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Relative reduction numbers, regularity of blowup algebras, and Ulrich modules

por

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Fevereiro de 2022

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sob orientação do

Prof. Dr. Cleto Brasileiro Miranda Neto

Tese apresentada ao Corpo Docente do Programa de Pós-Graduação em Matemática da UFPB, como requisito parcial para obtenção do título de Doutor em Matemática.

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Resumo

Provamos novos resultados relativos à conexão entre o número de redução (relativo) e a regularidade de Castelnuovo-Mumford de álgebras e módulos de blowup. Nossa ferramenta básica é o fecho de Ratliff-Rush (relativo). Inicialmente, respondemos em dois casos particulares uma pergunta feita por M. E. Rossi, D. T. Trung e N. V. Trung sobre a álgebra de Rees de ideais em anéis locais de Buchsbaum bidimensionais, e perguntamos se uma dessas situações sempre é válida. Em outro teorema, generalizamos um resultado de A. Mafi sobre ideais em anéis locais Cohen-Macaulay bidimensionais, estendendo-o para dimensão arbitrária e permitindo o contexto relativo a um módulo Cohen-Macaulay. Derivamos uma série de aplicações, incluindo progresso na teoria dos ideais e módulos de Ulrich generalizados e melhorias em resultados de outros autores.

Palavras-chave: Regularidade de Castelnuovo-Mumford; Número de redução; Álgebra de blowup; Fecho de Ratliff-Rush; Ideais e módulos de Ulrich generalizados.

Abstract

We prove new results concerning the connection between (relative) reduction numbers and the Castelnuovo-Mumford regularity of blowup algebras and blowup modules. A key basic tool is the operation of (relative) Ratliff-Rush closure. First, we answer in two particular cases a question of M. E. Rossi, D. T. Trung, and N. V. Trung about Rees algebras of ideals in two-dimensional Buchsbaum local rings, and we even ask whether one of such situations always holds. In another theorem we generalize a result of A. Mafi on ideals in two-dimensional Cohen-Macaulay local rings, by extending it to arbitrary dimension and allowing for the setting relative to a Cohen-Macaulay module. We derive a number of applications, including progress on the theory of generalized Ulrich ideals and modules and improvements of results by other authors.

Keywords: Castelnuovo-Mumford regularity; Reduction number; Blowup algebra; Ratliff-Rush closure; Generalized Ulrich ideals and modules.

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Contents

Introduction

The notion of Castelnuovo-Mumford regularity appeared informally in the 1893 paper “Sui multipli di una serie lineare di gruppi di punti appartenente ad una curva algebrica” by Guido Castelnuovo [8]. Seventy years later, inspired by the work of Castelnuovo, David Mumford [40] formally defined the so-called Castelnuovo-Mumford regularity of a coherent sheaf over a projective space. Since then numerous authors have been studying properties of the regularity both geometrically and algebraically.

Another important invariant is the reduction number of an ideal, which makes use of the concept of reductions, for which we refer to the work of Northcott and Rees [42]. This concept has been widely studied. For example, it plays a relevant role in the study of the Cohen-Macaulayness of associated graded rings. One of the most famous questions, raised by Sally [54] and which has triggered several subsequent works, is as follows:

Question 0.1. If $(R, \mathfrak{m})$ is a Cohen Macaulay local ring having an infinite residue field, then is the reduction number $\mathrm{r}(\mathfrak{m})$ independent of the choice of minimal reduction?

These subjects have been studied for decades and the literature about them is extensive; we mention, e.g., Aberbach, Huneke, and Trung [1], Brodmann and Linh [5], Corso, Polini, and Rossi [10], Dung and Hoa [11], Herzog, Popescu, and Trung [21], Johnson and Ulrich [27], Linh [29], Mafi [31], Marley [33], Polini and Xie [46], Puthenpurakal [48], Rossi, Trung, and Trung [51], Rossi, Trung, and Valla [52], Strunk [57], Trung [62], Vasconcelos [65], Wu [66], and Zamani [68].

This thesis aims at investigating the interplay between reduction numbers, eventually taken relative to a given module, and the Castelnuovo-Mumford regularity of Rees and associated graded algebras and modules. Note that blowup modules generalize blowup algebras of ideals (a classical topic in commutative algebra and algebraic geometry) and are less elaborated than the modern theory of blowup rings of modules, including the notion of reduction number of modules ([56], also [35], [36]). To this end, we shall begin by studying the well-known operation of Ratliff-Rush closure, which goes back to classical work of Ratliff and Rush [49], and also Heinzer, Johnston, Lantz, and Shah [19] in the relative case (i.e., with respect to a module). Then we will turn to explore the class of generalized Ulrich ideals and modules, introduced by Goto et al. in [16] (motivated by the pioneering paper of Ulrich [63] in the 80's) and studied by several authors such as Celikbas [9], Goto, Isobe and Kumashiro [14], Goto, Isobe and Taniguchi [15], Goto et al. [17], Goto, Takahashi and Taniguchi [18], Isobe [25], and Numata [43].

We now proceed to a more detailed description of the contents of the thesis. Chapter 1 is devoted to the preliminary facts that will be used throughout. In the first part, we present some basic definitions and standard results from commutative algebra. Next, we introduce blowup algebras and blowup modules, as well as the notions of reduction number and Castelnuovo-Mumford regularity, which are crucial in this work.

In Chapter 2 we study the Ratliff-Rush closure of an ideal I with respect to a module M. In particular, we introduce and investigate the number

$$s^*(I, M) := \min\{n \in \mathbb{N} \mid \widetilde{I^i_M} = I^i M \text{ for all } i \geq n\}.$$

In another section, we study the asymptotic behavior of certain sets of associated prime ideals, where the main result shows that, under suitable hypotheses, there is an equality

$$\mathrm{Ass}_R\left(M/\widetilde{I^n_M}\right) = \mathrm{Ass}_R\left(\widetilde{I^{n-1}_M}/\widetilde{I^n_M}\right).$$

In Chapter 3, we consider the relations between Castelnuovo-Mumford regularity of blowup rings and modules and the reduction number. The chapter provides the main results of this thesis, such as the ones proved in Miranda-Neto and Queiroz [37]. The first part focuses on furnishing a partial answer to the following question, suggested by Rossi, Trung and Trung in [51].

Question 0.2. Let $(R, \mathfrak{m})$ be a two-dimensional Buchsbaum local ring with depth $R > 0$. Let I be an $\mathfrak{m}$-primary ideal which is not a parameter ideal, and let J be a minimal reduction of I. Is it true that

$$\operatorname{reg}\mathcal{R}(I) = \min\{n \geq \mathrm{r}_J(I) \mid I^{n+1} : I = I^n\}?$$

Concerning this problem, we were able to prove that, under the hypotheses of question 0.2, there is an equality

$$\operatorname{reg}\mathcal{R}(I) = \min\{n \geq \mathrm{r}_J(I) \mid I^{m+1} : I = I^m \ \textit{for all}\ \ m \geq n\},$$

from which we derived a positive answer to the question in two particular situations.

In the second part of Chapter 3 we prove the following theorem:

Theorem 0.3. *Let $(R, \mathfrak{m})$ be a local ring with infinite residue field, M a Cohen-Macaulay R-module of dimension $s \geq 1$, I an $\mathfrak{m}$-primary ideal, and $J = (x_1, \ldots, x_s)$ a minimal reduction of I relative to M. Set $\mathrm{r} = \mathrm{r}_J(I, M)$ and $M_j = M/(x_1, \ldots, x_{j-1})M$ with $M_1 = M$. Assume that either $\mathrm{r} = 0$, or $\mathrm{r} \geq 1$ and $\widetilde{I^{\mathrm{r}}_{M_j}} = I^{\mathrm{r}} M_j$ for $j = 1, \ldots, s-1$ (if $s \geq 2$). Then*

$$\operatorname{reg} \mathcal{R}(I, M) \;=\; \operatorname{reg} \mathcal{G}(I, M) \;=\; \mathrm{r}_J(I, M).$$

In particular, $\mathrm{r}(I, M)$ is independent of the choice of J.

This theorem generalizes Mafi [31, Proposition 2.6] for arbitrary dimension and in addition allows for the context relative to a (Cohen-Macaulay) module. Then we close the chapter by deriving a number of applications of this theorem.

We finish the thesis with Chapter 4, where we continue to give applications of the main results obtained so far in this work. We start by recalling the definitions of generalized Ulrich ideals and modules. For such an ideal I, we prove that all powers I^n are Ratliff-Rush closed, and we characterize the regularity of the Rees and associated graded algebras of I. Also, we describe the Hilbert-Samuel polynomial of I as follows:

Proposition 0.4. *Let R be a Cohen-Macaulay local ring of dimension $d \geq 1$ and with infinite residue field, and let $\mathscr{I}$ be an Ulrich ideal of R minimally generated by $\nu(\mathscr{I})$ elements. Then*

$$P_{\mathscr{I}}(n) \;=\; \lambda(R/\mathscr{I}) \left[(\nu(\mathscr{I}) - d + 1) \binom{n+d-1}{d} - (\nu(\mathscr{I}) - d) \binom{n+d-2}{d-1} \right].$$

Furthermore, $\rho(\mathscr{I}) = -d$ if I is a parameter ideal, and $\rho(\mathscr{I}) = 1 - d$ otherwise. In particular, $H_{\mathscr{I}}(n) = P_{\mathscr{I}}(n)$ for all $n \geq 1$.

As for Ulrich modules, we study in particular their relations to modules of minimal multiplicity. Among the proven results, we highlight the following:

Proposition 0.5. *Suppose R is a Cohen-Macaulay local ring with infinite residue field. Then, every Ulrich R-module with respect to $\mathscr{I}$ has minimal multiplicity with respect to $\mathscr{I}$.*

The results in this chapter were originally obtained in Miranda-Neto and Queiroz [37, Section 6], and in Miranda-Neto, Queiroz, and Souza [38].

Chapter 1

Preliminaries

1.1 Commutative algebra background

In this first section, we recall some basic concepts and facts from commutative algebra that will be useful throughout this thesis.

Conventions. By *ring* we mean commutative, Noetherian, unital ring. If R is a ring, then by a *finite* R-module we mean a finitely generated R-module.

1.1.1 Integral closure

Let I be an ideal in a ring R. An element $b \in R$ is said to be *integral over* I if there exist $n \in \mathbb{Z}_+$ and elements $a_i \in I^i$, $i = 1, \ldots, n$, such that

$$b^n + a_1 b^{n-1} + \cdots + a_{n-1} b + a_n = 0.$$

Such a relation is called an *equation of integral dependence of* b *over* I (of degree n).

The set of all elements that are integral over I, denoted $\overline{I}$, is the so-called *integral closure de* I. If $I = \overline{I}$, then I is named *integrally closed.* If $I \subseteq J$ are ideals, we say that J is *integral* over I if $J \subseteq \overline{I}$. Some properties are listed below.

Remark 1.1. (i) $I \subseteq \overline{I}$.

(ii) If $I \subseteq J$ then $\overline{I} \subseteq \overline{J}$.

(iii) $\overline{I} \subseteq \sqrt{I}$.

The notion of integral closure defined here is entirely linked to the notion of Ratliff-Rush closure which will be introduced in Chapter 2. As we will see in the next chapter, all integrally closed ideals are Ratliff-Rush closed.

1.1.2 Associated primes

Let R be a ring and M an R-module. An ideal $\mathfrak{p} \in \operatorname{Spec}(R)$ is called an *associated prime ideal* to M if there exists an element $m \in M \setminus \{0\}$ such that $P = 0 : m$. The set of all associated primes of M is denoted $\operatorname{Ass}_R(M)$ or simply $\operatorname{Ass}(M)$. Note that $\operatorname{Ass}(M) \subseteq \operatorname{Spec}(R)$, and that $M = 0$ implies $\operatorname{Ass}(M) = \emptyset$. We present now some basic properties of this set.

Lemma 1.1. *Let R be a ring and M an non-zero R-module. Then,*

$$\operatorname{Ass}(M) \neq \emptyset.$$

Proof. See Matsumura [34, Theorem 6.1(i)]. □

Lemma 1.2. *Let R be a ring, M an R-module and N an R-submodule of M. Then,*

$$\operatorname{Ass}(N) \subseteq \operatorname{Ass}(M) \subseteq \operatorname{Ass}(N) \cup \operatorname{Ass}\left(\frac{M}{N}\right).$$

Proof. See Altman and Kleiman [2, Proposition 17.5]. □

If we take an exact sequence of R-modules

$$0 \longrightarrow M' \xrightarrow{\varphi} M \xrightarrow{\beta} M'' \longrightarrow 0$$

then, as a consequence of Lemma 1.2 above, we have

$$\operatorname{Ass}(M') \subseteq \operatorname{Ass}(M) \subseteq \operatorname{Ass}(M') \cup \operatorname{Ass}(M'').$$

1.1.3 Cohen-Macaulay modules

Let R be a ring and M an R-module. We say that $x \in R$ is an *M-regular element* whenever an equality $xm = 0$ (for some $m \in M$) implies $m = 0$, or in other words, if x is not a zero-divisor on M.

A sequence $\mathbf{x} = x_1, \ldots, x_n$ of elements of R is called an *M-regular sequence* or simply an *M-sequence* if the following conditions are satisfied:

(i) The element x_i is $\dfrac{M}{(x_1, \ldots, x_{i-1})M}$-regular for all $i = 1, \ldots, n$;

(ii) $\dfrac{M}{(\mathbf{x})M} \neq 0$.

Now if I is an ideal of R such that $IM \neq M$, it is standard to write grade(I, M) for the maximal length of an M-sequence contained in I. If $M = R$, the notation is simplified to grade I. Note that grade(I, M) is just the I-depth of M; in particular, when $(R, \mathfrak{m})$ is a local ring, we have grade$(\mathfrak{m}, M) = \operatorname{depth} M$.

We finish this subsection by introducing the notions of Cohen-Macaulay and Maximal Cohen-Macaulay modules, which are essential in this thesis; for more details, see Bruns and Herzog [7]. For example, we will see in Chapter 4 that the definition of Ulrich module requires this property.

Definition 1.1. Let R be a local ring. A non-zero finite R-module M is a *Cohen-Macaulay module* if $\operatorname{depth} M = \dim M$. If R itself is a Cohen-Macaulay module, then it is called a *Cohen-Macaulauy ring.* A *maximal Cohen-Macaulay module* is a Cohen-Macaulay module M such that $\dim M = \dim R$.

1.2 Blowup algebras and modules

Let I be an ideal of a ring R. The *Rees ring of I* is the blowup algebra

$$\mathcal{R}(I) = \bigoplus_{n\geq 0} I^n t^n$$

where, as usual, we put $I^0 = R$, This is the standard graded subalgebra $R[It] \subset R[t]$, where t is an indeterminate over R. For a finite R-module M, the *Rees module of I relative to M* is the blowup module

$$\mathcal{R}(I, M) \ = \ \bigoplus_{n\geq 0} I^n M,$$

which in particular is a finite graded module over $\mathcal{R}(I, R) = \mathcal{R}(I)$.

Let us now introduce another blowup module. The *associated graded module of I relative to M* is defined as

$$\mathcal{G}(I, M) \ = \ \bigoplus_{n\geq 0} \frac{I^n M}{I^{n+1} M} \ = \ \mathcal{R}(I, M) \otimes_R R/I,$$

which is a finite graded module over the *associated graded ring of I*,

$$\mathcal{G}(I) \ = \ \mathcal{G}(I,\, R) \ = \ \bigoplus_{n\geq 0} I^n/I^{n+1} \ = \ \mathcal{R}(I) \otimes_R R/I,$$

which in turn is standard graded over $\mathcal{G}(I)_0 = R/I$. We denote $\mathcal{G}(I)_+ = \bigoplus_{n\geq 1} I^n/I^{n+1}$.

It is important to mention that, under a certain set of hypotheses, there is a connection between the Cohen-Macaulayness of $\mathcal{G}(I, M)$ and the property $\mathrm{r}(I, M) \leq 1$, which we will define in the next section. We suggest Polini and Xie [46, Theorem 3.8]. In the following we give a useful result about blowup modules.

Lemma 1.3. *Let $(R, \mathfrak{m})$ be a local ring with infinite residue field, I an ideal of R, and M a finite R-module. (Given $x \in R$ denote by x^* the image of x in I^n/I^{n+1}, where n is the biggest integer such that $x \in I^n \setminus I^{n+1}$). Let $x \in \mathfrak{m} \setminus I$ be such that x^* is a $\mathcal{G}(I, M)$-regular element in $\mathcal{G}(I)$. Then,*

$$\mathcal{R}(I, M)/x\mathcal{R}(I, M) = \mathcal{R}((I + (x))/(x), M/xM).$$

Proof. See Zamani [67, Lemma 2.3]. □

1.3 Reduction number

Let I be a proper ideal of a ring R and let M be a non-zero finite R-module. An ideal $J \subseteq I$ is called a *reduction of I relative to M* if $JI^nM = I^{n+1}M$ for some integer $n \geq 0$. The reduction J is said to be *minimal* if it is minimal with respect to inclusion. In the classical case where $M = R$ we say that J is a *reduction of I*. If J is a reduction of I relative to M, we define the *reduction number of I with respect to J relative to M* as the number

$$\mathrm{r}_J(I, M) := \min\{m \in \mathbb{N} \mid JI^mM = I^{m+1}M\},$$

and the *reduction number of I relative to M* as

$$\mathrm{r}(I, M) := \min\{\mathrm{r}_J(I, M) \mid J \text{ is a minimal reduction of } I \text{ relative to } M\}.$$

We say that $\mathrm{r}(I, M)$ is *independent* if $\mathrm{r}_J(I, M) = \mathrm{r}(I, M)$ for every minimal reduction J of I relative to M. When $M = R$, we write $\mathrm{r}_J(I)$ (resp. $\mathrm{r}(I)$) instead of $\mathrm{r}_J(I, R)$ (resp. $\mathrm{r}(I, R)$), which is the *reduction number of I with respect to J* (resp. *reduction number of I*). The next result will be useful in section 3.4.

Lemma 1.4. *Let $(R, \mathfrak{m})$ be a local ring with infinite residue field, I an ideal of R, and M a finite R-module. (Given $x \in R$ denote by x^* the image of x in I^n/I^{n+1}, where n is the biggest integer such that $x \in I^n \setminus I^{n+1}$). Let $x \in \mathfrak{m} \setminus I$ be such that x^* is a $\mathcal{G}(I, M)$-regular element in $\mathcal{G}(I)$. Then,*

$$\mathrm{r}((I + (x))/(x), M/xM) = \mathrm{r}(I, M).$$

Proof. See Zamani [67, Lemma 2.2]. □

1.4 Castelnuovo-Mumford regularity

Let $S = \bigoplus_{n\geq 0} S_n$ be a finitely generated standard graded algebra over a ring S_0. *Standard* means that S is generated by S_1 as an S_0-algebra. We write $S_+ = \bigoplus_{n\geq 1} S_n$ for the ideal generated by all elements of S of positive degree. Consider a finite graded S-module $N \neq 0$. A sequence $y_1, \ldots, y_s$ of homogeneous elements of S is said to be an *N-filter regular sequence* if

$$y_i \notin P \quad \text{for all} \quad P \in \operatorname{Ass}_S\left(\frac{N}{(y_1, \ldots, y_{i-1})N}\right) \setminus V(S_+)$$

with $i = 1, \ldots, s$.

Now, for a graded S-module $A = \bigoplus_{n\in\mathbb{Z}} A_n$ satisfying $A_n = 0$ for all $n \gg 0$, we let $a(A) = \max\{n \in \mathbb{Z} \mid A_n \neq 0\}$ if $A \neq 0$, and $a(A) = -\infty$ if $A = 0$. For an integer $j \geq 0$, we use the notation

$$a_j(N) \;:=\; a(H^j_{S_+}(N)),$$

where $H^j_{S_+}(-)$ is the j-th local cohomology functor with respect to S_+. It is well-known that $H^j_{S_+}(N)$ is a graded module with $H^j_{S_+}(N)_n = 0$ for all $n \gg 0$; see Brodmann and Sharp [6, Proposition 15.1.5(ii)]. Hence $a_j(N) < +\infty$ (even if j is such that $H^j_{S_+}(N) = 0$). We can now define one of the main numerical invariants studied in this thesis.

Definition 1.2. Keep the above notations. The *Castelnuovo-Mumford regularity* of the S-module N is defined as

$$\operatorname{reg} N \;:=\; \max\{a_j(N) + j \mid j \geq 0\}.$$

It is well-known that $\operatorname{reg} N$ controls the complexity of the graded structure of N and is of great significance in commutative algebra and algebraic geometry, for example in the study of degrees of syzygies. The literature on the subject (including open problems) is extensive; we refer, e.g., to Bayer and Stillman [3], Brodmann and Sharp [6, Chapter 15], Eisenbud and Goto [12], and Trung [60].

We will be interested in the relations between $\operatorname{reg} \mathcal{R}(I, M)$ and the reduction number of I with respect to J relative to M. The following fact generalizes Ooishi [44, Lemma 4.8], also Johnson and Ulrich [27, Proposition 4.1].

Lemma 1.5. *Let R be a ring, I an ideal of R and M a finite R-module. Then* $\operatorname{reg}\mathcal{R}(I,M) = \operatorname{reg}\mathcal{G}(I,M)$.

Proof. See Zamani [68, Corollary 3]. □

Other useful properties are the following ones:

Lemma 1.6. *Let $(R,\mathfrak{m})$ be a local ring of dimension d, and I an $\mathfrak{m}$-primary ideal of R . Then, for any minimal reduction J of I,*

$$a_d(\mathcal{G}(I)) + d \leq r_J(I) \leq \operatorname{reg}\mathcal{G}(I).$$

Proof. See Trung [61, Proposition 3.2] or Marley [33, Lemma 1.2]. □

Lemma 1.7. *Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring, I an $\mathfrak{m}$-primary ideal of R and $t = \operatorname{grade}\mathcal{G}(I)_+$. Then,*

$$a_t(\mathcal{G}(I)) < a_{t+1}(\mathcal{G}(I)).$$

Proof. See Marley [33, Theorem 2.1(a)]. □

Lemma 1.8. *Let R be a ring, I an ideal of R and M a finite R-module. Let $x_1,\dots,x_s$ be elements of I. Then, $x_1t,\dots,x_st$ is an $\mathcal{R}(I,M)$-filter regular sequence if and only if*

$$[(x_1,\dots,x_{i-1})I^nM :_M x_i] \cap I^nM \;=\; (x_1,\dots,x_{i-1})I^{n-1}M \tag{1.1}$$

for $i = 1,\dots,s$ and all $n \gg 0$.

Proof. See Giral and Planas-Vilanova [13, Lemma 4.5]. □

Lemma 1.9. *Let R be a ring, I an ideal of R and M a finite R-module. Let $J = (x_1,\dots,x_s)$ be a reduction of I relative to M such that $x_1t,\dots,x_st$ is an $\mathcal{R}(I,M)$-filter regular sequence. Then*

$$\operatorname{reg}\mathcal{R}(I,M) \;=\; \min\{\ell \geq \mathrm{r}_J(I,M) \mid \textit{equality in (1.1) holds for all } \; n \geq \ell+1\}.$$

Proof. See Giral and Planas-Vilanova [13, Proposition 4.6]. □

Chapter 2

On Ratliff-Rush closure

Our goal in this chapter it is to study the Ratliff-Rush closure of an ideal with respect to a module. We start by introducing and characterizing the invariant $s^*(I, M)$, which will be a key tool in Subsection 3.2 in the case $M = R$. In addition, we study the asymptotic behavior of some sets of associated prime ideals related to the Ratliff-Rush closure.

2.1 Properties of Ratliff-Rush closure

This section follows essentially the exposition given in [37, Section 2].

In their investigation of reductions of ideals, Ratliff and Rush [49] introduced the concept of *Ratliff-Rush closure* $\widetilde{I}$ of a given ideal I in a ring R. Precisely,

$$\widetilde{I} = \bigcup_{n \geq 1} I^{n+1} : I^n.$$

It is well known (see [50]) that

$$I \subseteq \widetilde{I} \subseteq \overline{I} \subseteq \sqrt{I}.$$

Therefore, $\widetilde{I}$ is an ideal of R containing I which refines the integral closure of I, so that $\widetilde{I} = I$ whenever I is integrally closed. Now suppose I contains a regular element (i.e., a non-zero-divisor on R). Then $\widetilde{I}$ is the largest ideal that shares with I the same sufficiently high powers. Moreover, if M is an R-module, Heinzer, Johnston, Lantz, and Shah [19, Section 6] defined the *Ratliff-Rush closure of I with respect to M* as

$$\widetilde{I_M} = \bigcup_{n \geq 1} I^{n+1}M :_M I^n = \{m \in M \mid I^n m \subseteq I^{n+1}M \text{ for some n} \geq 1\}.$$

Of course, this retrieves the classical definition by letting $M = R$. Furthermore, $\widetilde{I_M}$ is an R-submodule of M, and it is easy to see that $IM \subseteq \widetilde{I}M \subseteq \widetilde{I_M}$. If the equality $IM = \widetilde{I_M}$ holds, the ideal I is said to be *Ratliff-Rush closed with respect to* M; in case $M = R$, we simply say that I is *Ratliff-Rush closed* (some authors also use the expression *Ratliff-Rush ideal*).

Lemma 2.1. (Naghipour [41, Proposition 2.2])*Let R be a ring and M be a non-zero finite R-module. Assume that I is an ideal of R containing an M-regular element (i.e., a non-zero-divisor on M) and such that $IM \neq M$ (e.g., if R is local). Then the following conditions hold:*

(i) *For an integer $n \geq 1$, $\widetilde{I^n_M}$ is the eventual stable value of the increasing sequence*

$$(I^{n+1}M :_M I) \subseteq (I^{n+2}M :_M I^2) \subseteq (I^{n+3}M :_M I^3) \subseteq \cdots$$

(ii) $\widetilde{I_M} \supseteq \widetilde{I^2_M} \supseteq \widetilde{I^3_M} \supseteq \cdots \supseteq \widetilde{I^n_M} = I^nM$ *for all $n \gg 0$, and hence I^n is Ratliff-Rush closed with respect to M.*

Proof. By definition $\widetilde{I^n_M} = \bigcup_{k\geq 1} I^{n+nk}M :_M I^{nk}$. As

$$\bigcup_{k\geq 1} I^{n+nk}M :_M I^{nk} = \bigcup_{k\geq 1} I^{n+k}M :_M I^k,$$

item (i) follows.

To prove item (ii), note that for every integer $n \geq 1$ we have

$$\bigcup_{k\geq 1} I^{n+k+1}M :_M I^k \subseteq \bigcup_{k\geq 1} I^{n+k}M :_M I^k$$

and so part (i) gives $\widetilde{I^{n+1}_M} \subseteq \widetilde{I^n_M}$. If we let $n \gg 0$, then $I^nM = I^{n(k+1)}M :_M I^{nk}$ for all integers $k \geq 1$. Hence the definition yields that $\widetilde{I^n_M} = I^nM$ for all large n, as desired. □

Note that, if the ideal I contains an M-regular element, Lemma 2.1 (ii) enables us to define the number

$$s^*(I, M) := \min\{n \in \mathbb{N} \mid \widetilde{I^i_M} = I^iM \text{ for all } i \geq n\},$$

which we simply write $s^*(I)$ if $M = R$. Since the equality $\widetilde{I^i_M} = I^iM$ holds trivially for $i = 0$, we have that $s^*(I, M) \geq 0$ if and only if $s^*(I, M) \geq 1$. Thus we can establish that, throughout the entire thesis,

$$s^*(I,M) \geq 1.$$

Definition 2.1. Let R be a ring, I an ideal of R, and M an R-module. We say that $x \in I$ is an *M-superficial element of I* if there exists $c \in \mathbb{N}$ such that

$$(I^{n+1}M :_M x) \cap I^c M = I^n M,$$

for all $n \geq c$. If x is R-superficial, we simply say that x is a *superficial element of I*. A sequence $x_1, \ldots, x_s$ in I is said to be an *M-superficial sequence of I* if x_1 is an M-superficial element of I and, for all $i = 2, \ldots, s$, the image of x_i in $I/(x_1, \ldots, x_{i-1})$ is an $M/(x_1, \ldots, x_{i-1})M$-superficial element of $I/(x_1, \ldots, x_{i-1})$.

If R is local, the above concept allows for an alternative, useful expression for $s^*(I,M)$. Indeed, in this case, Puthenpurakal [48, Corollary 2.7] shows that

$$s^*(I,M) = \min\{n \in \mathbb{N} \mid I^{i+1}M :_M x = I^i M \text{ for all } i \geq n\},$$

for any given M-superficial element x of I.

Before establishing the main result of this section, we recall the relation between regular elements and superficial elements, for which we refer to Swanson and Huneke [58].

Lemma 2.2. (Swanson and Huneke [58, Lemma 8.5.3]) *Let R be a ring, I an ideal of R, M a finite R-module, and $x \in I$ an M-regular element. Then, x is an M-superficial element of I if and only if $I^nM :_M x = I^{n-1}M$ for all $n \gg 0$.*

Proof. The condition $I^nM :_M x = I^{n-1}M$ for all $n \gg 0$ clearly implies that x is superficial, say by taking $c = 0$. Now assume that x is superficial. By the Artin-Rees Lemma there exists an integer k such that, for all $n \geq k$,

$$I^nM \cap xM = I^{n-k}(I^kM \cap xM) \subseteq xI^{n-k}M.$$

As $I^nM \cap xM = x(I^nM :_M x)$ and as x is a non-zerodivisor on M, it follows that $I^nM :_M x \subseteq I^{n-k}M$. Let $n \geq k+c$, then $I^nM :_M x = (I^nM :_M x) \cap I^cM = I^{n-1}M$ by the assumption on superficiality, which proves the lemma. □

The following is a partial converse:

Lemma 2.3. (Swanson and Huneke [58, Lemma 8.5.4])*Let R be a ring, I an ideal of R, and M a finite R-module. Assume that $\bigcap_n I^nM = 0$ and that I contains an M-regular element. Then every M-superficial element of I is M-regular.*

Proof. Let x be a superficial element of I. Let c be an integer such that for all integers $n \geq c$, $(I^{n+1}M :_M x) \cap I^cM = I^nM$. Then $(0 :_M x)I^c \subseteq I^nM$ for all n. Thus $(0 :_M x)I^c = 0$, so that, as I contains a non-zerodivisor on M, $0 :_M x = 0$. □

The condition $\bigcap_n I^nM = 0$ is satisfied whenever R is a local ring, as guaranteed by the well-known Krull's intersection theorem, so in order to use Lemma 2.3 we just need to find an M-regular element in I.

Lemma 2.2 and Lemma 2.3 put us in a position to prove Proposition 2.4 below, which, according to Mafi [31, Lemma 2.2], is known when $M = R$. Moreover, in our general context, part (ii) has been stated in Puthenpurakal [48, Theorem 2.2(2)]. However, the proofs are omitted in both situations, so for completeness we provide them here.

Proposition 2.4. *Let R be a local ring, M a finite R-module, and I an ideal of R containing an M-regular element. Then, for all $m \in \mathbb{N}$,*

(i) $\widetilde{I^{m+1}_M} :_M I = \widetilde{I^m_M}$;

(ii) $\widetilde{I^{m+1}_M} :_M x = \widetilde{I^m_M}$, *for every M-superficial element x of I.*

Proof. Fix an arbitrary $m \in \mathbb{N}$. By Lemma 2.1 (i), we can write

$$\widetilde{I^m_M} = I^{m+j}M :_M I^j = I^{m+j+1}M :_M I^{j+1}$$

and

$$\widetilde{I^{m+1}_M} = I^{m+j+1}M :_M I^j,$$

for all $j \gg 0$. In particular, we get

$$\widetilde{I^{m+1}_M} :_M I = (I^{m+j+1}M :_M I^j) :_M I = I^{m+j+1}M :_M I^{j+1} = \widetilde{I^m_M},$$

which proves (i). Now, let x be an M-superficial element of I. By Lemma 2.3, x is an M-regular element, and hence, in view of Lemma 2.2, we can choose j such that

$$I^{m+j+1}M :_M x = I^{m+j}.$$

It follows that

$$\widetilde{I^{m+1}_M} :_M x \;=\; (I^{m+j+1}M :_M I^j) :_M x \;=\; (I^{m+j+1}M :_M x) :_M I^j \;=\; I^{m+j}M :_M I^j \;=\; \widetilde{I^m_M},$$

thus showing (ii). □

As a consequence of item (i) we detect the following fact, which is the main result of this section and will be a key tool in this thesis.

Proposition 2.5. *Let R be a local ring and M a non-zero finite R-module. Let I be an ideal of R containing an M-regular element. Then*

$$s^*(I,M) \;=\; \min\{m \geq 1 \mid I^{n+1}M :_M I = I^nM \;\; \textit{for all} \;\; n \geq m\}.$$

Proof. Write $t = \min\{m \geq 1 \mid I^{n+1}M :_M I = I^nM$ for all $n \geq m\}$ and $s = s^*(I,M)$. Let $n \geq s$. From the definition of s we get $\widetilde{I^n_M} = I^nM$ and $\widetilde{I^{n+1}_M} = I^{n+1}M$. On the other hand, Proposition 2.4 (i) gives $\widetilde{I^{n+1}_M} :_M I = \widetilde{I^n_M}$. Hence,

$$I^{n+1}M :_M I \;=\; I^nM$$

for all $n \geq s$, which forces $s \geq t$. Suppose by way of contradiction that $s > t$. Then $s-1 \geq t \geq 1$ and, using Proposition 2.4 (i) again, we obtain

$$\widetilde{I^{s-1}_M} \;=\; \widetilde{I^s_M} :_M I \;=\; I^sM :_M I \;=\; I^{s-1}M,$$

thus violating the minimality of s. We conclude that $s = t$, as needed. □

2.2 The asymptotic stability of $\mathrm{Ass}_R\left(M/\widetilde{I^n_M}\right)$

Let R be a ring, $I \subseteq R$ an ideal and M a non-zero finite R-module. In [55] Sedghi showed, in two different results, that the sequences of associated primes $\mathrm{Ass}_R\left(M/\widetilde{I^n_M}\right)$ and $\mathrm{Ass}_R\left(\widetilde{I^n_M}/\widetilde{I^{n+1}_M}\right)$, $n \geq 1$, are increasing and eventually constant. He used this fact to prove Mirbagheri and Ratliff [39, Theorem 3.1] which, among other things, says that

$$\mathrm{Ass}_R\left(R/\widetilde{I^n}\right) = \mathrm{Ass}_R\left(\widetilde{I^{n-1}}/\widetilde{I^n}\right)$$

for all $n \gg 0$ and both sequences are increasing and eventually stable. In what follows we will generalize the theorem of Mirbagheri and Ratliff to the context of modules by means of a simpler proof. To this end, we need to (slightly) improve Sedghi [55, Proposition 2.4]. It is obvious that $Im \subseteq \widetilde{I_M}$ for any $m \in M$, so the proof of Proposition 2.6 is essentially the same as that of [55, Proposition 2.4]. Let $\widetilde{I^0_M} = M$.

Proposition 2.6. *Let R be a ring, I an ideal of R and M a non-zero finite R-module. Then the sequence*

$$\operatorname{Ass}_R\left(\widetilde{I^{n-1}_M}/\widetilde{I^n_M}\right),\ n \geq 1$$

is increasing and eventually stable.

Proof. Pick $\mathfrak{p} \in \operatorname{Ass}_R\left(\widetilde{I^{n-1}_M}/\widetilde{I^n_M}\right)$. Then, there exists $m \in \widetilde{I^{n-1}_M}$ such that $\mathfrak{p} = \widetilde{I^n_M} :_R m$, and in view of Proposition 2.4 we have

$$\mathfrak{p} = \widetilde{I^n_M} :_R m = (\widetilde{I^{n+1}_M} :_M I) :_R m = \widetilde{I^{n+1}_M} :_R Im.$$

Now if we write generators $I = (x_1, \ldots, x_t)$, then

$$\mathfrak{p} = \widetilde{I^{n+1}_M} :_R Im = \bigcap_{i=1}^{t} \widetilde{I^{n+1}_M} :_R x_i m$$

and hence $\mathfrak{p} = \widetilde{I^{n+1}_M} :_R x_j m$ for some j. Since $x_j m \in Im \subseteq \widetilde{I^n_M}$, we have $\mathfrak{p} \in \operatorname{Ass}_R\left(\widetilde{I^n_M}/\widetilde{I^{n+1}_M}\right)$ and so the increasing property follows.

To prove the second part, notice that Lemma 2.1 gives $\widetilde{I^n_M} = I^{n+s}M :_M I^s$ for $s \gg 0$. Therefore

$$\mathfrak{p} = (I^{n+s}M :_M I^s) :_R m = I^{n+s}M :_R I^s m,$$

so that $\mathfrak{p} \in \operatorname{Ass}_R(M/I^{n+s}M)$. Consequently,

$$\bigcup_{n\geq 1} \operatorname{Ass}_R\left(\widetilde{I^{n-1}_M}/\widetilde{I^n_M}\right) \subseteq \bigcup_{k\geq 1} \operatorname{Ass}_R\left(M/I^k M\right)$$

and the desired result follows from main theorem of Brodmann [4]. □

The next theorem is a generalization of the above-mentioned result of Mirbagheri and Ratliff (see [39, Theorem 3.1]).

Theorem 2.7. *Let R be a ring, I an ideal of R and M a non-zero finite R-module. Then,*

$$\operatorname{Ass}_R\left(M/\widetilde{I^n_M}\right) = \operatorname{Ass}_R\left(\widetilde{I^{n-1}_M}/\widetilde{I^n_M}\right)$$

for all $n \geq 1$.

Proof. We will proceed by induction on n. If $n = 1$ we have $\widetilde{I^0_M} = M$ and the result follows trivially. Now suppose $\operatorname{Ass}_R\left(M/\widetilde{I^n_M}\right) = \operatorname{Ass}_R\left(\widetilde{I^{n-1}_M}/\widetilde{I^n_M}\right)$ and consider the short exact sequence

$$0 \longrightarrow \frac{\widetilde{I^n_M}}{\widetilde{I^{n+1}_M}} \longrightarrow \frac{M}{\widetilde{I^{n+1}_M}} \longrightarrow \frac{M}{\widetilde{I^n_M}} \longrightarrow 0,$$

which then yields

$$\mathrm{Ass}_R\left(\widetilde{I^n_M}/\widetilde{I^{n+1}_M}\right) \subseteq \mathrm{Ass}_R\left(M/\widetilde{I^{n+1}_M}\right) \subseteq \mathrm{Ass}_R\left(\widetilde{I^n_M}/\widetilde{I^{n+1}_M}\right) \cup \mathrm{Ass}_R\left(\widetilde{I^{n-1}_M}/\widetilde{I^n_M}\right).$$

Using Proposition 2.6 above, we deduce that

$$\mathrm{Ass}_R\left(M/\widetilde{I^{n+1}_M}\right) = \mathrm{Ass}_R\left(\widetilde{I^n_M}/\widetilde{I^{n+1}_M}\right)$$

and the result follows by induction. □

Theorem 2.7 enables us to recover Sedghi [55, Theorem 2.2] as a corollary.

Corollary 2.8. *Let R be a ring, I an ideal of R and M a non-zero finite R-module. Then the sequence*

$$\mathrm{Ass}_R\left(M/\widetilde{I^n_M}\right),\ n \geq 1$$

is increasing and eventually stable.

An interesting problem is when $\mathrm{Ass}_R(M/I^nM) = \mathrm{Ass}_R\left(I^{n-1}M/I^nM\right)$ for all $n \geq 1$. With an additional hypothesis, Theorem 2.7 above gives us the following fact.

Corollary 2.9. *Let R be a ring, I an ideal of R and M a non-zero finite R-module. If $s^*(I, M) \leq 1$, then*

$$\mathrm{Ass}_R(M/I^nM) = \mathrm{Ass}_R\left(I^{n-1}M/I^nM\right)$$

for all $n \geq 1$.

Proof. Recall that the condition $s^*(I, M) \leq 1$ implies $\widetilde{I^n_M} = I^nM$ for all $n \geq 1$. Now the result follows by Theorem 2.7. □

The fact that $\mathrm{Ass}_R\left(I^{n-1}M/I^nM\right)$ is eventually stable is a consequence of Lemma 2.1(ii) along with Proposition 2.6. If we assume that $\mathrm{Ass}_R\left(I^{n-1}M/I^nM\right)$ is increasing, then using a similar argument as in Theorem 2.7 we get the equality

$$\mathrm{Ass}_R(M/I^nM) = \mathrm{Ass}_R\left(I^{n-1}M/I^nM\right)$$

for all $n \geq 1$ without the additional hypothesis required in Corollary 2.9. Therefore, it seems natural to raise the following question:

Question 2.10. Let R be a ring, I an ideal of R and M a non-zero finite R-module. Can we say that the sequence

$$\mathrm{Ass}_R\left(I_M^{n-1}/I_M^n\right),\ n \geq 1$$

is increasing?

Chapter 3

Relations between regularity and reduction number

This chapter, essentially taken from Miranda-Neto and Queiroz [37], is the main part of this thesis.

3.1 First results

We begin recalling some useful facts. The usefulness of the proposition below will be made clear in Remark 3.1 and Remark 3.2. Notice that the case $M = R$ recovers Trung [62, Proposition 4.7(i)]; in fact, the proof is essentially the same and we give it in more details.

Proposition 3.1. *Let R be a ring, I an ideal of R and M a finite R-module. Let $J = (x_1, \ldots, x_s)$ be a reduction of I relative to M such that*

$$[(x_1, \ldots, x_{i-1})M :_M x_i] \cap I^{\ell+1}M = (x_1, \ldots, x_{i-1})I^{\ell}M, \qquad i = 1, \ldots, s.$$

for a fixed $\ell \geq \mathrm{r}_J(I, M)$. Then, for all $n \geq \ell + 1$,

$$[(x_1, \ldots, x_{i-1})M :_M x_i] \cap I^n M = (x_1, \ldots, x_{i-1})I^{n-1}M, \qquad i = 1, \ldots, s.$$

Proof. We may take $n > \ell + 1$ as the case $n = \ell + 1$ holds by hypothesis. Since $I^n M = JI^{n-1}M = (x_1, \ldots, x_{s-1})I^{n-1}M + x_s I^{n-1}M$, and $(x_1, \ldots, x_{s-1})I^{n-1}M \subset (x_1, \ldots, x_{s-1})M \subset (x_1, \ldots, x_{s-1})M :_M x_s$, we can write

$$[(x_1, \ldots, x_{s-1})M :_M x_s] \cap I^n M = [(x_1, \ldots, x_{s-1})M :_M x_s] \cap [(x_1, \ldots, x_{s-1})I^{n-1}M + x_s I^{n-1}M]$$

$$= (x_1, \ldots, x_{s-1})I^{n-1}M + \{[(x_1, \ldots, x_{s-1})M :_M x_s] \cap x_s I^{n-1}M\}$$

$$= (x_1, \ldots, x_{s-1})I^{n-1}M + x_s\{[(x_1, \ldots, x_{s-1})M :_M x_s^2] \cap I^{n-1}M\}.$$

Induction on n yields

$$[(x_1, \ldots, x_{s-1})M :_M x_s] \cap I^{n-1}M = (x_1, \ldots, x_{s-1})I^{n-2}M.$$

Thus,

$$[(x_1, \ldots, x_{s-1})M :_M x_s^2] \cap [I^{n-1}M :_M x_s] = \{[(x_1, \ldots, x_{s-1})M :_M x_s] \cap I^{n-1}M\} :_M x_s$$

$$= (x_1, \ldots, x_{s-1})I^{n-2}M :_M x_s \subseteq (x_1, \ldots, x_{s-1})M :_M x_s.$$

Intersecting with $I^{n-1}M$, we get

$$[(x_1, \ldots, x_{s-1})M :_M x_s^2] \cap I^{n-1}M \subseteq [(x_1, \ldots, x_{s-1})M :_M x_s] \cap I^{n-1}M = (x_1, \ldots, x_{s-1})I^{n-2}M.$$

Therefore,

$$x_s\{[(x_1, \ldots, x_{s-1})M :_M x_s^2] \cap I^{n-1}M\} \subseteq x_s[(x_1, \ldots, x_{s-1})I^{n-2}M] \subseteq (x_1, \ldots, x_{s-1})I^{n-1}M.$$

Putting these facts together, we obtain

$$[(x_1, \ldots, x_{s-1})M :_M x_s] \cap I^nM = (x_1, \ldots, x_{s-1})I^{n-1}M.$$

Now, for $i < s$, we can use induction on n and on i to obtain

$$[(x_1, \ldots, x_{i-1})M :_M x_i] \cap I^nM = \{[(x_1, \ldots, x_{i-1})M :_M x_i] \cap I^{n-1}M\} \cap I^nM$$

$$= [(x_1, \ldots, x_{i-1})I^{n-2}M] \cap I^nM \subseteq [(x_1, \ldots, x_i)M] \cap I^nM$$

$$\subseteq [(x_1, \ldots, x_i)M :_M x_{i+1}] \cap I^nM = (x_1, \ldots, x_i)I^{n-1}M = (x_1, \ldots, x_{i-1})I^{n-1}M + x_iI^{n-1}M.$$

It follows that

$$[(x_1, \ldots, x_{i-1})M :_M x_i] \cap I^nM = (x_1, \ldots, x_{i-1})I^{n-1}M + x_i\{[(x_1, \ldots, x_{i-1})M :_M x_i^2] \cap I^{n-1}M\}.$$

Now the result follows similarly as in the case $i = s$. □

Remark 3.1. In the context of Proposition 3.1, a consequence is the following inclusion of submodules of I^nM,

$$[(x_1,\dots,x_{i-1})I^nM :_M x_i] \cap I^nM \subseteq (x_1,\dots,x_{i-1})I^{n-1}M$$

for all $n \geq \ell+1$ and $i = 1,\dots,s$. But this is then an equality, because clearly $x_i[(x_1,\dots,x_{i-1})I^{n-1}M] \subseteq (x_1,\dots,x_{i-1})I^nM$. It follows that (1.1) holds for all $n \geq \ell+1$ and therefore, by Lemma 1.8 and Lemma 1.9, we have $\operatorname{reg}\mathcal{R}(I,M) \leq \ell$. On the other hand, by Lemma 1.5, there is a general equality $\operatorname{reg}\mathcal{R}(I,M) = \operatorname{reg}\mathcal{G}(I,M)$. We conclude that

$$\operatorname{reg}\mathcal{R}(I,M) = \operatorname{reg}\mathcal{G}(I,M) \leq \ell,$$

which thus generalizes Trung [62, Proposition 4.7(iv)].

Remark 3.2. Let $k \geq 1$ be a positive integer. Making $\ell = \mathrm{r}_J(I,M)+k-1$ in Proposition 3.1, we retrieve Giral and Planas-Vilanova [13, Proposition 5.2]. The inequality $\mathrm{r}_J(I,M) \leq \operatorname{reg}\mathcal{R}(I,M)$ follows easily from Lemma 1.9, whereas the bound

$$\operatorname{reg}\mathcal{R}(I,M) \leq \mathrm{r}_J(I,M)+k-1$$

is a consequence of Remark 3.1. In particular, the case $k=1$ yields $\operatorname{reg}\mathcal{R}(I,M) = \mathrm{r}_J(I,M)$.

Next, we present a result which provides another set of conditions under which the equality $\operatorname{reg}\mathcal{R}(I,M) = \mathrm{r}_J(I,M)$ holds. It will be a key ingredient in the proof of Theorem 3.10. The next result is a particular case of Giral and Planas-Vilanova [13, Theorem 5.3].

Lemma 3.2. *Let R be a ring, I an ideal of R and M a finite R-module. Let $J = (z_1,\dots,z_s)$ be an M-reduction of I with reduction number $\mathrm{r}_J(I,M) = r$. Suppose:*

(i) *$z_1,\dots,z_s$ is an M-sequence;*

(ii) *$(z_1,\dots,z_i)M \cap I^{r+1}M = (z_1,\dots,z_i)I^rM$ for all $i = 1,\dots,s-1$.*

Then, $\operatorname{reg}\mathcal{R}(I,M) = \mathrm{r}_J(I,M)$.

Proof. By hypothesis,

$$[(z_1,\dots,z_{i-1})M : z_i] \cap I^{r+1}M = (z_1,\dots,z_{i-1})M \cap I^{r+1}M = (z_1,\dots,z_{i-1})I^rM$$

for all $i = 1,\dots,s$. Thus, applying Proposition 3.1, we obtain

$$[(z_1,\dots,z_{i-1})M : z_i] \cap I^nM = (z_1,\dots,z_{i-1})I^{n-1}M$$

for all $i = 1, \ldots, s$ and $n \geq r+1$, which clearly implies

$$[(z_1, \ldots, z_{i-1})I^n M : z_i] \cap I^n M = (z_1, \ldots, z_{i-1})I^{n-1}M$$

for all $i = 1, \ldots, s$ and $n \geq r+1$. Now the result follows by the case $k = 1$ of Remark 3.2. □

3.2 On a question of Rossi, Trung, and Trung

Here we focus on the interesting question raised by Rossi, Trung, and Trung [51, Remark 2.5] as to whether, for a ring R and an ideal I having a minimal reduction J, the formula

$$\operatorname{reg} \mathcal{R}(I) \;=\; \min\,\{n \geq \mathrm{r}_J(I) \;|\; I^{n+1} : I = I^n\}$$

holds under the hypotheses of Lemma 3.3 below. This lemma is a crucial ingredient in the proof of Theorem 3.4, which will lead us to partial answers to the question (this will be explained in Remark 3.3 and recorded in Corollary 3.5).

Lemma 3.3. *Let $(R, \mathfrak{m})$ be a two-dimensional Buchsbaum local ring with* depth $R > 0$. *Let I be an $\mathfrak{m}$-primary ideal which is not a parameter ideal, and let J be a minimal reduction of I. Then*

$$\operatorname{reg} \mathcal{R}(I) \;=\; \max\,\{\mathrm{r}_J(I),\, s^*(I)\} = \min\,\{n \geq \mathrm{r}_J(I) \;|\; I^n = \widetilde{I^n}\}.$$

Proof. See Rossi, Trung, and Trung [51, Theorem 2.4] □

Theorem 3.4. *For R, I, and J exactly as in Lemma 3.3, there is an equality*

$$\operatorname{reg} \mathcal{R}(I) \;=\; \min\,\{n \geq \mathrm{r}_J(I) \;|\; I^{m+1} : I = I^m \;\; \textit{for all} \;\; m \geq n\}.$$

Proof. By Lemma 3.3, $\operatorname{reg} \mathcal{R}(I) = \max\{\mathrm{r}_J(I), s^*(I)\}$. Assume first that $s^*(I) \leq \mathrm{r}_J(I)$. Note that, being an $\mathfrak{m}$-primary ideal, I contains a regular element since depth $R > 0$. Thus, Proposition 2.4 (i) yields $I^{m+1} : I = \widetilde{I^{m+1}} : I = \widetilde{I^m} = I^m$ for all $m \geq \mathrm{r}_J(I)$, and therefore

$$\operatorname{reg} \mathcal{R}(I) \;=\; \mathrm{r}_J(I) \;=\; \min\,\{n \geq \mathrm{r}_J(I) \;|\; I^{m+1} : I = I^m \text{ for all } m \geq n\}.$$

Now suppose $\mathrm{r}_J(I) < s^*(I)$. Thus $\operatorname{reg} \mathcal{R}(I) = s^*(I)$. On the other hand, Proposition 2.5 gives $s^*(I) = \min\{m \geq 1 \mid I^{n+1} : I = I^n \text{ for all } n \geq m\}$. Hence we can write $\operatorname{reg} \mathcal{R}(I) = \min\{m \geq \mathrm{r}_J(I) \mid I^{n+1} : I = I^n \text{ for all } n \geq m\}$, as needed. □

Remark 3.3. We point out that if $s^*(I) \leq \mathrm{r}_J(I) + 1$ then our theorem (with the crucial description of $s^*(I)$ given by Proposition 2.5) settles affirmatively the problem of Rossi-Trung-Trung. This is easily seen to be true in the situation $s^*(I) \leq \mathrm{r}_J(I)$. Now suppose $s^*(I) = \mathrm{r}_J(I) + 1$, and write $\mathrm{r}_J(I) = r$. Then necessarily

$$I^{r+1} : I \neq I^r.$$

It thus follows that Theorem 3.4 solves the problem once again. Another situation where our result answers affirmatively the question is when R is Cohen-Macaulay and $\widetilde{I^r} = I^r$; this holds because, by Mafi [31, Remark 2.5], we must have $s^*(I) \leq \mathrm{r}_J(I)$ in this case, and then we are done by a previous comment. Below we record such facts as a corollary.

Corollary 3.5. *Let R, I, and J be exactly as in Lemma 3.3. Suppose in addition any one of the following conditions:*

(i) $s^*(I) \leq \mathrm{r}_J(I) + 1$;

(ii) *R is Cohen-Macaulay and $\widetilde{I^r} = I^r$, where $r = \mathrm{r}_J(I)$.*

Then the Rossi-Trung-Trung question has an affirmative answer.

As far as we know, under the hypotheses of Lemma 3.3 there is no example satisfying $s^*(I) > \mathrm{r}_J(I) + 1$. It is thus natural to raise the following question, to which an affirmative answer would imply the definitive solution of the Rossi-Trung-Trung problem.

Question 3.6. Under the hypotheses of Lemma 3.3, is it true that $s^*(I) \leq \mathrm{r}_J(I) + 1$?

Regarding the setting relative to a module, some questions are in order.

Question 3.7. Let R be exactly as in Lemma 3.3. Let I be an $\mathfrak{m}$-primary ideal and let J be a minimal reduction of I relative to a Buchsbaum R-module M, satisfying $\mathrm{r}_J(I, M) \geq 1$. We raise the question of whether any of the following equalities

(i) $\operatorname{reg} \mathcal{R}(I, M) = \max \{\mathrm{r}_J(I, M),\ s^*(I, M)\}$,

(ii) $\operatorname{reg} \mathcal{R}(I, M) = \min \{n \geq \mathrm{r}_J(I, M) \mid I^{m+1}M :_M I = I^m M \text{ for all } m \geq n\}$

is true. Notice that the assertions (i) and (ii) are in fact equivalent; the implication (i)⇒(ii) can be easily seen by using a combination of Theorem 3.10 (to be proven in the next section) and Proposition 2.5, while (ii)⇒(i) follows easily with the aid of Proposition 2.5. We might even conjecture that such equalities hold in the particular case where R and M are Cohen-Macaulay.

3.3 A generalization of a result of Mafi

Our main goal in this section is to establish Theorem 3.10, which concerns the interplay between the numbers $\mathrm{r}_J(I,M)$ and $\operatorname{reg}\mathcal{R}(I,M)$ in a suitable setting. As we shall explain, our theorem generalizes a result due to Mafi from [31] (see Corollary 3.13) and gives us a number of other applications, to be described in Section 3.4 and also in Section 4. As a matter of notation, it is standard to write $\operatorname{grade}(I,M)$ for the maximal length of an M-sequence contained in the ideal I of the local ring $(R,\mathfrak{m})$. If $M=R$, the notation is simplified to $\operatorname{grade} I$. Note that $\operatorname{grade}(I,M)$ is just the I-depth of M; in particular, $\operatorname{grade}(\mathfrak{m},M)=\operatorname{depth} M$.

First we recall two general lemmas.

Lemma 3.8. Let R be a local ring, M a finite R-module of positive dimension, and I a proper ideal. Let $x_1,\ldots,x_s$ be an M-superficial sequence of I. Then $x_1,\ldots,x_s$ is an M-sequence if and only if $\operatorname{grade}(I,M)\geq s$.

Proof. See Rossi and Valla [53, Lemma 1.2]. □

Lemma 3.9. Let $(R,\mathfrak{m})$ be a local ring with infinite residue field, M a finite R-module of positive dimension, I an $\mathfrak{m}$-primary ideal, and J a minimal reduction of I relative to M. Then J can be generated by an M-superficial sequence of I, which is also a system of parameters of M.

Proof. See Rossi and Valla [53, p. 12]. □

Our theorem (which also holds in an appropriate graded setting) is as follows.

Theorem 3.10. *Let $(R,\mathfrak{m})$ be a local ring with infinite residue field, M a Cohen-Macaulay R-module of dimension $s\geq 1$, I an $\mathfrak{m}$-primary ideal, and $J=(x_1,\ldots,x_s)$ a minimal reduction of I relative to M. Set $\mathrm{r}=\mathrm{r}_J(I,M)$ and $M_j=M/(x_1,\ldots,x_{j-1})M$ with $M_1=M$. Assume that either $\mathrm{r}=0$, or $\mathrm{r}\geq 1$ and $\widetilde{I^{\mathrm{r}}_{M_j}}=I^{\mathrm{r}}M_j$ for $j=1,\ldots,s-1$ (if $s\geq 2$). Then*

$$\operatorname{reg}\mathcal{R}(I,M) \;=\; \operatorname{reg}\mathcal{G}(I,M) \;=\; \mathrm{r}_J(I,M).$$

In particular, $\mathrm{r}(I,M)$ is independent.

Proof. First, by Lemma 1.5, the equality $\operatorname{reg}\mathcal{R}(I,M)=\operatorname{reg}\mathcal{G}(I,M)$ holds. Thus our objective is to prove that $\operatorname{reg}\mathcal{R}(I,M)=\mathrm{r}$.

Because M is Cohen-Macaulay and I is $\mathfrak{m}$-primary, we have $s=\operatorname{depth} M=\operatorname{grade}(I,M)$ and then $x_1,\ldots,x_s$ must be in fact an M-sequence by Lemma 3.8. As a consequence, conditions (a) and (b) of Lemma 3.2 are satisfied. We shall prove that

$$(x_1,\dots,x_i)M \,\cap\, I^{\mathrm{r}+1}M \;=\; (x_1,\dots,x_i)I^{\mathrm{r}}M, \qquad i=1,\dots,s-1 \tag{3.1}$$

and then the desired formula $\operatorname{reg}\mathcal{R}(I,M)=\mathrm{r}$ will follow immediately by Lemma 3.2. Notice that the condition is vacuous if $s=1$ (both sides of (3.1) are regarded as zero), so we can suppose $s\geq 2$ from now on.

The case $\mathrm{r}=0$ is now trivial because $(x_1,\dots,x_i)M\subseteq IM$, or what amounts to the same, $(x_1,\dots,x_i)M\cap IM=(x_1,\dots,x_i)M$, for all $i=1,\dots,s-1$.

So let us assume that $\mathrm{r}\geq 1$ and $\widetilde{I^{\mathrm{r}}_{M_j}}=I^{\mathrm{r}}M_j$ for $j=1,\dots,s-1$. We proceed by induction on i. First, pick an element $f\in(x_1)M\cap I^{\mathrm{r}+1}M$. Then there exists $m\in M$ such that $f=x_1m\in I^{\mathrm{r}+1}M$. By Proposition 2.4(b) and the hypothesis that $\widetilde{I^{\mathrm{r}}_{M}}=I^{\mathrm{r}}M$, we have

$$m \,\in\, I^{\mathrm{r}+1}M :_M x_1 \,\subseteq\, \widetilde{I^{\mathrm{r}+1}_{M}} :_M x_1 \;=\; \widetilde{I^{\mathrm{r}}_{M}} \;=\; I^{\mathrm{r}}M,$$

so that $f=x_1m\in(x_1)I^{\mathrm{r}}M$, which shows $(x_1)M\cap I^{\mathrm{r}+1}M=(x_1)I^{\mathrm{r}}M$. Now let $i\geq 2$ and suppose

$$(x_1,\dots,x_{i-1})M \,\cap\, I^{\mathrm{r}+1}M \;=\; (x_1,\dots,x_{i-1})I^{\mathrm{r}}M. \tag{3.2}$$

We claim that

$$[I^{\mathrm{r}+1}M+(x_1,\dots,x_{i-1})M] \,\cap\, (x_1,\dots,x_i)M \;=\; (x_1,\dots,x_{i-1})M+x_iI^{\mathrm{r}}M. \tag{3.3}$$

The inclusion $(x_1,\dots,x_{i-1})M+x_iI^{\mathrm{r}}M\subseteq[I^{\mathrm{r}+1}M+(x_1,\dots,x_{i-1})M]\,\cap\,(x_1,\dots,x_i)M$ is clear. Now pick

$$g \,\in\, [I^{\mathrm{r}+1}M+(x_1,\dots,x_{i-1})M]\cap(x_1,\dots,x_i)M.$$

Then, there exist $h\in I^{\mathrm{r}+1}M$ and $m_l,n_k\in M$, with $l=1,\dots,i-1$ and $k=1,\dots,i$, such that

$$g \;=\; h+\sum_{l=1}^{i-1}x_lm_l \;=\; \sum_{k=1}^{i}x_kn_k.$$

Hence $h-x_in_i\in(x_1,\dots,x_{i-1})M$. Now, denote $(I^n)_i=(I^n+(x_1,\dots,x_{i-1}))/(x_1,\dots,x_{i-1})$ for any given $n\geq 0$, which agrees with $(I_i)^n$. Since $M_i=M/(x_1,\dots,x_{i-1})M$, we have

$$\widetilde{I^{n}_{M_i}} \;=\; \widetilde{(I_i)^{n}_{M_i}} \tag{3.4}$$

for all $n \geq 1$ by [48, Observation 2.3]. Modulo $(x_1, \ldots, x_{i-1})M$, we can write

$$\overline{x_i n_i} = \overline{h} \in I^{\mathrm{r}+1} M_i. \tag{3.5}$$

Note that $\mathrm{grade}(I_i, M_i) > 0$ as the image $\overline{x_i} \in I_i$ of x_i is M_i-regular. Moreover, $\overline{x_i}$ is M_i-superficial. Applying (3.4), (3.5), Proposition 2.4(b) and our hypotheses, we get

$$\overline{n_i} \in I^{\mathrm{r}+1} M_i :_{M_i} \overline{x_i} \subseteq \widetilde{I^{\mathrm{r}+1}_{M_i}} :_{M_i} \overline{x_i} = \widetilde{(I_i)^{\mathrm{r}+1}_{M_i}} :_{M_i} \overline{x_i} = \widetilde{(I_i)^{\mathrm{r}}_{M_i}} = \widetilde{I^{\mathrm{r}}_{M_i}} = I^{\mathrm{r}} M_i, \tag{3.6}$$

where in particular we are using that $\widetilde{I^{\mathrm{r}}_{M_s}} = I^{\mathrm{r}} M_s$, which we claim to hold and will confirm in the last part of the proof. So, (3.6) yields $n_i \in I^{\mathrm{r}} M + (x_1, \ldots, x_{i-1})M$, and this implies

$$g = \left(\sum_{k=1}^{i-1} x_k n_k \right) + x_i n_i \in (x_1, \ldots, x_{i-1})M + x_i I^{\mathrm{r}} M,$$

which thus proves (3.3). Now, using (3.3) and (3.2) we obtain

$$\begin{aligned}
I^{\mathrm{r}+1} M \cap (x_1, \ldots, x_i)M &= I^{\mathrm{r}+1} M \cap (I^{\mathrm{r}+1} M + (x_1, \ldots, x_{i-1})M) \cap (x_1, \ldots, x_i)M \\
&= I^{\mathrm{r}+1} M \cap [(x_1, \ldots, x_{i-1})M + x_i I^{\mathrm{r}} M] \\
&= [I^{\mathrm{r}+1} M \cap (x_1, \ldots, x_{i-1})M] + x_i I^{\mathrm{r}} M \\
&= (x_1, \ldots, x_{i-1}) I^{\mathrm{r}} M + x_i I^{\mathrm{r}} M \\
&= (x_1, \ldots, x_i) I^{\mathrm{r}} M,
\end{aligned}$$

which gives (3.1), as needed.

To finish the proof, it remains to verify $\widetilde{I^{\mathrm{r}}_{M_s}} = I^{\mathrm{r}} M_s$. First observe that, for all $n \geq \mathrm{r}$,

$$I^{n+1} M = J I^n M = (x_1, \ldots, x_s) I^n M = x^s I^n M + (x_1, \ldots, x_{s-1}) I^n M.$$

Adding $(x_1, \ldots, x_{s-1})M$ to both sides, we have

$$\begin{aligned}
& x_s I^n M + (x_1, \ldots, x_{s-1})M &= I^{n+1} M + (x_1, \ldots, x_{s-1})M \\
\Rightarrow\ & x_s J I^{n-1} M + (x_1, \ldots, x_{s-1})M &= I^{n+1} M + (x_1, \ldots, x_{s-1})M \\
\Rightarrow\ & x_s [x_s I^{n-1} M + (x_1, \ldots, x_{s-1}) I^{n-1} M] + (x_1, \ldots, x_{s-1})M &= I^{n+1} M + (x_1, \ldots, x_{s-1})M \\
\Rightarrow\ & x_s^2 I^{n-1} M + (x_1, \ldots, x_{s-1})M &= I^{n+1} M + (x_1, \ldots, x_{s-1})M.
\end{aligned}$$

Proceeding with the same argument, we deduce

$$x_s^{n-\mathrm{r}+1} I^{\mathrm{r}} M + (x_1, \ldots, x_{s-1})M = I^{n+1} M + (x_1, \ldots, x_{s-1})M.$$

Using this equality, the standard definitions and the fact that x_s is $M/(x_1,\ldots,x_{s-1})M$-regular, we can take $k \gg 0$ so that

$$\begin{array}{rclcl} \widetilde{I^{\mathrm{r}}_{M_s}} & = & \widetilde{(I_s)^{\mathrm{r}}_{M_s}} & = & \left(\frac{I}{(x_1,\ldots,x_{s-1})}\right)^{\mathrm{r}+k} M_s :_{M_s} \left(\frac{I}{(x_1,\ldots,x_{s-1})}\right)^{k} \\ & = & \frac{I^{\mathrm{r}+k}M+(x_1,\ldots,x_{s-1})M}{(x_1,\ldots,x_{s-1})M} :_{M_s} \frac{I^k+(x_1,\ldots,x_{s-1})}{(x_1,\ldots,x_{s-1})} & \subseteq & \frac{I^{\mathrm{r}+k}M+(x_1,\ldots,x_{s-1})M}{(x_1,\ldots,x_{s-1})M} :_{M_s} \overline{x_s^k} \\ & = & \frac{x_s^k I^{\mathrm{r}}M+(x_1,\ldots,x_{s-1})M}{(x_1,\ldots,x_{s-1})M} :_{M_s} \overline{x_s^k} & = & \frac{I^{\mathrm{r}}M+(x_1,\ldots,x_{s-1})M}{(x_1,\ldots,x_{s-1}))M} \\ & = & I^{\mathrm{r}}M_s. & & \end{array}$$

□

It seems natural to raise the following question.

Question 3.11. In Theorem 3.10 (assuming $s \geq 3$ and $\mathrm{r} \neq 0$), can we replace the set of hypotheses $\widetilde{I^{\mathrm{r}}_{M_j}} = I^{\mathrm{r}}M_j,\ j = 1,\ldots,s-1$, with the single condition $\widetilde{I^{\mathrm{r}}_M} = I^{\mathrm{r}}M$?

The result below (the case $s = 2$) is an immediate byproduct of Theorem 3.10.

Corollary 3.12. Let $(R,\mathfrak{m})$ be a local ring with infinite residue field, M a Cohen-Macaulay R-module with $\dim M = 2$, I an $\mathfrak{m}$-primary ideal, and J a minimal reduction of I relative to M. Setting $\mathrm{r} = \mathrm{r}_J(I,M)$, assume that either $\mathrm{r} = 0$, or $\mathrm{r} \geq 1$ and $\widetilde{I^{\mathrm{r}}_M} = I^{\mathrm{r}}M$. Then

$$\operatorname{reg}\mathcal{R}(I,M) \;=\; \operatorname{reg}\mathcal{G}(I,M) \;=\; \mathrm{r}_J(I,M).$$

We are now able to recover an interesting result of Mafi (see [31, Proposition 2.6]) about zero-dimensional ideals in two-dimensional Cohen-Macaulay local rings. More precisely, by taking $M = R$ in Corollary 3.12 we readily derive the following fact.

Corollary 3.13. Let $(R,\mathfrak{m})$ be a two-dimensional Cohen-Macaulay local ring with infinite residue field. Let I be an $\mathfrak{m}$-primary ideal and J a minimal reduction of I. Setting $\mathrm{r} = \mathrm{r}_J(I)$, assume that either $\mathrm{r} = 0$, or $\mathrm{r} \geq 1$ and $\widetilde{I^{\mathrm{r}}} = I^{\mathrm{r}}$. Then

$$\operatorname{reg}\mathcal{R}(I) \;=\; \operatorname{reg}\mathcal{G}(I) \;=\; \mathrm{r}_J(I).$$

Example 3.1. Consider the ideal $I = (x^6, x^4y^2, x^3y^3, y^6)$ of the formal power series ring $R = k[[x,y]]$, where k is a field. By [23, Example 3.2], we have $\mathrm{r}(I) = 3$ and $\operatorname{grade}\mathcal{G}(I)_+ > 0$. This latter fact is equivalent to all powers of I being Ratliff-Rush closed (see [19, Fact 9]), i.e., $s^*(I) = 1$. In particular, $\widetilde{I^3} = I^3$, and therefore Corollary 3.13 gives

$$\operatorname{reg}\mathcal{R}(I) \;=\; \operatorname{reg}\mathcal{G}(I) \;=\; 3.$$

Let us also mention that, in this case, it is easy to alternatively confirm any of these equalities, say $\operatorname{reg}\mathcal{R}(I) = 3$, by first regarding (for simplicity) I as an ideal of the polynomial ring $k[x,y]$ and then noting that the Rees algebra of I can be presented as $\mathcal{R}(I) = S/\mathcal{J} = k[x,y,Z,W,T,U]/\mathcal{J}$, where now x,y are seen in degree 0 and Z,W,T,U are indeterminates of degree 1 over $k[x,y]$. Explicitly, $\mathcal{J}$ is the homogeneous S-ideal

$$\mathcal{J} \;=\; (T^2 - ZU,\; yW - xT,\; W^3 - Z^2U,\; y^2Z - x^2W,\; xW^2 - yZT,\; y^3T - x^3U).$$

A computation gives the shifts of the graded S-free modules along a (length 3) resolution of $\mathcal{J}$, and we can finally use a standard device (e.g., [6, Exercise 15.3.7(iv)]) to get $\operatorname{reg}\mathcal{J} = 4$, hence $\operatorname{reg}\mathcal{R}(I) = \operatorname{reg} S/\mathcal{J} = \operatorname{reg}\mathcal{J} - 1 = 3$, as desired.

To conclude the section, we illustrate that Corollary 3.13 is no longer valid if we remove the hypothesis involving the Ratliff-Rush closure.

Example 3.2. Consider the monomial ideal

$$I \;=\; (x^{157},\; x^{35}y^{122},\; x^{98}y^{59},\; y^{157}) \;\subset\; k[x,y],$$

where $k[x,y]$ is a (standard graded) polynomial ring over a field k. By [59, Proposition 3.3 and Remark 3.4], the ideal $J = (x^{157}, y^{157})$ is a minimal reduction of I satisfying

$$\mathrm{r}_J(I) \;=\; 20 \;<\; 21 \;=\; \operatorname{reg}\mathcal{R}(I) \;=\; s^*(I),$$

where the last equality follows by Lemma 3.3. Thus, $\mathrm{r}_J(I) = s^*(I) - 1 \neq \operatorname{reg}\mathcal{R}(I)$. Finally notice that, as $s^*(I) = \mathrm{r}_J(I) + 1 = 20 + 1$, we have $\widetilde{I^{20}} \neq I^{20}$.

3.4 First applications

We now describe some applications of the results obtained in the preceding section. With the exception of Subsection 3.4.3, we shall focus on the two-dimensional case.

3.4.1 Hyperplane sections of Rees modules

The first application is the corollary below, which can be of particular interest for inductive arguments in dealing with Castelnuovo-Mumford regularity of Rees modules.

Corollary 3.14. Let $(R,\mathfrak{m})$ be a local ring with infinite residue field, I an $\mathfrak{m}$-primary ideal, and M a Cohen-Macaulay R-module with $\dim M = 2$. Suppose $\widetilde{I^{\mathrm{r}}_M} = I^{\mathrm{r}}M$ with $\mathrm{r} = \mathrm{r}(I,M)$. Let $x \in \mathfrak{m} \setminus I$ be an M-regular element whose initial form in $\mathcal{G}(I)$ is

$\mathcal{G}(I,M)$-regular. If

$$\left(\widetilde{\frac{I+(x)}{(x)}}\right)_{\frac{M}{xM}} = \frac{IM+xM}{xM}$$

then $\operatorname{reg}\mathcal{R}(I,M)/x\mathcal{R}(I,M) = \operatorname{reg}\mathcal{R}(I,M)$.

Proof. It follows from [67, Lemma 2.3] that

$$\operatorname{reg}\mathcal{R}(I,M)/x\mathcal{R}(I,M) \;=\; \operatorname{reg}\mathcal{R}((I+(x))/(x),\; M/xM).$$

Since M/xM is a Cohen-Macaulay $R/(x)$-module of positive dimension (equal to 1) and the ideal $(I+(x))/(x)$ is $\mathfrak{m}/(x)$-primary, we can apply Corollary 3.12 so as to obtain

$$\operatorname{reg}\mathcal{R}((I+(x))/(x),\; M/xM) \;=\; \mathrm{r}((I+(x))/(x),\; M/xM).$$

On the other hand, [67, Lemma 2.2] yields $\mathrm{r}((I+(x))/(x),\; M/xM) = \mathrm{r}(I,\; M)$. Now, again by Corollary 3.12, $\mathrm{r}(I,M) = \operatorname{reg}\mathcal{R}(I,M)$. Therefore, the asserted equality is true. □

3.4.2 The role of postulation numbers

This subsection focuses on the classical case $M = R$ and investigates connections of postulation numbers with the Castelnuovo-Mumford regularity of blowup algebras and reduction numbers. First recall that, if $(R,\mathfrak{m})$ is a local ring and I is an $\mathfrak{m}$-primary ideal of R, then the corresponding *Hilbert-Samuel function* is given by

$$H_I(n) \;=\; \lambda\,(R/I^n)$$

for any integer $n \geq 1$, and $H_I(n) = 0$ if $n \leq 0$. The symbol $\lambda(-)$ denotes length of R-modules. It is well-known that $H_I(n)$ coincides, for all sufficiently large integers n, with a polynomial $P_I(n)$ – the *Hilbert-Samuel polynomial of* I.

Definition 3.1. The *postulation number* of I is the integer

$$\rho(I) \;=\; \sup\{n \in \mathbb{Z} \mid H_I(n) \neq P_I(n)\}.$$

Here it is worth recalling that the functions $H_I(n)$ and $P_I(n)$ are defined for all integers n, so $\rho(I)$ can be – and often is – negative (as emphasized in [33, Introduction, p. 335]).

In the application below we furnish a characterization of $\operatorname{reg}\mathcal{R}(I)$ (which by [44, Lemma 4.8] agrees with $\operatorname{reg}\mathcal{G}(I)$), in terms, in particular, of $\rho(I)$. Notice that our statement makes no explicit use of the concept of postulation number $p(\mathcal{G}(I))$ for the ring $\mathcal{G}(I)$ (see [57, Remark 1.1]), which we only use in the proof.

Corollary 3.15. Let $(R, \mathfrak{m})$ be a two-dimensional Cohen-Macaulay local ring with infinite residue field, and let I be an $\mathfrak{m}$-primary ideal with a minimal reduction J. If $\operatorname{grade} \mathcal{G}(I)_+ = 0$, then

$$\operatorname{reg} \mathcal{R}(I) = \max\{\mathrm{r}_J(I),\, \rho(I) + 1\}.$$

Proof. We have $\dim \mathcal{G}(I) = 2$ and $H^j_{\mathcal{G}(I)_+}(\mathcal{G}(I)) = 0$ for all $j > 2$, so that $a_j(\mathcal{G}(I)) = -\infty$ whenever $j > 2$. Moreover, $\operatorname{grade} \mathcal{G}(I)_+ = 0$ and then $a_0(\mathcal{G}(I)) < a_1(\mathcal{G}(I))$ by virtue of [33, Theorem 2.1(a)]. Therefore,

$$\operatorname{reg} \mathcal{G}(I) \;=\; \max\{a_1(\mathcal{G}(I)) + 1,\, a_2(\mathcal{G}(I)) + 2\}.$$

Now let us consider the subcase where $a_1(\mathcal{G}(I)) \leq a_2(\mathcal{G}(I))$, so that $\operatorname{reg} \mathcal{G}(I) = a_2(\mathcal{G}(I)) + 2$. On the other hand, [33, Lemma 1.2] yields

$$a_2(\mathcal{G}(I)) + 2 \;\leq\; \mathrm{r}_J(I) \;\leq\; \operatorname{reg} \mathcal{G}(I),$$

and it follows that $\operatorname{reg} \mathcal{G}(I) = \mathrm{r}_J(I)$. Finally, if $a_2(\mathcal{G}(I)) < a_1(\mathcal{G}(I))$, then $a_2(\mathcal{G}(I)) + 2 \leq a_1(\mathcal{G}(I)) + 1$ and hence $\operatorname{reg} \mathcal{G}(I) = a_1(\mathcal{G}(I)) + 1$. In addition, [57, Proof of Theorem 3.10] gives $\rho(I) = p(\mathcal{G}(I))$, and on the other hand, applying [5, Corollary 2.3(2)] we can write $p(\mathcal{G}(I)) = a_1(\mathcal{G}(I))$ since $a_1(\mathcal{G}(I))$ is strictly bigger than both $a_0(\mathcal{G}(I))$ and $a_2(\mathcal{G}(I))$. Thus, $\operatorname{reg} \mathcal{G}(I) = \rho(I) + 1$. □

Next we provide a different proof (in fact an improvement) of [22, Proposition 3.7]. Notice that Hoa's $c(I)$ is just $s^*(I) - 1$.

Corollary 3.16. Let $(R, \mathfrak{m})$ be a two-dimensional Cohen-Macaulay local ring with infinite residue field, and let I be an $\mathfrak{m}$-primary ideal. If

$$\rho(I) \neq s^*(I) - 1$$

then $\mathrm{r}_J(I) = \operatorname{reg} \mathcal{R}(I)$ for any minimal reduction J of I. In particular, $\mathrm{r}(I)$ is independent.

Proof. Set $\mathrm{r} = \mathrm{r}(I)$. By virtue of Corollary 3.13, we can assume that $\mathrm{r}(I) \geq 1$ (in particular, I cannot be a parameter ideal; see Subsection 3.4.3 below). Let us consider first the case $\operatorname{grade} \mathcal{G}(I)_+ > 0$. According to [19, Fact 9], all powers of I must be

Ratliff-Rush closed, and hence $s^*(I) = 1$. Note $\mathrm{r}(I) \geq 1$ implies $\mathrm{r}_J(I) \geq 1 = s^*(I)$ for every minimal reduction J of I. By Lemma 3.3, the asserted equality follows.

So we can suppose $\operatorname{grade} \mathcal{G}(I)_+ = 0$. As verified in the proof of Corollary 3.15, if $a_1(\mathcal{G}(I)) \leq a_2(\mathcal{G}(I))$ then for any minimal reduction J of I we can write $\mathrm{r}_J(I) = \operatorname{reg} \mathcal{G}(I)$, which as we know coincides with $\operatorname{reg} \mathcal{R}(I)$. Moreover, we have seen that if $a_2(\mathcal{G}(I)) < a_1(\mathcal{G}(I))$ then

$$\operatorname{reg} \mathcal{G}(I) = \rho(I) + 1,$$

so that $\operatorname{reg} \mathcal{G}(I) \neq s^*(I)$, which is tantamount to saying that $\operatorname{reg} \mathcal{R}(I) \neq s^*(I)$. Now Lemma 3.3 yields $\operatorname{reg} \mathcal{R}(I) = \mathrm{r}_J(I)$ whenever J is a minimal reduction of I, as needed. □

3.4.3 Ideals of linear type

For an ideal I of a ring R, there is a canonical homomorphism

$$\pi \colon \mathcal{S}(I) \longrightarrow \mathcal{R}(I)$$

from the symmetric algebra $\mathcal{S}(I)$ of I onto its Rees algebra $\mathcal{R}(I)$. The ideal I is said to be of *linear type* if π is an isomorphism. To see what this means a bit more concretely, we can make use of some (any) R-free presentation

$$R^m \xrightarrow{\varphi} R^\nu \longrightarrow I \longrightarrow 0.$$

Letting $S = R[T_1, \ldots, T_\nu]$ be a standard graded polynomial ring in variables $T_1, \ldots, T_\nu$ over $R = S_0$, we can identify $\mathcal{S}(I) = S/\mathcal{L}$, where $\mathcal{L}$ is the ideal generated by the m linear forms in the T_i's given by the entries of the product $(T_1 \cdots T_\nu) \cdot \varphi$, where φ is regarded as a $\nu \times m$ matrix taken with respect to the canonical bases of R^ν and R^m. We can also write

$$\mathcal{R}(I) = S/\mathcal{J}$$

for a certain ideal $\mathcal{J}$ containing $\mathcal{L}$. Precisely, expressing $\mathcal{R}(I) = R[It]$ (where t is an indeterminate over R), then $\mathcal{J}$ is the kernel of the natural epimorphism $S \to \mathcal{R}(I)$. Now, the above-mentioned map π can be simply interpreted as the surjection

$$\pi \colon S/\mathcal{L} \longrightarrow S/\mathcal{J}.$$

It follows that I is of linear type if and only if $\mathcal{J} = \mathcal{L}$. For instance, if I is generated by a regular sequence then I is of linear type (see [58, 5.5]).

Now assume $(R, \mathfrak{m})$ is either local with infinite residue field or a standard graded algebra over an infinite field. It follows from [42, Theorem 4(ii)] that if I is of linear type (or if I is an $\mathfrak{m}$-primary parameter ideal), then $\mathrm{r}(I) = 0$. However, there exist examples of ideals with reduction number zero that are not of linear type, so the natural question remains as to under what conditions the property $\mathrm{r}(I) = 0$ forces I to be of linear type. The corollary below is a quick application of Theorem 3.10 and contributes to the problem. For us it is plausible to believe that the result is known to the experts.

Corollary 3.17. Let R be a standard graded polynomial ring over an infinite field, and let I be a zero-dimensional ideal of R. If $\mathrm{r}(I) = 0$, then I is of linear type.

Proof. Applying the graded analogue of Theorem 3.10 with $M = R$, we obtain

$$\operatorname{reg} \mathcal{R}(I) = 0.$$

Using the above notations, we get $\operatorname{reg} S/\mathcal{J} = 0$, so that $\operatorname{reg} \mathcal{J} = 1$, which implies (by a well-known property of the regularity over graded polynomial rings) that the homogeneous S-ideal $\mathcal{J}$ admits no minimal generator of degree greater than 1. In other words, we must have $\mathcal{J} = \mathcal{L}$. As already clarified, this means that I is of linear type. □

Chapter 4

Ulrich ideals and modules

This last chapter provides some applications concerning the theory of generalized Ulrich ideals and modules. Sections 4.1 and 4.2 are essentially taken from Miranda-Neto and Queiroz [37], and section 4.3 from Miranda-Neto, Queiroz, and Souza [38].

Here is the setup and central notion of this chapter.

Convention 4.1. Henceforth, we adopt the following convention. Whenever $(R, \mathfrak{m})$ is a d-dimensional Cohen-Macaulay local ring, we will let $\mathscr{I}$ (to be distinguished from the notation I) stand for an $\mathfrak{m}$-primary ideal that contains a parameter ideal

$$Q = (\mathbf{x}) = (x_1, \ldots, x_d)$$

as a reduction. As is well-known, any $\mathfrak{m}$-primary ideal of R has this property provided that the residue class field $R/\mathfrak{m}$ is infinite, or that R is analytically irreducible with $d = 1$.

Next, we recall the general notions of Ulrich ideal and Ulrich module as introduced by Goto et al in [16]. As will be made clear, the latter (Definition 4.2 below) generalizes the classical concept of Ulrich Modules – or Maximally Generated Maximal Cohen-Macaulay Modules – originally introduced by Ulrich [63].

Definition 4.1. (Goto, Ozeki, Takahashi, Watanabe and Yoshida [16, Definition 1.1]) Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring and $\mathscr{I}$ as above. We say that $\mathscr{I}$ is a *generalized Ulrich ideal* – *Ulrich ideal*, for short – if $\mathscr{I}$ satisfies the following properties:

(i) $\mathrm{r}_Q(\mathscr{I}) \leq 1$;

(ii) $\mathscr{I}/\mathscr{I}^2$ is a free $R/\mathscr{I}$-module.

Remark 4.1. Let us recall a couple of interesting basic facts about Ulrich ideals. First, since R is Cohen-Macaulay, it is clear that every parameter ideal is Ulrich. Second, if R and $\mathscr{I}$ are as above and $R/\mathfrak{m}$ is infinite, then by Valla [64, Lemma 1 and Theorem 1] the associated graded ring $\mathcal{G}(\mathscr{I})$ must be Cohen-Macaulay whenever $\mathrm{r}_Q(\mathscr{I}) \leq 1$. Therefore, if $\mathscr{I}$ is Ulrich then $\mathcal{G}(\mathscr{I})$ is Cohen-Macaulay.

Definition 4.2. (Goto, Ozeki, Takahashi, Watanabe and Yoshida [16, Definition 1.1]) Let R be a Cohen-Macaulay local ring and let M be a finite R-module. We say that M is *Ulrich with respect to* $\mathscr{I}$ if the following conditions hold:

(i) M is a maximal Cohen-Macaulay R-module;

(ii) $\mathscr{I}M = QM$;

(iii) $M/\mathscr{I}M$ is a free $R/\mathscr{I}$-module.

4.1 Regularity of blowup algebras

In order to determine the Castelnuovo-Mumford regularity of the blowup algebras of an Ulrich ideal, we first recall one of the auxiliary results.

Lemma 4.1. *If* $(R, \mathfrak{m})$ *is a Cohen-Macaulay local ring of positive dimension and* I *is an* $\mathfrak{m}$*-primary ideal with* $\mathrm{r}(I) \leq 1$*, then every power of* I *is Ratliff-Rush closed.*

Proof. See Heinzer, Lantz, and Shah [20, (1.3)]. □

This lemma immediately gives the following fact, which will be useful in the proof of Corollary 4.3.

Corollary 4.2. *Let* R *be a Cohen-Macaulay local ring of positive dimension with infinite residue field and let* $\mathscr{I}$ *an Ulrich ideal. Then*

$$\widetilde{\mathscr{I}^n} = \mathscr{I}^n, \qquad \forall\, n \geq 1.$$

Therefore, for any parameter ideal Q *which is a reduction of* I*, we have either* $\mathrm{r}_Q(\mathscr{I}) = 0$ *or* $s^*(\mathscr{I}) = \mathrm{r}_Q(\mathscr{I}) = 1$*.*

In addition, Corollary 4.2 is particularly useful to test for ideals that are not Ulrich, as illustrated in the next two examples.

Example 4.1. Let k be an infinite field and $R = k[[x, y]]$. Then, by Corollary 4.2, the ideal $I = (x^4, x^3y, xy^3, y^4)$ is not Ulrich, since $x^2y^2 \in \widetilde{I} \setminus I$.

Example 4.2. Let k be a field and $R = k[[t^3, t^{10}, t^{11}]]$, which is a 1-dimensional complete local domain. As observed by Heinzer, Lantz, and Shah [20, Example 1.16], the ideal $I = (t^9, t^{10}, t^{14})$ is not Ratliff-Rush closed (precisely, $t^{11} \in \widetilde{I} \setminus I$). Hence, Corollary 4.2 gives that I is not Ulrich.

The next example shows that the converse of Corollary 4.2 is not true.

Example 4.3. Let k be an infinite field. Consider the zero-dimensional ideal

$$I = (x^6,\, x^4y^2,\, x^3y^3,\, x^2y^4,\, xy^5,\, y^6) \subset R = k[[x,y]].$$

According to Mafi [30, Example 2.15] we have depth $\mathcal{G}(I) = 1$ and hence $\mathcal{G}(I)$ is not Cohen-Macaulay. By Remark 4.1, the ideal I cannot be Ulrich. On the other hand, since grade $\mathcal{G}(I)_+ > 0$ we have $\widetilde{I^n} = I^n$ for all $n \geq 1$.

Now we are able to find the regularity of the Rees algebra (hence of the associated graded ring) of an Ulrich ideal.

Corollary 4.3. *Let R be a Cohen-Macaulay local ring of positive dimension with infinite residue field, and let $\mathscr{I}$ be an Ulrich ideal of R. Then,*

$$\operatorname{reg} \mathcal{R}(\mathscr{I}) = \operatorname{reg} \mathcal{G}(\mathscr{I}) \leq 1,$$

with equality if and only if $\mathscr{I}$ is not a parameter ideal. Furthermore, $\mathrm{r}(\mathscr{I})$ *is independent.*

Proof. Pick a minimal reduction $Q = (x_1, \ldots, x_d)$ of $\mathscr{I}$ (notice that Q is necessarily a parameter ideal). Since $\mathscr{I}$ is Ulrich, [16, Lemma 2.3] yields that $\mathscr{I}/Q$ is a free $R/\mathscr{I}$-module. For $i = 1, \ldots, d$, set $R_i = R/(x_1, \ldots, x_{i-1})$ (with $R_1 = R$), $\mathscr{I}_i = \mathscr{I} R_i$ and $Q_i = QR_i$. As $Q\mathscr{I} = \mathscr{I}^2$, we have

$$Q_i \mathscr{I}_i = (\mathscr{I}_i)^2.$$

Moreover, $\mathscr{I}_i/Q_i \cong \mathscr{I}/Q$ and $R_i/\mathscr{I}_i \cong R/\mathscr{I}$, which gives that $\mathscr{I}_i/Q_i$ is a free $R_i/\mathscr{I}_i$-module. Applying [16, Lemma 2.3] once again, we get that $\mathscr{I}_i$ is an Ulrich ideal of R_i for all i. Now, by Corollary 4.2 (with $n = 1$) we have

$$\widetilde{\mathscr{I}_i} = \mathscr{I}_i = \mathscr{I} R_i$$

for all i. On the other hand, it is easy to see that $\widetilde{\mathscr{I}_i} = \widetilde{\mathscr{I}_{R_i}}$ in the notation of Theorem 3.10 (with $M = R$), which therefore gives $\operatorname{reg} \mathcal{R}(\mathscr{I}) = \operatorname{reg} \mathcal{G}(\mathscr{I}) = \mathrm{r}_Q(\mathscr{I}) \leq 1$, as asserted. Observe that this also shows the independence of $\mathrm{r}(\mathscr{I})$.

Now the characterization of equality can be rephrased as $\mathrm{r}_Q(\mathscr{I}) = 0$ if and only if $\mathscr{I}$ is a parameter ideal. Obviously, $\mathrm{r}_Q(\mathscr{I}) = 0$ means $\mathscr{I} = Q$, which is a parameter ideal. Conversely, if $\mathscr{I}$ is a parameter ideal then $\mathrm{r}(\mathscr{I}) = 0$ (see Subsection 3.4.3) and hence $\mathrm{r}_Q(\mathscr{I}) = 0$ by independence. □

Example 4.4. Let k be an infinite field. Given $\ell \geq 2$, consider the zero-dimensional ideal

$$\mathscr{I} \;=\; (t^4,\, t^6) \;\subset\; R \;=\; k[[t^4, t^6, t^{4\ell-1}]].$$

According to Goto, Ozeki, Takahashi, Watanabe and Yoshida [16, Example 2.7(1)], this ideal is Ulrich and strictly contains a parameter ideal $Q = (t^4)$ as a reduction, so that $\mathrm{r}_Q(\mathscr{I}) \geq 1$. Applying Corollary 4.3 we obtain

$$\operatorname{reg} \mathcal{R}(\mathscr{I}) \;=\; \operatorname{reg} \mathcal{G}(\mathscr{I}) \;=\; 1.$$

4.2 Hilbert-Samuel polynomial and postulation number

Our next goal is to determine the Hilbert-Samuel coefficients and the postulation number of an Ulrich ideal. First recall that if $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring of dimension $d \geq 1$ and I is an $\mathfrak{m}$-primary ideal of R, then it is a classical fact that the Hilbert-Samuel polynomial of I can be expressed as

$$P_I(n) \;=\; \sum_{i=0}^{d} (-1)^i \mathrm{e}_i(I) \binom{n+d-i-1}{d-i}, \tag{4.1}$$

where $\mathrm{e}_0(I), \ldots, \mathrm{e}_d(I)$ are the so-called *Hilbert-Samuel coefficients of I*. The number $\mathrm{e}_0(I)$ (which is always positive) is the multiplicity while $\mathrm{e}_1(I)$ is dubbed *Chern number* of I.

For the connection between postulation and reduction numbers, the following fact will be useful.

Lemma 4.4. *Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring with infinite residue field, and let I be an $\mathfrak{m}$-primary ideal with* $\operatorname{grade} \mathcal{G}(I)_+ \geq d-1$. *Then*

$$\mathrm{r}(I) = \rho(I) + d.$$

Proof. See Marley [32, Theorem 2]. □

Proposition 4.5. Let R be a Cohen-Macaulay local ring of dimension $d \geq 1$ and with infinite residue field, and let $\mathscr{I}$ be an Ulrich ideal of R minimally generated by $\nu(\mathscr{I})$ elements. Then

$$P_{\mathscr{I}}(n) \;=\; \lambda(R/\mathscr{I})\left[(\nu(\mathscr{I})-d+1)\binom{n+d-1}{d}-(\nu(\mathscr{I})-d)\binom{n+d-2}{d-1}\right].$$

Furthermore, $\rho(\mathscr{I})=-d$ if I is a parameter ideal, and $\rho(\mathscr{I})=1-d$ otherwise. In particular, $H_{\mathscr{I}}(n)=P_{\mathscr{I}}(n)$ for all $n\geq 1$.

Proof. Let Q be a minimal reduction of $\mathscr{I}$ (note Q is a parameter ideal). Since $\mathscr{I}^2=\mathscr{I}Q$, [24, Theorem 2.1] gives that the Chern number of $\mathscr{I}$ is given by

$$\mathrm{e}_1(\mathscr{I})=\mathrm{e}_0(\mathscr{I})-\lambda(R/\mathscr{I})$$

and in addition $\mathrm{e}_j(\mathscr{I})=0$ for all $j=2,\dots,d$. Moreover, by [64, Lemma 1],

$$\lambda(\mathscr{I}/\mathscr{I}^2) \;=\; \mathrm{e}_0(\mathscr{I})+(d-1)\lambda(R/\mathscr{I}). \tag{4.2}$$

On the other hand, $\mathscr{I}/\mathscr{I}^2$ is a free $R/\mathscr{I}$-module by hypothesis, and clearly the minimal number of generators of $\mathscr{I}/\mathscr{I}^2$ coincides with $\nu:=\nu(\mathscr{I})$. Thus, setting $\lambda:=\lambda(R/\mathscr{I})$, we get $\lambda(\mathscr{I}/\mathscr{I}^2)=\lambda((R/\mathscr{I})^{\nu})=\nu\lambda$. Therefore, by (4.2),

$$\mathrm{e}_0(\mathscr{I}) \;=\; \nu\lambda-(d-1)\lambda \;=\; \lambda(\nu-d+1)$$

and hence $\mathrm{e}_1(\mathscr{I})=\mathrm{e}_0(\mathscr{I})-\lambda=\lambda(\nu-d)$. Using (4.1), the formula for $P_{\mathscr{I}}(n)$ follows.

Finally, note that [64, Theorem 1] yields grade $\mathcal{G}(\mathscr{I})_+=d$. It now follows from Lemma 4.4 that $\rho(\mathscr{I})=\mathrm{r}(\mathscr{I})-d$. By Corollary 4.3 and its proof, we get $\mathrm{r}(\mathscr{I})\leq 1$ with $\mathrm{r}(\mathscr{I})=1$ if and only if $\mathscr{I}$ is not a parameter ideal, so the assertions about $\rho(\mathscr{I})$ hold. In particular, $\rho(\mathscr{I})\leq 0$, which gives $H_{\mathscr{I}}(n)=P_{\mathscr{I}}(n)$ for all $n\geq 1$. □

Next we remark that Ulrich ideals that are not parameter ideals are the same as Ulrich ideals with non-zero Chern number.

Remark 4.2. Maintain the setting and hypotheses of Proposition 4.5. We have shown in particular that the Chern number of $\mathscr{I}$ is given by

$$\mathrm{e}_1(\mathscr{I}) \;=\; \lambda(R/\mathscr{I})(\nu(\mathscr{I})-d).$$

As a consequence, the Ulrich ideal $\mathscr{I}$ is a parameter ideal if and only if $\mathrm{e}_1(\mathscr{I})=0$.

Example 4.5. Let us revisit the Ulrich ideal $\mathscr{I} \subset R$ of Example 4.4. Note that R is a domain with $d = 1$, hence R is Cohen-Macaulay. Moreover, we have $\nu(\mathscr{I}) = 2$ and $\lambda(R/\mathscr{I}) = 2$. Using Proposition 4.5 we get $\mathrm{e}_0(\mathscr{I}) = 4$, $\mathrm{e}_1(\mathscr{I}) = 2$, $\rho(\mathscr{I}) = 0$, and then

$$P_{\mathscr{I}}(n) = \lambda(R/\mathscr{I}^n) \;=\; 4n - 2, \qquad \forall n \geq 1.$$

Example 4.6. Consider the so-called E_8-singularity

$$R = \mathbb{C}[[x, y, z]]/(x^3 + y^5 + z^2),$$

which is a rational double point. By Goto, Ozeki, Takahashi, Watanabe and Yoshida [17, Example 7.2], the ideal

$$\mathscr{I} = (x, y^2, z)R$$

is an Ulrich ideal. Here we have $d = 2$, $\nu(\mathscr{I}) = 3$ and $\lambda(R/\mathscr{I}) = 2$, so that $\mathrm{e}_0(\mathscr{I}) = 4$, $\mathrm{e}_1(\mathscr{I}) = 2$. Also, $\rho(I) = -1$. By Proposition 4.5, we can write

$$P_{\mathscr{I}}(n) = \lambda(R/\mathscr{I}^n) \;=\; 4\binom{n+1}{2} - 2n \;=\; 2n^2, \qquad \forall n \geq 0.$$

Example 4.7. Fix integers $a \geq b \geq c \geq 2$, and consider the local ring

$$R \;=\; \mathbb{C}[[t, x, y, z]]/(xy - t^{a+b},\, xz - t^{a+c} + zt^a,\, yz - yt^c + zt^b)$$

which is a rational surface singularity (hence Cohen-Macaulay), more precisely a rational triple point. Given an integer ℓ with $1 \leq \ell \leq c$, consider the ideal

$$\mathscr{I} \;=\; (x,\, y,\, z,\, t^\ell)R.$$

By Goto, Ozeki, Takahashi, Watanabe and Yoshida [17, Example 7.5], the ideal $\mathscr{I}$ is Ulrich, with $\lambda(R/\mathscr{I}) = \ell$. Moreover, $d = 2$ and $\nu(\mathscr{I}) = 4$, so that $\mathrm{e}_0(\mathscr{I}) = 3\ell$ and $\mathrm{e}_1(\mathscr{I}) = 2\ell$. We also have $\rho(I) = -1$. Proposition 4.5 thus yields

$$P_{\mathscr{I}}(n) \;=\; \lambda(R/\mathscr{I}^n) \;=\; 3\ell\binom{n+1}{2} - 2\ell n \;=\; \frac{\ell n}{2}(3n-1), \qquad \forall n \geq 0.$$

Example 4.8. Fix integers $m \geq 1$, $d \geq 2$ and $n_1, \ldots, n_d \geq 2$. Given an infinite field K, consider the d-dimensional diagonal hypersurface ring

$$R \;=\; K[[x_0, x_1, \ldots, x_d]]/(x_0^{2m} + x_1^{n_1} + \ldots + x_d^{n_d}).$$

Now, fix integers $k_1, \ldots, k_d$ with $1 \leq k_i \leq \lfloor n_i/2 \rfloor$ for all $i = 1, \ldots, d$. By Goto, Ozeki, Takahashi, Watanabe and Yoshida [17, Example 2.4], the ideal

$$\mathscr{I} \;=\; (x_0^m, x_1^{k_1}, \ldots, x_d^{k_d})R$$

is Ulrich. In this example we have $\nu(\mathscr{I}) = d+1$ and $\lambda(R/\mathscr{I}) = mk_1 \cdots k_d$, so that $\mathrm{e}_0(\mathscr{I}) = 2mk_1 \cdots k_d$ and $\mathrm{e}_1(\mathscr{I}) = mk_1 \cdots k_d$. Therefore, by Proposition 4.5,

$$P_{\mathscr{I}}(n) \;=\; \lambda(R/\mathscr{I}^n) \;=\; 2mk_1 \cdots k_d \binom{n+d-1}{d} - mk_1 \cdots k_d \binom{n+d-2}{d-1}, \ \forall\, n \geq 2-d.$$

After some routine calculations, we obtain

$$P_{\mathscr{I}}(n) \;=\; \lambda(R/\mathscr{I}^n) \;=\; \frac{mk_1 \cdots k_d(2n+d-2)(n+d-2)!}{d!(n-1)!}, \qquad \forall\, n \geq 2-d.$$

In particular, for $d = 3$, we have

$$P_{\mathscr{I}}(n) = \lambda(R/\mathscr{I}^n) = \frac{mk_1k_2k_3}{6}(2n+1)(n+1)n \qquad \forall\, n \geq -1.$$

Note that $P_{\mathscr{I}}(-2) = -mk_1k_2k_3 \neq 0 = H_{\mathscr{I}}(-2)$, while $P_{\mathscr{I}}(-1) = 0 = H_{\mathscr{I}}(-1)$, $P_{\mathscr{I}}(0) = 0 = H_{\mathscr{I}}(0)$, $P_{\mathscr{I}}(1) = mk_1k_2k_3 = \lambda(R/\mathscr{I}) = H_{\mathscr{I}}(1)$, and so on.

4.2.1 Further results

In this part we shall provide a correction of a proposition of Mafi as well as improvements of independent results by other authors.

Let $(R, \mathfrak{m})$ be a two-dimensional Cohen-Macaulay local ring with infinite residue field, and let I be an $\mathfrak{m}$-primary ideal satisfying $\widetilde{I} = I$. Let J be a minimal reduction of I. Then Mafi [30, Proposition 2.6] states that $\mathrm{r}_J(I) = 2$ if and only if

$$H_I(n) = P_I(n), \qquad n = 1,\, 2.$$

However, if we take I as being an Ulrich ideal, then $\mathrm{r}_J(I) \leq 1$ and we have seen in Corollary 4.2 that $\widetilde{I} = I$; moreover, our Proposition 4.5 yields in particular $H_I(n) = P_I(n)$ for $n = 1, 2$. Any such I is therefore a counter-example to Mafi's claim.

We shall establish the correct statement in Proposition 4.7. First, we need a lemma.

Lemma 4.6. *Let $(R, \mathfrak{m})$ be a two-dimensional Cohen-Macaulay local ring with infinite residue field, I an $\mathfrak{m}$-primary ideal and J a minimal reduction of I with $\mathrm{r}_J(I) \leq 2$. Then, $\widetilde{I} = I$ if and only if* $\operatorname{grade} \mathcal{G}(I)_+ \geq 1$.

Proof. From [19, Fact 9] we have $\operatorname{grade} \mathcal{G}(I)_+ \geq 1$ if and only if all powers of I are Ratliff-Rush closed. In particular, if $\operatorname{grade} \mathcal{G}(I)_+ \geq 1$ then $\widetilde{I} = I$. Conversely, suppose $\widetilde{I} = I$. First, if $\mathrm{r}_J(I) \leq 1$ then $\mathrm{r}(I) \leq 1$ and it follows from Lemma 4.1 that $\widetilde{I^n} = I^n$ for all $n \geq 1$. Now if $\mathrm{r}_J(I) = 2$, with say $J = (x, y)$ (note that by letting $M = R$ in Lemma 3.9, or alternatively by [51, Lemma 1.2], we can take $\{x, y\}$ as being a superficial sequence for I), then using Proposition 2.4(b) we can write

$$I \subseteq I^2 : x \subseteq \widetilde{I^2} : x = \widetilde{I} = I,$$

which gives $I^2 : x = I$. By [30, Corollary 2.3] (which requires $\mathrm{r}_J(I) = 2$) we get $\widetilde{I^n} = I^n$ for all $n \geq 1$. Therefore, $\operatorname{grade} \mathcal{G}(I)_+ \geq 1$ in both cases. □

Proposition 4.7. *Let $(R, \mathfrak{m})$ be a two-dimensional Cohen-Macaulay local ring with infinite residue field, I an $\mathfrak{m}$-primary ideal with $\widetilde{I} = I$ and J a minimal reduction of I. Then the following assertions are equivalent:*

(i) $\mathrm{r}_J(I) \leq 2$;

(ii) $H_I(n) = P_I(n)$ *for all* $n \geq 1$;

(iii) $H_I(n) = P_I(n)$ *for* $n = 1, 2$.

Proof. Assume (i). By Lemma 4.6 we get $\operatorname{grade} \mathcal{G}(I)_+ \geq 1$, which as we know is equivalent to $\widetilde{I^n} = I^n$ for all $n \geq 1$. Then (iii) follows from Itoh [26, Proposition 16], which also gives the implication (iii) $\Rightarrow$ (i). Thus (i) and (iii) are equivalent. The implication (ii) $\Rightarrow$ (iii) is obvious. Finally let us show that (i) $\Rightarrow$ (ii). If $\mathrm{r}_J(I) \leq 2$ then Lemma 4.6 yields $\operatorname{grade} \mathcal{G}(I)_+ \geq 1 = d - 1$, so we can apply Lemma 4.4 and obtain

$$\mathrm{r}(I) = \rho(I) + d = \rho(I) + 2.$$

Therefore,

$$\rho(I) = \mathrm{r}(I) - 2 \leq \mathrm{r}_J(I) - 2 \leq 2 - 2 = 0,$$

which gives (ii). □

Remark 4.3. Besides correcting Mafi's proposition (as explained above), our Proposition 4.7 also sharpens independent results by Hoa, Huneke, and Itoh (see [22, Theorem 3.3], [24, Theorem 2.11], and [26, Proposition 16], respectively), where additional hypotheses are required.

4.3 Minimal multiplicity and the Ulrich property

Let $(R, \mathfrak{m})$ be a local ring with residue field k. For a proper ideal I of R, recall that the *fiber cone* of I is the special fiber ring of $\mathcal{R}(I)$, i.e., the standard graded k-algebra $\mathcal{F}(I) = \bigoplus_{n\geq 0} I^n/\mathfrak{m}I^n = \mathcal{R}(I) \otimes_R k$. We can also consider the finite graded $\mathcal{F}(I)$-module

$$\mathcal{F}(I, M) = \bigoplus_{n\geq 0} \frac{I^n M}{\mathfrak{m}I^n M} = \mathcal{R}(I, M) \otimes_R k,$$

whose Krull dimension, denoted $\mathrm{s}_M(I)$ herein, is by definition the *analytic spread* of I relative to M.

If $x_1, \ldots, x_s$ is an M-superficial sequence of I as in Definition 2.1, we say that $x_1, \ldots, x_s$ is *maximal* if $I^n M \subseteq (x_1, \ldots, x_s)M$ for all $n \gg 0$ and $I^n M \nsubseteq (x_1, \ldots, x_{s-1})M$ for all $n \gg 0$.

The lemma below detects a useful connection between minimal M-reductions and (*maximal*) *M-superficial sequences.*

Lemma 4.8. *Let $(R, \mathfrak{m})$ be a local ring with infinite residue field and let I be an $\mathfrak{m}$-primary ideal. Let M be a finite R-module of positive dimension. Then, every minimal M-reduction of I can be generated by a maximal M-superficial sequence of I. Conversely, an ideal generated by a maximal M-superficial sequence of I is necessarily a minimal M-reduction of I.*

Proof. See Rossi and Valla ([53, p. 12]). □

Next we introduce a central notion in this section – which we will relate to the concept of Ulrich module – as well as a useful lemma.

Definition 4.3. (Puthenpurakal [47, Definition 2.2]) Let $(R, \mathfrak{m})$ be a local ring, M a Cohen-Macaulay R-module of dimension t and I a proper ideal of R such that $\mathfrak{m}^n M \subset IM$ for some $n > 0$. Then M has *minimal multiplicity with respect to I* if

$$\mathrm{e}_I^0(M) = (1-t)\lambda(M/IM) + \lambda(IM/I^2 M).$$

Notice that, by taking $M = R$ and $I = \mathfrak{m}$, we recover the usual definition of a ring of minimal multiplicity.

Lemma 4.9. *Let $(R, \mathfrak{m})$ be a local ring, M a Cohen-Macaulay R-module of dimension t and I a proper ideal of R such that $\mathfrak{m}^n M \subset IM$ for some $n > 0$. The following conditions are equivalent:*

(i) M has minimal multiplicity with respect to I;

(ii) *For every maximal M-superficial sequence $z_1, \ldots, z_t$, there is an equality*

$$(z_1, \ldots, z_t)IM = I^2M;$$

(iii) *For some maximal M-superficial sequence $z_1, \ldots, z_t$ there is an equality*

$$(z_1, \ldots, z_t)IM = I^2M;$$

(iv) $\mathrm{e}_I^1(M) = \mathrm{e}_I^0(M) - \lambda(M/IM)$.

Proof. See Puthenpurakal [47, Theorem 2.4] □

Our first result in this part is the following. As in the previous sections, we let $Q = (x_1, \ldots, x_d) \subset \mathscr{I}$ be as in Convention 4.1.

Proposition 4.10. *Suppose R is a Cohen-Macaulay local ring with infinite residue field. Then, every Ulrich R-module with respect to $\mathscr{I}$ has minimal multiplicity with respect to $\mathscr{I}$.*

Proof. Let M be an Ulrich module with respect to $\mathscr{I}$. In particular, M is maximal Cohen-Macaulay. Let $\mathrm{grade}(\mathscr{I}, M)$ denote the maximal length of an M-sequence contained in $\mathscr{I}$. By [28, Lemma 1.3 and Lemma 1.6], we have

$$\mathrm{grade}(\mathscr{I}, M) \leq \mathrm{s}_M(\mathscr{I}) \leq \dim M.$$

As $\mathscr{I}$ is $\mathfrak{m}$-primary, $\mathrm{grade}(\mathscr{I}, M) = \operatorname{depth} M = d$, where $d = \dim R$. Hence

$$\mathrm{s}_M(\mathscr{I}) = d = \nu(Q),$$

where $\nu(-)$ stands for minimal number of generators. As is well-known (see, e.g., [68, p. 117]), this implies that Q is a minimal M-reduction of $\mathscr{I}$, and therefore Lemma 4.8 gives that $x_1, \ldots, x_d$ is in fact a maximal M-superficial sequence of $\mathscr{I}$. On the other hand, because M is Ulrich, we have $QM = \mathscr{I}M$ and so

$$Q\mathscr{I}M = \mathscr{I}^2M.$$

We conclude, by Lemma 4.9, that M has minimal multiplicity with respect to $\mathscr{I}$. □

Remark 4.4. If R is a Cohen-Macaulay local ring and M is a maximal Cohen-Macaulay R-module, then

$$\mathrm{e}_{\mathscr{I}}^0(M) = \mathrm{e}_Q^0(M) = \lambda(M/QM) \geq \lambda(M/\mathscr{I}M),$$

so that condition (ii) of Definition 4.2 is equivalent to saying that $\mathrm{e}^0_{\mathscr{I}}(M) = \lambda(M/\mathscr{I}M)$. In particular, if $\mathscr{I} = \mathfrak{m}$, condition (ii) is the same as $\mathrm{e}(M) = \nu(M)$.

The following consequence gives a generalization of [45, Corollary 1.3(1)].

Corollary 4.11. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with infinite residue field, and let M be a maximal Cohen-Macaulay R-module of positive dimension. Then $\mathrm{e}^1_{\mathscr{I}}(M) \geq 0$, and the following assertions are equivalent:*

(i) M is an Ulrich R-module with respect to $\mathscr{I}$;

(ii) $M/\mathscr{I}M$ is a free $R/\mathscr{I}$-module and $\mathrm{e}^1_{\mathscr{I}}(M) = 0$.

Proof. By [47, Proposition 2.3], we have $\mathrm{e}^1_{\mathscr{I}}(M) \geq \mathrm{e}^0_{\mathscr{I}}(M) - \lambda(M/\mathscr{I}M) \geq 0$. If M is Ulrich with respect to $\mathscr{I}$ then, by definition, the $R/\mathscr{I}$-module $M/\mathscr{I}M$ is free and in addition $\mathrm{e}^0_{\mathscr{I}}(M) = \lambda(M/\mathscr{I}M)$ by Remark 4.4. On the other hand, Proposition 4.10 ensures that M has minimal multiplicity with respect to $\mathscr{I}$, and therefore, by Lemma 4.9,

$$\mathrm{e}^1_{\mathscr{I}}(M) = \mathrm{e}^0_{\mathscr{I}}(M) - \lambda(M/\mathscr{I}M) = 0.$$

Conversely, suppose (ii). Since M is already assumed to be maximal Cohen-Macaulay, it remains to show that $\mathscr{I}M = QM$. Using Remark 4.4 once again, this is equivalent to $\mathrm{e}^0_{\mathscr{I}}(M) = \lambda(M/\mathscr{I}M)$. But this follows from

$$0 \leq \mathrm{e}^0_{\mathscr{I}}(M) - \lambda(M/\mathscr{I}M) \leq \mathrm{e}^1_{\mathscr{I}}(M) = 0.$$

This concludes the proof. □

Our next result, Theorem 4.12 below, is the main result of this section and provides a characterization of modules of minimal multiplicity in terms of reduction number and Castelnuovo-Mumford regularity (of blowup modules). In particular, it will lead us to a byproduct concerning Ulrich modules. This theorem also gives a generalization of [45, Proposition 1.2], where the situation $I = \mathfrak{m}$ was treated.

Theorem 4.12. *Let $(R, \mathfrak{m})$ be a local ring with infinite residue field, M a Cohen-Macaulay R-module of dimension $t > 0$ and I an $\mathfrak{m}$-primary ideal of R. Let $J = (z_1, \ldots, z_t)$ be a minimal M-reduction of I. The following assertions are equivalent:*

(i) *M has minimal multiplicity with respect to I;*

(ii) $\operatorname{reg} \mathcal{R}(I, M) = \operatorname{reg} \mathcal{G}(I, M) = \mathrm{r}_J(I, M) \leq 1$*;*

(iii) $\mathrm{r}_J(I, M) \leq 1$.

Proof. The core of the proof is the implication (i) $\Rightarrow$ (ii), so assume first that (i) holds. In general, we have $\operatorname{reg}\mathcal{R}(I, M) = \operatorname{reg}\mathcal{G}(I, M)$ by Lemma 1.5 and so it remains to prove that $\operatorname{reg}\mathcal{R}(I, M) = \mathrm{r}_J(I, M)$, which we shall accomplish by means of Lemma 3.2. Notice that $z_1, \ldots, z_t$ is a (maximal) M-superficial sequence of I by Lemma 4.8. As a consequence, being M Cohen-Macaulay and I $\mathfrak{m}$-primary, $z_1, \ldots, z_t$ must be in fact an M-sequence, according to [53, Lemma 1.2].

Moreover, since $z_1, \ldots, z_t$ is maximal M-superficial, Lemma 4.9 yields $JIM = I^2M$, i.e., $\mathrm{r}_J(I, M) \leq 1$. Now, to simplify notation, set

$$\mathbf{z}_i = z_1, \ldots, z_i, \quad i = 1, \ldots, t-1$$

(note we can assume $t > 1$). Since clearly $(\mathbf{z}_i)M \cap IM = (\mathbf{z}_i)M$ for all $i = 1, \ldots, t-1$, the case $\mathrm{r}_J(I, M) = 0$ is trivial by virtue of Lemma 3.2. Now suppose $\mathrm{r}_J(I, M) = 1$. Again in view of Lemma 3.2, all we need to prove is that

$$(\mathbf{z}_i)M \cap I^2M = (\mathbf{z}_i)IM \quad \text{for all} \quad i = 1, \ldots, t-1.$$

First, it is clear that $(\mathbf{z}_i)IM \subset (\mathbf{z}_i)M \cap I^2M$. To show the other inclusion, take an arbitrary $f \in (\mathbf{z}_i)M \cap I^2M$. Because $JIM = I^2M$, we have

$$f = z_1m_1 + \cdots + z_im_i = z_1a_1m_1' + \cdots + z_ta_tm_t'$$

with $m_j, m_k' \in M$ and $a_k \in I$. Hence

$$\overline{z_ta_tm_t'} = \overline{0} \in M/(\mathbf{z}_{t-1})M,$$

and since the sequence is regular on M, we have $\overline{a_tm_t'} = \overline{0} \in M/(\mathbf{z}_{t-1})M$, that is,

$$a_tm_t' = z_1w_{t,1} + \cdots + z_{t-1}w_{t,t-1}$$

with $w_{t,j} \in M$. Therefore, f can be expressed as

$$z_1m_1 + \cdots + z_im_i = z_1(a_1m_1' + z_tw_{t,1}) + \cdots + z_{t-1}(a_{t-1}m_{t-1}' + z_tw_{t,t-1}). \tag{4.3}$$

Next, by reducing modulo $(\mathbf{z}_{t-2})M$ and applying an analogous argument to the term

$z_{t-1}(a_{t-1}m'_{t-1} + z_t w_{t,t-1})$, we obtain

$$a_{t-1}m'_{t-1} + z_t w_{t,t-1} = z_1 w_{t-1,1} + \ldots + z_{t-2} w_{t-1,t-2} \tag{4.4}$$

with $w_{t-1,j} \in M$. Thus, by (4.3) and (4.4),

$$f = z_1(a_1 m'_1 + z_t w_{t,1} + z_{t-1} w_{t-1,1}) + \cdots + z_{t-2}(a_{t-2} m'_{t-2} + z_t w_{t,t-2} + z_{t-1} w_{t-1,t-2}).$$

Continuing with the argument, we get an equality

$$f = z_1(a_1 m'_1 + z_t w_{t,1} + \cdots + z_{i+1} w_{i+1,1}) + \cdots + z_i(a_i m'_i + z_t w_{t,i} + \cdots + z_{i+1} w_{i+1,i}).$$

Since $a_1, \ldots, a_i, z_{i+1}, \ldots, z_t \in I$, it follows that $f \in (\mathbf{z}_i)IM$, as needed.

The implication (ii) $\Rightarrow$ (iii) is obvious. Finally, suppose (iii) holds. Then $JIM = I^2M$, and we have seen that $z_1, \ldots, z_t$ is a maximal M-superficial sequence. By Lemma 4.9, we conclude that M has minimal multiplicity with respect to I. $\square$

As a consequence of Theorem 4.12, we determine the regularity of blowup modules of $\mathscr{I}$ relative to an Ulrich module. Also, taking $\mathscr{I} = \mathfrak{m}$ the result retrieves part of [45, Proposition 1.1]. Note Q is an M-reduction of $\mathscr{I}$ for any finite R-module M, so the number $\mathrm{r}_Q(\mathscr{I}, M)$ makes sense.

Corollary 4.13. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with infinite residue field. If M is an Ulrich R-module with respect to $\mathscr{I}$, then*

$$\operatorname{reg} \mathcal{R}(\mathscr{I}, M) = \operatorname{reg} \mathcal{G}(\mathscr{I}, M) = \mathrm{r}_Q(\mathscr{I}, M) = 0.$$

The converse holds in case M is maximal Cohen-Macaulay and $M/\mathscr{I}M$ is $R/\mathscr{I}$-free.

Proof. Because M is Ulrich with respect to $\mathscr{I}$, we have $QM = \mathscr{I}M$, i.e., $\mathrm{r}_Q(\mathscr{I}, M) = 0$. On the other hand, Proposition 4.10 and its proof ensure that M has minimal multiplicity with respect to $\mathscr{I}$ and that Q is in fact a minimal M-reduction of $\mathscr{I}$, and so we can apply Theorem 4.12 to obtain $\operatorname{reg} \mathcal{R}(\mathscr{I}, M) = \operatorname{reg} \mathcal{G}(\mathscr{I}, M) = r_Q(\mathscr{I}, M)$. As to the converse, the argument is clear. $\square$

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