**Universidade Federal da Paraíba**
**Programa de Pós-Graduação em Matemática**
**Doutorado em Matemática**

# On Minimax and Cominimax Modules Relative to a Good Family of Ideals

**por**

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**Fevereiro/2020**

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Tese apresentada ao Corpo Docente do Programa de Pós-Graduação em Matemática - UFPB, como requisito parcial para obtenção do título de Doutor em Matemática.

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**Fevereiro/2020**

†Este trabalho contou com apoio financeiro da CAPES.

# Abstract

This work develops a study of the class of minimax modules relative to a good family of ideals and introduces the collection of the $(\mathcal{S}, I, \beta)$-cominimax modules, where $\mathcal{S}$ is a Serre class in the $R$-modules category. Also, it addresses a generalized local cohomology module and ideal transforms with support into a good family of ideals. In addition, some results of minimaximality are presented for generalized local cohomology modules and generalized ideal transforms.

**Keywords:** Minimax modules; Cominimaximality; Generalized local cohomology; Ideal transforms.

# Agradecimentos

A minha mãe Maria Aparecida da Silva Bezerra, aos meus irmãos Maria Aparecida da Silva Silvestre, Maria José Bezerra Silvestre, João Bezerra da Silva, Rosângela Bezerra Silvestre, Simone Bezerra Silvestre, Elizângela Bezerra Silvestre, e aos demais familiares, por estarem ao meu lado incondicionalmente.

Aos Professores Ronaldo Freire de Lima, Odirlei Silva Jesus, Flank David Morais Bezerra, Uberlandio Batista Severo, Pedro Antonio Gómes Venegas, Aurélio Menegon Neto, Everaldo Souto de Medeiros, Pedro Antonio Hinojosa Vera e Eraldo Almeida Lima Junior por todo o incentivo e apoio intelectual antes e durante essa caminhada.

Aos professores Roberto Callejas Bedregal e Napoleón Caro Tuesta, por toda a paciência e excelente orientação. Agradeço ainda a estes, pelo ótimo acolhimento que contribuiu de forma significativa para aumentar o meu interesse pela área de Álgebra.

Agradeço aos meus amigos da graduação, em especial a Jacileide Adelino da Silva, Jéssica Patrícia Garcia de Araújo, Francisco Guedes de Moura, José Adilson Félix de Freitas, Ronaldo César Duarte, Geilson Ferreira Germano, Ruan Barbosa Fernandes e Antônio Djackson Alves da Silva, pelo apoio nos momentos bons e complicados.

A todos os amigos da Pós-graduação, em especial a Sérgio Romero Vital dos Santos Filho, Rubens Neves e Silva e Marcos Aurélio Guimarães Monteiro pelo simples fato existirem e pela amizade durante as fases boas e também nas fases difíceis desse trajeto. Quero ainda agradecer aos amigos Douglas Magno Gomes de Lima, Geovane de Souza Ferreira Júnior, Johnatan da Silva Costa, Victor Camillo Batista Pinheiro, Fernanda Roing, Diego Dias Félix, Caio Ilan Ferreira Rodrigues, Lucas Araújo Santos, Guilherme Francisco do Nascimento, Jonison Lucas dos Santos Carvalho, Thyago Santos de Souza, Francisco Calvi da Cruz Junior, Laise Dias Alves Araújo, Oslenne Nogueira de Araújo, Thiago Fiel da Costa Cabral, Dayane Santos de Lira, Lucas de Carvalho Nascimento, Lorena Maria Augusto Pequeno Silva, Douglas de Souza Queiroz, Ginaldo de Santana

Sá, Kelvin Jhonson Ribeiro de Sousa Almeida Silva, Janiely Maria da Silva e Lindinês Coleta da Silva pelos intensos momentos de estudos, conquistas, trabalhos em equipe durante o curso e pelo apoio em momentos difíceis.

Agradeço a CAPES, pelo apoio financeiro.

*"O conhecimento torna a alma jovem e diminui a amargura da velhice. Colhe, pois, a sabedoria. Armazena suavidade para o amanhã."*

*Leonardo da Vinci*

# Dedicatória

Aos meus familiares e a todos os meus professores.

**Catalogação na publicação**
**Seção de Catalogação e Classificação**

S587o Silvestre, Renato Bezerra.
On minimax and cominimax modules relative to a good family of ideals / Renato Bezerra Silvestre. - João Pessoa, 2020.
88 f.

Orientação: Roberto Callejas Bedregal.
Coorientação: Napoleón Caro Tuesta.
Tese (Doutorado) - UFPB/CCEN.

1. Matemática. 2. Álgebra. 3. Matemática aplicada. I. Bedregal, Roberto Callejas. II. Tuesta, Napoleón Caro. III. Título.

UFPB/BC CDU 51(043)

Elaborado por Larissa Silva Oliveira de Mesquita - CRB-15/746

**Universidade Federal da Paraíba**

**Programa de Pós-Graduação em Matemática**

**Doutorado em Matemática**

Área de Concentração: Álgebra

Aprovada em 27 de Fevereiro de 2020

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Tese apresentada ao Corpo Docente do Programa de Pós-Graduação em Matemática - UFPB como requisito parcial para obtenção do título de Doutor em Matemática.

**Fevereiro/2020**

# Contents

# Introduction

The first studies of local cohomology originated in Grothendieck- Hartshorne's notes published between 1955 and 1967, from a geometric initiative: they took groups of cohomology of a topological space X with coefficients in an abelian sheaf on X and support in a locally closed subspace. Moreover, even having its roots in Algebraic Geometry, the study of local cohomology serves for general purposes in calculations of invariants in Commutative Algebra. Over the years, many authors have presented generalizations of this concept in commutative algebra (see [2, 5, 17, 27]).

In [27] Takahashi, Yoshino and Yoshizawa introduce a local cohomology module with respect to a pair of ideals (I,J). This structure is a generalization of the usual local cohomology module. More precisely, let R be a commutative Noetherian ring. Let $\tilde{\mathcal{F}}(I,J) = \{\mathfrak{p} \in Spec(R) | I^n \subseteq \mathfrak{p} + J$ for some positive integer $n\}$ and $\mathcal{F}(I,J)$ denotes the set of ideals $\mathfrak{a}$ of $R$ such that $I^n \subseteq \mathfrak{a} + J$ for some positive integer $n$. For an $R$-module $M$, we consider the $(I,J)$- torsion submodule $\Gamma_{I,J}(M)$ of $M$ which consists of all elements $x$ belong to $M$ such that $\mathrm{Supp}_R(Rx) \subseteq \tilde{\mathcal{F}}(I,J)$. Furthermore, for an integer $i$, the local cohomology functor $\mathrm{H}^i_{I,J}$ with respect to $(I,J)$ is defined to be the $i$-th right derived functor of $\Gamma_{I,J}$. The module $\mathrm{H}^i_{I,J}(M)$ is called the $i$-th local cohomology module of $M$ with respect to $(I,J)$.

Recently some authors approached the study of properties of theses extended modules, see for example [9, 10, 24, 28]. In [2] Alba-Sarria presented an even more general approach than that discussed in [27]. He defined a local cohomology module with respect to a good family of ideals $\alpha$. In his work, Sarria also addressed many properties of these new modules. Such properties generalized many results studied in [27].

In 1970, J. Herzog defined in [17] a generalized local cohomology module in the local case with support in the maximal ideal by

$$\mathrm{H}^j_{\mathfrak{m}}(M,N)=\varinjlim_n \mathrm{Ext}^j_R\left(\frac{M}{\mathfrak{m}^n M},N\right).$$

Then, in the year 1980, M. H. Bijan-Zadeh introduced the generalized local cohomology module, supported by a system of ideals (see [5]).

During this process of generalizing the local cohomology module, we came across several important problems in commutative algebra related to that module. An important problem in commutative algebra is determining when is finite the $R$-module $Hom_R(R/I,\mathrm{H}^i_I(M))$. In [15], Grothendieck conjectured the following:
If $R$ is a Noetherian ring, then for any ideal $I$ of $R$ and any finite $R$-module $M$, the modules $Hom_R(R/I,\mathrm{H}^i_I(M))$ are finite for all $i\geq 0$.
It is well-known that if $R$ is a local Noetherian ring with maximal ideal $\mathfrak{m}$, then an $R$-module $M$ is Artinian if and only if $\mathrm{Supp}_R(M)\subseteq\{\mathfrak{m}\}$ and $\mathrm{Ext}^j_R(R/\mathfrak{m},M)$ is finitely generated for all $j\geq 0$.

Motivated by this result, Hartshorne [16] gave a counterexample which show that this Grothendieck's conjecture is false even when $R$ is regular, and where he defined an $R$-module $M$ to be cofinite with respect to $I$ (abbreviated as $I$-cofinite) if the support of $M$ is contained in $V(I)$ and $\mathrm{Ext}^j_R(R/I,M)$ is finitely generated for all $j$. On the other hand, Brodmann and Lashgari showed in [7] that if, for a finitely generated $R$-module $M$ and an integer $t$, the local cohomology modules $\mathrm{H}^0_I(M),\mathrm{H}^1_I(M),\cdots,\mathrm{H}^{t-1}_I(M)$ are finitely generated, then $R$-module $\mathrm{Hom}_R(R/I,\mathrm{H}^t_I(M))$ is finitely generated and for any finitely generated submodule $N$ of $\mathrm{H}^t_I(M)$ the set $\mathrm{Ass}_R(\mathrm{H}^t_I(M)/N)$ is finite.

In [32] H. Zöschinger, introduced the interesting class of minimax modules, and he has in [32, 33] given many equivalent conditions for a module to be minimax. The R-module $M$ is said to be minimax, if there is a finitely generated submodule $N$ of $M$, such that $M/N$ is Artinian. The class of minimax modules thus includes all finitely generated and all Artinian modules. It was shown by T. Zink [31] and by E. Enochs [13] that a module over a complete local ring is minimax if and only if it is Matlis reflexive.

Posteriorly, the authors J. Azami, R. Naghipour and B. Vakili, in a paper published in 2009 [3], presented two classes of modules: the class of $I$-minimax modules and $I$-cominimax modules, which generalized the classes of minimax modules and $I$-cofinite modules, respectively. In their paper, the authors refined the result presented by Brodmann and Lashgari as follow:

**Theorem 0.0.1** Let $R$ be a Noetherian ring, $I$ an ideal of $R$ and $M$ an $I$-minimax $R$-module. Let $t$ be an non-negative integer such that $\mathrm{H}^i_I(M)$ is $I$-minimax for all $i < t$. Then for any $I$-minimax submodule $N$ of $\mathrm{H}^t_I(M)$ the $R$-module $\mathrm{Hom}_R(R/I, \mathrm{H}^t_I(M)/N)$ is $I$-minimax. In particular, the Goldie dimension of $\mathrm{H}^t_I(M)/N$ is finite, and so the set $\mathrm{Ass}_R(\mathrm{H}^t_I(M)/N)$ is finite.

Later, more precisely in 2010, Tehranian and Talemi introduced in [28] the concept of $(I, J)$-cofinite modules. An $R$-module $M$ is called $(I, J)$-cofinite when $\mathrm{Supp}_R(M) \subseteq \tilde{\mathcal{F}}(I, J)$ and $\mathrm{Ext}^j_R(R/I, M)$ is finitely generated for any $j \geq 0$. This definition generalized the concept presented by Hartshorne [16]. Throughout their work, Talemi and Tehranian, presented conditions to know when the $R$-module of homomorphisms $Hom_R(R/I, \mathrm{H}^t_{I,J}(M))$ is finitely generated for some $t$. An answer to this question was presented in the following result:

**Theorem 0.0.2** Let $t$ be a non-negative integer. Let $M$ be an $R$-module such that $\mathrm{Ext}^t_R(R/I, M)$ is a finite $R$-module and $\mathrm{H}^i_{I,J}(M)$ is $(I, J)$-cofinite, for every $i < t$. If $N \subseteq \mathrm{H}^t_{I,J}(M)$ is such that $\mathrm{Ext}^1_R(R/I, N)$ is finite, then $\mathrm{Hom}_R(R/I, \mathrm{H}^t_{I,J}(M)/N)$ is a finite $R$-module.

In addition to this result, they also studied the finiteness conditions of $R$-module $\mathrm{Ext}^i_R(R/I, \mathrm{H}^j_{I,J}(M))$ for $i = 1, 2$.

Recently Kh. Ahmadi-Amoli and M. Y. Sadeghi [1] defined the $(I, J)$-minimax $R$-modules and studied some properties of them. An $R$-module $M$ is called $(I, J)$-minimax when any quotient module of $M$ has finite $(I, J)$-relative Goldie dimension. One of the results interesting presented in [1] proves that the $(I, J)$-minimax class is a Serre class which contains the $I$-minimax modules (see page 10). On the other hand, considering an arbitrary Serre class of $R$-modules $\mathcal{S}$, instead of finitely generated, the authors defined the $(\mathcal{S}, I, J)$-cominimax $R$-modules. An $R$-module $M$ is called $(\mathcal{S}, I, J)$-cominimax whenever $\mathrm{Supp}_R(M) \subseteq \tilde{\mathcal{F}}(I, J)$ and $\mathrm{Ext}^j_R(R/I, M) \in \mathcal{S}$ for each $j \geq 0$. This concept of $R$-modules can be as a generalization of $I$-cofinite $R$-modules [16], $I$-cominimax $R$-modules [3], and $(I, J)$-cofinite $R$-modules [28]. Also, the main result of [1] is more general than that of [7] and [3]. They proved the following:

**Theorem 0.0.3** Let $\mathfrak{a} \in \mathcal{F}(I,J)$ be an ideal. Let $t$ be a non-negative integer such that $\mathrm{Ext}^t_R(R/\mathfrak{a}, M) \in \mathcal{S}$ and $\mathrm{Ext}^j_R(R/\mathfrak{a}, \mathrm{H}^i_{I,J}(M)) \in \mathcal{S}$ for all $i < t$ and all $j$ non-negative. Then for any submodule $N$ of $\mathrm{H}^t_{I,J}(M)$ such that $\mathrm{Ext}^1_R(R/\mathfrak{a}, N) \in \mathcal{S}$, we have $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{I,J}(M)/N) \in \mathcal{S}$.

It is interesting to note that in the works mentioned above, the authors approached relationships between the concepts of minimaximality, cofiniteness, cominimaximality and local cohomology module, adapting to each context. An interesting question about previous concepts is whether there is some relationship between them and the generalized local cohomology module. The answer to this question is yes. In [18] K. Khashyarmanesh and M. Yassi proved that for any non-zero principal ideal $I$ of $R$, the $R$-module $\mathrm{H}^t_I(M,N)$ is an $I$-cofinite module for all $t \geq 0$. Also, A. Mafi and H. Saremi [19] showed that the generalized local cohomology modules $\mathrm{H}^t_I(M,N)$ are $I$-cofinite for all $t \geq 0$, in the following cases:
(i) $cd(I) = 1$, where $cd$ is the cohomological dimension of $I$ in $R$;
(ii) $dim R \leq 2$.
Additionally, they proved that if $cd(I) = 1$ then $\mathrm{Ext}^i_R(M, \mathrm{H}^t_I(N))$ is $I$- cofinite for all $i, t \geq 0$.

Next, H. Saremi [26] presented a result more general than that in [19]. He got conditions for the generalized local cohomology module $\mathrm{H}^t_I(M,N)$ to be $I$-cominimax. His article had as its main result the following:

**Theorem 0.0.4** Let $M$ be a finitely generated $R$-module and $N$ be a minimax $R$-module. Then the following statements hold:
(i) If $cd(I) = 1$, then $\mathrm{H}^j_I(M,N)$ and $\mathrm{Ext}^i_R(M, \mathrm{H}^j_I(N))$ are $I$-cominimax for all $i, j$.
(ii) If $J$ is an ideal of $R$ with $J \subseteq I$ and $cd(I) = 1$, then $\mathrm{H}^i_I(\mathrm{H}^j_J(N))$ is $I$- cominimax for all $i, j$.
(iii) If $t$ is a non-negative integer such that $\mathrm{H}^j_I(M,N)$ is $I$-minimax for all $j < t$, then for any $I$-minimax $R$-submodule $L$ of $\mathrm{H}^t_I(M,N)$ the R-module $\mathrm{Hom}_R(R/I, \mathrm{H}^t_I(M,N)/L)$ is $I$-minimax. As a consequence it follows that the Goldie dimension of $\mathrm{H}^t_I(M,N)/L$ is finite, and so the associated primes of $\mathrm{H}^t_I(M,N)/L$ is finite.

This work presents more general versions of the concepts of minimax modules, cominimax modules, local cohomology modules and Ideal transforms, considering a good family of ideals as their support. In addition, fundamental properties of each of these structures are proved and relationships established between them.

The first chapter defines Goldie dimension of a $R$-module $M$ relative to a good family of ideals and presents $\alpha$-minimax modules, proving that this class of modules is a Serre class. Next we study some properties involving Serre classes in the $R$-modules category and then some relationships between the $R$-modules minimax and $(I, \beta)$-minimax classes are proved.

The second chapter presents a class of $R$-modules called modules $(\mathcal{S}, I, \beta)$ - cominimax, where $S$ denotes a Serre class in the $R$-modules category. This concept generalizes those presented in [16], [3], [28] and [1]. Then some relations between the $(\mathcal{S}, I, \beta)$ - cominimax modules and the local cohomogy module $\mathrm{H}^i_{I,\beta}(M)$ are presented and demonstrated, motivated by the main results of Brodmann and Lashgari [7]; J. Azami, R. Naghipour and B. Vakili [3]; Tehranian and Talemi [28] and Kh. Ahmadi-Amoli and M. Y. Sadeghi [1].

Later, the third chapter introduces the concept of generalized local cohomology module $\mathrm{H}^i_\alpha(M, N)$ of $R$-modules $M, N$ supported by a good family of ideals $\alpha$ and proves some basic properties. Also, vanishing conditions are presented for the module $\mathrm{H}^i_\alpha(M, N)$ considering the projective dimension of $M$ and the Krull dimension of $N$. After these vanishing results, some statements that relate the $\alpha$-minimax modules as the generalized local cohomology module are shown. This chapter concludes by presenting some results that relate the generalized local comology module to the $(I, \beta)$-cominimax modules, using cohomological dimension and spectral sequences.

Next, the chapter four takes an approach to the ideal tranforms $\mathcal{D}_\alpha(M)$ of an $R$-module $M$ with respect to a good family of ideals and some relationships between right derived functor $\mathbf{R}^i\mathcal{D}_\alpha(-)$ modules and the local cohomology functor $\mathrm{H}^i_\alpha(-)$ are proven. In this chapter conditions are presented for the functor $\mathcal{D}_\alpha(-)$ to be an exact functor. Also, is proved that the $R$-module $\mathrm{Hom}_R(R/I, \mathrm{H}^1_\alpha(M))$ is finitely generated in the case that $M$ has finite projective dimension. The chapter closes by dedicating its last two sections to the study of the generalized ideal transform $\mathcal{D}_\alpha(M, N)$ of $R$-modules $M, N$ with respect to a good family of ideals. These sections related the functors $\mathbf{R}^i\mathcal{D}_\alpha(M, -)$ and $\mathrm{H}^i_\alpha(M, -)$ and generalize the properties of minimax modules to ideal transforms.

# Chapter 1

# Serre classes and $\alpha$-minimax modules

Throughout this chapter $R$ will denote a commutative Noetherian ring with identity. Recall that for an $R$-module $M$, the Goldie dimension of $M$ is defined as the cardinality of the set of indecomposable submodules of $E(M)$, the injective hull of $M$, which appears in the decomposition of $E(M)$ into the direct sum of indecomposable submodules. Therefore, $M$ is said to have finite Goldie dimension if $M$ does not contain an infinite direct sum of non-zero submodules, or equivalently $E(M)$ decomposes as a finite direct sum of indecomposable submodules. We shall use $GdimM$ to denote the Goldie dimension of $M$. It is clear by the definition of the Goldie dimension that

$$GdimM = \sum_{\mathfrak{p}\in Spec(R)} \mu^0(\mathfrak{p}, M) = \sum_{\mathfrak{p}\in \mathrm{Ass}(M)} \mu^0(\mathfrak{p}, M).$$

Also, in [24], the $(I, J)$-relative Goldie dimension of $M$ is defined as

$$Gdim_{I,J}M := \sum_{\mathfrak{p}\in \tilde{\mathcal{F}}(I,J)} \mu^0(\mathfrak{p}, M),$$

where the set $\tilde{\mathcal{F}}(I, J)$ is defined as

$$\tilde{\mathcal{F}}(I, J) = \{\mathfrak{p} \in Spec(R) : I^n \subseteq \mathfrak{p} + J, \text{ para algum } n \in \mathbb{N}\}.$$

In this chapter we will define a Goldie dimension and minimax $R$-modules with respect to family of ideals more general than the family $\tilde{\mathcal{F}}(I, J)$ and we will show that many of the properties already known, both of Goldie dimension and minimax modules, can be generalized when we consider this new family of ideals. Moreover, In section 2 will present some results on Serre classes in the category of R-modules and what the relation of the minimax modules with these classes.

## 1.1 The $\alpha$-minimax modules

In this section we define the Goldie dimension of a module relative to a good family of ideals and then we introduce the concept of minimax modules with respect to these good families, along with the properties of such modules.

Recall that an $R$-module $M$ is minimax when there exists a finite submodule $N$ of $M$ such that o quotient module $M/N$ is Artinian. It is known that when $R$ is a Noetherian ring, an $R$-module $M$ is minimax if and only if any homomorphic image of $M$ has finite Goldie dimension (see [14, 31, 32] ). Moreover, an $R$-module $M$ is $(I, J)$-minimax when any quotient module of $M$ has finite $(I, J)$-relative Goldie dimension (see [1]).

**Definition 1.1.1** A non-empty set $\alpha$ of ideals in $R$ is a *good family* when the following conditions holds:
$(i)$ If $I$ and $J$ are ideals of $R$ such that $J \subset I$ and $J \in \alpha$, then $I \in \alpha$;
$(ii)$ If $I$ and $J$ belong to $\alpha$, then $IJ \in \alpha$.

**Example 1.1.2** $(i)$ Let $\beta$ be an arbitrary non-empty collection of ideals on $R$, the set $\langle\beta\rangle = \{K \trianglelefteq R : I_1 \dots I_t \subseteq K$, for some $I_j \in \beta, j = 1, \dots, t\}$ is a good family of ideals in $R$. When $\beta = \varnothing$ we put $\langle\beta\rangle = \{R\}$. In particular, when $\beta = \{I\}$ is a single set we use the notation $\langle I\rangle$ instead of $\langle\{I\}\rangle$. So $\langle I\rangle = \{K \trianglelefteq R : I^n \subseteq K$, for some natural$n\}$;
$(ii)$ The set $\mathcal{F}(I, \beta) = \{K \trianglelefteq R : K + J \in \langle I\rangle, \forall J \in \beta\}$ is also a good family of ideals. When $\beta = \{J\}$ we call $\mathcal{F}(I, J)$ instead of $\mathcal{F}(I, \beta)$. Moreover the set $\mathcal{F}(I, J)$ coincides with set of ideals studied in [1].

**Remark 1.1.3** Note that $V(I) \subset \tilde{\mathcal{F}}(I, \beta)$ can be a strict inclusion. Indeed, let $R = \mathbb{Z}$ be the ring of integer numbers. Consider $I = 4\mathbb{Z}$, $\beta = \{\mathbb{Z}, 13\mathbb{Z}\}$ and $\mathfrak{p} = 3\mathbb{Z}$. Note that $\mathfrak{p} \in \tilde{\mathcal{F}}(I, \beta)$, since $\mathfrak{p} + \mathbb{Z} \supseteq 4\mathbb{Z} \in \langle I\rangle$ and $\mathfrak{p} + 13\mathbb{Z} \supseteq 16\mathbb{Z} \in \langle I\rangle$. But $\mathfrak{p} \notin V(I) = \{2\mathbb{Z}\}$. Therefore, we have $V(I) \subsetneq \tilde{\mathcal{F}}(I, \beta)$.

Now, recall that, for any $R$-module $M$, the $i$-th Bass number of $M$ with respect to prime ideal $\mathfrak{p}$, denoted by $\mu^i(\mathfrak{p}, M)$, it is defined as the number of copies of the injective hull $E_R(R/\mathfrak{p})$ of $R/\mathfrak{p}$ over $R$ occurring as direct summands in the $i$-th injective module of a minimal injective resolution of $M$. Moreover, we have

$$\mu^i(\mathfrak{p}, M) = dim_{k(\mathfrak{p})}\left(\operatorname{Ext}^i_{R_{\mathfrak{p}}}(k(\mathfrak{p}), M_{\mathfrak{p}})\right),$$

where $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is the fraction field of $R/\mathfrak{p}$.

**Definition 1.1.4** Let $\alpha$ be a good family of ideals in $R$ and $M$ an $R$-module. The *$\alpha$-relative Goldie dimension* of $M$ is defined as

$$Gdim_\alpha M := \sum_{\mathfrak{p}\in\alpha\cap Spec(R)} \mu^0(\mathfrak{p}, M).$$

When $\alpha = \mathcal{F}(I,\beta)$, we shall use $\tilde{\mathcal{F}}(I,\beta)$ to denote $\mathcal{F}(I,\beta)\cap Spec(R)$. In this case, we will simply call the $(I,\beta)$ - relative Goldie dimension of M and we will denote by

$$Gdim_{I,\beta} M := \sum_{\mathfrak{p}\in\tilde{\mathcal{F}}(I,\beta)} \mu^0(\mathfrak{p}, M).$$

**Example 1.1.5** $(i)$ We consider $R = \mathbb{C}[X,Y]$ and the $R$-module

$$M = \bigoplus_{(a,b)\in\mathbb{C}^2} \frac{\mathbb{C}[X,Y]}{(X-a,Y-b)}.$$

Let

$$\beta = \left\{(X-a,Y-b) \mid (a,b)\in\mathbb{C}^2 \text{ and } \mathfrak{Im}(a) = \mathfrak{Im}(b) = 0\right\} \text{ and } \alpha = \langle\beta\rangle\,.$$

We know that

$$\mathrm{Ass}_R(M) \supseteq \bigcup_{(a,b)\in\mathbb{C}^2} \mathrm{Ass}_R \frac{\mathbb{C}[X,Y]}{(X-a,Y-b)} = \left\{(X-a,Y-b) \mid (a,b)\in\mathbb{C}^2\right\} := B.$$

Note that $B$ is a infinite set and consequently $\mathrm{Ass}_R(M)$ is also. Since $\alpha\cap B$ is infinite, it follows that $\alpha\cap\mathrm{Ass}_R(M)$ is also. On the other hand $\mu^0(\mathfrak{p}, M) \geq 1$ if and only if $\mathfrak{p}\in\mathrm{Ass}_R(M)$. Therefore $Gdim_\alpha(M)$ is infinite.
$(ii)$ Let $M$ be an $R$-module finitely generated and $\alpha$ a good family. Then $M$ is Noetherian module and hence $\mathrm{Ass}_R(M)$ is a finite set. By [21, Theorem 18.7] we have $\mu^0(\mathfrak{p}, M) < \infty$ for all $\mathfrak{p}\in\mathrm{Ass}_R(M)$. Therefore $Gdim_\alpha(M)$ is finite.

**Definition 1.1.6** Let $\alpha$ be a good family of ideals in $R$ and $M$ an $R$-module. The *$\alpha$-torsion module* of $M$ is defined by

$$\Gamma_\alpha(M) := \{x\in M : Ix = 0, \text{ for some } I\in\alpha\}.$$

Note that $\Gamma_\alpha(M)$ is a submodule of $M$. Moreover, we have

$$\Gamma_\alpha(M) = \bigcup_{I\in\alpha}\Gamma_I(M) = \varinjlim_{I\in\alpha}\Gamma_I(M),$$

where $\alpha$ is seen as a direct set with the partial order $I \leq J$ if and only if $J \subseteq I$, in the second equality.

For a homomorphism $f : M \to N$ of $R$-modules, it is easy see that the inclusion $f(\Gamma_\alpha(M)) \subseteq \Gamma_\alpha(N)$, and hence the mapping $\Gamma_\alpha(f) : \Gamma_\alpha(M) \to \Gamma_\alpha(N)$ is defined so that it agrees with $f$ on $\Gamma_\alpha(M)$.

Thus $\Gamma_\alpha$ becomes an additive covariant functor from the category of $R$-modules to itself. In [2] this functor is called $\alpha$-torsion functor. When $\alpha = \mathcal{F}(I, \beta)$ we will denote it by $\Gamma_{I,\beta}$ instead of $\Gamma_{\mathcal{F}(I,\beta)}$ and we call it $(I, \beta)$-torsion functor.

Note that, if $\beta = \{J\}$ then the $(I, \beta)$-torsion functor $\Gamma_{I,\beta}$ coincides with $(I, J)$-torsion functor $\Gamma_{I,J}$ studied in [27].

**Lemma 1.1.7** Let $\alpha$ be a good family of ideals in $R$ and $M$ an $R$-module. Then

$$Gdim_\alpha M = Gdim \Gamma_\alpha(M).$$

***Proof*** Let $\mathfrak{p} \in Spec(R)$ be, and let $E(R/\mathfrak{p})$ be the injective hull of $R/\mathfrak{p}$. In [21, Theorem 18.4] is proved with $E(R/\mathfrak{p})$ is an $\mathfrak{p}$-torsion $R$-module and with, if $r \in R \setminus \mathfrak{p}$, then the multiplication by $r$ induces an automorphism of $E(R/\mathfrak{p})$. Therefore, if $\mathfrak{p} \supseteq I$, for some $I \in \alpha$ and $u \in E(R/\mathfrak{p})$, then $\mathfrak{p}^n u = 0 \Rightarrow I^n u = 0 \Rightarrow u \in \Gamma_\alpha(E(R/\mathfrak{p}))$. So $E(R/\mathfrak{p})$ is an $\alpha$-torsion. On the other hand, if $\mathfrak{p} \not\supseteq I$, for any $I \in \alpha$ and $u \in \Gamma_\alpha(E(R/\mathfrak{p}))$, then there exists $I \in \alpha$ such that $Iu = 0$. Since $I \nsubseteq \mathfrak{p}$, there exists $y \in I \setminus \mathfrak{p}$ and the multiplication by $y$ is an automorphism of $E(R/\mathfrak{p})$. So $u = 0$ and consequently, $\Gamma_\alpha(E(R/\mathfrak{p})) = 0$. Therefore, $\Gamma_\alpha(E(M))$ is can be decomposed as

$$\begin{aligned} \Gamma_\alpha(E(M)) &\cong \bigoplus_{\mathfrak{p} \in \alpha \cap Spec(R)} \Gamma_\alpha(E(R/\mathfrak{p}))^{\mu^0(\mathfrak{p},M)} \bigoplus \bigoplus_{\mathfrak{p} \notin \alpha \cap Spec(R)} \Gamma_\alpha(E(R/\mathfrak{p}))^{\mu^0(\mathfrak{p},M)} \\ &= \bigoplus_{\mathfrak{p} \in \alpha \cap Spec(R)} E(R/\mathfrak{p})^{\mu^0(\mathfrak{p},M)}. \end{aligned}$$

Note that $\Gamma_\alpha(E(M))$ is an essential extension of $\Gamma_\alpha(M)$.Indeed, if $x \in \Gamma_\alpha(E(M)) \subseteq E(M)$ and $x \neq 0$, then $Ix = 0$, for some $I \in \alpha$ and there exists $a \in R$ such that $ax \in M \setminus \{0\}$, because $E(M)$ is an essential extension of $M$. Since $Iax = 0$, follow that $ax \in \Gamma_\alpha(M) \setminus \{0\}$ and hence, $\Gamma_\alpha(E(M))$ is an essential extension of $\Gamma_\alpha(M)$. Also, we know that

$$\Gamma_\alpha(E(M)) = \varinjlim_{I \in \alpha} \Gamma_I(E(M))$$

and $\Gamma_I(E(M))$ is an injective $R$-module, for any $I \in \alpha$, we deduce, by [20, Proposition 1.2(1)], that $\Gamma_\alpha(E(M))$ is an injective $R$-module and consequently, $\Gamma_\alpha(E(M)) \cong E(\Gamma_\alpha(M))$. Therefore,

$$Gdim_\alpha M = \sum_{\mathfrak{p} \in \alpha \cap Spec(R)} \mu^0(\mathfrak{p}, M) = Gdim \Gamma_\alpha(M),$$

as required. ■

**Definition 1.1.8** Let $\alpha$ be a good family of ideals in $R$ and let $M$ be an $R$-module. We say that $M$ is *$\alpha$-minimax (or minimax with respect to $\alpha$)* when any quotient module of $M$ has finite $\alpha$-relative Goldie dimension.

**Remark 1.1.9** $(i)$ For the family $\alpha = \mathcal{F}(I, \beta)$ we simply called of $(I, \beta)$-minimax module instead of $\mathcal{F}(I, \beta)$-minimax module;
$(ii)$ By inclusions $V(I) \subseteq \tilde{\mathcal{F}}(I, \beta) \subseteq \tilde{\mathcal{F}}(I, J)$ for all $J \in \beta$, we have the inequalities

$$Gdim_I M \leq Gdim_{I,\beta} M \leq Gdim_{I,J} M \leq Gdim M$$

and consequently, the classes of $(I, \beta)$-minimax $R$-modules contain the classes of $(I, J)$-minimax $R$-modules, for any $J \in \beta$.

**Example 1.1.10** $(i)$ If $I = 0$, then $\tilde{\mathcal{F}}(I, \beta) = Spec(R) = V(I)$ and therefore, an $R$-module $M$ is minimax if, and only if, it is $(I, \beta)$-minimax if, and only if, it is $(I, J)$-minimax, for all $J \in \beta$ if, and only if, it is $I$-minimax;
$(ii)$ If $\beta$ is the set of all ideals of $R$, then $\tilde{\mathcal{F}}(I, \beta) = V(I)$. So $M$ is $(I, \beta)$-minimax if, and only if, it is $I$-minimax;
$(iii)$ Let $M$ be an $\alpha$-torsion module. Then $M$ is $\alpha$-minimax if, and only if, $M$ is minimax;
$(iv)$ If $M$ is an $I$-torsion module, then $M$ is minimax if, and only if, it is $(I, \beta)$-minimax if, and only if, it is $(I, J)$-minimax, for each $J \in \beta$ if, and only if, it is $I$-minimax.

**Proposition 1.1.11** Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ an short exact sequence of $R$-modules and $\alpha$ a good family of ideals in $R$. Then $M$ is $\alpha$-minimax if, and only if, $M'$ and $M''$ are both $\alpha$-minimax.

***Proof*** We suposse that $M$ is $\alpha$-minimax and let is make the identifications $M' \leq M$ and $M'' = M/M'$. Then, it follows directly that $M'$ and $M''$ are $\alpha$-minimax modules. Conversely, assume that $M'$ and $M''$ are both $\alpha$-minimax and let $N$ be a submodule of $M$. Let $\mathfrak{p} \in \mathrm{Ass}_R(M/N) \cap \alpha$. The exact sequence of $R$-modules

$$0 \longrightarrow \frac{M' + N}{N} \longrightarrow \frac{M}{N} \longrightarrow \frac{M}{M' + N} \longrightarrow 0$$

induces the exact sequence of $R_{\mathfrak{p}}$-modules

$$0 \longrightarrow \operatorname{Hom}_{R_{\mathfrak{p}}}\left(k(\mathfrak{p}), \frac{M'_{\mathfrak{p}}}{M'_{\mathfrak{p}} \cap N_{\mathfrak{p}}}\right) \longrightarrow \operatorname{Hom}_{R_{\mathfrak{p}}}\left(k(\mathfrak{p}), \frac{M_{\mathfrak{p}}}{N_{\mathfrak{p}}}\right) \longrightarrow \operatorname{Hom}_{R_{\mathfrak{p}}}\left(k(\mathfrak{p}), \frac{M_{\mathfrak{p}}}{M'_{\mathfrak{p}} + N_{\mathfrak{p}}}\right),$$

where $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. We know that

$$(M' + N)/N \cong M'/M' \cap N \text{ and } M/(M' + N) \cong (M/M')/(M' + N)/M'.$$

By $\alpha$-minimaximality of $M'$ and $M''$, we have

$$\mu^0\left(\mathfrak{p}, \frac{M' + N}{N}\right) \leq Gdim_\alpha\left(\frac{M' + N}{N}\right) < \infty$$

$$\mu^0\left(\mathfrak{p}, \frac{M}{M' + N}\right) \leq Gdim_\alpha\left(\frac{M}{M' + N}\right) < \infty.$$

Therefore

$$\mu^0\left(\mathfrak{p}, \frac{M}{N}\right) \leq \mu^0\left(\mathfrak{p}, \frac{M' + N}{N}\right) + \mu^0\left(\mathfrak{p}, \frac{M}{M' + N}\right) < \infty.$$

Moreover, the sets $\operatorname{Ass}((M' + N)/N) \cap \alpha$ and $\operatorname{Ass}(M/(M' + N)) \cap \alpha$ are both finite. Since $\operatorname{Ass}(M/N) \subset \operatorname{Ass}((M'+N)/N) \cup \operatorname{Ass}(M/(M'+N))$, it follows that $\operatorname{Ass}(M/N) \cap \alpha$ is finite and so $M$ is $\alpha$-minimax, as required. ■

**Corollary 1.1.12** Let $\alpha$ be a good family of ideals of $R$. Then any quotient of an $\alpha$-minimax module and any finite direct sum of $\alpha$-minimax modules are also $\alpha$-minimax.

## 1.2 Serre classes

In this section, we prove some results related to Serre classes, or Serre subcategory, in the category of $R$-modules.

**Definition 1.2.1** Let $Mod(R)$ the category of $R$-modules and $\mathcal{S}$ a class in $Mod(R)$. We say that $\mathcal{S}$ is a *Serre Class (or Serre Subcategory)* when the following property is satisfied: Given a short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of $R$-modules. Then $M \in \mathcal{S}$ if, and only if, $M' \in \mathcal{S}$ and $M'' \in \mathcal{S}$.

**Example 1.2.2** The following classes of $R$-modules are Serre Classes.
$(i)$ The class of Noetherian $R$-modules ;
$(ii)$ The class of Artinian $R$-modules;
$(iii)$ The class of $R$-modules with finite support;

$(iv)$ The class of $R$-modules with $dim_R M \leq n$, where $n$ is a non-negative integer;
$(v)$ The class of all $I$-minimax $R$-modules;
$(vi)$ The class of $\alpha$-torsion $R$-modules;
$(vii)$ The class of $\alpha$-minimax $R$-modules.

**Remark 1.2.3** During the section we will use the following notations:
$\mathcal{S}$ for an arbitrary Serre Class in the category $Mod(R)$;
$\mathcal{S}_0$ denotes the class of minimax $R$-modules;
$\mathcal{S}_I$ for the class of $I$-minimax $R$-modules;
$\mathcal{S}_{I,J}$ for the class of $(I, J)$-minimax $R$-modules;
$\mathcal{S}_\alpha$ denotes the class of $\alpha$-minimax $R$-modules. For the case $\alpha = \mathcal{F}(I, \beta)$, we denotes by $\mathcal{S}_{I,\beta}$ the class of $(I, \beta)$-minimax $R$-modules. Using the notation before, we obtains, from Remark 1.1.9 that $\mathcal{S}_0 \subseteq \mathcal{S}_{I,J} \subseteq \mathcal{S}_{I,\beta} \subseteq \mathcal{S}_I$, for all $J \in \beta$.

**Proposition 1.2.4** Let $I, I'$ be ideals of $R$, $\beta, \beta'$ two non-empty sets of ideals of $R$ and $M$ an $R$-module. Then
$(i)$ $\mathcal{S}_{I,\beta} = \mathcal{S}_{\sqrt{I},\beta}$.
$(ii)$ If $\beta \subseteq \langle\{I\}\rangle$, then $\mathcal{S}_0 = \mathcal{S}_{I,\beta}$.
$(iii)$ If $I^n \subseteq \sqrt{I'}$, for some $n \in \mathbb{N}$, then $\mathcal{S}_{I,\beta} \subseteq \mathcal{S}_{I',\beta}$.
$(iv)$ If $\beta' \subseteq \beta$, then $\mathcal{S}_{I,\beta'} \subseteq \mathcal{S}_{I,\beta}$.
$(v)$ If $I^n \subseteq \sqrt{I'}$, for some $n \in \mathbb{N}$ and $M$ is $(I', \beta)$-torsion, then $M \in \mathcal{S}_{I,\beta}$ if, and only if, $M \in \mathcal{S}_0$ if, and only if, $M \in \mathcal{S}_{I',\beta}$.
$(vi)$ If $\beta' \subseteq \beta$ and $M$ is $(I, \beta)$-torsion, then $M \in \mathcal{S}_{I,\beta}$ if, and only if, $M \in \mathcal{S}_0$ if, and only if, $M \in \mathcal{S}_{I,\beta'}$.

***Proof*** (i) Immediate.

(ii) Since $\beta \subseteq \langle I \rangle$, it follows that $\tilde{\mathcal{F}}(I, \beta) = Spec(R)$ and, therefore, $\mathcal{S}_0 = \mathcal{S}_{I,\beta}$.

(iii) Let $M \in \mathcal{S}_{I,\beta}$. Since $I^n \subseteq \sqrt{I'}$, we get $\tilde{\mathcal{F}}(\sqrt{I'}, \beta) \subseteq \tilde{\mathcal{F}}(I, \beta)$. Since $M$ is $(I, \beta)$-minimax $R$-module, it follows that $Gdim_{\sqrt{I},\beta} M/N \leq Gdim_{I,\beta} M/N < \infty$, for any submodule $N$ of $M$. Therefore $M \in \mathcal{S}_{\sqrt{I'},\beta} = \mathcal{S}_{I',\beta}$.

(iv) Let $M \in \mathcal{S}_{I,\beta'}$. By hypothesis, $\beta' \subseteq \beta$. So $\tilde{\mathcal{F}}(I, \beta) \subseteq \tilde{\mathcal{F}}(I, \beta')$. This implies that $Gdim_{I,\beta} M/N \leq Gdim_{I,\beta'} M/N < \infty$ for all submodule $N$ of $M$. Therefore, $M \in \mathcal{S}_{I,\beta}$.

(v) By proof of item (iii) we have $\tilde{\mathcal{F}}(I', \beta) \subseteq \tilde{\mathcal{F}}(I, \beta)$, so $M = \Gamma_{I',\beta}(M) \subseteq \Gamma_{I,\beta}(M)$, that implies $M$ is $(I, \beta)$-torsion. Consequently, we get

$$GdimM/N = Gdim_{I',\beta} M/N = Gdim_{I,\beta} M/N,$$

for all submodule $N$ of $M$. Therefore, it follows the result.

(vi) The proof is analogous to that of item (v). ■

**Proposition 1.2.5** If $N \in \mathcal{S}$ and $M$ is a finitely generated $R$-module, then, for any submodule $H$ of $\mathrm{Ext}^i_R(M,N)$ and $T$ of $\mathrm{Tor}^R_i(M,N)$, we have $\mathrm{Ext}^i_R(M,N)/H \in \mathcal{S}$ and $\mathrm{Tor}^R_i(M,N)/T \in \mathcal{S}$, for all $i \geq 0$.

***Proof*** Since $R$ is Noetherian and $M$ is finitely generated, it follows that $M$ has a free resolution

$$F_\bullet : \cdots \longrightarrow F_i \longrightarrow F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0,$$

where each $F_i$ has finite rank. Applying the functor $\mathrm{Hom}_R(-,N)$ in $F_\bullet$, we get the complex

$$\mathrm{Hom}_R(F_\bullet, N) : 0 \longrightarrow \mathrm{Hom}_R(F_0,N) \xrightarrow{\delta^0} \mathrm{Hom}_R(F_1,N) \xrightarrow{\delta^1} \cdots$$

have in mind that $\mathrm{Ext}^i_R(M,N) = H^i(\mathrm{Hom}_R(F_\bullet,N))$, so when $i=0$, we have

$$\mathrm{Ext}^0_R(M,N) = \mathrm{Hom}_R(M,N)$$

and we know that there is an injection $\mathrm{Hom}_R(M,N) \hookrightarrow N^k$, where $k$ is the number of generators of $M$. Since $N \in \mathcal{S}$, it follows from Definition 1.2.1 that $N^k \in \mathcal{S}$ and, therefore, $\mathrm{Ext}^0_R(M,N) \in \mathcal{S}$.

For the case $i > 0$, we know that $\mathrm{Ext}^i_R(M,N) = Ker\delta^i/Im\delta^{i-1}$. Since $Ker\delta^i$ is a submodule of $\mathrm{Hom}_R(F_i,N) \cong N^{n_i}$, where $n_i$ is the rank of $F_i$, and $N^{n_i} \in \mathcal{S}$, it follows that $Ker\delta^i \in \mathcal{S}$. Moreover, the sequence

$$0 \longrightarrow Im\delta^{i-1} \hookrightarrow ker\delta^i \twoheadrightarrow \mathrm{Ext}^i_R(M,N) \longrightarrow 0$$

is exact. So $\mathrm{Ext}^i_R(M,N) \in \mathcal{S}$ for all $i > 0$. By Definition 1.2.1, we have $\mathrm{Ext}^i_R(M,N)/H \in \mathcal{S}$ for any non-negative integer $i$. The proof of $\mathrm{Tor}^i_R(M,N)/T \in \mathcal{S}$, for all non-negative integer $i$, is similar to the previous one. ■

**Theorem 1.2.6** Let $M$ be a finitely generated $R$-module, $N$ an arbitrary $R$-module and $t$ a non-negative integer. Then the following statements are equivalent:
$(i)$ $\mathrm{Ext}^i_R(M,N) \in \mathcal{S}$, for all $i \leq t$.
$(ii)$ For any finitely generated $R$-module $H$ with $\mathrm{Supp}(H) \subseteq \mathrm{Supp}(M)$, we get $\mathrm{Ext}^i_R(H,N) \in \mathcal{S}$, for all $i \leq t$.

***Proof*** (i)⇒(ii) Since Supp$H \subseteq$ Supp$M$, by Gruson Theorem (see [29]), there exists a filtration of $R$-modules

$$0 = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_k = H$$

such that each factor $H_j/H_{j-1}$ is a quotient of a finite direct sum of copies of $M$. As a consequence we have the exact sequences

$$0 \longrightarrow K \longrightarrow M^n \longrightarrow H_1 \longrightarrow 0$$

$$0 \longrightarrow H_1 \longrightarrow H_2 \longrightarrow H_2/H_1 \longrightarrow 0$$

$$\vdots$$

$$0 \longrightarrow H_{k-1} \longrightarrow H_k \longrightarrow H_k/H_{k-1} \longrightarrow 0$$

and hence, for each $j$, a long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}_R^{i-1}(H_{j-1}, N) \longrightarrow \mathrm{Ext}_R^i(H_j/H_{j-1}, N) \longrightarrow \mathrm{Ext}_R^i(H_j, N) \longrightarrow$$

$\longrightarrow \mathrm{Ext}_R^i(H_{j-1}, N) \longrightarrow \cdots$

and applying induction on $k$, is sufficient we prove the result for $k = 1$. So, when $k = 1$, there is a short exact sequence

$$0 \longrightarrow K \longrightarrow M^n \longrightarrow H \longrightarrow 0, \tag{1.1}$$

for some $n \in \mathbb{N}$ and some finitely generated $R$-module $K$.

Now, we use induction on $t$. If $t = 0$, $\mathrm{Hom}_R(H, N)$ is a submodule of $R$-module $\mathrm{Hom}_R(M^n, N) \cong \bigoplus_n \mathrm{Hom}_R(M, N) \in \mathcal{S}$. Suppose now that $t > 0$ and $\mathrm{Ext}_R^j(H', N) \in \mathcal{S}$ for all finitely generated $R$-module $H'$ with Supp$H' \subseteq$ Supp$M$ and for any $j \leq t - 1$. Then, the exact sequence (1.1) induces the long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}_R^{i-1}(K, N) \xrightarrow{\delta^{i-1}} \mathrm{Ext}_R^i(H, N) \xrightarrow{\delta^i} \mathrm{Ext}_R^i(M^n, N) \longrightarrow \cdots .$$

Since Supp$K \subseteq$ Supp$M^n =$ Supp$M$, it follows from induction hypothesis that

$$\mathrm{Ext}_R^{i-1}(K, N) \in \mathcal{S}, \text{ for all } i \leq t.$$

On the other hand, $\mathrm{Ext}_R^i(M^n, N) \cong \bigoplus_n \mathrm{Ext}_R^i(M, N) \in \mathcal{S}$. Then, we have

$$Ker\delta^i = Im\delta^{i-1} \in \mathcal{S} \text{ and } Im\delta^i = Ker\delta^{i+1} \in \mathcal{S}.$$

The result follows from the exact sequence

$$0 \longrightarrow Ker\delta^i \longrightarrow \mathrm{Ext}^i_R(H,N) \longrightarrow Im\delta^i \longrightarrow 0.$$

(ii)⇒(i) Immediate. ■

**Corollary 1.2.7** Let $r$ a non-negative integer. Then, for any $R$-module $M$, the following statements are equivalent:
*(i)* $\mathrm{Ext}^i_R(R/I,M) \in \mathcal{S}$ for all $i \leq r$.
*(ii)* For all ideal $\mathfrak{a}$ of $R$ with $\mathfrak{a} \supseteq I$, $\mathrm{Ext}^i_R(R/\mathfrak{a},M) \in \mathcal{S}$ for all $i \leq r$.
*(iii)* For any finitely generated $R$-module $N$ with $\mathrm{Supp}(N) \subseteq V(I)$, $\mathrm{Ext}^i_R(N,M) \in \mathcal{S}$ for all $i \leq r$.
*(iv)* For any $\mathfrak{p} \in Min(I)$, $\mathrm{Ext}^i_R(R/\mathfrak{p},M) \in \mathcal{S}$ for all $i \leq r$.

***Proof*** The implications (i)⇒(ii) ⇒(iii)⇒(iv) follow directly from Theorem 1.2.6. So, remains to prove the implication (iv)⇒(i). Let $\mathfrak{p}_1,\ldots,\mathfrak{p}_n$ be the minimal primes ideals of $R$. By hypothesis, $\mathrm{Ext}^i_R(R/\mathfrak{p}_j,M) \in \mathcal{S}$, for each $j = 1,\ldots,n$. Then, we have $\mathrm{Ext}^i_R(\oplus_{j=1}^n R/\mathfrak{p}_j,M) \cong \oplus_{j=1}^n \mathrm{Ext}^i_R(R/\mathfrak{p}_j,M) \in \mathcal{S}$. Moreover

$$\mathrm{Supp}(R/I) = V(I) = V(\sqrt{I}) = V\left(\bigcap_{j=1}^n \mathfrak{p}_j\right) = \bigcup_{j=1}^n V(\mathfrak{p}_j) = \mathrm{Supp}(\oplus_{j=1}^n R/\mathfrak{p}_j).$$

Therefore, by Theorem 1.2.6, $\mathrm{Ext}^i_R(R/I,M) \in \mathcal{S}$, for all $i \leq t$. ■

# Chapter 2

# Properties of cominimaximality and finiteness of local cohomology modules

In [2], Alba-Sarria introduces the local cohomology modules with respect to a good family $\alpha$ which generalizes the local cohomology modules studied in [27]. For an integer $i$, the local cohomology functor $\mathrm{H}^i_\alpha$ with respect to $\alpha$ is defined to be the $i$-th right derived functor of $\Gamma_\alpha$. Also $\mathrm{H}^i_\alpha(M)$ is called the $i$-th local cohomology module of $M$ with respect to $\alpha$. In [2], Sarria approached the study of properties of these extended modules.

In this chapter, we will define in section 1 what are $(\mathcal{S}, I, \beta)$-cominimax modules and will prove some properties of the local cohomology modules $\mathrm{H}^i_{I,\beta}(M)$ related to this definition. In addition, starting from what was seen in section 1, we will see in section 2 when the set of the associated primes of $\mathrm{H}^i_{I,\beta}(M)$ will be finite. These results will conclude the chapter.

## 2.1 $(\mathcal{S}, I, \beta)$-Cominimax modules

Given an ideal $\mathfrak{a}$, in [16] is defined that an $R$-module $M$ is $\mathfrak{a}$-cofinite when $M$ has support in $V(\mathfrak{a})$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ is a finitely generated $R$-module for each $i$. In [3], Azami, Naghipour and Vakili define the $\mathfrak{a}$-cominimax $R$-modules. More precisely, we say that an $R$-module is $\mathfrak{a}$-cominimax when $\mathrm{Supp}_R(M) \subseteq V(\mathfrak{a})$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ is an $\mathfrak{a}$-minimax $R$-module, for all $i$. This last definition generalizes the concept of $\mathfrak{a}$-cofinite

$R$-modules. Furthermore, it is in [1] that Ahmadi-Amoli and Sadeghi present an even more general concept than the previous ones. These authors defined the $(\mathcal{S}, I, J)$-cominimax $R$-modules, where $\mathcal{S}$ is a Serre Class and $I, J$ are ideals of $R$. We say that an $R$-module $M$ is $(\mathcal{S}, I, J)$-cominimax if $\mathrm{Supp}_R(M) \subseteq \tilde{\mathcal{F}}(I, J)$ and $\mathrm{Ext}^i_R(R/I, M) \in \mathcal{S}$ for each $i \geq 0$. In all these cases, properties related to the concept of cominimaximality were presented for the local cohomology modules $\mathrm{H}^i_I(M)$ and $\mathrm{H}^i_{I,J}(M)$ respectively.

In this section we introduce the class of $(\mathcal{S}, I, \beta)$-Cominimax $R$-modules and some relationships between these modules and local cohomology module $\mathrm{H}^i_{I,\beta}(M)$.

**Definition 2.1.1** Let $I$ be an ideal of $R$ and $\beta$ a non-empty collection of ideals in $R$. We say that an $R$-module $M$ is *$(I, \beta)$-cofinite* when $\mathrm{Supp}_R(M) \subseteq \tilde{\mathcal{F}}(I, \beta)$ and $\mathrm{Ext}^i_R(R/I, M)$ is a finitely generated $R$-module, for all $i \geq 0$.

Recall that for an integer $i$, the $i$-th right derived functor of $\Gamma_\alpha$ is denoted by $\mathrm{H}^i_\alpha$ and will be referred to as the $i$-th local cohomology functor with respect to $\alpha$.

For an $R$-module $M$, the module $\mathrm{H}^i_\alpha(M)$ is called the $i$-th local cohomology module of $M$ with respect to good family $\alpha$.

When $\alpha = \mathcal{F}(I, \beta)$ we use $\mathrm{H}^i_{I,\beta}(M)$ to denote the $i$-th local cohomology $R$-module instead of $\mathrm{H}^i_{\mathcal{F}(I,\beta)}(M)$.

It is easy to see that if $\beta = \{J\}$, then $\mathrm{H}^i_{I,\beta}(M)$ coincides with the local cohomology functor $\mathrm{H}^i_{I,J}$ defined in [27].

**Remark 2.1.2** Notice that $\Gamma_I(M) \subseteq \Gamma_\alpha(M)$, for $I \in \alpha$. So, if $\Gamma_\alpha(M) = 0$, then $\Gamma_I(M) = 0$, for any $I \in \alpha$. Now, let $\bar{M} = M/\Gamma_\alpha(M)$ and let $E = E_R(\bar{M})$. Consider $L = E/\bar{M}$. Since $\Gamma_\alpha(\bar{M}) = 0$, it follows that $\Gamma_\alpha(E) = 0$ and $\Gamma_I(\bar{M}) = 0 = \Gamma_I(E)$, for each $I \in \alpha$. In particular, $\mathrm{Hom}_R(R/I, E) = 0$, for all $I \in \alpha$. On the other hand, given an exact sequence

$$0 \longrightarrow \bar{M} \longrightarrow E \longrightarrow L \longrightarrow 0$$

and applying the functors $\mathrm{Hom}_R(R/I, -)$ and $\Gamma_\alpha(-)$, we have the following isomorphisms:
$(i)$ $\mathrm{Ext}^i_R(R/I, L) \cong \mathrm{Ext}^{i+1}_R(R/I, \bar{M})$, $\forall I \in \alpha$ and $i \geq 0$.
$(ii)$ $\mathrm{H}^i_\alpha(L) \cong \mathrm{H}^{i+1}_\alpha(M)$, for all $i \geq 0$.

**Definition 2.1.3** Let $I$ be an ideal of $R$ and $\beta$ an arbitrary family of ideals in $R$. For a Serre Class $\mathcal{S}$, an $R$-module $M$ is called $(\mathcal{S}, I, \beta)$-*Cominimax* when $\mathrm{Supp}_R M \subseteq \tilde{\mathcal{F}}(I, \beta)$ and $\mathrm{Ext}^i_R(R/I, M) \in \mathcal{S}$, for all $i \geq 0$.

**Remark 2.1.4** $(i)$ Note that, if we consider $\mathcal{S}$ be the class of finitely generated $R$-modules in Definition 2.1.3, then we recover the Definition 2.1.1;
$(ii)$ When $\mathcal{S} = \mathcal{S}_{I,\beta}$, we called that the $R$-module is $(I, \beta)$-Cominimax;
$(iii)$ In the case $\beta = \{J\}$ and $\mathcal{S}$ be the class of finitely generated $R$-modules we recuper the Definition 2.1 in [28].

**Notation 2.1.5** For an ideal $I$ of $R$, $\beta$ an non-empty collection of ideals in $R$ and $\mathcal{S}$ a Serre Class, we use $\mathcal{C}(\mathcal{S}, I, \beta)$ to denote the class of all $(\mathcal{S}, I, \beta)$-Cominimax $R$-modules.

**Example 2.1.6** Let $N \in \mathcal{S}$ be such that $\mathrm{Supp}_R(N) \subseteq \tilde{\mathcal{F}}(I, \beta)$. Then, it follows from Proposition 1.2.5 that $N \in \mathcal{C}(\mathcal{S}, I, \beta)$.

**Proposition 2.1.7** Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be a short exact sequence of $R$-modules such that two of them are $(\mathcal{S}, I, \beta)$-Cominimax. Then the third module is also $(\mathcal{S}, I, \beta)$-Cominimax.

***Proof*** The result follows from the equality $\mathrm{Supp}(M) = \mathrm{Supp}(M') \cup \mathrm{Supp}(M'')$, the long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^i_R(R/I, M) \longrightarrow \mathrm{Ext}^i_R(R/I, M'') \longrightarrow \mathrm{Ext}^{i+1}_R(R/I, M') \longrightarrow \cdots$$

and Proposition 1.2.5. ■

**Proposition 2.1.8** Let $I$, $I'$ be ideals of $R$ and $\beta$, $\beta'$ non-empty collections of ideals in $R$. Then:
$(i)$ $M \in \mathcal{C}(\mathcal{S}, I, \beta)$ if and only if $M \in \mathcal{C}(\mathcal{S}, \sqrt{I}, \beta)$.
$(ii)$ If $M$ is $I$-Cominimax, then $M \in \mathcal{C}(\mathcal{S}_{I,\beta}, I, \beta)$.
$(iii)$ If $\mathrm{Min}(M) \subseteq \tilde{\mathcal{F}}(I', \beta)$, $I^n \subseteq \sqrt{I'}$ for some $n \in \mathbb{N}$ and $\mathrm{Ext}^i_R(R/I, M) \in \mathcal{S}_{I,\beta}$ for all $i \geq 0$, then $M \in \mathcal{C}(\mathcal{S}_{I,\beta}, I, \beta)$ and $M \in \mathcal{C}(\mathcal{S}_{I',\beta}, I', \beta)$.
$(iv)$ If $\mathrm{Min}(M) \subseteq \tilde{\mathcal{F}}(I, \beta)$ and $\beta' \subseteq \beta$, then $M \in \mathcal{C}(\mathcal{S}_{I,\beta}, I, \beta)$ if and only if $M \in \mathcal{C}(\mathcal{S}_{I,\beta'}, I, \beta')$.

***Proof*** (i) Note that $\tilde{\mathcal{F}}(I, \beta) = \tilde{\mathcal{F}}(\sqrt{I}, \beta)$. So $M$ is $(I, \beta)$-torsion if and only if is $(\sqrt{I}, \beta)$-torsion. Since $\mathcal{S}_{I,\beta} = \mathcal{S}_{\sqrt{I},\beta}$,the result follows by Proposition 1.2.4.

(ii) Since $M$ is $I$-Cominimax, we have

$$\mathrm{Supp}(M) \subseteq V(I) \subseteq \tilde{\mathcal{F}}(I, \beta) \text{ and } \mathrm{Ext}^i_R(R/I, M) \in \mathcal{S}_I,$$

for all $i \in \mathbb{N}_0$. Moreover, $\mathrm{SuppExt}^i_R(R/I, M) \subseteq V(I)$ for any $i \in \mathbb{N}_0$. So, $\mathrm{Ext}^i_R(R/I, M)$ is $I$-torsion, for each $i \geq 0$ and, consequently, $\mathrm{Ext}^i_R(R/I, M) \in \mathcal{S}_{I,\beta}$ for all $i$.

(iii) Knowing that $Min(M) \subseteq \tilde{\mathcal{F}}(I', \beta)$ and $I^n \subseteq \sqrt{I'}$, we get $\mathrm{Supp}(M) \subseteq \tilde{\mathcal{F}}(I', \beta) \subseteq \tilde{\mathcal{F}}(I, \beta)$. Since $\mathrm{Ext}^i_R(R/I, M) \in \mathcal{S}_{I,\beta}$ for all $i \geq 0$, it follows that $M \in \mathcal{C}(\mathcal{S}_{I,\beta}, I, \beta)$. Moreover, $\mathcal{S}_{I,\beta} \subseteq \mathcal{S}_{I',\beta}$, by Proposition 1.2.4(iii). Therefore, $M \in \mathcal{C}(\mathcal{S}_{I',\beta}, I', \beta)$.

(iv) Suppose that $M \in \mathcal{C}(\mathcal{S}_{I,\beta}, I, \beta)$. Since $\beta' \subseteq \beta$, we have $\mathrm{Supp}(M) \subseteq \tilde{\mathcal{F}}(I, \beta) \subseteq \tilde{\mathcal{F}}(I, \beta')$ and, consequently, $\mathrm{Ext}^i_R(R/I, M)$ is $(I, \beta')$-torsion, for all $i \geq 0$. So,

$$\mathrm{Ext}^i_R(R/I, M) \in \mathcal{S}_{I,\beta'}, \text{ for any } i \in \mathbb{N}_0.$$

Therefore $M \in \mathcal{C}(\mathcal{S}_{I,\beta'}, I, \beta')$. Conversely, suppose that $M \in \mathcal{C}(\mathcal{S}_{I,\beta'}, I, \beta')$. Then $\mathrm{Supp}(M) \subseteq \tilde{\mathcal{F}}(I, \beta')$ and $\mathrm{Ext}^i_R(R/I, M) \in \mathcal{S}_{I,\beta'}$ for each $i \geq 0$. Since $\beta' \subseteq \beta$, it follows from Proposition 1.2.4(iv) that $\mathcal{S}_{I,\beta'} \subseteq \mathcal{S}_{I,\beta}$. Therefore $M \in \mathcal{C}(\mathcal{S}_{I,\beta}, I, \beta)$. ■

**Proposition 2.1.9** Let $t$ be a non-negative integer such that $\mathrm{H}^i_{I,\beta}(M) \in \mathcal{C}(\mathcal{S}, I, \beta)$ for all $i < t$. Then, $\mathrm{Ext}^i_R(R/I, M) \in \mathcal{S}$ for any $i < t$.

***Proof*** We use induction on $t$. If $t = 0$, there is nothing to prove. For $t = 1$ we have $\mathrm{Hom}_R(R/I, \Gamma_{I,\beta}(M)) = \mathrm{Hom}_R(R/I, M)$. Since $\Gamma_{I,\beta}(M)$ is $(\mathcal{S}, I, \beta)$-cominimax, it follows that $\mathrm{Hom}_R(R/I, M) \in \mathcal{S}$.

Now, suppose that $t \geq 2$ and that for $t-1$ the result is hold. Using the notation of Remark 2.1.2 for $\alpha = \mathcal{F}(I, \beta)$, we have the short exact sequence

$$0 \longrightarrow \Gamma_{I,\beta}(M) \longrightarrow M \longrightarrow \bar{M} \longrightarrow 0$$

which induces the long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^i_R(R/I, \Gamma_{I,\beta}(M)) \longrightarrow \mathrm{Ext}^i_R(R/I, M) \longrightarrow \mathrm{Ext}^i_R(R/I, \bar{M}) \longrightarrow \cdots.$$

Since $\Gamma_{I,\beta}(M) \in \mathcal{C}(\mathcal{S}, I, \beta)$, we have $\mathrm{Ext}^i_R(R/I, \Gamma_{I,\beta}(M)) \in \mathcal{S}$ for all $i \geq 0$. Therefore, is sufficient to show that $\mathrm{Ext}^i_R(R/I, \bar{M}) \in \mathcal{S}$ for any $i < t$. By Remark 2.1.2 we have the isomorphisms

$$\mathrm{Ext}^i_R(R/I, L) \cong \mathrm{Ext}^{i+1}_R(R/I, \bar{M}) \text{ and } \mathrm{H}^i_{I,\beta}(L) \cong \mathrm{H}^{i+1}_{I,\beta}(M), \text{ para todo } i \geq 0.$$

Assuming that $\mathrm{H}^{i+1}_{I,\beta}(M) \in \mathcal{C}(\mathcal{S}, I, \beta)$ for all $i < t-1$, we get that $\mathrm{H}^i_{I,\beta}(L) \in \mathcal{C}(\mathcal{S}, I, \beta)$ for any $i < t-1$. So, applying the inductive hypothesis in the $R$-module $L$, we get $\mathrm{Ext}^i_R(R/I, L) \in \mathcal{S}$ and therefore, $\mathrm{Ext}^{i+1}_R(R/I, \bar{M}) \in \mathcal{S}$ for all $i < t-1$. ■

**Corollary 2.1.10** If $\mathrm{H}^i_{I,\beta}(M) \in \mathcal{C}(\mathcal{S}, I, \beta)$, for all $i \geq 0$, then $\mathrm{Ext}^i_R(R/I, M) \in \mathcal{S}$ for any $i \geq 0$.

**Theorem 2.1.11** Suppose that $\mathrm{Ext}^i_R(R/I, M) \in \mathcal{S}$ for all $i \geq 0$. Let $t$ be a non-negative integer such that $\mathrm{H}^i_{I,\beta}(M) \in \mathcal{C}(\mathcal{S}, I, \beta)$, for any $i \neq t$, then $\mathrm{H}^t_{I,\beta}(M) \in \mathcal{C}(\mathcal{S}, I, \beta)$.

***Proof*** We use induction on $t$. If $t = 0$, we must show that $\mathrm{H}^0_{I,\beta}(M) = \Gamma_{I,\beta}(M)$, that is, we must show that $\mathrm{Ext}^i_R(R/I, \Gamma_{I,\beta}(M)) \in \mathcal{S}$ for all $i \geq 0$. By long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^{i-1}_R(R/I, \bar{M}) \longrightarrow \mathrm{Ext}^i_R(R/I, \Gamma_{I,\beta}(M)) \longrightarrow \mathrm{Ext}^i_R(R/I, M) \longrightarrow \cdots$$

and by hypothesis, it is sufficient to show that $\mathrm{Ext}^i_R(R/I, \bar{M}) \in \mathcal{S}$ for each $i \geq 0$. By Remark 2.1.2, we get $\mathrm{H}^i_{I,\beta}(L) \in \mathcal{C}(\mathcal{S}, I, \beta)$, for all $i \geq 0$. Therefore, $\mathrm{Ext}^{i+1}_R(R/I, \bar{M}) \in \mathcal{S}$ fro each $i \geq 0$, by Corollary 2.1.10. Moreover, $\mathrm{Ext}^0_R(R/I, \bar{M}) = \mathrm{Hom}_R(R/I, \bar{M}) = \mathrm{Hom}_R(R/I, \Gamma_{I,\beta}(\bar{M})) = 0$. So, $\mathrm{Ext}^i_R(R/I, \bar{M}) \in \mathcal{S}$, for any $i \in \mathbb{N}_0$.

Suppose, inductively, that $t > 0$ and that the result holds for $t - 1$. By Remark 2.1.2 we have:

$$\mathrm{Ext}^i_R(R/I, L) \cong \mathrm{Ext}^{i+1}_R(R/I, \bar{M}) \in \mathcal{S} \text{ for any } i \geq 0$$

$$\mathrm{H}^i_{I,\beta}(L) \cong \mathrm{H}^{i+1}_{I,\beta}(M) \in \mathcal{C}(\mathcal{S}, I, \beta) \text{ for all } i \neq t-1.$$

By the inductive hypothesis, $\mathrm{H}^{t-1}_{I,\beta}(L) \in \mathcal{C}(\mathcal{S}, I, \beta)$ which implies that $\mathrm{H}^t_{I,\beta}(M) \in \mathcal{C}(\mathcal{S}, I, \beta)$. ■

**Corollary 2.1.12** Let $M \in \mathcal{S}$ and $t$ be a non-negative integer such that $\mathrm{H}^i_{I,\beta}(M)$ is $(\mathcal{S}, I, \beta)$-cominimax, for all $i \neq t$. Then $\mathrm{H}^t_{I,\beta}(M)$ is $(\mathcal{S}, I, \beta)$-cominimax.

***Proof*** Since $M \in \mathcal{S}$, we have $\mathrm{Ext}^i_R(R/I, M) \in \mathcal{S}$, for all $i \in \mathbb{N}_0$. Therefore, applying Theorem 2.1.11, we get the result. ■

Now, we have conditions to prove the main Theorem of this section, which is a generalization of one of the results in [1, Theorem 3.13].

**Theorem 2.1.13** Let $\mathfrak{a} \in \mathcal{F}(I, \beta)$. Let $t$ be a non-negative integer such that

$$\mathrm{Ext}^t_R(R/\mathfrak{a}, M) \in \mathcal{S} \text{ and } \mathrm{Ext}^j_R(R/\mathfrak{a}, \mathrm{H}^i_{I,\beta}(M)) \in \mathcal{S}$$

for any $i < t$ and all $j \geq 0$. Then, for any submodule $N$ of $\mathrm{H}^t_{I,\beta}(M)$ such that $\mathrm{Ext}^1_R(R/\mathfrak{a}, N) \in \mathcal{S}$, we have that $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{I,\beta}(M)/N) \in \mathcal{S}$; in particular, for $\mathfrak{a} = I$.

***Proof*** The short exact sequence

$$0 \longrightarrow N \longrightarrow \mathrm{H}^t_{I,\beta}(M) \longrightarrow \mathrm{H}^t_{I,\beta}(M)/N \longrightarrow 0$$

induces the long exact sequence

$$\cdots \longrightarrow \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{I,\beta}(M)) \longrightarrow \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{I,\beta}(M)/N) \longrightarrow \mathrm{Ext}^1_R(R/\mathfrak{a}, N) \longrightarrow \cdots .$$

Since $\mathrm{Ext}^1_R(R/\mathfrak{a}, N) \in \mathcal{S}$, it is sufficient to show that $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{I,\beta}(M)) \in \mathcal{S}$. For this, we use induction on $t$. When $t = 0$, we have

$$\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^0_{I,\beta}(M)) = \mathrm{Hom}_R(R/\mathfrak{a}, \Gamma_{I,\beta}(M)) = \mathrm{Hom}_R(R/\mathfrak{a}, M) \in \mathcal{S}.$$

Now, assume that $t > 0$ and that the result holds for $t - 1$. Then, considering the long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^t_R(R/\mathfrak{a}, M) \longrightarrow \mathrm{Ext}^t_R(R/\mathfrak{a}, \bar{M}) \longrightarrow \mathrm{Ext}^{t+1}_R(R/\mathfrak{a}, \Gamma_{I,\beta}(M)) \longrightarrow \cdots$$

we have, by hypothesis, that

$$\mathrm{Ext}^t_R(R/\mathfrak{a}, M) \in \mathcal{S} \text{ e } \mathrm{Ext}^{t+1}_R(R/\mathfrak{a}, \Gamma_{I,\beta}(M)) = \mathrm{Ext}^{t+1}_R(R/\mathfrak{a}, \mathrm{H}^0_{I,\beta}(M)) \in \mathcal{S}.$$

So $\mathrm{Ext}^t_R(R/\mathfrak{a}, \bar{M}) \in \mathcal{S}$. By Remark 2.1.2 we have

$$\mathrm{Ext}^{t-1}_R(R/\mathfrak{a}, L) \cong \mathrm{Ext}^t_R(R/\mathfrak{a}, \bar{M}) \in \mathcal{S} \text{ e} \mathrm{Ext}^j_R(R/\mathfrak{a}, \mathrm{H}^i_{I,\beta}(L)) \cong \mathrm{Ext}^j_R(R/\mathfrak{a}, \mathrm{H}^{i+1}_{I,\beta}(M)) \in \mathcal{S},$$

for each $j \geq 0$ and all $i < t - 1$. Therefore, by induction hypothesis,

$$\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^{t-1}_{I,\beta}(L)/N) \in \mathcal{S}$$

and consequently, $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{I,\beta}(M)/N) \in \mathcal{S}$. ■

**Corollary 2.1.14** Let $t$ be a non-negative integer such that $\mathrm{Ext}^t_R(R/I, M) \in \mathcal{S}$ and $\mathrm{H}^i_{I,\beta}(M) \in \mathcal{C}(\mathcal{S}, I, \beta)$, for all $i < t$. Then, for any submodule $N$ of $\mathrm{H}^t_{I,\beta}(M)$ and all finitely generated $R$-module $M'$ with $\mathrm{Supp}(M') \subseteq V(I)$ and $\mathrm{Ext}^1_R(M', N) \in \mathcal{S}$, we have $\mathrm{Hom}_R(M', \mathrm{H}^t_{I,\beta}(M)/N) \in \mathcal{S}$.

***Proof*** Knowing that $\mathrm{H}^i_{I,\beta}(M) \in \mathcal{C}(\mathcal{S}, I, \beta)$, for all $i < t$, we get $\mathrm{Ext}^j_R(R/I, \mathrm{H}^i_{I,\beta}(M)) \in \mathcal{S}$ for any $i < t$ and all $j \geq 0$. If $N$ is a submodule of $\mathrm{H}^t_{I,\beta}(M)$ and $M'$ is a finitely generated $R$-module such that $\mathrm{Supp}(M') \subseteq V(I)$ and $\mathrm{Ext}^1_R(M', N) \in \mathcal{S}$, then

$\mathrm{Ext}_R^1(R/I, N) \in \mathcal{S}$, by Corollary 1.2.7. By before Theorem, $\mathrm{Hom}_R(R/I, \mathrm{H}_{I,\beta}^t(M)/N) \in \mathcal{S}$. Since $M'$ is finitely generated and $\mathrm{Supp}(M') \subseteq V(I)$, it follows that

$$\mathrm{Hom}_R(M', \mathrm{H}_{I,\beta}^t(M)/N) \in \mathcal{S}.$$

■

**Proposition 2.1.15** Let $t$ be a non-negative integer such that $\mathrm{H}_{I,\beta}^i(M) \in \mathcal{C}(\mathcal{S}, I, \beta)$ for all $i < t$. Then the following statements holds:
$(i)$ If $\mathrm{Ext}_R^t(R/I, M) \in \mathcal{S}$, then $\mathrm{Hom}_R(R/I, \mathrm{H}_{I,\beta}^t(M)) \in \mathcal{S}$.
$(ii)$ If $\mathrm{Ext}_R^{t+1}(R/I, M) \in \mathcal{S}$, then $\mathrm{Ext}_R^1(R/I, \mathrm{H}_{I,\beta}^t(M)) \in \mathcal{S}$.
$(iii)$ If $\mathrm{Ext}_R^i(R/I, M) \in \mathcal{S}$ for all $i \geq 0$, then $\mathrm{Hom}_R(R/I, \mathrm{H}_{I,\beta}^{t+1}(M)) \in \mathcal{S}$ if, and only if, $\mathrm{Ext}_R^2(R/I, \mathrm{H}_{I,\beta}^t(M)) \in \mathcal{S}$.

***Proof*** (i) By hypothesis, $\mathrm{H}_{I,\beta}^i(M) \in \mathcal{C}(\mathcal{S}, I, \beta)$ for each $i < t$. So $\mathrm{Ext}_R^j(R/I, \mathrm{H}_{I,\beta}^i(M)) \in \mathcal{S}$ for all $j \geq 0$ and each $i < t$. Knowing that $\mathrm{Ext}_R^t(R/I, M) \in \mathcal{S}$ and taking $N = 0$ in the Theorem 2.1.13, we get $\mathrm{Hom}_R(R/I, \mathrm{H}_{I,\beta}^t(M)) \in \mathcal{S}$.

(ii) The proof will be done by induction on $t$. Suppose that $t = 0$. Then, by the long exact sequence

$$0 \longrightarrow \mathrm{Ext}_R^1(R/I, \Gamma_{I,\beta}(M)) \longrightarrow \mathrm{Ext}_R^1(R/I, M) \longrightarrow \mathrm{Ext}_R^1(R/I, \bar{M})$$

$$\longrightarrow \mathrm{Ext}_R^2(R/I, \Gamma_{I,\beta}(M)) \longrightarrow \mathrm{Ext}_R^2(R/I, M) \longrightarrow \mathrm{Ext}_R^2(R/I, \bar{M})$$

$$\vdots$$

$$\longrightarrow \mathrm{Ext}_R^i(R/I, M) \longrightarrow \mathrm{Ext}_R^i(R/I, \bar{M}) \longrightarrow \mathrm{Ext}_R^{i+1}(R/I, \Gamma_{I,\beta}(M)) \longrightarrow \cdots \qquad (2.1)$$

and by $\mathrm{Ext}_R^1(R/I, M) \in \mathcal{S}$, we have $\mathrm{Ext}_R^1(R/I, \mathrm{H}_{I,\beta}^0(M)) = \mathrm{Ext}_R^1(R/I, \Gamma_{I,\beta}(M)) \in \mathcal{S}$.

Now, suppose that $t > 0$ and assume that the result holds for $t - 1$. Then $\mathrm{H}_{I,\beta}^0(M) = \Gamma_{I,\beta}(M) \in \mathcal{C}(\mathcal{S}, I, \beta)$, which implies in $\mathrm{Ext}_R^i(R/I, \Gamma_{I,\beta}(M)) \in \mathcal{S}$, for all $i \geq 0$. Since $\mathrm{Ext}_R^{t+1}(R/I, M)$ and $\mathrm{Ext}_R^{t+2}(R/I, \Gamma_{I,\beta}(M))$ belong to $\mathcal{S}$, it follows from exact sequence (2.1) that $\mathrm{Ext}_R^{t+1}(R/I, \bar{M}) \in \mathcal{S}$. Using the Remark 2.1.2 we conclude that $\mathrm{H}_{I,\beta}^i(L) \in \mathcal{C}(\mathcal{S}, I, \beta)$ for any $i < t - 1$ and $\mathrm{Ext}_R^t(R/I, L) \in \mathcal{S}$. By the inductive hypothesis, $\mathrm{Ext}_R^1(R/I, \mathrm{H}_{I,\beta}^{t-1}(L)) \in \mathcal{S}$ and therefore, $\mathrm{Ext}_R^1(R/I, \mathrm{H}_{I,\beta}^t(M)) \in \mathcal{S}$.

(iii) Suppose that $\operatorname{Hom}_R(R/I, \mathrm{H}^{t+1}_{I,\beta}(M)) \in \mathcal{S}$. Now, we use induction on $t$. If $t=0$, then $\operatorname{Hom}_R(R/I, \mathrm{H}^1_{I,\beta}(M)) \in \mathcal{S}$. On the other hand, by Remark 2.1.2 we have

$$\begin{aligned}\operatorname{Hom}_R(R/I, \mathrm{H}^1_{I,\beta}(M)) &\cong \operatorname{Hom}_R(R/I, \Gamma_{I,\beta}(L)) \\ &\cong \operatorname{Hom}_R(R/I, L) \\ &\cong \operatorname{Ext}^1_R(R/I, \bar{M}) \in \mathcal{S}.\end{aligned}$$

Since $\operatorname{Ext}^2_R(R/I, M) \in \mathcal{S}$, it follows from exact sequence (2.1) that $\operatorname{Ext}^2_R(R/I, \Gamma_{I,\beta}(M)) \in \mathcal{S}$.

Suppose that $t>0$ and assume that the result holds for $t-1$. Since $\Gamma_{I,\beta}(M) \in \mathcal{C}(\mathcal{S}, I, \beta)$, we have $\operatorname{Ext}^i_R(R/I, \Gamma_{I,\beta}(M)) \in \mathcal{S}$, for any $i \geq 0$. So, the exactness of sequence (2.1) implies that $\operatorname{Ext}^i_R(R/I, \bar{M}) \in \mathcal{S}$, for all $i \geq 0$. Now, using the Remark 2.1.2 we have $\operatorname{Ext}^i_R(R/I, L) \in \mathcal{S}$, for all $i \geq 0$, and $\operatorname{Hom}_R(R/I, \mathrm{H}^t_{I,\beta}(L)) \in \mathcal{S}$. Therefore, by the inductive hypothesis, $\operatorname{Ext}^2_R(R/I, \mathrm{H}^{t-1}_{I,\beta}(L)) \in \mathcal{S}$ and consequently $\operatorname{Ext}^2_R(R/I, \mathrm{H}^t_{I,\beta}(M)) \in \mathcal{S}$.

Conversely, we use induction on $t$. If $t=0$, then $\operatorname{Ext}^2_R(R/I, \Gamma_{I,\beta}(M)) \in \mathcal{S}$. So, we must to show that $\operatorname{Hom}_R(R/I, \mathrm{H}^1_{I,\beta}(M)) \in \mathcal{S}$. Since $\operatorname{Ext}^1_R(R/I, M) \in \mathcal{S}$, it follows from exact sequence

$$\operatorname{Ext}^1_R(R/I, M) \longrightarrow \operatorname{Ext}^1_R(R/I, \bar{M}) \longrightarrow \operatorname{Ext}^2_R(R/I, \Gamma_{I,\beta}(M))$$

that $\operatorname{Ext}^1_R(R/I, \bar{M}) \cong \operatorname{Hom}_R(R/I, \mathrm{H}^1_{I,\beta}(M)) \in \mathcal{S}$.

Assume that $t>0$ and that the result holds for $t-1$. Then $\Gamma_{I,\beta}(M) \in \mathcal{C}(\mathcal{S}, I, \beta)$ and consequently $\operatorname{Ext}^i_R(R/I, \Gamma_{I,\beta}(M)) \in \mathcal{S}$, for all $i \geq 0$. By the exact sequence

$$\operatorname{Ext}^i_R(R/I, M) \longrightarrow \operatorname{Ext}^i_R(R/I, \bar{M}) \longrightarrow \operatorname{Ext}^{i+1}_R(R/I, \Gamma_{I,\beta}(M))$$

we get $\operatorname{Ext}^i_R(R/I, \bar{M}) \in \mathcal{S}$ for all $i \geq 0$. Note that $\operatorname{Ext}^i_R(R/I, L) \in \mathcal{S}$, for all $i \geq 0$, and $\operatorname{Ext}^2_R(R/I, \mathrm{H}^{i-1}_{I,\beta}(L)) \in \mathcal{S}$. By the inductive hypothesis, $\operatorname{Hom}_R(R/I, \mathrm{H}^t_{I,\beta}(L)) \in \mathcal{S}$ and so, $\operatorname{Hom}_R(R/I, \mathrm{H}^{t+1}_{I,\beta}(M)) \in \mathcal{S}$. ■

## 2.2 Properties of associated primes of $\mathrm{H}^t_{I,\beta}(M)$

**Lemma 2.2.1** Let $\mathfrak{a}$ be an ideal of $R$. Let $M$ be an $R$-module such that $\operatorname{Supp} M \subseteq V(\mathfrak{a})$ and $(0 :_M \mathfrak{a})$ has finite Goldie dimension. Then $M$ has finite Goldie dimension.

***Proof*** We know that $(0 :_M \mathfrak{a}) \cong \mathrm{Hom}_R(R/\mathfrak{a}, M)$. Then

$$\mathrm{Ass}_R(0 :_M \mathfrak{a}) = V(\mathfrak{a}) \cap \mathrm{Ass}_R(M).$$

Since $\mathrm{Ass}_R(M) \subseteq \mathrm{Supp}_R(M) \subseteq V(\mathfrak{a})$ it follows that $\mathrm{Ass}_R(0 :_M \mathfrak{a}) = \mathrm{Ass}_R(M)$ and consequently the set $\mathrm{Ass}_R(M)$ is finite. On the other hand, for any $\mathfrak{p} \in \mathrm{Ass}_R M$ we have

$$\mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}) \cong \mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), (0 :_{M_\mathfrak{p}} \mathfrak{a}R_\mathfrak{p})),$$

as $k(\mathfrak{p})$-vector spaces, where $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. Therefore $\mu^0(\mathfrak{p}, M)$ is finite and hence $GdimM < \infty$. ■

**Theorem 2.2.2** Let $M$ be an $R$-module. Let $t$ be a non-negative integer such that $\mathrm{H}^i_{I,\beta}(M)$ is $(I,\beta)$-cominimax for all $i < t$, and $\mathrm{Ext}^t_R(R/I, M)$ is $(I,\beta)$-minimax. Then for any $(I,\beta)$-minimax submodule $N$ of $\mathrm{H}^t_{I,\beta}(M)$ and for any finitely generated $R$-module $L$ with $\mathrm{Supp}_R(L) \subseteq V(I)$, the $R$-module $\mathrm{Hom}_R(L, \mathrm{H}^t_{I,\beta}(M)/N)$ is $(I,\beta)$-minimax.

***Proof*** The exact sequence

$$0 \longrightarrow N \longrightarrow \mathrm{H}^t_{I,\beta}(M) \longrightarrow \mathrm{H}^t_{I,\beta}(M)/N \longrightarrow 0$$

provides the following exact sequence

$$\cdots \longrightarrow \mathrm{Hom}_R(L, \mathrm{H}^t_{I,\beta}(M)) \longrightarrow \mathrm{Hom}_R(L, \mathrm{H}^t_{I,\beta}(M)/N) \longrightarrow \mathrm{Ext}^1_R(L, N) \longrightarrow \cdots .$$

Since by Proposition 1.2.5, $\mathrm{Ext}^1_R(L, N)$ is $(I,\beta)$-minimax, so in view of Definition 1.2.1 it is sufficient to prove that the $R$-module $\mathrm{Hom}_R(L, \mathrm{H}^t_{I,\beta}(M))$ is $(I,\beta)$-minimax. To this end, in view of Corollary 1.2.7, it is enough to prove that the $R$-module $\mathrm{Hom}_R(R/I, \mathrm{H}^t_{I,\beta}(M))$ is $(I,\beta)$-minimax.

We use induction on $t$. When $t = 0$, the $R$-module $\mathrm{Hom}_R(R/I, M)$ is $(I,\beta)$-minimax, by assumption. Since

$$\mathrm{Hom}_R(R/I, \mathrm{H}^0_{I,\beta}(M)) \cong \mathrm{Hom}_R(R/I, \Gamma_{I,\beta}(M)) \cong \mathrm{Hom}_R(R/I, M),$$

it follows that $\mathrm{Hom}_R(R/I, \mathrm{H}^0_{I,\beta}(M))$ is $(I,\beta)$-minimax.

Now suppose, inductively, that $t > 0$ and that the result has been proved for $t-1$. Since $\Gamma_{I,\beta}(M)$ is $(I,\beta)$-cominimax, it follows that $\mathrm{Ext}^i_R(R/I, \Gamma_{I,\beta}(M))$ is $(I,\beta)$-minimax for all $i \geq 0$. On the other hand, the exact sequence

$$0 \longrightarrow \Gamma_{I,\beta}(M) \longrightarrow M \longrightarrow M/\Gamma_{I,\beta}(M) \longrightarrow 0$$

induces the exact sequence

$$\longrightarrow \mathrm{Ext}^t_R(R/I, M) \longrightarrow \mathrm{Ext}^t_R(R/I, M/\Gamma_{I,\beta}(M)) \longrightarrow \mathrm{Ext}^{t+1}_R(R/I, \Gamma_{I,\beta}(M)) \longrightarrow \cdots.$$

Hence, by Definition 1.2.1 and the assumption, the $R$-module $\mathrm{Ext}^t_R(R/I, M/\Gamma_{I,\beta}(M))$ is $(I,\beta)$-minimax. Also since $\mathrm{H}^0_{I,\beta}(M/\Gamma_{I,\beta}(M)) = 0$ and $\mathrm{H}^i_{I,\beta}(M/\Gamma_{I,\beta}(M)) \cong \mathrm{H}^i_{I,\beta}(M)$ for all $i > 0$, it is follows that $\mathrm{H}^i_{I,\beta}(M/\Gamma_{I,\beta}(M))$ is $(I,\beta)$-cominimax for all $i < t$. Therefore we may assume that $M$ is $(I,\beta)$-torsion-free. Let $E$ be an injective envelope of $M$ and put $M_1 = E/M$. Then also $\Gamma_{I,\beta}(E) = 0$ and $\mathrm{Hom}_R(R/I, E) = 0$. Consequently, $\mathrm{Ext}^i_R(R/I, M_1) \cong \mathrm{Ext}^{i+1}_R(R/I, M)$ and $\mathrm{H}^i_{I,\beta}(M_1) \cong \mathrm{H}^{i+1}_{I,\beta}(M)$ for all $i \geq 0$. The induction hypothesis applied to $M_1$ yields that $\mathrm{Hom}_R(R/I, \mathrm{H}^{t-1}_{I,\beta}(M_1))$ is $(I,\beta)$-minimax. Hence $\mathrm{Hom}_R(R/I, \mathrm{H}^t_{I,\beta}(M))$ is $(I,\beta)$-minimax. ■

**Theorem 2.2.3** Let $M$ be an $(I,\beta)$-minimax $R$-module. Let $t$ be a non-negative integer such that $\mathrm{H}^i_{I,\beta}(M)$ is $(I,\beta)$-minimax for all $i < t$. Then for any $(I,\beta)$-minimax submodule $N$ of $\mathrm{H}^t_{I,\beta}(M)$ with $\mathrm{Supp}_R(\mathrm{H}^t_{I,\beta}(M)/N) \subseteq V(I)$, the $R$-module $\mathrm{Hom}_R(R/I, \mathrm{H}^t_{I,\beta}(M)/N)$ is $(I,\beta)$-minimax. In particular, the Goldie dimension of $\mathrm{H}^t_{I,\beta}(M)/N$ is finite, and so the set $\mathrm{Ass}_R(\mathrm{H}^t_{I,\beta}(M)/N)$ is finite.

***Proof*** Apply Theorem 2.2.2 and Lemma 2.2.1. ■

**Corollary 2.2.4** Let $M$ be a finitely generated $R$-module. Let $\mathcal{N}$ (resp. $\mathcal{A}$) denote the category of all Noetherian (resp. Artinian) $R$-modules and $R$-homomorphisms. Let $t$ be a non-negative integer such that $\mathrm{H}^i_{I,\beta}(M) \in Obj(\mathcal{N}) \cup Obj(\mathcal{A})$ for all $i < t$. Then the $R$-module $\mathrm{Hom}_R(R/I, \mathrm{H}^t_{I,\beta}(M))$ is $(I,\beta)$-minimax, and so the set $\mathrm{Ass}_R(\mathrm{H}^t_{I,\beta}(M))$ is finite.

***Proof*** Apply Theorem 2.2.2 and the fact that the class of $(I,\beta)$-minimax $R$-modules contains all Noetherian and Artinian modules. ■

**Proposition 2.2.5** Let $t$ be a non-negative integer such that $\mathrm{Ext}^t_R(R/I, M) \in \mathcal{S}_{I,\beta}$ and $\mathrm{H}^i_{I,\beta}(M) \in \mathcal{C}(\mathcal{S}_{I,\beta}, I, \beta)$, for all $i < t$. Let $N$ an $R$-submodule of $\mathrm{H}^t_{I,\beta}(M)$ such that $\mathrm{Ext}^1_R(R/I, N) \in \mathcal{S}_{I,\beta}$. If $\mathrm{Supp}(\mathrm{H}^t_{I,\beta}(M)/N) \subseteq V(I)$, then $Gdim(\mathrm{H}^t_{I,\beta}(M)/N) < \infty$ and so, the set of associated primes to $\mathrm{H}^t_{I,\beta}(M)/N$ is finite.

***Proof*** Using the Theorem 2.1.13 for the class $\mathcal{S}_{I,\beta}$, we have $\mathrm{Hom}_R(R/I, \mathrm{H}^t_{I,\beta}(M)/N) \in \mathcal{S}_{I,\beta}$, and this implies that $\mathrm{Hom}_R(R/I, \mathrm{H}^t_{I,\beta}(M)/N) \in \mathcal{S}_I$. Note that

$$\mathrm{Supp}\mathrm{Hom}_R(R/I, \mathrm{H}^t_{I,\beta}(M)/N) = \mathrm{Supp}(\mathrm{H}^t_{I,\beta}(M)/N) \subseteq V(I).$$

So $\mathrm{Hom}_R(R/I, \mathrm{H}^t_{I,\beta}(M)/N)$ is $I$-torsion and consequently, minimax. Therefore,

$$Gdim\mathrm{Hom}_R(R/I, \mathrm{H}^t_{I,\beta}(M)/N) < \infty.$$

By isomorphism

$$\mathrm{Hom}_R(R/I, \mathrm{H}^t_{I,\beta}(M)/N) \cong (0 :_{\mathrm{H}^t_{I,\beta}(M)/N} I),$$

it follows from Lemma 2.2.1 that $Gdim\mathrm{H}^t_{I,\beta}(M)/N < \infty$. ■

**Corollary 2.2.6** Let $t$ be a non-negative integer such that $\mathrm{Ext}^t_R(R/I, M)$ and $\mathrm{H}^i_{I,\beta}(M)$ are $(I, \beta)$-minimax, for all $i < t$. Let $N$ be a submodule of $\mathrm{H}^t_{I,\beta}(M)$ such that

$$\mathrm{Supp}(\mathrm{H}^t_{I,\beta}(M)/N) \subseteq V(I)$$

and $\mathrm{Ext}^1_R(R/I, N)$ is $(I, \beta)$-minimax. Then the set $\mathrm{Ass}_R(\mathrm{H}^t_{I,\beta}(M)/N)$ is finite.

***Proof*** Since $\mathrm{H}^i_{I,\beta}(M)$ is $(I, \beta)$-torsion, for all $i < t$, we have $\mathrm{Supp}(\mathrm{H}^i_{I,\beta}(M)) \subseteq \tilde{\mathcal{F}}(I, \beta)$, for any $i < t$. So, by Example 2.1.6, $\mathrm{H}^i_{I,\beta}(M) \in \mathcal{C}(\mathcal{S}_{I,\beta}, I, \beta)$ for any $i < t$. Therefore, by before Proposition, the set $\mathrm{Ass}_R(\mathrm{H}^t_{I,\beta}(M)/N)$ is finite. ■

**Corollary 2.2.7** Let $t$ be a non-negative integer such that $\mathrm{Ext}^t_R(R/I, M)$ and $\mathrm{H}^i_{I,\beta}(M)$ are $(I, \beta)$-minimax, for all $i < t$. If $\mathrm{Supp}(\mathrm{H}^t_{I,\beta}(M)) \subseteq V(I)$, then the set $\mathrm{Ass}_R(\mathrm{H}^t_{I,\beta}(M))$ is finite.

***Proof*** Set $N = 0$ in Corollary 2.2.6. ■

# Chapter 3

# Generalized local cohomology with respect to good family of ideals

In [23], Tran Tuan Nam and Nguyen Minh Tri introduced the generalized local cohomology modules with respect to a pair of ideals $(I, J)$ as follows: for two $R$-modules $M$ and $N$ the module $\Gamma_{I,J}(M,N)$ is the $(I,J)$-torsion submodule of $\operatorname{Hom}_R(M,N)$, where for any $R$-module $L$

$$\Gamma_{I,J}(L) = \{x \in L \mid I^n x \subseteq Jx \text{ for some } n \geq 1\}.$$

For each fixed $R$-module $M$, there is a covariant functor $\Gamma_{I,J}(M,-)$ from the category of $R$-modules to itself. The $i$-th generalized local cohomology functor $\mathrm{H}^i_{I,J}(M,-)$ with respect to pair of ideals $(I,J)$ is defined to be the $i$-th right derived functor of $\Gamma_{I,J}(M,-)$.

Another definition of generalized local cohomology functors was introduced by Zamani in [30] as follow

$$\mathrm{H}^i_{I,J}(M,N) = H^i(\operatorname{Hom}_R(M, \Gamma_{I,J}(E^\bullet)))$$

for all $i \geq 0$, where $E^\bullet$ is a minimal injective resolution of $R$-module $N$.

Inspired by this, we introduce a module $\Gamma_\alpha(M,N)$ as follows: given two $R$-modules $M$ and $N$ we define $\Gamma_\alpha(M,N)$ to be the $\alpha$-torsion submodule of $\operatorname{Hom}_R(M,N)$, where the $\alpha$-torsion module is defined by Alba-Sarria in [2] by: for any $R$-module $L$

$$\Gamma_\alpha(L) = \{x \in L \mid \operatorname{Supp}_R(Rx) \subseteq \alpha\}.$$

If $M$ is a fixed $R$-module, then there exists a covariant functor $\Gamma_\alpha(M,-)$ from the category of $R$-modules to itself. The $i$-th generalized local cohomology functor $\mathrm{H}^i_\alpha(M,-)$ with the respect to good family $\alpha$ is the $i$-th right derived functor of $\Gamma_\alpha(M,-)$. This definition is really a generalization of the local cohomology funtors $\mathrm{H}^i_\alpha$ with respect to $\alpha$ and it is also a generalization of the generalized local cohomology functors $\mathrm{H}^i_{I,J}(M,-)$.

The organization of the chapter is as follows. In the first section, we study some elementary properties of generalized local cohomology modules with respect to good family $\alpha$. We also show some vanishing results concerning these modules.

The second section is devoted to study the $\alpha$-minimaximality of local cohomology modules $\mathrm{H}^i_\alpha(M,N)$ and some results of $(I,\beta)$-cominimaximality of modules $\mathrm{H}^i_{I,\beta}(M,N)$.

## 3.1 Definition and properties of the generalized local cohomology module

**Definition 3.1.1** Let $\alpha$ be a good family of ideals of $R$. For $M$ and $N$ two $R$-modules, we define the *$\alpha$-torsion module module of $M$ and $N$* by

$$\Gamma_\alpha(M,N) := \Gamma_\alpha(\mathrm{Hom}_R(M,N)).$$

For the case $\alpha = \mathcal{F}(I,\beta)$, we will denote by $\Gamma_{I,\beta}(M,N)$ and we will call of $(I,\beta)$-torsion $R$-module of $M$ and $N$. When $M = R$, $\Gamma_\alpha(R,N) = \Gamma_\alpha(N)$, the $\alpha$-torsion $R$-module of $N$.

For each $R$-module $M$, $\Gamma_\alpha(M,-)$ is a left exact covariant functor from the category of $R$-modules to itself.

Let us denote by $\mathrm{H}^i_\alpha(M,-)$ the $i$-th right derived functor of $\Gamma_\alpha(M,-)$ and we call the $i$-th generalized local cohomology functor with the respect to $\alpha$.

**Theorem 3.1.2** Let $M$ be a finitely generated $R$-module and $N$ an $R$-module. Then

$$\Gamma_\alpha(M,N) = \mathrm{Hom}_R(M,\Gamma_\alpha(N)).$$

***Proof*** If $f \in \Gamma_\alpha(M,N)$, there exists an ideal $J \in \alpha$ such that $Jf(x) = 0$, for all $x \in M$. Since $f(x) \in N$, for any $x \in M$, we get $f(M) \subseteq \Gamma_\alpha(N)$. So $f \in \mathrm{Hom}_R(M,\Gamma_\alpha(N))$. Conversely, let $g \in \mathrm{Hom}_R(M,\Gamma_\alpha(N))$. Let $x_1,\dots,x_k$ generators of $M$. Then, for each

$i = 1, \ldots, k$ there exists $J_i \in \alpha$ such that $J_i g(x_i) = 0$. Taking $J = J_1 \ldots J_k$, we have $J \in \alpha$ and $Jg(x) = 0$, for all $x \in M$. Therefore $g \in \Gamma_\alpha(M, N)$. ■

**Corollary 3.1.3** Let $M$ be a fixed finitely generated $R$-module and $N$ an arbitrary $R$-module. If $E^\bullet$ is a minimal injective resolution of $N$, then

$$\mathrm{H}^i_\alpha(M, N) = H^i(\mathrm{Hom}_R(M, \Gamma_\alpha(E^\bullet)))$$

for any $i$.

***Proof*** Since $E^\bullet$ is a minimal injective resolution of $N$ and $\mathrm{H}^i_\alpha(M, -) := \mathbf{R}^i\Gamma_\alpha(M, -)$ we have

$$\mathrm{H}^i_\alpha(M, N) = \mathbf{R}^i\Gamma_\alpha(M, N) = H^i(\Gamma_\alpha(M, E^\bullet)).$$

By Theorem 3.1.2

$$\Gamma_\alpha(M, E^\bullet) = \mathrm{Hom}_R(M, \Gamma_\alpha(E^\bullet))$$

and hence

$$\mathrm{H}^i_\alpha(M, N) = H^i(\mathrm{Hom}_R(M, \Gamma_\alpha(E^\bullet)))$$

as required. ■

**Corollary 3.1.4** Let $M$ be a finitely generated $R$-module and $N$ an $R$-module. Then

$$\mathrm{Ass}_R(\Gamma_\alpha(M, N)) = \mathrm{Supp}(M) \cap \mathrm{Ass}_R(N) \cap \alpha.$$

***Proof*** Since $M$ is finitely generated, we have by Theorem 3.1.2

$$\begin{aligned} \mathrm{Ass}_R(\Gamma_\alpha(M, N)) &= \mathrm{Ass}_R(\mathrm{Hom}_R(M, \Gamma_\alpha(N))) \\ &= \mathrm{Supp}(M) \cap \mathrm{Ass}_R(\Gamma_\alpha(N)) \\ &= \mathrm{Supp}(M) \cap \mathrm{Ass}_R(N) \cap \alpha. \end{aligned}$$

■

**Proposition 3.1.5** Let $M$ be a finitely generated $R$-module and $N$ an $R$-module. Let $I, I'$ be ideals of $R$ and $\beta, \beta'$ non-empty collections of ideals in $R$. Then:
*(i)* $\Gamma_{I,\beta}(\Gamma_{I',\beta'}(M, N)) = \Gamma_{I',\beta'}(\Gamma_{I,\beta}(M, N))$.
*(ii)* If $I \subseteq I'$, then $\Gamma_{I,\beta}(M, N) \supseteq \Gamma_{I',\beta}(M, N)$.
*(iii)* If $\beta' \subseteq \beta$, then $\Gamma_{I,\beta}(M, N) \subseteq \Gamma_{I,\beta'}(M, N)$.
*(iv)* $\Gamma_{I,\beta}(\Gamma_{I',\beta}(M, N)) = \Gamma_{I+I',\beta}(M, N)$.
*(v)* $\Gamma_{I,\beta}(\Gamma_{I,\beta'}(M, N)) = \Gamma_{I,\beta\cup\beta'}(M, N)$.
*(vi)* If $\sqrt{I} = \sqrt{I'}$, then $\mathrm{H}^i_{I,\beta}(M, N) = \mathrm{H}^i_{I',\beta}(M, N)$, for all $i \geq 0$. In particular, $\mathrm{H}^i_{I,\beta}(M, N) = \mathrm{H}^i_{\sqrt{I},\beta}(M, N)$, for each $i \geq 0$.
*(vii)* If $\beta$ and $\beta'$ are cofinals, then $\mathrm{H}^i_{I,\beta}(M, N) = \mathrm{H}^i_{I,\beta'}(M, N)$, for any $i \geq 0$.

***Proof*** (i) Since $M$ is finitely generated, we get

$$\begin{aligned}\Gamma_{I,\beta}(\Gamma_{I',\beta'}(M,N)) &= \Gamma_{I,\beta}(\mathrm{Hom}_R(M,\Gamma_{I',\beta'}(N)))\\ &= \mathrm{Hom}_R(M,\Gamma_{I,\beta}(\Gamma_{I',\beta'}(N)))\\ &= \mathrm{Hom}_R(M,\Gamma_{I',\beta'}(\Gamma_{I,\beta}(N))) \text{ by Proposition B.3(i)}\\ &= \Gamma_{I',\beta'}(\Gamma_{I,\beta}(M,N)).\end{aligned}$$

(ii) By inclusion $I \subseteq I'$, it follows from Proposition B.2(i) that $\mathcal{F}(I',\beta) \subseteq \mathcal{F}(I,\beta)$. So $\Gamma_{I',\beta}(M,N) \subseteq \Gamma_{I,\beta}(M,N)$.

(iii) Since $\beta' \subseteq \beta$, we have $\mathcal{F}(I,\beta) \subseteq \mathcal{F}(I,\beta')$ by Proposition B.2(ii). Therefore $\Gamma_{I,\beta}(M,N) \subseteq \Gamma_{I,\beta'}(M,N)$.

(iv) By hypóthesis of $M$ to be finitely generated, we get

$$\begin{aligned}\Gamma_{I+I',\beta}(M,N) &= \mathrm{Hom}_R(M,\Gamma_{I+I',\beta}(N))\\ &= \mathrm{Hom}_R(M,\Gamma_{I,\beta}(\Gamma_{I',\beta}(N))) \text{ by Proposition B.3(iv)}\\ &= \Gamma_{I,\beta}(\Gamma_{I',\beta}(M,N)).\end{aligned}$$

(v) Note that

$$\begin{aligned}\Gamma_{I,\beta\cup\beta'}(M,N) &= \mathrm{Hom}_R(M,\Gamma_{I,\beta\cup\beta'}(N))\\ &= \mathrm{Hom}_R(M,\Gamma_{I,\beta}(\Gamma_{I,\beta'}(N))) \text{ by Proposition B.3(v)}\\ &= \Gamma_{I,\beta}(\Gamma_{I,\beta'}(M,N)).\end{aligned}$$

(vi) Since $\sqrt{I} = \sqrt{I'}$, it follows that $\langle I\rangle = \langle I'\rangle$. Logo $\mathcal{F}(I,\beta) = \mathcal{F}(I',\beta)$ and consequently, $\Gamma_{I,\beta}(M,N) = \Gamma_{I',\beta}(M,N)$. Therefore $\mathrm{H}^i_{I,\beta}(M,N) = \mathrm{H}^i_{I',\beta}(M,N)$, for all $i \geq 0$.

(vii) Knowing that $\beta$ and $\beta'$ are cofinals, we have $\langle\beta\rangle = \langle\beta'\rangle$. Then $\mathrm{H}^i_{I,\beta}(M,N) = \mathrm{H}^i_{I,\beta'}(M,N)$ for all $i \geq 0$. ■

**Proposition 3.1.6** If $M$ is finitely generated and $N$ is $\alpha$-torsion, then

$$\mathrm{H}^i_{\alpha}(M,N) \cong \mathrm{Ext}^i_R(M,N),$$

for all $i \geq 0$.

***Proof*** Since $N$ is $\alpha$-torsion, there exists, by [2, Corollary 1.26], a minimal injective resolution $E^\bullet$ of $N$ formed by $\alpha$-torsion $R$-modules. By Theorem 3.1.2, we get

$$\begin{aligned} \mathrm{H}^i_\alpha(M,N) &= H^i(\mathrm{Hom}_R(M,\Gamma_\alpha(E^\bullet))) \\ &= H^i(\mathrm{Hom}_R(M,E^\bullet)) \\ &= \mathrm{Ext}^i_R(M,N), \end{aligned}$$

for any $i \geq 0$ ∎

**Proposition 3.1.7** Let $\beta$ be a non-empty set of ideals of $R$. If $M$ is a finitely generated $R$-module and $N$ is $J$-torsion, for some $J \in \beta$, then

$$\mathrm{H}^i_{I,\beta}(M,N) \cong \mathrm{H}^i_I(M,N),$$

for all $i \geq 0$ and any ideal $I$ of $R$.

***Proof*** It is clear that $\Gamma_I(N) \subseteq \Gamma_{I,\beta}(N)$. Conversely, if $x \in \Gamma_{I,\beta}(N)$, then there is $K \in \mathcal{F}(I,\beta)$ such that $Kx = 0$. Since $N$ is $J$-torsion, there exists $n \in \mathbb{N}$ such that $J^n x = 0$. Moreover, $\mathcal{F}(I,\beta) = \mathcal{F}(I,\langle\beta\rangle)$. So $K + J^n \in \langle I\rangle$ and $(K + J^n)x = 0$, which implies that $x \in \Gamma_I(N)$. Therefore, by Theorem 3.1.2, we get

$$\begin{aligned} \Gamma_{I,\beta}(M,N) &= \mathrm{Hom}_R(M,\Gamma_{I,\beta}(N)) \\ &= \mathrm{Hom}_R(M,\Gamma_I(N)) \\ &= \Gamma_I(M,N). \end{aligned}$$

On the other hand, since $N$ is $J$-torsion $R$-module there exists, by [6, Corollary 2.1.6], a minimal resolution of $N$ formed by $J$-torsion $R$-modules. Therefore

$$\mathrm{H}^i_{I,\beta}(M,N) \cong \mathrm{H}^i_I(M,N),$$

for all $i \geq 0$ as required. ∎

**Theorem 3.1.8** Let $M$ be a finitely generated $R$-module and $\alpha$ a good family of ideals. Then there exists a natural isomorphism

$$\mathrm{H}^i_\alpha(M,-) \cong \varinjlim_{\mathfrak{a}\in\alpha} \mathrm{H}^i_{\mathfrak{a}}(M,-),$$

for all $i \geq 0$.

***Proof*** First we show that

$$\operatorname{Hom}_R(M, \Gamma_\alpha(N)) = \varinjlim_{\mathfrak{a}\in\alpha} \operatorname{Hom}_R(M, \Gamma_{\mathfrak{a}}(N)).$$

Note that the right side of above statement equals $\bigcup_{\mathfrak{a}\in\alpha} \operatorname{Hom}_R(M, \Gamma_{\mathfrak{a}}(N))$. So, it is enough to show that

$$\operatorname{Hom}_R(M, \Gamma_\alpha(N)) = \bigcup_{\mathfrak{a}\in\alpha} \operatorname{Hom}_R(M, \Gamma_{\mathfrak{a}}(N)).$$

For this, let $f \in \operatorname{Hom}_R(M, \Gamma_\alpha(N))$. Then $f(M) \subseteq \Gamma_\alpha(N)$. Since $M$ is finitely generated, there exists $x_1, \ldots, x_k$ such that $M = \langle x_1, \ldots, x_k \rangle$. So, for each $i = 1, \ldots, k$ there is $\mathfrak{a}_i \in \alpha$ such that $\mathfrak{a}_i f(x_i) = 0$. Taking $\mathfrak{a} = \mathfrak{a}_1 \ldots \mathfrak{a}_k$, we get $\mathfrak{a} \in \alpha$, because $\alpha$ is a good family, and $\mathfrak{a} f(x) = 0$, for all $x \in M$. Therefore $f \in \operatorname{Hom}_R(M, \Gamma_{\mathfrak{a}}(N))$. Conversely, if $g \in \bigcup_{\mathfrak{a}\in\alpha} \operatorname{Hom}_R(M, \Gamma_{\mathfrak{a}}(N))$, then there exists $\mathfrak{a} \in \alpha$ such that $g \in \operatorname{Hom}_R(M, \Gamma_{\mathfrak{a}}(N))$. So, $\mathfrak{a}^n g(x) = 0$, for some $n \in \mathbb{N}$ and all $x \in M$. Since $\mathfrak{a}^n \in \alpha$, it follows that $g(x) \in \Gamma_\alpha(N)$, for any $x \in M$. So $g \in \operatorname{Hom}_R(M, \Gamma_\alpha(N))$.

Now, let $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ be a short exact sequence of $R$-modules. Then, this sequence induces a long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^0_{\mathfrak{a}}(M, X) \longrightarrow \mathrm{H}^0_{\mathfrak{a}}(M, Y) \longrightarrow \mathrm{H}^0_{\mathfrak{a}}(M, Z) \longrightarrow \\ \longrightarrow \mathrm{H}^1_{\mathfrak{a}}(M, X) \longrightarrow \mathrm{H}^1_{\mathfrak{a}}(M, Y) \longrightarrow \mathrm{H}^1_{\mathfrak{a}}(M, Z) \longrightarrow \cdots, \end{aligned}$$

for all $\mathfrak{a} \in \alpha$. Since the direct limits is exact, we get a long exact sequence

$$\begin{aligned} 0 \longrightarrow \varinjlim_{\mathfrak{a}\in\alpha} \mathrm{H}^0_{\mathfrak{a}}(M, X) \longrightarrow \varinjlim_{\mathfrak{a}\in\alpha} \mathrm{H}^0_{\mathfrak{a}}(M, Y) \longrightarrow \varinjlim_{\mathfrak{a}\in\alpha} \mathrm{H}^0_{\mathfrak{a}}(M, Z) \longrightarrow \\ \longrightarrow \varinjlim_{\mathfrak{a}\in\alpha} \mathrm{H}^1_{\mathfrak{a}}(M, X) \longrightarrow \varinjlim_{\mathfrak{a}\in\alpha} \mathrm{H}^1_{\mathfrak{a}}(M, Y) \longrightarrow \varinjlim_{\mathfrak{a}\in\alpha} \mathrm{H}^1_{\mathfrak{a}}(M, Z) \longrightarrow \cdots. \end{aligned}$$

Since $\varinjlim_{\mathfrak{a}\in\alpha} \mathrm{H}^i_{\mathfrak{a}}(M, E) = 0$ for all $i > 0$, whenever $E$ is injective, it follows from [6, Theorem 1.3.5], that

$$\mathrm{H}^i_\alpha(M, -) \cong \varinjlim_{\mathfrak{a}\in\alpha} \mathrm{H}^i_{\mathfrak{a}}(M, -)$$

for each $i \in \mathbb{N}_0$. ■

Let $I \in \alpha$ an ideal of $R$. In [30] it is shown that, for any exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

and any $R$-module $N$ we have the long exact sequence

$$0 \longrightarrow \mathrm{H}_I^0(Z,N) \longrightarrow \mathrm{H}_I^0(Y,N) \longrightarrow \mathrm{H}_I^0(X,N) \longrightarrow \cdots .$$

Applying the direct limits on $\alpha$, we get the long exact sequence

$$0 \longrightarrow \mathrm{H}_\alpha^0(Z,N) \longrightarrow \mathrm{H}_\alpha^0(Y,N) \longrightarrow \mathrm{H}_\alpha^0(X,N) \longrightarrow \cdots ,$$

by Theorem 3.1.8.

**Theorem 3.1.9** Let $(R,\mathfrak{m})$ be a local ring and $M,N$ finitely generated $R$-modules such that $M$ has finite projective dimension $p$. Then

$$\mathrm{H}_\alpha^i(M,N) = 0,$$

for any $i > p + dimN$.

***Proof*** We prove by induction on $p \geq 0$. If $p = 0$, then $M$ is a finitely generated free $R$-module $R^t$, for some $t \in \mathbb{N}$. So, by Theorem 3.1.8 and by [2, Lemma 1.38], we have

$$\mathrm{H}_\alpha^i(M,N) \cong \varinjlim_{\mathfrak{a}\in\alpha} \mathrm{H}_\mathfrak{a}^i(M,N) \cong \varinjlim_{\mathfrak{a}\in\alpha} \mathrm{H}_\mathfrak{a}^i(R^t,N) \cong \bigoplus_t \mathrm{H}_\alpha^i(N) = 0,$$

for all $i > p + dimN$.

Assume that $p > 0$ and that the result holds for $p-1$. Then there exists a finitely generated free $R$-module $F$ and a submodule $L$ of $F$ such that $L$ has projective dimension $p-1$ and the sequence

$$0 \longrightarrow L \longrightarrow F \longrightarrow M \longrightarrow 0$$

is exact. The above exact sequence induces the long exact sequence

$$\mathrm{H}_\alpha^{i-1}(L,N) \longrightarrow \mathrm{H}_\alpha^i(M,N) \longrightarrow \mathrm{H}_\alpha^i(F,N).$$

By inductive hypothesis and Theorem 3.1.8 we get $\mathrm{H}_\alpha^{i-1}(L,N) = 0$ and $\mathrm{H}_\alpha^i(F,N) = 0$, for each $i > p + dimN$. Therefore $\mathrm{H}_\alpha^i(M,N) = 0$, for any $i > p + dimN$. ■

**Corollary 3.1.10** Let $(R,\mathfrak{m})$ a local ring and $M$ a finitely generated $R$-module with finite projective dimension $p$. Then, $\mathrm{H}_\alpha^i(M,N) = 0$, for $i > p + dimR$ for any (not necessarily finitely generated) $R$-module $N$.

***Proof*** By [25, Example 5.32(iii)] , we can write $N = \varinjlim_{\lambda} N_\lambda$ where each $N_\lambda$ is a finitely generated submodule of $N$. Note that, for each $i \in \mathbb{N}_0$

$$\begin{aligned}
\mathrm{H}^i_\alpha(M,N) &= \varinjlim_{\mathfrak{a}\in\alpha} \mathrm{H}^i_{\mathfrak{a}}(M,N) \\
&= \varinjlim_{\mathfrak{a}\in\alpha} \mathrm{H}^i_{\mathfrak{a}}(M, \varinjlim_{\lambda} N_\lambda) \\
&= \varinjlim_{\mathfrak{a}\in\alpha} \varinjlim_{\lambda} \mathrm{H}^i_{\mathfrak{a}}(M, N_\lambda) \\
&= \varinjlim_{\lambda} \varinjlim_{\mathfrak{a}\in\alpha} \mathrm{H}^i_{\mathfrak{a}}(M, N_\lambda) \\
&= \varinjlim_{\lambda} \mathrm{H}^i_\alpha(M, N_\lambda).
\end{aligned}$$

Since $dimR \geq dimN$, it follows from above equality and of Theorem 3.1.9 that $\mathrm{H}^i_\alpha(M,N) = 0$ for any $i > p + dimR$. ■

The next proposition was taken from [30] and will serve as a tool for the posterior theorem.

**Proposition 3.1.11** Let $(R, \mathfrak{m})$ be a local ring with Krull dimension $d$. Assume that $M, N$ are finitely generated $R$-modules and $M$ has finite projective dimension. Then, for each ideal $\mathfrak{a}$ of $R$, $\mathrm{H}^i_{\mathfrak{a}}(M,N) = 0$ for all $i > d$.

***Proof*** See [11, Theorem 3.1]. ■

The following Theorem provides a better quota than that of Corollary 3.1.10 for the vanishing of generalized local cohomology module supported in $\alpha$.

**Theorem 3.1.12** Let $(R, \mathfrak{m})$ be a local ring with Krull dimension $d$. Assume that $M$ is a finitely generated $R$-module and has finite projective dimension. Then

$$\mathrm{H}^i_\alpha(M,N) = 0$$

for all $i > d$ and any $R$-module $N$.

***Proof*** We can write $N = \varinjlim_{\lambda} N_\lambda$ where each $N_\lambda$ is a finitely generated submodule of $N$. Given $\mathfrak{a} \in \alpha$ and $N_\lambda$, we have by Proposition 3.1.11

$$\mathrm{H}^i_{\mathfrak{a}}(M, N_\lambda) = 0,$$

for all $i > d$. Therefore, by Theorem 3.1.8

$$\mathrm{H}^i_\alpha(M,N) \cong \varinjlim_{\lambda} \varinjlim_{\mathfrak{a}\in\alpha} \mathrm{H}^i_{\mathfrak{a}}(M, N_\lambda) = 0,$$

for each $i > d$. ■

**Corollary 3.1.13** Let $(R, \mathfrak{m})$ be a local ring. Assume that $M$ is a finitely generated $R$-module and has finite projective dimension $p$. Then, for any $R$-module $N$ finitely generated $\mathrm{H}^i_\alpha(M,N) = 0$ for all $i > min\{dimR,\ p + dimN\}$.

***Proof*** The result follows from Theorem 3.1.9 and Theorem 3.1.12. ■

## 3.2 Some results of $\alpha$-minimaximality and $(I, \beta)$-cominimaximality

**Theorem 3.2.1** Let $M$ a finitely generated $R$-module and $N$ an $R$-module. Let $t$ a positive integer. If $\mathrm{H}^i_\alpha(N)$ is $\alpha$-minimax, for all $i < t$, then
$(i)$ $\mathrm{H}^i_\alpha(M,N)$ is $\alpha$-minimax, for any $i < t$.
$(ii)$ $\mathrm{Ext}^i_R(R/\mathfrak{a}, N)$ is $\alpha$-minimax, for any $i < t$ and all $\mathfrak{a} \in \alpha$.

***Proof*** (i) We use induction on $t$. If $t = 1$, then $\mathrm{H}^0_\alpha(N) = \Gamma_\alpha(N)$ is $\alpha$-minimax. Since

$$\mathrm{H}^0_\alpha(M,N) = \Gamma_\alpha(M,N) = \mathrm{Hom}_R(M, \Gamma_\alpha(N))$$

and $M$ is finitely generated, it follows that $\mathrm{H}^0_\alpha(M,N)$ is $\alpha$-minimax.

Suppose that $t > 1$ and that the result holds for $t-1$. Applying the functors $\Gamma_\alpha(-)$ and $\Gamma_\alpha(M,-)$ to short exact sequence

$$0 \longrightarrow N \longrightarrow E(N) \longrightarrow E(N)/N \longrightarrow 0 \tag{3.1}$$

we get the isomorphisms

$$\mathrm{H}^i_\alpha(E(N)/N) \cong \mathrm{H}^{i+1}_\alpha(N)$$

$$\mathrm{H}^i_\alpha(M, E(N)/N) \cong \mathrm{H}^{i+1}_\alpha(M,N),$$

for all $i > 0$. By hypothesis, $\mathrm{H}^i_\alpha(N)$ is $\alpha$-minimax, for any $i < t$. So $\mathrm{H}^i_\alpha(E(N)/N)$ is $\alpha$-minimax, for each $i < t-1$. Applying the induction hypothesis in $E(N)/N$ we get $\mathrm{H}^i_\alpha(M, E(N)/N)$ is $\alpha$-minimax, for each $i < t-1$. Therefore $\mathrm{H}^j_\alpha(M,N)$ is $\alpha$-minimax, for all $j < t$.

(ii) The proof is by induction on $t$. If $t = 1$, then $\Gamma_\alpha(N)$ is $\alpha$-minimax. Moreover, given $\mathfrak{a} \in \alpha$, we have $\Gamma_\mathfrak{a}(N) \subseteq \Gamma_\alpha(N)$. So $\Gamma_\mathfrak{a}(N)$ is $\alpha$-minimax. Since $M$ is finitely generated and

$$\mathrm{Hom}_R(R/\mathfrak{a}, \Gamma_\mathfrak{a}(N)) \cong \mathrm{Hom}_R(R/\mathfrak{a}, N),$$

it follows that $\mathrm{Ext}^0_R(R/\mathfrak{a}, N)$ is $\alpha$-minimax.

Now, suppose that $t > 1$ and that the proof holds for $t-1$. By the short exact sequence (3.1) we get the isomorphisms

$$\mathrm{H}^i_\alpha(E(N)/N) \cong \mathrm{H}^{i+1}_\alpha(N)$$

$$\mathrm{Ext}^i_R(R/\mathfrak{a}, E(N)/N) \cong \mathrm{Ext}^{i+1}_R(R/\mathfrak{a}, N),$$

for all $i > 0$. Since $\mathrm{H}^{i+1}_\alpha(N)$ is $\alpha$-minimax for all $i < t-1$, it follows that $\mathrm{H}^i_\alpha(E(N)/N)$ is also $\alpha$-minimax, for each $i < t-1$. Applying the induction hypothesis in $E(N)/N$, we conclude that $\mathrm{Ext}^i_R(R/\mathfrak{a}, E(N)/N)$ is $\alpha$-minimax for all $i < t-1$. Therefore $\mathrm{Ext}^j_R(R/\mathfrak{a}, N)$ is $\alpha$-minimax, for any $j < t$. ■

**Theorem 3.2.2** Let $M, N$ be two finitely generated $R$-modules and $t$ a positive integer such that $\mathrm{H}^t_\alpha(M, R/\mathfrak{p})$ is $\alpha$-minimax, for all $\mathfrak{p} \in \mathrm{Supp}(N)$. Then $\mathrm{H}^t_\alpha(M, N)$ is $\alpha$-minimax.

***Proof*** As $N$ is finitely generated, there exists a filtration of $N$

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_k = N$$

such that $N_i/N_{i-1} \cong R/\mathfrak{p}_i$, for some $\mathfrak{p}_i \in \mathrm{Supp}(N)$.

If $i = 1$, then $\mathrm{H}^t_\alpha(M, N_1) \cong \mathrm{H}^t_\alpha(M, R/\mathfrak{p}_1)$ and therefore $\alpha$-minimax. For each $2 \leq i \leq k$, there is a short exact sequence

$$0 \longrightarrow N_{i-1} \longrightarrow N_i \longrightarrow R/\mathfrak{p}_i \longrightarrow 0. \tag{3.2}$$

For $i = 2$, consider the following part of the long exact sequence induced by (3.2)

$$\cdots \longrightarrow \mathrm{H}^t_\alpha(M, N_{i-1}) \longrightarrow \mathrm{H}^t_\alpha(M, N_i) \longrightarrow \mathrm{H}^t_\alpha(M, R/\mathfrak{p}_i) \longrightarrow \cdots. \tag{3.3}$$

Since $i = 2$, it follows that $\mathrm{H}^t_\alpha(M, N_1)$ is $\alpha$-minimax and by hypothesis $\mathrm{H}^t_\alpha(M, R/\mathfrak{p}_2)$ is also $\alpha$-minimax. Then, by (3.3) we have $\mathrm{H}^t_\alpha(M, N_2)$ is $\alpha$-minimax. Proceeding recursively, we conclude that $\mathrm{H}^t_\alpha(M, N)$ is $\alpha$-minimax. ■

**Corollary 3.2.3** Let $M, N$ be a finitely generated $R$-modules and $t$ a positive integer. Assume that $\mathrm{H}^t_\alpha(M, R/\mathfrak{p})$ is $\alpha$-minimax, for all $\mathfrak{p} \in \mathrm{Supp}(N)$.
($i$) If $L$ is finitely generated $R$-module such that $\mathrm{Supp}(L) \subseteq \mathrm{Supp}(N)$, then $\mathrm{H}^t_\alpha(M, L)$ is $\alpha$-minimax.
($ii$) If $\mathfrak{a}$ is an ideal of $R$ such that $V(\mathfrak{a}) \subseteq \mathrm{Supp}(N)$, then $\mathrm{H}^t_\alpha(M, R/\mathfrak{a})$ is $\alpha$-minimax.
($iii$) If $\alpha \cap Spec(R) \subseteq \mathrm{Supp}(N)$, then $\mathrm{H}^t_\alpha(M, R/\mathfrak{a})$ is $\alpha$-minimax, for all $\mathfrak{a} \in \alpha$.

***Proof*** (i) Since $\mathrm{Supp}(L) \subseteq \mathrm{Supp}(N)$, it follows that $\mathrm{H}^t_\alpha(M, R/\mathfrak{p})$ is $\alpha$-minimax, for each $\mathfrak{p} \in \mathrm{Supp}(L)$. By Theorem 3.2.2 $\mathrm{H}^t_\alpha(M, L)$ is $\alpha$-minimax.

(ii) knowing that $V(\mathfrak{a})$ is in bijection with $\mathrm{Supp}(R/\mathfrak{a})$, we fall back on the hypothesis of item (i). So $\mathrm{H}^t_\alpha(M, R/\mathfrak{a})$ is $\alpha$-minimax.

(iii) Given $\mathfrak{a} \in \alpha$, since $\alpha$ is a good family, it follows that $V(\mathfrak{a}) \subseteq \alpha \cap Spec(R) \subseteq \mathrm{Supp}(N)$. Therefore, by item (ii), $\mathrm{H}^t_\alpha(M, R/\mathfrak{a})$ is $\alpha$-minimax, for any $\mathfrak{a} \in \alpha$. ■

**Lemma 3.2.4** Assume that $(R, \mathfrak{m})$ is a local ring. Let $M$ be a finitely generated $R$-module with Krull dimension $d$. Then $\mathrm{H}^d_\alpha(M)$ is Artinian.

***Proof*** See [2, Theorem 2.2]. ■

**Theorem 3.2.5** Let $(R, \mathfrak{m})$ be a local ring and $M, N$ be two finitely generated $R$-modules such that $M$ has projective dimension $r$ and $N$ has Krull dimension $d$. Then

$$\mathrm{H}^{r+d}_\alpha(M, N) \cong \mathrm{Ext}^r_R(M, \mathrm{H}^d_\alpha(N)).$$

Moreover, $\mathrm{H}^{r+d}_\alpha(M, N)$ is an Artinian $R$-module.

***Proof*** Let $\mathcal{G}(-) = \Gamma_\alpha(-)$ and $\mathcal{F}(-) = \mathrm{Hom}_R(M, -)$ be functors from category of $R$-modules to itself. Then $\mathcal{FG} = \mathrm{Hom}_R(M, \Gamma_\alpha(-)) = \Gamma_\alpha(M, -)$, since $M$ is finitely generated. Moreover, $\mathcal{F}$ is left exact. Note that, given an injective module $E$ we have

$$\mathbf{R}^i\mathcal{F}(\mathcal{G}(E)) = \mathbf{R}^i\mathrm{Hom}_R(M, \Gamma_\alpha(E)) = 0,$$

for all $i > 0$, since $\Gamma_\alpha(E)$ is injective. By Theorem A.9, there exists a Grothedieck spectral sequence

$$E^{p,q}_2 = \mathrm{Ext}^p_R(M, \mathrm{H}^q_\alpha(N)) \Longrightarrow_p \mathrm{H}^{p+q}_\alpha(M, N).$$

Now, consider the homomorphisms of spectral

$$E^{r-k,d+k-1}_k \longrightarrow E^{r,d}_k \longrightarrow E^{r+k,d+1-k}_k.$$

Note que $\mathrm{H}^q_\alpha(N) = 0$ for all $q > d$, By [2, Lemma 1.38]. Then $E^{p,q}_2 = 0$, for any $p > r$ or $q > d$. Moreover, $E^{r-k,d+k-1}_k = E^{r+k,d+1-k}_k = 0$, for each $k \geq 2$. So

$$E^{r,d}_2 = E^{r,d}_3 = \cdots = E^{r,d}_\infty.$$

We affirm that $E_\infty^{r,d} \cong \mathrm{H}_\alpha^{r+d}(M,N)$. Indeed, there is a filtration $\Phi$ of $H^{r+d} = \mathrm{H}_\alpha^{r+d}(M,N)$ such that

$$0 = \Phi^{r+d+1}H^{r+d} \subseteq \Phi^{r+d}H^{r+d} \subseteq \cdots \subseteq \Phi^1 H^{r+d} \subseteq \Phi^0 H^{r+d} = \mathrm{H}_\alpha^{r+d}(M,N) \text{ and holds}$$

$$E_\infty^{i,r+d-i} = \Phi^i H^{r+d}/\Phi^{i+1}H^{r+d} \quad \text{for } 0 \leq i \leq r+d. \tag{3.4}$$

Note that $E_2^{i,r+d-i} = \mathrm{Ext}_R^i(M, \mathrm{H}_\alpha^{r+d-i}(N)) = 0$, for any $i \neq r$. So

$$E_2^{i,r+d-i} = E_3^{i,r+d-i} = \cdots = E_\infty^{i,r+d-i} = 0,$$

for all $i \neq r$. Applying $i = r+1, \ldots, r+d$ and $i = 0, \ldots, r-1$ in (3.4) we get

$$\Phi^{r+1}H^{r+d} = \Phi^{r+2}H^{r+d} = \cdots = \Phi^{r+d+1}H^{r+d} = 0$$

$$\Phi^r H^{r+d} = \Phi^{r-1}H^{r+d} = \cdots = \Phi^0 H^{r+d} = \mathrm{H}_\alpha^{r+d}(M,N).$$

So

$$E_\infty^{r,d} \cong \Phi^r H^{r+d}/\Phi^{r+1}H^{r+d} \cong \mathrm{H}_\alpha^{r+d}(M,N).$$

Therefore $\mathrm{Ext}_R^r(M, \mathrm{H}_\alpha^d(N)) \cong \mathrm{H}_\alpha^{r+d}(M,N)$. Finally, by Lemma 3.2.4 the module $\mathrm{H}_\alpha^d(N)$ is Artinian. Since $M$ is finitely generated, it follows that $\mathrm{H}_\alpha^{r+d}(M,N)$ is also Artinian. ■

**Lemma 3.2.6** Let $M$ be an $\alpha$-minimax $R$-module such that $\mathrm{Ass}_R(M) \subseteq \alpha$. Then $\mathrm{H}_\alpha^i(M)$ is $\alpha$-minimax for all $i \geq 0$.

***Proof*** If $i = 0$, then $\mathrm{H}_\alpha^0(M) = \Gamma_\alpha(M)$ is a submodule of $M$, and hence $\Gamma_\alpha(M)$ is $\alpha$-minimax. Since $\mathrm{Ass}_R(M) \subseteq \alpha$ we have $M = \Gamma_\alpha(M)$. Consequently, we get $\mathrm{H}_\alpha^i(M) = 0$ for all $i > 0$, and so $\mathrm{H}_\alpha^i(M)$ is $\alpha$-minimax for each $i \geq 0$. ■

**Theorem 3.2.7** Let $M$ be a finitely generated $R$-module with finite projective dimension $p$ and $N$ an $\alpha$-minimax $R$-module with $\mathrm{Ass}_R(N) \subseteq \alpha$. Then $\mathrm{H}_\alpha^i(M,N)$ is $\alpha$-minimax for all $i \geq 0$.

***Proof*** We proceed by induction on $p$. If $p = 0$, then $M$ is a finitely generated free module $R^r$ and so

$$\mathrm{H}_\alpha^i(M,N) \cong \mathrm{H}_\alpha^i(R^r,N) \cong \bigoplus_r \mathrm{H}_\alpha^i(N).$$

Since each $\mathrm{H}^i_\alpha(N)$ is $\alpha$-minimax by Lemma 3.2.6, it follows that $\mathrm{H}^i_\alpha(M,N)$ is also $\alpha$-minimax, for all $i \geq 0$.

Now, suppose that $p > 0$. Assume that the result is true for $p-1$. There exists an exact sequence

$$0 \longrightarrow L \longrightarrow R^k \longrightarrow M \longrightarrow 0, \tag{3.5}$$

where $L$ is finitely generated with projective dimension $p-1$. From the exact sequence (3.5), we get the following long exact sequence

$$\cdots \longrightarrow \mathrm{H}^{i-1}_\alpha(L,N) \longrightarrow \mathrm{H}^i_\alpha(M,N) \longrightarrow \mathrm{H}^i_\alpha(N)^k \longrightarrow \cdots. \tag{3.6}$$

By induction hypothesis, $\mathrm{H}^{i-1}_\alpha(L,N)$ is $\alpha$-minimax for all $i \geq 1$. Moreover, by Lemma 3.2.6, $\mathrm{H}^i_\alpha(N)^k$ is $\alpha$-minimax for each $i \geq 1$ (and also $i = 0$). Therefore, we conclude by (3.6) that $\mathrm{H}^i_\alpha(M,N)$ is $\alpha$-minimax, for all $i \geq 1$. For the case $i = 0$, we have

$$\mathrm{H}^0_\alpha(M,N) = \Gamma_\alpha(M,N) = \mathrm{Hom}_R(M, \Gamma_\alpha(N)).$$

Since $\Gamma_\alpha(N)$ is $\alpha$-minimax and $M$ is finitely generated, it follows that $\mathrm{H}^0_\alpha(M,N)$ is $\alpha$-minimax, as required. ■

**Definition 3.2.8** Let $(R,\mathfrak{m})$ be local ring, $I$ an ideal of $R$ and $\beta$ a non-empty collection of ideals of $R$. We define the *$\beta$-relative cohomological dimension of $I$* by

$$cd_\beta(I) = inf\left\{n \in \mathbb{N}_0 \ ; \ \mathrm{H}^i_{I,\beta}(M) = 0 \ for\ all\ i > n\ and\ all\ R-module\ M\right\}$$

Note that the set $\left\{n \in \mathbb{N}_0 \ ; \ \mathrm{H}^i_{I,\beta}(M) = 0 \ for\ all\ i > n\ and\ all\ R-module\ M\right\}$ is non-empty, by Theorem 3.1.12. Now, we have conditions to prove the next results that are related to $(I,\beta)$-cominimaximality of $\mathrm{H}^i_{I,\beta}(M,N)$.

**Theorem 3.2.9** Let $(R,\mathfrak{m})$ a local ring and $I$ an ideal of $R$ such that $cd_\beta(I) = 1$. Let $N$ be an $R$-module $(I,\beta)$-minimax. Then, for all finitely generated $R$-module $M$ and all $j \in \mathbb{N}_0$, $\mathrm{H}^j_{I,\beta}(M,N)$ is $(I,\beta)$-cominimax.

***Proof*** Let $\mathcal{F} = \Gamma_{I,\beta}(-)$ and $\mathcal{G} = \mathrm{Hom}_R(M,-)$ be functors from category of $R$-modules to itself. Then $\mathcal{F}\mathcal{G} = \Gamma_{I,\beta}(M,-) = \mathrm{Hom}_R(M, \Gamma_{I,\beta}(-))$, since $M$ is finitely generated. Futhermore, $\mathcal{F}$ is left exact. See that, given an injective module $E$ we have

$$\mathbf{R}^i\mathcal{F}(\mathcal{G}(E)) = \mathbf{R}^i\mathrm{Hom}_R(M, \Gamma_{I,\beta}(E)) = 0$$

for any $i > 0$. So, by Theorem A.9 there exists a Grothendieck spectral sequence

$$E_2^{p,q} = \mathrm{H}_{I,\beta}^p(\mathrm{Ext}_R^q(M, N)) \overset{p}{\Rightarrow} \mathrm{H}_{I,\beta}^{p+q}(M, N).$$

Hence, for all $j \geq 0$, there is a finite filtration of the module $H^j = \mathrm{H}_{I,\beta}^j(M, N)$

$$0 = \Phi^{j+1}H^j \subseteq \Phi^j H^j \subseteq \cdots \subseteq \Phi^1 H^j \subseteq \Phi^0 H^j = H^j$$

such that

$$E_\infty^{p,j-p} \cong \Phi^p H^j / \Phi^{p+1} H^j$$

for all $0 \leq p \leq j$.

By hypothesis, $E_2^{p,q} = 0$ for all $p \geq 2$ and all $q \geq 0$. Moreover, since $E_\infty^{p,q}$ is a subquotient of $E_2^{p,q}$ for all $p, q \geq 0$, it implies that $E_\infty^{p,q} = 0$, for all $p \geq 2$ and all $q \geq 0$. It therefore follows

$$0 = \Phi^{j+1}H^j = \Phi^j H^j = \cdots = \Phi^2 H^j.$$

On the other hand, by homomorphisms of spectral

$$0 = E_2^{-1,j} \to E_2^{1,j-1} \to E_2^{3,j-2} = 0$$
$$0 = E_2^{-1,j+1} \to E_2^{0,j} \to E_2^{2,j-1} = 0$$

we obtain the equalities

$$E_2^{1,j-1} = E_3^{1,j-1} = \cdots = E_\infty^{1,j-1} \text{ and } E_2^{0,j} = E_3^{0,j} = \cdots = E_\infty^{0,j}.$$

The above equalities and the isomorphisms

$$E_2^{1,j-1} = E_\infty^{1,j-1} \cong \Phi^1 H^j \text{ and } E_2^{0,j} = E_\infty^{0,j} \cong \Phi^0 H^j / E_\infty^{1,j-1} = \Phi^0 H^j / E_2^{1,j-1}$$

give us a short exact sequence

$$0 \longrightarrow E_2^{1,j-1} \longrightarrow \mathrm{H}_{I,\beta}^j(M, N) \longrightarrow E_2^{0,j} \longrightarrow 0.$$

Since $E_2^{1,j-1}$ and $E_2^{0,j}$ are both $(I, \beta)$-cominimax, it follows from Proposition 2.1.7 that $\mathrm{H}_{I,\beta}^j(M, N)$ is also $(I, \beta)$-cominimax. ■

**Corollary 3.2.10** Let $(R, \mathfrak{m})$ be local ring, $I$ an ideal of $R$ with $cd_\beta(I) = 1$ and let $N$ be an $(I, \beta)$-minimax $R$-module. Then, for all $j \geq 0$, $\mathrm{H}_{I,\beta}^j(N)$ is $(I, \beta)$-cominimax.

**Theorem 3.2.11** Let $I$ be an ideal of local ring $(R, \mathfrak{m})$ such that $cd_{\beta}(I) = 1$ and let $N$ be an $(I, \beta)$-minimax $R$-module. Then, for all finitely generated $R$-module $M$ and all $i, j \geq 0$, $\mathrm{Ext}^i_R(M, \mathrm{H}^j_{I,\beta}(N))$ is $(I, \beta)$-cominimax.

***Proof*** By Theorem A.9, we consider the Grothendieck spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_R(M, \mathrm{H}^q_{I,\beta}(N)) \overset{p}{\Rightarrow} \mathrm{H}^{p+q}_{I,\beta}(M, N). \tag{3.7}$$

By hypothesis $\mathrm{H}^j_{I,\beta}(N) = 0$, for all $j > 1$ and $E_2^{i,0}$ is $(I, \beta)$-minimax for all $i$; thus it suffices to show that $E_2^{i,1}$ is $(I, \beta)$-cominimax for all $i$. For all $p \geq 2$, we consider the homomorphisms of spectral

$$0 = E_{p+1}^{i-p-1,1+p} \longrightarrow E_{p+1}^{i,1} \longrightarrow E_{p+1}^{i+p+1,1-p} = 0.$$

Since $E_{p+1}^{i,1} = kerd_p^{i,1}/Imd_p^{i-p,p}$, we obtain

$$kerd_2^{i,1} \cong E_3^{i,1} \cong \cdots \cong E_\infty^{i,1}$$

for all $i \geq 0$. By using (3.7), there is a finite filtration of the module $H^{i+1} = \mathrm{H}^{i+1}_{I,\beta}(M, N)$

$$0 = \Phi^{i+2}H^{i+1} \subseteq \Phi^{i+1}H^{i+1} \subseteq \cdots \subseteq \Phi^1 H^{i+1} \subseteq \Phi^0 H^{i+1} = H^{i+1}$$

such that

$$E_\infty^{p,i+1-p} \cong \Phi^p H^{i+1}/\Phi^{P+1}H^{i+1}$$

for all $0 \leq p \leq i+1$. It therefore follows that

$$\Phi^i H^{i+1} = \Phi^{i-1}H^{i+1} = \cdots = \Phi^1 H^{i+1} = \Phi^0 H^{i+1} = \mathrm{H}^{i+1}_{I,\beta}(M, N).$$

Now, the exact sequence

$$0 \longrightarrow E_\infty^{i+1,0} \longrightarrow \mathrm{H}^{i+1}_{I,\beta}(M, N) \longrightarrow E_\infty^{i,1} \longrightarrow 0$$

in conjunction with $E_\infty^{i+1,0}$ is $(I, \beta)$-minimax and $\mathrm{H}^{i+1}_{I,\beta}(M, N)$ is $(I, \beta)$-cominimax by Theorem 3.2.9, yielding that $E_\infty^{i,1}$ is $(I, \beta)$-cominimax and so is $kerd_2^{i,1}$. Furthermore, considering the exact sequence

$$0 \longrightarrow kerd_2^{i,1} \longrightarrow E_2^{i,1} \longrightarrow Imd_2^{i,1} \longrightarrow 0$$

and knowing that $Imd_2^{i,1} \subseteq E_2^{i+2,0}$ is $(I, \beta)$-cominimax, it follows that $E_2^{i,1}$ is $(I, \beta)$-cominimax, as required. ■

# Chapter 4

# Ideal transforms with respect to a good family of ideals

Let $I$ be an ideal of $R$ and $M$ be an $R$-module. In [6], the authors defined the *ideal transform* $\mathcal{D}_I(M)$ of $M$ with respect to $I$ by

$$\mathcal{D}_I(M) = \varinjlim_{n} \mathrm{Hom}_R(I^n, M).$$

Ideal transform turns out to be a powerful tool in various fields of commutative algebra and it is an important algebraic tool in studying local cohomology modules with respect to an ideal. One extensions of $\mathcal{D}_I(M)$ is the *generalized ideal transform* $\mathcal{D}_I(M, N)$ of two $R$-modules $M$,$N$ with respect to $I$ which was defined and studied in [12] and [22].

In this chapter we introduce the notion of *ideal transform* $\mathcal{D}_\alpha(M)$ *of an $R$-module $M$ with respect to a good family $\alpha$ of ideals of $R$* (*generelized ideal transform* $\mathcal{D}_\alpha(M, N)$ *of two modules $M$ and $N$ with respect to a good family $\alpha$ of ideals of $R$*) and we explore their properties and its relation with local cohomology modules $\mathrm{H}^i_\alpha(M)$ (generalized local cohomology $\mathrm{H}^i_\alpha(M, N)$, respectively).

## 4.1 Basic properties of ideals transforms

In this chapter $\alpha$ will be denote a good family of ideals of $R$.

**Definition 4.1.1** The *$\alpha$-transform functor with respect to $\alpha$* is defined by

$$\mathcal{D}_\alpha(-) := \varinjlim_{I\in\alpha} \mathrm{Hom}_R(I, -).$$

Notice that $\mathcal{D}_\alpha(-)$ is an $R$-linear left exact functor from the category of $R$-modules to itself. Given an $R$-module $M$,

$$\mathcal{D}_\alpha(M) = \varinjlim_{I\in\alpha} \operatorname{Hom}_R(I, M).$$

is called *ideal transform of $M$ with respect to $\alpha$ (or $\alpha$-transform of $M$).*

For any non-negative integer $i$, the $i$-th right derived functor of $\mathcal{D}_\alpha$ is denoted by $\mathbf{R}^i\mathcal{D}_\alpha$.

**Lemma 4.1.2** For each $i \geq 0$, there is a natural isomorphism

$$\mathbf{R}^i\mathcal{D}_\alpha(-) \cong \varinjlim_{I\in\alpha} \operatorname{Ext}^i_R(I, -).$$

***Proof*** By definition 4.1.1 we have $\mathcal{D}_\alpha(-) = \varinjlim_{I\in\alpha} \operatorname{Hom}_R(I, -)$. On the other hand, the short exact sequence $0 \to M \to N \to P \to 0$ induces the long exact sequence

$$0 \to \operatorname{Hom}_R(I, M) \to \operatorname{Hom}_R(I, N) \to \operatorname{Hom}_R(I, P) \to \cdots$$

$$\cdots \to \operatorname{Ext}^i_R(I, M) \to \operatorname{Ext}^i_R(I, N) \to \operatorname{Ext}^i_R(I, P) \to \cdots$$

for all $I \in \alpha$. Since direct limits are exact functors, we have a long exact sequence

$$0 \to \varinjlim_{I\in\alpha} \operatorname{Hom}_R(I, M) \to \varinjlim_{I\in\alpha} \operatorname{Hom}_R(I, N) \to \varinjlim_{I\in\alpha} \operatorname{Hom}_R(I, P) \to \cdots$$

$$\cdots \to \varinjlim_{I\in\alpha} \operatorname{Ext}^i_R(I, M) \to \varinjlim_{I\in\alpha} \operatorname{Ext}^i_R(I, N) \to \varinjlim_{I\in\alpha} \operatorname{Ext}^i_R(I, P) \to \cdots.$$

Finally, for all $i > 0$ and any injective $R$-module $E$, $\varinjlim_{I\in\alpha} \operatorname{Ext}^i_R(I, E) = 0$ , because $\operatorname{Ext}^i_R(I, E) = 0$. ■

**Lemma 4.1.3** There is a functorial exact sequence

$$0 \to \Gamma_\alpha(-) \to \operatorname{Id}(-) \to \mathcal{D}_\alpha(-) \to \operatorname{H}^1_\alpha(-) \to 0,$$

where Id is the identity functor. Moreover, for each $i \geq 1$, there exists a natural isomorphism

$$\mathbf{R}^i\mathcal{D}_\alpha(-) \cong \operatorname{H}^{i+1}_\alpha(-).$$

***Proof*** Let $I$ and $J$ two ideals in $\alpha$ such that $I \geq J$ (i.e., $I \subseteq J$). Let $j^I_J : I \to J$ be the inclusion map and $h^I_J : R/I \to R/J$ be the natural epimorphism. Consider a homomorphism $f : M \to N$ of $R$-modules $M$ and $N$.

The diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & R & \longrightarrow & R/I & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle j^I_J} & & \|{\scriptstyle 1_R} & & \downarrow{\scriptstyle h^I_J} & & \\ 0 & \longrightarrow & J & \longrightarrow & R & \longrightarrow & R/J & \longrightarrow & 0 \end{array}$$

is commutative (in which the rows are the canonical exact sequences). Such diagram induces a chain map of the long exact sequence of $\mathrm{Ext}^\bullet_R(-, M)$ modules. Since $R$ is a projective $R$-module, we obtain the following commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{Hom}_R(R/J, M) & \longrightarrow & M & \longrightarrow & \mathrm{Hom}_R(J, M) & \longrightarrow & \mathrm{Ext}^1_R(R/J, M) & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Hom}_R(R/I, M) & \longrightarrow & M & \longrightarrow & \mathrm{Hom}_R(I, M) & \longrightarrow & \mathrm{Ext}^1_R(R/I, M) & \longrightarrow & 0 \end{array}$$

and for each $i \geq 1$, the commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^i_R(J, M) & \xrightarrow[\cong]{\beta^i_{J,M}} & \mathrm{Ext}^{i+1}_R(R/J, M) \\ {\scriptstyle \mathrm{Ext}^i_R(j^I_J, M)}\downarrow & & \downarrow{\scriptstyle \mathrm{Ext}^{i+1}_R(h^I_J, M)} \\ \mathrm{Ext}^i_R(I, M) & \xrightarrow[\cong]{\beta^i_{I,M}} & \mathrm{Ext}^{i+1}_R(R/I, M). \end{array}$$

Now passing the direct limits, we obtain the exact sequence

$$0 \longrightarrow \Gamma_\alpha(M) \xrightarrow{\xi_M} M \xrightarrow{\eta_M} \mathcal{D}_\alpha(M) \xrightarrow{\zeta^0_M} \mathrm{H}^1_\alpha(M) \longrightarrow 0$$

and, for each $i \geq 1$ the isomorphism

$$\beta^i_M : \varinjlim_{I \in \alpha} \mathrm{Ext}^i_R(I, M) \xrightarrow{\cong} \varinjlim_{I \in \alpha} \mathrm{Ext}^{i+1}_R(R/I, M).$$

Moreover, since the following diagrams

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{Hom}_R(R/I, M) & \longrightarrow & M & \longrightarrow & \mathrm{Hom}_R(I, M) & \longrightarrow & \mathrm{Ext}^1_R(R/I, M) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Hom}_R(R/I, N) & \longrightarrow & N & \longrightarrow & \mathrm{Hom}_R(I, N) & \longrightarrow & \mathrm{Ext}^1_R(R/I, N) & \longrightarrow & 0 \end{array}$$

$$\begin{array}{ccc} \mathrm{Ext}^i_R(I, M) & \xrightarrow[\cong]{\beta^i_{I,M}} & \mathrm{Ext}^{i+1}_R(R/I, M) \\ \downarrow & & \downarrow \\ \mathrm{Ext}^i_R(I, N) & \xrightarrow[\cong]{\beta^i_{I,N}} & \mathrm{Ext}^{i+1}_R(R/I, N) \end{array}$$

are commutative, it follows that $\xi$, $\eta$, $\zeta^0$ and $\beta^i$ constitute natural transformations. ■

An interesting question related to the ideal transform is about its exactness. Before to answer a similar question for the $\alpha$-transform functor we prove a few results.

**Lemma 4.1.4** Let $M$ be an $R$-module. Then the following statements hold:
*(i)* If $M$ is an $\alpha$-torsion module, then $\mathbf{R}^i\mathcal{D}_\alpha(M) = 0$ for all $i \geq 0$;
*(ii)* $\mathbf{R}^i\mathcal{D}_\alpha(M) \cong \mathbf{R}^i\mathcal{D}_\alpha(M/\Gamma_\alpha(M))$ for all $i \geq 0$;
*(iii)* $\mathbf{R}^i\mathcal{D}_\alpha(M) \cong \mathbf{R}^i\mathcal{D}_\alpha(\mathcal{D}_\alpha(M))$ for all $i \geq 0$;
*(iv)* $\Gamma_\alpha(\mathcal{D}_\alpha(M)) = 0 = \mathrm{H}^1_\alpha(\mathcal{D}_\alpha(M))$;
*(v)* $\mathrm{H}^i_\alpha(M) \cong \mathrm{H}^i_\alpha(\mathcal{D}_\alpha(M))$ for all $i > 1$;
*(vi)* Let $f : M \to N$ be an homomorphism of $R$-modules such that $Kerf$ and $Cokerf$ are both $\alpha$-torsion. Then $\mathcal{D}_\alpha(M) \cong \mathcal{D}_\alpha(N)$;
*(vii)* $\mathcal{D}_\alpha(\mathcal{D}_I(M)) \cong \mathcal{D}_\alpha(M)$ for all $I \in \alpha$.

***Proof*** (i) By Lemma 4.1.3 there is an exact sequence

$$0 \longrightarrow \Gamma_\alpha(M) \longrightarrow M \longrightarrow \mathcal{D}_\alpha(M) \longrightarrow \mathrm{H}^1_\alpha(M) \longrightarrow 0.$$

Since $M$ is an $\alpha$-torsion $R$-module, it follows that $\mathrm{H}^i_\alpha(M) = 0$ for all $i > 0$. Thus $\mathcal{D}_\alpha(M) \cong M/\Gamma_\alpha(M) = 0$ and $\mathbf{R}^i\mathcal{D}_\alpha(M) \cong \mathrm{H}^{i+1}_\alpha(M) = 0$ for all $i > 0$.

(ii) The short exact sequence

$$0 \longrightarrow \Gamma_\alpha(M) \longrightarrow M \longrightarrow M/\Gamma_\alpha(M) \longrightarrow 0$$

yields a long exact sequence

$$0 \longrightarrow \mathcal{D}_\alpha(\Gamma_\alpha(M)) \longrightarrow \mathcal{D}_\alpha(M) \longrightarrow \mathcal{D}_\alpha(M/\Gamma_\alpha(M)) \longrightarrow \cdots$$

$$\longrightarrow \mathbf{R}^i\mathcal{D}_\alpha(\Gamma_\alpha(M)) \longrightarrow \mathbf{R}^i\mathcal{D}_\alpha(M) \longrightarrow \mathbf{R}^i\mathcal{D}_\alpha(M/\Gamma_\alpha(M)) \longrightarrow \cdots.$$

Since $\mathbf{R}^i\mathcal{D}_\alpha(\Gamma_\alpha(M)) = 0$ by (i), we have

$$\mathbf{R}^i\mathcal{D}_\alpha(M) \cong \mathbf{R}^i\mathcal{D}_\alpha(M/\Gamma_\alpha(M))$$

for all $i \geq 0$.

(iii) The exact sequence

$$0 \longrightarrow \Gamma_\alpha(M) \longrightarrow M \longrightarrow \mathcal{D}_\alpha(M) \longrightarrow \mathrm{H}^1_\alpha(M) \longrightarrow 0$$

gives rise to an exact sequence

$$0 \longrightarrow M/\Gamma_\alpha(M) \longrightarrow \mathcal{D}_\alpha(M) \longrightarrow \mathrm{H}^1_\alpha(M) \longrightarrow 0.$$

By applying the functor $\mathcal{D}_\alpha$ to the above exact sequence, we get a long exact sequence

$$\cdots \longrightarrow \mathbf{R}^i\mathcal{D}_\alpha(M/\Gamma_\alpha(M)) \longrightarrow \mathbf{R}^i\mathcal{D}_\alpha(\mathcal{D}_\alpha(M) \longrightarrow \mathbf{R}^i\mathcal{D}_\alpha(\mathrm{H}^1_\alpha(M)) \longrightarrow \cdots.$$

Since $\mathrm{H}^1_\alpha(M)$ is $\alpha$-torsion, it follows from (i) that $\mathbf{R}^i\mathcal{D}_\alpha(\mathrm{H}^1_\alpha(M)) = 0$ for all $i \geq 0$. This implies that $\mathbf{R}^i\mathcal{D}_\alpha(M/\Gamma_\alpha(M)) \cong \mathbf{R}^i\mathcal{D}_\alpha(\mathcal{D}_\alpha(M))$ and then $\mathbf{R}^i\mathcal{D}_\alpha(M) \cong \mathbf{R}^i\mathcal{D}_\alpha(\mathcal{D}_\alpha(M))$ for all $i \geq 0$ by (ii).

(iv) and (v) The short exact sequence

$$0 \longrightarrow M/\Gamma_\alpha(M) \longrightarrow \mathcal{D}_\alpha(M) \longrightarrow \mathrm{H}^1_\alpha(M) \longrightarrow 0$$

induces a long exact sequence

$$\cdots \longrightarrow \mathrm{H}^i_\alpha(M/\Gamma_\alpha(M)) \longrightarrow \mathrm{H}^i_\alpha(\mathcal{D}_\alpha(M)) \longrightarrow \mathrm{H}^i_\alpha(\mathrm{H}^1_\alpha(M)) \longrightarrow \cdots .$$

Note that

$$\Gamma_\alpha(\mathrm{H}^1_\alpha(M)) = \mathrm{H}^1_\alpha(M),\ \Gamma_\alpha(M/\Gamma_\alpha(M)) = 0 = \mathrm{H}^i_\alpha(\mathrm{H}^1_\alpha(M)) \text{ for all } i > 1$$

and $\mathrm{H}^1_\alpha(M/\Gamma_\alpha(M)) \cong \mathrm{H}^1_\alpha(M)$. We thus $\Gamma_\alpha(\mathcal{D}_\alpha(M)) = 0 = \mathrm{H}^1_\alpha(\mathcal{D}_\alpha(M))$. Moreover, we have $\mathrm{H}_\alpha(M) \cong \mathrm{H}^i_\alpha(\mathcal{D}_\alpha(M))$ for all $i > 1$.

($vi$) Applying the functor $\mathcal{D}_\alpha$ to the following exact sequences

$$0 \longrightarrow Kerf \longrightarrow M \longrightarrow Imf \longrightarrow 0 \text{ and } 0 \longrightarrow Imf \longrightarrow M' \longrightarrow Cokerf \longrightarrow 0$$

together with (i) yields

$$\mathcal{D}_\alpha(M) \cong \mathcal{D}_\alpha(Imf) \cong \mathcal{D}_\alpha(M').$$

($vii$) Let $I \in \alpha$. Note that, if $K$ is an $I$-torsion $R$-module, then $K$ is $\alpha$-torsion. Applying (vi) in the exact sequence

$$0 \longrightarrow \Gamma_I(M) \longrightarrow M \longrightarrow \mathcal{D}_I(M) \longrightarrow \mathrm{H}^1_I(M) \longrightarrow 0$$

we obtain the isomorphism $\mathcal{D}_\alpha(M) \cong \mathcal{D}_\alpha(\mathcal{D}_I(M))$. The proof is complete. ■

**Lemma 4.1.5** Let $M$ be an $R$-module such that $\mathrm{Hom}_R(R/I, M) = 0$ for all $I \in \alpha$. Then $\mathrm{Ass}_R(M) \cap \alpha = \varnothing$. In particular $M$ is $\alpha$-torsion-free.

***Proof*** By the hypothesis,

$$V(I) \cap \mathrm{Ass}_R(M) = \mathrm{Ass}_R(\mathrm{Hom}_R(R/I, M)) = \varnothing$$

for all $I \in \alpha$. Note that

$$\bigcup_{I \in \alpha} V(I) = \alpha \cap Spec(R)$$

and then

$$\varnothing = (\bigcup_{I\in\alpha} V(I)) \cap \mathrm{Ass}_R(M) = \alpha \cap Spec(R) \cap \mathrm{Ass}_R(M) = \alpha \cap \mathrm{Ass}_R(M) = \mathrm{Ass}_R(\Gamma_\alpha(M)),$$

as required. ■

**Lemma 4.1.6** Let $M$ be an $R$-module. Then the following statements are equivalent:
$(i)$ $\mathrm{H}^i_\alpha(M) = 0$ for all $i \geq 0$;
$(ii)$ $\mathrm{Ext}^i_R(R/I, M) = 0$ for any $i \geq 0$ and all $I \in \alpha$.

***Proof*** Let $\mathcal{F} = \mathrm{Hom}_R(R/I, -)$ and $\mathcal{G} = \Gamma_\alpha(-)$ be functors from the category of $R$-modules to itself. We see that $\mathcal{FG}(M) = \mathrm{Hom}_R(R/I, M)$ for any $R$-module $M$. If $E$ is an injective $R$-module, then $\mathcal{G}(E) = \Gamma_\alpha(E)$ is also injective. Hence $\mathcal{G}(E)$ is right $\mathcal{F}$-acyclic. By Theorem A.9, there is a Grothendieck spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_R(R/I, \mathrm{H}^q_\alpha(M)) \overset{p}{\Rightarrow} \mathrm{Ext}^{p+q}_R(R/I, M).$$

For $n \geq 0$, we have a filtration of submodules of $H^n = \mathrm{Ext}^n_R(R/I, M)$

$$0 = \Phi^{n+1}H^n \subseteq \Phi^n H^n \subseteq \cdots \subseteq \Phi^0 H^n = H^n$$

such that

$$E_\infty^{i,n-i} \cong \Phi^i H^n / \Phi^{i+1} H^n$$

for all $i \leq n$.

(i) $\Rightarrow$ (ii) If $\mathrm{H}^q_\alpha(M) = 0$ for all $q \geq 0$, then $E_\infty^{p,q} = 0$ for all $p, q \geq 0$. This implies that

$$0 = \Phi^{n+1}H^n = \Phi^n H^n = \cdots = \Phi^1 H^n = \Phi^0 H^n = \mathrm{Ext}^n_R(R/I, M)$$

for all $n \geq 0$.

(ii)$\Rightarrow$(i) We prove $\mathrm{H}^n_\alpha(M) = 0$ by induction on $n$. Let $n = 0$, since $\mathrm{Hom}_R(R/I, M) = 0$ for all $I \in \alpha$, it follows from Lemma 4.1.5 that $\varnothing = \mathrm{Ass}_R(\Gamma_\alpha(M))$ and then $\Gamma_\alpha(M) = 0$. Assume that $\mathrm{H}^i_\alpha(M) = 0$ for all $i < n$. The homomorphisms of spectral sequence

$$0 \longrightarrow E_2^{0,n} \xrightarrow{d^{0,n}} E_2^{2,n-1} = 0$$

induces that

$$E_2^{0,n} = E_3^{0,n} = \cdots = E_\infty^{0,n}.$$

Since $E_\infty^{0,n} \cong \Phi^0 H^n/\Phi^1 H^n$ and $H^n = \mathrm{Ext}_R^n(R/I, M) = 0$, it follows that $0 = E_\infty^{0,n} = E_2^{0,n}$. So, we have $\mathrm{Hom}_R(R/I, \mathrm{H}_\alpha^n(M)) = 0$ for all $I \in \alpha$. In view of Lemma 4.1.5, we get $\mathrm{H}_\alpha^n(M) = 0$. ■

**Lemma 4.1.7** Let $M$ be a finitely generated $R$-module with finite projective dimension $d$. If $\mathrm{H}_\alpha^n(R) = 0$ for all $n \geq 2$, then $\mathrm{H}_\alpha^n(M) = 0$ for all $n \geq 2$.

***Proof*** Since $M$ is finitely generated, we have a free resolution

$$0 \longrightarrow F_d \xrightarrow{\delta_d} \cdots \xrightarrow{\delta_3} F_2 \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\delta_0} M \longrightarrow 0$$

where $F_i$ have finite rank, for each $0 \leq i \leq d$. Now, we prove by induction on $d$. If $d = 0$, then $M \cong F_0$ and therefore $\mathrm{H}_\alpha^n(M) = 0$ for all $n \geq 2$. When $d = 1$, the exact sequence

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

induces a long exact sequence

$$\cdots \longrightarrow \mathrm{H}_\alpha^n(F_0) \longrightarrow \mathrm{H}_\alpha^n(M) \longrightarrow \mathrm{H}_\alpha^{n+1}(F_1) \longrightarrow \cdots .$$

Since $\mathrm{H}_\alpha^n(F_0) = 0 = \mathrm{H}_\alpha^{n+1}(F_1)$, it follows that $\mathrm{H}_\alpha^n(M) = 0$.

Assume that $d > 1$ and the result is true for $d-1$. In the free resolution

$$0 \longrightarrow F_d \longrightarrow \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

we obtain the exacts sequences

$$0 \longrightarrow F_d \longrightarrow F_{d-1} \longrightarrow M_{d-1} \longrightarrow 0 \tag{4.1}$$

$$0 \longrightarrow M_i \longrightarrow F_{i-1} \longrightarrow M_{i-1} \longrightarrow 0\,, \quad 0 < i \leq d-1 \tag{4.2}$$

$$0 \longrightarrow M_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0, \tag{4.3}$$

where $M_i = Ker\delta_{i-1}$, for $1 < i \leq d-1$, and $M_d = F_d$. The exact sequence (4.1) induces a long exact sequence

$$\cdots \longrightarrow \mathrm{H}_\alpha^n(F_{d-1}) \longrightarrow \mathrm{H}_\alpha^n(M_{d-1}) \longrightarrow \mathrm{H}_\alpha^{n+1}(F_d) \longrightarrow \cdots .$$

Since $\mathrm{H}^n_\alpha(F_{d-1}) = 0 = \mathrm{H}^{n+1}_\alpha(F_d)$, it follows that $\mathrm{H}^n_\alpha(M_{d-1}) = 0$ for all $n \geq 2$. On the other hand, assuming $i = d-1$ in the exact sequence (4.2), we obtain the long exact sequence

$$\cdots \longrightarrow \mathrm{H}^n_\alpha(F_{d-2}) \longrightarrow \mathrm{H}^n_\alpha(M_{d-2}) \longrightarrow \mathrm{H}^{n+1}_\alpha(M_{d-1}) \longrightarrow \cdots$$

and consequently $\mathrm{H}^n_\alpha(M_{d-1}) = 0$. Proceeding recursively, we conclude that $\mathrm{H}^n_\alpha(M_i) = 0$ for each $1 \leq i \leq d$ and all $n \geq 2$.

Finally, by exact sequence (4.3), we have the induced long exact sequence

$$\cdots \longrightarrow \mathrm{H}^n_\alpha(F_0) \longrightarrow \mathrm{H}^n_\alpha(M) \longrightarrow \mathrm{H}^{n+1}_\alpha(M_1) \longrightarrow \cdots$$

and therefore $\mathrm{H}^n_\alpha(M) = 0$ for all $n \geq 2$, as required. ■

**Theorem 4.1.8** Let $M$ be a finitely generated $R$-module with finite projective dimension $p$. Then the following statements are equivalent:
*(i)* $\mathcal{D}_\alpha$ is an exact functor;
*(ii)* $\mathrm{H}^n_\alpha(R) = 0$ for all $n \geq 2$;
*(iii)* $\mathrm{H}^n_\alpha(M) = 0$ for all $n \geq 2$;
*(iv)* $\mathrm{H}^n_\alpha(\mathcal{D}_\alpha(M)) = 0$ for all $n \geq 0$;
*(v)* $\mathrm{Ext}^n_R(R/I, \mathcal{D}_\alpha(M)) = 0$ for any $n \geq 0$ and all $I \in \alpha$;
*(vi)* $\mathrm{Tor}^R_n(R/I, \mathcal{D}_\alpha(M)) = 0$ for any $n \geq 0$ and all $I \in \alpha$.

***Proof*** (i)⇒(ii) It follows from Lemma 4.1.3.

(ii)⇒(iii) Since $p < \infty$ and $\mathrm{H}^n_\alpha(R) = 0$ for all $n \geq 2$, is follows from Lemma 4.1.7 that $\mathrm{H}^n_\alpha(M) = 0$ for any $n \geq 2$.

(iii)⇔(iv) this is immediate from Lemma 4.1.4(iv and v).

(iv)⇔(v) It follows from Lemma 4.1.6.

(v)⇔(vi) See [4, Lemma 3.1].

(iii)⇒(i) By Lemma 4.1.3 and Lemma 4.1.4(v) we have

$$\mathbf{R}^i\mathcal{D}_\alpha(M) \cong \mathrm{H}^{i+1}_\alpha(M) \cong \mathrm{H}^{i+1}_\alpha(\mathcal{D}_\alpha(M)) \cong \varinjlim_{I\in\alpha} \mathrm{Ext}^{i+1}_R(R/I, \mathcal{D}_\alpha(M)) = 0$$

for all $i > 0$. The proof is complete. ■

**Corollary 4.1.9** Let $M$ be a finitely generated $R$-module with finite projective dimension. If $\mathcal{D}_\alpha$ is an exact functor, then $I\mathcal{D}_\alpha(M) = \mathcal{D}_\alpha(M)$ for all $I \in \alpha$.

***Proof*** By Theorem 4.1.8(vi) we have $\mathrm{Tor}^R_0(R/I, M) = 0$ for all $I \in \alpha$. Therefore $I\mathcal{D}_\alpha(M) = \mathcal{D}_\alpha(M)$ for any $I \in \alpha$. ■

**Proposition 4.1.10** Let $M$ be an $J$-torsion $R$-module for some $J \in \beta$. Then $\mathcal{D}_{I,\beta}(M) \cong \mathcal{D}_I(M)$.

***Proof*** Since $M$ is an $J$-torsion module, we have $\mathrm{H}^i_{I,\beta}(M) \cong \mathrm{H}^i_I(M)$ for all $i \geq 0$. There is a commutative diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & \Gamma_I(M) & \longrightarrow & M & \longrightarrow & \mathcal{D}_I(M) & \longrightarrow & \mathrm{H}^1_I(M) & \longrightarrow 0 \\
& \downarrow \cong & & \downarrow 1_M & & \downarrow & & \downarrow \cong & \\
0 \longrightarrow & \Gamma_{I,\beta}(M) & \longrightarrow & M & \longrightarrow & \mathcal{D}_{I,\beta}(M) & \longrightarrow & \mathrm{H}^1_{I,\beta}(M) & \longrightarrow 0
\end{array}$$

where two rows are exact sequences by Lemma 4.1.3. This implies that $\mathcal{D}_I(M) \cong \mathcal{D}_{I,\beta}(M)$. ■

**Proposition 4.1.11** Let $M$ be a finitely generated $R$-module with finite projective dimension. Then $\mathrm{Hom}_R(R/I, \mathrm{H}^1_\alpha(M))$ is finitely generated.

***Proof*** The short exact sequence

$$0 \longrightarrow M/\Gamma_\alpha(M) \longrightarrow \mathcal{D}_\alpha(M) \longrightarrow \mathrm{H}^1_\alpha(M) \longrightarrow 0$$

induces an exact sequence

$$\mathrm{Hom}_R(R/I, \mathcal{D}_\alpha(M)) \longrightarrow \mathrm{Hom}_R(R/I, \mathrm{H}^1_\alpha(M)) \longrightarrow \mathrm{Ext}^1_R(R/I, M/\Gamma_\alpha(M)).$$

Since $\mathcal{D}_\alpha(M)$ is $I$-torsion-free, we see that $\mathrm{Hom}_R(R/I, \mathcal{D}_\alpha(M)) = 0$. The proof is complete by the assumption. ■

## 4.2 Generalized ideal transforms

In this section, we proceed with the study of the generalized ideal transform $\mathcal{D}_\alpha(M, N)$ with respect to $\alpha$ which is an extension of the ideal transform $\mathcal{D}_\alpha(N)$. Theorem 4.2.5 shows $\mathbf{R}^i\mathcal{D}_\alpha(M, N) = 0$ provided that $M$ is a finitely generated $R$-module and $N$ is $\alpha$-torsion.

**Definition 4.2.1** Let $M$ be $R$-module. The *generalized $\alpha$-transform functor* is defined by

$$\mathcal{D}_\alpha(M, -) := \varinjlim_{I \in \alpha} \mathrm{Hom}_R(IM, -).$$

We see that $\mathcal{D}_\alpha(M,-)$ is an $R$-linear and left exact functor. The $i$-th right derived functor of $\mathcal{D}_\alpha(M,-)$ will be denoted by $\mathbf{R}^i\mathcal{D}_\alpha(M,-)$. Given an $R$-module $N$, we call $\mathcal{D}_\alpha(M,N)$ the *generalized ideal transform of $M$ and $N$ with respect to $\alpha$ (or generalized $\alpha$-transform of $M$ and $N$).*

**Proposition 4.2.2** Let $M$ be a finitely generated $R$-module. Then there exists a natural isomorphism

$$\mathbf{R}^i\mathcal{D}_\alpha(M,-) \cong \varinjlim_{I\in\alpha} \mathrm{Ext}^i_R(IM,-)$$

for all $i \geq 0$.

***Proof*** We know that $\mathcal{D}_\alpha(M,-) = \varinjlim_{I\in\alpha} \mathrm{Hom}_R(IM,-)$ by Definition 4.2.1. On the other hand, the short exact sequence $0 \to X \to Y \to Z \to 0$ induces the long exact sequence

$$0 \to \mathrm{Hom}_R(IM,X) \to \mathrm{Hom}_R(IM,Y) \to \mathrm{Hom}_R(IM,Z) \to \cdots$$

$$\cdots \to \mathrm{Ext}^i_R(IM,X) \to \mathrm{Ext}^i_R(IM,Y) \to \mathrm{Ext}^i_R(IM,Z) \to \cdots$$

for all $I \in \alpha$. Since direct limits are exact functors, we have a long exact sequence

$$0 \to \varinjlim_{I\in\alpha} \mathrm{Hom}_R(IM,X) \to \varinjlim_{I\in\alpha} \mathrm{Hom}_R(IM,Y) \to \varinjlim_{I\in\alpha} \mathrm{Hom}_R(IM,Z) \to \cdots$$

$$\cdots \to \varinjlim_{I\in\alpha} \mathrm{Ext}^i_R(IM,X) \to \varinjlim_{I\in\alpha} \mathrm{Ext}^i_R(IM,Y) \to \varinjlim_{I\in\alpha} \mathrm{Ext}^i_R(IM,Z) \to \cdots .$$

Moreover, for all $i > 0$ and any injective $R$-module $E$, we get $\varinjlim_{I\in\alpha} \mathrm{Ext}^i_R(IM,E) = 0$, because $\mathrm{Ext}^i_R(IM,E) = 0$. The proof is complete. ∎

In chapter 3, we studied the generalized local cohomology modules with respect to family $\alpha$ of ideals $\mathrm{H}^i_\alpha(M,N)$. The following lemma gives relationship between $\mathrm{H}^i_\alpha(M,N)$ and $\mathbf{R}^i\mathcal{D}_\alpha(M,N)$.

**Lemma 4.2.3** Let $M$ be a finitely generated $R$-module with finite projective dimension $p$. Then there is a functorial long exact sequence

$$0 \longrightarrow \Gamma_\alpha(M,-) \longrightarrow \mathrm{Hom}_R(M,-) \longrightarrow \mathcal{D}_\alpha(M,-) \longrightarrow \mathrm{H}^1_\alpha(M,-) \longrightarrow \cdots .$$

$$\cdots \longrightarrow \mathrm{H}^i_\alpha(M,-) \longrightarrow \mathrm{Ext}^i_R(M,-) \longrightarrow \mathbf{R}^i\mathcal{D}_\alpha(M,-) \longrightarrow \mathrm{H}^{i+1}_\alpha(M,-) \longrightarrow \cdots$$

Moreover, for each $i > p$, there exists a natural isomorphism

$$\mathbf{R}^i\mathcal{D}_\alpha(M,-) \cong \mathrm{H}^{i+1}_\alpha(M,-).$$

***Proof*** Let $I, J \in \alpha$ be two ideals with $I \geq J$. Let $j_J^I : IM \to JM$ be the inclusion map and $h_J^I : M/IM \to M/JM$ be the natural epimorphism. Consider a homomorphism $f : N \to P$ of $R$-modules $N$ and $P$. We know that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & IM & \longrightarrow & M & \longrightarrow & M/IM & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle j_J^I} & & \|{\scriptstyle 1_M} & & \downarrow{\scriptstyle h_J^I} & & \\ 0 & \longrightarrow & JM & \longrightarrow & M & \longrightarrow & M/JM & \longrightarrow & 0 \end{array}$$

is commutative. This diagram induces a chain map of the long exact sequence of $\mathrm{Ext}_R^\bullet(-, N)$ modules and we obtain the followings commutative diagrams for $0 \leq i \leq p$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_R(M/JM, N) & \longrightarrow & \mathrm{Hom}_R(M, N) & \longrightarrow & \mathrm{Hom}_R(JM, N) & \longrightarrow & \mathrm{Ext}_R^1(M/JM, N) \\ & & \downarrow & & \| & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Hom}_R(M/IM, N) & \longrightarrow & \mathrm{Hom}_R(M, N) & \longrightarrow & \mathrm{Hom}_R(IM, N) & \longrightarrow & \mathrm{Ext}_R^1(M/IM, N) \end{array}$$

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \mathrm{Ext}_R^i(M/JM, N) & \longrightarrow & \mathrm{Ext}_R^i(M, N) & \longrightarrow & \mathrm{Ext}_R^i(JM, N) & \longrightarrow & \mathrm{Ext}_R^{i+1}(M/JM, N) \\ & & \downarrow & & \| & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & \mathrm{Ext}_R^i(M/IM, N) & \longrightarrow & \mathrm{Ext}_R^i(M, N) & \longrightarrow & \mathrm{Ext}_R^i(IM, N) & \longrightarrow & \mathrm{Ext}_R^{i+1}(M/IM, N) \end{array}$$

and for each $i > p$, the commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}_R^i(JM, N) & \xrightarrow[\cong]{\beta_{J,N}^i} & \mathrm{Ext}_R^{i+1}(M/JM, N) \\ {\scriptstyle \mathrm{Ext}_R^i(j_J^I, N)}\downarrow & & \downarrow{\scriptstyle \mathrm{Ext}_R^{i+1}(h_J^I, N)} \\ \mathrm{Ext}_R^i(IM, N) & \xrightarrow[\cong]{\beta_{I,N}^i} & \mathrm{Ext}_R^{i+1}(M/IM, N). \end{array}$$

It is known that the direct limits is an exact functor. By applying the diect limits over $\alpha$, we obtain the exact long sequence

$$0 \longrightarrow \mathrm{H}_\alpha^0(M, N) \longrightarrow \mathrm{Hom}_R(M, N) \longrightarrow \mathcal{D}_\alpha(M, N) \longrightarrow \mathrm{H}_\alpha^1(M, N) \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathrm{H}_\alpha^i(M, N) \longrightarrow \mathrm{Ext}_R^i(M, N) \longrightarrow \mathbf{R}^i\mathcal{D}_\alpha(M, N) \longrightarrow \mathrm{H}_\alpha^{i+1}(M, N) \longrightarrow \cdots$$

and for any $i > p$ an isomorphism $\mathbf{R}^i\mathcal{D}_\alpha(M, N) \cong \mathrm{H}_\alpha^{i+1}(M, N)$. Moreover, the following diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_R(M/IM, N) & \longrightarrow & \mathrm{Hom}_R(M, N) & \longrightarrow & \mathrm{Hom}_R(IM, N) & \longrightarrow & \mathrm{Ext}_R^1(M/IM, N) \\ & & \downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_R(M,f)} & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Hom}_R(M/IM, P) & \longrightarrow & \mathrm{Hom}_R(M, P) & \longrightarrow & \mathrm{Hom}_R(IM, P) & \longrightarrow & \mathrm{Ext}_R^1(M/IM, P) \end{array}$$

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \mathrm{Ext}^i_R(M/IM,N) & \longrightarrow & \mathrm{Ext}^i_R(M,N) & \longrightarrow & \mathrm{Ext}^i_R(IM,N) & \longrightarrow & \mathrm{Ext}^{i+1}_R(M/IM,N) \\
& \downarrow & & \downarrow{\scriptstyle \mathrm{Ext}^i_R(M,f)} & & \downarrow & & \downarrow \\
\cdots \longrightarrow & \mathrm{Ext}^i_R(M/IM,P) & \longrightarrow & \mathrm{Ext}^i_R(M,P) & \longrightarrow & \mathrm{Ext}^i_R(IM,P) & \longrightarrow & \mathrm{Ext}^{i+1}_R(M/IM,P)
\end{array}$$

$$\begin{array}{ccc}
\mathrm{Ext}^i_R(IM,N) & \xrightarrow[\cong]{\beta^i_{I,N}} & \mathrm{Ext}^{i+1}_R(M/IM,N) \\
\downarrow & & \downarrow \\
\mathrm{Ext}^i_R(IM,P) & \xrightarrow[\cong]{\beta^i_{I,P}} & \mathrm{Ext}^{i+1}_R(M/IM,P)
\end{array}$$

are commutative and hence the proof is complete. ■

**Corollary 4.2.4** If $\mathcal{D}_\alpha(M,-)$ is exact, then $\mathrm{H}^i_\alpha(M,N) \cong \mathrm{Ext}^i_R(M,N)$ for all $i > 1$.

***Proof*** By Lemma 4.2.3 we have the long exact sequence

$$\cdots \longrightarrow \mathbf{R}^{i-1}\mathcal{D}_\alpha(M,N) \longrightarrow \mathrm{H}^i_\alpha(M,N) \longrightarrow \mathrm{Ext}^i_R(M,N) \longrightarrow \mathbf{R}^i\mathcal{D}_\alpha(M,N) \longrightarrow \cdots .$$

If $i > 1$, then $\mathbf{R}^{i-1}\mathcal{D}_\alpha(M,N) = 0 = \mathbf{R}^i\mathcal{D}_\alpha(M,N)$. Therefore $\mathrm{H}^i_\alpha(M,N) \cong \mathrm{Ext}^i_R(M,N)$ for all $i > 1$. ■

**Theorem 4.2.5** Let $M$ be a finitely generated $R$-module and $N$ an $\alpha$-torsion $R$-module. Then $\mathbf{R}^i\mathcal{D}_\alpha(M,N) = 0$ for all $i \geq 0$.

***Proof*** We first prove that $\mathcal{D}_\alpha(M,N) = 0$. Consider, for each $I \in \alpha$, the injections

$$\lambda_I : \mathrm{Hom}_R(IM,N) \to \bigoplus_{J\in\alpha} \mathrm{Hom}_R(JM,N)$$

and the homomorphisms

$$\varphi^I_J : \mathrm{Hom}_R(IM,N) \to \mathrm{Hom}_R(JM,N)$$

such that $\varphi^I_J(f_I) = f_I|_{JM}$ for all $I \leq J$.

Let $T$ be an $R$-submodule of $\bigoplus_{J\in\alpha} \mathrm{Hom}_R(JM,N)$ which is generated by elements $\lambda_J\varphi^I_J(f_I) - \lambda_I f_I$, where $f_I \in \mathrm{Hom}_R(IM,N)$ and $I \leq J$. Then

$$\mathcal{D}_\alpha(M,N) = \varinjlim_{I\in\alpha} \mathrm{Hom}_R(IM,N) = (\bigoplus_{I\in\alpha} \mathrm{Hom}_R(IM,N))/T.$$

For any $u \in \mathcal{D}_\alpha(M,N)$, we have $u = \sum_{K\in\alpha} \lambda_K f_K + T$, where $f_K \in \mathrm{Hom}_R(KM,N)$. Since $KM$ is a finitely generated $R$-module and $N$ is an $\alpha$-torsion $R$-module, there exists $\mathfrak{a} \in \alpha$ such that $\mathfrak{a} f_K(KM) = 0$. This implies that $f_K(\mathfrak{a}KM) = 0$ and so $\varphi^K_{\mathfrak{a}K}(f_K) = 0$. Therefore $\lambda_K f_K + T = 0$, for any $K$, by [25, Theorem 2.17(ii)]. It follows that $u = 0$ and then $\mathcal{D}_\alpha(M, N) = 0$.

The proof will be complete if we show $\mathbf{R}^i\mathcal{D}_\alpha(M, N) = 0$ for all $i > 0$.

As $N$ is $\alpha$-torsion, there is an injective resolution $E^\bullet$ of $N$ such that each term of the resolution is an $\alpha$-torsion $R$-module. It is known that

$$\mathbf{R}^i\mathcal{D}_\alpha(M, N) = H^i(\mathcal{D}_\alpha(M, E^\bullet)).$$

By the above proof, we have

$$\mathcal{D}_\alpha(M, E^i) = \varinjlim_{I \in \alpha} \operatorname{Hom}_R(IM, E^i) = 0$$

for all $i \geq 0$. Therefore $\mathbf{R}^i(\mathcal{D}_\alpha(M, N)) = 0$ for all $i \geq 0$. ■

**Corollary 4.2.6** Let $M$ be a finitely generated $R$-module and $N$ an $R$-module such that $\mathcal{D}_\alpha(N) = 0$. Then $\mathbf{R}^i\mathcal{D}_\alpha(M, N) = 0$ for all $i \geq 0$.

***Proof*** We consider the exact sequence

$$0 \longrightarrow \Gamma_\alpha(N) \longrightarrow N \longrightarrow \mathcal{D}_\alpha(N) \longrightarrow \mathrm{H}^1_\alpha(N) \longrightarrow 0.$$

From the hypothesis, we have $\Gamma_\alpha(N) \cong N$ that means $N$ is $\alpha$-torsion. By Theorem 4.2.5 we have the conclusion. ■

**Corollary 4.2.7** Let $M$ be a finitely generated $R$-module and $N$ an $\alpha$-torsion $R$-module. Then

$$\mathrm{H}^i_\alpha(M, N) \cong \operatorname{Ext}^i_R(M, N)$$

for all $i \geq 0$.

***Proof*** It follows from Lemma 4.2.3 and Theorem 4.2.5. ■

**Remark 4.2.8** If $M$ is a finitely generated $R$-module, then there is an exact sequence

$$R^r \longrightarrow M \longrightarrow 0$$

for some $r \in \mathbb{N}$. It induces an exact sequence

$$IR^r \longrightarrow IM \longrightarrow 0$$

for all $I \in \alpha$ and then we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(IM, N) \longrightarrow \mathrm{Hom}_R(IR^r, N).$$

Note that $\mathrm{Hom}_R(IR^r, N) \cong \mathrm{Hom}_R(I, N)^r$. Passing direct limits on $\alpha$, we get the following exact sequence

$$0 \longrightarrow \mathcal{D}_\alpha(M, N) \longrightarrow \mathcal{D}_\alpha(N)^r.$$

If $f : N \to N'$ is an $R$-module homomorphism such that $Kerf$ and $Cokerf$ are both $\alpha$-torsion $R$-modules, then $\mathbf{R}^i\mathcal{D}_\alpha(N) \cong \mathbf{R}^i\mathcal{D}_\alpha(N')$ for all $i \geq 0$. We have a similar property in case of the generalized ideal transforms.

**Proposition 4.2.9** Let $M$ be a finitely generated $R$-module and $f : N \to N'$ a homomorphism of $R$-modules such that $Kerf$ and $Cokerf$ are both $\alpha$-torsion $R$-modules. Then

$$\mathbf{R}^i\mathcal{D}_\alpha(M, N) \cong \mathbf{R}^i\mathcal{D}_\alpha(M, N')$$

for all $i \geq 0$.

***Proof*** Two short exact sequences

$$0 \longrightarrow Kerf \longrightarrow N \longrightarrow Imf \longrightarrow 0 \text{ and } 0 \longrightarrow Imf \longrightarrow N' \longrightarrow Cokerf \longrightarrow 0$$

induces two long exact sequences

$$0 \longrightarrow \mathcal{D}_\alpha(M, Kerf) \longrightarrow \mathcal{D}_\alpha(M, N) \longrightarrow \mathcal{D}_\alpha(M, Imf) \longrightarrow \mathbf{R}^1\mathcal{D}_\alpha(M, Kerf) \longrightarrow \cdots$$

$$0 \longrightarrow \mathcal{D}_\alpha(M, Imf) \longrightarrow \mathcal{D}_\alpha(M, N') \longrightarrow \mathcal{D}_\alpha(M, Cokerf) \longrightarrow \mathbf{R}^1\mathcal{D}_\alpha(M, Imf) \longrightarrow \cdots.$$

Since $Kerf$ and $Cokerf$ are both $\alpha$-torsion $R$-modules,

$$\mathbf{R}^i\mathcal{D}_\alpha(M, Kerf) = 0 = \mathbf{R}^i\mathcal{D}_\alpha(M, Cokerf)$$

for all $i \geq 0$. Hence $\mathbf{R}^i\mathcal{D}_\alpha(M, N) \cong \mathbf{R}^i\mathcal{D}_\alpha(M, Imf) \cong \mathbf{R}^i\mathcal{D}_\alpha(M, N')$. ■

**Proposition 4.2.10** Let $M$ be a finitely generated $R$-module and $N$ an $R$-module. The following statements hold:
$(i)$ $\mathcal{D}_\alpha(M, N)$ is an $\alpha$-torsion-free $R$-module;
$(ii)$ $\mathbf{R}^i\mathcal{D}_\alpha(M, N) \cong \mathbf{R}^i\mathcal{D}_\alpha(M, N/\Gamma_\alpha(N))$ for all $i \geq 0$;
$(iii)$ $\mathbf{R}^i\mathcal{D}_\alpha(M, N) \cong \mathbf{R}^i\mathcal{D}_\alpha(M, \mathcal{D}_\alpha(N))$ for all $i \geq 0$;
$(iv)$ $\mathcal{D}_\alpha(\mathcal{D}_\alpha(M, N)) \cong \mathcal{D}_\alpha(M, N)$;
$(v)$ $\mathcal{D}_\alpha(\mathrm{Hom}_R(M, N)) \cong \mathrm{Hom}_R(M, \mathcal{D}_\alpha(N))$;
$(vi)$ $\mathcal{D}_\alpha(\mathrm{Hom}_R(M, N)) \cong \mathcal{D}_\alpha(M, N)$.

***Proof*** (i) We have the following exact sequence by remark 4.2.8

$$0 \longrightarrow \Gamma_\alpha(\mathcal{D}_\alpha(M,N)) \longrightarrow \Gamma_\alpha(\mathcal{D}_\alpha(N))^r.$$

Since $\Gamma_\alpha(\mathcal{D}_\alpha(N)) = 0$, we get $\Gamma_\alpha(\mathcal{D}_\alpha(M,N)) = 0$.

(ii) The short exact sequence

$$0 \longrightarrow \Gamma_\alpha(N) \longrightarrow N \longrightarrow N/\Gamma_\alpha(N) \longrightarrow 0$$

gives rise to a long exact sequence

$$0 \longrightarrow \mathcal{D}_\alpha(M,\Gamma_\alpha(N)) \longrightarrow \mathcal{D}_\alpha(M,N) \longrightarrow \mathcal{D}_\alpha(M,N/\Gamma_\alpha(N)) \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathbf{R}^i\mathcal{D}_\alpha(M,N) \longrightarrow \mathbf{R}^i\mathcal{D}_\alpha(M,N/\Gamma_\alpha(N)) \longrightarrow \mathbf{R}^{i+1}\mathcal{D}_\alpha(M,\Gamma_\alpha(N)) \longrightarrow \cdots .$$

Then $\mathbf{R}^i\mathcal{D}_\alpha(M,N) \cong \mathbf{R}^i\mathcal{D}_\alpha(M,N/\Gamma_\alpha(N))$ for all $i \geq 0$, as $\mathbf{R}^i\mathcal{D}_\alpha(M,\Gamma_\alpha(N)) = 0$.

(iii) The short exact sequence

$$0 \longrightarrow N/\Gamma_\alpha(N) \longrightarrow \mathcal{D}_\alpha(N) \longrightarrow \mathrm{H}^1_\alpha(N) \longrightarrow 0$$

induces a long exact sequence

$$0 \longrightarrow \mathcal{D}_\alpha(M,N/\Gamma_\alpha(N)) \longrightarrow \mathcal{D}_\alpha(M,\mathcal{D}_\alpha(N)) \longrightarrow \mathcal{D}_\alpha(M,\mathrm{H}^1_\alpha(N)) \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathbf{R}^i\mathcal{D}_\alpha(M,N/\Gamma_\alpha(N)) \longrightarrow \mathbf{R}^i\mathcal{D}_\alpha(M,\mathcal{D}_\alpha(N)) \longrightarrow \mathbf{R}^i\mathcal{D}_\alpha(M,\mathrm{H}^1_\alpha(N)) \longrightarrow \cdots .$$

As $\mathbf{R}^i\mathcal{D}_\alpha(M,\mathrm{H}^1_\alpha) = 0$, we obtains

$$\mathbf{R}^i\mathcal{D}_\alpha(M,N/\Gamma_\alpha(N)) \cong \mathbf{R}^i\mathcal{D}_\alpha(M,\mathcal{D}_\alpha(N))$$

for all $i \geq 0$. Therefore

$$\mathbf{R}^i\mathcal{D}_\alpha(M,N) \cong \mathbf{R}^i\mathcal{D}_\alpha(M,\mathcal{D}_\alpha(N))$$

for all $i \geq 0$, by (ii).

(iv) We have

$$\begin{aligned}\mathcal{D}_\alpha(\mathcal{D}_\alpha(M,N)) &= \varinjlim_{I\in\alpha} \mathrm{Hom}_R(I,\mathcal{D}_\alpha(M,N))\\ &\cong \varinjlim_{I\in\alpha}\varinjlim_{J\in\alpha} \mathrm{Hom}_R(I,\mathrm{Hom}_R(JM,N))\\ &\cong \varinjlim_{J\in\alpha}\varinjlim_{I\in\alpha} \mathrm{Hom}_R(I\otimes_R JM,N)\\ &\cong \varinjlim_{J\in\alpha}\varinjlim_{I\in\alpha} \mathrm{Hom}_R(JM,\mathrm{Hom}_R(I,N))\\ &\cong \varinjlim_{J\in\alpha} \mathrm{Hom}_R(JM,\mathcal{D}_\alpha(N))\\ &\cong \mathcal{D}_\alpha(M,\mathcal{D}_\alpha(N)) \cong \mathcal{D}_\alpha(M,N).\end{aligned}$$

(v) Note that

$$\begin{aligned}\mathcal{D}_\alpha(\mathrm{Hom}_R(M,N)) &= \varinjlim_{I\in\alpha} \mathrm{Hom}_R(I,\mathrm{Hom}_R(M,N))\\ &\cong \varinjlim_{I\in\alpha} \mathrm{Hom}_R(M,\mathrm{Hom}_R(I,N))\\ &\cong \mathrm{Hom}_R(M,\mathcal{D}_\alpha(N)),\end{aligned}$$

as required.

(vi) The long exact sequence

$$0 \longrightarrow \Gamma_\alpha(M,N) \longrightarrow \mathrm{Hom}_R(M,N) \longrightarrow \mathcal{D}_\alpha(M,N) \xrightarrow{f} \mathrm{H}^1_\alpha(M,N) \longrightarrow \cdots$$

induces an exact sequence

$$0 \longrightarrow \Gamma_\alpha(M,N) \longrightarrow \mathrm{Hom}_R(M,N) \longrightarrow \mathcal{D}_\alpha(M,N) \xrightarrow{f} Imf \longrightarrow 0.$$

Note that $Imf$ is an $R$-submodule of $\mathrm{H}^1_\alpha(M,N)$, then $Imf$ is an $\alpha$-torsion $R$-module.

Since $\Gamma_\alpha(M,N)$ and $Imf$ are both $\alpha$-torsion $R$-modules, there are isomorphisms

$$\mathcal{D}_\alpha(\mathrm{Hom}_R(M,N)) \cong \mathcal{D}_\alpha(\mathcal{D}_\alpha(M,N)) \cong \mathcal{D}_\alpha(M,N)$$

and the proof is complete. ■

## 4.3 Results of minimaximality and finiteness of associated primes set for ideals transforms

In this section, we concerned with the associated primes of $\mathbf{R}^i\mathcal{D}_\alpha(M,N)$ and some results of $\alpha$-minimaximality. If $M$ is finitely generated and $N$ is minimax, then Theorem 4.3.7 shows that the associated primes of $\mathcal{D}_\alpha(N)$ and $\mathcal{D}_\alpha(M,N)$ are finite. By using Grothendieck spectral sequences, we get some results on the finiteness of $\mathrm{Ass}_R(\mathbf{R}^i\mathcal{D}_\alpha(M,N))$. This section is closed by Theorem 4.3.10 which says that if $M,N$ are finitely generated and $\mathrm{Supp}_R(\mathrm{Ext}^i_R(M,N))$ is finite for all $i<t$, then $\mathrm{Ass}_R(\mathbf{R}^t\mathcal{D}_\alpha(M,N))$ is finite.

**Proposition 4.3.1** Let $M$ be a finitely generated $R$-module with finite projective dimension $p$. If $\mathcal{D}_\alpha(M,-)$ is a exact functor and $N$ is an $\alpha$-minimax $R$-module, then $\mathrm{H}^i_\alpha(M,N)$ is $\alpha$-minimax for all $i>1$.

***Proof*** By hypothesis $\mathcal{D}_\alpha(M,-)$ is exact. Then we have by Corollary 4.2.4 the isomorphism

$$\mathrm{H}^i_\alpha(M,N)\cong \mathrm{Ext}^i_R(M,N)$$

for all $i>1$. Since $N$ is $\alpha$-minimax it follows that $\mathrm{H}^i_\alpha(M,N)$ is also $\alpha$-minimax for any $i>1$. ■

**Proposition 4.3.2** Let $M$ be a finitely generated $R$-module and $N$ an $R$-module such that $\mathcal{D}_\alpha(N)$ is $\alpha$-minimax. Then $\mathcal{D}_\alpha(M,N)$ is $\alpha$-minimax.

***Proof*** By Remark 4.2.8 we get the exact sequence

$$0\longrightarrow \mathcal{D}_\alpha(M,N)\longrightarrow \mathcal{D}_\alpha(N)^r \tag{4.4}$$

for some integer $r$. Since $\mathcal{D}_\alpha(N)$ is $\alpha$-minimax, we have $\mathcal{D}_\alpha(N)^r$ is also $\alpha$-minimax. The result follows by (4.4). ■

**Theorem 4.3.3** Let $M,N$ be two finitely generated $R$-modules and $t$ a positive integer such that $\mathbf{R}^t\mathcal{D}_\alpha(M,R/\mathfrak{p})$ is $\alpha$-minimax for each $\mathfrak{p}\in\mathrm{Supp}_R(N)$. Then $\mathbf{R}^t\mathcal{D}_\alpha(M,N)$ is $\alpha$-minimax.

***Proof*** As $N$ is finitely generated there exists a filtration of $N$

$$0=N_0\subsetneq N_1\subsetneq\cdots\subsetneq N_k=N$$

such that $N_i/N_{i-1} \cong R/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \mathrm{Supp}_R(N)$.

If $i = 1$, then $\mathbf{R}^t\mathcal{D}_\alpha(M, N_1) \cong \mathbf{R}^t\mathcal{D}_\alpha(M, R/\mathfrak{p}_1)$ and consequently is $\alpha$-minimax. When $2 \leq i \leq k$ we get a short exact sequence

$$0 \to N_{i-1} \to N_i \to R/\mathfrak{p}_i \to 0. \tag{4.5}$$

For $i = 2$, consider the following part of the exact sequence induced by (4.5)

$$\cdots \to \mathbf{R}^t\mathcal{D}_\alpha(M, N_1) \to \mathbf{R}^t\mathcal{D}_\alpha(M, N_2) \to \mathbf{R}^t\mathcal{D}_\alpha(M, R/\mathfrak{p}_2) \to \cdots. \tag{4.6}$$

Since $\mathbf{R}^t\mathcal{D}_\alpha(M, N_1)$ and $\mathbf{R}^t\mathcal{D}_\alpha(M, R/\mathfrak{p}_2)$ are both $\alpha$-minimax, it follows from (4.6) that $\mathbf{R}^t\mathcal{D}_\alpha(M, N_2)$ is also. Proceeding recursively, we conclude that $\mathbf{R}^t\mathcal{D}_\alpha(M, N)$ is an $\alpha$-minimax $R$-module. ■

**Corollary 4.3.4** Let $M, N$ be two finitely generated $R$-modules and $t$ a positive integer. Assume that $\mathbf{R}^t\mathcal{D}_\alpha(M, R/\mathfrak{p})$ is $\alpha$-minimax for any $\mathfrak{p} \in \mathrm{Supp}_R(N)$.
($i$) If $L$ is a finitely $R$-module such that $\mathrm{Supp}_R(L) \subseteq \mathrm{Supp}_R(N)$, then $\mathbf{R}^t\mathcal{D}_\alpha(M, L)$ is $\alpha$-minimax;
($ii$) If $I$ is an ideal such that $V(I) \subseteq \mathrm{Supp}_R(N)$ then $\mathbf{R}^t\mathcal{D}_\alpha(M, R/I)$ is $\alpha$-minimax.

***Proof*** (i) Since $\mathrm{Supp}_R(L) \subseteq \mathrm{Supp}_R(N)$, it follows that $\mathbf{R}^t\mathcal{D}_\alpha(M, R/\mathfrak{p})$ is $\alpha$-minimax for each $\mathfrak{p} \in \mathrm{Supp}_R(L)$. By Theorem 4.3.3 $\mathbf{R}^t\mathcal{D}_\alpha(M, L)$ is $\alpha$-minimax.
(ii) We know that there is a bijection between $V(I)$ and $\mathrm{Supp}_R(R/I)$. Therefore, the result follows by item (i). ■

**Lemma 4.3.5** Assume that $(R, \mathfrak{m})$ is a local ring. Let $M$ be a finitely generated $R$-module with dimension $d > 1$. Then $\mathbf{R}^{d-1}\mathcal{D}_\alpha(M)$ is an Artinian $R$-module.

***Proof*** By Lemma 4.1.3 we get a isomorphism

$$\mathbf{R}^{d-1}\mathcal{D}_\alpha(M) \cong \mathrm{H}^d_\alpha(M),$$

because $d > 1$. On the other hand, the Lemma 3.2.4 says that $\mathrm{H}^d_\alpha(M)$ is Arinian. Therefore the $R$-module $\mathbf{R}^{d-1}\mathcal{D}_\alpha(M)$ is also Artinian. ■

**Theorem 4.3.6** Let $(R, \mathfrak{m})$ be a local ring and $M, N$ be two finitely generated $R$-modules such that $M$ has projective dimension $p$ and $N$ has dimension $d > 1$. Then

$$\mathbf{R}^{p+d-1}\mathcal{D}_\alpha(M, N) \cong \mathrm{Ext}^p_R(M, \mathbf{R}^{d-1}\mathcal{D}_\alpha(N)).$$

Moreover, the $R$-module $\mathbf{R}^{p+d-1}\mathcal{D}_\alpha(M, N)$ is Artinian.

***Proof*** As $p+d-1>p$ it follows from Lemma 4.2.3 that

$$\mathbf{R}^{p+d-1}\mathcal{D}_\alpha(M,N)\cong \mathrm{H}^{p+d}_\alpha(M,N).$$

On the other hand, we get the following isormorphisms

$$\mathrm{H}^{p+d}_\alpha(M,N)\cong \mathrm{Ext}^p_R(M,\mathrm{H}^d_\alpha(M))$$

$$\mathbf{R}^{d-1}\mathcal{D}_\alpha(N)\cong \mathrm{H}^d_\alpha(N)$$

by Theorem 3.2.5 and Lemma 4.1.3 respectively. Therefore

$$\mathbf{R}^{p+d-1}\mathcal{D}_\alpha(M,N)\cong \mathrm{Ext}^p_R(M,\mathbf{R}^{d-1}\mathcal{D}_\alpha(N)).$$

Since the Lemma 4.3.5 says that $\mathbf{R}^{d-1}\mathcal{D}_\alpha(N)$ is Artinian, it follows that $\mathbf{R}^{p+d-1}\mathcal{D}_\alpha(M,N)$ is Artinian, as required. ■

Now, we show some results on the associated primes of $\mathcal{D}_\alpha(N)$ and $\mathcal{D}_\alpha(M,N)$. It is well-known that $\mathrm{Ass}_R(\mathcal{D}_I(N))=\mathrm{Ass}_R(N)\setminus V(I)$. The followings theorem extends this property.

**Theorem 4.3.7** Let $M$ be a finitely generated $R$-module and $N$ an $R$-module. The following statements hold:
*(i)* $\mathrm{Ass}_R(\mathcal{D}_\alpha(N))=\mathrm{Ass}_R(N)\setminus\alpha$;
*(ii)* $\mathrm{Ass}_R(\mathcal{D}_\alpha(M,N))=\mathrm{Supp}_R(M)\cap(\mathrm{Ass}_R(N)\setminus\alpha)$;
*(iii)* If $N$ is a minimax $R$-module, then $\mathrm{Ass}_R(\mathcal{D}_\alpha(N))$ and $\mathrm{Ass}_R(\mathcal{D}_\alpha(M,N))$ are finite.

***Proof*** (i) From the short exact sequence

$$0\longrightarrow N/\Gamma_\alpha(N)\longrightarrow \mathcal{D}_\alpha(N)\longrightarrow \mathrm{H}^1_\alpha(N)\longrightarrow 0$$

we have

$$\begin{aligned}\mathrm{Ass}_R(\mathcal{D}_\alpha(N))&\subseteq \mathrm{Ass}_R(N/\Gamma_\alpha(N))\cup\mathrm{Ass}_R(\mathrm{H}^1_\alpha(N))\\&\subseteq \mathrm{Ass}_R(N/\Gamma_\alpha(N))\cup\alpha.\end{aligned}$$

Let $\mathfrak{p}\in\mathrm{Ass}_R(\mathcal{D}_\alpha(N))$, there is a monomorphism $R/\mathfrak{p}\hookrightarrow\mathcal{D}_\alpha(N)$. Since $\mathcal{D}_\alpha(N)$ is $\alpha$-torsion-free, so is $R/\mathfrak{p}$ and then $\mathfrak{p}\notin\alpha$. This implies that

$$\mathrm{Ass}_R(\mathcal{D}_\alpha(N))=\mathrm{Ass}_R(N/\Gamma_\alpha(N)).$$

Moreover,

$$\begin{aligned}\mathrm{Ass}_R(N/\Gamma_\alpha(N)) &\subseteq \mathrm{Ass}_R(N) \cup \mathrm{Supp}_R(\Gamma_\alpha(N)) \\ &\subseteq \mathrm{Ass}_R(N) \cup \alpha.\end{aligned}$$

Consequently, we can conclude that

$$\mathrm{Ass}_R(\mathcal{D}_\alpha(N)) \subseteq \mathrm{Ass}_R(N) \setminus \alpha.$$

On the other hand, let $\mathfrak{p} \in \mathrm{Ass}_R(N) \setminus \alpha$, then $\mathfrak{p} \notin \mathrm{Ass}_R(\Gamma_\alpha(N))$. Since

$$\mathrm{Ass}_R(N) \subseteq \mathrm{Ass}_R(\Gamma_\alpha(N)) \cup \mathrm{Ass}_R(N/\Gamma_\alpha(N)),$$

it follows that $\mathfrak{p} \in \mathrm{Ass}_R(N/\Gamma_\alpha(N))$. Therefore

$$\mathrm{Ass}_R(\mathcal{D}_\alpha(N)) = \mathrm{Ass}_R(N) \setminus \alpha.$$

(ii) By Proposition 4.2.10(v) and (vi) yields

$$\mathcal{D}_\alpha(M, N) \cong \mathrm{Hom}_R(M, \mathcal{D}_\alpha(N)).$$

Hence,

$$\begin{aligned}\mathrm{Ass}_R(\mathcal{D}_\alpha(M, N)) &= \mathrm{Supp}_R(M) \cap \mathrm{Ass}_R(\mathcal{D}_\alpha(N)) \\ &= \mathrm{Supp}_R(M) \cap (\mathrm{Ass}_R(N) \setminus \alpha).\end{aligned}$$

(iii) Since $N$ is minimax, it follows that $\mathrm{Ass}_R(N)$ is a finite set. This implies that

$$\mathrm{Ass}_R(\mathcal{D}_\alpha(M, N)) = \mathrm{Supp}_R(M) \cap (\mathrm{Ass}_R(N) \setminus \alpha) \subseteq \mathrm{Ass}_R(N)$$

is also a finite set, as required. ■

**Remark 4.3.8** If $E$ is an injective $R$-module, then $\Gamma_\alpha(E)$ is also injective and $\mathrm{H}^1_\alpha(E) = 0$. Hence, the short exact sequence

$$0 \longrightarrow \Gamma_\alpha(E) \longrightarrow E \longrightarrow \mathcal{D}_\alpha(E) \longrightarrow 0$$

is split. This implies that $\mathcal{D}_\alpha(E)$ is an injective $R$-module.

**Theorem 4.3.9** Let $M$ be a finitely generated $R$-module and $N$ an $R$-module. Let $t$ be a non-negative integer. Then the following statements hold:
$(i)$ There is a Grothendieck spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_R(M, \mathbf{R}^q\mathcal{D}_\alpha(N)) \overset{p}{\Rightarrow} \mathbf{R}^{p+q}\mathcal{D}_\alpha(M, N);$$

$(ii)$ If $\mathbf{R}^i\mathcal{D}_\alpha(N)$ is $\alpha$-minimax for all $i < t$, then $\mathbf{R}^i\mathcal{D}_\alpha(M,N)$ is $\alpha$-minimax for any $i < t$. In particular, if $\mathbf{R}^i\mathcal{D}_\alpha(N)$ is minimax for each $i < t$, then $\mathrm{Ass}_R(\mathbf{R}^i\mathcal{D}_\alpha(M,N))$ is finite for all $i < t$;
$(iii)$ If $\mathbf{R}^i\mathcal{D}_\alpha(M) = 0$ for all $i < t$ and $\mathbf{R}^t\mathcal{D}_\alpha(M)$ is $\alpha$-minimax, then $\mathbf{R}^t\mathcal{D}_\alpha(M,N)$ is $\alpha$-minimax;
$(iv)$ $\mathrm{Ass}_R(\mathbf{R}^t\mathcal{D}_\alpha(M,N)) \subseteq \bigcup\limits_{i=0}^{t} \mathrm{Ass}_R(E_{t+2}^{i,t-i})$;
$(v)$ $\mathrm{Supp}_R(\mathbf{R}^t\mathcal{D}_\alpha(M,N)) \subseteq \bigcup\limits_{i=0}^{t} \mathrm{Supp}_R(\mathrm{Ext}_R^i(M, \mathbf{R}^{t-i}\mathcal{D}_\alpha(N)))$.

***Proof*** (i) Let $\mathcal{F} = \mathrm{Hom}_R(M, -)$ and $\mathcal{G} = \mathcal{D}_\alpha(-)$ be two functors from the category of $R$-modules to itself. If $E$ is an injective $R$-module, then $\mathcal{G}(E) = \mathcal{D}_\alpha(E)$ is also injective by Remark 4.3.8. Hence $\mathcal{G}(E)$ is right $\mathcal{F}$-acyclic. Moreover, $\mathcal{F}$ is a left exact functor and $\mathcal{F}\mathcal{G} \cong \mathcal{D}_\alpha(M, -)$. By Theorem A.9 we have the Grothendieck spectral sequence

$$E_2^{p,q} = \mathrm{Ext}_R^p(M, \mathbf{R}^q\mathcal{D}_\alpha(N)) \overset{p}{\Rightarrow} \mathbf{R}^{p+q}\mathcal{D}_\alpha(M,N).$$

(ii) Let $n < t$, there is a filtration $\Phi$ of $H^n = \mathbf{R}^n\mathcal{D}_\alpha(M,N)$

$$0 = \Phi^{n+1}H^n \subseteq \Phi^n H^n \subseteq \cdots \subseteq \Phi^1 H^n \subseteq \Phi^0 H^n = H^n$$

such that

$$E_\infty^{i,n-i} \cong \Phi^i H^n / \Phi^{i+1} H^n$$

for all $i \leq n$. By hypothesis, $E_2^{i,n-i}$ is $\alpha$-minimax for all $i \leq n$. Therefore, $E_\infty^{i,n-i}$ is $\alpha$-minimax for any $i \leq n$, because $E_\infty^{i,n-i}$ is a subquotient of $E_2^{i,n-i}$. Hence $\Phi^n H^n$, $\Phi^{n-1}H^n$, $\ldots$, $\Phi^1 H^n$, $\Phi^0 H^n$ are all $\alpha$-minimax. In particular, $\mathbf{R}^n\mathcal{D}_\alpha(M,N)$ is also. In the case that $\mathbf{R}^i\mathcal{D}_\alpha(N)$ is minimax, we conclude that $\mathbf{R}^i\mathcal{D}_\alpha(M,N)$ is minimax for any $i < t$ and hence $\mathrm{Ass}_R(\mathbf{R}^i\mathcal{D}_\alpha(M,N))$ is finite for all $i < t$.

(iii) In the same manner of the proof of (ii), we can prove that $E_\infty^{i,t-i} = 0$ for all $i \leq t$. This implies that

$$0 = \Phi^{t+1}H^t = \Phi^t H^t = \cdots = \Phi^1 H^t.$$

Note that

$$E_\infty^{0,t} \cong \Phi^0 H^t / \Phi^1 H^t = \mathbf{R}^t\mathcal{D}_\alpha(M,N).$$

We now consider the homomorphisms of spectral sequence

$$0 \longrightarrow E_r^{0,t} \xrightarrow{d_r^{0,t}} E_r^{r,t-r+1}$$

for all $r \geq 2$. By the assumption, $E_r^{r,t-r+1} = 0$ for all $r \geq 2$ and then

$$E_2^{0,t} = E_3^{0,t} = \cdots = E_\infty^{0,t}.$$

Therefore,

$$\mathbf{R}^t\mathcal{D}_\alpha(M,N) \cong E_2^{0,t} = \operatorname{Hom}_R(M, \mathbf{R}^t\mathcal{D}_\alpha(N)).$$

Since $\mathbf{R}^t\mathcal{D}_\alpha(N)$ is $\alpha$-minimax, it follows that $\mathbf{R}^t\mathcal{D}_\alpha(M,N)$ is $\alpha$-minimax.

(iv) By the isomorphisms

$$E_\infty^{i,t-i} \cong \Phi^i H^t/\Phi^{i+1}H^t$$

for all $i \leq t$, we conclude that

$$\begin{aligned}
\operatorname{Ass}_R(\mathbf{R}^t\mathcal{D}_\alpha(M,N)) &\subseteq \operatorname{Ass}_R(E_\infty^{0,t}) \cup \operatorname{Ass}_R(\Phi^1 H^t)\\
&\subseteq \operatorname{Ass}_R(E_\infty^{0,t}) \cup \operatorname{Ass}_R(E_\infty^{1,t-1}) \cup \operatorname{Ass}_R(\Phi^2 H^t)\\
&\cdots\\
&\subseteq \bigcup_{i=0}^{t} \operatorname{Ass}_R(E_\infty^{i,t-i}).
\end{aligned}$$

If we prove that $\operatorname{Ass}_R(E_\infty^{i,t-i}) \subseteq \operatorname{Ass}_R(E_{t+2}^{i,t-i})$ for all $0 \leq i \leq t$, then the assertion follows. Now, the homomorphisms of spectral sequence

$$0 = E_{t+2}^{i-t-2,2t-i+1} \longrightarrow E_{t+2}^{i,t-i} \longrightarrow E_{t+2}^{t+i+2,-i-1} = 0$$

yield

$$E_{t+2}^{i,t-i} = E_{t+3}^{i,t-i} = \cdots = E_\infty^{i,t-i}$$

for all $0 \leq i \leq t$.

(v) Analysis similar to that in the proof of (iv) shows that

$$\operatorname{Supp}_R(\mathbf{R}^t\mathcal{D}_\alpha(M,N)) \subseteq \bigcup_{i=0}^{t} \operatorname{Supp}_R(E_\infty^{i,t-i})$$

and

$$E_{t+2}^{i,t-i} = E_{t+3}^{i,t-i} = \cdots = E_\infty^{i,t-i}.$$

Thus $E_\infty^{i,t-i}$ is a subquotient of $E_2^{i,t-i}$ and then

$$\operatorname{Supp}_R(E_\infty^{i,t-i}) \subseteq \operatorname{Supp}_R(E_2^{i,t-i}) = \operatorname{Supp}_R(\operatorname{Ext}_R^i(M, \mathbf{R}^{t-i}\mathcal{D}_\alpha(N))).$$

This is finishes the proof. ■

If $M, N$ are finitely generated $R$-modules and $\mathrm{Ass}_R(\mathbf{R}^i\mathcal{D}_\alpha(M,N))$ is finite, then we can conclude that $\mathrm{Ass}_R(\mathrm{H}^{i+1}_\alpha(M,N))$ is finite by the long exact sequence in Lemma 4.2.3. Therefore, the finiteness of $\mathrm{Ass}_R(\mathrm{H}^i_\alpha(M,N))$ can be implied when we study the set $\mathrm{Ass}_R(\mathbf{R}^i\mathcal{D}_\alpha(M,N))$.

**Theorem 4.3.10** Let $M, N$ be $R$-modules and $t$ a non-negative integer. The following statements hold:
*(i)* There is a Grothendieck spectral sequence

$$E_2^{p,q} = \mathbf{R}^p\mathcal{D}_\alpha(\mathrm{Ext}^q_R(M,N)) \stackrel{p}{\Rightarrow} \mathbf{R}^{p+q}\mathcal{D}_\alpha(M,N);$$

*(ii)* If $\mathrm{Supp}_R(\mathrm{Ext}^i_R(M,N))$ is finite for all $i < t$, then

$$\mathrm{Supp}_R(\mathbf{R}^i\mathcal{D}_\alpha(M,N)) \text{ and } \mathrm{Supp}_R(\mathrm{H}^i_\alpha(M,N))$$

are finite for all $i < t$;
*(iii)* If $M, N$ are finitely generated and $\mathrm{Supp}_R(\mathrm{Ext}^i_R(M,N))$ is finite for all $i < t$, then $\mathrm{Ass}_R(\mathbf{R}^t\mathcal{D}_\alpha(M,N))$ is finite.

***Proof*** (i) Let $\mathcal{F} = \mathcal{D}_\alpha(-)$ and $\mathcal{G} = \mathrm{Hom}_R(M,-)$ be functors from the category of $R$-modules to itself. It follows from Proposition 4.2.10 that $\mathcal{F}\mathcal{G}(N) = \mathcal{D}_\alpha(M,N)$ for any $R$-module $N$. Let $E$ be an injective $R$-module, we will show that $\mathbf{R}^i\mathcal{F}(\mathcal{G}(E)) = 0$ for all $i > 0$. Assume that

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

is a free resolution of $M$ in which each $F_i$ is finitely generated. Note that

$$0 \longrightarrow \mathrm{Hom}_R(M,E) \longrightarrow \mathrm{Hom}_R(F_0,E) \longrightarrow \mathrm{Hom}_R(F_1,E) \longrightarrow \cdots$$

is an injective resolution of $\mathrm{Hom}_R(M,E) = \mathcal{G}(E)$. On the other hand, according to Remark 4.3.8, $\mathcal{D}_\alpha(E)$ is an injective $R$-module. By applying the functor $\mathrm{Hom}_R(-,\mathcal{D}_\alpha(E))$ to the free resolution of $M$, we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(M,\mathcal{D}_\alpha(E)) \longrightarrow \mathrm{Hom}_R(F_0,\mathcal{D}_\alpha(E)) \longrightarrow \mathrm{Hom}_R(F_1,\mathcal{D}_\alpha(E)) \longrightarrow \cdots .$$

Since $F_i$ is finitely generated free for all $i \geq 0$, we can conclude by Proposition 4.2.10 that $\mathrm{Hom}_R(F_i,\mathcal{D}_\alpha(E)) \cong \mathcal{D}_\alpha(\mathrm{Hom}_R(F_i,E))$ is injective for all $i \geq 0$. Consequently, there is an exact sequence

$$0 \longrightarrow \mathcal{D}_\alpha(\mathrm{Hom}_R(M,E)) \longrightarrow \mathcal{D}_\alpha(\mathrm{Hom}_R(F_0,E)) \longrightarrow \mathcal{D}_\alpha(\mathrm{Hom}_R(F_1,E)) \longrightarrow \cdots .$$

This implies that $\mathbf{R}^i\mathcal{F}(\mathrm{Hom}_R(M,E)) = 0$ for all $i > 0$. It follows from Theorem A.9 that there is a Grothendieck spectral sequence

$$E_2^{p,q} = \mathbf{R}^p\mathcal{D}_\alpha(\mathrm{Ext}_R^q(M,N)) \overset{p}{\Rightarrow} \mathbf{R}^{p+q}\mathcal{D}_\alpha(M,N).$$

(ii) By the hypothesis, $\mathrm{Supp}_R(E_2^{p,q})$ is finite for all $q < t$. Since $E_\infty^{p,q}$ is a subquotient of $E_2^{p,q}$, we see that $\mathrm{Supp}_R(E_\infty^{p,q})$ is finite for all $q < t$. Let $n < t$, we consider a filtration $\Phi$ os submodules of $H^n = \mathbf{R}^n\mathcal{D}_\alpha(M,N)$

$$0 = \Phi^{n+1}H^n \subseteq \Phi^n H^n \subseteq \cdots \subseteq \Phi^1 H^n \subseteq \Phi^0 H^n = H^n$$

such that

$$E_\infty^{i,n-i} \cong \Phi^i H^n/\Phi^{i+1}H^n$$

for all $i \leq n$. This implies that the supports of $\Phi^n H^n$, $\Phi^{n-1}H^n$,…, $\Phi^1 H^n$, $\Phi^0 H^n$ are finite. In particular $\mathrm{Supp}_R(\mathbf{R}^n\mathcal{D}_\alpha(M,N))$ is finite. The finiteness of $\mathrm{Supp}_R(\mathrm{H}_\alpha^n(M,N))$ follows from Lemma 4.2.3.

(iii) By an argument analogous to that in the proof of (ii) we conclude that the supports of $\Phi^t H^t$, $\Phi^{t-1}H^t$, …, $\Phi^1 H^t$ are finite where $H^t = \mathbf{R}^t\mathcal{D}_\alpha(M,N)$. The isomorphism

$$E_\infty^{0,t} \cong \mathbf{R}^t\mathcal{D}_\alpha(M,N)/\Phi^1 H^t$$

gives

$$\mathrm{Ass}_R(\mathbf{R}^t\mathcal{D}_\alpha(M,N)) \subseteq \mathrm{Ass}_R(E_\infty^{0,t}) \cup \mathrm{Ass}_R(\Phi^1 H^t).$$

Note that $E_\infty^{0,t}$ is a submodule of $E_2^{0,t} = \mathcal{D}_\alpha(\mathrm{Ext}_R^t(M,N))$. Now, combining the assumption with Theorem 4.3.7(iii) we see that $\mathrm{Ass}_R(\mathcal{D}_\alpha(\mathrm{Ext}_R^t(M,N)))$ is finite. This implies that $\mathrm{Ass}_R(E_\infty^{0,t})$ is finite and the proof is complete. ■

# Appendices

# Appendix A

# Spectral sequences

**Definition A.1** A *spectral sequence* is a sequence $\{E^r, d^r\}_{r\geq 1}$ of bigraded modules and maps of bidegree $(-r, r-1)$ with $d^r d^r = 0$ such that

$$E^{r+1}_{p,q} = Ker d^r_{p,q} / Im d^r_{p+r,q-r+1}$$

and then

$$E^{r+1} = H(E^r, d^r)$$

as bigraded modules.

**Definition A.2** Let $M$ be an $R$-module. A *subquotient* of $M$ is a module of the form $M'/M''$, where $M'$ and $M''$ are submodules of $M$.

An example of a subquotient module is homology: if $\mathcal{C}_\bullet$ is a complex, then $H_i(\mathcal{C}_\bullet)$ is a subquotient of $\mathcal{C}_i$.

**Remark A.3** Let $Z^r_{p,q} = Ker d^r_{p,q}$ and $B^r_{p,q} = Im d^r_{p+r,q-r+1}$. In spectral sequence each term $E^r$ is a subquotient of any earlier term. Set $Z^r = \left\{Z^r_{p,q}\right\}_{p,q}$ and $B^r = \left\{B^r_{p,q}\right\}_{p,q}$. Write $E^2 = Z^2/B^2$. Since $E^3 = Z^3/B^3$ is a subquotient of $E^2$, the third isomorphism theorem allows us to assume

$$0 \subset B^2 \subset B^3 \subset Z^3 \subset Z^2 \subset E^1.$$

Iterating, we get

$$0 \subset B^2 \subset \cdots \subset B^r \subset B^{r+1} \subset \cdots \subset Z^{r+1} \subset Z^r \subset \cdots \subset Z^2 \subset E^1.$$

**Definition A.4** Let $Z^\infty_{p,q} = \bigcap_r Z^r_{p,q}$; $B^\infty_{p,q} = \bigcup_r B^r_{p,q}$; $E^\infty_{p,q} = Z^\infty_{p,q}/B^\infty_{p,q}$. The *limit term* of the spectral sequence $\{E^r\}_r$ is the bigraded module $E^\infty = \left\{E^\infty_{p,q}\right\}_{p,q}$.

Note that, as $r$ gets large, the terms $E^r$ do "approximate" the limit term.

**Definition A.5**
*(i)* Let $\mathcal{C}$ a complex. A *filtration* of $\mathcal{C}$ is a family of subcomplexes $\{F^p\mathcal{C}\}_{p\in\mathbb{Z}}$ with $F^{p-1}\mathcal{C} \subset F^p\mathcal{C}$ for all $p$.
*(ii)* A *filtration of a graded module* $H = \{H_n\}_{n\in\mathbb{Z}}$ is a family of graded submodules $\{F^pH\}_{p\in\mathbb{Z}}$ with $F^{p-1}H \subset F^pH$ for each $p$.

**Theorem A.6** Every filtration $\{F^p\mathcal{C}\}$ of a complex $\mathcal{C}$ determines a spectral sequence with $E^1_{p,q} = H_{p+q}(F^p\mathcal{C}/F^{p-1}\mathcal{C})$.

***Proof*** See [25, Corollary 11.12]. ■

**Definition A.7** A filtration $\{F^pH\}$ of a graded module $H$ is *bounded* if for each $n$ there exist integers $s = s(n)$ and $t = t(n)$ such that

$$F^sH_n = 0 \text{ and } F^tH_n = H_n.$$

Note that if $\{F^pH\}$ is a bounded filtration, then for each $n$ we get $F^pH_n = 0$ for any $p \le s$ and $F^pH_n = H_n$ for all $p \ge t$. Then there is a finite chain

$$0 = F^sH_n \subset F^{s+1}H_n \subset \cdots \subset F^tH_n = H_n.$$

**Definition A.8** A spectral sequence $\{E^r\}_r$ *converges* to a graded module $H$ ( denoted by $E^2_{p,q} \overset{p}{\Rightarrow} H_n$ ) if there is some bounded filtration $\{\Phi^pH\}$ of $H$ such that

$$E^\infty_{p,q} \cong \Phi^pH_n/\Phi^{p-1}H_n$$

for all $p, q$ with $p + q = n$.

**Theorem A.9 (Grothendieck)** Let $\mathcal{G} : \mathfrak{U} \to \mathfrak{B}$ and $\mathcal{F} : \mathfrak{B} \to \mathfrak{C}$ be two functors with $\mathcal{F}$ left exact such that $E$ injective in $\mathfrak{U}$ implies $\mathcal{G}E$ is right $\mathcal{F}$-acyclic. Then for each module $M$ in $\mathfrak{U}$, there exists a third quadrant spectral sequence with

$$E_2^{p,q} = \mathbf{R}^p\mathcal{F}(\mathbf{R}^q\mathcal{G}(M)) \overset{p}{\Rightarrow} \mathbf{R}^{p+q}(\mathcal{F}\mathcal{G})(M).$$

***Proof*** See [25, Theorem 11.38]. ■

# Appendix B

# Results of Local Cohomology

Let $\alpha$ a good family of ideals of $R$.

**Lemma B.1** The $\alpha$-torsion functor $\Gamma_\alpha$ is a left exact functor on the category of all $R$-modules.

**Proposition B.2** Let $I, I'$ be ideals of $R$ and $\beta, \beta'$ families of ideals of $R$.
$(i)$ If $I \subseteq I'$, then $\mathcal{F}(I,\beta) \supseteq \mathcal{F}(I',\beta)$;
$(ii)$ If $\beta' \subseteq \beta$, then $\mathcal{F}(I,\beta) \subseteq \mathcal{F}(I,\beta')$;
$(iii)$ $\mathcal{F}(I+I',\beta) = \mathcal{F}(I,\beta) \cap \mathcal{F}(I',\beta)$;
$(iv)$ $\mathcal{F}(I,\beta) \cap \mathcal{F}(I,\beta') = \mathcal{F}(I,\beta \cup \beta')$;
$(v)$ $\mathcal{F}(I,\beta) = \mathcal{F}(\sqrt{I},\beta)$.

***Proof*** (i) If $K \in \mathcal{F}(I',\beta)$, then $K + J \in \langle I' \rangle$ for all $J \in \beta$. Since $I \subseteq I'$, this implies that $\langle I' \rangle \subseteq \langle I \rangle$. So $K + J \in \langle I \rangle$ and therefore, $K \in \mathcal{F}(I,\beta)$.

(ii) Let $K \in \mathcal{F}(I,\beta)$. This implies that $K + J \in \langle I \rangle$ for all $J \in \beta$. By $\beta' \subseteq \beta$, we have $K + J \in \langle I \rangle$ for all $J \in \beta'$. Thus $K \in \mathcal{F}(I,\beta')$.

(iii) Knowing that $I \subseteq I + I'$ and $I' \subseteq I + I'$ we have $\mathcal{F}(I+I',\beta) \subseteq \mathcal{F}(I,\beta) \cap \mathcal{F}(I',\beta)$, by (i). On the other hand, if $K \in \mathcal{F}(I,\beta) \cap \mathcal{F}(I',\beta)$, then $K + J \in \langle I \rangle \cap \langle I' \rangle$ for all $J \in \beta$. Since $\langle I \rangle \cap \langle I' \rangle = \langle I + I' \rangle$, it follows that $K \in \mathcal{F}(I+I',\beta)$.

(iv) Since $\beta \subseteq \beta \cup \beta'$ and $\beta' \subseteq \beta \cup \beta'$ it follows that $\mathcal{F}(I,\beta \cup \beta') \subseteq \mathcal{F}(I,\beta) \cap \mathcal{F}(I,\beta')$, by item (ii). Conversely, if $K \in \mathcal{F}(I,\beta) \cap \mathcal{F}(I,\beta')$, then $K + J'' \in \langle I \rangle$ for all $J'' \in \beta$ and $K + J' \in \langle I \rangle$ for any $J' \in \beta'$. So $K + J \in \langle I \rangle$ for each $J \in \beta \cup \beta'$. Therefore $K \in \mathcal{F}(I,\beta \cup \beta')$.

(v) The proof follows directly from the equality $\langle I\rangle = \left\langle\sqrt{I}\right\rangle$. ■

**Proposition B.3** Let $I, I'$ be ideals of $R$ and $\beta, \beta'$ families of ideals in $R$. Let $M$ be an $R$-module.
*(i)* $\Gamma_{I,\beta}(\Gamma_{I',\beta'}(M)) = \Gamma_{I',\beta'}(\Gamma_{I,\beta}(M))$;
*(ii)* If $I \subseteq I'$, then $\Gamma_{I,\beta}(M) \supseteq \Gamma_{I',\beta}(M)$;
*(iii)* If $\beta' \subseteq \beta$, then $\Gamma_{I,\beta}(M) \subseteq \Gamma_{I,\beta'}(M)$;
*(iv)* $\Gamma_{I,\beta}(\Gamma_{I',\beta}(M)) = \Gamma_{I+I',\beta}(M)$;
*(v)* $\Gamma_{I,\beta}(\Gamma_{I,\beta'}(M)) = \Gamma_{I,\beta\cup\beta'}(M)$;
*(vi)* If $\sqrt{I} = \sqrt{I'}$, then $\mathrm{H}^i_{I,\beta}(M) = \mathrm{H}^i_{I',\beta}(M)$, for all $i \geq 0$. In particular, $\mathrm{H}^i_{I,\beta}(M) = \mathrm{H}^i_{\sqrt{I},\beta}(M)$, for each $i \geq 0$;
*(vii)* If $\beta$ and $\beta'$ are cofinals, then $\mathrm{H}^i_{I,\beta}(M) = \mathrm{H}^i_{I,\beta'}(M)$, for any $i \geq 0$.

***Proof*** The statements from (i) to (iii) follow from the definitions.

(iv) Let $x \in \Gamma_{I,\beta}(\Gamma_{I',\beta}(M))$. Then, there exists $K \in \mathcal{F}(I,\beta)$ such that $Kx = 0$. Since $x \in \Gamma_{I',\beta}(M)$, there is $K' \in \mathcal{F}(I',\beta)$ such that $K'x = 0$. On the other hand, $K+K' \in \mathcal{F}(I,\beta)\cap\mathcal{F}(I',\beta) = \mathcal{F}(I+I',\beta)$, by Proposition B.2(iii), and $(K+K')x = 0$. So, $x \in \Gamma_{I+I',\beta}(M)$. The other inclusion is analogous.

(v) The proof is similar to proof of item (iv).

(vi) Since $\langle I\rangle = \left\langle\sqrt{I}\right\rangle = \left\langle\sqrt{I'}\right\rangle = \langle I'\rangle$, we have $\mathcal{F}(I,\beta) = \mathcal{F}(I',\beta)$. Therefore,

$$\Gamma_{I,\beta}(M) = \Gamma_{\sqrt{I},\beta}(M) = \Gamma_{\sqrt{I'}}(M) = \Gamma_{I',\beta}(M)$$

and consequently

$$\mathrm{H}^i_{I,\beta}(M) = \mathrm{H}^i_{\sqrt{I},\beta}(M) = \mathrm{H}^i_{\sqrt{I'},\beta}(M) = \mathrm{H}^i_{I',\beta}(M)$$

for all $i \geq 0$.

(vii) Knowing that $\beta, \beta'$ are cofinals, we have $\langle\beta\rangle = \langle\beta'\rangle$. Thus $\mathcal{F}(I,\beta) = \mathcal{F}(I,\beta')$ and therefore $\mathrm{H}^i_{I,\beta}(M) = \mathrm{H}^i_{I,\beta'}(M)$ for all $i \geq 0$. ■

**Proposition B.4** For an $R$-module $M$, the following are equivalent.
*(i)* $M$ is $\alpha$-torsion $R$-module;
*(ii)* $\mathrm{Min}(M) \subseteq \alpha$;
*(iii)* $\mathrm{Ass}_R(M) \subseteq \alpha$;
*(iv)* $\mathrm{Supp}_R(M) \subseteq \alpha$.

***Proof*** The proof of the implications (iv)⇒(iii)⇒(ii) is trivial.

(ii)⇒(iv) For $\mathfrak{p} \in \mathrm{Supp}_R(M)$, there exists $\mathfrak{q} \in \mathrm{Min}(M)$ such that $\mathfrak{p} \supseteq \mathfrak{q}$. Since $\mathfrak{q} \in \alpha$, it follows that $\mathfrak{p} \in \alpha$.

(i)⇒(iii) If $\mathfrak{p} \in \mathrm{Ass}_R(M)$ then $\mathfrak{p} = Ann(x)$ for some $x \in M$. Since $M$ is an $\alpha$-torsion $R$-module, there exists $I \in \alpha$ such that $Ix = 0$. So $I \subseteq \mathfrak{p}$. Therefore $\mathfrak{p} \in \alpha$.

(iv)⇒(i) To show that $M$ is $\alpha$-torsion, it suffices to prove that $M \subseteq \Gamma_\alpha(M)$. Let $x \in M$ and set $\mathrm{Min}(Rx) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$. Since $\mathrm{Min}(Rx) \subseteq \mathrm{Supp}_R(M) \subseteq \alpha$, we have $\mathfrak{p}_1 \cdots \mathfrak{p}_k \in \alpha$. On the other hand, $\sqrt{Ann(x)} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k \supseteq \mathfrak{p}_1 \cdots \mathfrak{p}_k$ which implies that $(\mathfrak{p}_1 \cdots \mathfrak{p}_k)^n x = 0$ for some $n \in \mathbb{N}$. Since $(\mathfrak{p}_1 \cdots \mathfrak{p}_k)^n \in \alpha$, it follows that $x \in \Gamma_\alpha(M)$ and the proof is completed. ■

**Corollary B.5** *(i)* For $x \in M$, the followings conditions are equivalent.

*(a)* $x \in \Gamma_\alpha(M)$*;*

*(b)* $\mathrm{Supp}(Rx) \subseteq \alpha$;

*(ii)* Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence of $R$-modules. Then $M$ is an $\alpha$-torsion modules if and only if $L$ and $N$ are $\alpha$-torsion modules.

***Proof*** (i):*(a)* ⇒ *(b)* The assumption implies that $\Gamma_\alpha(Rx) = Rx$. Thus by Proposition B.4 we get $\mathrm{Supp}(Rx) \subseteq \alpha$.

*(b)* ⇒ *(a)* By Proposition B.4 we get $x \in Rx = \Gamma_\alpha(Rx) \subseteq \Gamma_\alpha(M)$.

(ii) The proof follows from Proposition B.4 and by the equality $\mathrm{Supp}_R(M) = \mathrm{Supp}_R(L) \cup \mathrm{Supp}_R(N)$. ■

**Corollary B.6** If $M$ is an $(I, \beta)$-torsion $R$-module, then $M/JM$ is an $I$-torsion $R$-module for any $J \in \beta$. The converse holds if $M$ is finitely generated.

***Proof*** Since $M$ is an $(I, \beta)$-torsion $R$-module, we have $\mathrm{Supp}_R(M) \subseteq \tilde{\mathcal{F}}(I, \beta)$. For each $J \in \beta$, we get

$$\mathrm{Supp}_R(M/JM) \subseteq \mathrm{Supp}_R(M) \cap V(J) \subseteq \tilde{\mathcal{F}}(I, \beta) \cap V(J).$$

On the other the hand, if $\mathfrak{p} \in \tilde{\mathcal{F}}(I, \beta) \cap V(J)$ then $\mathfrak{p} + J \in \langle I \rangle$. Since $\mathfrak{p} \in V(J)$, we have $\mathfrak{p} \supseteq J$ and so $\mathfrak{p} \in \langle I \rangle$. Therefore, $\mathfrak{p} \in V(I)$ and consequently $M/JM$ is $I$-torsion module.

Suppose that $M$ is a finitely generated $R$-module. If $x \in M$, then, by Artin-Rees lemma, for each $J \in \beta$, there exists $n_J \geq 0$ such that $J^{n_J} M \cap Rx \subseteq Jx$. Since $M/JM$ is $I$-torsion, we have $\mathrm{Supp}_R(M/J^{n_J} M) = \mathrm{Supp}_R(M/JM) \subseteq V(I)$, therefore $M/J^{n_J} M$

is $I$-torsion as well. Thus there is an integer $m \geq 0$ with $I^m x \subseteq J^{n_J} M$. Hence it follows that $I^m x \subseteq J^{n_J} M \cap Rx \subseteq Jx$. Using the above information and taking $\mathfrak{a} = Ann(x)$, we get $\mathfrak{a} + J \in \langle I \rangle$ for all $J \in \beta$. This implies that $\mathfrak{a} \in \mathcal{F}(I, \beta)$. Therefore $x \in \Gamma_{I,\beta}(M)$. ■

**Proposition B.7** Let $M$ be an $R$-module. Then the equality

$$\mathrm{Ass}_R(M) \cap \alpha = \mathrm{Ass}_R(\Gamma_\alpha(M))$$

holds. In particular, $\Gamma_\alpha(M) \neq 0$ if and only if $\mathrm{Ass}_R(M) \cap \alpha \neq \varnothing$.

***Proof*** Since $\Gamma_\alpha(M)$ is an $\alpha$-torsion $R$-module, we get $\mathrm{Ass}_R(\Gamma_\alpha(M)) \subseteq \alpha$, by Proposition B.4. This implies that $\mathrm{Ass}_R(\Gamma_\alpha(M)) \subseteq \mathrm{Ass}_R(M) \cap \alpha$.

Now, take $\mathfrak{p} \in \mathrm{Ass}_R(M) \cap \alpha$. Then there is an non-zero element $x \in M$ such that $\mathfrak{p} = Ann(x)$. Moreover, $\mathfrak{p} \in \alpha$ which implies in $x \in \Gamma_\alpha(M)$. Since $\mathfrak{p} = Ann(x)$, it follows that $\mathfrak{p} \in \mathrm{Ass}_R(\Gamma_\alpha(M))$. ■

**Proposition B.8** Let $\mathfrak{p} \in Spec(R)$ and $\alpha$ a good family in $R$. If $\mathfrak{p} \in \alpha$, then $E(R/\mathfrak{p})$ is an $\alpha$-torsion $R$-module. On the other hand, if $\mathfrak{p} \notin \alpha$ then $E(R/\mathfrak{p})$ is an $\alpha$-torsion-free $R$-module.

***Proof*** If $\mathfrak{p} \in \alpha$, then $\mathrm{Ass}_R(E(R/\mathfrak{p})) = \{\mathfrak{p}\} \subseteq \alpha$. Consequently $\Gamma_\alpha(E(R/\mathfrak{p})) = E(R/\mathfrak{p})$ by Proposition B.4. Now, if $\mathfrak{p} \notin \alpha$, then $\mathrm{Ass}_R(E(R/\mathfrak{p})) \cap \alpha = \{\mathfrak{p}\} \cap \alpha = \varnothing$. Therefore, by Proposition B.7, we have $\Gamma_\alpha(E(R/\mathfrak{p})) = 0$. ■

**Proposition B.9** Let $M$ be an $\alpha$-torsion $R$-module. Then there exists an injective resolution of $M$ in which each term is an $\alpha$-torsion $R$-module.

***Proof*** Indeed, let $E^0$ be the injective hull of $M$. Since $M$ is $\alpha$-torsion, we have $\mathrm{Ass}_R(E^0) = \mathrm{Ass}_R(M) \subseteq \alpha$, by Proposition B.4. Then $E^0$ is $\alpha$-torsion module. Thus we see that $M$ can be embedded in an $\alpha$-torsion injective $R$-module $E^0$.

Suppose, inductively, we have constructed an exact sequence

$$0 \longrightarrow M \longrightarrow E^0 \longrightarrow \cdots \longrightarrow E^{p-1} \xrightarrow{d^{n-1}} E^n$$

of $R$-modules in which $E^0$, $\ldots$, $E^{n-1}$, $E^n$ are $\alpha$-torsion injective $R$-modules. Let $C = Coker d^{n-1}$. Since $E^n$ is an $\alpha$-torsion module, $C$ is $\alpha$-torsion as well by Corollary B.5(ii). Applying the argument in the first paragraph to $C$, we can embed $C$ into an $\alpha$-torsion injective $R$-module $E^{n+1}$. This completes the proof. ■

**Corollary B.10** Let $M$ be an $R$-module. Then the following statements hold.
(*i*) If $M$ is an $\alpha$-torsion $R$-module, then $\mathrm{H}^i_\alpha(M) = 0$ for all $i > 0$;
(*ii*) $\mathrm{H}^i_\alpha(\Gamma_\alpha(M)) = 0$ for all $i > 0$;
(*iii*) $M/\Gamma_\alpha(M)$ is an $\alpha$-torsion-free $R$-module;
(*iv*) There is an isomorphism $\mathrm{H}^i_\alpha(M) \cong \mathrm{H}^i_\alpha(M/\Gamma_\alpha(M))$ for any $i > 0$;
(*v*) $\mathrm{H}^i_\alpha(M)$ is an $\alpha$-torsion $R$-module for each $i \geq 0$.

***Proof*** (i) Follows from Proposition B.9.

(ii) Since $\Gamma_\alpha(M)$ is an $\alpha$-torsion $R$-module, it follows from (i) that $\mathrm{H}^i_\alpha(\Gamma_\alpha(M)) = 0$ for all $i > 0$.

(iii) and (iv) From the exact sequence

$$0 \longrightarrow \Gamma_\alpha(M) \longrightarrow M \longrightarrow M/\Gamma_\alpha(M) \longrightarrow 0$$

we get an exact sequence

$$0 \longrightarrow \Gamma_\alpha(\Gamma_\alpha(M)) \longrightarrow \Gamma_\alpha(M) \longrightarrow \Gamma_\alpha(M/\Gamma_\alpha(M)) \longrightarrow 0.$$

in which we obtain $\Gamma_\alpha(M/\Gamma_\alpha(M)) = 0$ and isomorphisms

$$\mathrm{H}^i_\alpha(M) \cong \mathrm{H}^i_\alpha(M/\Gamma_\alpha(M))$$

for all $i \geq 1$.

(v) Since $\mathrm{H}^i_\alpha(M)$ is an subquotient of an $\alpha$-torsion module, for all $i \geq 0$, it is also $\alpha$-torsion by Corollary B.5 and the proof is completed. ∎

**Theorem B.11** Let $M$ be an $R$-module. Then there is a natural isomorphism

$$\mathrm{H}^i_\alpha(M) \cong \varinjlim_{I \in \alpha} \mathrm{H}^i_I(M)$$

for any $i \geq 0$.

***Proof*** Firstly, we know that $\Gamma_\alpha(M) = \varinjlim_{I \in \alpha} \Gamma_I(M)$. Now, we take an exact sequence of $R$-modules

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0.$$

For each $I \in \alpha$, we get a long exact sequence

$$0 \longrightarrow \mathrm{H}^0_I(L) \longrightarrow \mathrm{H}^0_I(M) \longrightarrow \mathrm{H}^0_I(N) \longrightarrow \mathrm{H}^1_I(L) \longrightarrow \mathrm{H}^1_I(M) \longrightarrow \cdots.$$

Knowing that the direct limits is an exact functor and applying the direct limits on $\alpha$, we obtain the long exact sequence

$$0 \longrightarrow \varinjlim_{I\in\alpha} \mathrm{H}^0_I(L) \longrightarrow \varinjlim_{I\in\alpha} \mathrm{H}^0_I(M) \longrightarrow \varinjlim_{I\in\alpha} \mathrm{H}^0_I(N) \longrightarrow$$

$$\longrightarrow \varinjlim_{I\in\alpha} \mathrm{H}^1_I(L) \longrightarrow \varinjlim_{I\in\alpha} \mathrm{H}^1_I(M) \longrightarrow \varinjlim_{I\in\alpha} \mathrm{H}^1_I(N) \longrightarrow \cdots .$$

On the other hand, for any injective $R$-module $E$ and any integer $i > 0$, we have $\mathrm{H}^i_I(E) = 0$ for each $I \in \alpha$. Therefore,

$$\varinjlim_{I\in\alpha} \mathrm{H}^i_I(E) = 0$$

for all $i > 0$ and the proof is completed. ■

The next result shows that the local cohomology functor, with respect to $\alpha$, commutes with direct limits.

**Proposition B.12** Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a direct system of $R$-modules. Then there is a isomorphism

$$\mathrm{H}^i_\alpha(\varinjlim_{\lambda\in\Lambda} M_\lambda) \cong \varinjlim_{\lambda\in\Lambda} \mathrm{H}^i_\alpha(M_\lambda)$$

for all $i \geq 0$.

***Proof*** By Theorem B.11 we have

$$\mathrm{H}^i_\alpha(\varinjlim_{\lambda\in\Lambda} M_\lambda) \cong \varinjlim_{I\in\alpha} \mathrm{H}^i_I(\varinjlim_{\lambda\in\Lambda} M_\lambda)$$

for all $i \geq 0$. On the other hand, for each $I \in \alpha$ and by [6, Theorem 3.4.10] there is an isomorphism

$$\mathrm{H}^i_I(\varinjlim_{\lambda\in\Lambda} M_\lambda) \cong \varinjlim_{\lambda\in\Lambda} \mathrm{H}^i_I(M_\lambda)$$

for all $i \geq 0$. This implies that

$$\begin{aligned} \mathrm{H}^i_\alpha(\varinjlim_{\lambda\in\Lambda} M_\lambda) &\cong \varinjlim_{I\in\alpha} \varinjlim_{\lambda\in\Lambda} \mathrm{H}^i_I(M_\lambda) \\ &\cong \varinjlim_{\lambda\in\Lambda} \varinjlim_{I\in\alpha} \mathrm{H}^i_I(M_\lambda) \\ &\cong \varinjlim_{\lambda\in\Lambda} \mathrm{H}^i_\alpha(M_\lambda) \end{aligned}$$

for any $i \geq 0$, as required. ■

**Theorem B.13** For any finitely generated $R$-module $M$ we have the equality

$$inf\left\{i \mid \mathrm{H}^i_\alpha(M) \neq 0\right\} = inf\left\{depthM_\mathfrak{p} \mid \mathfrak{p} \in \alpha \cap Spec(R)\right\}.$$

***Proof*** We set $n = inf\{depthM_\mathfrak{p} \mid \mathfrak{p} \in \alpha \cap Spec(R)\}$, and let $E^\bullet(M)$ be a minimal injective resolution of $M$.

Given $\mathfrak{p} \in \alpha$ a prime ideal, then $n \leq depthM_\mathfrak{p} = inf\{i \mid \mu_i(\mathfrak{p}, M) \neq 0\}$. Hence we obtain

$$\Gamma_\alpha(E^i(M)) = \bigoplus_{\mathfrak{p}\in\alpha} E(R/\mathfrak{p})^{\mu_i(\mathfrak{p},M)} = 0, \tag{B.1}$$

for each integer $i < n$ and also $\Gamma_\alpha(E^n(M)) \neq 0$. It follows that $\mathrm{H}^i_\alpha(M) = 0$ if $i < n$.

It suffices to show that $\mathrm{H}^n_\alpha(M) \neq 0$. We see from equality B.1 that the complex $\Gamma_\alpha(E^\bullet(M))$ starts from its $n$-th term. Thus we have a commutative

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{H}^n_\alpha(M) & \longrightarrow & \Gamma_\alpha(E^n(M)) & \longrightarrow & \Gamma_\alpha(E^{n+1}(M)) \\ & & & \downarrow & & \downarrow \\ E^{n-1}(M) & \xrightarrow{d^{n-1}} & & E^n(M) & \xrightarrow{d^n} & E^{n+1}(M) \end{array}$$

with exact rows. Since $kerd^n = Imd^{n-1} \subseteq E^n(M)$ is an essential extension, it follows that $H^n(M) = \Gamma_\alpha(E^n(M)) \cap kerd^n \neq 0$. ■

**Corollary B.14** Let $M$ be a finitely generated module over a local ring $R$ with maximal ideal $\mathfrak{m}$. Then the following statements are equivalent:
$(i)$ $M$ is $\alpha$-torsion $R$-module;
$(ii)$ $\mathrm{H}^i_\alpha(M) = 0$ for all integers $i > 0$.

***Proof*** (i)⇒(ii) Follows from Corollary B.10(i).

(ii)⇒(i) Let us denote $N = M/\Gamma_\alpha(M)$. We will show that $N = 0$. Suppose $N \neq 0$. From Corollary B.10(iii) and (iv), we have $\Gamma_\alpha(N) = 0$ and $\mathrm{H}^i_\alpha(N) \cong \mathrm{H}^i_\alpha(M) = 0$ if $i > 0$. On the other hand, since $\mathfrak{m} \in \alpha$, we get the inequality

$$inf\left\{depthN_\mathfrak{p} \mid \mathfrak{p} \in \alpha\right\} \leq depthN_\mathfrak{m} = depthN < \infty$$

holds. Thus $\mathrm{H}^i_\alpha(N) \neq 0$ for an integer $i \leq depthN$ by Theorem B.13. This is a contradiction. Therefore $N = 0$. ■

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