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Macdonald dual in the semilocal case
and applications to local homology
and local cohomology

por

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Março/2021

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sob orientação do

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e coorientação do

Prof. Dr. Napoleón Caro Tuesta

Tese apresentada ao Corpo Docente do Programa de Pós-Graduação em Matemática - UFPB, como requisito parcial para obtenção do título de Doutor em Matemática.

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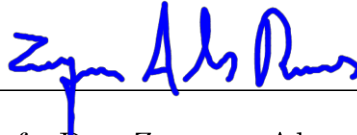
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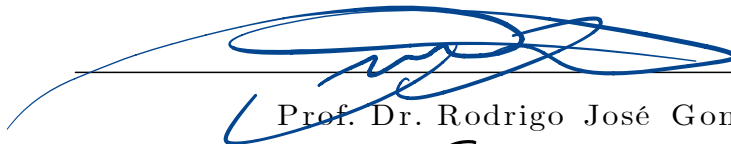
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Resumo

Neste trabalho estudamos alguns aspectos topológicos da álgebra comutativa. De forma mais precisa, estudamos caracterizações e propriedades dos módulos linearmente compactos sobre anéis semilocais e provamos, por exemplo, que esses módulos satisfazem uma bidualidade topológica. Além disso, existe uma dualidade entre módulos de homologia e cohomologia local para esses módulos. Outro resultado importante deste trabalho é estender os teoremas de anulamento e não-anulamento de Grothendieck para módulos linearmente compactos e semi-discretos sobre anéis semilocais, bem como provar a artinianidade dos módulos de cohomologia local e noetherianidade dos módulos de homologia local.

Palavras-chave: linearmente topologizado; linearmente compacto; semi-discreto; topologia J-ádica; Dual de Macdonal; cohomologia local e homologia local.

Abstract

In this work we study some topological aspects of commutative algebra. More precisely, we study characterizations and properties of linearly compact modules on semilocal rings and prove, for example, that these modules satisfy a topological duality. In addition, there is a duality between homology and local cohomology modules for these modules. Another important result of this work is to extend Grothendieck's vanishing and non-vanishing theorems to linearly compact and semi-discrete modules on semilocal rings, as well as to prove the artinianess of local cohomology modules and the noetherianess of local homology modules.

Keywords: linearly topologized; linearly compact; semi-discrete; J-adic topology; Macdonald Dual; local cohomology and local homology.

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“Cada escolha, uma renúncia, isso é a vida...”

Charlie Brown Jr

Dedicatória

Ao Antonival pai

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Introduction

What do you expect to find in an introduction to a thesis work? In general, the answers would revolve around: a description or presentation of the content of the work. Indeed, however this space can also be used to show a little between the lines, the soul of the work. Some feelings like: expectations, frustrations, achievements, anxieties, relief. Such feelings, which often may not be noticeable only in the cold, objective lines of the mathematical text. Furthermore, this space can be used to feel a little bit of the essence of author, his writing style and his personal involvement with the theme. After all, they are years of academic life that are having a stage closed with a doctoral conclusion work.

For a moment, I excuse the reader to write a little in the first person. I've been reading old works of mine with the intention of serving as inspiration for the current work, but I realized that I'm not the same anymore. In the graduation work we found a text with jokes, good humor and playful stories. In the work for completing the master's degree we found a cleaner text and at the same time a more excited author.

Today, I feel a little more down to earth. A little bit more tired too. I think that professional goals and the erosion of academic life have left animation aside, even so, the current feeling is positive, it is a feeling of relief, of accomplishment and hope that academic life will continue.

Anyway, I thought I would write little this time, but apparently the desire to express myself spoke louder. At this point I realized that this introduction to the introduction has already said a lot of what I would like to say about the relationship of the author with the work over the time of its construction. I Thank the patient reader who has so far heard sentimental words instead of reading about the mathematics

present in this work. Now, let's talk more specifically about mathematical content.

It was the end of 2017, we started the thesis research and the direction was to surf the wave of change of support of the local homology module. Namely, the intention was to change the support in the local homology module $H_i^I(M)$ to $H_i^\alpha(M)$, where α represented a family of specific ideals. One day while trying to prove a result of artinianess of local cohomology modules $H_\alpha^i(M)$ when M is finitely generated over a local ring, an idea came up. We would like the result sought to apply to linearly compact modules on *semilocal* rings, that is, with a finite number of maximal ideals, currently the Theorem (4.5.11). This result was proved by N. T. Cuong and T. T. Nam in the paper *A local homology theory for linearly compact modules* ([C-N1], 2008) in the local case. We can say that this theorem guided the path that was followed in the thesis and we can also say that from that day on, our direction has changed. We decided to change the local hypothesis to semilocal instead of working with the support of local homology and local cohomology modules.

We realized then that a topological isomorphism would be necessary, namely $(\varprojlim M_i)^* \cong \varinjlim M_i^*$ when M_i is a family of R -linearly compact modules in the semilocal context, currently the Theorem (3.5.14). This isomorphism was shown by I. G. Macdonald in the paper *Duality over Complete Local Rings* ([Mac], 1962), in the local case. After that, we went back to the study of the artinianess results described above.

To highlight the importance of this hypothesis change, we will mention some examples of semilocal rings: artinian rings; finite rings; finite-dimensional algebras over a field; finite direct product of local rings; finite direct sum of fields; the ring $End_R(M)$, where M is an artinian module over a ring R ; the classical ring of quotients of a noetherian ring R ; the ring R_S , where $\mathfrak{p}_1 \dots \mathfrak{p}_n$ are primes and $S = R \setminus (\mathfrak{p}_1 \cup \dots \cup \mathfrak{p}_n)$, with maximal ideals among $\mathfrak{p}_1 R_S, \dots, \mathfrak{p}_n R_S$. Thus, for example, all rational numbers r/s with s prime to 30 form a semilocal ring, with maximal ideals generated respectively by 2, 3, and 5. If R is a semilocal ring we have as following semilocal rings: $M_n(R)$; R -algebra finitely generated as R -module; R/I , where I is any ideal in R . The quotient $\mathbb{Z}/m\mathbb{Z}$, in particular, if m is a prime power, then $\mathbb{Z}/m\mathbb{Z}$ is a local ring; the ring of 2×2 upper triangular matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $a, d \in \mathbb{Q}$ and $b \in \mathbb{R}$. Some of the demonstrations of these facts can be found in [Lam] and [M-P].

Chapter one provides many definitions and results of topological algebra. In this chapter are presented: topological rings, topological modules, topologies used in submodules and some compatibility between algebraic and topological structures.

Chapter two is intended to present the linearly topologized modules and the linearly compact modules, as well as some results that serve as a reference for the other chapters.

In the third chapter many results are constructed with the main objective of studying the Macdonald dual. Namely, the paper [Mac] is generalized from the local case to the semilocal case. This chapter shows results of linear compactness, artinianess, noetherianess, topological isomorphisms and results of reflexivity and duality of some modules in relation to Macdonald dual.

Chapter four presents applications from the third chapter, including some generalizations of results from Grothendieck's theory of local cohomology, such results were extended from finitely generated modules to linearly compact semi-discrete modules by Cuong and Nam in the paper [C-N1] to the local case and we extend to the semilocal case. Verily, this paper is generalized in this sense.

Appendix A provides some important results on I -adic topology. In Appendix B, some results involving the Matlis dual are presented, including an important result by A. Ooish [Ooi] on this dual in the semilocal case and some results from chapter 10 of the book *Local Cohomology: An Algebraic Introduction with Geometric Applications* [B-S], by M. P. Brodmann and R. Y. Sharp are generalized to the semilocal context.

The most experienced reader can start reading from chapter three. Chapters one and two and appendix A bring only known results. Good reading.

Notation and terminology

Throughout this work, all rings are commutative with non-zero identity; the terminology “isomorphism” means “algebraic isomorphism”, also “topological isomorphism” means algebraic isomorphism where the homomorphisms (and its inverse) are continuous.

- $(R, \mathfrak{m}_1, \dots, \mathfrak{m}_n)$ denote a noetherian semilocal ring, where $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ are the maximal ideals of R .
- $\mathfrak{J} = \bigcap_{i=1}^n \mathfrak{m}_i$ denote the Jacobson radical of R .
- $\widehat{R} = \varprojlim \frac{R}{I^n}$ denote the I -adic completion of R for some ideal I of R . Note that the notation does not emphasize the ideal. This will not be a problem, as we will work mainly with the \mathfrak{J} -adic topology.
- $E(M)$ denote the injective envelope of a R -module M . Also, $R^* = E(R/\mathfrak{J})$.
- $D(M) = {}^A\text{Hom}_R(M, R^*)$ denote the module of all (algebraic) homomorphisms from M into R^* and $M^* = \text{Hom}_R(M, R^*)$, the module of all *continuous* homomorphisms from M into R^* (with the exception of Appendix B, where $D(M) = \text{Hom}_R(M, R^*)$ denote the module of all (algebraic) homomorphisms from M into R^*).

Chapter 1

Topological Modules

This chapter is basically a subset of chapter one of [War], with the exception of Remark (1.3.6), Corollary (1.3.7) and Example (1.3.8), which were included so that there was a compatibility with Chapter 2. Here the reader will find many definitions and results of topological algebra. In this chapter are presented: topological rings, topological modules, topologies used in submodules and some compatibility between algebraic and topological structures.

1.1 Topological Rings

In this introductory chapter we shall define and give examples of topological rings and modules, and show how they may be introduced by specifying the neighborhoods of zero.

A topological ring is simply a ring furnished with a topology for which its algebraic operations are continuous:

Definition 1.1.1 *A topology \mathcal{T} on a ring R is a **ring topology** and R , furnished with \mathcal{T} , is a **topological ring** if the following conditions hold:*

$$(i) \quad (x, y) \rightarrow x + y \text{ is a continuous function from } R \times R \text{ to } R; \quad (TR1)$$

$$(ii) \quad x \rightarrow -x \text{ is a continuous function from } R \text{ to } R; \quad (TR2)$$

$$(iii) \quad (x, y) \rightarrow xy \text{ is a continuous function from } R \times R \text{ to } R. \quad (TR3)$$

where $R \times R$ is provided with the product topology determined by \mathcal{T} .

A ring topology on a ring R induces a ring topology on any subring of R , and unless the contrary is indicated, we shall assume that a subring of a topological ring is furnished with its induced topology.

A way to construct topological rings is by considering norms functions on a ring.

Definition 1.1.2 A function N from a ring R to $\mathbb{R}_{\geq 0}$ (the set of non-negative real numbers) is a **norm** if the following conditions hold for all $x, y \in R$:

$$(i) \ N(x) = 0 \text{ if and only if } x = 0;$$

$$(ii) \ N(x + y) \leq N(x) + N(y);$$

$$(iii) \ N(-x) = N(x);$$

$$(iv) \ N(xy) \leq N(x)N(y).$$

If N is a norm on a ring R then the function d defined by $d(x, y) = N(x - y)$ for all $x, y \in R$ is a metric. Indeed, (i) above implies that $d(x, y) = 0$ if and only if $x = y$, (iii) implies that $d(x, y) = d(y, x)$, and (ii) yields the triangle inequality, since:

$$\begin{aligned} d(x, z) &= N(x - z) = N((x - y) + (y - z)) \\ &\leq N(x - y) + N(y - z) = d(x, y) + d(y, z). \end{aligned}$$

We say that N is a complete norm if the associated metric d is complete.

Often norms are denoted by the symbol $\| \cdot \|$.

Proposition 1.1.3 Let N be a norm on a ring. The topology given by the metric d induced by N is a ring topology.

We say that a topological ring is *normable* if its topology is defined by a norm. A *normed ring* is simply a ring furnished with a norm and hence with the topology defined by that norm.

Example 1.1.4 Let X be a set and $\mathcal{B}(X)$ be the ring of all bounded real-valued (or complex-valued) functions on X (a function f is bounded if $N(f) < +\infty$, where $N(f) = \sup\{|f(x)| : x \in X\}$). The function N just defined is a complete norm on $\mathcal{B}(X)$, hence $\mathcal{B}(X)$ and each of its subrings is a topological ring for the topology induced by N . Special cases: (a) The ring of all bounded continuous functions on a topological space X . (b) The ring of all continuous functions on a compact space X .

Example 1.1.5 Let $I \subseteq R$ be an ideal and R be a commutative ring equipped with the **I -adic topology**: a subset U of R is open if and only if for every $x \in U$ there exists a natural number n such that $x + I^n \subseteq U$ (see appendix [A]). This turn R into a topological ring (see Example (1.3.8)). The I -adic topology is Hausdorff if and only if the intersection of all powers of I is the ideal (0) (see [War] 3.4).

More examples can be found in Chapter 1 of [War].

Proposition 1.1.6 *The cartesian product of a family $(R_\lambda)_{\lambda \in L}$ of a topological rings is a topological ring with the product topology (the open sets in the product topology are unions, finite or infinite, of sets of the form $\prod_{\lambda \in L} U_\lambda$, where each U_λ is open in R_λ and $U_\lambda \neq R_\lambda$ for only finitely many λ).*

1.2 Topological Modules

A topological module is simply a module over a topological ring furnished with a topology for which its algebraic operations are continuous:

Definition 1.2.1 *Let R be a topological ring and M be an R -module. A topology \mathcal{T} on M is an **R -module topology** (or simply a **module topology** if no confusion results) and M , furnished with \mathcal{T} , is a **topological R -module** (or simply a **topological module**) if the following conditions hold:*

$$(i) \quad (x, y) \rightarrow x + y \text{ is a continuous function from } M \times M \text{ to } M; \quad (TM1)$$

$$(ii) \quad x \rightarrow -x \text{ is continuous function from } M \text{ to } M; \quad (TM2)$$

$$(iii) \quad (\lambda, x) \rightarrow \lambda x \text{ is a continuous function from } R \times M \text{ to } M. \quad (TM3)$$

where $R \times M$ is provided with the product topology determined by the topology of R and \mathcal{T} .

Any topological ring R may be regarded as a topological R -module, where scalar multiplication is given by the ring multiplication.

A module topology on a R -module M induces a module topology on any submodule of M , and unless the contrary is indicated, we shall assume that a submodule of a topological module is furnished with its induced topology.

If M is a topological R -module and if S is a subring of R , then M can be regarded as S -module by restricting scalar multiplication to S , in this case it becomes a topological S -module. Also, if M is a topological R -module then M , with its given topology, is still a topological module over the ring R furnished with a stronger ring topology.

Definition 1.2.2 A topology \mathcal{T} on a group G (not necessarily abelian), denoted additively, is a **group topology** and G , furnished with \mathcal{T} , is a **topological group** if the following conditions hold:

$$(i) \quad (x, y) \rightarrow xy \text{ is continuous function from } G \times G \text{ to } G; \quad (TG1)$$

$$(ii) \quad x \rightarrow -x \text{ is continuous function from } G \text{ to } G. \quad (TG2)$$

where $G \times G$ is provided with the cartesian product topology determined by \mathcal{T} .

The additive group of a topological ring or module is a commutative topological group, since $(TR1) - (TR2)$ and $(TM1) - (TM2)$ become $(TG1) - (TG2)$ in additive notation. A group topology on a group G induces a group topology on any subgroup of G , and unless the contrary is indicated, we shall assume that a subgroup of a topological group is furnished with its induced topology.

Proposition 1.2.3 Let G be a topological group, and let $a \in G$. The functions $x \rightarrow -x$, $x \rightarrow a + x$, and $x \rightarrow x + a$ are homeomorphisms from G to G . Consequently, for any subset X of G , $\overline{-X} = -\overline{X}$, $\overline{a + X} = a + \overline{X}$, $\overline{X + a} = \overline{X} + a$, (where $\overline{\cdot}$ denotes the topological closure) and for any open (closed) subset P of G , $-P$ and $a + P$ are open (closed).

A **neighborhood** of a point (respectively of a subset) of a topological space T is any subset of T that contains an open subset containing that point (respectively of a subset). Thus a subset of T is open if and only if it is a neighborhood of each of its points. A neighborhood V of zero of a Group is said to be symmetric if $-V = V$.

Proposition 1.2.4 Let G be a topological group.

$$(i) \quad \text{If } V \text{ is a neighborhood of zero, then there is a neighborhood } U \text{ of zero such that } U + U \subseteq V.$$

$$(ii) \quad \text{If } V \text{ is a neighborhood of zero, then so is } -V.$$

$$(iii) \quad \text{Every (open) neighborhood } U \text{ of zero contains a symmetric (open) neighborhood of zero, namely, } U \cap (-U).$$

Proposition 1.2.5 Let R be a topological ring.

$$(i) \quad \text{For each } b \in R, \text{ the function } x \rightarrow xb \text{ from } R \text{ to } R \text{ is continuous; if } b \text{ is invertible, it is an homeomorphism.}$$

(ii) If f and g are functions from a topological space T to R that are continuous at $t \in T$, then $f + g$, $-f$, and fg are continuous at t .

Proposition 1.2.6 *Let M be a topological R -module.*

(i) For each $b \in M$, the function $\lambda \rightarrow \lambda b$ from R to M is continuous, and for each $\beta \in R$, the function $x \rightarrow \beta x$ from M to M is continuous; if β is invertible, the function $x \rightarrow \beta x$ is an homeomorphism.

(ii) If f is a function from a topological space T to M that is continuous at $t \in T$, then for each $\lambda \in R$, λf is continuous at t .

Proposition 1.2.7 *If a topology \mathcal{T} on a ring R satisfies (TR1) and (TR2) of Definition (1.1.1), then \mathcal{T} satisfies (TR3) if and only if it satisfies the following two conditions:*

(i) The function $(x, y) \rightarrow xy$ from $R \times R$ into R is continuous at $(0, 0)$;
(TR4)

(ii) For each $b \in R$, the function $x \rightarrow xb$ from R into R is continuous at zero.
(TR5)

Proposition 1.2.8 *Let R be a topological ring and let M be an R -module. If a topology \mathcal{T} on M satisfies (TM1) and (TM2) of Definition (1.2.1), then \mathcal{T} satisfies (TM3) if and only if it satisfies the following three conditions:*

(i) The function $(\lambda, x) \rightarrow \lambda x$ from $R \times M$ into M is continuous at $(0, 0)$;
(TM4)

(ii) For each $b \in M$, the function $\lambda \rightarrow \lambda b$ from R into M is continuous at zero;
(TM5)

(iii) For each $\beta \in R$, the function $x \rightarrow \beta x$ from M into M is continuous at zero.
(TM6)

Proposition 1.2.9 *The cartesian product of a family of topological R -modules is a topological R -module.*

1.3 Neighborhoods of Zero

We recall that a family \mathcal{F} of subsets of a set E is a **filter** on E if $E \in \mathcal{F}$, $\emptyset \notin \mathcal{F}$, the intersection of any two members of \mathcal{F} again belongs to \mathcal{F} , and any subset of E

containing a member of \mathcal{F} also belongs to \mathcal{F} . For example, in a topological space \mathcal{T} , the set of all neighborhoods of a point (subset) of \mathcal{T} is a filter.

A family \mathcal{B} of subsets of E is a **filter base** on E if the set of all subsets F of E for which there exists $B \in \mathcal{B}$ such that $B \subseteq F$ is a filter, called the **filter generated by \mathcal{B}** . Thus \mathcal{B} is a filter base if and only if $\mathcal{B} \neq \emptyset$, $\emptyset \notin \mathcal{B}$, and the intersection of two members of \mathcal{B} contains a member of \mathcal{B} . Consequently, a filter base on E is also a filter base on any set containing E . In a topological space T , a **fundamental system of neighborhoods of a point (subset)** of T is any filter base generating the filter of neighborhoods of that point (subset). For example, the open neighborhoods of a point in a topological space form a fundamental system of neighborhoods of that point.

If \mathcal{V} is the filter of neighborhoods of zero for a group topology on a group G , then by Proposition (1.2.3), for each $a \in G$, $a + \mathcal{V}$ is the filter of neighborhoods of a . Consequently, a group topology is completely determined by the filter of neighborhoods of zero; that is, distinct group topologies have distinct filters of neighborhoods of zero. For commutative groups, the following theorem gives necessary and sufficient conditions for a filter to be the filter of neighborhoods of zero for a group topology on G :

Proposition 1.3.1 *If \mathcal{V} is the filter of neighborhoods of zero for group topology on a group G , then*

(i) *For each $V \in \mathcal{V}$, there exists $U \in \mathcal{V}$ such that $U + U \subseteq V$;* (TGN 1)

(ii) *If $V \in \mathcal{V}$, then $-V \in \mathcal{V}$.* (TGN 2)

Conversely, if \mathcal{V} is a filter on a commutative group G satisfying (TGN 1) and (TGN 2), then there is a unique group topology on G for which \mathcal{V} is the filter of neighborhoods of zero.

Corollary 1.3.2 *If \mathcal{V} is a fundamental system of neighborhoods of zero for group topology on a group G , then*

(i) *For each $V \in \mathcal{V}$, there exists $U \in \mathcal{V}$ such that $U + U \subseteq V$;* (TGB 1)

(ii) *For each $V \in \mathcal{V}$, there exists $U \in \mathcal{V}$ such that $U \subseteq -V$.* (TGB 2)

Conversely, if \mathcal{V} is a filter base on a commutative group G such that (TGB 1) and (TGB 2) hold, then there is a unique group topology on G for which \mathcal{V} is a fundamental system of neighborhoods of zero.

To specify a group topology on a commutative group, we need only specify a filter base \mathcal{V} satisfying (TGB 1) and (TGB 2). This method, in fact, is the most frequent way of defining a group topology. For example, if \mathcal{V} is a filter base of subgroups of a commutative group G , then \mathcal{V} is a fundamental system of neighborhoods of zero for a group topology on G .

Proposition 1.3.3 *Let G be a topological group, let \mathcal{V} be a fundamental system of neighborhoods of zero, and let $A \subseteq G$.*

- (i) *The open symmetric neighborhoods of zero form a fundamental system of neighborhoods of zero;*
- (ii) *For any (open) neighborhood V of zero, $A + V$ is a (open) neighborhood of A ;*
- (iii) *$\overline{A} = \bigcap \{A + V : V \in \mathcal{V}\}$; in particular, $\overline{\{0\}} = \bigcap \{V : V \in \mathcal{V}\}$;*
- (iv) *The closed symmetric neighborhoods of zero form a fundamental system of neighborhoods of zero.*

Proposition 1.3.4 *Let R be a ring. If \mathcal{V} is a fundamental system of neighborhoods of zero for a ring topology on R , then \mathcal{V} satisfies (TGB1), (TGB2) of the Corollary (1.3.2) and the following conditions:*

- (i) *For each $V \in \mathcal{V}$, there exists $U \in \mathcal{V}$ such that $UU \subseteq V$; (TRN 1)*
- (ii) *For each $V \in \mathcal{V}$ and each $b \in R$ there exists $U \in \mathcal{V}$ such that $Ub \subseteq V$. (TRN 2)*

Conversely, if \mathcal{V} is a filter base on R satisfying (TGB1), (TGB2), (TRN 1) and (TRN 2), then there is a unique ring topology on R for which \mathcal{V} is a fundamental system of neighborhoods of zero.

The most frequent way of defining a ring topology on a ring R is to specify a filter base satisfying the conditions of (1.3.4). Those conditions are satisfied, for example, by a filter base of ideals of R . Any ring topology having a fundamental system of neighborhoods of zero consisting of ideals is called an **ideal topology** or **linear topology**. In (1.3.8) we will see an example.

Proposition 1.3.5 *Let R be a ring and M be an R -module. If \mathcal{V} is a fundamental system of neighborhoods of zero for an R -module topology on M , then \mathcal{V} satisfies (TGB1), (TGB2) of the Corollary (1.3.2) and the following conditions:*

- (i) *For each $V \in \mathcal{V}$ there exist a neighborhood T of zero in R and $U \in \mathcal{V}$ such that $TU \subseteq V$; (TMN 1)*

(ii) For each $V \in \mathcal{V}$ and each $b \in M$ there exists a neighborhood T of zero in R such that $Tb \subseteq V$; (TMN 2)

(iii) For each $V \in \mathcal{V}$ and $\beta \in R$ there exist $U \in \mathcal{V}$ such that $U\beta \subseteq V$. (TMN3)

Conversely, if \mathcal{V} is a filter base on M satisfying (TGB 1), (TGB 2), (TMN 1), (TMN 2) and (TMN 3), then there is a unique R -module topology on M for which \mathcal{V} is a fundamental system of neighborhoods of zero.

Let M be a topological R -module and \mathcal{M} be a fundamental system of neighborhoods of zero of M . If N is a submodule of M which contains a neighborhood of zero then N is open (and therefore closed) in M and M/N is discrete. Furthermore, M is Hausdorff if and only if the intersection of all the elements of \mathcal{M} is 0 (see [War] 3.4).

Remark 1.3.6 If \mathcal{M} is a family of submodules of a R -module M then \mathcal{M} will be a filter base on M if and only if:

(i) Given N_1 and N_2 in \mathcal{M} there is N_3 in \mathcal{M} such that $N_3 \subseteq N_1 \cap N_2$.

Now suppose \mathcal{M} satisfies (i). Then \mathcal{M} will be a fundamental system of neighborhoods of zero for a R -topology on M if and only if:

(ii) Given $x \in M$ and $N \in \mathcal{M}$ there is a neighborhood U of zero in R such that $Ux \subseteq N$.

Indeed, the items (TGB 1), (TGB 2), (TMN 1) and (TMN 2) are automatically satisfied by the fact that \mathcal{M} consists of submodules (just take $U = V$). Therefore, we have the following result:

Corollary 1.3.7 *Let R be a ring and M be an R -module. Let \mathcal{M} be a fundamental system of neighborhoods of zero on M consisting of submodules. Then the elements of \mathcal{M} are open submodules of M and \mathcal{M} satisfies the following conditions:*

(i) Given N_1 and N_2 in \mathcal{M} there is N_3 in \mathcal{M} such that $N_3 \subseteq N_1 \cap N_2$; (TBS 1)

(ii) Given $x \in M$ and $N \in \mathcal{M}$ there is a neighborhood U of zero in R such that $Ux \subseteq N$. (TBS 2)

Conversely, if M is an R -module and \mathcal{M} is a family of submodules of M , then \mathcal{M} is a fundamental system of neighborhoods of zero on M provided (TBS 1) and (TBS 2) are satisfied, and the topology of M is then uniquely determined.

The term fundamental system of neighborhoods of zero is called nuclear base in [Mac].

Example 1.3.8 (*I*-adic topology on a Module) If I is an ideal of a ring R , we may consider in R the topology defined by taking the set of all powers I^n ($n \geq 0$) as a fundamental system of neighborhoods of zero. Note that the set $\{I^n\}_{n \geq 0}$ satisfies (TBS 1) and (TBS 2) hence, with this topology, R is a topological ring. We call this topology the *I*-adic topology or simply the *I*-topology and we say that R is an *I*-adic ring or *I*-ring. The *I*-adic topology is an example of linear topology. The ***I*-adic topology** on a R -module M is defined by taking the *I*-topology on R and the topology given by all the submodules $I^n M$ ($n \geq 0$) of M ; $\{I^n M\}$ satisfies (TBS 1) and (TBS 2), hence with this topology M is a topological module. This topology is an example of linear topology (ideal topology).

The reader can find more on *I*-adic topology in Appendix A.

1.4 Subrings, Ideals, Submodules and Subgroups

Proposition 1.4.1 *If H is a subgroup of a topological group G , then \overline{H} is a subgroup.*

Proposition 1.4.2 *If B is a subring (ideal) of a topological ring A , so is \overline{B} . If A is a dense subring of a topological ring A' and if J is an ideal of A , then the closure of J in A' is an ideal.*

Proposition 1.4.3 *If N is a submodule of a topological module M , so is \overline{N} .*

Proposition 1.4.4 *An open subgroup H of a topological group G is closed.*

Proposition 1.4.5 *If a subgroup H of a topological group G has an interior point, then H is open.*

We conclude with a useful result relating the neighborhoods of zero in a topological group to those in a dense subgroup.

Proposition 1.4.6 *If G is a dense subgroup of a Hausdorff group G_1 , the closures in G_1 of a fundamental system of neighborhoods of zero in G form a fundamental system of neighborhoods of zero in G_1 .*

1.5 Quotients of Rings and Modules

Let f be a function from a topological space S to a set T . Of all the topologies on T for which f is continuous there is a strongest one, namely $\{O \subseteq T : f^{-1}(O) \text{ is open in } S\}$, for that collection of subsets of T it is easily seen to be a topology on T .

Definition 1.5.1 *Let J be an ideal of a topological ring R . The **quotient topology** is the strongest topology on R/J for which the canonical epimorphism ϕ_J from R to R/J is continuous.*

We similarly define the quotient topology of M/N where N is a submodule (subgroup) of a topological module (group) M . The following theorems stated for quotient rings determined by ideals of topological rings are also valid (with essentially the same proof) for quotient modules (groups) determined by submodules (normal subgroups) of topological modules (groups). A function f from a topological space S into a topological space T is open if for every open subset O of S , $f(O)$ is open in T .

Proposition 1.5.2 *If J is an ideal of a topological ring R , the canonical epimorphism π_J from R to R/J is continuous and open.*

Proposition 1.5.3 *Let R , S , and T be topological spaces, let h be a continuous open surjection from R to S , and let q be a function from S to T . If $q \circ h$ is continuous (open), then q is continuous (open).*

Proposition 1.5.4 *If J is an ideal of a topological ring R , the quotient topology of R/J is a ring topology.*

Proposition 1.5.5 *If \mathcal{V} is a fundamental system of neighborhoods of zero in a topological ring R and if J is an ideal of R , then $\pi_J(\mathcal{V})$ is a fundamental system of neighborhoods of zero for the quotient topology of R/J .*

Corollary 1.5.6 *If the topology of a topological ring R is an ideal topology, then for any ideal J of R , the quotient topology of R/J is an ideal topology.*

Proposition 1.5.7 *Let J be an ideal of a topological ring R .*

(i) *R/J is Hausdorff if and only if J is closed.*

(ii) *R/J is discrete if and only if J is open.*

If S is a subring and J an ideal of a ring R such that $J \subseteq S$, then the quotient ring S/J is actually a subring of R/J .

Proposition 1.5.8 *Let S be a subring and J be an ideal of a topological ring R such that $J \subseteq S$. The quotient topology of S/J coincides with the topology induced on the subring S/J of R/J by the quotient topology of R/J .*

Corollary 1.5.9 *Let S be a subring and J be an ideal of a topological ring R . The quotient topology of $(S + J)/J$ coincides with the topology induced on it by the quotient topology of R/J .*

Definition 1.5.10 *Let f be a function from a topological ring (module, group) R to another S . The function f is a topological isomorphism if f is both an isomorphism and a homeomorphism; f is a topological homomorphism if f is a continuous homomorphism and is also an open mapping from R onto its image $f(R)$; f is a topological epimorphism (monomorphism) if f is a surjective (injective) topological homomorphism.*

Thus f is a topological epimorphism if and only if f is a continuous open epimorphism. If J is an ideal of a topological ring R , π_J is a topological epimorphism from R to R/J by Proposition (1.5.2). If f is a homomorphism from R to S and if f_1 is the epimorphism obtained from f by restricting its counterdomain to its image, then clearly f is a topological homomorphism if and only if f_1 is a topological epimorphism.

Proposition 1.5.11 *Let f be a homomorphism from a topological ring R to a topological ring S , and let J be an ideal contained in the kernel K of f . The homomorphism g from R/J to S satisfying $g \circ \pi_J = f$ is continuous (open, a topological homomorphism) if and only if f is. In particular, if $J = K$, g is a topological isomorphism (monomorphism) if and only if f is a topological epimorphism (homomorphism).*

Corollary 1.5.12 *If H and J are ideals of a topological ring R such that $J \subseteq H$, then the canonical epimorphism from R/J to R/H is a topological epimorphism.*

Corollary 1.5.13 *If H and J are ideals of a topological ring R such that $J \subseteq H$, the canonical isomorphism g from $(R/J)/(H/J)$ to R/H is a topological isomorphism.*

Proposition 1.5.14 *Let f be a homomorphism from a topological group G to a topological group H , and let \mathcal{V} be a fundamental system of neighborhoods of zero in G .*

- (i) f is continuous if and only if f is continuous at zero.
- (ii) f is open if and only if for every $V \in \mathcal{V}$, $f(V)$ is a neighborhood of zero in H .

Chapter 2

Linearly Topologized and Linearly Compact Modules

This chapter is basically a subset of Sections 2 and 3 of [Mac], with the exception of Example (2.2.2), which was inspired by Proposition 5 of [Zel]. Here are presented the topologically linearized modules and the linearly compact modules, as well as some results that serve as a reference for the other chapters.

2.1 Linearly Topologized Modules

Definition 2.1.1 *Let R be a topological ring and M be a topological R -module. M is said to be **linearly topologized** if M has a fundamental system of neighborhoods of zero \mathcal{M} consisting of submodules. For example, a discrete topological R -module is linearly topologized.*

The elements of \mathcal{M} are open submodules of M and \mathcal{M} satisfies the conditions (TBS 1) and (TBS 2) of Corollary (1.3.7). Conversely, if M is an R -module and \mathcal{M} is a family of submodules of M , then \mathcal{M} is a fundamental system of neighborhoods of zero on M provided (TBS 1) and (TBS 2) are satisfied, and the topology of M is then uniquely determined.

Remark 2.1.2 (Submodules and quotient modules) If M is a linearly topologized R -module, N a submodule of M , then N (with the induced topology) and M/N (with the quotient topology) are linearly topologized R -modules. M/N is Hausdorff if and only if N is closed in M .

Remark 2.1.3 (Direct Product) If $\{M_i\}_{i \in I}$ is a family of linearly topologized R -modules then $M = \prod_{i \in I} M_i$ (with the product topology) is a linearly topologized R -module. The submodules $\prod U_i$, where for each $i \in I$, U_i is an open submodule of M_i and $U_i = M_i$ for almost all i , form a fundamental system of neighborhoods of zero of M . If each M_i is Hausdorff, so is M . If L is a linearly topologized R -module and, for each index i , $u_i : L \rightarrow M_i$ is a continuous homomorphism, then there is a unique continuous homomorphism $u : L \rightarrow M$ such that $u_i = p_i \circ u$, for each $i \in I$, where $p_i : M \rightarrow M_i$ is the projection. Thus we have an R -module isomorphism

$$\alpha : \prod_{i \in I} \text{Hom}_R(L, M_i) \rightarrow \text{Hom}_R(L, \prod_{i \in I} M_i)$$

where Hom_R denotes the continuous R -module homomorphisms.

Remark 2.1.4 (Inverse Limit) If $\{M_i, \varphi_{ji}\}$ is an inverse system of linearly topologized R -modules M_i , indexed by a direct set I and continuous homomorphisms $\varphi_{ji} : M_j \rightarrow M_i$ ($j \geq i$) then $M = \varprojlim M_i$ is a submodule of $\prod M_i$. We give M the induced topology so that M becomes a linearly topologized R -module. If each M_i is Hausdorff, then so is M and M is closed in $\prod M_i$. The projections $\varphi_i : M \rightarrow M_i$ are continuous homomorphisms.

If L is a linearly topologized R -module and, for each index i , $u_i : L \rightarrow M_i$ is a continuous homomorphism such that $\varphi_{ji} \circ u_j = u_i$ whenever $j \geq i$, then there is a unique continuous homomorphism $u : L \rightarrow M$ such that $u_i = \varphi_i \circ u$ for all $i \in I$. Thus we have an R -module isomorphism

$$\alpha : \varprojlim_{i \in I} \text{Hom}_R(L, M_i) \rightarrow \text{Hom}_R(L, \varprojlim_{i \in I} M_i).$$

Remark 2.1.5 (Direct Sum) Let $\{M_i\}_{i \in I}$ be a family of linearly topologized R -modules and $M = \bigoplus_{i \in I} M_i$ be its direct sum. Let \mathcal{M} be the family of all submodules $\bigoplus_{i \in I} U_i$ of M , where U_i is an open submodule of M_i for each $i \in I$. \mathcal{M} satisfies properties (TBS 1) and (TBS 2) of Corollary (1.3.7) and therefore defines a linear topology on M ; this topology is the finest linear topology on M for which the injection $q_i : M_i \rightarrow M$ are continuous. If each M_i is Hausdorff so is M . If N is a linearly topologized R -module and, for each index i , $u_i : M_i \rightarrow N$ is a continuous homomorphism, then there is a unique continuous homomorphism $u : M \rightarrow N$ such that $u_i = u \circ q_i$, for each $i \in I$. Hence we have an R -module isomorphism

$$\gamma : \prod_{i \in I} \text{Hom}_R(M_i, N) \rightarrow \text{Hom}_R\left(\bigoplus_{i \in I} M_i, N\right).$$

There is a canonical monomorphism $\mu : \bigoplus M_i \rightarrow \prod M_i$ (the “sum-to-product map”) which is continuous, but not necessarily open if the index set I is infinite. If I is finite then μ is a topological isomorphism, so that there is no distinction between finite direct sums and finite direct products; and in any case $\mu(\bigoplus M_i)$ is dense in $\prod M_i$.

Remark 2.1.6 (Direct Limit) *If $\{M_i, \psi_{ij}\}$ is a direct system of linearly topologized R -modules M_i , indexed by a direct set I and continuous homomorphisms $\psi_{ij} : M_i \rightarrow M_j$ ($i \leq j$) then $M = \varinjlim M_i$ is a quotient module of $\bigoplus M_i$ and we give it the quotient topology so that it becomes a linearly topologized R -module. If all the M_i are Hausdorff, M will not necessarily be Hausdorff. The canonical maps $\psi_i : M_i \rightarrow M$ are continuous homomorphisms.*

If N is a linearly topologized R -module and, for each index i , $u_i : M_i \rightarrow N$ is a continuous homomorphism such that $u_j \circ \psi_{ij} = u_i$ whenever $i \leq j$, then there is a unique continuous homomorphism $u : M \rightarrow N$ such that $u_i = u \circ \psi_i$ for all $i \in I$. Thus we have an R -module isomorphism

$$\delta : \varprojlim_{i \in I} \text{Hom}_R(M_i, N) \rightarrow \text{Hom}_R(\varinjlim_{i \in I} M_i, N).$$

Remark 2.1.7 *Let M and N be linearly topologized R -modules, $M \otimes_R N$ their tensor product. The family of submodules of $M \otimes_R N$ of the form $U \otimes N + M \otimes V$, as U runs through the open submodules of M and V runs through the open submodules of N , satisfies (TBS 1) and (TBS 2) of Corollary (1.3.7) and hence defines a linear topology on $M \otimes N$. The tensor product, topologized in this way, has the usual associativity properties. If M', N' are linearly topologized R -modules and $f : M \rightarrow M'$ and $g : N \rightarrow N'$ are continuous homomorphisms, then $f \otimes g : M \otimes N \rightarrow M' \otimes N'$ is a continuous homomorphism.*

Proposition 2.1.8 *Let $\{M_i\}_{i \in I}$ and N be linearly topologized R -modules. Then the canonical map*

$$(\bigoplus_{i \in I} M_i) \otimes N \rightarrow \bigoplus_{i \in I} (M_i \otimes N)$$

is a topological isomorphism.

Proposition 2.1.9 *Let $\{M_i\}_{i \in I}$ be a direct system of linearly topologized R -modules and continuous homomorphisms and let N be a linearly topologized R -module. Then the canonical map*

$$(\varinjlim M_i) \otimes N \rightarrow \varinjlim (M_i \otimes N)$$

is a topological isomorphism.

Proposition 2.1.10 *Let M be a linearly topologized R -module and \mathcal{M} be a fundamental system of neighborhoods of zero of M consisting of submodules. Then the canonical homomorphisms $M \rightarrow M/U$ ($U \in \mathcal{M}$) induce by Remark (2.1.4) a homomorphism*

$$\eta : M \rightarrow \widehat{M} := \varprojlim M/U,$$

which is continuous and open, and $\eta(M)$ is dense in \widehat{M} . If M is Hausdorff, η is a monomorphism.

2.2 Linear Compact Modules

Let R be a topological ring. From now on we shall have to deal exclusively with Hausdorff linearly topologized R -modules, and henceforth “ R -module” shall mean “Hausdorff linearly topologized R -module”.

Definition 2.2.1 Remember that a (closed, open) coset is of the form $m + N$ where $m \in M$ and N is a (closed, open) submodule of M . An R -module M is **linearly compact** if M has the following property: if \mathcal{F} is a family of closed cosets in M which has the finite intersection property (the intersection over any finite subcollection of \mathcal{F} is non-empty), then the cosets in \mathcal{F} have a non-empty intersection.

Proposition 2.2.2 If M is a topological R -module with minimum condition (every non-empty subset of M has a minimal element) on closed submodules, then M is linearly compact.

Proof. Let $\{F_\alpha\}$ be a collection of closed cosets of the form $F_\alpha = m_\alpha + N_\alpha$ which satisfies the finite intersection property. Let N be a submodule minimal among all finite intersections of the N_α 's, say $N = \bigcap N_{\alpha_i}$, and let $F = \bigcap F_{\alpha_i}$. Since F is not empty it is a coset of the form $F = b + N$. Then for every F_β , $F_\beta \cap F$ is a coset of $N_\beta \cap N$; the latter equals N because N is minimal. Indeed, we have

$$\begin{aligned} F_\beta \cap F &= (a_\beta + N_\beta) \cap (b + N) \\ &= c + (N_\beta \cap N) \\ &= c + N. \end{aligned}$$

since $F_\beta \cap F \subseteq F$, follow that $c + N \subseteq b + N$. But, we have that $b + N \subseteq c + N$. Indeed, given $c + n' \in c + N$, follow that $c + n' \in b + N$ and $c + n' = b + n''$, for any $n'' \in N$. Thus, $b = c + (n' - n'') \in c + N$. Hence $b = c + n'''$ for any $n''' \in N$. Thus, $b + N = c + n''' + N \subseteq c + N$. Therefore $F \subseteq F_\beta \cap F$ and we have $F = F_\beta \cap F$ for all β . Thus, every $F_\beta \supseteq F$ and $\bigcap_\alpha F_\alpha$ is not empty. ■

Remark 2.2.3 In particular, every artinian R -module is linearly compact.

The following results can be found in Chapter 3 of [Mac]:

Proposition 2.2.4 If M is an R -module and N a linearly compact submodule of M then, N is closed in M .

Proposition 2.2.5 *If M is a linearly compact R -module and N a closed submodule of M then, N is a linearly compact.*

Proposition 2.2.6 *If M and N are R -modules, M linearly compact and $f : M \rightarrow N$ a continuous homomorphism then, $f(M)$ is linearly compact and hence is a closed map.*

Proposition 2.2.7 *If M is an R -module and N a closed submodule of M then, M is linearly compact if and only if N and M/N are linearly compact.*

Proposition 2.2.8 *The direct product of a family of linearly compact R -modules is linearly compact.*

Proposition 2.2.9 *The inverse limit of a family of linearly compact R -modules and continuous homomorphisms is linearly compact.*

Proposition 2.2.10 *If M is an R -module and N_1, \dots, N_r are linearly compact submodules of M then, $N_1 + \dots + N_r$ is linearly compact.*

Proposition 2.2.11 *If M is an R -module which satisfies the descending chain condition on closed submodules, then M is linearly compact and discrete. Thus the only linear topology that an artinian R -module can carry is the discrete topology.*

Proposition 2.2.12 *Let M be a linearly compact R -module. Then the map $\eta : M \rightarrow \widehat{M}$ of Proposition (2.1.10) is a topological isomorphism.*

Chapter 3

Macdonald Duality Over Complete Semilocal Rings

In this chapter we study the Macdonald Dual of a R -module in context the R is semilocal. The Macdonald dual of a topological R -module M , in which case R is local, is given by $M^* = \text{Hom}_R(M, E)$ where $E = E(R/\mathfrak{m})$ is the injective envelope of residual field and we consider only continuous homomorphisms. The study in this local case was well developed by I. G. Macdonald in his paper *Duality over complete local rings* [Mac] of 1964. The Matlis Dual is given by $D(M) = \text{Hom}_R(M, E)$ where considers all homomorphisms, including those that are not continuous. A. Ooishi, in his paper *Matlis duality and the width of a module* of 1976, studied such a dual in the semilocal case, where $E = E(R/\mathfrak{m})$ is replaced by $E(\bigoplus_{i=1}^n R/\mathfrak{m}_i)$. Here we study the Macdonald dual in which case E is replaced in the same way. The main result is the Theorem (3.5.14), which says that the algebraic isomorphism $(\varprojlim M_i)^* \cong \varinjlim M_i^*$ is a topological isomorphism in the case where $\{M_i\}$ is an inverse system of linearly compact modules. Throughout the chapter there are results of linear compactness, artinianess, noetherianess, topological isomorphisms and results of reflexivity and duality of some modules in relation to Macdonald dual. In this chapter “ R -module” means “Hausdorff linearly topologized R -module”.

3.1 Linearly Compact Modules Over a Semilocal Ring

Let $(R, \mathfrak{m}_1, \dots, \mathfrak{m}_n)$ be a noetherian semilocal ring and $J = \bigcap_{i=1}^n \mathfrak{m}_i$ its Jacobson ideal. Considering R with the J -adic topology it becomes a Hausdorff topological ring. Let $\widehat{R} = \varprojlim \frac{R}{J^n}$ be the J -adic completion of R : \widehat{R} is a complete noetherian semilocal ring, and its topology is the \widehat{J} -adic topology, where \widehat{J} is the extension of J to \widehat{R} via the natural homomorphism $\eta : R \rightarrow \widehat{R}$.

If N is a closed R -submodule of \widehat{R} then N is an ideal of \widehat{R} . Indeed, if $\widehat{a} \in \widehat{R}$ and $x \in N$, then for each positive integer n_0 there is an element a_{n_0} of R such that $\widehat{a} - (a_{n_0}) \in J^{n_0}\widehat{R}$; where (a_{n_0}) denotes the constant sequence. Hence $\widehat{a}x = a_{n_0}x + [\widehat{a} - (a_{n_0})]x \in N + J^{n_0}\widehat{R}$. Since n_0 was chosen arbitrarily, $\widehat{a}x \in \bigcap_n (N + J^n\widehat{R}) = N$. Furthermore, all ideals of \widehat{R} are closed in \widehat{R} (see [A-K] 22.36 solution).

Let M be an R -module (linearly topologized and Hausdorff, as usual). If each element of M is annihilated by a power of J , we say that M is **J -torsion**. If M is a J -torsion R -module then, M have structure of \widehat{R} -module. Indeed, let $a \in M$ with $J^t a = 0$ and

$$\widehat{r} = (r_n + J^n)_n = (r'_n + J^n)_n \in \varprojlim_n R/J^n = \widehat{R}$$

such that $r_n - r'_n \in J^n$ for all n and $r_{n+h} - r_n \in J^n$ for all n and h . Thus

$$r_{t+h} - r_t \in J^t \quad \text{and} \quad (r_{t+h} - r_t)a = 0$$

$$r'_{t+h} - r'_t \in J^t \quad \text{and} \quad (r'_{t+h} - r'_t)a = 0$$

$$r_t - r'_t \in J^t \quad \text{and} \quad (r_t - r'_t)a = 0,$$

hence $r_{t+h}a = r_t a = r'_t a = r'_{t+h}a$ for all h .

We have that

$$\begin{aligned} \widehat{R} \times M &\rightarrow M \\ (\widehat{r}, a) &\mapsto r_t a \end{aligned}$$

is well defined. Note that, if we regard this \widehat{R} -module as an R -module by means of the natural ring homomorphism

$$\begin{aligned} \varphi : R &\rightarrow \widehat{R} \\ x &\mapsto (x)_n \end{aligned}$$

then we recover the original R -module structure in M . Indeed, given $a \in M$, define $a \cdot x := a \cdot (x)_n = a \cdot x_t = a \cdot x$. Note also that a subset of M is an R -submodule if and only if it is an \widehat{R} -submodule.

The injective envelope $E(R/\mathbf{J})$ of R/\mathbf{J} will play an important role in what follows. We shall denote $E(R/\mathbf{J})$ by R^* .

Remark 3.1.1 R^* is an artinian \mathbf{J} -torsion R -module. Indeed, by Chinese Remainder Theorem, we have

$$\frac{R}{\mathbf{J}} \cong \frac{R}{\mathfrak{m}_1} \oplus \dots \oplus \frac{R}{\mathfrak{m}_n}.$$

Hence R^* is artinian (see [B–S] 10.1.4 and 10.2.5). Now, since R^* is artinian, we have that there exists $t \in \mathbb{N}$ (the set of natural numbers) such that $\mathbf{J}^t x = 0$, with $x \in R^*$. Indeed, if $x \neq 0$ then, since the R -module Rx is artinian, $R/(0 :_R x)$ is an artinian ring, and so its Jacobson radical is nilpotent.

Furthermore R^* (with the discrete topology) is a linearly topologized R -module. In fact, is an \widehat{R} -module and as such its lattice of \widehat{R} -submodules is identical with its lattice of R -submodules. From Proposition (2.2.11) the only linear topology that R^* can carry is the discrete topology satisfying (TBS 1) of Corollary (1.3.7). Since R^* is \mathbf{J} -torsion the condition (TBS 2) of Corollary (1.3.7) is satisfied and therefore R^* (with the discrete topology) is a linearly topologized R -module.

Lemma 3.1.2 *Let A be a ring and let N and M be A -modules such that $N \leq M$. Then $E(M)$ is isomorphic to submodule of $E(N) \oplus E(\frac{M}{N})$.*

Proof. Consider the exact sequence $0 \rightarrow N \rightarrow M \rightarrow \frac{M}{N} \rightarrow 0$. Let

$$0 \rightarrow N \rightarrow E'_1 = E(N) \rightarrow E'_2 \rightarrow \dots$$

and

$$0 \rightarrow \frac{M}{N} \rightarrow E''_1 = E(\frac{M}{N}) \rightarrow E''_2 \rightarrow \dots$$

be minimal injective resolutions of N and $\frac{M}{N}$. We have the following diagram:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \\
 & & E'_2 & & E''_2 & & \\
 & & \uparrow & & \uparrow & & \\
 & & E(N) & & E\left(\frac{M}{N}\right) & & \\
 & & \uparrow & & \uparrow & & \\
 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & \frac{M}{N} \longrightarrow 0 \\
 & & \uparrow & & & & \uparrow \\
 & & 0 & & & & 0
 \end{array}$$

By Horseshoe lemma (see [Rot] 6.14) there is an injective resolution of M :

$$0 \rightarrow M \rightarrow E_1 \rightarrow E_2 \rightarrow \dots$$

such that $E_i = E'_i \oplus E''_i$ and we have the following commutative diagram:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & E'_2 & \longrightarrow & E'_2 \oplus E''_2 & \longrightarrow & E''_2 \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & E(N) & \longrightarrow & E(N) \oplus E\left(\frac{M}{N}\right) & \longrightarrow & E\left(\frac{M}{N}\right) \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & \frac{M}{N} \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

Thus, $M \hookrightarrow E(N) \oplus E(\frac{M}{N})$, with $E(N) \oplus E(\frac{M}{N}) \cong E(N \oplus \frac{M}{N})$ injective module. Hence, $E(M) \hookrightarrow E(N) \oplus E(\frac{M}{N})$. ■

If M is an R -module and every submodule of M is closed we say that M is *semi-discrete*. Thus a discrete R -module is semi-discrete. The class of semi-discrete linearly compact modules contains all artinian modules.

Moreover, a finitely generated R -module M over a complete semilocal ring is a semi-discrete linearly compact module. Indeed, since $R = \varprojlim_t R/J^t$ and each R/J^t is artinian by the Chinese Remainder Theorem, R is semi-discrete linearly compact with the J -adic topology. Since M is a finitely generated R -module we have an open continuous epimorphism $R^t \rightarrow M$. Therefore M is semi-discrete and linearly compact (see Proposition (3.3.3) for another proof of this fact). Thus the class of semi-discrete linearly compact modules contains also all finitely generated modules over a complete ring.

Theorem 3.1.3 *Let M be a linearly compact semi-discrete J -torsion R -module. Then M is artinian.*

Proof. Firstly, we prove the result supposing that M is finitely generated. Let $N = (0 :_M J)$ be the annihilator of J in M . N is closed in M , hence N is linearly compact. Since $JN = 0$, N is an R/J -module and as such it is still semi-discrete, linearly compact and finitely generated. Let S be the sum of all the simple submodules of M and set $K_i = R/\mathfrak{m}_i$. Note that every simple submodule of M is isomorphic to K_i for some $i = 1, \dots, n$. Indeed, if M' is a simple submodule of M , there is $m \in M'$ such that $R/(0 :_R m) \cong M'$. Since M' is simple, $R/(0 :_R m)$ is too and has just two ideals, hence it is a field. Since R is semilocal, we have that $R/(0 :_R m) \cong K_i$ for some i . Thus $JS = 0$, so that $S \subset N$ and therefore S is a finitely generated R/J -module hence it has finite length since R/J is artinian. Let $S = S_r \supseteq S_{r-1} \supseteq \dots \supseteq S_1 \supseteq S_0 = 0$ be a composition series of S with $l(S) = r$. By Lemma (3.1.2), $E(S) \hookrightarrow E(\frac{S_r}{S_{r-1}}) \oplus \dots \oplus E(\frac{S_2}{S_1}) \oplus E(S_1)$. Since $\frac{S_j}{S_{j-1}}$ is simple for all j , we have that $\frac{R}{\mathfrak{m}_{i_j}} \cong \frac{S_j}{S_{j-1}}$ for some $i_j = 1, \dots, n$ and for each j . Thus, $E(\frac{S_j}{S_{j-1}})$ is artinian for all j , hence $E(S)$ is artinian.

We show next that M is an essential extension of S . Let $x \in M$, $x \neq 0$. Since M is J -torsion we have $J^n x = 0$ for some $n > 0$, hence $J^n \subseteq (0 :_R x)$ and Rx has finite

length since it is a finitely generated R/J^n -module and R/J^n is artinian. It follows that Rx has a simple submodule, hence $Rx \cap S \supsetneq 0$. Thus M is an essential extension of S and hence M is contained in the injective envelope $E(S)$ of S . Thus M is a submodule of an artinian module. Therefore M is artinian.

Now, let prove the result in the general case, that is when M is not necessarily finitely generated. By main Theorem in [Zös], there is a finitely generated submodule B of M such that M/B is an artinian module. Since M is semi-discrete B is closed thus, B is linearly compact, semi-discrete and J -torsion R -module. By the previous case, B is artinian.

Considering the exact sequence

$$0 \rightarrow B \rightarrow M \rightarrow M/B \rightarrow 0$$

we have that M is artinian. ■

Proposition 3.1.4 *Let M be a discrete R -module. Then M is semi-discrete and J -torsion.*

Proof. It is clear that M is semi-discrete. Let $x \in M$, $x \neq 0$. $\{0\}$ is open in M , hence there is a kernel U of R such that $Ux \subseteq \{0\}$, that is $Ux = 0$. But $U \supseteq J^n$ for some $n > 0$ because $\{J^n\}_n$ is fundamental system of neighborhoods of zero. Thus, $J^n x = 0$.

■

Every continuous homomorphism of a compact topological group is an open map. The analogous statement for linearly compact topological modules is false: for example, if A is a complete local ring, let A_1 denote A with the discrete topology, and let A_2 denote A with the topology defined by the powers of the maximal ideal; then A_1 is linearly compact and A_2 is a linearly compact A_1 -module, and the identity map $A_1 \rightarrow A_2$ is continuous but not open [Zel]. Following Leptin ([Lep], p. 246) we say that an A -module M is **strictly linearly compact** if M is linearly compact and every continuous homomorphism from M into any linearly topologized A -module is an open map.

Proposition 3.1.5 *Let M be an R -module. Then the following statements are equivalent:*

- (a) M is linearly compact;
- (b) M is an inverse limit of artinian R -modules;
- (c) M is strictly linearly compact.

Proof. (a) \Rightarrow (b). Let \mathfrak{B} be a fundamental system of neighborhoods of zero of M consisting of submodules. By Proposition (2.2.12), $M = \varprojlim_{U \in \mathfrak{B}} M/U$; M/U is discrete and linearly compact, hence artinian by Proposition (3.1.4) and Theorem (3.1.3). Thus M is an inverse limit of artinian R -modules.

(b) \Rightarrow (a). An artinian R -module is linearly compact by Proposition (2.2.2) thus, M is an inverse limit of linearly compact R -modules hence, it is linearly compact by Proposition (2.2.9).

(b) \Leftrightarrow (c) by [Mac] 4.7. ■

If M is an R -module, a **character** of M is a continuous homomorphism from M into R^* .

Proposition 3.1.6 *Let M be an R -module. Then M has enough characters, that is given any $x \neq 0$ in M there is a character u of M such that $u(x) \neq 0$.*

Proof. Since M is Hausdorff it has an open submodule U which does not contains x . Let π_U be the canonical projection from M into M/U and let \bar{x} be the image of x in M/U . We have that $\bar{x} \neq 0$, hence $(0 :_R \bar{x}) \subseteq \bigcup_{i=1}^n \mathfrak{m}_i$. Indeed, let $b \in R$ be such that $b \notin \bigcup_{i=1}^n \mathfrak{m}_i$ and $b \cdot \bar{x} = 0$. Thus $b \cdot x \in U$ and b is an unity in R and hence $x \in U$. Therefore, $(0 :_R \bar{x}) \subseteq \bigcup_{i=1}^n \mathfrak{m}_i$. By Prime Avoidance, there is \mathfrak{m}_j such that $(0 :_R \bar{x}) \subseteq \mathfrak{m}_j$. Define the homomorphism

$$\begin{aligned} \psi : \langle \bar{x} \rangle &\rightarrow \bigoplus_{i=1}^n R/\mathfrak{m}_i \\ a \cdot \bar{x} &\mapsto (0 + \mathfrak{m}_1, \dots, a + \mathfrak{m}_j, \dots, 0 + \mathfrak{m}_n). \end{aligned}$$

Note that ψ is well defined. Indeed,

$$\begin{aligned} b \cdot \bar{x} = a \cdot \bar{x} &\Rightarrow (a - b) \cdot \bar{x} = 0 \Rightarrow a - b \in (0 :_R \bar{x}) \Rightarrow a - b \in \mathfrak{m}_j \\ &\Rightarrow a + \mathfrak{m}_j = b + \mathfrak{m}_j. \end{aligned}$$

Now, since $\bigoplus_{i=1}^n R/\mathfrak{m}_i \cong R/\mathfrak{J}$ and denoting $i : R/\mathfrak{J} \hookrightarrow E(R/\mathfrak{J}) = R^*$, we have the commutative diagram

$$\begin{array}{ccc} & R^* & \\ & \uparrow & \swarrow \bar{u} \\ 0 & \longrightarrow \langle \bar{x} \rangle & \xrightarrow{j} M/U \end{array}$$

where j is the inclusion and there is \bar{u} since R^* is injective. Note that \bar{u} is continuous since M/U is discrete. Set $u = \bar{u} \circ \pi_U$. We have $u \in \text{Hom}_R(M, R^*)$ and

$$u(x) = \bar{u} \circ \pi_U(x) = \bar{u}(\bar{x}) = \bar{u}(j(\bar{x})) = i(\psi(\bar{x})).$$

Since $\psi(\bar{x}) \neq 0$, $u(x) \neq 0$. ■

Proposition 3.1.7 *Let M be an R -module and $u : M \rightarrow R^*$ be an homomorphism. Then the following statements are equivalent:*

- (a) u is a continuous;
- (b) $\ker u$ is open;
- (c) $\ker u$ is closed.

Proof. (a) \Leftrightarrow (b): since R^* is discrete and u is continuous in zero.

(b) \Rightarrow (c): since an open submodule is closed.

(c) \Rightarrow (b): u induces an (algebraic) isomorphism of $\frac{M}{\ker u}$ onto a submodule of R^* . Since $\ker u$ is closed, $\frac{M}{\ker u}$ is a Hausdorff linearly compact R -module and artinian, hence discrete by Proposition (2.2.11). Therefore $\ker u$ is open. ■

Proposition 3.1.8 *Let M be an R -module. Then, M is semi-discrete if and only if every homomorphism from M into R^* is continuous.*

Proof. If M is semi-discrete and $u : M \rightarrow R^*$ is a homomorphism then $\ker u$ is closed, hence u is continuous by Proposition (3.1.7). Conversely, let N be a submodule of M with $N \neq M$. Let $x \in M$, $x \notin N$ and let \bar{x} be the image of x in M/N . We have that $\bar{x} \neq 0$, hence by Proposition (3.1.6) there is a homomorphism $\bar{u}_x : M/N \rightarrow R^*$ such that $\bar{u}_x \neq 0$. Hence we have a character u_x of M such that $u_x(x) \neq 0$ and $u_x(N) = 0$. Consequently $N = \bigcap_{x \notin N} \ker u_x$ and hence N is closed. ■

Let \mathcal{C} be the category of linearly compact R -modules and continuous homomorphisms. By [Mac] 4.8 and Proposition (3.1.5), \mathcal{C} is an abelian category satisfying Grothendieck's axiom (AB5*) (see [Gro2]), thus Proposition (3.1.6) shows that R^* is a cogenerator of \mathcal{C} . Also R^* is an injective object of \mathcal{C} . More generally we have:

Proposition 3.1.9 *Let M be an R -module and be N a closed submodule of M . Then every character of N can be extended to a character of M .*

Proof. Let u be a character of N . Then M has an open submodule U such that $U \cap N \subseteq \text{Ker}(u)$ and $\text{Ker}(u)$ is open by Proposition (3.1.7). Also u induces $\bar{u} : N/N \cap U \rightarrow R^*$ given by $\bar{u}(\bar{x}) = u(x)$. Since $N/N \cap U \cong (N + U)/U$ and R^* is injective, we have the following diagram:

$$\begin{array}{ccccc} & & R^* & & \\ & & \uparrow & \swarrow \bar{v} & \\ & & \bar{u} & & \\ (N + U)/U & \xrightarrow{i} & M/U & \xleftarrow{\pi} & M \end{array}$$

and \bar{u} can be extended to a homomorphism \bar{v} which is continuous since M/U is discrete. Thus \bar{v} gives rise to a character $v : M \rightarrow R^*$ given by $v = \bar{v} \circ \pi$ and v extends u . ■

As a consequence of the above result and by [Gro2] 1.9.1 we have the following.

Proposition 3.1.10 *Every linearly compact R -module is topologically isomorphic to a closed submodule of a direct product of copies of R^* .*

Since the theory of essential extensions and injective envelopes is valid in any abelian category which satisfies (AB5) of [Gro2] and has a generator (see [Gab]) it follows that the dual of this theory is valid in \mathcal{C} . The essential features are as follows:

Let M be a linearly compact R -module. A **covering** of M is a continuous epimorphism $f : Q \rightarrow M$ where Q is a linearly compact R -module. f is an **essential covering** of M if for every proper closed submodule N of Q we have $N + \text{ker } f \subsetneq Q$.

Proposition 3.1.11

(i) M has a maximal essential covering $\varepsilon : P \rightarrow M$, that is if $\rho : Q \rightarrow M$ is any essential covering of M then there is a continuous homomorphism $\varphi : P \rightarrow Q$ such that $\varepsilon = \rho \circ \varphi$.

(ii) P is unique up to isomorphism, that is if $\varepsilon_0 : P_0 \rightarrow M$ is another maximal essential covering of M then we have a topological isomorphism $\alpha : P \rightarrow P_0$ such that $\varepsilon = \varepsilon_0 \circ \alpha$.

(iii) P is a projective object of \mathcal{C} and $\varepsilon : P \rightarrow M$ is the smallest projective covering of M , that is if $\rho : Q \rightarrow M$ is a continuous epimorphism and Q is a projective object of \mathcal{C} then there is a continuous homomorphism $\psi : Q \rightarrow P$ such that $\rho = \varepsilon \circ \psi$.

The unique P satisfying the above properties is called the **projective envelope** of M .

Proposition 3.1.12 *Let P be a linearly compact R -module. Then P is a projective object of \mathcal{C} if and only if for every quotient module M of R^* and every continuous*

homomorphism $P \rightarrow M$ there is a continuous homomorphism $P \rightarrow R^*$ such that the diagram

$$\begin{array}{ccccc} R^* & \longrightarrow & M & \longrightarrow & 0 \\ & \swarrow & \nearrow & & \\ & P & & & \end{array}$$

is commutative. (This is the dual of [Gro2] p. 136, Lemma 1 applied to \mathcal{C}).

Proposition 3.1.13 *The projective envelope of R/\mathbf{J} in \mathcal{C} is \widehat{R} .*

Proof. Note that \widehat{R} is linearly compact for it is the inverse limit of the artinian modules R/\mathbf{J}^n .

The natural homomorphism $\widehat{R} \rightarrow R/\mathbf{J}$ is an essential covering of R/\mathbf{J} . For suppose N is a closed R -submodule of \widehat{R} and $N + \widehat{\mathbf{J}} = \widehat{R}$. Since N is closed it is an ideal of \widehat{R} ; since $(\widehat{\cap_{i=1}^n \mathbf{m}_i}) = \cap_{i=1}^n \widehat{\mathbf{m}_i}$ by [A-K] 22.53 solution, we must have $N = \widehat{R}$.

We claim that \widehat{R} is a projective object of \mathcal{C} . If M is a quotient module of R^* , M is an \widehat{R} -module by Remark (3.1.1). Let $f : \widehat{R} \rightarrow M$ be any continuous R -homomorphism. Then f is an \widehat{R} -homomorphism, for if $f(1) = \overline{m} \in M$ and $\widehat{x} \in \widehat{R}$, let (x_n) be a Cauchy sequence in R whose limit is \widehat{x} , then $f(x_n) = x_n \overline{m}$ for all n , and $\widehat{x} \overline{m} = x_n \overline{m}$ for all sufficiently large n . Hence $f(\widehat{x}) = \lim f(x_n) = \widehat{x} \overline{m}$.

The diagram

$$\begin{array}{ccccc} R^* & \longrightarrow & M & \longrightarrow & 0 \\ & & \nearrow f & & \\ & & \widehat{R} & & \end{array}$$

is therefore a diagram of \widehat{R} -modules and \widehat{R} -homomorphisms. Therefore f can be lifted to an \widehat{R} -homomorphism $g : \widehat{R} \rightarrow R^*$. Since $\widehat{R}/\ker g$ is artinian, $\ker g \supseteq \widehat{\mathbf{J}}^n$ for some n , thus $\ker g$ is open and therefore g is continuous. Hence by Proposition (3.1.12) \widehat{R} is a projective object of \mathcal{C} . The result follows from Proposition (3.1.11). ■

3.2 Linearly Discrete Modules Over a Semilocal Ring

In this section $(R, \mathbf{m}_1, \dots, \mathbf{m}_n)$ have the same meanings as in the previous section. An R -module M is **linearly discrete** if every \mathbf{J} -torsion quotient of M is discrete.

Proposition 3.2.1 *Let M be an R -module. Then, M is discrete if and only if M is linearly discrete and \mathbf{J} -torsion.*

Proof. If M is discrete then M is linearly discrete and J-torsion by Proposition (3.1.4). The converse is obvious. ■

Proposition 3.2.2 *If M is a linearly discrete R -module, then every homomorphism $M \rightarrow R^*$ is continuous (and hence M is semi-discrete by Proposition (3.1.8)).*

Proof. For if $u : M \rightarrow R^*$ is a homomorphism, then $M/\ker u$ is (algebraically) isomorphic to a submodule of R^* , hence is J-torsion by Remark (3.1.1). Consequently $M/\ker u$ is discrete and therefore $\ker u$ is open. Hence u is continuous by Proposition (3.1.7). ■

Proposition 3.2.3 *If M is a linearly discrete R -module and N is a submodule of M , then M/N is a linearly discrete R -module.*

Proof. By Proposition (3.2.2) M is semi-discrete, hence N is closed in M and therefore M/N is Hausdorff, that is an R -module. Set $\overline{M} = M/N$ and $\overline{K} = K/N$ with K submodule of M . The result follows since $\overline{M}/\overline{K}$ is topologically isomorphic to M/K . ■

Proposition 3.2.4 *R is linearly discrete as R -module.*

Proof. If R/\mathfrak{a} is J-torsion where \mathfrak{a} is an ideal of R , then $J^n \subseteq \mathfrak{a}$ for some $n \in \mathbb{N}$. Hence \mathfrak{a} is an open ideal and therefore R/\mathfrak{a} is discrete. ■

Proposition 3.2.5 *If M, N are R -modules, M is linearly discrete and $f : M \rightarrow N$ is an open continuous homomorphism, then $f(M)$ is linearly discrete.*

Proof. For $f(M)$ is topologically isomorphic to $M/\ker(f)$, which is linearly discrete by Proposition (3.2.3). ■

Proposition 3.2.6 *If $\{M_i\}_{i \in I}$ is a family of linearly discrete R -modules, then $\bigoplus_{i \in I} M_i$ is linearly discrete.*

Proof. Let N be a submodule of $M = \bigoplus M_i$ such that $\frac{M}{N}$ is J-torsion. Then $\frac{M_i}{M_i \cap N} \cong \frac{M_i + N}{N}$ is J-torsion too. Hence $M_i \cap N$ is open in M_i , and therefore $\bigoplus (M_i \cap N)$ is an open submodule of M . Since $N \supseteq \bigoplus (M_i \cap N)$, N is open in M . Thus $\frac{M}{N}$ is discrete. ■

Proposition 3.2.7 *The direct limit of a direct system of linearly discrete R -modules and continuous homomorphisms is linearly discrete.*

Proof. This follows from Propositions (3.2.6) and (3.2.3). ■

Proposition 3.2.8 *Let M and N be R -modules and let $f : M \rightarrow N$ be an homomorphism.*

(i) *If M is linearly discrete, f is continuous.*

(ii) *If N is linearly discrete and f is a epimorphism then f is open.*

Proof. Suppose M is linearly discrete. Let V be an open submodule of N and set $U = f^{-1}(V)$. Thus f induces an (algebraic) monomorphism $M/U \rightarrow N/V$. Since N/V is discrete it is J -torsion by Proposition (3.1.4). Hence M/U is J -torsion and therefore it is discrete by Proposition (3.2.1), so that U is open and consequently f is continuous. Thus proves (i). Suppose on the other hand that N is linearly discrete. Let U be an open submodule of M , $V = f(U)$. Then f induces an epimorphism $M/U \rightarrow N/V$. Since M/U is discrete it is J -torsion by Proposition (3.1.4). Consequently N/V is J -torsion and therefore it is discrete. Thus V is open and therefore f is an open map, which proves (ii). ■

3.3 J-Separated Modules Over a Complete Semilocal Ring

Henceforth $(R, \mathfrak{m}_1, \dots, \mathfrak{m}_n)$ shall be a *complete*, with the J -adic topology, (noetherian semilocal, as usual) ring, where $J = \bigcap_{i=1}^n \mathfrak{m}_i$. In this setting, for $i \neq j$, \mathfrak{m}_i^r and \mathfrak{m}_j^r are coprime for all $i, j = 1, \dots, n$ and $r > 0$ (see [A-M] 1.16). Thus, by the Chinese Remainder Theorem we have that

$$\frac{R}{J^r} \cong \frac{R}{\mathfrak{m}_1^r} \oplus \dots \oplus \frac{R}{\mathfrak{m}_n^r}.$$

Hence $\frac{R}{J^r}$ is artinian for all $r > 0$.

Since R is complete it is a linearly compact R -module by Proposition (3.1.5), because $R \cong \varprojlim_n \frac{R}{J^n}$, it is an inverse limit of artinian modules. If M is an R -module, which is **Hausdorff** in the J -adic topology, that is, $\bigcap_n J^n M = 0$, we say that M is **J-separated**.

Proposition 3.3.1 *Let M, N be R -modules, M be linearly compact and J -separated and $f : M \rightarrow N$ be a continuous homomorphism. Then $f(M)$ is linearly compact and J -separated.*

Proof. $f(M)$ is linearly compact by Proposition (2.2.6). Each $J^n M$ is a closed submodule of M by [Mac] 3.14, and $J^n f(M) = f(J^n M)$. Hence we have that by [Mac] 3.12

$$\bigcap_n J^n f(M) = \bigcap_n f(J^n M) = f\left(\bigcap_n J^n M\right) = f(0) = 0.$$

■

Proposition 3.3.2 *If M is artinian and J -separated then M is of finite length.*

Proof. Since M is artinian and $\bigcap_n J^n M = 0$, we must have $J^n M = 0$ for some $n \geq 0$. Let $D(M) = {}^A Hom_R(M, R^*)$ be the R -module of all homomorphisms (not necessarily continuous) from M into R^* . Since M is artinian, $D(M)$ is noetherian by Theorem (B1) (iii) and $J^n D(M) = 0$. Suppose $D(M)$ is generated by y_1, \dots, y_r . Then each Ry_i is annihilated by J^n , hence is of finite length hence $D(M) = \sum_{i=1}^r Ry_i$ is of finite length. Hence by Theorem (B1) (ii),(iii) and (v) M is of finite length. ■

Proposition 3.3.3 *Let M be an R -module. Then the following statements are equivalent:*

- (a) M is noetherian;
- (b) M is linearly compact, linearly discrete and J -separated;
- (c) M is linearly compact, semi-discrete and J -separated.

Proof. (a) \Rightarrow (b): Remember that R is linearly compact since it is complete in the J -adic topology. Also by Proposition (3.2.4) R is linearly discrete. Hence for each $n > 0$, R^n is linearly discrete and linearly compact by Proposition (3.2.6) and Proposition (2.2.8). If M is a noetherian R -module, we have a continuous epimorphism $R^n \rightarrow M$. This epimorphism is open because R^n is strictly linearly compact by Proposition (3.1.5). Hence M is linearly compact by Proposition (2.2.6) and linearly discrete by (3.2.5). Furthermore, by Krull's Intersection Theorem (A.5), $\bigcap_n J^n M = 0$.

(b) \Rightarrow (c): Follows by Proposition (3.2.2).

(c) \Rightarrow (a): Since M is linearly compact and semi-discrete there is N a finitely generated submodule of M such that M/N is artinian module, by [Zös] Theorem. Since R is noetherian we have that N is noetherian. By Proposition (3.3.1) and Proposition (3.3.2) M/N is of finite length. Thus, M/N is noetherian and considering the exact sequence

$$0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$$

follows that M is noetherian. ■

Proposition 3.3.4 *If M is noetherian or artinian then every homomorphism $M \rightarrow R^*$ is continuous.*

Proof. If M is noetherian this follows by Propositions (3.3.3) and (3.1.8). If M is artinian, this follows by Propositions (2.2.11) and (3.1.8). ■

Proposition 3.3.5 *Let M be an R -module. Then, M is linearly discrete if and only if M is a direct limit of noetherian R -modules.*

Proof. Suppose M is linearly discrete and let N_i be the family of noetherian submodules of M . We have an (algebraic) isomorphism $\epsilon : \varinjlim N_i \rightarrow M$. Each N_i is linearly discrete by Proposition (3.3.3) and hence $\varinjlim N_i$ is linearly discrete by Proposition (3.2.7). It follows that ϵ is continuous and open by Proposition (3.2.8) and therefore M is (topologically) a direct limit of noetherian R -modules.

Conversely, a noetherian R -module is linearly discrete by Proposition (3.3.3) and hence M is linearly discrete by Proposition (3.2.7). ■

Proposition 3.3.6 *Let M and N be linearly discrete R -modules. Then $M \otimes_R N$ is linearly discrete.*

Proof. By Proposition (3.3.5) we can write $M = \varinjlim M_i$ and $N = \varinjlim N_j$ where the M_i and N_j are noetherian. Hence $M \otimes N = (\varinjlim M_i) \otimes (\varinjlim N_j)$ is topologically isomorphic to $\varinjlim_{i,j} (M_i \otimes N_j)$ by Proposition (2.1.9) and each $M_i \otimes N_j$ is noetherian. Thus $M \otimes N$ is a direct limit of noetherian R -modules and therefore it is linearly discrete by Proposition (3.3.5). ■

Proposition 3.3.7 *Let M be an R -module. Let N and N' be linearly compact submodules of M . Then*

$$\bigcap_n \mathfrak{J}^n N + \bigcap_n \mathfrak{J}^n N' = \bigcap_n \mathfrak{J}^n (N + N').$$

Proof. Let r be a positive integer, then for all $n \geq r$ we have $\mathfrak{J}^n N' \subseteq \mathfrak{J}^r N'$. Thus

$$\mathfrak{J}^n (N + N') \subseteq \mathfrak{J}^n N + \mathfrak{J}^n N' \subseteq \mathfrak{J}^n N + \mathfrak{J}^r N'.$$

Hence,

$$\bigcap_n \mathfrak{J}^n (N + N') = \bigcap_{n \geq r} \mathfrak{J}^n (N + N') \subseteq \bigcap_{n \geq r} (\mathfrak{J}^n N + \mathfrak{J}^r N') \quad (i);$$

each $J^n N$ and $J^n N'$ is a closed submodule of $N + N'$ by [Mac] 3.14, and $N + N'$ is linearly compact by Proposition (2.2.10). Hence by [Mac] 3.13 we have that

$$\bigcap_{n \geq r} (J^n N + J^n N') = J^r N' + \bigcap_{n \geq r} J^n N = J^r N' + \bigcap_n J^n N \quad (ii)$$

for all $n \geq r$. Thus, by equations (i) and (ii), we have that

$$\bigcap_n J^n (N + N') \subseteq J^r N' + \bigcap_n J^n N$$

for all $n \geq r$. Hence

$$\bigcap_n J^n (N + N') \subseteq \bigcap_{n \geq r} (J^r N' + \bigcap_n J^n N).$$

Since n grows unlimited, r too and the last inclusion is true for all $r > 0$. Therefore

$$\bigcap_n J^n (N + N') \subseteq \bigcap_r (J^r N' + \bigcap_n J^n N) = \bigcap_r (J^r N' + \bigcap_n J^n N)$$

by [Mac] 3.13 again.

On the other hand, if $x \in \bigcap_n J^n N + \bigcap_n J^n N'$ then $x = x_1 + x_2$, where $x_1 \in \bigcap_n J^n N$ and $x_2 \in \bigcap_n J^n N'$. Thus $x_1 \in \bigcap_n J^n (N + N')$, because $J^n N \subseteq J^n (N + N')$ for all n . Analogously $x_2 \in \bigcap_n J^n (N + N')$. Since $\bigcap_n J^n (N + N')$ is a module, $x = x_1 + x_2 \in \bigcap_n J^n (N + N')$. ■

Proposition 3.3.8 *Let M be an R -module and let N and N' be linearly compact J -separated submodules of M . Then $N + N'$ and $N \cap N'$ are linearly compact and J -separated.*

Proof. For $N + N'$, this follows from Propositions (3.3.7) and (2.2.10). Since $N \cap N'$, since N and N' are linearly compact they are closed in M , hence $N \cap N'$ is closed in N and therefore it is linearly compact and it is naturally J -separated because

$$\bigcap_n J^n (N \cap N') \subseteq \bigcap_n (J^n N \cap J^n N') = \left(\bigcap_n J^n N \right) \cap \left(\bigcap_n J^n N' \right) = 0.$$

■

3.4 $Hom_R(M, N)$

Let $(R, \mathfrak{m}_1, \dots, \mathfrak{m}_n)$ be a J -adically complete ring and let M and N be R -modules. Let $\mathbf{Hom}_R(M, N)$ be the set of all *continuous* homomorphisms from M into N and note that $Hom_R(M, N)$ can be provided with a natural R -module structure.

Remark 3.4.1 We provide next $\text{Hom}_R(M, N)$ with a topological structure. Let \mathcal{M} denote the set of all linearly compact J-separated submodules of M and let \mathcal{N} be the set of all open submodules of N . If $C \in \mathcal{M}$ and $V \in \mathcal{N}$, let $\nu(C, V)$ denote the set of all $u \in \text{Hom}_R(M, N)$ such that $u(C) \subseteq V$. We have that $\nu(C, V)$ is a submodule of $\text{Hom}_R(M, N)$ and we will show that the set of all $\nu(C, V)$ as C runs through \mathcal{M} and V runs through \mathcal{N} forms a fundamental system of neighborhoods of zero of a linear topology on $\text{Hom}_R(M, N)$. We have to verify (TBS 1) and (TBS 2) of Corollary (1.3.7). Firstly if $C, C' \in \mathcal{M}$ and $V, V' \in \mathcal{N}$, then $\nu(C + C', V \cap V') \subseteq \nu(C, V) \cap \nu(C', V')$ and by (3.3.8) $C + C' \in \mathcal{M}$ and $V \cap V' \in \mathcal{N}$ and hence (TBS 1) is satisfied. Next, let $u \in \text{Hom}_R(M, N)$ and $\nu(C, V)$ as above. Let $W = C \cap u^{-1}(V)$. Note that $\frac{C}{W}$ is linearly compact and J-separated by Proposition (3.3.1) and it is discrete, because W is open, hence $\frac{C}{W}$ is artinian by Proposition (3.1.4) and Theorem (3.1.3). Consequently we have that $J^n(\frac{C}{W}) = 0$ for some $n > 0$, since $\frac{C}{W}$ is J-separated. Thus

$$J^n C + W = W \Rightarrow J^n C \subseteq u^{-1}(V) \Rightarrow J^n u(C) \subseteq V \Rightarrow J^n u \subseteq \nu(C, V).$$

Hence (TBS 2) is satisfied. Finally, $\text{Hom}_R(M, N)$ is Hausdorff. Indeed, if $C \in \mathcal{M}$ and $u \in \bigcap_{V \in \mathcal{N}} \nu(C, V)$, then $u(C) \subseteq V$ for all $V \in \mathcal{N}$, since N is Hausdorff, $0 = \overline{\{0\}} = \bigcap_{V \in \mathcal{N}} V$ that is, $u(C) = 0$. Hence if $u \in \bigcap_{C \in \mathcal{M}} \bigcap_{V \in \mathcal{N}} \nu(C, V)$ we have that $u(C) = 0$ for every $C \in \mathcal{M}$. Now if $x \in M$ then $Rx \in \mathcal{M}$ by Proposition (3.3.3), hence $u(x) = 0$ for all $x \in M$ and thus $u = 0$. Consequently $\bigcap_{C \in \mathcal{M}} \bigcap_{V \in \mathcal{N}} \nu(C, V) = 0$, so that $\text{Hom}_R(M, N)$ is Hausdorff.

Proposition 3.4.2 *If M, M' and N are R -modules and $u : M \rightarrow M'$ and $v : N \rightarrow N'$ are continuous homomorphisms, then the induced map $\text{Hom}_R(M', N) \rightarrow \text{Hom}_R(M, N')$ is a continuous homomorphism.*

Proof. Let $\varphi : \text{Hom}_R(M', N) \rightarrow \text{Hom}_R(M, N')$ be given by $f \mapsto v \circ f \circ u$. It is enough to show that $u^* : \text{Hom}_R(M', N) \rightarrow \text{Hom}_R(M, N)$ given by $f \mapsto f \circ u$ and $v_* : \text{Hom}_R(M, N) \rightarrow \text{Hom}_R(M, N')$ given by $g \mapsto v \circ g$ are continuous, since $\varphi = v_* \circ u^*$. Let $C \in \mathcal{M}$ and $V \in \mathcal{N}$. It is easy to verify that $u^{*-1}(\nu(C, V)) = \nu(u(C), V)$ and that $u(C)$ is linearly compact and J-separated by Proposition (3.3.1), hence u^* is continuous.

Let $V' \in \mathcal{N}'$ be an open submodule of N' . We find that $v_*^{-1}(\nu(C, V')) = \nu(C, v^{-1}(V'))$ from which follows that v_* is continuous. ■

Proposition 3.4.3 *Let M, N be R -modules and let N' be a submodule of N . Then the injection $i : N' \rightarrow N$ induces a topological isomorphism i_* of $\text{Hom}_R(M, N')$ onto a submodule of $\text{Hom}_R(M, N)$.*

Proof. Note that i_* is continuous by Proposition (3.4.2) and it is a monomorphism. We have to verify that i_* is open. Let $C \in \mathcal{M}$ and $V' \in \mathcal{N}$. Then there is an open submodule V of N such that $V' \supseteq V \cap N'$ and we have $i_*(\nu(C, V')) \supseteq \nu(C, V) \cap i_*(\text{Hom}_R(M, N'))$. Indeed, if $u \in \nu(C, V) \cap i_*(\text{Hom}_R(M, N'))$ then $u(C) \subseteq V$ and $u(M) \subseteq N'$, so that $u(C) \subseteq V \cap N' \subseteq V'$ and thus $u \in \nu(C, V')$. Hence the image of a kernel of $\text{Hom}_R(M, N')$ is open in the image of $\text{Hom}_R(M, N')$ (see [War] 5.18), so that i_* is an open map. ■

Proposition 3.4.4 *Let $\{N_i\}$ be a family of R -modules and M be an R -module. Then the isomorphism*

$$\alpha : \prod_{i \in I} \text{Hom}_R(M, N_i) \rightarrow \text{Hom}_R(M, \prod_{i \in I} N_i)$$

of Remark (2.1.3) is a topological isomorphism.

Proof. A kernel of $\prod_{i \in I} \text{Hom}_R(M, N_i)$ is a product $W = \prod \nu(C_i, V_i)$ where each C_i is a linearly compact J -separated submodule of M and almost all C_i are zero. Each V_i is an open submodule of N_i and $V_i = N_i$ for almost all i . Set $C = \sum C_i$. Since this is a finite sum, C is linearly compact and J -separated by Proposition (3.3.8) and we have that $\alpha(W) \supseteq \nu(C, \prod V_i)$. Indeed, let $f \in \nu(C, \prod V_i)$ and $(f_i)_{i \in I} \in W$ given by $f_i = \pi_i \circ f$. We have that $\alpha((f_i)_{i \in I}) = f$. Consequently α is open. On the other hand if C is a linearly compact J -separated submodule of M then a kernel of $\text{Hom}_R(M, \prod N_i)$ is of the form $W' = \nu(C, \prod V_i)$, where the V_i are as before. We have that $\alpha^{-1}(W') = \prod \nu(C, V_i)$. Indeed, note that

$$(f_i)_{i \in I} \in \prod \text{Hom}(M, N_i) \text{ if and only if } \alpha((f_i)_{i \in I}) = f$$

where $f_i = \pi_i \circ f$ and

$$f \in \nu(C, \prod v_i) \text{ if and only if } f_i : C \rightarrow V_i$$

for all i . Which shows that α is continuous. ■

Proposition 3.4.5 *Let $\{N_i\}$ be an inverse system of R -modules and continuous homomorphisms and let M be an R -module. Then the isomorphism*

$$\beta : \varprojlim_{i \in I} \text{Hom}_R(M, N_i) \rightarrow \text{Hom}_R(M, \varprojlim_{i \in I} N_i)$$

of Remark (2.1.4) is a topological isomorphism.

Proof. We have a comutative diagram

$$\begin{array}{ccc}
\varprojlim_{i \in I} \text{Hom}_R(M, N_i) & \xrightarrow{\beta} & \text{Hom}_R(M, \varprojlim_{i \in I} N_i) \\
\downarrow i & & \downarrow j_* \\
\prod_{i \in I} \text{Hom}_R(M, N_i) & \xrightarrow{\alpha} & \text{Hom}_R(M, \prod_{i \in I} N_i)
\end{array}$$

where i is the natural embedding and j_* is induced by the embedding $\varprojlim_{i \in I} N_i \rightarrow \prod_{i \in I} N_i$. By Proposition (3.4.3) j_* is a topological isomorphism into $\text{Hom}_R(M, \prod_{i \in I} N_i)$ and α is a topological isomorphism by Proposition (3.4.4). Hence β is a topological isomorphism. ■

Proposition 3.4.6 *If M is semi-discrete then the topology of $\text{Hom}_R(M, N)$ coincides with that induced on it as a submodule of N^M (where $N^M = \prod_{x \in M} N_x$, each N_x being a copy of N).*

Proof. Firstly we will describe the structure of $\text{Hom}_R(M, N)$ as a submodule of N^M .

Let ϕ be defined by

$$\begin{aligned}
\phi : \text{Hom}_R(M, N) &\rightarrow N^M \\
\varphi &\mapsto (\varphi(x))_{x \in M}
\end{aligned}$$

and note that $\text{Ker}(\phi) = 0$.

Now, since M is semi-discrete every submodule of M is semi-discrete. Therefore if $C \in \mathcal{M}$ it is noetherian by Proposition (3.3.3) and hence it is finitely generated, say $C = \langle x_1, \dots, x_r \rangle$. Note that a sub-system of neighborhoods of zero of $\text{Hom}_R(M, N)$ consists of all $\nu(x, V)$ where $x \in M$ and $V \in \mathcal{N}$ since $\bigcap_{i=1}^r \nu(x_i, V) = \nu(C, V)$. Considering the product topology in N^M , an open in the induced topology in $\text{Hom}_R(M, N)$ has the form $\prod_{x \in M} V_x \cap \text{Hom}_R(M, N)$ such that V_x is open in N_x for each x and $V_x \neq N_x$ just for a finite number. Set $\Lambda = \{x \in M; V_x \neq N_x\}$. Now, identifying $\text{Hom}_R(M, N)$ with $\phi(\text{Hom}_R(M, N))$, we have that

$$\begin{aligned}
\prod_{x \in M} V_x \cap \text{Hom}_R(M, N) &= \{\varphi \in \text{Hom}_R(M, N); (\varphi(x))_{x \in M} \in \prod_{x \in M} V_x\} \\
&= \bigcap_{x \in M} \nu(x, V_x) \\
&= \bigcap_{x \in \Lambda} \nu(x, V_x),
\end{aligned}$$

since $\nu(x, N_x) = \{\varphi \in \text{Hom}_R(M, N); \varphi(x) \in N\} = \text{Hom}_R(M, N)$. ■

Proposition 3.4.7 *Let $\{M_i\}$ be a family of linearly discrete R -modules and let N be an R -module. Then the isomorphism*

$$\gamma : \prod_{i \in I} \text{Hom}_R(M_i, N) \rightarrow \text{Hom}_R\left(\bigoplus_{i \in I} M_i, N\right)$$

of Remark (2.1.5) is a topological isomorphism.

Proof. Since each M_i is linearly discrete, $\bigoplus M_i$ is linearly discrete by Proposition (3.2.6). Hence, if C is a linearly compact J -separated submodule of $\bigoplus M_i$, C is semi-discrete since $\bigoplus M_i$ is also semi-discrete. By Proposition (3.3.3) C is noetherian and hence C has a finite base. The homogeneous components of such a basis of C generate a direct sum $\bigoplus C_i$, where each C_i is a noetherian submodule M_i and $C_i = 0$ for almost all i . Furthermore, $C \subseteq \bigoplus C_i$ since the generators of C are contained. If V is an open submodule of N then $\gamma^{-1}(\nu(C, V)) \supseteq \prod \nu(C_i, V)$. This shows that γ is continuous. Conversely a kernel of $\prod \text{Hom}_R(M_i, N)$ is of the form $W = \prod \nu(C_i, V_i)$ where for each $i \in I$, C_i is a noetherian submodule of M_i and V_i is an open submodule of N and for almost all i we have that $C_i = 0$ and $V_i = N$. We have that $\gamma(W) \supseteq \nu(\bigoplus C_i, \bigcap V_i)$. Furthermore, $\bigoplus C_i$ is noetherian and hence linearly compact and J -separated by Proposition (3.3.3), and $\bigcap V_i$ is open since $V_i = N$ for almost all $i \in I$. Hence $\gamma(W)$ contains a kernel of $\text{Hom}(\bigoplus M_i, N)$ and therefore γ is an open map. ■

Proposition 3.4.8 *Let M, N be R -modules and let M'' be a semi-discrete quotient of M . The natural monomorphism ε^* from $\text{Hom}_R(M'', N)$ into $\text{Hom}_R(M, N)$, induced by the canonical epimorphism $\varepsilon : M \rightarrow M''$, is a topological monomorphism.*

Proof. ε^* is continuous by Proposition (3.4.2) and it is a monomorphism. We have to show that ε^* is open. Let C'' be a linearly compact J -separated submodule of M'' . Then C'' is semi-discrete (since it is a submodule of M'') and hence it is noetherian by Proposition (3.3.3). Take a finite basis of C'' and choose counter-images in M of the basis elements. These generate a noetherian submodule C of M , which by Proposition (3.3.3) is linearly compact and J -separated, and $\varepsilon(C) = C''$. Let V be an open submodule of N . We have that

$$\varepsilon^*(\nu(C'', V)) \supseteq \nu(CV) \cap \text{Im}(\varepsilon^*).$$

Hence $\varepsilon^*(\nu(C'', V))$ contains a kernel of $\text{Im}(\varepsilon^*)$ and therefore ε^* is an open map. ■

Proposition 3.4.9 *Let $\{M_i\}$ be a direct system of linearly discrete R -modules and continuous homomorphism and let N be an R -module. Then the isomorphism*

$$\delta : \varprojlim Hom_R(M_i, N) \rightarrow Hom_R(\varinjlim M_i, N)$$

of Remark (2.1.6) is a topological isomorphism.

Proof. This follows from Propositions (3.4.7) and (3.4.8) in the same way that the Proposition (3.4.5) follows from Propositions (3.4.3) and (3.4.4), we have a commutative diagram

$$\begin{array}{ccc} \prod Hom_R(M_i, N) & \xrightarrow{\gamma} & Hom_R(\oplus M_i, N) \\ \uparrow i & & \uparrow \varepsilon^* \\ \varprojlim Hom_R(M_i, N) & \xrightarrow{\delta} & Hom_R(\varinjlim M_i, N). \end{array}$$

■

Proposition 3.4.10 *If M is a linearly discrete R -module and N is a linearly compact R -module then $Hom_R(M, N)$ is linearly compact.*

Proof. By Proposition (3.4.6) we can identify $Hom_R(M, N)$ with a submodule Q of N^M . Since N is linearly compact, so is N^M by Proposition (2.2.8), hence it is enough to show that Q is closed in N^M . Let $v = (v_x)_{x \in M}$ be an element of N^M , $v \notin Q$. Then v is not an homomorphism for by Proposition (3.2.8) every homomorphism of M into N is continuous and therefore lies in Q . From this fact we can construct a neighborhood of $v \in N^M$ which does not meet Q , since N is Hausdorff. It follows that Q is closed in N^M and hence linearly compact. ■

3.5 Duality

Let $(R, \mathfrak{m}_1, \dots, \mathfrak{m}_n)$ be a J -adically complete ring. If M is any (linearly topologized, Hausdorff) R -module, then the **Macdonald Dual** of M is $M^* = Hom_R(M, R^*)$, the module of all continuous homomorphisms of M into R^* , the **character-module** of M , topologized as in Remark (3.4.1).

If S is any subset of M , S^0 denotes the annihilator of S in M^* , that is, the set of all $u \in M^*$ such that $u(S) = 0$. If T is any subset of M^* , 0T denotes the annihilator

of T in M , that is, the set of all $x \in M$ such that $u(x) = 0$ for all $u \in T$. Note that S^0 is a submodule of M^* and 0T is a submodule of M .

Since R^* is discrete, a fundamental system of neighborhoods of zero of M^* can be formed by the submodules C^0 where C runs through the set of linearly compact J-separated submodules of M .

Proposition 3.5.1 *S^0 is a closed submodule of M^* and 0T is a closed submodule of M .*

Proof. Let $x \in S$. Since $\langle x \rangle$ is noetherian it is linearly compact and J-separated by Proposition (3.3.3). Hence $\langle x \rangle^0$ is open and therefore closed in M^* . Notice that $S^0 = \bigcap_{x \in S} (\langle x \rangle)^0$ and hence it is closed in M^* . Analogously, ${}^0T = \bigcap_{u \in T} \text{Ker}(u)$ is likewise an intersection of closed submodules, hence it is closed in M . ■

Proposition 3.5.2 *Let M be an R -module and let N be a closed submodule of M . Then*

- (i) N^* is topologically isomorphic to M^*/N^0 ;
- (ii) $(M/N)^*$ is continuously isomorphic to N^0 , and topologically isomorphic if M/N is semi-discrete.

Proof. (i) Define $\rho : M^* \rightarrow N^*$ by $\rho(u) = u|N$. By Proposition (3.1.9) ρ is an epimorphism and clearly its kernel is N^0 . We have to show that ρ is continuous and open. Let D be a linearly compact J-separated submodule of N , D_1^0 its annihilator in N^* , D^0 its annihilator in M^* . We have $D^0 = \rho^{-1}(D_1^0)$, hence ρ is continuous. Let C be a linearly compact J-separated submodule of M then $C \cap N$ is J-separated, since C is also and $C \cap N$ is linearly compact, since it is closed in C . If $(C \cap N)_1^0$ is its annihilator in N^* then we have $\rho(C^0) = (C \cap N)_1^0$. Hence ρ is an open map.

(ii) The projection $\varepsilon : M \rightarrow M/N$ induces $\varepsilon^* : (M/N)^* \rightarrow M^*$ and $\text{Im}(\varepsilon^*) = N^0$. It follows from the proof of Proposition (3.4.2) that ε^* induces a continuous isomorphism $(M/N)^* \rightarrow N^0$, which by Proposition (3.4.8) is a topological isomorphism if M/N is semi-discrete. ■

Proposition 3.5.3 *If M is a semi-discrete R -module (in particular if is linearly discrete) then M^* is linearly compact.*

Proof. If M is linearly discrete, then M^* is linearly compact by Proposition (3.4.10) since R^* for being artinian it is linearly compact. If M is semi-discrete the proof is

analogous to that of Proposition (3.4.10). In fact, we can identify M^* with a submodule Q of $(R^*)^M$ and it is enough to show that Q is a closed submodule. This follows from the fact that if $v \in (R^*)^M$ and $v \notin Q$ then v is not an homomorphism, since by Proposition (3.1.8) every homomorphism $v : M \rightarrow R^*$ is continuous, that is, it is in Q . ■

Proposition 3.5.4 *If M is discrete then M^* is linearly compact and J -separated. If M is linearly compact and J -separated then M^* is discrete.*

Proof. If M is discrete then it is J -torsion by Proposition (3.1.4), that is if $x \in M$ we have that $J^n x = 0$ for some $n \in \mathbb{N}$ and therefore $J^n M^*$ annihilates x . Thus $\bigcap J^n M^*$ annihilates every element of M and hence it is 0. Therefore M^* is J -separated and it is linearly compact by Proposition (3.5.3).

If M is linearly compact and J -separated then $M^0 = 0$ is open in M^* and hence M^* is discrete. ■

Remark 3.5.5 Let M be an R -module. Each $x \in M$ defines a homomorphism $\omega(x) : M^* \rightarrow R^*$ by the rule $\omega(x)(u) = u(x)$. The kernel of $\omega(x)$ is $\langle x \rangle^0$, which is open in M^* since $\langle x \rangle$ is linearly compact and J -separated and therefore $\omega(x)$ is a character of M^* . Hence we have a homomorphism

$$\omega : M \rightarrow M^{**}.$$

If $\omega(x) = 0$ then $u(x) = 0$ for all $u \in M^*$, hence $x = 0$ by Proposition (3.1.6) and consequently ω is an (algebraic) monomorphism. We will prove that ω is an open isomorphism for any R -module M and we will characterize those R -modules for which ω is a topological isomorphism. The proof is in several steps.

Proposition 3.5.6 *Let M be a linearly compact J -separated R -module and let N be a submodule of finite length in M^* . Then $({}^0N)^0 = N$.*

Proof. Let n be the length of N . We will use induction on n . Consider first $n = 1$. Then $N \cong R/\mathfrak{m}_j = K_j$ for some j . Indeed, since N is simple, there is $x \in N$ such that $a \mapsto ax$, where $a \in R$, induces the isomorphism $R/Ann(x) \cong N$ and $R/Ann(x)$ is a simple R -module, hence it is a field and $Ann(x) = \mathfrak{m}_j$ for some j . Now, if $x \in M$ then $\omega(x)|N$ is a character of N , hence $x \mapsto \omega(x)|N$ is a homomorphism $M \rightarrow N^*$ whose kernel is 0N . It follows that $M/{}^0N$ is isomorphic to a submodule of N^* . Since $N \cong K_j$ and by [Ooi] 1.1 (2) we have that $K_j \cong K_j^*$ then $N^* \cong K_j$ and

therefore $M/{}^0N \cong K_j$ (otherwise we should have $M = {}^0N$ and thus $N = 0$). Hence $(M/{}^0N)^* \cong K_j$ and therefore by Proposition (3.5.2) $({}^0N)^0 \cong K_j$. Since $N \subseteq ({}^0N)^0$ we have that $N = ({}^0N)^0$. Now suppose the result is true for all submodules of length $< n$ in M^* ($n > 1$). If N is a submodule of length n in M^* we have that it has a submodule P of length $n-1$. By the inductive hypothesis $({}^0P)^0 = P$ and hence $({}^0P)^* \cong M^*/P$ by Proposition (3.5.2). If we identify $({}^0P)^*$ and M^*/P by this isomorphism, the annihilator ${}^0(N/P)$ of (N/P) in 0P is 0N . Since N/P is of length 1 and 0P is linearly compact and J-separated (since 0P is closed in M , by Proposition (3.5.1)) we have that, by the case $n = 1$, $({}^0(N/P))^0 = N/P$. But $({}^0(N/P))^0 = ({}^0N)^0/P$ and therefore $({}^0N)^0 = N$. ■

Proposition 3.5.7 *If M is linearly compact and J-separated then $\omega(M) = M^{**}$.*

Proof. Let N be a noetherian submodule of M^* . By Proposition (3.5.4) M^* is discrete and hence N is discrete. Furthermore N is linearly compact by Proposition (3.3.3) and it is artinian by Theorem (3.1.3) and Proposition (3.1.4). So N is of finite length and therefore $N = ({}^0N)^0 \cong (M/{}^0N)^*$ by Proposition (3.5.2) and Proposition (3.5.6). Consider the homomorphism $\omega_N : M \rightarrow N^*$ given by $x \mapsto \omega(x)|_N$, then $\text{Ker}(\omega_N) = {}^0N$ and hence $(M/{}^0N)$ is a submodule of N^* which is artinian by Theorem (B1)(ii). Hence, by Theorem (B1) (v), $(M/{}^0N)^{**} \cong M/{}^0N \cong N^*$. Also ω_N is an epimorphism, that is, if $u \in N^*$ then $u|_N \in N^*$ and there is $x_N \in M$ such that $\omega(x_N)|_N = u|_N$. Consider the cosets $x_N + {}^0N$ of M . As N runs through the noetherian submodules of M^* these are closed cosets by Proposition (3.5.1) and they have the finite intersection property. Indeed, for if N_1, \dots, N_r are noetherian submodules of M^* then $N_1 + \dots + N_r$ is noetherian, hence there is $x \in M$ such that $\omega(x)|_{N_1 + \dots + N_r} = u|_{N_1 + \dots + N_r}$. For each i there is $x_{N_i} \in M$ such that $\omega(x_{N_i})|_{N_i} = u|_{N_i}$, also $\omega(x)|_{N_i} = u|_{N_i}$, since $N_i \subseteq N_1 + \dots + N_r$. Hence $\omega(x)|_{N_i} = \omega(x_{N_i})|_{N_i}$ and $x - x_{N_i} \in \text{Ker}(\omega_{N_i}) = {}^0N_i$, that is, $x \in \bigcap_{i=1}^r (x_{N_i} + {}^0N_i)$.

Since M is linearly compact it follows that there is an element y of M such that $y \in x_N + {}^0N$ for all noetherian submodules N of M^* . Hence $\omega(y)$ agrees with u on every noetherian submodule of M^* and therefore $\omega(y) = u$. Thus ω is an epimorphism. ■

Proposition 3.5.8 *For every R -module M , $\omega(M) = M^{**}$.*

Proof. Let $u \in M^{**}$ and note that $\text{Ker}(u)$ is open in M^* . Hence $\text{Ker}(u) \supseteq C^0$ for some linearly compact J-separated submodule C of M . Note that u induces $u' : M^*/C^0 \rightarrow R^*$ given by $u'(\bar{f}) = u(f)$ where $f \in M^*$. Also, let $\rho : M^* \rightarrow C^*$ given by $\rho(f) = f|_C$. Let $\rho' : M^*/C^0 \rightarrow C^*$ be the homomorphism induced by ρ . Then ρ' is an isomorphism by Proposition (3.5.2). Hence, u induces $\bar{u} \in C^{**}$ given by $\bar{u}(f|_C) = u' \circ (\rho')^{-1}(f|_C)$. By Proposition (3.5.7) $\omega(C) = C^{**}$, that is, there is $x \in C$ such that $\bar{u} = \omega(x)$.

Notice if $f \in M^*$ we have that $u(f) = \bar{u}(f|_C) = \omega(x)(f)$. Thus $u = \omega(x)$ and therefore ω is an epimorphism. ■

Proposition 3.5.9 *Let M be an R -module and let N be a closed submodule of M . Then $\omega(N) = N^{00}$.*

Proof. Remember that $N^0 \leq M^*$ and

$$N^{00} = \{\varphi \in M^{**}; \varphi(f) = 0 \text{ for all } f \in N^0\}.$$

Hence, if $x \in N$, $\omega(x) \in N^{00}$, since $\omega(x)(f) = f(x) = 0$ for all $f \in N^0$.

Conversely, any element of M^{**} is of the form $\omega(y)$ for some $y \in M$, by Proposition (3.5.8). If $y \notin N$ then applying Proposition (3.1.6) to M/N we have that there is $\bar{u} \in (M/N)^*$ such that $\bar{u}(y + N) \neq 0$. Let $\varphi : (M/N)^* \rightarrow N^0$ be the isomorphism constructed in the proof of Proposition (3.5.2) (ii). Set $u = \varphi(\bar{u}) \in N^0$. Then $u(y) \neq 0$. Hence $\omega(y)(u) \neq 0$ and therefore $\omega(y) \notin N^{00}$. ■

Proposition 3.5.10 *For every R -module M , $\omega : M \rightarrow M^{**}$ is an open isomorphism.*

Proof. Let U be an open submodule of M . Thus M/U is discrete and by Proposition (3.5.2) U^0 is topologically isomorphic to $(M/U)^*$. Therefore U^0 is linearly compact and J-separated by Proposition (3.5.4). Hence U^{00} is open in M^{**} . But $U^{00} = \omega(U)$ by Proposition (3.5.9), hence ω is an open map and ω is an isomorphism by Proposition (3.5.8). ■

Proposition 3.5.11 *Let M be an R -module and let N be a submodule of M . Then*

- (i) N is closed in M if and only if $\omega(N)$ is closed in M^{**} ;
- (ii) N is linearly compact if and only if $\omega(N)$ is linearly compact.

Proof. (i) Suppose N is closed in M . Then $\omega(N) = N^{00}$ by Proposition (3.5.9) and hence it is closed in M^{**} by Proposition (3.5.1). Conversely suppose that $\omega(N)$ is

closed in M^{**} and let $x \in M$ with $x \notin N$. Then $\omega(x) \notin \omega(N)$. Since $\omega(N)$ is closed in M^{**} then, by Proposition (3.1.6) applied to $M^{**}/\omega(N)$ in composition with the projection $M^{**} \rightarrow M^{**}/\omega(N)$, there is a character ν of M^{**} such that $\nu(\omega(N)) = 0$ and $\nu(\omega(x)) \neq 0$. Let $\alpha : M^* \rightarrow M^{***}$ be given by $\alpha(u)(\varphi) = \varphi(u)$. By Proposition (3.5.10) we have that α is an open isomorphism. Since $\nu \in M^{***}$ there is a unique $u \in M^*$ such that $\alpha(u) = \nu$. Thus, if $y \in M$ we have that $\nu(\omega(y)) = \omega^*(u)(\omega(y)) = \omega(y)(u) = u(y)$. Hence $u(N) = 0$ and $u(x) \neq 0$. It follows that $N \subseteq \text{Ker}(u)$ and $x \notin \text{Ker}(u)$. Since $\text{Ker}(u)$ is closed, $x \notin \overline{N}$ and therefore N is closed in M .

(ii) It follows from (i) that ω transforms closed cosets into closed cosets in each direction and the result follows. ■

Proposition 3.5.12 *The following statements are equivalent for any R -module M :*

- (a) $\omega : M \rightarrow M^{**}$ is a topological isomorphism;
- (b) Every J -torsion semi-discrete quotient of M is discrete;
- (c) M is topologically isomorphic to N^* for some R -module N .

Proof. (a) \Rightarrow (b): Let M/N be an J -torsion semi-discrete quotient of M , then in particular the zero submodule of M/N is closed and therefore N is a closed submodule of M . Since M/N is semi-discrete, N^0 is topologically isomorphic to $(M/N)^*$ by Proposition (3.5.2) and therefore it is linearly compact by Proposition (3.5.3). Also, since M/N is J -torsion, $(M/N)^*$ is J -separated. Hence N^0 is linearly compact and J -separated and therefore N^{00} is open in M^{**} . But $N = \omega^{-1}(N^{00})$ by Proposition (3.5.9). Thus N is open and therefore M/N is discrete.

(b) \Rightarrow (a): Let U' be an open submodule of M^{**} and set $U = \omega^{-1}(U')$. Every submodule P' of M^{**} which contains U' is open and therefore closed, hence every submodule $P = \omega^{-1}(P')$ of M which contains U is closed by Proposition (3.5.11) and therefore M/U is semi-discrete. Since M^{**}/U' is discrete it is J -torsion by Proposition (3.2.1), hence M/U is J -torsion as well as semi-discrete and therefore discrete by hypothesis. Consequently U is open in M and so ω is continuous. By Proposition (3.5.10) it follows that ω is a topological isomorphism.

(a) \Rightarrow (c): Take $N = M^*$.

(c) \Rightarrow (a): We have that $N^{**} \rightarrow N$ is a continuous isomorphism by Proposition (3.5.10), hence its transpose $\theta : N^* \rightarrow N^{***}$ is a continuous isomorphism by Proposition (3.4.2). But θ is open by Proposition (3.5.10) and thus is a topological isomorphism.

■

An R -module M is **locally linearly compact** if M has a linearly compact submodule N such that M/N is linearly discrete.

Proposition 3.5.13 *If M is locally linearly compact, then so is M^* and $\omega : M \rightarrow M^{**}$ is a topological isomorphism. If M is linearly discrete (resp. linearly compact) then M^* is linearly compact (resp. linearly discrete).*

Proof. Firstly we will show that if M is locally linearly compact then so is M^* . Let N be a linearly compact submodule of M such that M/N is linearly discrete. Then by Proposition (3.5.2) N^0 is topologically isomorphic to $(M/N)^*$ and hence it is linearly compact by Proposition (3.5.3). By Proposition (3.5.2) M^*/N^0 is topologically isomorphic to N^* and therefore it is linearly discrete. This shows that M^* is locally linearly compact.

We will show next that if M is locally linearly compact then M satisfies condition (b) of Proposition (3.5.12). Let P be a submodule of M such that M/P is semi-discrete and J -torsion. Then P is closed in M . $(N+P)/P$ is a continuous homomorphic image of N and therefore is linearly compact. Also $(N+P)/P$ is a submodule of M/P , therefore it is semi-discrete and J -torsion. Hence $(N+P)/P$ is artinian by Proposition (3.1.3) and thus it is discrete by Proposition (2.2.11). Consequently P is open in $N+P$.

Consider $M/(N+P)$. Since it is a continuous homomorphic image of M/P , it is J -torsion. Also, since $M/(N+P)$ is a continuous homomorphic image of M/N , it is linearly discrete by Proposition (3.2.5). Hence by Proposition (3.2.1) $M/(N+P)$ is discrete and consequently $N+P$ is open in M . It follows that P is open in M , so that M/P is discrete and hence by Proposition (3.5.12) $\omega : M \rightarrow M^{**}$ is a topological isomorphism.

If M is linearly discrete, M^* is linearly compact by Proposition (3.5.3).

If M is linearly compact then by Proposition (3.1.5) M is an inverse limit of artinian R -modules, say $M = \varprojlim M_i$, where each M_i is artinian. By Matlis duality $N_i = M_i^*$ is noetherian, and therefore linearly discrete by Proposition (3.3.3), and $N_i^* \cong M_i$.

Hence by Proposition (3.4.9) $M = \varprojlim N_i^* \cong (\varinjlim N_i)^* = N^*$ say, where $N = \varinjlim N_i$ is linearly discrete by Proposition (3.2.7). Hence $M^* \cong N^{**} \cong N$ by the first part of the proof of Proposition (3.5.13), and therefore M^* is linearly discrete. ■

Theorem 3.5.14 *If $\{M_i\}$ is a family of linearly compact R -modules, we have topological isomorphisms*

$$\begin{aligned} \left(\prod_{i \in I} M_i\right)^* &\cong \bigoplus_{i \in I} M_i^*, \\ \left(\varprojlim M_i\right)^* &\cong \varinjlim M_i^*. \end{aligned}$$

Proof. From Proposition (3.5.13) we have that M_i^* is linearly discrete, hence by Proposition (3.4.7) $(\bigoplus M_i^*)^* \cong \prod M_i^{**} \cong \prod M_i$. It follows that $(\prod M_i)^* \cong (\bigoplus M_i^*)^{**}$ by Proposition (3.5.13), since $\bigoplus M_i^*$ is linearly discrete. Similarly for the other isomorphism. ■

Chapter 4

Applications to local homology and local cohomology modules

In this chapter, we present applications from the third chapter, namely, we will generalize for the semilocal case a local homology theory for linearly compact modules introduced by N. T. Cunog and T. T. Nam in the paper *A local homology theory for linearly compact modules* of 2008 [C-N1], which is in some sense dual to the local cohomology theory of A. Grothendieck [Gro1]. Some basic properties such as the noetherianess, the vanishing and non-vanishing of local homology modules of linearly compact modules are proved. A duality theory between local homology and local cohomology modules of linearly compact modules is developed by using Matlis duality and Macdonald duality. As consequences we obtain some generalizations of well-known results from Grothendieck's theory of local cohomology. These results were extended from finitely generated modules to linearly compact semi-discrete modules in [C-N1] and we extend these results to the semilocal case.

4.1 Local homology modules of linearly compact modules

Let I be an ideal of R . The *i th local homology* modules $H_i^I(M)$ of an R -module M with respect to I are defined by (see [C-N2] 3.1)

$$H_i^I(M) = \varprojlim_t \operatorname{Tor}_i^R(R/I^t, M).$$

Note that $H_0^I(M) \cong \Lambda_I(M)$, in which $\Lambda_I(M) = \varprojlim_t M/I^t M$ is the I -adic completion of M .

We say that a module M is **I -separated** if $\bigcap_{t>0} I^t M = 0$. We say also that M is **complete** with respect to the I -adic topology if $\Lambda_I(M) \cong M$.

Remark 4.1.1 (See [C-N1] 3.1).

(i) As $I^t \operatorname{Tor}_i^R(R/I^t, M) = 0$, for each $t > 0$, $\operatorname{Tor}_i^R(R/I^t, M)$ has a natural structure as a module over the ring R/I^t . Then $H_i^I(M) = \varprojlim_t \operatorname{Tor}_i^R(R/I^t, M)$ has a natural structure as a module over the ring $\Lambda_I(R) = \varprojlim_t R/I^t$.

(ii) If M is a finitely generated R -module, then $H_i^I(M) = 0$ for all $i > 0$.

Lemma 4.1.2 (See [C-N1] 3.2) Let I be a finitely generated ideal. Then for all $i \geq 0$, $H_i^I(M)$ is I -separated.

Let M be a linearly compact R -module. Then $\operatorname{Tor}_i^R(R/I^t, M)$ is also a linearly compact R -module by the topology defined as in [C-N1] 2.6, so we have an induced topology on the local homology module $H_i^I(M)$.

Theorem 4.1.3 (See [C-N1] 3.8) Let M be a linearly compact R -module. The following statements are equivalent:

- (a) M is I -separated.
- (b) M is complete with respect to the I -adic topology.
- (c) $H_0^I(M) \cong M$, $H_i^I(M) = 0$ for all $i > 0$.

From Theorem (4.1.3) we have the following criterion for a finitely generated module over a semilocal noetherian ring to be linearly compact. Remember that for us $(R, \mathfrak{m}_1, \dots, \mathfrak{m}_n)$ denotes a semilocal noetherian ring.

Corollary 4.1.4 Let $(R, \mathfrak{m}_1, \dots, \mathfrak{m}_n)$ be a ring and let M be a finitely generated R -module. Then M is a linearly compact R -module if and only if M is complete with respect to the J -adic topology, where $J = \bigcap_{i=1}^n \mathfrak{m}_i$.

Proof. Since M is a finitely generated R -module, M is \mathbf{J} -separated by Theorem (A.5). Thus, if M is a linearly compact R -module, $\Lambda_{\mathbf{J}}(M) \cong M$ by Theorem (4.1.3). Conversely, if M is complete in the \mathbf{J} -adic topology, we have that $M \cong \varprojlim_t M/\mathbf{J}^t M$. Since $M/\mathfrak{m}_i^t M$ are artinian R -modules for all $t > 0$ and for all $i = 1, \dots, n$ we have that $M/\mathbf{J}^t M$ are artinian R -modules for all $t > 0$, by the Chinese Remainder Theorem. Therefore M is linearly compact R -module by Proposition (2.2.9). ■

4.2 Vanishing and non-vanishing of local homology modules

Denote by $L(M)$ the sum of all artinian submodules of M . Let $Soc(M)$, the socle of M , be the sum of all simple submodules of M . The I -torsion functor Γ_I is defined by $\Gamma_I(M) = \bigcup_{t>0} (0 :_M I^t)$.

Lemma 4.2.1 (See [C-N1] 4.3) *Let M be a semi-discrete linearly compact R -module. Then there are only finitely many distinct maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ of R such that*

$$L(M) = \bigoplus_{j=1}^r \Gamma_{\mathfrak{m}_j}(M).$$

Lemma 4.2.2 (See [C-N1] 4.4) *Let M be a semi-discrete linearly compact R -module. Then there are only finitely many distinct maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ of R such that*

$$H_i^I(M) \cong \bigoplus_{j=1}^r H_i^I(\Gamma_{\mathfrak{m}_j}(M))$$

for all $i > 0$.

Proposition 4.2.3 *Let $(R, \mathfrak{m}_1, \dots, \mathfrak{m}_n)$ be a ring, \mathbf{J} the Jacobson radical of R and M a semi-discrete linearly compact R -module. Then*

$$L(M) = L(\Gamma_{\mathbf{J}}(M))$$

and

$$H_i^{\mathbf{J}}(M) \cong H_i^{\mathbf{J}}(\Gamma_{\mathbf{J}}(M))$$

for all $i > 0$.

Proof. By [B-S] 1.1.2, we have that $\Gamma_{\mathfrak{m}_j}(\Gamma_{\mathbf{J}}(M)) = \Gamma_{\mathfrak{m}_j}(M)$ for each $j = 1, \dots, n$. Since by [Sha] 1.4 $\Gamma_{\mathfrak{m}_j}(M) \neq 0$ if and only if $\Gamma_{\mathfrak{m}_j}(\Gamma_{\mathbf{J}}(M)) \neq 0$ it follows by Lemma (4.2.1)

that $L(M) = L(\Gamma_J(M))$. Furthermore, by Lemma (4.2.2), follows that $H_i^J(M) \cong H_i^J(\Gamma_J(M))$. ■

We now recall the concept of *noetherian dimension* of an R -module M denoted by $Ndim M$. Note that the notion of noetherian dimension was introduced first by R.N. Roberts [Rob] by the name Krull dimension. Later D. Kirby [Kir] changed this terminology due to Roberts and referred to as noetherian dimension to avoid confusion with well-known Krull dimension of finitely generated modules. Let M be an R -module. When $M = 0$ we put $Ndim M = -1$. Then by induction, for any ordinal α , we put $Ndim M = \alpha$ when (i) $Ndim M < \alpha$ is false, and (ii) for every ascending chain $M_0 \subseteq M_1 \subseteq \dots$ of submodules of M , there exists a positive integer m_0 such that $Ndim(M_{m+1}/M_m) < \alpha$ for all $m \geq m_0$. Thus M is non-zero and finitely generated if and only if $Ndim M = 0$. If $0 \rightarrow M'' \rightarrow M \rightarrow M' \rightarrow 0$ is a short exact sequence of R -modules, then $Ndim M = \max\{Ndim M'', Ndim M'\}$.

Remark 4.2.4

(i) In case M is an artinian module, $Ndim M < \infty$ (see [Kir] 2.6). More generally, if M is a semi-discrete linearly compact module, there is a short exact sequence $0 \rightarrow N \rightarrow M \rightarrow A \rightarrow 0$, where N is finitely generated and A is artinian (See [Zös] Theorem). Hence $Ndim M = \max\{Ndim M'', Ndim M'\} < \infty$.

(ii) (See [C-N1] 4.6) We recall the concept of co-associated primes of a module. A prime ideal \mathfrak{p} is called **co-associated** to a non-zero R -module M if there is an artinian homomorphic image L of M with $\mathfrak{p} = Ann_R L$. The set of all co-associated primes to M is denoted by $Coass_R(M)$ and M is called **\mathfrak{p} -coprimary** if $Coass_R(M) = \{\mathfrak{p}\}$. If M is an artinian R -module or more generally, a semi-discrete linearly compact R -module, then $Ndim M \leq \max\{dim R/\mathfrak{p} \mid \mathfrak{p} \in Coass(M)\}$.

Theorem 4.2.5 (See [C-N1] 4.8) Let M be a linearly compact R -module with $Ndim M = d$. Then

$$H_i^I(M) = 0$$

for all $i > d$.

A sequence of elements x_1, \dots, x_r in R is said to be an M -**coregular** sequence (see [Ooi], 3.1) if $(0 :_M (x_1, \dots, x_r)) \neq 0$ and $(0 :_M (x_1, \dots, x_{i-1})) \xrightarrow{x_i} (0 :_M (x_1, \dots, x_{i-1}))$ is surjective for $i = 1, \dots, r$. We denote by $width_I(M)$ the supremum of the lengths of all maximal M -coregular sequences in the ideal I . When M is a

semi-discrete linearly compact R -module we have by Remark (4.2.4) (i) and by [C-N1] 4.7 that

$$\text{width}_I(M) \leq N \dim M < \infty.$$

Theorem 4.2.6 (See [C-N1] 4.11) *Let M be a semi-discrete linearly compact R -module and let I be an ideal of R such that $(0 :_M I) \neq 0$. Then all maximal M -coregular sequences in I have the same length. Moreover*

$$\text{width}_I(M) = \inf\{i \mid H_i^I(M) \neq 0\}.$$

In Section (4.5) we will prove some theorems involving $N \dim M$ and $\text{width}_I(M)$ when M is a semi-discrete linearly compact R -module.

4.3 Noetherian Local homology modules

First, the following criterion for a module to be noetherian is useful for the investigation of the noetherian property of local homology modules.

Lemma 4.3.1 (See [C-N1] 5.1) *Let I be a finitely generated ideal of a ring R such that R is complete with respect to the I -adic topology. Let M be an R -module. If M/IM is a noetherian R -module and M is I -separated, then M is a noetherian R -module.*

Theorem 4.3.2 *Let $(R, \mathfrak{m}_1, \dots, \mathfrak{m}_n)$ be a ring, M be a semi-discrete linearly compact R -module and \mathfrak{J} be the Jacobson radical of R . Then $H_i^{\mathfrak{J}}(M)$ is a noetherian \widehat{R} -module for all $i \geq 0$.*

Proof. We will prove the result by induction on i . If $i = 0$, we have that $H_0^{\mathfrak{J}}(M) \cong \Lambda_{\mathfrak{J}}(M)$. As M is a semi-discrete linearly compact R -module, $M/\mathfrak{J}M$ is also a semi-discrete linearly compact R -module. By virtue of Proposition (3.3.3), $M/\mathfrak{J}M$ is a Noetherian R -module and also as R/\mathfrak{J} -module. Then $\Lambda_{\mathfrak{J}}(M)$ is a noetherian \widehat{R} -module by [D-G] 7.2.9. Let $i > 0$. Combining [C-N1] 3.10 with [C-N1] 4.1 we may assume, by replacing M with $\bigcap_{t>0} \mathfrak{J}^t M$, that there is an element $x \in \mathfrak{J}$ such that $xM = M$. Then, by [C-N1] 3.7, the short exact sequence of linearly compact modules

$$0 \rightarrow (0 :_M x) \rightarrow M \xrightarrow{x} M \rightarrow 0$$

gives rise to a long exact sequence of local homology modules

$$\cdots \rightarrow H_i^{\mathfrak{J}}(M) \xrightarrow{x} H_i^{\mathfrak{J}}(M) \xrightarrow{\delta} H_{i-1}^{\mathfrak{J}}(0 :_M x) \rightarrow \cdots .$$

If $(0 :_M x) = 0$, then $H_i^J(M) = xH_i^J(M) = \cap_{t>0} x^t H_i^J(M) = 0$ for all $i \geq 0$ by [C-N1] 3.2 (ii). We now assume that $(0 :_M x) \neq 0$. By the inductive hypothesis, $H_{i-1}^J(0 :_M x)$ is a noetherian \widehat{R} -module. Set $H = H_i^J(M)$, we have that $H/xH \cong \text{Im}(\delta) \subseteq H_{i-1}^J(0 :_M x)$. It follows that H/xH is a noetherian \widehat{R} -module. Thus $H/\widehat{J}H$ is also a noetherian \widehat{R} -module. Moreover, $\cap_{t>0} \widehat{J}^t H = \cap_{t>0} J^t H_i^J(M) = 0$. Therefore H is a noetherian \widehat{R} -module by (4.3.1). ■

Theorem 4.3.3 *Let $(R, \mathfrak{m}_1, \dots, \mathfrak{m}_n)$ be a ring and let M be a semi-discrete linearly compact R -module with $\text{Ndim } M = d$. Then $H_d^I(M)$ is a noetherian module over $\Lambda_I(R)$.*

Proof. We argue by induction on d . If $d = 0$, M is a finitely generated R -module, and so it is I -separated (see Theorem (A.5)). By Theorem (4.1.3), $H_0^I(M) \cong \Lambda_I(M) \cong M$ and therefore $H_0^I(M)$ is a noetherian $\Lambda_I(R)$ -module. Let $d > 0$. From [C-N1] 3.10 we have $H_d^I(M) \cong H_d^I(\cap_{t>0} I^t M)$. If $\text{Ndim}(\cap_{t>0} I^t M) < d$, then $H_d^I(M) = 0$ by Theorem (4.2.5) and then there is nothing to prove. If $\text{Ndim}(\cap_{t>0} I^t M) = d$, by [C-N1] 3.10 and [C-N1] 4.1 we may assume, by replacing M with $\cap_{t>0} I^t M$, that there is an element $x \in I$ such that $xM = M$. Then, from the short exact sequence of linearly compact modules

$$0 \rightarrow 0 :_M x \rightarrow M \xrightarrow{x} M \rightarrow 0$$

we get an exact sequence of local homology modules

$$H_d^I(M) \xrightarrow{x} H_d^I(M) \xrightarrow{\delta} H_{d-1}^I(0 :_M x).$$

By [C-N1] 4.7 we have that $\text{Ndim}(0 :_M x) \leq d - 1$. If $\text{Ndim}(0 :_M x) < d - 1$, then $H_{d-1}^I(0 :_M x) = 0$ by Theorem (4.2.5) and therefore

$$H_d^I(M) = xH_d^I(M) = \bigcap_{t>0} x^t H_d^I(M) = 0$$

by (4.1.2). Assume that $\text{Ndim}(0 :_M x) = d - 1$. It follows by the inductive hypothesis that $H_{d-1}^I(0 :_M x)$ is a noetherian $\Lambda_I(R)$ -module. On the other hand, we have that $H_d^I(M)/xH_d^I(M) \cong \text{Im}\delta \subseteq H_{d-1}^I(0 :_M x)$. Thus $H_d^I(M)/xH_d^I(M)$ is a noetherian $\Lambda_I(R)$ -module. Therefore $H_d^I(M)/LH_d^I(M)$ is a noetherian $\Lambda_I(R)$ -module, where $L = I\Lambda_I(R)$. Moreover, since $\cap_{t>0} L^t H_d^I(M) = \cap I^t H_d^I(M) = 0$ and $\Lambda_I(R)$ is complete in the L -adic topology, $H_d^I(M)$ is a noetherian $\Lambda_I(R)$ -module by Theorem (4.3.1) as required. ■

4.4 Macdonald duality

In this section let $R := (R, \mathfrak{m}_1, \dots, \mathfrak{m}_n)$ be a semilocal noetherian ring and let R^* be the injective envelope of R/\mathfrak{J} , where $\mathfrak{J} = \bigcap_{i=1}^n \mathfrak{m}_i$. Suppose that the topology on R is the \mathfrak{J} -adic topology. If M is any (linearly topologized, Hausdorff) R -module, the module $D(M) = {}^A\text{Hom}(M, R^*)$, of all (algebraic) homomorphisms from M into R^* , is called **Matlis dual** of M . Recall that **Macdonald dual** of M is $M^* = \text{Hom}_R(M, R^*)$ as defined in Section (3.5). In case R is complete the topology on M^* is defined as in Remark (3.4.1). Moreover, if M is semi-discrete, then the topology of M^* coincides with that induced on it as a submodule of $(R^*)^M$, where $(R^*)^M = \prod_{x \in M} (R^*)_x$, $(R^*)_x = R^*$ for all $x \in M$ (see Proposition (3.4.6)).

Remark 4.4.1 In Section (3.5) was established a generalization of the duality between linearly discrete and linearly compact modules as follows (see propositions (3.5.3), (3.5.12) and (3.5.13)). If R is a complete ring then

(i) If M is linearly compact, then M^* is linearly discrete (hence semi-discrete). If M is semi-discrete, then M^* is linearly compact.

(ii) If M is linearly compact or linearly discrete, then we have a topological isomorphism $\omega : M \rightarrow M^{**}$.

Lemma 4.4.2 *Let*

$$0 \rightarrow M \xrightarrow{f} N \xrightarrow{g} P \rightarrow 0$$

be a short exact sequence of linearly compact R -modules in which the homomorphisms f and g are continuous. Then the induced short sequence

$$0 \rightarrow P^* \xrightarrow{g^*} N^* \xrightarrow{f^*} M^* \rightarrow 0$$

is exact.

Proof. By Proposition (3.1.5) f is an open mapping. Hence by replacing M by $f(M)$ we may assume that M is a closed submodule of N . Therefore, by Proposition (3.1.9), for any continuous homomorphism $h : M \rightarrow E$ there is a continuous homomorphism $\varphi : N \rightarrow E$ which extends h . Thus f^* is surjective. We have that g^* is injective and $\text{Im } g^* \subseteq \text{Ker } f^*$. So it remains to show that $\text{Ker } f^* \subseteq \text{Im } g^*$. Let $\psi \in \text{Ker } f^*$, we have that $\psi(\text{ker } g) = \psi(f(M)) = 0$. Then ψ induces an homomorphism $\phi : P \rightarrow E$ such that $\phi \circ g = \psi$. It follows that $\text{ker } \phi = g(\text{Ker } \psi)$. Since ψ is continuous, $\text{Ker } \psi$ is

open by Proposition (3.1.7). Moreover, g is open, so $\text{Ker } \phi$ is also open. Therefore ϕ is continuous by Proposition (3.1.7). Thus $\psi \in \text{Im } g^*$. ■

Note that submodules and homomorphic images of a semi-discrete module are also semi-discrete. The following consequence shows that the converse is also true in the category of linearly compact R -modules.

Corollary 4.4.3 *Let*

$$0 \rightarrow M \xrightarrow{f} N \xrightarrow{g} P \rightarrow 0$$

be a short exact sequence of linearly compact R -modules with continuous homomorphisms f and g . If M and P are semi-discrete, then N is also semi-discrete.

Proof. It follows from Proposition (3.1.8) and the hypothesis that $P^* = D(P)$ and $M^* = D(M)$. We now have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P^* & \xrightarrow{g^*} & N^* & \xrightarrow{f^*} & M^* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow j & & \downarrow & & \\ 0 & \longrightarrow & D(P) & \xrightarrow{D(g)} & D(N) & \xrightarrow{D(f)} & D(M) & \longrightarrow & 0 \end{array}$$

in which j is an inclusion and rows are exact by Lemma (4.4.2) and [Rot] 3.25. It follows that $N^* = D(N)$, thus N is semi-discrete by Proposition (3.1.8). ■

Lemma 4.4.4 *Let N be a finitely generated R -module and let M be a linearly compact R -module. Then*

$$\begin{aligned} (\text{Tor}_i^R(N, M))^* &\cong \text{Ext}_R^i(N, M^*), \\ \text{Tor}_i^R(N, M^*) &\cong (\text{Ext}_R^i(N, M))^* \end{aligned}$$

for all $i \geq 0$.

Proof. Let

$$\mathbf{F}_\bullet : \cdots \rightarrow F_i \rightarrow F_{i-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow N \rightarrow 0$$

be a free resolution of N , in which the free R -modules F_i are finitely generated. Consider $\mathbf{F}_\bullet \otimes_R M$ as a complex of linearly compact R -modules and the continuous homomorphisms. By Lemma (4.4.2) and Theorem 1 of [Nor] 6.1 it follows by that

$$(H_i(\mathbf{F}_\bullet \otimes_R M))^* \cong H^i((\mathbf{F}_\bullet \otimes_R M)^*).$$

On the other hand, by virtue of Remark (2.1.5) we have that

$$(\mathbf{F}_\bullet \otimes_R M)^* \cong \text{Hom}_R(\mathbf{F}_\bullet, M^*).$$

Therefore

$$\begin{aligned} (\text{Tor}_i^R(N, M))^* &\cong (H_i(\mathbf{F}_\bullet \otimes_R M))^* \\ &\cong H^i(\text{Hom}_R(\mathbf{F}_\bullet, M))^* \\ &\cong \text{Ext}_R^i(N, M^*). \end{aligned}$$

The proof of the second isomorphism is similar. ■

If M is a linearly topologized R -module, then the module $\text{Ext}_R^i(R/I^t, M)$ is also a linearly topologized R -module by topology defined in [C-N1] 2.5. Since the local cohomology module $H_I^i(M) = \varinjlim_t \text{Ext}_R^i(R/I^t, M)$ is a quotient module of $\bigoplus_t \text{Ext}_R^i(R/I^t, M)$ it becomes a linearly topologized R -module with the quotient topology.

Lemma 4.4.5 *Let R be a complete ring. If M is a semi-discrete linearly compact R -module, then M is linearly discrete and therefore the local cohomology modules $H_I^i(M)$ are linearly discrete R -modules for all $i \geq 0$.*

Proof. We first show that if M is a semi-discrete linearly compact R -module, then M is linearly discrete. Indeed, since M is semi-discrete, M^* is linearly compact and hence M^{**} is linearly discrete by Remark (4.4.1) (i). On the other hand, since M is a linearly compact R -module, we have by Remark (4.4.1) (ii) a topological isomorphism $M \cong M^{**}$. Therefore M is linearly discrete. Now, by same argument as in the proof of [C-N1] 2.5 we can prove that $\text{Ext}_R^i(R/I^t, M)_t$ is a direct system of semi-discrete linearly compact R -modules with the continuous homomorphisms, and therefore it is a direct system of linearly discrete modules. Thus, by Proposition (3.2.7) $H_I^i(M) = \varinjlim_t \text{Ext}_R^i(R/I^t, M)$ are linearly discrete for all $i \geq 0$. ■

The following duality theorem between local homology and local cohomology modules is the main result of this section. Let L_i^I be the i th left derived functor of the I -adic completion functor Λ_I .

Theorem 4.4.6

(i) *Let M be an R -module. Then for all $i \geq 0$,*

$$L_i^I(D(M)) \cong D(H_I^i(M)) \cong H_i^I(D(M)).$$

(ii) If M is a linearly compact R -module, then for all $i \geq 0$,

$$H_i^I(M^*) \cong (H_i^I(M))^*.$$

Moreover, if R is a complete ring, then

$$H_i^I(M^*) \cong (H_i^I(M))^*.$$

(iii) If R is a complete ring and M is a semi-discrete linearly compact R -module, then we have topological isomorphisms of R -modules for all $i \geq 0$,

$$H_i^I(M^*) \cong (H_i^I(M))^*,$$

$$H_i^I(M^*) \cong (H_i^I(M))^*.$$

Proof. (i) From Remark (2.1.6) and [Sim] 4.1,

$$\begin{aligned} H_i^I(D(M)) &\cong \varprojlim_t \operatorname{Tor}_i^R\left(\frac{R}{I^t}; D(M)\right) \\ &\cong \varprojlim_t D(\operatorname{Ext}_R^i\left(\frac{R}{I^t}; M\right)) \\ &\cong D(\varinjlim_t \operatorname{Ext}_R^i\left(\frac{R}{I^t}; M\right)) \\ &= D(H_i^I(M)). \end{aligned}$$

The result is completed by [Sim] 5.6.

(ii) Note by Remark (2.1.6) that for direct system M_t of Hausdorff linearly topologized R -modules with the continuous homomorphisms we have an isomorphism $\varprojlim_t M_t^* \cong (\varinjlim_t M_t)^*$. Moreover, since $\{\operatorname{Ext}_R^i(R/I^t, M)\}_t$ forms a direct system of linearly compact R -modules with continuous homomorphisms by [C-N1] 2.5, we get by (4.4.4) that

$$\begin{aligned} H_i^I(M^*) &= \varprojlim_t \operatorname{Tor}_i^R(R/I^t, M^*) \\ &\cong \varprojlim_t (\operatorname{Ext}_R^i(R/I^t, M))^* \\ &\cong (\varinjlim_t \operatorname{Ext}_R^i(R/I^t, M))^* \\ &= (H_i^I(M))^*. \end{aligned}$$

To prove the second isomorphism, note by Theorem (3.5.14) that for an inverse system $\{M_t\}$ of linearly compact modules over complete local noetherian ring with continuous homomorphisms we have an isomorphism $(\varprojlim_t M_t)^* \cong \varinjlim_t M_t^*$, and that

$\{\mathrm{Tor}_i^R(R/I^t, M)\}_t$ forms an inverse system of linearly compact R -modules with continuous homomorphisms by [C-N1] 2.6. It follows by Lemma (4.4.4) that

$$\begin{aligned} H_I^i(M^*) &= \varinjlim_t \mathrm{Ext}_R^i(R/I^t, M^*) \\ &\cong \varinjlim_t (\mathrm{Tor}_i^R(R/I^t, M))^* \\ &\cong (\varprojlim_t \mathrm{Tor}_R^i(R/I^t, M))^* \\ &= (H_i^I(M))^*. \end{aligned}$$

(iii) Let us prove the first isomorphism. From (ii), it is the algebraic isomorphism. Thus, by Proposition (3.2.8), we only need to show that both $H_I^i(M^*)$ and $(H_i^I(M))^*$ are linearly discrete. Indeed, it follows by [C-N1] 3.3 that $H_i^I(M)$ is linearly compact and follows from Remark (4.4.1) (i) that $(H_i^I(M))^*$ is linearly discrete. On the other hand, since M is semi-discrete linearly compact, M^* is linearly compact and linearly discrete. Therefore the local cohomology modules $H_I^i(M^*)$ are linearly discrete by Lemma (4.4.5). Hence the first topological isomorphism is proved. The second topological isomorphism follows applying the first to M^* instead of M and from Remark (4.4.1) (ii):

$$H_I^i(M) \cong H_I^i(M^{**}) \cong (H_i^I(M^*))^*$$

and

$$H_I^i(M)^* \cong (H_i^I(M^*))^{**} \cong (H_i^I(M^*)).$$

■

Corollary 4.4.7 *Let R be a complete ring.*

(i) If M is linearly compact R -module, then for all $i \geq 0$,

$$H_I^i(M) \cong (H_i^I(M^*))^*,$$

$$H_i^I(M) \cong (H_I^i(M^*))^*.$$

(ii) If M is a semi-discrete linearly compact R -module, then we have topological isomorphism of R -modules for all $i \geq 0$,

$$H_I^i(M) \cong (H_i^I(M^*))^*,$$

$$H_i^I(M) \cong (H_I^i(M^*))^*.$$

Proof. (i) follows from Theorem (4.4.6) (ii) and Remark (4.4.1) (ii).

(ii) follows from Theorem (4.4.6) (iii) and Remark (4.4.1) (ii). ■

4.5 Local cohomology of semi-discrete linearly compact modules

In this section $R := (R, \mathfrak{m}_1, \dots, \mathfrak{m}_n)$ have the same meanings as in the previous section. We denote by $(\widehat{R}, \widehat{\mathfrak{m}}_1, \dots, \widehat{\mathfrak{m}}_n)$ the J -adic completion of R with maximal ideals $\widehat{\mathfrak{m}}_1, \dots, \widehat{\mathfrak{m}}_n$ and \widehat{M} the J -adic completion of the module M . Recall that an artinian R -module A has a natural structure of module over \widehat{R} as following: Let $\widehat{a} = (a_n) \in \widehat{R}$ and $x \in A$; since $J^k x = 0$ for some positive integer k , $a_n x$ is constant for all large n , and we define $\widehat{a}x$ to be this constant value. Then we have the following generalization of this fact for linearly compact R -modules.

Lemma 4.5.1 *Let M be a linearly compact R -module. Then the following statements are true.*

(i) *M has a natural structure of linearly compact module over \widehat{R} . Moreover, a subset N of M is a linearly compact \widehat{R} -submodule if and only if N is a closed R -submodule.*

(ii) *Assume in addition that M is a semi-discrete R -module. Then M is also a semi-discrete linearly compact \widehat{R} -module.*

Proof. (i) Assume that $\{U_i\}_{i \in \Lambda}$ is a fundamental system of neighborhoods of zero of M consisting of submodules. Then $M \cong \varprojlim_{i \in \Lambda} M/U_i$, in which M/U_i is an artinian R -module for all $i \in \Lambda$ by Proposition (2.2.12), [Mac] 4.1 and Proposition (3.1.5). It should be noted by Section (3.1) that an artinian module over a semilocal noetherian ring $(R, \mathfrak{m}_1, \dots, \mathfrak{m}_n)$ has a natural structure of artinian module over \widehat{R} so that a subset of M is an R -module if and only if it is an \widehat{R} -submodule. Thus $\{M/U_i\}_{i \in \Lambda}$ can be regard as an inverse system of artinian \widehat{R} -modules with \widehat{R} -homomorphisms. Therefore, by passing to the inverse limits, M has a natural structure of linearly compact module over \widehat{R} . It is clear that a linearly compact \widehat{R} -submodule of M is a closed R -submodule. Now, if N is a closed R -module of M , then $N \cong \varprojlim_{i \in \Lambda} N/(N \cap U_i)$. Since

$$N/(N \cap U_i) \cong (N + U_i)/U_i \subseteq M/U_i,$$

$N/(N \cap U_i)$ can be considered as an artinian R -submodule of M/U_i , so it is an artinian \widehat{R} -submodule. Moreover, the homomorphisms of the inverse system $\{N/(N \cap U_i)\}_{i \in \Lambda}$ are induced from the inverse system $\{M/U_i\}_{i \in \Lambda}$. Therefore, by Proposition (2.2.9) N is a linearly compact \widehat{R} -submodule of M .

(ii) follows from (i) by the fact that all submodules of a semi-discrete linearly compact module are closed. ■

Remember that the (**Krull**) **dimension** $\dim_R M$ of a non-zero R -module M is the supremum of lengths of chains of primes in the support of M if this supremum exists, and ∞ otherwise. If M is finitely generated, then $\dim_R M = \max\{\dim_R R/\mathfrak{p} \mid \mathfrak{p} \in \text{Ass} M\}$. For convenience, we set $\dim_R M = -1$ if $M = 0$. Note that if A is a artinian module, then $\dim_R A = 0$, since support of A consists only of maximal ideals. Indeed, assume there exist $\mathfrak{p} \in \text{Supp}(A)$ and that \mathfrak{p} is not maximal. Let $a \in A$ be an element such that its image under the natural map $A \rightarrow A_{\mathfrak{p}}$ is non-zero. Then $\text{Ann}(a) \subseteq \mathfrak{p}$. Thus $\langle a \rangle$ surjects onto R/\mathfrak{p} . Since R/\mathfrak{p} is the homomorphic image of a submodule of an artinian module, it is artinian. However, by our assumption R/\mathfrak{p} is ring of positive dimension. Thus it cannot be artinian. From this contradiction it follows that $\text{Supp}(A)$ consists only of maximal ideals.

Corollary 4.5.2 *Let M be a semi-discrete linearly compact R -module. Then*

$$(i) \quad N\dim_R M = N\dim_{\widehat{R}} M;$$

$$(ii) \quad \dim_R M = \dim_{\widehat{R}} M.$$

Proof. (i) Follows from Lemma (4.5.1)(ii) and from definition of noetherian dimension.

(ii) In the special case M is a finitely generated R -module, by [C-N1] 3.8 and from (A.5) we have that $M \cong \Lambda_J(M) = \widehat{M}$ and therefore $\dim_R M = \dim_{\widehat{R}} M$ by [B-S] 6.1.3, which is valid in the semilocal case by [G-S] 4.6 and by [G-S] 7.5. For any semi-discrete linearly compact module M , by main theorem in [Zös], there is a short exact sequence $0 \rightarrow N \rightarrow M \rightarrow A \rightarrow 0$, where N is finitely generated and A is artinian. Since $\dim_R A = \dim_{\widehat{R}} A = 0$, $\dim_R M = \max\{\dim_R N, \dim_R A\}$ and $\dim_R N = \dim_{\widehat{R}} N$, follows that $\dim_R M = \dim_{\widehat{R}} M$. ■

Remark 4.5.3

(i) Denote by \mathcal{C} the category of semi-discrete linearly compact R -modules. It is well known that the category \mathcal{C} contains the category of artinian R -modules and

also the category of finitely generated R -modules if R is complete (see Section 3.1). However, there are many semi-discrete linearly compact R -modules which are neither artinian nor finitely generated. An example for this conclusion can be found in [C-N1] 4.9. More generally, let R be complete ring, A an artinian R -module with $Ndim A > 0$ and N a finitely generated R -module with $dim N > 0$. Then $M = A \oplus N$ is semi-discrete linearly compact. Further, let Q be a quotient module of M , then Q is also a semi-discrete linearly compact R -module.

(ii) If $M \in \mathcal{C}$, then by Proposition (3.1.8) the Matlis dual $D(M)$ and the Macdonald dual M^* are the same. Moreover, M is linearly discrete by Proposition (4.4.5) and can be regarded by Lemma (4.5.1) as an \widehat{R} -module, therefore the Macdonald dual functor $(-)^*$ is a functor from \mathcal{C} to itself and we have by Remark (4.4.1) a topological isomorphism $\omega : M \rightarrow M^{**}$. Thus, $(-)^*$ is an equivalent functor on the category \mathcal{C} .

Proposition 4.5.4 *Let M be a semi-discrete linearly compact R -module. Then*

$$Ndim_R M^* = dim_R M \quad \text{and} \quad Ndim_R M = dim_R M^*.$$

Proof. From Lemma (4.5.1) (ii) and from Corollary (4.5.2) we may assume that R is a complete ring. If M is finitely generated R -module, M^* is artinian. Keep in mind in our case that $D(M) = M^*$. Firstly, we will show that $Ndim M^* = dim M$ when M is finitely generated. Since M^* is artinian, we have that $M^* = \bigoplus_{i=1}^r A_i$ where A_i is a R_{m_i} -module for each i , by [Sha] 1.4 and 1.7. By [C-N1] 7.4, $Ndim A_i = dim A_i^*$. Thus,

$$\begin{aligned} Ndim M^* &= \max\{Ndim A_i; i = 1, \dots, r\} \\ &= \max\{dim A_i^*; i = 1, \dots, r\} \\ &= dim M^{**} \\ &= dim \widehat{M} \\ &= dim M \end{aligned}$$

where the last two equalities occur because of Theorem (B.1)(iv) and by [C-N1] 3.8. If M is artinian then $Ndim M^* = dim M = 0$.

Suppose now that M is semi-discrete linearly compact. By main theorem in [Zös] there is a short exact sequence $0 \rightarrow N \rightarrow M \rightarrow A \rightarrow 0$ in which N is finitely generated and A is artinian. Thus we get by Macdonald duality an exact sequence

$0 \rightarrow A^* \rightarrow M^* \rightarrow N^* \rightarrow 0$, where N^* is artinian and A^* is finitely generated. Therefore

$$\begin{aligned} Ndim M^* &= \max\{Ndim N^*, Ndim A^*\} \\ &= \max\{dim N, dim A\} \\ &= dim M. \end{aligned}$$

The second equality follows from Remark (4.5.3) (ii). ■

The following lemma was inspired by [LRN] 3.16.

Lemma 4.5.5 *Let M be a non-zero artinian R -module with $Ndim M = d$ and J the Jacobson radical of R . Then $H_d^J(M) \neq 0$.*

Proof. From Lemma (4.5.1)(ii), Corollary (4.5.2) and by Independence Theorem ([B-S] 4.2.1) we may assume that R is a complete ring. By [Ooi] 1.6 the Matlis dual $D(M)$ is a non-zero finitely generated and $dim D(M) = d$ by Proposition (4.5.4). Hence $H_J^d(D(M)) \neq 0$, otherwise, $H_{\mathfrak{m}R_{\mathfrak{m}}}^d(D(M)_{\mathfrak{m}}) = H_{JR_{\mathfrak{m}}}^d(D(M)_{\mathfrak{m}}) = 0$ for any maximal \mathfrak{m} of R and this contradicts Grothendieck's Non-Vanishing Theorem ([B-S] 6.1.4). Therefore, from Theorem (4.4.6) and by [Ooi] 1.6 we have that

$$H_d^J(M) \cong H_d^J D(D((M))) \cong D(H_J^d(D(M))) \neq 0.$$

■

In [C-N1] 4.10 it was shown that $Ndim M = \max\{i \mid H_i^{\mathfrak{m}}(M) \neq 0\}$ if $Ndim M \neq 1$ when (R, \mathfrak{m}) is a local noetherian ring and M is a non-zero semi-discrete linearly compact module. We generalize this result for semilocal case. In [C-N1] 4.9 can be found an example which shows that the result is false when $Ndim M = 1$.

Theorem 4.5.6 *Let R be a ring, J the Jacobson radical of R and M a non-zero semi-discrete linearly compact R -module. Then*

$$(i) \ Ndim \Gamma_J(M) = \max\{i \mid H_i^J(M) \neq 0\} \text{ if } \Gamma_J(M) \neq 0;$$

$$(ii) \ Ndim M = \max\{i \mid H_i^J(M) \neq 0\} \text{ if } Ndim M \neq 1.$$

Proof. (i) By Theorem (3.1.3) $\Gamma_J(M)$ is an artinian R -module. From Theorem (4.2.5) and from Lemma (4.5.5) we have that

$$Ndim \Gamma_J(M) = \max\{i \mid H_i^J(\Gamma_J(M)) \neq 0\}.$$

The result follows by Proposition (4.2.3).

(ii) First, note by virtue of [Zös] 1.6(a) and from Remark (4.2.4)(ii) that if $Soc(M) = 0$ then $Ndim M \leq 1$. If $\Gamma_J(M) = 0$ then $Soc(M) = 0$ by Proposition (4.2.3). So we have from the hypothesis that $Ndim M = 0$. It follows that M is a finitely generated R -module and $H_0^J(M) \cong \widehat{M} \neq 0$, where \widehat{M} is the J -adic completion of M . Thus (ii) is proved in this case. Assume now that $\Gamma_J(M) \neq 0$. By (i) we have only to show that $Ndim M = Ndim \Gamma_J(M)$. Indeed, it is trivial for the case $Ndim M = 0$. Let $Ndim M > 1$. From the short exact sequence $0 \rightarrow \Gamma_J(M) \rightarrow M \rightarrow M/\Gamma_J(M) \rightarrow 0$ we have that

$$Ndim M = \max\{Ndim \Gamma_J(M), Ndim M/\Gamma_J(M)\}.$$

Since $Soc(M/\Gamma_J(M)) = 0$, $Ndim(M/\Gamma_J(M)) \leq 1$. Thus $Ndim M = Ndim \Gamma_J(M)$ as required. ■

In [C-N1] 4.12 there is a example which shows that the condition $\Gamma_J(M) \neq 0$ in Theorem (4.5.6)(i) is needful.

Corollary 4.5.7 *Let R be a ring, J the Jacobson radical of R and M a non-zero semi-discrete linearly compact module. Then $H_i^J(M) = 0$ for all $i \geq 0$ if and only if there exists an element $x \in J$ such that $xM = M$ and $(0 :_M x) = 0$.*

Proof. Suppose that $H_i^J(M) = 0$ for all $i \geq 0$. We obtain by [C-N1] 4.1 that $xM = M$ for some $x \in J$. On the other hand, it follows from the short exact sequence $0 \rightarrow (0 :_M x) \rightarrow M \xrightarrow{x} M \rightarrow 0$ that $H_i^J(0 :_M x) = 0$ for all $i \geq 0$. Since $(0 :_M x)$ is artinian by [Zös] corollary 1, follows that $(0 :_M x) = 0$ by Lemma (4.5.5). Conversely, suppose that $xM = M$ and $(0 :_M x) = 0$. From Lemma (4.1.2) we have that

$$H_i^J(M) = xH_i^J(M) = \bigcap_{t>0} x^t H_i^J(M) = 0$$

for all $i \geq 0$. ■

In [C-N1] 7.5 Cuong and Nam extend well-known results in Grothendieck's local cohomology theory of finitely generated R -modules for semi-discrete linearly compact modules. Here we generalize to the semilocal case.

Theorem 4.5.8 *Let R be a ring, J the Jacobson radical of R and M a non-zero semi-discrete linearly compact R -module. Then*

$$(i) \dim_R M = \max\{i \mid H_J^i(M) \neq 0\} \text{ if } \dim_R M \neq 1;$$

(ii) $\dim_R(\widehat{M}) = \max\{i \mid H_J^i(M) \neq 0\}$ if $\widehat{M} \neq 0$.

Proof. (i) Note by Lemma (4.5.1)(ii) and Corollary (4.5.2) that M is a semi-discrete linearly compact \widehat{R} -module with $\dim_R M = \dim_{\widehat{R}} M$. Moreover, the natural homomorphism $f : R \rightarrow \widehat{R}$ gives by [B-S] 4.2.1 an isomorphism $H_J^d(M) \cong H_J^d(M)$. Thus, we may assume without any loss of generality that R is a complete semilocal noetherian ring. Since M is a semi-discrete linearly compact R -module, M^* is also a semi-discrete linearly compact R -module by Remark (4.4.1)(i). Recall by Proposition (3.1.6) that $M^* = 0$ if and only if $M = 0$. Hence, since $1 \neq \dim M = N \dim M^*$, it follows from Proposition (4.5.4), Theorem (4.5.6)(ii) and Theorem (4.4.6)(i) that

$$\begin{aligned} \dim M = N \dim M^* &= \max\{i \mid H_i^J(M^*) \neq 0\} \\ &= \max\{i \mid (H_J^i(M))^* \neq 0\} \\ &= \max\{i \mid H_J^i(M) \neq 0\}. \end{aligned}$$

(ii) The continuous epimorphisms $M \rightarrow M/J^t M$ for all $t > 0$ induce by [C-N1] 2.4 a continuous epimorphism $\eta : M \rightarrow \widehat{M}$. Moreover η is the open homomorphism by Proposition (3.1.5). Thus \widehat{M} is also a semi-discrete linearly compact R -module. It follows from Corollary (4.5.2) that $\dim_R \widehat{M} = \dim_{\widehat{R}} \widehat{M}$. Hence, as in the proof of (i) we may assume without any loss of generality that R is a complete semilocal noetherian ring. Note that $H_0^I(M) = \Lambda_I(M)$ and $H_I^0(M) \cong \Gamma_I(M)$, hence by Theorem (4.4.6)(ii) we have that

$$0 \neq (\widehat{M})^* = (H_0^J(M))^* \cong \Gamma_J(M^*).$$

Thus, by virtue of Proposition (4.5.4), Theorem (4.5.6)(i) and Theorem (4.4.6)(ii) we have that

$$\begin{aligned} \dim \widehat{M} = N \dim (\widehat{M})^* = N \dim \Gamma_J(M^*) &= \max\{i \mid H_i^J(M^*) \neq 0\} \\ &= \max\{i \mid (H_J^i(M))^* \neq 0\} \\ &= \max\{i \mid H_J^i(M) \neq 0\}. \end{aligned}$$

■

In [C-N1] 7.6 there are examples which show that the conditions $\dim M \neq 1$ and $\widehat{M} \neq 0$ in Theorem (4.5.8) are needed.

In [C-N1] 7.7 was showed the unusual behavior on the vanishing theorem of local cohomology for semi-discrete linearly compact modules, with a characterization of semi-discrete linearly compact modules, whose all local cohomology modules are vanished. The next result generalize this result to the semilocal case.

Corollary 4.5.9 *Let R be a ring, J the Jacobson radical of R and M a non-zero semi-discrete linearly compact module. Then $H_J^i(M) = 0$ for all $i \geq 0$ if and only if there exists an element $x \in J$ such that $xM = M$ and $(0 :_M x) = 0$.*

Proof. The conclusion follows from Corollary (4.5.7) by using Theorem (4.4.6)(ii) and Remark (4.5.3)(ii). ■

Recall that a sequence of elements x_1, \dots, x_r in R is said to be an M -**regular** sequence if $M/(x_1, \dots, x_r)M \neq 0$ and $M/(x_1, \dots, x_{i-1})M \xrightarrow{x_i} M/(x_1, \dots, x_{i-1})M$ is injective for $i = 1, \dots, r$. Denote by $depth_I(M)$ the supremum of the lengths of all maximal M -regular sequences in the ideal I . Then we have the following result.

Theorem 4.5.10 *Let M be a semi-discrete linearly compact R -module and let I be an ideal of R such that $M/IM \neq 0$. Then*

$$depth_I(M) = \inf\{i \mid H_I^i(M) \neq 0\}.$$

Proof. Note by [Ooi] paragraph 3 that x_1, \dots, x_r is an M -regular sequence if and only if it is a $D(M)$ -coregular sequence. Since M is semi-discrete linearly compact, $D(M) = M^*$ and therefore $depth_I(M) = width_I(M^*)$. On the other hand, M^* is a semi-discrete linearly compact R -module by (4.4.1)(i) and $0 :_{M^*} I \cong (M/IM)^* \neq 0$. Thus the conclusion follows by virtue of Theorem (4.2.6) and Theorem (4.4.6)(ii). ■

Theorem 4.5.11 *Let M be a semi-discrete linearly compact R -module with $\dim_R M = d$. Then the following statements are true.*

(i) *The local cohomology modules $H_J^i(M)$ are artinian R -modules for all $i \geq 0$.*

(ii) *The local cohomology module $H_I^d(M)$ is artinian.*

Proof. Note first that if A is an artinian \widehat{R} -module, then A is an artinian R -module (see Section 3.1). Therefore, from the independence of the base ring of local cohomology ([B-S] 4.2.1) and by Lemma (4.5.1) we may assume without loss of generality that R is complete. Then, by applying the duality between local homology and local cohomology

Theorem (4.4.6), the statement (i) follows from Theorem (4.3.2) and the statement (ii) from Theorem (4.3.3). ■

As an consequence of Theorem (4.5.11) we get the following result.

Corollary 4.5.12 (See [B–S] 7.1.3 and 7.1.6) *Let M be a finitely generated R -module with $\dim_R M = d$. Then the local cohomology modules $H_{\mathfrak{J}}^i(M)$ and $H_{\mathfrak{I}}^d(M)$ are artinian R -modules for all $i \geq 0$.*

Appendices

Appendix A

I-adic Topology

This appendix is virtually a subset of paragraphs one and two of [G-S].

Definition A.1 (*I*-adic topology) *If I is an ideal of a ring R , we may consider in R the topology defined by taking the set of all powers I^n ($n \geq 0$) as a fundamental system of neighborhoods of 0. Note that the set $\{I^n\}_{n \geq 0}$ satisfies (TBS 1) and (TBS 2) of (1.3.7), namely, denoting $\{I^n\}_{n \geq 0} = \mathcal{M}$, we have that*

- (i) *Given L_1 and L_2 in \mathcal{M} there is L_3 in \mathcal{M} such that $L_3 \subseteq L_1 \cap L_2$; (TBS 1)*
- (ii) *Given $x \in R$ and $L \in \mathcal{M}$ there is a neighborhood U of zero in R such that $Ux \subseteq L$. (TBS 2)*

*Hence with this topology R is a topological ring. We call this topology the ***I*-adic topology**, or simply the *I*-topology, and we say that R is an *I*-adic ring, or *I*-ring. The *I*-adic topology is an example of linear topology.*

*The *I*-topology on a R -module M is defined by taking the *I*-topology on R and the topology given by all the submodules $I^n M$ ($n \geq 0$) of M . Note that $\{I^n M\}$ satisfies (TBS 1) and (TBS 2) of Corollary (1.3.7), namely, denoting $\{I^n M\}_{n \geq 0} = \mathcal{M}$, we have that*

- (i) *Given N_1 and N_2 in \mathcal{M} there is N_3 in \mathcal{M} such that $N_3 \subseteq N_1 \cap N_2$; (TBS 1)*
- (ii) *Given $x \in M$ and $N \in \mathcal{M}$ there is a neighborhood U of zero in R such that $Ux \subseteq N$. (TBS 2)*

Hence with this topology M is a topological module.

Definition A.2 (Filtration) Let M be an R -module. A chain $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n \supseteq \cdots$, where the M_n are submodules of M is called a **filtration** of M . A filtration is associated in a natural way with a topology in M , namely the one we get by taking $\{M_n\}$ as a fundamental system of neighborhoods of 0.

Let (M_n) be a filtration of M and I an ideal of R such that the following condition is verified:

$$IM_n \subseteq M_{n+1} \quad \text{for all } n \geq 0,$$

or equivalently:

$$I^k M_n \subseteq M_{n+k} \quad \text{for all } n, k \geq 0,$$

then we call (M_n) an **I -filtration**. Thus the I -topology of M is finer than (but not necessarily equal to) the topology defined by any I -filtration of M .

We say that the I -filtration (M_n) is **stable** if there exists $n_0 \geq 0$ such that the following equivalent conditions are verified:

$$M_{n+1} = IM_n \quad \text{for } n > n_0,$$

$$M_n = I^{n-n_0} M_{n_0} \quad \text{for } n > n_0,$$

$$M_{n+q} = I^q M_n \quad \text{for } n > n_0, q \geq 0.$$

The filtration $(I^n M)$, which defines the I -adic topology in M , is a stable I -filtration. Conversely the next lemma says that any stable I -filtration of M defines the I -topology.

Lemma A.3 If (M_n) and (M'_n) are stable I -filtrations of M , then they have bounded differences, that is, there exists n_0 such that $M_{n+n_0} \subseteq M'_n$ and $M'_{n+n_0} \subseteq M_n$ for all $n \geq 0$. Hence all stable I -filtrations determine the same topology on M , namely the I -topology.

The next three results are tools that are widely used throughout the work.

Theorem A.4 (Artin-Rees lemma) (See [C-N1] 1.4) Let R be a noetherian ring, I an ideal of R , M a finitely generated R -module, (M_n) a stable I -filtration of M and N a submodule of M . Then

(i) $(M_n \cap N)$ is a stable I -filtration of N ;

(ii) There exists an integer n_0 such that

$$(I^n M) \cap N = I^{n-n_0}((I^{n_0} M) \cap N)$$

for all $n \geq n_0$;

(iii) The filtrations $(I^n N)$ and $((I^n M) \cap N)$ have bounded differences; in particular the I -topology of N coincides with the topology induced on N by the I -topology of M .

Statement (ii) is what is usually known as the Artin-Rees Lemma, while (iii) is a topological but weaker version of that result. The Theorem (A.4) may be false if R is not noetherian. The reader can find examples in [G-S] Remark 2 of Theorem 1.4.

Now we recall some standard definitions. A ring R is local (resp. semilocal) if it has only one maximal ideal (resp. a finite number of maximal ideals). The intersection of all maximal ideals of a ring R is the **Jacobson radical** of R and is written $\text{rad } R$.

Theorem A.5 (Krull's intersection theorem) (See [C-N1] 1.5). *Let R be a noetherian ring, I an ideal of R and M a finitely generated R -module. Set $F = \bigcap I^n M$. Then $x \in F$ if and only if there exists $a \in I$ such that $(1 - a)x = 0$.*

In particular we have $F = 0$, if at least one of the following hypotheses is verified:

- (i) $I \subseteq \text{rad } R$;
- (ii) R is a local ring;
- (iii) R is an integral domain.

The Theorem (A.5) is false if R is not noetherian. An example can be found in [Ati] 10.17 Remark 2).

Proposition A.6 *Let R be a noetherian ring and let I be an ideal of R . The following are equivalent:*

- (a) $I \subseteq \text{rad } R$;
- (b) every finitely generated R -module is I -Hausdorff;
- (c) if M is any finitely generated R -module, every submodule of M is closed for the I -topology of M ;
- (d) every maximal ideal of R is closed for the I -topology.

Definition A.7 *A noetherian topological ring is said to be a **Zariski ring** if it is an I -adic ring for some I satisfying the equivalent conditions of Proposition (A.6).*

Example A.8 If $(R, \mathfrak{m}_1, \dots, \mathfrak{m}_n)$ is a semilocal noetherian ring and $J = \text{rad } R$, then R is a Zariski ring provided with the J -adic topology.

Let G be a filtered abelian group (that is a filtered \mathbb{Z} -module) and let (G_n) be its filtration. We can define a mapping

$$\nu : G \rightarrow \mathbb{N} \cup \{\infty\}$$

in the following way: $\nu(x) = \sup\{n \in \mathbb{N} | x \in G_n\}$. It is clear that $\nu(x) = \infty$ if and only if $x \in \bigcap G_n$, that is $x \in \overline{\{0\}}$ ([War] 3.3). The mapping allows us to define a *pseudometric* in G : let

$$d : G \times G \rightarrow \mathbb{R}$$

be the mapping defined by $d(x, y) = e^{-\nu(x-y)}$ (we agree that $e^{-\infty} = 0$). Then it is easy to see that

$$d(x, y) \leq \sup\{d(x, z), d(y, z)\}$$

and that d defines in G the topology induced by the filtration (G_n) .

Note that d is a metric if and only if $\bigcap G_n = \{0\}$, that is if and only if G is Hausdorff ([War] 3.4). It is also clear that d induces a metric on $G / \bigcap G_n$.

Thus if G is a filtered group we can define limits and Cauchy sequences. We say that a sequence (x_n) of elements of G is *convergent* if there is an $x \in G$ such that $\lim d(x, x_n) = 0$; we say also that x is a *limit* for (x_n) and write $x = \lim x_n$. A convergent sequence is always a *Cauchy sequence*, that is $\lim d(x_n, x_m) = 0$ as $m, n \rightarrow \infty$. The converse is not always true.

Definition A.9 A filtered group G is *complete* if and only if the following equivalent conditions are satisfied:

- (a) G is Hausdorff and any Cauchy sequence of elements of G is convergent;
- (b) Every Cauchy sequence of elements of G has a unique limit (in G).

Now we want to associate with any filtered group a complete one in a natural way. We begin with some preliminaries.

Definition A.10 A continuous homomorphism $f : G \rightarrow G'$ of topological groups is said to be *strict* if the quotient topology of $f(G)$ coincides with the topology induced by G' .

Note that if G' is any non discrete topological group and G is the same group but with the discrete topology, the identity map $G \rightarrow G'$ is continuous but not strict.

The next proposition gives two equivalent conditions we shall use to define a completion.

Proposition A.11 *Let G be a filtered group, (G_n) its filtration, G' a complete filtered group and $f : G \rightarrow G'$ a continuous homomorphism. Then the following conditions are equivalent:*

- (a) *$\text{Ker } f = \cap G_n$, f is strict and $f(G)$ is dense in G' ;*
- (b) *For any complete filtered group H and any continuous homomorphism $g : G \rightarrow H$ there is a unique continuous homomorphism $g' : G' \rightarrow H$ such that $g' \circ f = g$.*

Definition A.12 *Let G be a filtered group. A **completion** of G is a pair (G', f) , where G' is a complete filtered group and $f : G \rightarrow G'$ is a continuous homomorphism, satisfying the equivalent conditions of Proposition (A.11). We say also that G' is a completion of G , if the map f is understood.*

Before we show that a completion exists, we give some easy consequences of the above results.

Corollary A.13 *Let (G', f) be a completion of the filtered group G . Then*

- (i) *f is injective if and only if G is Hausdorff.*
- (ii) *f is bijective if and only if G is complete.*

Corollary A.14 *Let $u : G_1 \rightarrow G_2$ be a continuous homomorphism of filtered groups and let (G'_i, f_i) be a completion of G_i ($i = 1, 2$). Then there is a unique continuous homomorphism $u' : G'_1 \rightarrow G'_2$ such that $u' \circ f_1 = f_2 \circ u$. Moreover if u is the identity so is u' and if $v : G_2 \rightarrow G_3$ is a second continuous homomorphism of filtered groups then $(v \circ u)' = v' \circ u'$.*

The next corollary shows that *completion is unique (up to isomorphisms)*.

Corollary A.15 *Let G be a filtered group and let (G', f') and (G'', f'') be two completions of G . Then there is a unique isomorphism $u : G' \rightarrow G''$, which is also a homeomorphism, such that $u \circ f' = f''$.*

Now we want to show that any filtered group has a completion. Here we give the classical construction, based upon Cauchy sequences. Let \mathcal{G} be the set of all Cauchy

sequences of elements of G , and let \mathcal{N} be the set of the zero-sequences, that is of sequences having 0 as a limit. Clearly \mathcal{G} is a group and \mathcal{N} is a subgroup of \mathcal{G} . The quotient group $\widehat{G} = \mathcal{G}/\mathcal{N}$ can be given a metric by setting

$$d((\overline{x_n}), (\overline{y_n})) = \inf d(x_n, y_n)$$

where the upper bar denotes reduction modulo \mathcal{N} . It is not difficult to see that such d is a well defined metric on \widehat{G} and that \widehat{G} is a topological group with respect to d .

Let now $f : G \rightarrow \widehat{G}$ be the canonical map, which sends each $x \in G$ to the class of the constant sequence (x, x, \dots) . We claim that (\widehat{G}, f) is a completion of G . More precisely we have the follows result.

Proposition A.16 *Let G be a filtered group, (G_n) its filtration and $f : G \rightarrow \widehat{G}$ the canonical map. Then*

- (i) \widehat{G}_n can be canonically identified with a subgroup of \widehat{G} ($n = 0, 1, \dots$).
- (ii) The topology of \widehat{G} is induced by the filtration (\widehat{G}_n) .
- (iii) The filtered group \widehat{G} , together with the map f , is a completion of G .

Now we want to show that the completion of a filtered group can be obtained by means of inverse limits.

Definition A.17 *We start with the definition of an inverse system (or projective system) of groups and homomorphisms. Let (I, \leq) be a directed set. Let $(A_i)_{i \in I}$ be a family of groups and suppose we have a family of homomorphisms $f_{ji} : A_j \rightarrow A_i$ for all $j \geq i$ (note the order), with the following properties:*

- (i) f_{ii} is the identity on A_i ;
- (ii) $f_{ki} = f_{ji} \circ f_{kj}$ for all $i \leq j \leq k$.

Then the pair $(A_i, f_{ji})_{j \geq i \in I}$ is called an **inverse system** of groups and morphisms over I and has an **inverse limit** $\varprojlim A_i = A$ defined by:

$$A = \{(a_i) \in \prod A_i \mid f_{ji}(a_j) = a_i \text{ for all } j \geq i \in I\}.$$

This is a subgroup of $\prod A_i$ and comes equipped with natural projections $\pi_i : A \rightarrow A_i$ which pick out the i th component of the direct product for each $i \in I$, satisfying $\pi_i = f_{ji} \circ \pi_j$ for all $j \geq i$.

If G is a filtered group and (G_i) is its filtration, we have the inverse systems $(G/G_i, f_{ji})$ and, for fixed i , $(G_i/G_{i+k}, f_{j(i+k)})$, where f_{ji} and $f_{j(i+k)}$ are the canonical mappings. Set $\tilde{G} = \varprojlim G/G_i$ and $\tilde{G}_i = \varprojlim G_i/G_{i+k}$. Note that (\tilde{G}_i) is a filtration of (\tilde{G}) and that there is an homomorphism $g : G \rightarrow \tilde{G}$ defined by $g(x) = (x_0, x_1, \dots)$ where x_i is the class of $x \in G$ modulo G_i .

Proposition A.18 *The filtered group \tilde{G} , together with the homomorphism $g : G \rightarrow \tilde{G}$, is a completion of G .*

Proposition A.19 *Let R be a filtered ring, (a_n) its filtration and (\hat{R}, f) the completion of R as a filtered group. Then*

- (i) \hat{R} can be given a unique topological ring structure, such that f is a continuous ring homomorphism.
- (ii) The completion \hat{a}_n of each a_n can be embedded in \hat{R} as an ideal and the topology of \hat{R} is induced by the filtration (\hat{a}_n) .

Proposition A.20 *Let R be a filtered ring, M a filtered R -module and (M_n) its filtration. Let \hat{R} be the completion of R , and \hat{M}, \hat{M}_n the completions of M and M_n as filtered groups. Then, if M is a topological R -module we have that*

- (i) \hat{M} has a unique topological \hat{R} -module structure, such that the canonical homomorphism $M \rightarrow \hat{M}$ is R -linear.
- (ii) Each \hat{M}_n is canonically embedded in \hat{M} as a R -submodule and the topology of \hat{M} is induced by the filtration (\hat{M}_n) .

Proposition A.21 *Let R be an I -adic ring and let M be an R -module. Then*

- (i) The I -completions \hat{R} and \hat{M} exist, and \hat{M} is a topological \hat{R} -module.
- (ii) $\hat{R} \cong \varprojlim R/I^n$ and $\hat{M} \cong \varprojlim M/I^n M$.
- (iii) $R/I \cong \hat{R}/\hat{I}$.
- (iv) The (R/I) -modules M/IM and $\hat{M}/(\hat{IM})$ are isomorphic.

More results over I -adic topology can be found in [G-S].

Appendix B

Matlis dual in semilocal case

This appendix contains some results involving the Matlis dual. An important result by A. Ooishi ([Ooi]) is remembered and some results from Chapter 10 of [B-S] are generalized to the semilocal context.

The inspiration for this appendix was the Theorem (B.3), that is a generalization of 10.2.3 of [B-S]. Here the ring is semilocal, while in result of [B-S] the ring is local. This proposition was developed initially with the intention of showing that $(K_j)^* \cong K_j$, for any $i = 1, \dots, n$, where $(R, \mathfrak{m}_1, \dots, \mathfrak{m}_n)$ is a semilocal noetherian ring and $K_j = R/\mathfrak{m}_j$. Some time later, it was noticed that Ooishi already talked about such a result in [Ooi] 1.1. Anyway, we decided to include in this appendix the result of Ooishi and our proposition with an alternative demonstration for the most curious reader. Furthermore, some results are achieved as consequences of Theorem (B.3).

Set $E = E_R(\oplus_i R/\mathfrak{m}_i)$, where $(R, \mathfrak{m}_1, \dots, \mathfrak{m}_n)$ is a (semilocal noetherian, as usual) ring and $\mathfrak{J} = \bigcap_{i=1}^n \mathfrak{m}_i$. Here, differently of the rest of the work, $D(M) = \text{Hom}_R(M, E)$ denote the set of all homomorphisms (not necessarily continuous) of M into E , this is usual notation for Matlis dual.

Note that $D(R)$ is naturally isomorphic to E , and that $D(E) = \text{Hom}_R(E, E)$ is just the R -endomorphism ring of E considered as an R -module in the natural way. For each R -module M , let $\mu_M : M \rightarrow DD(M) = \text{Hom}_R(\text{Hom}_R(M, E), E)$ be the natural R -homomorphism given by $(\mu_M(x))(f) = f(x)$ for all $x \in M$ and $f \in \text{Hom}_R(M, E)$.

Theorem B.1 (See [Ooi] 1.6) *Let $(R, \mathfrak{m}_1, \dots, \mathfrak{m}_n)$ be a semilocal commutative noetherian ring, M a R -module and $\text{Mod}^F(R)$ (resp. $\text{Mod}^A(R)$) the category of finitely generated (resp. artinian) R -modules. The statements are true:*

- (i) *$\text{Hom}_R(E, E)$ is isomorphic (as R -algebra) to \widehat{R} , the completion of R with respect to its Jacobson radical.*
- (ii) *If M is finitely generated R -module, then $D(M)$ is an artinian R -module.*
- (iii) *Assume that R is complete. If M is an artinian R -module, then $D(M)$ is finitely generated R -module.*
- (iv) *If M is finitely generated R -module, then $DD(M)$ is isomorphic to $\widehat{M} = M \otimes_R \widehat{R}$.*
- (v) *If M is an artinian R -module, then $DD(M)$ is isomorphic to M .*
- (vi) *Assume that R is complete. Then D is an exact, R -linear anti-equivalence from $\text{Mod}^F(R)$ to $\text{Mod}^A(R)$, namely, D is a contravariant, exact, R -linear functor which is faithful, full and representative.*
- (vii) *$\text{Mod}^A(R)$ and $\text{Mod}^F(\widehat{R})$ are equivalent to each other.*
- (viii) *For any R -module M , $\text{Ann}_R(M) = \text{Ann}_R(D(M))$. In particular, $\dim(R/\text{Ann}_R M) = \dim(R/\text{Ann}_R D(M))$ as rings.*

From Theorem (B.1)(viii), $\dim_R(M) = \dim_R(D(M))$ as R -modules when M is finitely generated, agreeing with the definition of dimension of a module adopted in (4.5).

Lemma B.2 (See [B-S] 10.1.16) *Suppose that the R -module M is annihilated by the ideal \mathfrak{b} of R . We can regard M and $(0 :_{E_R(M)} \mathfrak{b})$ as modules over R/\mathfrak{b} in natural ways: when this is done, $(0 :_{E_R(M)} \mathfrak{b}) \cong E_{R/\mathfrak{b}}(M)$.*

Theorem B.3 *Suppose that $(R, \mathfrak{m}_1, \dots, \mathfrak{m}_n)$ is artinian. Then*

- (i) *$D(K_j) \cong K_j$ (as R -modules) for all j ;*
- (ii) *for each finitely generated R -module M (that is, for each R -module M of finite length), the Matlis dual $D(M)$ is also finitely generated and $\ell(D(M)) = \ell(M)$;*
- (iii) *$E = E_R(\oplus_i R/\mathfrak{m}_i)$ is finitely generated (and so artinian), and $l(E) = l(R)$;*
- (iv) *for each finitely generated R -module M (that is, for each artinian R -module M), the map $\mu_M : M \rightarrow DD(M)$ is an isomorphism; and*
- (v) *for each $f \in \text{Hom}_R(E, E)$, there is a unique $r_f \in R$ such that $f(x) = r_f x$ for all $x \in E$.*

Proof. (i) We have

$$\begin{aligned}
D(K_j) &= \text{Hom}_R(R/\mathfrak{m}_j, E(R/\mathfrak{J})) \\
&\cong \text{Hom}_R(R/\mathfrak{m}_j, \bigoplus_{i=1}^n E(R/\mathfrak{m}_i)) \\
&\cong \bigoplus_{i=1}^n \text{Hom}_R(R/\mathfrak{m}_j, E(K_i)) \\
&\cong \bigoplus_{i=1}^n (0 :_{E(K_i)} \mathfrak{m}_j) \tag{I}
\end{aligned}$$

Remember that the 0 in (I) is the null element of $E(K_i)$. Since K_i is submodule of $E(K_i)$, this 0 represents the class of elements of R contained in \mathfrak{m}_i . Now, we will study $(0 :_{E(K_i)} \mathfrak{m}_j)$ for each i separating in two cases.

Firstly, we will prove that $(0 :_{E(K_i)} \mathfrak{m}_j) = 0$ whenever $i \neq j$. Indeed, suppose there is $x \in (0 :_{E(K_i)} \mathfrak{m}_j)$ with $x \neq 0$, that is, $xb = 0$ for all $b \in \mathfrak{m}_j$. Since $x \in E(K_i)$, there is $a \in R$ such that $ax \in K_i$ with $ax \neq 0$. Therefore, $a(xb) = 0$ and hence $(ax)b = 0$ for all $b \in \mathfrak{m}_j$. Set $\bar{y} = ax \in R/\mathfrak{m}_i$, where $y \in R$. Thus $\bar{y}b = 0$ and hence $yb \in \mathfrak{m}_i$ for all $b \in \mathfrak{m}_j$. Since $y \notin \mathfrak{m}_i$, we have that $b \in \mathfrak{m}_i$ for all $b \in \mathfrak{m}_j$, that is, $\mathfrak{m}_j \subseteq \mathfrak{m}_i$ and hence $\mathfrak{m}_j = \mathfrak{m}_i$. Contradiction.

Otherwise, if $i = j$, by Lemma (B.2), $(0 :_{E(K_j)} \mathfrak{m}_j) \cong E_{K_j}(K_j)$.

Therefore, from (I) $D(K_j) \cong E_{K_j}(K_j) \cong K_j$, for all j .

(ii) We will use induction on length. Suppose $\ell(M) = 1$. Thus $M \cong K_j$ for some j and $D(M) \cong D(K_j) \cong K_j$. Hence $D(M)$ is finitely generated and $\ell(D(M)) = \ell(K) = \ell(M)$.

Now, supposed the statement valid for $\ell(M) = n-1$. Consider the decomposition series $M = M_n \supseteq \cdots \supseteq M_0 = 0$ and the exact sequence

$$0 \rightarrow M_{n-1} \rightarrow M_n \rightarrow M_n/M_{n-1} \rightarrow 0$$

that induces

$$0 \rightarrow D(M_{n-1}) \rightarrow D(M_n) \rightarrow D(M_n/M_{n-1}) \rightarrow 0.$$

Thus we have the following equalities

$$\ell(M_n) = \ell(M_{n-1}) + \ell(M_n/M_{n-1})$$

$$\ell(D(M_n)) = \ell(D(M_{n-1})) + \ell(D(M_n/M_{n-1})).$$

By induction hypotheses $\ell(D(M_n)) = \ell(M_n)$ and $D(M_n)$ is finitely generated.

(iii) Follows from (ii), since $E \cong D(R)$.

(iv) Let M be a finitely generated R -module. Two uses of part (ii) show that $DD(M)$ is also finitely generated and has $\ell(DD(M)) = \ell(M)$. Since μ_M is injective, it follows that $M \cong \text{Im}(\mu_M)$ is submodule of $DD(M)$. Hence $M \cong DD(M)$.

(v) By part (iv) the map

$$\begin{aligned} \mu_R : R &\rightarrow \text{Hom}_R(\text{Hom}_R(R, E)E) \\ a &\mapsto \mu_R(a) : \text{Hom}_R(R, E) \rightarrow E \\ &g \mapsto \mu_R(a)(g) = g(a) \end{aligned}$$

is an isomorphism.

We have also the isomorphism

$$\begin{aligned} \varphi : E &\rightarrow \text{Hom}_R(R, E) \\ x &\mapsto \varphi(x) = \varphi_x : R \rightarrow E \\ &a \mapsto \varphi_x(a) = ax \end{aligned}$$

whose inverse map is

$$\begin{aligned} \psi : \text{Hom}_R(R, E) &\rightarrow E \\ f &\mapsto f(1) \end{aligned}$$

and the induced map

$$\begin{aligned} \varphi^* : \text{Hom}_R(\text{Hom}_R(R, E), E) &\rightarrow \text{Hom}_R(E, E) \\ h &\mapsto h \circ \varphi \end{aligned}$$

whose inverse map is ψ^* .

Now, set $\theta = \varphi^* \circ \mu_R$ the isomorphism

$$\begin{aligned} \theta : R &\rightarrow \text{Hom}_R(E, E) \\ a &\mapsto (\varphi^* \circ \mu_R)(a). \end{aligned}$$

Note that $\theta(a)$ is the endomorphism of E given by multiplication by a . Indeed

$$\begin{aligned} (\varphi^* \circ \mu_R)(a) = \mu_R(a) \circ \varphi : E &\rightarrow E \\ x &\mapsto \mu_R(a) \circ \varphi(x) = \mu_R(a)(\varphi_x) = \varphi_x(a) = ax. \end{aligned}$$

■

Since K_j is semi-discrete R -module for any j by Proposition (3.3.4), we have that $K_j = D(K_j) = (K_j)^*$ (the Macdonal dual) when R is artinian.

Now we will generalize some more results that are in chapter 10 of [B-S].

Proposition B.4 *Let $(R, \mathfrak{m}_1, \dots, \mathfrak{m}_n)$ be a ring. By Remark (3.1.1) the R -module $E = E_R(\oplus_i R/\mathfrak{m}_i)$ is artinian and has a natural structure as an \widehat{R} -module.*

The natural R -homomorphism $\theta : \widehat{R} \rightarrow \text{Hom}_R(E, E)$ for which $\theta(\widehat{r}) = \widehat{r}\text{Id}_E$ for all $\widehat{r} \in \widehat{R}$ is an isomorphism. Thus, for each $f \in \text{Hom}_R(E, E)$ there is a unique $\widehat{r} \in \widehat{R}$ such that $f(x) = \widehat{r}_f x$ for all $x \in E$.

Proof. For each $n \in \mathbb{N}$, set $E_n := (0 :_E \mathfrak{J}^n)$. Let $f \in \text{Hom}_R(E, E)$ and $t \in \mathbb{N}$. Of course, R/\mathfrak{J} is annihilated by \mathfrak{J}^t . By Theorem (B.3), there is an isomorphism of R/\mathfrak{J}^t -modules $E_t \cong E_{R/\mathfrak{J}^t}((R/\mathfrak{J}^t)/(\mathfrak{J}/\mathfrak{J}^t))$. Note that $f(E_t) \subseteq E_t$ and that R/\mathfrak{J}^t is an artinian ring. Therefore, by Theorem (B.3)(v), there exists $r_t \in R$ such that $f(e) = r_t e$ for all $e \in E_t$, and moreover, if r'_t is any other element of R such that $f(e) = r'_t e$ for all $e \in E_t$, then the uniqueness aspect Theorem of (B.3)(v) ensures that $r_t + \mathfrak{J}^t = r'_t + \mathfrak{J}^t$, that is, $r_t - r'_t \in \mathfrak{J}^t$.

We can proceed as above for each $t \in \mathbb{N}$ and so construct a uniquely determined sequence $(r_n + \mathfrak{J}^n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} R/\mathfrak{J}^n$ with the property that, for every $n \in \mathbb{N}$, we have that $f(e) = r_n e$ for all $e \in E_n$. Furthermore, for n and $h \in \mathbb{N}$ we have that $E_n \subseteq E_{n+h}$ and so, since $f(e) = r_n e = r_{n+h} e$ for all $e \in E_n$, it follows from the Section (3.1) that $r_n - r_{n+h} \in \mathfrak{J}^n$.

We have therefore found a uniquely determined sequence

$$(r_n + \mathfrak{J}^n)_{n \in \mathbb{N}} \in \varprojlim_{n \in \mathbb{N}} R/\mathfrak{J}^n = \widehat{R}$$

such that for every $n \in \mathbb{N}$, $f(e) = r_n e$ for all $e \in E_n$. Since $E = \bigcup_{n \in \mathbb{N}} E_n$ by Remark (3.1.1), it follows that there is exactly one $\widehat{r} \in \widehat{R}$ such that $f(x) = \widehat{r}x$ for all $x \in E$. ■

Proposition B.5 *Suppose that $(R, \mathfrak{m}_1, \dots, \mathfrak{m}_n)$ is complete.*

- (i) *For each $f \in \text{Hom}_R(E, E)$ there is a unique $r_f \in R$ such that $f(x) = r_f x$ for all $x \in E$.*
- (ii) *Whenever N is a finitely generated R -module, the natural homomorphism $\mu_N : N \rightarrow DD(N)$ is an isomorphism and $D(N)$ is artinian.*
- (iii) *Whenever A is an artinian R -module, the natural homomorphism $\mu_A : A \rightarrow DD(A)$ is an isomorphism and $D(A)$ is noetherian.*

Proof. (i) This is immediate from Proposition (B.4).

(ii) The composition

$$\theta : R \xrightarrow{\mu_R} \text{Hom}_R(\text{Hom}_R(R, E), E) \xrightarrow{\cong} \text{Hom}_R(E, E),$$

in which the second map is the obvious natural isomorphism, is such that $\theta(r) = r\text{Id}_E$ for all $r \in R$. We have seen in part (i) that θ is an isomorphism and therefore μ_R is an isomorphism.

The identity functor and DD are both additive and the result of application of an additive functor to a split short exact sequence is again a split short exact sequence. Also, μ is a natural transformation of functors. We can therefore deduce, by induction on rank, that μ_F is an isomorphism whenever F is a finitely generated free R -module.

Let N be an arbitrary finitely generated R -module. Then N can be included in an exact sequence $F_1 \rightarrow F_0 \rightarrow N \rightarrow 0$ in which F_1 and F_0 are finitely generated free R -modules. Since the functor DD is additive and exact and μ is a natural transformation of functors, the above exact sequence induces a commutative diagram

$$\begin{array}{ccccccccc} F_1 & \longrightarrow & F_0 & \longrightarrow & N & \longrightarrow & 0 & \longrightarrow & 0 \\ \cong \downarrow & & \cong \downarrow & & \mu_N \downarrow & & \downarrow & & \downarrow \\ DD(F_1) & \longrightarrow & DD(F_0) & \longrightarrow & DD(N) & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

with exact rows. It therefore follows from the Five Lemma (see [Rot] 2.72) that μ_N is an isomorphism.

Finally, the application of the contravariant, exact functor D to the exact sequence $F_0 \rightarrow N \rightarrow 0$ shows that $D(N)$ is isomorphic to a submodule of $D(F_0)$. Since $D(F_0)$ is isomorphic to a direct sum of finitely many copies of $D(R)$ and $D(R) \cong E$, it follows from Remark (3.1.1) that $D(N)$ is artinian.

(iii) The composition

$$E \xrightarrow{\mu_E} \text{Hom}_R(\text{Hom}_R(E, E), E) \xrightarrow{\text{Hom}_R(\theta, \text{Id}_E)} \text{Hom}_R(R, E) \xrightarrow{\cong} E,$$

where θ is the isomorphism used in the proof of part (ii) above, is just the identity map and therefore μ_E is an isomorphism. We can now use the additivity of DD and the natural transformation μ to deduce, by induction on t , that μ_{E^t} is an isomorphism for all $t \in \mathbb{N}$.

Let A be an arbitrary artinian R -module. Two uses of Corollary [S-V] 3.21 show that there is an exact sequence $0 \rightarrow A \rightarrow E^{n_0} \rightarrow E^{n_1}$ for suitable positive integers n_0 and n_1 . We can now use the exactness of the functor DD together with the natural transformation μ , as we did in the above proof of part (ii), to obtain from this exact sequence a commutative diagram with exact rows and another application of the Five Lemma will yield the desired conclusion that μ_A is an isomorphism.

Finally, the application of the contravariant, exact functor D to the exact sequence $0 \rightarrow A \rightarrow E^{n_0}$ shows that $D(A)$ is a homomorphic image of $D(E^{n_0}) \cong (D(E))^{n_0}$. Since $\theta : R \rightarrow D(E)$ is an isomorphism, $D(A)$ is a homomorphic image of a finitely generated free R -module and so it is noetherian. ■

Proposition B.6 Partial Matlis Duality Theorem. *Let $(R, \mathfrak{m}_1, \dots, \mathfrak{m}_n)$ be a ring. By Remark (3.1.1) the R -module $E = E_R(\bigoplus_i R/\mathfrak{m}_i)$ is artinian and has a natural structure as an \widehat{R} -module.*

- (i) *For each $f \in \text{Hom}_R(E, E)$ there is a unique $\widehat{r} \in \widehat{R}$ such that $f(x) = \widehat{r}x$ for all $x \in E$.*
- (ii) *Whenever N is a finitely generated R -module, the natural homomorphism $\mu_N : N \rightarrow DD(N)$ is injective, $D(N)$ is artinian and $N \otimes_R \widehat{R} \cong DD(N)$.*
- (iii) *Whenever A is an artinian R -module, it has a natural structure as an \widehat{R} -module, it is artinian as such and $D(A)$ is noetherian \widehat{R} -module \widehat{R} -isomorphic to the Matlis dual of A over \widehat{R} .*

Proof. (i) Was showed in Proposition (B.4).

(ii) Let ν be the natural transformation from the identity functor to the functor $(-) \otimes_R \text{Hom}_R(E, E)$ (from category of R -modules to itself) given by the formula $\nu_M(m) = m \otimes \text{Id}_E$ for each R -module M and each $m \in M$.

Let N be a finitely generated R -module. Then, since N is a homomorphic image of R^h for some $h \in \mathbb{N}$, the Matlis dual $D(N)$ is isomorphic to a submodule of $D(R^h)$, since $D(R^h) \cong D(R)^h \cong E^h$, we deduce from Remark (3.1.1) that $D(N)$ is artinian. Furthermore, it is straightforward to check that the diagram

$$\begin{array}{ccc}
 N & \xrightarrow{\nu_N} & N \otimes_R \text{Hom}_R(E, E) \\
 & \searrow \mu_N & \downarrow \cong \\
 & & DD(N)
 \end{array}
 ,$$

where the vertical isomorphism is given by [B-S] 10.2.16, commutes. Now, by Proposition (3.1.6), the natural map μ_N is injective. By Proposition (B.5), $\text{Hom}_R(E, E) \cong \widehat{R}$ under an isomorphism which maps Id_E to $1_{\widehat{R}}$ and so, speaking loosely, we can regard the duality map $\mu_N : N \rightarrow DD(N)$ as the embedding of N into its completion.

(iii) Let A be an artinian R -module. The Section (3.1) shows that A can be regarded as an artinian \widehat{R} -module in a natural way. By Remark (3.1.1) the R -module E is artinian and so it too can be regarded as an artinian \widehat{R} -module in a natural way. By [Lea] 3.1.2 there is an \widehat{R} -homomorphism $E(R/\mathfrak{J}) \cong E_{\widehat{R}}(\widehat{R}/\widehat{\mathfrak{J}})$. By Section (3.1) $D(A) = \text{Hom}_R(A, E) = \text{Hom}_{\widehat{R}}(A, E)$ and so $D(A)$ has a natural (unambiguous) \widehat{R} -module structure. It follows from these observations that, as \widehat{R} -module, $D(A) \cong \text{Hom}_{\widehat{R}}(A, E_{\widehat{R}}(\widehat{R}/\widehat{\mathfrak{J}}))$. Finally, Proposition (B.5) shows that this is a noetherian \widehat{R} -module. ■

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