Universidade Federal da Paraíba
Programa de Pós-Graduação em Matemática
Doutorado em Matemática

Fractional powers approach of operators for abstract evolution equations of third order in time

by

Lucas Araújo Santos

João Pessoa - PB

June, 2020

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under the supervision of

Prof. Dr. Flank David Morais Bezerra

Tese apresentada ao Corpo Docente do Programa de Pós-Graduação em Matemática-UFPB, como requisito parcial para obtenção do título de Doutor em Matemática.

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Prof. Dr. Flank David Morais Bezerra

Orientador

Prof. Dr. Alexandre Nolasco de Carvalho

Prof. Dr. José María Arrieta Algarra

Prof. Dr. Marcelo José Dias Nascimento

Prof. Dr. Severino Horácio da Silva

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Resumo

Neste trabalho estudamos equações de evolução lineares de terceira ordem no tempo sob a perspectiva da teoria de semigrupos fortemente contínuos. Consideramos suas aproximações de ordem fracionária via teoria das potências fracionárias de operadores fechados e densamente definidos por fórmulas do tipo Balakrishnan. Sobre aplicações, analisamos equações do tipo Moore-Gibson-Thompson com amortecimentos fracionários.

Palavras-chave: aproximações fracionárias; equações de evolução lineares de terceira ordem no tempo; equações do tipo Moore-Gibson-Thompson; potências fracionárias.

Abstract

In this work we study third order linear evolution equations in time, in the sense of theory of strongly continuous one-parameter semigroups, and approximations them of fractional order via theory of the fractional powers of closed and densely defined operator tand ype Balakrishnan formula. As applications, we present approximations of the Moore-Gibson-Thompson type equations with fractional damped.

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Keywords: fractional approximations; third order linear evolution equations in time; Moore-Gibson-Thompson type equations; fractional powers.

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"A vida é como andar de bicicleta. Para se equilibrar é preciso estar em movimento".

Albert Einstein

Dedication

À minha família

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Introduction

In this work we consider the following abstract linear evolution equation of third order in time

$$\partial_t^3 u + Au = 0 \tag{1}$$

with initial conditions given by

$$u(0) = u_0 \in X^{\frac{2}{3}},\ \partial_t u(0) = u_1 \in X^{\frac{1}{3}},\ \partial_t^2 u(0) = u_2 \in X, \tag{2}$$

where X is a separable Hilbert space and $A : D(A) \subset X \to X$ is a linear, closed, densely defined, self-adjoint and positive definite unbounded operator with compact resolvent. We wish to study the fractional powers of Λ, the matricial operator obtained by rewriting (1)-(2) as a first order abstract system as follows:

We will consider the phase space

$$Y = X^{\frac{2}{3}} \times X^{\frac{1}{3}} \times X$$

which is a Banach space equipped with the norm given by

$$\| \cdot \|_Y^2 = \| \cdot \|_{X^{\frac{2}{3}}}^2 + \| \cdot \|_{X^{\frac{1}{3}}}^2 + \| \cdot \|_X^2$$

and we write the problem (1)-(2) as a Cauchy problem on Y, letting $v = \partial_t u$, $w = \partial_t^2 u$ and $\mathbf{u} = \left[\begin{smallmatrix} u \\ v \\ w \end{smallmatrix}\right]$ and the initial value problem

$$\begin{cases} \dfrac{d\mathbf{u}}{dt} + \Lambda \mathbf{u} = 0,\ t > 0 \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \tag{3}$$

where the unbounded linear operator $\Lambda : D(\Lambda) \subset Y \to Y$ is defined by

$$D(\Lambda) = D(A) \times D(A^{\frac{2}{3}}) \times D(A^{\frac{1}{3}}), \tag{4}$$

and

$$\varLambda\mathbf{u} = \begin{bmatrix} 0 & -I & 0 \\ 0 & 0 & -I \\ A & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} := \begin{bmatrix} -v \\ -w \\ Au \end{bmatrix}, \ \forall \mathbf{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \in D(\varLambda). \tag{5}$$

As we will see in Lemma 2.1.3, the problem (1)-(2) is ill-posed in the sense that the unbounded linear operator $-\varLambda$ is not the infinitesimal generator of a strongly continuous semigroup. Commonly, fractional powers theory is applied for densely closed operators whose negative generates a semigroup. According to Balakrishnan [4] "Fractional powers of closed linear operators were first constructed by Bochner [8] and subsequently Feller [22], for the Laplacian operator. These constructions depend in an essential way on the fact that Laplacian generates a semigroup..." (p. 419).

As far as we know, Balakrishnan, in [4], was the first to obtain a new construction for fractional powers in which it is not required that the base operator generate a semigroup. His ideas evolved into the current definition of fractional powers of closed operators widely used in the literature. It is well know from the semigroup theory that if B is a linear operator which is possible to calculate its fractional powers and $-B$ generates a strongly continuous semigroup of contractions, then $-B^\alpha$ generates an analytic semigroup for $\alpha \in (0,1)$, see Lemma 1.5.12 . However, what can one say about $-B^\alpha$ if $-B$ does not generate a strongly continuous semigroup? In general $-B^\alpha$ generates a strongly continuous analytic semigroup for $0 < \alpha \leqslant \frac{1}{2}$, see Remark 1.5.11.

One of the main results of this work is to give a complete answer to the above question for the operator $-\varLambda$ in (4)-(5). First, we prove that though the negative of $\varLambda$ is not an infinitesimal generator of a strongly continuous semigroup, it is possible to define its fractional powers $\varLambda^\alpha$, for $0 < \alpha < 1$. After calculating by an explicit formula the fractional powers of order $\alpha \in (0,1)$ of $\varLambda$, motivated by what was discussed in the last paragraph, we conclude that $-\varLambda^\alpha$ generates a strongly continuous semigroup on Y if and only if $0 < \alpha \leqslant \frac{3}{4}$ and it generates strongly continuous analytic semigroup on the open interval $0 < \alpha < \frac{3}{4}$.

As an application, we present parabolic approximations governed by the fractional powers of the Moore-Gibson-Thompson equations in a smooth bounded domain of Euclidean spaces. subject to Dirichlet boundary condition continuing the mathematical analysis of these models developed in Abadias, Lizama and Murillo [1], Caixeta, Lasiecka, and Cavalcanti [9], Conejero, Lizama and Rodenas [16], Dell'Oro, Lasiecka and Pata [19], Kaltenbacher, Lasiecka and Marchand [26], Kaltenbacher, Lasiecka and Pospieszalska [27], Knapp [30], Lasiecka and Wang [32], Marchand, McDevitt and Triggiani [33], Pellicer and Said-Houari [37], Pellicera and Solà-Morales [38] and references therein.

As a consequence of the fractional equation obtained by the calculation of $-\Lambda^{\alpha}$, for $0 < \alpha < 1$, we also study the problem:

$$\partial_t^3 u + 3A^{\frac{1}{3}}\partial_t^2 u + 3A^{\frac{2}{3}}\partial_t u + Au = 0 \tag{6}$$

with the initial conditions given by

$$u(0) = \varphi \in X^{\frac{2}{3}},\ \partial_t u(0) = \psi \in X^{\frac{1}{3}},\ \partial_t^2 u(0) = \xi \in X, \tag{7}$$

We show that the term $3A^{\frac{1}{3}}\partial_t^2 u + 3A^{\frac{2}{3}}\partial_t u$ is strong enough not only to make the problem (1)-(2) well-posed but also to ensure that the operator associated with this new problem is sectorial. Thus, the term $3A^{\frac{1}{3}}\partial_t^2 u + 3A^{\frac{2}{3}}\partial_t u$ behaves like the damping term $2A^{\frac{1}{2}}\partial_t u$ for the strongly damped wave equation which has been extensively studied for many authors, see for instance Carvalho and Cholewa [10] and Chen and Triggiani [12, 13].

Fractional powers approach of operators for the dissipativity of evolution equations has been divulged in the literature in the last years, in Bezerra, Carvalho, Cholewa, and Nascimento [5] the authors study parabolic approximations governed by the fractional powers of order $\alpha \in (0, 1)$ of the wave operator; in Bezerra, Carvalho, Dłotko, and Nascimento [6] the authors study a fractional Schrödinger equation of order $\alpha \in (0, 1)$ and the problem of solvability, asymptotic behaviour and connection with classical Schrödinger equation, Carvalho and Piskarev [11], where the authors study asymptotic dynamics of abstract parabolic problems in the sense of attractors, see also Cholewa and Dłotko [15], Hale [23], and references therein. To our best knowledge, there is no fractional powers approach for operators of third order in time evolution equations.

The thesis is organized as follows. In Chapter 1 entitled 'Preliminaries' we have compiled some basic facts on the semigroups of bounded linear operators theory.

In Chapter 2 entitled 'Third order differential equation on a time scale' our main results are stated and proved: we study the spectral properties of Λ and Λ^{α}, for $0 <$

$\alpha < 1$, we determine for which value of $\alpha \in (0,1)$ does the negative of the operator Λ^α generate a strongly continuous semigroup. We obtain a fractional differential equation from Λ^α and we state an approximation result as $\alpha \nearrow \frac{3}{4}$. Finally, we consider the strongly damped third order problem (6)-(7).

The results in the first four sections of the Chapter 2 constitute an article entitled '*Fractional powers approach of operators for abstract evolution equations of third order in time*' by myself and Flank D. M. Bezerra accepted for publication by the Journal of Differential Equations on 6 April, 2020. Its online version is already available in the link *https://doi.org/10.1016/j.jde.2020.04.020*.

The Chapter 3 entitled 'Future research directions' provides future research directions. We present partial results and some conjectures for nth order problem that generalizes the problem (1)-(2). Currently, this section is contained in a preprint by myself and Flank D. M. Bezerra and this manuscript will be submitted for publication soon. We use the calculation of the fractional powers of operators to give an alternative way to obtain the Euler-Rodrigues formula for three-dimensional rotations. Currently, this section is contained in a paper entitled 'Fractional powers of operators approach to Euler-Rodrigues formula for three-dimensional rotation' by myself and Flank D. M. Bezerra and this manuscript is submitted for publication.

Finally, in Appendix A entitled 'Chebyshev polynomials of the second kind' we give a very brief exposition of the Chebyshev polynomials of the second kind.

Notation and terminology

- Throughout this work, X denotes a Banach space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.
- We will denoted by $\mathcal{L}(X)$ the space of linear operators defined from the whole space X to itself endowed with the norm

$$\|S\|_{\mathcal{L}(X)} := \sup_{x \in X,\ x \neq 0} \frac{\|Sx\|_X}{\|x\|_X},\ \forall S \in \mathcal{L}(X).$$

- The domain of a linear operator A will be denoted by $D(A)$ and the image of A will be denoted by $R(A)$.
- The closure of a set $B \subset X$ will be denoted by $\overline{B}$.
- The Banach dual space of X will be denoted by X^* and the Banach adjoint operator of an operator A will be denoted by A^*.
- If $A : D(A) \subset X \to X$ is a linear operator, then the resolvent set of the operator A, denoted by the $\rho(A)$, is given by

  $\rho(A) = \{\lambda \in \mathbb{C} : \overline{R(\lambda I - A)} = X,\ (\lambda I - A)^{-1}\text{exists and is bounded on } R(\lambda I - A)\}$

- The set $\sigma(A) = \mathbb{C} \backslash \rho(A)$ is called the spectrum of the operator A. It consists of the point spectrum

$$\sigma_p(A) = \{\lambda \in \mathbb{C} : (\lambda I - A)^{-1} \text{ does not exist}\},$$

  the residual spectrum

$$\sigma_r(A) = \{\lambda \in \mathbb{C} : (\lambda I - A)^{-1} \text{ exists},\ \overline{R(\lambda I - A)} \neq X\}$$

and the continuous spectrum

$$\sigma_c A = \{\lambda \in \mathbb{C} : (\lambda I - A)^{-1} \text{ exists}, \ \overline{R(\lambda I - A)} = X, \ (\lambda I - A)^{-1} \text{ is not bounded}\}.$$

Chapter 1

Preliminaries

In this chapter we review some of standard facts on semigroups of bounded linear operators theory. The chapter is intended to make the work as self-contained as possible. We summarize without proofs the relevant material on semigroups of bounded linear operators theory for the reader who is not familiar with this theory. There is no intention whatsoever to rewrite or bring new results to this theory. For a deeper discussion of semigroups of bounded linear operators we refer the reader to Amann [3], Balakrishnan [4], Czaja [17] and Pazy [36].

1.1 Semigroups of bounded operators

For the proofs in this section we refer the reader to Pazy [36, Chapter 1].

Definition 1.1.1. *A one-parameter family* $\{T(t) : t \in [0, \infty)\} \subset \mathcal{L}(X)$ *of bounded operators is a* ***semigroup of bounded operators on*** X *if*

(i) $T(0) = I_X$

(ii) $T(t + s) = T(t)T(s)$ *for all* $t, s \in [0, \infty)$.

Definition 1.1.2. *A semigroup* $\{T(t) : t \in [0, \infty)\} \subset \mathcal{L}(X)$ *of bounded operators on* X *is a* ***uniformly continuous semigroup*** *if*

$$\lim_{t \to 0^+} \|T(t) - I_X\| = 0.$$

Definition 1.1.3. *A semigroup $\{T(t) : t \in [0, \infty)\} \subset \mathcal{L}(X)$ of bounded operators on X is a **strongly continuous semigroup** (or C^0**-semigroup** for short) if*

$$\lim_{t\to 0^+} T(t)x = x \text{ for all } x \in X.$$

Definition 1.1.4. *The linear operator $A : D(A) \subset X \to X$ defined by*

$$D(A) = \left\{ x \in X : \lim_{t\to 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

and

$$Ax = \lim_{t\to 0^+} \frac{T(t)x - x}{t} \text{ for all } x \in D(A)$$

*is the **infinitesimal generator of the semigroup** $\{T(t) : t \in [0, \infty)\}$ **on** X**.***

Theorem 1.1.5. *Let $\{T(t) : t \in [0, \infty)\}$ be a C_0-semigroup on X. Then there exist constants $\omega \geqslant 0$ and $M \geqslant 1$ such that*

$$\|T(t)\| \leqslant Me^{\omega t} \text{ for all } t \in [0, \infty).$$

Corollary 1.1.6. *If $\{T(t) : t \in [0, \infty)\}$ is a C_0-semigroup on X, then the function*

$$[0, \infty) \times X \ni (t, x) \longmapsto T(t)x \in X$$

is continuous.

Theorem 1.1.7. *Assume that $\{T(t) : t \in [0, \infty)\}$ is a C_0-semigroup on X and let $A : D(A) \subset X \to X$ be its infinitesimal generator. Then*

(a) $T(t)x \in D(A)$ for $x \in D(A)$ and $t \in [0, \infty)$. Moreover, for $x \in D(A)$ the function $[0, \infty) \ni t \longmapsto T(t)x \in X$ is differentiable and

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax. \tag{1.1}$$

In particular, $T(\cdot)x \in C([0, \infty), X^1) \cap C^1([0, \infty), X)$ for all $x \in D(A)$

(b) For $x \in D(A)$ and $0 \leqslant s \leqslant t < \infty$,

$$T(t)x - T(s)x = \int_t^s T(\tau)Axd\tau = \int_s^t AT(\tau)xd\tau$$

(c) $\bigcap_{n\geqslant 1} D(A^n)$ is dense on X.

(d) If $\|T(t)\| \leqslant Me^{\omega t}$, $t \in [0, \infty)$, for some $M \geqslant 1$ and $\omega \in \mathbb{R}$, then for all $x \in X$ and $\lambda \in \mathbb{C}$ with $Re\ \lambda > \omega$ we have

$$(\lambda I_X - A)^{-1}x = \int_0^\infty e^{-\lambda t}T(t)xdt.$$

Definition 1.1.8. *A C_0-semigroup $\{T(t) : t \in [0, \infty)\}$ of bounded operators is called a* ***semigroup of contractions*** *if*

$$\|T(t)\| \leqslant 1 \text{ for all } t \in [0, \infty).$$

Theorem 1.1.9. *Let $A : D(A) \subset X \to X$ be the infinitesimal generator of two C_0-semigroups $\{T(t) : t \in [0, \infty)\}$ and $\{S(t) : t \in [0, \infty)\}$. Then*

$$T(t) = S(t) \text{ for all } t \in [0, \infty)$$

Theorem 1.1.10. *A linear operator $A : D(A) \subset X \to X$ is the infinitesimal generator of a uniformly continuous semigroup if and only if A is bounded on X. Moreover, if $A \in \mathcal{L}(X)$, then A is the generator of the uniformly continuous semigroup $\{T(t) : t \in [0, \infty)\}$ given by*

$$T(t) = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} \text{ for all } t \in [0, \infty),$$

where the series is convergent in the operator norm.

1.2 Existence of semigroups

In this section we present classic results on generation of semigroups of bounded linear operators. For the proofs in this section we refer the reader to Pazy [36, Chapter 1].

Theorem 1.2.1. ***(Hille-Yosida)*** *If $A : D(A) \subset X \to X$ is a linear operator, then the following conditions are equivalent:*

(a) A is the infinitesimal generator of a C_0-semigroup of contractions,

(b) (i) A is closed and $\overline{D(A)} = X$,

(ii) The resolvent set $\rho(A)$ of A contains $(0, \infty)$ and for every $\lambda > 0$

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \leqslant \frac{1}{\lambda} \tag{1.2}$$

(c) (i) A is closed and $\overline{D(A)} = X$,

(ii) The resolvent set $\rho(A)$ of A contains the half plane $\{\lambda \in \mathbb{C} : Re\ \lambda > 0\}$ and for such λ

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \leqslant \frac{1}{Re\ \lambda} \tag{1.3}$$

Theorem 1.2.2. ***(Feller-Miyadera-Phillips)*** *If $A : D(A) \subset X \to X$ is a linear operator and $M \geqslant 1$, $\omega \in \mathbb{R}$ are constants, then the following conditions are equivalent:*

(a) A is the infinitesimal generator of a C_0-semigroup $\{T(t) : t \in [0, \infty)\}$ such that

$$\|T(t)\| \leqslant Me^{\omega t} \text{ for all } t \in [0, \infty)$$

(b) (i) A is closed and $\overline{D(A)} = X$,

(ii) The resolvent set $\rho(A)$ of A contains (ω, ∞) and for every $\lambda > \omega$ and $n \in \mathbb{N}$

$$\|(\lambda - A)^{-n}\|_{\mathcal{L}(X)} \leqslant \frac{M}{(\lambda - \omega)^n} \tag{1.4}$$

(c) (i) A is closed and $\overline{D(A)} = X$,

(ii) The resolvent set $\rho(A)$ of A contains the half plane $\{\lambda \in \mathbb{C} : Re\ \lambda > \omega\}$ and for such λ and $n \in \mathbb{N}$

$$\|(\lambda - A)^{-n}\|_{\mathcal{L}(X)} \leqslant \frac{M}{(Re\ \lambda - \omega)^n} \tag{1.5}$$

Let X be a Banach space and let X' be its dual. For every $x \in X$ we define the duality set $J(x) \subset X'$ by

$$J(x) = \{\varphi : \varphi \in X' \text{ and } \varphi(x) = \|x\|^2 = \|\varphi\|^2\}.$$

It follows from Hahn-Banach theorem that $J(x)$ is a nonempty set for every $x \in X$.

Definition 1.2.3. *A linear operator $A : D(A) \subset X \to X$ is* ***dissipative*** *if for every $x \in D(A)$ there exists a $\varphi \in J(x)$ such that $Re\ \varphi(Ax) \leqslant 0$.*

Theorem 1.2.4. *A linear operator $A : D(A) \subset X \to X$ is dissipative if and only if*

$$\|(\lambda I_X - A)x\| \geqslant \lambda\|x\|$$

for all $x \in D(A)$ and $\lambda > 0$.

Theorem 1.2.5. ***(Lumer-Phillips)*** *Let $A : D(A) \subset X \to X$ be a linear operator with $\overline{D(A)} = X$.*

(i) If A is dissipative and there exists a $\lambda_0 > 0$ such that

$$R(\lambda_0 I_X - A) = X,$$

then A is the infinitesimal generator of a C_0-semigroup of contractions on X.

(ii) If A is the infinitesimal generator of a C_0-semigroup of contraction on X then A is dissipative and

$$R(\lambda I_X - A) = X$$

for all $\lambda > 0$. Moreover, for every $x \in D(A)$ and every $\varphi \in J(x)$

$$Re\ \varphi(Ax) \leqslant 0.$$

Remark 1.2.6. *If X is a Hilbert space with scalar product $\langle \cdot, \cdot \rangle_X$, from the Riesz' representation theorem we have*

$$J(x) = \{\langle \cdot, x \rangle_X\}$$

for $x \in X$. In this case a linear operator $A : D(A) \subset X \to X$ is dissipative if and only if

$$Re\, \langle Ax, x \rangle_X \leqslant 0, \quad \text{for every } x \in X. \tag{1.6}$$

1.3 Analytic semigroups

For the proofs in this section we refer the reader to Pazy [36, Chapter 1].

Definition 1.3.1. *A C_0-semigroup $\{T(t) : t \in [0, \infty)\}$ is called* ***an analytic (strongly continuous) semigroup*** *if there exist a sector on the complex plane*

$$\Delta_\phi = \{z \in \mathbb{C} : |argz| < \phi\} \text{ with } 0 < \phi \leqslant \frac{\pi}{2}$$

and a family of bounded operators $\{T(z) : z \in \Delta_\phi\}$ which coincide with $T(t)$ for $t \in [0, \infty)$, such that

(i) the mapping $z \longmapsto T(z)$ is analytic in $\Delta_\phi \backslash \{0\}$,

(ii) $\lim_{z \to 0,\ z \in \Delta_\phi} T(z)x = x$ for all $x \in X$.

(iii) $T(z_1 + z_2) = T(z_1)T(z_2)$ for all $z_1, z_2 \in \Delta_\phi$.

Definition 1.3.2. *Let $0 < \phi < \frac{\pi}{2}$, $M \geqslant 1$ and $a \in \mathbb{R}$. We say that an operator $A : D(A) \subset X \to X$ is* ***sectorial*** *if*

(i) A is a densely defined closed operator.

(ii) the resolvent set $\rho(A)$ contains the sector

$$S_{a,\phi} = \{\lambda \in \mathbb{C} : \phi \leqslant |arg(\lambda - a)| \leqslant \pi,\ \lambda \neq a\}$$

and the estimate

$$\|(\lambda I_X - A)^{-1}\| \leqslant \frac{M}{|\lambda - a|}$$

holds for all $\lambda \in S_{a,\phi}$.

Theorem 1.3.3. *Let $A : D(A) \subset X \to X$ be a linear operator. Then the following conditions are equivalent:*

(i) A is the infinitesimal generator of an analytic semigroup.

(ii) $-A$ is a sectorial operator in X.

1.4 Homogeneous abstract equations

Let X be a Banach space and let $A : D(A) \subset X \to X$ be a linear operator. The abstract Cauchy problem for A with initial data $x \in X$ consists of finding a solution $u(t)$ to the initial value problem

$$\begin{cases} \dfrac{du(t)}{dt} = Au, \ t > 0 \\ u(0) = x, \end{cases} \tag{1.7}$$

where what we mean by a solution is described by the following definition

Definition 1.4.1. *A function $u : [0, \infty) \to X$ is called a **solution** of the problem (1.7) if*

$$u \in C([0, \infty), X) \cap C^1((0, \infty), X),$$

$u(t) \in D(A)$ for all $t > 0$ and u satisfies (1.7) in X.

For the proofs of the following theorems see Pazy [36, Chapter 4].

Theorem 1.4.2. *Let A be a densely defined linear operator with a nonempty resolvent set $\rho(A)$. The initial value problem (1.7) has a unique solution $u(t)$, which is continuously differentiable on $[0, \infty)$, for every initial value $x \in D(A)$ if and only if A is the infinitesimal generator of a C_0-semigroup $T(t)$*

Theorem 1.4.3. *If A is the infinitesimal generator of a differentiable semigroup then for every $x \in X$ the initial value problem (1.7) has a unique solution.*

Remark 1.4.4. *The solution mentioned in the above theorems is $u(t) = T(t)x$. If $\{T(t) : t \in [0, \infty)\}$ is a semigroup which is not differentiable and $x \notin D(A)$ then the initial value problem (1.7) does not have a solution. However $u(t) = T(t)x$ is the unique **weak solution** of (1.7), that is, $u \in C([0, \infty), X)$, $u(0) = x$ and for all $\varphi \in D(A^*)$, the function $t \longmapsto \varphi(u(t)) \in \mathbb{K}$ is differentiable and*

$$\frac{d}{dt}\varphi(u(t)) = A^*\varphi(u(t)), \ t \geqslant 0.$$

1.5 Fractional powers of positive operators

For the results in this section we refer the reader to Pazy [36, Section 2.2.6] and Amann [3, Section 3.4.6].

Definition 1.5.1. *A linear operator $A : D(A) \subset X \to X$ is of* ***positive type*** $K \geqslant 1$ *if it is closed, densely defined, $[0, \infty) \subset \rho(-A)$ and*

$$\|(\lambda I + A)^{-1}\|_{\mathcal{L}(X)} \leqslant \frac{K}{1+\lambda}, \text{ for all } \lambda \geqslant 0. \tag{1.8}$$

We call an operator A of ***positive type*** *if it is of positive type K for some $K \geqslant 1$. These are the operators that one can define the fractional power.*

Lemma 1.5.2. *If $A : D(A) \subset X \to X$ is a positive operator of type $K \geqslant 1$, then*

$$S(K) := \left\{\lambda \in \mathbb{C} : |arg\ \lambda| \leqslant \arcsin \frac{1}{2K}\right\} \cup \left\{|\lambda| \leqslant \frac{1}{2K}\right\} \subset \rho(-A) \tag{1.9}$$

and

$$(1 + |\lambda|)\|(\lambda I + A)^{-1}\| \leqslant 2K + 1, \text{ for all } \lambda \in S(K). \tag{1.10}$$

For an operator A of positive type and $\alpha > 0$ we define

$$A^{-\alpha} = \frac{1}{2\pi i}\int_C \lambda^{-\alpha}(A - \lambda I)^{-1} d\lambda \tag{1.11}$$

where the path C runs in the resolvent set of A from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$, $\omega < \theta < \pi$, avoiding the negative real axis and the origin and $\lambda^{-\alpha}$ is taken to be positive for real positive values of λ.

Lemma 1.5.3. *The formula* (1.11) *defines a bounded linear operator $A^{-\alpha}$. Moreover for $\alpha = n$ the definition* (1.11) *coincides with the classical definition of $(A^{-1})^n$.*

Lemma 1.5.4. *If A of positive type, then*

(i)

$$A^{-\alpha} = \frac{\sin \alpha\pi}{2\pi}\int_0^\infty \lambda^{-\alpha}(\lambda I + A)^{-1} d\lambda \qquad 0 < \alpha < 1. \tag{1.12}$$

(ii) For $\alpha, \beta \geqslant 0$

$$A^{-(\alpha+\beta)} = A^{-\alpha}A^{-\beta} \tag{1.13}$$

(iii) There exists a constant C such that

$$\|A^{-\alpha}\| \leqslant C \quad \text{for all } 0 \leqslant \alpha \leqslant 1.$$

(iv) $A^{-\alpha}$ is one-to-one.

(v) The family $\{A^{-t}; t \geqslant 0\}$ is a strongly continuous semigroup on X.

We denote its infinitesimal generator by

$$-\log A$$

which defines the logarithm of A. Then the intuitive formula $A^{-t} = e^{-t\log A}$, $t \geqslant 0$, is valid.

Definition 1.5.5. *For every $\alpha > 0$ we define*

$$A^{\alpha} = (A^{-\alpha})^{-1} \tag{1.14}$$

For $\alpha = 0, A^{\alpha} = I$

Lemma 1.5.6. *If α, β are real then*

$$A^{\alpha+\beta} = A^{\alpha}A^{\beta} \tag{1.15}$$

Lemma 1.5.7 (Balakrishnan formula)**.** *Let $0 < \alpha < 1$. If $x \in D(A)$ then*

$$A^{\alpha}x = \frac{\sin \alpha\pi}{\pi}\int_0^{\infty} \lambda^{\alpha-1}A(\lambda I + A)^{-1}xd\lambda \tag{1.16}$$

Lemma 1.5.8. *If $0 < \alpha < 1$ and β is real then A^{α} is of positive type and*

$$(A^{\alpha})^{\beta} = A^{\alpha\beta} \tag{1.17}$$

Lemma 1.5.9. *Assume that $A : D(A) \subset X \to X$ is a linear operator of positive type and $\alpha \in (0,1)$. Then $A^{\alpha} : D(A^{\alpha}) \subset X \to X$ is of positive type. In fact, if there exists $\theta \in (0, \pi)$ such (1.8) is satisfied for $\lambda \in \mathbb{C}$ with $|arg\ \lambda| \leqslant \theta$ then*

$$\{\lambda \in \mathbb{C} : |arg\ z| < \pi - (\pi - \theta)\alpha\} \cup \{0\} \subset \rho(-A^{\alpha})$$

and, given $\theta' \in (0, \theta)$,

$$(1 + |\lambda|)\|(\lambda I_X + A^{\alpha})^{-1}\| \leqslant K, \quad |arg\ \lambda| \leqslant \pi - (\pi - \theta')\alpha. \tag{1.18}$$

Corollary 1.5.10. *Suppose that $A : D(A) \subset X \to X$ is a linear operator of positive type and there exists $\theta \in (0, \pi)$ such that (1.8) is satisfied for $\lambda \in \mathbb{C}$ with $|arg\ \lambda| \leqslant \theta$. If $\alpha \in (0, 1)$ satisfies $\alpha < \pi/2(\pi - \theta)$ then $-A^{\alpha}$ generates a strongly continuous analytic semigroup on X.*

Remark 1.5.11. *Note that (1.9) ensures the existence of such $\theta \in (0, 1)$ above. But $0 < \theta < 1$ implies that*

$$\frac{\pi}{2(\pi - \theta)} > \frac{1}{2}.$$

This implies that $-A^{\alpha}$ generates a strongly continuous analytic semigroup on X whenever A is of positive type and $0 < \alpha \leqslant 1/2$.

We will make a proof of the next lemma because we will present a proof using our arguments.

Lemma 1.5.12. *Assume that $A : D(A) \subset X \to X$ is a linear operator of positive type $K \geqslant 1$ and $-A$ generates a C_0-semigroup of contractions on X. Then $-A^{\alpha}$ generates a strongly continuous analytic semigroup on X for $0 < \alpha < 1$.*

Proof: Fix $\alpha \in \mathbb{R}$ with $0 < \alpha < 1$ and choose $\theta \in (0, \pi)$ such that

$$\pi - \frac{\pi}{2\alpha} < \theta < \frac{\pi}{2}.$$

For such θ we have

$$\alpha < \frac{\pi}{2(\pi - \theta)} < 1.$$

Thus, from Corollary 1.5.10, it is sufficient to show that (1.8) is satisfied for $\lambda \in S_\theta := \{\lambda \in \mathbb{C} : |arg\ \lambda| \leqslant \theta\}$. By Lemma 1.5.2, we only need to consider the case $\lambda \in S_\theta \backslash S(K)$. It follows from Hille-Yosida Theorem 1.2.1 that

$$(1 + |\lambda|)\|(\lambda + A)^{-1}\| \leqslant \frac{1 + |\lambda|}{Re\ \lambda} \leqslant \frac{1}{Re\ \lambda} + \frac{1}{\cos\theta} \leqslant \frac{\cos^2\theta + 2K}{2K\cos\theta}$$

for $\lambda \in S_\theta \backslash S(K)$. □

If $A : D(A) \subset X \to X$ is a linear operator of positive type, then we will denote by X^α, for $\alpha \in [0, \infty)$ (taking $A^0 := I$ on $X^0 := X$ when $\alpha = 0$), the space $D(A^\alpha)$ with the norm

$$\|\cdot\|_{X^\alpha} := \|A^\alpha \cdot\|_X$$

It is easily seen that the fractional power space X^α is a Banach space.

Lemma 1.5.13. *If* $0 \leqslant \alpha \leqslant \beta$, *then* X_β *is a dense subset of* X_α *and the identity map* $X_\beta \ni x \longmapsto x \in X_\alpha$ *is continuous.*

Theorem 1.5.14. *(Moment Inequality) Let* $A : D(A) \subset X \to X$ *be a linear operator of positive type. If* $\alpha \in [0, \infty)$, *then there exists a constant* $K > 0$ *such that*

$$\|x\|_{X^\alpha} \leqslant \frac{K}{2}(\rho^\alpha\|x\| + \rho^{\alpha-1}\|x\|_{X^1}), \tag{1.19}$$

$$\|x\|_{X^\alpha} \leqslant K\|x\|_X^{1-\alpha}\|x\|_{X^1}^\alpha \tag{1.20}$$

Theorem 1.5.15. *If* $A : D(A) \subset H \to H$ *is a positive definite self-adjoint operator in a Hilbert space* H, *then the operator* $A^\alpha : D(A^\alpha) \subset H \to H$ *is positive definite self-adjoint for each* $\alpha > 0$.

Chapter 2

Third order differential equation on a time scale

As mentioned earlier, this chapter contains the main results of the thesis. The results in the first four sections constitute an article entitled '*Fractional powers approach of operators for abstract evolution equations of third order in time*' by myself and Flank D. M. Bezerra accepted for publication by the Journal of Differential Equations on 6 April, 2020. Its online version is already available in the link *https://doi.org/10.1016/j.jde.2020.04.020*.

We consider, abusing notation, the following abstract linear evolution equation of third order in time

$$\partial_t^3 u + Au = 0 \tag{2.1}$$

with initial conditions given by

$$u(0) = u_0 \in X^{\frac{2}{3}},\ \partial_t u(0) = u_1 \in X^{\frac{1}{3}},\ \partial_t^2 u(0) = u_2 \in X, \tag{2.2}$$

where X is a separable Hilbert space and $A : D(A) \subset X \to X$ is a linear, closed, densely defined, self-adjoint and positive definite operator. We wish to study the fractional powers of Λ, the matricial operator obtained by rewriting (2.1)-(2.2) as a first order abstract system. For this purpose we will consider the phase space

$$Y = X^{\frac{2}{3}} \times X^{\frac{1}{3}} \times X$$

which is a Banach space equipped with the norm given by

$$\|\cdot\|_Y^2 = \|\cdot\|^2_{X^{\frac{2}{3}}} + \|\cdot\|^2_{X^{\frac{1}{3}}} + \|\cdot\|^2_X.$$

We can write the problem (2.1)-(2.2) as a Cauchy problem on Y, letting $v = \partial_t u$, $w = \partial_t^2 u$ and $\mathbf{u} = \left[\begin{smallmatrix} u \\ v \\ w \end{smallmatrix}\right]$ and the initial value problem

$$\begin{cases} \dfrac{d\mathbf{u}}{dt} + \Lambda\mathbf{u} = 0, \ t > 0 \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \tag{2.3}$$

where the unbounded linear operator $\Lambda : D(\Lambda) \subset Y \to Y$ is defined by

$$D(\Lambda) = D(A) \times D(A^{\frac{2}{3}}) \times D(A^{\frac{1}{3}}), \tag{2.4}$$

and

$$\Lambda\mathbf{u} = \left[\begin{smallmatrix} 0 & -I & 0 \\ 0 & 0 & -I \\ A & 0 & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} u \\ v \\ w \end{smallmatrix}\right] := \left[\begin{smallmatrix} -v \\ -w \\ Au \end{smallmatrix}\right], \ \forall \mathbf{u} = \left[\begin{smallmatrix} u \\ v \\ w \end{smallmatrix}\right] \in D(\Lambda). \tag{2.5}$$

From now on, we denote

$$Y^1 = X^1 \times X^{\frac{2}{3}} \times X^{\frac{1}{3}},$$

equipped with the norm

$$\|\cdot\|^2_{Y^1} = \|\cdot\|^2_{X^1} + \|\cdot\|^2_{X^{\frac{2}{3}}} + \|\cdot\|^2_{X^{\frac{1}{3}}}.$$

In Section 2.1 we study the spectral properties of the operator Λ proving that though the negative of Λ is not an infinitesimal generator of a strongly continuous semigroup (see Lemma 2.1.3), it is possible to define its fractional powers Λ^α (Lemma 2.1.6). In Section 2.2 we study the spectral properties of the fractional powers Λ^α, for $0 < \alpha < 1$, what lead us to the main result of this chapter: $-\Lambda^\alpha$ generates a strongly continuous semigroup on Y if and only if $0 < \alpha \leqslant \frac{3}{4}$ and it generates strongly continuous analytic semigroup on the open interval $0 < \alpha < \frac{3}{4}$ (see Theorem 2.2.2). Section 2.3 is devoted to the study of the fractional differential equation obtained by the explicit representation of Λ^α. In Section 2.4 we will be concerned with an approximation result for the semigroups generated by $-\Lambda^\alpha$ on parameter $0 < \alpha \leqslant \frac{3}{4}$ as $\alpha \nearrow \frac{3}{4}$. Finally, Section 2.5 provides, as an application, the well posedness of a strogly damped third order equation in time.

2.1 Spectral properties of the operator Λ

In this section we study the unbounded linear operator Λ, in the sense of the theory of closed and densely defined operators.

Lemma 2.1.1. *The linear operator $\Lambda : D(\Lambda) \subset Y \to Y$ is closed and densely defined.*

Proof: Consider $\mathbf{u}_n = \left[\begin{smallmatrix} u_n \\ v_n \\ w_n \end{smallmatrix}\right] \in D(\Lambda)$ with $\mathbf{u}_n \to \mathbf{u} = \left[\begin{smallmatrix} u \\ v \\ w \end{smallmatrix}\right]$ in Y as $n \to \infty$, and $\Lambda \mathbf{u}_n \to \varphi$ in Y as $n \to \infty$, where $\varphi = \left[\begin{smallmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{smallmatrix}\right]$, then

$$v_n \to -\varphi_1 \text{ in } X^{\frac{2}{3}} \hookrightarrow X^{\frac{1}{3}} \text{ as } n \to \infty$$

and consequently, $v = -\varphi_1 \in D(A^{\frac{2}{3}})$. As well as, we have

$$w_n \to -\varphi_2 \text{ in } X^{\frac{1}{3}} \hookrightarrow X \text{ as } n \to \infty$$

and consequently, $w = -\varphi_2 \in D(A^{\frac{1}{3}})$. Finally, since A is a closed operator, we have $u \in D(A)$ and $Au = \varphi_3$; that is, $\mathbf{u} \in D(\Lambda)$ and $\Lambda \mathbf{u} = \varphi$.

Secondly, $D(\Lambda) = D(A) \times D(A^{\frac{2}{3}}) \times D(A^{\frac{1}{3}})$ is dense in $Y = X^{\frac{2}{3}} \times X^{\frac{1}{3}} \times X$ because the inclusions $X^{\alpha} \subset X^{\beta}$ are dense for $\alpha \geqslant \beta \geqslant 0$, by Lemma 1.5.13.

In this subsection, we study spectral properties of the operator Λ.

Lemma 2.1.2. *The resolvent set of $-\Lambda$ is given by*

$$\rho(-\Lambda) = \{\lambda \in \mathbb{C} : \lambda^3 \in \rho(-A)\}. \tag{2.6}$$

Proof: Suppose that $\lambda \in \mathbb{C}$ is such that $\lambda^3 \in \rho(-A)$. We claim that $\lambda \in \rho(-\Lambda)$. Indeed, since $-\Lambda$ is a closed operator, we only need to show that

$$\lambda I_Y + \Lambda : D(\Lambda) \subset Y \to Y$$

is bijective. For injectivity consider $\mathbf{u} = \left[\begin{smallmatrix} u \\ v \\ w \end{smallmatrix}\right] \in D(\Lambda)$ and $(\lambda I_Y + \Lambda)\mathbf{u} = 0$, then

$$\begin{bmatrix} \lambda I_X & -I_X & 0 \\ 0 & \lambda I_X & -I_X \\ A & 0 & \lambda I_X \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

It follows that

$$\begin{cases} \lambda u - v = 0 \\ \lambda v - w = 0 \\ Au + \lambda w = 0. \end{cases} \tag{2.7}$$

From (2.7) we have

$$(\lambda^3 I_X + A)u = 0. \tag{2.8}$$

Since $\lambda^3 \in \rho(-A)$, we conclude that $u = 0$ and consequently $\mathbf{u} = 0$. For surjectivity given $\varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix} \in Y$ we take $\mathbf{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$ with

$$\begin{aligned} u &= (\lambda^3 I + A)^{-1}(\lambda^2\varphi_1 + \lambda\varphi_2 + \varphi_3) \\ v &= (\lambda^3 I + A)^{-1}(\lambda^3\varphi_1 + \lambda^2\varphi_2 + \lambda\varphi_3) - \varphi_1 \\ w &= (\lambda^3 I + A)^{-1}(\lambda^4\varphi_1 + \lambda^3\varphi_2 + \lambda^2\varphi_3) - \lambda\varphi_1 - \varphi_2 \end{aligned} \tag{2.9}$$

Note that u,v and w are well defined since $\lambda^3 \in \rho(-A)$. Moreover, $u \in D(A)$, $v \in D(A^{\frac{2}{3}})$ because $\varphi_1 \in X^{\frac{2}{3}}$, $w \in D(A^{\frac{1}{3}})$ because $\varphi_2 \in X^{\frac{1}{3}}$. Then we have $\mathbf{u} \in D(\Lambda)$ and

$$(\lambda I_Y + \Lambda)\mathbf{u} = \varphi.$$

Now suppose that $\lambda \in \rho(-\Lambda)$. If $u \in D(A)$ is such that $(\lambda^3 I_X + A)u = 0$, taking $\mathbf{u} = \begin{bmatrix} u \\ \lambda u \\ \lambda^2 u \end{bmatrix} \in D(\Lambda)$ we have

$$(\lambda I_Y + \Lambda)\mathbf{u} = 0. \tag{2.10}$$

Since $\lambda \in \rho(-\Lambda)$, it follows that $\mathbf{u} = 0$ and consequently $u = 0$, which proves the injectivity of $\lambda^3 I_X + A$. Given $f \in X$, consider $\varphi = \begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix} \in Y$. By the surjectivity of $\lambda I_Y + \Lambda$ there exists $\mathbf{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \in D(\Lambda)$ such that

$$(\lambda I_Y + \Lambda)\mathbf{u} = \varphi \tag{2.11}$$

which gives $(\lambda^3 I_X + A)u = f$, and the proof is complete.

□

We shall show that it is possible to calculate explicitly the fractional power Λ^α of the operator Λ for $0 < \alpha < 1$, and with this, we will consider the fractional approximations of (2.3) given by

$$\begin{cases} \dfrac{d\mathbf{u}^\alpha}{dt} + \Lambda^\alpha \mathbf{u}^\alpha = 0, \ t > 0, \ 0 < \alpha < 1, \\ \mathbf{u}^\alpha(0) = \mathbf{u}_0^\alpha. \end{cases} \tag{2.12}$$

Here, $\Lambda^\alpha : D(\Lambda^\alpha) \subset Y \to Y$ denotes the fractional power operator of Λ to be defined by $\Lambda^\alpha = (\Lambda^{-\alpha})^{-1}$, where $\Lambda^{-\alpha}$ is given by the formula in (1.12) with domain $D(\Lambda^\alpha)$ characterized by complex interpolation methods, see e.g. Amann [3] and Cholewa and Dłotko [15].

Lemma 2.1.3. *The unbounded linear operator $-\Lambda$ with $\Lambda : D(\Lambda) \subset Y \to Y$ defined in* (2.4)-(2.5) *is not the infinitesimal generator of a strongly continuous semigroup on Y.*

Proof: If $-\Lambda$ generates a strongly continuous semigroup $\{e^{-\Lambda t} : t \geqslant 0\}$ on Y, it follows from Theorem 1.1.5 that there exist constants $\omega \geqslant 0$ and $M \geqslant 1$ such that

$$\|e^{-\Lambda t}\|_{\mathcal{L}(Y)} \leqslant Me^{\omega t} \quad \text{for all } 0 \leqslant t < \infty. \tag{2.13}$$

Moreover, from Theorem 1.2.2 we have

$$\{\lambda \in \mathbb{C} : Re\lambda > \omega\} \subset \rho(-\Lambda) \tag{2.14}$$

where $\rho(-\Lambda)$ denotes the resolvent set of the operator $-\Lambda$.

Let $\mathbf{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$ be a nontrivial element of $D(\Lambda)$. We shall consider the eigenvalue problem for the operator $-\Lambda$

$$-\Lambda\mathbf{u} = \lambda\mathbf{u}.$$

A straightforward calculation implies

$$\sigma_p(-\Lambda) = \{\lambda \in \mathbb{C} : \lambda^3 \in \sigma_p(-A)\}.$$

Where $\sigma_p(-\Lambda)$ and $\sigma_p(-A)$ denote the point spectrum set of $-\Lambda$ and $-A$, respectively. Since $\sigma_p(-A) = \{-\mu_n : n \in \mathbb{N}\}$ with $\mu_n \in \sigma_p(A)$ for each $n \in \mathbb{N}$ and $\mu_n \to \infty$ as $n \to \infty$, we conclude that

$$\sigma_p(-\Lambda) \cap \{\lambda \in \mathbb{C} : Re\lambda > \omega\} \neq \varnothing$$

This contradicts the equation (3.13) and therefore $-\Lambda$ can not be the infinitesimal generator of a strongly continuous semigroup on Y. □

Remark 2.1.4. *We note that $-\Lambda$ is not a dissipative operator on Y, according to* (1.6). *Indeed, if u is a non-trivial element in X^1 and $\mathbf{u} = \begin{bmatrix} u \\ 0 \\ -u \end{bmatrix}$, then*

$$\langle -\Lambda\mathbf{u}, \mathbf{u}\rangle_Y = \left\langle \begin{bmatrix} 0 \\ -u \\ -Au \end{bmatrix}, \begin{bmatrix} u \\ 0 \\ -u \end{bmatrix} \right\rangle_Y = \langle Au, u\rangle_X = \|u\|^2_{X^{\frac{1}{2}}} > 0.$$

Explicitly, this means that $-\Lambda$ is not an infinitesimal generator of a strongly continuous semigroup of contractions on Y. Nevertheless, the statement in lemma 2.1.3 is more precise because it says that $-\Lambda$ cannot be the infinitesimal generator of a strongly continuous semigroup of any type on Y.

Lemma 2.1.5. *Let $B : D(B) \subset E \to E$ be a linear operator of positive type. If $\alpha \in [0,1]$ and $\lambda \geqslant 0$, then $B^\alpha(\lambda I_X + B)^{-1} \in \mathcal{L}(X)$ and*

$$\|B^\alpha(\lambda I_X + B)^{-1}\|_{\mathcal{L}(E)} \leqslant \frac{K}{(1+\lambda)^{1-\alpha}} \tag{2.15}$$

for some $K \geqslant 1$.

Proof: Here, K will denote a positive constant, not necessarily the same one. We first observe that

$$B(\lambda I_E + B)^{-1} = I_E - \lambda(\lambda I_E + B)^{-1}.$$

This and the fact that B is of positive type give

$$\|B(\lambda I_E + B)^{-1}\|_{\mathcal{L}(E)} \leqslant 1 + \lambda\|(\lambda I_E + B)^{-1}\|_{\mathcal{L}(E)} \leqslant 1 + K. \tag{2.16}$$

Now, for $x \in E$, from the inequality (1.20) we have

$$\begin{aligned} \|B^\alpha(\lambda I_X + B)^{-1}x\|_E &\leqslant K\|(\lambda I_E + B)^{-1}x\|_E^{1-\alpha}\|B(\lambda I_E + B)^{-1}x\|_E^{\alpha} \\ &\leqslant \frac{K}{(1+\lambda)^{1-\alpha}} \end{aligned}$$

for some $K \geqslant 1$. In the last inequality we use the fact that B is of positive type and that $\|B(\lambda I_E + B)^{-1}x\|_E^{\alpha}$ is bounded by (2.16).

Lemma 2.1.6. *The unbounded linear operator Λ defined in* (2.4)-(2.5) *is of positive type $K \geqslant 1$.*

Proof: We have already seen in Lemma 2.1.1 that Λ is a closed and densely defined operator. That $[0,\infty) \subset \rho(-\Lambda)$ follows from (3.6). Finally, for $\lambda \geqslant 0$ we have

$$(\lambda I + \Lambda)^{-1}\mathbf{u} = \varphi$$

If and only if

$$\begin{aligned} \varphi_1 &= (\lambda^3 I + A)^{-1}(\lambda^2 u + \lambda v + w) \\ \varphi_2 &= (\lambda^3 I + A)^{-1}(\lambda^3 u + \lambda^2 v + \lambda w) - u \\ \varphi_3 &= (\lambda^3 I + A)^{-1}(\lambda^4 u + \lambda^3 v + \lambda^2 w) - \lambda u - v \end{aligned} \tag{2.17}$$

In order to verify the equation (1.5.1) for Λ it is sufficient to show that for $\|\mathbf{u}\|_Y \leqslant 1$ there exists a constant $K_\Lambda \geqslant 1$ such that

$$\|\varphi_1\|_{X^{\frac{2}{3}}} + \|\varphi_2\|_{X^{\frac{1}{3}}} + \|\varphi_3\|_X \leqslant \frac{K_\Lambda}{1+\lambda} \tag{2.18}$$

Note that

$$
\begin{aligned}
\|\varphi_1\|_{X^{\frac{2}{3}}} &\leqslant \lambda^2\|(\lambda^3 I + A)^{-1}u\|_{X^{\frac{2}{3}}} + \lambda\|A^{\frac{1}{3}}(\lambda^3 I + A)^{-1}v\|_{X^{\frac{1}{3}}} + \|A^{\frac{2}{3}}(\lambda^3 I + A)^{-1}w\|_X \\
\|\varphi_2\|_{X^{\frac{1}{3}}} &\leqslant \|A^{\frac{2}{3}}(\lambda^3 I + A)^{-1}u\|_{X^{\frac{2}{3}}} + \lambda^2\|(\lambda^3 I + A)^{-1}v\|_{X^{\frac{1}{3}}} + \lambda\|A^{\frac{1}{3}}(\lambda^3 I + A)^{-1}w\|_X \\
\|\varphi_3\|_X &\leqslant \lambda\|A^{\frac{1}{3}}(\lambda^3 I + A)^{-1}u\|_{X^{\frac{2}{3}}} + \|A^{\frac{2}{3}}(\lambda^3 I + A)^{-1}v\|_{X^{\frac{1}{3}}} + \lambda^2\|(\lambda^3 I + A)^{-1}w\|_X.
\end{aligned}
$$

Applying Lemma 2.1.5 we obtain a constant $K \geqslant 1$ such that

$$
\begin{aligned}
\|\varphi_1\|_{X^{\frac{2}{3}}} + \|\varphi_2\|_{X^{\frac{1}{3}}} + \|\varphi_3\|_X &\leqslant \frac{\lambda^2 K}{1+\lambda^3} + \frac{\lambda K}{(1+\lambda^3)^{\frac{2}{3}}} + \frac{K}{(1+\lambda^3)^{\frac{1}{3}}} \\
&\leqslant \frac{K_\Lambda}{1+\lambda},
\end{aligned}
$$

whereas $K_\Lambda \geqslant 1$ is sufficiently large. □

Remark 2.1.7. *Let $S : D(S) \subset E \to E$ be a linear operator of positive type on some Banach space E, if $-S$ generates a strongly continuous semigroup of contractions on E, then $-S^\alpha$ generates an analytic semigroup for $\alpha \in (0,1)$, see Lemma 1.5.12 . However, what can one say about $-S^\alpha$ if $-S$ does not generate a strongly continuous semigroup? In general if S is of positive type on E (see Definition 1.5.1) then $-S^\alpha$ generates a strongly continuous analytic semigroup on E for $0 < \alpha \leqslant \frac{1}{2}$, see Remark 1.5.11.*

2.2 Spectral properties of the fractional powers Λ^α

In this section we study spectral properties of the fractional powers operators Λ^α for $\alpha \in (0,1)$.

Theorem 2.2.1. *If A and Λ are as in (2.4)-(2.5), respectively, then we have all the following.*

i) $0 \in \rho(\Lambda)$ *and*

$$
\Lambda^{-1} = \begin{bmatrix} 0 & 0 & A^{-1} \\ -I & 0 & 0 \\ 0 & -I & 0 \end{bmatrix}.
$$

ii) Fractional powers Λ^α can be defined for $\alpha \in (0,1)$ through

$$
\Lambda^\alpha = \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{\alpha-1}\Lambda(\lambda I + \Lambda)^{-1} d\lambda. \tag{2.19}
$$

iii) Given any $\alpha \in (0,1)$ we have $\Lambda^\alpha : D(\Lambda^\alpha) \subset Y \to Y$ is given by

$$
\Lambda^\alpha = \begin{bmatrix} k_{\alpha,0}A^{\frac{\alpha}{3}} & -k_{\alpha,2}A^{\frac{\alpha-1}{3}} & k_{\alpha,1}A^{\frac{\alpha-2}{3}} \\ -k_{\alpha,1}A^{\frac{\alpha+1}{3}} & k_{\alpha,0}A^{\frac{\alpha}{3}} & -k_{\alpha,2}A^{\frac{\alpha-1}{3}} \\ k_{\alpha,2}A^{\frac{\alpha+2}{3}} & -k_{\alpha,1}A^{\frac{\alpha+1}{3}} & k_{\alpha,0}A^{\frac{\alpha}{3}} \end{bmatrix} \tag{2.20}
$$

where

$$D(\Lambda^\alpha) = D(A^{\frac{\alpha+2}{3}}) \times D(A^{\frac{\alpha+1}{3}}) \times D(A^{\frac{\alpha}{3}}),$$

and

$$k_{\alpha,j} = \frac{1}{3}\left(2\cos\tfrac{2\pi(\alpha+j)}{3} + 1\right), \quad \textit{for } j \in \{0,1,2\}. \tag{2.21}$$

Moreover, these coefficients satisfy the following properties

$$\det\begin{bmatrix} k_{\alpha,0} & -k_{\alpha,2} & k_{\alpha,1} \\ -k_{\alpha,1} & k_{\alpha,0} & -k_{\alpha,2} \\ k_{\alpha,2} & -k_{\alpha,1} & k_{\alpha,0} \end{bmatrix} = k_{\alpha,0}^3 + k_{\alpha,1}^3 + k_{\alpha,2}^3 - 3k_{\alpha,0}k_{\alpha,1}k_{\alpha,2} = 1, \tag{2.22}$$

$$k_{\alpha,0} + k_{\alpha,1} + k_{\alpha,2} = 1, \tag{2.23}$$

and

$$\begin{cases} k_{\alpha,0}^2 - k_{\alpha,1}k_{\alpha,2} = k_{\alpha,0} \\ k_{\alpha,1}^2 - k_{\alpha,0}k_{\alpha,2} = k_{\alpha,1} \\ k_{\alpha,2}^2 - k_{\alpha,0}k_{\alpha,1} = k_{\alpha,2}. \end{cases} \tag{2.24}$$

iv) Let $\alpha \in (0,1]$. Then $0 \in \rho(\Lambda^\alpha)$ and Λ^α has compact resolvent.

v) For each $\alpha \in (0,1]$ the spectrum of $-\Lambda^\alpha$ is such that the point spectrum consisting of eigenvalues

$$\left\{(\mu_n)^{\frac{\alpha}{3}} e^{i\pi} : n \in \mathbb{N}\right\} \cup \left\{(\mu_n)^{\frac{\alpha}{3}} e^{i\frac{\pi(3-2\alpha)}{3}} : n \in \mathbb{N}\right\} \cup \left\{(\mu_n)^{\frac{\alpha}{3}} e^{i\frac{\pi(3+2\alpha)}{3}} : n \in \mathbb{N}\right\} \tag{2.25}$$

where $\{\mu_n\}_{n\in\mathbb{N}}$ denotes the ordered sequence of eigenvalues of A including their multiplicity.

Proof: Part (i) immediately follows from the definition of Λ.

Part (ii) is a consequence of the fact that Λ is of positive type operator, see Lemma 1.5.7.

For part (iii) note that given $\lambda \in \mathbb{C}$ we have

$$\lambda I + \Lambda = \begin{bmatrix} \lambda I & -I & 0 \\ 0 & \lambda I & -I \\ A & 0 & \lambda I \end{bmatrix}$$

and

$$(\lambda I + \Lambda)^{-1} = \begin{bmatrix} \lambda^2(\lambda^3 I+A)^{-1} & \lambda(\lambda^3 I+A)^{-1} & (\lambda^3 I+A)^{-1} \\ -A(\lambda^3 I+A)^{-1} & \lambda^2(\lambda^3 I+A)^{-1} & \lambda(\lambda^3 I+A)^{-1} \\ -\lambda A(\lambda^3 I+A)^{-1} & -A(\lambda^3 I+A)^{-1} & \lambda^2(\lambda^3 I+A)^{-1} \end{bmatrix}, \text{ for all } \lambda \in \rho(-\Lambda).$$

Consequently,

$$\Lambda(\lambda I + \Lambda)^{-1} = \begin{bmatrix} A(\lambda^3 I+A)^{-1} & -\lambda^2(\lambda^3 I+A)^{-1} & -\lambda(\lambda^3 I+A)^{-1} \\ \lambda A(\lambda^3 I+A)^{-1} & A(\lambda^3 I+A)^{-1} & -\lambda^2(\lambda^3 I+A)^{-1} \\ \lambda^2 A(\lambda^3 I+A)^{-1} & \lambda A(\lambda^3 I+A)^{-1} & A(\lambda^3 I+A)^{-1} \end{bmatrix}, \text{ for all } \lambda \in \rho(-\Lambda).$$

Now using (2.19), applying in each entry of the above matrix the fractional formula for A

$$A^\alpha = \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty \lambda^{\alpha-1}A(\lambda I + A)^{-1}d\lambda,$$

and after the change of variable $\mu = \lambda^3$ we obtain

$$\frac{\sin(\alpha\pi)}{\pi}\int_0^\infty \lambda^{\alpha-1}\lambda^j A(\lambda^3 I + A)^{-1}d\lambda = \frac{(-1)^j}{3}\left(2\cos\tfrac{2(\alpha+j)\pi}{3} + 1\right)A^{\frac{\alpha+j}{3}} \tag{2.26}$$

where $j \in \{0,1,2\}$. Hence

$$\Lambda^\alpha = \begin{bmatrix} k_{\alpha,0}A^{\frac{\alpha}{3}} & -k_{\alpha,2}A^{\frac{\alpha-1}{3}} & k_{\alpha,1}A^{\frac{\alpha-2}{3}} \\ -k_{\alpha,1}A^{\frac{\alpha+1}{3}} & k_{\alpha,0}A^{\frac{\alpha}{3}} & -k_{\alpha,2}A^{\frac{\alpha-1}{3}} \\ k_{\alpha,2}A^{\frac{\alpha+2}{3}} & -k_{\alpha,1}A^{\frac{\alpha+1}{3}} & k_{\alpha,0}A^{\frac{\alpha}{3}} \end{bmatrix} \tag{2.27}$$

where

$$k_{\alpha,j} = \frac{1}{3}\left(2\cos\tfrac{2\pi(\alpha+j)}{3} + 1\right), \quad \text{for } j \in \{0,1,2\} \tag{2.28}$$

Part (iv) follows from the existence of bounded inverse operator $\Lambda^{-\alpha} : Y \to Y$

$$\Lambda^{-\alpha} = \begin{bmatrix} k_{\alpha,0}A^{-\frac{\alpha}{3}} & -k_{\alpha,1}A^{-\frac{\alpha+1}{3}} & k_{\alpha,2}A^{-\frac{\alpha+2}{3}} \\ -k_{\alpha,2}A^{-\frac{\alpha-1}{3}} & k_{\alpha,0}A^{-\frac{\alpha}{3}} & -k_{\alpha,1}A^{-\frac{\alpha+1}{3}} \\ k_{\alpha,1}A^{-\frac{\alpha-2}{3}} & -k_{\alpha,2}A^{-\frac{\alpha-1}{3}} & k_{\alpha,0}A^{-\frac{\alpha}{3}}, \end{bmatrix}$$

which takes bounded subsets of $Y = X^{\frac{2}{3}} \times X^{\frac{1}{3}} \times X$ into bounded subsets of $Y^\alpha := X^{\frac{\alpha+2}{3}} \times X^{\frac{\alpha+1}{3}} \times X^{\frac{\alpha}{3}}$, the latter space is compactly embedded in Y because the inclusions

$$X^\beta \subset X^\gamma, \beta > \gamma \geqslant 0$$

are compact provided that A has compact resolvent.

Concerning part (v) observe that $\lambda \in \mathbb{C}$ is an eigenvalue of $-\Lambda^\alpha$ if and only if there exists a nontrivial solution of

$$\begin{cases} -k_{\alpha,0}A^{\frac{\alpha}{3}}u + k_{\alpha,2}A^{\frac{\alpha-1}{3}}v - k_{\alpha,1}A^{\frac{\alpha-2}{3}}w = \lambda u \\ k_{\alpha,1}A^{\frac{\alpha+1}{3}}u - k_{\alpha,0}A^{\frac{\alpha}{3}}v + k_{\alpha,2}A^{\frac{\alpha-1}{3}}w = \lambda v \\ -k_{\alpha,2}A^{\frac{\alpha+2}{3}}u + k_{\alpha,1}A^{\frac{\alpha-2}{3}}v - k_{\alpha,0}A^{\frac{\alpha}{3}}w = \lambda w \end{cases}$$

which in turn holds, using (2.22) and (2.24), if and only if

$$\lambda^3 I + 3\lambda^2 k_{\alpha,0}A^{\frac{\alpha}{3}} + 3\lambda k_{\alpha,0}A^{\frac{2\alpha}{3}} + A^\alpha = \left(\lambda I + A^{\frac{\alpha}{3}}\right)\left(\lambda I - e^{i\frac{\pi(3-2\alpha)}{3}}A^{\frac{\alpha}{3}}\right)\left(\lambda I - e^{i\frac{\pi(3+2\alpha)}{3}}A^{\frac{\alpha}{3}}\right)$$

is not injective, however it happens if and only if

$$\lambda \in \left\{(\mu_n)^{\frac{\alpha}{3}} e^{i\pi} \,:\; n \in \mathbb{N}\right\} \cup \left\{(\mu_n)^{\frac{\alpha}{3}} e^{i\frac{\pi(3-2\alpha)}{3}} \,:\; n \in \mathbb{N}\right\} \cup \left\{(\mu_n)^{\frac{\alpha}{3}} e^{i\frac{\pi(3+2\alpha)}{3}} \,:\; n \in \mathbb{N}\right\}.$$

where $\{\mu_n\}_{n\in\mathbb{N}}$ denotes the ordered sequence of eigenvalues of A including their multiplicity. □

We include three figures that illustrate the position of the spectrum of $-\Lambda^\alpha$ in the complex plane.

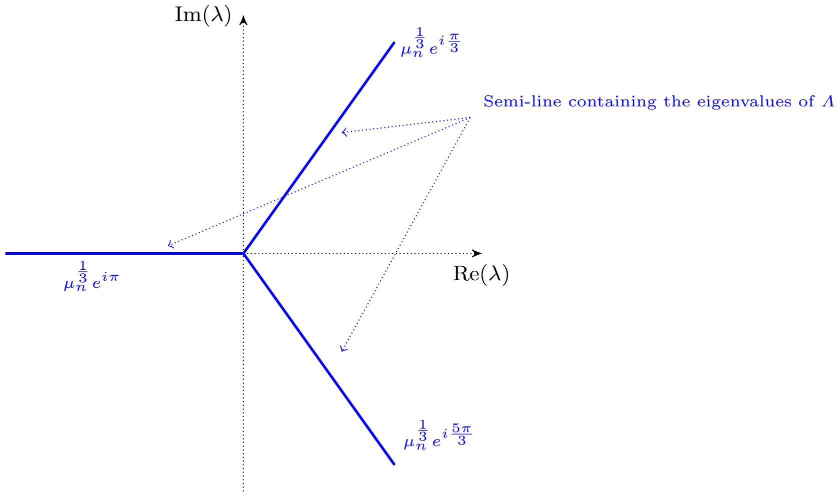

Figure 2.1: Location of the eigenvalues for $n = 3$ and $\alpha = 1$

Observe that for $3/4 < \alpha \leqslant 1$ the Figure 2.1 reflects, in particular, the ill-posedness of the Cauchy problem (2.12) in the sense that $-\Lambda^\alpha$ does not generate a strongly continuous semigroup on the state space Y. See Lemma 2.1.3.

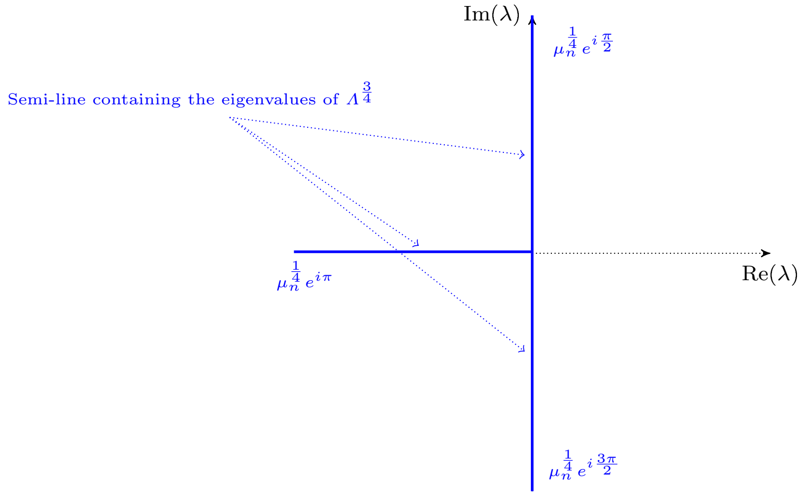

Figure 2.2: Location of the eigenvalues for $n = 3$ and $\alpha = \frac{3}{4}$

For $\alpha = 3/4$, the spectrum begins to reach a region where the generation of a strongly continuous semigroup is possible.

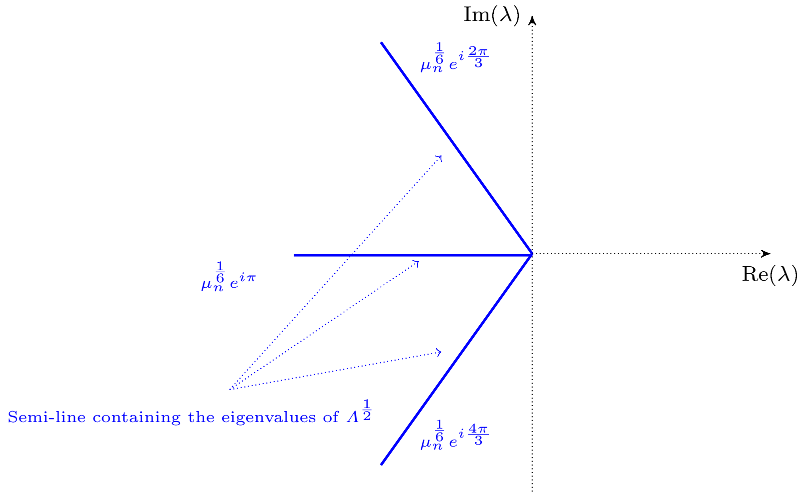

Figure 2.3: Location of the eigenvalues for $n = 3$ and $\alpha = \frac{1}{2}$

Finally, Figure 2.3 suggests the gain of the sectoriality property for Λ^α when $0 < \alpha < 3/4$. We are thus led to the following theorem, one of the main results in this paper.

Theorem 2.2.2. *The negative of the operator* Λ^α *in* (2.12) *is the generator of a strongly continuous semigroup on* Y *if and only if* $\alpha \in \left(0, \frac{3}{4}\right]$. *Moreover* $-\Lambda^\alpha$ *generates a strongly continuous analytic semigroup on* Y *for* $\alpha \in \left(0, \frac{3}{4}\right)$.

Proof:

Case I: $\alpha \in (\frac{3}{4}, 1)$.

By the computation of the spectrum of $-\Lambda^{\alpha}$ in item (v) of Theorem 2.2.1, it follows that

$$\sigma_p(-\Lambda^{\alpha}) \cap \{\lambda \in \mathbb{C} : Re\lambda > \omega\} \neq \varnothing$$

for any $\omega \geqslant 0$. Thus, the necessary condition for the generation of a strongly continuous semigroup is violated as in Lemma 2.1.3.

Case II: $\alpha = \frac{3}{4}$.

In this case we show that $-\Lambda^{\frac{3}{4}}$ is dissipative and there is $\lambda_0 > 0$ such that the range, $R(\lambda_0 I + \Lambda^{\frac{3}{4}})$, of $\lambda_0 I + \Lambda^{\frac{3}{4}}$ is Y. Then the result follows from Lumer-Phillips Theorem 1.2.5. Indeed for $\mathbf{u} = \left[\begin{smallmatrix} u \\ v \\ w \end{smallmatrix}\right] \in D(\Lambda^{\frac{3}{4}}) = D(A^{\frac{11}{12}}) \times D(A^{\frac{7}{12}}) \times D(A^{\frac{3}{12}})$ we have

$$\begin{aligned}
Re\left\langle \Lambda^{\frac{3}{4}}\mathbf{u}, \mathbf{u}\right\rangle_Y &= Re\Bigg(\left\langle \frac{1}{3}A^{\frac{3}{12}}u - \frac{1}{3}(1+\sqrt{3})A^{-\frac{1}{12}}v + \frac{1}{3}(1-\sqrt{3})A^{-\frac{5}{12}}w, u\right\rangle_{X^{\frac{2}{3}}} \\
&+ \left\langle -\frac{1}{3}(1-\sqrt{3})A^{\frac{7}{12}}u + \frac{1}{3}A^{\frac{3}{12}}v - \frac{1}{3}(1+\sqrt{3})A^{-\frac{1}{12}}w, v\right\rangle_{X^{\frac{1}{3}}} \\
&+ \left\langle \frac{1}{3}(1+\sqrt{3})A^{\frac{11}{12}}u - \frac{1}{3}(1-\sqrt{3})A^{\frac{7}{12}}v + \frac{1}{3}A^{\frac{3}{12}}w, w\right\rangle_X \Bigg) \\
&= \frac{1}{3}\left(\|A^{\frac{19}{24}}u\|_X^2 + \|A^{\frac{11}{24}}v\|_X^2 + \|A^{\frac{3}{24}}w\|_X^2\right) \\
&- \frac{2}{3}Re\left(\left\langle A^{\frac{11}{24}}v, A^{\frac{19}{24}}u\right\rangle_X - \left\langle A^{\frac{19}{24}}u, A^{\frac{3}{24}}w\right\rangle_X + \left\langle A^{\frac{11}{24}}v, A^{\frac{3}{24}}w\right\rangle_X\right) \\
&= \frac{1}{3}\|A^{\frac{19}{24}}u - A^{\frac{11}{24}}v + A^{\frac{3}{24}}w\|_X^2 \geqslant 0.
\end{aligned}$$

Which gives the dissipativity of $-\Lambda^{\frac{3}{4}}$. Now if we choose $\lambda_0 > 0$ such that $\|\Lambda^{-\frac{3}{4}}\|_{\mathcal{L}(Y)} < \lambda_0$, then $\lambda_0 \in \rho(-\Lambda^{-\frac{3}{4}})$ and

$$\lambda_0 \Lambda^{-\frac{3}{4}}(\lambda_0 I + \Lambda^{-\frac{3}{4}})^{-1}(\lambda_0^{-1} I + \Lambda^{\frac{3}{4}}) = I$$

This implies that $(\lambda_0^{-1}I + \Lambda^{\frac{3}{4}})^{-1} = \lambda_0\Lambda^{-\frac{3}{4}}(\lambda_0 I + \Lambda^{-\frac{3}{4}})^{-1}$, $\lambda_0^{-1} \in \rho(-\Lambda^{\frac{3}{4}})$ and consequently $R(\lambda_0^{-1} I + \Lambda^{\frac{3}{4}}) = Y$.

Case III: $\alpha \in (0, \frac{3}{4})$.

Finally, by Lemma 1.5.8, $\Lambda^{\frac{3}{4}}$ is also of positive type and we can study Λ^{α}, for $0 < \alpha < \frac{3}{4}$, considering the fractional powers $(\Lambda^{\frac{3}{4}})^{\beta} = \Lambda^{\frac{3\beta}{4}}$ for $0 < \beta < 1$. Since $-\Lambda^{\frac{3}{4}}$ is the infinitesimal generator of a strongly continuous semigroup of contractions, it follows from Lemma 1.5.12 that $-\Lambda^{\frac{3\beta}{4}}$ is the infinitesimal generator of an analytic semigroup for $0 < \beta < 1$ or, in other words, $-\Lambda^\alpha$ is the infinitesimal generator of an analytic semigroup for $0 < \alpha < \frac{3}{4}$.

Remark 2.2.3. *Note that Λ^α having compact resolvent (see item (iv) of Theorem 2.2.1) implies the semigroups $\{e^{-\Lambda^\alpha t} : 0 < \alpha \leqslant 3/4\}$ are compact.*

2.3 Fractional partial differential equation

Using the results of Section 2.2, we can consider a fractional formulation of the initial value problem in (2.1) and (2.2) as well as Bezerra, Carvalho, Cholewa, and Nascimento [5] proposed for damped wave equations, and Bezerra, Carvalho, Dłotko and Nascimento [6] proposed for Schrödinger equations. This fractional formulation and its well-posedness are established by our next theorem.

Theorem 2.3.1. *Let $0 < \alpha < 3/4$. Then for every $\left[\begin{smallmatrix} \varphi \\ \psi \\ \xi \end{smallmatrix}\right] \in X^{\frac{2}{3}} \times X^{\frac{2-\alpha}{3}} \times X^{\frac{2-2\alpha}{3}}$ the problem*

$$\partial_t^3 u + 3k_{\alpha,0} A^{\frac{\alpha}{3}} \partial_t^2 u + 3k_{\alpha,0} A^{\frac{2\alpha}{3}} \partial_t u + A^\alpha u = 0 \tag{2.29}$$

with the initial conditions given by

$$u(0) = \varphi,\ \partial_t u(0) = \psi,\ \partial_t^2 u(0) = \xi, \tag{2.30}$$

where $k_{\alpha,0}$ is given as in (2.21), has a unique solution in the class

$$C([0,\infty), X^{\frac{2}{3}}) \cap C^1((0,\infty), X^{\frac{2}{3}}) \cap C([0,\infty), D(A^{\frac{2+\alpha}{3}}))$$

Proof: This problem is equivalent to the first order system:

$$\frac{d}{dt}\left[\begin{smallmatrix} u \\ v \\ w \end{smallmatrix}\right] + \Lambda^\alpha \left[\begin{smallmatrix} u \\ v \\ w \end{smallmatrix}\right] = 0,\ t > 0,\ 0 < \alpha < 3/4, \tag{2.31}$$

subject to the initial conditions

$$\begin{cases} u(0) = \varphi \in X^{\frac{2}{3}} \\ v(0) = \dfrac{1}{k_{\alpha,1}^3 - k_{\alpha,0}^3}\left[k_{\alpha,1}(k_{\alpha,2} - k_{\alpha,0}) A^{\frac{1}{3}}\varphi - (k_{\alpha,2} + 3k_{\alpha,0}k_{\alpha,1}) A^{\frac{1-\alpha}{3}}\psi - k_{\alpha,1} A^{\frac{1-2\alpha}{3}}\xi\right] \in X^{\frac{1}{3}} \\ w(0) = \dfrac{1}{k_{\alpha,2}^3 - k_{\alpha,1}^3}\left[k_{\alpha,2}(k_{\alpha,0} - k_{\alpha,1}) A^{\frac{2}{3}}\varphi + (k_{\alpha,1} + 3k_{\alpha,0}k_{\alpha,2}) A^{\frac{2-\alpha}{3}}\psi - k_{\alpha,2} A^{\frac{2-2\alpha}{3}}\xi\right] \in X \end{cases}$$

Here, $\Lambda^\alpha : D(\Lambda^\alpha) \subset Y \to Y$ is the fractional power operator of Λ given in (2.20). In fact, we can see that (2.31) is equivalent to the system

$$\begin{cases} \partial_t u + k_{\alpha,0}A^{\frac{\alpha}{3}}u - k_{\alpha,2}A^{\frac{\alpha-1}{3}}v + k_{\alpha,1}A^{\frac{\alpha-2}{3}}w = 0, \\ \partial_t v - k_{\alpha,1}A^{\frac{\alpha+1}{3}}u + k_{\alpha,0}A^{\frac{\alpha}{3}}v - k_{\alpha,2}A^{\frac{\alpha-1}{3}}w = 0, \\ \partial_t w - k_{\alpha,2}A^{\frac{\alpha+2}{3}}u - k_{\alpha,1}A^{\frac{\alpha+1}{3}}v + k_{\alpha,0}A^{\frac{\alpha}{3}}w = 0, \end{cases} \tag{2.32}$$

and after some manipulations, we can obtain the partial differential equation (2.29) with initial conditions given in (2.30). The result follows from the fact that $-\Lambda^\alpha$ generates a strongly continuous analytic semigroup on Y for $\alpha \in (0, \frac{3}{4})$, see Theorem 2.2.2.

Remark 2.3.2. *As $-\Lambda^\alpha$ generates a strongly continuous analytic semigroup on Y for $\alpha \in (0, \frac{3}{4})$ we have*

$$\begin{bmatrix} u(t) \\ v(t) \\ w(t) \end{bmatrix} \in D(\Lambda^\alpha) = D(A^{\frac{\alpha+2}{3}}) \times D(A^{\frac{\alpha+1}{3}}) \times D(A^{\frac{\alpha}{3}}), \quad \textit{for } t > 0 \textit{ and } 0 < \alpha < \tfrac{3}{4},$$

and consequently by system (2.32)

$$u(t) \in D(A^{\frac{\alpha+2}{3}}),\ \partial_t u(t) \in D(A^{\frac{2}{3}}),\ \partial_t^2 u(t) \in D(A^{\frac{2-\alpha}{3}}), \quad \textit{for } t > 0 \textit{ and } 0 < \alpha < \tfrac{3}{4} \tag{2.33}$$

Remark 2.3.3 (Energy functional associated with perturbed problems)**.** *It is well known from the theory of sectorial operators that the negative of a positive sectorial operator generates an analytic semigroup of bounded linear operators that decays exponentially. Nevertheless, we would like to present an explicit formula for the energy functional associated with perturbed problems. If we multiply* (2.29) *by $A^{\frac{2-\alpha}{3}}\partial_t u$ in the sense of X, we get*

$$\frac{d}{dt}\Big[(\partial_t^2 u, A^{\frac{2-\alpha}{3}}\partial_t u)_X + \frac{3k_{\alpha,0}}{2}\|A^{\frac{1}{3}}\partial_t u\|_X^2 + \frac{1}{2}\|A^{\frac{\alpha+1}{3}}u\|_X^2\Big] = \|A^{\frac{2-\alpha}{6}}\partial_t^2 u\|^2 - 3k_{\alpha,0}\|A^{\frac{\alpha+2}{6}}\partial_t u\|_X^2.$$

Multiplying (2.29) *by $A^{\frac{2-2\alpha}{3}}\partial_t^2 u$ in the sense of X, we obtain*

$$\frac{d}{dt}\Big[(A^\alpha u, A^{\frac{2-2\alpha}{3}}\partial_t u)_X + \frac{3k_{\alpha,0}}{2}\|A^{\frac{1}{3}}\partial_t u\|_X^2 + \frac{1}{2}\|A^{\frac{1-\alpha}{3}}\partial_t^2 u\|_X^2\Big] = \|A^{\frac{\alpha+2}{6}}\partial_t u\|^2 - 3k_{\alpha,0}\|A^{\frac{2-\alpha}{6}}\partial_t^2 u\|_X^2.$$

Combining these equations we deduce that

$$\frac{d}{dt}\Big[(\partial_t^2 u, A^{\frac{2-\alpha}{3}}\partial_t u)_X + (A^\alpha u, A^{\frac{2-2\alpha}{3}}\partial_t u)_X + 3k_{\alpha,0}\|A^{\frac{1}{3}}\partial_t u\|_X^2 + \frac{1}{2}\|A^{\frac{1-\alpha}{3}}\partial_t^2 u\|_X^2 + \frac{1}{2}\|A^{\frac{\alpha+1}{3}}u\|_X^2\Big]$$
$$= (1 - 3k_{\alpha,0})\Big(\|A^{\frac{\alpha+2}{6}}\partial_t u\|_X^2 + \|A^{\frac{2-\alpha}{6}}\partial_t^2 u\|_X^2\Big).$$

If we define the function $\mathcal{L}_\alpha\left(\left[\begin{smallmatrix} u \\ \partial_t u \\ \partial_t^2 u\end{smallmatrix}\right]\right)$ *by*

$$\begin{aligned}\mathcal{L}_\alpha\left(\left[\begin{smallmatrix} u \\ \partial_t u \\ \partial_t^2 u\end{smallmatrix}\right]\right) &= (\partial_t^2 u, A^{\frac{2-\alpha}{3}}\partial_t u)_X + (A^\alpha u, A^{\frac{2-2\alpha}{3}}\partial_t u)_X + 3k_{\alpha,0}\|A^{\frac{1}{3}}\partial_t u\|_X^2 + \frac{1}{2}\|A^{\frac{1-\alpha}{3}}\partial_t^2 u\|_X^2 \\ &+ \frac{1}{2}\|A^{\frac{\alpha+1}{3}}u\|_X^2\end{aligned}$$

then the following differential equation is satisfied

$$\frac{d}{dt}\mathcal{L}_\alpha\left(\left[\begin{smallmatrix} u \\ \partial_t u \\ \partial_t^2 u\end{smallmatrix}\right]\right) = (1-3k_{\alpha,0})\Big(\|A^{\frac{\alpha+2}{6}}\partial_t u\|_X^2 + \|A^{\frac{2-\alpha}{6}}\partial_t^2 u\|_X^2\Big) < 0.$$

The last inequality follows from

$$1 < 2\cos\tfrac{2\pi\alpha}{3} + 1 = 3k_{\alpha,0}, \text{ for any } 0 < \alpha < \frac{3}{4}.$$

thanks to (2.21).

What was said above leads to the consideration of $\mathcal{L}_\alpha$ *as an energy functional associated with* (2.29) *defined on the domain*

$$D(\mathcal{L}_\alpha) = D(A^{\frac{\alpha+2}{3}}) \times D(A^{\frac{2}{3}}), \ \times D(A^{\frac{2-\alpha}{3}}), \quad 0 < \alpha < \tfrac{3}{4} \tag{2.34}$$

2.4 Parabolic approximations

As we see in Theorem 2.2.2, $-\Lambda^\alpha$ generates a strongly continuous analytic semigroup for $0 < \alpha < \frac{3}{4}$ whereas $-\Lambda^{\frac{3}{4}}$ generates only a strongly continuous semigroup ($\Lambda^{\frac{3}{4}}$ can not be a sectorial operator due to the position of its spectrum on imaginary axis, see Figure 2.2 and (2.25)). In this section we study, roughly speaking, the continuous dependence of the semigroup generated by $-\Lambda^\alpha$ on parameter $0 < \alpha \leqslant \frac{3}{4}$.

Theorem 2.4.1. *If* A *and* Λ *are as in* (2.4)-(2.5)*, respectively, then we have all the following.*

i) For every $\mathbf{u} \in Y$ *we have*

$$\Lambda^{-\alpha}\mathbf{u} \text{ converges to } \Lambda^{-\frac{3}{4}}\mathbf{u} \text{ in } Y \text{ as } \alpha \nearrow \frac{3}{4}.$$

ii) For every $\mathbf{u} \in D(\Lambda^{\frac{3}{4}})$ *we have*

$$\Lambda^{\alpha}\mathbf{u} \text{ converges to } \Lambda^{\frac{3}{4}}\mathbf{u} \text{ in } Y \text{ as } \alpha \nearrow \frac{3}{4}.$$

Proof: For part *i*) note that

$$\|\Lambda^{-\alpha}\mathbf{u} - \Lambda^{-\frac{3}{4}}\mathbf{u}\|_Y = \|\Lambda^{-\alpha}(I - \Lambda^{\alpha-\frac{3}{4}})\mathbf{u}\|_Y \leqslant \|\Lambda^{-\alpha}\|_{\mathcal{L}(Y)}\|(I - \Lambda^{\alpha-\frac{3}{4}})\mathbf{u}\|_Y$$

The result follows from the uniform boundedness of $\|\Lambda^{-\alpha}\|_{\mathcal{L}(Y)}$ for $0 \leqslant \alpha \leqslant 1$, and the convergence of $\Lambda^{-\beta}\mathbf{u}$ to $\mathbf{u}$ in Y for $0 \leqslant \beta \leqslant 1$ as $\beta \searrow 0$, Lemma 1.5.4 items (iii) and (v), respectively. Part *ii*) follows from the same argument above, since for each $\mathbf{u} \in D(\Lambda^{\frac{3}{4}})$

$$\|\Lambda^{\alpha}\mathbf{u} - \Lambda^{\frac{3}{4}}\mathbf{u}\|_Y = \|(\Lambda^{\alpha-\frac{3}{4}} - I)\Lambda^{\frac{3}{4}}\mathbf{u}\|_Y.$$

□

Proposition 2.4.2. *Let $\lambda \in \mathbb{C}$ with $Re\lambda \geqslant 0$ and $\alpha \in (0, \frac{3}{4})$. For each $\mathbf{u} \in Y$ we have*

$$(\lambda I + \Lambda^{\alpha})^{-1}\mathbf{u} \textit{ converges to } (\lambda I + \Lambda^{\frac{3}{4}})^{-1}\mathbf{u} \textit{ in } Y, \textit{ as } \alpha \textit{ tends to } 3/4.$$

Proof: Note first that for $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda \geqslant 0$ the following identity hold

$$(\lambda I + \Lambda^{\alpha})^{-1} - (\lambda I + \Lambda^{\frac{3}{4}})^{-1} = \Lambda^{\alpha}(\lambda I + \Lambda^{\alpha})^{-1}[\Lambda^{-\alpha} - \Lambda^{-\frac{3}{4}}]\Lambda^{\frac{3}{4}}(\lambda I + \Lambda^{\frac{3}{4}})^{-1}. \quad (2.35)$$

Now, observe that the equality

$$\Lambda^{\alpha}(\lambda I + \Lambda^{\alpha})^{-1} = I - \lambda(\lambda I + \Lambda^{\alpha})^{-1}$$

with the fact that Λ^{α} is of positive type K_{Λ} implies that

$$\|\Lambda^{\alpha}(\lambda I + \Lambda^{\alpha})^{-1}\|_{\mathcal{L}(Y)} \leqslant 1 + |\lambda|\|(\lambda I + \Lambda^{\alpha})^{-1}\|_{\mathcal{L}(Y)} \leqslant 1 + K_{\Lambda}, \text{ for all } \alpha \in (0,1). \quad (2.36)$$

The result follows from the equations (2.35), (2.36) and the pointwise convergence of $\Lambda^{-\alpha}$ to $\Lambda^{-\frac{3}{4}}$ as α tends to 3/4 given in item *i*) of Theorem (2.4.1).

Theorem 2.4.3. *Let $\{e^{-\Lambda^{\alpha}t} : t \geqslant 0\}$ be the semigroups generated by $-\Lambda^{\alpha}$, for $0 < \alpha \leqslant \frac{3}{4}$. Then for every $\mathbf{u} \in Y$ and $t \geqslant 0$, $e^{-\Lambda^{\alpha}t}\mathbf{u} \to e^{-\Lambda^{\frac{3}{4}}t}\mathbf{u}$ in Y as $\alpha \nearrow \frac{3}{4}$. Moreover, the convergence is uniform on bounded t-intervals.*

Proof: The result is consequence of Proposition (2.4.2) and Pazy [36, Theorem 3.4.2].

Remark 2.4.4. *If we consider $\alpha = \frac{3}{4}$ in equation (2.29) we obtain the initial value problem*

$$\partial_t^3 u + A^{\frac{1}{4}}\partial_t^2 u + A^{\frac{1}{2}}\partial_t u + A^{\frac{3}{4}}u = 0, \quad (2.37)$$

with initial conditions given by

$$u(0) = \varphi \in X^{\frac{2}{3}},\ \partial_t u(0) = \psi \in X^{\frac{5}{12}},\ \partial_t^2 u(0) = \xi \in X^{\frac{1}{6}}. \tag{2.38}$$

Then theorem 2.4.3 says that the problem (2.29) *for* $0 < \alpha < \frac{3}{4}$ *can be seen as parabolic approximation of the problem* (2.37) *and if we denote* $\mathbb{A} = A^{\frac{3}{4}}$ *then*

$$\partial_t^3 u + \mathbb{A}^{\frac{1}{3}} \partial_t^2 u + \mathbb{A}^{\frac{2}{3}} \partial_t u + \mathbb{A} u = 0,$$

and by this we understand that the fractional term $\mathbb{A}^{\frac{1}{3}} \partial_t^2 u + \mathbb{A}^{\frac{2}{3}} \partial_t u$ *provides a good framework to equation*

$$\partial_t^3 u + \mathbb{A} u = 0,$$

in the sense of the existence and uniqueness of global solution.

Moreover, note that by (2.34) *with* $\alpha = \frac{3}{4}$ $(3k_{\frac{3}{4},0} = 1)$ *an energy functional associated with* (2.37)-(2.38) *is given by*

$$\begin{aligned} \mathcal{L}_{\frac{3}{4}}\left(\begin{bmatrix} u \\ \partial_t u \\ \partial_t^2 u \end{bmatrix}\right) &= (\partial_t^2 u, A^{\frac{5}{12}} \partial_t u)_X + (A^{\frac{3}{4}} u, A^{\frac{1}{6}} \partial_t u)_X + \|A^{\frac{1}{3}} \partial_t u\|_X^2 + \frac{1}{2} \|A^{\frac{1}{12}} \partial_t^2 u\|_X^2 \\ &\quad + \frac{1}{2} \|A^{\frac{7}{12}} u\|_X^2 \end{aligned}$$

then the following differential equation is satisfied

$$\frac{d}{dt} \mathcal{L}_{\frac{3}{4}}\left(\begin{bmatrix} u \\ \partial_t u \\ \partial_t^2 u \end{bmatrix}\right) = (1 - 3k_{\frac{3}{4},0}) \left(\|A^{\frac{11}{24}} \partial_t u\|_X^2 + \|A^{\frac{5}{24}} \partial_t^2 u\|_X^2\right) = 0. \tag{2.39}$$

defined on the domain

$$D\left(\mathcal{L}_{\frac{3}{4}}\right) = X^{\frac{3}{4}} \times X^{\frac{11}{24}} \times X^{\frac{5}{24}}.$$

So (2.39) *implies that for* $\alpha = \frac{3}{4}$ *the energy is conserved.*

Remark 2.4.5. *One of our motivations to consider the class of problems in* (2.1)-(2.2) *are the initial-boundary value problems associated with Moore-Gibson-Thompson equations, these equations arise from modeling high-frequency ultrasound wave, for details see Moore and Gibson [35] and Thompson [41]. More precisely, let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain with smooth (at least* $\mathcal{C}^{2,\alpha}$*) boundary* $\partial\Omega$*, the Moore-Gibson-Thompson equations are evolution equations of third order in time of the type*

$$\tau \partial_t^3 u + \alpha \partial_t^2 u - \Delta u - \beta \Delta \partial_t u = 0,$$

where $\tau \geqslant 0$ *and* $\alpha, \beta \in \mathbb{R}$.

If we consider $\tau = 1$ *and* $\alpha = \beta = 0$*, then thanks to results of the previous sections we can set the "fractional Moore-Gibson-Thompson equations" associated with the third order linear evolution equation on the time*

$$\partial_t^3 u - \Delta u = 0, \tag{2.40}$$

subject to zero Dirichlet boundary condition and initial conditions

$$\begin{cases} u(x,t) = 0, & x \in \partial\Omega, t \geqslant 0, \\ u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1(x),\ \partial_t^2 u(x,0) = u_2(x), & x \in \partial\Omega. \end{cases} \tag{2.41}$$

If we consider $X = L^2(\Omega)$ *and the negative Laplacian operator*

$$A_D u = -\Delta u,$$

with domain

$$D(A_D) = H^2(\Omega) \cap H_0^1(\Omega),$$

then A_D *is a linear, closed, densely defined, self-adjoint and positive definite operator. There exists* $\zeta > 0$ *such that* $Re\sigma(A_D) > \zeta$*, that is,* $Re\lambda > \zeta$ *for all* $\lambda \in \sigma(A)$*, and therefore,* A_D *is a sectorial operator in the sense of Henry [25, Definition 1.3.1], with the eigenvalues* $\{\nu_n\}_{n\in\mathbb{N}}$*:*

$$0 < \nu_1 \leqslant \nu_2 \leqslant \cdots \leqslant \nu_n \leqslant \dots, \quad \nu_n \to +\infty \quad (as\ n \to +\infty).$$

This allows us to define the fractional power $A_D^{-\alpha}$ *of order* $\alpha \in (0,1)$ *according to Amann [3, Formula 4.6.9] and Henry [25, Theorem 1.4.2], as a closed linear operator on its domain* $D(A_D^{-\alpha})$ *with inverse* A_D^{α}*. Denote by* $X^\alpha = D(A_D^\alpha)$ *for* $\alpha \in [0,1)$*. The fractional power space* X^α *endowed with graphic norm*

$$\|\cdot\|_{X^\alpha} := \|A_D^\alpha \cdot\|_X$$

is a Banach space; namely,

$$X^1 = H^2(\Omega) \cap H_0^1(\Omega), \quad X^{\frac{1}{2}} = H_0^1(\Omega), \quad X^0 = X = L^2(\Omega).$$

With this notation, we have $X^{-\alpha} = (X^\alpha)'$ *for all* $\alpha > 0$*, see Amann [3] and Triebel [42] for the characterization of the negative scale. In particular,*

$$X' = (L^2(\Omega))' = L^2(\Omega) = X, \quad X^{-\frac{1}{2}} = (H_0^1(\Omega))' = H^{-1}(\Omega).$$

The scale of fractional power spaces $\{X^\alpha\}_{\alpha\in\mathbb{R}}$ *associated with* A_D *safisty*

$$X^\alpha \subset H^{2\alpha}(\Omega), \quad \alpha \geqslant 0,$$

where $H^{2\alpha}(\Omega)$ *are the potential Bessel spaces, see Cholewa and Dłotko [15]. From Sobolev embedding theorem, we obtain*

$$X^\alpha \subset L^r(\Omega),\ for\ r \leqslant \frac{2N}{N-4\alpha},\ 0 \leqslant \alpha < \frac{N}{4},$$

$$X = L^2(\Omega),$$

$$L^s(\Omega) \subset X^\alpha, \ \textit{for}\ s \geqslant \frac{2N}{N-4\alpha}, \ -\frac{N}{4} < \alpha \leqslant 0,$$

with continuous embeddings.

It is not difficult to show that A_D^α is the generator of a strongly continuous analytic semigroup on X, that we will denote by $\{e^{-tA_D^\alpha} : t \geqslant 0\}$, see Kreĭn [31] and Tanabe [40] for any $\alpha \in [0,1]$.

We recall that the fractional powers of the negative Laplacian operator can to be calculated through the spectral decomposition: since $X = L^2(\Omega)$ is a Hilbert space and $A_D = -\Delta$ with zero Dirichlet boundary condition in Ω is a self-adjoint operator and is the infinitesimal generator of a C_0-semigroup of contractions on X, it follows that there exists an orthonormal basis composed by eigenfunctions $\{\varphi_n, n \geqslant 1\}$ of A_D. Let ν_n be the eigenvalues of $A_D = -\Delta$, then $(\nu_n^\alpha, \varphi_n)$ are the eigenvalues and eigenfunctions of $A_D^\alpha = (-\Delta)^\alpha$, also with zero Dirichlet boundary condition, respectively.

It is well know that the fractional Laplacian $A_D^\alpha : D(A_D^\alpha) \subset X \to X$ is well defined in the space

$$D(A_D^\alpha) = X^\alpha = \Big\{ u = \sum_{n=1}^{\infty} a_n \varphi_n \in L^2(\Omega) : \sum_{n=1}^{\infty} a_n^2 \nu_n^{2\alpha} < \infty \Big\},$$

where

$$A_D^\alpha u = \sum_{n=1}^{\infty} \nu_n^\alpha a_n \varphi_n, \qquad u \in D(A_D^\alpha) = X^\alpha.$$

Finally, we apply all our results from previous sections to boundary value problem (2.40)-(2.41) *to obtain a track in α in which we can present a result of solubility and passage to the limit at $\alpha \nearrow \frac{3}{4}$ for fractional problems associated with* (2.40)-(2.41)*.*

2.5 Strongly damped third order equation

In Section 2.3 we considered, for $0 < \alpha < \frac{3}{4}$, the problem

$$\partial_t^3 u + 3k_{\alpha,0} A^{\frac{\alpha}{3}} \partial_t^2 u + 3k_{\alpha,0} A^{\frac{2\alpha}{3}} \partial_t u + A^\alpha u = 0 \tag{2.42}$$

and we obtained the well-posedness of this problem from the fact that $-\Lambda^\alpha$ generates a strongly continuous analytic semigroup on Y for $0 < \alpha < \frac{3}{4}$. If we denote $\mathbb{A} = A^\alpha$, then the equation (2.42) becomes

$$\partial_t^3 u + 3k_{\alpha,0} \mathbb{A}^{\frac{1}{3}} \partial_t^2 u + 3k_{\alpha,0} \mathbb{A}^{\frac{2}{3}} \partial_t u + \mathbb{A} u = 0 \tag{2.43}$$

what led us to consider the following problem:

$$\partial_t^3 u + 3A^{\frac{1}{3}} \partial_t^2 u + 3A^{\frac{2}{3}} \partial_t u + Au = 0 \tag{2.44}$$

with the initial conditions given by

$$u(0) = \varphi \in X^{\frac{2}{3}},\ \partial_t u(0) = \psi \in X^{\frac{1}{3}},\ \partial_t^2 u(0) = \xi \in X, \tag{2.45}$$

We understand that the fractional term $A^{\frac{1}{3}}\partial_t^2 u + A^{\frac{2}{3}}\partial_t u$ provides a good framework to the equation

$$\partial_t^3 u + Au = 0,$$

in the sense of the gain not only of the existence and uniqueness of global solution but also the sectorial property for the operator which represents this problem as we will see in Theorem 2.5.2.

We would like to establish an important analogy that brings us to the case of the wave equation. The second order differential equation in the space $X^{\frac{1}{2}} \times X$

$$\partial_t u + 2\eta A^\theta \partial_t u + Au = 0, \quad \textit{for } \eta > 0 \quad \textit{and} \quad \theta \in [1/2, 1] \tag{2.46}$$

has been extensively studied by many authors, see for instance Carvalho and Cholewa [10], Chen and Triggiani [12, 13]. In [12, 13] the authors prove the sectoriality of the operators associated with (2.46), $\Lambda_\theta : D(\Lambda_\theta) \subset X^{\frac{1}{2}} \times X \to X^{\frac{1}{2}} \times X$, for $\theta \in [1/2, 1]$, where

$$\Lambda_\theta = \begin{bmatrix} 0 & -I \\ A & 2\eta A^\theta \end{bmatrix}. \tag{2.47}$$

We believe that the study of the equation (2.44) can be seen as the first step to obtaining for the third order equation a series of analogous results to the second-order case as in [10, 12, 13]. To our best knowledge, there is no linear analysis, in the sense of geometric theory of linear parabolic equation as in Henry [25], of Moore-Gibson-Thompson type equations with fractional damping.

We consider the same phase space

$$Y = X^{\frac{2}{3}} \times X^{\frac{1}{3}} \times X$$

which is a Banach space equipped with the norm given by

$$\|\cdot\|_Y^2 = \|\cdot\|_{X^{\frac{2}{3}}}^2 + \|\cdot\|_{X^{\frac{1}{3}}}^2 + \|\cdot\|_X^2.$$

We can write the problem (2.44)-(2.45) as a Cauchy problem on Y, letting $v = \partial_t u$, $w = \partial_t^2 u$ and $\mathbf{u} = \left[\begin{smallmatrix} u \\ v \\ w \end{smallmatrix}\right]$ and the initial value problem

$$\begin{cases} \dfrac{d\mathbf{u}}{dt} + \Lambda \mathbf{u} = 0, \ t > 0 \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \tag{2.48}$$

where $\mathbf{u}_0 = (\varphi, \psi, \xi)$ and the unbounded linear operator $\Lambda : D(\Lambda) \subset Y \to Y$ is defined by

$$D(\Lambda) = D(A) \times D(A^{\frac{2}{3}}) \times D(A^{\frac{1}{3}}), \tag{2.49}$$

and

$$\Lambda \mathbf{u} = \left[\begin{smallmatrix} 0 & -I & 0 \\ 0 & 0 & -I \\ A & 3A^{\frac{2}{3}} & 3A^{\frac{1}{3}} \end{smallmatrix}\right] \left[\begin{smallmatrix} u \\ v \\ w \end{smallmatrix}\right] := \left[\begin{smallmatrix} -v \\ -w \\ Au + 3A^{\frac{2}{3}} v + 3A^{\frac{1}{3}} w \end{smallmatrix}\right], \ \forall \mathbf{u} = \left[\begin{smallmatrix} u \\ v \\ w \end{smallmatrix}\right] \in D(\Lambda). \tag{2.50}$$

In this section we study the resolution of the problem (2.44)-(2.45) and the spectral properties of the linear operator Λ. As the main result, we show that Λ is a sectorial operator. In this section, we denote

$$Y^1 = X^1 \times X^{\frac{2}{3}} \times X^{\frac{1}{3}}.$$

Proposition 2.5.1. *Let Λ be the unbounded linear operator defined in* (2.49)-(2.50). *Then the following assumptions hold.*

i) Λ is closed and densely defined;

ii) $0 \in \rho(\Lambda)$ and

$$\Lambda^{-1} = \left[\begin{smallmatrix} 3A^{-\frac{1}{3}} & 3A^{-\frac{2}{3}} & A^{-1} \\ -I & 0 & 0 \\ 0 & -I & 0 \end{smallmatrix}\right]$$

moreover, if A has compact resolvent, then Λ^{-1} is a compact operator on Y.

Proof: For (i) note that the inclusion $X^1 \times X^{\frac{2}{3}} \times X^{\frac{1}{3}} \subset X^{\frac{2}{3}} \times X^{\frac{1}{3}} \times X$ is dense (the inclusions $X^\alpha \subset X^\beta$ are dense for $\alpha \geqslant \beta \geqslant 0$). Secondly, we show that the operator Λ is closed. Indeed, if $\mathbf{u}_n = \left[\begin{smallmatrix} u_n \\ v_n \\ w_n \end{smallmatrix}\right] \in D(\Lambda)$ with $\mathbf{u}_n \to \mathbf{u} = \left[\begin{smallmatrix} u \\ v \\ w \end{smallmatrix}\right]$ in Y as $n \to \infty$, and $\Lambda \mathbf{u}_n \to \varphi = \left[\begin{smallmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{smallmatrix}\right]$ in Y as $n \to \infty$, then

$$v_n \to -\varphi_1 \text{ in } X^{\frac{2}{3}} \text{ as } n \to \infty$$

and consequently, $v = -\varphi_1 \in X^{\frac{2}{3}}$. As well as, we have

$$w_n \to -\varphi_2 \text{ in } X^{\frac{1}{3}} \text{ as } n \to \infty$$

and consequently, $w = -\varphi_2 \in X^{\frac{1}{3}}$.

Next, we have

$$A(u_n + 3A^{-\frac{1}{3}}v_n + 3A^{-\frac{2}{3}}w_n) = Au_n + 3A^{\frac{2}{3}}v_n + 3A^{\frac{1}{3}}w_n \to \varphi_3 \text{ in } X \text{ as } n \to \infty,$$

and

$$X^1 \ni u_n + 3A^{-\frac{1}{3}}v_n + 3A^{-\frac{2}{3}}w_n \to u + 3A^{-\frac{1}{3}}v + 3A^{-\frac{2}{3}}w \in X \text{ in } X \text{ as } n \to \infty,$$

and consequently, since A is a closed operator, we conclude $u + 3A^{-\frac{1}{3}}v + 3A^{-\frac{2}{3}}w \in X^1$ and $A(u + 3A^{-\frac{1}{3}}v + 3A^{-\frac{2}{3}}w) = \varphi_3$. Therefore, $\begin{bmatrix} u \\ v \\ w \end{bmatrix} \in D(\Lambda)$ and $\Lambda \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix}$.

Item (ii) follows immediately from the definition of Λ^{-1} which takes bounded subsets of Y into bounded subsets of Y^1, the latter space being compactly embedded in Y. □

Theorem 2.5.2. *The unbounded linear operator Λ defined in* (2.49)-(2.50) *is a sectorial operator.*

Proof: In this proof, K will denote a positive constant, not necessarily the same one. First, we note that the operator $A^{\frac{1}{3}} : D(A^{\frac{1}{3}}) \subset X \to X$ is a positive sectorial operator; that is, there exist $\phi \in (0, \frac{\pi}{2})$ and $M > 0$ such that the resolvent set $\rho(A^{\frac{1}{3}})$ contains the sector

$$\Sigma_\phi = \{\lambda \in \mathbb{C}; \phi \leqslant |\arg(\lambda)| \leqslant \pi\}$$

and for any $\lambda \in \Sigma_\phi$

$$\|(\lambda I - A^{\frac{1}{3}})^{-1}\|_{\mathcal{L}(X)} \leqslant \frac{M}{|\lambda|}.$$

It follows that, for each $n = 1, 2, 3, \ldots$, $(\lambda I - A^{\frac{1}{3}})^{-n}$ is a bounded linear operator on X and

$$\|(\lambda I - A^{\frac{1}{3}})^{-n}\|_{\mathcal{L}(X)} \leqslant \frac{M^n}{|\lambda|^n}, \tag{2.51}$$

for any $\lambda \in \Sigma_\phi$. Moreover, for each $\lambda \in \Sigma_\phi$, we have the following identities

$$A^{\frac{1}{3}}(\lambda I - A^{\frac{1}{3}})^{-3} = -(\lambda I - A^{\frac{1}{3}})^{-2} + \lambda(\lambda I - A^{\frac{1}{3}})^{-3},$$

and

$$A^{\frac{2}{3}}(\lambda I - A^{\frac{1}{3}})^{-3} = (\lambda I - A^{\frac{1}{3}})^{-1} - 2\lambda(\lambda I - A^{\frac{1}{3}})^{-2} + \lambda^2(\lambda I - A^{\frac{1}{3}})^{-3}.$$

Thus $A^{\frac{1}{3}}(\lambda I - A^{\frac{1}{3}})^{-3}$ and $A^{\frac{2}{3}}(\lambda I - A^{\frac{1}{3}})^{-3}$ are bounded linear operators on X, and

$$\|A^{\frac{1}{3}}(\lambda I - A^{\frac{1}{3}})^{-3}\|_{\mathcal{L}(X)} \leqslant \frac{M^2}{|\lambda|^2} + \frac{\lambda M^3}{|\lambda|^3} \leqslant \frac{K}{|\lambda|^2}, \tag{2.52}$$

and

$$\|A^{\frac{2}{3}}(\lambda I - A^{\frac{1}{3}})^{-3}\|_{\mathcal{L}(X)} \leqslant \frac{M}{|\lambda|} + \frac{2\lambda M^2}{|\lambda|^2} + \frac{\lambda^2 M^3}{|\lambda|^3} \leqslant \frac{K}{|\lambda|}. \tag{2.53}$$

Now, we observe that $\Sigma_\phi \subset \rho(\Lambda)$ and for $\lambda \in \rho(\Lambda)$, $(\lambda I - \Lambda)^{-1}$ is given by

$$(\lambda I - \Lambda)^{-1} = \begin{bmatrix} (\lambda^2 - 3\lambda A^{\frac{1}{3}} + 3A^{\frac{2}{3}})(\lambda I - A^{\frac{1}{3}})^{-3} & (-\lambda I + 3A^{\frac{1}{3}})(\lambda I - A^{\frac{1}{3}})^{-3} & (\lambda I - A^{\frac{1}{3}})^{-3} \\ -A(\lambda I - A^{\frac{1}{3}})^{-3} & (\lambda^2 - 3\lambda A^{\frac{1}{3}})(\lambda I - A^{\frac{1}{3}})^{-3} & -\lambda(\lambda I - A^{\frac{1}{3}})^{-3} \\ \lambda A(\lambda I - A^{\frac{1}{3}})^{-3} & (3\lambda A^{\frac{2}{3}} - A)(\lambda I - A^{\frac{1}{3}})^{-3} & \lambda^2(\lambda I - A^{\frac{1}{3}})^{-3} \end{bmatrix}.$$

Finally, we will prove that Λ is a sectorial operator using the same sector from the sectoriality of the operator $A^{\frac{1}{3}}$. If $\lambda \in \Sigma_\phi$ and $\mathbf{u} = \left[\begin{smallmatrix} u \\ v \\ w \end{smallmatrix}\right] \in Y$ with $\|\mathbf{u}\|_Y \leqslant 1$, then writing

$$(\lambda I + \Lambda)^{-1}\mathbf{u} = \left[\begin{smallmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{smallmatrix}\right]$$

where

$$\begin{aligned} \varphi_1 &= (\lambda^2 - 3\lambda A^{\frac{1}{3}} + 3A^{\frac{2}{3}})(\lambda I - A^{\frac{1}{3}})^{-3}u + (-\lambda I + 3A^{\frac{1}{3}})(\lambda I - A^{\frac{1}{3}})^{-3}v + (\lambda I - A^{\frac{1}{3}})^{-3}w, \\ \varphi_2 &= -A(\lambda I - A^{\frac{1}{3}})^{-3}u + (\lambda^2 - 3\lambda A^{\frac{1}{3}})(\lambda I - A^{\frac{1}{3}})^{-3}v - \lambda(\lambda I - A^{\frac{1}{3}})^{-3}w, \\ \varphi_3 &= \lambda A(\lambda I - A^{\frac{1}{3}})^{-3}u + (3\lambda A^{\frac{2}{3}} - A)(\lambda I - A^{\frac{1}{3}})^{-3}v + \lambda^2(\lambda I - A^{\frac{1}{3}})^{-3}w. \end{aligned}$$

In order to conclude the proof it is sufficient to show that

$$\left\|\left[\begin{smallmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{smallmatrix}\right]\right\|_Y \leqslant \frac{K}{|\lambda|}.$$

for some $K > 0$. Note that

$$\begin{aligned} \|\varphi_1\|_{X^{\frac{2}{3}}} \leqslant{}& \|\lambda^2(\lambda I - A^{\frac{1}{3}})^{-3}u\|_{X^{\frac{2}{3}}} + \|3\lambda A^{\frac{1}{3}}(\lambda I - A^{\frac{1}{3}})^{-3}u\|_{X^{\frac{2}{3}}} + \|3A^{\frac{2}{3}}(\lambda I - A^{\frac{1}{3}})^{-3}u\|_{X^{\frac{2}{3}}} \\ &+ \|\lambda A^{\frac{1}{3}}(\lambda I - A^{\frac{1}{3}})^{-3}v\|_{X^{\frac{1}{3}}} + \|3A^{\frac{2}{3}}(\lambda I - A^{\frac{1}{3}})^{-3}v\|_{X^{\frac{1}{3}}} + \|A^{\frac{2}{3}}(\lambda I - A^{\frac{1}{3}})^{-3}w\|_X, \end{aligned}$$

this with (2.51), (2.52) and (2.53) implies that

$$\|\varphi_1\|_{X^{\frac{2}{3}}} \leqslant \frac{K}{|\lambda|}.$$

Note that

$$\begin{aligned} \|\varphi_2\|_{X^{\frac{1}{3}}} \leqslant{}& \|A^{\frac{2}{3}}(\lambda I - A^{\frac{1}{3}})^{-3}u\|_{X^{\frac{2}{3}}} + \|\lambda^2(\lambda I - A^{\frac{1}{3}})^{-3}v\|_{X^{\frac{1}{3}}} \\ &+ \|3\lambda A^{\frac{1}{3}}(\lambda I - A^{\frac{1}{3}})^{-3}v\|_{X^{\frac{1}{3}}} + \|\lambda A^{\frac{1}{3}}(\lambda I - A^{\frac{1}{3}})^{-3}w\|_X, \end{aligned}$$

and

$$\begin{aligned} \|\varphi_3\|_X \leqslant{}& \|\lambda A^{\frac{1}{3}}(\lambda I - A^{\frac{1}{3}})^{-3}u\|_{X^{\frac{2}{3}}} + \|3\lambda A^{\frac{1}{3}}(\lambda I - A^{\frac{1}{3}})^{-3}v\|_{X^{\frac{1}{3}}} \\ &+ \|A^{\frac{2}{3}}(\lambda I - A^{\frac{1}{3}})^{-3}v\|_{X^{\frac{1}{3}}} + \|\lambda^2(\lambda I - A^{\frac{1}{3}})^{-3}w\|_X, \end{aligned}$$

From (2.51), (2.52) and (2.53) again we have

$$\|\varphi_2\|_{X^{\frac{1}{3}}} \leqslant \frac{K}{|\lambda|}.$$

and

$$\|\varphi_3\|_X \leqslant \frac{K}{|\lambda|}.$$

Therefore,

$$\|\varphi_1\|_{X^{\frac{2}{3}}} + \|\varphi_2\|_{X^{\frac{1}{3}}} + \|\varphi_3\|_X \leqslant \frac{K}{|\lambda|}.$$

□

As a consequence of this last result, we have

Corollary 2.5.3. *The unbounded linear operator Λ defined in* (2.49)-(2.50) *is such that $-\Lambda$ is the infinitesimal generator of an analytic semigroup on Y.*

Proof: Follows immediately from Theorem 1.3.3.

Chapter 3

Future research directions

The aim of this chapter is to stablish partial results and indicate future research directions based on the study of the previous chapter.

3.1 Nth order differential equation on a time scale

It is natural to try to generalize the problem (2.1)-(2.2) considering for $n \geqslant 2$ the following abstract evolution equation of n-th order in time, abusing notation

$$\partial_t^n u + Au = 0 \tag{3.1}$$

with initial conditions given by

$$u(0) = u_0 \in X^{\frac{n-1}{n}},\ \partial_t u(0) = u_1 \in X^{\frac{n-2}{n}},\ \partial_t^2 u(0) = u_2 \in X^{\frac{n-3}{n}}, \ldots,\ \partial_t^{n-1} u(0) = u_{n-1} \in X,$$

that is,

$$\partial_t^i u(0) = u_i \in X^{\frac{n-(i+1)}{n}}, \quad i \in \{0, 1, \ldots, n-1\}, \tag{3.2}$$

where $A : D(A) \subset X \to X$ is the linear operator as in Chapter 2.

The idea is to consider analogous results obtained to the case of third order. When considering this problem, we are faced with several difficulties produced by the order of the equation. We managed to get some partial results: We prove that the operator associated with the equation (3.1) is of positive type and we compute its fractional powers. For $n \geqslant 3$, we show that the problem given by (3.1) and (3.2) is ill-posed. Finally, we conjecture one result of generation of semigroup for the negative of the fractional power of the operator associated with (3.1) and (3.2).

We will rewrite (3.1)-(3.2) as a first order abstract system. Consider the phase space

$$Y = X^{\frac{n-1}{n}} \times X^{\frac{n-2}{n}} \times X^{\frac{n-3}{n}} \times \cdots \times X$$

which is a Banach space equipped with the norm given by

$$\| \cdot \|_Y^2 = \| \cdot \|^2_{X^{\frac{n-1}{n}}} + \| \cdot \|^2_{X^{\frac{n-2}{n}}} + \| \cdot \|^2_{X^{\frac{n-3}{n}}} + \cdots + \| \cdot \|^2_X.$$

We can write the problem (3.1)-(3.2) as a Cauchy problem on Y, letting $v_1 = u$, $v_2 = \partial_t u$, $v_3 = \partial_t^2 u$, $\ldots$, $v_n = \partial_t^{n-1} u$ and

$$\mathbf{u} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix}$$

and the initial value problem

$$\begin{cases} \dfrac{d\mathbf{u}}{dt} + \Lambda \mathbf{u} = 0, \ t > 0 \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \tag{3.3}$$

where the unbounded linear operator $\Lambda : D(\Lambda) \subset Y \to Y$ is defined by

$$D(\Lambda) = D(A) \times D(A^{\frac{n-1}{n}}) \times D(A^{\frac{n-2}{n}}) \times \cdots \times D(A^{\frac{1}{n}}), \tag{3.4}$$

equipped with the norm given by

$$\| \cdot \|^2 = \| \cdot \|^2_{X^1} + \| \cdot \|^2_{X^{\frac{n-1}{n}}} + \| \cdot \|^2_{X^{\frac{n-2}{n}}} + \cdots + \| \cdot \|^2_{X^{\frac{1}{n}}}.$$

and

$$\Lambda \mathbf{u} = \begin{bmatrix} 0 & -I & 0 & \cdots & 0 & 0 \\ 0 & 0 & -I & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -I \\ A & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n-1} \\ v_n \end{bmatrix} := \begin{bmatrix} -v_2 \\ -v_3 \\ -v_4 \\ \vdots \\ -v_n \\ Av_1 \end{bmatrix}, \ \forall \mathbf{u} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} \in D(\Lambda). \tag{3.5}$$

From now on, we denote

$$Y^1 = D(\Lambda) = X^1 \times X^{\frac{n-1}{n}} \times X^{\frac{n-2}{n}} \times \cdots \times X^{\frac{1}{n}},$$

equipped with the norm

$$\| \cdot \|^2_{Y^1} = \| \cdot \|^2_{X^1} + \| \cdot \|^2_{X^{\frac{n-1}{n}}} + \| \cdot \|^2_{X^{\frac{n-2}{n}}} + \cdots + \| \cdot \|^2_{X^{\frac{1}{n}}}.$$

In the following we prove that though the negative of Λ is not an infinitesimal generator of a strongly continuous semigroup for $n \geqslant 3$, it is possible to define its fractional powers Λ^α, for $0 < \alpha < 1$, and one can ask what is the maximum subinterval of $(0,1)$ where α is taken such that the negative of Λ^α is a generator. We conjecture that $-\Lambda^\alpha$ generates a strongly continuous semigroup on Y if and only if $0 < \alpha \leqslant \frac{n}{2(n-1)}$ and it generates strongly continuous analytic semigroup on the open interval $0 < \alpha < \frac{n}{2(n-1)}$. Note that this agrees with the case of the wave operator, $n = 2$, and the case $n = 3$ studied in the last chapter.

Currently, this section is contained in a preprint by myself and Flank D. M. Bezerra and this manuscript will be submitted for publication soon.

3.1.1 Spectral properties of the operator Λ for nth order equation

In this section, we study spectral properties of the operator Λ.

Lemma 3.1.1. *The resolvent set of $-\Lambda$ is given by*

$$\rho(-\Lambda) = \{\lambda \in \mathbb{C} : \lambda^n \in \rho(-A)\}. \tag{3.6}$$

Proof: Suppose that $\lambda \in \mathbb{C}$ is such that $\lambda^n \in \rho(-A)$. We claim that $\lambda \in \rho(-\Lambda)$. Indeed, since $-\Lambda$ is a closed operator, we only need to show that

$$\lambda I_Y + \Lambda : D(\Lambda) \subset Y \to Y$$

is bijective. For injectivity consider $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \in D(\Lambda)$ and $(\lambda I_Y + \Lambda)\mathbf{u} = 0$, then

$$\begin{cases} \lambda u_i - u_{i+1} = 0, \quad \text{for } 1 \leqslant i \leqslant n-1 \\ A u_1 + \lambda u_n = 0. \end{cases} \tag{3.7}$$

From (3.7) we have

$$(\lambda^n I_X + A)u_1 = 0. \tag{3.8}$$

Since $\lambda^n \in \rho(-A)$, we conclude that $u_1 = 0$ and consequently $\mathbf{u} = 0$. For surjectivity given $\varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_n \end{bmatrix} \in Y$ we take $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ with

$$u_i = (\lambda^n I_X + A)^{-1} \left(\sum_{j=1}^{n} \lambda^{n+i-j-1} \varphi_j \right) - \sum_{j=1}^{i-1} \lambda^{i-j-1} \varphi_j \tag{3.9}$$

for $1 \leqslant i \leqslant n$. Note that, for $1 \leqslant i \leqslant n$, u_i is well defined since $\lambda^n \in \rho(-A)$. Moreover, $u_i \in D(A^{\frac{n-i+1}{n}})$. Then we have $\mathbf{u} \in D(\Lambda)$ and

$$(\lambda I_Y + \Lambda)\mathbf{u} = \varphi.$$

Now suppose that $\lambda \in \rho(-\Lambda)$. If $u_1 \in D(A)$ is such that $(\lambda^n I_X + A)u_1 = 0$, taking $\mathbf{u} = \begin{bmatrix} u \\ \lambda u_1 \\ \vdots \\ \lambda^{n-1} u_1 \end{bmatrix} \in D(\Lambda)$ we have

$$(\lambda I_Y + \Lambda)\mathbf{u} = 0. \tag{3.10}$$

Since $\lambda \in \rho(-\Lambda)$, it follows that $\mathbf{u} = 0$ and consequently $u_1 = 0$, which proves the injectivity of $\lambda^n I_X + A$. Given $f \in X$, consider $\varphi = \begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix} \in Y$. By the surjectivity of $\lambda I_Y + \Lambda$ there exists $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \in D(\Lambda)$ such that

$$(\lambda I_Y + \Lambda)\mathbf{u} = \varphi \tag{3.11}$$

which gives

$$(\lambda^n I_X + A)u_1 = f$$

and the proof is complete.

□

Lemma 3.1.2. *If $n \geqslant 3$, then the unbounded linear operator $-\Lambda$ with $\Lambda : D(\Lambda) \subset Y \to Y$ defined in* (3.4)-(3.5) *is not the infinitesimal generator of a strongly continuous semigroup on Y.*

Proof: If $-\Lambda$ generates a strongly continuous semigroup $\{e^{-\Lambda t} : t \geqslant 0\}$ on Y, it follows from Theorem 1.1.5 that there exist constants $\omega \geqslant 0$ and $M \geqslant 1$ such that

$$\|e^{-\Lambda t}\|_{\mathcal{L}(Y)} \leqslant M e^{\omega t} \text{ for all } 0 \leqslant t < \infty. \tag{3.12}$$

Moreover, from Theorem 1.2.2 we have

$$\{\lambda \in \mathbb{C} : Re\lambda > \omega\} \subset \rho(-\Lambda) \tag{3.13}$$

where $\rho(-\Lambda)$ denotes the resolvent set of the operator $-\Lambda$.

Let $\mathbf{u}$ be a nontrivial element of $D(\Lambda)$. We shall consider the eigenvalue problem for the operator $-\Lambda$

$$-\Lambda\mathbf{u} = \lambda\mathbf{u}.$$

A straightforward calculation implies

$$\sigma_p(-\Lambda) = \{\lambda \in \mathbb{C} : \lambda^n \in \sigma_p(-A)\}.$$

Where $\sigma_p(-\Lambda)$ and $\sigma_p(-A)$ denote the point spectrum set of $-\Lambda$ and $-A$, respectively. Since $\sigma_p(-A) = \{-\mu_j : j \in \mathbb{N}\}$ with $\mu_j \in \sigma_p(A)$ for each $j \in \mathbb{N}$ and $\mu_j \to \infty$ as $j \to \infty$, we conclude that

$$\sigma_p(-\Lambda) \cap \{\lambda \in \mathbb{C} : Re\lambda > \omega\} \neq \varnothing$$

This contradicts the equation (3.13) and therefore $-\Lambda$ can not be the infinitesimal generator of a strongly continuous semigroup on Y. □

Remark 3.1.3. *We note that if $n \geqslant 3$, then $-\Lambda$ is not a dissipative operator on Y. Indeed, if u is a non-trivial element in X^1 and*

$$\mathbf{u} = \begin{bmatrix} u \\ 0 \\ 0 \\ \vdots \\ 0 \\ -u \end{bmatrix}$$

then

$$\langle -\Lambda\mathbf{u}, \mathbf{u}\rangle_Y = \left\langle \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ -u \\ -Au \end{bmatrix}, \begin{bmatrix} u \\ 0 \\ 0 \\ \vdots \\ 0 \\ -u \end{bmatrix} \right\rangle_Y = \langle Au, u\rangle_X = \|u\|^2_{X^{\frac{1}{2}}} > 0.$$

Explicitly, this means that $-\Lambda$ is not an infinitesimal generator of a strongly continuous semigroup of contractions on Y. Nevertheless, the statement in Lemma 3.1.2 is more precise because it says that $-\Lambda$ cannot be the infinitesimal generator of a strongly continuous semigroup of any type on Y.

Lemma 3.1.4. *The unbounded linear operator Λ defined in* (3.4)-(3.5) *is of positive type $K \geqslant 1$.*

Proof: Firstly, we show that the operator Λ is closed. Indeed, if $\mathbf{u}_j = \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ \vdots \\ u_{n,j} \end{bmatrix} \in D(\Lambda)$ with $\mathbf{u}_j \to \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix}$ in Y as $j \to \infty$, and $\Lambda\mathbf{u}_j \to \varphi$ in Y as $j \to \infty$, where $\varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \vdots \\ \varphi_n \end{bmatrix}$, then

$$\begin{cases} u_{i,j} \to -\varphi_{i-1} \text{ in } X^{\frac{n-i+1}{n}} \hookrightarrow X^{\frac{n-i}{n}} \text{ as } j \to \infty, \text{ for } 2 \leqslant i \leqslant n \\ Au_{1,j} \to \varphi_n \text{ in } X \text{ as } j \to \infty \end{cases}$$

and consequently, $u_i = -\varphi_{i-1} \in X^{\frac{n-i+1}{n}}$, for $2 \leqslant i \leqslant n$. Finally, using the fact that A is a closed operator, we have $u_1 \in D(A)$ and $Au_1 = \varphi_n$; that is, $\mathbf{u} \in D(\Lambda)$ and $\Lambda\mathbf{u} = \varphi$.

Secondly, $D(\varLambda) = D(A) \times D(A^{\frac{n-1}{n}}) \times D(A^{\frac{n-2}{n}}) \times \cdots \times D(A^{\frac{1}{n}})$ is dense in $Y = X^{\frac{n-1}{n}} \times X^{\frac{n-2}{n}} \times X^{\frac{n-3}{n}} \times \cdots \times X$ since the inclusions $X^\alpha \subset X^\beta$ are dense for $\alpha > \beta \geqslant 0$.

Finally, since the operator $\varLambda$ is closed, $\lambda \in \rho(-\varLambda)$ if and only if the operator $\lambda I + \varLambda$ is bijective. From Lemma 3.1.1 it follows easily that $[0, \infty) \subset \rho(-\varLambda)$. For $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix}$, $\varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \vdots \\ \varphi_n \end{bmatrix}$ in Y and any $\lambda \geqslant 0$ we have

$$(\lambda I + \varLambda)^{-1}\mathbf{u} = \varphi.$$

If and only if

$$\varphi_i = (\lambda^n I_X + A)^{-1} \left(\sum_{j=1}^{n} \lambda^{n+i-j-1} u_j \right) - \sum_{j=1}^{i-1} \lambda^{i-j-1} u_j \tag{3.14}$$

for $1 \leqslant i \leqslant n$. Note that, for $1 \leqslant i \leqslant n$, φ_i is well defined since $\lambda^n \in \rho(-A)$. Moreover, $\varphi_i \in D(A^{\frac{n-i+1}{n}})$. In order to verify the equation (1.8) for $\varLambda$, it is sufficient to show that for $\|\mathbf{u}\|_Y \leqslant 1$ there exists a constant $K_\varLambda \geqslant 1$ such that

$$\|\varphi_1\|_{X^{\frac{n-1}{n}}} + \|\varphi_2\|_{X^{\frac{n-2}{n}}} + \|\varphi_3\|_{X^{\frac{n-3}{n}}} + \cdots + \|\varphi_n\|_X \leqslant \frac{K_\varLambda}{1+\lambda} \tag{3.15}$$

Note that, for $1 \leqslant i \leqslant n$, we have

$$\|\varphi_i\|_{X^{\frac{n-i}{n}}} \leqslant \sum_{j=1}^{i} \lambda^{i-j} \|A^{\frac{j}{n}} (\lambda^n I + A)^{-1} u_j\|_{X^{\frac{n-j}{n}}} + \sum_{j=i}^{n} \lambda^{n-j+i-1} \|A^{\frac{j-i}{n}} (\lambda^n I + A)^{-1} u_j\|_{X^{\frac{n-j}{n}}} \tag{3.16}$$

Applying Lemma 2.1.5 we obtain a constant $K \geqslant 1$ such that

$$\sum_{i=1}^{n} \|\varphi_i\|_{X^{\frac{n-i}{n}}} \leqslant \sum_{i=1}^{n} \frac{\lambda^{n-i} K}{(1+\lambda^n)^{\frac{n-i+1}{n}}} \leqslant \frac{K_\varLambda}{1+\lambda},$$

whereas $K_\varLambda \geqslant 1$ is sufficiently large. □

3.1.2 Spectral properties of the fractional power operators $\varLambda^\alpha$ for nth order equation

In this subsection we study spectral properties of the fractional power operators $\varLambda^\alpha$ for $\alpha \in (0, 1)$.

Theorem 3.1.5. *If A and $\varLambda$ are as in* (3.4)-(3.5), *respectively, then we have all the following.*

i) $0 \in \rho(\Lambda)$ *and*

$$\Lambda^{-1} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & A^{-1} \\ -I & 0 & 0 & \cdots & 0 & 0 \\ 0 & -I & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & -I & 0 \end{bmatrix}.$$

ii) *Fractional powers* Λ^α *can be defined for* $\alpha \in (0,1)$ *by*

$$\Lambda^\alpha = \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty \lambda^{\alpha-1}\Lambda(\lambda I + \Lambda)^{-1} d\lambda, \tag{3.17}$$

iii) *Given any* $\alpha \in (0,1)$ *we have that* $\Lambda^\alpha : D(\Lambda^\alpha) \subset Y \to Y$*, where*

$$D(\Lambda^\alpha) = D(A^{\frac{\alpha+n-1}{n}}) \times D(A^{\frac{\alpha+n-2}{n}}) \times D(A^{\frac{\alpha+n-3}{n}}) \times \cdots \times D(A^{\frac{\alpha}{n}}),$$

is given by

$$\Lambda^\alpha = \left[\frac{(-1)^{i+j}}{n} U_{n-1}\left(cos\left(\frac{(\alpha+i-j)\pi}{n} \right) \right) A^{\frac{\alpha+i-j}{n}} \right]_{ij} \tag{3.18}$$

where $U_n(x)$ *is the* n*-th Chebyshev polynomial of second kind, see Appendix A.*

Proof: Part (i) immediately follows from the definition of Λ. Part (ii) is a consequence of the fact that Λ is of positive type operator, see Lemma 1.5.7. Concerning part (iii) note that

$$\Lambda(\lambda I + \Lambda)^{-1} = \begin{cases} [A(\lambda^n I + A)^{-1}\lambda^{i-j} I]_{ij}, \text{if } i \geqslant j \\ [-A(\lambda^n I + A)^{-1}\lambda^{n+i-j} A^{-1}]_{ij}, \text{if } i < j \end{cases}. \tag{3.19}$$

Now we apply in each entry the fractional formula for A

$$A^\alpha = \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty \lambda^{\alpha-1} A(\lambda I + A)^{-1} d\lambda,$$

and after the change of variable $\mu = \lambda^n$ and using the property (A.1), we obtain for $i \geqslant j$

$$A(\lambda^n I + A)^{-1}\lambda^{i-j} I = \frac{(-1)^{i+j}}{n} U_{n-1}\left(cos\left(\frac{(\alpha+i-j)\pi}{n} \right) \right) A^{\frac{\alpha+i-j}{n}} \tag{3.20}$$

and for $i < j$, we have

$$-A(\lambda^n I + A)^{-1}\lambda^{n+i-j} A^{-1} = \frac{(-1)^{n+i-j-1}}{n} U_{n-1}\left(cos\left(\frac{(\alpha+n+i-j)\pi}{n} \right) \right) A^{\frac{\alpha+n+i-j}{n}-1}. \tag{3.21}$$

In the last equation we use the property (A.3), which leads to (3.18). □

Remark 3.1.6. *Analyzing what occurred in the case $n = 3$ in Theorem 3.1.5 (v), one may conjecture that for each $\alpha \in (0,1]$ the spectrum of $-\Lambda^\alpha$ is such that the point spectrum consists of eigenvalues*

$$\bigcup_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \left(\left\{ (\mu_j)^{\frac{\alpha}{n}} e^{i\frac{\pi(n-(n-2k-1)\alpha)}{n}} \ : \ j \in \mathbb{N} \right\} \cup \left\{ (\mu_j)^{\frac{\alpha}{n}} e^{i\frac{\pi(n+(n-2k-1)\alpha)}{n}} \ : \ j \in \mathbb{N} \right\} \right) \tag{3.22}$$

where $\{\mu_j\}_{j\in\mathbb{N}}$ denotes the ordered sequence of eigenvalues of A including their multiplicity and $\lfloor x \rfloor := \max\{z \in \mathbb{Z} \mid z \leqslant x\}$.

We include four figures that illustrate the position of the spectrum of $-\Lambda^\alpha$ in the complex plane.

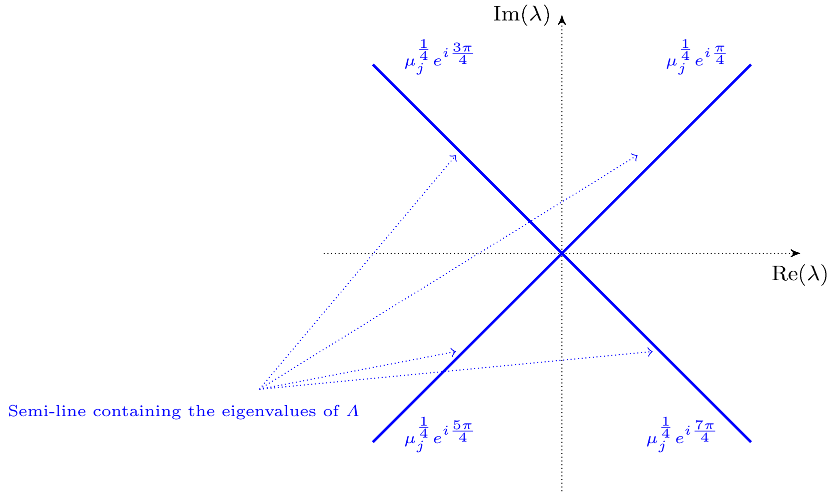

Figure 3.1: Location of the eigenvalues for $n = 4$ and $\alpha = 1$

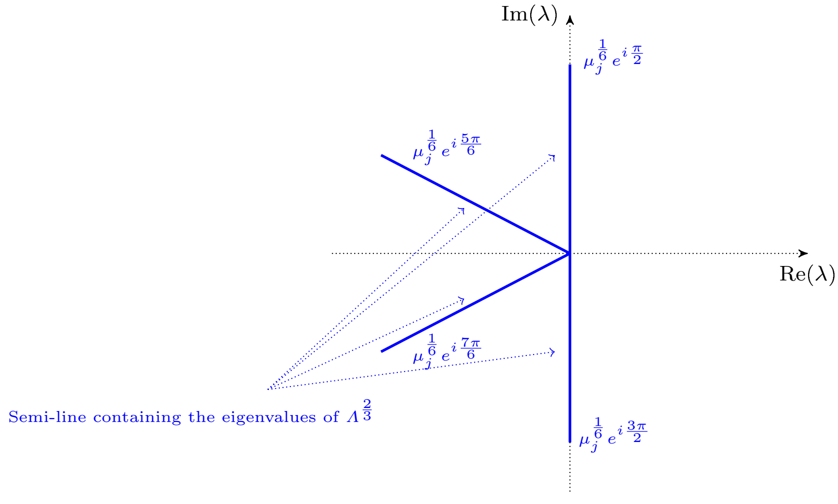

Figure 3.2: Location of the eigenvalues for $n = 4$ and $\alpha = \frac{2}{3}$

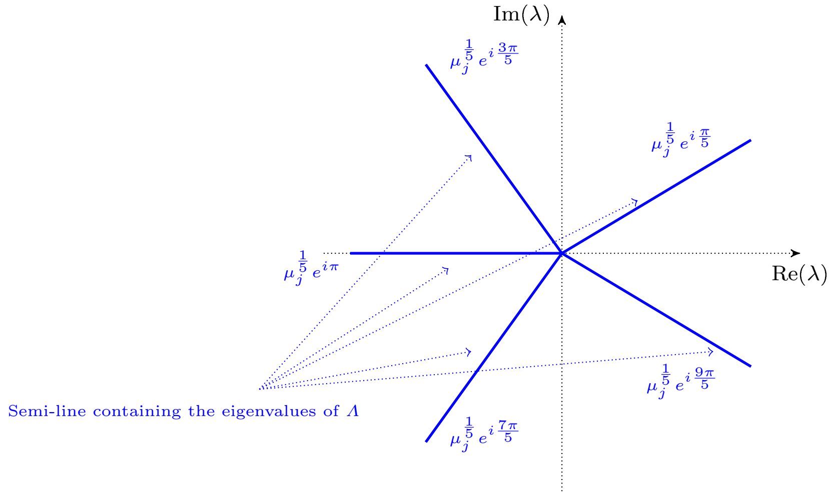

Figure 3.3: Location of the eigenvalues for $n = 5$ and $\alpha = 1$

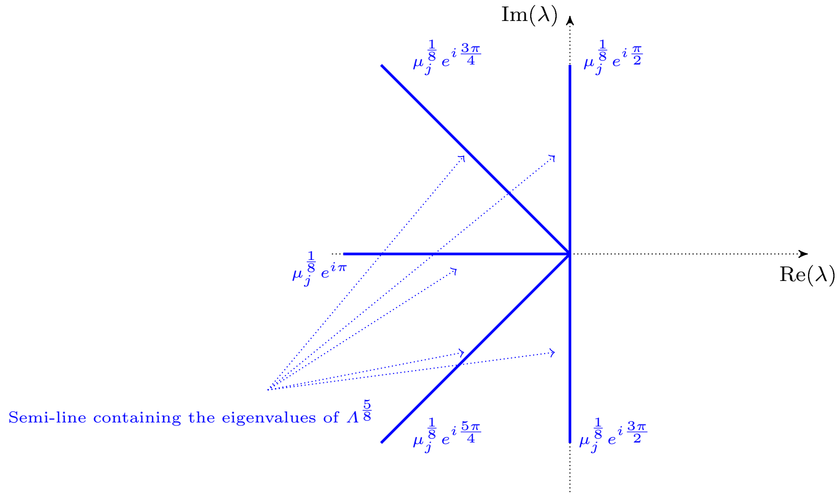

Figure 3.4: Location of the eigenvalues for $n = 5$ and $\alpha = \frac{5}{8}$

Observe that for $\alpha = 1$ Figure 3.1 and Figure 3.3 reflect, in particular, the ill-posedness of the Cauchy problem (3.1) in the sense that $-\Lambda^{\alpha}$ does not generate a strongly continuous semigroup on the state space Y. See Lemma 3.1.2. For $n = 4; \alpha = 2/3$ and $n = 5; \alpha = 5/8$, Figure 3.2 and Figure 3.4 show that the spectrum of $-\Lambda^{\alpha}$ begins to reach a region where the generation of a strongly continuous semigroup is possible. We are thus led to the following conjecture

Conjectura 3.1.7. *The negative of the operator Λ^{α} in* (2.12) *is the generator of a strongly continuous semigroup on Y if and only if $\alpha \in \left(0, \frac{n}{2(n-1)}\right]$. Moreover $-\Lambda^{\alpha}$ generates a strongly continuous analytic semigroup on Y for $\alpha \in \left(0, \frac{n}{2(n-1)}\right)$.*

For $\alpha \in \left(\frac{n}{2(n-1)}, 1\right)$, the computation of the spectrum of $-\Lambda^\alpha$ in remark 3.1.6 gives

$$\sigma_p(-\Lambda^\alpha) \cap \{\lambda \in \mathbb{C} : Re\lambda > \omega\} \neq \varnothing$$

for any $\omega \geqslant 0$. Thus, the necessary condition for the generation of a strongly continuous semigroup is violated as in Lemma 3.1.2.

when $\alpha = \frac{n}{2(n-1)}$ our idea is to show that $-\Lambda^{\frac{n}{2(n-1)}}$ is dissipative and there is $\lambda_0 > 0$ such that the range, $R(\lambda_0 I + \Lambda^{\frac{n}{2(n-1)}})$, of $\lambda_0 I + \Lambda^{\frac{n}{2(n-1)}}$ is Y. Then the result would follow from Lumer-Phillips Theorem 1.2.5.

For $\alpha \in \left(0, \frac{n}{2(n-1)}\right)$ we can obtain the sectorial property of $-\Lambda^\alpha$ by the same argument as in the case $n = 3$. Indeed, $\Lambda^{\frac{n}{2(n-1)}}$ is also of positive type $K \geqslant 1$ and we can consider their fractional powers. Therefore we can study Λ^α, for $0 < \alpha < \frac{n}{2(n-1)}$, considering the fractional powers $(\Lambda^{\frac{n}{2(n-1)}})^\beta = \Lambda^{\frac{\beta n}{2(n-1)}}$ for $0 < \beta < 1$. Since $-\Lambda^{\frac{n}{2(n-1)}}$ is the infinitesimal generator of a strongly continuous contraction semigroup, it follows from Lemma 1.5.12 that $-\Lambda^{\frac{\beta n}{2(n-1)}}$ is the infinitesimal generator of an analytic semigroup for $0 < \beta < 1$ or, in other words, $-\Lambda^\alpha$ is the infinitesimal generator of an analytic semigroup for $0 < \alpha < \frac{n}{2(n-1)}$.

3.2 Fractional powers of operators approach to Euler-Rodrigues formula for three-dimensional rotation

In this section, we review the Euler-Rodrigues formula for three-dimensional rotation with fractional powers of operators approach. The Euler-Rodrigues formula describes the rotation of a vector in three dimensions, it was first discovered by Euler [20] and later rediscovered independently by Rodrigues [39] and it is related a number of interesting problems in computer graphics, dynamics, kinematics, mathematics, and robotics, see Cheng and Gupta [14] and references therein.

Reviews of the Euler-Rodrigues formula in different mathematical forms can be found in the literature, see e.g., Dai [18], Kahvecí, Yayli and Gök [29] and Mebius [34]. Here, we explored the geometric aspect of the classical Balakrishnan formula in [4] to obtain a new algorithm for the generation of three-dimensional rotation matrix.

The idea of exploring the geometry of the spectral behavior of the fractional powers of operators has been explored in recent years in the infinite-dimensional dynamical systems, see e.g., Bezerra *et al.* [7, 5, 6] and Cholewa and Carvalho [10].

Currently, this section is contained in a paper entitled 'Fractional powers of operators approach to Euler-Rodrigues formula for three-dimensional rotation' by myself and Flank D. M. Bezerra and this manuscript is submitted for publication.

Here, the matricial representations of linear operators on $\mathbb{R}^3$ are considered using the standard basis of $\mathbb{R}^3$, and $\hat{\mathbf{n}} = (n_1, n_2, n_3)$ denotes a vector in $\mathbb{R}^3$ with $n_1^2+n_2^2+n_3^2 = 1$.

Lemma 3.2.1. *The matrix which represents the rotation by an angle* $\pi/2$ *about the axis* $\hat{\mathbf{n}} = (n_1, n_2, n_3)$ *is given by*

$$A(\hat{\mathbf{n}}, \pi/2) = \begin{bmatrix} n_1^2 & n_1n_2 - n_3 & n_1n_3 + n_2 \\ n_1n_2 + n_3 & n_2^2 & n_2n_3 - n_1 \\ n_1n_3 - n_2 & n_2n_3 + n_1 & n_3^2 \end{bmatrix}. \tag{3.23}$$

Proof: Choose two vectors, $\hat{\mathbf{l}}$ and $\hat{\mathbf{m}}$, such that $\{\hat{\mathbf{l}}, \hat{\mathbf{m}}, \hat{\mathbf{n}}\}$ is a right-handed orthonormal basis. Let $u = a\hat{\mathbf{l}} + b\hat{\mathbf{m}} + c\hat{\mathbf{n}}$, with $a, b, c \in \mathbb{R}$, be any vector to be rotated by an angle $\pi/2$ counterclockwise about the axis $\hat{\mathbf{n}}$. The resulting vector u' is the vector u with its component in the $\hat{\mathbf{l}}, \hat{\mathbf{m}}$ plane rotated by $\pi/2$

$$\begin{aligned} u' &= -b\hat{\mathbf{l}} + a\hat{\mathbf{m}} + c\hat{\mathbf{n}} \\ &= \hat{\mathbf{n}} \times u + \langle u, \hat{\mathbf{n}} \rangle \hat{\mathbf{n}}. \end{aligned}$$

Consider the standard basis $\{\hat{\mathbf{e_1}}, \hat{\mathbf{e_2}}, \hat{\mathbf{e_3}}\}$ of $\mathbb{R}^3$. If u is written as

$$u = u_1\hat{\mathbf{e_1}} + u_2\hat{\mathbf{e_2}} + u_3\hat{\mathbf{e_3}},$$

then

$$\begin{aligned} u' &= \hat{\mathbf{n}} \times u + \langle u, \hat{\mathbf{n}} \rangle \hat{\mathbf{n}} \\ &= (n_2u_3 - n_3u_2 + u_1n_1^2 + u_2n_1n_2 + u_3n_1n_3)\hat{\mathbf{e_1}} + \\ &\quad (n_3u_1 - n_1u_3 + u_1n_1n_2 + u_2n_2^2 + u_3n_2n_3)\hat{\mathbf{e_2}} + \\ &\quad (n_1u_2 - n_2u_1 + u_1n_1n_3 + u_2n_2n_3 + u_3n_3^2)\hat{\mathbf{e_3}}. \end{aligned}$$

Therefore, the matrix representation of this rotation is

$$A(\hat{\mathbf{n}}, \pi/2) = \begin{bmatrix} n_1^2 & n_1n_2 - n_3 & n_1n_3 + n_2 \\ n_1n_2 + n_3 & n_2^2 & n_2n_3 - n_1 \\ n_1n_3 - n_2 & n_2n_3 + n_1 & n_3^2 \end{bmatrix}. \square$$

Remark 3.2.2. *Thanks to the characterization* (3.23) *of the matrix which represents the rotation by an angle $\pi/2$ about the axis $\hat{\mathbf{n}} = (n_1, n_2, n_3)$ we can obtain a matrix characterization of the linear semigroup generated by $A(\hat{\mathbf{n}}, \pi/2)$, namely the uniformly continuous semigroup of bounded linear operators generated by $A(\hat{\mathbf{n}}, \pi/2)$, denoted by $T(t)$, has the following explicit representation*

$$T(t) = e^{tA(\hat{\mathbf{n}},\pi/2)} = \sum_{n=0}^{\infty} \frac{(tA(\hat{\mathbf{n}}, \pi/2))^n}{n!} =$$

$$= \begin{bmatrix} n_1^2(e^t - \cos t) + \cos t & n_1n_2(e^t - \cos t) - n_3 \sin t & n_1n_3(e^t - \cos t) + n_2 \sin t \\ n_1n_2(e^t - \cos t) + n_3 \sin t & n_2^2(e^t - \cos t) + \cos t & n_2n_3(e^t - \cos t) - n_1 \sin t \\ n_1n_3(e^t - \cos t) - n_2 \sin t & n_2n_3(e^t - \cos t) + n_1 \sin t & n_3^2(e^t - \cos t) + \cos t \end{bmatrix}$$

for any $t \geqslant 0$.

Remark 3.2.3. *An explicit formula for the matrix elements of a general 3×3 rotation matrix can be find in Rodrigues [39]; namely, if $R(\hat{\mathbf{n}}, \theta)$ denotes the a rotation by an angle θ about an axis $\hat{\mathbf{n}} = (n_1, n_2, n_3)$ $(n_1^2 + n_2^2 + n_3^2 = 1)$, whose elements are denoted by $R_{ij}(\hat{\mathbf{n}}, \theta)$, then we have the Rodrigues formula*

$$R_{ij}(\hat{\mathbf{n}}, \theta) = \cos(\theta)\delta_{ij} + (1 - \cos(\theta))n_in_j - \sin(\theta)\epsilon_{ijk}n_k, \tag{3.24}$$

where δ_{ij} denotes the Kronecker delta, i.e.,

$$\delta_{ij} = \begin{cases} 1, & \textit{if } i = j, \\ 0, & \textit{if } i \neq j, \end{cases}$$

and ϵ_{ijk} denotes the Levi-Civita tensor, i.e.,

$$\epsilon_{ijk} = \begin{cases} 1, & \textit{if } (i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\}, \\ -1, & \textit{if } (i,j,k) \in \{(3,2,1),(1,3,2),(2,1,3)\}, \\ 0, & \textit{if } i = j, \ \textit{or } j = k, \ \textit{or } k = i, \end{cases}$$

which is called the angle-and-axis parameterization of the three-dimensional rotation matrix.

We wish to derive all the rotations by any angle $\theta \in \mathbb{R}$ through the rotation by $\pi/2$ and its fractional powers. In order to get this result we first explicit, in the following theorem, the fractional power, for $0 \leqslant \alpha \leqslant 1$, of the rotation $A(\hat{\mathbf{n}}, \pi/2)$ in Lemma 3.2.1. It is one of the main results of this section.

Theorem 3.2.4. *For $0 \leqslant \alpha \leqslant 1$, the fractional power of the rotation $A(\hat{\mathbf{n}}, \pi/2)$ in Lemma 3.2.1 is given by*

$$A^{\alpha}(\hat{\mathbf{n}}, \pi/2) =$$
$$\begin{bmatrix} n_1^2(1-\cos\frac{\alpha\pi}{2}) + \cos\frac{\alpha\pi}{2} & n_1 n_2(1-\cos\frac{\alpha\pi}{2}) - n_3\sin\frac{\alpha\pi}{2} & n_1 n_3(1-\cos\frac{\alpha\pi}{2})) + n_2\sin\frac{\alpha\pi}{2} \\ n_1 n_2(1-\cos\frac{\alpha\pi}{2}) + n_3\sin\frac{\alpha\pi}{2} & n_2^2(1-cos\frac{\alpha\pi}{2}) + cos\frac{\alpha\pi}{2} & n_2 n_3(1-cos\frac{\alpha\pi}{2}) - n_1\sin\frac{\alpha\pi}{2} \\ n_1 n_3(1-cos\frac{\alpha\pi}{2}) - n_2\sin\frac{\alpha\pi}{2} & n_2 n_3(1-\cos\frac{\alpha\pi}{2}) + n_1\sin\frac{\alpha\pi}{2} & n_3^2(1-cos\frac{\alpha\pi}{2}) + \cos\frac{\alpha\pi}{2} \end{bmatrix}.$$

Proof: The proof consists in the explicit calculation of the fractional power of the operator $A(\hat{\mathbf{n}}, \pi/2)$ through the formula (1.16) for $0 < \alpha < 1$.

$$A(\hat{\mathbf{n}}, \pi/2)^{\alpha} = \frac{\sin\alpha\pi}{\pi}\int_0^{\infty} \lambda^{\alpha-1} A(\hat{\mathbf{n}}, \pi/2)(\lambda I + A(\hat{\mathbf{n}}, \pi/2))^{-1} d\lambda,\ 0 < \alpha < 1. \qquad (3.25)$$

Simple computations give

$$(\lambda I + A(\hat{\mathbf{n}}, \pi/2))^{-1} =$$
$$= \frac{1}{(\lambda+1)(\lambda^2+1)} \begin{bmatrix} a^2(1-\lambda) + \lambda(1+\lambda) & ab(1-\lambda) + c(1+\lambda) & ac(1-\lambda) - b(1+\lambda) \\ ab(1-\lambda) - c(1+\lambda) & b^2(1-\lambda) + \lambda(1+\lambda) & bc(1-\lambda) + a(1+\lambda) \\ ac(1-\lambda) + b(1+\lambda) & bc(1-\lambda) - a(1+\lambda) & c^2(1-\lambda) + \lambda(1+\lambda) \end{bmatrix}$$

and

$$A(\hat{\mathbf{n}}, \pi/2)(\lambda I + A(\hat{\mathbf{n}}, \pi/2))^{-1} =$$
$$\frac{1}{(\lambda+1)(\lambda^2+1)} \begin{bmatrix} a^2\lambda(\lambda-1) + 1 + \lambda & ab\lambda(\lambda-1) - c\lambda(1+\lambda) & ac\lambda(\lambda-1) + b\lambda(1+\lambda) \\ ab\lambda(\lambda-1) + c\lambda(1+\lambda) & b^2\lambda(\lambda-1) + 1 + \lambda & bc\lambda(\lambda-1) - a\lambda(1+\lambda) \\ ac\lambda(\lambda-1) - b\lambda(1+\lambda) & bc\lambda(\lambda-1) + a\lambda(1+\lambda) & c^2\lambda(\lambda-1) + 1 + \lambda \end{bmatrix}.$$

Since

$$\begin{aligned} \frac{\lambda(\lambda-1)}{(\lambda+1)(\lambda^2+1)} &= \frac{1}{\lambda+1} - \frac{1}{\lambda^2+1} \\ \frac{\lambda+1}{(\lambda+1)(\lambda^2+1)} &= \frac{1}{\lambda^2+1} \\ \frac{\lambda(\lambda+1)}{(\lambda+1)(\lambda^2+1)} &= \frac{\lambda}{\lambda^2+1} \end{aligned}$$

computing the formula in the right-hand side of the equation (3.25) for each entry of the matrix $A(\hat{\mathbf{n}},\pi/2)(\lambda I + A(\hat{\mathbf{n}},\pi/2))^{-1}$ and applying (1.16) for real numbers we obtain

$$A^{\alpha}(\hat{\mathbf{n}},\pi/2) =$$
$$\begin{bmatrix} n_1^2(1-\cos\frac{\alpha\pi}{2})+\cos\frac{\alpha\pi}{2} & n_1n_2(1-\cos\frac{\alpha\pi}{2})-n_3\sin\frac{\alpha\pi}{2} & n_1n_3(1-\cos\frac{\alpha\pi}{2}))+n_2\sin\frac{\alpha\pi}{2} \\ n_1n_2(1-\cos\frac{\alpha\pi}{2})+n_3\sin\frac{\alpha\pi}{2} & n_2^2(1-cos\frac{\alpha\pi}{2})+cos\frac{\alpha\pi}{2} & n_2n_3(1-cos\frac{\alpha\pi}{2})-n_1\sin\frac{\alpha\pi}{2} \\ n_1n_3(1-cos\frac{\alpha\pi}{2})-n_2\sin\frac{\alpha\pi}{2} & n_2n_3(1-\cos\frac{\alpha\pi}{2})+n_1\sin\frac{\alpha\pi}{2} & n_3^2(1-cos\frac{\alpha\pi}{2})+\cos\frac{\alpha\pi}{2} \end{bmatrix}.$$

Finally, the cases $\alpha = 0$ and $\alpha = 1$ are immediate, and the proof is complete. ▫

Corollary 3.2.5. *The fractional power $A^{\alpha}(\hat{\mathbf{n}},\pi/2)$ coincides with the matrix $R(\hat{\mathbf{n}},\frac{\alpha\pi}{2}) = [R_{ij}(\hat{\mathbf{n}},\frac{\alpha\pi}{2})]$, where $R_{ij}(\hat{\mathbf{n}},\frac{\alpha\pi}{2})$ is given by* (3.24)*, for $0 \leqslant \alpha \leqslant 1$.*

We are now in a position to give our definition for the rotation matrix by an angle θ through fractional powers of the rotation by $\pi/2$.

Definition 3.2.6. *The rotation by $\theta \in \mathbb{R}$, denoted by $A(\hat{\mathbf{n}},\theta)$, is defined to be*

$$A(\hat{\mathbf{n}},\theta) := A^{\frac{2\theta}{\pi}}(\hat{\mathbf{n}},\pi/2). \tag{3.26}$$

Note that $A(\hat{\mathbf{n}},\pi/2)$ being of positive type K implies that the fractional power $A^{\alpha}(\hat{\mathbf{n}},\pi/2)$ is well defined by (1.11) and (1.14) for $\alpha \in \mathbb{R}$.

Theorem 3.2.4 states that the definition in (3.26) agrees with the classical one given by *Rodrigues formula* in (3.24) for $0 \leqslant \theta \leqslant \pi/2$. The following theorem extends this result for $\theta \in \mathbb{R}$.

Theorem 3.2.7. *Let $A(\hat{\mathbf{n}},\theta)$ be the rotation defined in* (3.26)*. Then*

$$A(\hat{\mathbf{n}},\theta) = R(\hat{\mathbf{n}},\theta) \tag{3.27}$$

for any $\theta \in \mathbb{R}$.

Proof: Firstly for $\theta \geqslant 0$, it is sufficient to show that (3.27) is satisfied for

$$\frac{(n-1)\pi}{2} \leqslant \theta \leqslant \frac{n\pi}{2},$$

for $n \in \mathbb{N}$. We proceed by induction. The case $n = 1$ follows from Teorema 3.2.4. Assuming (3.27) to hold for n, we will prove it for $n+1$. Consider

$$\frac{n\pi}{2} \leqslant \theta \leqslant \frac{(n+1)\pi}{2}$$

we have

$$\frac{(n-1)\pi}{2} \leqslant \theta - \frac{\pi}{2} \leqslant \frac{n\pi}{2}.$$

Note that

$$A(\hat{\mathbf{n}},\theta) = A^{\frac{2\theta}{\pi}}(\hat{\mathbf{n}},\pi/2) = A^{\frac{2\theta}{\pi}-1}(\hat{\mathbf{n}},\pi/2)A(\hat{\mathbf{n}},\pi/2) = A(\hat{\mathbf{n}},\theta-\pi/2)A(\hat{\mathbf{n}},\pi/2) \quad (3.28)$$

and by induction hypothesis

$$A(\hat{\mathbf{n}},\theta-\pi/2) = R(\hat{\mathbf{n}},\theta-\pi/2) \quad (3.29)$$

combining (3.28) with (3.29) we obtain

$$\begin{aligned} A(\hat{\mathbf{n}},\theta) &= R(\hat{\mathbf{n}},\theta-\pi/2)A(\hat{\mathbf{n}},\pi/2) \\ &= R(\hat{\mathbf{n}},\theta)R(\hat{\mathbf{n}},-\pi/2)R(\hat{\mathbf{n}},\pi/2) \\ &= R(\hat{\mathbf{n}},\theta) \end{aligned}$$

above we use some basic properties of the Euler-Rodrigues formula.

Secondly, for $-\pi/2 \leqslant \theta \leqslant 0$, thanks to (1.11) and proceeding analogously to the proof of Theorem (3.2.4) we can obtain the expression

$$A^{-\alpha}(\hat{\mathbf{n}},\pi/2) = \begin{bmatrix} n_1^2(1-\cos\frac{\alpha\pi}{2})+\cos\frac{\alpha\pi}{2} & n_1n_2(1-\cos\frac{\alpha\pi}{2})+n_3\sin\frac{\alpha\pi}{2} & n_1n_3(1-\cos\frac{\alpha\pi}{2}))-n_2\sin\frac{\alpha\pi}{2} \\ n_1n_2(1-\cos\frac{\alpha\pi}{2})-n_3\sin\frac{\alpha\pi}{2} & n_2^2(1-\cos\frac{\alpha\pi}{2})+\cos\frac{\alpha\pi}{2} & n_2n_3(1-\cos\frac{\alpha\pi}{2})+n_1\sin\frac{\alpha\pi}{2} \\ n_1n_3(1-\cos\frac{\alpha\pi}{2})+n_2\sin\frac{\alpha\pi}{2} & n_2n_3(1-\cos\frac{\alpha\pi}{2})-n_1\sin\frac{\alpha\pi}{2} & n_3^2(1-\cos\frac{\alpha\pi}{2})+\cos\frac{\alpha\pi}{2} \end{bmatrix}$$

and so the definition in (3.26) agrees with the classical one given by *Euler-Rodrigues formula* in (3.24) for $-\pi/2 \leqslant \theta \leqslant 0$. Finally, an analogous argument of induction as in the first part of this proof shows that (3.26) agrees with the *Euler-Rodrigues formula* in (3.24) for $\theta \leqslant 0$. □

Corollary 3.2.8. *The family* $\{A(\hat{\mathbf{n}},\theta); \theta \in \mathbb{R}\}$, *where*

$$A(\hat{\mathbf{n}},\theta) = \begin{bmatrix} n_1^2(1-\cos(\theta))+\cos(\theta) & n_1n_2(1-\cos(\theta))-n_3\sin(\theta) & n_1n_3(1-\cos(\theta))+n_2\sin(\theta) \\ n_1n_2(1-\cos(\theta))+n_3\sin(\theta) & n_2^2(1-\cos(\theta))+\cos(\theta) & n_2n_3(1-\cos(\theta))-n_1\sin(\theta) \\ n_1n_3(1-\cos(\theta))-n_2\sin(\theta) & n_2n_3(1-\cos(\theta))+n_1\sin(\theta) & n_3^2(1-\cos(\theta))+\cos(\theta) \end{bmatrix}$$

is a uniformly continuous group on $\mathbb{R}^3$ *with infinitesimal generator* $G : \mathbb{R}^3 \to \mathbb{R}^3$ *given by*

$$G = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}.$$

Proof: That the family $\{A(\hat{\mathbf{n}},\theta);\theta\in\mathbb{R}\}$ is a group is a immediate consequence of the definition of $A(\hat{\mathbf{n}},\theta)$ in (3.26) and Lemma 1.5.6. We obtain G easily from the definition of infinitesimal generator of a group

$$D(G)=\left\{u\in\mathbb{R}^3;\lim_{\theta\to 0}\frac{A(\hat{\mathbf{n}},\theta)u-u}{\theta}\text{ exists}\right\}$$

and

$$Gu=\lim_{\theta\to 0}\frac{A(\hat{\mathbf{n}},\theta)u-u}{\theta}\text{ for }u\in D(G).$$

Since G is a bounded linear operator, we conclude that $\{A(\hat{\mathbf{n}},\theta);\theta\in\mathbb{R}\}$ is a uniformly continuous group on $\mathbb{R}^3$. ▫

Remark 3.2.9. *In particular, we can obtain the explicit expression of the logarithm of rotations $A(\hat{\mathbf{n}},\theta)$ thanks to the fact that the logarithm is the infinitesimal generator of the uniformly continuous group $\{A^\alpha(\hat{\mathbf{n}},\theta);\alpha\in\mathbb{R}\}$ on $\mathbb{R}^3$; namely, we have*

$$\log A(\hat{\mathbf{n}},\theta)=\begin{bmatrix}0&-\theta n_3&\theta n_2\\\theta n_3&0&-\theta n_1\\-\theta n_2&\theta n_1&0\end{bmatrix}.$$

Apêndices

Appendix A

Chebyshev polynomials of the second kind

In this appendix we give a very brief exposition of the Chebyshev polynomials of the second kind. We introduce only the definition of these polynomials and state the two properties used in Chapter 3. See Abramowitz and Stegun [2] for the complete bibliography. The Chebyshev polynomials of the second kind, $U_n : \mathbb{C} \to \mathbb{C}$ for $n = 0, 1, 2, \ldots$, are defined by the recurrence relation

$$\begin{aligned} U_0(x) &= 1 \\ U_1(x) &= 2x \\ U_{n+1}(x) &= 2xU_n(x) - U_{n-1}(x). \end{aligned}$$

They arise in the development of four-dimensional spherical harmonics in angular momentum theory. However, our interest in them is due to their connection with trigonometric multiple-angle formulas. Namely, the polynomials of the second kind satisfy:

$$U_{n-1}(\cos\theta)\sin\theta = \sin n\theta \tag{A.1}$$

or

$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta} \tag{A.2}$$

They also satisfy the following symmetry property:

$$U_n(-x) = (-1)^n U_n(x). \tag{A.3}$$

Bibliography

[1] L. Abadias, C. Lizama and M. Murillo, Hölder regularity for the Moore-Gibson-Thompson equation with infinite delay, Communications on Pure and Applied Analysis, **17** (1) (2018) 243–265.

[2] M. Abramowitz and C. A. Stegun, (Eds.). "Orthogonal Polynomials." Ch. 22 in Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, **9** (1972), 771–802.

[3] H. Amann, Linear and quasilinear parabolic problems. Volume I: Abstract Linear Theory, Birkhäuser Verlag, Basel, 1995.

[4] A. V. Balakrishnan, Fractional powers of closed operators and the semigroups generated by them. Pacific J. Math. **10** (1960), no. 2, 419–437.

[5] F. D. M. Bezerra, A. N. Carvalho, J. Cholewa, and M. J. D. Nascimento, Parabolic approximation of damped wave equations via fractional powers: fast growing nonlinearities and continuity of the dynamics, Journal of Mathematical Analysis and Applications, **450** (1) (2017) 377–405.

[6] F. D. M. Bezerra, A. N. Carvalho, T. Dłotko and M. J. D. Nascimento, Fractional Schrödinger equation; solvability, asymptotic behaviour and connection with classical Schrödinger equation, J. Math. Anal. Appl., **457** (1) (2018) 336–360.

[7] F. D. M. Bezerra and M. J. D. Nascimento, Non-autonomous approximations governed by the fractional powers of damped wave operators, Electron. J. Differential Equations, **2019** (2019), No. 72, 1–19.

[8] S. Bochner, Diffusion equation and stochastic processes, Proc. Nat. Acad. Sciences, U.S.A., 35 (1949), 368-370.

[9] A. H. Caixeta, I. Lasiecka, and V. N. D. Cavalcanti, On long time behavior of Moore-Gibson-Thompson equation with molecular relaxation, Evolution Equations & Control Theory, **05** (2016) 661.

[10] A. N. Carvalho and J. W. Cholewa, Local well posedness for strongly damped wave equations with critical nonlinearities. Bull. Austral. Math. Soc. **66** (2002), 443–463.

[11] A. N. Carvalho and S. Piskarev, A general approximation scheme for attractors of abstract parabolic problems. Numer. Funct. Anal. Optim. **27** (2006) 785–829.

[12] S. Chen and R. Triggiani. Proof of two conjectures of G. Chen and D.L. Russellon structural damping for elastic systems. In Lecture Notes in Mathematics, Springer-Verlag, **1354** (1988), 234–256.

[13] S. Chen and R. Triggiani. Proof of extension of two conjectures on structural damping for elastic systems: The case $\frac{1}{2} \leqslant \alpha \leqslant 1$, Pacific J. Math. **10** (136), 15–55.

[14] H. Cheng and K. C. Gupta, An historical note on finite rotations, Journal of Applied Mechanics, **56** (1989) 139–145.

[15] J. W. Cholewa, and T. Dłotko, Global Attractors in abstract parabolic problems, London Mathematical Society, Lecture Note Series, Cambridge University Press, 278, 2000.

[16] J. A. Conejero, C. Lizama and F. Rodenas, Chaotic behaviour of the solutions of the Moore-Gibson-Thompson equation, Applied Mathematics and Information Sciences, **09** (5) (2015) 2233–2238.

[17] R. Czaja, Differential equations with sectorial operator, Wydawnictwo Uniwersytetu Slaskiego, Katowice, 2002.

[18] J. S. Dai, Euler-Rodrigues formula variations, quaternion conjugation and intrinsic connections, Mechanism and Machine Theory, **92** (2015) 144–152.

[19] F. Dell'Oro, I. Lasiecka and V. Pata, The Moore-Gibson-Thompson equation with memory in the critical case, Journal of Differential Equations, **261** 7 (2016) 4188–4222.

[20] L. Euler, Problema algebraicum ob affectiones prorsus singulares memorabile. *Commentatio 407 indicis Enestrœmiani, Novi commentarii academiæscientiarum Petropolitanæ*, 15, (1770), 1771, p.75-106, reprinted in L.Euleri Opera Omnia, 1st series, 6, 287–315.

[21] H. O. Fattorini, The Cauchy Problem. In Encyclopedia of Mathematics and its Applications, Vol. 18. Addison-Wesley Publishing Company: Reading, Massachusetts, 1983.

[22] W. Feller, On a generalization of Marcel Riesz' potentials and the semigroups generated by them, Communications du seminaire mathematique de ?universite de Lund, tome supplimentaire (1952), dedie' a Marcel Riesz.

[23] J. K. Hale, *Asymptotic Behavior of Dissipative Systems (Mathematical Surveys and Monographs).* American Mathematical Society, Providence, 1988.

[24] M. Hasse, The Functional Calculus for Sectorial Operators. Birkhäuser Verlag, 2006.

[25] D. Henry, Geometric theory of semilinear parabolic equations, Lecture Notes in Mathematics 840, Springer-Verlag, Berlin, 1981.

[26] B. Kaltenbacher, I. Lasiecka and R. Marchand, Well-posedness and exponential decay rates for the Moore-Gibson-Thompson equation arising in high intensity ultrasound, Control Cybernet., **40** (4) (2011), 971–988.

[27] B. Kaltenbacher, I. Lasiecka and M. Pospieszalska, Well-posedness and exponential decay of the energy in the nonlinear Jordan-Moore-Gibson-Thompson equation arising in high intensity ultrasound, Math. Models Methods Appl. Sci., 22 (1) (2012).

[28] T. Kato, Note on fractional powers of linear operators, Proc. Japan Acad., **36** (1960), 94–96.

[29] D. Kahvecí, Y. Yayli and I. Gök, The geometrical and algebraic interpretations of Euler-Rodrigues formula in Minkowski 3-space, International Journal of Geometric Methods in Modern Physics, **13** 1650116 (2016) 1–10.

[30] J. D. Knapp, Stability and convergence of approximate solutions to the Moore-Gibson-Thompson equation, Ph. D. Thesis, University of Virginia, 2014.

[31] S. G. Kreĭn, Linear Differential Equations in a Banach Space. Transl. Mathem. Monogr., 29, Amer. Math. Soc., 1971.

[32] I. Lasiecka and X. Wang, Moore-Gibson-Thompson equation with memory, part I: Exponential decay of energy, Zeitschrift für angewandte Mathematik und Physik ZAMP, **67** (05) (2015).

[33] R. Marchand, T. McDevitt and R. Triggiani, An abstract semigroup approach to the third-order Moore-Gibson-Thompson partial differential equation arising in high-intensity ultrasound: structural decomposition, spectral analysis, exponential stability, Math. Methods Appl. Sci. **35** (15) (2012) 1896–1929.

[34] J. E. Mebius, Derivation of the Euler-Rodrigues formula for three-dimensional rotations from the general formula for four-dimensional rotations, Mathematics, 2007.

[35] F. K. Moore and W. E. Gibson, Propagation of weak disturbances in a gas subject to relaxation effects, Journal of Aerospace Sciences and Technologies 27 (1960), 117-127.

[36] A. Pazy, Semigroup of linear operators and applications to partial differential equations, *Springer-Verlag.* New York, 1983.

[37] M. Pellicer and B. Said-Houari, Wellposedness and Decay Rates for the Cauchy Problem of the Moore-Gibson-Thompson Equation Arising in High Intensity Ultrasound, Applied Mathematics & Optimization, (2017) 1–32.

[38] M. Pellicera and J. Solà-Morales, Optimal scalar products in the Moore-Gibson-Thompson equation, Evolution Equations and Control Theory, (2019) **8**(1): 203–220.

[39] O. Rodrigues, Des lois géométriques qui régissent les déplacements d'un système solide dans l'espace, et de la variation des coordonnées provenant de ces déplacements considérés indépendamment des causes qui peuvent les produire. Journal de Mathématiques pures et appliquées, 1^{re} série, 5, 1840, 380–440.

[40] H. Tanabe, Equations of Evolution. Pitman, London, 1979.

[41] P. A. Thompson, Compressible-fluid Dynamics, McGraw-Hill, New York, 1972.

[42] H. Triebel, Interpolation theory, function spaces, differential operators, Veb Deutscher, 1978.