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Doutorado em Matemática

On some class of planar semilinear elliptic problems

por

Jonison Lucas dos Santos Carvalho

João Pessoa - PB
Julho de 2021

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sob orientação do

Prof. Dr. Everaldo Souto de Medeiros

Tese apresentada ao Corpo Docente do Programa de Pós-Graduação em Matemática UFPB como requisito parcial para obtenção do título de Doutor em Matemática.

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ATA DE DEFESA DE DOUTORADO JUNTO AO PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA DA UNIVERSIDADE FEDERAL DA PARAÍBA, REALIZADA NO DIA 30 DE JULHO DE 2021.

Ao trigésimo dia de julho de dois mil e vinte e um, às 10:00 horas, por meio da plataforma virtual *Google Meet*, através do link: http://meet.google.com/kqo-yemn-tzx, em conformidade com a portaria PRPG nº 02/2021, que dispõe sobre a regulamentação, em caráter excepcional e temporário, das atividades da pós-graduação da Universidade Federal da Paraíba durante o período de isolamento social imposto pela pandemia de coronavírus (COVID-19), foi aberta a sessão pública de Defesa de tese intitulada "**Sobre a existência de solução para algumas classes de problemas elípticos semilineares no plano**", do aluno **Jonison Lucas dos Santos Carvalho**, que havia cumprido, anteriormente, todos os requisitos para a obtenção do grau de Doutor em Matemática, sob a orientação do Prof. Dr. Everaldo Souto de Medeiros. A Banca Examinadora, indicada pelo Colegiado do Programa de Pós-Graduação em Matemática, foi composta pelos professores: Everaldo Souto de Medeiros (Orientador), João Marcos Bezerra Do Ó (UFPB), Disson Soares Dos Prazeres (UFS), Emerson Alves Mendonça de Abreu (UFMG), Giovany de Jesus Malcher Figueiredo (UnB) e Lucas Catão de Freitas Ferreira (UNICAMP). O professor Everaldo Souto de Medeiros, em virtude da sua condição de orientador, presidiu os trabalhos e, depois das formalidades de apresentação, convidou o aluno a discorrer sobre o conteúdo da tese. Concluída a explanação, o candidato foi arguido pela banca examinadora que, em seguida, sem a presença do aluno, finalizando os trabalhos, reuniu-se para deliberar tendo concedido a menção: APROVADO. E, para constar, foi lavrada a presente ata que será assinada pelos membros da Banca Examinadora.

João Pessoa, 30 de julho de 2021.

Everaldo Souto de Medeiros ______________________

João Marcos Bezerra Do Ó ______________________

Disson Soares Dos Prazeres ______________________

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Abstract

In this thesis, we address the existence of solutions for some class of planar semilinear elliptic problems involving subcritical and critical growth. To do this, we establish some new weighted Trudinger-Moser type inequalities in weighted Sobolev spaces including the radial and nonradial cases.

Keywords: Trudinger-Moser type inequality, Weighted Sobolev space, Weighted Lebesgue space, Semilinear elliptic problem.

Resumo

Nesta tese, abordamos a existência de soluções para alguma classe de problemas elípticos semilineares planos envolvendo crescimento subcrítico e crítico. Para fazer isso, estabelecemos algumas desigualdades do tipo Trudinger-Moser ponderadas em espaços de Sobolev ponderados, incluindo os casos radiais e não radiais.

Palavras-chave: Desigualdade do tipo Trudinger-Moser, Espaço de Sobolev com peso, Espaço de Lebesgue com peso, Problema elíptico semilinear.

Contents

Notation

We select here some notations used throughout the work.

Spaces

- $L^p(\mathbb{R}^2;\omega) := \left\{u : \mathbb{R}^2 \to \mathbb{R} \text{ measurable} : \|u\|_{L^p(\mathbb{R}^2;\omega)} := \left(\int_{\mathbb{R}^2} \omega(x)|u|^p dx\right)^{1/p} < +\infty\right\}$;
- $L^\infty(\Omega) = \{u : \Omega \to \mathbb{R} :\ u \text{ is bounded and mensurable}\}$;
- $\mathbb{R}^2$ denotes the usual euclidean space with the norm $|x| = \left(\sum_{j=1}^2 x_j^2\right)^{1/2}$, $x \in \mathbb{R}^2$;
- $H^1(\mathbb{R}^2)$ denotes the usual Sobolev space;
- $C^0(\Omega)$ denotes the space of continuous real functions in $\Omega \subset \mathbb{R}^2$;
- For an integer $k \geq 1$, $C^k(\Omega)$ denotes the space of k-times continuously differentiable real functions in $\Omega \subset \mathbb{R}^2$;
- $C^\infty(\Omega) = \bigcap\limits_k C^k(\Omega)$;
- $C_0^\infty(\Omega)$ denotes the space of infinitely differentiable real functions whose support is compact in $\Omega \subset \mathbb{R}^2$;
- E' denotes the topological dual of the Banach space E;

Norms

- For $1 \leq p < +\infty$, the standard norm in $L^p(\mathbb{R}^2;\omega)$ is denoted by $\|\cdot\|_{L^p(\mathbb{R}^2;\omega)}$;

Other Notation

- $|A|$ denotes the Lebesgue measure of a set $A \subset \mathbb{R}^2$;
- suppφ denotes the support of function φ;
- C, C_0, C_1, C_2, C_3, $\ldots$ denote positive constants possibly different;

- $C(s)$ denotes constant which depends of s;
- $o_n(1)$ denotes a sequence which converges to 0 as $n \to \infty$;
- $\rightharpoonup$ denotes weak convergence in a normed space;
- $\to$ denotes strong convergence in a normed space;
- $\hookrightarrow$ denotes continuous embedding;
- For $R > 0$ and $y \in \mathbb{R}^2$, we denote by $B_R(x)$ the open ball $\{x \in \mathbb{R}^2 : |y - x| < R\}$. If $x = 0$, we write only B_R;
- *Weight functions* are functions measurable and positive almost everywhere (a.e.)

Introduction

The purpose of this thesis is to address the existence of solutions for some class of semilinear elliptic problems in the euclidean space $\mathbb{R}^2$. Precisely, we consider four class of problems that we will be describe next.

Firstly, we study semilinear elliptic equations of the form

$$-\Delta u + V(|x|)u = \lambda Q(|x|)f(u), \quad x \in \mathbb{R}^2, \tag{1}$$

where $\lambda > 0$ is a parameter, $V, Q : (0, \infty) \to \mathbb{R}$ are radial weights, and the nonlinearity f has exponential critical growth, which will be explained below.

After, we will look for solutions to the Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(|x|)u + \eta\phi K(|x|)u = \lambda Q(|x|)f(u), & x \in \mathbb{R}^2, \\ \Delta\phi = K(|x|)u^2, & x \in \mathbb{R}^2, \end{cases} \tag{2}$$

where $\eta, \lambda > 0$, the potential $K : (0, \infty) \to \mathbb{R}$ is radial function. Moreover, we consider a Hartree-Fock type system in presence of a Coulomb interacting term, driven by a suitable parameter $\beta \geq 0$, namely

$$\begin{cases} -\Delta u + (1+\phi)u = |u|^{2p-2}u + \beta|v|^p|u|^{p-2}u, & \text{in } \mathbb{R}^2, \\ -\Delta v + (1+\phi)v = |v|^{2p-2}v + \beta|u|^p|v|^{p-2}v, & \text{in } \mathbb{R}^2, \\ \Delta\phi = 2\pi(u^2+v^2), & \text{in } \mathbb{R}^2, \end{cases} \tag{3}$$

with $2 \leq p < \infty$.

Finally, we investigate the existence of solutions for the following Choquard type equation

$$-\Delta u + V(x)u = \frac{1}{2\pi}\left[\log\frac{1}{|x|} * \Big(K(x)F(u)\Big)\right] Q(x)f(u), \quad x \in \mathbb{R}^2, \tag{4}$$

where V, K, Q are nonradial potentials.

As is well known, in bounded planar domains $\Omega \subset \mathbb{R}^2$, the Sobolev embedding theorem assures that the embedding

$$H_0^1(\Omega) \hookrightarrow L^q(\Omega) \quad \text{for any} \quad 1 \leq q < \infty,$$

is continuous, and it does not holds when $q = \infty$.

In view of this feature, Yudovic [85], Pohozaev [67] and Trudinger [81], states an alternative

Sobolev inequality. Precisely, they proved that there exists a positive constant $\alpha > 0$ such that the embedding

$$H_0^1(\Omega) \hookrightarrow L_\phi(\Omega),$$

where $L_\phi(\Omega)$ is the Orlicz space determined by the Young function $\phi(s) = e^{\alpha s^2} - 1$. Later, Moser in [63] sharpened this result by finding the best constant α in the embedding above. More precisely, he proved that there exists a constant $C > 0$ such that

$$\sup_{\|\nabla u\|_{L^2(\Omega)} \le 1} \int_\Omega e^{\alpha u^2} dx \le C|\Omega|,$$

for any $0 < \alpha \le 4\pi$. Moreover, the constant 4π is sharp in the sense that if $\alpha > 4\pi$, then the supremum above will become infinity.

In the whole space $\mathbb{R}^2$, by using using Schwarz symmetrization, D. Cao in [27] proved the following version of Trudinger-Moser inequality in the space $H^1(\mathbb{R}^2)$: There exists a constant $C = C(M, \alpha) > 0$ such that

$$\sup_{\left\{\|\nabla u\|^2_{L^2(\mathbb{R}^2)} \le m < 1,\ \|u\|_{L^2(\mathbb{R}^2)} < M\right\}} \int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1\right) dx \le C(m, M),$$

for any $0 < \alpha \le 4\pi$. Later B. Ruf proved in [73] that 4π is a critical exponent. See also [1, 41, 82] for an equivalent version.

Therefore, to study problems (1), (2), and (4) we need to establish some weighted Trudinger-Moser type inequalities of the form

$$\sup_{\|u\|_E \le 1} \int_{\mathbb{R}^2} \omega(x) \Phi_{\alpha, j_0}(u) dx < \infty,$$

considering radial and nonradial positive weights functions $\omega \in L^1_{\text{loc}}(\mathbb{R}^2)$, where E is a Sobolev space that will be defined later and $\Phi_{\alpha,j_0}(s)$ is a Young function of the form

$$\Phi_{\alpha,j_0}(s) := e^{\alpha s^2} - \sum_{j=0}^{j_0-1} \frac{\alpha^j}{j!} s^{2j}, \quad s \in \mathbb{R},$$

with $\alpha > 0$, $j_0 := \lceil \gamma/2 \rceil = \inf\{j \in \mathbb{N} : j \ge \gamma/2\}$, $\gamma := \max\{2, 2(2 + 2b - a)/(a + 2)\}$, and $a, b \in \mathbb{R}$.

Next we will describe the content of this thesis which is written in five chapters as follows.

In **Chapter 1**, we will focus on the problem (1). In this case, we consider the Hilbert space

$$E := \left\{ u \in L^2_{\text{loc}}(\mathbb{R}^2) : |\nabla u| \in L^2(\mathbb{R}^2) \text{ and } \int_{\mathbb{R}^2} V(|x|) u^2 dx < \infty \right\},$$

endowed with inner product

$$\langle u, v \rangle_E := \int_{\mathbb{R}^2} [\nabla u \nabla v + V(|x|)uv]\, dx.$$

On the other hand, we will assume the following assumptions on the radial functions V and Q:

(V) $V : (0, \infty) \to \mathbb{R}$ is continuous, $V > 0$ and there are $a_0, a > -2$ such that

$$\limsup_{r \to 0^+} V(r) r^{-a_0} < \infty \quad \text{and} \quad \liminf_{r \to \infty} V(r) r^{-a} > 0.$$

(Q) $Q : (0, \infty) \to \mathbb{R}$ is continuous, $Q > 0$ and there are $b_0, b > -2$ such that

$$0 < D_0 := \liminf_{r \to 0^+} Q(r) r^{-b_0} \leq \limsup_{r \to 0^+} Q(r) r^{-b_0} < \infty, \quad \limsup_{r \to \infty} Q(r) r^{-b} < \infty.$$

Moreover, we assume that $f(s)$ is continuous and has exponential critical growth at infinity, i.e., there exists $\alpha_0 > 0$ such that

$$\lim_{|s| \to \infty} \frac{|f(s)|}{e^{\alpha s^2}} = \begin{cases} 0 & \text{if} \quad \alpha > \alpha_0, \\ \infty & \text{if} \quad \alpha < \alpha_0. \end{cases} \tag{5}$$

Also, we consider the following assumptions:

(f_1) $f(s) = o(|s|^{\gamma-1})$ as $s \to 0$, where

$$\gamma := \max\{2, 2(2+2b-a)/(a+2)\} = \begin{cases} 2 & \text{if} \quad -2 < b \leq a, \\ 2(2+2b-a)/(a+2) & \text{if} \quad -2 < a < b; \end{cases}$$

(f_2) there exists $\theta > \gamma$ such that $0 < \theta F(s) \leq f(s)s$ for all $s \neq 0$, where $F(s) := \int_0^s f(t)dt$;

(f_3) the following limit holds: $\lim_{|s| \to \infty} f(s)s/F(s) = \infty$.

Using the condition (Q), there exists $r_0 > 0$ such that $Q(r) \geq D_0 r^{b_0}/2$ for all $0 < r \leq r_0$. For this fixed r_0 we will assume the following assumption:

(f_4) there exists $\beta_0 > 2(b_0+2)^2/D_0 \alpha_0 r_0^{b_0+2}$ such that $\liminf_{|s| \to \infty} f(s) s e^{-\alpha_0 s^2} \geq \beta_0$.

Our existence result for problem (1) can be stated as follows:

Theorem 0.0.1. *Assume that* (5), $(f_1) - (f_4)$, (V) *and* (Q) *hold. Then, for each* $\lambda > 0$, *the problem* (1) *possesses a nonzero weak solution* $u_\lambda \in E$ *satisfying,*

$$0 \leq u_\lambda(x) \leq c_0 \exp\left(-c_1 |x|^{(a+2)/4}\right), \quad x \in \mathbb{R}^2,$$

for some constants $c_0, c_1 > 0$ *depending only on* λ.

In addition if we assume the local hypothesis

(f_5) there exists $\nu > \gamma$ such that $\liminf_{s\to 0} F(s)|s|^{-\nu} > 0$,

our multiplicity result is stated as follows

Theorem 0.0.2. *Assume that* (5), $(f_1)-(f_5)$, (V) *and* (Q) *hold. If in addition, f is odd, there exists a sequence* $(\lambda_k) \subset \mathbb{R}_+$ *with* $\lambda_k \to \infty$ *such that for all* $\lambda > \lambda_k$, (1) *has at least k pairs of weak solutions in E.*

In **Chapter 2**, we investigate the existence of solutions to the system (2). To this, we will assume conditions $(V), (Q)$ and the potential K satisfies

(K) $K : (0,\infty) \to \mathbb{R}$ is continuous, $K > 0$ and there are $l_0 > -3/2$, $-2 < l < \min\{a, (a-1)/2\}$ such that

$$\limsup_{r\to 0^+} \frac{K(r)}{r^{l_0}} < \infty \quad \text{and} \quad \limsup_{r\to\infty} \frac{K(r)}{r^{l}} < \infty.$$

We suppose that f satisfy (5) and (f_1). Furthermore, we also assume the following assumptions on f:

$(\widetilde{f_2})$ there exists $\theta > \max\{\gamma, 4\}$ such that $0 < \theta F(s) \leq f(s)s$ for all $s \geq 0$;

$(\widetilde{f_3})$ there exists $r > \max\{\gamma, 4\}$ such that $\liminf_{s\to 0^+} F(s)/s^r > 0$;

$(\widetilde{f_4})$ the function $s \mapsto f(s)/s^3$ is increasing for $s > 0$.

In the past few years, many authors have considered the 3-dimensional case assuming different conditions on the potentials and the nonlinearity f. We could cite [14, 33, 37, 48, 53, 55, 74] and references therein. A common aspect in most of the works is the variational approach. It essentially consists in impose some regularity condition on K, use Lax-Milgram Theorem to solve the second equation and obtain ϕ as the convolution $\phi = \Gamma_3 * (Ku^2)$, where Γ_3 is the fundamental solution of the Laplacian in $\mathbb{R}^3$, namely $\Gamma_3(x) = (-1/4\pi)|x|^{-1}$.

For the planar case, we can use the same idea to conclude that

$$\phi_u(x) := \frac{1}{2\pi}\int_{\mathbb{R}^2} \log(|x-y|)K(y)u^2(y)\mathrm{d}y,$$

where we have used that the fundamental solution in $\mathbb{R}^2$ is given by $\Gamma_2(x) := (1/2\pi)\log|x|$. Hence, we are leading to consider the nonlocal equation

$$-\Delta u + V(|x|)u + \frac{\eta}{2\pi}[\log|\cdot| * (Ku^2)](x)K(|x|)u = Q(|x|)f(u), \quad x \in \mathbb{R}^2. \tag{6}$$

The main existence result for problem (6) can be stated as follows:

Theorem 0.0.3. *Assume that* $(V),(K),(Q)$, (1.1), (f_1), *and* $(\widetilde{f}_2)-(\widetilde{f}_4)$ *hold. Then, equation* (6) *possesses a nonzero weak solution* $u_\lambda \in W$ *with minimal energy (or ground state solution) if*

$$\lambda \geq \bar{\lambda} := \max\left\{ \frac{\lambda_0}{\theta C_0}, \left[\frac{4\alpha_0 \|Q\|_{L^1(B_{1/2})} \lambda_0^{\frac{r}{r-2}}}{\alpha_2} \left(\left(\frac{2}{r}\right)^{\frac{2}{r-2}} - \left(\frac{2}{r}\right)^{\frac{r}{r-2}} \right) \right]^{\frac{r-2}{2}} \right\},$$

where

$$\lambda_0 := \frac{4\pi + \|V\|_{L^1(B_1)} + \log 3 \|K\|^2_{L^1(B_1)}}{\|Q\|_{L^1(B_{1/2})}} \quad \text{and} \quad \alpha_2 := 4\pi(1 + b_0/2).$$

As a byproduct of Theorem 0.0.3, we prove the existence of solutions to the system (2).

Theorem 0.0.4. *Assume the conditions of Theorem 0.0.3 and let* u_λ *be the solution obtained in Theorem 0.0.3. In addition, suppose that* $K \in C^\sigma_{\text{loc}}(\mathbb{R}^2)$ *for some* $\sigma \in (0,1)$. *Then, the pair* $(u_\lambda, \phi_{u_\lambda})$ *is a weak solution of system* (2), *where* $\phi_{u_\lambda} = \Gamma_2 * K u_\lambda^2$.

In **Chapter 3**, we will show the existence of semitrivial and vectorial solutions to system (3) depending on the parameters involved. Defining the logarithmic potential

$$\phi_{u,v}(x) := \int_{\mathbb{R}^2} \log(|x-y|) \left(u^2(y) + v^2(y)\right) dy,$$

we can to consider the following auxiliary system with the nonlocal term $\phi_{u,v}$

$$\begin{cases} -\Delta u + (1+\phi_{u,v})u = |u|^{2p-2}u + \beta |v|^p |u|^{p-2} u, & \text{in } \mathbb{R}^2, \\ -\Delta v + (1+\phi_{u,v})v = |v|^{2p-2}v + \beta |u|^p |v|^{p-2} v, & \text{in } \mathbb{R}^2. \end{cases} \tag{7}$$

In this context we now formulate our main result, concerning systems (7) and (3).

Theorem 0.0.5. *Assume that* $2 \leq p < \infty$. *Then, for any* $\beta \geq 0$ *the coupled system* (7) *possesses a least energy solution* $(u,v) \in \widetilde{W}_{\text{rad}} \times \widetilde{W}_{\text{rad}}$, *where* $\widetilde{W}_{\text{rad}}$ *is an appropriated subspace of* $H^1(\mathbb{R}^2)$, *with* $u, v \geq 0$ *satisfying the following statements:*

- *(i) for every* $\beta > 2^{p-1} - 1$ *the pair* (u,v) *is a vector solution, i.e.,* $u \neq 0$, $v \neq 0$ *and in this case* $u, v > 0$;
- *(ii) for* $0 \leq \beta < 2^{p-1} - 1$ *the least energy solution is semi-trivial, i.e.,* $u = 0$ *or* $v = 0$.

Furthermore, the triple $(u, v, \phi_{u,v})$ *is a weak solution of system* (3).

In **Chapter 4**, we will prove the embeddings results, involving the potential V, K and Q, such that:

(VKQ) $V, K, Q \in C(\mathbb{R}^2)$ and there exist $\widetilde{\gamma} \leq 2 < \widetilde{\beta}$ and positive constants a_0, b_0 such that

$$\frac{a_0}{(1+|x|)^{\widetilde{\gamma}}} \leq V(x), \quad 0 < K(x), Q(x) \leq \frac{b_0}{(1+|x|)^{\widetilde{\beta}}}, \quad \text{for all } x \in \mathbb{R}^2.$$

Considering the auxiliary weight function $\widetilde{w} \in L^1_{\text{loc}}(\mathbb{R}^2)$, satisfying

$$\widetilde{\omega}(x) \leq C_0 \cdot \begin{cases} 1 & \text{if} \quad |x| \leq 1 \\ \log(1+|x|)Q(x) & \text{if} \quad |x| > 1, \end{cases}$$

for some $C_0 > 0$, we have the following result.

Proposition 0.0.6. *Assume (VKQ). Then, for any $2 \leq p < \infty$, the weighted Sobolev embedding $E \hookrightarrow L^p(\mathbb{R}^2; \widetilde{\omega})$ is continuous and compact.*

Thus, it is natural to study embedding from E into Orlicz space. To this end, we will prove a version of Trudinger-Moser type inequality, case nonradial.

Theorem 0.0.7. *For any $\alpha > 0$ and $u \in E$, the function $\widetilde{\omega}(\cdot)\Phi_{\alpha,1}(u)$ belongs to $L^1(\mathbb{R}^2)$. Moreover, there exists $\alpha_* \in (0, 4\pi)$ such that*

$$\sup_{u \in E, \, \|u\|_E \leq 1} \int_{\mathbb{R}^2} \widetilde{\omega}(x)\Phi_{\alpha,1}(u)dx < \infty,$$

for any $0 < \alpha \leq \alpha_*$.

This chapter ends with the study of system 2, case nonradial, assuming that f satisfies the following assumptions:

($\mathfrak{f}_1$) $f(s) = o(|s|)$ as $s \to 0$;

($\mathfrak{f}_2$) there exists $\theta > 4$ such that $0 < \theta F(s) \leq f(s)s$ for all $s \neq 0$;

($\mathfrak{f}_3$) there exists $\zeta > 0$ such that $F(s) \geq \zeta |s|^4$, for all $s \in \mathbb{R}$;

($\mathfrak{f}_4$) the function $s \mapsto f(s)/|s|^3$ is increasing in $|s| > 0$.

The main existence result for problem (6) can be stated as follows:

Theorem 0.0.8. *Suppose that (VKQ), (5), and $(\mathfrak{f}_1) - (\mathfrak{f}_4)$ hold. Then, there exists $\alpha_* \in (0, 4\pi)$ such that problem (6) has a nonzero small energy solution provided*

$$\zeta > S_4^2(Q) \max \left\{ \frac{1}{S_2(K)}, \frac{\alpha_0}{2\alpha_*} \right\}.$$

As consequence of Theorem 0.0.8, we can give a contribution concerning the existence of solutions to the system (2), namely

Theorem 0.0.9. *Suppose the same hypotheses of Theorem 0.0.8 and let $u \in W$ be the solution obtained in that theorem. Then, the pair (u, ϕ_u) is a weak solution of system (2), where $\phi_u = \Gamma_2 * (Ku^2)$.*

Finally, **Chapter 5** contains our study of the problem (4), using the embeddings results and Trudinger-Moser type inequality, obtained in the Chapter 4. To this end, we shall consider that the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is differentiable, $f(s) = 0$ for all $s \leq 0$ and $f(s) > 0$ for all $s > 0$, (1.1) holds, and satisfies the following conditions:

($\mathfrak{f}_1$) $f(s) = o(|s|)$ as $s \to 0$;

($\widetilde{\mathfrak{f}_2}$) there exists $\delta \in (0,1)$ such that

$$\delta \leq \frac{F(s)f'(s)}{f^2(s)}, \quad \forall\, s > 0;$$

($\widetilde{\mathfrak{f}_3}$) there exist $\xi > 0$ and $\kappa > 2$ such that $F(s) \geq \xi s^\kappa$, for all $s \in (0,1]$.

Under this hypotheses, our main result can be stated as follows.

Theorem 0.0.10. *Suppose that* (VKQ), (5), ($\mathfrak{f}_1$), *and* ($\widetilde{\mathfrak{f}_2}$) *hold. Then there exists* $\xi^* > 0$ *such that if* ($\widetilde{\mathfrak{f}_3}$) *holds with* $\xi \geq \xi^*$, (4) *has a nontrivial weak solution which is nonnegative.*

Chapter 1

Nonlinear Schrödinger equations involving exponential critical growth

This chapter is devoted to present the results of paper [28], where we proved a Trudinger-Moser type inequality in the radial case and as a consequence we established some results of existence and multiplicity to the semilinear Schröringer equation.

1.1 Main results

In this chapter, we are concerned with semilinear elliptic equations of the form

$$-\Delta u + V(|x|)u = \lambda Q(|x|)f(u), \quad x \in \mathbb{R}^2, \tag{$\mathcal{P}$}$$

where $\lambda > 0$ is a parameter, $V, Q : (0, \infty) \to \mathbb{R}$ are radial weights, which can by singular at the origin, unbounded or decaying at infinity and the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is continuous and has exponential critical growth at infinity, i.e., there exists $\alpha_0 > 0$ such that

$$\lim_{|s|\to\infty} \frac{|f(s)|}{e^{\alpha s^2}} = \begin{cases} 0 & \text{if} \quad \alpha > \alpha_0, \\ \infty & \text{if} \quad \alpha < \alpha_0. \end{cases} \tag{1.1}$$

The study of the stationary equation ($\mathcal{P}$) is motivated by the study of standing wave solutions of the nonlinear Schrödinger equation, see e.g., [24, 68, 72, 77] and references therein. Our starting point here are the works [79, 80], where the authors proved some weighted Sobolev embedding theorems, and there is a growing recent interest in applications of these results in the study of partial differential equations, see for example [3, 17, 26, 59, 75].

We will assume the following assumptions on the radial functions V and Q:

(V) $V : (0, \infty) \to \mathbb{R}$ is continuous, $V > 0$ and there are $a_0, a > -2$ such that

$$\limsup_{r\to 0^+} V(r)r^{-a_0} < \infty \quad \text{and} \quad \liminf_{r\to\infty} V(r)r^{-a} > 0.$$

(Q) $Q : (0, \infty) \to \mathbb{R}$ is continuous, $Q > 0$ and there are $b_0, b > -2$ such that

$$0 < D_0 := \liminf_{r\to 0^+} Q(r)r^{-b_0} \le \limsup_{r\to 0^+} Q(r)r^{-b_0} < \infty, \quad \limsup_{r\to\infty} Q(r)r^{-b} < \infty.$$

We also comment here that Ambrosetti-Felli-Malchiodi [11] and do Ó-Sani-Zhang [42] studied equation $(\mathcal{P})$ by assuming that V and Q satisfy the following assumption: there are $A_1, A_2, A_3 > 0$ such that

$$A_1(1+|x|^\alpha)^{-1} \le V(x) \le A_2 \qquad \text{and} \qquad 0 < Q(x) \le A_3(1+|x|^\beta)^{-1},$$

where $0 < \alpha < 2$ and $\beta > 0$. Thus, when V and Q are radial we are considering a more general class of potentials than the one in [11, 42].

In the papers [79, 80], the authors studied the existence of solutions for the equation $(\mathcal{P})$ when $f(u) = |u|^{p-2}u$ with $2 < p < 2N/(N-2)$ if $N \ge 3$ and $2 < p < \infty$ if $N = 2$. Our main purpose is to obtain solutions when the nonlinearity f has exponential critical growth. Precisely, besides the critical growth condition (1.1), we also assume the following conditions:

(f_1) $f(s) = o(|s|^{\gamma-1})$ as $s \to 0$, where

$$\gamma := \max\{2, 2(2+2b-a)/(a+2)\} = \begin{cases} 2 & \text{if} \quad -2 < b \le a, \\ 2(2+2b-a)/(a+2) & \text{if} \quad -2 < a < b; \end{cases}$$

(f_2) there exists $\theta > \gamma$ such that $0 < \theta F(s) \le f(s)s$ for all $s \ne 0$, where $F(s) := \int_0^s f(t)dt$;

(f_3) the following limit holds: $\lim_{|s|\to\infty} f(s)s/F(s) = \infty$.

In view of hypothesis (Q), there exists $r_0 > 0$ such that $Q(r) \ge D_0 r^{b_0}/2$ for all $0 < r \le r_0$. For this fixed r_0 we will assume the following assumption:

(f_4) there exists $\beta_0 > 2(b_0+2)^2/D_0\alpha_0 r_0^{b_0+2}$ such that $\liminf_{|s|\to\infty} f(s)se^{-\alpha_0 s^2} \ge \beta_0$.

It is worthwhile to mention that similar issues have been addressed in the paper [3], where the authors proved the existence of positive solutions for equation $(\mathcal{P})$ by assuming similar hypotheses (V) and (Q) with a, b in the range $-2 < b < (a-2)/2$ and f with exponential critical growth. To improve this condition, inspired by the paper [69], we used a change of variables to obtain a sharp weighted Trudinger-Moser type inequality. We also mention that our hypotheses on f include the ones in [3].

In order to present our main results, we need some notations. As usual, we denote by $C_0^\infty(\mathbb{R}^2)$ the space of infinitely differentiable functions with compact support. Moreover, given a positive function $\omega \in L^1_{\text{loc}}(\mathbb{R}^2)$ and $1 \le p < \infty$, we define the weighted Lebesgue space

$$L^p(\mathbb{R}^2;\omega) := \left\{ u : \mathbb{R}^2 \to \mathbb{R} \ \text{ measurable} : \|u\|_{L^p(\mathbb{R}^2;\omega)} := \left(\int_{\mathbb{R}^2} \omega(x)|u|^p dx \right)^{1/p} < \infty \right\}.$$

As in the paper [42], we consider the space

$$E := \left\{ u \in L^2_{\text{loc}}(\mathbb{R}^2) : |\nabla u| \in L^2(\mathbb{R}^2) \text{ and } \int_{\mathbb{R}^2} V(|x|)u^2 dx < \infty \right\},$$

which is a Hilbert space (see [6]) when endowed with inner product

$$\langle u, v \rangle_E := \int_{\mathbb{R}^2} [\nabla u \nabla v + V(|x|)uv] \, dx,$$

and its correspondent norm $\|u\|_E := \langle u, u \rangle_E^{1/2}$.

Using standard arguments one can prove that $C_0^\infty(\mathbb{R}^2)$ is dense in E. Furthermore, the subspace

$$E_{\text{rad}} := \Big\{ u \in E : u \text{ is radial} \Big\}$$

is closed in E and thus it is a Hilbert space itself. In this context, by a *weak solution* of $(\mathcal{P})$ we understand a function $u \in E$ such that

$$\int_{\mathbb{R}^2} [\nabla u \nabla \varphi + V(|x|)u\varphi] \, dx = \lambda \int_{\mathbb{R}^2} Q(|x|) f(u) \varphi dx, \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^2).$$

Remark 1.1.1. *We observe that condition $a, b > -2$ seems to be necessary to the existence of weak solutions to equation $(\mathcal{P})$. To illustrate, let $u \in E$ be a solution of the model equation*

$$-\Delta u + |x|^a u = |x|^b f(u), \quad x \in \mathbb{R}^2, \tag{1.2}$$

and for $\mu > 0$, consider $u_\mu(x) = u(\mu x)$ the continuous path in E converging to u, as $\mu \to 1$. Since u is a critical point of the energy functional

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} |x|^a u^2 dx - \int_{\mathbb{R}^2} |x|^b F(u) dx,$$

it should satisfies $\frac{dJ}{d\mu}(u_\mu)\Big|_{\mu=1} = 0$, and taking into account that $\|\nabla u_\mu\|_{L^2(\mathbb{R}^2)}$ is constant, by performing a change of variables we get

$$\frac{(a+2)}{2} \int_{\mathbb{R}^2} |x|^a u^2 dx = (b+2) \int_{\mathbb{R}^2} |x|^b F(u) dx. \tag{1.3}$$

Therefore, equation (1.2) has no nonzero weak solution if (a, b) belongs to the region $\mathcal{R} = \Big((-\infty, -2] \times (-2, \infty)\Big) \cup \Big((-2, \infty) \times (-\infty, -2]\Big)$. Furthermore, if u is a nonzero weak solution of equation (1.2) we see that

$$\int_{\mathbb{R}^2} |\nabla u|^2 dx + \int_{\mathbb{R}^2} |x|^a u^2 dx = \int_{\mathbb{R}^2} |x|^b f(u) u dx,$$

which combined with (1.3), yields

$$\int_{\mathbb{R}^2} |\nabla u|^2 dx = \int_{\mathbb{R}^2} |x|^b \left(f(u)u - \frac{2(b+2)}{(a+2)} F(u) \right) dx.$$

Consequently, equation (1.2) has no nonzero weak solution if $f(u)u \leq 2(b+2)/(a+2)F(u)$, and hence $\theta > \gamma \geq \gamma_0 := 2(b+2)/(a+2)$ in the hypothesis (f_2) is a necessary condition for the *existence when $a=b$.*

In this setting our first result can be stated as follows.

Theorem 1.1.2. *Assume that* (1.1), $(f_1)-(f_4)$, (V) *and* (Q) *hold. Then, for each* $\lambda>0$, *the problem* $(\mathcal{P})$ *possesses a nonzero weak solution* $u_\lambda \in E_{\mathrm{rad}}$ *satisfying,*

$$0 \le u_\lambda(x) \le c_0 \exp\left(-c_1|x|^{(a+2)/4}\right), \quad x \in \mathbb{R}^2, \tag{1.4}$$

for some constants $c_0, c_1 > 0$ *depending only on* λ.

We quote that conditions (f_3) and (f_4) have already been considered in others works, see for instance [46, 64]. The crucial ingredient to prove Theorem 1.1.2 and Theorem 1.1.3 is a weighted Trudinger-Moser type inequality. This inequality, combined with a suitable estimate of the minimax level, yields compactness of the Palais-Smale sequence.

Our second main result concerns the multiplicity of solutions to equation $(\mathcal{P})$ for large $\lambda>0$. To this purpose, we shall assume in addition the following local hypothesis:

(f_5) there exists $\nu>\gamma$ such that $\liminf_{s\to 0} F(s)|s|^{-\nu}>0$.

Our multiplicity result is stated as follows.

Theorem 1.1.3. *Assume that* (1.1), $(f_1)-(f_5)$, (V) *and* (Q) *hold. If in addition,* f *is odd, there exists a sequence* $(\lambda_k) \subset \mathbb{R}_+$ *with* $\lambda_k \to \infty$ *such that for all* $\lambda > \lambda_k$, $(\mathcal{P})$ *has at least* k *pairs of weak solutions in* E_{rad}.

For instance, one can check that Theorem 1.1.2 and Theorem1.1.3 apply for the model equation

$$-\Delta u + |x|^a u = \lambda |x|^b \left(|u|^{\nu-2}u + |u|^{q-2}u(e^{u^2}-1)\right), \quad x \in \mathbb{R}^2,$$

with $a,b>-2$, $\lambda>0$, $\nu>\gamma$ and $q\ge\gamma$, where γ is defined in (f_1). Thus, this class of equations includes the Henon and singular equations ones, which correspond to $a,b>0$ and $a,b<0$, respectively.

The remainder of the chapter is organized as follows. In Section 1.2, we introduce our variational setting and prove a weighted Trudinger-Moser type inequality. Finally, in Section 1.3, we present the proofs of Theorem 1.1.2 and Theorem 1.1.3.

1.2 A sharp Trudinger-Moser type inequality

In this section we introduce the variational framework and prove a weighted Trudinger-Moser type inequality, which is a key ingredient in the proof of Theorem 1.1.2 and Theorem 1.1.3. For the proof, we borrow some ideas of [27, 69, 73]. We start off by collecting some well-known results that we shall use throughout.

Lemma 1.2.1. [79, Lemma 4] *Assume* (V). *If* $u \in E$ *and* $R>0$, *then* $u \in H^1(B_R)$ *and* $\|u\|_{H^1(B_R)} \le C_R \|u\|_E$ *with* $C_R>0$.

Lemma 1.2.2. [79, Theorem 2] *Assume that* (V) *and* (Q) *hold. Then the embedding* $E_{\mathrm{rad}} \hookrightarrow L^p(\mathbb{R}^2; Q)$ *is continuous for all* $\gamma \leq p < \infty$. *Furthermore, if* $-2 < b < a$ *the embedding is compact for* $\gamma \leq p < \infty$ *and if* $-2 < a \leq b$ *the embedding is compact for all* $\gamma < p < \infty$.

For easy reference, it follows from assumptions (V) and (Q) that for every $0 < R_0 < R_1$ there are positive constants C_0, C_1, C_3, C_4 such that

$$\begin{cases} V(|x|) \leq C_0|x|^{a_0}, \quad C_1|x|^{b_0} \leq Q(|x|) \leq C_2|x|^{b_0} & \text{if} \quad 0 < |x| < R_0 \\ C_3|x|^a \leq V(|x|), \quad Q(|x|) \leq C_4|x|^b & \text{if} \quad |x| > R_1. \end{cases} \tag{1.5}$$

In view of Lemma 1.2.2, it is natural to look for a weighted Trudinger-Moser inequality on the space E_{rad} determined by the Young function

$$\Phi_{\alpha,j_0}(s) := e^{\alpha s^2} - \sum_{j=0}^{j_0-1} \frac{\alpha^j}{j!} s^{2j}, \quad s \in \mathbb{R}, \tag{1.6}$$

where $\alpha > 0$ and $j_0 := [\gamma/2] = \inf\{j \in \mathbb{N} : j \geq \gamma/2\}$. For this purpose, the next lemma plays an important role in the proof of the optimal exponent of our weighted Trudinger-Moser type inequality.

Lemma 1.2.3. *Assume* (V) *and consider the so-called Moser's sequence (see e.g.,* [63]*) given by*

$$M_n(x,r) = \frac{1}{(2\pi)^{1/2}} \begin{cases} (\log n)^{1/2} & \textit{if} \quad |x| \leq r/n, \\ \dfrac{\log(r/|x|)}{(\log n)^{1/2}} & \textit{if} \quad r/n \leq |x| \leq r, \\ 0 \quad \textit{if} \quad |x| \geq r. \end{cases}$$

Then $\|M_n\|_E^2 \leq 1 + o_n(1)$, *where* $o_n(1)$ *denotes a quantity which goes to zero as* $n \to \infty$.

Proof. First, one can easily check that $\|\nabla M_n\|_{L^2(\mathbb{R}^2)} = 1$. On the other hand, we can write

$$\int_{\mathbb{R}^2} V(|x|)M_n^2 dx = \int_{B_{r/n}} V(|x|)M_n^2 dx + \int_{B_r \setminus B_{r/n}} V(|x|)M_n^2 dx.$$

By (1.5), with $R_0 = r$ we get

$$\int_{B_{r/n}} V(|x|)M_n^2 dx \leq C_0 \log n \int_0^{r/n} s^{a_0+1} ds = \frac{C_0 \log n}{(a_0+2)} \left(\frac{r}{n}\right)^{a_0+2} = o_n(1).$$

Considering the change of variables $t = \log(r/s)$ we get

$$\int_{B_r \setminus B_{r/n}} V(|x|)M_n^2 dx \leq \frac{C_0}{\log n} \int_{r/n}^r \log^2(r/s) s^{a_0+1} ds = \frac{C_0 r^{a_0+2}}{\log n} \int_0^{\log n} t^2 e^{-(a_0+2)t} dt,$$

and integrating by parts twice, we obtain

$$\int_0^{\log n} t^2 e^{-(a_0+2)t}dt \;=\; \left(-\frac{\log^2 n}{(a_0+2)n^{a_0+2}} - \frac{2\log n}{(a_0+2)^2 n^{a_0+2}} - \frac{2}{(a_0+2)^3 n^{a_0+2}} + \frac{2}{(a_0+2)^3}\right),$$

which implies the desired result. □

Now we prove a weighted Trudinger-Moser type inequality in balls.

Lemma 1.2.4. *Let $R>0$ be fixed and assume that (V) and (Q) hold. Then for all $\alpha>0$ and $u\in E_{\mathrm{rad}}$, it holds that $Q(|x|)e^{\alpha u^2}\in L^1(B_R)$. Moreover,*

$$L(\alpha,V,Q,R) := \sup_{\|u\|_E\le 1}\int_{B_R} Q(|x|)e^{\alpha u^2}dx<\infty,$$

if and only if $0<\alpha\le\alpha_2 := 4\pi(1+b_0/2)$.

Proof. Let $\alpha>0$ and $R_0>0$ to be chosen later. We shall split the proof into two cases.
Case 1: Assume $b_0\le a_0$. For each $u\in E_{\mathrm{rad}}$ and $a_0>-2$, inspired by the paper [69] we consider the function

$$w(r) := \left(\frac{a_0+2}{2}\right)^{1/2} u(H(r)), \quad \text{for all} \quad r\ge 0,$$

where $H(r)=\left(\frac{a_0+2}{2}\right)^{2/(a_0+2)} r^{2/(a_0+2)}$. Carrying out a straightforward computation one has

$$\int_{B_{R_0}}|\nabla w|^2dx=\int_{B_{H(R_0)}}|\nabla u|^2dx \quad\text{and}\quad \int_{B_{R_0}}w^2dx=\int_{B_{H(R_0)}}|x|^{a_0}u^2dx. \tag{1.7}$$

Moreover, there exists $C_1=C_1(a_0,b_0)>0$ such that

$$\int_{B_{R_0}}|x|^{\delta}e^{(\frac{2\alpha}{a_0+2})w^2}dx=C_1\int_{B_{H(R_0)}}|x|^{b_0}e^{\alpha u^2}dx, \tag{1.8}$$

where $\delta=-2(a_0-b_0)/(a_0+2)\in(-2,0]$ since $b_0\le a_0$. By (1.5), there exists $C_2>0$ such that $C_2|x|^{a_0}\le V(|x|)$ for all $0<|x|\le R_0$. Thus, by (1.7) we get

$$\int_{B_{R_0}}|\nabla w|^2dx+C_2\int_{B_{R_0}}w^2dx\le\int_{B_{H(R_0)}}|\nabla u|^2dx+\int_{B_{H(R_0)}}V(|x|)u^2dx,$$

and consequently $w\in H^1(B_{R_0})$. Now following a scheme similar to the one in [73] we define $\overline{w}\in H_0^1(B_{R_0})$ by

$$\overline{w}(x):=\begin{cases} w(|x|)-w(R_0) & \text{if} \quad 0\le|x|\le R_0,\\ 0 \quad \text{if} \quad |x|\ge R_0,\end{cases}$$

and using Young's inequality we see that

$$w^2(x)\le\overline{w}^2(x)+\overline{w}^2(x)w^2(R_0)+1+w^2(R_0)=v^2(x)+c^2,$$

with

$$v(x) := \overline{w}(x)\left(1+w^2(R_0)\right)^{1/2} \in H_0^1(B_{R_0}) \quad \text{and} \quad c := \left(1+w^2(R_0)\right)^{1/2}.$$

According to (1.8) and [2, Theorem 2.1] we have

$$C_1 \int_{B_{H(R_0)}} |x|^{b_0} e^{\alpha u^2} dx = \int_{B_{R_0}} |x|^{\delta} e^{(\frac{2\alpha}{a_0+2})w^2} dx \leq e^{c^2} \int_{B_{R_0}} |x|^{\delta} e^{(\frac{2\alpha}{a_0+2})v^2} dx < \infty. \tag{1.9}$$

By (1.5) with $R_1 = H(R_0)$, $Q(|x|) \leq C_3|x|^{b_0}$ if $0 < |x| \leq H(R_0)$. Consequently, $Q(|x|)e^{\alpha u^2} \in L^1(B_R)$ for R_0 sufficiently large such that $H(R_0) > R$. On the other hand, observe that, by (1.7) if $\|u\| \leq 1$ we have

$$\int_{B_{R_0}} |\nabla w|^2 dx + \int_{\mathbb{R}^2} V(|x|)u^2 dx = \int_{B_{H(R_0)}} |\nabla u|^2 dx + \int_{\mathbb{R}^2} V(|x|)u^2 dx \leq \|u\|_E^2 \leq 1,$$

that is,

$$\int_{B_{R_0}} |\nabla \overline{w}|^2 dx = \int_{B_{R_0}} |\nabla w|^2 dx \leq 1 - \int_{\mathbb{R}^2} V(|x|)u^2 dx.$$

Therefore, this inequality and the definition of v gives us the estimate

$$\int_{B_{R_0}} |\nabla v|^2 dx \leq \left(1+w^2(R_0)\right)\left(1 - \int_{\mathbb{R}^2} V(|x|)u^2 dx\right) \leq 1 - \int_{\mathbb{R}^2} V(|x|)u^2 dx + w^2(R_0).$$

Thus, by the definition of w we get

$$\int_{B_{R_0}} |\nabla v|^2 dx \leq 1 - \int_{\mathbb{R}^2} V(|x|)u^2 dx + \left(\frac{a_0+2}{2}\right) u^2(H(R_0)).$$

Then, using the following version of the so-called Radial Lemma (see e.g., [3, Lemma 2.1]) due to Strauss [77]

$$|u(x)| \leq C_0\|u\||x|^{-(a+2)/4}, \quad \text{if } |x| \gg 1, \tag{1.10}$$

we have that $\|\nabla v\|_{L^2(B_{R_0})} \leq 1$ for R_0 sufficiently large. Now, we observe that $\alpha \leq 4\pi(1+b_0/2)$ if and only if $(2/(a_0+2))\alpha \leq 4\pi(1+\delta/2)$ and hence by [2, Theorem 2.1] we have

$$\sup_{\|v\|_{H_0^1(B_{R_0})}\leq 1} \int_{B_{R_0}} |x|^{\delta} e^{(\frac{2\alpha}{a_0+2})v^2} dx < \infty.$$

Once again using that $Q(|x|) \leq C_2|x|^{b_0}$ for all $0 < |x| \leq H(R_0)$, from the last estimate and (1.9) we infer that

$$L(\alpha, V, Q, R) \leq L(\alpha, V, Q, H(R_0)) = \sup_{\|u\|_E \leq 1} \int_{B_{H(R_0)}} Q(|x|)e^{\alpha u^2} dx < \infty,$$

for R_0 sufficiently large such that $H(R_0) > R$.

Case 2: Assume $a_0 < b_0$. In this case, we consider the function

$$w(r) := \left(\frac{b_0+2}{2}\right)^{1/2} u(H(r)), \quad \text{for all} \quad r \geq 0,$$

where $H(r) = \left(\frac{b_0+2}{2}\right)^{2/(b_0+2)} r^{2/(b_0+2)}$. Once again, a straightforward computation shows that

$$\int_{B_{R_0}} |\nabla w|^2 dx = \int_{B_{H(R_0)}} |\nabla u|^2 dx \quad \text{and} \quad \int_{B_{R_0}} w^2 dx = \int_{B_{H(R_0)}} |x|^{b_0} u^2 dx.$$

Moreover, there exists $C_3 = C_3(a_0, b_0) > 0$ such that

$$\int_{B_{R_0}} e^{(\frac{2\alpha}{b_0+2})w^2} dx = C_1 \int_{B_{H(R_0)}} |x|^{b_0} e^{\alpha u^2} dx.$$

Then, using that $a_0 < b_0$ and (V) we find $C_4 > 0$ such that

$$\int_{B_{R_0}} |\nabla w|^2 dx + C_4 \int_{B_{R_0}} w^2 dx \leq \int_{B_{H(R_0)}} |\nabla u|^2 dx + \int_{B_{H(R_0)}} V(|x|) u^2 dx.$$

Now, repeating the same argument as in the proof of **Case 1** and applying the classical Trudinger-Moser inequality we conclude that $L(\alpha, V, Q, R) < \infty$ if $2\alpha/(b_0+2) \leq 4\pi$, that is, $\alpha \leq 4\pi(1 + b_0/2)$. Next, we will prove that $L(\alpha, V, Q, R) = \infty$ whenever $\alpha > \alpha_2$. In fact, setting $\tilde{M}_n = M_n/\|M_n\|_E$ we see that $\tilde{M}_n \in E_{\text{rad}}$ and $\|\tilde{M}_n\|_E = 1$. By Lemma 1.2.3 if $|x| \leq r/n$ we have

$$\tilde{M}_n^2(x, r) = \frac{M_n^2}{\|M_n\|_E^2} = \frac{\log n}{2\pi \|M_n\|_E^2} \geq \frac{\log n}{2\pi(1 + o_n(1))}. \tag{1.11}$$

Once again, by assumption (Q), there exists $C_5 > 0$ such that $Q(|x|) \geq C_5 |x|^{b_0}$ for all $0 < |x| \leq r$. Thus, for large n

$$\int_{B_R} Q(|x|) e^{\alpha \tilde{M}_n^2} dx \geq C_6 r^{b_0+2} n^{\alpha(2\pi)^{-1}(1+o_n(1))^{-1} - (b_0+2)},$$

which goes to infinity, since $\alpha(2\pi)^{-1} - (b_0+2) > 0$ implies that $\alpha(2\pi)^{-1}(1+o_n(1))^{-1} - (b_0+2) > 0$ for large n and this completes the proof. □

We are now ready to prove our sharp weighted Trudinger-Moser type inequality in the whole space $\mathbb{R}^2$.

Theorem 1.2.5. *Let $j_0 = [\gamma/2]$ and assume that (V) and (Q) hold. Then, for all $\alpha > 0$ and $u \in E_{\text{rad}}$, it holds that $Q(|x|)\Phi_{\alpha, j_0}(u) \in L^1(\mathbb{R}^2)$. Moreover,*

$$L(\alpha, V, Q, \infty) := \sup_{\|u\|_E \leq 1} \int_{\mathbb{R}^2} Q(|x|) \Phi_{\alpha, j_0}(u) dx < \infty,$$

if and only if $0 < \alpha \leq \alpha_2 := 4\pi(1 + b_0/2)$.

Proof. For $R > 0$ and $u \in E_{\text{rad}}$ we split the integral as

$$\int_{\mathbb{R}^2} Q(|x|)\Phi_{\alpha,j_0}(u)dx = \int_{B_R} Q(|x|)\Phi_{\alpha,j_0}(u)dx + \int_{B_R^c} Q(|x|)\Phi_{\alpha,j_0}(u)dx. \tag{1.12}$$

Since $Q(|x|)\Phi_{\alpha,j_0}(u) \leq Q(|x|)e^{\alpha u^2}$, by Lemma 1.2.4 it is enough to estimate the second integral on the right-hand side of (1.12). For that, by (1.5) there exists $C_1 > 0$ such that $Q(|x|) \leq C_1|x|^b$ for all $|x| \geq R$ and according to (1.10) we have $u^{2j}(x) \leq (C\|u\|_E)^{2j}|x|^{-j(a+2)/2}$ for $|x| \geq R$. If $2j_0 \geq \gamma$ we can choose $j_1 \in \mathbb{N}, j_1 \geq 1$ such that $(b+2) - \frac{j(a+2)}{2} < -1$, for $j \geq j_1 > j_0$. Consequently, we can estimate

$$\int_{B_R^c} Q(|x|)\Phi_{\alpha,j_0}(u)dx \leq \sum_{j=j_0}^{j_1-1} \frac{\alpha^j}{j!} \int_{B_R^c} Q(|x|)u^{2j}dx + C_2R^{-1}e^{\alpha C^2\|u\|_E^2}. \tag{1.13}$$

Since $2j_0 \geq \gamma$, by the continuous embedding $E_{\text{rad}} \hookrightarrow L^p(\mathbb{R}^2; Q)$ with $\gamma \leq p < \infty$, we get

$$\int_{B_R^c} Q(|x|)\Phi_{\alpha,j_0}(u)dx \leq C_3 \sum_{j=j_0}^{j_1-1} \|u\|_E^{2j} + C_2R^{-1}e^{\alpha C\|u\|_E^2} < \infty,$$

and taking the supremum over $u \in E_{\text{rad}}$ with $\|u\|_E \leq 1$ we conclude the proof. □

We quote here that Theorem 1.2.5 improves the Trudinger-Moser inequality proved in [3] in the case that $\gamma = 2$, i.e., $-2 < b \leq a$ where the authors obtain a similar result for $0 < \alpha < \alpha_2 := \min\left\{4\pi, 4\pi(1 + b_0/2)\right\}$.

As a consequence of Theorem 1.2.5 we have the following version of a convergence result due to Lions [54].

Corollary 1.2.6. *Let $j_0 = [\gamma/2]$ and assume that (V) and (Q) hold. Let $(v_n) \subset E_{\text{rad}}$ with $\|v_n\|_E = 1$ and suppose that $v_n \rightharpoonup v$ in E_{rad} with $\|v\|_E < 1$. Then, for each $0 < \beta < \alpha_2(1 - \|v\|_E^2)^{-1}$, up to a subsequence, it holds that*

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^2} Q(|x|)\Phi_{\beta,j_0}(v_n)dx < \infty.$$

Proof. Since $v_n \rightharpoonup v$ in E_{rad} and $\|v_n\|_E = 1$ we see that

$$\|v_n - v\|_E^2 = 1 - 2\langle v_n, v\rangle_E + \|v\|_E^2 \to 1 - \|v\|_E^2 < \frac{\alpha_2}{\beta}, \quad \text{as} \quad n \to \infty.$$

Thus, for large $n \in \mathbb{N}$ we have $\beta\|v_n - v\|_E^2 < \beta' < \alpha_2$ for some $\beta' > 0$. Moreover, observing that $\beta v_n^2 \leq \beta(1+\varepsilon^2)(v_n - v)^2 + \beta\left(1 + \frac{1}{\varepsilon^2}\right)v^2$ and applying Young's inequality with $1/r_1 + 1/r_2 = 1$ and $r_1 > 1$ such that $r_1\beta(1+\varepsilon^2)\|v_n - v\|_E^2 \leq \alpha_2$ one has

$$e^{\beta v_n^2}dx \leq \frac{1}{r_1}e^{r_1\beta(1+\varepsilon^2)\|v_n-v\|_E^2\left(\frac{v_n-v}{\|v_n-v\|_E}\right)^2} + \frac{1}{r_2}e^{r_2\beta(1+\frac{1}{\varepsilon^2})v^2}.$$

For every $R > 0$, multiplying the above inequality by $Q(x)$ and invoking Lemma 1.2.4 we obtain

$$\sup_{n\in\mathbb{N}} \int_{B_R} Q(|x|)\Phi_{\beta,j_0}(v_n)dx \le \sup_{n\in\mathbb{N}} \int_{B_R} Q(|x|)e^{\beta v_n^2}dx < \infty.$$

On the other hand, as $2j_0 \ge \gamma$, we can use inequality (1.13) with v_n to conclude that

$$\sup_{n\in\mathbb{N}} \int_{B_R^c} Q(|x|)\Phi_{\beta,j_0}(v_n)dx < \infty,$$

and hence the proof is complete. □

The proof Theorem 1.1.2 will be reached by using variational approach. For this purpose, we start off by considering $\alpha > \alpha_0$, as in the hypothesis (1.1) and $q \ge 1$. Thus, from (f_1), for any given $\varepsilon > 0$, there exist constants $C_1, C_2 > 0$ such that

$$|f(s)| \le \varepsilon|s|^{\gamma-1} + C_1|s|^{q-1}\Phi_{\alpha,j_0}(s), \quad \text{for all } s \in \mathbb{R}, \tag{1.14}$$

and

$$|F(s)| \le \frac{\varepsilon}{2}|s|^{\gamma} + C_2|s|^{q}\Phi_{\alpha,j_0}(s), \quad \text{for all } s \in \mathbb{R}. \tag{1.15}$$

Consider the energy functional associated with equation $(\mathcal{P})$ given by

$$J_\lambda(u) = \frac{1}{2}\|u\|_E^2 - \lambda \int_{\mathbb{R}^2} Q(|x|)F(u)dx, \quad \text{for all } u \in E_{\text{rad}}.$$

By using that, for all $r \ge 1$ the elementary inequality (see e.g., [83, Lemma 2.1])

$$(\Phi_{\alpha,j_0}(s))^r \le \Phi_{r\alpha,j_0}(s), \quad \text{for all } s \in \mathbb{R}, \tag{1.16}$$

holds, it follows from (1.15), Lemma 1.2.2 and Theorem 1.2.5 that J_λ is well defined and standard arguments show that $J_\lambda \in C^1(E_{\text{rad}}, \mathbb{R})$ with derivative given by:

$$J_\lambda'(u)v = \int_{\mathbb{R}^2} (\nabla u \nabla v + V(|x|)uv)\, dx - \lambda \int_{\mathbb{R}^2} Q(|x|)f(u)v dx, \quad \text{for all } v \in E_{\text{rad}}.$$

Remark 1.2.7. *Since the value of $\lambda > 0$ is not relevant in the proof of Theorem 1.1.2, we restrict our analysis to $\lambda = 1$ and to simplify notation we denote J_1 by J.*

Inspired by [16, Lemma 5.1], we have the following version of Palais' Principle of Symmetric Criticality due to Palais [65].

Proposition 1.2.8. *Every critical point of J in E_{rad} is a weak solution of $(\mathcal{P})$.*

Proof. Let $u \in E_{\text{rad}}$ and consider the linear functional $T_u : E \to \mathbb{R}$ defined by

$$T_u(w) := \int_{\mathbb{R}^2} \nabla u \nabla w dx + \int_{\mathbb{R}^2} V(|x|)uw dx - \int_{\mathbb{R}^2} Q(|x|)f(u)w dx.$$

First, we are going to check that T_u is well defined and continuous on E, which is enough to estimate the last term. For each $w \in E$, by (1.14) with $q = \gamma + 1$ and $\varepsilon = 1$ we get a constant $C_1 > 0$ such that

$$\int_{\mathbb{R}^2} Q(|x|)|f(u)w|dx \leq \int_{\mathbb{R}^2} Q(|x|)|u|^{\gamma-1}|w|dx + C_1 \int_{\mathbb{R}^2} Q(|x|)|u|^{\gamma}\Phi_{\alpha,j_0}(u)|w|dx. \tag{1.17}$$

Let us to analyze the last two integrals above. For any $R > 0$, we can write

$$\int_{\mathbb{R}^2} Q(|x|)|u|^{\gamma-1}|w|dx = \int_{B_R} Q(|x|)|u|^{\gamma-1}|w|dx + \int_{B_R^c} Q(|x|)|u|^{\gamma-1}|w|dx.$$

Now using Hölder's inequality and Lemma 1.2.2 we get $C_2 > 0$ such that

$$\int_{B_R} Q(|x|)|u|^{\gamma-1}|w|dx \leq C_2\|u\|_E^{\gamma-1}\left(\int_{B_R} Q(|x|)|w|^{\gamma}dx\right)^{1/\gamma}. \tag{1.18}$$

Choosing $p_1 > 1$ such that $p_1 b_0 > -2$, we see that $|x|^{p_1 b_0} \in L^1(B_R)$, and hence we can use (Q) together with Hölder's inequality with $1/p_1 + 1/p_2 = 1$ to get

$$\int_{B_R} Q(|x|)|w|^{\gamma}dx \leq C_3\left(\int_{B_R} |x|^{p_1 b_0}dx\right)^{1/p_1}\left(\int_{B_R} |w|^{p_2\gamma}dx\right)^{1/p_2} \leq C_4\left(\int_{B_R} |w|^{p_2\gamma}dx\right)^{1/p_2}.$$

From this and the continuous embedding $E \hookrightarrow L^{p_2\gamma}(B_R)$ (see Lemma 2.2.2), we deduce that

$$\int_{B_R} Q(|x|)|w|^{\gamma}dx \leq C_5\|w\|_E, \tag{1.19}$$

which combined with (1.18) implies that

$$\int_{B_R} Q(|x|)|u|^{\gamma-1}|w|dx \leq C_6\|w\|_E. \tag{1.20}$$

On the other hand, by (1.5), inequality (1.10) and the fact that $b - (\gamma-2)(a+2)/4 \leq a$, for $|x| > R$ one has

$$Q(|x|)|u|^{\gamma-1} \leq C_7\|u\|_E^{\gamma-2}|x|^{b-(\gamma-2)(a+2)/4}|u| \leq C_8|x|^a|u| \ \leq C_9 V(|x|)|u|.$$

Consequently, we get

$$\int_{B_R^c} Q(|x|)|u|^{\gamma-1}|w|dx \leq C_9\|u\|_E\|w\|_E = C_{10}\|w\|_E.$$

This, combined with (1.20), implies

$$\int_{\mathbb{R}^2} Q(|x|)|u|^{\gamma-1}|w|dx \leq C_{11}\|w\|_E. \tag{1.21}$$

Next, we estimate the second integral on the hight-hand side of (1.17). For $R > 0$ we also split

$$\int_{\mathbb{R}^2} Q(|x|)|u|^{\gamma}\Phi_{\alpha,j_0}(u)|w|dx = T_1(w) + T_2(w),$$

where

$$T_1(w) := \int_{B_R} Q(|x|)|u|^{\gamma}\Phi_{\alpha,j_0}(u)|w|dx \quad \text{and} \quad T_2(w) := \int_{B_R^c} Q(|x|)|u|^{\gamma}\Phi_{\alpha,j_0}(u)|w|dx.$$

Invoking Hölder's inequality, (1.16), Lemma 1.2.2, Lemma 1.2.4, and (1.19), we see that

$$\begin{aligned}|T_1(w)| &\leq \left(\int_{B_R} Q(|x|)|u|^{q_1\gamma}dx\right)^{1/q_1}\left(\int_{B_R} Q(|x|)\Phi_{q_2\alpha,j_0}(u)dx\right)^{1/q_2}\left(\int_{B_R} Q(|x|)|w|^{q_3}dx\right)^{1/q_3}\\ &\leq C_{12}\|w\|_E,\end{aligned}$$

for $q_1, q_2, q_3 > 1$ satisfying $1/q_1 + 1/q_2 + 1/q_3 = 1$ and $q_3 = \gamma$. On the other hand, using that $b - \gamma(a+2)/2 \leq a$, from Hölder's inequality, (V), (Q), (1.10), (1.16) and Theorem 1.2.5 we get

$$\begin{aligned}|T_2(w)| &\leq C_{13}\left(\int_{B_R^c} |x|^b|u|^{2\gamma}w^2dx\right)^{1/2}\left(\int_{B_R^c} Q(|x|)\Phi_{2\alpha,j_0}(u)dx\right)^{1/2}\\ &\leq C_{14}\|u\|_E^{\gamma}\left(\int_{B_R^c} |x|^{b-\gamma(a+2)/2}w^2dx\right)^{1/2} \leq C_{15}\left(\int_{B_R^c} |x|^a w^2dx\right)^{1/2} \leq C_{16}\|w\|_E.\end{aligned}$$

Therefore,

$$\int_{\mathbb{R}^2} Q(|x|)|u|^{\gamma}\Phi_{\alpha,j_0}(u)|w|dx \leq C_{17}\|w\|_E.$$

This, together with (1.17) and (1.21), implies that T_u is continuous. Now, suppose that $u \in E_{\text{rad}}$ is a critical point of J, i.e., $T_u(w) = 0$ for all $w \in E_{\text{rad}}$. By the Riesz Representation Theorem in the space E, there exists a unique $\overline{u} \in E$ such that $T_u(\overline{u}) = \|\overline{u}\|_E^2 = \|T_u\|_{E'}^2$, where E' denotes the dual space of E. Let $\mathcal{O}(2)$ denotes the group of orthogonal transformations in $\mathbb{R}^2$. Since V, Q and u are radial, by using change of variables, one has for each $w \in E$

$$T_u(gw) = T_u(w) \quad \text{and} \quad \|gw\|_E = \|w\|_E, \quad \text{for all} \quad g \in \mathcal{O}(2).$$

Applying this with $w = \overline{u}$, by uniqueness, $g\overline{u} = \overline{u}$, for all $g \in \mathcal{O}(2)$, which means that $\overline{u} \in E_{\text{rad}}$ and consequently $T_u(\overline{u}) = 0$, that is, $\|T_u\|_{E'} = 0$ which implies that $T_u(w) = 0$, for all $w \in E$ and this concludes the proof. □

1.3 Proofs of Theorem 1.1.2 and Theorem 1.1.3

In view of Proposition 1.2.8 we are going to get solutions of $(\mathcal{P})$ looking for critical points of J. We first prove that J satisfies the Mountain Pass geometry.

Lemma 1.3.1. *Assume that* (1.1), $(f_1)-(f_2)$, (V) *and* (Q) *hold. Then*

(i) there exist $\tau, \rho > 0$ *such that* $J(u) \geq \tau$ *if* $\|u\|_E = \rho$;

(ii) there exists $e \in E_{\rm rad}$, *with* $\|e\|_E > \rho$, *such that* $J(e) < 0$.

Proof. For every $q > \gamma \geq 2$, by Hölder's inequality with exponents $1/r_1 + 1/r_2 = 1$ together with (1.16) we get

$$\int_{\mathbb{R}^2} Q(|x|)|s|^q \Phi_{\alpha,j_0}(u)dx \leq \|u\|^q_{L^{qr_1}(\mathbb{R}^2;Q)} \left(\int_{\mathbb{R}^2} Q(|x|)\Phi_{r_2\alpha\|u\|_E,j_0}\left(\frac{u}{\|u\|_E}\right)dx\right)^{1/r_2}.$$

Choosing $\|u\|_E = \rho < (\alpha_2/r_2\alpha)^{1/2}$, we can apply Theorem 1.2.5 and use inequality (1.15) to get $C_3 > 0$ such that

$$\int_{\mathbb{R}^2} Q(|x|)F(u)dx \leq \frac{\varepsilon}{2}\|u\|^\gamma_{L^\gamma(\mathbb{R}^2;Q)} + C_3\|u\|^q_{L^{qr_1}(\mathbb{R}^2;Q)},$$

for every $\varepsilon > 0$. Hence according to Lemma 1.2.2, there exist constants $C_4, C_5 > 0$ such that

$$J(u) \geq \frac{1}{2}\|u\|^2_E - \frac{C_4\varepsilon}{2}\|u\|^\gamma_E - C_5\|u\|^q_E,$$

which gives us (i), if $\gamma > 2$. In case that $\gamma = 2$, we obtain the result by choosing $\varepsilon > 0$ sufficiently small.

To prove (ii), we consider a function $\varphi \in C^\infty_{0,\rm rad}(\mathbb{R}^2) \setminus \{0\}$ and denote its support by suppφ. From (f_2) there exist $\theta > \gamma \geq 2$ and constants $C_6, C_7 > 0$ such that $F(s) \geq C_6|s|^\theta - C_7$, for all $s \in \mathbb{R}$. Thus, for all $t > 0$, it holds that

$$J(t\varphi) \leq \frac{t^2}{2}\|\varphi\|^2_E - C_6 t^\theta \int_{\rm supp\varphi} Q(|x|)|\varphi|^\theta dx + C_7\int_{\rm supp\varphi} Q(|x|)dx.$$

Since $\theta > 2$, we obtain (ii) by taking $e = t\varphi$ with $t > 0$ sufficiently large and this concludes the proof. □

In view of Lemma 1.3.1 the minimax level

$$c = \inf_{g\in\Gamma}\max_{t\in[0,1]} J(g(t)),$$

with $\Gamma = \{g \in C([0,1], E_{\rm rad}) : g(0) = 0 \text{ and } J(g(1)) < 0\}$ is positive. According to the Mountain Pass Theorem without the Palais-Smale condition (see e.g., [23]) we obtain a Palais-Smale sequence ($(PS)_c$ for short) $(u_n) \subset E_{\rm rad}$ at the level c, that is,

$$J(u_n) \to c \quad \text{and} \quad \|J'(u_n)\|_{E^*} \to 0.$$

Whenever $(u_n) \subset E_{\rm rad}$ is a $(PS)_c$ sequence, we will show next that, up to a subsequence, $u_n \rightharpoonup u$ in $E_{\rm rad}$. In order to prove that u is a weak solution of $(\mathcal{P})$ we will need the following compactnesses result:

Lemma 1.3.2. *Assume* (1.1), $(f_1)-(f_3)$, (V) *and* (Q). *If* $(u_n)\subset E_{\text{rad}}$ *is a* $(PS)_c$ *sequence to* J *then* (u_n) *is bounded in* E_{rad} *and, up to a subsequence, it holds that:*

(i) $Q(|x|)f(u_n)\to Q(|x|)f(u)$ *in* $L^1_{\text{loc}}(\mathbb{R}^2)$*;*

(ii) $Q(|x|)F(u_n)\to Q(|x|)F(u)$ *in* $L^1(\mathbb{R}^2)$.

Proof. By hypothesis, we have

$$\frac{1}{2}\|u_n\|_E^2-\int_{\mathbb{R}^2}Q(|x|)F(u_n)dx=c+o_n(1)$$

and

$$\|u_n\|_E^2-\int_{\mathbb{R}^2}Q(|x|)f(u_n)u_ndx=o_n(\|u_n\|_E).$$

Thus, for every $\theta>\gamma\geq 2$ we get constants $C_1,C_2>0$ such that

$$1+C_1+C_2\|u_n\|_E\geq\left(\frac{1}{2}-\frac{1}{\theta}\right)\|u_n\|_E^2+\int_{\mathbb{R}^2}Q(|x|)\left(\frac{1}{\theta}f(u_n)u_n-F(u_n)\right)dx,\tag{1.22}$$

which yields that (u_n) is bounded by (f_2). As a consequence we have the estimate

$$\int_{\mathbb{R}^2}Q(|x|)f(u_n)u_ndx\leq C_3,\quad\text{for all } n\in\mathbb{N}.$$

Then, up to a subsequence, $u_n\rightharpoonup u$ in E_{rad} and $Q(|x|)u_n\to Q(|x|)u$ in $L^1_{\text{loc}}(\mathbb{R}^2)$ by Lemma 2.2.2. From (1.14), Lemma 2.2.2 and Theorem 1.2.5 we see that $Q(|x|)f(u)\in L^1_{\text{loc}}(\mathbb{R}^2)$. Therefore, thanks to [44, Lemma 2.1] we conclude that (i) holds.

In order to prove (ii), for every $R>0$ we write

$$\int_{\mathbb{R}^2}Q(|x|)(F(u_n)-F(u))dx=I_n(B_R)+I_n(B_R^c),$$

where for $\Omega=B_R$ or $\Omega=B_R^c$,

$$I_n(\Omega):=\int_\Omega Q(|x|)(F(u_n)-F(u))dx.$$

First we check that, for all $R>0$ fixed we have

$$\lim_{n\to\infty}I_n(B_R)=0.\tag{1.23}$$

In fact, for any $\varepsilon>0$, according to Egoroff's Theorem there exists a measurable set $\Omega\subset B_R$ with $|\Omega|<\varepsilon$ such that $u_n(x)\to u(x)$ uniformly in $B_R\setminus\Omega$, and consequently

$$|I_n(B_R)|\leq\int_\Omega Q(|x|)F(u)dx+\int_\Omega Q(|x|)F(u_n)dx+o_n(1).\tag{1.24}$$

From (1.15), for $q \geq \gamma$ we see that

$$\int_\Omega Q(|x|)F(u)dx \leq \frac{\varepsilon}{2}\int_\Omega Q(|x|)|u|^\gamma dx + C_2\int_\Omega Q(|x|)|u|^q\Phi_{\alpha,j_0}(u)dx.$$

On the other hand, by Lemma 1.2.2, (1.16) and Theorem 1.2.5 we have

$$\begin{aligned}\int_\Omega Q(|x|)|u|^q\Phi_{\alpha,j_0}(u)dx &\leq \left(\int_\Omega Q(|x|)dx\right)^{1/q_3}\|u\|^q_{L^{qq_1}(\mathbb{R}^2;Q)}\left(\int_\Omega Q(|x|)\Phi_{q_2\alpha,j_0}(u)dx\right)^{1/q_2}\\ &\leq C_3\left(\int_\Omega Q(|x|)dx\right)^{1/q_3},\end{aligned}$$

whenever $q_1, q_2, q_3 > 1$ satisfy $1/q_1 + 1/q_2 + 1/q_3 = 1$. From hypothesis (Q), there exists $C_4 > 0$ such that $Q(|x|) \leq C_4|x|^{b_0}$, for all $0 < |x| \leq R$. Since $b_0 > -2$ we can choose $r_1 > 1$ with $1/r_1 + 1/r_2 = 1$ such that $r_1b_0 > -2$ and hence

$$\int_\Omega Q(|x|)dx \leq C_5\left(\int_\Omega |x|^{r_1b_0}dx\right)^{1/r_1}|\Omega|^{1/r_2} \leq C_6\varepsilon^{1/r_2}.$$

Thus, we conclude that

$$\int_\Omega Q(|x|)F(u)dx \leq C_7\varepsilon^{1/r_2q_3}. \tag{1.25}$$

Next, we estimate the second integral on the right-hand side of (2.27). To do this, from (1.22), it follows that

$$\int_{\mathbb{R}^2} Q(|x|)\left(\frac{1}{\theta}f(u_n)u_n - F(u_n)\right)dx \leq C_8. \tag{1.26}$$

For any $\varepsilon > 0$, we can choose $\theta_0 > \theta$ such that

$$0 < \frac{\theta C_8}{(\theta_0 - \theta)} < \varepsilon. \tag{1.27}$$

By hypothesis (f_3) there exists $s_0 > 0$ such that $\theta_0 F(s) \leq f(s)s$ for any $|s| \geq s_0$. Furthermore, by (1.26) we infer that

$$\int_{\{|u_n|\geq s_0\}} Q(|x|)\left(f(u_n)u_n - \theta F(u_n)\right)dx \leq \theta C_8,$$

and consequently by (f_2) we get

$$\begin{aligned}(\theta_0-\theta)\int_{\{|u_n|\geq s_0\}} Q(|x|)F(u_n)dx &= \int_{\{|u_n|\geq s_0\}} Q(|x|)\left(\theta_0F(u_n) - f(u_n)u_n + f(u_n)u_n - \theta F(u_n)\right)dx\\ &\leq \theta C_8,\end{aligned}$$

which combined with (1.27) implies that

$$\int_{\{|u_n|\geq s_0\}} Q(|x|)F(u_n)dx < \varepsilon. \tag{1.28}$$

On the other hand, since $Q(|x|)f(u_n) \to Q(|x|)f(u)$ in $L^1(B_R)$ there exists $g \in L^1(B_R)$ such that $Q(|x|)|f(u_n)| \le g$ a.e in B_R. From assumption (f_2), we have

$$Q(|x|)F(u_n) \le \frac{1}{\theta}Q(|x|)f(u_n)(u_n) \le \frac{1}{\theta}gs_0 \quad \text{a.e.} \quad \text{in} \quad \Omega \cap \{|u_n| < s_0\}.$$

Then, by applying the Lebesgue Dominated Convergence Theorem and using (1.25) we obtain

$$\lim_{n\to\infty} \int_{\Omega\cap\{|u_n|<s_0\}} Q(|x|)F(u_n)dx = \int_{\Omega\cap\{|u|<s_0\}} Q(|x|)F(u)dx < C_7\varepsilon^{1/r_2q_3}.$$

Since

$$\int_\Omega Q(|x|)F(u_n)dx = \int_{\Omega\cap\{|u_n|\ge s_0\}} Q(|x|)F(u_n)dx + \int_{\Omega\cap\{|u_n|<s_0\}} Q(|x|)F(u_n)dx,$$

from (1.28) we find

$$\limsup_{n\to\infty} \int_\Omega Q(|x|)F(u_n)dx \le \varepsilon + C_7\varepsilon^{1/r_2q_3}.$$

Since $\varepsilon > 0$ is arbitrary, the last estimate above together with (2.27), and (1.25) imply that (2.26) holds true.

Next, we will prove that for $R > 0$ sufficiently large $\lim_{n\to\infty} I_n(B_R^c) = 0$. For this, using that $Q(|x|)F(u) \in L^1(\mathbb{R}^2)$ for any $\varepsilon > 0$ we can choose $R > 0$ sufficiently large such that

$$|I_n(B_R^c)| \le \int_{B_R^c} Q(|x|)F(u)dx + \int_{B_R^c} Q(|x|)F(u_n)dx < \varepsilon + \int_{B_R^c} Q(|x|)F(u_n)dx. \tag{1.29}$$

To estimate the last integral above, since (u_n) is bounded, using Hölder's inequality with $1/r_1 + 1/r_2 = 1$ we get

$$\begin{aligned}\int_{B_R^c} Q(|x|)|u_n|^q\Phi_{\alpha,j_0}(u_n)dx \le &\left(\int_{B_R^c} Q(|x|)|u_n|^{r_1q}dx\right)^{1/r_1}\\ &\times\left(\int_{B_R^c} Q(|x|)\Phi_{r_2\alpha,j_0}(u_n)dx\right)^{1/r_2}.\end{aligned} \tag{1.30}$$

This together with inequality (1.15) implies that

$$\begin{aligned}\int_{B_R^c} Q(|x|)F(u_n)dx &\le \frac{\varepsilon}{2}\int_{B_R^c} Q(|x|)|u_n|^\gamma dx + C_9\int_{B_R^c} Q(|x|)|u_n|^q\Phi_{\alpha,j_0}(u_n)dx\\ &\le C_{10}\varepsilon + C_{11}\left(\int_{B_R^c} Q(|x|)\Phi_{r_2\alpha,j_0}(u_n)dx\right)^{1/r_2},\end{aligned} \tag{1.31}$$

for every $\varepsilon > 0$ and $n \in \mathbb{N}$. Now using inequality (1.13) with u replaced to u_n we get

$$\int_{B_R^c} Q(|x|)\Phi_{r_2\alpha,j_0}(u_n)dx \le \sum_{j=j_0}^{j_1-1}\frac{\alpha^j}{j!}\int_{B_R^c} Q(|x|)u_n^{2j}dx + C_{12}R^{-1}e^{r_2\alpha C^2\|u_n\|_E^2}.$$

Since (u_n) is bounded by Lemma 1.2.2 we deduce that

$$\limsup_{n\to\infty} \int_{B_R^c} Q(|x|)\Phi_{r_2\alpha,j_0}(u_n)dx \le \sum_{j=j_0}^{j_1-1} \frac{\alpha^j}{j!} \int_{B_R^c} Q(|x|)u^{2j}dx + C_{12}R^{-1}e^{r_2\alpha C^2 C_{13}},$$

that goes to zero as $R \to \infty$. This in combination with (1.29) and (1.31) implies that the convergence $\lim_{n\to\infty} I_n(B_R^c) = 0$ holds, and so the lemma is proved. □

As a consequence of the previous lemma, we have the following local compactness result:

Proposition 1.3.3. *Assume* (1.1), $(f_1)-(f_3)$, (V) *and* (Q). *Then the functional J satisfies the $(PS)_c$ condition for every $c \in (0, \alpha_2/2\alpha_0)$.*

Proof. Let $(u_n) \subset E_{\text{rad}}$ be a $(PS)_c$. By Lemma 1.3.2 we can assume, up to a subsequence, that $u_n \rightharpoonup u$ weakly in E_{rad} and for all $\varphi \in C^\infty_{0,\text{rad}}(\mathbb{R}^2)$ we have

$$\int_{\mathbb{R}^2} (\nabla u_n \nabla\varphi + V(|x|)u_n\varphi)\, dx - \int_{\mathbb{R}^2} Q(|x|)f(u_n)\varphi dx = o_n(1)\|\varphi\|_E. \tag{1.32}$$

Passing to the limit and using Lemma 1.3.2 we get

$$\int_{\mathbb{R}^2} (\nabla u\nabla\varphi + V(|x|)u\varphi)\, dx - \int_{\mathbb{R}^2} Q(|x|)f(u)\varphi dx = 0, \quad \text{for all } \varphi \in C^\infty_{0,\text{rad}}(\mathbb{R}^2). \tag{1.33}$$

Next, we are going to check that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} Q(|x|)f(u_n)u_n dx = \int_{\mathbb{R}^2} Q(|x|)f(u)udx. \tag{1.34}$$

If this is true, from (1.32) and (1.33) it follows that

$$\lim_{n\to\infty} \|u_n\|_E^2 = \lim_{n\to\infty} \int_{\mathbb{R}^2} Q(|x|)f(u_n)u_n dx = \int_{\mathbb{R}^2} Q(|x|)f(u)udx = \|u\|_E^2$$

and this finishes the proof. Thus, it remains to prove (1.34). To this end, we first observe that by Lemma 1.3.2 $Q(|x|)F(u_n) \to Q(|x|)F(u)$ in $L^1(\mathbb{R}^2)$ and hence from the fact that $J(u_n) = c + o_n(1)$ we get

$$\lim_{n\to\infty} \|u_n\|_E^2 = 2\left(c + \int_{\mathbb{R}^2} Q(|x|)F(u)dx\right) > 0. \tag{1.35}$$

Defining $v_n := u_n/\|u_n\|_E$, by the weak convergence of (u_n) we have

$$v_n \rightharpoonup v := u/\lim_{n\to\infty}\|u_n\|_E \quad \text{weakly in } E_{\text{rad}},$$

with $\|v\|_E \le 1$. If $\|v\|_E = 1$ we finish the proof. Otherwise, it follows from (f_2) and (1.33) that

$$J(u) = J(u) - \frac{1}{\theta}J'(u)u = \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u\|_E^2 + \int_{\mathbb{R}^2} Q(|x|)\left(\frac{1}{\theta}f(u)u - F(u)\right)dx \ge 0.$$

Setting $A := \left(c + \int_{\mathbb{R}^2} Q(|x|)F(u)dx\right)(1 - \|v\|_E^2)$, it follows from the definition of v that $A = c - J(u)$. Thus, from (1.35) we reach

$$\frac{1}{2}\lim_{n\to\infty}\|u_n\|_E^2 = \frac{A}{1-\|v\|_E^2} = \frac{c - J(u)}{1-\|v\|_E^2} \leq \frac{c}{1-\|v\|_E^2} < \frac{\alpha_2}{2\alpha_0(1-\|v\|_E^2)}.$$

Consequently, for large $n \in \mathbb{N}$ there are $q_1 > 1$ sufficiently close to 1, $\alpha > \alpha_0$ close to α_0 and $\beta > 0$ such that $q_1\alpha\|u_n\|_E^2 \leq \beta < \alpha_2(1-\|v\|_E^2)^{-1}$. Therefore, by Corollary 1.2.6 there exists $C_1 > 0$ such that

$$\int_{\mathbb{R}^2} Q(|x|)\Phi_{q_1\alpha,j_0}(u_n)dx \leq C_1. \tag{1.36}$$

Now we observe that

$$\left|\int_{\mathbb{R}^2} Q(|x|)\left(f(u_n)u_n - f(u)u\right)dx\right| \leq L_1(n) + L_2(n),$$

where

$$L_1(n) := \int_{\mathbb{R}^2} Q(|x|)|f(u_n)(u_n - u)|dx \quad \text{and} \quad L_2(n) := \int_{\mathbb{R}^2} Q(|x|)|f(u_n) - f(u)||u|dx.$$

We are going to check that $\lim_{n\to\infty} L_i(n) = 0$ for $i = 1, 2$. To this, by inequality (1.14) with $q = 1$, for every $\varepsilon > 0$ there exists $C_2 >$ such that Hölder's inequality implies

$$I_1(n) \leq \varepsilon\|u_n\|_{L^\gamma(\mathbb{R}^2;Q)}^{\gamma-1}\|u_n - u\|_{L^\gamma(\mathbb{R}^2;Q)} + C_2\int_{\mathbb{R}^2} Q(|x|)|u_n - u|\Phi_{\alpha,j_0}(u_n)dx.$$

Now, using (1.16) and Hölder's inequality with exponents $1/q_1 + 1/q_2 = 1$, q_1 close to 1 and q_2 sufficiently large we get

$$\int_{\mathbb{R}^2} Q(|x|)\Phi_{\alpha,j_0}(u_n)|u_n - u|dx \leq \|u_n - u\|_{L^{q_2}(\mathbb{R}^2;Q)}\left(\int_{\mathbb{R}^2} Q(|x|)\Phi_{q_1\alpha,j_0}(u_n)dx\right)^{1/q_1}.$$

This combined with (1.36) and the compact embedding in Lemma 1.2.2 implies that the convergence $\lim_{n\to\infty} L_1(n) = 0$ holds. Next we will check that $\lim_{n\to\infty} L_2(n) = 0$. For this purpose, since $C_{0,\text{rad}}^\infty(\mathbb{R}^2)$ is dense in E_{rad}, for each $\varepsilon > 0$, there exists $v \in C_{0,\text{rad}}^\infty(\mathbb{R}^2)$ such that $\|u - v\|_E < \varepsilon$. Now, notice that

$$\begin{aligned} L_2(n) \leq \int_{\mathbb{R}^2} Q(|x|)|f(u_n)(u - v)|dx &+ \int_{\mathbb{R}^2} Q(|x|)|f(u)(v - u)|dx \\ &+ \|v\|_{L^\infty(\mathbb{R}^2)}\int_{\text{supp } v} Q(|x|)|f(u_n) - f(u)|dx. \end{aligned}$$

Since (u_n) is bounded, applying (1.32) with $\varphi = u - v$ we find $C_3 > 0$ such that

$$\left|\int_{\mathbb{R}^2} Q(|x|)f(u_n)(u - v)dx\right| \leq o_n(1)\|u - v\|_E + \|u_n\|_E\|u - v\|_E \leq C_3\varepsilon.$$

In a similar way, from (1.33) we get $C_4 > 0$ such that

$$\int_{\mathbb{R}^2} Q(|x|)|f(u)(u-v)|dx < C_4\varepsilon$$

and according to Lemma 1.3.2 we have

$$\|v\|_{L^\infty(\mathbb{R}^2)} \int_{\text{supp } v} Q(|x|)|f(u_n) - f(u)|dx = o_n(1).$$

Therefore, $\lim_{n\to\infty} L_2(n) = 0$ and this completes the proof. □

Next, we will obtain an estimate for the minimax level.

Lemma 1.3.4. *Assume* $(f_2), (f_4)$, (V) *and* (Q). *Then, there exists* $n \in \mathbb{N}$ *such that*

$$\max_{t\geq 0} J(t\tilde{M}_n) = \max_{t\geq 0} \left\{ \frac{t^2}{2} - \int_{\mathbb{R}^2} Q(|x|)F(t\tilde{M}_n)dx \right\} < \frac{\alpha_2}{2\alpha_0},$$

where $\tilde{M}_n(x, r_0) := M_n(x, r_0)/\|M_n\|_E$ with r_0 given in condition (f_4).

Proof. We argue towards a contradiction, by supposing that the conclusion of the lemma fails. Then, for every $n \in \mathbb{N}$, there exists $t_n > 0$ such that

$$\frac{t_n^2}{2} - \int_{\mathbb{R}^2} Q(|x|)F(t_n\tilde{M}_n)dx \geq \frac{\alpha_2}{2\alpha_0}.$$

Since $Q(|x|) > 0$ and $F(s) \geq 0$, we have $t_n^2 \geq \alpha_2/\alpha_0$. Taking into account that $\frac{d}{dt}\left(J(t\tilde{M}_n)\right)\Big|_{t=t_n} = 0$ we infer that

$$t_n^2 = \int_{\mathbb{R}^2} Q(|x|)f(t_n\tilde{M}_n)t_n\tilde{M}_n dx. \tag{1.37}$$

Now we recall that by hypothesis (f_4), for all $0 < \varepsilon < \beta_0$ there exists $R = R(\varepsilon) > 0$ such that

$$f(s)s \geq (\beta_0 - \varepsilon)e^{\alpha_0 s^2} \quad \text{for all} \quad |s| \geq R, \tag{1.38}$$

where $\beta_0 > 2(b_0+2)^2/D_0\alpha_0 r_0^{b_0+2}$ with r_0 and D_0 satisfying

$$Q(|x|) \geq \frac{D_0}{2}|x|^{b_0} \quad \text{for all} \quad 0 < |x| \leq r_0. \tag{1.39}$$

Since $t_n^2 \geq \alpha_2/\alpha_0$, for large $n \in \mathbb{N}$ we obtain $t_n\tilde{M}_n(x, r_0) \geq R$ for all $0 \leq |x| \leq r_0/n$ and hence by (1.38) we obtain

$$f(t_n\tilde{M}_n(x, r_0))t_n\tilde{M}_n(x, r_0) \geq (\beta_0 - \varepsilon)e^{\alpha_0(t_n\tilde{M}_n(x,r_0))^2}.$$

On the other hand, from estimate (1.11) with $r = r_0$ we have

$$\tilde{M}_n^2(x, r_0) \geq \frac{\log n}{2\pi(1 + o_n(1))} \quad \text{if} \quad |x| \leq r_0/n.$$

Then, from the above estimates we obtain

$$t_n^2 \geq \frac{(\beta_0-\varepsilon)D_0}{2}\int_{|x|\leq r_0/n}|x|^{b_0}e^{\frac{\alpha_0 t_n^2\log n}{2\pi(1+o_n(1))}}dx \;=\frac{(\beta_0-\varepsilon)D_0\pi}{(b_0+2)}\left(\frac{r_0}{n}\right)^{b_0+2}e^{\frac{\alpha_0 t_n^2\log n}{2\pi(1+o_n(1))}},$$

which leads to

$$C_0 t_n^2 n^{b_0+2} \geq n^{\alpha_0 t_n^2/2\pi(1+o_n(1))} \quad with \quad C_0=(b_0+2)/(\beta_0-\varepsilon)D_0\pi r_0^{b_0+2}. \tag{1.40}$$

We claim that (t_n) is bounded. Indeed, suppose by contradiction that $t_n\to\infty$. From (1.40) we get

$$\log(C_0)+\log(t_n^2)+(b_0+2)\log n \geq \frac{\alpha_0}{2\pi(1+o_n(1))}t_n^2\log n. \tag{1.41}$$

Thus,

$$\frac{\log(C_0)}{t_n^2\log n}+\frac{\log(t_n^2)}{t_n^2\log n}+\frac{(b_0+2)}{t_n^2}\geq\frac{\alpha_0}{2\pi(1+o_n(1))},$$

and taking the limit we obtain a contradiction. Now we will show that

$$\lim_{n\to\infty}t_n^2=\frac{\alpha_2}{\alpha_0}.$$

Otherwise, there exists some $\delta>0$ such that for $n\in\mathbb{N}$ sufficiently large $t_n^2\geq\alpha_2/\alpha_0+\delta$. This, together with (1.41) implies

$$\frac{\log(C_0)}{\log n}+\frac{\log(t_n^2)}{\log n}+(b_0+2)\geq\frac{\alpha_0 t_n^2}{2\pi(1+o_n(1))}\geq\frac{\alpha_2+\alpha_0\delta}{2\pi(1+o_n(1))}.$$

Since (t_n) is bounded, taking the limit we obtain $2\pi(b_0+2)\geq(\alpha_2+\alpha_0\delta)$, which contradicts the fact that $\alpha_2=2\pi(b_0+2)$. Finally, we estimate β_0 to get a contradiction. It follows from (1.37) that

$$t_n^2=\int_{A_n}Q(|x|)f(t_n\tilde{M}_n)t_n\tilde{M}_n dx+\int_{B_n}Q(|x|)f(t_n\tilde{M}_n)t_n\tilde{M}_n dx, \tag{1.42}$$

where $A_n:=\{x\in\mathbb{R}^2:|t_n\tilde{M}_n|\leq R\}$ and $B_n:=\{x\in\mathbb{R}^2:|t_n\tilde{M}_n|\geq R\}$. Since $\tilde{M}_n\to 0$ a.e. in $\mathbb{R}^2$, by applying the Lebesgue Dominated Convergence Theorem we get

$$\lim_{n\to\infty}\int_{A_n}Q(|x|)f(t_n\tilde{M}_n)t_n\tilde{M}_n dx=0.$$

Thus, taking the limit in (1.42) we obtain

$$\frac{\alpha_2}{\alpha_0}=\lim_{n\to\infty}\int_{B_n}Q(|x|)f(t_n\tilde{M}_n)t_n\tilde{M}_n dx. \tag{1.43}$$

Using that $t_n^2\geq\alpha_2/\alpha_0$, from (1.38) and (1.39) it follows that

$$\int_{B_n}Q(|x|)f(t_n\tilde{M}_n)t_n\tilde{M}_n dx\geq\frac{(\beta_0-\varepsilon)D_0}{2}\int_{B_n}|x|^{b_0}e^{\alpha_2\tilde{M}_n^2}dx.$$

Now we observe that

$$\int_{B_{r_0}} |x|^{b_0} e^{\alpha_2 \tilde{M}_n^2} dx = \frac{2\pi r_0^{b_0+2}}{(b_0+2)} + \int_{\frac{r_0}{n} \le |x| \le r_0} |x|^{b_0} e^{\alpha_2 \tilde{M}_n^2} dx.$$

Performing a straightforward computation and doing the change of variables $r = r_0 e^{-\|\tilde{M}_n\|_E (\log n)^{1/2} s}$ we get

$$\begin{aligned} \int_{\frac{r_0}{n} \le |x| \le r_0} |x|^{b_0} e^{\alpha_2 \tilde{M}_n^2} dx &= 2\pi \int_{r_0/n}^{r_0} r^{b_0} e^{(b_0+2)[(\log n)^{-1/2} \|\tilde{M}_n\|_E^{-1} \log(r_0/r)]^2} r dr \\ &= 2\pi r_0^{b_0+2} \|\tilde{M}_n\|_E \sqrt[2]{\log n} \int_0^{\|\tilde{M}_n\|_E^{-1} (\log n)^{1/2}} e^{(b_0+2)[s^2 - \|\tilde{M}_n\|_E (\log n)^{1/2} s]} ds. \end{aligned}$$

Since $e^{(b_0+2)s^2} \ge 1$, after a simple computation we find

$$\int_{\frac{r_0}{n} \le |x| \le r_0} |x|^{b_0} e^{\alpha_2 \tilde{M}_n^2} dx \ge \frac{2\pi r_0^{b_0+2}}{(b_0+2)} \left(1 + \frac{1}{n^{b_0+2}}\right).$$

Therefore,

$$\liminf_{n\to\infty} \int_{B_{r_0}} |x|^{b_0} e^{\alpha_0 t_n^2 \tilde{M}_n^2} dx \ge \frac{4\pi}{(b_0+2)} r_0^{b_0+2}. \tag{1.44}$$

On the other hand, observing that

$$\int_{B_n} |x|^{b_0} e^{\alpha_2 \tilde{M}_n^2} dx = \int_{B_{r_0}} |x|^{b_0} e^{\alpha_2 \tilde{M}_n^2} dx - \int_{A_n} |x|^{b_0} e^{\alpha_2 \tilde{M}_n^2} dx,$$

and by applying the Lebesgue Dominated Convergence Theorem we obtain

$$\lim_{n\to\infty} \int_{A_n} |x|^{b_0} e^{\alpha_2 \tilde{M}_n^2} dx = \frac{2\pi}{(b_0+2)} r_0^{b_0+2}.$$

Then, estimate (1.44) yields

$$\liminf_{n\to\infty} \int_{B_n} |x|^{b_0} e^{\alpha_2 \tilde{M}_n^2} dx \ge \frac{2\pi}{(b_0+2)} r_0^{b_0+2}.$$

Therefore, from (1.42) and (1.43) we get

$$\frac{\alpha_2}{\alpha_0} \ge \frac{(\beta_0 - \varepsilon) D_0 \pi}{(b_0+2)} r_0^{b_0+2}.$$

Using that $\alpha_2 = 2\pi(b_0+2)$ and letting $\varepsilon \to 0$ we contradicts (f_4), and this completes the proof. □

We shall also need a basic regularity result, which will be used to prove (1.4).

Lemma 1.3.5. *Let $R > 0$ and $u \in H_0^1(B_R)$ be a weak solution of the semilinear elliptic problem*

$$\begin{cases} -\Delta u = h(x,u), & in \quad B_R, \\ \quad u = 0, & on \quad \partial B_R, \end{cases} \tag{1.45}$$

where $h : B_R \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

$$|h(x,s)| \le C_0 |x|^{b_0} e^{\alpha s^2}, \quad for\ a.e.\ x \in B_R,\ and\ s \in \mathbb{R},$$

with $C_0 > 0$, $b_0 > -2$, and $\alpha > 0$. Then, $u \in C^\sigma(\overline{B}_R)$ for some $\sigma \in (0,1)$.

Proof. Since $b_0 > -2$, there exists $p > 1$ such that $pb_0 > -2$. Similarly, we can choose $q_1, q_2 > 1$ satisfying $1/q_1 + 1/q_2 = 1$ and $q_1 p b_0 > -2$. Then, by Hölder's inequality one has

$$\int_{B_R} |h(x,u)|^p dx \le C_0^p \left(\int_{B_R} |x|^{q_1 p b_0} dx \right)^{1/q_1} \left(\int_{B_R} e^{q_2 p \alpha u^2} dx \right)^{1/q_2}.$$

Taking into account that $|x|^{q_1 p b_0} \in L^1(B_R)$, and by the classical Trudinger-Moser inequality (see [63],[81]) it holds that $e^{q_2 p \alpha u^2} \in L^1(B_R)$, we conclude that $|h(x,u)|^p \in L^1(B_R)$. Therefore, by classical elliptic regularity theory $u \in W^{2,p}(B_R) \hookrightarrow C^\sigma(\overline{B}_R)$ for some $\sigma \in (0,1)$ and this finishes the proof. □

We now present the proof of Theorem 1.1.2 with the aid of the previous results.

Proof of Theorem 1.1.2. By Proposition 1.2.8 it is sufficient to show that J has a critical point. By Proposition 1.3.3, Lemma 1.3.4 and the Mountain Pass Theorem J has a nonzero critical point. Moreover, we can assume that $f(s) = 0$ for $s \le 0$ and the above results are valid also for this modified nonlinearity. Thus, there exists $u \in E_{\text{rad}} \setminus \{0\}$ such that $J'(u) = 0$. Since $u^-(x) := \max\{-u(x), 0\}$ one has $0 = J'(u)u^- = -\|u^-\|_E^2$, which implies that $u \ge 0$ a.e. in $\mathbb{R}^2$. To conclude the proof it remains to prove (1.4). For this purpose, by the assumption (V) for all $R_0 > 0$ there exists $C_0 > 0$ such that $V(|x|) \ge C_0|x|^a$ for all $|x| \ge R_0$. Defining $\phi(x) = e^{-c_1|x|^{(a+2)/4}}$ with $c_1 = 2\sqrt{C_0}/(a+2) > 0$ and using a straightforward computation we see that

$$-\Delta\phi + V(|x|)\phi \ge \frac{C_0}{4}|x|^a \phi \quad \text{in} \quad |x| \ge R_0. \tag{1.46}$$

On the other hand, from assumption (Q) there exists $C_1 > 0$ such that $Q(|x|) \le C_1|x|^b$ for all $|x| \ge R_0$. Then, by inequality (1.10) and (f_1), for R_0 sufficiently large we have

$$Q(|x|)f(u) \le \frac{C_0}{4}|x|^b|u|^{\gamma-2}u \le \frac{C_0}{4}|x|^{b-(\gamma-2)(a+2)/4}u \quad \text{for all} \quad |x| \ge R_0.$$

Taking into account that $b - \frac{(\gamma-2)(a+2)}{4} \le a$ we get

$$Q(|x|)f(u) \le \frac{C_0}{4}|x|^a u, \quad \text{for all} \quad |x| \ge R_0.$$

This combined with (1.46) (where c_0 is a positive constant such that $u \leq c_0\phi$ on $|x| = R_0$) implies

$$\begin{cases} -\Delta(u - c_0\phi) + \left(V(|x|) - \dfrac{C_0}{4}|x|^a\right)(u - c_0\phi) \leq 0 \quad \text{in} \quad |x| > R_0, \\ u - c_0\phi \leq 0 \quad \text{on} \quad |x| = R_0. \end{cases}$$

Then, by the maximum principle we have that $u(x) \leq c_0\phi(x)$ if $|x| \geq R_0$. To complete the proof it is enough to show that $u \in C^\sigma(\overline{B}_{R_0})$ for some $\sigma \in (0,1)$. Defining $v(x) := u(x) - u(R_0)$ and using Lemma 2.2.2, $v \in H_0^1(B_{R_0})$. By the behavior of V and Q at the origin, we can assume that $V(|x|) = |x|^{a_0}$ and $Q(|x|) = |x|^{b_0}$ and so v is a weak solution of problem (1.45) with $R = R_0$ and $h(x,v) = |x|^{b_0}\left[f(v + u(R_0)) - |x|^{a_0 - b_0}(v + u(R_0)\right]$. Now using that $b_0 \leq a_0$ (similarly to $a_0 < b_0$), by (1.1) and the continuity of f we find $C_2 = C_2(R) > 0$ such that for all $x| \leq R_0$

$$|h(x,v)| \leq |x|^{b_0}\left[|f(v + u(R_0))| + R_0^{a_0 - b_0}|v + u(R_0)|\right] \leq C_2|x|^{b_0}e^{\alpha v^2}.$$

By applying Lemma 3.2.3, we conclude that $v \in C^\sigma(\overline{B}_{R_0})$. Therefore, $u = v + u(R_0) \in C^\sigma(\overline{B}_{R_0})$ and this completes the proof of Theorem 1.1.2. □

In order to prove our multiplicity result we shall use the following version of the Symmetric Mountain Pass Theorem (see, e.g., [13]).

Theorem 1.3.6. *Let E be a real Banach space and $I \in C^1(E,\mathbb{R})$ be an even functional satisfying $I(0) = 0$ and*

- (I_1) *there are constants $\rho, \tau > 0$ such that $I(u) \geq \tau$, for all $\|u\|_E = \rho$;*
- (I_2) *there exist $D > 0$ and a finite-dimensional subspace S of E such that $\max_{u \in S} I(u) \leq D$.*

If the functional I satisfies the $(PS)_c$ condition for $0 < c < D$, then it possesses at least $\dim S$ *pairs of nonzero critical points.*

Proof of Theorem 1.1.3. Since f is odd and satisfies (f_1), then J_λ is even and $J_\lambda(0) = 0$. Moreover, we observe that all the results proved in the previous sections holds for J_λ, for all $\lambda > 0$. Arguing as in the proof of the first theorem, we obtain that J_λ satisfies (I_1). From (f_3) and the local condition (f_5), there exists $C_0 > 0$ such that $F(s) \geq \frac{C_0}{\nu}|s|^\nu$, for all $s \in \mathbb{R}$. Consequently,

$$J_\lambda(u) \leq \frac{1}{2}\|u\|_E^2 - \frac{C_0\lambda}{\nu}\int_{\mathbb{R}^2} Q(|x|)|u|^\nu dx.$$

We now observe that, for any $k-$dimensional subspace S of E_{rad}, the norms are equivalent and hence

$$\max_{u \in S} J_\lambda(u) \leq \max_{u \in S}\left[\frac{1}{2}\|u\|_E^2 - c_k\frac{C_0\lambda}{\nu}\|u\|_E^\nu\right] = \left(\frac{1}{2} - \frac{1}{\nu}\right)\left(\frac{1}{c_k C_0}\right)^{\frac{2}{\nu-2}}\lambda^{\frac{2}{2-\nu}} =: D_k(\lambda).$$

Since $2/(2-\nu) < 0$, we have that $\lim_{\lambda\to\infty} D_k(\lambda) = 0$. Thus, there exists $\lambda_k > 0$ such that $D_k(\lambda) < \alpha_2/(2\alpha_0)$ for any $\lambda > \lambda_k$. Therefore, we can apply Proposition 1.3.3 and Theorem 1.3.6 to obtain k pairs of nonzero critical points of J_λ, which concludes the proof. □

Chapter 2

On a planar non-autonomous Schrödinger-Poisson system involving exponential critical growth

In this chapter, we present the results of the paper [4] where we investigate the existence of solutions for a class of planar non–autonomous Schrödinger-Poisson system. One of our basic tools consists in a Trudinger-Moser type inequality obtained in the Chapter 1.

2.1 Main results

Here, we are concerning with the existence of a solution to the planar non–autonomous Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(|x|)u + \eta\phi K(|x|)u = \lambda Q(|x|)f(u), & x \in \mathbb{R}^2, \\ \Delta\phi = K(|x|)u^2, & x \in \mathbb{R}^2, \end{cases} \tag{$\mathcal{S}$}$$

where $\eta, \lambda > 0$, the potentials $V, K, Q : \mathbb{R}^2 \to \mathbb{R}$ are radial functions, which can by singular at the origin, unbounded or decaying at infinity and $f(s)$ is a continuous function with exponential critical growth in the Trudinger-Moser sense.

It is well-known that the solutions of system ($\mathcal{S}$) are related to solitary wave solutions to the nonlinear Schrödinger-Poisson system

$$\begin{cases} i\psi_t - \Delta_x\psi + E(x)\psi + \eta\phi K(x)\psi = Q(x)f(\psi), & (x,t) \in \mathbb{R}^2 \times (0,\infty), \\ \Delta_x\phi = K(x)\psi^2, & (x,t) \in \mathbb{R}^2 \times (0,\infty), \end{cases} \tag{2.1}$$

where $\psi : \mathbb{R}^2 \times (0,\infty) \to \mathbb{C}$ is the wave function, $E(x) = V(x) - \xi$ with $\xi \in \mathbb{R}$ is a real-valued external potential, ϕ represents an internal potential for a nonlocal self-interaction of the wave function and the nonlinear term $f(s)$ describes the interaction effect among particles. If we look for a standing wave ansatz $\psi(x,t) = e^{-i\xi t}u(x)$, with $\xi \in \mathbb{R}$, the system (2.1) reduces to system ($\mathcal{S}$).

Similar problems in dimension $N \geq 2$ have been widely investigated due to the fact that they have a strong physical meaning, because they appear in quantum mechanics models and semiconductors (see e.g., [22, 32, 50, 53, 57] and references therein). In [20, 21], systems like $(\mathcal{S})$ have been introduced as a model describing solitary waves, for nonlinear stationary equations of Schrödinger type interacting with an electrostatic field, and are usually known as Schrödinger-Poisson systems. Due to this deep physical meaning, in dimension $N = 3$, the non–autonomous Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi K(x)u = \lambda Q(x)f(u), & x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{2.2}$$

has strongly attracted the attention of many researches (see e.g., [33, 34, 74] and references therein). In this case, we can take the advantage of the Sobolev space $D^{1,2}(\mathbb{R}^3)$ to obtain a solution to the Poisson equation in system (2.2). Roughly speaking, for each $u \in H^1(\mathbb{R}^3)$, thanks to the Lax-Milgram lemma the unique solution of the Poisson equation is given by the Newtonian potential of Ku^2, i.e.,

$$\phi(x) := \left[\Gamma_3 * Ku^2\right](x) = \int_{\mathbb{R}^3} \Gamma_3(x-y)K(|y|)u^2(y)dy,$$

where $\Gamma_3(x) = 1/(4\pi|x|)$ is the fundamental solution of the Laplacian in $\mathbb{R}^3$. Plugging this relation into the Schrödinger equation in (2.2) we get a nonlinear Schrödinger equation with a nonlocal term and, afterward, a solution of this equation is obtained by using different techniques. There is a vast literature dealing with system (2.2) under different assumptions on V, Q and f in the autonomous case, that is, $K \equiv 1$ and the non–autonomous. We can refer the reader to the papers [14, 33, 37–39, 48, 49, 53, 55, 62, 74] and references therein.

In dimension $N = 2$, motived by the papers [8, 36, 78] we will use a different strategy: Precisely, for any u in an appropriated Hilbert space we consider the Newton potential of Ku^2, that is,

$$\phi(x) := \left[\Gamma_2 * Ku^2\right](x) = \int_{\mathbb{R}^2} \Gamma_2(x-y)K(|y|)u^2(y)dy,$$

where $\Gamma_2(x) = (1/2\pi)\log(|x|)$ is the fundamental solution of the Laplacian in $\mathbb{R}^2$ and by choosing $\eta = 2\pi$ we obtain first the solution of the integrodifferential equation

$$-\Delta u + V(|x|)u + \left[\log * Ku^2\right](x)K(|x|)u = \lambda Q(|x|)f(u), \quad x \in \mathbb{R}^2. \tag{$\mathcal{E}$}$$

Afterward, we obtain a solution to the Poisson equation by using some regularity results. In this context, some mathematical difficulties appear different from the articles mentioned above and, therefore, the number of papers is scantier. The first difficulty that we face in dealing with the two-dimensional case is the fact that the integral kernel Γ_2 is sign-changing, differently from Γ_3 that is positive. To overcome this difficulty, we employ a similar argument to that developed in the papers [8, 36].

The weighted feature of V yields another difficulty that prevents us to work directly in

$H^1(\mathbb{R}^2)$. As performed in many papers, we use an appropriate Hilbert space. With this aim, our starting point here is the hypotheses of the weight functions V, K and Q which were firstly introduced in the works [79, 80], where the authors proved some weighted Sobolev embedding theorems. Furthermore, there is a recent growing interest in applications of these results in the study of partial differential equations, see for example [3, 12, 15, 17, 26].

Throughout this chapter we assume the conditions (V) and (Q) (see Chapter 1) on the radial potentials V and Q. Moreover, we assume the following assumptions on K:

(K) $K : (0, \infty) \to \mathbb{R}$ is continuous, $K > 0$ and there are $l_0 > -3/2$, $-2 < l < \min\{a, (a-1)/2\}$ such that

$$\limsup_{r \to 0^+} \frac{K(r)}{r^{l_0}} < \infty \quad \text{and} \quad \limsup_{r \to \infty} \frac{K(r)}{r^{l}} < \infty.$$

We consider in this chapter that $f(s)$ satisfy (1.1) and (f_1) (see Chapter 1). Furthermore, to perform a variational approach, recalling that $\gamma = \max\{2, 2(2+2b-a)/(a+2)\}$, we also assume the following assumptions on f:

$(\widetilde{f_2})$ there exists $\theta > \max\{\gamma, 4\}$ such that $0 < \theta F(s) \leq f(s)s$ for all $s \geq 0$;

$(\widetilde{f_3})$ there exists $r > \max\{\gamma, 4\}$ such that $\liminf_{s \to 0^+} F(s)/s^r > 0$;

$(\widetilde{f_4})$ the function $s \mapsto f(s)/s^3$ is increasing for $s > 0$.

In order to obtain a positive solution of system $(\mathcal{S})$, we look for a positive solution of equation $(\mathcal{E})$. For that, we observe that $(\mathcal{E})$ has, at least formally, a variational structure given by the energy functional defined by

$$I_\lambda(u) = \frac{1}{2}\|u\|_E^2 + \frac{1}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(|x-y|)K(|y|)u^2(y)K(|x|)u^2(x)dydx - \lambda\int_{\mathbb{R}^2} Q(|x|)F(u)dx.$$

Remark 2.1.1. *As we will see, it follows from the hypotheses on V, Q and $f(s)$ that the functional I_λ is well defined in E_{rad}, except possibly at the nonlocal term. Taking into account the elementary inequality*

$$|\log(|x-y|)| \leq \frac{1}{|x-y|} + \log(1+|x|) + \log(1+|y|),$$

and the Sobolev embedding $E_{\text{rad}} \hookrightarrow L^2(\mathbb{R}^2; K)$ (see Lemma 1.2.2), for each $u \in E_{\text{rad}} \setminus \{0\}$ we see that

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x|)K(|y|)u^2(y)K(|x|)u^2(x)dydx = \|u\|_{L^2(\mathbb{R}^2;K)}^2 \int_{\mathbb{R}^2} \log(1+|x|)K(|x|)u^2(x)dx$$

and

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|y|)K(|y|)u^2(y)K(|x|)u^2(x)dydx = \|u\|_{L^2(\mathbb{R}^2;K)}^2 \int_{\mathbb{R}^2} \log(1+|y|)K(|y|)u^2(y)dy,$$

which combined with the Hardy-Littlewood-Sobolev inequality (see Proposition 2.3.5) motivate the definition of our function space on which I_λ is well-defined.

Inspired by the paper [78] (see also [58]) and Remark 2.1.1 we consider the new Hilbert space defined by

$$W := \left\{ u \in E : \int_{\mathbb{R}^2} \log(1+|x|)K(|x|)u^2 dx < \infty \right\},$$

with the norm $\|u\|_W := \langle u, u\rangle_W^{1/2}$ induced by the scalar product

$$\langle u, v\rangle_W = \langle u, v\rangle_E + \int_{\mathbb{R}^2} \log(1+|x|)K(|x|)uvdx.$$

Using standard arguments one can prove that $C_0^\infty(\mathbb{R}^2)$, the space of infinitely differentiable functions with compact support, is dense in W. The proof that $(W, \|\cdot\|_W)$ is a Hilbert space is not direct and so it will be done in the next section.

Remark 2.1.2. *Naturally, the continuous embedding $(W, \|\cdot\|_W) \hookrightarrow (E, \|\cdot\|_E)$ holds true.*

Even if W provides a variational framework to equation $(\mathcal{E})$, some difficulties appear due to the following unpleasant facts. For example, the norm in W does not appear explicitly in the expression of the functional. Another obstacle is that the quadratic part of I_λ is not coercive on W. However, the condition $(\widetilde{f}_4)$ allows us to use the minimization arguments in the Nehari manifold.

Now we can introduce the concept of solution that we are interested in here. We say that $u \in W$ is a weak solution to equation $(\mathcal{E})$ if, for all $\varphi \in C_0^\infty(\mathbb{R}^2)$ it holds that

$$\langle u, \varphi\rangle_E + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(|x-y|)K(|y|)u^2(y)K(|x|)u(x)\varphi(x)dydx = \lambda \int_{\mathbb{R}^2} Q(|x|)f(u)\varphi dx.$$

Remark 2.1.3. *It follows from* (1.1) *and* $(\widetilde{f}_3)$ *that there exists $C_0 > 0$ such that $F(s) \geq C_0 s^r$ for all $s \geq 0$.*

Now, we are ready to state the main result concerning the existence of the solution to equation $(\mathcal{E})$. Our main result is the following:

Theorem 2.1.4. *Assume that $(V), (K), (Q)$, (1.1), (f_1), and $(\widetilde{f}_2) - (\widetilde{f}_4)$ hold. Then, equation $(\mathcal{E})$ possesses a nonzero weak solution $u_\lambda \in W$ with minimal energy (or ground state solution) if*

$$\lambda \geq \bar{\lambda} := \max\left\{ \frac{\lambda_0}{\theta C_0}, \left[\frac{4\alpha_0 \|Q\|_{L^1(B_{1/2})}\lambda_0^{\frac{r}{r-2}}}{\alpha_2} \left(\left(\frac{2}{r}\right)^{\frac{2}{r-2}} - \left(\frac{2}{r}\right)^{\frac{r}{r-2}} \right) \right]^{\frac{r-2}{2}} \right\}, \tag{2.3}$$

where

$$\lambda_0 := \frac{4\pi + \|V\|_{L^1(B_1)} + \log 3 \|K\|^2_{L^1(B_1)}}{\|Q\|_{L^1(B_{1/2})}} \quad \text{and} \quad \alpha_2 := 4\pi(1 + b_0/2). \tag{2.4}$$

As a byproduct of Theorem 1.1.2, under additional assumptions on the potential K, our contribution in the present paper concerns the existence of solutions to the system $(\mathcal{S})$ is the following:

Theorem 2.1.5. *Assume the conditions of Theorem 2.1.4 and let* u_λ *be the solution obtained in Theorem 2.1.4. In addition, suppose that* $K \in C^\sigma_{\text{loc}}(\mathbb{R}^2)$ *for some* $\sigma \in (0,1)$*. Then, the pair* $(u_\lambda, \phi_{u_\lambda})$ *is a weak solution of system* $(\mathcal{S})$*, where* $\phi_{u_\lambda} = \Gamma_2 * K u_\lambda^2$*.*

Remark 2.1.6. *Examples of functions satisfying the hypotheses* $(V), (K)$*, and* (Q) *are:*

i) $V(x) = |x|^a$*, with* $a > -2$*;*

ii) $K(x) = |x|^l$*, with* $l = l_0 > -2$*;*

iii) $Q(x) = |x|^b$*, with* $b = b_0 > -2$*.*

Furthermore, if $l \geq 1$ *then the functions* V, K*, and* Q *satisfy the assumptions of Theorem 2.1.5.*

The remainder of this chapter is organized as follows: In Section 2.2, we prove that $(W, \|\cdot\|_W)$ is a Hilbert space. In Section 2.3, we study the nonlocal term and establish the functional setting in which the problem will be posed. The two further sections are devoted to the proof of Theorem 2.1.4 and Theorem 2.1.5, respectively.

2.2 Preliminary results

In this section, we will establish some preliminary results used in the proof of our main theorems. We start by proving that $(W, \|\cdot\|_W)$ is a Hilbert space, whose proof is inspired by the paper [6].

Proposition 2.2.1. $(W, \|\cdot\|_W)$ *is a Hilbert space.*

Proof. Let $(u_n) \subset W$ be a Cauchy sequence in the norm $\|\cdot\|_W$. We can say that

$$\left(\frac{\partial u_n}{\partial x_i}\right)_n \ (i = 1,2), \quad \left(V^{1/2}(|x|)u_n\right)_n \quad \text{and} \quad \left([\log(1+|x|)K(|x|)]^{1/2}u_n\right)_n$$

are both Cauchy sequences in $L^2(\mathbb{R}^2)$. Consequently,

$$\frac{\partial u_n}{\partial x_i} \to u^i \ (i = 1,2), \quad V^{1/2}(|x|)u_n \to v \text{ and } [\log(1+|x|)K(|x|)]^{1/2}u_n \to z \ \text{ in } L^2(\mathbb{R}^2), \tag{2.5}$$

as $n \to \infty$. Hence, up to a subsequence,

$$\frac{\partial u_n}{\partial x_i} \to u^i \ (i = 1,2) \text{ and } u_n \to w := V^{-1/2}(|x|)v = [\log(1+|x|)K(|x|)]^{-1/2}\, z \ \text{ a.e. in } \mathbb{R}^2, \tag{2.6}$$

as $n \to \infty$. To complete the proof it is sufficient to show that $w \in W$ and $u_n \to w$ in W. First we check that $w \in L^2_{\text{loc}}(\mathbb{R}^2)$. Indeed, let $R > 0$ and consider $\varphi \in C^\infty_{0,\text{rad}}(\mathbb{R}^2)$ satisfying $\text{supp}\varphi \subset B_{R+1}$ and $\varphi \equiv 1$ in B_R. Thus, $\varphi(u_n - u_m) \in H^1_0(B_{R+1})$ and by Poincaré's inequality we get

$$
\begin{aligned}
\int_{B_R} |u_n - u_m|^2 dx &\le \int_{B_{R+1}} |\varphi(u_n - u_m)|^2 dx \le C_1 \int_{B_{R+1}} |\nabla(\varphi(u_n - u_m))|^2 dx \\
&\le C_2 \left(\int_{B_{R+1}} |\varphi|^2 |\nabla u_n - \nabla u_m|^2 dx + \int_{B_{R+1}\setminus B_R} |\nabla \varphi|^2 |u_n - u_m|^2 dx \right) \\
&\le C_2 \|\varphi\|^2_{L^\infty(\mathbb{R}^2)} \int_{B_{R+1}} |\nabla u_n - \nabla u_m|^2 dx \\
&\quad + \frac{C_2 \|\nabla \varphi\|^2_{L^\infty(\mathbb{R}^2)}}{M_R} \int_{B_{R+1}\setminus B_R} V(|x|) |u_n - u_m|^2 dx \\
&\le C_3 \|u_n - u_m\|^2_E \le C_3 \|u_n - u_m\|^2_W,
\end{aligned}
\tag{2.7}
$$

where $M_R := \min_{B_{R+1}\setminus B_R} V(|x|) > 0$. The last inequality gives us that $(u_n)_n$ is a Cauchy sequence in $L^2(B_R)$ and so there exists $u_R \in L^2(B_R)$ such that

$$
u_n \to u_R \quad \text{in } L^2(B_R) \quad \text{and} \quad u_n \to u_R \quad \text{a.e. in } B_R, \tag{2.8}
$$

as $n \to \infty$. This and (2.6) implies that $w = u_R \in L^2(B_R)$ and so $w \in L^2_{\text{loc}}(\mathbb{R}^2)$. Next, we prove that w has weak derivate and $|\nabla w| \in L^2(\mathbb{R}^2)$. In fact, let $\varphi \in C_0^\infty(\mathbb{R}^2)$ with supp$\varphi \subset B_R$. For each $n \in \mathbb{N}$, one has

$$
\int_{\mathbb{R}^2} u_n \frac{\partial \varphi}{\partial x_i} dx = - \int_{\mathbb{R}^2} \frac{\partial u_n}{\partial x_i} \varphi dx, \quad i = 1, 2.
$$

From (2.5) and (2.8) together with the fact that $u_R = w$ in B_R, we have

$$
\int_{\mathbb{R}^2} w \frac{\partial \varphi}{\partial x_i} dx = - \int_{\mathbb{R}^2} u^i \varphi dx, \quad i = 1, 2,
$$

guaranteeing the existence of weak derivative of w, with $\dfrac{\partial w}{\partial x_i} = u^i$, $i = 1, 2$. As the direct effect of the last equality and (2.5), we ensure that $|\nabla w| \in L^2(\mathbb{R}^2)$. Moreover, by (2.6)

$$
\int_{\mathbb{R}^2} V(|x|) w^2 dx = \int_{\mathbb{R}^2} v^2 dx < \infty \quad \text{and} \quad \int_{\mathbb{R}^2} \log(1 + |x|) K(|x|) w^2 dx = \int_{\mathbb{R}^2} z^2 dx < \infty,
$$

and hence $w \in W$. Finally, it remains to prove that $u_n \to w$ in W. Observe that from (2.5),(2.6) and since $\dfrac{\partial w}{\partial x_i} = u^i$, $i = 1, 2$, it follows that

$$
\int_{\mathbb{R}^2} |\nabla u_n - \nabla w|^2 dx \to 0, \quad \int_{\mathbb{R}^2} V(|x|) |u_n - w|^2 dx = \int_{\mathbb{R}^2} \left| V(|x|)^{1/2} u_n - v \right|^2 dx \to 0, \quad \text{as } n \to \infty,
$$

$$
\int_{\mathbb{R}^2} \log(1 + |x|) K(|x|) |u_n - w|^2 dx = \int_{\mathbb{R}^2} \left| [\log(1 + |x|) K(|x|)]^{1/2} u_n - z \right|^2 dx \to 0, \quad \text{as } n \to \infty,
$$

completing the proof. □

Remark 2.2.2. *By estimates (2.7), we observe that for each open ball $B_R \subset \mathbb{R}^2$, the space W is continuously immersed in $H^1(B_R)$. Thus, in particular, W is continuously immersed in $L^p(B_R)$*

for all $1 \le p < \infty$. *Similarly, we can also conclude this fact for the space* E.

2.3 The variational setting

2.3.1 Properties of the nonlocal term

We now collect some important properties of the nonlocal term. First, we prove that the nonlocal term is well-defined. For this purpose, we need to introduce the following subspace of W:

$$W_{\text{rad}} := \Big\{ u \in W : u \text{ is radial} \Big\}.$$

Taking $r = |x - y|$ in the elementary identity $\log r = \log(1 + r) - \log(1 + r^{-1})$, for each $u \in W_{\text{rad}}$ we can write the nonlocal term as:

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(|x-y|)K(|y|)u^2(y)K(|x|)u^2(x)dx = \mathcal{V}_1(u) - \mathcal{V}_2(u), \tag{2.9}$$

where

$$\mathcal{V}_1(u) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x-y|)K(|y|)u^2(y)K(|x|)u^2(x)dydx,$$

and

$$\mathcal{V}_2(u) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x-y|^{-1})K(|y|)u^2(y)K(|x|)u^2(x)dydx.$$

Remark 2.3.1. *It follows from Remark 2.1.2 and Lemma 1.2.2 that the embedding* $W_{\text{rad}} \hookrightarrow L^2(\mathbb{R}^2; K)$ *is compact.*

Lemma 2.3.2. *The functional* $\mathcal{V}_1 : W_{\text{rad}} \to [0, \infty)$ *is well-defined and the following two statements hold:*

i) $\mathcal{V}_1(u) \le 2\|u\|^2_{L^2(\mathbb{R}^2;K)}\|u\|^2_W$;

ii) $\mathcal{V}_1 \in C^1(W_{\text{rad}}, \mathbb{R})$ *and for all* $v \in W_{\text{rad}}$, *we have*

$$\mathcal{V}_1'(u)v = 4\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x-y|)K(|y|)u^2(y)K(|x|)u(x)v(x)dydx.$$

In particular, we have that $\mathcal{V}_1'(u)u = 4\mathcal{V}_1(u)$.

Proof. Since $1 + |x - y| \le (1 + |x|)(1 + |y|)$ for all $x, y \in \mathbb{R}^2$ and the increasing behaviour of the *log*-function, we get the elementary inequality

$$\log(1+|x-y|) \le \log((1+|x|)(1+|y|)) = \log(1+|x|) + \log(1+|y|). \tag{2.10}$$

This, together with Remark 2.3.1 yields

$$\begin{aligned} \mathcal{V}_1(u) &\le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \Big(\log(1+|x|) + \log(1+|y|)\Big)K(|y|)u^2(y)K(|x|)u^2(x)dydx \\ &\le 2\|u\|^2_{L^2(\mathbb{R}^2;K)}\|u\|^2_W < \infty. \end{aligned} \tag{2.11}$$

Hence, the first assertion holds. Now we taking a sequence (u_n) in W_{rad} such that $u_n \to u$ in W_{rad}. A simple computation shows that

$$\begin{aligned}\mathcal{V}_1(u_n) - \mathcal{V}_1(u) = &\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x-y|)K(|y|)K(|x|)u_n^2(y)\Big(u_n^2(x) - u^2(x)\Big)dydx \\ &+ \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x-y|)K(|y|)K(|x|)\Big(u_n^2(y) - u^2(y)\Big)u^2(x)dydx.\end{aligned}$$

This, (2.10) and Hölder's inequality imply that

$$\begin{aligned}|\mathcal{V}_1(u_n) - \mathcal{V}_1(u)| \leq &\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x|)K(|y|)K(|x|)u_n^2(y)|u_n(x)-u(x)||u_n(x)+u(x)|dydx \\ &+ \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|y|)K(|y|)K(|x|)u_n^2(y)|u_n(x)-u(x)||u_n(x)+u(x)|dydx \\ &+ \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x|)K(|y|)K(|x|)|u_n(y)-u(y)||u_n(y)+u(y)|u^2(x)dydx \\ &+ \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|y|)K(|y|)K(|x|)|u_n(y)-u(y)||u_n(y)+u(y)|u^2(x)dydx \\ \leq &\|u_n\|^2_{L^2(\mathbb{R}^2;K)}\|u_n-u\|_W\|u_n+u\|_W + \|u_n\|^2_W\|u_n-u\|_{L^2(\mathbb{R}^2;K)}\|u_n+u\|_{L^2(\mathbb{R}^2;K)} \\ &+ \|u_n-u\|_{L^2(\mathbb{R}^2;K)}\|u_n+u\|_{L^2(\mathbb{R}^2;K)}\|u\|^2_W + \|u_n-u\|_W\|u_n+u\|_W\|u\|^2_{L^2(\mathbb{R}^2;K)}.\end{aligned}$$

Since (u_n) is bounded in W_{rad}, by Remark 2.3.1, we derive that $\mathcal{V}_1(u_n)$ converges to $\mathcal{V}_1(u)$, as $n \to \infty$. For any $v \in W_{\text{rad}}$, we see that the Gateaux derivative of $\mathcal{V}_1$ at $u \in W_{\text{rad}}$ is given by

$$\mathcal{V}_1'(u)v = 4\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x-y|)K(|y|)u^2(y)K(|x|)u(x)v(x)dydx.$$

From (2.10), Hölder's inequality and Remark 2.3.1, one has

$$\begin{aligned}|\mathcal{V}_1'(u)v| \leq 4&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\Big(\log(1+|x|)+\log(1+|y|)\Big)K(|y|)u^2(y)K(|x|)|u(x)v(x)|dydx \\ \leq &\|u\|^2_{L^2(\mathbb{R}^2;K)}\|u\|_W\|v\|_W + \|u\|^2_W\|u\|_{L^2(\mathbb{R}^2;K)}\|v\|_{L^2(\mathbb{R}^2;K)} \leq C_1\|v\|_W\end{aligned}, \tag{2.12}$$

and hence $\mathcal{V}_1'(u) \in W'$. Now, for any sequence $(u_n) \subset W_{\text{rad}}$ such that $u_n \to u$ and $v \in W_{\text{rad}}$ we have

$$\begin{aligned}\mathcal{V}_1'(u_n)v - \mathcal{V}_1'(u)v &= 4\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x-y|)K(|y|)K(|x|)\Big(u_n^2(y)u_n(x) - u^2(y)u(x)\Big)v(x)dydx \\ &= 4\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x-y|)K(|y|)K(|x|)u_n^2(y)\Big(u_n(x) - u(x)\Big)v(x)dydx \\ &\quad + 4\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x-y|)K(|y|)K(|x|)\Big(u_n^2(y) - u^2(y)\Big)u(x)v(x)dydx.\end{aligned}$$

This combined with (2.10), Hölder's inequality and Remark 2.3.1 yields that

$$\begin{aligned}\frac{|\mathcal{V}_1'(u_n)v - \mathcal{V}_1'(u)v|}{4} &\le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x|)K(|y|)K(|x|)u_n^2(y)|u_n(x)-u(x)||v(x)|dydx \\ &+ \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|y|)K(|y|)K(|x|)u_n^2(y)|u_n(x)-u(x)||v(x)|dydx \\ &+ \int_{\mathbb{R}^2} \log(1+|x|)K(|y|)K(|x|)|u_n(y)-u(y)||u_n(y)+u(y)|u(x)v(x)|dydx \\ &+ \int_{\mathbb{R}^2} \log(1+|y|)K(|y|)K(|x|)|u_n(y)-u(y)||u_n(y)+u(y)|u(x)v(x)|dydx \\ &\le \|u_n\|^2_{L^2(\mathbb{R}^2;K)}\|u_n-u\|_W\|v\|_W + \|u_n\|_W^2\|u_n-u\|_{L^2(\mathbb{R}^2;K)}\|v\|_{L^2(\mathbb{R}^2;K)} \\ &+ \|u_n-u\|_{L^2(\mathbb{R}^2;K)}\|u_n+u\|_{L^2(\mathbb{R}^2;K)}\|u\|_W\|v\|_W \\ &+ \|u_n-u\|_W\|u_n+u\|_W\|u\|_{L^2(\mathbb{R}^2;K)}\|v\|_{L^2(\mathbb{R}^2;K)} = o_n(\|v\|_W).\end{aligned}$$

Thus, $\mathcal{V}_1$ is continuously differentiable on W_{rad} and this ends the proof. □

The next embedding result will be crucial in the course of the work.

Lemma 2.3.3. *Assume that* (V) *and* (K) *hold. Then the embedding* $E_{\text{rad}} \hookrightarrow L^{8/3}(\mathbb{R}^2; K^{4/3})$ *is continuous and compact.*

Proof. Let $R > 0$ to be chosen later. For any $u \in E_{\text{rad}}$, we can split the integral as

$$\int_{\mathbb{R}^2} K^{4/3}(|x|)|u|^{8/3}dx = \int_{B_R} K^{4/3}(|x|)|u|^{8/3}dx + \int_{B_R^c} K^{4/3}(|x|)|u|^{8/3}dx. \tag{2.13}$$

Our first task is to estimate the integral on the ball. For this purpose, we observe that by hypothesis (K), there exists $C_1 > 0$ such that $K(|x|) \le C_1|x|^{l_0}$, for any $0 < |x| \le R$. Since $l_0 > -3/2$, then $(4l_0)/3 > -2$ and thus it is possible to get $p_1 > 1$ such that $(4p_1l_0)/3 > -2$. Hence $|x|^{(4p_1l_0)/3} \in L^1(B_R)$ and consequently by Hölder's inequality and Remark 2.2.2, one has

$$\begin{aligned}\int_{B_R} K^{4/3}(|x|)|u|^{8/3}dx &\le C_2\int_{B_R} |x|^{4l_0/3}|u|^{8/3}dx \\ &\le C_2\left(\int_{B_R}|x|^{(4p_1l_0)/3}dx\right)^{1/p_1}\left(\int_{B_R}|u|^{(8p_2)/3}dx\right)^{1/p_2} \le C_3\|u\|_E^{8/3},\end{aligned} \tag{2.14}$$

with $1/p_1 + 1/p_2 = 1$. Now, we estimate the second integral on the right-hand side of (2.13). By (K) and (1.10), there exists $C_4 > 0$ such that $K(|x|) \le C_4|x|^l$ and $|u(x)| \le C_5\|u\|_E|x|^{-(a+2)/4}$, for any $|x| \ge R$. Since $l < (a-1)/2$ then $(4l)/3 - 2(a+2)/3 + 2 < 0$, and so we obtain

$$\int_{B_R^c} K^{4/3}(|x|)|u|^{8/3}dx \le C_6\|u\|_E^{8/3}\int_{B_R^c}|x|^{(4l)/3-2(a+2)/3}dx = C_7\|u\|_E^{8/3}R^{4l/3-2(a+2)/3+2}. \tag{2.15}$$

This combined with (2.14), implies that the embedding $E_{\text{rad}} \hookrightarrow L^{8/3}(\mathbb{R}^2; K^{4/3})$ is continuous. We shall next prove the compactness. To do this, let (u_n) be a sequence in E_{rad} such that $u_n \rightharpoonup 0$

in E_{rad}. From (2.14), Remark 2.2.2 and Rellich-Kondrachov theorem, we can conclude

$$\int_{B_R} K^{4/3}(|x|)|u_n|^{8/3}dx = o_n(1). \tag{2.16}$$

On the other hand, since $(4l)/3 - 2(a+2)/3 + 2 < 0$ and $(u_n)_n$ is bounded, for $\varepsilon > 0$ arbitrary, we can take $R > 0$ large enough in (2.15) such that

$$\int_{B_R^c} K^{4/3}(|x|)|u_n|^{8/3}dx \leq \varepsilon, \quad \text{for all } \in \mathbb{N},$$

and this together with (2.16) completes the proof of the lemma. □

Remark 2.3.4. *Combining Remark 2.1.2 and Lemma 2.3.3, we infer that the embedding* $W_{\text{rad}} \hookrightarrow L^{8/3}(\mathbb{R}^2; K^{4/3})$ *is continuous and compact.*

To make use later, let us recall the well-known Hardy-Littlewood-Sobolev inequality.

Proposition 2.3.5. [49] *Let* $s, r > 1$ *and* $0 < \mu < 2$ *with* $1/s + \mu/2 + 1/r = 2$, $g \in L^s(\mathbb{R}^2)$, *and* $h \in L^r(\mathbb{R}^2)$. *There exists a sharp constant* $C(s, \mu, r) > 0$, *independent of* g, h, *such that*

$$\left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{g(y)h(x)}{|x-y|^{\mu}} \mathrm{d}y\mathrm{d}x \right| \leq C(s, \mu, r)\|g\|_{L^s(\mathbb{R}^2)}\|h\|_{L^r(\mathbb{R}^2)}.$$

Lemma 2.3.6. *The functional* $\mathcal{V}_2 : W_{\text{rad}} \to [0, \infty)$ *is well-defined and the following two statements hold:*

i) there exists $C > 0$ *such that* $\mathcal{V}_2(u) \leq C\|u\|_E^4$ *for all* $u \in W_{\text{rad}}$;

ii) $\mathcal{V}_2 \in C^1(W_{\text{rad}}, \mathbb{R})$ *and for each* $v \in W_{\text{rad}}$ *we have*

$$\mathcal{V}_2'(u)v = 4\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1 + |x-y|^{-1})K(|y|)u^2(y)K(|x|)u(x)v(x)dydx, \quad \textit{for all} \, v \in W_{\text{rad}}.$$

In particular, we have that $\mathcal{V}_2'(u)u = 4\mathcal{V}_2(u)$.

Proof. From the elementary inequality

$$\log(1 + |x-y|^{-1}) \leq |x-y|^{-1} \tag{2.17}$$

we get

$$\mathcal{V}_2(u) \leq \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{K(|y|)u^2(y)K(|x|)u^2(x)}{|x-y|}dydx.$$

Now, we can use Remark 2.3.4 and Proposition 2.3.5, with $\mu = 1$, $s = r = 4/3$, to obtain

$$\mathcal{V}_2(u) \leq C\left(\int_{\mathbb{R}^2} (K(|x|)u^2)^{4/3}dx\right)^{3/4}\left(\int_{\mathbb{R}^2} (K(|x|)u^2)^{4/3}dx\right)^{3/4} \leq C\|u\|_{E_{\text{rad}}}^4 < \infty,$$

and $i)$ is proved. Considering $(u_n) \subset W_{\text{rad}}$, such that $u_n \to u$ in W_{rad}, and using (2.17) it follows that

$$
\begin{aligned}
|\mathcal{V}_2(u_n) - \mathcal{V}_2(u)| &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x-y|^{-1})K(|y|)K(|x|)\Big(u_n^2(y)u_n^2(x) - u^2(y)u^2(x)\Big)dydx \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x-y|^{-1})K(|y|)K(|x|)u_n^2(y)\Big(u_n^2(x) - u^2(x)\Big)dydx \\
&\quad + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x-y|^{-1})K(|y|)K(|x|)\Big(u_n^2(y) - u^2(y)\Big)u^2(x)dydx \\
&\le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{K(|y|)K(|x|)u_n^2(y)\,|u_n(x)-u(x)|\,|u_n(x)+u(x)|}{|x-y|}dydx \\
&\quad + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{K(|y|)K(|x|)\,|u_n(y)-u(y)|\,|u_n(y)+u(y)|\,u^2(x)}{|x-y|}dydx.
\end{aligned}
$$

Combining the above estimate, Remark 2.3.4, Proposition 2.3.5, with $\mu = 1$, $s = r = 4/3$ and Hölder's inequality, we get

$$
\begin{aligned}
|\mathcal{V}_2(u_n) - \mathcal{V}_2(u)| &\le C\|u_n\|^2_{L^{8/3}(\mathbb{R}^2;K^{4/3})}\|u_n - u\|_{L^{8/3}(\mathbb{R}^2;K^{4/3})}\|u_n + u\|_{L^{8/3}(\mathbb{R}^2;K^{4/3})} \\
&\quad + C\|u_n - u\|_{L^{8/3}(\mathbb{R}^2;K^{4/3})}\|u_n + u\|_{L^{8/3}(\mathbb{R}^2;K^{4/3})}\|u\|^2_{L^{8/3}(\mathbb{R}^2;K^{4/3})} \\
&\le C\|u_n - u\|_{L^{8/3}(\mathbb{R}^2;K^{4/3})}\|u_n + u\|_{L^{8/3}(\mathbb{R}^2;K^{4/3})} \\
&\quad \times \Big(\|u_n\|^2_{L^{8/3}(\mathbb{R}^2;K^{4/3})} + \|u\|^2_{L^{8/3}(\mathbb{R}^2;K^{4/3})}\Big) = o_n(1).
\end{aligned}
\tag{2.18}
$$

Hence $\mathcal{V}_2$ is continuous on W_{rad}. We can see that for any $v \in W_{\text{rad}}$ the Gateaux derivative of $\mathcal{V}_2$ at $u \in W_{\text{rad}}$ along v is given by

$$
\mathcal{V}_2'(u)v = 4\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x-y|^{-1})K(|y|)u^2(y)K(|x|)u(x)v(x)dydx.
$$

By applying Proposition 2.3.5 with $\mu = 1$ and $s = r = 4/3$, and using Remark 2.3.4 together with Lemma 2.3.3 we derive that

$$
\begin{aligned}
|\mathcal{V}_2'(u)v| &\le 4\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{K(|y|)u^2(y)K(|x|)|u(x)||v(x)|}{|x-y|}dydx \\
&\le 4C\left(\int_{\mathbb{R}^2}(K(|x|)u^2)^{4/3}dx\right)^{3/4}\left(\int_{\mathbb{R}^2}(K(|x|)|uv|)^{4/3}dx\right)^{3/4} \\
&\le 4C\|u\|^3_{L^{8/3}(\mathbb{R}^2;K^{4/3})}\|v\|_{L^{8/3}(\mathbb{R}^2;K^{4/3})} \le C_1\|u\|^3_W\|v\|_W.
\end{aligned}
\tag{2.19}
$$

Thus, we conclude that $\mathcal{V}_2'(u) \in W'$. Now, we will prove that $\mathcal{V}_2$ is continuously differentiable on W_{rad}. To this, we observe that for any sequence $u_n \to u$ in W_{rad} and $v \in W_{\text{rad}}$, we have

$$
\begin{aligned}
\mathcal{V}_2'(u_n)v - \mathcal{V}_2'(u)v &= 4\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x-y|^{-1})K(|y|)K(|x|)\Big(u_n^2(y)u_n(x) - u^2(y)u(x)\Big)v(x)dydx \\
&= 4\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x-y|^{-1})K(|y|)K(|x|)u_n^2(y)\Big(u_n(x) - u(x)\Big)v(x)dydx \\
&\quad + 4\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x-y|^{-1})K(|y|)K(|x|)\Big(u_n^2(y) - u^2(y)\Big)u(x)v(x)dydx.
\end{aligned}
$$

This identity, (2.17), Lemma 2.3.3, Proposition 2.3.5, with $\mu = 1$, $s = r = 4/3$ and Hölder's inequality, implies that

$$\begin{aligned}\frac{|\mathcal{V}_2'(u_n)v - \mathcal{V}_2'(u)v|}{4} &\leq \int_{\mathbb{R}^2} \frac{K(|y|)K(|x|)u_n^2(y)|u_n(x) - u(x)||v(x)|}{|x-y|} dydx \\ &+ \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{K(|y|)K(|x|)|u_n(y) - u(y)||u_n(y) + u(y)||u(x)||v(x)|}{|x-y|} dydx \\ &\leq C\|u_n\|^2_{L^{8/3}(\mathbb{R}^2;K^{4/3})}\|u_n - u\|_{L^{8/3}(\mathbb{R}^2;K^{4/3})}\|v\|_{L^{8/3}(\mathbb{R}^2;K^{4/3})} \\ &+ C\|u_n - u\|_{L^{8/3}(\mathbb{R}^2;K^{4/3})}\|u_n + u\|_{L^{8/3}(\mathbb{R}^2;K^{4/3})} \\ &\times \|u\|_{L^{8/3}(\mathbb{R}^2;K^{4/3})}\|v\|_{L^{8/3}(\mathbb{R}^2;K^{4/3})} = o_n(\|v\|_W).\end{aligned}$$

Therefore, $\mathcal{V}_2$ is continuous differentiable on W_{rad} and so $\mathcal{V}_2 \in C^1(W_{\text{rad}}, \mathbb{R})$, which shows the result. □

2.3.2 Critical points of I_λ are weak solutions of $(\mathcal{E})$

In this subsection, inspired by the paper [16], we show that critical points of I_λ are weak solutions of $(\mathcal{E})$.

By (2.9), we see that the functional associated to equation $(\mathcal{E})$ can be written as

$$I_\lambda(u) = \frac{1}{2}\|u\|_E^2 + \frac{1}{4}\mathcal{V}_1(u) - \frac{1}{4}\mathcal{V}_2(u) - \lambda\int_{\mathbb{R}^2} Q(|x|)F(u)dx.$$

From (1.15), Lemma 1.2.2, and Theorem 1.2.5 we have that $\int_{\mathbb{R}^2} Q(|x|)F(u)dx$ is well-defined. This together with Lemmas 2.3.2 and 2.3.6, infer that the functional I_λ is well-defined. Moreover, by using standard arguments we see that $I_\lambda \in C^1(W_{\text{rad}}, \mathbb{R})$ with

$$I_\lambda'(u)v = \langle u, v\rangle_E + \frac{1}{4}\mathcal{V}_1'(u)v - \frac{1}{4}\mathcal{V}_2'(u)v - \lambda\int_{\mathbb{R}^2} Q(|x|)f(u)vdx, \quad \text{for all } v \in W_{\text{rad}}.$$

Inspired by [16, Lemma 5.1], we have the following version of the Principle of Symmetric Criticality due to Palais [65].

Proposition 2.3.7. *Assume* $(V), (K), (Q)$, (1.1), *and* (f_1). *If* $u \in W_{\text{rad}}$ *is a critical point of* I_λ, *then* u *is a weak solution of equation* $(\mathcal{E})$.

Proof. Let $u \in W_{\text{rad}}$ be fixed. We claim that the linear functional $T_u : W \to \mathbb{R}$ defined by

$$T_u(w) := \langle u, w\rangle_E + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(|x-y|)K(|y|)u^2(y)K(|x|)u(x)w(x)dydx - \lambda\int_{\mathbb{R}^2} Q(|x|)f(u)wdx$$

is well-defined and continuous on W. In the Proposition 1.2.8, it has been proven that

$$\left|\int_{\mathbb{R}^2} Q(|x|)f(u)wdx\right| \leq C_1\|w\|_E \leq C_1\|w\|_W.$$

Now, to ensure that T_u is well-defined and continuous on W, it is sufficient to analyze the term

$$G(w) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(|x-y|)K(|y|)u^2(y)K(|x|)u(x)w(x)dydx.$$

By Hölder's inequality, (2.10) and (1.21) replacing γ by 2 and Q by K, we estimate

$$\begin{aligned}|G(w)| &\leq \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \Big(\log(1+|x|)+\log(1+|y|)\Big)K(|y|)u^2(y)K(|x|)u(x)w(x)dydx \\ &\leq \|u\|^2_{L^2(\mathbb{R}^2;K)}\|u\|_W\|w\|_W + \|u\|^2_W \int_{\mathbb{R}^2} K(|x|)|u(x)w(x)|dx \leq C_2\|w\|_W.\end{aligned}$$

Therefore $|T_u(w)| \leq C_3\|w\|_W$, for all $w \in W$ and the claim is proved.

Now, if $u \in W_{\text{rad}}$ is a critical point of I_λ then $T_u(w) = 0$ for all $w \in W_{\text{rad}}$. The Riesz Representation Theorem in the space W guarantees the existence of a unique $\overline{u} \in W$ such that $T_u(\overline{u}) = \|\overline{u}\|^2_W = \|T_u\|^2_{W'}$, where W' denotes the dual space of W. Let $\mathcal{O}(2)$ be the group of orthogonal transformations in $\mathbb{R}^2$. Then, by using a change of variables, for each $w \in W$ we get

$$T_u(gw) = T_u(w) \quad \text{and} \quad \|gw\|_W = \|w\|_W, \quad \text{for all } \ g \in \mathcal{O}(2),$$

whence, applying with $w = \overline{u}$, by uniqueness, $g\overline{u} = \overline{u}$, for all $g \in \mathcal{O}(2)$, which means that $\overline{u} \in W_{\text{rad}}$. Consequently, since $T_u(w) = 0$, for all $w \in W_{\text{rad}}$, we obtain $\|T_u\|_{W'} = 0$, which implies that $T_u(w) = 0$, for all $w \in W$. This concludes the proof of the proposition. □

2.4 Proof of Theorem 2.1.4

In what follows, we denote by $\mathcal{N}_\lambda$ the Nehari manifold associated to the functional I_λ, that is,

$$\mathcal{N}_\lambda := \{u \in W_{\text{rad}} \setminus \{0\} : I'_\lambda(u)u = 0\}.$$

We first prove that $\mathcal{N}_\lambda$ is not empty and that I_λ is bounded from below on $\mathcal{N}_\lambda$.

Lemma 2.4.1. *Assume* $(V), (K), (Q)$, (1.1), $(f_1), (\widetilde{f}_3)$, *and* $(\widetilde{f}_4)$. *For each* $u \in W_{\text{rad}} \setminus \{0\}$, *there exists a unique* $t = t(u) > 0$ *such that* $tu \in \mathcal{N}_\lambda$. *Furthermore,* $I_\lambda(u) > 0$ *for every* $u \in \mathcal{N}_\lambda$.

Proof. For each $u \in W_{\text{rad}} \setminus \{0\}$, defining $\gamma_u(t) = I_\lambda(tu)$ for $t > 0$, we see that

$$tu \in \mathcal{N}_\lambda \Leftrightarrow I'_\lambda(tu)(tu) = 0 \Leftrightarrow I'_\lambda(tu)u = 0 \Leftrightarrow \gamma'_u(t) = 0. \tag{2.20}$$

First, we shall prove that $\gamma_u(t) > 0$ for $t > 0$ sufficiently small and $\lim_{t\to\infty}\gamma_u(t) = -\infty$. For this purpose, from (1.15), given $\varepsilon > 0$, $\alpha > \alpha_0$ and $q > \gamma$, one has

$$\int_{\mathbb{R}^2} Q(|x|)F(tu)dx \leq \frac{\varepsilon}{2}t^\gamma \int_{\mathbb{R}^2} Q(|x|)|u|^\gamma dx + C_2 t^q \int_{\mathbb{R}^2} Q(|x|)|u|^q \Phi_{\alpha,j_0}(tu)dx.$$

This, together with Hölder's inequality, (1.16), and Lemma 1.2.2, implies that

$$\begin{aligned}\int_{\mathbb{R}^2} Q(|x|)F(tu)dx &\leq \varepsilon C_3 t^{\gamma}\|u\|_E^{\gamma} + C_2 t^q \left(\int_{\mathbb{R}^2} Q(|x|)|u|^{r_1 q}dx\right)^{1/r_1} \left(\int_{\mathbb{R}^2} Q(|x|)\Phi_{r_2\alpha,j_0}(tu)dx\right)^{1/r_2} \\ &\leq \varepsilon C_3 t^{\gamma}\|u\|_E^{\gamma} + C_4 t^q \|u\|_E^q \left(\int_{\mathbb{R}^2} Q(|x|)\Phi_{r_2\alpha\|tu\|_E^2,j_0}\left(\frac{tu}{\|tu\|_E}\right)dx\right)^{1/r_2},\end{aligned}$$

whenever $r_1, r_2 > 1$ satisfies $1/r_1 + 1/r_2 = 1$. Choosing $t_1 > 0$ sufficiently small such that $r_2\alpha\|t_1 u\|_E^2 < \alpha_2$ and applying Theorem 1.2.5 we obtain

$$\int_{\mathbb{R}^2} Q(|x|)F(tu)dx \leq \varepsilon C_3 t^{\gamma}\|u\|_E^{\gamma} + C_5 t^q\|u\|_E^q,$$

for all $t \in (0, t_1)$. Therefore, using that $\mathcal{V}_1 \geq 0$ and Lemma 2.3.6, we infer that

$$\gamma_u(t) \geq t^2\left[\frac{1}{2}\|u\|_E^2 - \frac{C}{4}t^2\|u\|_E^4 - \lambda\varepsilon C_3 t^{\gamma-2}\|u\|_E^{\gamma} - \lambda C_5 t^{q-2}\|u\|_E^q\right].$$

Assuming that $\gamma > 2$, since $q > 2$ we can get $\bar{t} > 0$ such that $\gamma_u(t) > 0$ for all $t \in (0, \bar{t})$. If $\gamma = 2$, we can choose $0 < \varepsilon < 1/(2\lambda C_3)$ and deduce that

$$\gamma_u(t) \geq \frac{1}{2}\|u\|_E^2 - \frac{C}{4}t^2\|u\|_E^4 - \lambda\varepsilon C_3\|u\|_E^2 - \lambda C_5 t^{q-2}\|u\|_E^q > 0,$$

and so $\gamma_u(t) > 0$ for $t > 0$ sufficiently small.

To check that $\lim_{t\to\infty}\gamma_u(t) = -\infty$, we note that by Remark 2.1.3 there exists $C_0 > 0$ such that $F(s) \geq C_0 s^r$ for all $s \geq 0$. Since $\mathcal{V}_2 \geq 0$, by Lemma 2.3.2 we get

$$\begin{aligned}\gamma_u(t) &\leq \frac{t^2}{2}\|u\|_E^2 + \frac{1}{4}\mathcal{V}_1(tu) - \lambda C_0 t^r \int_{\mathbb{R}^2} Q(|x|)|u|^r dx \\ &\leq \frac{t^2}{2}\|u\|_E^2 + \frac{1}{2}t^4\|u\|_{L^2(\mathbb{R}^2;Q)}^2\|u\|_W^2 - \lambda C_0 t^r \int_{\mathbb{R}^2} Q(|x|)|u|^r dx,\end{aligned}$$

which implies that $\lim_{t\to\infty}\gamma_u(t) = -\infty$, since $r > 4$. As a consequence, there exists $t = t(u) > 0$ with $\gamma(t(u)) = I_\lambda(t(u)u) > 0$ such that $t(u)u \in \mathcal{N}_\lambda$. We claim that $t(u)$ is unique. Indeed, by Lemmas 2.3.2 and 2.3.6 it follows that

$$\mathcal{V}_1'(tu)u = 4\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log(1 + |x-y|)K(|y|)(tu)^2(y)K(|x|)(tu)(x)u(x)dydx = 4t^3\mathcal{V}_1(u)$$

and

$$\mathcal{V}_2'(tu)u = 4\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log(1 + |x-y|^{-1})K(|y|)(tu)^2(y)K(|x|)(tu)(x)u(x)dydx = 4t^3\mathcal{V}_2(u).$$

Computing the derivative, we get

$$\begin{aligned}\gamma_u'(t) &= t\|u\|_E^2 + \frac{1}{4}\mathcal{V}_1'(tu)u - \frac{1}{4}\mathcal{V}_1'(tu)u - \lambda\int_{\mathbb{R}^2} Q(|x|)f(tu)u dx \\ &= t\|u\|_E^2 + t^3\mathcal{V}_1(u) - t^3\mathcal{V}_2(u) - \lambda\int_{\mathbb{R}^2} Q(|x|)f(tu)u dx \\ &= t^3\left(\frac{1}{t^2}\|u\|_E^2 + \mathcal{V}_1(u) - \mathcal{V}_2(u) - \lambda\int_{\mathbb{R}^2} Q(|x|)\frac{f(tu)}{(tu)^3}u^4 dx\right).\end{aligned}$$

By hypothesis $(\widetilde{f}_4)$, the function $t \mapsto f(t)/t^3$ is increasing for $t > 0$ and hence $\gamma_u'(t)/t^3$ is decreasing. Suppose by contradiction that there are $t_2 > t_1 > 0$ such that $t_1 u, t_2 u \in \mathcal{N}_\lambda$. Then, it follows from (2.20) that $\gamma_u'(t_1) = \gamma_u'(t_2) = 0$ and using that $\gamma_u'(t)/t^3$ is decreasing we get

$$0 = \frac{\gamma_u'(t_1)}{t_1^3} > \frac{\gamma_u'(t_2)}{t_2^3} = 0,$$

which is a contradiction.

Lastly, we prove that $I_\lambda(u) > 0$ for any $u \in \mathcal{N}_\lambda$. In fact, since for all $u \in \mathcal{N}_\lambda$ there exists a unique $t > 0$ such that $tu \in \mathcal{N}_\lambda$ and $I_\lambda(tu) > 0$. By uniqueness, $t = 1$ and hence $I_\lambda(u) = I_\lambda(tu) > 0$ and this completes the proof. □

Remark 2.4.2. *As a byproduct of the above proof, we see that the point t_u which projects u in the Nehari manifold is exactly the maximum point of γ_u. Since $\gamma_u > 0$ near the origin and it has a unique critical point, we conclude that γ_u' is positive in $(0, t_u)$ and negative in (t_u, ∞). In particular, we have that $t_u \in (0, 1]$ whenever $\gamma_u'(1) = I_\lambda'(u)u \leq 0$.*

In view of Lemma 2.4.1 the value

$$c_\lambda := \inf_{u\in\mathcal{N}_\lambda} I_\lambda(u) \tag{2.21}$$

is well-defined. Now, we will prove that a minimizing sequence for c_λ is bounded in the norm $\|\cdot\|_E$.

Lemma 2.4.3. *Assume $(V), (K), (Q)$, (1.1), and $(\widetilde{f}_2)$. If $(u_n) \subset \mathcal{N}_\lambda$ is a minimizing sequence for c_λ, then (u_n) is bounded in the norm $\|\cdot\|_E$.*

Proof. Let $(u_n) \subset \mathcal{N}_\lambda$ be a minimizing sequence for c_λ. Thus $I_\lambda(u_n) = c_\lambda + o_n(1)$ and $I_\lambda'(u_n)u_n = 0$. Using this with the fact that $\mathcal{V}_1'(u_n)u_n = 4\mathcal{V}_1(u_n)$ and $\mathcal{V}_2'(u_n)u_n = 4\mathcal{V}_2(u_n)$ (see Lemmas 2.3.2 and 2.3.6), and $(\widetilde{f}_2)$ we have

$$\begin{aligned}c_\lambda + o_n(1) &= I_\lambda(u_n) - \frac{1}{4}I_\lambda'(u_n)u_n \\ &= \frac{1}{4}\|u_n\|_E^2 + \frac{1}{4}\mathcal{V}_1(u_n) - \frac{1}{4}\mathcal{V}_2(u_n) + \lambda\int_{\mathbb{R}^2} Q(|x|)\left(\frac{1}{4}f(u_n)u_n - F(u_n)\right)dx \\ &\quad - \frac{1}{16}\mathcal{V}_1'(u_n)u_n + \frac{1}{16}\mathcal{V}_2'(u_n)u_n \\ &= \frac{1}{4}\|u_n\|_E^2 + \lambda\int_{\mathbb{R}^2} Q(|x|)\left(\frac{1}{4}f(u_n)u_n - F(u_n)\right)dx \geq \frac{1}{4}\|u_n\|_E^2,\end{aligned} \tag{2.22}$$

for large $n \in \mathbb{N}$ and completes the proof. □

In the next result we prove that sequences in $\mathcal{N}_\lambda$ cannot converge to 0.

Lemma 2.4.4. *Assume* $(V), (K), (Q)$, (1.1), *and* (f_1). *Then there exists a constant* $C > 0$ *such that*

$$0 < C \leq \|u\|_E, \quad \textit{for all}\, u \in \mathcal{N}_\lambda.$$

Proof. Otherwise, there exists a sequence $(u_n) \subset \mathcal{N}_\lambda$ such that $u_n \to 0$ strongly in E. Since

$$\|u_n\|_E^2 + \frac{1}{4}\mathcal{V}_1'(u_n)u_n - \frac{1}{4}\mathcal{V}_2'(u_n)u_n - \lambda \int_{\mathbb{R}^2} Q(|x|)f(u_n)u_n dx = 0,$$

and $\mathcal{V}_1'(u_n)u_n = 4\mathcal{V}_1(u_n) \geq 0$ (see Lemma 2.3.2), there holds

$$\|u_n\|_E^2 - \frac{1}{4}\mathcal{V}_2'(u_n)u_n - \lambda \int_{\mathbb{R}^2} Q(|x|)f(u_n)u_n dx \leq 0.$$

Combining (2.19) and the continuous embedding in Lemma 2.3.3, one has

$$|\mathcal{V}_2'(u_n)u_n| \leq 4C\|u_n\|^3_{L^{8/3}(\mathbb{R}^2;K^{4/3})}\|u_n\|_{L^{8/3}(\mathbb{R}^2;K^{4/3})} = o_n(1) \tag{2.23}$$

and hence

$$\|u_n\|_E^2 + o_n(1) \leq \lambda \int_{\mathbb{R}^2} Q(|x|)f(u_n)u_n \mathrm{d}x. \tag{2.24}$$

According to (1.14), with $q > \gamma$, Hölder's inequality with exponents $1/r_1 + 1/r_2 = 1$, (1.16) and Lemma 1.2.2, we have

$$\begin{aligned}\int_{\mathbb{R}^2} Q(|x|)f(u_n)u_n dx \leq\ & \varepsilon \int_{\mathbb{R}^2} Q(|x|)|u_n|^\gamma dx \\ & + C_1 \left(\int_{\mathbb{R}^2} Q(|x|)|u_n|^{r_1 q} dx\right)^{1/r_1} \left(\int_{\mathbb{R}^2} Q(|x|)\Phi_{r_2\alpha, j_0}(u_n) dx\right)^{1/r_2} \\ & \leq \varepsilon C_2 \|u_n\|_E^\gamma + C_3\|u_n\|_E^q \left(\int_{\mathbb{R}^2} Q(|x|)\Phi_{r_2\alpha\|u_n\|_E^2, j_0}\left(\frac{u_n}{\|u_n\|_E}\right) dx\right)^{1/r_2}.\end{aligned}$$

From the convergence $u_n \to 0$ in E_{rad}, we get $r_2\alpha\|u_n\|_E^2 < \alpha_2$ for large $n \in \mathbb{N}$. This, Proposition 1.2.5, (2.24) and the last estimate, implies that

$$\|u_n\|_E^2 + o_n(1) \leq \lambda\varepsilon C_2\|u_n\|_E^\gamma + \lambda C_4\|u_n\|_E^q.$$

Taking $0 < \varepsilon < 1/(\lambda C_2)$, case $\gamma = 2$ and using that $q > \gamma \geq 2$, the above inequality contradicts the fact that $u_n \to 0$ strongly in E_{rad} and we finish the proof. □

In order, we will need the following compactnesses result:

Lemma 2.4.5. *Assume* $(V), (K), (Q)$, (1.1), (f_1), *and* $(\widetilde{f_2})$. *If* $(u_n) \subset \mathcal{N}_\lambda$ *is a minimizing sequence for* $c_\lambda < \alpha_2/4\alpha_0$ *such that* $u_n \rightharpoonup u$ *in* E_{rad}, *then:*

i) $Q(|x|)f(u_n)u_n \to Q(|x|)f(u)u$ *in* $L^1(\mathbb{R}^2)$;

ii) $Q(|x|)F(u_n) \to Q(|x|)F(u)$ *in* $L^1(\mathbb{R}^2)$.

Proof. To prove i), for every $R > 0$ we can write

$$\int_{\mathbb{R}^2} Q(|x|)(f(u_n)u_n - f(u)u)dx = J_1^R(n) + J_2^R(n), \tag{2.25}$$

where

$$J_1^R(n) := \int_{B_R} Q(|x|)(f(u_n)u_n - f(u)u)dx \quad \text{and} \quad J_2^R(n) := \int_{B_R^c} Q(|x|)(f(u_n)u_n - f(u)u)dx.$$

First we check that, for all $R > 0$ fixed we have

$$\lim_{n\to\infty} J_1^R(n) = 0. \tag{2.26}$$

In fact, for any $\varepsilon > 0$, according to Egoroff's Theorem there exists a measurable set $\Omega \subset B_R$ with $|\Omega| < \varepsilon$ such that $u_n(x) \to u(x)$ uniformly in $B_R \setminus \Omega$, and consequently

$$|J_1^R(n)| \leq \int_\Omega Q(|x|)f(u_n)u_n dx + \int_\Omega Q(|x|)f(u)u dx + o_n(1). \tag{2.27}$$

From (1.14), for $q \geq \gamma$ we see that

$$\int_\Omega Q(|x|)f(u_n)u_n dx \leq \varepsilon \int_\Omega Q(|x|)|u_n|^\gamma dx + C_1 \int_\Omega Q(|x|)|u_n|^q \Phi_{\alpha,j_0}(u_n)dx. \tag{2.28}$$

By (3.14) and the inequality $c_\lambda < \alpha_2/4\alpha_0$, one has

$$\lim_{n\to\infty} \|u_n\|_E^2 \leq 4c_\lambda < \frac{\alpha_2}{\alpha_0}.$$

Thus, we can obtain $r_1 > 1$, $\alpha > \alpha_0$ and $0 < \beta < \alpha_2$ such that $r_1\alpha\|u_n\|_E^2 \leq \beta < \alpha_2$, for large $n \in \mathbb{N}$. Therefore, by (1.16), Lemma 1.2.2, Hölder's inequality with exponents $1/r_1 + 1/r_2 + 1/r_3 = 1$ such that $r_2 \geq \gamma$ and Proposition 1.2.5, we get

$$\begin{aligned}\int_\Omega Q(|x|)|u_n|^q \Phi_{\alpha,j_0}(u_n)dx &\leq \left(\int_\Omega Q(|x|)\Phi_{r_1\alpha\|u_n\|_E^2,j_0}\left(\frac{u_n}{\|u_n\|_E}\right)dx\right)^{1/r_1} \\ &\quad \times \left(\int_\Omega Q(|x|)|u_n|^{r_2 q}dx\right)^{1/r_2}\left(\int_\Omega Q(|x|)dx\right)^{1/r_3} \\ &\leq C_2\left(\int_\Omega Q(|x|)dx\right)^{1/r_3}.\end{aligned} \tag{2.29}$$

From hypothesis (Q), there exists $C_3 > 0$ such that $Q(|x|) \leq C_3|x|^{b_0}$, for all $0 < |x| \leq R$. Since

$b_0 > -2$ we can choose $q_1 > 1$ with $1/q_1 + 1/q_2 = 1$ such that $q_1 b_0 > -2$ and hence

$$\int_\Omega Q(|x|)dx \leq C_3 \left(\int_\Omega |x|^{q_1 b_0} dx\right)^{1/q_1} |\Omega|^{1/q_2} \leq C_4 \varepsilon^{1/q_2}.$$

This combined with (2.28) and (2.29) implies

$$\int_\Omega Q(|x|) f(u_n) u_n dx \leq C_5 \varepsilon^{1/q_2 r_3}.$$

Similarly

$$\int_\Omega Q(|x|) f(u) u dx \leq C_5 \varepsilon^{1/q_2 r_3}$$

and hence (2.26) is true.

Next, we will prove that for any $\varepsilon > 0$ there is $R > 0$ such that for all n large

$$\left|J_2^R(n)\right| < \varepsilon. \tag{2.30}$$

In fact, since (u_n) is bounded and $r_1 \alpha \|u_n\|_E^2 \leq \beta < \alpha_2$, for large $n \in \mathbb{N}$, from (1.14), for $q \geq \gamma$, Hölder's inequality, (1.16), Lemma 1.2.2, and Theorem 1.2.5 it follows that

$$\begin{aligned}
\int_{B_R^c} Q(|x|) f(u_n) u_n dx &\leq \varepsilon \int_{B_R^c} Q(|x|)|u_n|^\gamma dx + C_6 \int_{B_R^c} Q(|x|)|u_n|^q \Phi_{\alpha, j_0}(u_n) dx \\
&\leq C_7 \varepsilon + C_6 \left(\int_{B_R^c} Q(|x|)|u_n|^{r_2 q} dx\right)^{1/r_2} \\
&\quad \times \left(\int_{B_R^c} Q(|x|) \Phi_{r_1 \alpha \|u_n\|_E^2, j_0}\left(\frac{u_n}{\|u_n\|_E}\right) dx\right)^{1/r_1} \\
&\leq C_7 \varepsilon + C_8 \left(\int_{B_R^c} Q(|x|)|u_n|^{r_2 q} dx\right)^{1/r_2},
\end{aligned}$$

for all $n \in \mathbb{N}$. Invoking Lemma 1.2.2, there holds

$$\limsup_{n \to \infty} \int_{B_R^c} Q(|x|) f(u_n) u_n dx \leq C_7 \varepsilon + C_8 \left(\int_{B_R^c} Q(|x|)|u|^{r_2 q} dx\right)^{1/r_2} < C_7 \varepsilon + C_8 \varepsilon,$$

for $R > 0$ large enough. A similar argument provides $\int_{B_R^c} Q(|x|) f(u) u dx < \varepsilon$. Hence (2.30) holds. By using estimates (2.26) and (2.30), from (2.25) we infer that

$$\limsup_{n \to \infty} \left|\int_{\mathbb{R}^2} Q(|x|)(f(u_n) u_n - f(u) u) dx\right| \leq \varepsilon.$$

As this holds for any $\varepsilon > 0$, the desired result follows.

To prove *ii*), note that by item *i*), there exists $g \in L^1(\mathbb{R}^2)$ such that

$$F(u_n) \leq \frac{1}{\theta} f(u_n)u_n \leq \frac{1}{\theta} g, \quad \text{a.e. in } \mathbb{R}^2,$$

where we used $(\widetilde{f}_2)$. Now, since $F(u_n) \to F(u)$ a.e. in $\mathbb{R}^2$, we can apply the dominated convergence theorem to get *ii*), proving the lemma. □

The next result guarantees that the weak limit of a minimizing sequence for c_λ is nonzero.

Lemma 2.4.6. *Assume* $(V), (K), (Q)$*,* (1.1)*,* (f_1)*, and* $(\widetilde{f}_2)$*. If* $(u_n) \subset \mathcal{N}_\lambda$ *is a minimizing sequence for* $c_\lambda < \alpha_2/4\alpha_0$ *such that* $u_n \rightharpoonup u$ *in* E_{rad}*, then* $u \neq 0$*.*

Proof. Suppose by contradiction that $u = 0$. Since $u_n \rightharpoonup u$ in E_{rad}, we can apply Lemma 2.3.3 in (2.23) to get $\mathcal{V}_2'(u_n)u_n = o_n(1)$. Moreover, from Lemma 2.4.5, it follows that

$$\int_{\mathbb{R}^2} Q(|x|)f(u_n)u_n dx = o_n(1).$$

Now using that

$$\|u_n\|_E^2 + \frac{1}{4}\mathcal{V}_1'(u_n)u_n = \frac{1}{4}\mathcal{V}_2'(u_n)u_n + \lambda \int_{\mathbb{R}^2} Q(|x|)f(u_n)u_n dx, \tag{2.31}$$

recalling that $\mathcal{V}_1'(u_n)u_n = 4\mathcal{V}_1(u_n) \geq 0$, we obtain that $u_n \to 0$ which contradicts Lemma 2.4.4. □

Now, we state a boundedness result.

Lemma 2.4.7. *Assume* $(V), (K), (Q)$*,* (1.1)*,* (f_1)*, and* $(\widetilde{f}_2)$*. If* $(u_n) \subset \mathcal{N}_\lambda$ *is a minimizing sequence for* $c_\lambda < \alpha_2/4\alpha_0$ *then, up to a subsequence,* $\mathcal{V}_1'(u_n)u_n \leq C$*, for all* $n \in \mathbb{N}$*.*

Proof. By Lemma 2.4.3 the sequence (u_n) is bounded in the norm $\|\cdot\|_E$. Since $I_\lambda'(u_n)u_n = 0$, we obtain

$$\frac{1}{4}\mathcal{V}_1'(u_n)u_n \leq \|u_n\|_E^2 + \frac{1}{4}\mathcal{V}_1'(u_n)u_n = \frac{1}{4}\mathcal{V}_2'(u_n)u_n + \lambda \int_{\mathbb{R}^2} Q(|x|)f(u_n)u_n dx. \tag{2.32}$$

From Lemma 2.3.6, we see that

$$\mathcal{V}_2'(u_n)u_n = 4\mathcal{V}_2(u_n) \leq 4C\|u_n\|_E^2 \leq C_1,$$

for all $n \in \mathbb{N}$. Next we will estimate the integral on the right-hand side of (2.32). To do this, we observe that since (u_n) is bounded in the norm $\|\cdot\|_E$, then $u_n \rightharpoonup u$ in E_{rad}. Therefore, using that $c_\lambda < \alpha_2/4\alpha_0$, we can apply Lemma 2.4.5 to get

$$\int_{\mathbb{R}^2} Q(|x|)f(u_n)u_n dx \leq C_2,$$

for all $n \in \mathbb{N}$, concluding the proof. □

Next, we shall obtain a boundedness in the norm of W.

Lemma 2.4.8. *Assume* $(V), (K), (Q)$, (1.1), (f_1), $(\widetilde{f_2})$, *and* $(\widetilde{f_3})$. *If* $(u_n) \subset \mathcal{N}_\lambda$ *is a minimizing sequence for* $c_\lambda < \alpha_2/4\alpha_0$, *then* (u_n) *is bounded in the norm* $\|\cdot\|_W$.

Proof. By Lemmas 2.4.3 and 2.4.6, we can assume that $u_n \rightharpoonup u$ weakly in $E_{\mathrm{rad}} \setminus \{0\}$. Since

$$\|u_n\|_W^2 = \|u_n\|_E^2 + \int_{\mathbb{R}^2} \log(1+|x|)K(|x|)u_n^2 dx,$$

it remains to calculate $\int_{\mathbb{R}^2} \log(1+|x|)K(|x|)u_n^2 dx$. Note that for $x \in \mathbb{R}^2 \setminus B_{2R}$ and $y \in B_R$, we obtain

$$1+|x-y| \geq 1+|x|-|y| \geq 1+|x|-R \geq 1+\frac{|x|}{2} \geq \sqrt{1+|x|}. \tag{2.33}$$

From Lemma 2.3.2, we may then estimate

$$\begin{aligned} \mathcal{V}_1'(u_n)u_n &\geq 4\int_{\mathbb{R}^2\setminus B_{2R}}\int_{B_R} \log(1+|x-y|)K(|y|)u_n^2(y)K(|x|)u_n^2(x)dydx \\ &\geq 2\int_{\mathbb{R}^2\setminus B_{2R}}\int_{B_R} \log(1+|x|)K(|y|)u_n^2(y)K(|x|)u_n^2(x)dydx \\ &= 2\left(\int_{B_R} K(|y|)u_n^2(y)dy\right)\left(\int_{\mathbb{R}^2\setminus B_{2R}} \log(1+|x|)K(|x|)u_n^2(x)dx\right). \end{aligned} \tag{2.34}$$

In view of the convergence $u_n \rightharpoonup u$ in $E_{\mathrm{rad}} \setminus \{0\}$, by Lemma 1.2.2 we see that $K(|y|)u_n^2(y) \to K(|y|)u^2(y)$ a.e. in B_R and so we can use Fatou's Lemma to obtain $\delta > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R} K(|y|)u_n^2(y)dy \geq \int_{B_R} K(|y|)u^2(y)dy > \delta > 0. \tag{2.35}$$

On the other hand, recalling that $\log(1+|x|) \leq 1+|x|$, there holds

$$\int_{B_{2R}} \log(1+|x|)K(|x|)u_n^2(x)dx \leq \int_{B_{2R}} (1+|x|)K(|x|)u_n^2(x)dx \leq C_1\|u_n\|_E^2 \leq C_2.$$

This, combined with (2.34) and (2.35), yields

$$\begin{aligned} \liminf_{n\to\infty} \mathcal{V}_1'(u_n)u_n &\geq 2\delta\left(\int_{\mathbb{R}^2} \log(1+|x|)K(|x|)u_n^2(x)dx - \int_{B_{2R}} \log(1+|x|)K(|x|)u_n^2(x)dx\right) \\ &\geq 2\delta\left(\int_{\mathbb{R}^2} \log(1+|x|)K(|x|)u_n^2(x)dx - C_2\right). \end{aligned}$$

Therefore, by Lemma 2.4.7 we conclude that (u_n) is bounded in the norm $\|\cdot\|_W$, proving the desired result. □

The next result is an estimate from above to c_λ.

Lemma 2.4.9. *Assume* $(V), (K), (Q)$, (1.1), *and* $(\widetilde{f_2}) - (\widetilde{f_4})$. *If* λ *satisfies* (2.3), *then* $c_\lambda < \alpha_2/4\alpha_0$.

Proof. First, we are going to consider a function $\varphi \in C^\infty_{0,\mathrm{rad}}(\mathbb{R}^2)$, given by $\varphi(x) = 1$ if $|x| \leq 1/2$, $\varphi(x) = 0$ if $|x| \geq 1$, $0 \leq \varphi(x) \leq 1$ for all $x \in \mathbb{R}^2$ and $|\nabla\varphi(x)| \leq 2$ for all $x \in \mathbb{R}^2$. By assumption $(\widetilde{f}_2)$ and Remark 2.1.3, one deduces that $f(s)s \geq \theta F(s) \geq \theta C_0 s^r$. By the fact that $\mathcal{V}_1'(\varphi)\varphi = 4\mathcal{V}_1(\varphi)$, $\mathcal{V}_2'(\varphi)\varphi = 4\mathcal{V}_2(\varphi) \geq 0$ and $\lambda \geq \lambda_0/(\theta C_0)$ (see (2.3) for the definition of λ_0), there holds

$$\begin{aligned} I_\lambda'(\varphi)\varphi \leq & \int_{B_1} \left[|\nabla\varphi|^2 + V(|x|)\varphi^2\right] dx + \int_{B_1}\int_{B_1} \log(1+|x-y|)K(|y|)\varphi^2(y)K(|x|)\varphi^2(x)dydx \\ & - \lambda\int_{B_1} Q(|x|)f(\varphi)\varphi dx < 4\pi + \|V\|_{L^1(B_1)} + \log 3\|K\|^2_{L^1(B_1)} - \lambda_0\|Q\|_{L^1(B_{1/2})} = 0 \end{aligned}$$

where we used that $\log(1+|x-y|) \leq \log 3$ in B_1 and the definition of λ_0. In particular

$$\int_{B_1} \left[|\nabla\varphi|^2 + V(|x|)\varphi^2\right] dx < \lambda_0\|Q\|_{L^1(B_{1/2})} - \log 3\|K\|^2_{L^1(B_1)}. \tag{2.36}$$

From Lemma 2.4.1, since $I_\lambda'(\varphi)\varphi \leq 0$ there exists $t \in (0,1]$ such that $t\varphi \in \mathcal{N}_\lambda$. Combining this, (2.36) and the hypothesis on λ, a simple computation shows that for all $t \in (0,1]$

$$\begin{aligned} c_\lambda \leq I_\lambda(t\varphi) \leq & \left[\frac{t^2}{2}\left(\int_{B_1} \left[|\nabla\varphi|^2 + V(|x|)\varphi^2\right] dx\right) + \frac{t^4\log 3}{4}\|K\|^2_{L^1(B_1)} - \lambda t^r\|Q\|_{L^1(B_{1/2})}\right] \\ < & \left[t^2\left(\lambda_0\|Q\|_{L^1(B_{1/2})} - \frac{\log 3}{2}\|K\|^2_{L^1(B_1)}\right) + \frac{t^4\log 3}{4}\|K\|^2_{L^1(B_1)} - \lambda t^r\|Q\|_{L^1(B_{1/2})}\right]. \end{aligned}$$

Since $\frac{t^4\log 3}{4} \leq \frac{t^2\log 3}{2}$ for all $t \in (0,1]$, we get

$$c_\lambda < \|Q\|_{L^1(B_{1/2})} \max_{t>0}\left[\lambda_0 t^2 - \lambda t^r\right].$$

By carrying out a straightforward computation, we conclude that

$$c_\lambda < \|Q\|_{L^1(B_{1/2})}\frac{\lambda_0^{\frac{r}{r-2}}}{\lambda^{\frac{2}{r-2}}}\left(\left(\frac{2}{r}\right)^{\frac{2}{r-2}} - \left(\frac{2}{r}\right)^{\frac{r}{r-2}}\right) \leq \alpha_2/4\alpha_0,$$

for any

$$\lambda \geq \widetilde{\lambda} = \left[\frac{4\alpha_0\|Q\|_{L^1(B_{1/2})}\lambda_0^{\frac{r}{r-2}}}{\alpha_2}\left(\left(\frac{2}{r}\right)^{\frac{2}{r-2}} - \left(\frac{2}{r}\right)^{\frac{r}{r-2}}\right)\right]^{\frac{r-2}{2}}.$$

Therefore, the estimate holds for all $\lambda \geq \bar{\lambda} := \max\{\lambda_0/(\theta C_0), \widetilde{\lambda}\}$ and the proof is complete. □

Now, we are ready to present the proof of Theorem 2.1.4.

Proof of Theorem 2.1.4. First, observe that without loss of generality, we can assume that $f(s) = 0$ for $s \leq 0$ and the above results are valid also for this modified nonlinearity, again denoted by f. For λ as in hypothesis (2.3), it follows from Lemma 2.4.9 that $c_\lambda < \alpha_2/4\alpha_0$. Let $(u_n) \subset \mathcal{N}_\lambda$ be a minimizing sequence for c_λ. By Lemma 2.4.3 we have that $u_n \rightharpoonup u$ in E_{rad}. By Lemmas 2.4.6 and 3.3.4 we conclude that $u \neq 0$ and (u_n) is bounded in the norm $\|\cdot\|_W$, respectively. Thus

$u_n \rightharpoonup v$ in W_{rad} and hence by Lemma 1.2.2 and Remark 2.3.1, we have that $u_n(x) \to u(x)$ and $u_n(x) \to v(x)$ a.e. in $\mathbb{R}^2$ and consequently $u = v \in W_{\text{rad}}$.

Our next task is to show that there exists $t > 0$ such that $tu \in \mathcal{N}_\lambda$ and $I_\lambda(tu) = c_\lambda$ for to conclude that tu is a point critical of I_λ, thanks to [19, Proposition 3.1] and [45, Lemma 2.5]. Since $I'_\lambda(u_n)u_n = 0$, then

$$\|u_n\|_E^2 + \frac{1}{4}\mathcal{V}'_1(u_n)u_n - \frac{1}{4}\mathcal{V}'_2(u_n)u_n - \lambda \int_{\mathbb{R}^2} Q(|x|)f(u_n)u_n dx = 0. \tag{2.37}$$

We claim that

$$\liminf_{n\to\infty} \mathcal{V}'_1(u_n)u_n \geq \mathcal{V}'_1(u)u. \tag{2.38}$$

In fact, for any $R > 0$ we see that

$$\mathcal{V}'_1(u_n)u_n \geq 4\int_{B_R}\int_{B_R} \log(1+|x-y|)K(|y|)u^2(y)K(|x|)u^2(x)dydx + 4D_n(R), \tag{2.39}$$

where

$$D_n := \int_{B_R}\int_{B_R} \log(1+|x-y|)K(|y|)K(|x|)\Big(u_n^2(y)u_n^2(x) - u^2(y)u^2(x)\Big)dydx.$$

On the other hand, there exists $C_1 = C_1(R)$ such that $\log(1+|x-y|) \leq C_1$ in B_R. By doing a straightforward computation and Remark 2.3.1, one has

$$\begin{aligned}
|D_n| &\leq C_1 \int_{B_R}\int_{B_R} K(|y|)K(|x|)u_n^2(y)\Big(u_n^2(x) - u^2(x)\Big)dydx \\
&\quad + C_1 \int_{B_R}\int_{B_R} K(|y|)K(|x|)\Big(u_n^2(y) - u^2(y)\Big)u^2(x)dydx \\
&\leq C_1\|u_n\|^2_{L^2(\mathbb{R}^2;K)}\|u_n - u\|_{L^2(\mathbb{R}^2;K)}\|u_n + u\|_{L^2(\mathbb{R}^2;K)} \\
&\quad + C_1\|u_n - u\|_{L^2(\mathbb{R}^2;K)}\|u_n + u\|_{L^2(\mathbb{R}^2;K)}\|u\|^2_{L^2(\mathbb{R}^2;K)} = o_n(1),
\end{aligned}$$

which combined with (2.4) implies that

$$\liminf_{n\to\infty} \mathcal{V}'_1(u_n)u_n \geq 4\int_{B_R}\int_{B_R} \log(1+|x-y|)K(|y|)u^2(y)K(|x|)u^2(x)dydx.$$

Considering a sequence $R_m \to \infty$ and applying the monotone convergence theorem to the sequence of functions $f_m(x) = \Big(\int_{B_{R_m}} \log(1+|x-y|)K(|y|)u^2(y)K(|x|)u^2(x)dy\Big)\chi_{B_{R_m}}$ together with Lemma 2.3.2 to get

$$\lim_{R\to\infty} 4\int_{B_R}\int_{B_R} \log(1+|x-y|)K(|y|)u^2(y)K(|x|)u^2(x)dydx = 4\mathcal{V}_1(u) = \mathcal{V}'_1(u)u,$$

and we conclude that the claim holds true. Moreover, using the fact that $\mathcal{V}'_2(u_n)u_n = 4\mathcal{V}_2(u_n)$

and $\mathcal{V}_2'(u)u = 4\mathcal{V}_2(u)$ together with Remark 2.3.4 and (2.18) we get

$$\lim_{n\to\infty} \mathcal{V}_2'(u_n)u_n = \mathcal{V}_2'(u)u.$$

Thus, by (2.37), (2.38), Lemma 2.4.5 and using that the norm is weakly lower semicontinuous, we obtain

$$I_\lambda'(u)u = \|u\|_E^2 + \frac{1}{4}\mathcal{V}_1'(u)u - \frac{1}{4}\mathcal{V}_2'(u)u - \lambda \int_{\mathbb{R}^2} Q(|x|)f(u)udx \le 0.$$

This inequality implies that there exists $t \in (0,1]$ such that $tu \in \mathcal{N}_\lambda$ (see Remark 2.4.2). Consequently,

$$c_\lambda \le I_\lambda(tu) = I_\lambda(tu) - \frac{1}{4}I_\lambda'(tu)(tu) = \frac{1}{4}\|tu\|_E^2 + \lambda \int_{\mathbb{R}^2} Q(|x|)\left(\frac{1}{4}f(tu)(tu) - F(tu)\right)dx. \quad (2.40)$$

On the other hand, by hypothesis $(\widetilde{f_3})$ the function $\frac{1}{4}f(s)s - F(s)$ is increasing in $(0,\infty)$ (see [8, Lemma 2.4]) and hence

$$\begin{aligned}\int_{\mathbb{R}^2} Q(|x|)\left(\frac{1}{4}f(tu)(tu) - F(tu)\right)dx &= \int_{\{u>0\}} Q(|x|)\left(\frac{1}{4}f(tu)(tu) - F(tu)\right)dx,\\ &\le \int_{\{u>0\}} Q(|x|)\left(\frac{1}{4}f(u)u - F(u)\right)dx\\ &= \int_{\mathbb{R}^2} Q(|x|)\left(\frac{1}{4}f(u)u - F(u)\right)dx.\end{aligned}$$

Combining (3.14), (3.19), Lemma 2.4.5, and the fact that the norm is weakly lower semicontinuous, implies that

$$c_\lambda \le I_\lambda(tu) \le \liminf_{n\to\infty}\left[\frac{1}{4}\|u_n\|_E^2 + \lambda\int_{\mathbb{R}^2} Q(|x|)\left(\frac{1}{4}f(u_n)u_n - F(u_n)\right)dx\right] = \liminf_{n\to\infty} I_\lambda(u_n) = c_\lambda.$$

Therefore, $c_\lambda = I_\lambda(tu)$ and this completes the proof of Theorem 2.1.4. □

2.5 Proof of Theorem 2.1.5

This section is devoted to the proof of Theorem 2.1.5. See that, as an application of Lemma 1.3.5 we have the following regularity result.

Lemma 2.5.1. *Assume* $(V), (K), (Q)$, *and* (1.1). *Suppose that* u_λ *is a weak solution of* $(\mathcal{E})$, *then* $u_\lambda \in C^{\widetilde{\sigma}}_{\text{loc}}(\mathbb{R}^2)$ *for some* $\widetilde{\sigma} \in (0,1)$.

Proof. Let $R > 0$ and define $v(x) := u_\lambda(x) - u_\lambda(R)$ with $x \in B_R$. By Remark 2.2.2, $v \in H_0^1(B_R)$ and v is a weak solution of problem (3.2.3) with

$$h(x,v) = \lambda Q(|x|)f(v + u_\lambda(R)) - V(|x|)(v + u_\lambda(R)) - \left[\log * K u_\lambda^2\right](x)K(|x|)(v + u_\lambda(R)).$$

Given $x \in B_R$, we estimate

$$\left|\left[\log * K u_\lambda^2\right](x)\right| \leq \int_{|x-y|<1} |\log(|x-y|)|\, K(|y|) u_\lambda^2(y) dy + \int_{|x-y|\geq 1} \log(|x-y|) K(|y|) u_\lambda^2(y) dy.$$

From Hölder's inequality and Remark 2.3.4 we get $C_1 > 0$ such that

$$\begin{aligned}\int_{|x-y|<1} |\log(|x-y|)|\, K(|y|) u_\lambda^2(y) dy &\leq \left(\int_{|x-y|<1} |\log(|x-y|)|^4 \,dy\right)^{1/4} \\ &\quad \times \left(\int_{\mathbb{R}^2} K^{4/3}(|y|) |u_\lambda(y)|^{8/3} dy\right)^{3/4} \\ &\leq C_1 \|u_\lambda\|_W^2 =: C_2.\end{aligned}$$

On the other hand, using that $\log|x-y| \leq \log(|x|+|y|) \leq \log\left((1+|x|)(1+|y|)\right) = \log(1+|x|) + \log(1+|y|)$ we get

$$\begin{aligned}\int_{|x-y|\geq 1} \log(|x-y|) K(|y|) u_\lambda^2(y) dy &\leq \int_{\mathbb{R}^2} \Big(\log(1+|x|) + \log(1+|y|)\Big) K(|y|) u_\lambda^2(y) dy \\ &\leq \log(1+|x|) \int_{\mathbb{R}^2} K(|y|) u_\lambda^2(y) dy \\ &\quad + \int_{\mathbb{R}^2} \log(1+|y|) K(|y|) u_\lambda^2(y) dy.\end{aligned}$$

Now using that $\log(1+|x|) \leq C_3$ in B_R and the continuous embedding $W_{\text{rad}} \hookrightarrow L^2(\mathbb{R}^2; K)$, we obtain

$$\left|\left[\log * K u_\lambda^2\right](x)\right| \leq C_4 + C_5 \|u_\lambda\|_W^2 =: C_6. \tag{2.41}$$

By the assumptions (V), (K) and (Q), there are constants $C_7, C_8 > 0$ and $C_9 > 0$ that depend on R such that $V(|x|) \leq C_7 |x|^{a_0}$, $K(|x|) \leq C_8 |x|^{l_0}$ and $Q(|x|) \leq C_9 |x|^{b_0}$, for every $0 < |x| \leq R$. Consequently, considering $d_0 := \min\{a_0, l_0, b_0,\} > -2$, the estimate above together with (1.1) and the continuity of $f(s)$, we can find $C_{10}, C_{11} > 0$ such that

$$|h(x,v)| \leq C_{10} |x|^{d_0} \Big(f(v + u_\lambda(R)) + (v + u_\lambda(R)\Big) \leq C_{11} |x|^{d_0} e^{\alpha v^2}, \quad \text{for a.e. } x \in B_R.$$

It follows from (2.41) that $[\log * K u_\lambda^2](x) \in L^1(B_R)$ and hence $h(x,v)$ is measurable. This and the above estimate, combined with Lemma 1.3.5, imply that $v \in C^{\widetilde{\sigma}}(\overline{B}_R)$ for some $\widetilde{\sigma} \in (0,1)$ and so $u_\lambda = v + u_\lambda(R) \in C^{\widetilde{\sigma}}(\overline{B}_R)$ and this completes the proof. □

Now we are ready to prove Theorem 2.1.5.

Proof of Theorem 2.1.5. Let u_λ be the weak solution obtained in Theorem 2.1.4. By Lemma 2.5.1, $u_\lambda \in C^{\widetilde{\sigma}}_{\text{loc}}(\mathbb{R}^2)$, for some $\widetilde{\sigma} \in (0,1)$. Taking into account that the potential $K \in C^{\sigma}_{\text{loc}}(\mathbb{R}^2)$, for some $\sigma \in (0,1)$ we see that $K u_\lambda^2$ is locally Hölder continuous. Since ϕ_{u_λ} is the Newtonian potential of $K u_\lambda^2$, by elliptic regularity (see [47, Lemma 4.2]) we have that $\phi_{u_\lambda} \in C^2(\mathbb{R}^2)$ and $\Delta \phi_{u_\lambda} = K u_\lambda^2$, in B_R for any $R > 0$. Therefore, the pair $(u_\lambda, \phi_{u_\lambda})$ is a weak solution of system $(\mathcal{S})$ and this finishes the proof. □

Chapter 3

On a planar Hartree-Fock type system

This chapter is devoted to study the existence of solutions for a class of Hartree-Fock type system in the two dimensional Euclidean space. Our approach is variation and based on a minimization technique in the Nehari manifold. The main steps in the prove are some trick estimates from the sign-changing logarithm potential in an appropriate subspace of $H^1(\mathbb{R}^2)$. This chapter is in [29].

3.1 Main results

Here, we are concerned with the existence of solutions to the following class of planar Hartree-Fock system

$$\begin{cases} -\Delta u + (1+\phi)u = |u|^{2p-2}u + \beta|v|^p|u|^{p-2}u, & \text{in } \mathbb{R}^2, \\ -\Delta v + (1+\phi)v = |v|^{2p-2}v + \beta|u|^p|v|^{p-2}v, & \text{in } \mathbb{R}^2, \\ \Delta\phi = 2\pi(u^2+v^2), & \text{in } \mathbb{R}^2, \end{cases} \tag{$\mathcal{S}_\beta$}$$

where $2 \le p < \infty$ and $\beta \ge 0$ is a real parameter.

In higher dimension, system $(\mathcal{S}_\beta)$ appears in quantum mechanics model describing the nonrelativistic electrons interacting with static nuclei via Coulomb forces. For more details on Hartree and Hartree-Fock approximations, see [40, 51, 53] and the references therein.

Our motivation to study $(\mathcal{S}_\beta)$, comes from the study of L^2- normalized solutions of planar coupled Schrödinger-Poisson equations developed in the works [8, 35, 36, 78]. In fact, system $(\mathcal{S}_\beta)$ has a slight relation with the couple Schrödinger-Poisson system

$$\begin{cases} -\Delta u + (1+\phi)u = |u|^{2p-2}u, & \text{in } \mathbb{R}^n, \\ -\Delta\phi = c_n u^2, & \text{in } \mathbb{R}^n, \end{cases} \tag{$\mathcal{SP}$}$$

where $c_n = 2\pi$ if $n = 2$ and $c_n = n(n-2)\omega_n$ if $n \ge 3$, with ω_n denoting the volume of the unit ball in $\mathbb{R}^n$, which has been object of intense study in recent years. For instance, if $n = 3$ then system of the type $(\mathcal{SP})$ appeared in semiconductor theory and has been studied in [20, 21, 33, 37, 74], and many others.

In the planar case $n = 2$, if the pair (u, ϕ) is a solution from the Schrödinger–Poisson system

$(\mathcal{SP})$ then the triple $(u, 0, \phi)$ is a semi-trivial solution of system $(\mathcal{S}_\beta)$.

An essential ingredient to solve $(\mathcal{S}_\beta)$ in dimension 3 consists in to use the Lax-Milgram Theorem to solve the third equation and obtain ϕ as the convolution $\phi = \Gamma_3 * 4\pi(u^2+v^2)$, where Γ_3 is the fundamental solution of the Laplacian in $\mathbb{R}^3$, namely $\Gamma_3(x) = (-1/4\pi)|x|^{-1}$ (see [40]).

In dimension $n = 2$, we can not make use of the same idea and there are less results available. However, given $(u, v) \in \widetilde{W} \times \widetilde{W}$, where $\widetilde{W}$ is an appropriated subspace of $H^1(\mathbb{R}^2)$, we can define at least formally the logarithmic potential

$$\phi_{u,v}(x) := \int_{\mathbb{R}^2} \log(|x-y|) \left(u^2(y) + v^2(y)\right) dy, \tag{3.1}$$

and so we are lead to consider the following auxiliary system with the nonlocal term $\phi_{u,v}$

$$\begin{cases} -\Delta u + (1+\phi_{u,v})u = |u|^{2p-2}u + \beta|v|^p|u|^{p-2}u, & \text{in } \mathbb{R}^2, \\ -\Delta v + (1+\phi_{u,v})v = |v|^{2p-2}v + \beta|u|^p|v|^{p-2}v, & \text{in } \mathbb{R}^2, \end{cases} \tag{$\mathcal{A}_\beta$}$$

and after obtaining a weak solution, we can use regularity theory to prove that the triple $(u, v, \phi_{u,v})$ weakly solves $(\mathcal{S}_\beta)$.

In fact, systems involving this kind of power nonlinearities have motivated a large amount of works (see for instance [10, 43, 56, 76] and references therein).

When dealing with $(\mathcal{A}_\beta)$ via variational methods, the first difficulty occurs due to the logarithmic kernel, which is unbounded and has indefinite sign. It turns out that the formal energy functional associated to the difficult equation is not well defined in $H^1(\mathbb{R}^2)$. To overcome this, Stubbe [78] (see also Cingolani-Weth [36]) introduced a new space which is appropriated to deal with the nonlocal part of the energy functional, namely

$$\mathcal{T}(u,v) := \int_{\mathbb{R}^2} \phi_{u,v} \left(u^2(x) + v^2(x)\right) dx = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\left(|x-y|\right) \left(u^2(y)+v^2(y)\right)\left(u^2(x)+v^2(x)\right) dydx.$$

Precisely, inspired by the paper [78], we shall addresses the a variational frame work to deal with $(\mathcal{A}_\beta)$, within the subspace of $H^1(\mathbb{R}^2)$ defined by

$$\widetilde{W} := \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} \log(1+|x|)u^2 dx < \infty \right\},$$

endowed with the norm

$$\|u\|_{\widetilde{W}} := \left(\|u\|^2_{H^1(\mathbb{R}^2)} + \int_{\mathbb{R}^2} \log(1+|x|)u^2 dx \right)^{1/2}.$$

We observe that $(\mathcal{A}_\beta)$ has, at least formally, a variational structure given by an associated energy functional defined in $\widetilde{W}$

$$I_\beta(u,v) = \frac{1}{2}\left(\|u\|^2_{H^1(\mathbb{R})} + \|v\|^2_{H^1(\mathbb{R})}\right) + \frac{1}{4}\mathcal{T}(u,v) - \frac{1}{2p}\left(\|u\|^{2p}_{L^{2p}(\mathbb{R}^2)} + \|v\|^{2p}_{L^{2p}(\mathbb{R}^2)} + 2\beta\int_{\mathbb{R}^2} |uv|^p dx\right)$$

with derivative given by

$$I'_\beta(u,v)(\varphi,\psi) = \int_{\mathbb{R}^2} (\nabla u \nabla \varphi + u\varphi + \nabla v \nabla \psi + v\psi)\, dx + \int_{\mathbb{R}^2} \phi_{u,v}\,(u\varphi + v\psi)\, dx$$
$$- \int_{\mathbb{R}^2} \left(|u|^{2p-2}u\varphi + |v|^{2p-2}v\psi\right) dx - \beta \int_{\mathbb{R}^2} \left(|v|^p|u|^{p-2}u\varphi + |u|^p|v|^{p-2}v\psi\right) dx.$$

Actually, it is necessary to guarantee that the nonlocal term $\mathcal{T}(u,v)$ is well defined. Naturally, we have the continuous Sobolev embeddings

$$\widetilde{W} \hookrightarrow H^1(\mathbb{R}^2) \hookrightarrow L^r(\mathbb{R}^2), \quad \text{for all } 2 \leq q < \infty. \tag{3.2}$$

We say that a pair $(u,v) \in \widetilde{W} \times \widetilde{W}$ is a weak solution to system $(\mathcal{A}_\beta)$ if for all $\varphi, \psi \in C_0^\infty(\mathbb{R}^2)$, it holds that

$$\int_{\mathbb{R}^2} (\nabla u \nabla \varphi + u\varphi)\, dx + \int_{\mathbb{R}^2} \phi_{u,v} u\varphi dx = \int_{\mathbb{R}^2} [|u|^{2p-2}u + \beta|v|^p|u|^{p-2}u]\varphi dx,$$

and

$$\int_{\mathbb{R}^2} (\nabla v \nabla \psi + v\psi)\, dx + \int_{\mathbb{R}^2} \phi_{u,v} v\psi dx = \int_{\mathbb{R}^2} [|v|^{2p-2}v + \beta|u|^p|v|^{p-2}v]\psi dx.$$

Therefore, critical points of I_β are weak solutions of $(\mathcal{A}_\beta)$.

In order to overcome the loss of compactness, we will work reduce ourselves to the radial setting

$$\widetilde{W}_{\text{rad}} := \left\{u \in H^1_{\text{rad}}(\mathbb{R}^2) : \int_{\mathbb{R}^2} \log(1+|x|)u^2 dx < \infty\right\},$$

and we have the compact embedding (see [77])

$$\widetilde{W}_{\text{rad}} \hookrightarrow H^1_{\text{rad}}(\mathbb{R}^2) \hookrightarrow L^q(\mathbb{R}^2), \quad \text{for all } 2 < q < \infty. \tag{3.3}$$

We also will prove that the functional I_β restricted to $\widetilde{W}_{\text{rad}} \times \widetilde{W}_{\text{rad}}$ is well defined and critical points of I_β are weak solutions of $(\mathcal{A}_\beta)$.

Our main interest here is on the least energy solutions to systems $(\mathcal{A}_\beta)$. Precisely, let us denote by $\mathcal{N}$ the Nehari manifold associated to the functional I_β, namely

$$\mathcal{N} := \left\{(u,v) \in \left(\widetilde{W}_{\text{rad}} \times \widetilde{W}_{\text{rad}}\right) \setminus \{0,0\} : \Gamma_\beta(u,v) = 0\right\},$$

where $\Gamma_\beta(u,v) := I'_\beta(u,v)(u,v)$, i.e.,

$$\Gamma_\beta(u,v) = \left(\|u\|^2_{H^1(\mathbb{R}^2)} + \|v\|^2_{H^1(\mathbb{R}^2)}\right) + \mathcal{T}(u,v) - \left(\|u\|^{2p}_{L^{2p}(\mathbb{R}^2)} + \|v\|^{2p}_{L^{2p}(\mathbb{R}^2)} + 2\beta \int_{\mathbb{R}^2} |uv|^p dx\right).$$

We shall see that the least energy level

$$c_\beta := \inf_{(u,v)\in\mathcal{N}} I_\beta(u,v),$$

is well defined (see Lemma 3.3.1) and so we will consider solutions that are minimizers of c_β, also called least energy solutions.

In this context we now formulate our main result, concerning systems $(\mathcal{A}_\beta)$ and $(\mathcal{S}_\beta)$.

Theorem 3.1.1. *Assume that $2 \leq p < \infty$. Then, for any $\beta \geq 0$ the coupled system $(\mathcal{A}_\beta)$ possesses a least energy solution $(u, v) \in \widetilde{W}_{\text{rad}} \times \widetilde{W}_{\text{rad}}$ with $u, v \geq 0$ satisfying the following statements:*

(i) for every $\beta > 2^{p-1} - 1$ the pair (u, v) is a vector solution, i.e., $u \neq 0$, $v \neq 0$ and in this case $u, v > 0$;

(ii) for $0 \leq \beta < 2^{p-1} - 1$ the least energy solution is semi-trivial, i.e., $u = 0$ or $v = 0$.

Furthermore, the triple $(u, v, \phi_{u,v})$ is a weak solution of system $(\mathcal{S}_\beta)$.

We emphasize here that the explicitly value of β obtained in item (i) of Theorem 3.1.1 is the same one obtained in [56].

To prove Theorem 3.1.1 we adopt here some arguments introduced in [40], where the authors have studied the system $(\mathcal{S}_\beta)$ in dimension 3, and [36, 78], where the couple Schrödinger-Poisson system of the type $(\mathcal{SP})$ was consider.

This chapter is organized as follows: In Section 3.2, we study the nonlocal term and establish the functional setting in which the problem will be posed, as well as some regularity properties. The final section is devoted to the proof of our existence result.

3.2 Preliminary results

3.2.1 Properties of the nonlocal term

First we collect some important properties of the nonlocal term. Using that $\log r = \log(1 + r) - \log(1 + r^{-1})$ for any $r > 0$, we can write

$$\mathcal{T}(u, v) = \mathcal{T}_1(u, v) - \mathcal{T}_2(u, v),$$

where

$$\mathcal{T}_1(u, v) := \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log\left(1 + |x - y|\right) \left(u^2(y) + v^2(y)\right) \left(u^2(x) + v^2(x)\right) dydx,$$

and

$$\mathcal{T}_2(u, v) := \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log\left(1 + |x - y|^{-1}\right) \left(u^2(y) + v^2(y)\right) \left(u^2(x) + v^2(x)\right) dydx.$$

Hence, I_β can be rewritten as

$$I_\beta(u, v) = \frac{1}{2}\left(\|u\|^2_{H^1(\mathbb{R})} + \|v\|^2_{H^1(\mathbb{R})}\right) + \frac{1}{4}\left[\mathcal{T}_1(u, v) - \mathcal{T}_2(u, v)\right] - \frac{1}{2p}\left(\|u\|^{2p}_{L^{2p}(\mathbb{R}^2)} + \|v\|^{2p}_{L^{2p}(\mathbb{R}^2)}\right)$$
$$- \frac{\beta}{p} \int_{\mathbb{R}^2} |uv|^p dx.$$

As proved in [36, Lemma 2.2], the nonlocal terms $\mathcal{T}_1$, $\mathcal{T}_2$ are well defined and belong to $C^1(\widetilde{W}\times\widetilde{W},\mathbb{R})$. Furthermore, taking into account that $1+|x-y|\leq(1+|x|)(1+|y|)$, for any $x,y\in\mathbb{R}^2$, we have that

$$\log(1+|x-y|)\leq\log((1+|x|)(1+|y|))=\log(1+|x|)+\log(1+|y|). \tag{3.4}$$

By using a straightforward computation, we find

$$\mathcal{T}_1(u,v)\leq 2\left(\|u\|^2_{L^2(\mathbb{R}^2)}+\|v\|^2_{L^2(\mathbb{R}^2)}\right)\left(\|u\|^2_{\widetilde{W}}+\|v\|^2_{\widetilde{W}}\right),\quad \text{for all } (u,v)\in\widetilde{W}\times\widetilde{W}. \tag{3.5}$$

Now, we estimate $\mathcal{T}_2(u,v)$. Applying Proposition 2.3.5 with $\mu=1$, $q=s=4/3$, and using the elementary inequality $\log(1+r)\leq r$ for any $r>0$ together with the Sobolev embedding (3.3), one has

$$\begin{aligned}\mathcal{T}_2(u,v)&\leq\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{u^2(y)u^2(x)+u^2(y)v^2(x)+v^2(y)u^2(x)+v^2(y)v^2(x)}{|x-y|}dydx\\&\leq C_1\left(\|u\|^4_{L^{8/3}(\mathbb{R}^2)}+2\|u\|^2_{L^{8/3}(\mathbb{R}^2)}\|v\|^2_{L^{8/3}(\mathbb{R}^2)}+\|v\|^4_{L^{8/3}(\mathbb{R}^2)}\right)\\&\leq C_2\left(\|u\|^4_{\widetilde{W}}+\|u\|^2_{\widetilde{W}}\|v\|^2_{\widetilde{W}}+\|v\|^4_{\widetilde{W}}\right).\end{aligned} \tag{3.6}$$

3.2.2 Critical points of I_β are weak solutions of $(\mathcal{A}_\beta)$

Inspired by [16, Lemma 5.1], we have the following version of the Principle of Symmetric Criticality due to Palais [65].

Proposition 3.2.1. *Assume that $p\geq 2$ and $\beta\geq 0$. If $(u,v)\in\widetilde{W}_{\mathrm{rad}}\times\widetilde{W}_{\mathrm{rad}}$ is a critical point of I_β, then (u,v) is a weak solution of system $(\mathcal{A}_\beta)$.*

Proof. Let $(u,v)\in\widetilde{W}_{\mathrm{rad}}\times\widetilde{W}_{\mathrm{rad}}$ be a critical point and consider the linear functionals $T_u,T_v:\widetilde{W}\to\mathbb{R}$ defined by

$$T_u(w):=\int_{\mathbb{R}^2}(\nabla u\nabla w+uw)\,dx+\int_{\mathbb{R}^2}\phi_{u,v}uwdx-\int_{\mathbb{R}^2}[|u|^{2p-2}u+\beta|v|^p|u|^{p-2}u]wdx$$

and

$$T_v(w):=\int_{\mathbb{R}^2}(\nabla v\nabla w+vw)\,dx+\int_{\mathbb{R}^2}\phi_{u,v}vwdx-\int_{\mathbb{R}^2}[|v|^{2p-2}v+\beta|u|^p|v|^{p-2}v]wdx.$$

We claim that T_u,T_v are continuous on $\widetilde{W}$. To see this, using (3.4) and Hölder's inequality, one deduce

$$\begin{aligned}\left|\int_{\mathbb{R}^2}\phi_{u,v}uwdx\right|&\leq\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\Big(\log(1+|x|)+\log(1+|y|)\Big)(u^2(y)+v^2(y))u(x)w(x)dydx\\&\leq(\|u\|^2_{L^2(\mathbb{R}^2)}+\|v\|^2_{L^2(\mathbb{R}^2)})\|u\|_{\widetilde{W}}\|w\|_{\widetilde{W}}+(\|u\|^2_{\widetilde{W}}+\|v\|^2_{\widetilde{W}})\|u\|_{L^2(\mathbb{R}^2)}\|w\|_{L^2(\mathbb{R}^2)}\\&\leq C_1\|w\|_{\widetilde{W}},\end{aligned}$$

where $C_1 = C_1(u,v)$. On the other hand, by Hölder's inequality and the embedding (3.2) we get

$$\left|\int_{\mathbb{R}^2} |u|^{2p-2}uwdx\right| \leq \left(\int_{\mathbb{R}^2} |u|^{2(2p-1)}dx\right)^{1/2}\left(\int_{\mathbb{R}^2} w^2dx\right)^{1/2} \leq C_2\|w\|_{\widetilde{W}},$$

and

$$\left|\int_{\mathbb{R}^2} |v|^p|u|^{p-2}uwdx\right| \leq \left(\int_{\mathbb{R}^2} |v|^{3p}dx\right)^{1/3}\left(\int_{\mathbb{R}^2} |u|^{3(p-1)}dx\right)^{1/3}\left(\int_{\mathbb{R}^2} |w|^3dx\right)^{1/3} \leq C_3\|w\|_{\widetilde{W}}.$$

Therefore, $|T_u(w)| \leq C_4\|w\|_{\widetilde{W}}$, for all $w \in \widetilde{W}$. Similarly we get $|T_v(w)| \leq C_5\|w\|_{\widetilde{W}}$ for all $w \in \widetilde{W}$ and so the claim is proved. By the Riesz representation theorem in the Hilbert space $\widetilde{W}$ there exists an unique $\overline{u} \in \widetilde{W}$ such that $T_u(\overline{u}) = \|\overline{u}\|^2_{\widetilde{W}} = \|T_u\|^2_{(\widetilde{W})'}$, where $(\widetilde{W})'$ denotes the dual space of $\widetilde{W}$. Similarly, there exists an unique $\overline{v} \in \widetilde{W}$ such that $T_v(\overline{v}) = \|\overline{v}\|^2_{\widetilde{W}} = \|T_v\|^2_{(\widetilde{W})'}$. Let $\mathcal{O}(2)$ be the group of orthogonal transformations in $\mathbb{R}^2$. Then, by performing a change of variables, for each $w \in \widetilde{W}$ we find

$$T_u(gw) = T_u(w), \quad T_v(gw) = T_v(w), \text{ and } \quad \|gw\|_{\widetilde{W}} = \|w\|_{\widetilde{W}}, \quad \text{for all } \ g \in \mathcal{O}(2).$$

Applying this with $w = \overline{u}$ and $w = \overline{v}$, by uniqueness we concluded that $g\overline{u} = \overline{u}$ and $g\overline{v} = \overline{v}$, for all $g \in \mathcal{O}(2)$, which means that $(\overline{u},\overline{v}) \in \widetilde{W}_{\text{rad}} \times \widetilde{W}_{\text{rad}}$. Consequently, if $(u,v) \in \widetilde{W}_{\text{rad}} \times \widetilde{W}_{\text{rad}}$ is a critical point of I_β, i.e., $T_u(w) = T_v(w) = 0$ for all $w \in \widetilde{W}_{\text{rad}}$ we obtain that $\|T_u\|_{(\widetilde{W})'} = \|T_v\|_{(\widetilde{W})'} = 0$, which implies that $T_u(w) = T_v(w) = 0$, for all $w \in \widetilde{W}$ and this completes the proof. □

3.2.3 Regularity results and non triviality

In this subsection, we discuss the regularity and positivity of nonnegative solution of system $(\mathcal{A}_\beta)$. We also analyze when the pair (u,v) is vectorial or semi-trivial depending from the parameter $\beta \geq 0$.

Proposition 3.2.2. *If the pair (u,v) is a solution of couple system $(\mathcal{A}_\beta)$ then the triple $(u,v,\phi_{u,v})$, where $\phi_{u,v}$ is defined in* (3.1)*, is a weak solution of system $(\mathcal{S}_\beta)$*

Proof. Indeed, let $(u,v) \in \widetilde{W} \times \widetilde{W}$ be a solution of $(\mathcal{A}_\beta)$ and $\varphi \in C_0^\infty(\mathbb{R}^2)$ fixed. Consider $R > 0$ such that B_R contains the support of φ. Since $u,v \in H^1(B_R)$, then $u,v \in L^q(B_R)$, for every $q > 1$. From the classical potential theory (see [47, Theorem 9.9]) we derive that $\phi_{u,v} \in W^{2,q}(B_R)$ and $\Delta\phi_{u,v} = 2\pi(u^2+v^2)$ for a.e. $x \in B_R$. This and the Divergence Theorem imply that

$$-\int_{B_R} \nabla\phi_{u,v}\nabla\varphi\,dx = \int_{B_R} (\Delta\phi_{u,v})\varphi\,dx = 2\pi\int_{B_R}(u^2+v^2)\varphi\,dx.$$

Therefore, $(u,v,\phi_{u,v}) \in W \times W \times W^{2,q}_{\text{loc}}(\mathbb{R}^2)$ is a weak solution of system $(\mathcal{S}_\beta)$ and this completes the proof. □

Lemma 3.2.3. *If $(u,v) \in \widetilde{W} \times \widetilde{W}$ is a weak solution of system $(\mathcal{A}_\beta)$, then $u,v \in C^2(\mathbb{R}^2)$.*

Proof. First we observe that for every $R > 0$ and $\varphi \in C_0^\infty(B_R)$ we have

$$\int_{B_R} (\nabla u \nabla \varphi + u\varphi)\, dx = \int_{B_R} h_1 \varphi dx,$$

where $h_1 = -\phi_{u,v} u + |u|^{2p-2}u + \beta |v|^p |u|^{p-2} u$. By Proposition 3.2.2 we know that $\phi_{u,v} \in W^{2,q}_{\mathrm{loc}}(\mathbb{R}^2)$ for every $q > 1$ and consequently by the Sobolev embedding $\phi_{u,v} \in C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^2)$ for some $\alpha \in (0,1)$ and hence $|\phi_{u,v}| \leq C_1$ in B_R for some constant $C_1 > 0$. A simple computation shows that, for every $q \geq 2$ there exists $C_2 > 0$ such that

$$|h_1|^q \leq C_2 \left(|u|^q + |u|^{q(2p-1)} + |v|^{2qp} + |u|^{2q(p-1)} \right) \quad \text{in} \quad B_R.$$

Therefore, $h_1 \in L^q(B_R)$ and by classical elliptic regularity theory, it follows that $u \in W^{2,q}(B_R) \hookrightarrow C^{0,\alpha_1}(B_R)$. Similarly, we have that $v \in C^{0,\alpha_2}(B_R)$. Thus, we conclude that $h_1 \in C^{0,\alpha_3}(B_R)$ for some $\alpha_3 > 0$ and by the regularity theorem of Agmon-Douglas-Nirenberg $u \in C^{2,\alpha}(B_R)$. Similarly, one has $v \in C^{2,\alpha}(B_R)$ and this finishes the proof. □

Lemma 3.2.4. *Assume that $2 \leq p < \infty$ and $\beta \geq 0$. Let $(u,v) \in \widetilde{W} \times \widetilde{W}$ be a minimizer of c_β with $u \geq 0$ and $v \geq 0$. If $u \neq 0$ and $v \neq 0$ then $u, v > 0$.*

Proof. If (u,v) is a minimizer of c_β, then (u,v) is a weak solution of system $(\mathcal{A}_\beta)$, and hence by Lemma 3.2.3 $u, v \in C^2(\mathbb{R}^2)$. Since, for every $r > 0$ we have that $\log r = \log(1+r) - \log(1+r^{-1})$, we can write

$$\phi_{u,v}(x) = \int_{\mathbb{R}^2} \log(|x-y|) \left(u^2(y) + v^2(y)\right) dy = \psi_1(x) - \psi_2(x),$$

where

$$\psi_1(x) := \int_{\mathbb{R}^2} \log(1 + |x-y|) \left(u^2(y) + v^2(y)\right) dy \geq 0$$

and

$$\psi_2(x) := \int_{\mathbb{R}^2} \log(1 + |x-y|^{-1}) \left(u^2(y) + v^2(y)\right) dy \geq 0.$$

Thus,

$$-\Delta u + (1 + \psi_1) u = \psi_2 u + |u|^{2p-2} u + \beta |v|^p |u|^{p-2} u \geq 0,$$

and the result follows from the strong maximum principle. □

The following is a key lemma in our analysis:

Lemma 3.2.5. *Assume that $p \geq 2$ and $\beta > 2^{p-1} - 1$. If (u,v) is a minimizer of c_β then $u \neq 0$ and $v \neq 0$.*

Proof. Let $(u,v) \in \mathcal{N}$ be such that $I_\beta(u,v) = c_\beta$ and suppose by contradiction that $v = 0$. Considering the vectorial function $(\tilde{u}, \tilde{v}) := (u\cos\theta, u\sin\theta) \in \left(\widetilde{W}_{\mathrm{rad}} \times \widetilde{W}_{\mathrm{rad}}\right) \setminus \{(0,0)\}$, by using a simple computation one can check that

$$\|\tilde{u}\|^2_{H^1(\mathbb{R}^2)} + \|\tilde{v}\|^2_{H^1(\mathbb{R}^2)} = \cos^2\theta \|u\|^2_{H^1(\mathbb{R}^2)} + \sin^2\theta \|u\|^2_{H^1(\mathbb{R}^2)} = \|u\|^2_{H^1(\mathbb{R}^2)},$$

$$
\begin{aligned}
\mathcal{T}(\tilde{u},\tilde{v}) &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\left(|x-y|\right)\left(\cos^2\theta u^2(y)+\sin^2\theta u^2(y)\right)\left(\cos^2\theta u^2(x)+\sin^2\theta u^2(x)\right)dydx \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\left(|x-y|\right) u^2(y)u^2(x)dydx = \mathcal{T}(u,0),
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\tilde{u}\|^{2p}_{L^{2p}(\mathbb{R}^2)} + \|\tilde{v}\|^{2p}_{L^{2p}(\mathbb{R}^2)} + 2\beta\int_{\mathbb{R}^2}|\tilde{u}\tilde{v}|^p dx \\
&\qquad = \left((\cos^2\theta)^p + (1-\cos^2\theta)^p + 2\beta(\cos^2\theta)^{p/2}(1-\cos^2\theta)^{p/2}\right)\|u\|^{2p}_{L^{2p}(\mathbb{R}^2)}.
\end{aligned}
$$

In particular, if we choose $\theta=\pi/4$ we get

$$
\|\tilde{u}\|^{2p}_{L^{2p}(\mathbb{R}^2)} + \|\tilde{v}\|^{2p}_{L^{2p}(\mathbb{R}^2)} + 2\beta\int_{\mathbb{R}^2}|\tilde{u}\tilde{v}|^p dx = (\beta+1)2^{1-p}\|u\|^{2p}_{L^{2p}(\mathbb{R}^2)} > \|u\|^{2p}_{L^{2p}(\mathbb{R}^2)},
$$

whenever $\beta > 2^{p-1}-1$. By (Lemma 3.3.1 below) there exists $t_0>0$ such that $(t_0\tilde{u}, t_0\tilde{v}) \in \mathcal{N}$. Consequently, it holds

$$
\begin{aligned}
I_\beta(t_0u,0) &= \frac{t_0^2}{2}\|u\|^2_{H^1(\mathbb{R}^2)} + \frac{t_0^4}{4}\mathcal{T}(u,0) - \frac{t_0^{2p}}{2p}\|u\|^{2p}_{L^{2p}(\mathbb{R}^2)} \\
&> \frac{t_0^2}{2}\left(\|\tilde{u}\|^2_{H^1(\mathbb{R}^2)} + \|\tilde{v}\|^2_{H^1(\mathbb{R}^2)}\right) + \frac{t_0^4}{4}\mathcal{T}(\tilde{u},\tilde{v}) \\
&\quad - \frac{t_0^{2p}}{2p}\left(\|\tilde{u}\|^{2p}_{L^{2p}(\mathbb{R}^2)} + \|\tilde{v}\|^{2p}_{L^{2p}(\mathbb{R}^2)} + 2\beta\int_{\mathbb{R}^2}|\tilde{u}\tilde{v}|^p dx\right) \\
&= I_\beta(t_0\tilde{u}, t_0\tilde{v}) \geq c_\beta.
\end{aligned}
\tag{3.7}
$$

This, together with the fact that $(u,0)\in\mathcal{N}$, Lemma 3.3.1, imply that

$$
c_\beta = I_\beta(u,0) = \max_{t>0} I_\beta(tu,0) \geq I_\beta(t_0u,0) > c_\beta,
$$

which is a contradiction and this finishes the proof. □

We also need the following lemma taken from [40]. Here we present a simple proof.

Lemma 3.2.6. *Let $p\geq 2$ and $0\leq\beta<2^{p-1}-1$. Then the function defined by $h_\beta(s) := s^p+(1-s)^p+2\beta s^{p/2}(1-s)^{p/2}$ with $s\in[0,1]$ satisfies $h_\beta(s)<1$, for all $s\in(0,1)$.*

Proof. First we note that $h_\beta(0)=1=h_\beta(1)$,

$$
h_\beta(\frac{1}{2}) = \frac{1+\beta}{2^{p-1}} < 1 \quad\text{and}\quad h_\beta(\frac{1}{2}-s) = h_\beta(\frac{1}{2}+s) \quad\text{for all}\quad s\in(-\frac{1}{2},\frac{1}{2}).
$$

Thus, is enough to prove that $h_\beta(s)<1$ for $s\in(0,1/2)$. Since h_0 is strictly decreasing in $(0,1/2)$ the case $\beta=0$ is trivial, and so let us consider $\beta>0$. Now, observe that

$$
h'_\beta(s) = ps^{p-1} - p(1-s)^{p-1} + \beta ps^{p/2-1}(1-s)^{p/2} - \beta ps^{p/2}(1-s)^{p/2-1}. \tag{3.8}
$$

We will assume that $p > 2$, since the case $p = 2$ is immediate. Notice that

$$\frac{h'_\beta(s)}{p(1-s)^{p-1}} = \left(\frac{s}{1-s}\right)^{p-1} - 1 + \beta\left(\frac{s}{1-s}\right)^{p/2-1} - \beta\left(\frac{s}{1-s}\right)^{p/2}. \tag{3.9}$$

Since $y = s/(1-s) \in (0,1)$, the right hand side of (3.9) can be written as

$$g_\beta(y) := y^{p-1} - 1 + \beta y^{p/2-1} - \beta y^{p/2}, \quad y \in [0,1].$$

Thus, is enough to prove that $g_\beta(y) < 0$ for all $y \in (0,1)$. For that, we observe $g_\beta(0) = -1$, $g_\beta(1) = 0$ and the derive of g_β is given by

$$g'_\beta(y) = (p-1)y^{p-2} + \frac{\beta(p-2)}{2}y^{p/2-2} - \frac{\beta p}{2}y^{p/2-1},$$

which implies

$$\frac{2g'_\beta(y)}{y^{p/2-2}} = 2(p-1)y^{p/2} + \beta(p-2) - \beta p y =: f_\beta(y). \tag{3.10}$$

We observe that $f_\beta(0) = \beta(p-2) > 0$, $f_\beta(1) = 2(p-1-\beta)$ and

$$f'_\beta(y) = p(p-1)y^{p/2-1} - \beta p.$$

Consequently,

$$f'_\beta(y) = 0 \Leftrightarrow p(p-1)y^{\frac{p-2}{2}} = \beta p \Leftrightarrow y = \left(\frac{\beta}{p-1}\right)^{\frac{2}{p-2}} =: y_0.$$

Depending from the location of the critical point y_0 we will consider three case:

Case 1: If $0 < \beta < p-1$ we have that $f_\beta(1) > 0$. This together with the fact that f_β is strictly decreasing in $(0, y_0)$ and strictly increasing in $(y_0, 1)$ implies that y_0 is a local minimum and hence a straightforward calculation shows that

$$f_\beta(y) \geq f_\beta(y_0) = \frac{\beta(p-2)}{(p-1)^{2/(p-2)}}\left((p-1)^{2/(p-2)} - \beta^{2/(p-2)}\right) > 0 \quad \text{for all } y \in [0,1],$$

and this concludes the proof in this case.

Case 2: If $\beta = p-1$ we have that $f_\beta(0) > 0$, $f_\beta(1) = 0$, and f_β is is strictly decreasing in $(0,1)$ and hence $f_\beta(y) > 0$ for $y \in (0,1)$ and this also concludes the proof in this case.

Case 3: If $\beta > p-1$, we have that $f_\beta(0) > 0$, $f_\beta(1) < 0$, and f_β is (strictly) decreasing in $(0,1)$. Thus, f_β has an unique zero t_β which is the unique critical point of g_β (see (3.10)). Moreover, g_β is strictly increasing in $(0, t_\beta)$ and strictly decreasing in $(t_\beta, 1)$. Consequently, g_β has a unique zero in $(0,1)$ which gives us a unique critical point of h_β in $(0, 1/2)$. Since $h_\beta(0) = h_\beta(1) = 1$, $\lim_{s\to 0+} h'_\beta(s) = -p < 0$ (that is, h_β is strictly decreasing in a neighborhood of 0), and $h_\beta(1/2) = (1+\beta)/2^{p-1} < 1$, then $h_\beta(s) < 1$, for any $s \in (0, 1/2)$ and the lemma is proved. □

Lemma 3.2.7. *Assume $p \geq 2$ and $0 \leq \beta < 2^{p-1} - 1$. If $(u,v) \in \widetilde{W} \times \widetilde{W}$ is a minimizer of c_β*

then $u = 0$ *or* $v = 0$.

Proof. Let $(u, v) \in \mathcal{N}$ be such that $I_\beta(u, v) = c_\beta$ and assume by contradiction that $u \neq 0$ and $v \neq 0$. By Lemma 3.2.4 we have that $u, v > 0$ and using polar coordinates for the pair (u, v), namely we write

$$(u, v) = (\rho\cos\theta, \rho\sin\theta) \quad \text{where} \quad \rho^2 = u^2 + v^2 \quad \text{and} \quad \theta = \theta(x) \in (0, \pi/2).$$

It is straightforward to check that

$$\nabla u = \nabla\rho\cos\theta - \rho\nabla\theta\sin\theta \quad \text{and} \quad \nabla v = \nabla\rho\sin\theta + \rho\nabla\theta\cos\theta.$$

Hence

$$\begin{aligned}|\nabla u|^2 + |\nabla v|^2 &= \left(|\nabla\rho|^2\cos^2\theta - 2\rho\cos\theta\sin\theta\nabla\rho\nabla\theta + \rho^2|\nabla\theta|^2\sin^2\theta\right) \\ &\quad + \left(|\nabla\rho|^2\sin^2\theta + 2\rho\sin\theta\cos\theta\nabla\rho\nabla\theta + \rho^2|\nabla\theta|^2\cos^2\theta\right) \\ &= |\nabla\rho|^2 + \rho^2|\nabla\theta|^2,\end{aligned}$$

and so

$$\|u\|^2_{H^1(\mathbb{R}^2)} + \|v\|^2_{H^1(\mathbb{R}^2)} = \int_{\mathbb{R}^2}|\nabla\rho|^2dx + \int_{\mathbb{R}^2}\rho^2|\nabla\theta|^2dx + \int_{\mathbb{R}^2}\rho^2dx \geq \|\rho\|^2_{H^1(\mathbb{R}^2)}.$$

On the other hand,

$$\begin{aligned}\mathcal{T}(u, v) &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log(|x-y|)\left(\rho^2(y)\cos^2\theta(y) + \rho^2(y)\sin^2\theta(y)\right) \\ &\quad \times \left(\rho^2(x)\cos^2\theta(x) + \rho^2(x)\sin^2\theta(x)\right)dydx \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log(|x-y|)\,\rho^2(y)\rho^2(x)dydx = \mathcal{T}(\rho, 0).\end{aligned}$$

Since $\theta \in (0, \pi/2)$, then $0 < \cos^2\theta < 1$. Thus, we can apply Lemma 3.2.6 to obtain

$$\begin{aligned}&\|u\|^{2p}_{L^{2p}(\mathbb{R}^2)} + \|v\|^{2p}_{L^{2p}(\mathbb{R}^2)} + 2\beta\int_{\mathbb{R}^2}|uv|^pdx \\ &\qquad = \int_{\mathbb{R}^2}\left((\cos^2\theta)^p + (\sin^2\theta)^p + 2\beta(\cos^2\theta)^{p/2}(\sin^2\theta)^{p/2}\right)|\rho|^{2p}dx \\ &\qquad < \|\rho\|^{2p}_{L^{2p}(\mathbb{R}^2)}.\end{aligned}$$

Thus, there exists $t_0 > 0$ such that $(t_0\rho, 0) \in \mathcal{N}$ (see Lemma 3.3.1 below). Consequently, we obtain

$$\begin{aligned}I_\beta(t_0u, t_0v) &= \frac{t_0^2}{2}\left(\|u\|^2_{H^1(\mathbb{R}^2)} + \|v\|^2_{H^1(\mathbb{R}^2)}\right) + \frac{t_0^4}{4}\mathcal{T}(u, v) \\ &\quad - \frac{t_0^{2p}}{2p}\left(\|u\|^{2p}_{L^{2p}(\mathbb{R}^2)} + \|v\|^{2p}_{L^{2p}(\mathbb{R}^2)} + 2\beta\int_{\mathbb{R}^2}|uv|^pdx\right) \qquad (3.11) \\ &> \frac{t_0^2}{2}\|\rho\|^2_{H^1(\mathbb{R}^2)} + \frac{t_0^4}{4}\mathcal{T}(\rho, 0) - \frac{t_0^{2p}}{2p}\|\rho\|^{2p}_{L^{2p}(\mathbb{R}^2)} = I_\beta(t_0\rho, 0) \geq c_\beta.\end{aligned}$$

This, together with the fact that $(u,v)\in\mathcal{N}$ and Lemma 3.3.1 below imply that

$$c_\beta = I_\beta(u,v) = \max_{t>0} I_\beta(tu,tv) > I_\beta(t_0\rho,0) \geq c_\beta,$$

which is a contradiction and this concludes the proof. □

3.3 Proof of Theorems 3.1.1

The proof of Theorem 3.1.1 will be fulfilled in some lemmas. We first prove that $\mathcal{N}$ is not empty and I_β is bounded from below on $\mathcal{N}$. More precisely, we have

Lemma 3.3.1. *Assume that $p\geq 2$ and $\beta\geq 0$. Then, for each $(u,v)\in\left(\widetilde{W}_{\text{rad}}\times\widetilde{W}_{\text{rad}}\right)\setminus\{0,0\}$, there exists an unique $t_{u,v}>0$ such that $I_\beta(t_{u,v}u,t_{u,v}v)=\max_{t>0}I_\beta(tu,tv)$ and $(t_{u,v}u,t_{u,v}v)\in\mathcal{N}$. Furthermore, $I_\beta(u,v)>0$ for every $(u,v)\in\mathcal{N}$.*

Proof. Let $(u,v)\in\widetilde{W}_{\text{rad}}\times\widetilde{W}_{\text{rad}}\setminus\{0,0\}$ and for $t>0$ we define

$$\begin{aligned}\gamma(t) := I_\beta(tu,tv) &= \frac{t^2}{2}\left(\|u\|^2_{H^1(\mathbb{R})}+\|v\|^2_{H^1(\mathbb{R})}\right)+\frac{t^4}{4}\mathcal{T}(u,v)\\ &-\frac{t^{2p}}{2p}\left(\|u\|^{2p}_{L^{2p}(\mathbb{R}^2)}+\|v\|^{2p}_{L^{2p}(\mathbb{R}^2)}+2\beta\int_{\mathbb{R}^2}|uv|^p dx\right).\end{aligned}$$

A simple computation shows that

$$\begin{aligned}\Gamma_\beta(tu,tv) &= t^2\left(\|u\|^2_{H^1(\mathbb{R})}+\|v\|^2_{H^1(\mathbb{R})}\right)+t^4\mathcal{T}(u,v)\\ &- t^{2p}\left(\|u\|^{2p}_{L^{2p}(\mathbb{R}^2)}+\|v\|^{2p}_{L^{2p}(\mathbb{R}^2)}+2\beta\int_{\mathbb{R}^2}|uv|^p dx\right)=t\gamma'(t),\end{aligned}$$

and consequently

$$(tu,tv)\in\mathcal{N}\Leftrightarrow\gamma'(t)=0. \tag{3.12}$$

Taking into account that

$$\gamma(t)=\frac{t^2}{2}\left(\|u\|^2_{H^1(\mathbb{R})}+\|v\|^2_{H^1(\mathbb{R})}\right)-\frac{t^4}{4}\mathcal{T}(u,v)-\frac{t^{2p}}{2p}\left(\|u\|^{2p}_{L^{2p}(\mathbb{R}^2)}+\|v\|^{2p}_{L^{2p}(\mathbb{R}^2)}+2\beta\int_{\mathbb{R}^2}|uv|^p dx\right),$$

and using that $p\geq 2>1$ we deduce that there exists $t_1>0$ sufficiently small such that $\gamma(t)>0$ for any $t\in(0,t_1)$. On the other hand, using that $p\geq 2>1$ we see $\lim_{t\to\infty}\gamma(t)=-\infty$. So, the function γ achieves its maximum value at some $t_{u,v}>0$ such that $\gamma'(t_{u,v})=0$. Furthermore, $t_{u,v}$ is the unique critical point of γ. To prove this, we observe that

$$\begin{aligned}\gamma'(t) &= t\left(\|u\|^2_{H^1(\mathbb{R})}+\|v\|^2_{H^1(\mathbb{R})}\right)+t^3\mathcal{T}(u,v)\\ &- t^{2p-1}\left(\|u\|^{2p}_{L^{2p}(\mathbb{R}^2)}+\|v\|^{2p}_{L^{2p}(\mathbb{R}^2)}+2\beta\int_{\mathbb{R}^2}|uv|^p dx\right)=t^3h(t),\end{aligned} \tag{3.13}$$

where

$$h(t) := \frac{\|u\|^2_{H^1(\mathbb{R})} + \|v\|^2_{H^1(\mathbb{R})}}{t^2} + \mathcal{T}(u,v) - t^{2p-4}\left(\|u\|^{2p}_{L^{2p}(\mathbb{R}^2)} + \|v\|^{2p}_{L^{2p}(\mathbb{R}^2)} + 2\beta\int_{\mathbb{R}^2}|uv|^p dx\right).$$

Since $p \geq 2$ we see that h is decreasing and as a consequence the function $\gamma'(t)/t^3$ is decreasing. Now, suppose by contradiction that there exist $t_2 > t_1 > 0$ such that $\gamma'(t_1) = \gamma'(t_2) = 0$. Then, using that $\gamma'(t)/t^3$ is decreasing, we find

$$0 = \frac{\gamma'(t_1)}{t_1^3} > \frac{\gamma'(t_2)}{t_2^3} = 0,$$

which is a contradiction. Finally, we prove that $I_\beta(u,v) > 0$ for every $(u,v) \in \mathcal{N}$. In fact, if $(u,v) \in \mathcal{N}$ there exists a unique $t > 0$ such that $(tu,tv) \in \mathcal{N}$ and $\gamma(t) > 0$ and so by uniqueness $t = 1$. Therefore, $I_\beta(u,v) = \gamma(1) = \gamma(t) > 0$ and this completes the proof. □

Remark 3.3.2. *As a byproduct of the above proof, we see that the point $t_{u,v}$ which projects (u,v) in the Nehari manifold is exactly the maximum point of γ. Since $\gamma > 0$ near the origin and it has a unique critical point, we conclude that γ' is positive in $(0,t_{u,v})$ and negative in $(t_{u,v},\infty)$. In particular, we have that $t_{u,v} \in (0,1]$ whenever $\gamma'(1) = \Gamma_\beta(u,v) \leq 0$.*

Next, we will prove that any minimizing sequence for c_β is bounded in $H^1(\mathbb{R}^2)$.

Lemma 3.3.3. *Assume that $p \geq 2$ and $\beta \geq 0$. If $(u_n,v_n) \subset \mathcal{N}$ is a minimizing sequence for c_β, then the following conditions holds:*

(i) the sequences (u_n) and (v_n) are bounded in the norm $\|\cdot\|_{H^1(\mathbb{R}^2)}$;

(ii) up to a subsequence $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$, in $H^1(\mathbb{R}^2)$ with $(u,v) \neq (0,0)$;

(iii) there exists $C > 0$ such that $\mathcal{T}_1(u_n,v_n) \leq C$, for all $n \in \mathbb{N}$.

Proof. Since $\Gamma_\beta(u_n,v_n) = 0$ and $p \geq 2$, we see that

$$\begin{aligned} c_\beta + o_n(1) &= I_\beta(u_n,v_n) - \frac{1}{4}\Gamma_\beta(u_n,v_n) \\ &= \frac{1}{4}\left(\|u_n\|^2_{H^1(\mathbb{R}^2)} + \|v_n\|^2_{H^1(\mathbb{R}^2)}\right) \\ &\quad + \frac{(p-2)}{4p}\left(\|u_n\|^{2p}_{L^{2p}(\mathbb{R}^2)} + \|v_n\|^{2p}_{L^{2p}(\mathbb{R}^2)} + 2\beta\int_{\mathbb{R}^2}|u_nv_n|^p dx\right) \\ &\geq \frac{1}{4}\left(\|u_n\|^2_{H^1(\mathbb{R}^2)} + \|v_n\|^2_{H^1(\mathbb{R}^2)}\right), \end{aligned} \tag{3.14}$$

which implies (i). Thus, up to a subsequence we can assume that $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$, in $H^1(\mathbb{R}^2)$ and we claim that $(u,v) \neq (0,0)$. Otherwise, $\|u_n\|_{H^1(\mathbb{R}^2)} \to 0$ and $\|v_n\|_{H^1(\mathbb{R}^2)} \to 0$. By (3.6) and the embedding (3.3) with $r = 8/3$ it follows that

$$\mathcal{T}_2(u_n,v_n) \leq C_0\left(\|u_n\|^4_{H^1(\mathbb{R}^2)} + \|v_n\|^4_{H^1(\mathbb{R}^2)}\right). \tag{3.15}$$

Thus, using that $\Gamma_\beta(u_n, v_n) = 0$ and $\mathcal{T}_1(u_n, v_n) \geq 0$, we get

$$\|u_n\|^2_{H^1(\mathbb{R}^2)} + \|v_n\|^2_{H^1(\mathbb{R}^2)} \leq C_1 \left(\|u_n\|^{2p}_{H^1(\mathbb{R}^2)} + \|v_n\|^{2p}_{H^1(\mathbb{R}^2)} \right) + 2\beta \int_{\mathbb{R}^2} |u_n v_n|^p dx + \mathcal{T}_2(u_n, v_n).$$

On the other hand, using Hölder's inequality and the embedding (3.2) we obtain

$$\int_{\mathbb{R}^2} |u_n v_n|^p dx \leq \|v_n\|^p_{L^{2p}(\mathbb{R}^2)} \|u_n\|^p_{L^{2p}(\mathbb{R}^2)} \leq C_2 \left(\|v_n\|^{2p}_{H^1(\mathbb{R}^2)} + \|u_n\|^{2p}_{H^1(\mathbb{R}^2)} \right). \tag{3.16}$$

Combining this estimates we conclude that

$$\left(\|u_n\|^2_{H^1(\mathbb{R}^2)} + \|v_n\|^2_{H^1(\mathbb{R}^2)} \right) \leq C_0 \left(\|u_n\|^4_{H^1(\mathbb{R}^2)} + \|v_n\|^4_{H^1(\mathbb{R}^2)} \right) + (C_1 + C_2) \left(\|u_n\|^{2p}_{H^1(\mathbb{R}^2)} + \|v_n\|^{2p}_{H^1(\mathbb{R}^2)} \right),$$

from where we obtain a contradiction, since $p \geq 2 > 1$ and (ii) is proved. To see that (iii) holds, since $\Gamma_\beta(u_n, v_n) = 0$, i.e.,

$$\left(\|u_n\|^2_{H^1(\mathbb{R}^2)} + \|v_n\|^2_{H^1(\mathbb{R}^2)} \right) + \mathcal{T}_1(u_n, v_n) = \mathcal{T}_2(u_n, v_n) + \left(\|u_n\|^{2p}_{L^{2p}(\mathbb{R}^2)} + \|v_n\|^{2p}_{L^{2p}(\mathbb{R}^2)} + 2\beta \int_{\mathbb{R}^2} |u_n v_n|^p dx \right),$$

from estimates (3.15)-(3.16) and the Sobolev embedding we get

$$\mathcal{T}_1(u_n, v_n) \leq C_0 \left(\|u_n\|^4_{H^1(\mathbb{R}^2)} + \|v_n\|^4_{H^1(\mathbb{R}^2)} \right) + (C_1 + C_2) \left(\|u_n\|^{2p}_{H^1(\mathbb{R}^2)} + \|v_n\|^{2p}_{H^1(\mathbb{R}^2)} \right),$$

and hence (iii) follows from item (i), and this completes the proof. □

Next, we shall obtain boundedness in the norm of $\widetilde{W}$.

Lemma 3.3.4. *Assume that $p \geq 2$ and $\beta \geq 0$. If $(u_n, v_n) \subset \mathcal{N}$ is a minimizing sequence for c_β, then (u_n) and (v_n) are bounded in the norm $\| \cdot \|_{\widetilde{W}}$.*

Proof. First we observe that

$$\|u_n\|^2_{\widetilde{W}} = \|u_n\|^2_{H^1(\mathbb{R}^2)} + \int_{\mathbb{R}^2} \log(1 + |x|) u_n^2 dx \quad \text{and} \quad \|v_n\|^2_{\widetilde{W}} = \|v_n\|^2_{H^1(\mathbb{R}^2)} + \int_{\mathbb{R}^2} \log(1 + |x|) v_n^2 dx.$$

Thus, by item (i) of Lemma 3.3.3 it remains to prove that there exists $C > 0$ such that

$$\int_{\mathbb{R}^2} \log(1 + |x|)(u_n^2 + v_n^2) dx \leq C, \quad \forall n \in \mathbb{N}. \tag{3.17}$$

For this, by Lemma 3.3.3 there exists $R > 0$ such that

$$\int_{B_R} (u^2 + v^2)\, dx > 0.$$

For any $x \in \mathbb{R}^2 \setminus B_{2R}$ and $y \in B_R$, there holds

$$1+|x-y| \geq 1+|x|-|y| \geq 1+|x|-R \geq 1+\frac{|x|}{2} \geq \sqrt{1+|x|}.$$

Now using Lemma 3.3.3 we deduce that

$$\begin{aligned} C_1 \geq \mathcal{T}_1(u_n,v_n) &\geq \int_{\mathbb{R}^2\setminus B_{2R}} \int_{B_R} \log(1+|x-y|) \left(u_n^2(y)+v_n^2(y)\right) \left(u_n^2(x)+v_n^2(x)\right) dydx \\ &= \frac{1}{2}\left(\int_{B_R} \left(u_n^2(y)+v_n^2(y)\right) dy\right) \left(\int_{\mathbb{R}^2\setminus B_{2R}} \log(1+|x|) \left(u_n^2(x)+v_n^2(x)\right) dx\right). \end{aligned}$$

Taking the limit and using the compact embedding $H^1(B_R) \hookrightarrow L^2(B_R)$ we get

$$\limsup_{n\to\infty} \int_{\mathbb{R}^2\setminus B_{2R}} \log(1+|x|) \left(u_n^2(x)+v_n^2(x)\right) dx \leq 2C_1 \left(\int_{B_R} (u^2+v^2)\, dx\right)^{-1}.$$

On the other hand, using that $\log(1+|x|) \leq 1+|x|$ we see that

$$\int_{B_{2R}} \log(1+|x|) \left(u_n^2(x)+v_n^2(x)\right) dx \leq (1+2R) \left(\|u_n\|^2_{H^1(\mathbb{R}^2)} + \|v_n\|^2_{H^1(\mathbb{R}^2)}\right) \leq C_2.$$

Therefore, (3.17) holds and this completes the proof. $\square$

We collect some auxiliary compactnesses results in the following

Lemma 3.3.5. *Assume that $p \geq 2$ and $\beta \geq 0$. If $(u_n,v_n) \subset \mathcal{N}$ is a minimizing sequence for c_β, then the following statements hold true:*

(i) $\lim_{n\to\infty} \int_{\mathbb{R}^2} |u_n v_n|^p dx = \int_{\mathbb{R}^2} |uv|^p dx$;

(ii) $\liminf_{n\to\infty} \mathcal{T}_1(u_n,v_n) \geq \mathcal{T}_1(u,v)$;

(iii) $\lim_{n\to\infty} \mathcal{T}_2(u_n,v_n) = \mathcal{T}_2(u,v)$.

Proof. Note that by Hölder's inequality, one deduces

$$\begin{aligned} \left|\int_{\mathbb{R}^2} \left(|u_n v_n|^p - |uv|^p\right) dx\right| &\leq \int_{\mathbb{R}^2} |u_n|^p \left||v_n|^p - |v|^p\right| dx + \int_{\mathbb{R}^2} |v|^p \left||u_n|^p - |u|^p\right| dx \\ &\leq \|u_n\|^p_{L^{2p}(\mathbb{R}^2)} \left(\int_{\mathbb{R}^2} \left||v_n|^p - |v|^p\right|^2 dx\right)^{1/2} \\ &\quad + \|v\|^p_{L^{2p}(\mathbb{R}^2)} \left(\int_{\mathbb{R}^2} \left||u_n|^p - |u|^p\right|^2 dx\right)^{1/2}, \end{aligned} \tag{3.18}$$

and one can easily obtain (i) from the compact embedding (3.3). To prove (ii) we write

$$\mathcal{T}_1(u_n,v_n) = A_n^1 + 2A_n^2 + A_n^3,$$

where

$$A_n^1 := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x-y|)\, u_n^2(y)u_n^2(x)dydx, \quad A_n^2 := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x-y|)\, u_n^2(y)v_n^2(x)dydx,$$

and

$$A_n^3 := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x-y|)\, v_n^2(y)v_n^2(x)dydx.$$

Now, we observe that for any $R>0$ we have

$$A_n^1 \geq D_n(R) + \int_{B_R}\int_{B_R} \log(1+|x-y|)u^2(y)u^2(x)dydx,$$

where $D_n := D_n^1 + D_n^2$ and

$$D_n^1 := \int_{B_R}\int_{B_R} \log(1+|x-y|)u_n^2(y)\Big(u_n^2(x)-u^2(x)\Big)dydx,$$

$$D_n^2 := \int_{B_R}\int_{B_R} \log(1+|x-y|)u^2(x)\Big(u_n^2(y)-u^2(y)\Big)dydx.$$

Taking into account that $\log(1+|x-y|) \leq C_1$ for $x,y \in B_R$, it follows from Hölder's inequality and the compact embedding $H^1(B_R) \hookrightarrow L^2(B_R)$ that

$$|D_n^1| \leq C_1 \|u_n\|^2_{L^2(B_R)} \|u_n - u\|_{L^2(B_R)} \|u_n + u\|_{L^2(B_R)} = o_n(1).$$

Similarly, one has $D_n^2 = o_n(1)$ and hence by Fatou's lemma

$$\liminf_{n\to\infty} A_n^1 \geq \int_{B_R}\int_{B_R} \log(1+|x-y|)u^2(y)u^2(x)dydx.$$

In a similar way,

$$\liminf_{n\to\infty} A_n^2 \geq \int_{B_R}\int_{B_R} \log(1+|x-y|)u^2(y)v^2(x)dydx,$$

$$\liminf_{n\to\infty} A_n^3 \geq \int_{B_R}\int_{B_R} \log(1+|x-y|)v^2(y)v^2(x)dydx.$$

As a consequence, we get

$$\liminf_{n\to\infty} \mathcal{T}_1(u_n,v_n) \geq \int_{B_R}\int_{B_R} \log(1+|x-y|)\left(u^2(y)+v^2(y)\right)\left(u^2(x)+v^2(x)\right)dydx.$$

Letting $R\to\infty$ in the above expression and using the Monotone Convergence Theorem, (ii) holds.

To prove that (iii) holds true, we write

$$\mathcal{T}_2(u_n,v_n) = B_n^1 + 2B_n^2 + B_n^3,$$

where

$$B_n^1 := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\left(1+|x-y|^{-1}\right) u_n^2(y)u_n^2(x)dydx,$$

$$B_n^2 := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\left(1+|x-y|^{-1}\right) u_n^2(y)v_n^2(x)dydx,$$

and

$$B_n^3 := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\left(1+|x-y|^{-1}\right) v_n^2(y)v_n^2(x)dydx.$$

Using the elementary inequality $\log(1+r) \leq r$, for any $r > 0$ it follows that

$$\begin{aligned} C_n &:= \left| B_n^2 - \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\left(1+|x-y|^{-1}\right) u^2(y)u^2(x)dydx \right| \\ &\leq \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|u_n^2(y)u_n^2(x) - u^2(y)u^2(x)|}{|x-y|}dydx. \end{aligned}$$

Since

$$u_n^2(y)u_n^2(x) - u^2(y)v^2(x) = u_n^2(y)\left(u_n^2(x) - u^2(x)\right) + v^2(x)\left(u_n^2(y) - u^2(y)\right),$$

we find

$$\begin{aligned} C_n &\leq \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{u_n^2(y)|u_n(x)-u(x)||u_n(x)+u(x)|}{|x-y|}dydx \\ &+ \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{u^2(x)|u_n(y)-u(y)||u_n(y)+u(y)|}{|x-y|}dydx. \end{aligned}$$

Thus, applying Proposition 2.3.5 with $\mu = 1$, $q = s = 4/3$ we obtain

$$C_n \leq \|u_n\|^2_{L^{8/3}(\mathbb{R}^2)} \left(\int_{\mathbb{R}^2} \left(|u_n - u||v_n + u|\right)^{4/3} dx\right)^{3/4} + \|u\|^2_{L^{8/3}(\mathbb{R}^2)} \left(\int_{\mathbb{R}^2} \left(|u_n - u||u_n + u|\right)^{4/3} dx\right)^{3/4}.$$

Using Hölder's inequality and the compact embedding (3.3) with $q = 8/3$ we get

$$\begin{aligned} C_n &\leq \|u_n\|^2_{L^{8/3}(\mathbb{R}^2)} \|u_n - v\|_{L^{8/3}(\mathbb{R}^2)} \|u_n + u\|_{L^{8/3}(\mathbb{R}^2)} + \|u\|^2_{L^{8/3}(\mathbb{R}^2)} \|u_n - u\|_{L^{8/3}(\mathbb{R}^2)} \|u_n + u\|_{L^{8/3}(\mathbb{R}^2)} \\ &= o_n(1). \end{aligned}$$

Thus, we find

$$B_n^1 = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\left(1+|x-y|^{-1}\right) u^2(y)u^2(x)dydx + o_n(1).$$

Proceeding, in a similarly way we obtain

$$B_n^2 = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\left(1+|x-y|^{-1}\right) u^2(y)v^2(x)dydx + o_n(1),$$

and

$$B_n^3 = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\left(1+|x-y|^{-1}\right) v^2(y)v^2(x)dydx + o_n(1).$$

Therefore,

$$\begin{aligned}\mathcal{T}_2(u_n,v_n) &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\left(1+|x-y|^{-1}\right)\left(u^2(y)+v^2(y)\right)\left(u^2(x)+v^2(x)\right)dydx + o_n(1) \\ &= \mathcal{T}_2(u,v)+o_n(1),\end{aligned}$$

which implies (iii), completing the proof. □

We are now in the position to complete the proof of Theorem 3.1.1.

Finalizing the proof of Theorem 3.1.1. Let $(u_n,v_n)\subset\mathcal{N}$ be a minimizing sequence for c_β. By Lemma 3.3.3, we may assume that $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^2)$, with $(u,v)\neq(0,0)$. On the other hand, by Lemma 3.3.4, up to subsequence we may assume that $u_n \rightharpoonup \tilde{u}$ and $v_n \rightharpoonup \tilde{v}$ weakly in $\widetilde{W}_{\text{rad}}$. From the compact embedding (3.3), we conclude that $u_n(x)\to u(x)$ and $u_n(x)\to\tilde{u}(x)$ for a.e. $x\in\mathbb{R}^2$. Similarly $v_n(x)\to v(x)$ and $v_n(x)\to\tilde{v}(x)$ for a.e. $x\in\mathbb{R}^2$ and consequently $(u,v)=(\tilde{u},\tilde{v})\in(\widetilde{W}_{\text{rad}}\times\widetilde{W}_{\text{rad}})\setminus\{(0,0)\}$.

Let $t=t_{u,v}>0$ be such that $(tu,tv)\in\mathcal{N}$. Arguing as in (3.14), we conclude that

$$\begin{aligned}c_\beta \leq I_\beta(tu,tv) &= \frac{t^4}{4}\left(\|u\|^2_{H^1(\mathbb{R}^2)}+\|v\|^2_{H^1(\mathbb{R}^2)}\right) \\ &+\frac{(p-2)t^{2p}}{4p}\left(\|u\|^{2p}_{L^{2p}(\mathbb{R}^2)}+\|v\|^{2p}_{L^{2p}(\mathbb{R}^2)}+2\beta\int_{\mathbb{R}^2}|uv|^p dx\right).\end{aligned} \tag{3.19}$$

Now, recalling that the norm is weakly lower semicontinuous, and the compact embedding (3.3), we can use Lemma 3.3.5 to obtain

$$\begin{aligned}\Gamma_\beta(u,v) &= \left(\|u\|^2_{H^1(\mathbb{R}^2)}+\|v\|^2_{H^1(\mathbb{R}^2)}\right)+\mathcal{T}(u,v)-\left(\|u\|^{2p}_{L^{2p}(\mathbb{R}^2)}+\|v\|^{2p}_{L^{2p}(\mathbb{R}^2)}+2\beta\int_{\mathbb{R}^2}|uv|^p dx\right) \\ &\leq \liminf_{n\to\infty}\Gamma_\beta(u_n,v_n)=0,\end{aligned}$$

and hence by Remark 3.3.2 we conclude that $t\in(0,1]$. Consequently,

$$c_\beta \leq I_\beta(tu,tv) \leq \frac{1}{4}\left(\|u\|^2_{H^1(\mathbb{R}^2)}+\|v\|^2_{H^1(\mathbb{R}^2)}\right)+\frac{(p-2)}{4p}\left(\|u\|^{2p}_{L^{2p}(\mathbb{R}^2)}+\|v\|^{2p}_{L^{2p}(\mathbb{R}^2)}+2\beta\int_{\mathbb{R}^2}|uv|^p dx\right).$$

This, the item (i) of Lemma 3.3.5, the weak semicontinuity of the norm, and (3.14) imply that

$$\begin{aligned}c_\beta &\leq I_\beta(tu,tv) \\ &\leq \liminf_{n\to\infty}\frac{1}{4}\left(\|u_n\|^2_{H^1(\mathbb{R}^2)}+\|v_n\|^2_{H^1(\mathbb{R}^2)}\right) \\ &\quad+\liminf_{n\to\infty}\frac{(p-2)}{4p}\left(\|u_n\|^{2p}_{L^{2p}(\mathbb{R}^2)}+\|v_n\|^{2p}_{L^{2p}(\mathbb{R}^2)}+2\beta\int_{\mathbb{R}^2}|u_nv_n|^p dx\right) \\ &= \liminf_{n\to\infty} I_\beta(u_n,v_n)=c_\beta.\end{aligned}$$

Thus, $I_\beta(tu,tv)=c_\beta$ and by using a rather standard deformation argument as in [19, Proposition 3.1] (see also [9, pp. 1163]) we conclude that (tu,tv) is the desired solution. Noting that

$I_\beta(u,v) = I_\beta(|u|,|v|)$ and $\Gamma_\beta(u,v) = \Gamma_\beta(|u|,|v|)$, whenever $(u,v) \in \mathcal{N}$ we may assume that $u, v \geq 0$. Therefore, applying Lemmas 3.2.5, Lemma 3.2.7, and Proposition 3.2.2 we concludes the proof of Theorem 3.1.1. $\square$

Chapter 4

Embeddings results, Trudinger-Moser type inequality and an application

Finally, in this chapter we will present the results obtained in the paper [5], where we establish embedding results and a Trudinger-Moser type inequality involving potential nonradial. Moreover, as an application, we investigate the existence of solutions for a class of Schröringer-Poisson system similar to that presented in Chapter 2 in the nonradial case.

4.1 Embeddings results

Inspired in the paper [11] (see also [79, 80]), we will focus our study on embeddings results and Trudinger-Moser type inequality, involving the potential V, K and Q, such that:

(VKQ) $V, K, Q \in C(\mathbb{R}^2)$ and there exist $\widetilde{\gamma} \leq 2 < \widetilde{\beta}$ and positive constants a_0, b_0 such that

$$\frac{a_0}{(1+|x|)^{\widetilde{\gamma}}} \leq V(x), \quad 0 < K(x), Q(x) \leq \frac{b_0}{(1+|x|)^{\widetilde{\beta}}}, \quad \text{for all } x \in \mathbb{R}^2.$$

In order to formulate our main weighted Sobolev embedding we consider the auxiliary weight function $\widetilde{w} \in L^1_{\text{loc}}(\mathbb{R}^2)$, satisfying

$$\widetilde{\omega}(x) \leq C_0 \cdot \begin{cases} 1 & \text{if} \quad |x| \leq 1 \\ \log(1+|x|)Q(x) & \text{if} \quad |x| > 1, \end{cases}$$

for some $C_0 > 0$.

Example 4.1.1. *$\widetilde{w}(x) := Q(x)$ and $\widetilde{w}(x) := \log(1+|x|)Q(x)$, for any $x \in \mathbb{R}^2$.*

Next we prove a weighted Sobolev embedding which will play a fundamental role in our variational setting. For related results see for instance [70].

Proposition 4.1.2. *Assume (VKQ). Then, for any $2 \leq p < \infty$, the weighted Sobolev embedding $E \hookrightarrow L^p(\mathbb{R}^2; \widetilde{\omega})$ is continuous and compact.*

Proof. For any $u \in E$ we observe that

$$\int_{\mathbb{R}^2} \widetilde{\omega}(x)|u|^p dx \leq C_0 \left(\int_{B_1} |u|^p dx + \sum_{j=0}^{\infty} \int_{A_j} \log(1+|x|)Q(x)|u|^p dx \right), \tag{4.1}$$

where $A_j := \{z \in \mathbb{R}^2 : 2^j < |z| < 2^{j+1}\}$, for $j \in \mathbb{N} \cup \{0\}$. To estimate the first integral on the right-hand side of (4.1) we notice that by the embedding $E \hookrightarrow H^1(B_1) \hookrightarrow L^p(B_1)$, which holds for all $p \geq 2$ we obtain

$$\int_{B_1} |u|^p dx \leq C_1 \left(\int_{B_1} \left[|\nabla u|^2 + u^2 \right] dx \right)^{p/2} \leq C_2 \left(\int_{B_1} \left[|\nabla u|^2 + V(x)u^2 \right] dx \right)^{p/2}, \tag{4.2}$$

for some constants $C_1, C_2 > 0$, where we apply (VKQ).

Next we will estimate the second integral on the right-hand side of (4.1). For this, we observe that using hypothesis (VKQ) and performing a change of variables $y := 2^{-j}x$ we obtain

$$\int_{A_j} \log(1+|x|)Q(x)|u|^p dx \leq \frac{\log(1+2^{j+1})}{2^{\widetilde{\beta} j}} \int_{A_j} |u|^p dx = \frac{\log(1+2^{j+1})}{2^{(\widetilde{\beta}-2)j}} \int_{A_0} |u_j(y)|^p dy,$$

where $u_j(y) := u(2^j y)$. By the Sobolev embedding $H^1(A_0) \hookrightarrow L^p(A_0)$, there exists $C_3 > 0$ such that

$$\begin{aligned}\int_{A_0} |u_j(y)|^p dy &\leq C_3 \left(\int_{A_0} \left[|\nabla u_j(y)|^2 + u_j^2(y) \right] dy \right)^{p/2} \\ &= C_3 \left(\int_{A_j} \left[|\nabla u(x)|^2 + 2^{-2j} u^2(x) \right] dx \right)^{p/2}.\end{aligned}$$

Since $(1+2^{j+1}) \leq 2 \cdot 2^{j+1}$ and we may assume without loss of generality that $\widetilde{\gamma} \geq 0$, one deduce

$$\int_{A_j} 2^{-2j} u^2(x) dx \leq 2^{-2j}(1+2^{j+1})^{\widetilde{\gamma}} \int_{A_j} \frac{u^2(x)}{(1+|x|)^{\widetilde{\gamma}}} dx \leq 2^{2\widetilde{\gamma}+(\widetilde{\gamma}-2)j} \int_{A_j} V(x)u^2 dx.$$

Since $2 < \beta$, then $\lim_{j \to +\infty} \log(1+2^{j+1})/2^{(\beta-2)j} = 0$, and so we obtain $C_4 > 0$ such that $\log(1+2^{j+1})/2^{(\beta-2)j} \leq C_4$, for $j \in \mathbb{N} \cup \{0\}$. This, combined with the above estimates and the fact that $\gamma \leq 2$, we deduce that

$$\begin{aligned}\int_{A_j} \log(1+|x|)Q(x)|u|^p dx &\leq \frac{\log(1+2^{j+1})}{2^{(\widetilde{\beta}-2)j}} \left(\int_{A_j} \left[|\nabla u|^2 + 2^{2\widetilde{\gamma}+(\widetilde{\gamma}-2)j} V(x)u^2 \right] dx \right)^{p/2} \\ &\leq C_4 \left(\int_{A_j} \left[|\nabla u|^2 + V(x)u^2 \right] dx \right)^{p/2}.\end{aligned} \tag{4.3}$$

Thus, recalling that the function $s \mapsto s^{p/2}$ is super additive for $p \geq 2$, we conclude that

$$\sum_{j=0}^{\infty} \int_{A_j} \log(1+|x|)Q(x)|u|^p dx \leq C_4 \left(\int_{B_1^c} \left[|\nabla u|^2 + V(x)u^2 \right] dx \right)^{p/2}. \tag{4.4}$$

This, together with (4.2) and (4.1) implies the continuous embedding.

For the compactness result, we take $(u_n) \subset E$ such that $u_n \rightharpoonup 0$ weakly in E. From convergence $\lim_{j\to\infty} \log(1+2^{j+1})/2^{(\widetilde{\beta}-2)j} = 0$, for any $\varepsilon > 0$, there exists $j_0 \in \mathbb{N}$ such that $\log(1+2^{j+1})/2^{(\widetilde{\beta}-2)j} < \varepsilon$, whenever $j \geq j_0$. Since the embedding $E \hookrightarrow H^1(B_1) \hookrightarrow L^p(B_1)$ is compact we have that $\int_{B_1} |u_n|^p dx = o_n(1)$. Using this, (VKQ), (4.3), and that $s \mapsto s^{p/2}$ is super additive, we obtain

$$\sum_{j=j_0}^{\infty} \int_{A_j} \log(1+|x|)Q(x)|u_n|^p dx \leq \varepsilon C_4 \sum_{j=j_0}^{\infty} \left(\int_{A_j} \left[|\nabla u_n|^2 + V(x) u_n^2 \right] dx \right)^{p/2} \leq \varepsilon C_4 \|u_n\|_E^p,$$

and using that (u_n) is bounded and $\varepsilon > 0$ is arbitrary we obtain the compact embedding and this completes the proof. □

As a byproduct of Proposition 4.1.2 we have the following result.

Remark 4.1.3. *Suppose that (VKQ) holds. If $\omega(x) = Q(x)$, $\omega(x) = \log(1+|x|)Q$, $\omega(x) = K(x)$ or $\omega(x) = \log(1+|x|)K(x)$, then clearly we have that $\omega(x) \leq C_1 \widetilde{\omega}(x)$ for some constant $C_1 > 0$. Therefore, for any $2 \leq p < \infty$, the Sobolev embedding $E \hookrightarrow L^p(\mathbb{R}^2;\omega)$ is continuous and compact. Furthermore, take into account that*

$$\overline{Q}(x) := Q^{4/3}(x) \leq \frac{b_0^{4/4}}{(1+|x|)^{4\widetilde{\beta}/3}} \quad \textit{and} \quad \overline{K}(x) := K^{4/3}(x) \leq \frac{b_0^{4/3}}{(1+|x|)^{4\widetilde{\beta}/3}}$$

and $4\widetilde{\beta}/3 > 2$, we see that $\overline{Q}$ and $\overline{K}$ satisfy hypothesis (VKQ). Hence the embedding also holds when $\omega(x) = Q^{4/3}(x)$ or $\omega(x) = K^{4/3}(x)$.

4.2 Trudinger-Moser type inequality

In view of Proposition 4.1.2 the following Trudinger-Moser type inequality is natural on the space E determined by the Young function

$$\Phi_{\alpha,1}(s) := e^{\alpha s^2} - 1,$$

where $\alpha > 0$ (see (1.6) with $j_0 = 1$).

Theorem 4.2.1. *For any $\alpha > 0$ and $u \in E$, the function $\widetilde{\omega}(\cdot)\Phi_{\alpha,1}(u)$ belongs to $L^1(\mathbb{R}^2)$. Moreover, there exists $\alpha_* \in (0, 4\pi)$ such that*

$$\sup_{u\in E,\, \|u\|_E \leq 1} \int_{\mathbb{R}^2} \widetilde{\omega}(x)\Phi_{\alpha,1}(u)dx < \infty,$$

for any $0 < \alpha \leq \alpha_*$.

Before to present the proof of Theorem 4.2.1 we will need some auxiliary result. We start off by recalling a Trudinger-Moser inequality in the ball [82, Lemma 3.1].

Lemma 4.2.2. *Let $x_0 \in \mathbb{R}^2$ and $u \in H_0^1(B_R(x_0))$ be such that $\int_{B_R(x_0)} |\nabla u|^2 dx \leq 1$. Then there exists $C > 0$ such that*

$$\int_{B_R(x_0)} \left(e^{4\pi u^2} - 1\right) dx \leq C \cdot R^2 \int_{B_R(x_0)} |\nabla u|^2 dx.$$

The second auxiliary result is a version, for our functional space, of a previous result presented in [42]. In their proof, the authors used, among other things, Besicovitch covering lemma. The proof we present here is new and easier than the former one.

Lemma 4.2.3. *Suppose that (VKQ) holds. Then there exist $\widetilde{C} > 0$ and $\alpha_* \in (0, 4\pi)$ such that*

$$\int_{\mathbb{R}^2} \widetilde{\omega}(x)\Phi_{\alpha,1}(u)dx \leq \widetilde{C},$$

for any $0 < \alpha \leq \alpha_*$ and $u \in E$ verifying $\|u\|_E \leq 1$.

Proof. Let $u \in E$ be such that $\|u\|_E \leq 1$ and observe that

$$\begin{aligned} \int_{\mathbb{R}^2} \widetilde{\omega}(x)\left(e^{\alpha u^2} - 1\right) dx &\leq C_0 \int_{B_1} \left(e^{\alpha u^2} - 1\right) dx \\ &\quad + C_0 \sum_{j=0}^{\infty} \int_{A_j} \log(1 + |x|)Q(x)\left(e^{\alpha u^2} - 1\right) dx. \end{aligned} \tag{4.5}$$

In order to estimate the first integral on the right-hand side of (4.5) we consider $\varphi \in C_0^\infty(B_2)$ satisfying $\varphi \equiv 1$ in B_1 and $|\nabla\varphi| \leq 2$ in B_2. By (VKQ), we can estimate

$$\int_{B_2} |\nabla(\varphi u)|^2 dx \leq C_1 \int_{B_2} \left[|\nabla u|^2 + u^2\right] dx \leq C_2 \int_{B_2} \left[|\nabla u|^2 + V(x)u^2\right] dx.$$

Setting $v := (1/C_2)^{1/2}\varphi u$, we can apply Lemma 4.2.2 to obtain

$$\int_{B_2} \left(e^{4\pi v^2} - 1\right) dx \leq C \cdot 2^2 \int_{B_2} |\nabla v|^2 dx \leq C_3 \int_{B_2} \left[|\nabla u|^2 + V(x)u^2\right] dx.$$

Thus, for any $0 < \alpha \leq 4\pi/C_2$, one has

$$\begin{aligned} \int_{B_1} \left(e^{\alpha u^2} - 1\right) dx \leq C_4 \int_{B_2} \left(e^{\alpha(\varphi u)^2} - 1\right) dx &= C_4 \int_{B_2} \left(e^{\alpha C_2 v^2} - 1\right) dx \\ &\leq C_5\|u\|_E^2 \leq C_5. \end{aligned} \tag{4.6}$$

We claim that there exists $C_6 > 0$ and $\alpha_* > 0$ such that

$$\sum_{j=0}^{\infty} \int_{A_j} \log(1 + |x|)Q(x)\left(e^{\alpha u^2} - 1\right) dx \leq C_6. \tag{4.7}$$

for any $0 < \alpha \leq \alpha_*$. For this purpose, let us $j \in \mathbb{N} \cup \{0\}$ fixed. Performing the change of variables $y := 2^{-j}x$, (VKQ) and using the fact that $\lim_{j\to\infty} \log(1 + 2^{j+1})/2^{(\widetilde{\beta}-2)j} = 0$ together with (VKQ) we obtain

$$\begin{aligned}\int_{A_j} \log(1+|x|)Q(x)\left(e^{\alpha u^2}-1\right)dx &\leq \frac{b_0\log(1+2^{j+1})}{2^{\widetilde{\beta}j}}\int_{A_j}\left(e^{\alpha u^2}-1\right)dx\\ &= \frac{b_0\log(1+2^{j+1})}{2^{(\widetilde{\beta}-2)j}}\int_{A_0}\left(e^{\alpha u_j^2}-1\right)dy \\ &\leq C_7\int_{A_0}\left(e^{\alpha u_j^2}-1\right)dy,\end{aligned} \tag{4.8}$$

where $u_j(y) := u(2^j y)$. To estimate the last integral above, for $y \in A_0$, setting $R_y := \operatorname{dist}(y, \partial A_0)$ we see that $B_{R_y}(y) \subset A_0$. Moreover, from the compactness of $\overline{A_0}$, we obtain points $y_1, \ldots, y_k \in A_0$ such that $A_0 \subset \bigcup_{i=1}^k B_{R_i/2}(y_i)$, where $R_i := R_{y_i}$. For each $i = 1, \ldots, k$, we pick a function $\varphi_i \in C_0^\infty(B_{R_i}(y_i))$ such that $0 \leq \varphi_i \leq 1$ in $B_{R_i}(y_i)$, $\varphi_i \equiv 1$ in $B_{R_i/2}(y_i)$ and $|\nabla \varphi_i| \leq 4/R_i$ in $B_{R_i}(y_i)$. If we call $B^i := B_{R_i}(y_i)$, we have that

$$\begin{aligned}\int_{B^i}\left|\nabla\left(\varphi_i(y)u_j(y)\right)\right|^2 dy &\leq C_8\int_{A_0} 2^{2j}|\nabla u(2^j y)|^2 dy + C_8 R_i^{-2}\int_{A_0} u^2(2^j y)dy\\ &\leq C_8\int_{A_j}|\nabla u|^2 dx + \frac{C_8 R_i^{-2}}{2^{2j}}\int_{A_j} u^2 dx.\end{aligned}$$

Since $(1+2^{j+1})^{\widetilde{\gamma}} \leq 4^{\widetilde{\gamma}}\cdot 2^{\widetilde{\gamma}j}$ and we may assume without loss of generality that $\widetilde{\gamma} \geq 0$ and hence from (VKQ) one has

$$\int_{A_j} u^2 dx \leq 4^{\widetilde{\gamma}}\cdot 2^{\widetilde{\gamma}j}\int_{A_j}\frac{u^2}{(1+|x|)^{\widetilde{\gamma}}}dx \leq \frac{4^{\widetilde{\gamma}}\cdot 2^{\widetilde{\gamma}j}}{a_0}\int_{A_j} V(x)u^2 dx$$

Since $\widetilde{\gamma} \leq 2$, there holds

$$\int_{B^i}\left|\nabla\left(\varphi_i(y)u_j(y)\right)\right|^2 dy \leq C_9\int_{A_j}\left[|\nabla u|^2 + V(x)u^2\right]dx.$$

At this point we define

$$\alpha_* := \min\left\{\frac{4\pi}{C_2}, \frac{4\pi}{C_9}\right\}.$$

If $v_{i,j} := (1/C_7)^{1/2}\varphi_i u_j$, we can apply Lemma 4.2.2 to estimate

$$\int_{B^i}\left(e^{4\pi v_{i,j}^2}-1\right)dy \leq C\cdot R_i^2\int_{B^i}|\nabla v_{i,j}|^2 dy \leq C_{10}\int_{A_j}\left[|\nabla u|^2+V(x)u^2\right]dx.$$

Consequently, for all $0 < \alpha \leq \alpha_*$ it holds

$$\int_{B^i}\left(e^{\alpha(\varphi_i u_j)^2}-1\right)dy \leq C_{10}\int_{A_j}\left[|\nabla u|^2+V(x)u^2\right]dx,$$

Therefore,

$$\int_{A_0}\left(e^{\alpha u_j^2}-1\right)dy \le \sum_{i=1}^{k}\int_{B_{R_i/2}(y_i)}\left(e^{\alpha(\varphi_i u_j)^2}-1\right)dy \le C_{11}\int_{A_j}\left[|\nabla u|^2+V(x)u^2\right]dx.$$

Combining this inequality with (4.8) and summing up we obtain that (4.7) holds, since $\|u\|_E \le 1$. Finally, the desired result follows from (4.5) together with the estimates (4.6) and (4.7). □

We are ready to present the proof of our first main theorem.

Proof of Theorem 4.2.1. Let $\alpha > 0$ and $u \in E$. By density, there exists $u_0 \in C_0^\infty(\mathbb{R}^2)$ such that

$$\|u-u_0\|_E \le \delta,$$

with $\delta > 0$ to be chosen later. Since $u^2 \le 2(u-u_0)^2 + 2u_0^2$, we may estimate

$$\int_{\mathbb{R}^2}\widetilde{\omega}(x)\Phi_{\alpha,1}(u)dx = \int_{\mathbb{R}^2}\widetilde{\omega}(x)\left(e^{\alpha u^2}-1\right)dx \le \int_{\mathbb{R}^2}\widetilde{\omega}(x)\left(e^{2\alpha(u-u_0)^2}e^{2\alpha u_0^2}-1\right)dx.$$

Recalling the elementary inequality

$$ab-1 \le \frac{1}{2}(a^2-1)+\frac{1}{2}(b^2-1), \quad \forall\, a,b \ge 0,$$

setting $w := u-u_0$ and denoting by Ω_0 the support of u_0, we obtain

$$\begin{aligned} 2\int_{\mathbb{R}^2}\widetilde{\omega}(x)\left(e^{\alpha u^2}-1\right)dx &\le \int_{\mathbb{R}^2}\widetilde{\omega}(x)\left(e^{4\alpha w^2}-1\right)dx + \int_{\Omega_0}\widetilde{\omega}(x)\left(e^{4\alpha u_0^2}-1\right)dx \\ &\le \int_{\mathbb{R}^2}\widetilde{\omega}(x)\left(e^{4\alpha\|w\|_E^2\left(\frac{w}{\|w\|_E}\right)^2}-1\right)dx + C_1\int_{\Omega_0}1dx, \end{aligned}$$

with $C_1 := \|\widetilde{\omega}\|_{L^\infty(\Omega_0)}e^{4\alpha\|u_0\|^2_{L^\infty(\Omega_0)}}$. We now pick $\delta > 0$ in such way that

$$4\alpha\|w\|_E^2 \le 4\alpha\delta^2 \le \alpha_*$$

and we using Lemma 4.2.3 to conclude that

$$\int_{\mathbb{R}^2}\widetilde{\omega}(x)\left(e^{\alpha u^2}-1\right)dx \le \frac{C}{2}+\frac{C_1}{2}|\Omega_0| < \infty.$$

This proves the first statement of Theorem 4.2.1. The second one is a direct consequence of Lemma 4.2.3. □

Remark 4.2.4. *Assume (VKQ) holds. As a immediate consequence of Theorem 4.2.1, there exists $\alpha_* \in (0,4\pi)$ such that*

$$\sup_{u\in E,\,\|u\|_E\le 1}\int_{\mathbb{R}^2}\omega(x)\Phi_{\alpha,1}(u)dx < \infty,$$

for any $0 < \alpha \leq \alpha_*$ *whenever* ω *is one of the functions* $Q(x)$, $K(x)$, $\log(1+|x|)Q(x)$, $\log(1+|x|)K(x)$, $Q^{4/3}(x)$*or* $K^{4/3}(x)$.

4.3 Application

In this section, we are concerned with the existence of solution to the system $(\mathcal{S})$, where $\lambda = 1$, the potential V, K, Q are nonradial and satisfy (VKQ), and (1.1) holds.

Since $W \hookrightarrow E$, by Remark 4.1.3, we can define the numbers

$$S_4(Q) := \inf_{u \in W\setminus\{0\}} \frac{\|u\|_W^2}{\|u\|^2_{L^4(\mathbb{R}^2;Q)}}, \quad S_2(K) := \inf_{u \in W\setminus\{0\}} \frac{\|u\|_W^2}{\|u\|^2_{L^2(\mathbb{R}^2;K)}}.$$

On the other hand, we assume that f satisfies the following conditions:

($\mathfrak{f}_1$) $f(s) = o(|s|)$ as $s \to 0$;

($\mathfrak{f}_2$) there exists $\theta > 4$ such that $0 < \theta F(s) \leq f(s)s$ for all $s \neq 0$;

($\mathfrak{f}_3$) there exists $\zeta > 0$ such that $F(s) \geq \zeta |s|^4$, for all $s \in \mathbb{R}$;

($\mathfrak{f}_4$) the function $s \mapsto f(s)/|s|^3$ is increasing in $|s| > 0$.

The main existence result for problem $(\mathcal{E})$ can be stated as follows:

Theorem 4.3.1. *Suppose that* (VKQ), (1.1), *and* ($\mathfrak{f}_1$)$-$($\mathfrak{f}_4$) *hold. Then, there exists* $\alpha_* \in (0, 4\pi)$ *such that problem* $(\mathcal{E})$ *has a nonzero small energy solution provided*

$$\zeta > S_4^2(Q) \max\left\{\frac{1}{S_2(K)}, \frac{\alpha_0}{2\alpha_*}\right\}. \tag{4.9}$$

As a byproduct of Theorem 4.3.1, we can give a contribution concerning the existence of solutions to the system $(\mathcal{S})$, namely

Theorem 4.3.2. *Suppose the same hypotheses of Theorem 4.3.1 and let* $u \in W$ *be the solution obtained in that theorem. Then, the pair* (u, ϕ_u) *is a weak solution of system* $(\mathcal{S})$, *where* $\phi_u = \Gamma_2 * (Ku^2)$.

4.3.1 Existence results

Since the proof of Theorem 4.3.1 it is similar the proof of Theorem 2.1.4, in this subsection we will estimate the level $c_1 = \inf_{u \in \mathcal{N}_1} I_1(u)$ differently (see (2.21) and recall that in this case we are considering $\lambda = 1$).

We obtain in what follows the required estimate on the minimax level c_1.

Lemma 4.3.3. *Suppose that* ($\mathfrak{f}_3$) *holds and let* $\alpha_* \in (0, 4\pi)$ *be given by Theorem 4.2.1. If* ζ *satisfies* (4.9), *then* $c_1 < \alpha_*/(4\alpha_0)$.

Proof. Since $W \hookrightarrow E \hookrightarrow L^4(\mathbb{R}^2; Q)$ and this last embedding is compact (see Remark 4.1.3), there exists $\omega \in W \setminus \{0\}$ such that

$$\|\omega\|_W^2 = S_4(Q), \quad \int_{\mathbb{R}^2} Q(x)\omega^4 dx = 1.$$

We may assume $\omega \geq 0$, and therefore we obtain from Lemma 2.4.1 a number $t_\omega > 0$ such that $t_\omega \omega \in \mathcal{N}$. So, recalling that $\mathcal{V}_2 \geq 0$, using (3.5), $(\mathfrak{f}_3)$ and the above equalities, we obtain

$$\begin{aligned} c_1 \leq I(t_\omega \omega) &\leq \frac{t_\omega^2}{2} S_4(Q) + \frac{1}{4}\mathcal{V}_1(t_\omega \omega) - \int_{\mathbb{R}^2} Q(x)F(t_\omega \omega)dx \\ &\leq \frac{t_\omega^2}{2} S_4(Q) + \frac{t_\omega^4}{2}\|\omega\|^2_{L^2(\mathbb{R}^2;K)} S_4(Q) - t_\omega^4 \zeta. \end{aligned}$$

But the definition of $S_2(K)$ and (4.9) provide

$$\|\omega\|_{L^2(\mathbb{R}^2;K)} \leq \frac{1}{S_2(K)}\|\omega\|_W^2 = \frac{S_4(Q)}{S_2(K)} \leq \frac{\zeta}{S_4(Q)},$$

and therefore

$$\begin{aligned} c_1 &\leq \frac{t_\omega^2}{2} S_4(Q) + \frac{t_\omega^4}{2}\zeta - t_\omega^4 \zeta \\ &\leq \frac{1}{2}\max_{t>0}\left[t^2 S_4(Q) - t^4 \zeta\right] = \frac{1}{2}\left(\frac{S_4^2(Q)}{4\zeta}\right) = \frac{S_4^2(Q)}{8\zeta} < \frac{\alpha_*}{4\alpha_0}, \end{aligned}$$

where we have used (4.9) again in the last part. The proof is complete. □

Arguing as in the proof of Theorem 2.1.4 we can check that Theorem 4.3.1 holds. Using the solution obtained in Theorem 4.3.1 and elliptic regularity, we can easily obtain a weak solution for the system $(\mathcal{S})$, with $\lambda = 1$.

Proof of Theorem 4.3.2. Let $u \in W$ be the solution given by Theorem 4.3.1, $\varphi \in C_0^\infty(\mathbb{R}^2)$ and $R > 0$ be such that the support of φ is contained in B_R. For any $1 < p < \infty$, we have that

$$\int_{B_R} |K(x)u^2|^p dx \leq \|K\|^p_{L^\infty(\mathbb{R}^2)} \int_{B_R} |u|^{2p} dx < \infty,$$

since $W \hookrightarrow L^{2p}(B_R)$. It follows from the classical potential theory (see [47, Theorem 9.9]) that $\phi_u := \Gamma_2 * (Ku^2) \in W^{2,p}(B_R)$ and $\Delta\phi_u = K(x)u^2$ for a.e. $x \in B_R$. This and Divergence Theorem ensure that

$$-\int_{B_R} \nabla\phi_u \cdot \nabla\varphi dx = \int_{B_R} (\Delta\phi_u)\varphi dx = \int_{B_R} (K(x)u^2)\varphi dx.$$

Therefore, the pair $(u, \phi_u) \in W \times W^{2,p}_{\mathrm{loc}}(\mathbb{R}^2)$ is a weak solution of system $(\mathcal{S})$ and the theorem is proved. □

Chapter 5

On the planar Choquard equation with nonradial potencial and exponential critical growth

In this chapter is devoted to the paper [30], where we study a Choquard type equation in the whole plane involving the logarithmic kernel and the exponential nonlinearity. We will use results obtained in the Chapter 4.

5.1 Main results

In this chapter we investigate the existence of solutions for the equation

$$-\Delta u + V(x)u = \frac{1}{2\pi}\left[\log\frac{1}{|x|} * \Big(K(x)F(u)\Big)\right] Q(x)f(u), \quad x \in \mathbb{R}^2, \qquad (\mathcal{E}.\mathcal{C})$$

where V, K, Q are continuous potentials (see Chapter 4) and

$$\left[\log\frac{1}{|\cdot|} * \Big(KF(u)\Big)\right](x) = \int_{\mathbb{R}^2} \log\left(\frac{1}{|x-y|}\right) K(y)F(u(y))dy.$$

.

The Choquard equation appears in several physical contents, such as an approximation to the Hartree-'Fock theory for one component plasma in the paper Lieb-Simon [51] and the description by Pekar of the quantum physics of a polaron at rest [66]. For complete discussion and references on the nonlinear Choquard equation we refer the reader to [18, 49, 51, 52, 55, 62] and references therein.

In [51], the authors have addressed the classical Choquard equation

$$-\Delta u + u = (\mathcal{I}_2(x) * |u|^2)u \quad \mathbb{R}^3, \qquad (5.1)$$

where $\mathcal{I}_2(x) = |x|^{-1}$ is the Riesz potential.

The nonlinear Choquard equation in high dimension

$$-\Delta u + u = (\mathcal{I}_\alpha(x) * |u|^p)|u|^{p-2}u \quad \mathbb{R}^N, \quad N \geq 3, \tag{5.2}$$

when the potential $\mathcal{I}_\alpha(x) = |x|^{\alpha-N}$ (with $0 < \alpha < N$) is the Riesz potential, has been subject of interest by many authors in the last years (see for instance [7, 18, 60–62] and references therein).

We quote that the existence of solutions for the nonlinear Choquard equation in the planar case has been addressed in many papers such as [7, 31, 60, 71, 84]. In [7] the authors consider the equation when $\mathcal{I}_\alpha(x) = |x|^{-\alpha}$ (with $0 < \alpha < 2$) is the Riesz potential and V is periodic.

As in the papers [25, 36, 78], it is quite natural to consider potentials $\mathcal{I}_\alpha$ of logarithm type which have signal changes. In [18], the authors investigate the existence of solutions for the planar Choquard equation when the potential V interacts with $\mathcal{I}_\alpha(x)$ and the nonlinearity f has polynomial growth.

Motivated by the aforementioned results, our purpose here is to investigate the existence of solutions to problem $(\mathcal{E}.\mathcal{C})$ when the nonlinearity f has the maximal growth for which the energy functional associated is well defined.

In this chapter, we shall assume that the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is differentiable, $f(s) = 0$ for all $s \leq 0$ and $f(s) > 0$ for all $s > 0$, (1.1) holds, and satisfies the following conditions:

- $(\mathfrak{f}_1)$ $f(s) = o(|s|)$ as $s \to 0$;
- $(\widetilde{\mathfrak{f}_2})$ there exists $\delta \in (0, 1)$ such that

$$\delta \leq \frac{F(s)f'(s)}{f^2(s)}, \quad \forall\, s > 0;$$

- $(\widetilde{\mathfrak{f}_3})$ there exist $\xi > 0$ and $\kappa > 2$ such that $F(s) \geq \xi s^\kappa$, for all $s \in (0, 1]$.

A typical example of nonlinearity satisfying our assumptions is $F(s) = e^{\alpha_0 s^2} - 1$, that is, $f(s) = F'(s) = 2\alpha_0 s e^{\alpha_0 s^2}$.

Remark 5.1.1. *It follows from $(\widetilde{\mathfrak{f}_2})$ that f is monotone increasing and hence*

$$F(s) = \int_0^s f(t)dt \leq f(s)s, \quad \forall s > 0, \tag{5.3}$$

which improves the famous Ambrosetti-Rabinowitz condition. Furthermore,

$$\frac{d}{ds}\left(\frac{F(s)}{f(s)}\right) = \frac{f^2(s) - F(s)f'(s)}{f^2(s)} \leq 1 - \delta. \tag{5.4}$$

Consequently, for $s > 0$ fixed, if we choose $0 < \varepsilon < s$ arbitrary, one deduce

$$\int_\varepsilon^s \frac{d}{dt}\left(\frac{F(t)}{f(t)}\right) dt \leq \int_\varepsilon^s (1 - \delta)dt,$$

which implies

$$\frac{F(s)}{f(s)} - \frac{F(\varepsilon)}{f(\varepsilon)} \leq (1-\delta)(s-\varepsilon).$$

By (5.3), $\lim_{\varepsilon\to 0^+} F(\varepsilon)/f(\varepsilon) = 0$. *Thus, taking* $\varepsilon \to 0^+$ *in the last estimate, follows that*

$$F(s) \leq (1-\delta)f(s)s, \quad \forall s > 0. \tag{5.5}$$

We say that $u \in E$ is a weak solution for $(\mathcal{E}.\mathcal{C})$ if for any $\varphi \in C_0^\infty(\mathbb{R}^2)$ there holds

$$\int_{\mathbb{R}^2} [\nabla u \nabla \varphi + V(x)u\varphi]\, dx = \frac{1}{2\pi}\int_{\mathbb{R}^2}\left[\log\frac{1}{|x|} * \Big(K(x)F(u)\Big)\right] Q(x)f(u)\varphi dx. \tag{5.6}$$

Remark 5.1.2. *If* $u \in E$ *is a weak solution for equation* $(\mathcal{E}.\mathcal{C})$, *considering* $\varphi = u^- := \max\{0, -u\}$ *as a test function in* (5.6) *we obtain that* $u^- = 0$ *and consequently every weak solution of* $(\mathcal{E}.\mathcal{C})$ *is nonnegative.*

Our main existence result for problem $(\mathcal{E}.\mathcal{C})$ is state as follows.

Theorem 5.1.3. *Suppose that* (VKQ), (1.1), $(\mathfrak{f}_1)$, *and* $(\widetilde{\mathfrak{f}_2})$ *hold. Then there exists* $\xi^* > 0$ *such that if* $(\widetilde{\mathfrak{f}_3})$ *holds with* $\xi \geq \xi^*$, $(\mathcal{E}.\mathcal{C})$ *has a nontrivial weak solution which is nonnegative.*

The remainder of the chapter is organized as follows. In Section 5.2 we shows some properties of the nonlocal term which are fundamental in our approach. Finally, in Section **??** we prove Theorem 5.1.3.

5.2 Variational setting

This section is devoted to introduce the variational setting to study equation $(\mathcal{E}.\mathcal{C})$. To this purpose we observe that $(\mathcal{E}.\mathcal{C})$ has, at least formally, a variational structure given by the energy functional $\mathcal{J} : E \to \mathbb{R}$ defined by

$$\mathcal{J}(u) = \frac{1}{2}\|u\|_E^2 - \frac{1}{4\pi}\mathcal{G}(u),$$

where

$$\mathcal{G}(u) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left[\log\left(\frac{1}{|x-y|}\right)\Big(Q(y)F(u(y))\Big)\right] Q(x)F(u(x))dydx.$$

First let us show that the functional I is well defined. Since Q and K have the same growth, from now on we will assume that $Q \equiv K$. Using the elementary identity

$$\log\frac{1}{r} = \log\left(1+\frac{1}{r}\right) - \log(1+r),$$

we can write $\mathcal{G}(u) = \mathcal{G}_1(u) - \mathcal{G}_2(u)$, where

$$\mathcal{G}_1(u) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left[\log\left(1+\frac{1}{|x-y|}\right)\Big(Q(y)F(u(y))\Big)\right] Q(x)F(u(x))dydx$$

and

$$\mathcal{G}_2(u) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\Big[\log(1+|x-y|)\Big(Q(y)F(u(y))\Big)\Big]Q(x)F(u(x))dydx.$$

Hence

$$\mathcal{J}(u) = \frac{1}{2}\|u\|_E^2 - \frac{1}{4\pi}\left(\mathcal{G}_1(u) - \mathcal{G}_2(u)\right). \tag{5.7}$$

To show that the nonlocal term $\mathcal{G}_1(u)$ is well defined, we recall the relevant Hardy–Littlewood–Sobolev inequality (see for instance Proposition 2.3.5).

Since $\log(1+1/t) \leq 1/t$ holds for all $t > 0$, applying Proposition 2.3.5 with $\mu = 1$ and $s = r = 4/3$ one has

$$\begin{aligned}|\mathcal{G}_1(u)| &\leq \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{Q(y)F(u(y))Q(x)F(u(x))}{|x-y|}dydx \\ &\leq C\left(\int_{\mathbb{R}^2}Q^{4/3}(y)F^{4/3}(u)dy\right)^{3/4}\left(\int_{\mathbb{R}^2}Q^{4/3}(x)F^{4/3}(u)dx\right)^{3/4} \\ &= C\left(\int_{\mathbb{R}^2}Q^{4/3}(x)F^{4/3}(u)dx\right)^{3/2}.\end{aligned} \tag{5.8}$$

From (1.15) (with $\gamma = 2$) and (1.16) it follows that

$$\int_{\mathbb{R}^2}Q^{4/3}(x)F^{4/3}(u)dx \leq \varepsilon^{4/3}\int_{\mathbb{R}^2}Q^{4/3}(x)|u|^{8/3}dx + C_1\int_{\mathbb{R}^2}Q^{4/3}(x)|u|^{4q/3}\Phi_{(4\alpha/3)}, 1(u)dx.$$

On the other hand, by Hölder's inequality with exponents $1/r_1 + 1/r_2 = 1$ together with (1.16) we deduce

$$\begin{aligned}\int_{\mathbb{R}^2}Q^{4/3}(x)|u|^{4q/3}\Phi_{(4\alpha/3)}, 1(u)dx \leq &\left(\int_{\mathbb{R}^2}Q^{4/3}(x)|u|^{4r_2q/3}dx\right)^{1/r_2} \\ &\times\left(\int_{\mathbb{R}^2}Q^{4/3}(x)\Phi_{(4r_1\alpha/3)}, 1(u)dx\right)^{1/r_1}.\end{aligned}$$

Hence,

$$\begin{aligned}\int_{\mathbb{R}^2}Q^{4/3}(x)F^{4/3}(u)dx \leq C_2\|u\|^{8/3}_{L^{8/3}(\mathbb{R}^2;Q^{4/3})} &+ C_2\|u\|^{4q/3}_{L^{4r_2q/3}(\mathbb{R}^2;Q^{4/3})} \\ &\times\left(\int_{\mathbb{R}^2}Q^{4/3}(x)\Phi_{(4r_1\alpha/3)}, 1(u)dx\right)^{1/r_1}.\end{aligned} \tag{5.9}$$

This combined with and (5.8) yields

$$\begin{aligned}|\mathcal{G}_1(u)| \leq\ &C_3\|u\|^4_{L^{8/3}(\mathbb{R}^2;Q^{4/3})} \\ &+ C_3\|u\|^{2p}_{L^{4r_2q/3}(\mathbb{R}^2;Q^{4/3})}\left(\int_{\mathbb{R}^2}Q^{4/3}(x)\Phi_{(4r_1\alpha/3)}, 1(u)dx\right)^{3/2r_1}.\end{aligned} \tag{5.10}$$

Choosing $q \geq 2$, it follows from Remarks 4.1.3 and 4.2.4 that $\mathcal{G}_1(u)$ is well defined.

Now, we estimate $\mathcal{G}_2(u)$. Since $\log(1+|x-y|) \le \log(1+|x|) + \log(1+|y|)$, we get

$$\begin{aligned}
|\mathcal{G}_2(u)| &\le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left[\Big(\log(1+|x|) + \log(1+|y|)\Big)\Big(Q(y)F(u(y))\Big)\right] Q(x)F(u(x))dydx \\
&\le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x|)Q(y)F(u(y))Q(x)F(u(x))dydx \\
&\quad + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|y|)Q(y)F(u(y))Q(x)F(u(x))dydx \\
&= \left(\int_{\mathbb{R}^2} Q(y)F(u(y))dy\right)\left(\int_{\mathbb{R}^2} \log(1+|x|)Q(x)F(u(x))dx\right) \\
&\quad + \left(\int_{\mathbb{R}^2} \log(1+|y|)Q(y)F(u(y))dy\right)\left(\int_{\mathbb{R}^2} Q(x)F(u(x))dx\right) \\
&= 2\left(\int_{\mathbb{R}^2} \log(1+|x|)Q(x)F(u(x))dx\right)\left(\int_{\mathbb{R}^2} Q(x)F(u(x))dx\right).
\end{aligned}$$

By (1.15) (with $\gamma = 2$), one has

$$\begin{aligned}
\int_{\mathbb{R}^2} \log(1+|x|)Q(x)F(u)dx \le \varepsilon &\int_{\mathbb{R}^2} \log(1+|x|)Q(x)u^2dx \\
&+ C\int_{\mathbb{R}^2} \log(1+|x|)Q(x)|u|^q\Phi_{\alpha,1}(u)dx.
\end{aligned}$$

Applying Hölder's inequality with exponents $1/q_1 + 1/q_2 = 1$, together with Theorem 4.1.2, (1.16), and Theorem 4.2.1, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^2} \log(1+|x|)Q(x)F(u)dx &\le \varepsilon \int_{\mathbb{R}^2} \log(1+|x|)Q(x)u^2dx \\
&\quad + \left(\int_{\mathbb{R}^2} \log(1+|x|)Q(x)|u|^{q_1 q}dx\right)^{1/q_1} \\
&\quad \times \left(\int_{\mathbb{R}^2} \log(1+|x|)Q(x)\Phi_{q_2\alpha,1}(u)dx\right)^{1/q_2} \\
&\le C_4\|u\|_E^2 + C_4\|u\|_E^q\left(\int_{\mathbb{R}^2} \log(1+|x|)Q(x)\Phi_{q_2\alpha,1}(u)dx\right)^{1/q_2} \\
&< \infty.
\end{aligned}$$

Similarly, we can use Remark 4.1.3, (1.16) and Remark 4.2.4 to get

$$\begin{aligned}
\int_{\mathbb{R}^2} Q(x)F(u)dx &\le \varepsilon \int_{\mathbb{R}^2} Q(x)u^2dx + C_5\left(\int_{\mathbb{R}^2} Q(x)|u|^{q_1 q}dx\right)^{1/q_1} \\
&\quad \times \left(\int_{\mathbb{R}^2} Q(x)\Phi_{q_2\alpha,1}(u)dx\right)^{1/q_2} \\
&\le C_6\|u\|_E^2 + C_6\|u\|_E^q\left(\int_{\mathbb{R}^2} Q(x)\Phi_{q_2\alpha,1}(u)dx\right)^{1/q_2} \\
&< \infty.
\end{aligned} \tag{5.11}$$

From the above estimates we conclude that

$$\begin{aligned} |\mathcal{G}_2(u)| \leq C_7\|u\|_E^2 + C_7\|u\|_E^q \left(\int_{\mathbb{R}^2} \log(1+|x|)Q(x)\Phi_{q_2\alpha,1}(u)dx\right)^{1/q_2} \\ + C_7\|u\|_E^q \left(\int_{\mathbb{R}^2} Q(x)\Phi_{q_2\alpha,1}(u)dx\right)^{1/q_2} < \infty. \end{aligned} \tag{5.12}$$

Next, following the same steps proved in [31, Lemma 4.2] we can see that $\mathcal{G} \in C^1(E,\mathbb{R})$ and for all $u, v \in E$ it holds

$$\begin{aligned} \mathcal{J}'(u)v &= \langle u, v\rangle_E - \frac{1}{2\pi}\int_{\mathbb{R}^2}\left[\log\frac{1}{|x|} * \Big(Q(x)F(u)\Big)\right] Q(x)f(u)v dx \\ &= \langle u, v\rangle_E - \frac{1}{2\pi}\left(\mathcal{G}_1'(u)v - \mathcal{G}_2'(u)v\right), \end{aligned}$$

where

$$\mathcal{G}_1'(u)v = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left[\log\left(1+\frac{1}{|x-y|}\right)\Big(Q(y)F(u(y))\Big)\right] Q(x)f(u(x))v(x)dydx$$

and

$$\mathcal{G}_2'(u)v = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left[\log\left(1+|x-y|\right)\Big(Q(y)F(u(y))\Big)\right] Q(x)f(u(x))v(x)dydx.$$

Consequently, weak solutions of problem $(\mathcal{E}.\mathcal{C})$ are precisely the critical points of $\mathcal{J}$.

5.3 Proof of Theorem 5.1.3

This section is devoted to prove Theorem 5.1.3 which will be achieved by using a variational approach.

First, we show that the functional $\mathcal{I}$ satisfies the geometry required in the Mountain Pass Theorem.

Lemma 5.3.1. *Suppose that (VKQ), (1.1), and $(\mathfrak{f}_1)$ hold. Then there are constants $\rho, \tau > 0$ such that $\mathcal{J}(u) \geq \tau$, for any $\|u\|_E = \rho$. Furthermore, there exists $e \in E$ such that $\|e\|_E > \rho$ and $\mathcal{J}(e) < 0$.*

Proof. From (5.7), (5.12), and the fact that $\mathcal{G}_1 \geq 0$, we can use Remarks 4.2.4 and 4.1.3 to conclude that

$$\mathcal{J}(u) \geq C_1\|u\|_E^2 - C_2\|u\|_E^q,$$

whenever $\|u\|_E \leq \rho_1$, with $\rho_1 > 0$ satisfying $q_2\alpha\rho_1^2 \leq \alpha_*$. Taking $q > 2$ and $0 < \rho \leq \rho_1$ small enough, we can easily use the above estimate to obtain this first statement of the lemma.

In order to prove the second one, we fix $\varphi \in C_0^\infty(\mathbb{R}^2) \setminus \{0\}$, with supp$\varphi \subset B_{1/4}$. Taking into account that $|x-y|<1/2$ for any $x,y\in B_{1/4}$, we have, for $t>0$, that

$$\begin{aligned}\mathcal{G}(t\varphi) &= \int_{B_{1/4}}\int_{B_{1/4}} \log\frac{1}{|x-y|} Q(y)F(t\varphi(y))Q(x)F(t\varphi(x))dydx \\ &\geq \log 2\left(\int_{B_{1/4}} Q(x)F(t\varphi(x))dx\right)^2.\end{aligned}$$

By (1.1), for $q>1$ fixed there are constants $C_3, C_4>0$ such that $F(s)\geq C_3|s|^q - C_4$ for all $s\in\mathbb{R}$. Thus, for some constants $C_5, C_6>0$ we get

$$\mathcal{J}(t\varphi)\leq \frac{t^2}{2}\|\varphi\|_E^2 - \log 2\left(C_5 t^q\int_{B_{1/4}} Q(x)|\varphi|^q dx - C_7\int_{B_{1/4}} Q(x)dx\right)^2.$$

Since $2q>2$, the second statement holds for $e:=t\varphi$ with $t>0$ sufficiently large, completing the proof. □

In view of Lemma 5.3.1, the minimax level

$$c_{MP} := \inf_{g\in\Gamma}\max_{t\in[0,1]} \mathcal{J}(g(t)), \tag{5.13}$$

where $\Gamma := \{g\in C([0,1],E) : g(0)=0 \text{ and } \mathcal{J}(g(1))<0\}$ is well defined and positive.

The following result holds true.

Lemma 5.3.2. *Suppose that (VKQ) and $(\widetilde{\mathfrak{f}_2})$ hold. If $(u_n)\subset E$ is a $(PS)_c$ sequence for the functional $\mathcal{J}$, then (u_n) is bounded in E.*

Proof. If $(u_n)\subset E$ is a $(PS)_c$ sequence we have

$$\frac{1}{2}\|u_n\|_E^2 - \frac{1}{4\pi}\int_{\mathbb{R}^2}\left[\log\frac{1}{|x|} * \Big(Q(x)F(u_n)\Big)\right]Q(x)F(u_n)dx = c + o_n(1) \tag{5.14}$$

and

$$\left|\langle u_n, v\rangle_E - \frac{1}{2\pi}\int_{\mathbb{R}^2}\left[\log\frac{1}{|x|} * \Big(Q(x)F(u_n)\Big)\right]Q(x)f(u_n)v dx\right| \leq \tau_n\|v\|_E, \tag{5.15}$$

for any $v\in E$, where $\tau_n\to 0$ as $n\to\infty$. We claim that $v_n := F(u_n)/f(u_n)\in E$, for all $n\in\mathbb{N}$. Considering

$$T_n(v) := \int_{\mathbb{R}^2}\left[\log\frac{1}{|x|} * \Big(Q(x)F(u_n)\Big)\right]Q(x)f(u_n)v dx,$$

since $f(s)=0$, for all $s\leq 0$, one deduces

$$\begin{aligned}T_n(v) &= \int_{\{u_n>0\}}\left[\log\frac{1}{|x|} * \Big(Q(x)F(u_n)\Big)\right]Q(x)f(u_n)v dx \\ &\quad + \int_{\{u_n\leq 0\}}\left[\log\frac{1}{|x|} * \Big(Q(x)F(u_n)\Big)\right]Q(x)f(u_n)v dx \\ &= \int_{\{u_n>0\}}\left[\log\frac{1}{|x|} * \Big(Q(x)F(u_n)\Big)\right]Q(x)f(u_n)v dx.\end{aligned}$$

Thus, we can assume $u_n > 0$ and so $f(u_n) > 0$, implying that v_n is well defined. Now, to show that $v_n \in E$, for all $n \in \mathbb{N}$, let us notice that from (5.3) there holds

$$\int_{B_R} v_n^2 dx = \int_{B_R} \frac{F^2(u_n)}{f^2(u_n)} dx \leq \int_{B_R} u_n^2 dx < \infty,$$

for any $R > 0$, since $(u_n) \subset E$. Again by (5.3), we obtain

$$\int_{\mathbb{R}^2} V(x) v_n^2 dx \leq \int_{\mathbb{R}^2} V(x) u_n^2 dx < \infty. \tag{5.16}$$

On the other hand, observing that

$$\nabla v_n = \nabla u_n \frac{f^2(u_n) - F(u_n) f'(u_n)}{f^2(u_n)}, \tag{5.17}$$

from (5.4) it follows that

$$\int_{\mathbb{R}^2} |\nabla v_n|^2 dx \leq (1-\delta)^2 \int_{\mathbb{R}^2} |\nabla u_n|^2 dx < \infty.$$

Therefore, $(v_n) \subset E$ as claimed. So, we can apply (5.15) with v_n instead of v to get

$$-\langle u_n, v_n \rangle_E + \frac{1}{2\pi} \int_{\mathbb{R}^2} \left[\log \frac{1}{|x|} * \Big(Q(x) F(u_n) \Big) \right] Q(x) F(u_n) dx \leq \tau_n \|v_n\|_E.$$

This last inequality combined with (5.14) and the fact that $\|v_n\|_E \leq \|u_n\|_E$ infer that

$$\|u_n\|_E^2 = 2c + 2o_n(1) + \langle u_n, v_n \rangle_E + \tau_n \|u_n\|_E.$$

By the definition of v_n, (5.17), (5.4), and (5.5), we obtain

$$\begin{aligned} \langle u_n, v_n \rangle_E &= \int_{\mathbb{R}^2} |\nabla u_n|^2 \left(\frac{f^2(u_n) - F(u_n) f'(u_n)}{f^2(u_n)} \right) dx + \int_{\mathbb{R}^2} V(x) u_n \frac{F(u_n)}{f(u_n)} dx \\ &\leq (1-\delta) \int_{\mathbb{R}^2} |\nabla u_n|^2 dx + (1-\delta) \int_{\mathbb{R}^2} V(x) u_n^2 dx. \end{aligned}$$

From the above estimates and the fact that for $\varepsilon > 0$ arbitrary, one can choose $2o_n(1), \tau_n \leq \varepsilon$, for sufficiently large $n \in \mathbb{N}$, to conclude that

$$\delta \|u_n\|_E^2 \leq 2c + \varepsilon + \varepsilon \|u_n\|_E, \tag{5.18}$$

which implies the (u_n) is bounded in E and the proof is complete. □

Next, we need to establish the following compactness result:

Lemma 5.3.3. *Suppose that (VKQ), (1.1), $(\mathfrak{f}_1)$, and $(\widetilde{\mathfrak{f}_2})$ hold. Then there exists $c_0 > 0$ such that the functional $\mathcal{J}$ satisfies the $(PS)_c$ condition at any level $0 < c < c_0$.*

Proof. If $(u_n) \subset E$ is a $(PS)_c$ sequence, by Lemma 5.3.2, up to a subsequence, we can assume that $u_n \rightharpoonup u$ weakly in E. We claim that

$$D_n := \int_{\mathbb{R}^2} \left[\log \frac{1}{|x|} * \Big(Q(x) F(u_n) \Big) \right] Q(x) f(u_n)(u_n - u) dx = o_n(1). \tag{5.19}$$

If this is true, since $\lim_{n\to\infty} \mathcal{J}(u_n)(u_n - u) = 0$ and $u_n \rightharpoonup u$ weakly in E, we must have

$$\lim_{n\to\infty} \left(\|u_n\|_E^2 - \langle u_n, u \rangle_E \right) = \lim_{n\to\infty} \left(\|u_n\|_E^2 - \|u\|_E^2 \right) = 0.$$

Hence, this and the fact that $u_n \rightharpoonup u$ weakly in E, give us

$$\lim_{n\to\infty} \|u_n - u\|_E^2 = \lim_{n\to\infty} \left(\|u_n\|_E^2 - 2\langle u_n, u \rangle_E - \|u\|_E^2 \right) = \|u\|_E^2 - 2\langle u, u \rangle_E + \|u\|_E^2 = 0,$$

and this finishes the proof. Thus, it remains to prove (5.19).

To this end, note that

$$D_n = \mathcal{G}_1'(u_n)(u_n - u) + \mathcal{G}_2'(u_n)(u_n - u). \tag{5.20}$$

Taking into account that $\log(1 + 1/t) \leq 1/t$ for $t > 0$ and Proposition 2.3.5, with $\mu = 1$ and $r = s = 4/3$, we get

$$\mathcal{G}_1'(u_n)(u_n - u) \leq C_1 \left(\int_{\mathbb{R}^2} Q^{4/3}(x) F^{4/3}(u_n) dx \right)^{3/4} \left(\int_{\mathbb{R}^2} Q^{4/3}(x) f^{4/3}(u_n) |u_n - u|^{4/3} dx \right)^{3/4}.$$

From (5.9) and Remark 4.2.4, it follows that

$$\begin{aligned} \int_{\mathbb{R}^2} Q^{4/3}(x) F^{4/3}(u_n) dx &\leq C_2 \|u_n\|_E^{8/3} + C_2 \|u_n\|_E^{4q/3} \\ &\quad \times \left(\int_{\mathbb{R}^2} Q^{4/3}(x) \Phi_{(4r_1\alpha/3)\|u_n\|_E^2, 1} \left(\frac{u_n}{\|u_n\|_E} \right) dx \right)^{1/r_1}. \end{aligned} \tag{5.21}$$

On the other hand, since (u_n) is bounded in E, by (5.18) one deduces

$$\delta \lim_{n\to\infty} \|u_n\|_E^2 \leq 2c + \varepsilon + \varepsilon \lim_{n\to\infty} \|u_n\|_E \leq 2c + \varepsilon + \varepsilon C_3.$$

Thus, picking $c_1 > 0$ and $\varepsilon > 0$ small enough such that

$$\lim_{n\to\infty} \|u_n\|_E^2 \leq \frac{2c}{\delta} + \frac{\varepsilon}{\delta} + \frac{\varepsilon C_3}{\delta} < \left(\frac{2c_1}{\delta} + \frac{\varepsilon}{\delta} + \frac{\varepsilon C_3}{\delta} \right) < \frac{3\alpha_*}{4\alpha r_1}, \tag{5.22}$$

for any $0 < c < c_1$. So, we can conclude that $(4r_1\alpha/3)\|u_n\|_E^2 < \alpha_*$, for any $n \in \mathbb{N}$ large enough. Using Remark 4.1.3 in (5.21), there holds

$$\mathcal{G}_1'(u_n)(u_n - u) \leq C_4 \left(\int_{\mathbb{R}^2} Q^{4/3}(x) f^{4/3}(u_n) |u_n - u|^{4/3} dx \right)^{3/4}.$$

Invoking Hölder's inequality with exponents $2/3 + 1/3 = 1$, we get

$$\int_{\mathbb{R}^2} Q^{4/3}(x) f^{4/3}(u_n)|u_n - u|^{4/3} dx \leq \left(\int_{\mathbb{R}^2} Q^{4/3}(x) f^2(u_n) dx \right)^{2/3} \|u_n - u\|^{4/3}_{L^4(\mathbb{R}^2;Q^{4/3})}.$$

By Remark 4.2.4, we obtain the following convergence

$$\lim_{n\to\infty} \|u_n - u\|^{4/3}_{L^{4q_2/3}(\mathbb{R}^2;Q^{4/3})} = 0,$$

and so, to guarantee that $\lim_{n\to\infty} \mathcal{G}_1'(u_n)(u_n - u) = 0$, it is enough shows that the sequence $\left(\int_{\mathbb{R}^2} Q^{4/3}(x) f^2(u_n) dx\right)$ is bounded. In fact, from (1.14) with $q = 1$, Remark 4.2.4, and (1.16), we have

$$\begin{aligned} \int_{\mathbb{R}^2} Q^{4/3}(x) f^2(u_n) dx &\leq \varepsilon \int_{\mathbb{R}^2} Q^{4/3}(x) u_n^2 dx + C_5 \int_{\mathbb{R}^2} Q^{4/3}(x) \Phi_{2\alpha,1}(u_n) dx \\ &\leq C_6 + C_6 \int_{\mathbb{R}^2} Q^{4/3}(x) \Phi_{2\alpha\|u_n\|_E^2,1} \left(\frac{u_n}{\|u_n\|_E} \right) dx. \end{aligned} \tag{5.23}$$

From the first inequality (5.22), taking $c_2 > 0$ and $\varepsilon > 0$ small enough we get

$$\lim_{n\to\infty} \|u_n\|_E^2 \leq \frac{2c}{\delta} + \frac{\varepsilon}{\delta} + \frac{\varepsilon C_3}{\delta} < \left(\frac{2c_2}{\delta} + \frac{\varepsilon}{\delta} + \frac{\varepsilon C_3}{\delta} \right) < \frac{\alpha_*}{2\alpha},$$

for any $0 < c < c_2$. Thus, for any $n \in \mathbb{N}$ large

$$2\alpha \|u_n\|_E^2 < \alpha_*. \tag{5.24}$$

By Remark 4.2.4 and (5.23), the sequence $\left(\int_{\mathbb{R}^2} Q^{4/3}(x) f^2(u_n) dx\right)$ is bounded and therefore $\lim_{n\to\infty} \mathcal{G}_1'(u_n)(u_n - u) = 0$. This and (5.20) imply

$$\lim_{n\to\infty} D_n = \lim_{n\to\infty} \mathcal{G}_2'(u_n)(u_n - u).$$

Since $\log(1 + |x - y|) \leq \log(1 + |x|) + \log(1 + |y|)$ we have

$$\begin{aligned} \mathcal{G}_2'(u_n)(u_n - u) = &\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \Big[\log(1 + |x - y|) \Big(Q(y) F(u_n(y)) \Big) \Big] \\ &\times Q(x) f(u_n(x))(u_n(x) - u(x)) \\ \leq &\ A_n + B_n, \end{aligned}$$

where

$$A_n := \left(\int_{\mathbb{R}^2} Q(y) F(u_n(y)) dy \right) \left(\int_{\mathbb{R}^2} \log(1 + |x|) Q(x) f(u_n(x))(u_n(x) - u(x)) dx \right)$$

and

$$B_n := \left(\int_{\mathbb{R}^2} \log(1 + |y|) Q(y) F(u_n(y)) dy \right) \left(\int_{\mathbb{R}^2} Q(x) f(u_n(x))(u_n(x) - u(x)) dx \right).$$

From (5.11), we can estimate

$$\int_{\mathbb{R}^2} Q(x)F(u_n)dx \leq C_7\|u_n\|_E^2 + C_7\|u_n\|_E^q \left(\int_{\mathbb{R}^2} Q(x)\Phi_{q_2\alpha\|u_n\|_E^2,1}\left(\frac{u_n}{\|u_n\|_E}\right)dx\right)^{1/q_2}. \tag{5.25}$$

Once again, by (5.22) we can choose $c_3 > 0$ and $\varepsilon > 0$ small enough such that

$$\lim_{n\to\infty} \|u_n\|_E^2 \leq \left(\frac{2c}{\delta} + \frac{\varepsilon}{\delta} + \frac{\varepsilon C_3}{\delta}\right) < \left(\frac{2c_3}{\delta} + \frac{\varepsilon}{\delta} + \frac{\varepsilon C_3}{\delta}\right) < \frac{\alpha_*}{\alpha q_2},$$

for all $0 < c < c_3$. Thus, $q_2\alpha\|u_n\|_E^2 < \alpha_*$, for $n \in \mathbb{N}$ large. This, Remark 4.2.4, and (5.25) imply that

$$A_n \leq C_8 \left(\int_{\mathbb{R}^2} \log(1+|x|)Q(x)f(u_n(x))(u_n(x)-u(x))dx\right).$$

Invoking Hölder's inequality we conclude that

$$\int_{\mathbb{R}^2} \log(1+|x|)Q(x)f(u_n(x))(u_n(x)-u(x))dx \leq \left(\int_{\mathbb{R}^2} \log(1+|x|)Q(x)f^2(u_n)dx\right)^{1/2} \times \|u_n - u\|_{L^2(\mathbb{R}^2;Q)}.$$

Since, $\lim_{n\to\infty} \|u_n - u\|_{L^2(\mathbb{R}^2;Q)} = 0$ (see Theorem 4.1.2), if we show that the sequence

$$\left(\int_{\mathbb{R}^2} \log(1+|x|)Q(x)f^2(u_n)dx\right)$$

is bounded, we have that $\lim_{n\to\infty} A_n = 0$. In fact, from (1.14), with $q = 1$, (1.16), (5.24), Theorem 4.1.2, and Theorem 4.2.1, yield

$$\begin{aligned}\int_{\mathbb{R}^2} \log(1+|x|)Q(x)f^2(u_n)dx &\leq \varepsilon \int_{\mathbb{R}^2} \log(1+|x|)Q(x)u_n^2dx \\ &\quad + C_9 \int_{\mathbb{R}^2} \log(1+|x|)Q(x)\Phi_{2\alpha\|u_n\|_E^2,1}\left(\frac{u_n}{\|u_n\|_E}\right)dx \\ &\leq C_{10}.\end{aligned}$$

Similarly $\lim_{n\to\infty} B_n = 0$, that is, $\lim_{n\to\infty} \mathcal{G}_2'(u_n)(u_n-u) = 0$. Consequently, $\lim_{n\to\infty} D_n = 0$ (see (5.20)). Now choosing $0 < c_0 < \min\{c_1, c_2, c_3\}$ we obtain the desired result and this completes the proof. □

The next result is an estimate from above for the minimax level c_{MP} defined in (5.13).

Lemma 5.3.4. *Suppose that* (VKQ) *and* $(\mathfrak{f}_1)$ *hold. There exists* $\lambda^* > 0$ *such that if* $(\widetilde{\mathfrak{f}_3})$ *holds with* $\lambda \geq \lambda^*$*, then* $c < c_0$*, where* c_0 *is given in Lemma 5.3.3.*

Proof. First, we shall consider a function $\varphi \in C_0^\infty(\mathbb{R}^2)$, given by $\varphi(x) = 1$ if $|x| \leq 1/2$, $\varphi(x) = 0$

if $|x| \geq 1$, $0 \leq \varphi(x) \leq 1$ for all $x \in \mathbb{R}^2$ and $|\nabla\varphi(x)| \leq 2$ for all $x \in \mathbb{R}^2$. Thus,

$$\begin{aligned}
\mathcal{J}(\varphi) &= \frac{1}{2}\int_{B_1} \left(|\nabla\varphi|^2 + V(x)\varphi^2\right) dx - \frac{1}{4\pi}\left(\mathcal{G}_1(\varphi) - \mathcal{G}_2(\varphi)\right) \\
&\leq 4\pi + \frac{1}{2}\|V\|_{L^1(B_1)} \\
&\quad - \frac{1}{4\pi}\int_{B_1}\int_{B_1} \log\left(1 + \frac{1}{|x-y|}\right) Q(y)F(\varphi(y))Q(x)F(\varphi(x))dydx \\
&\quad + \frac{1}{4\pi}\int_{B_1}\int_{B_1} \log\left(1 + |x-y|\right) Q(y)F(\varphi(y))Q(x)F(\varphi(x))dydx.
\end{aligned}$$

For any $x, y \in B_1$ we see that $|x - y| \leq 2$ and hence

$$\frac{3}{2} = 1 + \frac{1}{2} \leq 1 + \frac{1}{|x-y|}.$$

Therefore, $\log(1 + 1/|x-y|) \geq \log(3/2)$. Moreover, $\log(1+|x-y|) \leq \log 3$ in B_1. Consequently,

$$\begin{aligned}
\mathcal{J}(\varphi) \leq 4\pi + \frac{1}{2}\|V\|_{L^1(B_1)} &- \frac{\log(3/2)}{4\pi}\int_{B_{1/2}}\int_{B_{1/2}} Q(y)F(\varphi(y))Q(x)F(\varphi(x))dydx \\
&+ \frac{\log 3}{4\pi}\int_{B_1}\int_{B_1} Q(y)F(\varphi(y))Q(x)F(\varphi(x))dydx.
\end{aligned}$$

On the one hand, by assumption $(\mathfrak{f}_1)$ there exists $C_1 > 0$ such that $F(s) \leq C_1 s^2$, for all $s \in [0,1]$. On the other hand, from $(\widetilde{\mathfrak{f}_3})$, $F(s) \geq \lambda|s|^\nu$, for all $s \in (0,1]$. So,

$$\mathcal{J}(\varphi) \leq 4\pi + \frac{1}{2}\|V\|_{L^1(B_1)} - \frac{\log(3/2)}{4\pi}\lambda^2\|Q\|^2_{L^1(B_{1/2})} + \frac{\log 3}{4\pi}C_1\|Q\|^2_{L^1(B_1)},$$

where we used that $\varphi(x) = 1$ if $|x| \leq 1/2$ and $0 \leq \varphi(x) \leq 1$ for all $x \in \mathbb{R}^2$. Since the right-hand side above goes to $-\infty$ as $\lambda \to \infty$, we can obtain $\lambda_1 > 0$ such that $\mathcal{J}(\varphi) < 0$, whenever $\lambda \geq \lambda_1$ and hence the path $g(t) := t\varphi$ belongs to Γ. Since $t^4 \leq t^2$ for $t \in [0,1]$, a simple computation shows that

$$\begin{aligned}
c_{MP} &\leq \max_{t\in[0,1]} I(t\varphi) \\
&\leq \max_{t\in[0,1]}\left[\frac{t^2}{2}\left(4\pi + \frac{1}{2}\|V\|_{L^1(B_1)}\right) - t^{2\nu}\frac{\log(3/2)}{4\pi}\lambda^2\|Q\|^2_{L^1(B_{1/2})} + t^4\frac{\log 3}{4\pi}C_1\|Q\|^2_{L^1(B_1)}\right] \\
&\leq \max_{t\in[0,1]}\left[\frac{t^2}{2}\left(4\pi + \frac{1}{2}\|V\|_{L^1(B_1)} + \frac{\log 3}{2\pi}C_1\|Q\|^2_{L^1(B_1)}\right) - t^{2\nu}\frac{\log(3/2)}{4\pi}\lambda^2\|Q\|^2_{L^1(B_{1/2})}\right] \\
&\leq \max_{t\geq 0}\left[\frac{t^2}{2}A - t^{2\nu}D(\lambda)\right],
\end{aligned}$$

where

$$A := 4\pi + \frac{1}{2}\|V\|_{L^1(B_1)} + \frac{\log 3}{2\pi}C_1\|Q\|^2_{L^1(B_1)}$$

and

$$D(\lambda) := \frac{\log(3/2)}{4\pi}\lambda^2\|Q\|^2_{L^1(B_{1/2})}.$$

By carrying out a straightforward computation, we conclude that

$$c \leq \frac{A^{2\nu/(2\nu-2)}}{(D(\lambda))^{2/(2\nu-2)}} \left(\frac{1}{2(2\nu)^{2/(2\nu-2)}} - \frac{1}{(2\nu)^{2\nu/(2\nu-2)}} \right).$$

Since $\lim_{\lambda\to\infty} D(\lambda) = \infty$, the right-hand side above goes to 0 as $\lambda \to \infty$, and hence we obtain $\lambda^* \geq \lambda_1$ such that the inequality $c < c_0$ is verified, for any $\lambda \geq \lambda^*$ and this concludes the proof. □

Finalizing the proof of Theorem 1.1.2. Let λ^* be given by the last lemma and suppose that $(\widetilde{\mathfrak{f}}_3)$ holds with $\lambda \geq \lambda^*$. It follows from all the above lemmas and the Mountain Pass Theorem [13] that $\mathcal{J}$ has a nonzero critical point $u \in E$ which is a weak solution for equation $(\mathcal{E}.\mathcal{C})$. Furthermore, u is nonnegative by Remark 5.1.2 and this finishes the proof. □

Bibliography

[1] S. Adachi, K. Tanaka, *Trudinger type inequalities in $\mathbb{R}^N$ and their best exponents,* Proc. Amer. Math. Soc. **128** (2000), 2051–2057. 2

[2] Adimurthi, K. Sandeep, *A singular Moser-Trudinger embedding and its applications*, Nonlinear Differential Equations Appl. **13** (2007), 585–603. 14

[3] F. S. B. Albuquerque, C. O. Alves, E. S. Medeiros, *Nonlinear Schrödinger equation with unbounded or decaying radial potentials involving exponential critical growth in* $\mathbb{R}^2$, J. Math. Anal. Appl. **409** (2014), 1021–103. 8, 9, 14, 16, 34

[4] F. S. B. Albuquerque, J. L. Carvalho, G. M. Figueiredo, E. S. Medeiros, *On a planar non-autonomous Schrödinger-Poisson system involving exponential critical growth*, Calc. Var. Partial Differential Equations 60 (2021), no. 1, Paper No. 40, 30 pp. 32

[5] F. S. B. Albuquerque, J. L. Carvalho, M. F. Furtado, E. S. Medeiros, *On a critical Schrödinger-Poisson system in* $\mathbb{R}^2$ *with potentials vanishing at infinity*, Submitted. 74

[6] F. S. B. Albuquerque, M. C. Ferreira, U. B. Severo, *Ground state solutions for a nonlocal equation in* $\mathbb{R}^2$ *involving vanishing potentials and exponential critical growth*, preprint, (2019). https://arxiv.org/abs/1911.05707 10, 36

[7] C. O. Alves, D. Cassani, C. Tarsi, M. Yang, *Existence and concentration of ground state solutions for a critical nonlocal Schrödinger equation in* $\mathbb{R}^2$, J. Differ. Equations 261, 1933–1972 (2016). 83

[8] C. O. Alves, G. M. Figueiredo, *Existence of positive solution for a planar Schrödinger-Poisson system with exponential growth*, J. Math. Phys. **60** (2019), 011–503. 33, 54, 56

[9] C. O. Alves, M. A. Souto, *Existence of least energy nodal solution for a Schrödinger-Poisson system in bounded domains*, Z. Angew. Math. Phys. **65** (2014), 1153–1166. 72

[10] A. Ambrosetti, E. Colorado, *Standing waves of some coupled nonlinear Schrödinger equations*, J. Lond. Math. Soc. **75** (2007), 67–82. 57

[11] A. Ambrosetti, V. Felli, A. Malchiodi, *Ground states of nonlinear Schrödinger equations with potentials vanishing at infinity*, J. Eur. Math. Soc. **7** (2005), 117–144. 9, 74

[12] A. Ambrosetti, A. Malchiodi, D. Ruiz, *Radial solutions concentrating on spheres of NLS with with vanishing potentials*, preprint. Proc. Roy. Soc. Edinburgh Sect. A. **136** (2006), 889–907. 34

[13] A. Ambrosetti, P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381. 30, 94

[14] A. Azzollini, A. Pomponio, *Ground state solutions for the nonlinear Schrödinger-Maxwell equations*, J. Math. Anal. Appl. **345** (2008), 90–108. 4, 33

[15] M. Badiale, M. Guida M, S. Rolando, *Compactness and existence results in weighted Sobolev spaces of radial functions. Part I: compactness*, Calc. Var. Partial Differential Equations. **54** (2015), 1061–1090. 34

[16] M. Badiale, L. Pisani, S. Rolando, *Sum of weighted Lebesgue spaces and nonlinear elliptic equations*, Nonlinear Differential Equations Appl. **18** (2011), 369–405. 17, 43, 60

[17] M. Badiale, F. Zaccagni, *Radial nonlinear elliptic problems with singular or vanishing potentials*, Adv. Nonlinear Stud. **18** (2018), 409–428. 8, 34

[18] L. Battaglia, J. V. Schaftingen, *Existence of groundstates for a class of nonlinear Choquard equations in the plane*. Adv. Nonlinear Stud. **17** (2017), 581–594. 82, 83

[19] T. Bartsch, T. Weth, M. Willem, *Partial symmetry of least energy nodal solutions to some variational problems*, J. Anal. Math. **96** (2005), 1–18. 53, 72

[20] V. Benci, D. Fortunato, *An eigenvalue problem for the Schrödinger–Maxwell equations*, Topol. Methods Nonlinear Anal. **11** (1998), 283–293. 33, 56

[21] V. Benci, D. Fortunato, *Solitary waves of the nonlinear Klein-Gordon equation coupled with Maxwell equations*, Rev. Math. Phys. **14** (2002), 409–420. 33, 56

[22] R. Benguria, H. Brezis, E. H. Lieb, *The Thomas-Fermi-von Weizsäcker theory of atoms and molecules*, Comm. Math. Phys. **79** (1981), 167–180. 33

[23] H. Brezis, L. Nirenberg, *Remarks on finding critical points*, Comm. Pure Appl. Math. **44** (1991), 939–963. 20

[24] H. Berestycki, P. L. Lions, *Nonlinear scalar field equations, I Existence of a ground state*, Arch. Rational Mech. Anal. **82** (1983), 313–345. 8

[25] D. Bonheure, S. Cingolani, J. V. Schaftingen, *The logarithmic Choquard equation: sharp asymptotics and nondegeneracy of the groundstate*, J. Funct. Anal. **272** (2017), 5255–5281. 83

[26] D. Bonheure, C. Mercuri, *Embedding theorems and existence results for nonlinear Schrödinger-Poisson systems with unbounded and vanishing potentials*, J. Differential Equations **251** (2011), 1056–1085. 8, 34

[27] D. M. Cao, *Nontrivial solution of semilinear elliptic equation with critical exponent in* $\mathbb{R}^2$, Comm. Partial Differential Equations **17** (1992), 407–435. 2, 11

[28] J. A. Cardoso, J. L. Carvalho, E. S. Medeiros, *Nonlinear Schrödinger equations involving exponential critical growth with unbounded or vanishing potentials*, Submitted. 8

[29] J. L. Carvalho, G. M. Figueiredo, L. A. Maia, E. S. Medeiros, *On a Planar Hartree-Fock type system*, Submitted. 56

[30] J. L. Carvalho, E. S. Medeiros, B. H. C. Ribeiro, *On the planar Choquard equation with nonradial potencial and exponential critical growth*, Submitted. 82

[31] D. Cassani, C. Tarsi, *Schrödinger-Newton equation in dimension two via a Pohozaev-Trudinger log-weighted inequality*, preprint, (2021). https://arxiv.org/abs/2104.04930 83, 87

[32] I. Catto, P. Lions, *Binding of atoms and stability of molecules in Hartree and Thomas-Fermi type theories*, Commun. Partial Differ. Equ. **18** (1993), 1149–1159. 33

[33] G. Cerami, G. Vaira, *Positive solutions for some non–autonomous Schrödinger–Poisson systems*, J. Differ. Equations. **248** (2010), 521–543. 4, 33, 56

[34] G. Cerami, R. Molle, *Positive bound state solutions for some Schrödinger-Poisson systems,* Nonlinearity **29** (2016), 3103–3119. 33

[35] S. T. Chen, X. H. Tang, *Axially symmetric solutions for the planar Schrödinger-Poissson system with critical exponential growth*, J. Differential Equations **269** (2020), 9144–9174. 56

[36] S. Cingolani, T. Weth, *On the planar Schrödinger-Poisson system*, Ann. Inst. H. Poincaré Anal. Non Linéaire. **33** (2016), 169–197. 33, 56, 57, 59, 60, 83

[37] T. D'Aprile, D. Mugnai, *Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations*, Proc. R. Soc. Edinb. Sect. A. **134** (2004), 893–906. 4, 33, 56

[38] T. D'Aprile, D. Mugnai, *Non-existence results for the coupled Klein-Gordon-Maxwell equations*, Adv. Nonlinear Stud. **4** (2004), 307–322. 33

[39] P. d'Avenia, *Non-radially symmetric solutions of nonlinear Schrödinger equation coupled with Maxwell equations*, Adv. Nonlinear Stud. **2** (2002), 177–192. 33

[40] P. d'Avenia, L. A. Maia, G. Siciliano, *Hartree–Fock type systems: existence of ground states and asymptotic behavior*. Preprint, (2020). https://arxiv.org/abs/2008.03191v2 56, 57, 59, 63

[41] J. M. B. do Ó, N-Laplacian equations in $\mathbb{R}^N$ with critical growth, Abstr. Appl. Anal. **2** (1997), 301–315. 2

[42] J. M. B. do Ó, F. Sani, J. Zhang, *Stationary nonlinear Schrödinger equations in* $\mathbb{R}^2$ *with potentials vanishing at infinity*, Ann. Mat. Pura Appl. **196** (2017), 363–393. 9, 77

[43] D. G. de Figueiredo, O. Lopes, *Solitary waves for some nonlinear Schrödinger systems*, Ann. Inst. H. Poincaré Anal. Non Linéaire **25** (2008), 149–161. 57

[44] D. G. de Figueiredo, O. H. Miyagaki, B. Ruf, *Elliptic equations in* $\mathbb{R}^2$ *with nonlinearities in the critical growth range*, Calc. Var. Partial Differential Equations. **3** (1995), 139–153. 21

[45] G. M. Figueiredo, F. B. M. Nunes, *Existence of positive solutions for a class of quasilinear elliptic problems with exponential growth via the Nehari manifold method*, Rev. Mat. Complut. **32** (2019), 1–18. 53

[46] M. F. Furtado, E. S. Medeiros, U. B. Severo, *On a class of semilinear elliptic eigenvalue problems in* $\mathbb{R}^2$, Proc. Edinb. Math. Soc. (2) **60** (2017), 107–126. 11

[47] D. Gilbarg, N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition. 55, 61, 81

[48] I. Ianni, G. Vaira, *On concentration of positive bound states for the Schrödinger-Poisson problem with potentials*, Adv. Nonlinear Stud. **8** (2008), 573–595. 4, 33

[49] E. H. Lieb, *Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation*, Stud. Appl. Math. **57** (1976/1977), 93–105. 33, 41, 82

[50] E. H. Lieb, *Thomas-Fermi and related theories of atoms and molecules*, Rev. Mod. Phys. **53** (1981), 263–301. 33

[51] E.H. Lieb, B. Simon, *The Hartree-Fock theory for Coulomb systems*, Commun. Math. Phys., **53** (1977), 185–194. 56, 82

[52] P. L. Lions, *The Choquard equation and related questions*, Nonlinear Anal. **4** (1980), 1063–1072. 82

[53] P. L. Lions, *Solutions of Hartree-Fock equations for Coulomb systems*, Commun. Math. Phys. **109** (1984), 33–97. 4, 33, 56

[54] P. L. Lions, *The concentration-compactness principle in the calculus of variations*, Part I. Rev. Mat. Ibero. **1** (2017), 145–201. 16

[55] L. Ma, L. Zhao, *Classification of positive solitary solutions of the nonlinear Choquard equation*, Arch. Ration. Mech. Anal. **195** (2010), 455–467. 4, 33, 82

[56] L. A. Maia, E. Montefusco, B. Pellacci, *Positive solutions for a weakly coupled nonlinear Schrödinger system*, J. Differential Equations **229** (2006), 743–767. 57, 59

[57] P. Markowich, C. Ringhofer, C. Schmeiser, *Semiconductor Equations*, Springer-Verlag, New York. 1990. 33

[58] S. Masaki, *Energy solution to a Schrödinger–Poisson system in the two-dimensional whole space*, SIAM J. Math. Anal., **43** (2011), 2719–2731. 35

[59] S. H. Monari Soares, Y. R. Santaria Leuyacc, *Singular Hamiltonian elliptic systems with critical exponential growth in dimension two*, Math. Nachr. **292** (2019), 137–158. 8

[60] V. Moroz, J. V. Schaftingen, *Groundstates of nonlinear Choquard equations: Existence, qualitative properties and decay asymptotics*, J. Funct. Anal. **265** (2013), 153–184. 83

[61] V. Moroz, J. V. Schaftingen, *Existence of groundstates for a class of nonlinear Choquard equations.* Trans. Amer. Math. Soc. **367** (2015), 6557–6579. 83

[62] V. Moroz, J. V. Schaftingen, *A guide to the Choquard equation.* Fixed Point Theory Appl. **19** (2017), 773–813. 33, 82, 83

[63] J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1971), 1077–1092. 2, 12, 29

[64] J. C. N. Pádua, E. A. B. Silva, E. A. B. Soares, S. H. Monari, *Positive solutions of critical semilinear problems involving a sublinear term at the origin*, Indiana Univ. Math. J. **55** (2006) , 1091–1111. 11

[65] R. S. Palais, *The principle of symmetric criticality*, Commun. Math. Phys. **69** (1979) 19–30. 17, 43, 60

[66] S. Pekar, *Untersuchungen über die Elektronentheorie der Kristalle.* Akademie–Verlag, Berlin (1954). 82

[67] S. I. Pohozaev, *The Sobolev Embedding in the Case $pl = n$*, Proc. Tech. Sci. Conf. on Adv. Sci. Research 1964-1965, Mathematics Section, Moskov. Energet. Inst. Moscow, 1965, 158–170. 1

[68] M. del Pino, P. Felmer, *Local mountain pass for semilinear elliptic problems in unbounded domains*, Calc. Var. Partial Differential Equations. **4** (1996), 121-137. 8

[69] V. H. Nguyen, F. Takahashi, *On a weighted Trudinger-Moser type inequality on the whole space and related maximizing problem*, Differential Integral Equations. **31** (2018), 785–806. 9, 11, 13

[70] B. Opic, A. Kufner, *Hardy-Type Inequalities*, Pitman Research Notes in Mathematics Series, 219. Longman Scientific and Technical, Harlow (1990) 74

[71] D. Qin, X. Tang, *On the planar Choquard equation with indefinite potential and critical exponential growth.* J. Differ. Equations. **285** (2021), 40–98. 83

[72] P. H. Rabinowitz, *On a class of nonlinear Schrödinger equations*, Z. Angew. Math. Phys. **43** (1992), 270–291. 8

[73] B. Ruf, *A sharp Trudinger-Moser type inequality for unbounded domains in $\mathbb{R}^2$*, J. Funct. Anal. **219** (1992), 340–367. 2, 11, 13

[74] D. Ruiz, *The Schrödinger-Poisson equation under the effect of a nonlinear local term*, J. Funct. Anal. **237** (2006), 655–674. 4, 33, 56

[75] U. B. Severo, G. M. de Carvalho, *Quasilinear Schrödinger equations with unbounded or decaying potentials.* Math. Nachr. **291** (2018), 492–517. 8

[76] B. Sirakov, *Least energy solitary waves for a system of nonlinear Schrödinger equations in $\mathbb{R}^n$*, Comm. Math. Phys. **271** (2007), 199–221. 57

[77] W. A. Strauss, *Existence of solitary waves in higher dimensions*, Comm. Math. Phys. **55** (1977), 149–162. 8, 14, 58

[78] J. Stubbe, *Bound States of Two-Dimensional Schrödinger-Newton Equations.* preprint, (2008). arxiv.org/abs/0807.4059 33, 35, 56, 57, 59, 83

[79] J. Su, Z. Q. Wang, M. Willem, *Nonlinear Schrödinger equations with unbounded and decaying radial potentials*, Commun. Contemp. Math. **9** (2007), 571–583. 8, 9, 11, 12, 34, 74

[80] J. Su, Z. Q. Wang, M. Willem, *Weighted Sobolev embedding with unbounded and decaying radial potentials*, J. Differential Equations **238** (2007), 201–219. 8, 9, 34, 74

[81] N. S. Trudinger, *On the embedding into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473–484. 1, 29

[82] Y. Yang, X. Zhu, *A new proof of subcritical Trudinger-Moser inequalities on the whole Euclidean space*, J. Partial Differ. Equ. **26** (2013), 300–304. 2, 76

[83] Y. Yang, *Existence of positive solutions to quasi-linear elliptic equations with exponential growth in the whole Euclidean space*, J. Funct. Anal. **262** (2012), 1679–1704. 17

[84] M. Yang, *Semiclassical ground state solutions for a Choquard type equation in* $\mathbb{R}^2$ *with critical exponential growth*, ESAIM Control Optim. Calc. Var. **24** (2018), 177–209. 83

[85] V. I. Yudovic, *Some estimates connected with integral operators and with solutions of elliptic equations,* Dokl. Akad. Nauk SSSR **138** (1961), 805–808. 1