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Ground state and nodal solutions for some elliptic equations involving the fractional Laplacian operator and Trudinger-Moser nonlinearity

por

Thiago Luiz de Oliveira do Rêgo

João Pessoa - PB
Julho/2020

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Thiago Luiz de Oliveira do Rêgo †

sob orientação

Orientador: Prof. Dr. Manassés Xavier de Souza
Coorientador: Prof. Dr. Uberlandio Batista Severo

Tese apresentada ao Corpo Docente do Programa de Pós-Graduação em Matemática - UFPB, como requisito parcial para obtenção do título de Doutor em Matemática.

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Resumo

Neste trabalho, estudamos existência de soluções ground state e soluções nodais de energia mínima para quatro classes de problemas envolvendo o operador Laplaciano fracionário com não linearidades que podem possuir crescimento exponencial crítico no sentido da desigualdade de Trudinguer-Moser. Provamos que as soluções ground state possuem sinal definido e mostramos que o nível nodal de energia mínima é maior que o dobro da energia ground state. O primeiro problema é definido num intervalo aberto e limitado de $\mathbb{R}$ e o segundo é definido em toda a reta real, ambos envolvendo o operador $1/2-$Laplaciano. O terceiro problema, também com o operador $1/2-$Laplaciano e definido em um intervalo limitado da reta real, é do tipo Kirchhoff-fracionário com função de Kirchhoff da forma $m_b(t) = a + bt$, com $a, b > 0$. Mostramos a existência de uma solução nodal de energia mínima, uma solução não negativa e uma solução não positiva, cada uma dessas possuindo energia mínima entre as soluções com sinal definido. Ainda neste caso, estudamos o comportamento assintótico das soluções nodais, quando $b \to 0^+$. O último problema abordado é definido em um domínio limitado $\Omega \subset \mathbb{R}^N$, $N \geq 2$, com fronteira Lipschitz $\partial\Omega$ e envolve o operador $N/s-$Laplaciano fracionário, $s \in (0, 1)$. Nesse caso, também encontramos uma solução nodal de energia mínima e soluções não triviais não negativa e não positiva ambas de menor energia entre as soluções com sinal definido. As principais ferramentas usadas nesse trabalho são: desigualdades do tipo Trudiguer-Moser, métodos variacionais, lema da deformação e teoria do grau.

Palavras-chave: Laplaciano fracionário, Problemas de Kirchhoff fracionário, Soluções nodais, Soluções de energia mínima, Desigualdade de Trudinger-Moser.

Abstract

In this work, we study the existence of ground state and least energy nodal solutions for four classes of problems involving the fractional Laplacian operator with nonlinearities that may have critical exponential growth in the sense of the Trudinguer-Moser inequality. We prove that ground state solutions have a defined signal and we show that the least energy nodal level is greater than twice the ground state level. The first problem is defined in an open bounded interval of $\mathbb{R}$ and the second one is defined in the whole real line, both involving the $1/2-$Laplacian operator. The third problem, also with the $1/2-$Laplacian operator and defined in an open bounded interval, is of Kirchhoff-fractional type with Kirchhoff function of the form $m_b(t) = a + bt$, with $a, b > 0$. We show the existence of a least energy nodal solution, a nonnegative solution and a nonpositive solution, each of which has minimum energy between the solutions with defined signal. In this case, we also study the asymptotic behavior of nodal solutions, when $b \to 0^+$. The last problem addressed is defined in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with Lipschitz boundary $\partial\Omega$ and involves the fractional $N/s-$Laplacian operator, $s \in (0, 1)$. In this case, we also found a least energy nodal solution and nontrivial nonnegative and nonpositive solutions, which have minimum energy between the solutions with defined signal. The main tools used in this study are: Trundiguer-Moser type inequalities, variational methods, deformation lemma and degree theory.

Keywords: Fractional Laplacian, Fractional Kirchhoff problems, Nodal solutions, Ground state solutions, Trudinger-Moser inequality.

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Para minha esposa, Adeline Gomes, e
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Introduction

In this thesis, our main purpose is to study the existence of nodal (sing-changing) solutions and ground state solutions with defined signal for four classes of problems involving the fractional Laplacian operator and the nonlinear terms may have critical or subcritical exponential growth in the Trudinger-Moser sense.

In recent years, we have seen an increasing interest in studying fractional Sobolev space and problems involving fractional type operators. The motivation to study fractional Sobolev space arises naturally when we deal with the characterization of $H^k(\mathbb{R}^N)$, for $k = 1, 2, \ldots$, by means of the Fourier transform. Using Fourier transform, the fractional Sobolev space $H^s(\mathbb{R}^N)$, for $s \in (0,1)$, can be described as

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (1 + |\xi|^{2s}) |\mathcal{F}u(\xi)|^2 \mathrm{d}\xi < \infty \right\}.$$

The fractional s-Laplacian operator $(-\Delta)^s$ is closely related to the space $H^s(\mathbb{R}^N)$, which for a function $u \in C_0^\infty(\mathbb{R}^N)$, it is defined by

$$(-\Delta)^s u(x) = -\frac{1}{2} C(N,s) \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} \mathrm{d}y, \quad \text{for all} \quad x \in \mathbb{R}^N$$

where $C(N,s)^{-1} = \int_{\mathbb{R}^N} \frac{1-\cos(\zeta_1)}{|\zeta|^{N+2s}} \mathrm{d}\zeta$ is a positive normalization constant. With respect to the Fourier transform, $(-\Delta)^s$ can be describe as

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F}u)$$

and the following relation holds

$$\|(-\Delta)^{s/2} u\|^2_{L^2(\mathbb{R}^N)} = \frac{1}{2} C(N,s) [u]^2_{H^s(\mathbb{R}^N)} = \frac{1}{2} C(N,s) \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x \mathrm{d}y,$$

where $[u]^2_{H^s(\mathbb{R}^N)}$ is the so called Gagliardo seminorm of u in $H^s(\mathbb{R}^N)$. The space $H^s(\mathbb{R}^N) \subset L^2(\mathbb{R}^N)$ is a Hilbert space with the norm $\|u\|_{H^s(\mathbb{R}^N)}$ defined by

$$\|u\|^2_{H^s(\mathbb{R}^N)} = \|u\|_2^2 + [u]^2_{H^s(\mathbb{R}^N)}.$$

An interesting fact about the fractional s-Laplacian operator is that, for any function $u \in C_0^\infty(\mathbb{R}^N)$, the classical Laplacian operator $-\Delta$ can be recovered by $(-\Delta)^s$ in the sense that

$$\lim_{s\to 0^+} (-\Delta)^s u(x) = u(x) \quad \text{and} \quad \lim_{s\to 1^-} (-\Delta)^s u(x) = -\Delta u(x).$$

The general fractional Sobolev space $W^{s,p}(\Omega) \subset L^p(\Omega)$, for $s \in (0,1)$ and $p \in [1,\infty)$, where Ω is an open set in $\mathbb{R}^N$, is defined by

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) : [u]_{W^{s,p}(\Omega)} < \infty \right\}$$

and when we consider the norm $\|u\|^p_{W^{s,p}(\Omega)} := \|u\|^p_{L^p(\Omega)} + [u]^p_{W^{s,p}(\Omega)}$ in $W^{s,p}(\Omega)$, it is a Banach space, where $[u]_{W^{s,p}(\Omega)}$ is the Gagliardo seminorm of u given by

$$[u]_{W^{s,p}(\Omega)} = \left(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{p}}.$$

This space has been extensively studied by several researchers. For an introduction to the basic theory of fractional Sobolev space, we suggest the survey of Di Nezza *et al.* in [32]. If $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $W^{s,p}(\Omega)$ is an intermediary Banach space between $L^p(\Omega)$ and the classical Sobolev space $W^{1,p}(\Omega)$ (see [32, Proposition 2.2]). Related to the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$, the fractional p-Laplacian operator, denoted by $(-\Delta)^s_p$, is a natural generalization of the fractional s-Laplacian operator, which is defined by

$$(-\Delta)^s_p u(x) := 2 \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} \,\mathrm{d}y, \; x \in \mathbb{R}^N,$$

for $u \in C_0^\infty(\mathbb{R}^N)$.

The fractional s-Laplacian operator also plays an important role in different areas of sciences. For example, fractional Schrödinger equations of the form

$$i\frac{\partial\psi}{\partial t} = (-\Delta)^s\psi + (V(x)+\omega)\psi - f(x,\psi), (x,t) \in \mathbb{R}^N \times \mathbb{R}$$

where $\omega \in \mathbb{R}$, $s \in (0,1)$, $V : \mathbb{R}^N \to \mathbb{R}$ is an external potential function and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuous function, are of interest in quantum mechanics. In fact, fractional operators are involved in many areas of sciences such as Biology, Chemistry, Finance or Physics (for more physical motivation, see [32] and [54, 55] and their references).

Motivated by physical or purely mathematical aspects, recently problems involving the fractional Laplacian have attracted the attention of many researchers and topics like existence, regularity, symmetry, uniqueness and stability were studied, see for example [33, 38, 39].

In [19], Caffarelli and Silvestre developed a method, called s-harmonic extension method, which expressed the nonlocal operator $(-\Delta)^s$ as a Dirichlet-Neumann operator in the domain $\mathbb{R}^{N+1}_+ = \{(x,t) \in \mathbb{R}^{N+1} : t > 0\}$. The techniques developed in [19] were widely used in several studies of equations involving fractional Laplacian operators, see for example [2, 8, 15, 35, 49].

As we said, our main goal in this work is to study the existence of nodal and ground state solutions for four classes of equations involving the fractional Laplacian operator in different contexts. More explicitly, for each of the problems addressed, we prove the existence of at least one nodal solution. We also show the existence of ground state solutions and that these solutions have defined signal. In each of the situations, we relate the least energy nodal level and the ground state level. Moreover, we are interested in looking for solutions when the nonlinearity involved has exponential growth, which is the maximal growth that allows us to treat the problems by variational methods. We emphasize that, in this thesis, we do not use the extension method in [19] and we prefer to analyze directly the problem by exploring the properties of the fractional Laplacian operator.

This thesis is divided into four chapters and one appendix. In what follows, we describe each of the chapters.

In Chapter 1, we study the existence of a least energy nodal solution and ground state solution for the following class of problems:

$$\begin{cases} (-\Delta)^{\frac{1}{2}} u + V(x)u = f(u) & \text{in} \quad \Omega \\ u = 0 & \text{in} \quad \mathbb{R} \setminus \Omega, \end{cases} \tag{1}$$

where $\Omega = (a,b)$ is a bounded open interval, $V : [a,b] \to [0,\infty)$ is a continuous function, the nonlinear term $f : \mathbb{R} \to \mathbb{R}$ is of class C^1 and behaves like $\exp(t^2)$ as

$t \to \infty$. In fact, exponential growth like $\exp(t^2)$ is the maximal growth that allows us to apply variational methods to treat problem (1). Next, we recall some known facts involving the limiting Sobolev embedding theorem in one-dimension. If $s \in (0, 1/2)$ then the Sobolev embedding states that $H^s(\mathbb{R}) \hookrightarrow L^{2_s^*}(\mathbb{R})$, where $2_s^* := 2/(1-2s)$ is the fractional critical Sobolev exponent. Moreover, this same result ensures that $H^{1/2}(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$ for any $q \in [2, \infty)$, but $H^{1/2}(\mathbb{R})$ is not continuously embedded in $L^\infty(\mathbb{R})$ (for more details see [32, 61]). Thus, if $s \in (0, 1/2)$ then the maximal growth on the nonlinearity $f(t)$, which lets us to work with (1) by considering a variational approach in $H^s(\mathbb{R})$, is given by $|t|^{2_s^*-1}$ as $|t| \to \infty$. On the other hand, in the limiting case $s = 1/2$, the maximal growth on $f(t)$, which allows us to study (1) by applying a variational framework involving the space $H^{1/2}(\mathbb{R})$, it is motivated by the Trudinger-Moser inequality proved by Ozawa [61] and improved by Kozono *et al.* in [53] and Takahashi in [69]. Precisely, by combining some of the results contained in [53, 61, 69], it was established that

$$\sup_{\{u \in H^{1/2}(\mathbb{R}) \,:\, \|u\|_{1/2,2} \leq 1\}} \int_{\mathbb{R}} (e^{\alpha u^2} - 1) \mathrm{d}x < \infty, \tag{2}$$

for any $0 \leq \alpha \leq \pi$, where

$$\|u\|_{1/2,2} := \left(\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x-y|^2} \mathrm{d}x\mathrm{d}y + \|u\|_2^2 \right)^{1/2},$$

is the so-called full Sobolev norm on $H^{1/2}(\mathbb{R})$. Motivated by (2), we say that $f(t)$ has *exponential critical growth* if there exists $\alpha_0 > 0$ such that

$$\lim_{|t| \to \infty} f(t) e^{-\alpha |t|^2} = \begin{cases} 0, & \text{for all } \alpha > \alpha_0, \\ \infty, & \text{for all } \alpha < \alpha_0, \end{cases} \tag{3}$$

and we say that $f(t)$ has *exponential subcritical growth* if

$$\lim_{|t| \to \infty} f(t) e^{-\alpha |t|^2} = 0, \text{ for all } \alpha > 0. \tag{4}$$

Based on this notion of criticality, many papers have been developed in order to study issues related to the existence of solutions for problems involving the fractional Laplacian operator and nonlinearities with exponential growth. For example, existence and multiplicity of solutions for similar problems to (1) were treated by different methods in [49, 50, 63]. By exploiting (2) and the Mountain-Pass Theorem, Iannizzotto and Squassina [50] proved the existence and multiplicity of solutions for the class of one-dimensional nonlocal equations

$$\begin{cases} (-\Delta)^{\frac{1}{2}}u = f(u) \quad \text{in} \quad (a,b), \\ u = 0 \quad \text{in} \quad \mathbb{R}\setminus(a,b), \end{cases}$$

when $f(t)$ is $o(|t|)$ at the origin and behaves like $e^{\alpha t^2}$ as $|t| \to +\infty$, for some $\alpha > 0$. Giacomoni *et al.* [49] studied the problem

$$\begin{cases} (-\Delta)^{\frac{1}{2}}u = \lambda g(x)|u|^{q-2}u + u^p e^{u^\beta} \quad \text{in} \quad (a,b), \\ u > 0 \quad \text{in} \quad (a,b) \\ u = 0 \quad \text{in} \quad \mathbb{R}\setminus(a,b), \end{cases}$$

where $1 < q < 2$, $p > 1$, $0 \leq \beta \leq 2$, $\lambda > 0$ and the function $g \in L^{\frac{p+q+\beta}{p+q+\beta-1}}(a,b)$. The authors showed the existence of mountain-pass solution when the nonlinearity is concave near at origin and has exponential growth at infinity. Furthermore, they showed the existence of multiple solutions for a suitable range of λ, by analyzing the fibering maps and the corresponding Nehari manifold.

We point out that none of the previous papers treated the existence of sign-changing solution (nodal solution). After a bibliographic review, we did not find works that study nodal solutions for similar problems to (1), even in the case that the nonlinearity has exponential subcritical growth. Motivated by this fact, our goal is to prove the existence of least energy nodal solutions for problem (1) when the nonlinearity has exponential growth in the Trudinger-Moser sense.

In order to find nodal solutions and ground state solutions for problem (1), we assume the following assumptions on the nonlinearity f:

(H_1) $f \in C^1(\mathbb{R})$ and there exists $C_0 > 0$ such that

$$|f(t)| \leq C_0 e^{\pi t^2}, \text{ for all } t \in \mathbb{R};$$

(H_2) $\displaystyle\lim_{t\to 0}\frac{f(t)}{t} = 0;$

(H_3) there exists $\theta > 2$ such that

$$0 < \theta F(t) := \theta \int_0^t f(s)\mathrm{d}s \leq tf(t), \text{ for all } t \in \mathbb{R}\setminus\{0\};$$

(H_4) the function $\dfrac{f(t)}{|t|}$ is strictly increasing for $t \neq 0$;

(H_5) there exist constants $p > 2$ and $C_p > 0$ such that

$$\text{sgn}(t)f(t) \geq C_p|t|^{p-1}, \text{ for all } t \in \mathbb{R}.$$

We observe that the hypothesis (H_1) allows us to consider nonlinearities with subcritical or critical growth in the sense defined in (3) and (4).

In this context, due to the critical growth on the nonlinearity f, a well-known difficulty to study the class of problems (1) is the loss of compactness of the energy functional associated. By analyzing an auxiliary polynomial problem involving the function $|t|^{p-2}t$, we will consider an estimate from below for the constant $C_p > 0$ in (f_5). Thus, we will obtain a suitable estimate for minimum energy of nodal solutions of (1) in way to overcome the lack of compactness. Under the hypotheses $(H_1) - (H_5)$, we will prove that the problem possesses a least energy nodal solution and a ground state solution. We also will show that the ground state solution is a nonpositive or a nonnegative function. Moreover, the energy of any sign-changing solution is strictly larger than twice the ground state energy (see Theorem 1.1.2 and Theorem 1.1.3). This property is the so-called energy doubling by Weth [71]. We emphasize that the results of this chapter were published in the article [29].

In Chapter 2, we deal with the following class of problems:

$$(-\Delta)^{\frac{1}{2}}u + V(x)u = K(x)f(u) \quad \text{in} \quad \mathbb{R}, \tag{5}$$

where $V, K : \mathbb{R} \to \mathbb{R}_+$ are continuous potentials and $f : \mathbb{R} \to \mathbb{R}$ has exponential growth in the sense of the Trudinger-Moser as in (3) and (4). Our goal in this chapter is to show that, under appropriate conditions in f, V and K, problem (5) has a least energy nodal solution and a nodal solution, which are distinct.

By exploiting the Trudinger-Moser embedding due to Ozawa [61] and the Mountain-Pass Theorem, do Ó *et al.* in [36] proved the existence of ground state solutions for the following class of nonlinear scalar field equations:

$$\begin{cases} (-\Delta)^{\frac{1}{2}}u + u = f(u) \quad \text{in} \quad \mathbb{R}, \\ u(x) \to 0, \quad \text{as} \quad |x| \to \infty, \end{cases}$$

when $f(t) = o(|t|)$ at the origin and behaves like $e^{\alpha t^2}$ as $|t| \to +\infty$, for some $\alpha > 0$. In [27], Souza and Araújo considered a perturbation of this problem by a general potential $V(x)$, namely,

$$\begin{cases} (-\Delta)^{\frac{1}{2}}u + V(x)u = f(u) \quad \text{in} \quad \mathbb{R}, \\ u(x) \to 0, \quad \text{as} \quad |x| \to \infty, \end{cases}$$

where $V(x)$ is a nonnegative function which is asymptotically periodic at infinity. See also [2, 23, 28, 34, 35, 48] for others investigations.

We would like to point out that recently Miyagaki and Pucci [60] have considered a nonlocal Kirchhoff problem of the form

$$-M(\|u\|)((-\Delta)^{\frac{1}{2}}u + V(x)u) = K(x)f(u) \quad \text{in} \quad \mathbb{R}, \tag{6}$$

where $M : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous Kirchhoff function, V and K are continuous positive potentials satisfying the conditions introduced in [34] and f is a nonlinearity with exponential critical growth with respect to the Trudinger-Moser inequality established by Ozawa [61]. In this work, by applying suitable variational methods, in order to overcome the lack of compactness due to the unboundedness of the domain and the Trudinger-Moser inequality, the authors have obtained the existence of nontrivial solutions for (6).

Again, we point out that none of the previous works treated the existence of sign-changing solution (nodal solution).

In this chapter, we assume the following assumptions on the functions V and K:

(V_1) $V, K : \mathbb{R} \to [0, \infty)$ are continuous and $K \in L^\infty(\mathbb{R})$;

(V_2) there exist $b_0, R_0 > 0$ such that

$$V(x) \geq b_0, \ \text{ for } |x| \geq R_0;$$

We emphasize that, assumptions $(V_1) - (V_2)$ allow that the potential V can be zero in a bounded interval. Since problem (5) is set on the whole real line, we face loss of compactness. Here, motivated by do Ó *et al.* in [35], in order to overcome this difficulty, we assume the following assumption on K:

(K_1) if $\{A_n\}$ is a sequence of Borel sets of $\mathbb{R}$ with $\sup\limits_{n \in \mathbb{N}} |A_n| \leq R$, for some $R > 0$, then

$$\lim_{r \to \infty} \int_{A_n \cap B_r^c(0)} K(x)\,\mathrm{d}x = 0,$$

uniformly with respect to $n \in \mathbb{N}$.

With respect to the nonlinearity f, we suppose that f satisfies conditions $(H_2)-(H_5)$ and the hypothesis

(H_1') $f \in C^1(\mathbb{R})$ and there exist $C_0, t_0 > 0$ such that

$$|f(t)| \leq C_0 \left(e^{\pi t^2} - 1\right), \quad \text{for all} \quad |t| \geq t_0;$$

By exploring the hypotheses (K_1) and (H_5), we handle with the lack of compactness due the unboundedness of the domain and the critical behavior of the nonlinearity. In fact, under theses hypotheses, we will show similar results to Chapter 1. Our goal is to show that problem (5) has a least energy nodal solution and a ground state, which are distinct. Moreover, we also show that the energy of any nodal solution is strictly larger than twice the ground state energy (see Theorem 2.2.3 and Theorem 2.2.4).

In Chapter 3, we study the existence of least energy nodal solution and ground state solutions for the following class of fractional Kirchhoff-type problems

$$\begin{cases} (a + b\|u\|^2)\left[(-\Delta)^{1/2}u + V(x)u\right] &= f(u) \quad \text{in} \quad \Omega, \\ u &= 0 \quad\quad \text{in} \quad \mathbb{R} \setminus \Omega, \end{cases} \tag{7}$$

where $a > 0$, $b \geq 0$, $\Omega \subset \mathbb{R}$ is a bounded open interval, $V : \overline{\Omega} \to [0, \infty)$ is a continuous potential, $f \in C^1(\mathbb{R})$ is a function that may have the exponential subcritical or critical growth in the Trudinger-Moser sense as in (3) and (4). Here, the function u belongs to an appropriate functional space and the norm $\|u\|$ is defined by

$$\|u\| = \left(\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x - y|^2} \mathrm{d}x\mathrm{d}y + \int_\Omega V(x)|u|^2 \mathrm{d}x\right)^{1/2}.$$

The motivation to study problem (7) comes from Kirchhoff equations of the type

$$-(a + b\|\nabla u\|_2^2)\,\Delta u = g(x, u) \quad \text{in} \quad \Theta, \tag{8}$$

where $\Theta \subset \mathbb{R}^N$ is a bounded domain, $N \geq 2$, u satisfies some boundary conditions and $g(x, u)$ satisfies some suitable assumptions. This class of problems is related to the stationary problem of a model introduced by Kirchhoff (see [52]) in the study on transverse vibrations of elastic strings proposed by the hyperbolic equation

$$\rho\,\frac{\partial^2 u}{\partial t^2} - \left(\frac{\tau_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2} = 0, \quad (x, t) \in [0, L] \times [0, +\infty), \tag{9}$$

where the parameters in the equation have the following meanings: L is the length of the string, h is the area of cross-section, E is the Young modulus of the material, ρ is the mass density and τ_0 is the initial tension. Eq. (9) is a generalization of the classical d'Alembert's wave equation, by considering the effects of the changes in the length of the strings during the vibrations. See [14, 58, 64] for classical studies of Kirchhoff-type problems.

Recently, Fiscella and Valdinoci [43] proposed a stationary Kirchhoff model driven by the fractional Laplacian by taking into account the nonlocal aspect of the tension, see [43, Appendix A] for more details. The Problem (7) is a version of (8) for the fractional Laplacian operator.

Similar problems to (7) have attracted a lot of attention of many researchers and some existence and multiplicity results have been obtained. Using variational methods in higher dimensions, the Kirchhoff problem for the fractional operator involving non-linearities of the type subcritical or critical power, in the sense of Sobolev, have been investigated, for example, by [7, 20, 41, 42, 43] and references therein. For fractional Kirchhoff problems in unbounded domains, Cheng and Gao [21] studied the existence of least energy nodal solution for the following equation:

$$(a + b[u]_{N,s})\,(-\Delta)^s u + V(x)u = f(x,u) \quad \text{in} \quad \mathbb{R}^N$$

where $N > 2s$, f is a Carathéodory function, $f(x,s) = o(|s|^3)$ as $|s| \to 0$, $f(x,s) = o(|s|^{p-1})$ as $|s| \to \infty$ and $f(x,s)/|s|^3$ is nondeacreasing on $\mathbb{R} \setminus \{0\}$.

A Kirchhoff type problem involving exponential growth was treated by Giacomoni *et al.* in [47], by using the Nehari method. Mingqi *et al.* in [59] proved the existence and multiplicity of solutions for a class of fractional Kirchhoff-type problems for the p-fractional Laplace operator. None of the previous papers treated the existence of sign-changing solution (nodal solution) for problem (7) when the nonlinearity has exponential growth. For the our knowledge, one of the first results in this direction is due to [29] (which was treated in our first chapter), where the authors have considered problem (7) with $a > 0$ and $b = 0$.

In Chapter 3, we will assume that f satisfies $(H_1) - (H_2)$ and the conditions

(H_3') there exists $\theta > 4$ such that

$$0 < \theta F(t) := \theta \int_0^t f(\tau)\mathrm{d}\tau \leq tf(t), \text{ for all } t \in \mathbb{R} \setminus \{0\};$$

(H_4') the function $\dfrac{f(t)}{|t|^3}$ is strictly increasing, for $t \neq 0$.

(H_5') there exist $p > 4$ and $C_p > 0$ such that

$$sgn(t)f(t) \geq C_p|t|^{p-1}, \text{ for all } t \in \mathbb{R}.$$

Due to the critical exponential growth on f, we need to overcome the loss of compactness of the energy functional associated to (7). As in Problem (1) of Chapter 1, the key to overcome this difficult is to exploit suitably the constant $C_p > 0$ in (H_5). However, the Kirchhoff term in (7) produces many additional difficulties in this study. By using an auxiliary Kirchhoff problem involving the polynomial function $|t|^{p-2}t/2$, we will find an estimate from below for C_p and therefore, under theses hypotheses, we prove the existence of at least three nontrivial solutions: a least energy nodal solution, one nonpositive and one nonnegative ground state solution. We will also study the asymptotic behavior of the nodal solutions as $b \to 0^+$. Explicitly, we will show that if $(b_n) \subset \mathbb{R}$ is a sequence such that $b_n > 0$ and $b_n \to 0^+$, then problem (P_{a,b_n}) has a least nodal solution u_{b_n} and, up to a subsequence, this sequence converge strongly (in an appropriated subspace of $H^{1/2}(\mathbb{R})$) to u_0, where u_0 is a least energy nodal solution of problem $(P_{a,0})$ (see Theorem 3.1.2, Theorem 3.1.3 and Theorem 3.1.4).

Finally, in Chapter 4, we study the existence and multiplicity solutions for the following class of problems involving the fractional N/s-Laplacian operator:

$$\begin{cases} (-\Delta)^s_{N/s}u + V(x)|u|^{\frac{N}{s}-2}u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{10}$$

where $\lambda > 0$, $s \in (0,1)$, $\Omega \subset \mathbb{R}^N$ is a bonded domain with Lipschitz boundary $\partial\Omega$, $N \geq 2$, $V : \overline{\Omega} \to \mathbb{R}$ is a continuous and nonnegative potential, and the nonlinearity f can have the maximal exponential growth, which allows us to study (10) by means of variational methods. We assume that the nonlinearity f satisfies the conditions

$(\widetilde{H_1})$ $f(x,t)$ is continuous and continuously differentiable in the variable t, and there exist $C_0, \alpha_0 > 0$ such that

$$|f(x,t)| \leq C_0 e^{\alpha_0 |t|^{\frac{N}{N-s}}}, \quad \text{for all} \quad (x,t) \in \Omega \times \mathbb{R};$$

$(\widetilde{H_2})$ $\displaystyle\lim_{t\to 0} \frac{f(x,t)}{|t|^{\frac{N}{s}-2}t} = 0$ uniformly in $x \in \Omega$;

$(\widetilde{H}_3)$ there exists $\theta > \frac{N}{s}$ such that

$$0 < \theta F(x,t) := \theta \int_0^t f(x,\tau)\,\mathrm{d}\tau \le t f(x,t), \quad \text{uniformly in } \Omega, \quad \text{for all } t \neq 0;$$

$(\widetilde{H}_4)$ the function $t \mapsto f(x,t)/|t|^{\frac{N}{s}-2}t$ is strictly increasing on $(0,\infty)$ and strictly decreasing on $(-\infty,0)$, uniformly in $x \in \Omega$;

$(\widetilde{H}_5)$ there exist $p > \frac{N}{s}$ and $C > 0$ such that

$$\operatorname{sgn}(t) f(x,t) \ge C|t|^{p-1}, \quad \text{for all } t \in \mathbb{R}, \text{ uniformly in } x \in \Omega.$$

Recently, Parini and Ruf in [62] proved a Trudinger-Moser inequality type for the fractional Sobolev space $\widetilde{W}_0^{s,N/s}(\Omega) \subset W^{s,N/s}(\Omega)$, defined as the closure of $C_0^\infty(U)$ with respect to the norm

$$u \mapsto \left([u]_{s,N/s}^{N/s} + \|u\|_{L^{N/s}(\Omega)}^{N/s}\right)^{\frac{s}{N}}.$$

In our context, the space $\widetilde{W}_0^{s,N/s}(\Omega)$ can be described by

$$\widetilde{W}_0^{s,N/s}(\Omega) = \{u \in L^{N/s}(\Omega) : u \equiv 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega \quad \text{and} \quad [u]_{s,N/s} < \infty\},$$

which is the appropriate functional space to treat problem (10) (for more details, see Section 4.3). In fact, they proved that there exists $\alpha_* > 0$ such that

$$\sup_{\{u \in \widetilde{W}_0^{s,N/s}(\Omega)\,:\,[u]_{s,N/s} \le 1\}} \int_\Omega e^{\alpha |u|^{N/(N-s)}} \mathrm{d}x < \infty \tag{11}$$

for all $\alpha \in [0,\alpha_*)$ and there exists $\alpha_{s,N}^*$ such that the supremum in (11) is infinity for all $\alpha > \alpha_{s,N}^*$ (see also [16, 17]).

In [63], Perera and Squassina, by using a suitable topological argument based on cohomological linking and by exploiting the Trudinger-Moser inequality, have studied the existence of multiple solutions for the following problem:

$$\begin{cases} (-\Delta)_{N/s}^s u = & \lambda |u|^{(N-2s)/s} \exp(|u|^{N/(N-s)}) \quad \text{in} \quad \Omega, \\ u = & 0 \quad \text{in} \quad \mathbb{R} \setminus \Omega, \end{cases}$$

where $\lambda > 0$ is a parameter. Mingqi *et al.* in [59] investigated the existence of solutions for folloowing class of fractional Kirchhoff-type problems:

$$\begin{cases} M\left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{N/s}}{|x-y|^{2N}} \mathrm{d}x\mathrm{d}y\right)(-\Delta)_{N/s}^s u &= f(x,u) \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{in} \quad \mathbb{R} \setminus \Omega, \end{cases}$$

where $N \geq 1$, $\Omega \subset \mathbb{R}^N$ is a bounded domaim with Lipschitz boundary $\partial\Omega$, $M : [0,\infty) \to [0,\infty)$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ are continuous functions with f that behaves like $\exp(\alpha|t|^{\frac{N}{N-s}})$ as $t \to \infty$. They proved the existence of a ground state solution with positive energy and the existence of nonnegative solutions with negative energy. By exploting a suitable Trudinger-Moser inequality for fractional Sobolev spaces in unbounded domains and a fixed point theorem, M. de Souza in [26] proved the existence of solution for the following fractional p-Laplacian equation:

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u = f(x,u) + \lambda h \quad \text{in} \quad \mathbb{R}^N,$$

where the nonlinear term f has exponential growth.

Ghosh *at al.* in [46] proved the existence of least energy nodal solution for the following fractional p-Laplacian problem:

$$\begin{cases} (-\Delta)_p^s u = & \lambda g(u) + f(x,u) \quad \text{in} \quad \Omega, \\ u = & 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $g(s)$ is singular at the origin and f is a power nonlinearity. For others similar problems, in the context of fractional Kirchhoff operators dealing with nodal solutions, see also [21, 45].

As we know, condition $(\widetilde{H_1})$ is the maximal growth which allows us to treat (10) variationally. Under assumptions $(\widetilde{H_1}) - (\widetilde{H_5})$, we show that (10) has one least energy nodal solution, one nonnegative and one nonpositive ground state solutions (see Theorem 4.2.2 and Theorem 4.2.3). We point out that our results complete the study presented in [21, 45, 46, 59, 63], since we work with nonlinearities that have maximal exponential growth and because we are interested in looking for nodal solutions.

In what follows, we will present a fundamental difference between problems involving local and nonlocal operators. In the special case of the stationary Schrödinger equation

$$-\Delta u + V(x)u = f(u) \quad \text{in} \quad \mathbb{R}^N, \tag{12}$$

there are several ways in the literature to obtain sign-changing solutions (see [3, 4, 9, 12, 44, 72]). However, the methods used in these works heavily rely on the following two decompositions:

$$J(u) = J(u^+) + J(u^-), \tag{13}$$

$$J'(u)u^+ = J'(u^+)u^+ \quad \text{and} \quad J'(u)u^- = J'(u^-)u^-, \tag{14}$$

where J is the energy functional associated to (12) given by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\mathrm{d}x - \int_{\mathbb{R}^N} F(u)\mathrm{d}x.$$

However, due to the Gagliardo seminorm $[u]_{s,q}$, the energy functional does not satisfy the decompositions as in (13) and (14). For example, if I is the energy functional associated to problem (1) and $u^{\pm} \neq 0$, then

$$I(u) > I(u^+) + I(u^-),$$

$$I'(u)u^+ > I'(u^+)u^+ \text{ and } I'(u)u^- > I'(u^-)u^-.$$

(see Lemma 1.2.3 and Lemma 4.3.6). This fact shows a great difference with the local operators case. Thus, the methods used to find nodal solutions for the local problems (such as problems involving the Laplacian operator) usually rely on these decompositions. Therefore, these methods seem not be applicable for our problem. Furthermore, since the nonlinear term f has exponential critical growth, we have the difficulty of the loss of compactness of the energy functional.

Finalizing this introduction, we emphysize that the main tools used in this work are the following: Trudinger-Moser inequality, constraint variation methods, quantitative deformation lemma, Montain-Pass Theorem and results of the degree theory.

In order to we do not resort to the Introduction and for the sake of independence of the chapters, we will present again, in each chapter, the main results and the related assumptions.

Notation and terminology

- C, C_i, $i = 1, 2, ...$, denote positive (possibly different) constants;
- $C(\varepsilon)$ denotes positive constant which depends on the parameter ε;
- $B_R(x)$ denotes the ball centered at $x \in \mathbb{R}^N$ and radius R and $B_R = B_R(0)$;
- for a subset $\Omega \subset \mathbb{R}^N$, we denote by $\partial\Omega$, $\overline{\Omega}$, $|\Omega|$ and Ω^c, the boundary, the closure, the Lebesgue measure and the complement of Ω in $\mathbb{R}^N$, respectively;
- χ_Ω denotes the characteristic function of a set $\Omega \subset \mathbb{R}^N$, that is, $\chi_\Omega(x) = 1$ if $x \in \Omega$ and $\chi_\Omega(x) = 0$ if $x \in \Omega^c$;
- $u^+(x) = \max\{u(x), 0\}$ and $u^-(x) = \min\{u(x), 0\}$;
- $o_n(1)$ denotes a sequence that converges to zero;
- for $1 \leq p \leq \infty$, the standard norm in $L^p(\mathbb{R}^N)$ is denoted by $\| \cdot \|_p$;
- $u_n \rightharpoonup u$ and $u_n \to u$ denote weak and strong convergence, respectively, in a normed space;
- $C_0^\infty(\Omega)$ denotes the space of infinitely differentiable real functions whose support is compact in $\Omega \subset \mathbb{R}^N$.

Chapter 1

Ground state and nodal solutions for a class of fractional equations involving exponential growth in a bounded domain

In the present chapter we study the existence of least energy nodal solution and ground state solution for a Dirichlet problem in an open bounded interval $\Omega = (a, b)$ driven by the $\frac{1}{2}$−Laplacian operator with the nonlinearity that grows like $\exp(t^2)$ as $t \to \infty$. By using the constraint variational method and quantitative deformation lemma, we obtain a least energy nodal solution u for the given problem. Moreover, we show that the energy of u is strictly larger than twice the ground state energy. The results of this chapter were published in the article [29].

1.1 Introduction and main results

In this section, we consider the existence and multiplicity of weak solutions for the following class of equations:

$$\begin{cases} (-\Delta)^{\frac{1}{2}} u + V(x)u = f(u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R} \setminus \Omega, \end{cases} \tag{1.1}$$

where $\Omega = (a,b)$ is a bounded open interval, $V : [a,b] \to [0,\infty)$ is a continuous and nonnegative function, the nonlinear term $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function that may have a subcritical or critical exponential growth in the Trudinger-Moser sense due to Ozawa [61] (see (1.4) and (1.5)), $(-\Delta)^{\frac{1}{2}}$ is the 1/2-Laplacian operator which, for $u \in C_0^\infty(\mathbb{R})$, is defined as

$$(-\Delta)^{\frac{1}{2}} u(x) = -\frac{1}{2\pi} \int_{\mathbb{R}} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^2} \, \mathrm{d}y. \tag{1.2}$$

In order to study variationally the problem (1.1), we consider a suitable subspace of the fractional Sobolev space $H^{1/2}(\mathbb{R})$. The fractional Sobolev space $H^{1/2}(\mathbb{R})$ is defined as the space

$$H^{1/2}(\mathbb{R}) = \left\{ u \in L^2(\mathbb{R}) : \int_{\mathbb{R}^2} \frac{(u(x)-u(y))^2}{|x-y|^2} \mathrm{d}x\mathrm{d}y < \infty \right\}$$

and it is equipped with the norm

$$\|u\|_{H^{1/2}(\mathbb{R})} := \left([u]_{1/2}^2 + \|u\|_2^2 \right)^{1/2},$$

where

$$[u]_{1/2}^2 = \int_{\mathbb{R}^2} \frac{|u(x)-u(y)|^2}{|x-y|^2} \mathrm{d}x\mathrm{d}y$$

is the Gagliardo seminorm of u.

We are interested in study the problem (1.1) in the case that the nonlinearity $f(t)$ has the maximal growth which allows us to treat problem (1.1) variationally in $H^{1/2}(\mathbb{R})$. In order to improve the presentation of the hypotheses on f, we recall some well-known facts involving the limiting Sobolev embedding theorem in one-dimension. If $s \in (0, 1/2)$, the Sobolev embedding states that $H^s(\mathbb{R}) \hookrightarrow L^{2_s^*}(\mathbb{R})$, where $2_s^* := 2/(1-2s)$ (the critical Sobolev exponent). Moreover, this same result ensures that $H^{1/2}(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$ for any $q \in [2,\infty)$ but $H^{1/2}(\mathbb{R})$ is not continuously embedded in $L^\infty(\mathbb{R})$ (for more details see [32, 61]). Thus, if $s \in (0, 1/2)$, the maximal growth on the nonlinearity $f(t)$, which lets us to work with (1.1) by considering a variational approach in $H^s(\mathbb{R})$, is given by $|t|^{2_s^*-1}$ as $|t| \to \infty$. On the other hand, in the limiting case $s = 1/2$, the maximal growth on $f(t)$, which allows us to study (1.1) by applying a variational framework involving the space $H^{1/2}(\mathbb{R})$, it is motivated by the Trudinger-Moser inequality proved by Ozawa [61] and improved by Kozono *et al.* in [53] and

Takahashi in [69]. Precisely, by combining some of the results contained in [53, 61, 69], it is established that

$$\sup_{\{u\in H^{1/2}(\mathbb{R})\,:\,\|u\|_{1/2,2}\le 1\}} \int_{\mathbb{R}} (e^{\alpha u^2}-1)\mathrm{d}x < \infty, \tag{1.3}$$

for any $0 \le \alpha \le \pi$, where

$$\|u\|_{1/2,2} := \left(\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{|u(x)-u(y)|^2}{|x-y|^2} \mathrm{d}x\mathrm{d}y + \|u\|_2^2 \right)^{1/2}.$$

Motivated by (1.3), we say that $f(t)$ has *exponential critical growth* if there exists $\alpha_0 > 0$ such that

$$\lim_{|t|\to\infty} f(t)e^{-\alpha|t|^2} = \begin{cases} 0, & \text{for all } \alpha > \alpha_0, \\ \infty, & \text{for all } \alpha < \alpha_0, \end{cases} \tag{1.4}$$

and we say that $f(t)$ has *exponential subcritical growth* if

$$\lim_{|t|\to\infty} f(t)e^{-\alpha|t|^2} = 0, \text{ for all } \alpha > 0. \tag{1.5}$$

Based on this notion of criticality, many papers have been developed in order to study issues related to the existence of solutions for problems involving the fractional Laplacian operator and nonlinearities with exponential growth. For example, existence and multiplicity of solutions of the problems similar to (1.1) were treated by different methods in [49, 50, 63]. We also mention [27, 28, 34, 36, 48] for others investigations in the one dimensional case on the whole space $\mathbb{R}$. However, we point out that none of the previous papers treated the existence of nodal solution (sign-changing solution). Motivated by this fact, our goal in this chapter is proving the existence nodal solutions for problem (1.1) when the nonlinearity has exponential critical or subcritical growth in the Trudinger-Moser sense as (1.5) and (1.4).

In order to reach this goal, we assume the following assumption on the potential V:

(V_1) $V : \overline{\Omega} \to \mathbb{R}$ is continuous and nonnegative, where $\Omega = (a, b)$ is a bounded open interval.

On the nonlinearity f, we assume the following assumptions:

(f_1) $f \in C^1(\mathbb{R})$ and there exists $C_0 > 0$ such that

$$|f(t)| \leq C_0 e^{\pi t^2}, \text{ for all } t \in \mathbb{R};$$

(f_2) $\lim_{t \to 0} \frac{f(t)}{t} = 0$;

(f_3) there exists $\theta > 2$ such that

$$0 < \theta F(t) := \theta \int_0^t f(s) \mathrm{d}s \leq t f(t), \text{ for all } t \in \mathbb{R} \setminus \{0\};$$

(f_4) the function $\frac{f(t)}{|t|}$ is strictly increasing for $t \neq 0$;

(f_5) there exist constants $p > 2$ and $C_p > 0$ such that

$$sgn(t) f(t) \geq C_p |t|^{p-1}, \text{ for all } t \in \mathbb{R}.$$

Example 1.1.1 *If $p > 2$, the nonlinearity*

$$f(t) = C_p |t|^{p-2} t + |t|^{p-2} t e^{t^2}$$

satisfies the assumptions $(f_1) - (f_5)$.

In order to study variationally the problem (1.1), we consider a suitable subspace of the fractional Sobolev space $H^{1/2}(\mathbb{R})$ defined as follows

$$X := \{u \in H^{1/2}(\mathbb{R}) : u = 0 \text{ in } \mathbb{R} \setminus \Omega\}, \tag{1.6}$$

equipped with inner product

$$\langle u, v \rangle = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^2} \mathrm{d}x\mathrm{d}y + \int_\Omega V(x) uv \, \mathrm{d}x \tag{1.7}$$

and the corresponding norm

$$\|u\| = \left(\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x - y|^2} \mathrm{d}x\mathrm{d}y + \int_\Omega V(x) |u|^2 \mathrm{d}x \right)^{1/2}$$

By (V_1), Proposition 2.2 and Proposition 2.3 of [50], X is a Hilbert space and X is continuous and compactly embedded in $L^q(\mathbb{R})$, see Lemma 1.2.1.

In this context, we say $u \in X$ is a weak solution of (1.1) (or simply solution) if

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^2} \mathrm{d}x\mathrm{d}y + \int_\Omega V(x) uv \, \mathrm{d}x - \int_\Omega f(u) v \mathrm{d}x = 0,$$

for all $v \in X$. If u is a weak solution of (1.1) such that $u^{\pm} \neq 0$, we say that u is a nodal solution, where $u^{+}(x) = \max\{u(x), 0\}$ and $u^{-}(x) = \min\{u(x), 0\}$.

As we shall see in Section 2, the space X has nice properties. In particular, I given by

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\Omega} F(u)\mathrm{d}x \tag{1.8}$$

is well defined in X, where $F(t) := \int_0^t f(\tau)\mathrm{d}\tau$. Moreover, $I \in C^1(X, \mathbb{R})$ and its critical points are weak solutions of (1.1).

Now, we define the Nehari sets associated to I and their respective minimums energy level by:

- The Nehari set and the ground state level

$$\mathcal{N} = \{u \in X \setminus \{0\} : I'(u)u = 0\} \quad \text{and} \quad c_{\mathcal{N}} := \inf_{u \in \mathcal{N}} I(u); \tag{1.9}$$

- The nodal Nehari set and the nodal level

$$\mathcal{M} := \{u \in X : u^{\pm} \neq 0 \text{ and } I'(u)u^{\pm} = 0\} \quad \text{and} \quad c_{\mathcal{M}} := \inf_{u \in \mathcal{M}} I(u). \tag{1.10}$$

We say that a nonzero critical point $w \in X$ of I is a least energy solution (ground state solution) if w achieves the minimum $c_{\mathcal{N}}$. Note that, if u is a solution of (1.1), taking u^{+} and u^{-} as test functions, we get

$$I'(u)u^{+} = 0 \quad \text{and} \quad I'(u)u^{-} = 0.$$

Then, any sing-changing solution to (1.1) belongs to $\mathcal{M}$. If $w \in \mathcal{M}$ is a solution of (1.1) such that $I(w) = c_{\mathcal{M}}$ we say that w is a least energy nodal solution (1.1).

Our main result of this chapter is the following:

Theorem 1.1.2 *Suppose that* (V_1) *and* $(f_1) - (f_5)$ *are satisfied. Then problem* (1.1) *possesses a least energy nodal solution, provided that*

$$C_p > \left[\frac{2\theta c_{\mathcal{M}^p}}{\theta - 2}\right]^{(p-2)/2},$$

where

$$c_{\mathcal{M}^p} = \inf_{u \in \mathcal{M}^p} I_p(u), \;\; \mathcal{M}^p = \{u \in X : u^{\pm} \neq 0, I_p'(u)u^{\pm} = 0\}$$

and

$$I_p(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p}\int_{\Omega} |u|^p \mathrm{d}x.$$

Another goal of this paper is to prove that the energy of any sign-changing solution of (1.1) is strictly larger than twice the ground state energy. This property is the so called energy doubling by Weth [71].

Theorem 1.1.3 *Suppose that* (V_1) *and* $(f_1)-(f_5)$ *are satisfied. Then* $c_{\mathcal{N}} > 0$ *is achieved for a solution of* (1.1) *and*

$$I(w) > 2c_{\mathcal{N}}, \tag{1.11}$$

where w *is the least energy nodal solution obtained in Theorem 1.1.2. In particular,* $c_{\mathcal{N}}$ *is achieved either by a nonnegative or a nonpositive function.*

It is interesting to note that in the last decades the existence and multiplicity of positive and nodal solutions of classical elliptic problems have been widely investigated, see [3, 4, 9, 10, 11, 12, 44, 72] and references therein. Specially, some results on nodal solutions of nonlinear elliptic equations involving different operators have been obtained by combining minimax method with invariant sets of descending flow, such as Laplacian operator [9, 11, 12], $p-$Laplacian operator [10] and Schrödinger operator [3, 4, 44].

In the special case of the stationary equation of Schrödinger

$$-\Delta u + V(x)u = f(u) \text{ in } \mathbb{R}^N, \tag{1.12}$$

there are several ways in the literature to obtain sign-changing solution (see [3, 4, 9, 12, 44, 72]). However, the methods used in these works heavily rely on the following two decompositions:

$$J(u) = J(u^+) + J(u^-), \tag{1.13}$$

$$J'(u)u^+ = J'(u^+)u^+ \text{ and } J'(u)u^- = J'(u^-)u^-, \tag{1.14}$$

where J is the energy functional of (1.12) given by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)u^2)\mathrm{d}x - \int_{\mathbb{R}^N} F(u)\mathrm{d}x.$$

In the case of problem (1.1), the functional associated does not possess the same decompositions as (1.13) and (1.14). Indeed, since $\langle u^+, u^-\rangle > 0$ when $u^\pm \neq 0$, a straightforward computation yields that (see Lemma 1.2.3)

$$I(u) > I(u^+) + I(u^-),$$

$$I'(u)u^+ > I'(u^+)u^+ \text{ and } I'(u)u^- > I'(u^-)u^-,$$

where I is defined in (1.8). Therefore, the methods used to obtain sign-changing solutions for the local problem (1.12) seem not be applicable to problem (1.1). Note that $\mathcal{M} \subset \mathcal{N}$. If $u \in \mathcal{M}$, then

$$u^+ \notin \mathcal{N} \quad \text{and} \quad u^- \notin \mathcal{N}.$$

This a big difference between nonlocal and local problems.

Furthermore, a second well-known difficulty for the class of problems (1.1) is the loss of compactness due to the critical growth on the nonlinearity f.

In order to overcome these difficulties, we define the following constrained set

$$\mathcal{M} = \{u \in X : u^\pm \neq 0 \text{ and } I'(u)u^\pm = 0\}$$

and consider a minimization problem of I on $\mathcal{M}$. Borrowing ideas from [21], we prove $\mathcal{M} \neq \emptyset$ via modified Miranda's theorem (see Lemma 1.3.5 and Lemma 1.3.6). Combining the ideas developed in [3, 4, 11, 21], we prove that the minimizer of the constrained problem is also a sign-changing solution via the quantitative deformation lemma and degree theory (see Section 1.3 and Section 1.4).

Remark 1.1.4 *Using the regularity results due to Servadei and Valdinoci [68], we have that the weak solutions of problem* (1.1) *obtained in Theorems 1.1.2 and 1.1.3 belong to* $C(\mathbb{R})$.

Remark 1.1.5 *In the hypothesis* (f_1) *we assume that* $|f(t)| \leq C_0 e^{\pi t^2}$. *This growth condition allows us to consider nonlinearities with critical growth in the sense defined in* (1.4) *with an exponent* $\alpha_0 = \pi$ *and with subcritical growth in the sense defined in* (1.5). *More generally, we can consider an exponent* α_0 *different from* π. *In this more general case, this new constant would imply a normalization of the constant* C_p *defined in Theorem 1.1.2.*

Remark 1.1.6 *We point out that the results of this chapter were published in [29] and complement the works [48, 49, 50, 63] in the sense that we prove the existence of sign-change solutions and the work [21] in the sense that we consider exponential growth on the nonlinearity. Furthermore, our results extend for the fractional Laplacian some of the results contained in [3, 4, 72].*

The outline of this chapter is as follows: Section 1.2 contains some notations and it is established a version of the Trudinger-Moser inequality for the class of problem

(1.1). In addition, the variational framework is presented. Section 1.3 is dedicated to the study of the nodal set and the nodal level, the main goal is to prove that the nodal level is attained. In Section 1.4, we complete the proof of Theorem 1.1.2 and Section 1.5 is proved Theorem 1.1.3.

1.2 Variational formulation and preliminary results

Next we shall prove the first lemma of this work.

Lemma 1.2.1 *Under the assumption (V_1), the embedding $X \hookrightarrow L^q(\mathbb{R})$ is continuous and compact for all $q \in [1, \infty)$.*

Proof . From [50, Proposition 2.2], we obtain that the embedding $X \hookrightarrow H^{1/2}(\Omega)$ is continuous, and from [32, Theorem 6.9], [32, Theorem 6.10] and [32, Theorem 7.1], the embedding $H^{1/2}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous and compact for all $q \in [1, \infty)$. This completes the proof. ■

The main tool to study problems involving exponential growth in the fractional Sobolev space is the fractional Trudinger-Moser inequality due to Ozawa [61]. In this work, combining the results due to [61] and [69] we prove a version of this inequality for the space X in the next lemma.

Lemma 1.2.2 *If $0 \leq \alpha \leq \pi$, it holds*

$$\sup_{\{u \in X \,:\, \|u\| \leq 1\}} \int_\Omega e^{\alpha u^2} \,\mathrm{d}x < \infty. \tag{1.15}$$

Moreover, for any $\alpha > 0$ and $u \in X$, we have

$$\int_\Omega e^{\alpha u^2} \,\mathrm{d}x < \infty. \tag{1.16}$$

Proof . The first statement of the result follows from [32, Proposition 3.3] and [69, Proposition 1]. For the second part, let $u \in X$. By density, (see [32, Theorem 2.4]) given $\varepsilon > 0$, there exists $\varphi \in C_0^\infty(\Omega)$ such that $\|u - \varphi\| < \varepsilon$. Using Young's inequality we have

$$e^{\alpha u^2} \leq e^{2\alpha((u-\varphi)^2 + \varphi^2)} \leq \frac{1}{2} e^{4\alpha(u-\varphi)^2} + \frac{1}{2} e^{4\alpha\varphi^2}.$$

Then

$$\int_\Omega e^{\alpha u^2} \mathrm{d}x \leq \frac{1}{2} \int_\Omega e^{4\alpha\|u-\varphi\|^2 (\frac{u-\varphi}{\|u-\varphi\|})^2} \mathrm{d}x + \frac{1}{2} \int_\Omega e^{4\alpha\varphi^2} \mathrm{d}x.$$

Choosing $\varepsilon > 0$ such that $4\alpha\varepsilon^2 \leq \pi$, by (1.15) we have

$$\int_\Omega e^{\alpha u^2} \mathrm{d}x \leq C + \frac{1}{2} \int_{\mathrm{supp}(\varphi)} e^{4\alpha\varphi^2} \mathrm{d}x < \infty$$

and this completes the proof of the lemma. ■

As a consequence of Lemma 1.2.2 and (f_1) the energy functional $I : X \longrightarrow \mathbb{R}$ given by

$$I(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F(u)\mathrm{d}x \tag{1.17}$$

is well defined. Moreover, by Lemma 1.2.2, it is standard to show that $I \in C^1(X, \mathbb{R})$ and, for every $u, v \in X$,

$$I'(u)v = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^2} \mathrm{d}x\mathrm{d}y + \int_\Omega V(x)uv\mathrm{d}x - \int_\Omega f(u)v\mathrm{d}x.$$

Therefore, a critical point of I is a weak solution of (1.1) and reciprocally.

Our goal in this paper is to show that problem (1.1) has a nodal solution. As we saw in the Introduction one of the difficulties is the fact that the functional I does not possess the decompositions (1.13) and (1.14). In fact, inspired by [21], we have:

Lemma 1.2.3 *Let $u \in X$. Then,*

(i) $\langle u, u^\pm \rangle = \langle u^\pm, u^\pm \rangle + \dfrac{1}{\pi} \displaystyle\int_{\mathbb{R}^2} \frac{u^+(x)(-u^-(y))}{|x - y|^2} \mathrm{d}x\mathrm{d}y,$

(ii) *if* $u^\pm \neq 0$, $\langle u^+, u^- \rangle > 0$,

(iii) $I(u) > I(u^+) + I(u^-)$,

(iv) $I'(u)u^+ > I'(u^+)u^+$ *and* $I'(u)u^- > I'(u^-)u^-$.

Proof . By density (see [32, Theorem 2.4]) we can assume that u is continuous. We define

$$\Omega_+ = \{x \in \Omega : u(x) \geq 0\}, \Omega_- = \{x \in \Omega : u(x) \leq 0\}$$
$$U(x, y) = \frac{u(x) - u(y)}{|x - y|^2} \quad \text{and} \quad U(x^+, y^+) = \frac{u^+(x) - u^+(y)}{|x - y|^2}.$$

Using the above notation, we have

$$\int_{\mathbb{R}^2} \frac{(u(x) - u(y))(u^+(x) - u^+(x))}{|x - y|^2} \mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^2} U(x, y)(u^+(x) - u^+(x))\mathrm{d}x\mathrm{d}y.$$

Now, since $u = 0$ in Ω^c, we get

$$\begin{aligned}\int_{\mathbb{R}^2} U(x,y)(u^+(x) - u^+(x))\mathrm{d}x\mathrm{d}y &= \int_{\Omega\times\Omega^c} U(x,y)(u^+(x) - u^+(x))\mathrm{d}x\mathrm{d}y \\ &+ \int_{\Omega\times\Omega} U(x,y)(u^+(x) - u^+(x))\mathrm{d}x\mathrm{d}y \\ &+ \int_{\Omega^c\times\Omega} U(x,y)(u^+(x) - u^+(x))\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Moreover, since $\Omega \times \Omega^c = (\Omega_+ \times \Omega^c) \cup (\Omega_- \times \Omega^c)$ and $\Omega^c \times \Omega = (\Omega^c \times \Omega_+) \cup (\Omega^c \times \Omega_-)$, we can deduce that

$$\begin{aligned}\int_{\mathbb{R}^2} U(x,y)(u^+(x) - u^+(x))\mathrm{d}x\mathrm{d}y &= \int_{\Omega\times\Omega} U(x,y)(u^+(x) - u^+(x))\mathrm{d}x\mathrm{d}y \\ &+2\int_{\Omega_+\times\Omega^c} U(x,y)(u^+(x) - u^+(x))\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{1.18}$$

Similarly, we can show that

$$\begin{aligned}\int_{\mathbb{R}^2} U(x^+,y^+)(u^+(x) - u^+(x))\mathrm{d}x\mathrm{d}y &= \int_{\Omega\times\Omega} U(x^+,y^+)(u^+(x) - u^+(x))\mathrm{d}x\mathrm{d}y \\ &+2\int_{\Omega_+\times\Omega^c} U(x^+,y^+)(u^+(x) - u^+(x))\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{1.19}$$

By the expression of $U(x,y)$ and $U(x^+,y^+)$, we can easily check that $U(x,y)|_{\Omega_+\times\Omega^c} = U(x^+,y^+)$ and so

$$D_{u^+} = \int_{\Omega_+\times\Omega^c} U(x,y)(u^+(x) - u^+(x))\mathrm{d}x\mathrm{d}y = \int_{\Omega_+\times\Omega^c} U(x^+,y^+)(u^+(x) - u^+(x))\mathrm{d}x\mathrm{d}y. \tag{1.20}$$

Therefore, by (1.18), (1.19) and (1.20), we have

$$\begin{aligned}\int_{\mathbb{R}^2} U(x,y)(u^+(x) - u^+(x))\mathrm{d}x\mathrm{d}y &= \int_{\mathbb{R}^2} U(x^+,y^+)(u^+(x) - u^+(x))\mathrm{d}x\mathrm{d}y \\ &+ \int_{\Omega\times\Omega} (U(x,y) - U(x^+,y^+))(u^+(x) - u^+(x))\mathrm{d}x\mathrm{d}y\end{aligned} \tag{1.21}$$

Now, since $\Omega \times \Omega = (\Omega_+ \times \Omega_+) \cup (\Omega_+ \times \Omega_-) \cup (\Omega_- \times \Omega_+) \cup (\Omega_- \times \Omega_-)$ and again by the expression of U, we get

$$\begin{aligned}&\int_{\Omega\times\Omega} (U(x,y) - U(x^+,y^+))(u^+(x) - u^+(x))\mathrm{d}x\mathrm{d}y = \\ &2\int_{\Omega_+\times\Omega_-} (U(x,y) - U(x^+,y^+))(u^+(x) - u^+(x))\mathrm{d}x\mathrm{d}y = 2\int_{\Omega_+\times\Omega_-} \frac{u^+(x)(-u^-(y))}{|x-y|^2}\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{1.22}$$

Thus, by (1.7), (1.21) and (1.22), we obtain that

$$\begin{aligned}\langle u, u^+\rangle &= \frac{1}{2\pi}\int_{\mathbb{R}^2} U(x,y)(u^+(x) - u^+(x))\mathrm{d}x\mathrm{d}y + \int_{\Omega} V(x)|u^+|^2\mathrm{d}x \\ &= \langle u^+, u^+\rangle + \frac{1}{\pi}\int_{\Omega_+\times\Omega_-} \frac{u^+(x)(-u^-(y))}{|x-y|^2}\mathrm{d}x\mathrm{d}y,\end{aligned}$$

and **(i)** is proved.

Now, since $\langle u^+, u^-\rangle = \langle u, u^+\rangle - \langle u^+, u^+\rangle$, the item **(ii)** follows from **(i)**.

Moreover, since $I(u) = \langle u^+, u^-\rangle + I(u^+) + I(u^-)$, $I'(u)u^+ = \langle u^+, u^-\rangle + I'(u^+)u^+$ and $I'(u)u^- = \langle u^+, u^-\rangle + I'(u^-)u^-$, the proof of **(iii)** and **(iv)** follows from **(ii)**. ■

Corollary 1.2.4 *If $u \in X$ then*

$$\|u\|^2 \geq \|u^+\|^2 + \|u^-\|^2.$$

Proof . By Lemma 1.2.3, we have

$$\|u\|^2 = \|u^+\|^2 + 2\langle u^+, u^-\rangle + \|u^-\|^2 \geq \|u^+\|^2 + \|u^-\|^2$$

which implies the desired inequality. ■

1.3 Constrained minimization problem

In order to obtain nodal solutions for (1.1), we define the Nehari manifold and nodal set associated to functional I by

$$\mathcal{N} = \{u \in X \setminus \{0\} : I'(u)u = 0\} \tag{1.23}$$

and

$$\mathcal{M} = \{u \in X : u^\pm \neq 0 \text{ and } I'(u)u^\pm = 0\}. \tag{1.24}$$

The ground state level is defined by

$$c_{\mathcal{N}} := \inf_{u\in\mathcal{N}} I(u) \tag{1.25}$$

and the nodal level by

$$c_{\mathcal{M}} := \inf_{u\in\mathcal{M}} I(u). \tag{1.26}$$

Note that since $\mathcal{M} \subset \mathcal{N}$ we have $c_{\mathcal{N}} \leq c_{\mathcal{M}}$.

In the following, we shall study some properties of $\mathcal{N}$ and $\mathcal{M}$. First, we observe that by $(f_1)-(f_2)$, given $\varepsilon > 0$ and $q > 2$, there exists a positive constant C_ε such that

$$|f(t)| \leq \varepsilon|t| + C_\varepsilon|t|^{q-1}e^{\pi t^2}, \text{ for all } t \in \mathbb{R} \tag{1.27}$$

and, by (f_3), we have

$$|F(t)| \leq \varepsilon|t|^2 + C_\varepsilon|t|^q e^{\pi t^2}, \text{ for all } t \in \mathbb{R}. \tag{1.28}$$

Moreover, by (f_3), we can find positive constants C_1 and C_2 such that

$$F(t) \geq C_1|t|^\theta - C_2, \text{ for all } t \in \mathbb{R}. \tag{1.29}$$

Lemma 1.3.1 *Assume that (V_1) and $(f_1)-(f_4)$ are satisfied. Then, given $u \in X\setminus\{0\}$, there exists a unique $t = t(u) > 0$ such that $tu \in \mathcal{N}$. In addition, the number t satisfies*

$$I(tu) = \max_{s\geq 0} I(su). \tag{1.30}$$

Proof . Given $u \in X \setminus \{0\}$, we define $h(s) := I(su)$ for $s \geq 0$. By (1.28), we get

$$h(s) \geq \frac{s^2}{2}\|u\|^2 - \varepsilon s^2 \int_\Omega |u|^2 \mathrm{d}x - C_\varepsilon s^q \int_\Omega |u|^q e^{\pi s^2 u^2} \mathrm{d}x. \tag{1.31}$$

If $s \in [0, 1]$, we have $e^{\pi s^2 u^2} \leq e^{\pi u^2}$. Using Hölder's inequality, Lemma 1.2.1 and Lemma 1.2.2, we have

$$\int_\Omega |u|^q e^{\pi s^2 u^2} \mathrm{d}x \leq \left(\int_\Omega |u|^{2q} \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega e^{2\pi u^2} \mathrm{d}x\right)^{\frac{1}{2}} < \infty$$

whenever $s \in [0, 1]$. This together with (1.31) and Lemma 1.2.1 implies that there exist positive constants C_1 and $C_2 = C_2(u)$, which do not depend on s, such that

$$h(s) \geq s^2 \left(\frac{1}{2} - \varepsilon C_1\right) \|u\|^2 - C_2 s^q \tag{1.32}$$

for all $s \in [0, 1]$. Now, choosing $\varepsilon > 0$ such that $\frac{1}{2} - \varepsilon C_1 > 0$, it follows from (1.32) that

$$h(s) > 0 \text{ for } s > 0 \text{ small enough.} \tag{1.33}$$

On the other hand, using (1.29) we get

$$h(s) \leq \frac{s^2}{2}\|u\|^2 - C_1 s^\theta \int_\Omega |u|^\theta \mathrm{d}x + C_2(b - a).$$

Hence, since $\theta > 2$

$$h(s) \to -\infty, \text{ as } s \to \infty. \tag{1.34}$$

Therefore, from (1.33) and (1.34), there exists $t = t(u) > 0$ such that

$$I(tu) = \max_{s \geq 0} I(su),$$

and, consequently, $tu \in \mathcal{N}$.

Now, if $s > 0$ is such that $su \in \mathcal{N}$, we have

$$s^2 \|u\|^2 = \int_\Omega f(su) su \, dx$$

and since it also holds $t^2\|u\|^2 = \int_\Omega f(tu)tu \, dx$, it follows that

$$\int_\Omega \left(\frac{f(tu)}{tu} - \frac{f(su)}{su} \right) u^2 dx = 0. \tag{1.35}$$

By (f_4) and since $u \neq 0$, it follows from (1.35) that $t = s$. Thus, we finish the proof. ■

Lemma 1.3.2 *Assume that* (V_1) *and* $(f_1)-(f_2)$ *are satisfied. Then there exists* $m_0 > 0$ *such that* $\|u\|^2 \geq m_0$, *for all* $u \in \mathcal{N}$.

Proof . In order to obtain a contradiction, suppose that there exists $(u_n) \subset \mathcal{N}$ such that $\|u_n\| \to 0$ as $n \to \infty$. By definition we know that

$$\|u_n\|^2 = \int_\Omega f(u_n) u_n dx. \tag{1.36}$$

On the other hand, using (1.27), Hölder's inequality and Lemma 1.2.1, we get

$$\begin{aligned} \int_\Omega f(u_n)u_n dx &\leq \varepsilon \int_\Omega |u_n|^2 dx + C_\varepsilon \int_\Omega |u_n|^q e^{\pi u_n^2} dx \\ &\leq \varepsilon \|u_n\|_2^2 + C_\varepsilon \left(\int_\Omega |u_n|^{2q} dx \right)^{\frac{1}{2}} \left(\int_\Omega e^{2\pi u_n^2} dx \right)^{\frac{1}{2}} \\ &\leq \varepsilon \|u_n\|_2^2 + C_\varepsilon \|u_n\|_{2q}^q \left(\int_\Omega e^{2\pi \|u_n\|^2 \left(\frac{u_n}{\|u_n\|}\right)^2} dx \right)^{\frac{1}{2}} \\ &\leq \varepsilon C \|u_n\|^2 + C\|u_n\|^q \left(\int_\Omega e^{2\pi \|u_n\|^2 \left(\frac{u_n}{\|u_n\|}\right)^2} dx \right)^{\frac{1}{2}}. \end{aligned} \tag{1.37}$$

Since $\|u_n\| \to 0$ as $n \to \infty$, we can find $n_0 \in \mathbb{N}$ such that $2\pi\|u_n\|^2 \leq \pi$ for all $n \geq n_0$. Hence, it follows from Lemma 1.2.2 that

$$\int_\Omega e^{2\pi \|u_n\|^2 \left(\frac{u_n}{\|u_n\|}\right)^2} dx \leq C. \tag{1.38}$$

From (1.36)-(1.38) we can find positive constants C_1 and C_2 such that

$$\|u_n\|^2 \leq \varepsilon C_1 \|u_n\|^2 + C_2 \|u_n\|^q.$$

Choosing $\varepsilon > 0$ such that $1 - \varepsilon C_1 > 0$ and since $(u_n) \subset \mathcal{N}$, we get

$$0 < \left(\frac{1 - \varepsilon C_1}{C_2} \right) \leq \|u_n\|^{q-2}.$$

But as $q > 2$ this contradicts the fact that $\|u_n\| \to 0$. This completes the proof of the lemma. ■

Corollary 1.3.3 *Assume that* (V_1) *and* $(f_1) - (f_3)$ *are satisfied. Then there exists* $\delta_0 > 0$ *such that* $I(u) \geq \delta_0$, *for all* $u \in \mathcal{N}$. *In particular,*

$$0 < \delta_0 \leq c_{\mathcal{N}} \leq c_{\mathcal{M}}.$$

Proof . Since $I'(u)u = 0$, by Lemma 1.3.2 and (f_3), we have

$$\begin{aligned} I(u) &= I(u) - \frac{1}{\theta} I'(u)u \\ &= \left(\frac{1}{2} - \frac{1}{\theta} \right) \|u\|^2 + \frac{1}{\theta} \int_\Omega (f(u)u - \theta F(u))\, \mathrm{d}x \\ &\geq \left(\frac{1}{2} - \frac{1}{\theta} \right) \|u\|^2 \geq \left(\frac{1}{2} - \frac{1}{\theta} \right) m_0 := \delta_0, \end{aligned}$$

which is the desired inequality. ■

Lemma 1.3.4 *Assume that* (V_1) *and* $(f_1) - (f_2)$ *are satisfied. Then there exists* $m_0' > 0$ *such that* $\|u^\pm\|^2 \geq m_0'$, *for all* $u \in \mathcal{M}$.

Proof . The proof is similar to that of Lemma 1.3.2, so it is sufficient to prove an estimating similar to (1.37) for u^+ and u^-. Since $u \in \mathcal{M}$ we have $u^+ \neq 0$ and $\langle u, u^+ \rangle = \int_\Omega f(u^+)u^+ \mathrm{d}x$. Now, by Lemma 1.2.3, we have $\|u^+\|^2 \leq \langle u^+, u^+ \rangle + \langle u^-, u^+ \rangle = \langle u, u^+ \rangle$. Thus, using (1.27) we obtain

$$\|u^+\|^2 \leq \int_\Omega f(u^+)u^+ \mathrm{d}x \leq \varepsilon \int_\Omega |u^+|^2 \mathrm{d}x + C_\varepsilon \int_\Omega |u^+|^q e^{\pi |u^+|^2} \mathrm{d}x.$$

Similarly, we have

$$\|u^-\|^2 \leq \int_\Omega f(u^-)u^- \mathrm{d}x \leq \varepsilon \int_\Omega |u^-|^2 \mathrm{d}x + C_\varepsilon \int_\Omega |u^-|^q e^{\pi |u^-|^2} \mathrm{d}x$$

and the proof of the lemma is done. ■

Using Lemma 1.2.3, we observe that Lemma 1.3.1 can not be applied to show that $\mathcal{M} \neq \emptyset$. In order to obtain some results on the nodal set $\mathcal{M}$, we shall use the so-called Poincaré-Miranda Theorem (see [70]).

Lemma 1.3.5 *Let $h : P \subset \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a continuous function, where $P = \prod_{i=1}^{N}[a_i, b_i]$ is a N-dimensional block in $\mathbb{R}^N$, with $a_i \neq b_i$, for $i = 1, \dots, N$. Let $P_i^- = \{x \in P : x_i = a_i\}$ and $P_i^+ = \{x \in P : x_i = b_i\}$. Assume that the coordinates functions of h satisfy:*

(i) *$h_i(x) \geq 0$, for all $x \in P_i^-$,*

(ii) *$h_i(x) \leq 0$, for all $x \in P_i^+$.*

Then there exists $x_0 \in P$ such that $h(x_0) = 0$.

As application of Lemma 1.3.5, we shall show that $\mathcal{M} \neq \emptyset$.

Lemma 1.3.6 *Assume that (V_1) and $(f_1) - (f_4)$ are satisfied. Given $u \in X$ with $u^{\pm} \neq 0$, there exists a unique pair (t, s) of positive numbers such that $tu^+ + su^- \in \mathcal{M}$.*

Proof . Let $u \in X$ with $u^{\pm} \neq 0$, we define the continuous vector field $g : (0, \infty) \times (0, \infty) \longrightarrow \mathbb{R}^2$ by

$$g(t, s) = \left(I'(tu^+ + su^-)tu^+, I'(tu^+ + su^-)su^-\right).$$

Initially we want to find $(t, s) \in (0, \infty) \times (0, \infty)$ such that $g(t, s) = (0, 0)$. The first step is to show that for t and s sufficiently small the coordinates functions are positive. Note that by (1.27) we have

$$|f(\xi)\xi| \leq \varepsilon|\xi|^2 + C_\varepsilon|\xi|^q e^{\pi\xi^2}, \text{ for all } \xi \in \mathbb{R}.$$

Hence,

$$\begin{aligned} I'(tu^+ + su^-)tu^+ &= t^2\|u^+\|^2 + ts\langle u^+, u^-\rangle - \int_\Omega f(tu^+)tu^+ \mathrm{d}x \\ &\geq t^2\|u^+\|^2 + ts\langle u^+, u^-\rangle - \varepsilon t^2 \int_\Omega |u^+|^2 \mathrm{d}x - C_\varepsilon t^q \int_\Omega |u^+|^q e^{\pi t^2|u^+|^2} \mathrm{d}x. \end{aligned}$$

Lemma 1.2.1 implies

$$I'(tu^+ + su^-)tu^+ \geq t^2\|u^+\|^2 + ts\langle u^+, u^-\rangle - \varepsilon C t^2\|u^+\|^2 - C_\varepsilon t^q \int_\Omega |u^+|^q e^{\pi t^2|u^+|^2} \mathrm{d}x.$$

Now, if $t \in [0, 1]$, using Hölder's inequality, Lemma 1.2.2 and Lemma 1.2.1, we have

$$\int_\Omega |u^+|^q e^{\alpha t^2|u^+|^2} \mathrm{d}x \leq \left(\int_\Omega |u^+|^{2q} \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega e^{2\alpha|u^+|^2} \mathrm{d}x\right)^{\frac{1}{2}} \leq C'\|u^+\|^q,$$

Thus, we can find $C = C(u) > 0$ such that

$$I'(tu^+ + su^-)tu^+ \geq t^2\|u^+\|^2 + ts\langle u^+, u^-\rangle - \varepsilon t^2\|u^+\|_2^2 - Ct^q\|u^+\|^q.$$

By Lemma 1.2.3 we have $\langle u^+, u^-\rangle > 0$. Then there exists $r > 0$ small enough such that

$$I'(ru^+ + su^-)ru^+ > 0, \text{ for all } s > 0.$$

Analogously, there exists $r > 0$ small enough such that

$$I'(tu^+ + ru^-)ru^- > 0, \text{ for all } t > 0.$$

Now, we shall show that for t and s large enough the coordinates functions are negative. In fact, using (1.29), we can find positive constants C_1 and C_2 such that

$$\int_\Omega f(tu^+)tu^+\mathrm{d}x \geq \theta \int_\Omega F(tu^+)\mathrm{d}x \geq C_1\|u^+\|_\theta^\theta - C_2.$$

Thus

$$\begin{aligned} I'(tu^+ + su^-)tu^+ &= t^2\|u^+\|^2 + ts\langle u^+, u^-\rangle - \int_\Omega f(tu^+)tu^+\mathrm{d}x \\ &\leq t^2\|u^+\|^2 + ts\langle u^+, u^-\rangle - C_1t^\theta\|u^+\|_\theta^\theta + C_2. \end{aligned}$$

Since $\theta > 2$, there exists $R > r$ large enough such that

$$I'(Ru^+ + su^-)Ru^+ < 0, \text{ for all } 0 \leq s \leq R.$$

Analogously, there exists $R > r$ small enough such that

$$I'(tu^+ + Ru^-)Ru^- < 0, \text{ for all } 0 \leq t \leq R.$$

Hence, from Lemma 1.3.5, there exists $(t, s) \in [r, R] \times [r, R]$ such that $g(t, s) = (0, 0)$. Therefore, $tu^+ + su^- \in \mathcal{M}$.

Finally we shall prove the uniqueness of the pair (t, s). First, we assume that $u = u^+ + u^- \in \mathcal{M}$ and $(t, s) \in (0, \infty) \times (0, \infty)$ is such that $tu^+ + su^- \in \mathcal{M}$. In this case we need to show that $(t, s) = (1, 1)$. Note that

$$\|u^+\|^2 + \langle u^+, u^-\rangle = \int_\Omega f(u^+)u^+\mathrm{d}x, \tag{1.39}$$

$$\|u^-\|^2 + \langle u^+, u^-\rangle = \int_\Omega f(u^-)u^-\mathrm{d}x, \tag{1.40}$$

$$t^2\|u^+\|^2 + ts\langle u^+, u^-\rangle = \int_\Omega f(tu^+)tu^+\mathrm{d}x, \tag{1.41}$$

and

$$s^2\|u^-\|^2 + ts\langle u^+, u^-\rangle = \int_\Omega f(su^-)su^-\mathrm{d}x. \tag{1.42}$$

We can assume, without loss of generality, that $t \le s$. Then, using $\langle u^+, u^-\rangle > 0$ we have

$$\|u^+\|^2 + \langle u^+, u^-\rangle \le \int_\Omega \frac{f(tu^+)}{t}u^+\mathrm{d}x.$$

Thus, it follows from (1.39) that

$$\int_\Omega \left(\frac{f(tu^+)}{tu^+} - \frac{f(u^+)}{u^+}\right)(u^+)^2\mathrm{d}x \ge 0.$$

Hence, by (f_4) and since $u^+ \ne 0$ we obtain $t \ge 1$.

On the other hand, since $t/s \le 1$ and $\langle u^+, u^-\rangle > 0$, we get

$$\|u^-\|^2 + \langle u^+, u^-\rangle \ge \int_\Omega \frac{f(su^+)}{s}u^-\mathrm{d}x.$$

This together with (1.40) implies

$$\int_\Omega \left(\frac{f(su^-)}{su^-} - \frac{f(u^-)}{u^-}\right)(u^-)^2\mathrm{d}x \le 0$$

and consequently $s \le 1$. Thus we conclude the proof of the uniqueness of the pair $(1,1)$.

For the general case, we suppose that u does not necessarily belong to $\mathcal{M}$. Let $(t,s),(t',s') \in (0,\infty)\times(0,\infty)$ are such that $tu^+ + su^-$ and $t'u^+ + s'u^-$ belongs to $\mathcal{M}$. We define $v = v^+ + v^-$, where $v^+ = tu^+$ and $v^- = su^-$. Then, we have that $v \in \mathcal{M}$ and

$$\frac{t'}{t}v^+ + \frac{s'}{s}v^- = t'u^+ + s'u^- \in \mathcal{M}.$$

Hence, using the first case we have $t'/t = 1$ and $s'/s = 1$, which completes the proof. ■

The following two lemmas will be used in the proof of Theorem 1.1.2 in the next section.

Lemma 1.3.7 *Assume that* (V_1) *and* $(f_1)-(f_4)$ *are satisfied. Let* $u \in X$ *be a function such that* $u^\pm \ne 0$, $I'(u)u^+ \le 0$ *and* $I'(u)u^- \le 0$. *Then the unique pair* (t,s) *given in Lemma* 1.3.6 *satisfies* $0 < t, s \le 1$.

Proof . We can assume, without loss of generality, that $s \geq t > 0$ and $tu^+ + su^- \in \mathcal{M}$. Now, since $I'(u)u^- \leq 0$ and $I'(tu^+ + su^-)su^- = 0$, we have

$$\|u^-\|^2 + \langle u^+, u^- \rangle \leq \int_\Omega f(u^-)u^- \mathrm{d}x$$

and

$$\|u^-\|^2 + \frac{t}{s}\langle u^+, u^- \rangle = \int_\Omega \frac{f(su^-)}{s} u^- \mathrm{d}x.$$

Thus by Lemma 1.2.3 we get

$$\begin{aligned}\int_\Omega \left(\frac{f(u^-)}{u^-} - \frac{f(su^-)}{su^-} \right) (u^-)^2 \mathrm{d}x &= \int_\Omega f(u^-)u^- \mathrm{d}x - \|u^-\|^2 - \frac{t}{s}\langle u^+, u^- \rangle \\ &\geq \|u^-\|^2 + \langle u^+, u^- \rangle - \|u^-\|^2 - \frac{t}{s}\langle u^+, u^- \rangle \\ &\geq \left(1 - \frac{t}{s}\right) \langle u^+, u^- \rangle \geq 0.\end{aligned}$$

Using this inequality, (f_4) and the fact that $u^- \neq 0$, we obtain $s \leq 1$ and so $t \leq s \leq 1$. The case $t \geq s > 0$ is analogous and we finish the proof of the lemma. ■

Lemma 1.3.8 *Assume that* (V_1) *and* $(f_1) - (f_4)$ *are satisfied. Let* $u \in X$ *be a function such that* $u^\pm \neq 0$ *and* (t, s) *be the unique pair of positive numbers given in Lemma 1.3.6. Then* (t, s) *is the unique maximum point of the function* $\phi : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ *defined by* $\phi(\alpha, \beta) = I(\alpha u^+ + \beta u^-)$.

Proof . In the demonstration of Lemma 1.3.6, we saw that (t, s) is the unique critical point of ϕ in $(0, \infty) \times (0, \infty)$. Note that, using (1.29), we get

$$\begin{aligned}\phi(\alpha, \beta) &= \frac{1}{2}\left\|\alpha u^+ + \beta u^-\right\|^2 - \int_\Omega F(\alpha u^+ + \beta u^-)\mathrm{d}x \\ &\leq \frac{1}{2}\left\|\alpha u^+ + \beta u^-\right\|^2 - \int_\Omega \left(C_1|\alpha u^+ + \beta u^-|^\theta - C_2\right) \mathrm{d}x \\ &\leq \frac{(\alpha+\beta)^2}{2} \left\| \left(\frac{\alpha}{\alpha+\beta}\right) u^+ + \left(\frac{\beta}{\alpha+\beta}\right) u^- \right\|^2 \\ &\quad - C_1 (\alpha+\beta)^\theta \left\| \left(\frac{\alpha}{\alpha+\beta}\right) u^+ + \left(\frac{\beta}{\alpha+\beta}\right) u^- \right\|_\theta^\theta - C_2(b-a).\end{aligned}$$

Hence $\phi(\alpha, \beta) \to -\infty$ as $|(\alpha, \beta)| \to \infty$. In particular, there exists $R > 0$ such that $\phi(\alpha, \beta) < \phi(t, s)$ for all $(\alpha, \beta) \in (0, \infty) \times (0, \infty) \setminus \overline{B_R(0)}$, where $\overline{B_R(0)}$ is a closure of the ball of radius R in $\mathbb{R}^2$.

In order to finalize the proof, we shall show that the maximum of ϕ does not occur in the boundary of $\mathbb{R}_+ \times \mathbb{R}_+$. Suppose, by contradiction, that $(0, \beta)$ is a maximum point of ϕ. Given $\alpha \geq 0$, it is easy to see that

$$\phi(\alpha,\beta) = \frac{\alpha^2}{2}\|u^+\|^2 + \alpha\beta\langle u^+, u^-\rangle - \int_\Omega F(\alpha u^+)\mathrm{d}x + \phi(0,\beta).$$

Using similar arguments to the proof of Lemma 1.3.1, we obtain that

$$\frac{\alpha^2}{2}\|u^+\|^2 + \alpha\beta\langle u^+, u^-\rangle - \int_\Omega F(\alpha u^+)\mathrm{d}x > 0$$

for $\alpha > 0$ small enough. But this contradicts the assumption that $(0,\beta)$ is a maximum point of ϕ. The case $(\alpha, 0)$ is similar and we complete the proof. ■

Remark 1.3.9 *Note that the point (t,s) given in Lemma 1.3.6 satisfies $\phi(t,s) = I(tu^+ + su^-) > 0$ since $\phi(\alpha,\beta) > 0$ for $\alpha,\beta > 0$ small enough.*

Now, we shall prove an upper bound for the nodal level $c_{\mathcal{M}}$ defined in (1.26).

Lemma 1.3.10 *Assume that (V_1) and $(f_1)-(f_5)$ are satisfied. If θ is the constant given by (f_3), then*

$$c_{\mathcal{M}} < \frac{\theta-2}{2\theta}. \tag{1.43}$$

Proof . By Theorem B.1.9 (see Appendix) there exists $w \in \mathcal{M}^p$ such that $I_p(w) = c_{\mathcal{M}^p}$ and $I_p'(w)w^\pm = 0$. Consequently, we have

$$\frac{1}{2}\|w\|^2 - \frac{1}{p}\|w\|_p^p = c_{\mathcal{M}^p}, \tag{1.44}$$

$$\|w^\pm\|^2 = \|w^\pm\|_p^p - \langle w^+, w^-\rangle \tag{1.45}$$

and

$$\|w\|^2 = \|w\|_p^p. \tag{1.46}$$

Hence, by (1.46) and (1.44), we get

$$\left(\frac{1}{2} - \frac{1}{p}\right)\|w\|_p^p = c_{\mathcal{M}^p}. \tag{1.47}$$

Since $w^\pm \neq 0$, by Lemma 1.3.6, there exist $t,s > 0$ such that $tw^+ + sw^- \in \mathcal{M}$. Consequently, we obtain

$$\begin{aligned} c_{\mathcal{M}} \leq I(tw^+ + sw^-) =& \frac{t^2}{2}\|w^+\|^2 + ts\langle w^+, w^-\rangle + \frac{s^2}{2}\|w^-\|^2 \\ &- \int_\Omega F(tw^+)\mathrm{d}x - \int_\Omega F(sw^-)\mathrm{d}x. \end{aligned}$$

This together with (f_5) implies

$$c_{\mathcal{M}} \leq \frac{t^2}{2}\|w^+\|^2 + ts\langle w^+, w^-\rangle + \frac{s^2}{2}\|w^-\|^2 - \frac{C_p t^p}{p}\int_\Omega |w^+|\mathrm{d}x - \frac{C_p s^p}{p}\int_\Omega |w^-|\mathrm{d}x.$$

Using (1.45) and Lemma 1.2.3, we have

$$\begin{aligned} c_{\mathcal{M}} &\leq \frac{t^2}{2}\|w^+\|_p^p - \frac{t^2}{2}\langle w^+, w^-\rangle + ts\langle w^+, w^-\rangle + \frac{s^2}{2}\|w^-\|_p^p \\ &\quad - \frac{s^2}{2}\langle w^+, w^-\rangle - \frac{C_p t^p}{p}\|w^+\|_p^p - \frac{C_p s^p}{p}\|w^-\|_p^p \\ &= \left(\frac{t^2}{2} - \frac{C_p t^p}{p}\right)\|w^+\|_p^p + \left(\frac{s^2}{2} - \frac{C_p s^p}{p}\right)\|w^-\|_p^p - \frac{1}{2}(t-s)^2\langle w^+, w^-\rangle \\ &\leq \max_{\xi\geq 0}\left(\frac{\xi^2}{2} - \frac{C_p\xi^p}{p}\right)\|w\|_p^p. \end{aligned}$$

By elementary calculus, it is easy to see that

$$\max_{\xi\geq 0}\left(\frac{\xi^2}{2} - \frac{C_p\xi^p}{p}\right) = C_p^{\frac{2}{2-p}}\left(\frac{1}{2} - \frac{1}{p}\right).$$

Hence, by (1.47) it follows that

$$c_{\mathcal{M}} \leq C_p^{\frac{2}{2-p}}\left(\frac{1}{2} - \frac{1}{p}\right)\|w\|_p^p = C_p^{\frac{2}{2-p}} c_{\mathcal{M}^p}.$$

Therefore, by the definition of C_p given in Theorem 1.1.2, we obtain (1.43). ■

Remark 1.3.11 *By Corollary 1.3.3 and Lemma 1.3.10, we have*

$$0 < \delta_0 \leq c_{\mathcal{N}} \leq c_{\mathcal{M}} < \frac{\theta - 2}{2\theta}.$$

The next step is to obtain a minimizing sequence for the nodal level $c_{\mathcal{M}}$ with a special behavior. For this, we start by defining the set

$$\widetilde{S}_\lambda = \{u \in \mathcal{M} : I(u) < c_{\mathcal{M}} + \lambda\} \text{ for } \lambda > 0.$$

Lemma 1.3.12 *Assume that (V_1) and (f_1)–(f_5) are satisfied. For $\lambda > 0$ small enough, there exists $m_\lambda \in (0,1)$ such that*

$$0 < m_0' \leq \|u^\pm\|^2 < \|u\|^2 \leq m_\lambda,$$

for any $u \in \widetilde{S}_\lambda$.

Proof . Let $u \in \widetilde{S}_\lambda$. By Lemma 1.3.4 and using $\langle u^+, u^- \rangle > 0$, we have $m_0' \leq \|u^\pm\|^2 < \|u\|^2$. On the other hand, by (f_3) and since $I'(u)u = 0$, we obtain

$$\begin{aligned} c_{\mathcal{M}} + \lambda > I(u) &= I(u) - \frac{1}{\theta} I'(u)u \\ &= \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u\|^2 - \int_\Omega \left(F(u) - \frac{1}{\theta} f(u)u\right) \mathrm{d}x \\ &\geq \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u\|^2. \end{aligned}$$

By Lemma 1.3.10, we can take $\lambda > 0$ such that $c_{\mathcal{M}} + \lambda < \left(\dfrac{\theta - 2}{2\theta}\right)$. Consequently, it follows that

$$\|u\|^2 \leq \frac{2\theta}{\theta - 2}(c_{\mathcal{M}} + \lambda) =: m_\lambda < 1,$$

for all $u \in \widetilde{S}_\lambda$. This completes the proof of the lemma. ■

Lemma 1.3.13 *Assume that $(V_1), (f_1)$ and (f_5) are satisfied. Let (u_n) be a sequence in X such that $u_n \rightharpoonup u$ weakly in X, $b := \sup_{n\in\mathbb{N}} \|u_n\|^2 < 1$. Then, up to a subsequence, for all $v \in X$, we have*

$$\lim_{n\to\infty} \int_\Omega f(u_n)u_n \mathrm{d}x = \int_\Omega f(u)u \mathrm{d}x; \tag{1.48}$$

$$\lim_{n\to\infty} \int_\Omega f(u_n)u_n^\pm \mathrm{d}x = \int_\Omega f(u)u^\pm \mathrm{d}x; \tag{1.49}$$

$$\lim_{n\to\infty} \int_\Omega f(u_n)v \mathrm{d}x = \int_\Omega f(u)v \mathrm{d}x \tag{1.50}$$

and

$$\lim_{n\to\infty} \int_\Omega F(u_n) \mathrm{d}x = \int_\Omega F(u) \mathrm{d}x. \tag{1.51}$$

Proof . Since $b < 1$ and by using (f_1) and (f_3), Hölder's inequality and Lemma 1.2.1, it is easy to see that the integrals

$$\int_\Omega |f(u_n)u_n||u_n| \mathrm{d}x, \ \int_\Omega |f(u_n)u_n^\pm||u_n| \mathrm{d}x, \ \int_\Omega |f(u_n)v||u_n| \mathrm{d}x \text{ and } \int_\Omega |F(u_n)|u_n| \mathrm{d}x$$

are uniformly bounded. Thus the convergences (1.48)-(1.51) follow from Lemma 2.1 of [25]. ■

From now on, we will write $\widetilde{S}_\lambda$ with $\lambda > 0$ given in Lemma 1.3.12.

Lemma 1.3.14 *Assume that (V_1) and $(f_1) - (f_5)$ are satisfied. For any $q > 2$, there exists $\delta_q > 0$ such that*

$$0 < \delta_q \leq \int_\Omega |u^\pm|^q \mathrm{d}x < \int_\Omega |u|^q \mathrm{d}x,$$

for each $u \in \widetilde{S}_\lambda$.

Proof . If $u \in \widetilde{S}_\lambda$ and $q > 2$, then

$$\|u^\pm\|^2 + \langle u^+, u^-\rangle = \int_\Omega f(u^\pm)u^\pm \mathrm{d}x.$$

By Lemma 1.2.3 we know that $\langle u^+, u^-\rangle > 0$. Thus by Lemma 1.3.4 we get

$$0 < m_0' \le \|u^\pm\|^2 < \int_\Omega f(u^\pm)u^\pm \mathrm{d}x.$$

Hence, using (f_1) and Hölder's inequality, we have

$$m_0' \le C_0 \int_\Omega |u^\pm| e^{\pi|u^\pm|^2}\mathrm{d}x \le C_0 \left(\int_\Omega |u^\pm|^t \mathrm{d}x\right)^{1/t} \left(\int_\Omega e^{\pi t'|u^\pm|^2}\mathrm{d}x\right)^{1/t'}$$

where $t, t' > 1$ and satisfy $1/t + 1/t' = 1$. By Lemma 1.3.12, we know that $\|u^\pm\|^2 < \|u\|^2 \le m_\lambda$ with $m_\lambda \in (0,1)$. Now, we can take $t' > 1$ sufficiently close to 1, $t > q$ and such that $\pi t'\|u^\pm\|^2 \le \pi t' m_\lambda \le \pi$. Consequently, by Lemma 1.2.2 we get

$$m_0' \le C_0 \|u^\pm\|_t \left(\int_\Omega e^{\pi t'\|u^\pm\|^2 \left(\frac{|u^\pm|}{\|u^\pm\|}\right)^2} \mathrm{d}x\right)^{1/t'} \le C_1 \|u^\pm\|_t.$$

Hence

$$0 < \frac{m_0'}{C_1} \le \|u^\pm\|_t, \text{ for all } u \in \widetilde{S}_\lambda. \tag{1.52}$$

We suppose, by contradiction, that there exists $(u_n) \subset \widetilde{S}_\lambda$ such that $\|u_n^\pm\|_q \to 0$ as $n \to \infty$. From Lemma 1.3.12 and Lemma 1.2.1 we obtain that $(u_n^\pm)$ is bounded in $L^{2t}(\Omega)$. Consequently, since $q < t < 2t$, by the interpolation inequality we find that $\|u_n^\pm\|_t \to 0$ as $n \to \infty$, which is impossible in view of (1.52). ■

The next technical result will be used in the proof of Lemma 1.3.16.

Lemma 1.3.15 *Assume $(f_1) - (f_4)$. Then the function $H(t) := f(t)t - 2F(t)$ satisfies*

(i) *$H(0) = 0$ and $H(t) > 0$, for all $t \ne 0$;*

(ii) *$H(t_0) \le H(t_1)$ if $0 < t_0 \le t_1$;*

(iii) *$H(t_0) \ge H(t_1)$ if $t_0 \le t_1 < 0$.*

Proof . Item **(i)** is immediate from (f_2) and (f_3). For item **(ii)**, if $0 < t_0 \le t_1$ then by (f_4)

$$\begin{aligned} H(t_0) &= \frac{f(t_0)}{t_0} t_0^2 - 2F(t_1) + 2\int_{t_0}^{t_1} f(\tau)\mathrm{d}\tau \\ &\le \frac{f(t_1)}{t_1} t_0^2 - 2F(t_1) + 2\frac{f(t_1)}{t_1}\int_{t_0}^{t_1} \tau \mathrm{d}\tau \\ &\le \frac{f(t_1)}{t_1} t_0^2 - 2F(t_1) + \frac{f(t_1)}{t_1}(t_1^2 - t_0^2) = H(t_1), \end{aligned}$$

which implies the item **(ii)**. The proof of the item **(iii)** is similar. ■

Now we have all the results that will allow us to prove that the nodal level $c_{\mathcal{M}}$ is attained in a function with $u \in \mathcal{M}$.

Lemma 1.3.16 *Assume that* (V_1) *and* $(f_1) - (f_5)$ *are satisfied. Then there exists* $\widetilde{u} \in \mathcal{M}$ *such that* $I(\widetilde{u}) = c_{\mathcal{M}}$.

Proof . Let $(u_n) \subset \mathcal{M}$ be such that $I(u_n) \to c_{\mathcal{M}}$ as $n \to \infty$. We can assume that $u_n \in \widetilde{S}_\lambda$, for all $n \in \mathbb{N}$. Then, by (f_3) we have

$$\begin{aligned} c_{\mathcal{M}} + o_n(1) = I(u_n) &= I(u_n) - \frac{1}{\theta} I'(u_n)u_n \\ &= \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2 + \frac{1}{\theta}\int_\Omega (f(u_n)u_n - \theta F(u_n))\,\mathrm{d}x \\ &\geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2. \end{aligned}$$

Hence, (u_n) is bounded in X and consequently (u_n^+) and (u_n^-) are also bounded in X. Since X is a Hilbert space, up to a subsequence, there exists $u \in X$ such that $u_n^\pm \rightharpoonup u^\pm$ and $u_n \rightharpoonup u$ in X. Utilizing Lemma 1.2.1, up to a subsequence, we can assume that $u_n^\pm \to u^\pm$ in $L^q(\mathbb{R})$, for all $q \in [1, \infty)$, and $u_n^\pm(x) \to u^\pm(x)$ a.e. in $\mathbb{R}$ (see Lemma A.1.8). Now, by using Lemma 1.3.14, we obtain $u^\pm \neq 0$ in X. Now, from Lemma 1.3.6 there exist $t, s \in (0, \infty)$ such that $\widetilde{u} = tu^+ + su^- \in \mathcal{M}$. We claim that $I'(u)u^\pm \leq 0$. In fact, by the convergence (1.49) in Lemma 1.3.13, by **(i)** of Lemma 1.2.3 and by the Fatou's Lemma, it follows that

$$\begin{aligned} \langle u, u^+\rangle &= \langle u^+, u^+\rangle + \langle u^+, u^-\rangle \\ &\leq \liminf_{n\to\infty} \|u_n^+\|^2 + \frac{1}{\pi}\liminf_{n\to\infty}\int_{\mathbb{R}^2} \frac{u_n^+(x)(-u_n^-(y))}{|x-y|^2}\mathrm{d}x\mathrm{d}y \\ &\leq \liminf_{n\to\infty}\left(\|u_n^+\|^2 + \langle u_n^+, u_n^-\rangle\right) \\ &= \liminf_{n\to\infty}\int_\Omega f(u_n)u_n^+\mathrm{d}x \\ &= \int_\Omega f(u)u^+\mathrm{d}x. \end{aligned}$$

Hence, $I'(u)u^+ \leq 0$. Similarly, we get $I'(u)u^- \leq 0$. Then, by Lemma 1.3.7 we obtain $0 < t, s \leq 1$. In particular, $\|\widetilde{u}\|^2 \leq \|u\|^2$.

Now, in order to conclude the proof, note that using the convergences in Lemma 1.3.13 and Lemma 1.3.15, it holds

$$
\begin{aligned}
c_{\mathcal{M}} \le I(\widetilde{u}) &= I(\widetilde{u}) - \frac{1}{2}I'(\widetilde{u})\widetilde{u} \\
&= \frac{1}{2}\int_\Omega (f(\widetilde{u})\widetilde{u} - 2F(\widetilde{u}))\,\mathrm{d}x \\
&= \frac{1}{2}\int_\Omega H(tu^+)\mathrm{d}x + \frac{1}{2}\int_\Omega H(su^-)\mathrm{d}x \\
&\le \frac{1}{2}\int_\Omega H(u^+)\mathrm{d}x + \frac{1}{2}\int_\Omega H(u^-)\mathrm{d}x \\
&= \frac{1}{2}\int_\Omega (f(u)u - 2F(u))\,\mathrm{d}x = I(u_n) - \frac{1}{2}I'(u_n)u_n + o_n(1) = c_{\mathcal{M}}
\end{aligned}
$$

and this concludes the proof. ■

1.4 Proof of Theorem 1.1.2

First, we shall prove an auxiliary result and present some notations that will be used in the proof of Theorem 1.1.2. We consider $D = (\frac{1}{2}, \frac{3}{2}) \times (\frac{1}{2}, \frac{3}{2})$ and $g : \overline{D} \to X$ given by $g(\alpha, \beta) = \alpha\widetilde{u}^+ + \beta\widetilde{u}^-$, where $\widetilde{u}$ was obtained in Lemma 1.3.16.

Lemma 1.4.1 *Let $P = \{u \in X : u(x) \ge 0\, a.e.\ x \in \Omega\}$ and $-P = \{u \in X : u(x) \le 0\, a.e.\ x \in \Omega\}$. Then $d_0' = dist(g(\overline{D}), \Lambda) > 0$, where $\Lambda := P \cup (-P)$.*

Proof . We suppose, by contradiction, that $d_0' = dist(g(\overline{D}), \Lambda) = 0$. Then we can find $(v_n) \subset g(\overline{D})$ and $(w_n) \subset \Lambda$ such that $\|v_n - w_n\| \to 0$ as $n \to \infty$. We can assume, without loss of generality, that $w_n \ge 0$ a.e. in Ω. Now, since $v_n \in g(\overline{D})$, there exist $\alpha_n, \beta_n \in [\frac{1}{2}, \frac{3}{2}]$ such that $v_n = \alpha_n\widetilde{u}^+ + \beta_n\widetilde{u}^-$. Utilizing that (v_n) is bounded in X and Lemma 1.2.1, up to a subsequence, we find $(a_0, b_0) \in \overline{D}$ such that

$$v_n(x) \to a_0\widetilde{u}^+(x) + b_0\widetilde{u}^-(x) \text{ a.e. } x \in \Omega.$$

On the other hand, by the convergence $\|v_n - w_n\| \to 0$ and Lemma 1.2.1, up to a subsequence, we obtain that

$$w_n(x) \to a_0\widetilde{u}^+(x) + b_0\widetilde{u}^-(x) \text{ a.e. } x \in \Omega.$$

Since $\widetilde{u}^- \ne 0$, the convergence above produces a contradiction with the assumption that $w_n \ge 0$ a.e. in Ω, which completes the proof. ■

We are now able to complete the proof of Theorem 1.1.2.

By Lemma 1.3.16, we have found $\widetilde{u} \in \mathcal{M}$ such that $I(\widetilde{u}) = c_{\mathcal{M}}$. It remains to prove that $\widetilde{u}$ is a critical point of the functional I. Suppose, by contradiction, that $I'(\widetilde{u}) \neq 0$. Thus, by the continuity of I', there exist $\lambda, \delta > 0$ with $\delta \leq \frac{d_0'}{2}$, where d_0' is given in Lemma 1.4.1, such that

$$\|I'(v)\| \geq \lambda, \text{ for all } v \in B_{3\delta}(\widetilde{u}). \tag{1.53}$$

By Lemma 1.3.8 we have that the function $(I \circ g)(\alpha, \beta)$, for $(\alpha, \beta) \in \overline{D}$, has a strict maximum point $(1,1)$. In particular, we have that

$$m^* = \max_{(\alpha,\beta)\in\partial D} (I \circ g)(\alpha, \beta) < c_{\mathcal{M}}.$$

Let $\varepsilon > 0$ be such that $\varepsilon < \min\{(c_{\mathcal{M}} - m^*)/2, \lambda\delta/8\}$ and we define $S = B_\delta(\widetilde{u})$. By the choice of ε and by condition (1.53), if $v \in S_{2\delta} = B_{3\delta}(\bar{u})$ we have $\|I'(v)\| \geq \frac{8\varepsilon}{\delta}$. In particular,

$$\forall\, v \in I^{-1}([c_{\mathcal{M}} - 2\varepsilon, c_{\mathcal{M}} + 2\varepsilon]) \cap S_{2\delta}, \text{ it has to satisfy } \|I'(v)\| \geq \frac{8\varepsilon}{\delta}.$$

By the quantitative deformation lemma in [72, Lemma 2.3], there exists $\eta \in C([0,1] \times X, X)$ such that

(i) $\eta(t, u) = u$, if $t = 0$ or $u \notin I^{-1}([c_{\mathcal{M}} - 2\varepsilon, c_{\mathcal{M}} + 2\varepsilon]) \cap S_{2\delta}$;

(ii) $\eta(1, I^{c_{\mathcal{M}}+\varepsilon} \cap S) \subset I^{c_{\mathcal{M}}-\varepsilon}$;

(iii) $\eta(t, \cdot)$ is an homeomorphism of X, $\forall\, t \in [0,1]$;

(iv) $\|\eta(t, u) - u\| \leq \delta$, $\forall\, u \in X$, $\forall\, t \in [0,1]$;

(v) $I(\eta(\cdot, u))$ is non increasing, $\forall\, u \in X$;

(vi) $I(\eta(t, u)) < c_{\mathcal{M}}$, $\forall\, u \in I^{c_{\mathcal{M}}} \cap S_\delta$, $\forall\, t \in (0,1]$.

As an application, we get

$$\max_{(\alpha,\beta)\in\overline{D}} I(\eta(1, g(\alpha, \beta))) < c_{\mathcal{M}}. \tag{1.54}$$

In fact, if $(\alpha, \beta) \in D$ with $(\alpha, \beta) \neq (1, 1)$, using Lemma 1.3.8 we have $I(g(\alpha, \beta)) < c_{\mathcal{M}}$. Hence

$$I(\eta(1, g(\alpha, \beta))) \leq I(\eta(0, g(\alpha, \beta))) = I(g(\alpha, \beta)) < c_{\mathcal{M}}.$$

If $(\alpha, \beta) = (1, 1)$ then $g(1, 1) = \widetilde{u} \in I^{c_{\mathcal{M}}+\varepsilon} \cap S$. Thus $I(\eta(1, g(1, 1))) < c_{\mathcal{M}} - \varepsilon$, showing (1.54).

Now, let us define $h(\alpha, \beta) = \eta(1, g(\alpha, \beta))$. We claim that

$$h(\alpha, \beta) = g(\alpha, \beta) \text{ in } \partial D. \tag{1.55}$$

In fact, given $(\alpha, \beta) \in \partial D$, by the definition of m^* and by the choice of ε, we have

$$I(g(\alpha, \beta)) \leq m^* = c_{\mathcal{M}} - 2\frac{(c_{\mathcal{M}} - m^*)}{2} < c_{\mathcal{M}} - 2\varepsilon.$$

Hence $g(\alpha, \beta) \notin I^{-1}([c_{\mathcal{M}} - 2\varepsilon, c_{\mathcal{M}} + 2\varepsilon])$. So using the property **(i)** of the function η we get (1.55).

Claim 1.4.2 *We claim that $h(\alpha, \beta)^{\pm} \neq 0$, for all $(\alpha, \beta) \in \overline{D}$.*

In fact, let $v \in \Lambda$. By using the choice of $\delta > 0$ and Lemma 1.4.1, we have that

$$\begin{aligned} \|h(\alpha, \beta) - v\| &\geq \|g(\alpha, \beta) - v\| - \|h(\alpha, \beta) - g(\alpha, \beta)\| \\ &\geq \|g(\alpha, \beta) - v\| - \delta \\ &\geq d_0' - \frac{d_0'}{2} = \frac{d_0'}{2}. \end{aligned}$$

Consequently, $h^{\pm}(\alpha, \beta) \neq 0$ for all $(\alpha, \beta) \in \overline{D}$, concluding the statement.

Now, we consider the vector fields

$$\mathcal{F}(\alpha, \beta) = (I'(g(\alpha, \beta))\widetilde{u}^+, I'(g(\alpha, \beta))\widetilde{u}^-)$$

and

$$\mathcal{G}(\alpha, \beta) = (\frac{1}{\alpha} I'(h(\alpha, \beta))h(\alpha, \beta)^+, \frac{1}{\beta} I'(h(\alpha, \beta))h(\alpha, \beta)^-).$$

From (1.55), we have $\mathcal{F} = \mathcal{G}$ in ∂D. Hence, by the degree theory (see Lemma A.1.14), we have

$$\deg(\mathcal{F}, D, (0, 0)) = \deg(\mathcal{G}, D, (0, 0)). \tag{1.56}$$

Claim 1.4.3 $\deg(\mathcal{F}, D, (0, 0)) = 1.$

In fact, we consider

$$\mathcal{F}_1(\alpha,\beta)=I'(\alpha\widetilde{u}^++\beta\widetilde{u}^-)\widetilde{u}^+ \text{ and } \mathcal{F}_2(\alpha,\beta)=I'(\alpha\widetilde{u}^++\beta\widetilde{u}^-)\widetilde{u}^-$$

the coordinates functions of the vector field $\mathcal{F}$. Calculating the partial derivatives of $\mathcal{F}_1$ and $\mathcal{F}_2$, we get

$$\begin{cases} \dfrac{\partial\mathcal{F}_1}{\partial\alpha}(\alpha,\beta) &= \|\widetilde{u}^+\|^2-\displaystyle\int_\Omega f'(\alpha\widetilde{u}^+)(\widetilde{u}^+)^2\mathrm{d}x,\\ \dfrac{\partial\mathcal{F}_1}{\partial\beta}(\alpha,\beta) &= \dfrac{\partial\mathcal{F}_2}{\partial\alpha}(\alpha,\beta)=\langle\widetilde{u}^+,\widetilde{u}^-\rangle,\\ \dfrac{\partial\mathcal{F}_2}{\partial\beta}(\alpha,\beta) &= \|\widetilde{u}^-\|^2-\displaystyle\int_\Omega f'(\beta\widetilde{u}^-)(\widetilde{u}^-)^2\mathrm{d}x. \end{cases}$$

Now, for $(\alpha,\beta)=(1,1)$ in the above equations and using the condition $I'(\widetilde{u})\widetilde{u}^\pm=0$, we reach

$$\begin{cases} \dfrac{\partial\mathcal{F}_1}{\partial\alpha}(1,1) &= -\langle\widetilde{u}^+,\widetilde{u}^-\rangle+\displaystyle\int_\Omega G(\widetilde{u}^+)\widetilde{u}^+\mathrm{d}x,\\ \dfrac{\partial\mathcal{F}_1}{\partial\beta}(1,1) &= \dfrac{\partial\mathcal{F}_2}{\partial\alpha}(1,1)=\langle\widetilde{u}^+,\widetilde{u}^-\rangle,\\ \dfrac{\partial\mathcal{F}_2}{\partial\beta}(1,1) &= -\langle\widetilde{u}^+,\widetilde{u}^-\rangle+\displaystyle\int_\Omega G(\widetilde{u}^-)\widetilde{u}^-\mathrm{d}x, \end{cases}$$

where $G(t)=f(t)-f'(t)t$, for $t\in\mathbb{R}$. Using (f_4) and $\widetilde{u}^\pm\neq 0$, it is easy to see that

$$\int_\Omega G(\widetilde{u}^+)\widetilde{u}^+\mathrm{d}x<0 \text{ and } \int_\Omega G(\widetilde{u}^-)\widetilde{u}^-\mathrm{d}x<0. \tag{1.57}$$

Hence, using (1.57) and $\langle\widetilde{u}^+,\widetilde{u}^-\rangle>0$, it follows that

$$\det\begin{bmatrix} \dfrac{\partial\mathcal{F}_1}{\partial\alpha}(1,1) & \dfrac{\partial\mathcal{F}_1}{\partial\beta}(1,1)\\ \dfrac{\partial\mathcal{F}_2}{\partial\alpha}(1,1) & \dfrac{\partial\mathcal{F}_2}{\partial\beta}(1,1) \end{bmatrix}>0.$$

Thus, since $(1,1)$ is the unique solution of $\mathcal{F}(\alpha,\beta)=(0,0)$ in D, by the definition of topological degree (see Lemma A.1.15), we have $\deg(\mathcal{F},D,(0,0))=1$, showing (1.4.3).

Utilizing the Claim 1.4.3 and (1.56) we obtain $\deg(\mathcal{G},D,(0,0))=\deg(\mathcal{F},D,(0,0))=1$ and therefore there exists $(\alpha_0,\beta_0)\in D$ such that $\mathcal{G}(\alpha_0,\beta_0)=(0,0)$ (see Lemma A.1.13 and Lemma A.1.14), that is,

$$\begin{cases} I'(\eta(1,g(\alpha_0,\beta_0)))\eta(1,g(\alpha_0,\beta_0))^+ &= 0,\\ I'(\eta(1,g(\alpha_0,\beta_0)))\eta(1,g(\alpha_0,\beta_0))^- &= 0. \end{cases} \tag{1.58}$$

By Claim 1.4.2 we have that $h(\alpha_0,\beta_0)^\pm\neq 0$. Hence, system (1.58) implies that $h(\alpha_0,\beta_0)$ belongs to $\eta(1,g(D))\cap\mathcal{M}$. Thus, by the definition of $c_\mathcal{M}$,

$$I(h(\alpha_0,\beta_0))=I(\eta(1,g(\alpha_0,\beta_0))\geq c_\mathcal{M},$$

which is a contradiction in view of (1.54). Therefore, $I'(\widetilde{u}) = 0$ and this completes the proof of Theorem 1.1.2.

1.5 Proof of Theorem 1.1.3

Defining the set

$$\overline{S}_\lambda = \{u \in \mathcal{N} : I(u) < c_{\mathcal{N}} + \lambda\},$$

where $c_{\mathcal{N}}$ is the ground state level defined in (1.25) and λ is given in Lemma 1.3.12. By Remark 1.3.11 and (f_3), we get

$$c_{\mathcal{M}} + \lambda \geq c_{\mathcal{N}} + \lambda > I(u) \geq \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u\|^2, \quad \text{for all} \quad u \in \overline{S}_\lambda.$$

Hence, by Lemma 1.3.2 and Lemma 1.3.12, we have

$$0 < m_0 \leq \|u\|^2 < m_\lambda \quad \text{for all} \quad u \in \overline{S}_\lambda \tag{1.59}$$

where $m_\lambda \in (0, 1)$ is given in Lemma 1.3.12. Similar to Lemma 1.3.14, using (1.59), we can to show that for any $q > 2$ there exists $\delta_q > 0$ such that

$$0 < \delta_q \leq \int_\Omega |u|^q \mathrm{d}x, \quad \text{for all} \quad u \in \overline{S}_\lambda. \tag{1.60}$$

Let $(v_n) \subset \overline{S}_\lambda$ be a sequence such that $I(v_n) \to c_{\mathcal{N}}$. By (1.59), (v_n) is bounded sequence and X is a Hilbert space, up to a subsequence, there exists $v \in X$ such that $v_n \rightharpoonup v$. Utilizing Lemma 1.2.1, up to a subsequence, we can assume that $v_n \to v$ in $L^q(\mathbb{R})$, for all $q \in [1, \infty)$, and $v_n(x) \to v(x)$ a.e. in $\mathbb{R}$. Using (1.60), we infer that $v \neq 0$ in X. By Lemma 1.3.1, there exists $t > 0$ such that $\widetilde{v} = tv \in \mathcal{N}$. Considering (1.59), we can assume, without loss of generality, that the convergences in Lemma 1.3.13 hold for the sequence (v_n). Now, since $I'(v_n)v_n = 0$ for all $n \in \mathbb{N}$, by lower semicontinuity and using Lemma 1.3.13, we have

$$\|v\|^2 \leq \liminf_{n\to\infty} \|v_n\|^2 = \liminf_{n\to\infty} \int_\Omega f(v_n)v_n \mathrm{d}x = \int_\Omega f(v)v \mathrm{d}x \tag{1.61}$$

and so $I'(v)v \leq 0$. Analogously to Lemma 1.3.7, we can deduce that $t \leq 1$. Following similar ideas from of the proof of Lemma 1.3.16, we can show that $I(\widetilde{v}) = c_{\mathcal{N}}$. Moreover, utilizing the same steps of the proof of Theorema 1.1.2, we show that the function $\widetilde{v}$ satisfies that $I'(\widetilde{v}) = 0$. Thus, $\widetilde{v}$ is a ground state solution of problem (1.1). Now, in order to prove (1.11), we consider the function $\widetilde{u}$ obtained in Theorem 1.1.2. Since $\widetilde{u}^{\pm} \neq 0$, by Lemma 1.3.1, there exists a unique pair (t_1, t_2) such that $t_1\widetilde{u}^{+} \in \mathcal{N}$ and $t_2\widetilde{u}^{-} \in \mathcal{N}$. By Corollary 1.3.3, we have $c_{\mathcal{N}} > 0$. Now, by using the definition of $c_{\mathcal{N}}$, Lemma 1.2.3, Lemma 1.3.7 and Lemma 1.3.8, we have that

$$0 < 2c_{\mathcal{N}} \leq I(t_1\widetilde{u}^{+}) + I(t_2\widetilde{u}^{-}) < I(t_1\widetilde{u}^{+} + t_2\widetilde{u}^{-}) \leq I(\widetilde{u}^{+} + \widetilde{u}^{-}) = c_{\mathcal{M}},$$

showing (1.11). In particular, the inequality above shows that can not exist a nodal ground state solution of problem (1.1). Thus, the ground state solution $\widetilde{v}$ is nonpositive or nonnegative.

Chapter 2

Ground state and nodal solutions for a class of fractional equations involving exponential growth in a unbounded domain

In this chapter we consider the following fractional Schrödinger equation:

$$(-\Delta)^{\frac{1}{2}}u + V(x)u = K(x)f(u) \quad \text{in} \quad \mathbb{R}, \tag{2.1}$$

where $(-\Delta)^{\frac{1}{2}}$ is the 1/2-Laplacian operator defined in (1.2), $V, K : \mathbb{R} \to \mathbb{R}_+$ are functions satisfying appropriate conditions which will be introduced later and $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function that may have a subcritical or critical exponential growth in the Trudinger-Moser embedding sense. Since the problem is set on the whole real line one has to tackle compactness issues, which can be overcome by considering suitable assumptions of K at infinity. Similar to Chapter 1, our goal is to show that under appropriate conditions problem (2.1) has a ground state and a nodal solution u, which are distinct. Moreover, we show that the energy of u is strictly larger than twice the ground state energy. The results of this chapter were submitted for publication in article [31].

2.1 Introduction

As in Chapter 1, we are interested in looking for solutions of (2.1) when the nonlinearity $f(t)$ has exponential growth. The fractional Sobolev space $H^{1/2}(\mathbb{R})$ is continuously embedded into $L^q(\mathbb{R})$ for any $q \in [1, +\infty)$ and compactly embedded into $L^q(\Omega)$ for any $q \in [1, +\infty)$, for all $\Omega \subset \mathbb{R}$ bounded interval. But $H^{1/2}(\mathbb{R})$ is not continuously embedded in $L^\infty(\mathbb{R})$ (see [32, 61]). However, S. Iula, A. Maalaou and L. Martinazzi in [51] proved a Trudinger-Moser type inequality on $H^{1/2}(\mathbb{R})$ as:

$$\sup_{\{u \in H^{1/2}(\mathbb{R}) \,:\, \|u\|_{1/2,2} \leq 1\}} \int_{\mathbb{R}} (e^{\alpha u^2} - 1)\mathrm{d}x < \infty, \tag{2.2}$$

for any $0 \leq \alpha \leq \pi$, where

$$\|u\|_{1/2,2} := \left(\|(-\Delta)^{\frac{1}{4}} u\|_2^2 + \|u\|_2^2 \right)^{1/2} \quad \text{and} \quad \|(-\Delta)^{\frac{1}{4}} u\|_2^2 = \frac{1}{2\pi}[u]_{1/2}^2.$$

(see also [53, 61, 69]).

Thus the maximal growth on $f(t)$, which allows us to study (2.1) by applying a variational framework involving the space $H^{1/2}(\mathbb{R})$, is given by $e^{\alpha u^2}$ as $|u| \to +\infty$, for some $\alpha > 0$. Motivated by result, we say that $f(t)$ has *exponential critical growth* if there exists $\alpha_0 > 0$ such that

$$\lim_{|t| \to +\infty} f(t)e^{-\alpha|t|^2} = \begin{cases} 0, & \text{for all} \quad \alpha > \alpha_0, \\ +\infty, & \text{for all} \quad \alpha < \alpha_0, \end{cases} \tag{2.3}$$

and we say that $f(t)$ has *exponential subcritical growth* if

$$\lim_{|t| \to +\infty} f(t)e^{-\alpha|t|^2} = 0, \quad \text{for all} \quad \alpha > 0. \tag{2.4}$$

Motivated by Trudinger-Moser inequality, many papers have been developed in order to study issues related to the existence of solutions for problems involving the fractional Laplacian operator and nonlinearities with exponential growth. For example, by exploiting the Trudinger-Moser embedding due to Ozawa [61] and the Mountain Pass Theorem, J. M. do Ó, Miyagaki and Squassina [36] proved the existence of ground state solutions for the following class of nonlinear scalar field equations:

$$\begin{cases} (-\Delta)^{\frac{1}{2}} u + u = f(u) \quad \text{in} \quad \mathbb{R}, \\ u(x) \to 0, \quad \text{as} \quad |x| \to \infty, \end{cases}$$

when $f(t)$ is $o(|t|)$ at the origin and behaves like $e^{\alpha t^2}$ as $|t| \to +\infty$, for some $\alpha > 0$. In [27], Souza and Araújo considered a perturbation of this problem by a general potential $V(x)$, namely,

$$\begin{cases} (-\Delta)^{\frac{1}{2}} u + V(x)u = f(u) \quad \text{in} \quad \mathbb{R}, \\ u(x) \to 0, \quad \text{as} \quad |x| \to \infty, \end{cases}$$

where $V(x)$ is a nonnegative function which is asymptotically periodic at infinity. See also [2, 23, 28, 34, 48, 60] for others investigations.

However, none of the previous works treated the existence of sign-changing solution (nodal solution).

2.2 Assumptions and main results

In order to reach our goals, we assume the following assumptions on the functions V and K:

(V_1) $V, K : \mathbb{R} \to \mathbb{R}_+$ are continuous and $K \in L^\infty(\mathbb{R})$;

(V_2) there exist $b_0, R_0 > 0$ such that

$$V(x) \geq b_0, \;\; \text{for } |x| \geq R_0;$$

Since problem (2.1) is set on the whole real line, we face loss of compactness. Here, motivated by [35], in order to overcome this difficulty, we assume the following assumption on K:

(K_1) if $\{A_n\}$ is a sequence of Borel sets of $\mathbb{R}$ with $\sup_{n\in\mathbb{N}} |A_n| \leq R$, for some $R > 0$, then

$$\lim_{r\to\infty} \int_{A_n \cap B_r^c(0)} K(x)\, \mathrm{d}x = 0,$$

uniformly with respect to $n \in \mathbb{N}$.

On the nonlinearity f, we assume the following assumptions:

(f_1) $f \in C^1(\mathbb{R})$ and there exist $C_0, t_0 > 0$ such that

$$|f(t)| \leq C_0 \left(e^{\pi t^2} - 1\right), \quad \text{for all} \quad |t| \geq t_0;$$

(f_2) $\lim_{t\to 0}\frac{f(t)}{t}=0$;

(f_3) there exists $\theta>2$ such that

$$0<\theta F(t):=\theta\int_0^t f(s)\mathrm{d}s\le tf(t),\quad \text{for all}\quad t\in\mathbb{R}\setminus\{0\};$$

(f_4) the function $\frac{f(t)}{|t|}$ is strictly increasing for $t\neq 0$;

(f_5) there exist constants $p>2$ and $C_p>0$ such that

$$sgn(t)f(t)\ge C_p|t|^{p-1},\quad \text{for all}\quad t\in\mathbb{R}.$$

We point out that from (f_1) we can consider nonlinearities with exponential critical growth in the sense of (2.3) and with exponential subcritical growth in the sense of (2.4). Furthermore, by (V_2) the potential $V(x)$ may be zero on a bounded interval. For example, we may consider the potential

$$V(x)=\begin{cases}0, & \text{if}\quad |x|<1\\ x^2-1, & \text{if}\quad 1\le|x|\le 2\\ 3, & \text{if}\quad |x|\ge 2.\end{cases}$$

Example 2.2.1 *A function K satisfying (V_1) and (K_1) is $K(x)=e^{-x^2}$.*

Example 2.2.2 *If $p>2$, the nonlinearity*

$$f(t)=C_p|t|^{p-2}t+|t|^{p-2}t\left(e^{t^2}-1\right)$$

satisfies the assumptions $(f_1)-(f_5)$.

In order to apply variational methods to study (2.1) in $H^{1/2}(\mathbb{R})$, it is natural to work in the subspace of $H^{1/2}(\mathbb{R})$ defined as

$$X:=\left\{u\in H^{1/2}(\mathbb{R}):\int_{\mathbb{R}}V(x)u^2\mathrm{d}x<\infty\right\}. \tag{2.5}$$

From $(V_1)-(V_2)$ (see Lemma 2.3.1 and Proposition 2.3.2), we show that X is a Hilbert space when endowed with the inner product

$$\langle u,v\rangle:=\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^2}\mathrm{d}x\mathrm{d}y+\int_{\mathbb{R}}V(x)uv\mathrm{d}x \tag{2.6}$$

and the corresponding norm

$$\|u\| := \left(\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x-y|^2} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}} V(x)|u|^2 \mathrm{d}x \right)^{1/2}. \tag{2.7}$$

Throughout this chapter, we say $u \in X$ is a weak solution of (2.1) if

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^2} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}} V(x) uv \, \mathrm{d}x - \int_{\mathbb{R}} K(x) f(u) v \mathrm{d}x = 0,$$

for all $v \in X$.

In Section 2.3, we will show that the energy functional

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} K(x) F(u) \mathrm{d}x, \tag{2.8}$$

belongs to $C^1(X, \mathbb{R})$ and its critical points are weak solutions of (2.1).

In order to find nodal solutions for problem (2.1) by applying an appropriate minimization argument, we introduce:

- the Nehari manifold

$$\mathcal{N} = \{u \in X \setminus \{0\} : I'(u)u = 0\}; \tag{2.9}$$

- the nodal set

$$\mathcal{M} = \{u \in X : u^{\pm} \neq 0 \text{ and } I'(u)u^{\pm} = 0\}; \tag{2.10}$$

- the ground state level

$$c_{\mathcal{N}} := \inf_{u \in \mathcal{N}} I(u); \tag{2.11}$$

- the nodal level

$$c_{\mathcal{M}} := \inf_{u \in \mathcal{M}} I(u). \tag{2.12}$$

Since $\mathcal{M} \subset \mathcal{N}$ we have $c_{\mathcal{N}} \leq c_{\mathcal{M}}$. We say that a nonzero critical point $w \in X$ of I is a least energy solution (or ground state solution) if w achieves the infimum $c_{\mathcal{N}}$. One of our goals will be to show that the minimum $c_{\mathcal{M}}$ is reached by a critical point of I. If $w^{\pm} \neq 0$ is a critical point of I such that $I(w) = c_{\mathcal{M}}$ we say that w is a least energy nodal solution of (2.1).

Now we can state our main results.

Theorem 2.2.3 *Suppose that* $(V_1)-(V_2)$*,* (K_1) *and* $(f_1)-(f_5)$ *are satisfied. Then problem* (2.1) *possesses a least energy nodal solution, provided that*

$$C_p > \left[\frac{2\theta\kappa c_{\mathcal{M}^p}}{\theta-2}\right]^{(p-2)/2}, \tag{2.13}$$

where

$$c_{\mathcal{M}^p} := \inf_{u\in\mathcal{M}^p} I_p(u), \quad \mathcal{M}^p := \{u \in X : u^{\pm} \neq 0, I_p'(u)u^{\pm} = 0\} \tag{2.14}$$

and

$$I_p(u) := \frac{1}{2}\|u\|^2 - \frac{1}{p}\int_{\mathbb{R}} K(x)|u|^p \mathrm{d}x, \tag{2.15}$$

and $\kappa > 0$ *is the constant given in* (2.22)*.*

Another goal is to prove that the energy of any sign-changing solution of (2.1) is strictly larger than twice the ground state energy. This property is so-called energy doubling by Weth [71].

Theorem 2.2.4 *Suppose that* $(V_1)-(V_2)$*,* (K_1)*,* $(f_1)-(f_5)$ *and* (2.13) *are satisfied. Then problem* (2.1) *has a least energy solution and*

$$I(w) > 2c_{\mathcal{N}}, \tag{2.16}$$

where w *is the least energy sign-changing solution obtained in Theorem* 2.2.3*. In particular,* $c_{\mathcal{N}}$ *is achieved either by a nonnegative or a nonpositive function.*

Remark 2.2.5 *Note that if we assume that the function* f *is odd, then, using Theorem 2.2.4, it follows that problem* (2.1) *has at least one negative solution, one positive solution, and one nodal solution.*

Remark 2.2.6 *Using the regularity results due to Servadei and Valdinoci [68], we have that weak solutions of problem* (2.1) *belong to* $C(\mathbb{R})$*.*

As in Chapter 1, if $u^{\pm} \neq 0$ then $\langle u^+, u^- \rangle > 0$. Thus, if $u^{\pm} \neq 0$, the energy functional I in (2.8) satisfies

$$I(u) > I(u^+) + I(u^-),$$

$$I'(u)u^+ > I'(u^+)u^+ \text{ and } I'(u)u^- > I'(u^-)u^-.$$

(see Lemma 2.3.7.)

Therefore, the methods used to obtain sign-changing solutions for the local problems can not be applicable to problem (2.1). Moreover, since the problem (2.1) is set on the whole real line one has to tackle compactness issues. Furthermore, an other well-known difficulty is the loss of compactness due to the critical growth on the nonlinearity f.

The outline of this chapter is as follows: Section 2.3 contains some preliminary results related to functional I and the space X. In particular, we obtain a suitable compact injections for X in a weighted Banach space. Section 2.4 is dedicated to the study of the nodal set and the nodal level. Using adequate estimates at the nodal level, suitable compact immersions and tools like the Straus's compactness lemma, we prove that the nodal level is attained. In Section 2.5, we complete the proof of Theorem 1.1.2 and Section 2.6 is proved Theorem 1.1.3.

2.3 Preliminaries

First, we recall that

$$\|(-\Delta)^{\frac{1}{4}}u\|_2^2 = \frac{1}{2\pi}[u]_{1/2}^2 = \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{|u(x)-u(y)|^2}{|x-y|^2}\mathrm{d}x\mathrm{d}y, \text{ for all } u \in H^{1/2}(\mathbb{R}). \quad (2.17)$$

(see [32, Proposition 3.6]).

With this in mind, we prove the following result:

Lemma 2.3.1 *Assume that* $(V_1)-(V_2)$ *are satisfied. Then,*

$$\lambda_1 := \inf_{\substack{u\in X \\ \|u\|_2=1}}\left(\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{|u(x)-u(y)|^2}{|x-y|^2}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}}V(x)u^2\mathrm{d}x\right) > 0.$$

Proof . Suppose, by contradiction, that $\lambda_1 = 0$. Hence, there exists $(u_n) \subset X$ such that

$$\|u_n\|_2^2 = 1 \quad \text{and} \quad \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^2}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}}V(x)u_n^2\mathrm{d}x \to 0, \quad \text{as} \quad n\to\infty. \quad (2.18)$$

From [61], for any $1 < q < \infty$, there exists a constant $M > 0$ such that

$$\|v\|_q \le Mq^{1/2}\|(-\Delta)^{1/4}v\|_2^{1-2/q}\|v\|_2^{2/q}, \quad \text{for all} \quad v \in H^{1/2}(\mathbb{R}). \quad (2.19)$$

Combining (2.17), (2.18) and (2.19), for each $q > 2$, we obtain

$$\|u_n\|_q \le Mq^{1/2}\|(-\Delta)^{1/4}u_n\|_2^{1-2/q} \to 0, \quad \text{as} \quad n\to\infty.$$

Now, note that choosing $t > 1$ such that $2t = q$ and by using the Hölder inequality, we get

$$\|u_n\|^2_{L^2(B_{R_0})} \le |B_{R_0}|^{\frac{1}{t'}} \|u_n\|^2_{L^q(B_{R_0})} \to 0, \quad \text{as} \quad n \to \infty. \tag{2.20}$$

On the other hand, by (V_2) and (2.18), we have

$$\int_{B^c_{R_0}} u_n^2 \mathrm{d}x \le \frac{1}{b_0} \int_{B^c_{R_0}} V(x) u_n^2 \mathrm{d}x \to 0, \quad \text{as} \quad n \to \infty. \tag{2.21}$$

But, (2.20) and (2.21) imply that

$$1 = \|u_n\|^2_{L^2(B_{R_0})} + \|u_n\|^2_{L^2(B^c_{R_0})} \to 0,$$

as $n \to \infty$, which is absurd. Thus, we complete the proof. ■

From Lemma 2.3.1, we reach the following result:

Corollary 2.3.2 *Assume that $(V_1) - (V_2)$ are satisfied. Then the embedding $X \hookrightarrow H^{1/2}(\mathbb{R})$ is continuous and there exists $\kappa > 0$ such that*

$$\frac{1}{\kappa} := \inf_{\substack{u \in X \\ u \neq 0}} \frac{\|u\|^2}{\|u\|^2_{1/2,2}}. \tag{2.22}$$

In particular, X is a Hilbert space with the inner product (2.6) and the embedding $X \hookrightarrow L^q(\mathbb{R})$ is continuous and locally compact for all $q \in [2, +\infty)$.

Now, given $r \ge 1$, we define weighted Banach space

$$L^r_K := \left\{ u : \mathbb{R} \to \mathbb{R} \,:\, u \text{ is measurable and } \int_{\mathbb{R}} K(x)|u|^r \mathrm{d}x < \infty \right\}$$

endowed with the norm

$$\|u\|_{L^r_K} := \left(\int_{\mathbb{R}} K(x)|u|^r \mathrm{d}x \right)^{\frac{1}{r}}.$$

Note that, since $K \in L^\infty(\mathbb{R})$, the embedding $H^{1/2}(\mathbb{R}) \hookrightarrow L^q_K$ is continuous for all $q \ge 2$. Inspired by [35], we have the following result:

Lemma 2.3.3 *$H^{1/2}(\mathbb{R})$ is compactly embedded into L^q_K for all $q \in (2, +\infty)$.*

Proof . Given $q > 2, s > q$ and $\varepsilon > 0$. Since $K(x) \le C$, there exist $0 < t_0(\varepsilon) < t_1(\varepsilon)$ such that

$$K(x)|t|^q \le \varepsilon C(|t|^2 + |t|^s) + K(x)\chi_{[t_0(\varepsilon), t_1(\varepsilon)]}(|t|)|t|^q, \quad \text{for all} \quad t, x \in \mathbb{R}.$$

Hence, for any $v \in H^{1/2}(\mathbb{R})$, we have

$$K(x)|v|^q \leq \varepsilon C(|v|^2 + |v|^s) + K(x)\chi_{[t_0(\varepsilon),t_1(\varepsilon)]}(|v|)|v|^q, \quad \text{for all} \quad x \in \mathbb{R}.$$

Fixing $R > 0$, we reach

$$\int_{B_R^c(0)} K(x)|v|^q \mathrm{d}x \leq \varepsilon Q(v) + \int_{A^\varepsilon \cap B_R^c(0)} K(x)|v|^q \mathrm{d}x, \tag{2.23}$$

where

$$Q(v) := C(\|v\|_2^2 + \|v\|_s^s) \text{ and } A^\varepsilon := \{x \in \mathbb{R} : t_0(\varepsilon) \leq |v| \leq t_1(\varepsilon)\}.\mathbb{R}$$

Let $(u_n) \subset H^{1/2}(\mathbb{R})$ such that $u_n \rightharpoonup u$ weakly in $H^{1/2}(\mathbb{R})$. Then, up to a subsequence, there exists $M > 0$ such that

$$\|u_n - u\|_2^2 \leq M \quad \text{and} \quad \|u_n - u\|_s^s \leq M.$$

In particular,

$$Q(u_n - u) \leq 2CM, \quad \text{for all} \quad n \in \mathbb{N}. \tag{2.24}$$

Now, if $A_n^\varepsilon = \{x \in \mathbb{R} : t_0(\varepsilon) \leq |u_n - u| \leq t_1(\varepsilon)\}$, we get

$$t_0(\varepsilon)^2|A_n^\varepsilon| = \int_{A_n^\varepsilon} t_0(\varepsilon)^2 \mathrm{d}x \leq \int_{\mathbb{R}} |u_n - u|^2 \mathrm{d}x \leq M, \quad \text{for all} \quad n \in \mathbb{N}.$$

Then by (K_1), there exists $R(\varepsilon) > 0$ such that

$$\int_{A_n^\varepsilon \cap B_{R(\varepsilon)}^c(0)} K(x)|u_n - u|^q \mathrm{d}x < \varepsilon, \text{ for all } n \in \mathbb{N}. \tag{2.25}$$

Utilizing the estimates (2.24) and (2.25) in (2.23), we obtain

$$\int_{B_{R(\varepsilon)}^c(0)} K(x)|u_n - u|^q \mathrm{d}x \leq \varepsilon(2MC + 1), \text{ for all } n \in \mathbb{N}. \tag{2.26}$$

On the other hand, by using the compact embedding $H^{1/2}(\mathbb{R}) \hookrightarrow L^q(B_{R(\varepsilon)}(0))$, up to a subsequence, we get

$$\lim_{n \to +\infty} \int_{B_{R(\varepsilon)}(0)} K(x)|u_n - u|^q \mathrm{d}x = 0. \tag{2.27}$$

Therefore, from (2.26) and (2.27), we complete the proof. ■

As a consequence of Corollary 2.3.2 and Lemma 2.3.3, we obtain the following result:

Corollary 2.3.4 *Assume that $(V_1)-(V_2)$ are satisfied. The space X is continuous embedded in L^2_K and compactly embedded into L^q_K for all $q\in(2,+\infty)$.*

One of the main tools to study problems involving exponential growth in the fractional Sobolev spaces is the so-called fractional Trudinger-Moser inequality due to Ozawa [61]. Combining the results in [27, 51, 53, 61, 69], the Trudinger-Moser inequality due to Ozawa has been refined and can be stated as follows.

Lemma 2.3.5 *For any $u\in H^{1/2}(\mathbb{R})$ and $\alpha\geq 0$, we have*

$$\int_{\mathbb{R}}\left(e^{\alpha u^2}-1\right)\,\mathrm{d}x<\infty. \tag{2.28}$$

Furthermore, if $0\leq\alpha\leq\pi$, it holds

$$\sup_{\{u\in H^{1/2}(\mathbb{R})\,:\,\|u\|_{1/2,2}\leq 1\}}\int_{\mathbb{R}}\left(e^{\alpha u^2}-1\right)\,\mathrm{d}x<\infty \tag{2.29}$$

and if $0\leq\alpha<\pi$, there exists $C_\alpha>0$ such that

$$\int_{\mathbb{R}}\left(e^{\alpha u^2}-1\right)\,\mathrm{d}x\leq C_\alpha\|u\|_2^2, \tag{2.30}$$

whenever $u\in H^{1/2}(\mathbb{R})$ and $\|(-\Delta)^{\frac{1}{4}}u\|_2\leq 1$.

As an application of this inequality, we get the following convergence result:

Lemma 2.3.6 *Let $\alpha>0$ and $(u_n)\subset H^{1/2}(\mathbb{R})$ be such that $u_n\to u$ strongly in $H^{1/2}(\mathbb{R})$. Then*

$$\lim_{n\to+\infty}\int_{\mathbb{R}}\left(e^{\alpha u_n^2}-1\right)\mathrm{d}x=\int_{\mathbb{R}}\left(e^{\alpha u^2}-1\right)\mathrm{d}x.$$

Proof . By the Mean Value Theorem, for each $x\in\mathbb{R}$, there exists $a_n(x)$ between $u_n(x)$ and $u(x)$ such that

$$|(e^{\alpha u_n^2(x)}-1)-(e^{\alpha u^2(x)}-1)|=2\alpha|a_n(x)|e^{\alpha a_n^2(x)}||u_n(x)-u(x)|.$$

Now, since that

$$|a_n(x)|\leq|u_n(x)-u(x)|+|u(x)| \text{ and } (|u_n(x)-u(x)|+|u(x)|)^2\leq 2|u_n(x)-u(x)|^2+2|u(x)|^2,$$

we have

$$|(e^{\alpha u_n^2}-1)-(e^{\alpha u^2}-1)|\leq 2\alpha(|u_n-u|+|u|)e^{2\alpha|u_n-u|^2}e^{2\alpha|u|^2}|u_n-u|.$$

Adding and subtracting 1 one each of the factors $e^{\alpha|u_n-u|^2}$ and $e^{\alpha|u|^2}$ in the right-hand of inequality above we obtain four terms one of which is

$$2\alpha(|u_n-u|+|u|)(e^{2\alpha|u_n-u|^2}-1)(e^{2\alpha|u|^2}-1)|u_n-u|.$$

Applying the Hölder inequality with exponents $r_1, r_2 \geq 2$ and $r_3, r_4 > 1$, such that $1/r_1 + 1/r_2 + 1/r_3 + 1/r_4 = 1$, and using Lemma A.1.1, we get

$$(\|u_n-u\|_{r_1}+\|u\|_{r_1})\|u_n-u\|_{r_2}\left(\int_{\mathbb{R}}(e^{2\alpha r_3|u_n-u|^2}-1)\mathrm{d}x\right)^{\frac{1}{r_3}}\left(\int_{\mathbb{R}}(e^{2\alpha r_4|u|^2}-1)\mathrm{d}x\right)^{\frac{1}{r_4}}. \tag{2.31}$$

Now, using that $\|u_n-u\|_{H^{1/2}(\mathbb{R})} \to 0$ and the Lemma 2.3.5, there exists $C > 0$ such that

$$\left(\int_{\mathbb{R}}(e^{2\alpha r_3|u_n-u|^2}-1)\mathrm{d}x\right)^{\frac{1}{r_3}} \leq C \quad \text{and} \quad \left(\int_{\mathbb{R}}(e^{2\alpha r_4|u|^2}-1)\mathrm{d}x\right)^{\frac{1}{r_4}} \leq C.$$

Again by using the convergence $\|u_n-u\|_{H^{1/2}(\mathbb{R})} \to 0$ and the continuous embedding of $H^{1/2}(\mathbb{R})$ in $L^{r_2}(\mathbb{R})$, we have that $\|u_n-u\|_{r_2} \to 0$, as $n \to +\infty$. In this way, the quantity in (2.31) goes to zero as $n \to +\infty$. The other terms can be handled in similar fashion. ■

Now, note that by Lemma 2.3.5, Lemma 2.3.6 and the hypotheses on f and V, we obtain that the energy functional $I : X \to \mathbb{R}$ associated to problem (2.1) given by

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} K(x)F(u)\mathrm{d}x$$

is well defined and belongs to $C^1(X, \mathbb{R})$ with

$$I'(u)v = \langle u, v\rangle - \int_{\mathbb{R}} K(x)f(u)v\mathrm{d}x, \quad \text{for} \quad u, v \in X$$

and consequently critical points of I are precisely the weak solutions of (2.1).

As in Lemma 1.2.3 and Corollary 1.2.4, we have the following results:

Lemma 2.3.7 *Assume that* $(V_1)-(V_2)$ *are satisfied. Let* $u \in X$. *Then,*

(i) $\langle u, u^{\pm}\rangle = \langle u^{\pm}, u^{\pm}\rangle + \dfrac{1}{\pi}\displaystyle\int_{\mathbb{R}^2}\frac{u^+(x)(-u^-(y))}{|x-y|^2}\mathrm{d}x\mathrm{d}y$

(ii) *if* $u^{\pm} \neq 0$, $\langle u^+, u^-\rangle > 0$,

(iii) $I(u) > I(u^+) + I(u^-)$,

(iv) $I'(u)u^+ > I'(u^+)u^+$ *and* $I'(u)u^- > I'(u^-)u^-$.

Corollary 2.3.8 *If* $u \in X$ *then*

$$\|u\|^2 \geq \|u^+\|^2 + \|u^-\|^2.$$

2.4 Some properties of the Nehari manifold and nodal set

In order to prove some properties of $\mathcal{M}$ and $\mathcal{N}$, we observe that by $(f_1) - (f_2)$, given $\varepsilon > 0$ and $q \geq 1$, there is a positive constant C_ε such that

$$|f(t)| \leq \varepsilon|t| + C_\varepsilon|t|^{q-1}(e^{\pi t^2} - 1), \quad \text{for all} \quad t \in \mathbb{R} \tag{2.32}$$

and, by virtue of (f_3),

$$|F(t)| \leq \varepsilon|t|^2 + C_\varepsilon|t|^q(e^{\pi t^2} - 1), \quad \text{for all} \quad t \in \mathbb{R}. \tag{2.33}$$

Moreover, by (f_5), we have

$$|f(t)| \geq C_p|t|^{p-1}, \quad \text{for all} \quad t \in \mathbb{R} \tag{2.34}$$

and

$$F(t) \geq \frac{C_p}{p}|t|^p, \quad \text{for all} \quad t \in \mathbb{R}. \tag{2.35}$$

Lemma 2.4.1 *Assume that* $(V_1) - (V_2)$ *and* $(f_1) - (f_5)$ *are satisfied. Then, given* $u \in X \setminus \{0\}$, *there is a unique* $t = t(u) > 0$ *such that* $tu \in \mathcal{N}$. *In addition, the number* t *satisfies*

$$I(tu) = \max_{s \geq 0} I(su). \tag{2.36}$$

Proof . Given $u \in X \setminus \{0\}$, we define $h(s) := I(su)$ for $s \geq 0$. By (2.35) and since $p > 2$, we obtain

$$h(s) \leq \frac{s^2}{2}\|u\|^2 - \frac{C_p s^p}{p}\int_{\mathbb{R}} K(x)|u|^p \mathrm{d}x \to -\infty, \quad \text{as} \quad s \to \infty. \tag{2.37}$$

On the other hand, choosing $q > 2$, by using (2.33) and that $K(x) \leq C$, we get

$$h(s) \geq \frac{s^2}{2}\|u\|^2 - C\int_{\mathbb{R}} (\varepsilon s^2|u|^2 + C_\varepsilon s^q|u|^q(e^{\pi s^2 u^2} - 1))\mathrm{d}x. \tag{2.38}$$

If $s \in [0,1]$, we have $(e^{\pi s^2 u^2} - 1) \leq (e^{\pi u^2} - 1)$. Hence, by Proposition 2.3.2, we get

$$h(s) \geq s^2 \left(\frac{1}{2} - C_1\varepsilon\right) \|u\|^2 - C_{2,\varepsilon} s^q \int_{\mathbb{R}} |u|^q (e^{\pi u^2} - 1)\mathrm{d}x > 0 \tag{2.39}$$

for $s > 0$ small enough. Thus, from (2.37) and (2.39), there exists $t = t(u) > 0$ such that $I(tu) = \max_{s \geq 0} I(su)$ and, consequently, $tu \in \mathcal{N}$. Now, if $s > 0$ is such that $su \in \mathcal{N}$, we have

$$s^2\|u\|^2 = \int_{\mathbb{R}} f(su)su\,\mathrm{d}x$$

and since it also holds

$$t^2\|u\|^2 = \int_{\mathbb{R}} f(tu)tu\,\mathrm{d}x,$$

it follows that

$$\int_{\mathbb{R}} \left(\frac{f(tu)}{tu} - \frac{f(su)}{su}\right) u^2 \mathrm{d}x = 0. \tag{2.40}$$

By (f_4) and since $u \neq 0$, it follows from (2.40) that $t = s$. Thus, we finish the proof. ■

Lemma 2.4.2 *Assume that $(V_1)-(V_2)$ and $(f_1)-(f_2)$ are satisfied. Then, there exists $m_0 > 0$ such that $\|u\|^2 \geq m_0$ for all $u \in \mathcal{N}$.*

Proof . In order to obtain a contradiction, suppose that there exists $(u_n) \subset \mathcal{N}$ such that $\|u_n\| \to 0$ as $n \to \infty$. By definition, we know that

$$\|u_n\|^2 = \int_{\mathbb{R}} K(x) f(u_n) u_n \mathrm{d}x. \tag{2.41}$$

Since $K(x) \leq C$, utilizing (2.32) with $q > 2$, we get

$$\|u_n\|^2 \leq \int_{\mathbb{R}} K(x)|f(u_n)u_n|\mathrm{d}x \leq \varepsilon C \int_{\mathbb{R}} |u_n|^2 \mathrm{d}x + C_\varepsilon \int_{\mathbb{R}} |u_n|^q (e^{\pi u_n^2} - 1)\mathrm{d}x. \tag{2.42}$$

Now, from Lemma 2.3.5, by using the Hölder inequality and the assumptions $\|u_n\| \to 0$, we obtain that

$$\int_{\mathbb{R}} |u_n|^q (e^{\pi u_n^2} - 1)\mathrm{d}x \leq C\|u_n\|_{2q}^q \left(\int_{\mathbb{R}} (e^{2\pi\|u_n\|^2 \left(\frac{u_n}{\|u_n\|}\right)^2} - 1)\mathrm{d}x\right)^{\frac{1}{2}} \leq C_\pi \|u_n\|_{2q}^q \tag{2.43}$$

for $n \in \mathbb{N}$ sufficiently large. From Proposition 2.3.2, there exist $C_1, C_2 > 0$ such that $\|u_n\|_{2q}^q \leq C_1\|u_n\|^q$ and $\|u_n\|_2^2 \leq C_2\|u_n\|^2$. Hence, choosing $\varepsilon > 0$ and utilizing (2.41), (2.42) and (2.43), we have $0 < C_0 \leq \|u_n\|^{q-2}$, for $n \in \mathbb{N}$ sufficiently large. But, as $q > 2$, this contradicts the assumption $\|u_n\| \to 0$ and the proof of the lemma is complete. ■

Corollary 2.4.3 *Assume that* $(V_1) - (V_2)$, (f_1) *and* (f_3) *are satisfied. Then, there exists* $\delta_0 > 0$ *such that* $I(u) \geq \delta_0$ *for all* $u \in \mathcal{N}$. *In particular,*

$$0 < \delta_0 \leq c_{\mathcal{N}} \leq c_{\mathcal{M}}.$$

Proof . Since $I'(u)u = 0$, by Lemma 2.4.2 and (f_3), we have

$$\begin{aligned} I(u) = I(u) - \frac{1}{\theta} I'(u)u &= \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u\|^2 + \frac{1}{\theta} \int_{\mathbb{R}} K(x) \left[f(u)u - \theta F(u)\right] \mathrm{d}x \\ &\geq \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u\|^2 \geq \left(\frac{1}{2} - \frac{1}{\theta}\right) m_0 := \delta_0, \end{aligned}$$

which is the desired inequality. ■

Lemma 2.4.4 *Assume that* $(V_1) - (V_2)$ *and* $(f_1) - (f_2)$ *are satisfied. Then, there exists* $m_0' > 0$ *such that* $\|u^{\pm}\|^2 \geq m_0'$ *for all* $u \in \mathcal{M}$.

Proof . The proof is similar to Lemma 2.4.2. Hence, it is sufficient to prove a similar estimate to (2.42) for u^+ and u^-. Since $u \in \mathcal{M}$ we have $u^+ \neq 0$ and $\langle u, u^+ \rangle = \int_{\mathbb{R}} K(x) f(u^+) u^+ \mathrm{d}x$. Now, by Lemma 2.3.7, we have $\langle u^+, u^+ \rangle < \langle u, u^+ \rangle$. Thus, by using (2.32) we obtain

$$\|u^+\|^2 \leq \int_{\mathbb{R}} K(x) f(u^+) u^+ \mathrm{d}x \leq \varepsilon C \int_{\mathbb{R}} |u^+|^2 \mathrm{d}x + C_\varepsilon \int_{\mathbb{R}} |u^+|^q (e^{\pi |u^+|^2} - 1) \mathrm{d}x.$$

Similarly,

$$\|u^-\|^2 \leq \int_{\mathbb{R}} K(x) f(u^-) u^- \mathrm{d}x \leq \varepsilon C \int_{\mathbb{R}} |u^-|^2 \mathrm{d}x + C_\varepsilon \int_{\mathbb{R}} |u^-|^q (e^{\pi |u^-|^2} - 1) \mathrm{d}x,$$

and the proof of lemma is done. ■

Now, using Lemma 1.3.5, we shall show that $\mathcal{M} \neq \emptyset$.

Lemma 2.4.5 *Assume that* $(V_1) - (V_2)$, $(f_1) - (f_2)$ *and* $(f_4) - (f_5)$ *are satisfied. Then, given* $u \in X$ *with* $u^{\pm} \neq 0$, *there exists a unique pair* (t, s) *of positive numbers such that* $tu^+ + su^- \in \mathcal{M}$.

Proof . Let $u \in X$ be such that $u^{\pm} \neq 0$. We define the continuous vector field $g : (0, \infty) \times (0, \infty) \to \mathbb{R}^2$ by

$$g(t, s) = \left(I'(tu^+ + su^-) tu^+, I'(tu^+ + su^-) su^-\right).$$

Firstly, we want to find $(t,s) \in (0,\infty)\times(0,\infty)$ such that $g(t,s)=(0,0)$. The first step is to show that for t and s sufficiently small the coordinates functions are positive. Given $\varepsilon > 0$ and $q > 2$, by (2.32) and $K(x) \leq C$, we get

$$\begin{aligned} I'(tu^+ + su^-)tu^+ &= t^2\|u^+\|^2 + ts\langle u^+, u^-\rangle - \int_{\mathbb{R}} K(x)f(tu^+)tu^+\mathrm{d}x \\ &\geq t^2\|u^+\|^2 + ts\langle u^+, u^-\rangle - \varepsilon C t^2 \int_{\mathbb{R}} |u^+|^2\mathrm{d}x \\ &- C_\varepsilon C t^q \int_{\mathbb{R}} |u^+|^q (e^{\pi t^2 |u^+|^2} - 1)\mathrm{d}x. \end{aligned}$$

Hence, if $t \in [0,1]$, by using Proposition 2.3.2, there exists $C_1 > 0$ such that

$$\begin{aligned} I'(tu^+ + su^-)tu^+ \geq t^2\|u^+\|^2 + ts\langle u^+, u^-\rangle - \varepsilon C_1 C t^2 \|u^+\|^2 \\ - C_\varepsilon C t^q \int_{\mathbb{R}} |u^+|^q (e^{\pi |u^+|^2} - 1)\mathrm{d}x. \end{aligned}$$

By Lemma 2.3.7 we have $\langle u^+, u^-\rangle > 0$. Then there exists $r > 0$ small enough such that

$$I'(ru^+ + su^-)ru^+ > 0, \quad \text{for all} \quad s > 0.$$

Analogously, there exists $r > 0$ large enough such that

$$I'(tu^+ + ru^-)ru^- > 0, \quad \text{for all} \quad t > 0.$$

Now, we shall show that, for t and s large enough, the coordinates functions are negative. In fact, by (f_3) and (2.35), we have

$$\int_{\mathbb{R}} K(x)f(tu^+)tu^+\mathrm{d}x \geq \theta \int_{\mathbb{R}} K(x)F(tu^+)\mathrm{d}x \geq \frac{\theta C_p t^p}{p} \int_{\mathbb{R}} K(x)|u^+|^p\mathrm{d}x.$$

Thus,

$$\begin{aligned} I'(tu^+ + su^-)tu^+ &= t^2\|u^+\|^2 + ts\langle u^+, u^-\rangle - \int_{\mathbb{R}} K(x)f(tu^+)tu^+\mathrm{d}x \\ &\leq t^2\|u^+\|^2 + ts\langle u^+, u^-\rangle - \frac{\theta C_p t^p}{p}\|u^+\|^p_{L^p_K}. \end{aligned}$$

Since $p > 2$, there exists $R > r$ large enough such that

$$I'(Ru^+ + su^-)Ru^+ < 0, \text{ for all } 0 \leq s \leq R.$$

Analogously, there exists $R > r$ small enough such that

$$I'(tu^+ + Ru^-)Ru^- < 0, \text{ for all } 0 \leq t \leq R.$$

Hence, considering the block $P = [r,R] \times [r,R]$ and applying Lemma 1.3.5, there exists $(t,s) \in [r,R] \times [r,R]$ such that $g(t,s) = (0,0)$ and consequently, we have $tu^+ + su^- \in \mathcal{M}$.

Finally, we shall prove the uniqueness of the pair (t,s). First, we assume that $u = u^+ + u^- \in \mathcal{M}$ and $(t,s) \in (0,\infty) \times (0,\infty)$ is such that $tu^+ + su^- \in \mathcal{M}$. In this case, we need to show that $(t,s) = (1,1)$. Note that

$$\|u^+\|^2 + \langle u^+, u^- \rangle = \int_{\mathbb{R}} K(x) f(u^+) u^+ \mathrm{d}x \tag{2.44}$$

$$\|u^-\|^2 + \langle u^+, u^- \rangle = \int_{\mathbb{R}} K(x) f(u^-) u^- \mathrm{d}x \tag{2.45}$$

$$t^2\|u^+\|^2 + ts\langle u^+, u^- \rangle = \int_{\mathbb{R}} K(x) f(tu^+) tu^+ \mathrm{d}x \tag{2.46}$$

$$s^2\|u^-\|^2 + ts\langle u^+, u^- \rangle = \int_{\mathbb{R}} K(x) f(su^-) su^- \mathrm{d}x. \tag{2.47}$$

We can assume, without loss of generality, that $t \leq s$. Then, by using $\langle u^+, u^- \rangle > 0$ and (2.46), we have

$$\|u^+\|^2 + \langle u^+, u^- \rangle \leq \int_{\mathbb{R}} K(x) \frac{f(tu^+)}{t} u^+ \mathrm{d}x.$$

It follows from (2.44) that

$$\int_{\mathbb{R}} K(x) \left(\frac{f(tu^+)}{tu^+} - \frac{f(u^+)}{u^+} \right) (u^+)^2 \mathrm{d}x \geq 0.$$

Hence, by (f_4) and since $u^+ \neq 0$ we obtain $t \geq 1$. On the other hand, since $t/s \leq 1$ and $\langle u^+, u^- \rangle > 0$, we get

$$\|u^-\|^2 + \langle u^+, u^- \rangle \geq \int_{\mathbb{R}} K(x) \frac{f(su^+)}{s} u^- \mathrm{d}x.$$

This, together with (2.45), implies

$$\int_{\mathbb{R}} K(x) \left(\frac{f(su^-)}{su^-} - \frac{f(u^-)}{u^-} \right) (u^-)^2 \mathrm{d}x \leq 0$$

and consequently $s \leq 1$. Thus we conclude that $t = s = 1$.

For the general case, we suppose that u does not necessarily belong to $\mathcal{M}$. Let $(t,s), (t',s') \in (0,\infty) \times (0,\infty)$ be such that $tu^+ + su^-$ and $t'u^+ + s'u^-$ belongs to $\mathcal{M}$.

We define $v = v^+ + v^-$, where $v^+ = tu^+$ and $v^- = su^-$. Then, we have that $v \in \mathcal{M}$ and

$$\frac{t'}{t}v^+ + \frac{s'}{s}v^- = t'u^+ + s'u^- \in \mathcal{M}.$$

Hence, by the first case, we reach $t'/t = 1$ and $s'/s = 1$, which completes the proof. ■

Now, we shall present two technical lemmas that will be used in the next section.

Lemma 2.4.6 *Assume that* $(V_1) - (V_2)$, $(f_1) - (f_2)$ *and* $(f_4) - (f_5)$ *are satisfied. Let* $u \in X$ *be a function such that* $u^\pm \neq 0$, $I'(u)u^+ \leq 0$ *and* $I'(u)u^- \leq 0$. *Then the unique pair* (t, s) *given in Lemma 2.4.5 satisfies* $0 < t, s \leq 1$.

Proof . We can assume, without loss of generality, that $s \geq t > 0$ and $tu^+ + su^- \in \mathcal{M}$. Now, since $I'(u)u^- \leq 0$ and $I'(tu^+ + su^-)su^- = 0$, we have

$$\|u^-\|^2 + \langle u^+, u^- \rangle \leq \int_{\mathbb{R}} K(x)f(u^-)u^-\mathrm{d}x$$

and

$$\|u^-\|^2 + \frac{t}{s}\langle u^+, u^- \rangle = \int_{\mathbb{R}} K(x)\frac{f(su^-)}{s}u^-\mathrm{d}x.$$

By Lemma 2.3.7 we get

$$\begin{aligned}\int_{\mathbb{R}} K(x)\left(\frac{f(u^-)}{u^-} - \frac{f(su^-)}{su^-}\right)(u^-)^2\mathrm{d}x &= \int_{\mathbb{R}} K(x)f(u^-)u^-\mathrm{d}x - \|u^-\|^2 - \frac{t}{s}\langle u^+, u^- \rangle \\ &\geq \|u^-\|^2 + \langle u^+, u^- \rangle - \|u^-\|^2 - \frac{t}{s}\langle u^+, u^- \rangle \\ &\geq \left(1 - \frac{t}{s}\right)\langle u^+, u^- \rangle \geq 0.\end{aligned}$$

By using this inequality, (f_4) and the fact that $u^- \neq 0$, we obtain $s \leq 1$ and we finish the proof. ■

Lemma 2.4.7 *Assume that* $(V_1) - (V_2)$, $(f_1) - (f_2)$ *and* $(f_4) - (f_5)$ *are satisfied. Let* $u \in X$ *be a function such that* $u^\pm \neq 0$ *and* (t, s) *be the unique pair of positive numbers given in Lemma 2.4.5. Then* (t, s) *is the unique maximum point of the function* $\phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ *defined by* $\phi(\alpha, \beta) = I(\alpha u^+ + \beta u^-)$.

Proof . In the demonstration of Lemma 2.4.5, we saw that (t, s) is the unique critical point of ϕ in $(0, \infty) \times (0, \infty)$. Note that, by using (2.35), we get

$$\begin{aligned}\phi(\alpha, \beta) \leq& \frac{1}{2}\left\|\alpha u^+ + \beta u^-\right\|^2 - \frac{C_p}{p}\int_{\mathbb{R}} K(x)|\alpha u^+ + \beta u^-|^p\mathrm{d}x \\ =& \frac{(\alpha+\beta)^2}{2}\left\|(\frac{\alpha}{\alpha+\beta})u^+ + (\frac{\beta}{\alpha+\beta})u^-\right\|^2 \\ &- \frac{C_p}{p}(\alpha+\beta)^p\left\|(\frac{\alpha}{\alpha+\beta})u^+ + (\frac{\beta}{\alpha+\beta})u^-\right\|_{L^p_K}^p.\end{aligned}$$

Hence, since $p > 2$, $\phi(\alpha,\beta) \to -\infty$ as $|(\alpha,\beta)| \to \infty$. In particular, there exists $R > 0$ such that $\phi(\alpha,\beta) < \phi(t,s)$ for all $(\alpha,\beta) \in (0,\infty)\times(0,\infty)\setminus\overline{B_R}$, where $\overline{B_R}$ is the closure of the ball of radius R in $\mathbb{R}^2$. In order to finalize the proof, we shall show that the maximum of ϕ does not occur in the boundary of $\mathbb{R}_+\times\mathbb{R}_+$. Suppose, by contradiction, that $(0,\beta)$ is a maximum point of ϕ, given $\alpha \geq 0$, we have that

$$\phi(\alpha,\beta) = \frac{\alpha^2}{2}\|u^+\|^2 + \alpha\beta\langle u^+,u^-\rangle - \int_{\mathbb{R}} K(x)F(\alpha u^+)\mathrm{d}x + \phi(0,\beta).$$

Arguing similarly to Lemma 2.4.1, we get

$$\frac{\alpha^2}{2}\|u^+\|^2 + \alpha\beta\langle u^+,u^-\rangle - \int_{\mathbb{R}} K(x)F(\alpha u^+)\mathrm{d}x > 0$$

for $\alpha > 0$ small enough. But this contradicts the assumption that $(0,\beta)$ is a maximum point of ϕ. The case $(\alpha,0)$ is similar and we omit it. The proof is complete. ■

Now, we shall prove an upper bound for the nodal level $c_{\mathcal{M}}$ defined in (2.12).

Lemma 2.4.8 *Assume that* $(V_1)-(V_2)$, $(f_1)-(f_2)$ *and* $(f_4)-(f_5)$ *hold and* C_p *satisfies* (2.13). *If* θ *is the constant given by* (f_3) *and* κ *is given in* (2.22), *then*

$$c_{\mathcal{M}} < \frac{\theta-2}{2\theta\kappa}. \tag{2.48}$$

Proof . From Theorem B.2.8 (see Appendix), there exists $w \in \mathcal{M}^p$ such that $I_p(w) = c_{\mathcal{M}^p}$ and $I_p'(w)w^\pm = 0$, where $c_{\mathcal{M}^p}$ and $\mathcal{M}^p$ was defined in (2.14) and 2.15. Consequently,

$$\frac{1}{2}\|w\|^2 - \frac{1}{p}\|w\|^p_{L^p_K} = c_{\mathcal{M}^p}, \tag{2.49}$$

$$\|w^\pm\|^2 = \|w^\pm\|^p_{L^p_K} - \langle w^+,w^-\rangle \tag{2.50}$$

$$\|w\|^2 = \|w\|^p_{L^p_K}. \tag{2.51}$$

Hence, by (2.49) and (2.51), we get

$$\left(\frac{1}{2}-\frac{1}{p}\right)\|w\|^p_{L^p_K} = c_{\mathcal{M}^p}. \tag{2.52}$$

Since $w^\pm \neq 0$, by Lemma 2.4.5, there exist $t,s > 0$ such that $tw^+ + sw^- \in \mathcal{M}$. Consequently, we obtain

$$\begin{aligned} c_{\mathcal{M}} \leq I(tw^+ + sw^-) &= \frac{t^2}{2}\|w^+\|^2 + ts\langle w^+,w^-\rangle + \frac{s^2}{2}\|w^-\|^2 \\ &\quad - \int_{\mathbb{R}} K(x)F(tw^+)\mathrm{d}x - \int_{\mathbb{R}} K(x)F(sw^-)\mathrm{d}x. \end{aligned}$$

This together with (2.35) implies

$$c_{\mathcal{M}} \leq \frac{t^2}{2}\|w^+\|^2 + ts\langle w^+, w^-\rangle + \frac{s^2}{2}\|w^-\|^2 - \frac{C_p t^p}{p}\|w^+\|^p_{L^p_K} - \frac{C_p s^p}{p}\|w^-\|^p_{L^p_K}.$$

By (2.50) and Lemma 2.3.7, we have

$$\begin{aligned} c_{\mathcal{M}} &\leq \frac{t^2}{2}(\|w^+\|^p_{L^p_K} - \langle w^+, w^-\rangle) + ts\langle w^+, w^-\rangle + \frac{s^2}{2}(\|w^-\|^p_{L^p_K} - \langle w^+, w^-\rangle) \\ &\quad - \frac{C_p t^p}{p}\|w^+\|^p_{L^p_K} - \frac{C_p s^p}{p}\|w^-\|^p_{L^p_K} \\ &= \left(\frac{t^2}{2} - \frac{C_p t^p}{p}\right)\|w^+\|^p_{L^p_K} + \left(\frac{s^2}{2} - \frac{C_p s^p}{p}\right)\|w^-\|^p_{L^p_K} - \frac{1}{2}(t-s)^2\langle w^+, w^-\rangle \\ &\leq \max_{\xi \geq 0}\left(\frac{\xi^2}{2} - \frac{C_p \xi^p}{p}\right)\|w\|^p_{L^p_K}. \end{aligned}$$

On the other hand, it is easy to see that

$$\max_{\xi \geq 0}\left(\frac{\xi^2}{2} - \frac{C_p \xi^p}{p}\right) = C_p^{\frac{2}{2-p}}\left(\frac{1}{2} - \frac{1}{p}\right).$$

Hence, by (2.52) it follows that

$$c_{\mathcal{M}} \leq C_p^{\frac{2}{2-p}}\left(\frac{1}{2} - \frac{1}{p}\right)\|w\|^p_{L^p_K} = C_p^{\frac{2}{2-p}} c_{\mathcal{M}^p}.$$

Therefore, by the definition of C_p given in (2.13), we obtain (2.48). ■

The next step is to obtain a minimizing sequence for the nodal level $c_{\mathcal{M}}$ with a special behavior. For this, for $\lambda > 0$, we start by defining the set

$$\widetilde{S}_\lambda = \{u \in \mathcal{M} \,:\, I(u) < c_{\mathcal{M}} + \lambda\}.$$

Lemma 2.4.9 *Assume that $(V_1) - (V_2)$ and $(f_1) - (f_5)$ hold and C_p satisfies (2.13). For $\lambda > 0$ small enough, there exists $m_\lambda \in (0, \frac{1}{\kappa})$ such that*

$$0 < m_0' \leq \|u^\pm\|^2 < \|u\|^2 \leq m_\lambda,$$

for any $u \in \widetilde{S}_\lambda$.

Proof . Let $u \in \widetilde{S}_\lambda$. By Lemma 2.4.4 and by using $\langle u^+, u^-\rangle > 0$, we have $m_0' \leq \|u^\pm\|^2 < \|u\|^2$. On the other hand, by (f_3) and since $I'(u)u = 0$, we obtain

$$\begin{aligned} c_{\mathcal{M}} + \lambda > I(u) &= I(u) - \frac{1}{\theta}I'(u)u \\ &= \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u\|^2 + \frac{1}{\theta}\int_{\mathbb{R}} K(x)\,[f(u)u - \theta F(u)]\,\mathrm{d}x \geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u\|^2. \end{aligned}$$

By Lemma 2.4.8, we can take $\lambda > 0$ such that $c_{\mathcal{M}} + \lambda < \left(\frac{\theta - 2}{2\theta\kappa}\right)$. Consequently, it follows that

$$\|u\|^2 \leq \frac{2\theta}{\theta - 2}(c_{\mathcal{M}} + \lambda) =: m_\lambda < \frac{1}{\kappa},$$

for all $u \in \widetilde{S}_\lambda$ and this concludes the proof of the lemma. ■

Lemma 2.4.10 *Assume that* (K_1) *and* $(f_1) - (f_3)$ *are satisfied. Let* $(u_n) \subset H^{1/2}(\mathbb{R})$ *be such that* $u_n \rightharpoonup u$ *weakly in* $H^{1/2}(\mathbb{R})$ *and* $b := \sup_{n\in\mathbb{N}} \|u_n\|^2_{1/2,2} < 1$*. Then, up to a subsequence, one has*

$$\lim_{n\to+\infty} \int_{\mathbb{R}} K(x)f(u_n)u_n \mathrm{d}x = \int_{\mathbb{R}} K(x)f(u)u\mathrm{d}x; \tag{2.53}$$

$$\lim_{n\to+\infty} \int_{\mathbb{R}} K(x)f(u_n^\pm)u_n^\pm \mathrm{d}x = \int_{\mathbb{R}} K(x)f(u^\pm)u^\pm\mathrm{d}x; \tag{2.54}$$

$$\lim_{n\to+\infty} \int_{\mathbb{R}} K(x)F(u_n) \mathrm{d}x = \int_{\mathbb{R}} K(x)F(u)\mathrm{d}x; \tag{2.55}$$

$$\lim_{n\to+\infty} \int_{\mathbb{R}} K(x)f(u_n)v \mathrm{d}x = \int_{\mathbb{R}} K(x)f(u)v\mathrm{d}x, \quad \text{for all} \quad v \in H^{1/2}(\mathbb{R}). \tag{2.56}$$

Proof . We will prove only (2.53), since the proofs of (2.54)-(2.56) are similar and we will omit them. Let $\pi < \alpha < \pi/b^2$. Then, by using (f_1) and (f_2), we have

$$\lim_{|t|\to\infty} \frac{f(t)t}{e^{\alpha t^2} - 1} = 0 \quad \text{and} \quad \lim_{|t|\to 0} \frac{f(t)t}{t^2} = 0. \tag{2.57}$$

Hence, given $q > 2$ and $\varepsilon > 0$, there exists $0 < t_0(\varepsilon) < t_1(\varepsilon)$ and $C_\varepsilon > 0$ such that

$$K(x)|f(t)t| \leq \varepsilon C(|t|^2 + e^{\alpha t^2} - 1) + C_\varepsilon K(x)\chi_{[t_0(\varepsilon),t_1(\varepsilon)]}(|t|)|t|^q, \quad \text{for all} \quad t, x \in \mathbb{R}. \tag{2.58}$$

Now, from the continuous embedding $H^{1/2}(\mathbb{R}) \hookrightarrow L^s(\mathbb{R})$, for $s \geq 2$, and Lemma 2.3.5, we can find $M > 0$ such that

$$\int_{\mathbb{R}} |u_n|^2 \mathrm{d}x \leq M, \int_{\mathbb{R}} |u_n|^q \mathrm{d}x \leq M \quad \text{and} \quad \int_{\mathbb{R}} (e^{\alpha u_n^2} - 1)\mathrm{d}x \leq M, \quad \text{for all} \quad n \in \mathbb{N}. \tag{2.59}$$

Denoting $A_n^\varepsilon = \{x \in \mathbb{R} : t_0(\varepsilon) \leq |u_n(x)| \leq t_1(\varepsilon)\}$, we get

$$t_0(\varepsilon)^2 |A_n^\varepsilon| = \int_{A_n^\varepsilon} t_0(\varepsilon)^2 \mathrm{d}x \leq \int_{\mathbb{R}} |u_n|^2 \mathrm{d}x \leq M, \quad \text{for all} \quad n \in \mathbb{N}.$$

Thus, utilizing (K_1), there exists $r(\varepsilon) > 0$ such that

$$\int_{A_n^\varepsilon \cap B^c_{r(\varepsilon)}(0)} K(x)\mathrm{d}x < \frac{\varepsilon}{C_\varepsilon t_1(\varepsilon)^q}, \quad \text{for all} \quad n \in \mathbb{N} \tag{2.60}$$

and by using (2.59) and (2.60) in (2.58), we reach

$$\int_{B^c_{r(\varepsilon)}(0)} K(x)|f(u_n)u_n|\mathrm{d}x \le (2CM+1)\varepsilon, \quad \text{for all} \quad n\in\mathbb{N}. \tag{2.61}$$

On the other hand, using that $u_n \rightharpoonup u$ weakly in $H^{1/2}(\mathbb{R})$ and the locally compact embedding $H^{1/2}(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$, up to a subsequence, we have $u_n(x) \to u(x)$ a.e. in $\mathbb{R}$. Thus, $K(x)f(u_n(x))u_n(x) \to K(x)f(u(x))u(x)$ a.e. in $\mathbb{R}$ and according to (2.57), (2.59) and Strauss Lemma (Lemma A.1.10 applied with $P(t) = f(t)t$ and $Q(t) = e^{\alpha t^2} - 1$), one has

$$\lim_{n\to+\infty} \int_{B_{r(\varepsilon)}} K(x)f(u_n)u_n\mathrm{d}x = \int_{B_{r(\varepsilon)}} K(x)f(u)u\mathrm{d}x. \tag{2.62}$$

Combining (2.61) and (2.62),the proof of (2.53) follows. ■

From now on, we will write $\widetilde{S}_\lambda$ with $\lambda > 0$ given in Lemma 2.4.9.

Lemma 2.4.11 *Assume that $(V_1)-(V_2)$ and $(f_1)-(f_5)$ hold and C_p satisfies (2.13). Then for any $q > 2$, there exists $\delta_q > 0$ such that*

$$0 < \delta_q \le \int_{\mathbb{R}} K(x)|u^\pm|^q\mathrm{d}x < \int_{\mathbb{R}} K(x)|u|^q\mathrm{d}x,$$

for each $u \in \widetilde{S}_\lambda$.

Proof . Let $u \in \widetilde{S}_\lambda$ and $q > 2$. We know that

$$\|u^\pm\|^2 + \langle u^+, u^-\rangle = \int_{\mathbb{R}} K(x)f(u^\pm)u^\pm\mathrm{d}x.$$

By using Lemma 2.3.7 and Lemma 2.4.4, it follows that

$$0 < m_0' \le \|u^\pm\|^2 < \int_{\mathbb{R}} K(x)f(u^\pm)u^\pm\mathrm{d}x$$

and from (2.32), we have

$$m_0' \le \varepsilon \int_{\mathbb{R}} K(x)|u^\pm|^2\mathrm{d}x + C_\varepsilon \int_{\mathbb{R}} K(x)|u^\pm|(e^{\pi|u^\pm|^2} - 1)\mathrm{d}x.$$

Now, Corollary 2.3.4 and the fact that $u \in \widetilde{S}_\lambda$ imply that there exists $C_1 > 0$, independent of u, such that

$$\int_{\mathbb{R}} K(x)|u|^2\mathrm{d}x \le C_1.$$

Choosing $\varepsilon > 0$ such that $m_0' - \varepsilon C_1 > 0$, we obtain

$$0 < \frac{m_0' - \varepsilon C_1}{C_\varepsilon} \le \int_{\mathbb{R}} K(x)|u^\pm|(e^{\pi|u^\pm|^2} - 1)\mathrm{d}x. \tag{2.63}$$

Let $t' > 0$ sufficiently close to 1 such that $\pi t' m_\lambda \kappa \leq \pi$, with $1/t + 1/t' = 1$ and $t > q$. Utilizing the Hölder inequality, Lemma 2.4.9, $K(x) \leq C$ and Lemma 2.3.5, we reach

$$\begin{aligned}\int_{\mathbb{R}} K(x)|u^\pm|(e^{\pi|u^\pm|^2}-1)\mathrm{d}x &= \int_{\mathbb{R}} K(x)^{\frac{1}{t}}|u^\pm|K(x)^{\frac{1}{t'}}(e^{\pi|u^\pm|^2}-1)\mathrm{d}x \\ &\leq \left(\int_{\mathbb{R}} K(x)|u^\pm|^t \mathrm{d}x\right)^{\frac{1}{t}} \cdot \\ &\cdot \left(\int_{\mathbb{R}} K(x)(e^{\pi t'\|u^\pm\|^2_{1/2,2}\left(\frac{|u^\pm|}{\|u\|_{1/2,2}}\right)^2}-1)\mathrm{d}x\right)^{\frac{1}{t'}} \\ &\leq C^{\frac{1}{t'}}\left(\int_{\mathbb{R}} K(x)|u^\pm|^t \mathrm{d}x\right)^{\frac{1}{t}}\left(\int_{\mathbb{R}}(e^{\pi t' m_\lambda \kappa\left(\frac{|u^\pm|}{\|u^\pm\|_{1/2,2}}\right)^2}-1)\mathrm{d}x\right)^{\frac{1}{t'}} \\ &\leq C\|u^\pm\|_{L^t_K}.\end{aligned}$$

This last inequality and (2.63) implies that

$$0 < \frac{m_0' - \varepsilon C_1}{C_\varepsilon} \leq C\|u^\pm\|_{L^t_K}. \tag{2.64}$$

Now, we suppose, by contradiction, that there exists $(u_n) \subset \widetilde{S}_\lambda$ such that $\|u_n^\pm\|_{L^q_K} \to 0$ as $n \to \infty$. From Lemma 2.4.9 we obtain that $(u_n^\pm)$ is bounded in $L^{2t}(\mathbb{R})$. Consequently, since $q < t < 2t$, by the interpolation inequality we get that $\|u_n^\pm\|_{L^t_K} \to 0$ as $n \to \infty$, which is impossible in view of (2.64), concluding the proof. ■

The next technical result will be used in the proof of Lemma 2.4.13.

Lemma 2.4.12 *Assume (f_1) and $(f_3) - (f_4)$ are satisfied. Then the function $H(t) := f(t)t - 2F(t)$ satisfies*

(i) *$H(0) = 0$ and $H(t) > 0$, for all $t \neq 0$;*

(ii) *$H(t_0) \leq H(t_1)$ if $0 < t_0 \leq t_1$;*

(iii) *$H(t_0) \geq H(t_1)$ if $t_0 \leq t_1 < 0$.*

Proof . Let us show (iii). First we note that $H \in C^1(\mathbb{R})$ and $H'(t) = f'(t)t - f(t)$, for all $t \in \mathbb{R}$. From (f_4), we have

$$\frac{d}{\mathrm{d}t}\left(\frac{f(t)}{|t|}\right) \geq 0, \quad \text{for all} \quad t \in \mathbb{R} \setminus \{0\}.$$

If $t < 0$ then $f(t) - f'(t)t \geq 0$ and therefore $H'(t) \leq 0$ for all $t < 0$. Thus, $H(t)$ is decreasing for $t \leq 0$, which implies the item (iii). The proof of the item (ii) is similar.

■

Next, we have all the results that will allow us to prove that the nodal level $c_{\mathcal{M}}$ is attained in a function $u \in X$ with $u^{\pm} \neq 0$.

Lemma 2.4.13 *Assume that $(V_1)-(V_2)$, (K_1) and $(f_1)-(f_5)$ hold and C_p satisfies (2.13). Then there exists $\widetilde{u} \in \mathcal{M}$ such that $I(\widetilde{u}) = c_{\mathcal{M}}$.*

Proof . Let $(u_n) \subset \mathcal{M}$ be such that $I(u_n) \to c_{\mathcal{M}}$ as $n \to +\infty$. We can assume that $u_n \in \widetilde{S}_{\lambda}$, for all $n \in \mathbb{N}$. In particular, by Lemma 2.4.9, we have

$$m_0' \leq \|u_n^{\pm}\|^2 < \|u_n\|^2 \leq m_{\lambda}, \quad \text{for all} \quad n \in \mathbb{N}, \quad \text{with} \quad m_{\lambda} \in \left(0, \frac{1}{\kappa}\right).$$

Thus, $(u_n), (u_n^+)$ and (u_n^-) are bounded in X. Since X is a Hilbert space, up to a subsequence, there exists $u \in X$ such that $u_n^{\pm} \rightharpoonup u^{\pm}$ and $u_n \rightharpoonup u$ in X. Let $q > 2$. From Corollary 2.3.4, up to a subsequence, we have $u_n^{\pm} \to u^{\pm}$ in L_K^q and utilizing Lemma 2.4.11, there exists $\delta_q > 0$ such that

$$0 < \delta_q \leq \int_{\mathbb{R}} K(x)|u_n^{\pm}|^q \mathrm{d}x < \int_{\mathbb{R}} K(x)|u_n|^q \mathrm{d}x, \quad \text{for all} \quad n \in \mathbb{N}.$$

Hence $u^{\pm} \neq 0$ in X. Now, from Lemma 2.4.5 there exist $t, s \in (0, \infty)$ such that $\widetilde{u} = tu^+ + su^- \in \mathcal{M}$. We claim that $I'(u)u^{\pm} \leq 0$. Since $\sup_{n\in\mathbb{N}} \|u_n\|^2 \leq m_{\lambda}$ and $\|u_n\|_{1/2,2}^2 \leq \kappa \|u_n\|^2$, we have $\sup_{n\in\mathbb{N}} \|u_n\|_{1/2,2} \in (0,1)$. Moreover, since the embedding $X \hookrightarrow L_{loc}^2(\mathbb{R})$ is compact, up to a subsequence, we can assume that $u_n^{\pm}(x) \to u^{\pm}(x)$ a.e. in $\mathbb{R}$. By the convergence (2.54) in Lemma 2.4.10 and by the Fatou Lemma, it follows that

$$\begin{aligned}
\|u^+\|^2 + \langle u^+, u^- \rangle &\leq \liminf_{n\to+\infty} \left(\|u_n^+\|^2 + \langle u_n^+, u_n^- \rangle\right) \\
&= \liminf_{n\to+\infty} \int_{\mathbb{R}} K(x) f(u_n^+) u_n^+ \mathrm{d}x = \int_{\mathbb{R}} K(x) f(u^+) u^+ \mathrm{d}x.
\end{aligned}$$

Hence, $I'(u)u^+ \leq 0$. Similarly, we get $I'(u)u^- \leq 0$. Then, by Lemma 2.4.6, we obtain $0 < t, s \leq 1$. In particular, $\|\widetilde{u}\|^2 \leq \|u\|^2$. Now, in order to conclude the proof, note that using the convergence in Lemma 2.4.10 and Lemma 2.4.12, it holds

$$\begin{aligned}
c_{\mathcal{M}} \leq I(\widetilde{u}) = I(\widetilde{u}) - \frac{1}{2} I'(\widetilde{u})\widetilde{u} &= \frac{1}{2} \int_{\mathbb{R}} K(x) \left(f(\widetilde{u})\widetilde{u} - 2F(\widetilde{u})\right) \mathrm{d}x \\
&= \frac{1}{2} \int_{\mathbb{R}} K(x) H(tu^+) \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}} K(x) H(su^-) \mathrm{d}x
\end{aligned}$$

and therefore

$$c_{\mathcal{M}} \leq \frac{1}{2}\int_{\mathbb{R}} K(x)H(u^+)\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} K(x)H(u^-)\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}} K(x)\left(f(u)u - 2F(u)\right)\mathrm{d}x$$
$$= I(u_n) - \frac{1}{2}I'(u_n)u_n + o_n(1),$$
$$= I(u_n) + o_n(1) = c_{\mathcal{M}}$$

which concludes the proof. ■

Next, we consider $D = (\frac{1}{2}, \frac{3}{2}) \times (\frac{1}{2}, \frac{3}{2})$ and $g : \overline{D} \to X$ given by $g(\alpha, \beta) = \alpha\widetilde{u}^+ + \beta\widetilde{u}^-$, where $\widetilde{u}$ was obtained in Lemma 2.4.13. We shall prove an auxiliary result and present some notations that will be used in the proof of Theorem 2.2.3.

Lemma 2.4.14 *Let $P = \{u \in X : u(x) \geq 0\, a.e.\ x \in \mathbb{R}\}$ and $-P = \{u \in X : u(x) \leq 0\, a.e.\ x \in \mathbb{R}\}$. Then $d_0' = dist(g(\overline{D}), \Lambda) > 0$, where $\Lambda := P \cup (-P)$.*

Proof . We suppose, by contradiction, that $d_0' = dist(g(\overline{D}), \Lambda) = 0$. Hence, we can find $(v_n) \subset g(\overline{D})$ and $(w_n) \subset \Lambda$ such that $\|v_n - w_n\| \to 0$ as $n \to \infty$. We can assume, without loss of generality, that $w_n \geq 0$ a.e. in $\mathbb{R}$. Since $v_n \in g(\overline{D})$, there exist $\alpha_n, \beta_n \in [\frac{1}{2}, \frac{3}{2}]$ such that $v_n = \alpha_n\widetilde{u}^+ + \beta_n\widetilde{u}^-$. By compactness of $[\frac{1}{2}, \frac{3}{2}]$, up to a subsequence, we have $\alpha_n \to a_0$ and $\beta_n \to b_0$ as $n \to \infty$. Hence

$$v_n \to a_0\widetilde{u}^+ + b_0\widetilde{u}^- \quad \text{in} \quad X.$$

Thus, we obtain $w_n \to a_0\widetilde{u}^+ + b_0\widetilde{u}^-$ in X. Now, by Proposition 2.3.2, we have

$$w_n(x) \to a_0\widetilde{u}^+(x) + b_0\widetilde{u}^-(x) \quad \text{a.e. in} \quad \mathbb{R}.$$

Since $\widetilde{u}^- \neq 0$, the convergence above produces a contradiction with the assumption that $w_n \geq 0$ a.e. in $\mathbb{R}$, which completes the proof. ■

2.5 Proof of Theorem 2.2.3

The proof of this theorem is done in the same way as the Theorem 1.1.2 and we omit it.

2.6 Proof of Theorem 2.2.4

The proof of this theorem is done in the same way as the Theorem 1.1.3 and we omit it.

Chapter 3

Nodal and constant sign solutions for a class of fractional Kirchhoff-type problems involving exponential growth

In this chapter, we study the existence of nonnegative, nonpositive and nodal solutions of smaller energy for a fractional Kirchhoff problem with a nonlinear term that may have a exponential critical growth in the Trudinger-Moser sense. By using the constrained minimization in Nehari set, the quantitative deformation lemma and degree theory results, we obtain a least energy nodal solution. Then, by exploring estimates obtained in the first result and by using the Mountain Pass Theorem,we get one nonpositive and one nonnegative ground state solution. Moreover, we show that the energy of the nodal solution is strictly larger than twice the ground state level. When we regard b as a positive parameter, we study the asymptotic behavior of the nodal solutions as $b_n \to 0^+$. The results of this chapter were submitted for publication in article [30].

3.1 Introduction and main results

This chapter is devoted to study the existence of ground state and nodal solutions for the following class of fractional Kirchhoff-type problems:

$$\begin{cases} (a+b\|u\|^2)\left[(-\Delta)^{1/2}u+V(x)u\right] &= f(u) \quad \text{in} \quad \Omega, \\ u &= 0 \qquad \text{in} \quad \mathbb{R}\setminus\Omega, \end{cases} \tag{$P_{a,b}$}$$

where $a>0$, $b\geq 0$, $\Omega\subset\mathbb{R}$ is a bounded open interval, $V:\overline{\Omega}\to[0,\infty)$ is a continuous potential, $f\in C^1(\mathbb{R})$ may have exponential subcritical or critical growth in the Trudinger-Moser sense (see (1.4) and (1.5)). Here, $(-\Delta)^{1/2}$ is the $1/2-$Laplacian operator defined in (1.2) and the function u belongs to an appropriate subspace of $H^{1/2}(\mathbb{R})$ endowed with the norm

$$\|u\| = \left(\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{|u(x)-u(y)|^2}{|x-y|^2}\mathrm{d}x\mathrm{d}y+\int_\Omega V(x)|u|^2\mathrm{d}x\right)^{1/2}. \tag{3.1}$$

Motivated by physical or mathematical aspects, classes of problems like ($P_{a,b}$) have attracted a lot of attention of many researchers and some existence and multiplicity results have been obtained. A Kirchhoff type problem involving exponential growth was treated by J. Giacomoni *et al.* [47], by using the Nehari method. X. Mingqi *et al.* [59] proved the existence and multiplicity of solutions for a class of fractional Kirchhoff-type problems for the p-fractional Laplace operator.

None of the previous papers treated the existence of nodal solution for the problem ($P_{a,b}$) when the nonlinearity has exponential growth. In Chapter 1, we deal with the problem ($P_{a,b}$) when $a>0$ and $b=0$. Motivated by this fact, our goal in the present chapter is to study the existence of nodal solutions for the problem ($P_{a,b}$) when the nonlinearity has exponential growth as in (1.4) and (1.5).

Throughout this chapter we will assume the following hypotheses:

(V_1) $V:\overline{\Omega}\to[0,\infty)$ is a continuous function, where $\Omega\subset\mathbb{R}$ is a bounded open interval.

For the nonlinearity f we assume that:

(f_1) $f\in C^1(\mathbb{R})$ and there exists $C_0>0$ such that

$$|f(t)|\leq C_0e^{\pi t^2}, \text{ for all } t\in\mathbb{R};$$

(f_2) $\displaystyle\lim_{t\to 0}\frac{f(t)}{t}=0$;

(f_3) there exists $\theta>4$ such that

$$0<\theta F(t):=\theta\int_0^t f(\tau)\mathrm{d}\tau\leq tf(t),\ \text{for all } t\in\mathbb{R}\setminus\{0\};$$

(f_4) the function $\dfrac{f(t)}{|t|^3}$ is strictly increasing, for any $t\neq 0$;

(f_5) there exist $p>4$ and $C_p>0$ such that

$$sgn(t)f(t)\geq C_p|t|^{p-1},\ \text{ for all } t\in\mathbb{R}.$$

Example 3.1.1 *If $p>4$, the nonlinearity*

$$f(t)=C_p|t|^{p-2}t+|t|^{p-2}te^{t^2}$$

satisfies the assumptions $(f_1)-(f_5)$.

As in Chapter 1, to obtain weak solutions of $(P_{a,b})$, we consider the subspace X of $H^{1/2}(\mathbb{R})$ defined by

$$X:=\{u\in H^{1/2}(\mathbb{R}): u=0 \quad \text{in} \quad \mathbb{R}\setminus\Omega\},$$

which will be equipped with inner product

$$\langle u,v\rangle:=\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^2}\mathrm{d}x\mathrm{d}y+\int_\Omega V(x)uv\,\mathrm{d}x \tag{3.2}$$

and the corresponding norm given in (3.1). X is a Hilbert space and the embedding $X\hookrightarrow H^{1/2}(\mathbb{R})$ is continuous. Moreover, X is continuous and compactly embedded in $L^q(\mathbb{R})$ (see [50] and Lemma 1.2.1).

To simplify the notation, we consider the function $m_b(t)=a+bt$, and we rewrite $(P_{a,b})$ as

$$\begin{cases} m_b(\|u\|^2)\left[(-\Delta)^{1/2}u+V(x)u\right] &= f(u) \quad \text{in } \Omega,\\ u &= 0 \qquad \text{in } \mathbb{R}\setminus\Omega.\end{cases}$$

In this context, we say that $u\in X$ is a weak solution of $(P_{a,b})$, if

$$m_b(\|u\|^2)\langle u,v\rangle-\int_\Omega f(u)v\,\mathrm{d}x=0,\ \text{for all } v\in X. \tag{3.3}$$

Considering the functional $I_b : X \to \mathbb{R}$ given by

$$I_b(u) = \frac{1}{2} M_b(\|u\|^2) - \int_\Omega F(u)\, \mathrm{d}x, \tag{3.4}$$

where $M_b(t) := \int_0^t m_b(\tau)\mathrm{d}\tau = at + bt^2/2$, for $t \in \mathbb{R}$. Using the assumptions on f, by standard arguments, we can see that I_b is $C^1(X, \mathbb{R})$ and

$$I_b'(u)v = m_b(\|u\|^2)\langle u, v\rangle - \int_\Omega f(u)v\, \mathrm{d}x, \text{ for all } u, v \in X.$$

Thus, critical points of the functional I_b are weak solutions of ($P_{a,b}$) and reciprocally (see details in Section 3.2).

In order to present the main results of this chapter, we define the Nehari sets associated to I_b and their respective minimums energy level by:

- The Nehari set and the ground state level

$$\mathcal{N}_b = \{u \in X \setminus \{0\} : I_b'(u)u = 0\} \quad \text{and} \quad c_{\mathcal{N}_b} := \inf_{u \in \mathcal{N}_b} I_b(u); \tag{3.5}$$

- The set of nonnegative functions on the Nehari set

$$\mathcal{N}_b^+ = \{u \in \mathcal{N}_b : u^- = 0\} \quad \text{and} \quad c_{\mathcal{N}_b^+} := \inf_{u \in \mathcal{N}_b^+} I_b(u); \tag{3.6}$$

- The set of nonpositive functions on the Nehari set

$$\mathcal{N}_b^- = \{u \in \mathcal{N}_b : u^+ = 0\} \quad \text{and} \quad c_{\mathcal{N}_b^-} := \inf_{u \in \mathcal{N}_b^-} I_b(u); \tag{3.7}$$

- The nodal Nehari set and the nodal level

$$\mathcal{M}_b = \{u \in X : u^+ \neq 0,\ u^- \neq 0,\ I_b'(u)u^+ = 0 \text{ and } I_b'(u)u^- = 0\} \tag{3.8}$$

and

$$c_{\mathcal{M}_b} := \inf_{u \in \mathcal{M}_b} I_b(u).$$

Our first objective is to guarantee that the minimum $c_{\mathcal{M}_b}$ is achieved by a weak solution $w \in \mathcal{M}_b$ and, in this case, w will be called of least energy nodal solution (see also Remark 3.3.5). Notice that the set $\mathcal{M}_b$ is a subset of the nodal functions in $\mathcal{N}_b$.

Now we can state our first result.

Theorem 3.1.2 *Suppose that* (V_1) *and* $(f_1)-(f_5)$ *are satisfied. Furthermore, we assume*

$$C_p > \max\left\{\frac{1}{2}, \left[\frac{4\theta d_b^*(2^{\frac{2}{2-p}}\,p - 2^{\frac{p}{2-p}}\,4)}{a(\theta-2)(p-4)}\right]^{(p-2)/2}\right\}, \tag{3.9}$$

where

$$d_b^* := \inf_{u\in\mathcal{M}_b^p} J_b(u), \tag{3.10}$$

$$\mathcal{M}_b^p = \{u \in X : u^+ \neq 0,\ u^- \neq 0,\ J_b'(u)u^+ = 0 \quad \textit{and} \quad J_b'(u)u^- = 0\} \tag{3.11}$$

and

$$J_b(u) = \frac{1}{2}M_b(\|u\|^2) - \frac{1}{2p}\int_\Omega |u|^p \mathrm{d}x. \tag{3.12}$$

Then, the problem $(P_{a,b})$ *has a least energy nodal solution* u_b.

Our second result provides one nonnegative solution and one nonpositive solution of $(P_{a,b})$, which the energy is minimal between the solutions that have the signal defined. Moreover, we also show that the energy of any sign-changing solution of $(P_{a,b})$ is strictly larger than twice the ground state energy. This property is so-called energy doubling by Weth [71].

Theorem 3.1.3 *Suppose that* (V_1), $(f_1)-(f_5)$ *and* (3.9) *are satisfied. Then, there exist* $u_+ \in \mathcal{N}_b^+$ *with* $I_b(u_+) = c_{\mathcal{N}_b^+}$ *and* $u_- \in \mathcal{N}_b^-$ *with* $I_b(u_-) = c_{\mathcal{N}_b^-}$, *weak solutions of* $(P_{a,b})$. *Moreover, we have*

$$c_{\mathcal{M}_b} = I_b(u_b) > c_{\mathcal{N}_b^+} + c_{\mathcal{N}_b^-} \geq 2c_{\mathcal{N}_b}, \tag{3.13}$$

where u_b *is the least energy nodal solution obtained in Theorem* 3.1.2.

The third result is to study the asymptotic behavior of the least nodal solutions u_b as $b \to 0^+$. Precisely, we prove that:

Theorem 3.1.4 *Suppose that* (V_1), $(f_1)-(f_5)$ *and* (3.9) *are satisfied. Let* $(b_n) \subset \mathbb{R}$ *be a sequence such that* $0 \leq b_n \leq b$ *and* $b_n \to 0^+$, *as* $n \to \infty$. *Then, for any* $n \in \mathbb{N}$, *the problem* (P_{a,b_n}) *has a least energy nodal solution* u_{b_n} *and, up to a subsequence,* u_{b_n} *converges strongly to* u_0 *in* X, *where* u_0 *is a least energy nodal solution to the problem*

$$\begin{cases} a\,(-\Delta)^{1/2}u + a\,V(x)u &= f(u) \quad \textit{in} \quad \Omega, \\ u &= 0 \quad\quad\ \textit{in} \quad \mathbb{R}\setminus\Omega. \end{cases} \tag{$P_{a,0}$}$$

As in Chapter 1, the functional I_b associated to $(P_{a,b})$ does not possess the same decompositions as (1.13) and (1.14). Indeed, since $\langle u^+, u^- \rangle > 0$ whenever $u^+ \neq 0$ and $u^- \neq 0$, a straightforward computation yields that (see Lemma 3.2.3 and Corollary 3.2.4)

$$I_b(u) > I_b(u^+) + I_b(u^-),$$

$$I_b'(u)u^+ > I_b'(u^+)u^+ \quad \text{and} \quad I_b'(u)u^- > I_b'(u^-)u^-,$$

where I_b is defined in (3.4). Therefore, the methods used to obtain sign-changing solutions for the local problem like (1.12) seem not be applicable to the problem $(P_{a,b})$. Additionally, we have difficulties due to the presence of the non-local Kirchhoff term and the loss of the Palais Smale compactness condition due to the exponential growth on the nonlinearity. In order to overcome these difficulties, we define the constrained set $\mathcal{M}_b$ (see (3.8)) and consider a minimization problem of I_b on $\mathcal{M}_b$. Borrowing ideas from [45], we prove $\mathcal{M}_b \neq \emptyset$ via geometric properties of the functional of I_b (see Lemma 3.3.4). Combining the ideas developed in [3, 4, 11, 21], we prove that the minimizer of the constrained problem is also a sign-changing solution via the quantitative deformation lemma and degree theory (see Section 3.3).

Remark 3.1.5 *The hypothesis* (f_1) *allows us to consider nonlinearities with critical growth in the sense defined in* (1.4) *with an exponent* $\alpha_0 = \pi$ *and with subcritical growth as in* (1.5). *More generally, we can consider an exponent* α_0 *different from* π. *In this more general case, this new constant would imply a normalization of the constant* C_p *defined in* (3.9).

Remark 3.1.6 *We point out that the results of this chapter complement the works [48, 49, 50, 63] in the sense that we prove the existence of sign-changing solutions and the work [21] in the sense that we consider exponential growth on the nonlinearity. Furthermore, our results extend for the fractional Laplacian some of the results contained in [3, 4, 72].*

The outline of this chapter is as follows: Section 3.2 contains some auxiliaries results and the variational framework. Section 3.3 is dedicated to the study of the nodal set and the nodal level, the main goal is to prove that the nodal level is attained by a sign-changing weak solution of $(P_{a,b})$. In Section 3.4 is devoted to prove the existence of solutions that have signal defined and Section 3.5 we study the convergence of the nodal solutions as $b \to 0^+$.

3.2 Preliminary results

As in Lemma 1.2.1 and Lemma 1.2.2, we have the following results:

Lemma 3.2.1 *Under the assumption* (V_1)*, the embedding* $X \hookrightarrow L^q(\mathbb{R})$ *is continuous and compact for all* $q \in [1, +\infty)$.

Lemma 3.2.2 *If* $0 \le \alpha \le \pi$*, it holds*

$$\sup_{\{u \in X \,:\, \|u\| \le 1\}} \int_\Omega e^{\alpha u^2} \,\mathrm{d}x < \infty. \tag{3.14}$$

Moreover, for any $\alpha > 0$ *and* $u \in X$*, we have*

$$\int_\Omega e^{\alpha u^2} \,\mathrm{d}x < \infty. \tag{3.15}$$

As a consequence of Lemma 3.2.1, Lemma 3.2.2 and (f_1), the energy functional $I_b : X \longrightarrow \mathbb{R}$ given by

$$I_b(u) = \frac{1}{2} M_b(\|u\|^2) - \int_\Omega F(u) \,\mathrm{d}x$$

is well defined and belongs to $C^1(X, \mathbb{R})$. Moreover, by straightforward calculation, we have

$$I_b'(u)v = m_b(\|u\|^2)\langle u, v\rangle - \int_\Omega f(u)v \,\mathrm{d}x, \text{ for all } u, v \in X.$$

As in Lemma 1.2.3:

Lemma 3.2.3 *Let* $u \in X$*. It holds that*

(i) $\langle u, u^\pm\rangle = \langle u^\pm, u^\pm\rangle + \dfrac{1}{\pi}\displaystyle\int_{\mathbb{R}^2} \frac{u^+(x)(-u^-(y))}{|x-y|^2} \mathrm{d}x\mathrm{d}y$

(i) $\langle u^+, u^-\rangle \ge 0$,

(ii) $\|u\|^2 \ge \|u^+\|^2 + \|u^-\|^2$.

Moreover, if $u^+ \ne 0$ *and* $u^- \ne 0$*, these inequalities are strict.*

We now collect some estimates for the functions f and m_b. By the definition of m_b we have

$$m_b(t)/t \text{ is strictly decreasing, for all } t > 0; \tag{3.16}$$

$$m_b'(t)t < m_b(t), \text{ for all } t > 0; \tag{3.17}$$

$$\frac{1}{2}M_b(t) - \frac{1}{4}m_b(t)t \text{ is positive and increasing, for all } t > 0; \tag{3.18}$$

$$M_b(t+s) \geq M_b(t) + M_b(s), \text{ for all } t, s \geq 0; \tag{3.19}$$

and this inequality is strict, if t and s are positive.

By $(f_1) - (f_2)$, given $\varepsilon > 0$ and $q \geq 1$, there exists $C = C(\varepsilon, q) > 0$ such that

$$|f(t)| \leq \varepsilon|t| + C|t|^{q-1}e^{\pi t^2}, \text{ for all } t \in \mathbb{R}, \tag{3.20}$$

and by (f_3), we have

$$|F(t)| \leq \varepsilon|t|^2 + C|t|^q e^{\pi t^2}, \text{ for all } t \in \mathbb{R}. \tag{3.21}$$

Moreover, by (f_3), we can find positive constants C_1 and C_2 such that

$$F(t) \geq C_1|t|^\theta - C_2, \text{ for all } t \in \mathbb{R}. \tag{3.22}$$

By (f_5), it follows that

$$|f(t)| \geq C_p|t|^{p-1}, \text{ for all } t \in \mathbb{R}, \tag{3.23}$$

and consequently

$$F(t) \geq \frac{C_p}{p}|t|^p, \text{ for all } t \in \mathbb{R}. \tag{3.24}$$

We finish this section with the following consequence of Lemma 3.2.3 and (3.19).

Corollary 3.2.4 *Let $u \in X$. It holds that*

(i) $I_b(u) \geq I_b(u^+) + I_b(u^-)$,

(ii) $I_b'(u)u^+ \geq I_b'(u^+)u^+$ *and* $I_b'(u)u^- \geq I_b'(u^-)u^-$.

Moreover, if $u^+ \neq 0$ and $u^- \neq 0$, these inequalities are strict.

Proof . By Lemma 3.2.3, we have

$$\|u\|^2 = \|u^+\|^2 + 2\langle u^+, u^-\rangle + \|u^-\|^2 \geq \|u^+\|^2 + \|u^-\|^2.$$

Thus, by using (3.19), we get the desired inequalities. ■

3.3 Constrained minimization problem

We begin this section by introducing some notations. Given $u \in X$, we define $\varphi_u : [0,\infty) \to \mathbb{R}$ by

$$\varphi_u(t) = I_b(tu) = \frac{1}{2}M_b(\|tu\|^2) - \int_\Omega F(tu)\,\mathrm{d}x, \tag{3.25}$$

and $\psi_u : [0,\infty) \times [0,+\infty) \to \mathbb{R}$ by

$$\psi_u(t,s) = I_b(tu^+ + su^-), \tag{3.26}$$

and the vector field $\Psi_u : [0,\infty) \times [0,\infty) \to \mathbb{R}^2$ by

$$\Psi_u(t,s) = \left(I_b'(tu^+ + su^-)tu^+, I_b'(tu^+ + su^-)su^-\right). \tag{3.27}$$

Next we will show that the Nehari sets $\mathcal{N}_b, \mathcal{N}_b^+$ e $\mathcal{N}_b^-$ are not empty.

Lemma 3.3.1 *Assume that* (V_1) *and* $(f_1)-(f_4)$ *are satisfied. Then, given* $u \in X\setminus\{0\}$, *there exists a unique* $t = t(u) > 0$ *such that*

$$I_b(tu) = \max_{s\geq 0} I_b(su). \tag{3.28}$$

As a consequence, the Nehari sets $\mathcal{N}_b, \mathcal{N}_b^+$ *and* $\mathcal{N}_b^-$ *are not empty.*

Proof . Let $u \in X \setminus \{0\}$. Since $M_b(s) = as + bs^2/2$, we have

$$\varphi_u(s) = \frac{as^2}{2}\|u\|^2 + \frac{bs^4}{4}\|u\|^4 - \int_\Omega F(su)\,\mathrm{d}x.$$

By (3.22), we get

$$\varphi_u(s) \leq \frac{as^2}{2}\|u\|^2 + \frac{bs^4}{4}\|u\|^4 - C_1 s^\theta \|u\|_\theta^\theta + C_2|\Omega|.$$

Hence, since $\theta > 4$, we obtain

$$\varphi_u(s) \to -\infty, \quad \text{as} \quad s \to \infty. \tag{3.29}$$

On the other hand, given $\varepsilon > 0$ and $q > 2$, by using (3.21) we have

$$\varphi_u(s) \geq \frac{as^2}{2}\|u\|^2 + \frac{bs^4}{4}\|u\|^4 - \varepsilon s^2\|u\|_2^2 - Cs^q \int_\Omega |u|^q e^{\pi s^2 u^2}\mathrm{d}x.$$

If $s \in [0,1]$, we have $e^{\pi s^2 u^2} \leq e^{\pi u^2}$. Then, by Lemma 3.2.1 and Lemma 3.2.2, we can find $C_1, C_2 > 0$ such that

$$\varphi_u(s) \geq s^2\left(\frac{a}{2} - \varepsilon C_1\right)\|u\|^2 + \frac{bs^4}{4}\|u\|^4 - C(u)s^q. \tag{3.30}$$

Thus, choosing $\varepsilon > 0$ such that $\frac{a}{2} - \varepsilon C_1 > 0$, by using (3.30), we obtain

$$\varphi_u(s) > 0, \text{ for } s > 0 \text{ small enough.} \tag{3.31}$$

From (3.29) and (3.31), there exists $t = t(u) > 0$ satisfying (3.28).

Next we will show the uniqueness of $t = t(u)$. Suppose, by contradiction, that there exists $s > t$ such that $I_b'(su)su = 0$. By the definition of m_b, we have

$$at^2\|u\|^2 + bt^4\|u\|^4 = \int_\Omega f(tu)tu \,\mathrm{d}x$$

and

$$as^2\|u\|^2 + bs^4\|u\|^4 = \int_\Omega f(su)su \,\mathrm{d}x.$$

From these, it follows that

$$\int_\Omega \left(\frac{f(tu)}{(tu)^3} - \frac{f(su)}{(su)^3}\right) u^4 \mathrm{d}x = a\left(\frac{1}{t^2} - \frac{1}{s^2}\right)\|u\|^2.$$

Since $s > t$, this equality implies that

$$\int_\Omega \left(\frac{f(tu^+)}{(tu^+)^3} - \frac{f(su^+)}{(su^+)^3}\right)(u^+)^4\mathrm{d}x + \int_\Omega \left(\frac{f(tu^-)}{(tu^-)^3} - \frac{f(su^-)}{(su^-)^3}\right)(u^-)^4\mathrm{d}x > 0. \tag{3.32}$$

But, by using the assumption (f_4) and (3.32), we get a contradiction. The case $0 < s < t$ is similar and we omit it. Therefore, we obtain that $t = s$. This completes the proof. ■

The next result shows some geometric properties of functional I_b, which will be use to study the Nehari nodal set $\mathcal{M}_b$.

Lemma 3.3.2 *Assume that (V_1) and $(f_1) - (f_3)$ are satisfied. Then, the functional I_b satisfies the following geometric conditions:*

(i) *given $u \in X \setminus \{0\}$, we have*

$$I_b(tu^+ + su^-) \to -\infty, \quad \textit{as} \quad |(t,s)| \to \infty;$$

(ii) *there exists $r > 0$ such that*

$$I_b(u) \geq \frac{b}{4}\|u\|^4, \quad \textit{for all} \quad \|u\| \leq r.$$

Proof . By using (3.22) and $(x+y)^q \leq 2^{q-1}(x^q+y^q)$, for all $x,y\geq 0$, we obtain

$$\begin{aligned} I_b(tu^+ + su^-) \leq & \frac{a}{2}\|tu^+ + su^-\|^2 + \frac{b}{4}\|tu^+ + su^-\|^4 - C_1\|tu^+ + su^-\|_\theta^\theta + C_2|\Omega| \\ \leq & at^2\|u^+\| + as^2\|u^-\|^2 + 2bt^4\|u^+\| + 2bs^4\|u^-\|^4 - C_1|t|^\theta\|u^+\|_\theta^\theta \\ & + C_1|s|^\theta\|u^-\|_\theta^\theta + C_2|\Omega|. \end{aligned}$$

Thus, since $\theta > 4$ and $u \neq 0$, we get **(i)**.

Now, given $\varepsilon > 0$, $q > 2$ and using (3.21), we obtain that

$$I_b(u) \geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \varepsilon\|u\|_2^2 - C\int_\Omega |u|^q e^{\pi u^2}\mathrm{d}x.$$

If $\|u\|^2 \leq \frac{1}{2}$, by using the Hölder inequality, Lemma 3.2.1 and Lemma 3.2.2, we can find $C_1, C > 0$ such that

$$I_b(u) \geq \left(\frac{a}{2} - \varepsilon C_1\right)\|u\|^2 + \frac{b}{4}\|u\|^4 - C\|u\|^q. \tag{3.33}$$

Chosing $\varepsilon > 0$ such that $\frac{a}{2} - \varepsilon C_1 > 0$ and since $q > 2$, for $\|u\|$ small enough, we obtain that

$$\left(\frac{a}{2} - \varepsilon C_1\right)\|u\|^2 - C\|u\|^q > 0. \tag{3.34}$$

Therefore, from (3.33) and (3.34), there exists $0 < r \leq \frac{1}{2}$, such that **(ii)** holds. ■

Remark 3.3.3 *The energy functional associated to the problem* $(P_{a,0})$*, defined by* $I_0(u) = \frac{a}{2}\|u\|^2 - \int_\Omega F(u)\,\mathrm{d}x$*, it has similar geometric properties like to the previous lemma.*

Lemma 3.3.4 *Assume that* (V_1) *and* $(f_1)-(f_4)$ *are satisfied. Then, given* $u \in X$*, with* $u^+ \neq 0$ *and* $u^- \neq 0$*, there exists a unique pair of positive numbers* (t_u, s_u) *such that* $t_u u^+ + s_u u^- \in \mathcal{M}_b$*. Moreover, if* $(t,s) \neq (t_u, s_u)$*, with* $t,s \geq 0$*, we have*

$$I_b(tu^+ + su^-) < I_b(t_u u^+ + s_u u^-).$$

Proof . By Lemma 3.3.2, there exists $(t_u, s_u) \in [0,+\infty) \times [0,\infty)$ such that

$$I_b(t_u u^+ + s_u u^-) = \max_{[0,+\infty)\times[0,\infty)} I_b(tu^+ + su^-).$$

Next, we will show that $(t_u, s_u) \in (0,\infty) \times (0,\infty)$. Using **(ii)** of Lemma 3.3.2, we have

$$I_b(tu^+) > 0 \quad \text{and} \quad I_b(su^-) > 0 \quad \text{for} \quad t,s > 0 \quad \text{small enough.}$$

Thus, by using **(i)** of Corollary 3.2.4, for $t, s > 0$ small enough, we obtain

$$I_b(tu^+) < I_b(tu^+) + I_b(su^-) < I_b(tu^+ + su^-).$$

Hence, the pair $(t_u, s_u) \in (0, \infty) \times (0, \infty)$. In particular, we get that $t_u u^+ + s_u u^- \in \mathcal{M}_b$.

To show the uniqueness of the pair (t_u, s_u), it is sufficient to consider the case where $u \in \mathcal{M}_b$ and $tu^+ + su^- \in \mathcal{M}_b$, with $t, s > 0$, and to prove that implies in $(t, s) = (1, 1)$. In order to prove this claim notice that

$$m_b(\|u\|^2)\langle u, u^+\rangle = \int_\Omega f(u^+)u^+ \mathrm{d}x$$

and

$$m_b(\|tu^+ + su^-\|^2)\langle tu^+ + su^-, tu^+\rangle = \int_\Omega f(tu^+)tu^+ \mathrm{d}x.$$

We will suppose that $t \geq s$ (the case $s \geq t$ is similar and we will omit it), then

$$\langle tu^+ + su^-, tu^+\rangle = t^2\langle u^+, u^+\rangle + st\langle u^+, u^-\rangle \leq t^2\langle u, u^+\rangle \tag{3.35}$$

and

$$\|tu^+ + su^-\|^2 = t^2\|u^+\|^2 + 2ts\langle u^+, u^-\rangle + s^2\|u^-\|^2 \leq t^2\|u\|^2. \tag{3.36}$$

Hence, by using Lemma 3.2.3 and that m_b is a increasing function, by (3.35) and (3.36), we obtain that

$$\frac{1}{t^4}m_b(t^2\|u\|^2)t^2\langle u, u^+\rangle \geq \frac{1}{t^4}m_b(\|tu^+ + su^-\|^2)\langle tu^+ + su^-, u^+\rangle = \int_\Omega \frac{f(tu^+)}{t^3}u^+ \mathrm{d}x.$$

Thus, we get that

$$\left(\frac{m_b(\|tu\|^2)}{\|tu\|^2} - \frac{m_b(\|u\|^2)}{\|u\|^2}\right)\|u\|^2\langle u, u^+\rangle \geq \int_\Omega \left(\frac{f(tu^+)}{(tu^+)^3} - \frac{f(u^+)}{(u^+)^3}\right)(u^+)^4 \mathrm{d}x. \tag{3.37}$$

If $t > 1$, by (3.16) and Lemma 3.2.3, (3.37) implies that

$$\int_\Omega \left(\frac{f(tu^+)}{(tu^+)^3} - \frac{f(u^+)}{(u^+)^3}\right)(u^+)^4 \mathrm{d}x < 0,$$

and so, by (f_4), we obtain a contradiction. Then, we obtain that $0 < s \leq t \leq 1$. Arguing similarly by using the equations $I_b'(tu^+ + su^-)su^- = 0$ and $I_b'(u)u^- = 0$, we obtain that $1 \leq s \leq t$, which implies $t = s = 1$ and the proof is complete. ■

Remark 3.3.5 *Clearly, any nodal solution to* $(P_{a,b})$ *belongs to* $\mathcal{M}_b$*. Similarly, any nonnegative solution and nonpositive solution to* $(P_{a,b})$ *belongs to* $\mathcal{N}_b^+$ *and* $\mathcal{N}_b^-$*, respectively. Now, let* $u \in \mathcal{M}_b$*. By Lemma 3.3.1, there exist* $t, s > 0$ *such that* $tu^+ \in \mathcal{N}_b^+$ *and*

$su^- \in \mathcal{N}_b^-$. Now, by Lemma 3.3.4, we have $I_b(tu^+ + su^-) \le I_b(u)$. Thus, by Corollary 3.2.4, we reach that

$$2c_{\mathcal{N}_b} \le c_{\mathcal{N}_b^+} + c_{\mathcal{N}_b^-} \le I_b(tu^+) + I_b(su^-) < I_b(tu^+ + su^-) \le I_b(u).$$

Hence, taking the infimum in $u \in \mathcal{M}_b$, we obtain that

$$2c_{\mathcal{N}^b} \le c_{\mathcal{N}_b^+} + c_{\mathcal{N}_b^-} \le c_{\mathcal{M}_b}.$$

In particular, if $c_{\mathcal{M}_b}$ is achieved for some function in $\mathcal{M}_b$, then, we get $2c_{\mathcal{N}^b} \le c_{\mathcal{N}_b^+} + c_{\mathcal{N}_b^-} < c_{\mathcal{M}_b}$, as in (3.13), and $c_{\mathcal{N}^b} = c_{\mathcal{N}_b^+}$ or $c_{\mathcal{N}^b} = c_{\mathcal{N}_b^-}$.

Lemma 3.3.6 *Assume that (V_1) and $(f_1) - (f_4)$ are satisfied. Let $u \in X$ such that $u^+ \neq 0$, $u^- \neq 0$, $I_b'(u)u^+ \le 0$ and $I_b'(u)u^- \le 0$. Then the unique pair (t, s) given in Lemma 3.3.4 satisfies $0 < t, s \le 1$.*

Proof . Without loss of generality, we can assume $0 < s \le t$ and, by contradiction, that $t > 1$. Note that (3.35) and (3.36) remain valid. Thus, since $I_b'(tu^+ + su^-)tu^+ = 0$ and $I_b'(u)u^+ \le 0$, arguing as in Lemma 3.3.4, we have

$$\begin{aligned}
\int_\Omega \frac{f(tu^+)}{(tu^+)^3}(u^+)^4 \mathrm{d}x &= \frac{1}{t^4} m_b(\|tu^+ + su^-\|^2)\langle tu^+ + su^-, tu^+\rangle \\
&\le \frac{1}{t^4} m_b(\|tu\|^2)t^2\langle u, u^+\rangle = \frac{m_b(\|tu\|^2)}{\|tu\|^2}\|u\|^2\langle u, u^+\rangle \\
&< \frac{m_b(\|u\|^2)}{\|u\|^2}\|u\|^2\langle u, u^+\rangle \le \int_\Omega f(u^+)u^+ \mathrm{d}x,
\end{aligned}$$

and so

$$\int_\Omega \left(\frac{f(tu^+)}{(tu^+)^3} - \frac{f(u^+)}{(u^+)^3}\right)(u^+)^4 \mathrm{d}x < 0.$$

But, by (f_4), we obtain a contradiction. Therefore, we reach $t \le 1$, and the proof is complete. ■

Lemma 3.3.7 *Assume that (V_1) and $(f_1) - (f_4)$ are satisfied. Then, there exists $m_0 > 0$, independent of b, such that for any $u \in \mathcal{N}_b$ and for any $v \in \mathcal{M}_b$, we have*

$$m_0 \le \|u\|^2 \text{ and } m_0 \le \|v^+\|^2, \|v^-\|^2.$$

Proof . We will show the estimates only for $v \in \mathcal{M}_b$. Suppose, by contradiction, that $(v_n) \in \mathcal{M}_b$ and $\|v_n^+\| \to 0$ as $n \to \infty$. By using $I_b'(v_n)v_n^+ = 0$ and Lemma 3.2.3, we have

$$a\|v_n^+\|^2 < m_b(\|v_n\|^2)\langle v_n, v_n^+\rangle = \int_\Omega f(v_n^+)v_n^+ \mathrm{d}x, \quad \text{for all} \quad n \in \mathbb{N}.$$

Thus, given $\varepsilon > 0$ and $q > 2$, by using (3.20), we get

$$a\|v_n^+\|^2 < \varepsilon\|v_n^+\|_2^2 + C\int_\Omega |v_n^+|^q e^{\pi|v_n^+|^2}\mathrm{d}x. \tag{3.38}$$

Now, from the Hölder inequality, Lemma 3.2.2 and that since $\|v_n^+\| \to 0$, for $n \in \mathbb{N}$ large enough, we obtain that

$$\int_\Omega |v_n^+|^q e^{\pi|v_n^+|^2}\mathrm{d}x \le \left(\int_\Omega |v_n^+|^{2q}\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega e^{2\|v_n^+\|^2\pi\left(\frac{v_n^+}{\|v_n^+\|}\right)^2}\mathrm{d}x\right)^{\frac{1}{2}} \le C'\|v_n^+\|_{2q}^q.$$

By using this inequality and Lemma 3.2.1 in (3.38), we have

$$a\|v_n^+\|^2 < \varepsilon C_1\|v_n^+\|^2 + C_2\|v_n^+\|^q, \tag{3.39}$$

for all $n \in \mathbb{N}$ large enough. We can choose $\varepsilon > 0$ such that $a - \varepsilon C_1 > 0$. Thus, from (3.39), we have

$$0 < \frac{a - \varepsilon C_1}{C_2} < \|v_n^+\|^{q-2} \text{ for all } n \in \mathbb{N},$$

contrary to the assumption. Therefore, there exists $m_0 > 0$ with the desired property. ■

Corollary 3.3.8 *Assume that (V_1) and $(f_1) - (f_3)$ are satisfied. Then, there exists $\delta_0 > 0$, independent of b, such that $I_b(u) \ge \delta_0$, for all $u \in \mathcal{N}_b$. In particular*

$$\delta_0 \le c_{\mathcal{N}_b},\ \delta_0 \le c_{\mathcal{N}_b^+},\ \delta_0 \le c_{\mathcal{N}_b^-} \text{ and } \delta_0 \le c_{\mathcal{M}_b}.$$

Proof . Let $u \in \mathcal{N}^b$. Since $I_b'(u)u = 0$, from Lemma 3.3.7 and (f_3), we have

$$\begin{aligned}
I_b(u) &= I_b(u) - \frac{1}{\theta}I_b'(u)u \\
&= \frac{1}{2}M_b(\|u\|^2) - \frac{1}{\theta}m_b(\|u\|^2)\|u\|^2 + \frac{1}{\theta}\int_\Omega (f(u)u - \theta F(u))\,\mathrm{d}x \\
&\ge \frac{1}{2}M_b(\|u\|^2) - \frac{1}{\theta}m_b(\|u\|^2)\|u\|^2 \\
&= a\left(\frac{1}{2} - \frac{1}{\theta}\right)\|u\|^2 + b\left(\frac{1}{4} - \frac{1}{\theta}\right)\|u\|^4 \\
&\ge a\left(\frac{1}{2} - \frac{1}{\theta}\right)\|u\|^2 \ge a\left(\frac{1}{2} - \frac{1}{\theta}\right)m_0 := \delta_0,
\end{aligned}$$

which is the desired conclusion. ■

In the next result, we will obtain an important estimate for the nodal level $c_{\mathcal{M}_b}$. That will be a powerful tool in order to obtain an appropriate bound of the norm of a minimizing sequence for $c_{\mathcal{M}_b}$ in $\mathcal{M}_b$.

Lemma 3.3.9 *Assume that* (V_1), $(f_1)-(f_5)$ *and* (3.9) *are satisfied. It holds that*

$$c_{\mathcal{M}_b} < \frac{a(\theta-2)}{2\theta}. \tag{3.40}$$

Proof . From Theorem B.3.1 in the Appendix, there exists $w \in \mathcal{M}_b^p$ such that $J_b(w) = d_b^*$ and $J_b'(w) = 0$. Consequently, we get

$$M_b(\|w\|^2) - \frac{1}{p}\|w\|_p^p = 2d_b^* \tag{3.41}$$

and

$$m_b(\|w\|^2)\langle w, w^+\rangle = \frac{1}{2}\|w^+\|_p^p \quad \text{and} \quad m_b(\|w\|^2)\langle w, w^-\rangle = \frac{1}{2}\|w^-\|_p^p. \tag{3.42}$$

By Lemma 3.3.4, there exist $t, s > 0$ such that $tw^+ + sw^- \in \mathcal{M}_b$. By using (3.23) and that $C_p > 1/2$, we have

$$m_b(\|w\|^2)\langle w, w^\pm\rangle = \frac{1}{2}\|w\|_p^p = \frac{1}{2C_p}C_p\|w^\pm\|_p^p \le \int_\Omega f(w^\pm)w^\pm \mathrm{d}x.$$

Thus, $I_b'(w)w^+ \le 0$ and $I_b'(w)w^- \le 0$. Then, we can apply Lemma 3.3.6, in order to get that $0 < t, s \le 1$. By the definition of $c_{\mathcal{M}_b}$ and (3.24), we get

$$c_{\mathcal{M}_b} \le \frac{1}{2}M_b(\|tw^+ + sw^-\|^2) - \frac{C_p t^p}{p}\|w^+\|_p^p - \frac{C_p s^p}{p}\|w^-\|_p^p. \tag{3.43}$$

Now, from (3.42) and by the definition of M_b and m_b, we obtain

$$\frac{1}{2}M_b(\|tw^+ + sw^-\|^2) = \frac{at^2}{2}\|w^+\|^2 + ats\langle w^+, w^-\rangle + \frac{as^2}{2}\|w^-\|^2 + \frac{b}{4}\|tw^+ + sw^-\|^4,$$

and

$$a\|w^\pm\|^2 = \frac{1}{2}\|w^\pm\|_p^p - a\langle w^+, w^-\rangle - b\|w\|^2\langle w, w^\pm\rangle.$$

These estimates together with (3.43), imply that

$$\begin{aligned} c_{\mathcal{M}_b} \le & \frac{at^2}{2}\|w^+\|^2 + ats\langle w^+, w^-\rangle + \frac{as^2}{2}\|w^-\|^2 + \frac{b}{4}\|tw^+ + sw^-\|^4 \\ & - \frac{C_p t^p}{p}\|w^+\|_p^p - \frac{C_p s^p}{p}\|w^-\|_p^p \\ = & \frac{t^2}{4}\|w^+\|_p^p - \frac{at^2}{2}\langle w^+, w^-\rangle - \frac{bt^2}{2}\|w\|^2\langle w, w^+\rangle + ats\langle w^+, w^-\rangle \\ & + \frac{s^2}{4}\|w^-\|_p^p - \frac{as^2}{2}\langle w^+, w^-\rangle - \frac{bs^2}{2}\|w\|^2\langle w, w^-\rangle + \frac{b}{4}\|tw^+ + sw^-\|^4 \\ & - \frac{C_p t^p}{p}\|w^+\|_p^p - \frac{C_p s^p}{p}\|w^-\|_p^p. \end{aligned}$$

This implies that

$$
\begin{aligned}
c_{\mathcal{M}_b} \leq & \left(\frac{t^2}{4} - \frac{C_p t^p}{p}\right) \|w^+\|_p^p + \left(\frac{s^2}{4} - \frac{C_p s^p}{p}\right) \|w^-\|_p^p - \frac{a}{2}(t-s)^2\langle w^+, w^-\rangle \\
& - \frac{bt^2}{2}\|w\|^2\langle w, w^+\rangle - \frac{bs^2}{2}\|w\|^2\langle w, w^-\rangle + \frac{b}{4}\|tw^+ + sw^-\|^4.
\end{aligned}
$$

By Lemma 3.2.3, we have $\frac{a}{2}(t-s)^2\langle w^+, w^-\rangle \geq 0$ and so we deduce that

$$
c_{\mathcal{M}_b} \leq \left(\frac{t^2}{4} - \frac{C_p t^p}{p}\right) \|w^+\|_p^p + \left(\frac{s^2}{4} - \frac{C_p s^p}{p}\right) \|w^-\|_p^p + A(t, s, w, b), \tag{3.44}
$$

where

$$
A(t, s, w, b) := -\frac{bt^2}{2}\|w\|^2\langle w, w^+\rangle - \frac{bs^2}{2}\|w\|^2\langle w, w^-\rangle + \frac{b}{4}\|tw^+ + sw^-\|^4.
$$

We claim that $A(t, s, w, b) \leq 0$.

Indeed, notice that

$$
\begin{aligned}
\frac{1}{2}\|tw^+ + sw^-\|^4 =& \frac{t^4}{2}\|w^+\|^4 + 2t^3 s\|w^+\|^2\langle w^+, w^-\rangle + t^2 s^2\|w^+\|^2\|w^-\|^2 \\
& + 2t^2 s^2\langle w^+, w^-\rangle^2 + 2ts^3\|w^-\|^2\langle w^+, w^-\rangle + \frac{s^4}{2}\|w^-\|^4
\end{aligned}
$$

and

$$
\begin{aligned}
-t^2\|w\|^2\langle w, w^+\rangle - s^2\|w\|^2\langle w, w^-\rangle =& - t^2\|w^+\|^4 - (3t^2 + s^2)\|w^+\|\langle w^+, w^-\rangle \\
& - (t^2 + s^2)\|w^+\|^2\|w^-\|^2 - 2(t^2 + s^2)\langle w^+, w^-\rangle^2 \\
& - (3s^2 + t^2)\|w^-\|^2\langle w^+, w^-\rangle - s^2\|w^-\|^4.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
\frac{2}{b}A(t, s, w, b) =& \left(\frac{t^4}{2} - t^2\right) \|w^+\|^4 + (2t^3 s - 3t^2 - s^2)\|w^+\|^2\langle w^+, w^-\rangle \\
& + (t^2 s^2 - t^2 - s^2)\|w^+\|^2\|w^-\|^2 + 2(t^2 s^2 - t^2 - s^2)\langle w^+, w^-\rangle^2 \\
& + (2ts^3 - 3s^2 - t^2)\|w^-\|^2\langle w^+, w^-\rangle + \left(\frac{s^4}{2} - s^2\right) \|w^-\|^4.
\end{aligned}
$$

Now, since $0 < t, s \leq 1$, from Lemma 3.2.3, it is easy to see that $A(t, s, w, b) \leq 0$.

By using that $A(t, s, w, b) \leq 0$ and (3.44), we get

$$
c_{\mathcal{M}_b} < \max_{\xi \geq 0} \left(\frac{\xi^2}{4} - \frac{C_p \xi^p}{p}\right) \|w\|_p^p. \tag{3.45}
$$

It is simple to check that

$$
\max_{\xi \geq 0} \left(\frac{\xi^2}{4} - \frac{C_p \xi^p}{p}\right) = C_p^{\frac{2}{2-p}} \left(\frac{2^{\frac{2}{2-p}} p - 2^{\frac{p}{2-p}} 4}{p - 4}\right) \left(\frac{1}{4} - \frac{1}{p}\right). \tag{3.46}
$$

Note that by using (3.42), we get $m_b(\|w\|^2)\|w\|^2 = \frac{1}{2}\|w\|_p^p$ and consequently, we have

$$\frac{1}{4}\|w\|_p^p = \frac{1}{2}m_b(\|w\|^2)\|w\|^2 = \frac{a}{2}\|w\|^2 + \frac{b}{2}\|w\|^4.$$

This together with (3.41), implies that

$$\begin{aligned}\left(\frac{1}{4} - \frac{1}{p}\right)\|w\|_p^p =& \frac{1}{4}\|w\|_p^p - \frac{1}{p}\|w\|_p^p = \frac{a}{2}\|w\|^2 + \frac{b}{2}\|w\|^4 - \frac{1}{p}\|w\|_p^p \\ <& a\|w\|^2 + \frac{b}{2}\|w\|^4 - \frac{1}{p}\|w\|_p^p = M_b(\|w\|^2) - \frac{1}{p}\|w\|_p^p = 2d_b^*.\end{aligned} \tag{3.47}$$

Thus, by combining (3.45), (3.46) and (3.47), we obtain

$$c_{\mathcal{M}_b} < C_p^{\frac{2}{2-p}}\left(\frac{2^{\frac{2}{2-p}}p - 2^{\frac{p}{2-p}}4}{p-4}\right)2d_b^*. \tag{3.48}$$

Therefore, by (3.9) and (3.48), we obtain that (3.40) holds. ■

For the next result, consider the set $\widetilde{S}_\lambda^b = \{u \in \mathcal{M}_b \ : \ I_b(u) < c_{\mathcal{M}_b} + \lambda\}$ for $\lambda > 0$. As a consequence of Lemma 3.3.9, we will prove that:

Lemma 3.3.10 *Assume that (V_1), $(f_1)-(f_5)$ and (3.9) are satisfied. For $\lambda > 0$ small enough, there exists $\kappa = \kappa(\lambda) \in (0,1)$ such that*

$$0 < m_0 \le \|u^\pm\|^2 < \|u\|^2 \le \kappa,$$

for any $u \in \widetilde{S}_\lambda^b$.

Proof . From Lemma 3.3.9, we can choose $\lambda > 0$ such that $c_{\mathcal{M}_b} + \lambda < \frac{a(\theta-2)}{2\theta}$. Given $u \in \widetilde{S}_\lambda^b$, by Lemma 3.3.7 and by using $\langle u^+, u^-\rangle > 0$, we have $m_0 \le \|u^\pm\|^2 < \|u\|^2$. On the other hand, by (f_3) and since $I_b'(u)u = 0$, we obtain

$$\begin{aligned}c_{\mathcal{M}_b} + \lambda > I_b(u) &= I_b(u) - \frac{1}{\theta}I_b'(u)u \\ &= \frac{1}{2}M_b(\|u\|^2) - \frac{1}{\theta}m_b(\|u\|^2)\|u\|^2 + \frac{1}{\theta}\int_\Omega (f(u)u - \theta F(u))\,\mathrm{d}x \\ &\ge \frac{1}{2}M_b(\|u\|^2) - \frac{1}{\theta}m_b(\|u\|^2)\|u\|^2 \\ &= a\left(\frac{1}{2} - \frac{1}{\theta}\right)\|u\|^2 + b\left(\frac{1}{4} - \frac{1}{\theta}\right)\|u\|^4 \\ &\ge a\left(\frac{1}{2} - \frac{1}{\theta}\right)\|u\|^2.\end{aligned}$$

Consequently, it follows that

$$\|u\|^2 \le \frac{2\theta}{a(\theta-2)}(c_{\mathcal{M}_b} + \lambda) =: \kappa < 1,$$

for all $u \in \widetilde{S}_\lambda^b$, as desired. ■

From now on, we will write $\widetilde{S}_\lambda^b$ with λ given in Lemma 3.3.10.

Lemma 3.3.11 *Assume that* (V_1), $(f_1)-(f_5)$ *and* (3.9) *are satisfied. For any* $q \geq 1$, *there exists* $\delta_q > 0$ *such that*

$$0 < \delta_q \leq \int_\Omega |u^\pm|^q \mathrm{d}x < \int_\Omega |u|^q \mathrm{d}x,$$

for any $u \in \widetilde{S}^b_\lambda$.

Proof . By Lemma 3.3.7 and (f_1), for any $u \in \widetilde{S}^b_\lambda$, we have

$$0 < am_0 \leq a\|u^\pm\|^2 < m_b(\|u\|^2)\langle u, u^\pm\rangle = \int_\Omega f(u^\pm)u^\pm \mathrm{d}x \leq C_0 \int_\Omega |u^\pm| e^{\pi|u^\pm|^2} \mathrm{d}x.$$

Since $\kappa < 1$, we can choose $t' > 1$, with $\kappa t' < 1$ and $t > q$ such that $1/t' + 1/t = 1$. Now, by using the Hölder inequality, Lemma 3.3.10 and Lemma 3.2.2, we obtain that

$$\int_\Omega |u^\pm| e^{\pi|u^\pm|^2} \mathrm{d}x \leq \left(\int_\Omega |u^\pm|^t \mathrm{d}x\right)^{\frac{1}{t}} \left(\int_\Omega e^{\kappa t' \pi \left(\frac{u^\pm}{\|u^\pm\|}\right)^2} \mathrm{d}x\right)^{\frac{1}{t'}} \leq C' \|u^\pm\|_t.$$

Hence, for all $u \in \widetilde{S}^b_\lambda$, we get

$$0 < C \leq \|u^\pm\|_t. \tag{3.49}$$

We suppose, by contradiction, that there exists $(u_n) \subset \widetilde{S}^b_\lambda$ such that $\|u_n^\pm\|_q \to 0$ as $n \to \infty$. From Lemma 3.2.1 and Lemma 3.3.10, we obtain that $(u_n^\pm)$ is bounded in $L^{2t}(\Omega)$. Consequently, since $q < t < 2t$, by the interpolation inequality, we find that $\|u_n^\pm\|_t \to 0$ as $n \to \infty$, which is impossible in view of (3.49). ■

Lemma 3.3.12 *Assume that* (V_1) *and* $(f_1)-(f_5)$ *are satisfied. Let* (u_n) *be a sequence in* X *such that* $u_n \rightharpoonup u$ *weakly in* X *and* $B := \sup_{n\in\mathbb{N}} \|u_n\|^2 < 1$. *Then, for all* $v \in X$, *up to a subsequence, we have*

$$\lim_{n\to\infty} \int_\Omega f(u_n)u_n \mathrm{d}x = \int_\Omega f(u)u \mathrm{d}x; \tag{3.50}$$

$$\lim_{n\to\infty} \int_\Omega f(u_n)u_n^\pm \mathrm{d}x = \int_\Omega f(u)u^\pm \mathrm{d}x; \tag{3.51}$$

$$\lim_{n\to\infty} \int_\Omega f(u_n)v \mathrm{d}x = \int_\Omega f(u)v \mathrm{d}x \tag{3.52}$$

and

$$\lim_{n\to\infty} \int_\Omega F(u_n) \mathrm{d}x = \int_\Omega F(u) \mathrm{d}x. \tag{3.53}$$

Proof . Since $B < 1$, by using (f_1), the Hölder inequality and Lemma 3.2.1, it is easy to see that the integrals

$$\int_\Omega |f(u_n)u_n||u_n|\mathrm{d}x,\ \int_\Omega |f(u_n)u_n^{\pm}||u_n|\mathrm{d}x,\ \int_\Omega |f(u_n)v||u_n|\mathrm{d}x \quad \text{and} \quad \int_\Omega |F(u_n)|u_n|\mathrm{d}x$$

are uniformly bounded. Thus, by Lemma 3.2.1, the convergences (3.50)-(3.53) follow from Lemma 2.1 in [25]. ■

Lemma 3.3.13 *Assume that* $(f_1), (f_3)$ *and* (f_4) *are satisfied. Then the function* $H(t) = f(t)t - 4F(t)$ *satisfies*

(i) $H(0) = 0$ *and* $H(t) > 0$, *for all* $t \neq 0$;

(ii) $H(t)$ *is increasing for* $t > 0$ *and decreasing for* $t < 0$.

Proof . It is clear that the hypothesis (f_1) and (f_3) imply **(i)**. To get **(ii)**, it is enough to analyze the derivative of H together with the assumptions (f_1) and (f_4). ■

Next we will present a technical lemma that will be crucial in the proof of Theorem 3.1.2.

Lemma 3.3.14 *Assume that* $(V_1), (f_1), (f_3)$ *and* (f_4) *are satisfied. Then, for any* $u \in \mathcal{M}_b$, *we have*

$$\det J_{(1,1)}\Psi_u > 0,$$

where $J_{(1,1)}\Psi_u$ *is the Jacobian matrix of* Ψ_u *at the point* $(1, 1)$.

Proof . Let $\Psi_u^1(t, s) = I_b'(tu^+ + su^-)tu^+$ and $\Psi_u^2(t, s) = I_b'(tu^+ + su^-)su^-$ the coordinates functions of $\Psi_u(t, s)$, where Ψ_u is defined in (3.27). Calculating the partial derivatives of Ψ_u^1 and Ψ_u^2 at the point $(1, 1)$ and by using Lemma 3.2.3, we get

$$\begin{cases} \dfrac{\partial\Psi_u^1}{\partial t}(1,1) = 2m_b'(\|u\|^2)\langle u, u^+\rangle^2 + m_b(\|u\|^2)(2\|u^+\|^2 + \langle u^+, u^-\rangle) \\ \qquad\qquad - \displaystyle\int_\Omega f'(u^+)(u^+)^2 + f(u^+)u^+\mathrm{d}x; \\ \dfrac{\partial\Psi_u^1}{\partial s}(1,1) = \dfrac{\partial\Psi_u^2}{\partial t}(1,1) = 2m_b(\|u\|^2)\langle u, u^+\rangle\langle u, u^-\rangle + m_b(\|u\|^2)\langle u^+, u^-\rangle > 0; \\ \dfrac{\partial\Psi_u^2}{\partial s}(1,1) = 2m_b'(\|u\|^2)\langle u, u^-\rangle^2 + m_b(\|u\|^2)(2\|u^-\|^2 + \langle u^+, u^-\rangle) \\ \qquad\qquad - \displaystyle\int_\Omega f'(u^-)(u^-)^2 + f(u^-)u^-\mathrm{d}x. \end{cases}$$

Note that we can rewrite

$$\frac{\partial \Psi_u^1}{\partial t}(1,1) = 2m_b'(\|u\|^2)\|u\|^2\langle u,u^+\rangle - 2m_b'(\|u\|^2)\langle u,u^+\rangle\langle u,u^-\rangle$$
$$+ 2m_b(\|u\|^2)\langle u,u^+\rangle - m_b(\|u\|^2)\langle u^+,u^-\rangle - \int_\Omega f'(u^+)(u^+)^2 + f(u^+)u^+\mathrm{d}x.$$

Thus, by using (3.17), we have

$$\begin{aligned}\frac{\partial \Psi_u^1}{\partial t}(1,1) < 4m_b(\|u\|^2)\langle u,u^+\rangle - 2m_b'(\|u\|^2)\langle u,u^+\rangle\langle u,u^-\rangle \\ - m_b(\|u\|^2)\langle u^+,u^-\rangle - \int_\Omega f'(u^+)(u^+)^2 + f(u^+)u^+\mathrm{d}x.\end{aligned} \tag{3.54}$$

On the other hand, since $I_b'(u)u^+ = 0$, we have

$$m_b(\|u\|^2)\langle u,u^+\rangle = \int_\Omega f(u^+)u^+\mathrm{d}x. \tag{3.55}$$

Combining (3.54) and (3.55), we get

$$\frac{\partial \Psi_u^1}{\partial t}(1,1) < -2m_b'(\|u\|^2)\langle u,u^+\rangle\langle u,u^-\rangle - m_b(\|u\|^2)\langle u^+,u^-\rangle) - \int_\Omega H'(u^+)u^+\mathrm{d}x.$$

By the item **(ii)** of Lemma 3.3.13, we have $\int_\Omega H'(u^+)u^+\mathrm{d}x \geq 0$. Hence, we deduce that

$$\frac{\partial \Psi_u^1}{\partial t}(1,1) < -2m_b'(\|u\|^2)\langle u,u^+\rangle\langle u,u^-\rangle - m_b(\|u\|^2)\langle u^+,u^-\rangle = -\frac{\partial \Psi_u^1}{\partial s}(1,1) < 0. \tag{3.56}$$

Similarly, we can show that

$$\frac{\partial \Psi_u^2}{\partial s}(1,1) < -\frac{\partial \Psi_u^1}{\partial s}(1,1) < 0. \tag{3.57}$$

Hence, by (3.56) and (3.57), we have

$$\begin{aligned}\det J_{(1,1)}\Psi_u &= \frac{\partial \Psi_u^1}{\partial t}(1,1)\frac{\partial \Psi_u^2}{\partial s}(1,1) - \frac{\partial \Psi_u^1}{\partial s}(1,1)\frac{\partial \Psi_u^2}{\partial t}(1,1) \\ &= \frac{\partial \Psi_u^1}{\partial t}(1,1)\frac{\partial \Psi_u^2}{\partial s}(1,1) - \left(\frac{\partial \Psi_u^1}{\partial s}(1,1)\right)^2 \\ &> \left(\frac{\partial \Psi_u^1}{\partial s}(1,1)\right)^2 - \left(\frac{\partial \Psi_u^1}{\partial s}(1,1)\right)^2 = 0,\end{aligned}$$

as desired. ∎

Now, we have all the results that will allow us to prove that the nodal level $c_{\mathcal{M}_b}$ is attained in a function with $u \in \mathcal{M}_b$.

Lemma 3.3.15 *Assume that* (V_1), $(f_1) - (f_5)$ *and* (3.9) *are satisfied. Then, there exists* $u_b \in \mathcal{M}_b$ *such that* $I_b(u_b) = c_{\mathcal{M}_b}$.

Proof . Let $(u_n) \subset \widetilde{S}_\lambda^b$ be a sequence such that $I_b(u_n) \to c_{\mathcal{M}_b}$ as $n \to \infty$. By Lemma 3.3.10, we have that

$$0 < m_0 \leq \|u_n^\pm\|^2 < \|u_n\|^2 \leq \kappa < 1, \text{ for all } n \in \mathbb{N}.$$

Then, we can assume, without loss of generality, that the convergences in Lemma 3.3.12 hold for the sequence (u_n). Since X is a Hilbert space, there exists $u \in X$ such that $u_n^\pm \rightharpoonup u^\pm$ as $n \to +\infty$. By Lemma 3.2.1, up to a subsequence, we have $u_n^\pm \to u^\pm$ in $L^q(\mathbb{R})$ and $u_n^\pm(x) \to u(x)^\pm$ a.e. in $\mathbb{R}$, with $q \geq 1$ (see Lemma A.1.8). Now from Lemma 3.3.11, we can deduce that $u^+ \neq 0$ and $u^- \neq 0$ in X. Note that by Lemma 3.3.4, there exist $t, s > 0$ such that $u_b := tu^+ + su^- \in \mathcal{M}_b$. We claim that $0 < t, s \leq 1$. In order to prove this, by Lemma 3.3.6, it is sufficient to show that $I_b'(u)u^+ \leq 0$ and $I_b'(u)u^- \leq 0$. Indeed, by using Lemma 3.2.3, the Fatou's Lemma and by lower semicontinuity of the norm, we get

$$0 < \langle u, u^+\rangle \leq \liminf_{n\to\infty} \langle u_n, u_n^+\rangle \quad \text{and} \quad 0 < \|u\|^2 \leq \liminf_{n\to\infty} \|u_n\|^2.$$

Thus, by using the properties of lim inf, we get

$$\begin{aligned} m_b(\|u\|^2)\langle u, u^+\rangle &\leq \liminf_{n\to\infty} a\langle u_n, u_n^+\rangle + \liminf_{n\to\infty} b\|u_n\|^2\langle u_n, u_n^+\rangle \\ &\leq \liminf_{n\to\infty} \left(a\langle u_n, u_n^+\rangle + b\|u_n\|^2\langle u_n, u_n^+\rangle\right) = \liminf_{n\to\infty} m_b(\|u_n\|^2)\langle u_n, u_n^+\rangle. \end{aligned} \tag{3.58}$$

On the other hand, by using $I_b'(u_n)u_n^+ = 0$ and (3.51), we have

$$\lim_{n\to\infty} m_b(\|u_n\|^2)\langle u_n, u_n^+\rangle = \lim_{n\to\infty} \int_\Omega f(u_n^+)u_n^+ \mathrm{d}x = \int_\Omega f(u^+)u^+ \mathrm{d}x. \tag{3.59}$$

From (3.58) and (3.59) we deduce that $I_b'(u)u^+ \leq 0$, and similarly we can prove $I_b'(u)u^- \leq 0$. Therefore, $0 < t, s \leq 1$ and hence the claim is proved. Now, by using that $\|u_b\|^2 \leq \|u\|^2$ and again to lower semicontinuity of the norm, we have

$$\frac{1}{2}M_b(\|u_b\|^2) - \frac{1}{4}m_b(\|u_b\|^2)\|u_b\|^2 = a\left(\frac{1}{2} - \frac{1}{4}\right)\|u\|^2 \leq \liminf_{n\to\infty} a\left(\frac{1}{2} - \frac{1}{4}\right)\|u_n\|^2. \tag{3.60}$$

On the other hand, by using Lemma 3.3.13 and Lemma 3.3.12, we have

$$\begin{aligned} \frac{1}{4}\int_\Omega H(u_b)\,\mathrm{d}x &= \frac{1}{4}\int_\Omega H(tu^+)\,\mathrm{d}x + \frac{1}{4}\int_\Omega H(su^-)\,\mathrm{d}x \\ &\leq \frac{1}{4}\int_\Omega H(u^+)\,\mathrm{d}x + \frac{1}{4}\int_\Omega H(u^-)\,\mathrm{d}x \\ &= \frac{1}{4}\int_\Omega H(u)\,\mathrm{d}x = \lim_{n\to\infty} \frac{1}{4}\int_\Omega f(u_n)u_n - 4F(u_n)\,\mathrm{d}x \\ &= \frac{1}{4}\int_\Omega f(u)u - 4F(u)\,\mathrm{d}x. \end{aligned} \tag{3.61}$$

Thus, by (3.60) and (3.61), we deduce that

$$\begin{aligned}
c_{\mathcal{M}_b} \leq\ & I_b(u_b) = I_b(u_b) - \frac{1}{4}I_b'(u_b)u_b \\
=\ & \frac{1}{2}M_b(\|u_b\|^2) - \frac{1}{4}m_b(\|u_b\|^2)\|u_b\|^2 + \frac{1}{4}\int_\Omega H(u_b)\,\mathrm{d}x \\
\leq\ & \liminf_{n\to\infty} a\left(\frac{1}{2}-\frac{1}{4}\right)\|u_n\|^2 + \liminf_{n\to\infty}\frac{1}{4}\int_\Omega f(u_n)u_n - 4F(u_n)\,\mathrm{d}x \\
\leq\ & \liminf_{n\to\infty}\left(\frac{1}{2}M_b(\|u_n\|^2) - \frac{1}{4}m_b(\|u_n\|^2)\|u_n\|^2 + \frac{1}{4}\int_\Omega f(u_n)u_n - 4F(u_n)\,\mathrm{d}x\right) \\
=\ & \liminf_{n\to\infty}\left(I_b(u_n) - \frac{1}{4}I_b'(u_n)u_n\right) = c_{\mathcal{M}_b}.
\end{aligned}$$

Therefore, we get that $I_b(u_b) = c_{\mathcal{M}_b}$, which is the desired conclusion. ■

Next we will introduce some notations and a technical result that will be apply in the proof of Theorem 3.1.2.

Let $D = \left(\frac{1}{2}, \frac{3}{2}\right) \times \left(\frac{1}{2}, \frac{3}{2}\right)$ and $g : \overline{D} \to X$, given by $g(t,s) = tu_b^+ + su_b^-$, where u_b is given in Lemma 3.3.15. Then, as in Lemma 1.4.1, following basic result:

Lemma 3.3.16 *Let $P = \{u \in X : u(x) \geq 0$ a.e. $x \in \mathbb{R}\}$ and $-P = \{u \in X : u(x) \leq 0$ a.e. $x \in \mathbb{R}\}$. Then $d' = dist(g(\overline{D}), \Lambda) > 0$, where $\Lambda := P \cup (-P)$.*

3.3.1 Proof of Theorem 3.1.2

From Lemma 3.3.15, it remains to show that u_b is a critical point of I_b. Suppose, by contradiction, that $I_b'(u_b) \neq 0$. Thus, by the continuity of I_b', there exist $\gamma, \delta > 0$ with $\delta \leq \frac{d'}{2}$, such that

$$\|I_b'(v)\| \geq \gamma, \text{ for all } v \in B_{3\delta}(u_b), \tag{3.62}$$

where d' is given in Lemma 3.3.16. Since $u_b \in \mathcal{M}_b$, by using Lemma 3.3.4, the function $(I_b \circ g)(t,s)$, for $(t,s) \in \overline{D}$, has a strict maximum point $(1,1)$. Thus, we get

$$n_b^* = \max_{(t,s)\in\partial D}(I_b \circ g)(t,s) < c_{\mathcal{M}_b}.$$

Let $\varepsilon > 0$ be such that $\varepsilon < \min\{(c_{\mathcal{M}_b} - n_b^*)/2, \gamma\delta/8\}$ and we define $S = B_\delta(u_b)$. From this choice, for $(t,s) \in \partial D$, we have

$$(I_b \circ g)(t,s) \leq n_b^* = c_{\mathcal{M}_b} - 2(c_{\mathcal{M}_b} - n_b^*)/2 < c_{\mathcal{M}_b} - 2\varepsilon.$$

Hence, we deduce that

$$g(\partial D) \cap I_b^{-1}([c_{\mathcal{M}_b} - 2\varepsilon, c_{\mathcal{M}_b} + 2\varepsilon]) = \emptyset \tag{3.63}$$

and, by estimates in (3.62),

$$\|I_b'(v)\| \geq \frac{8\varepsilon}{\delta};\ \forall v \in I_b^{-1}([c_{\mathcal{M}_b} - 2\varepsilon, c_{\mathcal{M}_b} + 2\varepsilon]) \cap S_{2\delta}.$$

Thus, by the quantitative deformation lemma in [72, Lemma 2.3], there exists $\eta \in C([0,1] \times X, X)$ such that

(i) $\eta(t,u) = u$, if $t = 0$ or $u \notin I_b^{-1}([c_{\mathcal{M}_b} - 2\varepsilon, c_{\mathcal{M}_b} + 2\varepsilon]) \cap S_{2\delta}$;

(ii) $\eta(1, I_b^{c_{\mathcal{M}_b}+\varepsilon} \cap S) \subset I_b^{c_{\mathcal{M}_b}-\varepsilon}$;

(iii) $\eta(t,\cdot)$ is a homeomorphism of X, $\forall\, t \in [0,1]$;

(iv) $\|\eta(t,u) - u\| \leq \delta$, $\forall\, u \in X$, $\forall\, t \in [0,1]$;

(v) $I_b(\eta(\cdot,u))$ is non increasing, $\forall\, u \in X$;

(vi) $I_b(\eta(t,u)) < c_{\mathcal{M}_b}$, $\forall\, u \in I_b^{c_{\mathcal{M}_b}} \cap S_\delta$, $\forall\, t \in (0,1]$.

We claim that

$$\max_{(t,s)\in\overline{D}} I_b(\eta(1, g(t,s))) < c_{\mathcal{M}_b}. \tag{3.64}$$

Indeed, if $(t,s) \in D$ with $(t,s) \neq (1,1)$, by using Lemma 3.3.4 we have $I_b(g(t,s)) < c_{\mathcal{M}_b}$. Hence

$$I_b(\eta(1, g(t,s))) \leq I(\eta(0, g(t,s))) = I(g(t,s)) < c_{\mathcal{M}_b}.$$

If $(t,s) = (1,1)$ then $g(1,1) = u_b \in I_b^{c_{\mathcal{M}_b}} \cap S_\delta$ and so $I_b(\eta(1, g(1,1))) < c_{\mathcal{M}_b}$, showing (3.64).

Now, by the definition of $c_{\mathcal{M}_b}$ and (3.64), we get

$$\eta(1, g(\overline{D})) \cap \mathcal{M}_b = \emptyset. \tag{3.65}$$

Let us consider $h : \overline{D} \to X$, given by $h(t,s) = \eta(1, g(t,s))$. Using (3.63) and the properties of η, we get

$$h(t,s) = g(t,s) \quad \text{in} \quad \partial D. \tag{3.66}$$

Claim 3.3.17 *We claim that $h(t,s)^+ \neq 0$ and $h(t,s)^- \neq 0$, for all $(t,s) \in \overline{D}$.*

Indeed, let $v \in \Lambda$. By the choice of $\delta > 0$ and Lemma 3.3.16, we have that

$$\begin{aligned}\|h(t,s)-v\| &\geq \|g(t,s)-v\| - \|h(t,s)-g(t,s)\| \\ &\geq \|g(t,s)-v\| - \delta \\ &\geq d' - \frac{d'}{2} = \frac{d'}{2}.\end{aligned}$$

Thus, $h(t,s)^+ \neq 0$ and $h(t,s)^- \neq 0$ for all $(t,s) \in \overline{D}$, concluding the statement.

Now, let us consider the vector fields $\Psi_{u_b}, \mathcal{F} : \overline{D} \to \mathbb{R}^2$, where Ψ_{u_b} is given in (3.27) and

$$\mathcal{F}(t,s) = (I_b'(h(t,s))h(t,s)^+, I_b'(h(t,s))h(t,s)^-).$$

From (3.66), we have $\Psi_{u_b} = \mathcal{F}$ in ∂D. Hence, by the degree theory (see Lemma A.1.14), we get

$$\deg(\Psi_{u_b}, D, (0,0)) = \deg(\mathcal{F}, D, (0,0)). \tag{3.67}$$

But, by using again Lemma 3.3.4, we have that the point $(1,1)$ is a unique point in $\overline{D}$ such that $\Psi_{u_b}(t,s) = (0,0)$. Consequently, again by the degree theory (see Lemma A.1.15) and Lemma 3.3.14, we can deduce that

$$\deg(\Psi_{u_b}, D, (0,0)) = sgn(J_{(1,1)}\Psi_{u_b}) = 1. \tag{3.68}$$

Then, by (3.67), we get

$$\deg(\mathcal{F}, D, (0,0)) = 1.$$

Thus, by degree theory (see Lemma A.1.13), there exists a point $(t_0, s_0) \in D$ such that

$$I_b'(h(t_0,s_0))h(t_0,s_0)^+ = 0 \quad \text{and} \quad I_b'(h(t_0,s_0))h(t_0,s_0)^- = 0. \tag{3.69}$$

By Claim 3.3.17 we have that $h(t_0,s_0)^+ \neq 0$ and $h(t_0,s_0)^+ \neq 0$. Hence, (3.69) implies that $h(t_0,s_0)$ belongs to $\eta(1, g(\overline{D})) \cap \mathcal{M}_b$, which is a contradiction in view of (3.65), and the proof is complete.

3.4 Nonpositive solution and nonnegative solution of $(P_{a,b})$

First, we define the functionals $I_b^+ : X \to \mathbb{R}$ and $I_b^- : X \to \mathbb{R}$ by

$$I_b^{\pm}(u) = \frac{1}{2}M_b(\|u\|^2) - \int_\Omega F(u^{\pm})\,\mathrm{d}x. \tag{3.70}$$

By using the assumptions on f, we have that $I_b^\pm \in C^1(X,\mathbb{R})$ and, for any $u, v \in X$, one has

$$(I_b^\pm)'(u)v = m_b(\|u\|^2)\langle u, v\rangle - \int_\Omega f(u^\pm)v\,\mathrm{d}x. \tag{3.71}$$

Note that considering functions such that $u = u^+$ and $v = v^-$, we have

$$I_b^+(u) = I_b(u),\ I_b^-(v) = I_b(v),\ (I_b^+)'(u) = I_b'(u) \quad \text{and} \quad (I_b^-)'(v) = I_b'(v),$$

that is, the functionals $I_b^\pm$, and their derivatives, coincide with I_b, and their derivatives, in P and $-P$, respectively, where P is defined in Lemma 3.3.16. If $u \in X \setminus \{0\}$ is a critical point of I_b^+ then, taking u^- as a test function in (3.71), we deduce that

$$0 = (I_b^+)'(u)u^- = m_b(\|u\|^2)\langle u, u^-\rangle.$$

This implies $\langle u, u^-\rangle = 0$ and from Lemma 3.2.3 we obtain that $u^- = 0$. Therefore, nontrivial critical points u of I_b^+ are nonnegative solutions of ($P_{a,b}$) and, in particular, $u = u^+ \in \mathcal{N}_b^+$. Analogously, nontrivial critical points u of I_b^- are nonpositive solutions of ($P_{a,b}$).

The first result in this section proves that the functionals $I_b^\pm$ have the mountain pass geometry.

Lemma 3.4.1 *Assume that (V_1) and $(f_1) - (f_3)$ are satisfied. Then, the functionals $I_b^\pm$ have the following geometric properties:*

(i) *there exist $r > 0$ and $\tau > 0$ such that $I_b^\pm(u) \geq \tau$, for $\|u\| = r$;*

(ii) *there exists $e \in X$, with $\|e\| > r$, $I_b^\pm(e) < 0$.*

Proof . It is similar to Lemma 3.3.2 and we will omit it. ■

Let us consider the sets

$$\Gamma_b := \{\gamma \in C([0,1], X) : \gamma(0) = 0 \quad \text{and} \quad I_b(\gamma(1)) < 0\},$$

$$\Gamma_b^\pm := \{\gamma \in C([0,1], X) : \gamma(0) = 0 \quad \text{and} \quad I_b^\pm(\gamma(1)) < 0\}$$

and the respective minimax levels

$$c_b = \inf_{\gamma\in\Gamma_b} \max_{t\in[0,1]} I_b(\gamma(t)) \quad \text{and} \quad c_b^\pm = \inf_{\gamma\in\Gamma_b^\pm} \max_{t\in[0,1]} I_b^\pm(\gamma(t)).$$

As an application of Lemma 3.3.2, Lemma 3.4.1 and by the Mountain Pass Theorem, we obtain the following corollary:

Corollary 3.4.2 *Assume that* (V_1) *and* $(f_1)-(f_3)$ *are satisfied. There exist sequences* $(u_n), (u_{n,\pm}) \subset X$ *such that* (u_n) *is a* $(PS)_{c_b}$ *sequence for* I_b *and* $(u_{n,\pm})$ *are* $(PS)_{c_b^\pm}$ *sequences for* $I_b^\pm$.

The next lemma we will show that the minimax levels defined above c_b and $c_b^\pm$ are less or equal to the respective Nehari level in (3.6), (3.7) and (3.8) of functional I_b. This will be our main tool to show that the functionals I_b and $I_b^\pm$ satisfy the Palais-Smale condition at the levels c_b and $c_b^\pm$, respectively.

Lemma 3.4.3 *Assume that* (V_1) *and* $(f_1)-(f_4)$ *are satisfied. The following inequalties hold*

$$c_b \le c_{\mathcal{N}_b},\ c_b^+ \le c_{\mathcal{N}_b^+} \quad \textit{and} \quad c_b^- \le c_{\mathcal{N}_b^-}.$$

Proof . We will only show the inequality $c_b^+ \le c_{\mathcal{N}_b^+}$ (the proof of the other ones are similar). First, by using Lemma 3.3.1, we get that

$$c_{\mathcal{N}_b^+} = \inf_{u=u^+\neq 0} \max_{t\ge 0} I_b(tu^+).$$

Let $u = u^+ \neq 0$. By Lemma 3.3.2, we have $I_b(su^+) \to -\infty$ as $s \to \infty$. Thus there exists $C_{u^+} > 0$ large enough such that $I_b(su^+) < 0$, for all $s \ge C_{u^+}$. Now we consider the family of curves $\gamma_{u^+}^s : [0,1] \to X$, given by $\gamma_{u^+}^s(t) = stu^+$, for $s \ge C_{u^+}$. For any $u \in P \setminus \{0\}$, the family of curves so defined is such that $\{\gamma_{u^+}^s\}_{s\ge C_{u^+}} \subset \Gamma_b^+$. Thus, we have that

$$\begin{aligned} c_b^+ = \inf_{\gamma\in\Gamma_b^+} \max_{t\in[0,1]} I_b^+(\gamma(t)) &\le \inf_{\substack{\{\gamma_{u^+}^s\}_{s\ge C_{u^+}} \\ u=u^+\neq 0}} \max_{t\in[0,1]} I_b^+(\gamma_{u^+}^s(t)) \\ &\le \inf_{u=u^+\neq 0} \max_{t\ge 0} I_b(tu^+) = c_{\mathcal{N}_b^+} \end{aligned}$$

and so we finish the proof of the Lemma. ■

Remark 3.4.4 *From Remark 3.3.5, Theorem 3.1.2 and Lemma 3.4.3, it follows that*

$$c_b^+ + c_b^- \le c_{\mathcal{N}_b^+} + c_{\mathcal{N}_b^-} < c_{\mathcal{M}_b}. \tag{3.72}$$

Moreover, if u *is a critical point of* I_b^+ *such that* $I_b^+(u) = c_b^+$, *then* $u \in \mathcal{N}_b^+$ *is a nonnegative solution of* $(P_{a,b})$. *Thus, we have* $c_{\mathcal{N}_b^+} \le I_b(u) = I_b^+(u) = c_b^+$. *Therefore, we deduce that*

$$c_b^+ = c_{\mathcal{N}_b^+}. \tag{3.73}$$

Similarly, we have that $c_b = c_{\mathcal{N}_b}$ *and* $c_b^- = c_{\mathcal{N}_b^-}$.

3.4.1 Proof of Theorem 3.1.3

We will only show the existence of one nonnegative and nonzero solution (the case of one nonpositive and nonzero solution is similar and we will omit it). From Corollary 3.4.2, there exists $(u_n) \subset X$ such that $I_b^+(u_n) \to c_b^+$ and $(I_b^+)'(u_n) \to 0$ as $n \to +\infty$. Now, by (f_3), we have

$$\begin{aligned}
c_b^+ + o_n(1)\|u_n\| =& I_b^+(u_n) - \frac{1}{\theta}(I_b^+)'(u_n)u_n \\
=& \frac{1}{2}M_b(\|u_n\|^2) - \frac{1}{\theta}m_b(\|u_n\|^2)\|u_n\| + \frac{1}{\theta}\int_\Omega [f(u_n^+)u_n^+ - \theta F(u_n^+)]\,\mathrm{d}x \\
\geq& a\left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2
\end{aligned}$$

and so, it follows that (u_n) is bounded in X. Let $C > 0$ such that $\|u_n\| \leq C$, for all $n \in \mathbb{N}$. Let $n_0 \in \mathbb{N}$ such that $\|(I_b^+)'(u_n)\|_*\|u_n\| \leq o_n(1)C < \lambda$, for all $n \geq n_0$, where $\lambda > 0$ is given in Lemma 3.3.10. Thus, by the estimates above and (3.72), we get

$$\|u_n\|^2 < \frac{2\theta}{a(\theta-2)}(c_{\mathcal{M}_b} + \lambda) = \kappa < 1, \text{ for all } n \geq n_0.$$

Without loss of generality, we can assume that $\|u_n\|^2 \leq \kappa < 1$, for all $n \in \mathbb{N}$. Since X is a Hilbert space, there exists $u_+ \in X$ such that $u_n \rightharpoonup u_+$ in X as $n \to \infty$. By Lemma 3.2.1, up to a subsequence, we have $u_n \to u_+$ in $L^q(\mathbb{R})$, for all $q \geq 1$, $u_n(x) \to u_+(x)$ a.e. in $\mathbb{R}$ as $n \to \infty$. Moreover, we can also assume that the convergences in Lemma 3.3.12 hold for the sequence (u_n).

Now, since $M_b(t) = at + bt^2/2$ is a increasing function and by the lower semicontinuity of the norm, we have

$$\frac{1}{2}M_b(\|u_+\|^2) \leq \liminf_{n\to\infty} \frac{1}{2}M_b(\|u_n\|^2). \tag{3.74}$$

On the other hand, we have M_b is a convex function and so, by using properties of derivative of convex functions (see Lemma A.1.2), for any $n \in \mathbb{N}$, we get

$$\frac{1}{2}\left(M_b(\|u_+\|^2) - M_b(\|u_n\|^2)\right) \geq \frac{1}{2}M_b'(\|u_n\|^2)(u_+ - u_n).$$

By using (3.51), (3.52) and by inequality above, we have

$$\begin{aligned}
\frac{1}{2}\left(M_b(\|u_+\|^2) - M_b(\|u_n\|^2)\right) \geq& m_b(\|u_n\|^2)\langle u_n, u_+ - u_n\rangle \\
=& (I_b^+)'(u_n)(u_+ - u_n) + \int_\Omega f(u_n^+)(u_+ - u_n)\,\mathrm{d}x = o_n(1),
\end{aligned}$$

and so $o_n(1) + M_b(\|u_n\|^2) \leq M_b(\|u_+\|^2)$, and consequently, we get

$$\liminf_{n\to\infty} \frac{1}{2}M_b(\|u_n\|^2) \leq \frac{1}{2}M_b(\|u_+\|^2). \tag{3.75}$$

Hence, by (3.74) and (3.75), up to a subsequence, we have $M_b(\|u_n\|^2) \to M_b(\|u_+\|^2)$ as $n \to \infty$. Since M_b is a increasing function, we deduce that $\|u_n\| \to \|u_+\|$ as $n \to \infty$. Thus, as X is a Hilbert space, we have $u_n \to u_+$ strongly in X as $n \to \infty$. Therefore, u_+ is a critical point of I_b^+ and $I_b^+(u_+) = c_b^+ > 0$. Consequently, by Remark 3.4.4, we have $u_+ \in \mathcal{N}_b^+$ and $c_b^+ = c_{\mathcal{N}_b^+}$, and the proof is complete.

3.5 The asymptotic behavior of the nodal solutions

We started this section by proving some facts related to functional J_b and the nodal level d_b^*, which are defined in (3.10) and (3.12). Given $b' \geq 0$, we consider $M_{b'}(t) = at + b't^2/2$ and $J_{b'} : X \to \mathbb{R}$ defined by

$$J_{b'}(u) = \frac{1}{2}M_{b'}(\|u\|^2) - \frac{1}{2p}\int_\Omega |u|^p \mathrm{d}x \quad \text{and} \quad d_{b'}^* = \inf_{u\in\mathcal{M}_{b'}^p} J_{b'}(u),$$

where

$$\mathcal{M}_{b'}^p = \{u \in X : u^+ \neq 0,\ u^- \neq 0\ (J_{b'})'(u)u^+ = 0 \quad \text{and} \quad (J_{b'})'(u)u^- = 0\}.$$

Note that $J_{b'}$ is the energy functional of the problem

$$\begin{cases} m_{b'}(\|u\|^2)\left[(-\Delta)^{1/2}u + V(x)u\right] = \frac{1}{2}|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}\setminus\Omega, \end{cases} \tag{$\widetilde{P}_{a,b'}$}$$

where $m_{b'}(t) = (M_{b'})'(t)$. Note that $g(t) = |t|^{p-2}t/2$ satisfies the assumptions (f_1)–(f_5). Thus, the results from the previous sections are also valid for the functional $J_{b'}$.

First, we will show that the nodal levels associated to $(\widetilde{P}_{a,b'})$ are strictly increasing with relation to the constant b'.

Lemma 3.5.1 *If $0 \leq b_1 < b_2$, then $d_{b_1}^* < d_{b_2}^*$.*

Proof . Since $b_2 > b_1$, we have $M_{b_2}(t) > M_{b_1}(t)$, for all $t \neq 0$. Thus, for any $u \in X\backslash\{0\}$, we get

$$J_{b_2}(u) > J_{b_1}(u). \tag{3.76}$$

Let $u \in \mathcal{M}^p_{b_2}$ such that $J_{b_2}(u) = d^*_{b_2}$ (the existence of u is established in Theorem B.3.1, in the Appendix, see also Remark B.3.3). By Lemma 3.3.4, there exists $(t_1, s_1) \in (0,\infty) \times (0,\infty)$ such that $t_1u^+ + s_1u^- \in \mathcal{M}^p_{b_1}$. By applying again Lemma 3.3.4 and (3.76), we get

$$d^*_{b_2} = J_{b_2}(u) = J_{b_2}(1 \cdot u^+ + 1 \cdot u^-) \geq J_{b_2}(t_1u^+ + s_1u^-) > J_{b_1}(t_1u^+ + s_1u^-) \geq d^*_{b_1},$$

as desired. ■

Remark 3.5.2 *Note that by* (3.9) *and Lemma 3.5.1, for* $0 \leq b' \leq b$, *we have* $d^*_{b'} \leq d^*_b$ *and, consequently*

$$C_p > \max\left\{\frac{1}{2}, \left[\frac{4\theta d^*_{b'}(2^{\frac{2}{2-p}}\, p - 2^{\frac{p}{2-p}}\, 4)}{a(\theta-2)(p-4)}\right]^{(p-2)/2}\right\}.$$

Then, by applying Theorem 3.1.2, the problem

$$\begin{cases} m_{b'}(\|u\|^2)\left[(-\Delta)^{1/2}u + V(x)u\right] = f(u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}\setminus\Omega, \end{cases}$$

has a least energy nodal solution, which we will denote by $u_{b'}$. *Similar to Lemma 3.5.1, we can prove that the nodal level of the functional*

$$I_{b'}(u) = \frac{1}{2}M_{b'}(\|u\|^2) - \int_\Omega F(u)\mathrm{d}x$$

satisfies $c_{\mathcal{M}_{b_1}} \leq c_{\mathcal{M}_{b_2}}$, *whenever* $b_1 < b_2$.

3.5.1 Proof of Theorem 3.1.4

Let $(b_n) \subset [0, b]$ be a sequence such that $b_n \to 0^+$ as $n \to \infty$. Let $u_{b_n} \in \mathcal{M}_{b_n}$ be the respective least energy nodal solution of the problem (P_{a,b_n}). By Remark 3.5.2, for all $n \in \mathbb{N}$, we have $I_{b_n}(u_{b_n}) = c_{\mathcal{M}_{b_n}} \leq c_{\mathcal{M}_b}$.

Claim 3.5.3 *The sequence* $(u_{b_n}) \subset X$ *satisfies*

$$0 < m_0 \leq \|u^{\pm}_{b_n}\|^2 < \|u_{b_n}\|^2 \leq \kappa < 1. \tag{3.77}$$

Indeed, the lower estimate is similar to Lemma 3.3.7. To obtain the upper bound, by considering $\lambda > 0$ as in Lemma 3.3.10 and by (f_3), we have

$$\begin{aligned} \frac{a(\theta-2)}{2\theta} > c_{\mathcal{M}_b} + \lambda > c_{\mathcal{M}_{b_n}} &= I_{b_n}(u_{b_n}) - \frac{1}{\theta}I'_{b_n}(u_{b_n})u_{b_n} \\ &\geq \frac{1}{2}M_{b_n}(\|u_{b_n}\|^2) - \frac{1}{\theta}m_{b_n}(\|u_{b_n}\|^2)\|u_{b_n}\|^2 \\ &\geq a\left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_{b_n}\|^2. \end{aligned}$$

This implies that $\|u_{b_n}\|^2 \leq \kappa < 1$, as desired.

Therefore, since X is a Hilbert space, up to a subsequence, there exists $u_0 \in X$ such that $u_{b_n} \rightharpoonup u_0$ in X, as $n \to \infty$. By Lemma 3.2.1, without loss of generality, we can assume that $u_{b_n}^+ \to u_0^+$ and $u_{b_n}^- \to u_0^-$ in $L^q(\mathbb{R})$, for all $q > 1$, and $u_{b_n}^+(x) \to u_0^+(x)$ and $u_{b_n}^-(x) \to u_0^-(x)$ a.e. in $\mathbb{R}$, as $n \to \infty$. By (3.77) and Lemma 3.3.11, we can deduce that $u_0^+ \neq 0$ and $u_0^- \neq 0$ in X. Moreover, using (3.77), we can also assume that the convergences in Lemma 3.3.12 hold for the sequence (u_{b_n}).

We claim that u_0 is a nodal solution of the problem $(P_{a,0})$. Indeed, given $v \in X$, we must prove that

$$a\langle u_0, v\rangle = \int_\Omega f(u_0)v \,\mathrm{d}x. \tag{3.78}$$

Since u_{b_n} is a weak solution of (P_{a,b_n}), we have

$$m_{b_n}(\|u_{b_n}\|^2)\langle u_{b_n}, v\rangle = \int_\Omega f(u_{b_n})v \,\mathrm{d}x, \quad \text{for all} \quad n \in \mathbb{N}.$$

By using (u_{b_n}) is bounded, $u_{b_n} \rightharpoonup u_0$ in X and $b_n \to 0^+$, we get $m_{b_n}(\|u_{b_n}\|^2) \to a$ and $\langle u_{b_n}, v\rangle \to \langle u_0, v\rangle$ as $n \to \infty$. Thus, by (3.52), we have

$$a\langle u_0, v\rangle = \lim_{n\to\infty} m_{b_n}(\|u_{b_n}\|^2)\langle u_{b_n}, v\rangle = \lim_{n\to\infty} \int_\Omega f(u_{b_n})v \,\mathrm{d}x = \int_\Omega f(u_0)v \,\mathrm{d}x,$$

showing (3.78) and so $u_0 \in \mathcal{M}_0$ is a nodal solution of $(P_{a,0})$.

Next we will show that (u_{b_n}) converge strongly to u_0 in X. First, we have that

$$I'_{b_n}(u_0)(u_{b_n} - u_0) \to 0 \quad \text{as} \quad n \to \infty. \tag{3.79}$$

Indeed, since $b_n \to 0^+$ and $u_{b_n} \rightharpoonup u_0$ in X, we have $m_{b_n}(\|u_0\|^2)\langle u_0, u_{b_n} - u_0\rangle \to 0$ as $n \to \infty$. From (f_1), the Hölder inequality, Lemma 3.2.1 and Lemma 3.2.2, we get

$$\left|\int_\Omega f(u_0)(u_{b_n} - u_0)\,\mathrm{d}x\right| \leq C_0\|u_{b_n} - u_0\|_q \left(\int_\Omega e^{\pi q' u_0^2}\mathrm{d}x\right)^{\frac{1}{q'}} \to 0 \text{ as } n \to \infty,$$

and so we obtain the convergence in (3.79).

Arguing similarly to the previous convergence, and by using (3.77), we can deduce that

$$\int_\Omega (f(u_{b_n}) - f(u_0))(u_{b_n} - u_0)\,\mathrm{d}x \to 0 \quad \text{as} \quad n \to \infty. \tag{3.80}$$

Thus, since $I'_{b_n}(u_{b_n}) = 0$, by using (3.79), it easy to see that

$$(I'_{b_n}(u_{b_n}) - I'_{b_n}(u_0))(u_{b_n} - u_0) = o_n(1), \tag{3.81}$$

where $o_n(1) \to 0$ as $n \to \infty$. Hence, by (3.80) and (3.81), we get

$$\langle m_{b_n}(\|u_{b_n}\|^2)u_{b_n} - m_{b_n}(\|u_0\|^2)u_0, u_{b_n} - u_0\rangle = o_n(1). \tag{3.82}$$

Now, by using that $\|u_{b_n} - u_0\| \leq C$, $m_{b_n}(\|u_{b_n}\|^2), m_{b_n}(\|u_0\|^2) \to a$ as $n \to \infty$, and the Cauchy-Schwartz inequality, we get

$$\begin{aligned}
|\langle m_{b_n}(\|u_{b_n}\|^2)u_{b_n} - m_{b_n}(\|u_0\|^2)u_0, u_{b_n} - u_0\rangle - a\|u_{b_n} - u_0\|^2| = \\
|\langle (m_{b_n}(\|u_{b_n}\|^2) - a)u_{b_n} - (m_{b_n}(\|u_0\|^2) - a)u_0, u_{b_n} - u_0\rangle| \\
\leq \|(m_{b_n}(\|u_{b_n}\|^2) - a)u_{b_n} - (m_{b_n}(\|u_0\|^2) - a)u_0\| \cdot \|u_{b_n} - u_0\| \\
\leq \big(|m_{b_n}(\|u_{b_n}\|^2) - a|\|u_{b_n}\| + |m_{b_n}(\|u_0\|^2) - a|\|u_0\|\big) \cdot C \to 0,
\end{aligned} \tag{3.83}$$

as $n \to \infty$. From (3.82) and (3.83), we obtain that

$$a\|u_{b_n} - u_0\|^2 \to 0 \quad \text{as} \quad n \to \infty,$$

that is, $u_{b_n} \to u_0$ strongly in X.

To finish it remains to show that $u_0 \in \mathcal{M}_0$ is a least energy nodal solution of ($P_{a,0}$). Let $v_0 \in \mathcal{M}_0$ be a least energy nodal solution of ($P_{a,0}$). Since $v_0^+ \neq 0$ and $v_0^- \neq 0$, by Lemma 3.3.4, for each $n \in \mathbb{N}$, there exists $(t_n, s_n) \in (0,\infty) \times (0,\infty)$ such that $t_n v_0^+ + s_n v_0^- \in \mathcal{M}_{b_n}$.

Claim 3.5.4 *There exists $t_0, s_0 > 0$ such that, up to a subsequence, $(t_n, s_n) \to (t_0, s_0)$ as $n \to +\infty$.*

Indeed, by using $I'_{b_n}(t_n v_0^+ + s_n v_0^-)t_n v_0^+ = 0$, $I'_{b_n}(t_n v_0^+ + s_n v_0^-)s_n v_0^- = 0$ and (f_5), we get

$$C_p t_n^p \|v_0^+\|_p^p \leq \int_\Omega f(t_n v_0^+)t_n v_0^+ \mathrm{d}x = m_{b_n}(\|t_n v_0^+ + s_n v_0^-\|^2)\langle t_n v_0^+ + s_n v_0^-, t_n v_0^+\rangle \tag{3.84}$$

and

$$C_p s_n^p \|v_0^-\|_p^p \leq \int_\Omega f(s_n v_0^-)s_n v_0^- \mathrm{d}x = m_{b_n}(\|t_n v_0^+ + s_n v_0^-\|^2)\langle t_n v_0^+ + s_n v_0^-, s_n v_0^-\rangle. \tag{3.85}$$

Summing (3.85) and (3.84), and by the definition of m_{b_n}, we have

$$\begin{aligned}
C_p(t_n^p\|v_0^+\|_p^p + s_n^p\|v_0^-\|_p^p) \leq & (a + b_n\|t_n v_0^+ + s_n v_0^-\|^2)\|t_n v_0^+ + s_n v_0^-\|^2 \\
\leq & (a + 2b_n t_n^2\|v_0^+\|^2 + 2b_n s_n^2\|v_0^-\|^2)(2t_n^2\|v_0^+\|^2 + 2s_n^2\|v_0^-\|^2).
\end{aligned}$$

Thus, since $p > 4$, we deduce that the sequence $((t_n, s_n))$ is bounded. Let $M > 0$ such that $0 < t_n, s_n \leq M$, for all $n \in \mathbb{N}$. Now, up to a subsequence, we have $(t_n, s_n) \to (t_0, s_0)$ as $n \to \infty$, with $t_0, s_0 \geq 0$. On the other hand, given $q > 2$ and using (3.20), we have

$$\begin{aligned} m_{b_n}(\|t_n v_0^+ + s_n v_0^-\|^2)\langle t_n v_0^+ + s_n v_0^-, t_n v_0^+\rangle &= \int_\Omega f(t_n v_0^+) t_n v_0^+ \mathrm{d}x \\ &\leq \varepsilon t_n^2 \|v_0^+\|_2^2 + C t_n^q \int_\Omega |v_0^+|^q e^{\pi M^2 |v_0^+|^2} \mathrm{d}x. \end{aligned} \tag{3.86}$$

Moreover, by Lemma 3.2.3 and by defintion of m_{b_n}, we have

$$a t_n^2 \|v_0^+\|^2 \leq m_{b_n}(\|t_n v_0^+ + s_n v_0^-\|^2)\langle t_n v_0^+ + s_n v_0^-, t_n v_0^+\rangle. \tag{3.87}$$

Choosing $\varepsilon > 0$ such that $a\|v_0^+\|^2 - \varepsilon \|v_0^+\|_2^2 > 0$, since the integral that involves v_0^+ is positive, (3.86) and (3.87) imply that

$$(a\|v_0^+\|^2 - \varepsilon\|v_0^+\|_2^2) \leq C t_n^{q-2}, \quad \text{for all} \quad n \in \mathbb{N}.$$

This inequality implies that $t_0 > 0$. Analogously, we can show that $s_0 > 0$ and so the Claim 3.5.4 is established.

From Claim 3.5.4, we have $t_n v_0^+ \to t_0 v_0^+$ and $s_n v_0^- \to s_0 v_0^-$ strongly in X and so we obtain that

$$m_{b_n}(\|t_n v_0^+ + s_n v_0^-\|^2)\langle t_n v_0^+ + s_n v_0^-, t_n v_0^+\rangle \to a\langle t_0 v_0^+ + s_0 v_0^-, t_0 v_0^+\rangle;$$

$$m_{b_n}(\|t_n v_0^+ + s_n v_0^-\|^2)\langle t_n v_0^+ + s_n v_0^-, s_n v_0^-\rangle \to a\langle t_0 v_0^+ + s_0 v_0^-, s_0 v_0^-\rangle;$$

$$\int_\Omega f(t_n v_0^+) t_n v_0^+ \mathrm{d}x \to \int_\Omega f(t_0 v_0^+) t_0 v_0^+ \mathrm{d}x;$$

$$\int_\Omega f(s_n v_0^-) s_n v_0^- \mathrm{d}x \to \int_\Omega f(s_0 v_0^-) s_0 v_0^- \mathrm{d}x$$

On the other hand, since $I'_{b_n}(t_n v_0^+ + s_n v_0^-) t_n v_0^+ = 0$ and $I'_{b_n}(t_n v_0^+ + s_n v_0^-) s_n v_0^- = 0$, the convergences above imply that $I'_0(t_0 v_0^+ + s_0 v_0^-) t_0 v_0^+ = 0$ and $I'_0(t_0 v_0^+ + s_0 v_0^-) s_0 v_0^- = 0$. Thus, since $v_0 \in \mathcal{M}_0$ and by using the uniqueness of the pair given in Lemma 3.3.4, we deduce that $(t_0, s_0) = (1, 1)$. As a consequence, we have

$$I_0(v_0) = \lim_{n\to\infty} I_{b_n}(t_n v_0^+ + s_n v_0^-). \tag{3.88}$$

Now, since $b_n \to 0^+$ and $u_{b_n} \to u_0$ strongly in X as $n \to \infty$, it is easy to see that

$$I_0(u_0) = \lim_{n\to\infty} I_{b_n}(u_{b_n}). \tag{3.89}$$

Using (3.88) and (3.89), we conclude that

$$\begin{aligned} c_{\mathcal{M}_0} =& I_0(v_0) \leq I_0(u_0) \\ &= \lim_{n\to\infty} I_{b_n}(u_{b_n}) = \lim_{n\to\infty} c_{\mathcal{M}_{b_n}} \\ &\leq \lim_{n\to\infty} I_{b_n}(t_n v_0^+ + s_n v_0^-) = I_0(v_0) = c_{\mathcal{M}_0}. \end{aligned}$$

Therefore, u_0 is a least energy nodal solution of the problem ($P_{a,0}$) and furthermore we obtain the convergence $\lim_{n\to\infty} c_{\mathcal{M}_{b_n}} = c_{\mathcal{M}_0}$, which proves Theorem 3.1.4.

Chapter 4

Ground state and nodal solutions for a class of fractional N/s-Laplacian equations involving exponential growth

In this chapter, we prove the existence of at least three nontrivial solutions for a class of problems with fractional N/s-Laplacian operator:

$$\begin{cases} (-\Delta)^s_{N/s}u + V(x)|u|^{\frac{N}{s}-2}u &= \lambda f(x,u) \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{in} \quad \mathbb{R}\setminus\Omega, \end{cases}$$

where $\lambda > 0$, $s \in (0,1)$, $\Omega \subset \mathbb{R}^N$ is a bonded domain with Lipschitz boundary $\partial\Omega$, $N \geq 2$, $V(x)$ is a continuous and nonnegative potential, the nonlinearity f can have a subcritical and critical exponential growth in the Trudinger-Moser sense and satisfies appropriate conditions. As $N/s > 2$, the respective functional space to deal the problem with variational methods is not a Hilbert space and, because of that, the techniques applied to estimate the nodal level in the Chapter 1 seem not be applicable to this problem. By the study of asymptotic behavior of the nodal level, we will overcome this difficulties. We will show the existence of a least energy nodal solution and by means of the Mountain Pass Theorem, we get nonpositive and nonnegative ground state solution. Moreover, we show that the energy of the nodal solution is strictly larger than twice the ground state level. The results of this chapter are in the final stages of preparation for submission for publication.

4.1 Introduction and main results

In this chapter we consider the existence and multiplicity of solutions to the fractional N/s-Laplacian problem

$$\begin{cases} (-\Delta)^s_{N/s} u + V(x)|u|^{\frac{N}{s}-2}u = \lambda f(x,u) & \text{in} \quad \Omega, \\ u = 0 & \text{in} \quad \mathbb{R}^N \setminus \Omega, \end{cases} \tag{P_λ}$$

where λ is a positive parameter, $N \geq 2$, $s \in (0,1)$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $V : \overline{\Omega} \longrightarrow \mathbb{R}$ is continuous and nonnegative potential, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and C^1 in the second coordinate, and may have a subcritical or critical exponential growth in the Trudinger-Moser sense (see Definition 4.1.2), $(-\Delta)^s_{N/s}$ is the fractional $N/s-$Laplacian operator, which, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$, is defined as

$$(-\Delta)^s_{N/s}\varphi(x) := 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|\varphi(x) - \varphi(y)|^{\frac{N}{s}-2}(\varphi(x) - \varphi(y))}{|x-y|^{2N}} \, \mathrm{d}y.$$

The appropriate space to deal with the problem (P_λ), by using variational methods, is the space X defined as

$$X := \left\{ u \in L^{N/s}(\mathbb{R}^N) : \, u \equiv 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega \quad \text{and} \quad [u]_{s,N/s} < \infty \right\},$$

which will be endowed with the norm

$$\|u\| := \left([u]_{s,N/s}^{N/s} + \int_\Omega V(x)|u|^{N/s} \mathrm{d}x \right)^{\frac{s}{N}},$$

where $[u]_{s,N/s}$ is the Gagliardo seminorm given by

$$[u]_{s,N/s} := \left(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{N/s}}{|x-y|^{2N}} \mathrm{d}x\mathrm{d}y \right)^{\frac{s}{N}}.$$

It is well-known that X is a reflexive Banach space and is compactly embedded into $L^q(\mathbb{R}^N)$, for all $q \in [1, \infty)$, see Section 4.3 for more details.

We are interested in looking for solutions when the nonlinearity f has the maximal growth which allows us to treat the problem (P_λ) variationally in X. In order to better understanding of the critical growth on f, let us to recall some well-known facts involving the limiting Sobolev embedding. If $U \subset \mathbb{R}^N$ is a bounded extension domain (for example, if ∂U is Lipschitz) and $sq < N$, then the embedding $W^{s,q}(U) \hookrightarrow L^\nu(U)$ is continuous and compact, for any $\nu \in [1, q^*]$, where $q^* = q^*(s,N) = Nq/(N-sq)$ is the fractional critical Sobolev exponent. If $qs = N$ then the embedding $W^{s,q}(U) \hookrightarrow L^\nu(U)$ is continuous and compact, for any $\nu \in [1,\infty)$. However, $W^{s,q}(U)$ is not continuously embedded in $L^\infty(U)$ (see [61] and [32, Theorems 6.7, 6.9 and 7.1]). Then, a natural question is what is the maximum growth that we can consider on nonlinearity in order to apply a variational method to find solutions for the problem (P_λ).

In order to answer this question let $\widetilde{W}_0^{s,N/s}(\Omega)$ be the space defined as the closure of $C_0^\infty(U)$ with respect to the norm

$$u \mapsto \left([u]_{s,N/s}^{N/s} + \|u\|_{L^{N/s}(\Omega)}^{N/s}\right)^{\frac{s}{N}}.$$

This space has been extensively studied by many authors, in particular, see [16, 17, 62, 69]. For this space, Parini and Ruf [62] proved a fractional Trudinger-Moser type inequality. Precisely:

Lemma 4.1.1 *There exist $\alpha_* = \alpha_*(s,N,\Omega) > 0$ such that*

$$\ell(\alpha,N,s,\Omega) := \sup\left\{\int_\Omega e^{\alpha|u|^{\frac{N}{N-s}}}\mathrm{d}x : u \in \widetilde{W}_0^{s,N/s}(\Omega) \quad \textit{and} \quad [u]_{s,N/s} \le 1\right\} < \infty,$$

for all $\alpha \in [0,\alpha_*)$. Moreover, $\ell(\alpha,N,s,\Omega) = \infty$ for $\alpha \in (\alpha^*_{s,N},\infty)$, where

$$\alpha^*_{s,N} := N\left(\frac{2(N\omega_N)^2\Gamma(\frac{N}{s}+1)}{N!}\sum_{k=0}^\infty \frac{(N+k-1)!}{k!}\frac{1}{(N+2k)^{N/s}}\right)^{\frac{s}{N-s}}$$

and ω_N is the volume of N-dimensional unit ball.

As a consequence of this result, the maximum growth we can assume in order to apply a variational method in space X is of the exponential type given by Lemma 4.1.1. Motivated by this, we established the following definition:

Definition 4.1.2 *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function. We say that $g(x,t)$ has subcritical exponential growth, in the Trudinger-Moser sense, if*

$$\lim_{|t|\to\infty} \frac{g(x,t)}{e^{\gamma|t|^{\frac{N}{N-s}}}} = 0 \quad \textit{uniformily in} \quad x \in \Omega,$$

for every $\gamma > 0$, and has critical exponential growth, in the Trudinger-Moser sense, if there exist $\gamma_0 > 0$ such that

$$\lim_{|t|\to\infty} \frac{|g(x,t)|}{e^{\gamma|t|^{\frac{N}{N-s}}}} = \begin{cases} 0, & \textit{for all} \quad \gamma > \gamma_0, \\ \infty, & \textit{for all} \quad \gamma < \gamma_0, \end{cases}$$

uniformly in $x \in \Omega$.

Our main goal is to prove existence and multiplicity of weak solutions to the problem (P_λ). We show that (P_λ) has a nodal solution, a nonnegative and a nonpositive solutions when the nonlinearity $f(x,t)$ has subcritical or critical exponential growth.

In the following, we will present our hypotheses and main results.

4.2 Assumptions and main results

Throughout this chapter we will consider the following hypotheses on Ω, V and f:

(V_1) $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$ and $V : \overline{\Omega} \to \mathbb{R}$ is a continuous and nonnegative function.

(f_1) $f(x,t)$ is continuous and continuously differentiable on the variable t, and there exist $C_0, \alpha_0 > 0$ such that

$$|f(x,t)| \leq C_0 e^{\alpha_0|t|^{\frac{N}{N-s}}}, \quad \text{for all} \quad (x,t) \in \Omega \times \mathbb{R};$$

(f_2) $\displaystyle\lim_{t\to 0} \frac{f(x,t)}{|t|^{\frac{N}{s}-2}t} = 0$ uniformly in $x \in \Omega$;

(f_3) there exists $\theta > \frac{N}{s}$ such that

$$0 < \theta F(x,t) := \theta \int_0^t f(x,\tau)\,\mathrm{d}\tau \leq t f(x,t), \quad \text{uniformly in} \quad \Omega, \quad \text{for all} \quad |t| \neq 0;$$

(f_4) the function $t \mapsto f(x,t)/|t|^{\frac{N}{s}-2}t$ is strictly increasing on $(0,\infty)$ and strictly decreasing on $(-\infty, 0)$, uniformly $x \in \Omega$;

(f_5) there exist $p > \frac{N}{s}$ and $C > 0$ such that

$$sgn(t) f(x,t) \geq C|t|^{p-1}, \quad \text{for all} \quad t \in \mathbb{R}, \text{ uniformly in } x \in \Omega.$$

Example 4.2.1 *If $p > \frac{N}{s}$, the nonlinearity*

$$f(x,t) = C|t|^{p-2}t + |t|^{p-2}te^{|t|^{\frac{N}{N-s}}}$$

satisfies the assumptions $(f_1) - (f_5)$.

We note that the assumption (f_1) allows to treat nonlinearities $f(x,t)$ which may have subcritical or critical exponential growth. In this way, f can have the maximum growth that allows us to treat problem (P_λ) in the variational way. In this context, we say that $u \in X$ is a *weak solution* (or simply, *solution*) to the problem (P_λ), if

$$\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{\frac{N}{s}-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{2N}} \mathrm{d}x\mathrm{d}y + \int_\Omega V(x)|u|^{\frac{N}{s}-2}uv\, \mathrm{d}x$$
$$= \lambda \int_\Omega f(x,u)v\, \mathrm{d}x, \quad \text{for all} \quad v \in X.$$

If u is a solution of (P_λ), with $u^+ \neq 0$ and $u^- \neq 0$, we say that u is a nodal (sing-changing) solution of (P_λ).

Associated to the problem (P_λ), we have the energy functional $I_\lambda : X \to \mathbb{R}$ given by

$$I_\lambda(u) = \frac{s}{N}\|u\|^{N/s} - \lambda \int_\Omega F(x,u)\, \mathrm{d}x. \tag{4.1}$$

Using the assumptions on f, by standard arguments, we have $I_\lambda \in C^1(X,\mathbb{R})$ and its derivative is given by

$$I'_\lambda(u)v = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{\frac{N}{s}-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{2N}} \mathrm{d}x\mathrm{d}y + \int_\Omega V(x)|u|^{\frac{N}{s}-2}uv\, \mathrm{d}x$$
$$-\lambda \int_\Omega f(x,u)v\, \mathrm{d}x, \tag{4.2}$$

for all $u, v \in X$. Thus, solutions of the problem (P_λ) are precisely the critical points of I_λ and reciprocally.

In order to present the main results of this work, we define the Nehari sets associated to I_λ and their respective minimums energy level:

- the Nehari manifold and the ground state level

$$\mathcal{N}_\lambda = \{u \in X \setminus \{0\} : I'_\lambda(u)u = 0\} \quad \text{and} \quad c_{\mathcal{N}_\lambda} := \inf_{u \in \mathcal{N}_\lambda} I_\lambda(u); \tag{4.3}$$

- the set of nonnegative functions on Nehari manifold

$$\mathcal{N}_\lambda^+ = \{u \in \mathcal{N}_\lambda : u^- = 0\} \quad \text{and} \quad c_{\mathcal{N}_\lambda^+} := \inf_{u \in \mathcal{N}_\lambda^+} I_\lambda(u); \tag{4.4}$$

- the set of nonpositive functions on Nehari manifold

$$\mathcal{N}_\lambda^- = \{u \in \mathcal{N}_\lambda : u^+ = 0\} \quad \text{and} \quad c_{\mathcal{N}_\lambda^-} := \inf_{u \in \mathcal{N}_\lambda^-} I_\lambda(u); \tag{4.5}$$

- the nodal Nehari set and the nodal level

$$\mathcal{M}_\lambda = \{u \in X : u^+ \neq 0,\ u^- \neq 0,\ I_\lambda'(u)u^+ = 0 \quad \text{and} \quad I_\lambda'(u)u^- = 0\} \tag{4.6}$$

and

$$c_{\mathcal{M}_\lambda} := \inf_{u \in \mathcal{M}_\lambda} I_\lambda(u). \tag{4.7}$$

Since we are looking for nodal solutions, one of our goals will be to show that $c_{\mathcal{M}_\lambda}$ is a minimum of I_λ and the minimum point is a critical point of I_λ. If $u \in \mathcal{M}_\lambda$ is a solution of (P_λ) such that $I_\lambda(u) = c_{\mathcal{M}_\lambda}$ we say that u is a least energy nodal solution of (P_λ) (see Remark 4.4.5).

The first result of this chapter is:

Theorem 4.2.2 *Suppose that (V_1) and $(f_1) - (f_5)$ are satisfied. Then, there exists $\lambda^* > 0$ such that, for any $\lambda \geq \lambda^*$, the problem (P_λ) possesses a least energy nodal solution. Explicitly, for every $\lambda \geq \lambda^*$, there exists $\bar{u} \in \mathcal{M}_\lambda$ such that $I_\lambda'(\bar{u}) = 0$ and $I_\lambda(\bar{u}) = c_{\mathcal{M}_\lambda}$.*

In the second result we will prove that the problem (P_λ) have one nonnegative and one nonpositive solution, both nonzero, whose energy is minimal between the solutions that have a defined signal. Moreover, we also show that the energy of any nodal solution of (P_λ) is strictly larger than twice the ground state energy, see Remark 4.4.2 for details (in particular, this implies that $c_{\mathcal{N}_\lambda} = c_{\mathcal{N}_\lambda^+}$ or $c_{\mathcal{N}_\lambda} = c_{\mathcal{N}_\lambda^+}$). This property is so-called energy doubling by Weth [71].

Theorem 4.2.3 *Suppose that (V_1) and $(f_1) - (f_5)$ are satisfied and let $\lambda \geq \lambda^*$, where λ^* given in Theorem 4.2.2. Then, there exist $u_+ \in \mathcal{N}_\lambda^+$, with $I_\lambda(u_+) = c_{\mathcal{N}_\lambda^+}$, and $u_- \in \mathcal{N}_\lambda^-$, with $I_\lambda(u_-) = c_{\mathcal{N}_\lambda^-}$, solutions of (P_λ). Moreover, we have*

$$c_{\mathcal{M}_\lambda} = I_\lambda(\bar{u}) > c_{\mathcal{N}_\lambda^+} + c_{\mathcal{N}_\lambda^-} \geq 2c_{\mathcal{N}_\lambda}, \tag{4.8}$$

where $\bar{u}$ is the least energy nodal solution obtained in Theorem 4.2.2.

In order to understand the main difficulties in studying the existence of nodal solutions to the problem (P_λ), consider the Dirichlet value problem involving the $q-$ Laplacian operator:

$$\begin{cases} -\Delta_q u + V(x)|u|^{q-2}u &= \lambda f(x,u) \quad \text{in} \quad \Omega, \\ u &= 0 \quad\quad\quad \text{on} \quad \partial\Omega. \end{cases} \tag{4.9}$$

The energy functional $J : W_0^{1,q}(\Omega) \to \mathbb{R}$ associated to (4.9) is given by

$$J(u) = \frac{1}{q}\int_\Omega (|\nabla u|^q + V(x)|u|^q)\,\mathrm{d}x - \lambda\int_\Omega F(x,u)\mathrm{d}x,$$

which satisfies the following decompositions

$$J(u) = J(u^+) + J(u^-) \quad \text{and} \quad J'(u)u^\pm = J'(u^\pm)u^\pm. \tag{4.10}$$

Due to the Gagliardo seminorm of $u \in X$, the functional I_λ in (4.1) does not possess theses the same decompositions as (4.10). This fact implies that the standard methods to find nodal solutions for the local problem (4.9) can not be applicable to the problem (P_λ). In fact, when $u^+ \neq 0$ and $u^- \neq 0$, the functional I_λ satisfies

$$I_\lambda(u) > I_\lambda(u^+) + I_\lambda(u^-),$$

$$I'_\lambda(u)u^+ > I'_\lambda(u^+)u^+ \quad \text{and} \quad I'_\lambda(u)u^- > I'_\lambda(u^-)u^-,$$

see Lemma 4.3.7. In the problems involving a nonlocal operators many additional difficulties arise due to the fact that the decomposition (4.10) does not occur. Note that $\mathcal{N}_\lambda^\pm \subset \mathcal{N}_\lambda$ and $\mathcal{M}_\lambda \subset \mathcal{N}_\lambda$. If $u \in \mathcal{M}_\lambda$, then

$$u^+ \notin \mathcal{N}_\lambda^+ \quad \text{and} \quad u^- \notin \mathcal{N}_\lambda^- \tag{4.11}$$

see Corollary 4.3.7 and Remark 4.4.2. As we observer in the other chapters, this a big difference between nonlocal and local problems. Moreover, another well-known difficulty for the class of the problems (P_λ) is the loss of compactness due to the critical growth on the nonlinearity f.

We ended this section by mentioning that for problems involving fractional equations, critical nonlinearities and domains Ω of $\mathbb{R}^N$, with $N > 2s$, there is a large literature and we refer to [42, 65, 66, 67, 68], and to the references therein.

The outline of this chapter is as follows: Section 4.3 contains results about the space X and we make some important observations regarding the behavior of the norm in this space (in especial, see Lemma 4.3.6). Section 4.4 is dedicated to the study of the nodal set and the nodal level, the main goal is to prove that, for λ larger enough, the nodal level is attained by a sign-changing weak solution of (P_λ). In Section 4.5 is devoted to prove the existence of solutions that have signal defined.

4.3 Preliminaries

We will start this section by presenting some basic facts about the fractional Sobolev space $\widetilde{W}_0^{s,N/s}(\Omega)$, endowed with a suitable norm, this is a adequate space to deal with the problem (P_λ). For a more complete discussion of this space, we cite mainly [16, 17, 32, 62].

Let U be an open subset of $\mathbb{R}^N$. Given $s \in (0,1)$ and $q \in [1,\infty)$, the fractional Sobolev space $W^{s,q}(U)$ is defined by

$$W^{s,q}(U) = \left\{ u \in L^p(U) : [u]_{W^{s,q}(U)} < \infty \right\},$$

where $[u]_{W^{s,q}(U)}$ is the Gagliardo seminorm of u given by

$$[u]_{W^{s,q}(U)} := \left(\int_{U\times U} \frac{|u(x)-u(y)|^q}{|x-y|^{N+sq}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{q}}.$$

The space $W^{s,q}(U)$ endowed with the norm

$$\|u\|_{W^{s,q}(U)} := \left(\|u\|^q_{L^q(U)} + [u]^q_{W^{s,q}(U)} \right)^{\frac{1}{q}}$$

is a Banach space. For $u \in W^{s,q}(\mathbb{R}^N)$, we denote by $[u]_{s,q}$ the Gagliardo seminorm and the correspondent norm by $\|u\|_{s,q} = \left(\|u\|_q^q + [u]_{s,q}^q\right)^{1/q}$. Let U be a bounded domain. We consider the spaces $W_0^{s,q}(U)$ and $\widetilde{W}_0^{s,q}(U)$ as the completions of the space $C_0^\infty(U)$ with the norms

$$\varphi \mapsto \|\varphi\|^q_{W^{s,q}(U)} := \|\varphi\|^q_{L^q(U)} + [\varphi]^q_{W^{s,q}(U)} \quad \text{and} \quad \varphi \mapsto \|\varphi\|^q_{s,q} := \|\varphi\|^q_{L^q(U)} + [\varphi]^q_{s,q}$$

respectively. Since $C_0^\infty(U) \subset W^{s,q}(U)$ and $\|\cdot\|_{W^{s,q}(U)} \leq \|\cdot\|_{s,q}$ we have the following continuous embeddings

$$\widetilde{W}_0^{s,q}(U) \hookrightarrow W_0^{s,q}(U) \hookrightarrow W^{s,q}(U). \tag{4.12}$$

By [16, Remark 2.5], the space $\widetilde{W}_0^{s,q}(U)$ can be equivalently defined by completion of $C_0^\infty(U)$ with respect to the Gagliardo seminorm $[\cdot]_{s,q}$ and so the natural norm of $u \in \widetilde{W}_0^{s,q}(U)$ is given by

$$\|u\|_{\widetilde{W}_0^{s,q}(U)} := [u]_{s,q}.$$

If ∂U is Lipschitz, by [17, Proposition B.1], the space $\widetilde{W}_0^{s,q}(U)$ can be described as

$$\widetilde{W}_0^{s,q}(U) := \left\{u \in L^q(\mathbb{R}^N) : u \equiv 0 \text{ in } \mathbb{R}^N \setminus U \quad \text{and} \quad [u]_{s,q} < \infty\right\},$$

Moreover, in the case that $sq \neq 1$, we have

$$W_0^{s,q}(U) = \widetilde{W}_0^{s,q}(U),$$

see [16, Proposition B.1]. In the above condition, the norm in $W_0^{s,q}(U)$ also is given by the Gagliardo seminorm. The next result shows that $W_0^{s,q}(U)$ is a reflexive space.

Lemma 4.3.1 *Let $U \subset \mathbb{R}^N$ a bounded domain with Lipschitz boundary ∂U. Let $q > 1$, $s \in (0,1)$ such that $sq \neq 1$. Then $W_0^{s,q}(U)$ is uniformly convex and, hence, a reflexive Banach space.*

Proof . Let us consider $T : W_0^{s,q}(U) \to L^q(\mathbb{R}^{2N})$, by $T(u) = \widetilde{u}$, the linear transformation given by

$$\widetilde{u}(x,y) = \frac{u(x) - u(y)}{|x-y|^{\frac{N}{q}+s}}, (x,y) \in \mathbb{R}^{2N}, \ \textit{with } x \neq y.$$

We have

$$\|T(u)\|_{L^q(\mathbb{R}^{2N})} = \left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^q}{|x-y|^{N+sq}} \mathrm{d}x\mathrm{d}y\right)^{\frac{1}{q}} = \|u\|_{W_0^{s,q}(U)}$$

and so T is a linear isometric embedding. Therefore, the uniform convexity and, hence the reflexivity, $W_0^{s,q}(U)$ follow of the uniform convexity of $L^q(\mathbb{R}^{2N})$ (see Lemma A.1.9).

■

Next, we will present a Brézis-Lieb lemma in $W_0^{s,q}(U)$.

Lemma 4.3.2 *Let $U \subset \mathbb{R}^N$ a bounded domain with Lipschitz boundary ∂U. Let $q > 1$, $s \in (0,1)$ such that $sq \neq 1$. Let (u_n) is a bounded sequence in $W_0^{s,q}(U)$ such that $u_n(x) \to u(x)$ a.e in U. Then*

$$\lim_{n\to\infty} \left(\|u_n\|^q_{W_0^{s,q}(U)} - \|u_n - u\|^q_{W_0^{s,q}(U)}\right) = \|u\|^q_{W_0^{s,q}(U)}.$$

Proof . Let $\widetilde{u_n} = T(u_n)$ and $\widetilde{u} = T(u)$, where T was defined in Lemma 4.3.1. Now, since T is isometric and $\widetilde{u}_n(x, y) \to \widetilde{u}(x, y)$ a.e. in $\mathbb{R}^{2N}$, we can apply the Brézis-Lieb lemma for the $L^q(\mathbb{R}^{2N})$ space. Thus we finished the proof. ■

From now on, we will consider the space $W_0^{s,N/s}(\Omega)$, where Ω satisfies the assumptions in (V_1).

Lemma 4.3.3 *The embedding $W_0^{s,N/s}(\Omega) \hookrightarrow L^q(\mathbb{R}^N)$ is continuous and compact, for all $q \in [1, \infty)$.*

Proof . By (4.12) and using [32, Theorem 7.1], we have that the embedding $W_0^{s,N/s}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for any $q \in [1, N/s]$. Now, let $q > N/s$. We can choose $s' \in (0, s)$ such that

$$\frac{N(N/s)}{N - s'(N/s)} > q. \tag{4.13}$$

Since $s' < s$, by [32, Proposition 2.1], we have the continuous embedding

$$W^{s,N/s}(\Omega) \hookrightarrow W^{s',N/s}(\Omega). \tag{4.14}$$

Now, since $s'(N/s) < N$, by [32, Corollary 7.2], we obtain the following compact embedding

$$W^{s',N/s}(\Omega) \hookrightarrow L^\nu(\Omega), \quad \text{for any} \quad 1 \leq \nu < \frac{N(N/s)}{N - s'(N/s)}.$$

Thus, by (4.13), (4.14) and (4.12), we deduce that $W_0^{s,N/s}(\Omega) \hookrightarrow L^q(\Omega)$ is compact and the proof is complete. ■

As a consequence of Lemma 4.1.1, we obtain the following corollary:

Corollary 4.3.4 *Let $u \in W_0^{s,N/s}(\Omega)$. Then, for every $\alpha > 0$, we have*

$$\int_\Omega e^{\alpha|u|^{\frac{N}{N-s}}} \mathrm{d}x < \infty.$$

Proof . By density, let us choose $v \in C_0^\infty(\Omega)$ such that $[u - v]_{s,N/s} \leq \frac{1}{2}\left(\frac{\alpha_*}{2\alpha}\right)^{\frac{N-s}{N}}$. Let $w = u - v$. Using the triangle inequality and that $(a + b)^q \leq 2^{q-1}(a^q + b^q)$, for all $a, b \geq 0$, by the convexity, we have

$$\begin{aligned}
e^{\alpha|u|^{\frac{N}{N-s}}} &\leq e^{\alpha(|v|+|w|)^{\frac{N}{N-s}}} \\
&\leq e^{2^{\frac{N}{N-s}-1}\alpha(|v|^{\frac{N}{N-s}}+|w|^{\frac{N}{N-s}})} \\
&\leq \frac{1}{2}e^{2^{\frac{N}{N-s}}\alpha|v|^{\frac{N}{N-s}}} + \frac{1}{2}e^{2^{\frac{N}{N-s}}\alpha|w|^{\frac{N}{N-s}}}.
\end{aligned}$$

Since $v \in C_0^\infty(\Omega)$, then $e^{2^{\frac{N}{N-s}}\alpha|v|^{\frac{N}{N-s}}} \in L^1(\Omega)$. Moreover, using Lemma 4.1.1 and the estimate in the norm of w, we get

$$\begin{aligned} e^{2^{\frac{N}{N-s}}\alpha|w|^{\frac{N}{N-s}}} &= e^{2^{\frac{N}{N-s}}\alpha[w]_{s,N/s}^{\frac{N}{N-s}}(|w|/[w]_{s,N/s})^{\frac{N}{N-s}}} \\ &\le e^{\frac{\alpha_*}{2}(|w|/[w]_{s,N/s})^{\frac{N}{N-s}}} \in L^1(\Omega). \end{aligned}$$

This concludes the proof. ∎

We denote by X the space $W_0^{s,N/s}(\Omega)$ endowed with the norm

$$\|u\| := \left([u]_{s,N/s}^{N/s} + \int_\Omega V(x)|u|^{N/s}\mathrm{d}x\right)^{\frac{s}{N}}. \tag{4.15}$$

Clearly, by (V_1) and again by [16, Remark 2.5], $\|\cdot\|$ is a norm equivalent to norm $[\cdot]_{s,N/s}$ and $[u]_{s,N/s} \le \|u\|$ for every $u \in X$. In particular, by Lemma 4.3.1, X is a reflexive Banach space. Moreover, the result of Lemma 4.3.2 is true for X and, by Lemma 4.3.3, we obtain the following result:

Lemma 4.3.5 *The embedding $X \hookrightarrow L^q(\mathbb{R}^N)$ is continuous and compact, for all $q \in [1,\infty)$.*

Using Lemma 4.3.5, Corollary 4.3.4 and by the Hölder inequality, given $\alpha > 0$ and $u \in X$, the following integral is finite

$$\int_\Omega |u|^q e^{\alpha|u|^{\frac{N}{N-s}}}\mathrm{d}x < \infty. \tag{4.16}$$

Let us consider the operator $(-\Delta)_{N/s}^s : W_0^{s,N/s}(\Omega) \to (W_0^{s,N/s}(\Omega))'$ defined by

$$\langle(-\Delta)_{N/s}^s u, v\rangle := \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{\frac{N}{s}-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{2N}}\mathrm{d}x\mathrm{d}y, \tag{4.17}$$

for $u, v \in W_0^{s,N/s}(\Omega)$.

This operator is weak-to-weak continuous and the function

$$u \mapsto \langle(-\Delta)_{N/s}^s u, u\rangle = [u]_{s,N/s}^{N/s}$$

is convex and C^1, see [21, 26] for more details.

By Lemma 4.1.1, Corollary 4.3.4 and by Lemma 4.3.5, we can see that the functional $I_\lambda : X \to \mathbb{R}$, in (4.1), is well-defined and is C^1. Moreover, according to the notation in (4.17), its derivative can be write by

$$I_\lambda'(u)v = \langle(-\Delta)_{N/s}^s u, v\rangle + \int_\Omega V(x)|u|^{\frac{N}{s}-2}uv\,\mathrm{d}x - \lambda\int_\Omega f(x,u)v\,\mathrm{d}x \tag{4.18}$$

for every $u, v \in X$. Hence, weak solutions of the problem (P_λ) are the critical point of I_λ and reciprocally.

As we said in the Introduction, many difficulties arise due to the fact that, in general, I_λ does not satisfy (4.10). In fact, this difficulty arises because if $u^\pm \neq 0$, then

$$\|u\|^{N/s} \neq \|u^+\|^{N/s} + \|u^-\|^{N/s}.$$

Next, inspired by [21, 29, 45, 46] we will present a lemma that deal more precise with the behavior of norm u, u^+ and u^- in X and, consequently, we will obtain some estimates for the functional I_λ. In fact, considering the methods applied, this lemma is one of the main tools to obtain nodal solution for the problem (P_λ).

Lemma 4.3.6 *Let $u \in X$ and $\Omega_+ = \{x \in \Omega : u(x) \geq 0\}$ and $\Omega_- = \{x \in \Omega : u(x) \leq 0\}$. Then*

(i) $\langle (-\Delta)^s_{N/s} u, u^+ \rangle = A^+(u) + B^+(u), \langle (-\Delta)^s_{N/s} u, u^- \rangle = A^-(u) + B^-(u)$ *and*

$$\langle (-\Delta)^s_{N/s} u, u \rangle = A(u) + B(u),$$

where

$$A^+(u) := \int_{\Omega_-^c \times \Omega_-^c} \frac{|u^+(x) - u^+(y)|^{N/s}}{|x-y|^{2N}} \mathrm{d}x\mathrm{d}y, A^-(u) := \int_{\Omega_+^c \times \Omega_+^c} \frac{|u^-(x) - u^-(y)|^{N/s}}{|x-y|^{2N}} \mathrm{d}x\mathrm{d}y,$$

$$B^\pm(u) := 2 \int_{\Omega_+ \times \Omega_-} \frac{|u^+(x) - u^-(y)|^{\frac{N}{s}-1}(u^\pm(x) - u^\pm(y))}{|x-y|^{2N}} \mathrm{d}x\mathrm{d}y$$

and

$$A(u) = A^+(u) + A^-(u) \quad \textit{and} \quad B(u) = B^+(u) + B^-(u).$$

(ii) *if $\alpha, \beta \in (0, \infty)$, we have*

$$0 \leq A^+(\alpha u^+ + \beta u^-) = \alpha^{N/s} A^+(u^+) \quad \textit{and} \quad 0 \leq A^-(\alpha u^+ + \beta u^-) = \beta^{N/s} A^-(u^-).$$

(iii) *if $0 < \beta \leq \alpha$, then*

$$0 \leq B^+(\alpha u^+ + \beta u^-) \leq \alpha^{N/s} B^+(u), 0 \leq B^-(\alpha u^+ + \beta u^-) \leq \alpha^{N/s} B^-(u),$$

$$0 \leq \beta^{N/s} B^+(u) \leq B^+(\alpha u^+ + \beta u^-) \quad \textit{and} \quad 0 \leq \beta^{N/s} B^-(u) \leq B^-(\alpha u^+ + \beta u^-).$$

Similar inequalities holds if $0 < \alpha \leq \beta$.

(iv) *$\langle(-\Delta)^s_{N/s}u, u^\pm\rangle = \langle(-\Delta)^s_{N/s}u^\pm, u^\pm\rangle + 2C^\pm(u)$, with $C^\pm(u) \geq 0$, where*

$$C^+(u) = \int_{\Omega_+\times\Omega_-} \frac{|u^+(x) - u^-(y)|^{\frac{N}{s}-1}u^+(x) - |u^+(x)|^{\frac{N}{s}}}{|x-y|^{2N}}\mathrm{d}x\mathrm{d}y$$

and

$$C^-(u) = \int_{\Omega_+\times\Omega_-} \frac{|u^+(x) - u^-(y)|^{\frac{N}{s}-1}(-u^-(y)) - |-u^-(y)|^{\frac{N}{s}}}{|x-y|^{2N}}\mathrm{d}x\mathrm{d}y.$$

Moreover, if $u^\pm \neq 0$, then $C^\pm(u) > 0$.

(v) *$0 \leq \langle(-\Delta)^s_{N/s}u^\pm, u^\pm\rangle \leq \langle(-\Delta)^s_{N/s}u, u^\pm\rangle \leq \langle(-\Delta)^s_{N/s}u, u\rangle$. In particular, $\|u^\pm\| \leq \|u\|$ for all $u \in X$, where $\|\cdot\|$ is defined in (4.15). Moreover, if $u^\pm \neq 0$, this inequalities are strict.*

Proof . To show **(i)**, we define

$$U(x,y) = \frac{|u(x)-u(y)|^{\frac{N}{s}-2}(u(x)-u(y))}{|x-y|^{2N}}, \; U(x^+,y) = \frac{|u^+(x)-u(y)|^{\frac{N}{s}-2}(u^+(x)-u(y))}{|x-y|^{2N}},$$

$$U(x,y^+) = \frac{|u(x)-u^+(y)|^{\frac{N}{s}-2}(u(x)-u^+(y))}{|x-y|^{2N}}, \; U(x^+,y^+) = \frac{|u^+(x)-u^+(y)|^{\frac{N}{s}-2}(u^+(x)-u^+(y))}{|x-y|^{2N}},$$

and, in an analogous way, we consider $U(x^-,y), U(x,y^-), U(x^-,y^-), U(x^+,y^-)$ and $U(x^-,y^+)$. Considering this notation and (4.17), we can write

$$\langle(-\Delta)^s_{N/s}u, u^+\rangle = \int_{\mathbb{R}^{2N}} U(x,y)(u^+(x) - u^+(y))\mathrm{d}x\mathrm{d}y.$$

Since $\mathbb{R}^{2N} = (\Omega_-^c \times \Omega_-^c) \cup (\Omega_-^c \times \Omega_-) \cup (\Omega_- \times \Omega_-^c) \cup (\Omega_- \times \Omega_-)$, we have

$$\begin{aligned}\langle(-\Delta)^s_{N/s}u, u^+\rangle = &\int_{\Omega_-^c\times\Omega_-^c} U(x,y)(u^+(x)-u^+(y))\mathrm{d}x\mathrm{d}y + 2\int_{\Omega_-^c\times\Omega_-} U(x,y)(u^+(x)-u^+(y))\mathrm{d}x\mathrm{d}y \\ &+ \int_{\Omega_-\times\Omega_-} U(x,y)(u^+(x)-u^+(y))\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Thus

$$\begin{aligned}\langle(-\Delta)^s_{N/s}u, u^+\rangle = &\int_{\Omega_-^c\times\Omega_-^c} U(x,y)(u^+(x)-u^+(y))\mathrm{d}x\mathrm{d}y \\ &+2\int_{\Omega_-^c\times\Omega_-} U(x,y)(u^+(x)-u^+(y))\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{4.19}$$

We observe that, if $(x,y) \in \Omega_-^c \times \Omega_-^c$, then $u(x) = u^+(x)$ and $u(y) = u^+(y)$. Thus

$$U(x,y) = U(x^+,y^+) = \frac{|u^+(x)-u^+(y)|^{\frac{N}{s}-2}(u^+(x)-u^+(y))}{|x-y|^{2N}}$$

and so, by (4.19),

$$\langle(-\Delta)^s_{N/s}u, u^+\rangle = A^+(u) + 2\int_{\Omega_-^c\times\Omega_-} U(x,y)(u^+(x)-u^+(y))\mathrm{d}x\mathrm{d}y. \tag{4.20}$$

Now, since $\Omega_-^c \times \Omega_- = (\Omega^c \times \Omega_-) \cup (\Omega_+ \times \Omega_-)$ and $U(x,y)|_{\Omega^+ \times \Omega^-} = U(x^+, y^-)$, we get

$$\begin{aligned}\int_{\Omega_-^c \times \Omega_-} U(x,y)(u^+(x) - u^+(y))\mathrm{d}x\mathrm{d}y &= \int_{\Omega^c \times \Omega_-} U(x,y)(u^+(x) - u^+(y))\mathrm{d}x\mathrm{d}y \\ &+ \int_{\Omega_+ \times \Omega_-} U(x,y)(u^+(x) - u^+(y))\mathrm{d}x\mathrm{d}y \\ &= \int_{\Omega_+ \times \Omega_-} U(x^+,y^-)(u^+(x) - u^+(y))\mathrm{d}x\mathrm{d}y \\ &= \int_{\Omega_+ \times \Omega_-} \frac{|u^+(x) - u^-(y)|^{\frac{N}{s}-1}(u^+(x) - u^+(y))}{|x-y|^{2N}}\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Hence, by (4.20), we deduce that

$$\langle (-\Delta)^s_{N/s} u, u^+ \rangle = A^+(u) + B^+(u).$$

Similarly, we deduce that $\langle (-\Delta)^s_{N/s} u, u^- \rangle = A^-(u) + B^-(u)$, showing the item **(i)**.

The proof of **(ii)** follows from the expressions of $A^+(u)$ and $A^-(u)$.

Let $0 < \beta \leq \alpha$. Note that $0 \leq -\beta u^-(y) \leq -\alpha u^-(y)$, for all $y \in \mathbb{R}^N$. Hence, if $(x,y) \in \Omega_+ \times \Omega_-$, we have

$$\begin{aligned}0 \leq 2\frac{|\alpha u^+(x) - \beta u^-(y)|^{\frac{N}{s}-1}\alpha u^+(x)}{|x-y|^{2N}} &= 2\frac{|\alpha u^+(x) - \beta u^-(y)|^{\frac{N}{s}-1}(\alpha u^+(x) - \alpha u^+(y))}{|x-y|^{2N}} \\ &= 2\alpha^{N/s}\frac{|u^+(x) - \frac{\beta}{\alpha} u^-(y)|^{\frac{N}{s}-1}(u^+(x) - u^+(y))}{|x-y|^{2N}} \\ &\leq 2\alpha^{N/s}\frac{|u^+(x) - u^-(y)|^{\frac{N}{s}-1}(u^+(x) - u^+(y))}{|x-y|^{2N}}\end{aligned}$$

and so, integrating over $\Omega_+ \times \Omega_-$, we obtain

$$0 \leq B^+(\alpha u^+ + \beta u^-) \leq \alpha^{N/s} B^+(u).$$

Analogously, if $0 < \beta \leq \alpha$ and $(x,y) \in \Omega_+ \times \Omega_-$,

$$\begin{aligned}0 \leq 2\frac{|\alpha u^+(x) - \beta u^-(y)|^{\frac{N}{s}-1}(-\beta u^-(y))}{|x-y|^{2N}} &= 2\frac{|\alpha u^+(x) - \beta u^-(y)|^{\frac{N}{s}-1}(\beta u^-(x) - \beta u^-(y))}{|x-y|^{2N}} \\ &\leq 2\frac{|\alpha u^+(x) - \alpha u^-(y)|^{\frac{N}{s}-1}(\alpha u^-(x) - \alpha u^-(y))}{|x-y|^{2N}} \\ &= 2\alpha^{N/s}\frac{|u^+(x) - u^-(y)|^{\frac{N}{s}-1}(u^-(x) - u^-(y))}{|x-y|^{2N}}.\end{aligned}$$

Again, integrating over $\Omega_+ \times \Omega_-$, we deduce that

$$0 \leq B^-(\alpha u^+ + \beta u^-) \leq \alpha^{N/s} B^-(u).$$

In a similar way we can obtain the other inequalities of the item (iii), which completes the proof of (iii).

To show **(iv)**, we observe that, since $u = 0$ in $\Omega^c = \mathbb{R}^N \setminus \Omega$, the integration in (4.17) is calculated only on the set $Q = \mathbb{R}^{2N} \setminus (\Omega^c \times \Omega^c)$. However, Q can be written as

$$Q = (\Omega \times \Omega^c) \cup (\Omega \times \Omega) \cup (\Omega^c \times \Omega).$$

By using this decomposition and (4.17), we get

$$\begin{aligned}\langle(-\Delta)^s_{N/s}u, u^+\rangle = &\int_{\Omega\times\Omega^c} U(x,y)(u^+(x) - u^+(y))\mathrm{d}x\mathrm{d}y + \int_{\Omega\times\Omega} U(x,y)(u^+(x) - u^+(y))\mathrm{d}x\mathrm{d}y \\ &+ \int_{\Omega^c\times\Omega} U(x,y)(u^+(x) - u^+(y))\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Now, since

$$\Omega \times \Omega^c = (\Omega_+ \times \Omega^c) \cup (\Omega_- \times \Omega^c) \quad \text{and} \quad \Omega^c \times \Omega = (\Omega^c \times \Omega_+) \cup (\Omega^c \times \Omega_-)$$

we can deduce that

$$\begin{aligned}\langle(-\Delta)^s_{N/s}u, u^+\rangle = &\int_{\Omega_+\times\Omega^c} U(x^+,y)(u^+(x))\mathrm{d}x\mathrm{d}y + \int_{\Omega\times\Omega} U(x,y)(u^+(x) - u^+(y))\mathrm{d}x\mathrm{d}y \\ &+ \int_{\Omega^c\times\Omega_+} U(x,y^+)(-u^+(y))\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{4.21}$$

Reasoning in a similar way, one can see that

$$\begin{aligned}\langle(-\Delta)^s_{N/s}u^+, u^+\rangle = &\int_{\Omega_+\times\Omega^c} U(x^+,y^+)(u^+(x))\mathrm{d}x\mathrm{d}y + \int_{\Omega\times\Omega} U(x^+,y^+)(u^+(x) - u^+(y))\mathrm{d}x\mathrm{d}y \\ &+ \int_{\Omega^c\times\Omega_+} U(x^+,y^+)(-u^+(y))\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{4.22}$$

A straightforward computation shows that

$$\int_{\Omega_+\times\Omega^c} U(x^+,y)(u^+(x))\mathrm{d}x\mathrm{d}y = \int_{\Omega_+\times\Omega^c} U(x^+,y^+)(u^+(x))\mathrm{d}x\mathrm{d}y \tag{4.23}$$

and

$$\int_{\Omega^c\times\Omega_+} U(x,y^+)(-u^+(y))\mathrm{d}x\mathrm{d}y = \int_{\Omega^c\times\Omega_+} U(x^+,y^+)(-u^+(y))\mathrm{d}x\mathrm{d}y. \tag{4.24}$$

Using (4.21), (4.22), (4.23) and (4.24), we find

$$\begin{aligned}\langle(-\Delta)^s_{N/s}u, u^+\rangle = &\langle(-\Delta)^s_{N/s}u^+, u^+\rangle \\ &+ \int_{\Omega\times\Omega} \big(U(x,y) - U(x^+,y^+)\big)\,(u^+(x) - u^+(y))\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{4.25}$$

Since $\Omega\times\Omega = (\Omega_+\times\Omega_+)\cup(\Omega_+\times\Omega_-)\cup(\Omega_-\times\Omega_+)\cup(\Omega_-\times\Omega_-)$, by expression of U and by the Fubini's Theorem, we deduce that

$$\int_{\Omega\times\Omega}\big(U(x,y)-U(x^+,y^+)\big)(u^+(x)-u^+(y))\mathrm{d}x\mathrm{d}y$$
$$= 2\int_{\Omega_+\times\Omega_-}\big(U(x,y)-U(x^+,y^+)\big)(u^+(x)-u^+(y))\mathrm{d}x\mathrm{d}y.$$

Again by expression of U and since $u^+(y)=0$ for all $y\in\Omega_-$, we deduce that

$$\int_{\Omega\times\Omega}\big(U(x,y)-U(x^+,y^+)\big)(u^+(x)-u^+(y))\mathrm{d}x\mathrm{d}y =$$
$$2\int_{\Omega_+\times\Omega_-}\frac{|u^+(x)-u^-(y)|^{\frac{N}{s}-1}u^+(x)-|u^+(x)|^{\frac{N}{s}}}{|x-y|^{2N}}\mathrm{d}x\mathrm{d}y. \quad (4.26)$$

Then, by (4.25) and (4.26), we have

$$\langle(-\Delta)^s_{N/s}u,u^+\rangle = \langle(-\Delta)^s_{N/s}u^+,u^+\rangle + 2C^+(u).$$

Similarly, we can show that

$$\langle(-\Delta)^s_{N/s}u,u^-\rangle = \langle(-\Delta)^s_{N/s}u^-,u^-\rangle + 2C^-(u),$$

and **(iv)** is proved.

The proof of **(v)** follows from **(iv)**. ■

Using Lemma 4.3.6, we obtain the following corollary:

Corollary 4.3.7 *Let $u\in X$. Then, the following inequalities are satisfied*

(i) $I_\lambda(u^+)+I_\lambda(u^-)\le I_\lambda(u)$,

(ii) $I'_\lambda(u^+)u^+\le I'_\lambda(u)u^+$ *and* $I'_\lambda(u^-)u^-\le I'_\lambda(u)u^-$.

Moreover, if $u^\pm\neq 0$, the above inequalities are strict.

We end this section with some estimates that follow of the assumptions about f. By $(f_1)-(f_2)$, given $\varepsilon>0$ and $q\ge 1$, there exists a constant $C=C(\varepsilon,q)>0$ such that

$$|f(x,t)|\le\varepsilon|t|^{\frac{N}{s}-1}+C|t|^{q-1}e^{\alpha_0|t|^{\frac{N}{N-s}}}, \quad \text{for all} \quad (x,t)\in\Omega\times\mathbb{R}, \quad (4.27)$$

and so, by (f_3), we have

$$F(x,t)\le\varepsilon|t|^{\frac{N}{s}}+C|t|^{q}e^{\alpha_0|t|^{\frac{N}{N-s}}}, \quad \text{for all} \quad (x,t)\in\Omega\times\mathbb{R}. \quad (4.28)$$

By (f_3), we can find positive constants C_1 and C_2 such that

$$F(x,t) \geq C_1|t|^{\theta} - C_2, \quad \text{for all} \quad (x,t) \in \Omega \times \mathbb{R}. \tag{4.29}$$

From (f_5), we have

$$f(x,t)t \geq C|t|^{p}, \quad \text{for all} \quad (x,t) \in \Omega \times \mathbb{R}. \tag{4.30}$$

Moreover, by (f_4), we have

$$\left(\frac{N}{s} - 1\right) f(x,t)t - f'(x,t)t^2 < 0, \quad \text{for all} \quad x \in \Omega \quad \text{with} \quad |t| \neq 0, \tag{4.31}$$

where, for simplicity, $f'(x,t)$ denotes $\partial_t f(x,t)$.

4.4 Constrained minimization problem

In this section, we will study the Nehari sets associated to the functional I_λ. We will obtain estimates for each energy level and for the functions in these sets. Moreover, taking into account that the nonlinearity f may have a critical exponential growth, another goal in this section is to obtain estimates in the nodal level of I_λ and, this way, we may overcome the difficulties that appears from this behavior. This will be done by study of asymptotic properties of the nodal level $c_{\mathcal{M}_\lambda}$.

First, let us introduce some notations. We define $\varphi_u^\lambda : [0,\infty) \to \mathbb{R}$, for $u \in X \backslash \{0\}$, by

$$\varphi_u^\lambda(t) := I_\lambda(tu) = \frac{s}{N}\|tu\|^{N/s} - \lambda \int_\Omega F(x,tu)\,\mathrm{d}x. \tag{4.32}$$

Let $u \in X$, with $u^\pm \neq 0$, we define $\psi_u^\lambda : [0,\infty) \times [0,\infty) \to \mathbb{R}$, given by

$$\psi_u^\lambda(\alpha,\beta) := I_\lambda(\alpha u^+ + \beta u^-). \tag{4.33}$$

We also define the vector field $\Psi_u^\lambda : [0,\infty) \times [0,\infty) \to \mathbb{R}^2$ by

$$\Psi_u^\lambda(\alpha,\beta) := \left(I_\lambda'(\alpha u^+ + \beta u^-)\alpha u^+, I_\lambda'(\alpha u^+ + \beta u^-)\beta u^-\right). \tag{4.34}$$

The next lemma shows that the Nehari sets $\mathcal{N}_\lambda, \mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$ are not empty.

Lemma 4.4.1 *Assume that (V_1) and $(f_1)-(f_4)$ are satisfied. Then, given $u \in X \backslash \{0\}$, there exists a unique $t = t(u) > 0$ such that*

$$I_\lambda(tu) = \max_{s \geq 0} I_\lambda(su). \tag{4.35}$$

As a consequence, the Nehari sets $\mathcal{N}_\lambda, \mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$ are not empty.

Proof . Let $u \in X \setminus \{0\}$. By (4.29), we have

$$\varphi_u^\lambda(t) \le \frac{s}{N}\|tu\|^{N/s} - \lambda C_1 t^\theta \|u\|_\theta^\theta + \lambda C_2|\Omega|.$$

Since $\theta > \frac{N}{s}$, we have

$$\varphi_u^\lambda(t) \to -\infty \text{ as } t \to \infty. \tag{4.36}$$

On the other hand, given $\varepsilon > 0$ and $q > \frac{N}{s}$, by (4.28), we have

$$\varphi_u^\lambda(t) \ge \frac{s}{N} t^{\frac{N}{s}} \|u\|^{N/s} - \lambda\varepsilon t^{\frac{N}{s}} \|u\|_{N/s}^{N/s} - \lambda C t^q \int_\Omega |u|^q e^{\alpha_0 |tu|^{\frac{N}{N-s}}} \mathrm{d}x.$$

Using Lemma 4.3.5, we get

$$\varphi_u^\lambda(t) \ge \left(\frac{s}{N} - \lambda C_1\varepsilon\right) t^{\frac{N}{s}} \|u\|^{N/s} - \lambda C t^q \int_\Omega |u|^q e^{\alpha_0 |tu|^{\frac{N}{N-s}}} \mathrm{d}x.$$

Now, given $0 \le t \le 1$ and by (4.16), we deduce that

$$0 < \int_\Omega |u|^q e^{\alpha_0 |tu|^{\frac{N}{N-s}}} \mathrm{d}x \le \int_\Omega |u|^q e^{\alpha_0 |u|^{\frac{N}{N-s}}} \mathrm{d}x < \infty.$$

Hence, we obtain that

$$\varphi_u^\lambda(t) \ge \left(\frac{s}{N} - \lambda C_1\varepsilon\right) t^{\frac{N}{s}} \|u\|^{N/s} - \lambda C_2 t^q.$$

Choosing $\varepsilon > 0$ such that $\frac{s}{N} - \lambda C_1\varepsilon > 0$, since $q > \frac{N}{s}$, the previous estimates implies that

$$\varphi_u^\lambda(t) > 0 \text{ for } t > 0 \text{ small enough.} \tag{4.37}$$

From (4.36) and (4.37), there exists $t = t(u) > 0$ satisfying (4.35).

It remains now to show the uniqueness of $t > 0$ with this property. Suppose, by contradiction, that $s > t$ is such that $I'_\lambda(su)su = 0$. Thus, we have

$$\|tu\|^{N/s} = \lambda \int_\Omega f(x,tu)tu \, \mathrm{d}x \quad \text{and} \quad \|su\|^{N/s} = \lambda \int_\Omega f(x,su)su \, \mathrm{d}x.$$

Then, we obtain that

$$\int_\Omega \left(\frac{f(x,tu)}{|tu|^{\frac{N}{s}-2}tu} - \frac{f(x,su)}{|su|^{\frac{N}{s}-2}su} \right) |u|^{N/s} \mathrm{d}x = D_+ + D_- = 0 \tag{4.38}$$

where

$$D_\pm = \int_\Omega \left(\frac{f(x,tu^\pm)}{|tu^\pm|^{\frac{N}{s}-2}tu^\pm} - \frac{f(x,su^\pm)}{|su^\pm|^{\frac{N}{s}-2}su^\pm} \right) |u^\pm|^{N/s} \mathrm{d}x.$$

Now, since $u \in X \setminus \{0\}$, we have $u^+ \neq 0$ or $u^- \neq 0$. If $u^- \neq 0$, since $s > t$, we have that $su^- < tu^- < 0$ a.e in Ω_-. Hence, by (f_4), we deduce

$$D_- = \int_\Omega \left(\frac{f(x, tu^-)}{|tu^-|^{\frac{N}{s}-2} tu^-} - \frac{f(x, su^-)}{|su^-|^{\frac{N}{s}-2} su^-} \right) |u^-|^{N/s} \mathrm{d}x < 0.$$

Similarly, if $u^+ \neq 0$, we can deduce that $D_+ < 0$. But, in view of (4.38), the previous estimates leads to a contradiction. This completes the proof. ■

Remark 4.4.2 *Note that, if $u^\pm \neq 0$, by Lemma 4.4.1, there exist $t, s > 0$ such that $tu^+ \in \mathcal{N}_\lambda^+$ and $su^- \in \mathcal{N}_\lambda^-$. Now, using Corollary 4.3.7, we deduce*

$$0 = I'_\lambda(tu^+)tu^+ < I'_\lambda(tu^+ + su^-)tu^+ \quad \text{and} \quad 0 = I'_\lambda(su^-)su^- < I'_\lambda(tu^+ + su^-)su^-$$

and so $v = tu^+ + su^- \notin \mathcal{M}_\lambda$. Thus, the previous lemma cannot be used to show that $\mathcal{M}_\lambda \neq \emptyset$ (but this reasoning can be applied for the cases of local operators, as in (4.9)).

The next lemma deals with some geometric properties of the functional I_λ and, in particular, this result will be applied for show that the Nehari nodal set $\mathcal{M}_\lambda$ is no empty.

Lemma 4.4.3 *Assume that (V_1) and $(f_1) - (f_3)$ are satisfied. Then, the functional I_λ satisfies the following geometric conditions:*

(i) *given $u \in X \setminus \{0\}$, we have*

$$\psi_u^\lambda(\alpha, \beta) = I_\lambda(\alpha u^+ + \beta u^-) \to -\infty, \quad \text{as} \quad |(\alpha, \beta)| \to \infty;$$

(ii) *there exist $r > 0$ and $C > 0$ such that*

$$I_\lambda(u) \geq C\|u\|^{N/s}, \quad \text{for all} \quad \|u\| \leq r.$$

Proof . By (4.29), we get

$$I_\lambda(\alpha u^+ + \beta u^-) \leq \frac{s}{N}\|\alpha u^+ + \beta u^-\|^{N/s} - \lambda C_1 \|\alpha u^+ + \beta u^-\|_\theta^\theta + \lambda C_2 |\Omega|.$$

Now, using the triangle inequality and that $(a+b)^q \leq 2^{q-1}(a^q + b^q)$, for all $a, b \geq 0$, we have

$$\begin{aligned} I_\lambda(\alpha u^+ + \beta u^-) \leq{}& \frac{s}{N} 2^{\frac{N}{s}-1}(|\alpha|^{N/s}\|u^+\|^{N/s} + |\beta|^{N/s}\|u^-\|^{N/s}) \\ &- \lambda C_1(|\alpha|^\theta \|u^+\|_\theta^\theta + |\beta|^\theta \|u^-\|_\theta^\theta) + \lambda C_2 |\Omega|. \end{aligned}$$

Since $\theta > \frac{N}{s}$, the above inequality implies that

$$I_\lambda(\alpha u^+ + \beta u^-) \to -\infty, \quad \text{as} \quad |(\alpha,\beta)| \to \infty, \tag{4.39}$$

showing **(i)**.

Given $\varepsilon > 0$ and $q > \frac{N}{s}$, by (4.28), we have

$$I_\lambda(u) \geq \frac{s}{N}\|u\|^{N/s} - \lambda\varepsilon\|u\|_{N/s}^{N/s} - \lambda C \int_\Omega |u|^q e^{\alpha_0|u|^{\frac{N}{N-s}}} \mathrm{d}x.$$

By Lemma 4.3.3, we have

$$I_\lambda(u) \geq \left(\frac{s}{N} - \lambda C_1\varepsilon\right)\|u\|^{N/s} - \lambda C \int_\Omega |u|^q e^{\alpha_0|u|^{\frac{N}{N-s}}} \mathrm{d}x.$$

We can choose $\varepsilon > 0$ such that $C_2 = \frac{s}{N} - \lambda C_1\varepsilon > 0$. Now, if $\|u\|^{\frac{N}{N-s}} < \frac{\alpha_*}{4\alpha_0}$, then, by the Hölder inequality and by Lemma 4.1.1, we obtain that

$$\begin{aligned}
\int_\Omega |u|^q e^{\alpha_0|u|^{\frac{N}{N-s}}} \mathrm{d}x &\leq \left(\int_\Omega |u|^{2q}\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega e^{2\alpha_0|u|^{\frac{N}{N-s}}}\mathrm{d}x\right)^{\frac{1}{2}} \\
&= \|u\|_{2q}^q \left(\int_\Omega e^{2\alpha_0\|u\|^{\frac{N}{N-s}}(|u|/\|u\|)^{\frac{N}{N-s}}}\mathrm{d}x\right)^{\frac{1}{2}} \\
&\leq \|u\|_{2q}^q \left(\int_\Omega e^{\frac{\alpha_*}{2}(|u|/\|u\|)^{\frac{N}{N-s}}}\mathrm{d}x\right)^{\frac{1}{2}} \leq C_3\|u\|_{2q}^q.
\end{aligned} \tag{4.40}$$

Hence, for $\|u\|^{\frac{N}{N-s}} < \frac{\alpha_*}{4\alpha_0}$, by (4.40), we have

$$I_\lambda(u) \geq C_2\|u\|^{N/s} - C_4\|u\|_{2q}^q.$$

Then, using again Lemma 4.3.3, we find

$$I_\lambda(u) \geq C_2\|u\|^{N/s} - C_5\|u\|^q, \quad \text{for all} \quad \|u\|^{\frac{N}{N-s}} < \frac{\alpha_*}{4\alpha_0}. \tag{4.41}$$

Thus, since $q > \frac{N}{s}$, we can choosing $0 < r < \frac{\alpha_*}{4\alpha_0}$ small enough, such that

$$\|u\|^q \leq \frac{C_2}{2C_5}\|u\|^{N/s}, \quad \text{for all} \quad \|u\| \leq r.$$

Hence, by (4.41) and the inequality above we get

$$I_\lambda(u) \geq C\|u\|^{N/s}, \quad \text{for all} \quad \|u\| \leq r,$$

proving **(ii)**. ■

As an application of Lemma 4.4.3, we will show that $\mathcal{M}_\lambda \neq \emptyset$. Moreover, we will obtain an important geometric property of ψ_u^λ, with $u \in \mathcal{M}_\lambda$.

Lemma 4.4.4 *Assume that* (V_1) *and* $(f_1)-(f_4)$ *are satisfied. Let* $u \in X$ *with* $u^{\pm} \neq 0$. *Then, there exists a unique pair of positive number* (α_u, β_u) *such that* $\alpha_u u^+ + \beta_u u^- \in \mathcal{M}_\lambda$. *Moreover, if* $(\alpha, \beta) \neq (\alpha_u, \beta_u)$, *with* $\alpha, \beta \geq 0$, *then*

$$I_\lambda(\alpha u^+ + \beta u^-) < I_\lambda(\alpha_u u^+ + \beta_u u^-).$$

Proof . By Lemma 4.4.3, there exists $(\alpha_u, \beta_u) \in [0, \infty) \times [0, \infty)$ such that

$$I_\lambda(\alpha_u u^+ + \beta_u u^-) = \max_{[0,\infty)\times[0,\infty)} \psi_u^\lambda(\alpha, \beta).$$

Next we will show that $(\alpha_u, \beta_u) \in (0, \infty) \times (0, \infty)$. Since $u^{\pm} \neq 0$, using **(ii)** of Lemma 4.4.3, we have

$$I_\lambda(\alpha u^+) > 0 \quad \text{and} \quad I_\lambda(\beta u^-) > 0, \quad \text{for} \quad \alpha, s > 0 \quad \text{small enough.}$$

Thus, for $\alpha, \beta > 0$ small enough and by Corollary 4.3.7, we get

$$\psi_u^\lambda(\alpha, 0) = I_\lambda(\alpha u^+) < I_\lambda(\alpha u^+) + I_\lambda(\beta u^-) < I_\lambda(\alpha u^+ + \beta u^-) = \psi_u^\lambda(\alpha, \beta).$$

Thus, we deduce that $\alpha_u, \beta_u > 0$. Hence $\partial_\alpha \psi_u^\lambda(\alpha_u, \beta_u) = 0$ and $\partial_\beta \psi_u^\lambda(\alpha_u, \beta_u) = 0$. In particular, we have $\alpha_u u^+ + \beta_u u^- \in \mathcal{M}_\lambda$.

We will show that (α_u, β_u) is the unique in $(0, \infty) \times (0, \infty)$ with this property. It is sufficient to consider the case where $u \in \mathcal{M}_\lambda$ and $\alpha u^+ + \beta u^- \in \mathcal{M}_\lambda$, with $\alpha, \beta > 0$, and to prove that it implies in $(\alpha, \beta) = (1, 1)$. As $I'_\lambda(u)u^{\pm} = 0, I'_\lambda(\alpha u^+ + \beta u^-)\alpha u^+ = 0$ and $I'_\lambda(\alpha u^+ + \beta u^-)\beta u^- = 0$, we can write

$$\langle (-\Delta)^s_{N/s} u, u^{\pm} \rangle + \int_\Omega V(x)|u^{\pm}|^{N/s} \mathrm{d}x = \int_\Omega f(x, u^{\pm})u^{\pm} \mathrm{d}x, \tag{4.42}$$

$$\langle (-\Delta)^s_{N/s}(\alpha u^+ + \beta u^-), \alpha u^+ \rangle + \alpha^{N/s} \int_\Omega V(x)|u^+|^{N/s} \mathrm{d}x = \int_\Omega f(x, \alpha u^+)\alpha u^+ \mathrm{d}x \tag{4.43}$$

and

$$\langle (-\Delta)^s_{N/s}(\alpha u^+ + \beta u^-), \beta u^- \rangle + \beta^{N/s} \int_\Omega V(x)|u^-|^{N/s} \mathrm{d}x = \int_\Omega f(x, \beta u^-)\beta u^- \mathrm{d}x. \tag{4.44}$$

Without loss of generality, we can assume that $0 < \beta \leq \alpha$. Now, using **(i)**, **(ii)** and **(iii)** of Lemma 4.3.6, we get

$$\begin{aligned} \langle (-\Delta)^s_{N/s}(\alpha u^+ + \beta u^-), \alpha u^+ \rangle &= A^+(\alpha u^+ + \beta u^-) + B^+(\alpha u^+ + \beta u^-) \\ &= \alpha^{N/s} A^+(u) + B^+(\alpha u^+ + \beta u^-) \\ &\leq \alpha^{N/s} A^+(u) + \alpha^{N/s} B^+(u) \\ &= \alpha^{N/s} \langle (-\Delta)^s_{N/s} u, u^+ \rangle. \end{aligned} \tag{4.45}$$

Thus, by (4.43), (4.45) and (4.42), we get

$$\int_\Omega f(x,\alpha u^+)\alpha u^+ \mathrm{d}x \leq \alpha^{N/s}\langle(-\Delta)^s_{N/s}u,u^+\rangle + \alpha^{N/s}\int_\Omega V(x)|u^+|^{N/s}\mathrm{d}x$$
$$= \alpha^{N/s}\int_\Omega f(x,u^+)u^+\mathrm{d}x.$$

Then, we obtain that

$$\int_\Omega \left(\frac{f(x,\alpha u^+)}{|\alpha u^+|^{\frac{N}{s}-2}\alpha u^+} - \frac{f(x,u)}{|u^+|^{\frac{N}{s}-2}u^+}\right)|u^+|^{N/s}\mathrm{d}x \leq 0.$$

Hence, by (f_4), we have $\alpha \leq 1$ and so $0 < \beta \leq \alpha \leq 1$. Using again **(i)**, **(ii)** and **(iii)** of Lemma 4.3.6, we have

$$\langle(-\Delta)^s_{N/s}(\alpha u^+ + \beta u^-),\beta u^-\rangle \geq \beta^{N/s}\langle(-\Delta)^s_{N/s}u,u^-\rangle. \tag{4.46}$$

Thus, by (4.44), (4.46) and (4.42), we deduce that

$$\int_\Omega \left(\frac{f(x,\beta u^-)}{|\beta u^-|^{\frac{N}{s}-2}\beta u^-} - \frac{f(x,u)}{|u^-|^{\frac{N}{s}-2}u^-}\right)|u^-|^{N/s}\mathrm{d}x \geq 0.$$

Hence, again by (f_4), we obtain that $\beta \geq 1$ and so $\alpha = \beta = 1$. This concludes the proof of the lemma. ■

Remark 4.4.5 *Note that, any nodal solution to* (P_λ) *belongs to* $\mathcal{M}_\lambda$. *Similarly, any nonnegative solution and nonpositive solution to* $(P_{a,b})$ *belongs to* $\mathcal{N}^+_\lambda$ *and* $\mathcal{N}^-_\lambda$, *respectively. Let* $u \in \mathcal{M}_\lambda$. *By Lemma 4.4.1, there exist* $\alpha,\beta > 0$ *such that* $\alpha u^+ \in \mathcal{N}^+_\lambda$ *and* $\beta u^- \in \mathcal{N}^-_\lambda$. *Now, using Lemma 4.4.4, we have* $I_\lambda(\alpha u^+ + \beta u^-) \leq I_\lambda(u)$. *Thus, by definition of the levels in Nehari sets and by using Corollary 4.3.7, we infer that*

$$2c_{\mathcal{N}_\lambda} \leq c_{\mathcal{N}^+_\lambda} + c_{\mathcal{N}^-_\lambda} \leq I_\lambda(\alpha u^+) + I_\lambda(\beta u^-) < I_\lambda(tu^+ + su^-) \leq I_\lambda(u).$$

Hence, taking the infimum in $u \in \mathcal{M}_\lambda$, *we obtain that*

$$2c_{\mathcal{N}_\lambda} \leq c_{\mathcal{N}^+_\lambda} + c_{\mathcal{N}^-_\lambda} \leq c_{\mathcal{M}_\lambda}.$$

In particular, if $c_{\mathcal{M}_\lambda}$ *is achieved for some function in* $\mathcal{M}_\lambda$, *then, we get* $c_{\mathcal{M}_\lambda} > c_{\mathcal{N}^+_\lambda} + c_{\mathcal{N}^-_\lambda} \geq 2c_{\mathcal{N}_\lambda}$, *as in* (4.8), *and, in this case,* $c_{\mathcal{N}_\lambda} = c_{\mathcal{N}^+_\lambda}$ *and* $c_{\mathcal{N}_\lambda} = c_{\mathcal{N}^-_\lambda}$.

Lemma 4.4.6 *Assume that* (V_1) *and* $(f_1)-(f_4)$ *are satisfied. Let* $u \in X$ *such that* $u^\pm \neq 0$ *and* $I'_\lambda(u)u^\pm \leq 0$. *Then, the unique pair* (α,β) *given in Lemma 4.4.4 satisfies* $0 < \alpha,\beta \leq 1$.

Proof . We can suppose, without loss of generality, that $0 < \beta \leq \alpha$. Now, we have

$$\langle (-\Delta)^s_{N/s} u, u^+ \rangle + \int_\Omega V(x)|u^+|^{N/s}\mathrm{d}x \leq \int_\Omega f(x,u^+)u^+\mathrm{d}x$$

and

$$\langle (-\Delta)^s_{N/s}(\alpha u^+ + \beta u^-), \alpha u^+ \rangle + \alpha^{N/s} \int_\Omega V(x)|u^+|^{N/s}\mathrm{d}x = \int_\Omega f(x,\alpha u^+)\alpha u^+\mathrm{d}x.$$

Using **(i)**, **(ii)** and **(iii)** of Lemma 4.3.6, as in (4.45), we get

$$\begin{aligned}\int_\Omega f(x,\alpha u^+)\alpha u^+\mathrm{d}x &\leq \alpha^{N/s}\langle (-\Delta)^s_{N/s} u, u^+ \rangle + \alpha^{N/s}\int_\Omega V(x)|u^+|^{N/s}\mathrm{d}x \\ &= \alpha^{N/s}\int_\Omega f(x,u^+)u^+\mathrm{d}x.\end{aligned}$$

Thus, we have

$$\int_\Omega \left(\frac{f(x,\alpha u^+)}{|\alpha u^+|^{\frac{N}{s}-2}\alpha u^+} - \frac{f(x,u^+)}{|u^+|^{\frac{N}{s}-2}u^+} \right)|u^+|^{N/s}\mathrm{d}x \leq 0.$$

Thus, by (f_4), we obtain that $\alpha \leq 1$. Hence, we have $0 < \beta, \alpha \leq 1$. This completes the proof. ■

Lemma 4.4.7 *Assume that* (V_1) *and* (f_1) *are satisfied. There exists* $m_\lambda > 0$ *such that*

(i) $\|u\|^{N/s} \geq m_\lambda$ *for all* $u \in \mathcal{N}_\lambda$.

(ii) $\|u^\pm\|^{N/s} \geq m_\lambda$ *for all* $u \in \mathcal{M}_\lambda$.

Proof . We will show only the item **(ii)**. Suppose, by contradiction, that (u_n) is a sequence in $\mathcal{M}_\lambda$ such that $\|u_n^+\|^{N/s} \to 0$ as $n \to \infty$. Now $I'_\lambda(u_n)u_n^+ = 0$ and $u_n^\pm \neq 0$ for all $n \in \mathbb{N}$, by **(v)** of Lemma 4.3.6, and so

$$0 < \|u_n^+\|^{N/s} < \langle (-\Delta)^s_{N/s} u_n, u_n^+ \rangle + \int_\Omega V(x)|u_n^+|^{N/s}\mathrm{d}x = \lambda \int_\Omega f(x,u_n^+)u_n^+\mathrm{d}x,$$

for all $n \in \mathbb{N}$. Since $\|u_n^+\|^{N/s} \to 0$ as $n \to \infty$, given $q > \frac{N}{s}$, using (4.27), by the Hölder inequality and Lemma 4.1.1, as in (4.40), we get

$$\|u_n^+\|^{N/s} < \lambda C \|u_n^+\|^q_{2q}, \quad \text{for all} \quad n \quad \text{large enough.}$$

Thus, by Lemma 4.3.3, there exists $n_0 \in \mathbb{N}$ such that

$$\|u_n^+\|^{N/s} < \lambda C_1 \|u_n^+\|^q, \quad \text{for all} \quad n \geq n_0.$$

Since $u_n^+ \neq 0$ for all $n \in \mathbb{N}$, we obtain that

$$\frac{1}{\lambda C_1} \leq \|u_n^+\|^{q-\frac{N}{s}} \quad \text{for all} \quad n \geq n_0,$$

which is a contradiction with our assumption, and the proof is complete. ■

Corollary 4.4.8 *Assume that* (V_1), (f_1) *and* (f_3) *are satisfied. Then, there exists* $\delta_\lambda > 0$ *such that* $I_\lambda(u) \geq \delta_\lambda$ *for all* $u \in \mathcal{N}_\lambda$. *In particular*

$$\delta_\lambda \leq c_{\mathcal{N}_\lambda},\ \delta_\lambda \leq c_{\mathcal{N}_\lambda^\pm} \quad \textit{and} \quad \delta_\lambda \leq c_{\mathcal{M}_\lambda}.$$

Proof . Let $u \in \mathcal{N}_\lambda$. By Lemma 4.4.7 and by (f_3), we have

$$\begin{aligned} I_\lambda(u) &= I_\lambda(u) - \frac{1}{\theta} I_\lambda'(u)u \\ &= \left(\frac{s}{N} - \frac{1}{\theta}\right) \|u\|^{N/s} + \frac{\lambda}{\theta} \int_\Omega f(x,u)u - \theta F(x,u)\, \mathrm{d}x \\ &\geq \left(\frac{s}{N} - \frac{1}{\theta}\right) m_\lambda := \delta_\lambda > 0. \end{aligned}$$

This is the desired conclusion. ■

Lemma 4.4.9 *Assume that* $(V_1), (f_1)$ *and* (f_4) *are satisfied. Let* $u \in \mathcal{M}_\lambda$. *Then* $\det J_{(1,1)}\Psi_u^\lambda > 0$, *where* Ψ_u^λ *is defined in* (4.34) *and* $J_{(1,1)}\Psi_u^\lambda$ *is the Jacobian matrix of* Ψ_u^λ *at the point* $(1,1)$.

Proof . Let $u \in \mathcal{M}_\lambda$. Let us denote by

$$\Psi_u^{\lambda,1}(\alpha,\beta) = I_\lambda'(\alpha u^+ + \beta u^-)\alpha u^+ \quad \text{and} \quad \Psi_u^{\lambda,2}(\alpha,\beta) = I_\lambda'(\alpha u^+ + \beta u^-)\beta u^-$$

the components functions of the vector field Ψ_u^λ. Explicitly, by **(i)** and **(ii)** of Lemma 4.3.6, we have

$$\Psi_u^{\lambda,1}(\alpha,\beta) = \alpha^{N/s}A^+(u) + B^+(\alpha u^+ + \beta u^-) + \alpha^{N/s}\int_\Omega V(x)|u^+|\mathrm{d}x - \lambda \int_\Omega f(x,\alpha u^+)\alpha u^+ \mathrm{d}x$$

and

$$\Psi_u^{\lambda,2}(\alpha,\beta) = \beta^{N/s}A^-(u) + B^-(\alpha u^+ + \beta u^-) + \beta^{N/s}\int_\Omega V(x)|u^-|\mathrm{d}x - \lambda \int_\Omega f(x,\beta u^-)\beta u^- \mathrm{d}x.$$

By a straightforward computation, we get

$$\begin{aligned}\partial_\alpha \Psi_u^{\lambda,1}(1,1) =& \frac{N}{s}A^+(u) + B^+(u) + \frac{N}{s}\int_\Omega V(x)|u^+|^{N/s}\mathrm{d}x \\ &+2\left(\frac{N}{s}-1\right)\int_{\Omega_+\times\Omega_-}\frac{(u^+(x)-u^-(y))^{\frac{N}{s}-2}u^+(x)(u^+(x)-u^+(y))}{|x-y|^{2N}}\mathrm{d}x\mathrm{d}y \\ &-\lambda\int_\Omega f'(x,u^+)(u^+)^2 + f(x,u^+)u^+\mathrm{d}x;\end{aligned}$$

$$\begin{aligned}\partial_\beta \Psi_u^{\lambda,1}(1,1) =& \partial_\alpha \Psi_u^{\lambda,2}(1,1) \\ =& 2\left(\frac{N}{s}-1\right)\int_{\Omega_+\times\Omega_-}\frac{(u^+(x)-u^-(y))^{\frac{N}{s}-2}u^+(x)(-u^-(y))}{|x-y|^{2N}}\mathrm{d}x\mathrm{d}y > 0\end{aligned} \tag{4.47}$$

and

$$\begin{aligned}\partial_\beta \Psi_u^{\lambda,2}(1,1) =& \frac{N}{s}A^-(u) + B^-(u) + \frac{N}{s}\int_\Omega V(x)|u^-|^{N/s}\mathrm{d}x \\ &+2\left(\frac{N}{s}-1\right)\int_{\Omega_+\times\Omega_-}\frac{(u^+(x)-u^-(y))^{\frac{N}{s}-2}(-u^-(y))(u^-(x)-u^-(y))}{|x-y|^{2N}}\mathrm{d}x\mathrm{d}y \\ &-\lambda\int_\Omega f'(x,u^-)(u^-)^2 + f(x,u^-)u^-\mathrm{d}x.\end{aligned}$$

As $I'_\lambda(u)u^+ = 0$, using again **(i)** and **(ii)** of Lemma 4.3.6, we have

$$\frac{N}{s}A^+(u) + \frac{N}{s}B^+(u) + \frac{N}{s}\int_\Omega V(x)|u^+|^{N/s}\mathrm{d}x = \frac{N}{s}\lambda\int_\Omega f(x,u^+)u^+\mathrm{d}x.$$

Thus, we get

$$\begin{aligned}\partial_\alpha \Psi_u^{\lambda,1}(1,1) =& \frac{N}{s}\lambda\int_\Omega f(x,u^+)u^+\mathrm{d}x - \left(\frac{N}{s}-1\right)B^+(u) \\ &+2\left(\frac{N}{s}-1\right)\int_{\Omega_+\times\Omega_-}\frac{(u^+(x)-u^-(y))^{\frac{N}{s}-2}u^+(x)(u^+(x)-u^+(y))}{|x-y|^{2N}}\mathrm{d}x\mathrm{d}y \\ &-\lambda\int_\Omega f'(x,u^+)(u^+)^2 + f(x,u^+)u^+\mathrm{d}x.\end{aligned}$$

Hence, we have

$$\begin{aligned}\partial_\alpha \Psi_u^{\lambda,1}(1,1) =& \lambda\int_\Omega \left(\frac{N}{s}-1\right)f(x,u^+)u^+ - f'(x,u^+)(u^+)^2\mathrm{d}x - \left(\frac{N}{s}-1\right)B^+(u) \\ &+2\left(\frac{N}{s}-1\right)\int_{\Omega_+\times\Omega_-}\frac{(u^+(x)-u^-(y))^{\frac{N}{s}-2}u^+(x)(u^+(x)-u^+(y))}{|x-y|^{2N}}\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Then, since $u^+ \neq 0$, by (4.31) and by expression of $B^+(u)$, we deduce that

$$\begin{aligned}\partial_\alpha \Psi_u^{\lambda,1}(1,1) <& -\left(\frac{N}{s}-1\right) B^+(u)\\ &+2\left(\frac{N}{s}-1\right)\int_{\Omega_+\times\Omega_-}\frac{(u^+(x)-u^-(y))^{\frac{N}{s}-2}u^+(x)(u^+(x)-u^+(y))}{|x-y|^{2N}}\mathrm{d}x\mathrm{d}y\\ =& -\partial_\beta \Psi_u^{\lambda,1}(1,1) < 0.\end{aligned} \tag{4.48}$$

Similarly, we can deduce that

$$\partial_\beta \Psi_u^{\lambda,2}(1,1) < -\partial_\alpha \Psi_u^{\lambda,2}(1,1) < 0. \tag{4.49}$$

Hence, by (4.47), (4.48) and (4.49), we have

$$\begin{aligned}\det J_{(1,1)}\Psi_u^\lambda =& \partial_\alpha \Psi_u^{\lambda,1}(1,1)\partial_\beta \Psi_u^{\lambda,2}(1,1) - \partial_\beta \Psi_u^{\lambda,1}(1,1)\partial_\alpha \Psi_u^{\lambda,2}(1,1)\\ =& \partial_\alpha \Psi_u^{\lambda,1}(1,1)\partial_\beta \Psi_u^{\lambda,2}(1,1) - \left(\partial_\beta \Psi_u^{\lambda,1}(1,1)\right)^2\\ >& \left(\partial_\beta \Psi_u^{\lambda,1}(1,1)\right)^2 - \left(\partial_\beta \Psi_u^{\lambda,1}(1,1)\right)^2 = 0.\end{aligned}$$

■

As it was stated at the Introduction, the exponential critical growth in the nonlinearity f produces a lack of compactness in the operator I_λ. The next lemma will be a powerful tool to overcome this difficulty for a minimizing sequence associated to $c_{\mathcal{M}_\lambda}$ in $\mathcal{M}_\lambda$.

Lemma 4.4.10 *Assume that* (V_1) *and* $(f_1)-(f_5)$ *are satisfied. Then*

(i) $c_{\mathcal{M}_\lambda}$ *is nonincreasing in* $\lambda > 0$*;*

(ii) $\lim_{\lambda\to\infty} c_{\mathcal{M}_\lambda} = 0.$

Proof . Let $0 < \lambda_1 < \lambda_2$ and $u \in \mathcal{M}_{\lambda_1}$. By Lemma 4.4.4, there exist $\alpha_2, \beta_2 > 0$ such that $\alpha_2 u^+ + \beta_2 u^- \in \mathcal{M}_{\lambda_2}$. Now, using again Lemma 4.4.4, we have $\psi_u^{\lambda_1}(1,1) \geq \psi_u^{\lambda_1}(\alpha_2,\beta_2)$. Thus, by (f_3), we get

$$\begin{aligned}I_{\lambda_1}(u) \geq& I_{\lambda_1}(\alpha_2 u^+ + \beta_2 u^-)\\ =& I_{\lambda_2}(\alpha_2 u^+ + \beta_2 u^-) + (\lambda_2-\lambda_1)\int_\Omega F(x,\alpha_2 u^+ + \beta_2 u^-)\,\mathrm{d}x\\ >& I_{\lambda_2}(\alpha_2 u^+ + \beta_2 u^-) \geq c_{\mathcal{M}_{\lambda_2}}.\end{aligned}$$

Hence, taking the infimum in $u \in \mathcal{M}_{\lambda_1}$, we obtain that $c_{\mathcal{M}_{\lambda_1}} \geq c_{\mathcal{M}_{\lambda_2}}$, this completes the proof of **(i)**.

Let $u \in X$ with $u^{\pm} \neq 0$. By Lemma 4.4.4, for each $\lambda > 0$, there exist $\alpha_\lambda, \beta_\lambda > 0$ such that $\alpha_\lambda u^+ + \beta_\lambda u^- \in \mathcal{M}_\lambda$. Using again Remark 4.4.5, Corollary 4.4.8 and by (f_3), we have

$$0 < c_{\mathcal{M}_\lambda} \leq I_\lambda(\alpha_\lambda u^+ + \beta_\lambda u^-) \leq \frac{s}{N}\|\alpha_\lambda u^+ + \beta_\lambda u^-\|^{N/s}.$$

Thus, to show **(ii)**, it is enough to prove that $(\alpha_\lambda, \beta_\lambda) \to (0,0)$ as $\lambda \to \infty$. Now, by (4.30) and since $I'_\lambda(\alpha_\lambda u^+ + \beta_\alpha u^-)(\alpha_\lambda u^+ + \beta_\alpha u^-) = 0$, we have

$$\begin{aligned} \lambda C\|\alpha_\lambda u^+ + \beta_\lambda u^-\|_p^p &\leq \lambda \int_\Omega f(x, \alpha_\lambda u^+ + \beta_\lambda u^-)(\alpha_\lambda u^+ + \beta_\lambda u^-)\,\mathrm{d}x \\ &= \|\alpha_\lambda u^+ + \beta_\lambda u^-\|^{N/s}. \end{aligned}$$

Hence, since $p > \frac{N}{s}$, this inequality implies that $\{(\alpha_\lambda, \beta_\lambda)\}_{\lambda \geq 1}$ is bounded. Without loss of generality, we can assume that

$$(\alpha_n, \beta_n) \to (\alpha', \beta'), \quad \text{as} \quad \lambda_n \to \infty,$$

where $\alpha_n = \alpha_{\lambda_n}$and $\beta_n = \beta_{\lambda_n}$. In particular, we have $\alpha_n u^+ + \beta_n u^- \to \alpha' u^+ + \beta' u^-$ strongly in X, as $n \to \infty$. We claim that $\alpha' = \beta' = 0$. Indeed, if $\alpha' > 0$, then, by (f_3), we have

$$\int_\Omega f(x, \alpha_n u^+ + \beta_n u^-)(\alpha_n u^+ + \beta_n u^-)\,\mathrm{d}x \to \int_\Omega f(x, \alpha' u^+ + \beta' u^-)(\alpha' u^+ + \beta' u^-)\,\mathrm{d}x > 0 \tag{4.50}$$

as $n \to \infty$. But, for all $n \in \mathbb{N}$, we have

$$\|\alpha_n u^+ + \beta_n u^-\|^{N/s} = \lambda_n \int_\Omega f(x, \alpha_n u^+ + \beta_n u^-)(\alpha_n u^+ + \beta_n u^-)\,\mathrm{d}x. \tag{4.51}$$

Since $\lambda_n \to \infty$ and $(\|\alpha_n u^+ + \beta_n u^-\|)$ is a bounded sequence, by (4.50) and (4.51), we obtain a contradiction. Therefore $\alpha' = \beta' = 0$, and the prove is complete. ∎

Remark 4.4.11 *The same results of Lemma 4.4.10 holds for* $c_{\mathcal{N}_\lambda}, c^+_{\mathcal{N}_\lambda}$ *and* $c_{\mathcal{N}^-_\lambda}$.

As a consequence of Lemma 4.4.10 we obtain the following corollary:

Corollary 4.4.12 *Assume that* (V_1) *and* $(f_1) - (f_5)$ *are satisfied. Then, there exists* $\lambda^* > 0$ *such that*

$$c_{\mathcal{M}_\lambda} < \left(\frac{\theta s - N}{N\theta}\right)\left(\frac{\alpha_*}{2\alpha_0}\right)^{\frac{N-s}{s}}, \quad \textit{for all} \quad \lambda \geq \lambda^*.$$

In particular, the same inequalities apply to for $c_{\mathcal{N}_\lambda}, c_{\mathcal{N}^+_\lambda}$ *and* $c^-_{\mathcal{N}_\lambda}$.

From now on, we will consider that $\lambda \geq \lambda^*$, where λ^* is given in Corollary 4.4.12.

As a consequence of Corollary 4.4.12, we will obtain appropriate behavior for a minimizing sequence in $\mathcal{M}_\lambda$. In order to make this more precise, we consider the following subset of $\mathcal{M}_\lambda$:

$$\widetilde{S}_\rho = \{u \in \mathcal{M}_\lambda \,:\, I_\lambda(u) < c_{\mathcal{M}_\lambda} + \rho\} \quad \text{for} \quad \rho > 0.$$

Lemma 4.4.13 *Assume that* (V_1) *and* (f_1)*–*(f_5) *are satisfied. For* $\rho > 0$ *small enough, there exists* $m_\rho \in \left(0, \frac{\alpha^*}{2\alpha_0}\right)$ *such that*

$$0 < m_\lambda^{\frac{s}{N-s}} \leq \|u^\pm\|^{\frac{N}{N-s}} < \|u\|^{\frac{N}{N-s}} \leq m_\rho \quad \textit{for all} \quad u \in \widetilde{S}_\rho.$$

Proof . By Corollary 4.4.12, we can choose $\rho > 0$ small enough such that

$$c_{\mathcal{M}_\lambda} + \rho < \left(\frac{\theta s - N}{N\theta}\right)\left(\frac{\alpha_*}{2\alpha_0}\right)^{\frac{N-s}{s}}.$$

Let $u \in \widetilde{S}_\rho$. Then, by (f_3), we have

$$\begin{aligned} c_{\mathcal{M}_\lambda} + \rho > I_\lambda(u) &= I_\lambda(u) - \frac{1}{\theta} I'_\lambda(u)u \\ &= \left(\frac{s}{N} - \frac{1}{\theta}\right)\|u\|^{N/s} + \frac{\lambda}{\theta}\int_\Omega f(x,u)u - \theta F(x,u)\,\mathrm{d}x \\ &\geq \left(\frac{s}{N} - \frac{1}{\theta}\right)\|u\|^{N/s}. \end{aligned}$$

Hence, we get

$$\|u\|^{\frac{N}{N-s}} \leq \left[\left(\frac{N\theta}{\theta s - N}\right)(c_{\mathcal{M}_\lambda} + \rho)\right]^{\frac{s}{N-s}} := m_\rho < \frac{\alpha_*}{2\alpha_0}.$$

Therefore, by Lemma 4.4.7, we get $0 < m_\lambda^{\frac{s}{N-s}} \leq \|u^\pm\|^{\frac{N}{N-s}} < \|u\|^{\frac{N}{N-s}} \leq m_\rho$ for all $u \in \widetilde{S}_\rho$. ■

From now on, we will write $\widetilde{S}_\rho$, where ρ is given in Lemma 4.4.13.

The next lemma will be used to show that does no exist a sequence (u_n) in $\widetilde{S}_\rho$ convergent in $L^q(\Omega)$ to a nonnodal function, for $q \geq 1$.

Lemma 4.4.14 *Assume that* (V_1) *and* $(f_1) - (f_5)$ *are satisfied. For any* $q \geq 1$*, there exists* $\delta_q > 0$ *such that*

$$0 < \delta_q \leq \int_\Omega |u^\pm|^q \mathrm{d}x < \int_\Omega |u|^q \mathrm{d}x,$$

for any $u \in \widetilde{S}_\rho$.

Proof . Fix $q \geq 1$. Let $u \in \widetilde{S}_\rho$. Since $I'_\lambda(u)u^\pm = 0$, by Lemma 4.4.7 and by **(v)** of Lemma 4.3.6, we have

$$0 < m_\lambda \leq \|u^\pm\|^{N/s} \leq \lambda \int_\Omega f(x, u^\pm)u^\pm \mathrm{d}x.$$

Thus, by (f_1), we deduce that

$$0 < \frac{m_\lambda}{\lambda C_0} \leq \int_\Omega |u^\pm| e^{\alpha_0 |u^\pm|^{\frac{N}{N-s}}} \mathrm{d}x. \tag{4.52}$$

Let $r > q$ and $r' < 2$ such that $1/r + 1/r' = 1$. By the Hölder inequality, by Lemma 4.4.13 and by Lemma 4.1.1, we have

$$\begin{aligned} \int_\Omega |u^\pm| e^{\alpha_0 |u^\pm|^{\frac{N}{N-s}}} \mathrm{d}x &\leq \left(\int_\Omega |u^\pm|^r \mathrm{d}x\right)^{\frac{1}{r}} \left(\int_\Omega e^{r'\alpha_0 \|u^\pm\|^{\frac{N}{N-s}} (|u^\pm|/\|u^\pm\|)^{\frac{N}{N-s}}} \mathrm{d}x\right)^{\frac{1}{r'}} \\ &\leq \|u^\pm\|_r^r \left(\int_\Omega e^{(r'\alpha_*/2)(|u^\pm|/\|u^\pm\|)^{\frac{N}{N-s}}} \mathrm{d}x\right)^{\frac{1}{r'}} \\ &\leq C_1 \|u^\pm\|_r^r. \end{aligned}$$

Thus, by (4.52), we have

$$0 < C \leq \|u^\pm\|_r^r \quad \text{for all} \quad u \in \widetilde{S}_\rho. \tag{4.53}$$

We suppose, by contradiction, that there exists (u_n) in $\widetilde{S}_\rho$ such that $\|u_n^\pm\|_q \to 0$ as $n \to \infty$. From Lemma 4.4.13 and Lemma 4.3.5 we obtain that $(u_n^\pm)$ is bounded in $L^{2r}(\Omega)$. Now, since $q < r < 2r$, using the interpolation inequality, we obtain that

$$\|u_n^\pm\|_r \leq \|u_n^\pm\|_q^\xi \cdot \|u_n^\pm\|_{2r}^{1-\xi} \to 0, \quad \text{as} \quad n \to \infty,$$

where $\xi \in (0,1)$, but, in view of (4.53), the convergence above is impossible. This completes the proof of the lemma. ■

Lemma 4.4.15 *Assume that $(V_1), (f_1)$ and (f_3) are satisfied. Let (u_n) be a sequence in X such that $u_n \rightharpoonup u$ weakly in X and $b := \sup_{n\in\mathbb{N}} \|u_n\|^{\frac{N}{N-s}} \leq \frac{\alpha_*}{2\alpha_0}$. Then, for all $v \in X$, up to a subsequence, we have*

$$\lim_{n\to\infty} \int_\Omega f(x, u_n)u_n \,\mathrm{d}x = \int_\Omega f(x, u)u \,\mathrm{d}x; \tag{4.54}$$

$$\lim_{n\to\infty} \int_\Omega f(x, u_n^\pm)u_n^\pm \,\mathrm{d}x = \int_\Omega f(x, u^\pm)u^\pm \,\mathrm{d}x; \tag{4.55}$$

$$\lim_{n\to\infty} \int_\Omega f(x, u_n^\pm)v \,\mathrm{d}x = \int_\Omega f(x, u^\pm)v \,\mathrm{d}x \tag{4.56}$$

and

$$\lim_{n\to\infty} \int_\Omega F(x, u_n^\pm) \,\mathrm{d}x = \int_\Omega F(x, u^\pm) \,\mathrm{d}x. \tag{4.57}$$

Proof . Since $b \leq \frac{\alpha_*}{2\alpha_0}$, using (f_1), (f_3) and by the Hölder's inequality and Lemma 4.3.5, it is easy to see that the integrals

$$\int_\Omega |f(x,u_n)u_n||u_n|\mathrm{d}x, \int_\Omega |f(x,u_n)u_n^\pm||u_n|\mathrm{d}x, \int_\Omega |f(x,u_n)v||u_n|\mathrm{d}x$$

and

$$\int_\Omega |F(x,u_n)||u_n|\mathrm{d}x$$

are uniformly bounded. Thus, using again Lemma 4.3.5, up to a subsequence, the convergences (4.54)-(4.57) follow from Lemma 2.1 of [25]. ■

Lemma 4.4.16 *Assume that and* $(f_1), (f_3)$ *and* (f_4) *are satisfied. Let* $H(x,t) = f(x,t)t - \frac{N}{s}F(x,t)$. *Then*

(i) $H(x,\cdot)$ *is a* C^2 *function,* $H(x,0) = 0$ *and* $H(x,t) > 0$, *for all* $t \neq 0$ *and for all* $x \in \Omega$*;*

(ii) $H(x,\cdot)$ *is strictly increasing in* $(0,\infty)$ *and is strictly decreasing in* $(-\infty,0)$, *for all* $x \in \Omega$.

Proof . The proof of this lemma follows directly from (4.31) and (f_3). ■

Lemma 4.4.17 *Assume that* (V_1) *and* $(f_1) - (f_5)$ *are satisfied. Then, there exists* $\bar{u} \in \mathcal{M}_\lambda$ *such that* $I_\lambda(\bar{u}) = c_{\mathcal{M}_\lambda}$.

Proof . Let (u_n) a sequence in $\widetilde{S}_\rho$ such that $I_\lambda(u_n) \to c_{\mathcal{M}_\lambda}$ as $n \to \infty$. By Lemma 4.4.13, $(u_n^\pm)$ are bounded sequences in X. Thus, by using Lemma B.4.1 and by Lemma 4.3.1, there exists $u \in X$ such that $u_n^\pm \rightharpoonup u^\pm$ as $n \to \infty$. Using Lemma 4.3.5, we can assume that $u_n^\pm \to u^\pm$ in $L^{N/s}(\mathbb{R}^N)$ and $u_n^\pm(x) \to u^\pm(x)$ a.e. in $\mathbb{R}^N$, as $n \to \infty$. Hence, by Lemma 4.4.14, we have $u^\pm \neq 0$ in $L^q(\Omega)$ and so $u^\pm \neq 0$ in X. By Lemma 4.4.4, there exist $\alpha, \beta > 0$ such that $\bar{u} = \alpha u^+ + \beta u^- \in \mathcal{M}_\lambda$. Using again Lemma 4.4.13, without of loss generality, we can assume that the convergences in Lemma 4.4.15 holds to sequence (u_n). Since $u^\pm(x) \to u^\pm(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$, we get

$$0 \leq \frac{|u_n^+(x) - u_n^-(y)|^{\frac{N}{s}-1}u_n^+(x) - |u_n^+(x)|^{\frac{N}{s}}}{|x-y|^{2N}} \to \frac{|u^+(x) - u^-(y)|^{\frac{N}{s}-1}u^+(x) - |u^+(x)|^{\frac{N}{s}}}{|x-y|^{2N}}$$

as $n \to \infty$, for a.e. $(x,y) \in \mathbb{R}^{2N}$. Thus, from **(iv)** of Lemma 4.3.6, by semicontinuity of the seminorm and using Fatou's Lemma, we can deduce that

$$\begin{aligned}\langle(-\Delta)^s_{N/s}u, u^+\rangle &= \langle(-\Delta)^s_{N/s}u^+, u^+\rangle + 2C^+(u)\\ &\leq \liminf_{n\to\infty}\langle(-\Delta)^s_{N/s}u_n^+, u_n^+\rangle + \liminf_{n\to\infty} 2C^+(u_n).\\ &\leq \liminf_{n\to\infty}\left(\langle(-\Delta)^s_{N/s}u_n^+, u_n^+\rangle + 2C^+(u_n)\right)\\ &\leq \liminf_{n\to\infty}\langle(-\Delta)^s_{N/s}u_n, u_n^+\rangle.\end{aligned}$$

Thus, since $I'_\lambda(u_n)u_n^+ = 0$ for all $n \in \mathbb{N}$, and by (4.55), we get

$$\begin{aligned}\langle(-\Delta)^s_{N/s}u, u^+\rangle + \int_\Omega V(x)|u^+|^{N/s}\mathrm{d}x \leq& \liminf_{n\to\infty}\langle(-\Delta)^s_{N/s}u_n, u_n^+\rangle\\ &+ \liminf_{n\to\infty}\int_\Omega V(x)|u_n^+|\mathrm{d}x\\ &\leq \liminf_{n\to\infty}\left(\langle(-\Delta)^s_{N/s}u_n, u_n^+\rangle + \int_\Omega V(x)|u_n^+|\mathrm{d}x\right)\\ &= \liminf_{n\to\infty}\lambda\int_\Omega f(x,u_n^+)u_n^+\mathrm{d}x\\ &= \lambda\int_\Omega f(x,u^+)u^+\mathrm{d}x.\end{aligned}$$

Then $I'_\lambda(u)u^+ \leq 0$. Similarly, we can deduce that $I'_\lambda(u)u^- \leq 0$. Thus, by Lemma 4.4.6, we have $0 < \alpha, \beta \leq 1$. By Lemma 4.4.16 and again using Lemma 4.4.15, we have

$$\begin{aligned}c_{\mathcal{M}_\lambda} \leq I_\lambda(\bar{u}) &= I_\lambda(\alpha u^+ + \beta u^-) - \frac{s}{N}I'_\lambda(\alpha u^+ + \beta u^-)(\alpha u^+ + \beta u^-)\\ &= \frac{s}{N}\lambda\int_\Omega f(x,\alpha u^+ + \beta u^-)(\alpha u^+ + \beta u^-) - \frac{N}{s}F(x,\alpha u^+ + \beta u^-)\,\mathrm{d}x\\ &= \frac{s}{N}\lambda\left(\int_\Omega H(x,\alpha u^+)\,\mathrm{d}x + \int_\Omega H(x,\beta u^-)\,\mathrm{d}x\right)\\ &\leq \frac{s}{N}\lambda\int_\Omega H(x,u)\,\mathrm{d}x = \lim_{n\to\infty}\frac{s}{N}\lambda\int_\Omega H(x,u_n)\,\mathrm{d}x\\ &= \lim_{n\to\infty}\left(I_\lambda(u_n) - \frac{s}{N}I'_\lambda(u_n)u_n\right) = \lim_{n\to\infty} I_\lambda(u_n) = c_{\mathcal{M}_\lambda}.\end{aligned}$$

Therefore $I_\lambda(\bar{u}) = c_\lambda$, which is the desired conclusion. ■

Next we will introduce some notations and a technical result that will be apply in the proof of Theorem 4.2.2.

Let $D = \left(\frac{1}{2}, \frac{3}{2}\right) \times \left(\frac{1}{2}, \frac{3}{2}\right)$ and $g : \overline{D} \to X$, given by $g(\alpha,\beta) = \alpha\bar{u}^+ + \beta\bar{u}^-$, where $\bar{u}$ is given in Lemma 4.4.17.

Lemma 4.4.18 *Let $P = \{u \in X : u(x) \geq 0$ a.e. $x \in \mathbb{R}^N\}$ and $-P = \{u \in X : u(x) \leq 0$ a.e. $x \in \mathbb{R}^N\}$. Then $d' = dist(g(\overline{D}), \Lambda) > 0$, where $\Lambda := P \cup (-P)$.*

Proof . We suppose, by contradiction, that $d' = dist(g(\overline{D}), \Lambda) = 0$. Then, we can find $(v_n) \subset g(\overline{D})$ and $(w_n) \subset \Lambda$ such that $\|v_n - w_n\| \to 0$ as $n \to \infty$. We can assume, without loss of generality, that $w_n(x) \geq 0$ a.e. in $\mathbb{R}^N$. Now, since $v_n \in g(\overline{D})$, there exist $\alpha_n, \beta_n \in [\frac{1}{2}, \frac{3}{2}]$ such that $v_n = \alpha_n \bar{u}^+ + \beta \bar{u}^-$. By compactness of $[\frac{1}{2}, \frac{3}{2}]$, up to a subsequence, we have $\alpha_n \to \alpha'$ and $\beta_n \to \beta'$ as $n \to \infty$. Hence

$$v_n \to \alpha' \bar{u}^+ + \beta' \bar{u}^- \quad \text{in} \quad X \quad \text{as} \quad n \to \infty.$$

Now, by Lemma 4.3.5, we have $v_n(x) \to \alpha' \bar{u}^+(x) + \beta' \bar{u}^-(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$. Again by Lemma 4.3.5 and by uniqueness of limit, we have $w_n(x) \to \alpha' \bar{u}^+(x) + \beta' \bar{u}^-(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$. Since $\bar{u}^- \neq 0$, the convergence above produces a contradiction with the assumption that $w_n(x) \geq 0$ a.e. in $\mathbb{R}^N$ as $n \to \infty$, which completes the proof.
■

4.4.1 Proof of Theorem 4.2.2

By Lemma 4.4.17, we have $\bar{u} \in \mathcal{M}_\lambda$ and $I_\lambda(\bar{u}) = c_{\mathcal{M}_\lambda}$. Thus, it remains to show that $I'_\lambda(\bar{u}) = 0$. Suppose, by contradiction, that $I'_\lambda(\bar{u}) \neq 0$. By the continuity of I'_λ, there exist $\gamma, \delta > 0$ with $\delta \leq \frac{d'}{2}$, such that

$$\|I'_\lambda(v)\| \geq \gamma, \quad \text{for all} \quad v \in B_{3\delta}(\bar{u}), \tag{4.58}$$

where d' is given in Lemma 4.4.18. Since $\bar{u} \in \mathcal{M}_\lambda$, using Lemma 4.4.4, the function $(I_\lambda \circ g)(\alpha, \beta)$, for $(\alpha, \beta) \in \overline{D}$, has a strict maximum point $(1, 1)$. Hence

$$m^* = \max_{(\alpha,\beta)\in\partial D} (I_\lambda \circ g)(\alpha, \beta) < c_{\mathcal{M}_\lambda}.$$

Let $\varepsilon > 0$ be such that $\varepsilon < \min\{(c_{\mathcal{M}_\lambda} - m^*)/2, \gamma\delta/8\}$ and let $S = B_\delta(\bar{u})$. From this choice, for all $(\alpha, \beta) \in \partial D$, we have

$$(I_\lambda \circ g)(\alpha, \beta) \leq m^* = c_{\mathcal{M}_\lambda} - 2(c_{\mathcal{M}_\lambda} - m^*)/2 < c_{\mathcal{M}_\lambda} - 2\varepsilon.$$

Hence, we deduce that

$$g(\partial D) \cap I_\lambda^{-1}([c_{\mathcal{M}_\lambda} - 2\varepsilon, c_{\mathcal{M}_\lambda} + 2\varepsilon]) = \emptyset. \tag{4.59}$$

Moreover, by estimates in (4.58), we have

$$\|I'_b(v)\| \geq \frac{8\varepsilon}{\delta}; \, \forall v \in I_\lambda^{-1}([c_{\mathcal{M}_\lambda} - 2\varepsilon, c_{\mathcal{M}_\lambda} + 2\varepsilon]) \cap S_{2\delta}.$$

Thus, by the quantitative deformation lemma in [72, Lemma 2.3], there exists $\eta \in C([0,1]\times X, X)$ such that

(i) $\eta(t,u) = u$, if $t = 0$ or $u \notin I_\lambda^{-1}([c_{\mathcal{M}_\lambda} - 2\varepsilon, c_{\mathcal{M}_\lambda} + 2\varepsilon]) \cap S_{2\delta}$;

(ii) $\eta(1, I_\lambda^{c_{\mathcal{M}_\lambda}+\varepsilon} \cap S) \subset I_\lambda^{c_{\mathcal{M}_\lambda}-\varepsilon}$;

(iii) $\eta(t,\cdot)$ is an homeomorphism of X, $\forall\, t \in [0,1]$;

(iv) $\|\eta(t,u) - u\| \le \delta$, $\forall\, u \in X$, $\forall\, t \in [0,1]$;

(v) $I_\lambda(\eta(\cdot,u))$ is non increasing, $\forall\, u \in X$;

(vi) $I_\lambda(\eta(t,u)) < c_{\mathcal{M}_\lambda}$, $\forall\, u \in I_\lambda^{c_{\mathcal{M}_\lambda}} \cap S_\delta$, $\forall\, t \in (0,1]$.

Let $h : \overline{D} \to X$ defined by $h(\alpha,\beta) = \eta(1, g(\alpha,\beta))$. We claim that

$$\max_{(\alpha,\beta)\in\overline{D}} I_\lambda(h(\alpha,\beta)) < c_{\mathcal{M}_\lambda}. \tag{4.60}$$

Indeed, if $(\alpha,\beta) \in \overline{D}$ with $(\alpha,\beta) \neq (1,1)$, using Lemma 4.4.4, we have $I_\lambda(g(t,s)) < c_{\mathcal{M}_\lambda}$. Hence

$$I_\lambda(h(\alpha,\beta)) \le I(\eta(0, g(\alpha,\beta))) = I(g(\alpha,\beta)) < c_{\mathcal{M}_\lambda}.$$

If $(\alpha,\beta) = (1,1)$ then $g(1,1) = \bar{u} \in I_\lambda^{c_{\mathcal{M}_\lambda}} \cap S_\delta$ and so $I_\lambda(h(1,1)) < c_{\mathcal{M}_\lambda}$, showing (4.60). By using the definition of $c_{\mathcal{M}_\lambda}$ and (4.60), we deduce that

$$h(\overline{D}) \cap \mathcal{M}_\lambda = \emptyset. \tag{4.61}$$

Using (4.59) and the property **(i)** of η, we get

$$h(\alpha,\beta) = g(\alpha,\beta) \quad \text{in} \quad \partial D. \tag{4.62}$$

Claim 4.4.19 *We claim that $h(\alpha,\beta)^{\pm} \neq 0$ for all $(\alpha,\beta) \in \overline{D}$.*

In fact, let $v \in \Lambda$. By using the choice of $\delta > 0$ and Lemma 4.4.18, we have that

$$\begin{aligned}\|h(\alpha,\beta) - v\| &\ge \|g(\alpha,\beta) - v\| - \|h(\alpha,\beta) - g(\alpha,\beta)\| \\ &\ge \|g(\alpha,\beta) - v\| - \delta \\ &\ge d' - \frac{d'}{2} = \frac{d'}{2}.\end{aligned}$$

Hence, $h^{\pm}(\alpha,\beta) \neq 0$ for all $(\alpha,\beta) \in \overline{D}$, concluding the statement.

Now, let us consider the vector fields $\Psi_{\bar{u}}^{\lambda}, \mathcal{F} : \overline{D} \to \mathbb{R}^2$, where $\Psi_{\bar{u}}^{\lambda}$ is given in (4.34) and

$$\mathcal{F}(\alpha,\beta) = (I'_{\lambda}(h(\alpha,\beta))h(t,s)^{+}, I'_{\lambda}(h(\alpha,\beta))h(t,s)^{-}).$$

From (4.62), we have $\Psi_{\bar{u}}^{\lambda} = \mathcal{F}$ in ∂D. Hence, by the degree theory (see Lemma A.1.14), we have

$$\deg(\Psi_{\bar{u}}, D, (0,0)) = \deg(\mathcal{F}, D, (0,0)). \tag{4.63}$$

Moreover, by using again Lemma 4.4.4, we have that the point $(1,1)$ is a unique point in $\overline{D}$ such that $\Psi_{\bar{u}}^{\lambda}(\alpha,\beta) = (0,0)$. Consequently, by Lemma 4.4.9, we can deduce that

$$\deg(\Psi_{\bar{u}}^{\lambda}, D, (0,0)) = sgn(J_{(1,1)}\Psi_{\bar{u}}^{\lambda}) = 1.$$

(see Lemma A.1.15). Thus, by (4.63), we get

$$\deg(\mathcal{F}, D, (0,0)) = 1.$$

Then, by degree theory (see Lemma A.1.13), there exists a point $(\alpha',\beta') \in D$ such that

$$I'_{\lambda}(h(\alpha',\beta'))h(\alpha',\beta')^{+} = 0 \quad \text{and} \quad I'_{\lambda}(h(\alpha',\beta'))h(\alpha',\beta')^{-} = 0. \tag{4.64}$$

By Claim 4.4.19 and by (4.64), we obtain that $h(\alpha',\beta') \in \mathcal{M}_{\lambda}$, which is impossible in view of (4.61), which proves the theorem.

4.5 Nonnegative solution and nonpositive solution of problem (P_{λ})

Our goal in this section is to prove that, if $\lambda \geq \lambda^*$, where λ^* is given in Corollary 4.4.12, the problem (P_{λ}) has a nonnegative and a nonpositive solutions, both nonzero and of lowest energy in their respective classes. The main tool to prove this is to get a appropriate estimates to the minimax levels associate to the truncation of the problem (P_{λ}). We will apply the Mountain Pass together with estimates in the Nehari levels $c_{\mathcal{N}_b^+}$ and $c_{\mathcal{N}_b^-}$ given in Remark 4.4.5. The techniques that we apply are motivated by the work in [45].

We define the functionals $I_\lambda^\pm : X \to \mathbb{R}$ by

$$I_\lambda^\pm(u) = \frac{s}{N}\|u\|^{N/s} - \lambda \int_\Omega F(x, u^\pm)\,\mathrm{d}x. \tag{4.65}$$

From the assumptions in f and by considering Lemma B.4.1 in the Appendix, we have that $I_\lambda^\pm \in C^1(X, \mathbb{R})$ and

$$(I_\lambda^\pm)'(u)v = \langle(-\Delta)^s_{N/s}u, v\rangle + \int_\Omega V(x)|u|^{\frac{N}{s}-2}uv\,\mathrm{d}x - \lambda \int_\Omega f(x, u^\pm)v\,\mathrm{d}x, \tag{4.66}$$

for any $u, v \in X$. Note that, if $u = u^+$ and $v = v^-$, then

$$I_\lambda^+(u) = I_\lambda(u),\ I_\lambda^-(v) = I_\lambda(v),\ (I_\lambda^+)'(u) = I_\lambda'(u) \quad \text{and} \quad (I_\lambda^-)'(v) = I_\lambda'(v). \tag{4.67}$$

Thus, the functional I_λ^+ and its derivative $(I_\lambda^+)'$ coincides witch I_λ and I_λ' in nonnegative functions, respectively. Let $u \in X \setminus \{0\}$ a critical point of I_λ^+. Then, taking u^- as a test function in (4.66), we deduce that

$$0 = (I_\lambda^+)'(u)u^- = \langle(-\Delta)^s_{N/s}u, u^-\rangle + \int_\Omega V(x)|u^-|^{\frac{N}{s}}\mathrm{d}x$$

and so, by **(v)** Lemma 4.3.6, we have $u = u^+$. In particular, by (4.67), $u \in \mathcal{N}_\lambda^+$ is a solution of the problem (P_λ). Similarly to I_λ^-.

As a consequence of Lemma 4.4.3 we obtain that the functionals $I_\lambda^\pm$ have the montain pass geometry. Explicitly, we have following result:

Lemma 4.5.1 *Assume that* (V_1) *and* $(f_1) - (f_3)$ *are satisfied. Then, the functionals* $I_\lambda^\pm$ *has the following geometric properties:*

(i) *there exist* $r > 0$ *and* $\tau > 0$ *such that* $I_\lambda^\pm(u) \geq \tau$*, for* $\|u\| = r$*;*

(ii) *there exists* $e \in X$*, with* $\|e\| > r$*, such that* $I_\lambda^\pm(e) < 0$*.*

Let us consider the sets

$$\Gamma_\lambda := \{\gamma \in C([0,1], X) : \gamma(0) = 0 \quad \text{and} \quad I_\lambda(\gamma(1)) < 0\},$$

$$\Gamma_\lambda^\pm := \{\gamma \in C([0,1], X) : \gamma(0) = 0 \quad \text{and} \quad I_\lambda^\pm(\gamma(1)) < 0\}$$

and the respective minimax levels

$$c_\lambda = \inf_{\gamma\in\Gamma_\lambda}\max_{t\in[0,1]} I_\lambda(\gamma(t)) \quad \text{and} \quad c_\lambda^\pm = \inf_{\gamma\in\Gamma_\lambda^\pm}\max_{t\in[0,1]} I_\lambda^\pm(\gamma(t)).$$

By Lemma 4.4.3 and Lemma 4.5.1, by apply the Mountain Pass Theorem, we obtain the following corollary.

Corollary 4.5.2 *Assume that* (V_1) *and* $(f_1)-(f_3)$ *are satisfied. There exist sequences* $(u_n),(u_{n,\pm})\subset X$ *such that* (u_n) *is a* $(PS)_{c_\lambda}$ *sequence for* I_λ *and* $(u_{n,\pm})$ *are* $(PS)_{c^\pm_\lambda}$ *sequences for* $I^\pm_\lambda$*, respectively.*

The next Lemma we will show that the minimax levels defined above are less or equal, respectively, to the Nehari levels $c_{\mathcal{N}_\lambda}, c_{\mathcal{N}^+_\lambda}$ and $c_{\mathcal{N}^-_\lambda}$. In fact, this will be our main tool to show that the functionals I_λ and $I^\pm_\lambda$ satisfy the Palais-Smale condition at the levels c_λ and $c^\pm_\lambda$, respectively.

Lemma 4.5.3 *Assume that* (V_1) *and* $(f_1)-(f_4)$ *are satisfied. The following equalities holds*

$$c_\lambda \le c_{\mathcal{N}_\lambda}, c^+_\lambda \le c_{\mathcal{N}^+_\lambda} \quad \textit{and} \quad c^-_b \le c_{\mathcal{N}^-_\lambda}.$$

Proof . We will only show the inequality $c^-_\lambda \le c_{\mathcal{N}^-_\lambda}$. Note that, by Lemma 4.4.1 and (4.67), we get that

$$c_{\mathcal{N}^-_\lambda} = \inf_{u=u^-\neq 0} \max_{\alpha\ge 0} I^-_\lambda(\alpha u).$$

Let $u=u^-\neq 0$. Now, by Lemma 4.4.3, we have $I^-_\lambda(\alpha u)\to -\infty$ as $\alpha\to\infty$. Thus, there exists $C_u>0$ larger enough such that $I^-_\lambda(\alpha u)<0$ for all $\alpha\ge C_u$. Let us consider the family of the curves $\gamma^\alpha_u:[0,1]\to X$, given by $\gamma^\alpha_u(\beta)=\beta(\alpha u^-)$ for $\alpha\ge C_u$, where $u=u^-\neq 0$. The family of curves so defined is such that $\{\gamma^\alpha_u\}_{\alpha\ge C_u}\subset \Gamma^-_\lambda$, for $u\in -P\setminus\{0\}$. Then, we have that

$$\begin{aligned} c^-_\lambda = \inf_{\gamma\in\Gamma^-_\lambda}\max_{\beta\in[0,1]} I^-_\lambda(\gamma(t)) &\le \inf_{\substack{\{\gamma^\alpha_u\}_{\alpha\ge C_u}\\ u=u^-\neq 0}} \max_{\beta\in[0,1]} I^-_\lambda(\gamma^\alpha_u(\beta)) \\ &\le \inf_{u=u^-\neq 0}\max_{\alpha\ge 0} I^-_\lambda(\alpha u^+) = c_{\mathcal{N}^-_\lambda}. \end{aligned}$$

and so we finish the proof of the lemma. ■

Remark 4.5.4 *Using Lemma 4.5.3 and by Remark 4.4.5, we get*

$$c^+_\lambda + c^-_\lambda \le c_{\mathcal{N}^+_\lambda} + c_{\mathcal{N}^-_\lambda} < c_{\mathcal{M}_\lambda}. \tag{4.68}$$

Moreover, if u *is a critical point of* I^-_λ *such that* $I^-_\lambda(u)=c^-_\lambda$*, then* $u\in\mathcal{N}^-_\lambda$ *is a nonpositive solution of* (P_λ)*. Thus, we have* $c_{\mathcal{N}^-_\lambda}\le I_\lambda(u)=I^-_\lambda(u)=c^-_\lambda$*. Therefore, we deduce that*

$$c^-_\lambda = c_{\mathcal{N}^-_\lambda}. \tag{4.69}$$

Similarly to c_λ *and* c^+_λ*.*

4.5.1 Proof of Theorem 4.2.3

We will only show the existence of a nonpositive lest energy solution u_- of problem (P_λ). From Corollary 4.5.2, there exists $(u_n) \subset X$ such that $I_\lambda^-(u_n) \to c_\lambda^-$ and $(I_\lambda^-)'(u_n) \to 0$ as $n \to \infty$. By (f_3), we have

$$\begin{aligned} c_\lambda^- + o_n(1) + o_n\|u_n\| &= I_\lambda^-(u_n) - \frac{1}{\theta}(I_\lambda^-)'(u_n)u_n \\ &= \left(\frac{s}{N} - \frac{1}{\theta}\right)\|u_n\|^{N/s} + \frac{\lambda}{\theta}\int_\Omega f(x,u_n^-)u_n^- - \theta F(x,u_n^-)\,\mathrm{d}x \\ &\geq \left(\frac{s}{N} - \frac{1}{\theta}\right)\|u_n\|^{N/s} \end{aligned} \tag{4.70}$$

and so (u_n) is bounded in X. Let $C > 0$ such that $\|u_n\| \leq C$ for all $n \in \mathbb{N}$. Using (4.68), there exists $n_0 \in \mathbb{N}$ such that $o_n(1)\|u_n\| \leq o_n(1)C < \rho$ and $c_\lambda^- + o_n(1) < c_{\mathcal{M}_\lambda}$ for all $n \geq n_0$, where $\rho > 0$ is given in Lemma 4.4.13. Thus, by (4.70), we get

$$\|u_n\|^{N/s} < \left(\frac{N\theta}{s\theta - N}\right)(c_{\mathcal{M}_\lambda} + \rho), \quad \text{for all} \quad n \geq n_0.$$

Hence, as in Lemma 4.4.13, we have

$$\|u_n\|^{\frac{N}{N-s}} \leq m_\rho < \frac{\alpha_*}{2\alpha_0}, \quad \text{for all} \quad n \geq n_0. \tag{4.71}$$

Without loss of generality, we can assume that (4.71) holds for all $n \in \mathbb{N}$. Since X is a reflexive space, there exists $u_- \in X$ such that $u_n \rightharpoonup u_-$ in X as $n \to \infty$. By Lemma 4.3.5, up to a subsequence, we have $u_n \to u_-$ in $L^q(\mathbb{R}^N)$, for $q \geq 1$, and $u_n(x) \to u_-(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$. Using Lemma 4.3.5 and (4.71), up to a subsequence, we can also assume that the convergences in Lemma 4.4.15 holds for the sequence (u_n). By (4.55) and (4.56), we have

$$\lambda\int_\Omega f(x,u_n^-)(u_n - u_-)\,\mathrm{d}x = o_n(1). \tag{4.72}$$

Now, by the lower semicontinuity of the norm, we have

$$\|u_-\|^{N/s} \leq \liminf_{n\to\infty}\|u_n\|^{N/s}. \tag{4.73}$$

On the other hand, since the function $\|\cdot\|^{N/s}$ is convex, by using properties of the derivative of convex function (see Lemma A.1.2), we have that

$$\frac{s}{N}\left(\|u_-\|^{N/s} - \|u_n\|^{N/s}\right) \geq \langle(-\Delta)^s_{N/s}u_n, u_- - u_n\rangle + \int_\Omega V(x)|u_n|^{\frac{N}{s}-2}u_n(u_- - u_n)\,\mathrm{d}x.$$

Thus, since $I'_\lambda(u_n) = o_n(1)$ and by (4.72), we deduce that

$$\frac{s}{N}\left(\|u_-\|^{N/s} - \|u_n\|^{N/s}\right) \geq (I_\lambda^-)'(u_n)(u_n - u_-) + \lambda \int_\Omega f(x, u_n^-)(u_n - u_-)\,\mathrm{d}x = o_n(1).$$

Hence, taking the lim inf, we have

$$\liminf_{n\to\infty} \|u_n\|^{N/s} \leq \|u_-\|^{N/s}. \tag{4.74}$$

Thus, by (4.73) and (4.74), we have

$$\liminf_{n\to\infty} \|u_n\|^{N/s} = \|u_-\|^{N/s}.$$

Thus, by Lemma 4.3.2, up to a subsequence, we have $u_n \to u_-$ strongly in X as $n \to \infty$. Therefore, u_- is a critical point of I_λ^- and $I_\lambda^-(u_-) = c_\lambda^- > 0$. Consequently, by Remark 4.5.4, we have $u_- \in \mathcal{N}_\lambda^-$ is a nonpositive solution of (P_λ) and $c_\lambda^- = c_{\mathcal{N}_\lambda^-}$, and the proof is complete.

Appendix A

Auxiliary results

A.1 General auxiliary results

Lemma A.1.1 *Let $\alpha > 0$ and $r > 1$. Then, for each $\beta \geq r$, we have*

$$\left(e^{\alpha|s|^2} - 1\right)^r \leq \left(e^{\beta\alpha|s|^2} - 1\right) \quad \textit{for all} \quad s \in \mathbb{R}.$$

Proof . It is enough to prove that

$$\left(e^t - 1\right)^r \leq e^{rt} - 1 \quad \text{for all} \quad t \geq 0.$$

Let $\gamma = e^t - 1$, then, the above inequality is equivalent to

$$\gamma^r + 1 \leq (\gamma + 1)^r \quad \text{for all} \quad \gamma \geq 0.$$

Let $h(\gamma) = (\gamma+1)^r - \gamma^r - 1$. Then $h(0) = 0$ and $h'(\gamma) = r(\gamma+1)^{r-1} - r\gamma^{r-1} \geq 0$, for all $\gamma \geq 0$. Thus $h(\gamma) \geq 0$, for all $\gamma \geq 0$. Therefore, we get $\gamma^r + 1 \leq (\gamma+1)^r$, for all $\gamma \geq 0$, which completes the proof. ■

Lemma A.1.2 *Let X a normed vector space and $f : X \to \mathbb{R}$ a differentiable function. Then, the following condition are equivalent:*

(a) *f is convex in X;*

(b) *$f(y) - f(x) \geq f'(x)(y - x)$, for all $x, y \in X$;*

(b) *$(f'(y) - f'(x))(y - x) \geq 0$, for all $x, y \in X$.*

(see [22, Theorem 7.4].)

Lemma A.1.3 *(**Fatou's Lemma**) Let (f_n) be a sequence of functions in L^1 that satisfy*

(a) *for all n, $f_n \geq 0$ a.e.*

(b) $\sup_n \int f_n < \infty$.

For almost all $x \in \Omega$ we set $f(x) = \liminf_{n\to\infty} f_n(x) \leq \infty$. Then $f \in L^1$ and

$$\int f \leq \liminf_{n\to\infty} \int f_n.$$

(see [18, Lemma 4.1])

Lemma A.1.4 *(**dominated convergence theorem, Lebesgue**) Let (f_n) be a sequence of functions in L^1 that satisfy*

(a) *$f_n(x) \to f(x)$ a.e. on Ω,*

(b) *there is a function $g \in L^1$ such that for all n, $|f_n(x)| \leq g(x)$ a.e. on Ω.*

Then $f \in L^1$ and $\|f_n - f\|_1 \to 0$.

(see [18, Theorem 4.2])

Lemma A.1.5 *(**Hölder's inequality**) Assuma that $f \in L^p$ and $g \in L^{p'}$ with $1/p + 1/p' = 1$, $1 \leq p \leq \infty$. Then $fg \in L^1$ and*

$$\int |fg| \leq \|f\|_p \|g\|_{p'}.$$

(see [18, Theorem 4.6])

Lemma A.1.6 *If $f \in L^p \cap L^q$ with $1 \leq p \leq q \leq \infty$, then $f \in L^r$ for all $p \leq r \leq q$ and*

$$\|f\|_r \leq \|f\|_p^{\alpha} \|f\|_p^{1-\alpha}, \quad \textit{where} \quad \frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{p}, 0 \leq \alpha \leq 1.$$

(see [18])

Lemma A.1.7 *Let (f_n) be a sequence in L^p and $f \in L^p$ such that $\|f_n - f\|_p \to 0$. Then, there exist a subsequence (f_{n_k}) and a function $h \in L^p$ such that*

(a) *$f_{n_k}(x) \to f(x)$ a.e. on Ω,*

(b) *$|f_{n_k}(x)| \leq h(x)$, $\forall\, k$ a.e. on Ω.*

(see [18, Theorem 4.9])

Lemma A.1.8 *Let (u_n) be a sequence in $L^q(\Omega)$, with $q \geq 1$, such that $u_n \to u$ in $L^q(\Omega)$. Then $u_n^+ \to u^+$ and $u_n^- \to u^-$ in $L^q(\Omega)$.*

Proof . Since $u_n \to u$ in $L^q(\Omega)$, there exists $h \in L^q(\Omega)$ and a subsequence (u_{n_k}) of (u_n) such that $u_{n_k}(x) \to u(x)$ a.e. $x \in \Omega$ and $|u_{n_k}(x)| \leq h(x)$ a.e. in Ω. Let us define

$$A_0 = \{x \in \Omega : u(x) = 0\}, A^+ = \{x \in \Omega : u(x) > 0\} \quad \text{and} \quad A^- = \{x \in \Omega : u(x) < 0\}.$$

Let $x \in A^+$ such that $u_{n_k}(x) \to u(x) = u^+(x) > 0$ as $k \to \infty$. Then, there exists k_0, such that $u_{n_k}(x) = u_{n_k}^+(x)$ for all $k \geq k_0$ and so $u_{n_k}^+(x) \to u^+(x)$ as $k \to \infty$. Similarly, if $x \in A^-$ and $u_{n_k}(x) \to u(x) = u^-(x)$ as $k \to \infty$, we have $u_{n_k}^+(x) = 0$ for all k larger enough. Thus $u_{n_k}^+(x) \to u^+(x) = 0$ as $k \to \infty$. Finally, if $x \in A_0$ and $u_{n_k}(x) \to u(x) = 0$ as $k \to \infty$, since $|u_n^+| \leq |u_n|$, we have $u_{n_k}^+(x) \to u^+(x) = 0$ as $k \to \infty$. Therefore, by the Dominated Convergence Theorem, we have $u_{n_k}^+ \to u^+$ in $L^q(\Omega)$.

Now, if (u_n^+) does not converge to u^+ in $L^q(\Omega)$, then there exists a subsequence (u_{n_k}) of (u_n) such that $\|u_{n_k}^+ - u^+\|_q \geq \varepsilon > 0$ for all $k \in \mathbb{N}$. Then, applying the above argument to the sequence (u_{n_k}) we obtain a contradiction, which completes the proof.

■

Lemma A.1.9 *L^p is uniformly convex, and thus reflexive for any p, $1 < p < \infty$.*

(see [18, Theorem 4.10])

Lemma A.1.10 *(**Straus's lemma**) Let $P, Q : \mathbb{R} \to \mathbb{R}$ be two continuous functions satisfying*

$$\frac{P(s)}{Q(s)} \to 0 \quad \text{as} \quad |s| \to \infty.$$

Let $u_n : \mathbb{R}^N \to \mathbb{R}$ be a sequence of measurable functions such that

$$\sup_n \int_{\mathbb{R}^N} |Q(u_n(x))| \mathrm{d}x < \infty$$

and

$$P(u_n(x)) \to v(x) \quad \text{a.e. in} \quad \mathbb{R}^N, \quad \text{as} \quad n \to \infty.$$

Then for any bounded Borel set B one has

$$\int_B |P(u_n(x)) - v(x)| \mathrm{d}x \to 0 \quad \text{as} \quad n \to \infty.$$

If one further assumes that

$$\frac{P(s)}{Q(s)} \to 0 \quad as \quad s \to 0$$

and

$$u_n(x) \to 0 \quad as \quad |x| \to \infty, \quad \text{uniformly with respect to} \quad n,$$

then $P(u_n)$ *converges to* v *in* $L^1(\mathbb{R}^N)$ *as* $n \to \infty$.

(see [13, Theorem A.I])

Lemma A.1.11 ***(quantitative deformation lemma)*** *Let* X *be a Banach space,* $\varphi \in C^1(X, \mathbb{R}), S \subset X, c \in \mathbb{R}, \varepsilon, \delta >$ *such that*

$$\forall u \in \varphi^{-1}([c - 2\varepsilon, c + 2\varepsilon] \cap S_{2\delta}) : \|\varphi'(u)\| \geq 8\varepsilon/\delta.$$

Then there exits $\eta \in C([0, 1] \times X, X)$ *such that*

(*i*) $\eta(t, u) = u$, *if* $t = 0$ *or if* $u \notin \varphi^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap S_{2\delta}$;

(*ii*) $\eta(1, \varphi^{c+\varepsilon} \cap S) \subset \varphi^{c-\varepsilon}$;

(*iii*) $\eta(t, \cdot)$ *is an homeomorphism of* X, $\forall t \in [0, 1]$;

(*iv*) $\|\eta(t, u) - u\| \leq \delta$, $\forall u \in X$, $\forall t \in [0, 1]$;

(*v*) $\varphi(\eta(\cdot, u))$ *is non increasing,* $\forall u \in X$;

(*vi*) $\varphi(\eta(t, u)) < c$, $\forall u \in \varphi^c \cap S_\delta$, $\forall t \in (0, 1]$.

(see [72, Lemma 2.3])

Lemma A.1.12 ***(Mountain pass theorem)*** *Let* X *be a Banach space,* $\varphi \in C^1(X, \mathbb{R}), e \in X$ *and* $r > 0$ *be such that* $\|e\| > r$ *and*

$$b := \inf_{\|u\|=r} \varphi(u) > \varphi(0) \geq \varphi(e).$$

Then, there exists a sequence (u_n) *in* X *(a* $(PS)_c$ *sequence) such that* $\varphi(u_n) \to c$ *and* $\varphi'(u_n) \to 0$ *where*

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t))$$

$$\Gamma := \{\gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = e\}.$$

(see [72, Theorem 2.10])

Lemma A.1.13 *Let Ω a open bounded set in $\mathbb{R}^N$, with boundary $\partial\Omega$. Let $f : \overline{\Omega} \to \mathbb{R}^N$ a continuous function and $p \in \mathbb{R}^N$ such that $p \notin f(\partial\Omega)$. If $\deg(f, \Omega, p) \neq 0$ then there exists $z \in \Omega$ such that $f(z) = p$.*

(see [5, Section 3.1])

Lemma A.1.14 *Let Ω a open bounded set in $\mathbb{R}^N$, with boundary $\partial\Omega$. Let $f, g \in C(\overline{\Omega}, \mathbb{R}^N)$ be such that $f(x) = g(x)$ for all $x \in \Omega$ and let $p \notin f(\partial\Omega) = g(\partial\Omega)$. Then $\deg(f, \Omega, p) = \deg(g, \Omega, p)$.*

(see [5, Theorem 3.2])

Lemma A.1.15 *Let Ω a open bounded set in $\mathbb{R}^N$, with boundary $\partial\Omega$. Let $f \in C(\overline{\Omega}, \mathbb{R}^N) \cap C^1(\Omega, \mathbb{R}^N)$ and let $p \notin f(\partial\Omega)$ a regular value of f. Then*

$$\deg(f, \Omega, p) = \sum_{x \in f^{-1}(p)} sgn[J_f(x)].$$

(see [5, Corollary 3.15])

Appendix B

Auxiliary results of chapters

B.1 Appendix of Chapter 1

In this section we prove some auxiliary results that we used in Chapter 1.

We consider the problem

$$\begin{cases} (-\Delta)^{1/2}u + V(x)u = |u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R} \setminus \Omega, \end{cases} \tag{B.1}$$

where $\Omega = (a, b)$, $p > 2$ and V satisfies the condition (V_1). The functional $I_p : X \to \mathbb{R}$ associated to (B.1) is given by

$$I_p(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p}\|u\|_p^p.$$

We define the Nehari manifolds and nodal set associated to I_p and the respective ground state and nodal level by

$$\mathcal{N}^p = \{u \in X \setminus \{0\} : I_p'(u)u = 0\}, \tag{B.2}$$

$$\mathcal{M}^p = \{u \in X : u^\pm \neq 0 \text{ and } I_p'(u)u^\pm = 0\}, \tag{B.3}$$

$$c_{\mathcal{N}^p} = \inf_{u \in \mathcal{N}^p} I_p(u), \tag{B.4}$$

and

$$c_{\mathcal{M}^p} = \inf_{u \in \mathcal{M}^p} I_p(u). \tag{B.5}$$

We will show in this section that problem (B.1) has a nodal solution of least energy. The arguments used in this section are similar to those developed in Sections 1.3 and 1.4, so many will be omitted in order to avoid repetition.

Lemma B.1.1 *Given $u \in X \setminus \{0\}$, there exists a unique $t = t(u) > 0$ such that $tu \in \mathcal{N}^p$. In addition, t satisfies*

$$I_p(tu) = \max_{s \geq 0} I_p(su). \tag{B.6}$$

Proof . The proof of this result follows the same ideas of Lemma 1.3.1. ■

Corollary B.1.2 *Let $u \in X \setminus \{0\}$. Then $u \in \mathcal{M}$ if only if $I_p(u) = \max_{s \geq 0} I_p(su)$.*

Lemma B.1.3 *There exist $\beta_0 > 0$ and $k_0 > 0$ such that $I_p(u) \geq \beta_0$ and $\|u\|^2 \geq k_0$, for all $u \in \mathcal{N}^p$, and $\|u^{\pm}\|^2 \geq k_0$, for all $u \in \mathcal{M}^p$.*

Proof . The proof of this result follows the same ideas of Lemma 1.3.2. ■

The lemma above shows that the levels $c_{\mathcal{N}^p}$ and $c_{\mathcal{M}^p}$ are well defined and $c_p \geq c_p^* \geq \beta_0$, since $\mathcal{M}^p \subset \mathcal{N}^p$.

Lemma B.1.4 *Given $u \in X$ with $u^{\pm} \neq 0$, there exists a unique pair (t, s) of positive numbers such that $tu^+ + su^- \in \mathcal{M}^p$.*

Proof . The proof of this result follows the same ideas of Lemma 1.3.6. ■

Lemma B.1.5 *Let $u \in X$, with $u^{\pm} \neq 0$, such that $I_p'(u)u^+ \leq 0$ and $I_p'(u)u^- \leq 0$. Then the unique pair (t, s) given in Lemma B.1.4 satisfies that $0 < t, s \leq 1$.*

Proof . The proof of this result follows the same ideas of Lemma 1.3.7. ■

Lemma B.1.6 *Let $u \in X$, with $u^{\pm} \neq 0$, and (t, s) the unique pair of positive numbers given in Lemma B.1.4. Then (t, s) is the unique maximum point of the function $\phi_p : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ defined by $\phi_p(\alpha, \beta) = I_p(\alpha u^+ + \beta u^-)$.*

Proof . The proof of this result follows the same ideas of Lemma 1.3.8. ■

Now, we shall show that the nodal level $c_{\mathcal{M}^p}$ is attained.

Lemma B.1.7 *There exists $\bar{u} \in \mathcal{M}^p$ such that $I_p(\bar{u}) = c_{\mathcal{M}^p}$.*

Proof . Let $(u_n) \subset \mathcal{M}^p$ be such that $I_p(u_n) \to c_{\mathcal{M}^p}$. Now, since $u_n \in \mathcal{M}^p$, for all $n \in \mathbb{N}$, we have

$$c_{\mathcal{M}^p} + o_n(1) = I_p(u_n) = \frac{1}{2}\|u_n\|^2 - \frac{1}{p}\|u_n\|_p^p = \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2.$$

Hence, (u_n) is bounded in X. Therefore, (u_n^+) and (u_n^-) are also bounded in X. Since X is a Hilbert space, up to a subsequence, there exists $u \in X$ such that $u_n^\pm \rightharpoonup u^\pm$ in X. Utilizing Lemma 1.2.1, passing to a subsequence, we can assume that $u_n^\pm \to u^\pm$ in $L^q(\mathbb{R})$, for all $q \in [1, \infty)$, and $u_n^\pm(x) \to u^\pm(x)$ a.e. in $\mathbb{R}$.

We claim that $u^\pm \neq 0$. We suppose, by contradiction, that $u^+ = 0$ (similarly u^-). Since $u_n \in \mathcal{M}_{nod}$, we have $I_p'(u_n)u_n^+ = 0$. Thus

$$\langle u_n, u_n^+\rangle = \int_\Omega |u_n^+|^p \mathrm{d}x \to \int_\Omega |u^+|^p \mathrm{d}x = 0.$$

However, by Lemma 1.2.3 we have $\langle u_n, u_n^+\rangle \geq \|u_n^+\|^2$. This implies that $\|u_n^+\|^2 \to 0$, which is a contradiction in view of Lemma B.1.3.

Utilizing Lemma B.1.4, there exists a pair of positive numbers (t, s) such that $tu^+ + su^- \in \mathcal{M}^p$. Let $\bar{u} = tu^+ + su^-$. We will show that $I_p'(u)u^\pm \leq 0$. In fact, by Fatou's lemma, we have

$$\begin{aligned}
\|u^+\|^2 + \langle u^+, u^-\rangle &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{|u^+(x) - u^+(y)|^2}{|x-y|^2}\mathrm{d}x\mathrm{d}y + \int_\Omega V(x)|u^+|^2 \mathrm{d}x \\
&\quad + \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{u^+(x)(-u^-(y))}{|x-y|^2}\mathrm{d}x\mathrm{d}y \\
&\leq \liminf_{n\to+\infty} \left(\|u_n^+\|^2 + \langle u_n^+, u_n^-\rangle\right) \\
&= \liminf_{n\to+\infty} \int_\Omega |u_n^+|^p \mathrm{d}x = \int_\Omega |u^+|^p \mathrm{d}x = \|u^+\|_p^p.
\end{aligned}$$

Analogously, $I_p'(u)u^- \leq 0$. Hence, using Lemma B.1.5, we have that $0 < t, s \leq 1$. In this way, we have that $\|\bar{u}\|^2 \leq \|u\|^2$. Now, by using that $\bar{u} \in \mathcal{M}^p$ and the Fatou's lemma, we reach

$$\begin{aligned}
c_{\mathcal{M}^p} \leq I_p(\bar{u}) &= I_p(\bar{u}) - \frac{1}{p}I_p'(\bar{u})\bar{u} \\
&= \left(\frac{1}{2} - \frac{1}{p}\right)\|\bar{u}\|^2 \\
&\leq \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2 \\
&\leq \liminf_{n\to+\infty}\left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 = \liminf_{n\to+\infty}\left(\frac{1}{2}\|u_n\|^2 - \frac{1}{p}\|u_n\|_p^p\right) = c_{\mathcal{M}^p}
\end{aligned}$$

and this completes the proof. ■

We define $D = (\frac{1}{2}, \frac{3}{2}) \times (\frac{1}{2}, \frac{3}{2})$ and $g : \overline{D} \to X$ by $g(\alpha, \beta) = \alpha \bar{u}^+ + \beta \bar{u}^-$, where $\bar{u}$ was found in Lemma B.1.7. Before presenting the main result of this section, we will present the following lemma:

Lemma B.1.8 *Let $P = \{u \in X : u(x) \geq 0 \, a.e. \, x \in \Omega\}$ and $-P = \{u \in X : u(x) \leq 0 \, a.e. \, x \in \Omega\}$. Then $d_0 = dist(g(\overline{D}), \Lambda) > 0$, where $\Lambda = P \cup -P$.*

Proof . The proof of this result follows the same ideas of Lemma 1.4.1. ■

Now, we will now present the main result of this section.

Theorem B.1.9 *The function $\bar{u} \in \mathcal{M}^p$ found in Lemma B.1.7 is a nodal solution of least energy of problem (B.1).*

Proof . The proof of this result follows the same ideas used in Theorem 1.1.2 and we omit it. ■

B.2 Appendix of Chapter 2

In this section we prove some auxiliary results that used in Chapter 2.

We consider the problem

$$(-\Delta)^{1/2} u + V(x) u = K(x) |u|^{p-2} u \quad \text{in} \quad \mathbb{R}, \tag{B.7}$$

where $p > 2$, V and K are such that $(V_1) - (V_2)$ and (K_1) hold. The energy functional $I_p : X \to \mathbb{R}$ associated to (B.7) is given by

$$I_p(u) = \frac{1}{2} \|u\|^2 - \frac{1}{p} \|u\|^p_{L^p_K}.$$

We define the Nehari manifold and nodal set associated to I_p and the respective ground state and nodal levels by

$$\mathcal{N}^p = \{u \in X \setminus \{0\} : I'_p(u) u = 0\}, \tag{B.8}$$

$$\mathcal{M}^p = \{u \in X : u^{\pm} \neq 0, I'_p(u) u^{\pm} = 0\}, \tag{B.9}$$

$$c_{\mathcal{N}^p} = \inf_{u \in \mathcal{N}^p} I_p(u), \tag{B.10}$$

$$c_{\mathcal{M}^p} = \inf_{u \in \mathcal{M}^p} I_p(u). \tag{B.11}$$

We will show that problem (B.7) has a nodal solution of least energy. The steps to show this are the same of the Sections 2.3, 2.4 and 2.5. Thus, many computations will be omitted in order to avoid repetitions.

Lemma B.2.1 *Given $u \in X \setminus \{0\}$, there exists a unique $t = t(u) > 0$ such that $tu \in \mathcal{N}^p$. In addition, t satisfies*

$$I_p(tu) = \max_{s \geq 0} I_p(su). \tag{B.12}$$

Proof . Let $h(s) := I_p(su) = s^2\|u\|^2/2 - s^p\|u\|^p_{L^p_K}/p$, for $s \geq 0$. Since $p > 2$, we have $h(s) > 0$ for $s > 0$ small enough and $h(s) \to -\infty$ as $s \to \infty$. Hence, there exists a $t > 0$ satisfying (B.12). In particular, $tu \in \mathcal{N}^p$. Moreover, $h'(t) = 0$ if and only if $t = (\|u\|^2/\|u\|^p_{L^p_K})^{1/(p-2)}$. ■

Corollary B.2.2 *Let $u \in X \setminus \{0\}$. Then $u \in \mathcal{N}^p$ if only if $I_p(u) = \max_{s \geq 0} I_p(su)$.*

Lemma B.2.3 *There exist $\beta_0 > 0$ and $\ell_0 > 0$ such that $\|u\|^2 \geq \ell_0$, for all $u \in \mathcal{N}^p$, $\|u^{\pm}\|^2 \geq \ell_0$, for all $u \in \mathcal{M}^p$ and $I_p(u) \geq \beta_0$.*

Proof . The proof of this result follows by Corollary 2.3.4 and using the same ideas of Lemmas 2.4.2 and 2.4.4. ■

The lemma above shows that the levels $c_{\mathcal{N}^p}$ and $c_{\mathcal{M}^p}$ are well defined and $c_{\mathcal{M}^p} \geq c_{\mathcal{N}^p} \geq \beta_0$, since $\mathcal{M}^p \subset \mathcal{N}^p$. The proofs of the next three result follow the same ideas of Lemmas 2.4.5, 2.4.6 and 2.4.7, and we omit them.

Lemma B.2.4 *Given $u \in X$ with $u^{\pm} \neq 0$, there exists a unique pair (t, s) of positive numbers such that $tu^+ + su^- \in \mathcal{M}^p$.*

Lemma B.2.5 *Let $u \in X$, with $u^{\pm} \neq 0$, such that $I'_p(u)u^+ \leq 0$ and $I'_p(u)u^- \leq 0$. Then the unique pair (t, s) given in Lemma B.2.4 satisfies that $0 < t, s \leq 1$.*

Lemma B.2.6 *Let $u \in X$, with $u^{\pm} \neq 0$, and (t, s) the unique pair of positive numbers given in Lemma B.2.4. Then (t, s) is the unique maximum point of the function $\phi_p : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ defined by $\phi_p(\alpha, \beta) = I_p(\alpha u^+ + \beta u^-)$.*

Now, we shall show that the nodal level $c_{\mathcal{M}^p}$ is attained.

Lemma B.2.7 *There exists $\bar{u} \in \mathcal{M}^p$ such that $I_p(\bar{u}) = c_{\mathcal{M}^p}$.*

Proof . Let $(u_n) \subset \mathcal{M}^p$ be such that $I_p(u_n) \to c_{\mathcal{M}^p}$. Now, since $u_n \in \mathcal{M}^p$, for all $n \in \mathbb{N}$, we have

$$c_{\mathcal{M}^p} + o_n(1) = I_p(u_n) = \frac{1}{2}\|u_n\|^2 - \frac{1}{p}\|u_n\|^p_{L^p_K} = \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2.$$

Hence, (u_n) is bounded in X. Therefore, (u_n^+) and (u_n^-) are also bounded in X. Since X is a Hilbert space, up to a subsequence, there exists $u \in X$ such that $u_n^\pm \rightharpoonup u^\pm$ in X. Since $p > 2$, utilizing Proposition 2.3.2 and Corollary 2.3.4, passing to a subsequence, we can assume that $u_n^\pm \to u^\pm$ in L^p_K and $u_n^\pm(x) \to u^\pm(x)$ a.e. in $\mathbb{R}$.

We claim that $u^\pm \neq 0$. We suppose, by contradiction, that $u^+ = 0$ (similarly $u^- = 0$). Since $u_n \in \mathcal{M}_{nod}$, we have $I_p'(u_n)u_n^+ = 0$. Thus,

$$\langle u_n, u_n^+\rangle = \int_{\mathbb{R}} K(x)|u_n^+|^p \mathrm{d}x \to \int_{\mathbb{R}} K(x)|u^+|^p \mathrm{d}x = 0.$$

However, by Lemma 2.3.7 we have $\langle u_n, u_n^+\rangle \geq \|u_n^+\|^2$. This implies that $\|u_n^+\|^2 \to 0$, which is a contradiction in view of Lemma B.2.3. Utilizing Lemma B.2.4, there exists a pair of positive numbers (t, s) such that $tu^+ + su^- \in \mathcal{M}^p$. Let $\bar{u} = tu^+ + su^-$. We will show that $I_p'(u)u^\pm \leq 0$. In fact, by Fatou's lemma, we have

$$\begin{aligned}
\|u^+\|^2 + \langle u^+, u^-\rangle &\leq \liminf_{n\to+\infty} \left(\|u_n^+\|^2 + \langle u_n^+, u_n^-\rangle\right) \\
&= \liminf_{n\to+\infty} \int_{\mathbb{R}} K(x)|u_n^+|^p \mathrm{d}x = \int_{\mathbb{R}} K(x)|u^+|^p \mathrm{d}x = \|u^+\|^p_{L^p_K}.
\end{aligned}$$

Analogously, $I_p'(u)u^- \leq 0$. Hence, using Lemma B.2.5, we have $0 < t, s \leq 1$. In particular, $\|\bar{u}\|^2 \leq \|u\|^2$. Now, by using that $\bar{u} \in \mathcal{M}^p$ and by lower semicontinuity of norm, we reach

$$\begin{aligned}
c_{\mathcal{M}^p} \leq I_p(\bar{u}) &= I_p(\bar{u}) - \frac{1}{p} I_p'(\bar{u})\bar{u} \\
&= \left(\frac{1}{2} - \frac{1}{p}\right)\|\bar{u}\|^2 \leq \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2 \\
&\leq \liminf_{n\to+\infty} \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 = \liminf_{n\to+\infty} \left(\frac{1}{2}\|u_n\|^2 - \frac{1}{p}\|u_n\|^p_{L^p_K}\right) = c_{\mathcal{M}^p}
\end{aligned}$$

and this completes the proof. ■

Now, we will present the main result of this section.

Theorem B.2.8 *The function $\bar{u} \in \mathcal{M}_{nod}$ found in Lemma B.2.7 is a nodal solution of least energy of problem* (B.7).

Proof . It follows by applying the same ideas used in the proof of Theorem 2.2.3 and we omit it. ■

B.3 Appendix of Chapter 3

In this section we prove some auxiliary results that we used in Chapter 3. The first result is established as follows.

Theorem B.3.1 *Assume that* (V_1) *holds. Then, there exists* $w \in \mathcal{M}_b^p$ *such that* $J_b(w) = d_b^*$, *where* $d_b^* := \inf_{u \in \mathcal{M}_b^p} J_b(u)$.

Proof . The proof of this theorem is obtained by the following steps:

(1) For $u \in X$, with $u^+ \neq 0$ and $u^- \neq 0$, there exists a unique pair of positive numbers (t_u, s_u) such that $t_u u^+ + s_u u^- \in \mathcal{M}_b^p$ and $J_b(t_u u^+ + s_u u^-) > 0$. Moreover, if $(t,s) \neq (t_u, s_u)$, with $t, s \geq 0$, similar to Lemma 3.3.4, we have

$$J_b(tu^+ + su^-) < J_b(t_u u^+ + s_u u^-).$$

(2) There exists $\kappa_0 > 0$ such that $\|u^\pm\|^2 \geq \kappa_0$, for all $u \in \mathcal{M}_b^p$. This is similar to Lemma 3.3.7.

(3) If $u \in X$, with $u^+ \neq 0$ and $u^- \neq 0$, it is such that $J_b'(u)u^\pm \leq 0$. Then, simliar to Lemma 3.3.6, the unique pair (t_u, s_u) in Step (1) satisfies $0 < t_u, s_u \leq 1$.

(4) Now, let $(u_n) \subset \mathcal{M}_b^p$ be a sequence such that $J_b(u_n) \to d_b^*$. Similar to Lemma 3.3.15, we can to show that, up to a subsequence, $u_n \rightharpoonup \widetilde{w}$ in X. From Step (2), we show that $\widetilde{w}^+ \neq 0$ and $\widetilde{w}^- \neq 0$. Using the Steps (1), (3) and again similar to Lemma 3.3.15, we can find $w \in \mathcal{M}_b^p$ such that $J_b(w) = d_b^*$, as desired. ■

Theorem B.3.2 *Assume that* (V_1) *holds. The function w given in Theorem B.3.1 is a least energy nodal solution of the problem*

$$\begin{cases} m_b(\|u\|^2)\left[(-\Delta)^{1/2}u + V(x)u\right] = \dfrac{1}{2}|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R} \setminus \Omega. \end{cases}$$

Proof . It is easy to check that the same result of Lemma 3.3.14 holds to

$$\Phi_u(t,s) = \big(J_b'(tu^+ + su^-)tu^+, J_b'(tu^+ + su^-)su^-\big) \quad \text{for} \quad u \in \mathcal{M}_b^p.$$

The rest of the proof follows the same ideas used in the proof of Theorem 3.1.2 and we omit it. ■

Remark B.3.3 *Note that, for any $b' \geq 0$, there exists a least energy nodal solution for the problem*

$$\begin{cases} m_{b'}(\|u\|^2)\left[(-\Delta)^{1/2}u + V(x)u\right] = \dfrac{1}{2}|u|^{p-2}u & \text{in } \Omega,\\ u = 0 & \text{in } \mathbb{R}\setminus\Omega. \end{cases}$$

Explicitly, there exists $w_{b'} \in X$, with $w_{b'}^{\pm} \neq 0$, such that $w_{b'}$ is a critical point of the functional

$$J_{b'}(u) = \frac{1}{2}M_{b'}(\|u\|^2) - \frac{1}{2p}\int_\Omega |u|^p \mathrm{d}x \quad \text{and} \quad J_{b'}(w_{b'}) = d_{b'}^*$$

where

$$d_{b'}^* = \inf_{u\in\mathcal{M}_{b'}^p} J_{b'}(u) \quad \text{and} \quad \mathcal{M}_{b'}^p = \{u \in X : u^+ \neq 0,\ u^- \neq 0,\ (J_{b'})'(u)u^+ = 0 \text{ and } (J_{b'})'(u)u^- = 0\}.$$

B.4 Appendix of Chapter 4

In this section, the space X is defined as in Chapter 4. However, this space is a natural generalization for larger dimensions of the spaces defined in chapters 1 and 3. Hence, we emphasize that the following result can be applied to the contexts of the other chapters.

Lemma B.4.1 *Let $P^{\pm} : X \to X$ the operators given by $P^{\pm}(u) = u^{\pm}$. Then:*

(i) *if $u_n \rightharpoonup u$ in X as $n \to \infty$ then, up to a subsequence, $P^{\pm}(u_n) \rightharpoonup P^{\pm}(u)$ in X as $n \to \infty$;*

(ii) *$P^{\pm}$ are strongly continuous.*

Proof . By **(v)** of Lemma 4.3.6, we have $\|u^{\pm}\| \leq \|u\|$ and so the operators $P^{\pm}$ are well defined. Let $(u_n) \subset X$ such that $u_n \rightharpoonup u$ in X as $n \to \infty$. Since $(u_n^{\pm})$ is bonded, by Lemma 4.3.1, there exist $v_1, v_2 \in X$ such that, up to a subsequence, $u_n^+ \rightharpoonup v_1$ and $u_n^- \rightharpoonup v_2$ in X as $n \to \infty$. Using Lemma 4.3.3, up to a subsequence, we have $u_n \to u$, $u_n^+ \to v_1$ and $u_n^- \to v_2$ in $L^q(\Omega)$, for $q \geq 1$, as $n \to \infty$. However, by Lemma A.1.8, up to a subsequence, $u_n^+ \to u^+$ and $u_n^- \to u^-$ in $L^q(\Omega)$. Thus $v_1 = u^+$ and $v_2 = u^-$, showing the item **(i)**.

Let $(u_n) \subset X$ such that $u_n \to u$ in X as $n \to \infty$. Using again Lemma 4.3.3 and Lemma A.1.8, we have $u_n \to u$ and $u_n^\pm \to u^\pm$ in $L^{N/s}(\mathbb{R}^N)$ as $n \to \infty$. In particular, we have that $u_n^\pm(x) \to u^\pm(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$. Now, for $(x,y) \in \mathbb{R}^{2N}$ with $x \neq y$, we define

$$v_n(x,y) = \frac{|u_n(x) - u_n(y)|^{N/s}}{|x-y|^{2N}} \quad \text{and} \quad v(x,y) = \frac{|u(x) - u(y)|^{N/s}}{|x-y|^{2N}}. \tag{B.13}$$

By the strongly convergence, we get $v_n \to v$ in $L^1(\mathbb{R}^{2N})$ as $n \to \infty$. Thus, there exists $h \in L^1(\mathbb{R}^{2N})$ such that $0 \leq v_n(x,y) \leq h(x,y)$ a.e. in $\mathbb{R}^{2N}$. Similar to (B.13), we consider

$$v_{n,\pm}(x,y) = \frac{|u_n^\pm(x) - u_n^\pm(y)|^{N/s}}{|x-y|^{2N}} \quad \text{and} \quad v_\pm(x,y) = \frac{|u^\pm(x) - u^\pm(y)|^{N/s}}{|x-y|^{2N}}. \tag{B.14}$$

Thus we get $v_{n,\pm}(x,y) \to v_\pm(x,y)$ a.e in $\mathbb{R}^{2N}$. Now, let us consider the following decomposition

$$\mathbb{R}^{2N} = (\Omega_{n,-}^c \times \Omega_{n,-}^c) \cup (\Omega_{n,-}^c \times \Omega_{n,-}) \cup (\Omega_{n,-} \times \Omega_{n,-}^c) \cup (\Omega_{n,-} \times \Omega_{n,-}) \tag{B.15}$$

where

$$\Omega_{n,-} = \{x \in \Omega : u_n(x) \leq 0\}.$$

Using the decomposition (B.15), by a straightforward calculation, we can see that

$$0 \leq v_{n,-}(x,y) \leq v_n(x,y) \leq h(x,y).$$

Analogously, we can show that $0 \leq v_{n,+}(x,y) \leq v_n(x,y) \leq h(x,y)$. Hence, by the dominated convergence theorem, we obtain that

$$v_{n,\pm} \to v_\pm \quad \text{in} \quad L^1(\mathbb{R}^{2N}), \quad \text{as} \quad n \to \infty. \tag{B.16}$$

Moreover, as in Lemma A.1.8, it is easy to check that

$$\int_\Omega V(x)|u_n^\pm|^{N/s}\mathrm{d}x \to \int_\Omega V(x)|u^\pm|^{N/s}\mathrm{d}x, \quad \text{as} \quad n \to \infty. \tag{B.17}$$

Hence, by (B.16) and (B.17), we deduce that $\|u_n^\pm\| \to \|u^\pm\|$ as $n \to \infty$. Therefore, using Lemma 4.3.2, we have $P^\pm(u_n) \to P(u^\pm)$ as $n \to \infty$, which concludes the proof of **(ii)**. ■

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