold (M^{n+p}, g_M) with boundary ∂M (possibly empty) is said to be static if it admits a non-trivial solution $\rho \in C^\infty(M)$ to the equation

$$-(\Delta_{g_M}\rho)g_M + \nabla^2_{g_M}\rho - \rho Ric_{g_M} = 0 \ \text{ in } \operatorname{int}(M). \tag{4.6}$$

It is important to note that a solution of (4.6) in a manifold allows us to construct a spacetime satisfying the vacuum Einstein equations with a cosmological constant.

Corollary 4.4.3 *Let $\overline{M}^{n+p+1} = M^{n+p} \times_\rho \mathbb{R}_1$ be an Einstein standard static spacetime with zero cosmological constant, nonnegative Ricci curvature, whose Riemannian base M^{n+p} has nonnegative sectional curvature and such that the warping function ρ is subharmonic in M^{n+p}. For $0 \leq r < n$ even, consider a complete weakly past r-trapped submanifold $\Sigma^n \to M^{n+p} \times_\rho \mathbb{R}_1$ such that $L_r \leq \beta\Delta$, for some a positive constant β, and the second fundamental form is bounded. If Σ^n is bounded away from the future infinity and $\nabla\rho$ and ρ are bounded, then $|\vec{H}_{r+1}|$ cannot be away from zero. Particularly, if $\vec{H}_{r+1}$ is parallel then Σ^n is r-minimal.*

Under some mild hypotheses for the tensor T_r, condition (1.11) above guarantees that the maximum principle is valid for the operator L_r, we prove the result of non-existence that follows.

Theorem 4.4.1-B *Let* $\overline{M}^{n+p+1} = M^{n+p} \times_\rho \mathbb{R}_1$ *be a standard static spacetime such that* ρ *and* $\nabla\rho$ *are bounded and let* $\psi : \Sigma^n \longrightarrow \overline{M}^{n+p+1}$ *be a complete, non-compact spacelike submanifold with bounded second fundamental form and whose radial sectional curvature satisfies the condition* (1.11). *Moreover assume that* Σ^n *is bounded away from the future infinity and, for some* $0 \leq r \leq n-1$ *even, suppose that* $\sup_\Sigma trT_r < +\infty$, $T_r \geq 0$ *and* $H_r > 0$. *Then* Σ^n *cannot be past* r*-trapped nor marginally past* r*-trapped. Particularly, if* Σ^n *is weakly past* r*-trapped then* Σ^n *must be* r*-minimal.*

Proof. Since the second fundamental form is limited, it follows that

$$\begin{aligned} L_r(h) &= -2T_r(\nabla \ln(\rho), \nabla h) + \frac{1}{\rho^2}k(r)\langle \vec{H}_{r+1}, K\rangle \\ &\geq -\frac{2c_2}{\rho}|\nabla\rho||\nabla h| + \frac{1}{\rho^2}k(r)\langle \vec{H}_{r+1}, K\rangle. \end{aligned}$$

On the other hand, by lemma 1.5.2, we have that the Omori-Yau maximum principle holds on Σ^n for positive semi-definite operator L_r. So, on an Omori-Yau sequence $\{p_j\}_{j\in\mathbb{N}}$ we obtain

$$\begin{aligned} \inf_\Sigma \langle \vec{H}_{r+1}, K\rangle &\leq \langle \vec{H}_{r+1}, K\rangle(p_j) \\ &< \frac{\rho^2(p_j)}{jk(r)} + 2c_2\rho(p_j)|\nabla\rho(p_j)||\nabla h(p_j)| \end{aligned}$$

Consequently, taking the limit with $j \to \infty$ and noting that ρ and $\nabla\rho$ are bounded, it follows that $\inf_\Sigma \langle \vec{H}_{r+1}, K\rangle \leq 0$. ■

4.5 The Product Spacetime Case

We dedicate this section to study the particular case of some of the results in chapters 3 and 4 when the warping function satisfies $\rho \equiv 1$, i.e., we turn our attention to spacelike submanifolds immersed in the product manifold $-I \times M^{n+p}$ (which, in turn, is both a GRW and an SSST).

Note that the hypothesis about the sectional curvature of the ambient space in the Theorem 4.3.4 provides the use of equation (4.4). However, when the ambient space is a product spacetime $M^{n+p}\times\mathbb{R}_1$ (i.e, when $\rho\equiv 1$), we have that equation (4.2) provides

$$L_r(h)=k(r)\langle\vec{H}_{r+1},K\rangle, \tag{4.7}$$

regardless of whether the sectional curvature of the ambient space is constant or not. On the other hand, we have that the ϕ-divergent coincides with the divergent when ϕ is constant (consequently, $L_{r,\phi}=L_r$). Thus, in the case where $\rho\equiv 1$, we can dispense with the hypothesis about the sectional curvature of the ambient space and obtain the next two corollaries.

Corollary 4.5.1 *Let $\overline{M}^{n+p+1}=M^{n+p}\times\mathbb{R}_1$ a product spacetime and consider a complete spacelike submanifold Σ^n immersed in $\overline{M}^{n+p+1}$ bounded away from the future (or past) infinity and such that, for $1\le r<n$ even, $\langle\vec{H}_{r+1},\partial_t\rangle\ge 0$ (or ≤ 0). Moreover, assuming that T_r is positive definite and satisfying the assumptions* (1.13) *and* (1.15) *above. Let $\xi_+(d)$ be defined in* (1.13). *If*

$$\frac{1}{\left(\int_{\partial B_s}\xi_+(d)e^{-s}\right)^{1/\delta}}\notin L^1(+\infty),$$

where $d=d(x,o)$ is the geodesic distance in Σ^n from some fixed origin $o\in\Sigma^n$ and $B_R=B_R(o)$ is the geodesic ball centered at o with radius R. Then Σ^n is contained in $\{t_0\}\times\mathbb{R}^{n+p}$, for some $t_0\in\mathbb{R}$.

Particularly in the closed case we have the following

Corollary 4.5.2 *Let $\overline{M}^{n+p+1}=M^{n+p}\times\mathbb{R}_1$ be a product spacetime and $0\le r<n$ even. Then it holds the following:*

(i) There do not exist n-dimensional closed spacelike future (or past) r-trapped submanifolds immersed in $\overline{M}^{n+p+1}$;

(ii) There do not exist n-dimensional closed spacelike marginally future (or past) r-trapped submanifolds immersed in $\overline{M}^{n+p+1}$;

(iii) Every n-dimensional closed spacelike weakly future (or past) r-trapped submanifold immersed in $\overline{M}^{n+p+1}$ is r-minimal.

Using the equation (4.7), we get the following

Corollary 4.5.3 *Let $\overline{M}^{n+p+1}=M^{n+p}\times\mathbb{R}_1$ be a product spacetime whose Riemannian base M^{n+p} has nonnegative sectional curvature and, for $0\le r<n$ even, consider a complete weakly future (or past) r-trapped submanifold $\Sigma^n\to\overline{M}^{n+p+1}$ bounded away*

from the past (or future) infinity and such that $L_r \leq \beta\Delta$ for some a positive constant β. Then $|\vec{H}_{r+1}|$ cannot be away from zero. Particularly, if $\vec{H}_{r+1}$ is parallel then Σ^n is r-minimal.

For the product case we can simplify the Theorem **4.4.1-B** and get the corollary below.

Corollary 4.5.4 *Let $\overline{M}^{n+p+1} = M^{n+p} \times \mathbb{R}_1$ be a product spacetime and let $\psi : \Sigma^n \longrightarrow \overline{M}^{n+p+1}$ be a complete, non-compact spacelike submanifold whose radial sectional curvature satisfies the condition (1.11). Moreover assume that Σ^n is bounded away from the future (or past) infinity and, for some $0 \leq r \leq n-1$ even, suppose that $\sup_\Sigma trT_r < +\infty$, $T_r \geq 0$ and $H_r > 0$. Then Σ^n cannot be past (or future) r-trapped nor marginally past (or future) r-trapped. In particular, if Σ^n is weakly past r-trapped then Σ^n must be r-minimal.*

4.6 Examples

Example 4.6.1 *Let Σ^{2n+1} be an immersed spacelike hypersurface into a Lorentzian manifold M_1^{2n+2}. For each $t \in I$ fixed, the inclusion $\phi_t : \Sigma^{2n+1} \to I \times M_1^{2n+2}$ is such that*

$$\vec{H}_{2n+1} = H_{2n+1} N,$$

where $\vec{H}_{2n+1}$ stands for the $(2n+1)$-th mean curvature of Σ^{2n+1} in M_1^{2n+2} with respect to its timelike Gauss map N.

Now, assuming that $\Sigma^3 = \mathbb{R} \times \mathbb{H}^2$ and $M = \mathbb{L}^4$ endowed with standard metric, we obtain that $k_1 = k_2 = -1$ and $k_3 = 0$. Let $N(t,p) = (0,p)$ be the timelike Gauss map of $\Sigma^3 \subset \mathbb{L}^4 \subset I \times \mathbb{L}^4$ where p is the position vector in $\mathbb{H}^2$. This way we have

$$\vec{H}_3 = 0, \;\; H_2 = \frac{1}{3} \;\; \text{and} \;\; \vec{H}_1 = -\frac{2}{3}N.$$

This submanifold is trapped but it is not 2-trapped, in fact it is 2-minimal.

Example 4.6.2 *Considering the surface $\Gamma = \{(t,x,y) \in M; (t,x,y) = (a\ln y, x, y)\}$, for some $a \neq 0$, in the space $M = -\mathbb{R} \times \mathbb{H}^2$. Let us consider the smooth function $u_a : \mathbb{H}^2 \to \mathbb{R}$ given by $u_a(x,y) = a\ln y$. Henceforth,*

$$N = \frac{\partial_t + Du_a}{\sqrt{1-a^2}},$$

with a straightforward computation we obtain that

$$\vec{H}_\Gamma = \frac{1}{2}\frac{a}{\sqrt{1-a^2}}N.$$

Now consider $\alpha(X,Y) = \lambda(X,Y)N$, where $\lambda(X,Y) = \langle AX, Y\rangle$ is the shape operator associated to N. Therefore

$$A = \frac{a}{\sqrt{1-a^2}} e^1 \otimes e_1.$$

Where e_1 is a unit vector associated to the non zero eigenvalue of A and e^1 its dual, according to Example 4.4 in [21]. Consider the submanifold $\Sigma = \Gamma\times\mathbb{S}^2 \subset -\mathbb{R}\times\mathbb{H}^2\times\mathbb{R}^3$. Notice that the lifting to the product of the vector fields N and p, where p is the position vector in $\mathbb{S}^2$. They constitue an orthonormal frame for $\mathfrak{X}^\perp(\Sigma)$. In this case the second fundamental form is given by

$$\alpha(X,Y) = \frac{a}{\sqrt{1-a^2}}\langle \pi_1(X), \pi_1(Y)\rangle N + \langle \pi_*(X), \pi_*(Y)\rangle \vec{p},$$

where π_1 is the projection onto the direction of e_1 which is a unit vector associated to the non zero eigenvalue of A, we notice it is well defined despite the choice of e_1 and π is the projection $\pi : \Sigma \to \mathbb{S}^2$. Therefore

$$\vec{H}_\Sigma = \frac{1}{4}\left[\frac{a}{\sqrt{1-a^2}}N + 2\vec{p}\right] \quad \textit{and} \quad \vec{H}_{\Sigma,3} = \frac{1}{12}\left[\frac{a}{\sqrt{1-a^2}}N + 2\vec{p}\right].$$

Then this submanifold is simultaneously 0-trapped and 2-trapped for any $a \in \left(\frac{2}{\sqrt{5}}, 1\right)$.

Remark 4.6.3 *Following the same reasoning, but considering Σ as $\Gamma\times\mathbb{R}$ instead of $\Gamma\times\mathbb{S}^2$, it follows that*

$$\vec{H}_\Sigma = \frac{1}{3}\left[\frac{a}{\sqrt{1-a^2}}N\right] \quad \textit{and} \quad \vec{H}_{\Sigma,3} = 0.$$

Example 4.6.4 *Let $\Gamma^2 \hookrightarrow M_1^3$ be a spacelike surface in a Lorentzian manifold M_1^3 and $\Sigma^2 \hookrightarrow P^3$ a surface in a Riemannian manifold P^3. Let $A_\Gamma = (\mu_i)$ and $B_\Sigma = (\lambda_j)$ be the diagonalized second fundamental forms of Γ^2 and Σ^2 respectively. Now consider the product $\Gamma^2\times\Sigma^2 \hookrightarrow M_1^3 \times P^3$. In this case, taking into account the definition of r-th mean curvature, for $0 \le r \le 4$, and the fact that $\vec{H}_1 = \vec{H}$, we have to*

$$\vec{H} = \frac{1}{2}[H_\Gamma N + H_\Sigma \nu],$$

where $H_\Gamma = -\frac{\mu_1+\mu_2}{2}$ and $H_\Sigma = \frac{\lambda_1+\lambda_2}{2}$, and

$$\vec{H}_3 = \frac{1}{12}[K_G^\Sigma H_\Gamma N + K_G^\Gamma H_\Sigma \nu],$$

where $K_G^\Gamma = -\mu_1\mu_2$ and $K_G^\Sigma = \lambda_1\lambda_2$, K_G is the Gauss-Kronecker curvature given by $K_G = \varepsilon \det A$ where $\varepsilon = \langle \eta, \eta\rangle$ for η the Gauss map of the Surface Γ^2 or Σ^2. Therefore

$$\langle \vec{H}, \vec{H}\rangle = \frac{1}{4}\left(-H_\Gamma^2 + H_\Sigma^2\right)$$

and

$$\langle \vec{H}_3, \vec{H}_3 \rangle = \frac{1}{144}[(K_G^\Gamma)^2 H_\Sigma^2 - (K_G^\Sigma)^2 H_\Gamma^2].$$

If the mean curvartures are equal, that is, $|H_\Gamma| = |H_\Sigma|$ *the submanifold* $\Gamma \times \Sigma$ *is marginally* 0*-trapped or minimal and with a small pertubation it can be trapped or not. If they are such that* $(K_G^\Sigma)^2 \neq (K_G^\Gamma)^2 \neq 0$. *We have*

$$\langle \vec{H}_3, \vec{H}_3 \rangle = \frac{1}{144}[(K_G^\Gamma)^2 - (K_G^\Sigma)^2] \neq 0.$$

Then for the different choices we can make of the Gauss-Kronecker curvatures, we have that $\Gamma \times \Sigma$ *is* 0*-trapped and* 2*-trapped, only* 0*-trapped but not* 2*-trapped, vice-versa or none of them. Particularly, if* Σ^2 *is a Clifford hypersurface in the sphere* $\mathbb{S}^3$ *given by*

$$\Sigma^2 = \mathbb{S}^1(\sin\theta) \times \mathbb{S}^1(\cos\theta) \hookrightarrow \mathbb{S}^3,$$

where $\theta \in (0, \pi/2)$ *is a positive angle, we have that* $\lambda_1 = cotg\theta$, $\lambda_2 = -tg\theta$ *and, consequently,*

$$H_\Sigma = cotg 2\theta \quad and \quad K_G^\Sigma = -1.$$

On the other hand, if Γ^2 *is the hyperbolic space*

$$\Gamma^2 = \mathbb{H}^2\left(-\frac{1}{S}\right) \hookrightarrow \mathbb{L}^3,$$

then $\mu_1 = \mu_2 = 1/S$ *and, with this,*

$$H_\Gamma = -\frac{1}{S} \quad and \quad K_G^\Gamma = -\frac{1}{S^2}.$$

So, turning our attention to $\Gamma^2 \times \Sigma^2 \hookrightarrow \mathbb{S}^3 \times \mathbb{L}^3$, *we have that*

$$\langle \vec{H}, \vec{H} \rangle = \frac{1}{4}\left(cotg^2 2\theta - \frac{1}{S^4}\right)$$

and

$$\langle \vec{H}_3, \vec{H}_3 \rangle = \frac{1}{144}\frac{1}{S^2}\left(\frac{cotg^2 2\theta}{S^2} - 1\right).$$

Therefore, for $S \neq 1$, *we have*

$$\begin{cases} \textit{if} \ \ |cotg2\theta| < 1/S^2, \ \Gamma^2 \times \Sigma^2 \ \textit{is 0-trapped but not } 2-\textit{trapped}; \\ \textit{if} \ \ |cotg2\theta| < S, \ \Gamma^2 \times \Sigma^2 \ \textit{is 2-trapped but not } 0-\textit{trapped}; \end{cases}$$

Moreover, if $S \neq 1$ *and* $|cotg2\theta| < S$, *then*

$$\langle \vec{H}_3, N \rangle = -\frac{1}{12}K_G^\Sigma H_\Gamma = -\frac{1}{12}\frac{1}{S}$$

and, with this, $\Gamma^2 \times \Sigma^2$ *is past* 2*-trapped.*

Chapter 5

De Lellis-Topping inequalities on weighted manifolds with boundary

A well-known result in Riemannian geometry is given by Schur's Lemma (see Lemma 1.2.1). By studying the stability of this result, De Lellis and Topping [20] approached the case where the metric is close to be Einstein and relation with its scalar curvature. In this setting, they demonstrated that

Lemma 5.0.5 (De Lellis-Topping [20]) *Let (Σ^n, g) be a closed Riemannian manifold of dimension $n \geq 3$, with nonnegative Ricci curvature. Then*

$$\int_\Sigma (R - \overline{R})^2 dv_g \leq \frac{4n(n-1)}{(n-2)^2} \int_\Sigma |Ric - (R/n)g|^2 \, dv_g, \tag{5.1}$$

where $\overline{R} = \frac{1}{Vol(\Sigma)} \int_\Sigma R dv_g$ is the average value of R over Σ^n. Furthermore, the equality occurs if and only if (Σ^n, g) is an Einstein manifold.

In a strict sense, the authors showed that if a manifold, in the conditions of the above theorem, is close to be Einstein, in the L^2-norm sense, its scalar curvature is close to be constant in the respective norm. Furthermore, De Lellis and Topping also demonstrated in [20] that the coefficient of the right hand in (5.1) is optimal and the hypothesis of nonnegative Ricci curvature is crucial for attains the result in dimensions greater or equal than five (in fact, for dimensions 3 and 4, the same result occurs for a weaker hypothesis of nonnegative scalar curvature, see [23] and [24]). A thorough analysis of the demonstration of theorem 5.0.5, shows that a crucial step is the integral identity below

$$-\int_M \langle \nabla R_g, \nabla u \rangle dv = \frac{2n}{n-2} \int_M \langle \overset{\circ}{Ric}_g, \nabla^2 u \rangle dv,$$

where u is the solution of a PDE and $\overset{\circ}{Ric}_g$ is the traceless Ricci tensor. For this identity, in turn, a use of integration by parts and the second Bianchi's identity is required. In fact, it is understood as a special case of a famous Pohozaev-type identity, demonstrated by R. Shoen ([44]).

In this scenario, for example, Cheng showed these type inequalities for symmetric $(0, 2)$-tensors satisfying a second Bianchi type identity and the Ricci curvature is bounded from below by a negative constant, where the inequality (5.1) has to be modified by taking into account the first nonzero eigenvalue of the Laplacian (for more details, see Theorem 1.7 in [15], and [16]). On the other hand, this results has been studied in manifolds with a nonempty boundary. Still within this scope, Ho [28] got a similar De Lellis-Topping type inequality for manifolds with a totally geodesic boundary.

In this chapter we propose study these type inequalities on weighted manifolds with constraints in the Bakry-Émery Ricci tensor. In section 5.1 we enunciate and demonstrate the main results of this chapter (see Theorems 5.1.1 and 5.1.4) and, in addition, we obtain direct corollaries. Finally, in section 5.2, we provide some applications of the main results. More generally, we extend the results obtained by Cheng in [15] to weighted manifolds with convex boundary. As particular cases of these results, we obtain versions that extend, for example, the Ho [28] result for the of convex boundary case.

5.1 Main Results

Throughout this section, we will work with a weighted manifold $(\Sigma^n, g, e^{-f} dv)$. In addition, it is worth mentioning that we deal with weighted manifolds with a convex boundary (i.e., $A_{\partial\Sigma} \geq 0$, where $A_{\partial\Sigma}$ is the second fundamental form of the immersion $\partial\Sigma \hookrightarrow \Sigma$) and with Bakry-Émery Ricci tensor bounded from below by a negative constant and obtain some De Lellis-Topping type inequalities for symmetric $(0, 2)$-tensors. The main result of this chapter is the following:

Theorem 5.1.1 *Let $(\Sigma^n, g, e^{-f}dv)$ be a compact n-dimensional weighted manifold with $n \geq 3$, convex boundary $\partial\Sigma$ and $f : \Sigma^n \longrightarrow \mathbb{R}$ a smooth function such that $(\partial f/\partial\nu) \equiv 0$ on $\partial\Sigma$, where ν is the exterior unit normal vector field along $\partial\Sigma$. Let T be a symmetric $(0,2)$-tensor field such that $T(\nu,\cdot) \geq 0$ along the boundary and $\operatorname{div} T = c\nabla B$, where $c \in \mathbb{R}$ is a constant and $B = tr_g T$ denotes the trace of T with respect to g. If $Ric_f \geq -(n-1)K_1 g$, where $K_1 \geq 0$ is a constant, and $K_2 := \max_{x\in M} \Delta f(x)$, then*

$$(nc-1)^2 \int_\Sigma \left(B - \overline{B}\right)^2 e^{-f} dv \leq n^2 \left(\frac{(n-1)K_1 + K_2}{\lambda_1} + \frac{n-1}{n}\right) \int_\Sigma |\overset{\circ}{T}|^2 e^{-f} dv, \quad (5.2)$$

where $\overline{B} = \left(\int_\Sigma B e^{-f} dv\right) / \left(\int_\Sigma e^{-f} dv\right)$ is the weighted average value of the B over Σ^n, λ_1 is the first nonzero eigenvalue for weighted Laplacian with Neumann boundary condition and $\overset{\circ}{T} = T - \left(tr_g T/n\right) g$ denotes traceless part of the tensor field T. Moreover, assuming positivity of Ricci curvature, the equality holds if and only if f is constant and $\overset{\circ}{T} = 0$.

Proof. Initially, note that if $c = 1/n$, then

$$0 = (nc-1) \int_\Sigma \left(B - \overline{B}\right)^2 e^{-f} dv$$

and inequality (5.2) follows trivially. So, suppose that $c \neq 1/n$.

Now, let $u : \Sigma^n \longrightarrow \mathbb{R}$ be the only solution of following PDE with Neumann boundary condition

$$\begin{cases} \Delta_f u = B - \overline{B} & \text{in } \Sigma; \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Sigma. \end{cases} \quad (5.3)$$

Moreover, note that the condition $\operatorname{div} T = c\nabla B$ provides

$$\begin{aligned} \operatorname{div} \overset{\circ}{T} &= c\nabla B - \frac{1}{n}\nabla B \\ &= \frac{nc-1}{n}\nabla B. \end{aligned}$$

Since $(\partial u/\partial\nu) = 0$ and $T(\nu,\cdot) \geq 0$ on $\partial\Sigma$, we can use the equation (1.6) and obtain

$$\begin{aligned} -\int_\Sigma \left\langle \operatorname{div} \overset{\circ}{T}, \nabla u \right\rangle e^{-f} dv &= \int_\Sigma \left\langle \overset{\circ}{T}, \widetilde{\nabla}^2 u \right\rangle e^{-f} dv - \int_{\partial\Sigma} \overset{\circ}{T}(\nu, \nabla u) e^{-f} d\mu \\ &\leq \int_\Sigma \left\langle \overset{\circ}{T}, \widetilde{\nabla}^2 u \right\rangle e^{-f} dv. \end{aligned}$$

Here we use the fact that $\overset{\circ}{T}(\nu, \nabla u) = T(\nu, \nabla u) \geq 0$ on $\partial\Sigma$. Thus,

$$
\begin{aligned}
(nc-1)\int_\Sigma \big(B-\overline{B}\big)^2 e^{-f}dv &= (nc-1)\int_\Sigma \big(B-\overline{B}\big)\Delta_f u e^{-f}dv\\
&= (nc-1)\left(-\int_\Sigma \langle \nabla B, \nabla u\rangle\, e^{-f}dv + \int_{\partial\Sigma}\big(B-\overline{B}\big)\frac{\partial u}{\partial \nu}e^{-f}d\mu\right)\\
&= -n\int_\Sigma \Big\langle \mathrm{div}(\mathring{T}), \nabla u\Big\rangle\, e^{-f}dv\\
&\leq n\int_\Sigma \Big\langle \mathring{T}, \widetilde{\nabla}^2 u\Big\rangle\, e^{-f}dv\\
&= n\int_\Sigma \Big\langle \mathring{T}, \widetilde{\nabla}^2 u - \frac{\Delta_f u}{n}g\Big\rangle\, e^{-f}dv\\
&\leq n\left(\int_\Sigma |\,\mathring{T}\,|^2 e^{-f}dv\right)^{1/2}\left(\int_\Sigma \left|\widetilde{\nabla}^2 u - \frac{\Delta_f u}{n}g\right|^2 e^{-f}dv\right)^{1/2}.
\end{aligned}
$$

By Reilly formula (1.8), the hypothesis $A_{\partial\Sigma}\geq 0$ and $\big(\partial u/\partial\nu\big)=0$ on $\partial\Sigma$ provides

$$
\int_\Sigma |\nabla^2 u|^2 e^{-f}dv \leq \int_\Sigma \big(-Ric_f(\nabla u,\nabla u) + (\Delta_f u)^2\big)e^{-f}dv, \tag{5.4}
$$

and so

$$
\begin{aligned}
&\int_\Sigma |\widetilde{\nabla}^2 u - \frac{\Delta_f u}{n}g|^2 e^{-f}dv\\
= &\int_\Sigma \left(|\widetilde{\nabla}^2 u|^2 - \frac{(\Delta_f u)^2}{n}\right)e^{-f}dv\\
= &\int_\Sigma \left(|\nabla^2 u|^2 - 2\nabla^2 u(\nabla u,\nabla f) + \frac{|\nabla f|^2|\nabla u|^2 + \langle \nabla f,\nabla u\rangle^2}{2} - \frac{(\Delta_f u)^2}{n}\right)e^{-f}dv\\
\leq &\int_\Sigma \left(|\nabla^2 u|^2 - \big\langle \nabla f, \nabla|\nabla u|^2\big\rangle + |\nabla f|^2|\nabla u|^2 - \frac{(\Delta_f u)^2}{n}\right)e^{-f}dv\\
\leq &\int_\Sigma \left(\Big(1-\frac{1}{n}\Big)(\Delta_f u)^2 - Ric_f(\nabla u,\nabla u) - \big\langle \nabla f, \nabla|\nabla u|^2\big\rangle + |\nabla f|^2|\nabla u|^2\right)e^{-f}dv,
\end{aligned}
$$

where the first inequality follows from the fact that $\langle \nabla f,\nabla u\rangle \leq |\nabla f||\nabla u|$ and the last of (5.4). We also used the fact that $2\nabla^2 u(\nabla u,\nabla f) = \langle \nabla f, \nabla|\nabla u|^2\rangle$. Now, since

$$
-\big\langle \nabla f, \nabla|\nabla u|^2\big\rangle = |\nabla u|^2\Delta f - \mathrm{div}(|\nabla u|^2\nabla f),
$$

follows that

$$
\int_\Sigma -\big\langle \nabla f, \nabla|\nabla u|^2\big\rangle\, e^{-f}dv = \int_\Sigma |\nabla u|^2\Delta f e^{-f}dv - \int_\Sigma \mathrm{div}(|\nabla u|^2\nabla f)e^{-f}dv
$$

$$
\begin{aligned}
&= \int_\Sigma |\nabla u|^2 \Delta f e^{-f} dv - \int_\Sigma |\nabla u|^2 |\nabla f|^2 e^{-f} dv \\
&\quad - \int_\Sigma \mathrm{div}(e^{-f}|\nabla u|^2 \nabla f) dv \\
&= \int_\Sigma |\nabla u|^2 \Delta f e^{-f} dv - \int_\Sigma |\nabla u|^2 |\nabla f|^2 e^{-f} dv \\
&\quad - \int_{\partial\Sigma} |\nabla u|^2 \frac{\partial f}{\partial \nu} e^{-f} d\mu \\
&= \int_\Sigma |\nabla u|^2 \Delta f e^{-f} dv - \int_\Sigma |\nabla u|^2 |\nabla f|^2 e^{-f} dv.
\end{aligned}
$$

Furthermore, since the first nonzero eigenvalue for weighted Laplacian with Neumann boundary condition λ_1 on Σ^n is characterized by

$$
\lambda_1 = min \left\{ \frac{\int_\Sigma |\nabla \varphi|^2 e^{-f} dv}{\int_\Sigma \varphi^2 e^{-f} dv} ; \varphi \text{ is nontrivial and } \frac{\partial \varphi}{\partial \nu} = 0 \text{ on } \partial\Sigma \right\}, \tag{5.5}
$$

we have that

$$
\begin{aligned}
\int_\Sigma |\nabla u|^2 e^{-f} dv &= -\int_\Sigma u \Delta_f u e^{-f} dv + \int_{\partial\Sigma} u \frac{\partial u}{\partial \nu} e^{-f} d\mu \\
&= -\int_\Sigma u \Delta_f u e^{-f} dv \\
&= -\int_\Sigma u \big(B - \overline{B}\big) e^{-f} dv \\
&\leq \left(\int_\Sigma u^2 e^{-f} dv \right)^{1/2} \left(\int_\Sigma \big(B - \overline{B}\big)^2 e^{-f} dv \right)^{1/2} \\
&\leq \left(\frac{\int_\Sigma |\nabla u|^2 e^{-f} dv}{\lambda_1} \right)^{1/2} \left(\int_\Sigma \big(B - \overline{B}\big)^2 e^{-f} dv \right)^{1/2}.
\end{aligned}
$$

Consequently

$$
\int_\Sigma |\nabla u|^2 e^{-f} dv \leq \frac{1}{\lambda_1} \int_\Sigma \big(B - \overline{B}\big)^2 e^{-f} dv.
$$

Therefore,

$$
\begin{aligned}
&\int_\Sigma |\widetilde{\nabla}^2 u - \frac{\Delta_f u}{n} g|^2 e^{-f} dv \\
\leq\ & \int_\Sigma \left(-Ric_f(\nabla u, \nabla u) + \left(\frac{n-1}{n} \right) (\Delta_f u)^2 \right) e^{-f} dv \\
& - \int_\Sigma \Big(\big\langle \nabla f, \nabla |\nabla u|^2 \big\rangle - |\nabla f|^2 |\nabla u|^2 \Big) e^{-f} dv \\
=\ & \int_\Sigma \left(-Ric_f(\nabla u, \nabla u) + \left(\frac{n-1}{n} \right) (\Delta_f u)^2 \right) e^{-f} dv \\
& + \int_\Sigma |\nabla u|^2 \Delta f e^{-f} dv
\end{aligned}
$$

$$
\begin{aligned}
&\le \int_\Sigma \left((n-1)K_1|\nabla u|^2 + \left(\frac{n-1}{n}\right)(\Delta_f u)^2 + K_2|\nabla u|^2\right) e^{-f}dv \\
&= \Big((n-1)K_1 + K_2\Big)\int_\Sigma |\nabla u|^2 e^{-f}dv + \frac{n-1}{n}\int_\Sigma (\Delta_f u)^2 e^{-f}dv \\
&\le \left(\frac{(n-1)K_1+K_2}{\lambda_1}\right)\int_\Sigma \left(B-\overline{B}\right)^2 e^{-f}dv + \frac{n-1}{n}\int_\Sigma \left(B-\overline{B}\right)^2 e^{-f}dv \\
&= \left(\frac{(n-1)K_1+K_2}{\lambda_1} + \frac{n-1}{n}\right)\int_\Sigma \left(B-\overline{B}\right)^2 e^{-f}dv.
\end{aligned}
$$

Then, after a straightforward calculation, we get the desired inequality.

For the second part of the theorem, suppose that the equality in (5.2) holds. Hence, we must have to $\Delta f = K_2$. Thus, from classical divergence theorem and by the fact of $(\partial f/\partial\nu) = 0$ on $\partial\Sigma$, we have that $K_2 = 0$ and consequently f is constant. Therefore, $Ric_f = Ric$ and Reilly's formula (1.8) becomes

$$
\begin{aligned}
\int_\Sigma |\nabla^2 u|^2 e^{-f}dv &= \int_\Sigma (\Delta_f u)^2 e^{-f}dv \\
&\quad - \left(\int_\Sigma Ric_g(\nabla u,\nabla u)e^{-f}dv + \int_{\partial\Sigma} A_{\partial M}(\nabla u,\nabla u)e^{-f}d\mu\right)
\end{aligned}
$$

and equality in (5.2) only occurs when

$$Ric(\nabla u,\nabla u) = 0 \ \text{ and } \ A_{\partial\Sigma}(\nabla u,\nabla u) = 0.$$

But this is only possible when $\nabla u = 0$, since $Ric > 0$. Then u is constant and $\overset{\circ}{T}= 0$.

■

Equivalently, we have the following

Corollary 5.1.2 *With he same assumptions as for Theorem 5.1.1, we have that*

$$
\begin{aligned}
&(nc-1)^2\int_\Sigma \left|T - \frac{\overline{B}}{n}g\right|^2 e^{-f}dv \\
&\le \left[n(nc-1)^2 + n^2\left(\frac{(n-1)K_1+K_2}{\lambda_1} + \frac{n-1}{n}\right)\right]\int_\Sigma \left|T - \frac{B}{n}g\right|^2 e^{-f}dv.
\end{aligned}
$$

Moreover, assuming positivity of Ricci curvature, the equality holds if and only if f is constant and $\overset{\circ}{T}= 0$.

Proof. Just note that inequality (5.2) is equivalent to inequality (5.6). In fact, from (5.3) in the above demonstration, we have $\overline{B} = B - \Delta_f u$. Consequently,

$$
\begin{aligned}
|T - (\overline{B}/n)g|^2 &= |T - (B/n)g + (\Delta_f u/n)g|^2 \\
&= |T - (B/n)g|^2 + (1/n)(\Delta_f u)^2 \\
&= |T - (B/n)g|^2 + (1/n)(B-\overline{B})^2.
\end{aligned}
$$

Therefore,

$$\begin{aligned} n(nc-1)^2\int_\Sigma |T-(\overline{B}/n)g|^2 e^{-f}dv &= n(nc-1)^2\int_\Sigma |T-(B/n)g|^2 e^{-f}dv \\ &\quad +(nc-1)^2\int_\Sigma (B-\overline{B})^2 e^{-f}dv \end{aligned}$$

and the equivalence between their respective inequalities follows. ■

A slight change on the proof of Theorem 5.1.1 (that would evoke a solution to a analogous Dirichlet's problem to 5.3, in addition to the weighted Bochner identity) could give us an approach for closed manifolds.

Corollary 5.1.3 *Let $(\Sigma^n, g, e^{-f}dv)$ be a closed n-dimensional weighted manifold with $n \geq 3$. Let T be a symmetric $(0,2)$-tensor field such that* $\operatorname{div} T = c\nabla B$*, where $c \in \mathbb{R}$ is a constant and $B = tr_g T$ denotes the trace of T with respect to g. If $Ric_f \geq -(n-1)K_1 g$, where $K_1 \geq 0$ is a constant, and $K_2 := \max_{x\in\Sigma}\Delta f(x)$, then*

$$(nc-1)^2\int_\Sigma \left(B-\overline{B}\right)^2 e^{-f}dv \;\leq\; n^2\left(\frac{(n-1)K_1+K_2}{\lambda_1}+\frac{n-1}{n}\right)\int_\Sigma |\overset{\circ}{T}|^2 e^{-f}dv,$$

where $\overline{B} = \left(\int_\Sigma Be^{-f}dv\right)/\left(\int_\Sigma e^{-f}dv\right)$ is the weighted average value of the B over Σ^n, λ_1 is the first nonzero eigenvalue for weighted Laplacian with Dirichlet condition and $\overset{\circ}{T}= T - \left(tr_g T/n\right)g$ denotes traceless part of the tensor field T. Moreover, assuming positivity of Ricci curvature, the equality holds if and only if f is constant and $\overset{\circ}{T}= 0$.

As mentioned in the chapter 1 (see section 1.4), we can take a $(0,2)$-tensor T and, from there, consider the weighted tensor $T_f = T + \nabla^2 f$. Moreover, remember that

$$\tilde{\nabla}^2 u = \nabla^2 u - \frac{\nabla f\otimes\nabla u+\nabla u\otimes\nabla f}{2}.$$

In this context, we obtain a De Lellis-Topping type inequality with weighted objects. More precisely, we have:

Theorem 5.1.4 *Let $(\Sigma^n, g, e^{-f}dv)$ be a compact smooth metric measure space with $n \geq 3$, convex boundary $\partial\Sigma$ and $f : \Sigma^n \longrightarrow \mathbb{R}$ smooth and such that $(\partial f/\partial\nu) \equiv 0$ on $\partial\Sigma$. Let T be a symmetric $(0,2)$-tensor field such that* $\operatorname{div} T = c\nabla B$ *and $T(\nu,\cdot)$ along of the boundary, where $c \geq 0$ is a constant and $B = tr_g T$. If $Ric_f \geq (\Delta f - (n-1)K)g$, where $K \geq 0$ is a constant, then*

$$\begin{aligned} \int_\Sigma \left(B_f-\overline{B}_f\right)^2 e^{-f}dv \;&\leq\; \frac{n^2}{(nc-1)^2}\left(\frac{(n-1)K}{\lambda_1}+\frac{n-1}{n}\right)\int_\Sigma |\overset{\circ}{T}_f - \nabla^2 f|^2 e^{-f}dv \\ &\quad + \int_\Sigma (\Delta f)^2 e^{-f}dv, \end{aligned} \tag{5.6}$$

where $T_f = T + \nabla^2 f$ *and* $B_f = tr_g T_f$, $\overline{B_f} = \left(\int_\Sigma B_f e^{-f} dv\right) / \left(\int_\Sigma e^{-f} dv\right)$ *is the weighted average value of the* B_f *over* Σ^n, λ_1 *is the first nonzero eigenvalue for weighted Laplacian with Neumann boundary condition and* $\overset{\circ}{T} = T - \Big(tr_g T/n\Big) g$ *denotes traceless part of the tensor field* T. *Moreover, assuming positivity of Ricci curvature, the equality holds if and only if* f *is constant and* $\overset{\circ}{T} = 0$.

Proof. Note that

$$B_f = tr_g T_f \Longrightarrow \nabla B = \nabla B_f - \nabla \Delta f.$$

Thus,

$$\begin{aligned}
\operatorname{div}\big(\overset{\circ}{T}_f\big) &= \operatorname{div}\big(T_f - \frac{B_f}{n} g\big) \\
&= \operatorname{div}\big(T_f\big) - \operatorname{div}\big(\frac{B_f}{n} g\big) \\
&= \operatorname{div}\big(T\big) + \operatorname{div}(\nabla^2 f) - \frac{1}{n}\nabla B_f \\
&= c\nabla B + \operatorname{div}(\nabla^2 f) - \frac{1}{n}\nabla B_f \\
&= c\nabla B_f - c\nabla\Delta f + \operatorname{div}(\nabla^2 f) - \frac{1}{n}\nabla B_f \\
&= \frac{nc-1}{n}\nabla B_f - c\nabla\Delta f + \operatorname{div}(\nabla^2 f).
\end{aligned}$$

Let $u : M^n \longrightarrow \mathbb{R}$ be the smooth function given by

$$\begin{cases} \Delta_f u = B_f - \overline{B_f} & \text{in } \Sigma; \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Sigma, \end{cases}$$

we have

$$\begin{aligned}
&\int_\Sigma \big(B_f - \overline{B}_f\big)^2 e^{-f} dv \\
&= \int_\Sigma \big(B_f - \overline{B}_f\big)\Delta_f u e^{-f} dv \\
&= -\int_\Sigma \langle \nabla B_f, \nabla u\rangle\, e^{-f} dv + \int_{\partial\Sigma} \big(B_f - \overline{B}_f\big)\frac{\partial u}{\partial \nu} e^{-f} d\mu \\
&= -\frac{n}{nc-1}\int_\Sigma \Big\langle \operatorname{div}\big(\overset{\circ}{T}_f\big) + c\nabla\Delta f - \operatorname{div}(\nabla^2 f), \nabla u\Big\rangle\, e^{-f} dv.
\end{aligned}$$

However,

$$\begin{aligned}
\int_\Sigma \Big\langle \operatorname{div}\big(\overset{\circ}{T}_f\big), \nabla u\Big\rangle\, e^{-f} dv &= -\int_\Sigma \Big\langle \overset{\circ}{T}_f, \tilde{\nabla}^2 u\Big\rangle\, e^{-f} dv + \int_{\partial\Sigma} \overset{\circ}{T}_f (\nabla u, \nu) e^{-f} d\mu \\
&= -\int_\Sigma \Big\langle \overset{\circ}{T}_f, \tilde{\nabla}^2 u\Big\rangle\, e^{-f} dv + \int_{\partial\Sigma} T_f(\nabla u, \nu) e^{-f} d\mu,
\end{aligned}$$

$$\begin{aligned}\int_\Sigma \langle \nabla \Delta f, \nabla u\rangle e^{-f} dv &= -\int_M \Delta f \Delta_f u e^{-f} dv + \int_{\partial\Sigma} \Delta f \frac{\partial u}{\partial \nu} e^{-f} d\mu \\ &= -\int_\Sigma \Delta f \Delta_f u e^{-f} dv\end{aligned}$$

and

$$\int_\Sigma \left\langle \operatorname{div}(\nabla^2 f), \nabla u\right\rangle e^{-f} dv = -\int_\Sigma \left\langle \nabla^2 f, \tilde{\nabla}^2 u\right\rangle e^{-f} dv + \int_{\partial\Sigma} \nabla^2 f(\nabla u, \nu) e^{-f} d\mu.$$

Therefore

$$\begin{aligned}&\int_\Sigma \left(B_f - \overline{B}_f\right)^2 e^{-f} dv \\ = &\ \frac{n}{(nc-1)}\left(\int_\Sigma \left\langle \overset{\circ}{T}_f - \nabla^2 f, \tilde{\nabla}^2 u\right\rangle e^{-f} dv\right) \\ &- \frac{n}{(nc-1)}\left(\int_{\partial\Sigma} \left(T_f(\nabla u, \nu) - \nabla^2 f(\nabla u, \nu)\right) e^{-f} d\mu\right) \\ &+ \frac{n}{(nc-1)}\left(\int_\Sigma c\Delta f \Delta_f u e^{-f} dv\right) \\ = &\ \frac{n}{(nc-1)}\left(\int_\Sigma \left\langle \overset{\circ}{T}_f - \nabla^2 f, \tilde{\nabla}^2 u\right\rangle e^{-f} dv\right) \\ &- \frac{n}{(nc-1)}\left(\int_{\partial\Sigma} T(\nabla u, \nu) e^{-f} d\mu\right) \\ &+ \frac{n}{(nc-1)}\left(\int_\Sigma c\Delta f \Delta_f u e^{-f} dv\right).\end{aligned}$$

By hypothesis, we have that $T(\nabla u, \nu) \geq 0$ implies

$$-\frac{n}{(nc-1)}\left(\int_{\partial\Sigma} T(\nabla u, \nu) e^{-f} d\mu\right) \leq 0.$$

Thus, of the fact of

$$\left\langle \nabla^2 f, \frac{\Delta_f u}{n} g\right\rangle = \frac{1}{n}\Delta f \Delta_f u,$$

follows that

$$\begin{aligned}&\int_\Sigma \left(B_f - \overline{B}_f\right)^2 e^{-f} dv \\ \leq &\ \frac{n}{(nc-1)}\left(\int_\Sigma \left\langle \overset{\circ}{T}_f - \nabla^2 f, \tilde{\nabla}^2 u - \frac{\Delta_f u}{n} g\right\rangle e^{-f} dv\right) \\ &+ \frac{n}{(nc-1)}\left(\int_\Sigma c\Delta f \Delta_f u e^{-f} dv - \int_\Sigma \frac{1}{n}\Delta f \Delta_f u e^{-f} dv\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{n}{(nc-1)}\left(\int_\Sigma \left\langle \overset{\circ}{T}_f - \nabla^2 f, \tilde{\nabla}^2 u - \frac{\Delta_f u}{n} g \right\rangle e^{-f} dv\right) \\
&\quad + \frac{n}{(nc-1)}\left(\int_\Sigma \frac{(nc-1)}{n} \Delta f \Delta_f u e^{-f} dv\right) \\
&= \frac{n}{(nc-1)}\left(\int_\Sigma \left\langle \overset{\circ}{T}_f - \nabla^2 f, \tilde{\nabla}^2 u - \frac{\Delta_f u}{n} g \right\rangle e^{-f} dv\right) \\
&\quad + \int_\Sigma \Delta f \Delta_f u e^{-f} dv \\
&\leq \frac{n}{(nc-1)}\left(\int_\Sigma |\overset{\circ}{T}_f - \nabla^2 f|^2 e^{-f} dv\right)^{1/2} \left(\int_M |\tilde{\nabla}^2 u - \frac{\Delta_f u}{n} g|^2 e^{-f} dv\right)^{1/2} \\
&\quad + \left(\int_\Sigma (\Delta f)^2 e^{-f} dv\right)^{1/2} \left(\int_\Sigma (\Delta_f u)^2 e^{-f} dv\right)^{1/2}.
\end{aligned}
$$

By Rielly formula, we have that

$$
\int_\Sigma |\nabla^2 u|^2 e^{-f} dv \leq \int_\Sigma \big(- Ric_f(\nabla u, \nabla u) + (\Delta_f u)^2 \big) e^{-f} dv, \tag{5.7}
$$

and so

$$
\begin{aligned}
&\int_\Sigma |\tilde{\nabla}^2 u - \frac{\Delta_f u}{n} g|^2 e^{-f} dv \\
&= \int_\Sigma |\tilde{\nabla}^2 u|^2 - \frac{(\Delta_f u)^2}{n} e^{-f} dv \\
&= \int_\Sigma \left(|\nabla^2 u|^2 - 2\nabla^2 u(\nabla u, \nabla f) + \frac{|\nabla f|^2 |\nabla u|^2 + \langle \nabla f, \nabla u \rangle^2}{2} - \frac{(\Delta_f u)^2}{n} \right) e^{-f} dv \\
&\leq \int_\Sigma \left(|\nabla^2 u|^2 - \big\langle \nabla f, \nabla |\nabla u|^2 \big\rangle + |\nabla f|^2 |\nabla u|^2 - \frac{(\Delta_f u)^2}{n} \right) e^{-f} dv \\
&\leq \int_\Sigma \left(\Big(1 - \frac{1}{n}\Big)(\Delta_f u)^2 - Ric_f(\nabla u, \nabla u) - \big\langle \nabla f, \nabla |\nabla u|^2 \big\rangle + |\nabla f|^2 |\nabla u|^2 \right) e^{-f} dv,
\end{aligned}
$$

where the first inequality follows from the fact that $\langle \nabla f, \nabla u \rangle \leq |\nabla f||\nabla u|$ and the last of (5.7). Now, since

$$
\begin{aligned}
-\big\langle \nabla f, \nabla |\nabla u|^2 \big\rangle &= |\nabla u|^2 \operatorname{div}(\nabla f) - \operatorname{div}(|\nabla u|^2 \nabla f) \\
&= |\nabla u|^2 \Delta f - \operatorname{div}(|\nabla u|^2 \nabla f),
\end{aligned}
$$

we have that

$$
\operatorname{div}(|\nabla u|^2 \nabla f) e^{-f} = \operatorname{div}(e^{-f} |\nabla u|^2 \nabla f) + |\nabla u|^2 |\nabla f|^2 e^{-f}
$$

and, then

$$\begin{aligned}
\int_\Sigma -\left\langle \nabla f, \nabla|\nabla u|^2\right\rangle e^{-f} dv &= \int_\Sigma |\nabla u|^2 \Delta f e^{-f} dv - \int_M \operatorname{div}(|\nabla u|^2 \nabla f) e^{-f} dv \\
&= \int_\Sigma |\nabla u|^2 \Delta f e^{-f} dv - \int_M |\nabla u|^2 |\nabla f|^2 e^{-f} dv \\
&\quad - \int_\Sigma \operatorname{div}(e^{-f}|\nabla u|^2 \nabla f) dv \\
&= \int_\Sigma |\nabla u|^2 \Delta f e^{-f} dv - \int_M |\nabla u|^2 |\nabla f|^2 e^{-f} dv \\
&\quad - \int_{\partial\Sigma} |\nabla u|^2 \frac{\partial f}{\partial \nu} e^{-f} dv \\
&= \int_\Sigma |\nabla u|^2 \Delta f e^{-f} dv - \int_M |\nabla u|^2 |\nabla f|^2 e^{-f} dv.
\end{aligned}$$

Now, using the characterization of the first nonzero Neumann eigenvalue λ_1 given by (5.5) and proceeding as in the proof of Theorem 5.1.1, it follows that

$$\int_\Sigma |\nabla u|^2 e^{-f} dv \leq \frac{1}{\lambda_1} \int_\Sigma \left(B_f - \overline{B}_f\right)^2 e^{-f} dv.$$

Hence

$$\begin{aligned}
& \int_\Sigma |\tilde{\nabla}^2 u - \frac{\Delta_f u}{n} g|^2 e^{-f} dv \\
\leq\ & \int_\Sigma \left(-Ric_f(\nabla u, \nabla u) + \left(\frac{n-1}{n}\right) (\Delta_f u)^2 \right) e^{-f} dv \\
& - \int_\Sigma \Big(\left\langle \nabla f, \nabla |\nabla u|^2 \right\rangle + |\nabla f|^2 |\nabla u|^2 \Big) e^{-f} dv \\
=\ & \int_\Sigma \left(-Ric_f(\nabla u, \nabla u) + \left(\frac{n-1}{n}\right) (\Delta_f u)^2 \right) e^{-f} dv \\
& + \int_\Sigma |\nabla u|^2 \Delta f e^{-f} dv \\
\leq\ & \int_\Sigma \left(-\big(\Delta f - (n-1)K\big) |\nabla u|^2 + \left(\frac{n-1}{n}\right) (\Delta_f u)^2 + |\nabla u|^2 \Delta f \right) e^{-f} dv \\
=\ & (n-1)K \int_\Sigma |\nabla u|^2 e^{-f} dv + \frac{n-1}{n} \int_\Sigma (\Delta_f u)^2 e^{-f} dv \\
\leq\ & \frac{(n-1)K}{\lambda_1} \int_\Sigma \left(B_f - \overline{B}_f\right)^2 e^{-f} dv + \frac{n-1}{n} \int_M \left(B_f - \overline{B}_f\right)^2 e^{-f} dv \\
=\ & \left(\frac{(n-1)K}{\lambda_1} + \frac{n-1}{n} \right) \int_\Sigma \left(B_f - \overline{B}_f\right)^2 e^{-f} dv.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\int_\Sigma \left(B_f-\overline{B}_f\right)^2 e^{-f}dv\\
\leq\ & \frac{n}{(nc-1)}\left(\int_\Sigma |\overset{\circ}{T}_f-\nabla^2 f|^2 e^{-f}dv\right)^{1/2}\left(\left(\frac{(n-1)K}{\lambda_1}+\frac{n-1}{n}\right)\int_\Sigma \left(B_f-\overline{B}_f\right)^2 e^{-f}dv\right)^{1/2}\\
&+\left(\int_\Sigma (\Delta f)^2 e^{-f}dv\right)^{1/2}\left(\int_\Sigma (\Delta_f u)^2 e^{-f}dv\right)^{1/2}\\
=\ & \frac{n}{(nc-1)}\left(\int_\Sigma |\overset{\circ}{T}_f-\nabla^2 f|^2 e^{-f}dv\right)^{1/2}\left(\left(\frac{(n-1)K}{\lambda_1}+\frac{n-1}{n}\right)\int_\Sigma \left(B_f-\overline{B}_f\right)^2 e^{-f}dv\right)^{1/2}\\
&+\left(\int_\Sigma (\Delta f)^2 e^{-f}dv\right)^{1/2}\left(\int_\Sigma \left(B_f-\overline{B}_f\right)^2 e^{-f}dv\right)^{1/2}
\end{aligned}
$$

and, with this,

$$
\begin{aligned}
\int_\Sigma \left(B_f-\overline{B}_f\right)^2 e^{-f}dv \ \leq\ & \frac{n^2}{(nc-1)^2}\left(\frac{(n-1)K}{\lambda_1}+\frac{n-1}{n}\right)\int_\Sigma |\overset{\circ}{T}_f-\nabla^2 f|^2 e^{-f}dv\\
&+\int_\Sigma (\Delta f)^2 e^{-f}dv.
\end{aligned}
$$

■

In the closed case, we have the following

Corollary 5.1.5 *Let $(\Sigma^n, g, e^{-f}dv)$ be a closed n-dimensional weighted manifold with $n \geq 3$. Let T be a symmetric $(0,2)$-tensor field such that $\operatorname{div} T = c\nabla B$, where $c \in \mathbb{R}$ is a constant and $B = tr_g T$ denotes the trace of T with respect to g. If $Ric_f \geq (\Delta f - (n-1)K)g$, where $K \geq 0$ is a constant, then*

$$
\begin{aligned}
\int_\Sigma \left(B_f-\overline{B}_f\right)^2 e^{-f}dv \ \leq\ & \frac{n^2}{(nc-1)^2}\left(\frac{(n-1)K}{\lambda_1}+\frac{n-1}{n}\right)\int_\Sigma |\overset{\circ}{T}_f-\nabla^2 f|^2 e^{-f}dv\\
&+\int_\Sigma (\Delta f)^2 e^{-f}dv,
\end{aligned}
$$

where $T_f = T + \nabla^2 f$ and $B_f = tr_g T_f$, $\overline{B_f} = \left(\int_\Sigma B_f e^{-f}dv\right) / \left(\int_\Sigma e^{-f}dv\right)$ is the weighted average value of the B_f over Σ^n, λ_1 is the first nonzero eigenvalue for weighted Laplacian with Dirichlet condition and $\overset{\circ}{T} = T - \left(tr_g T/n\right)g$ denotes traceless part of the tensor field T. Moreover, assuming positivity of Ricci curvature, the equality holds if and only if f is constant and $\overset{\circ}{T} = 0$.

5.2 Applications

In this section we will provide some applications of the main results contained in the previous section.

5.2.1 Locally Conservative Tensors

In view of the definition of locally conservative tensor, we can reinterpret the condition $\operatorname{div} T = c\nabla B$, since that

$$\begin{cases} c = 0 \Rightarrow \ T \text{ is locally conserved;} \\ c \neq 0 \Rightarrow \ T - cBg \text{ is locally conserved.} \end{cases}$$

Therefore, provided that the boundary conditions are valid, the inequalities (5.2) and (5.6) follows for locally conservative symmetric $(0,2)$-tensor fields. For a better understanding of the definition, the motivation in conservation laws, as well as a source of examples of locally conservative tensors, the reader can consult [25]. However, we highlight the following examples:

Example 5.2.1 The second Bianchi's identity provides that the Einstein tensor

$$E_g = Ric_g - \frac{R_g}{2} g$$

is locally conserved. Notice that $\overset{\circ}{E}_g = \overset{\circ}{Ric}_g$.

Example 5.2.2 In a more general context, we can look at a spacelike hypersurface immersed in a semi-Riemannian manifold of index 1 (in this case, a Lorentz manifold) or 0 (that is, a Riemannian manifold). For a more attentive reader, it can be perceived that there is an abuse of language in the case where the hypersurface is immersed in a Riemannian ambiente, since the induced metric is already, in itself, Riemannian. With this in mind, take a Einstein semi-Riemannian manifold $\big(\overline{M}^{n+1}, \overline{g}, \overline{\nabla}\big)$, consider a hypersurface $\varphi : \big(\Sigma^n, g, \nabla\big) \to \big(\overline{M}^{n+1}, \overline{g}, \overline{\nabla}\big)$ and, to fix the notations, let $\{e_1, ..., e_n, e_{n+1}\}$ be a local orthonormal frame on $\overline{M}^{n+1}$ with $e_{n+1} = \nu$ the exterior unit normal vector field along Σ^n. In this setting, we have

$$\begin{cases} \text{If the index of } \overline{g} \text{ is } 1, \text{ then } \overline{g}(\nu,\nu) = -1; \\ \text{If the index of } \overline{g} \text{ is } 0, \text{ then } \overline{g}(\nu,\nu) = 1. \end{cases}$$

From Codazzi's equations, we obtain that

$$\big\langle \overline{R}_{VW}\nu, X \big\rangle = \left(\nabla_V II\right)(W,X) - \left(\nabla_W II\right)(V,X), \ \forall X, V, W \in \mathfrak{X}(\Sigma),$$

where $\overline{R}$ is the curvature endomorphism of $\overline{M}^{n+1}$ and $II(X,Y) = \overline{g}\big(\alpha(X,Y), \nu\big)$ for every X and Y in $\mathfrak{X}(\Sigma)$. Now, remember that the Ricci tensor $\overline{Ric}$ in $\overline{M}^{n+1}$ is given by

$$\overline{Ric}(X,Y) = tr\left(\overline{R}(X,\cdot,Y,\cdot)\right) = \sum_{i=1}^{n+1} \overline{g}^{ik}\overline{R}(X,e_i,Y,e_k) = \sum_{i=1}^{n+1} \epsilon_i \overline{R}(X,e_i,Y,e_i), \tag{5.8}$$

where $\epsilon_i = \langle e_i, e_i \rangle$. In this way, making $V = e_k$ (with $1 \leq k \leq n$), $W = X = e_i$ and taking the sum with $1 \leq i \leq n+1$, we get that

$$\begin{aligned} \overline{Ric}_{k(n+1)} &= \sum_{i=1}^{n+1} \epsilon_i \overline{R}(e_k, e_i, e_{n+1}, e_i) \\ &= \sum_{i=1}^{n} \epsilon_i \overline{R}(e_k, e_i, e_{n+1}, e_i) \\ &= \sum_{i=1}^{n} \overline{R}(e_k, e_i, e_{n+1}, e_i) \\ &= \sum_{i=1}^{n} \Big((\nabla_{e_k} II)(e_i, e_i) - (\nabla_{e_i} II)(e_k, e_i) \Big). \end{aligned} \tag{5.9}$$

Considering the equation (5.9) above and assuming that the ambient space $\overline{M}^{n+1}$ is Einstein, it follows that $\overline{Ric}_{k(n+1)} = (\overline{R}/n)\overline{g}_{k(n+1)} = 0$, where $\overline{R}$ denotes the scalar curvature of $\overline{M}^{n+1}$, and so

$$\sum_{i=1}^{n} (\nabla_{e_k} II)(e_i, e_i) = \sum_{i=1}^{n} (\nabla_{e_i} II)(e_k, e_i).$$

Therefore, considering that the mean curvature of Σ^n is given by $H = tr(II)$, it follows that

$$\begin{aligned} (\nabla H)_k &= \nabla H(e_k) \\ &= \nabla_{e_k} H \\ &= \nabla_{e_k} tr(II) \\ &= tr\left(\nabla_{E_k} B\right) \\ &= \sum_{i=1}^{n} (\nabla_{e_k} II)(e_i, e_i) \\ &= \sum_{i=1}^{n} (\nabla_{e_i} II)(e_k, e_i) \end{aligned}$$

provides

$$\operatorname{div} II = \nabla H, \tag{5.10}$$

Since

$$
\begin{aligned}
(\operatorname{div} II)_k &= (\operatorname{div} II)(e_k) \\
&= \sum_{i=1}^{n} \langle (\nabla_{e_i} II)(e_k), e_i \rangle \\
&= \sum_{i=1}^{n} (\nabla_{e_i} II)(e_k, e_i) \\
&= (\nabla H)_k .
\end{aligned}
$$

Therefore, in this case, the tensor $T = II - Hg$ is locally conserved.

Example 5.2.3 Le Σ^n be a spacelike submanifold immersed in a semi-Riemannian manifold $\overline{M}^{n+m}$ of index 1 or 0. Thus, in view of Lemma 2.2.3, if $\overline{M}^{n+m}$ has constant sectional curvature and $0 \leq r \leq n$ is even, then

$$
\operatorname{div} T_r = 0,
$$

and therefore the symmetric $(0,2)$-tensor T_r is locally conserved.

Remark 5.2.4 *Since Lemma 2.2.3 deals with the case where the ambient space is Lorentz, for the case where the ambient space is Riemannian, see Lemma* 2.1 *in [26].*

As the reader can see in the section 2.2 of the chapter 2, the hypothesis that $r \in \{1, 2, ..., n\}$ is even above can be replaced by $m = 1$, that is, for every $0 \leq r \leq n$ holds $\operatorname{div} T_r = 0$ for a hypersurface Σ^n immersed in a space form M^{n+1}.

5.2.2 Weighted Almost Schur

Taking in account the example 5.2.1, we can obtain a weighted versions of Theorem 5.0.5. In fact, applying Theorem 5.1.1, we get the following

Corollary 5.2.5 *Let $(\Sigma^n, g, e^{-f}dv)$ be a compact weighted manifold n-dimensional with $n \geq 3$, convex boundary $\partial\Sigma$ and $f : \Sigma^n \longrightarrow \mathbb{R}$ smooth such that $(\partial f/\partial \nu) \equiv 0$ on $\partial\Sigma$. If $Ric_g(\nu, \cdot) \geq 0$ along of the boundary and $Ric_f \geq -(n-1)K_1 g$, where $K_1 \geq 0$ is a constant, and $K_2 := \max_{x \in \Sigma} \Delta f(x)$, then*

$$
\int_\Sigma \left(R_g - \overline{R}_g\right)^2 e^{-f} dv \;\leq\; \frac{4n^2}{(n-2)^2} \left(\frac{(n-1)K_1 + K_2}{\lambda_1} + \frac{n-1}{n} \right) \int_\Sigma |\, \overset{\circ}{Ric}_g \,|^2 e^{-f} dv,
$$

where λ_1 is the first nonzero eigenvalue for weighted laplacian with Neumann boundary. Moreover, assuming positivity of Ricci curvature, the equality holds if and only if f is constant and M^n is Einstein.

And, applying Theorem 5.1.4, we get the corollary below.

Corollary 5.2.6 *Let $(\Sigma^n, g, e^{-f}dv)$ be a compact weighted manifold n-dimensional with $n \geq 3$, convex boundary $\partial\Sigma$ and $f : \Sigma^n \longrightarrow \mathbb{R}$ smooth such that $(\partial f/\partial\nu) \equiv 0$ on $\partial\Sigma$. If $Ric_g(\nu, \cdot) \geq 0$ along of the boundary and $Ric_f \geq (\Delta f - (n-1)K)g$, where $K_1 \geq 0$ is a constant, and $K_2 := \max_{x\in\Sigma} \Delta f(x)$, then*

$$\begin{aligned}\int_\Sigma \big(R_f - \overline{R}_f\big)^2 e^{-f}dv \;\; &\leq \;\; \frac{n^2}{(nc-1)^2}\left(\frac{(n-1)K}{\lambda_1} + \frac{n-1}{n}\right)\int_\Sigma |\, \overset{\circ}{Ric}_f - \nabla^2 f|^2 e^{-f}dv \\ &\quad + \int_\Sigma (\Delta f)^2 e^{-f}dv,\end{aligned}$$

where $R_f = tr\big(Ric_f\big)$. Moreover, the equality holds if and only if f is constant and M^n is Einstein.

5.2.3 Hypersurfaces immersed in Einstein manifolds

In [17], the authors deal with "nearly" umbilical hypersurfaces, obtaining De Lellis-Topping inequalities in this setting. We observe that, taking in account the Example 5.2.2, we can obtain a improvent of that results on hypersurfaces with boundary and constraints in the Bakry-Émery Ricci tensor. In fact, applying Theorem 5.1.1, we get the following stability result

Corollary 5.2.7 *Let $(M^{n+1}, g, e^{-f}dv)$ be a Einstein manifold. Let Σ^n be a compact hypersurface immersed in $(M^{n+1}, g, e^{-f}dv)$ with $n \geq 3$, convex boundary $\partial\Sigma$ and $(\partial f/\partial\nu) = 0$ on $\partial\Sigma$. If If $A(\nu, \cdot) \geq 0$ along of the boundary of Σ^n and $Ric_f^\Sigma \geq -(n-1)K_1 g$, where $K_1 \geq 0$ is a constant, and $K_2 := \max_{x\in M} \Delta f(x)$, then*

$$\int_\Sigma \big(H - \overline{H}\big)^2 e^{-f}dv \;\; \leq \;\; \frac{n^2}{(n-1)^2}\left(\frac{(n-1)K_1 + K_2}{\lambda_1} + \frac{n-1}{n}\right)\int_\Sigma |\, \overset{\circ}{A}\, |^2 e^{-f}dv,$$

where λ_1 is the first nonzero eigenvalue for weighted Laplacian with Neumann boundary. Moreover, assuming positivity of Ricci curvature, the equality holds if and only if f is constant and Σ^n is totally umbilical.

5.2.4 Submanifolds immersed in Space Forms

In view of example 5.2.3, we have the following result for submanifolds immersed in spatial forms:

Corollary 5.2.8 *Let $(M^m, g, e^{-f}dv)$ be a Einstein manifold. Let Σ^n be a compact hypersurface immersed in $(M^m, g, e^{-f}dv)$ with $m > n \geq 3$, convex boundary $\partial\Sigma$, $(\partial f/\partial\nu) = 0$ on $\partial\Sigma$ and*

1. *$2 \leq r \leq n$ ie even or*

2. $m = n + 1$*, i.e.,* Σ^n *is a hypersurface.*

If $T_r(\nu, \cdot) \geq 0$ *along of the boundary of* Σ^n *and* $Ric_f^\Sigma \geq -(n-1)K_1 g$*, where* $K_1 \geq 0$ *is a constant, and* $K_2 := \max_{x \in M} \Delta f(x)$*, then*

$$(n-r)^2 \int_\Sigma \left(H_r - \overline{H_r}\right)^2 e^{-f} dv \leq n^2 \left(\frac{(n-1)K_1 + K_2}{\lambda_1} + \frac{n-1}{n}\right) \int_\Sigma |T_r - \frac{(n-r)H_r}{n} g|^2 e^{-f} dv,$$

where $\overline{H_r} = \left(\int_\Sigma H_r e^{-f} dv\right) / \left(\int_\Sigma e^{-f} dv\right)$ *is the weighted average value of the* $H + r$ *over* Σ^n*,* λ_1 *is the first nonzero eigenvalue for weighted Laplacian with Neumann boundary condition. Moreover, assuming positivity of Ricci curvature, the equality holds if and only if* f *is constant and* $T_r - \frac{(n-r)H_r}{n} g = 0$*.*

Remark 5.2.9 *It is worth mentioning that, given the explanation of the examples mentioned in [25], the same inequalities, with different motivations, can be thought in other locally conserved tensors, such as those involving Newton transformations, Einstein Lovelock tensors, Q-curvature and many others.*

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