REMARKS ON THE DIRAC EQUATION IN A CLASS OF BLACK HOLES WITH A CLOUD OF STRINGS

By

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To the Faculty of Universidade Federal da Paraíba:

The members of the Committee appointed to examine the dissertation of SAULO SOARES DE ALBUQUERQUE FILHO find it satisfactory and recommend that it be accepted.

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Abstract

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The main goal of this dissertation was to obtain the metric of the spacetime generated by a rotating, charged gravitational body surrounded by a cloud of strings and analyze the Dirac equation and its solutions in the gravitational field generated by this body, which corresponds to a charged rotating black hole with a cloud of strings, i.e., the Kerr-Newman black hole with the addition of a cloud of strings as a source for the Einstein field equations. In order to do that, we firstly reviewed, in the second chapter, some fundamental physical concepts that were important for the further investigations, the physical concept of a black hole and the physical concept of a cloud of strings. We also obtained, for the sake of completeness, the metric for the Schwarzschild black hole surrounded by a cloud of strings (Letelier's spacetime) as well as the Reissner-Nordstrom's black hole surrounded by a cloud of strings. In order to illustrate the method developed by Janis and Newman [1], we obtained the Kerr metric and then used this method to find a new solution of the Einstein field equations, generated by a charged rotating gravitational body surrounded by a cloud of strings, which we called Kerr-Newman black hole with a cloud of strings. We aimed to obtain such background for our application of Dirac equation in the fourth chapter. In the third chapter, we reviewed the special relativistic wave equation for spin-$1/2$ particles, called Dirac equation, since its obtaining, up to its form invariance under Lorentz transformations. In this same third chapter, we discussed the procedure that generalizes this wave equation for curved spacetimes, obtaining the so-called general relativistic Dirac equation, which is covariant under both the group of general coordinate transformations for the entire manifold, and the group of local Lorentz transformations defined at each and every point of the manifold individually. Having obtained this Dirac equation for curved spacetimes, and getting to know how to formulate it once the spacetime metric is given, we could apply this equation, at the fourth chapter, for our desired spacetime configuration, which is the Kerr-Newman spacetime with a cloud of strings as obtained in chapter two. As the result of this application, we finally managed to separate the solution into the angular and radial parts and compare with recent studies performed by Kraniotis [2] for Kerr-Newman black hole and concluded that the equations are formally similar, and correspond to a Generalized Heun Equation. Therefore, the solutions for the Dirac equation in the background spacetime under consideration, are also given in terms of the Generalized Heun Functions as well. We analyzed these solutions and all particular cases, namely, uncharged, non rotating, and uncharged and non rotating black holes with cloud of strings, and pointed out the signature of the strings and consequently, their physical role. These results call attention to the application power of the Generalized Heun Equation, what may permit to obtain other results, such as analytical solutions for the Dirac equation in others spacetime configurations. We also analyzed the particular cases corresponding to Kerr, Reissner-Nordström and Schwarzschild black holes surrounded by cloud of strings, whose results are also given in terms of Generalized Heun Functions. The analytical solutions at the vicinity of some specific reference points, such as the Cauchy horizon, event horizon, and at the infinity were obtained and analyzed and the signature of the presence of the cloud of strings was pointed out.

Keywords: Dirac Equation in Curved Spacetimes. Black Holes with a Cloud of Strings. Rotating Charged Black Holes.

Resumo

O principal objetivo dessa dissertação foi obter a métrica do espaço-tempo gerado por um objeto gravitacional carregado, em rotação, e cercado por uma nuvem de cordas, e analisar a equação de Dirac e suas soluções no campo gravitacional gerado por esse corpo, que corresponde a um buraco negro carregado, em rotação, e com uma nuvem de cordas como fonte para as equações de campo de Einstein. De modo a fazer isso, nós primeiramente revisamos, no segundo capítulo, alguns conceitos físicos fundamentais que foram importantes para as investigações posteriores, tais como o conceito físico de um buraco negro e o conceito físico de uma nuvem de cordas. Nós também obtivemos, por motivos de completeza, a métrica para o buraco negro de Schwarzschild cercado por uma nuvem de cordas (espaço-tempo de Letelier) assim como o buraco negro de Reissner-Nordstrom cercado por uma nuvem de cordas. De maneira a ilustrar o método desenvolvido por Janis e Newman [1], nós obtivemos a métrica de Kerr e então usamos esse método para encontrar uma nova solução para as equações de campo de Einstein, gerada por um objeto gravitacional carregado, em rotação, e cercado por uma nuvem de cordas, que nós chamamos de buraco negro de Kerr-Newman com uma nuvem de cordas. Nós desejávamos obter tal plano de fundo para nossa aplicação da equação de Dirac no quarto capítulo. No terceiro capítulo, nós revisamos a equação de onda da relatividade especial para partículas de spin-1/2, chamada equação de Dirac, desde sua obtenção, até sua invariância em forma sob transformações de Lorentz. Neste mesmo terceiro capítulo, nós discutimos o procedimento que generaliza esta equação de onda para espaços-tempo curvos, obtendo a chamada equação de Dirac da relatividade geral, que é covariante sob ambos os grupos de transformações gerais de coordenadas sobre a variedade inteira, e o grupo de transformações locais de Lorentz definidas em cada ponto da variedade individualmente. Tendo obtido essa equação de Dirac para espaços-tempo curvos, e aprendendo como formular a mesma uma vez que a métrica do espaço-tempo é dada, nós pudemos aplicar essa equação, no quarto capítulo, para nossa desejada configuração do espaço-tempo,

que é o espaço-tempo de Kerr-Newman com uma nuvem de cordas, conforme obtida no capítulo dois. Como resultado dessa aplicação, nós finalmente conseguimos separar a solução em suas partes radiais e angular e compará-las com estudos recentes feitos por Kraniotis [2] para o buraco negro de Kerr-Newman, e concluímos que as equações são formalmente similares e correspondem a equações generalizadas de Heun. Portanto, as soluções para a equação de Dirac, no plano de fundo do espaço tempo sob consideração, também são dadas em termos de funções generalizadas de Heun. Nós analisamos essas soluções e todos os casos particulares, tais como buracos negros com nuvem de cordas mas sem carga, sem rotação, e sem carga e sem rotação, e apontamos a assinatura da nuvem de cordas e consequentemente seu papel físico. Esses resultados chamam nossa atenção para o poder de aplicação da equação generalizada de Heun, que pode nos permitir obter outros resultados, tais como soluções analíticas para a equação de Dirac em outras configurações do espaço-tempo. Nós também analisamos os casos particulares correspondentes aos buracos negros, cercados por nuvem de cordas, de Kerr, Reissner-Nordstrom e Schwarzschild , cujos resultados também são dados em termos de funções generalizadas de Heun. As soluções analíticas na vizinhança de alguns pontos específicos de referência, tais como o horizonte de Cauchy, horizonte de eventos, e no infinito são obtidas e analisadas, e a assinatura da presença da nuvem de cordas é apontada.
Palavras chave: Equação de Dirac em Espaços Tempo Curvos. Buracos Negros com uma Nuvem de Cordas. Buracos Negros Carregados em Rotação.

TABLE OF CONTENTS

Chapter One

INTRODUCTION

The investigations concerning the behavior of quantum systems in curved space-times goes back to the end of the 1920's and to the beginning of the 1930's [3], when the generalization of the Schrödinger and Dirac equations to curved spaces has been discussed, motivated by the proposal to formulate a theory combining quantum physics and general relativity. Since this time, and specially, in the last three decades, the study of the interaction of quantum systems with different gravitational fields has been an exciting subject of research.

Spinor fields and particles interacting with gravitational fields has been the subject of many investigations. Along this line of research we can mention those connected with quantum mechanics in different background space-times [4, 5, 6, 7, 8] and, in particular, the ones which consider the hydrogen atom [9, 10, 11, 12, 13, 14] in an arbitrary curved spacetimes.

The first experiment which showed the gravitational effect on a wave function was performed by Colella, Overhauser and Werner [15, 16] in which they measured the quantum mechanical phase difference of two neutron beams induced by a gravitational field.

The information about the states of a black hole can be completely obtained from the three parameters that characterizes it, namely, mass M, charge Q and angular momentum per unit mass, a. If fermions are placed, in the region surrounding the black hole, it will experience a perturbation, and a consequent changing in these three parameters. This motivates the study of fields on black hole space times in order to know about the black hole, using the impact of these perturbations.

The study of the single-particle states which are exact solutions of the generalized Dirac equation in curved space-times constitutes an important element to construct a theory that combines quantum physics and gravity, and thus it should be important from the theoretical, as well as observational point of view, and for this reason, the investigation of the behavior of relativistic particles, in the present case, fermions, is of considerable interest.

The pioneering work by Chandrasekhar [17], in the 1970's, showing an unexpected result that the Dirac equation can be separated in Kerr Geometry and written in terms of radial and angular equations, opens up a field of research concerning the study of Dirac equation in different background gravitational fields. In the same direction of Chandrasekhar, Page [18] showed that the Dirac equation is also separable in the Kerr-Newman black hole space-time.

Since that time, more precisely, during last forty-four years, the behavior of relativistic fermions in a class of black hole space-times was studied using different approaches [19, 20, 21, 22, 2, 23, 24, 25, 26, 27, 28, 29] , with the one adopted by Chandrasekhar being predominant.

Using the solutions of the Dirac equation in different gravitational fields, many physical processes, among them we mention, scattering of Dirac waves [30, 24, 31, 32] and existence of quasi-normal modes [33, 34, 35, 32] were investigated and some results on the physics of black holes were achieved.

The modern concept of black hole can be dated back to the 1930's, with the studies about gravitational collapse of neutron stars made by Oppenheimer and Volkoff [36]. They discussed about the limits of the mass of a neutron star such that the collapse should occur. At this time, this collapsed star was termed frozen star. Only three decades after, the term black hole was introduced by John Wheeler [37, 38]. These objects are uniquely characterized by its mass, angular momentum and charge. This statement is known as the "no hair conjecture" [38].

In the same year in which the final formulation of the field equations of general relativity was published, Schwarzschild published a solution of the Einstein's equations [39, 40] that a half century later was identified as describing a black hole [41, 42]. Therefore, it is the first and also the simplest solution of the Einstein vacuum equations describing a black hole and represents a space-time which is asymptotically flat, whose source is a static, spherically symmetric, uncharged gravitational body.

Later on, in 1963 Kerr generalized the static solution corresponding to the Schwarzschild black hole to rotating ones [43]. Two years later, E. T. Newman et al. [44, 1] presented a new solution of the Einstein-Maxwell equations which represents a charged rotating gravitational body . This solution was obtained using the Newman-Janis method [1] based on a complex coordinate transformation, which permits to obtain, in principle, the rotating solution from the static one.

In 1916 also, Einstein predicted the existence of gravitational waves [45], which travel with the speed of light, by working out his equations in the weak-field regime. Einstein concluded that gravitational-wave amplitudes would be very small. During a long time, there was some skepticism with relation to the physical reality of gravitational waves [46]. Presently, there is no doubt about the existence of black holes, which is well established [47, 48, 49, 50, 51] according to the observation of the orbital motion of the S-stars in the center of the Milky Way [52, 53, 54], more recently, through the observations of Laser Interferometer Gravitational-Wave Observatory (LIGO) [47, 48, 49, 50, 51, 52, 53, 54, 55] that observed the first gravitational-wave signal GW150914 from a binary black hole merger [48]. More recently, it came out another confirmation, from observation made by the Event Horizon Telescope of a black hole shadow in M87, which again confirms the existence of black holes [56].

As a conclusion, black holes are solutions of Einstein field equations of general relativity, as we know for a long time, and recently we also got to know that they exist according to the observations concerning the first detection of associated gravitational waves by LIGO–VIRGO collaboration [48].

Almost four decades ago, Letelier [57] proposed a model with cloud of strings in the framework of general relativity and used this cloud as a source of the gravitational field. Thus, he obtained a class of solutions of the Einstein equations with different symmetries, namely, plane-symmetric, spherically and cylindrically symmetric [57]. In the case of spherical symmetry, which is the object of our interest, the obtained solution is essentially a generalization of the Schwarzschild solution, in the sense that the solutions are similar, but with the horizon enlarged as compared with the Schwarzschild solution. This solution corresponding to a black hole surrounded by a spherically symmetric cloud of strings, we are calling Letelier space-time [58]. Later on, Letelier extended his model [59] in order to include the pressure, and thus a fluid of strings is considered rather than a cloud. In this case, the general solution for a fluid of strings with spherical symmetry was obtained.

The main motivation presented at that time, to construct those models was based on the fact that the universe can be better represented, in principle, by a collection of extended objects, like one-dimensional strings, rather than of point particles.

Otherwise, the great advantage to consider extended objects is the fact that they are potentially the best alternative to be used as the fundamental elements to describe physical phenomena which occur in the universe. From the gravitational point of view, it is important to investigate, for example, a black hole immersed in a cloud of strings due to the fact that these sources have astrophysical observable consequences [60, 61, 62]. Studies concerning different aspects associated with the physics of a cloud of strings [62, 63] and a fluid of strings [64] in the framework of general relativity have been performed during the last decades.

It is worth emphasize that strings have become a very important ingredient in many physical theories, and the idea of strings is fundamental in superstring theories [65]. The apparent relationship between counting string states and the entropy of the black hole horizon [61, 66] suggests an association of strings with black holes. Furthermore the intense level of activity in string theory has lead to the idea that many of the classic vacuum scenarios, such as the static Schwarzschild black hole, may have atmospheres composed by a strings [67].

In this dissertation, the solutions of the Dirac equation in the black hole space-times corresponding to a gravitating body, non-rotating and rotating, uncharged and charged, surrounded by a cloud of strings are considered, and the signal of the cloud of strings on the angular and radial solutions are analysed.

In Chapter 2 the solution corresponding to different black holes are discussed and the one corresponding to a rotating charged black hole surrounded by a cloud of string is obtained. Chapter 3 reviews the Dirac equation in curved space-times. Chapter 4 presents the angular and radial solutions of the Dirac equation in the Kerr-Newman black hole with a cloud of strings. These were obtained following the paper by Kraniotis [2]. Chapter 6 presents a the conclusions and possible future works.

Throughout this dissertation we use units where $G = c = 1$ and a metric signature $(+, -, -, -)$. We use Greek letters μ, ν,... to denote spacetime indices, Latin letters a, b, c ... denote tetrad indices and Roman letters i, j, k,... denote spatial indices $1, 2, 3$.

Chapter Two

BLACK HOLES WITH A CLOUD OF STRINGS

The introduction of a cloud of strings as a source of the gravitational field in the context of general relativity was considered by Letelier [57] who constructed a gauge invariant model of a cloud of geometric strings, which corresponds to strings without particles attached to them, along their extensions. In this framework, he obtained solutions of the Einstein field equations for different symmetries of the sources, namely, plane, cylindrical and spherical. Particularly, in the case of spherical symmetry, he obtained a generalization of the Schwarzschild black hole in the case that now, this black hole is surrounded by a cloud of strings, which is known in the literature as Letelier spacetime [58].

Letelier [59] also considered a generalization of his model of Cloud of Strings in order to take into account massive strings instead of massless, and as consequence, the pressure due to these strings was considered. In this case, instead of a cloud of strings, we have a fluid of strings. Taking into account this new scenario, Letelier [59] also obtained the solution of Einstein field equations, but now for the spherically symmetric case only.

These ideas developed by Letelier [57, 58] were considered by other authors [63, 62, 64] who assumed that cloud or/and fluid of strings can be sources of gravitational fields and performed some investigations in different scenarios [63, 62, 64].

In this chapter, we are interested to obtain the solution of the Eistein field equations for a rotating gravitational body, taking into account that the eletromagnetic field and a cloud of strings are sources of the gravitational field. This solution will be called Kerr-Newman black hole with a cloud of strings.

2.1 Static black holes with a cloud of strings

In 1916, Schwarzschild, assuming a spherical symmetry for the spatial configuration of a static spherically symmetric gravitational body, limited to a finite volume, derived the first solution to Einstein's field equation[39][40]. This solution was initially called Schwarzschild's solution and a few years later Schwarzschild's static black hole solution. After that, other solutions which shared the same spherical symmetry were obtained as well, generalising the simple situation of a static source of matter spatially limited, for more elaborated scenarios, for instance, Reissner-Nordstrom's solution [68, 69], for a charged black hole; and Letelier's [57], for a black hole surrounded by a cloud of strings.

In this framework, we shall work in this specific symmetry: the spherical symmetry. This study is much easier if we use the spherical coordinate system:

$$x^0 = t, \tag{2.1}$$

$$x^1 = r, \tag{2.2}$$

$$x^2 = \theta, \tag{2.3}$$

$$x^3 = \phi. \tag{2.4}$$

where we are using a unit system such that $c = 1$.

With this coordinate system, and by means of a convenient set of coordinate transformations, we can write the line element for the spacetime at the neighbourhood of a spherically symmetric source, without lost of generality, in the following way:

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2), \tag{2.5}$$

where $\nu = \nu(r,t)$ and $\lambda = \lambda(r,t)$. Note that with these dependencies, the metric depends on the time. But, we will assume that the source is necessarily static. Thus, these parameters depend only on the radial coordinate, i.e.:

$$\nu = \nu(r), \tag{2.6}$$

$$\lambda = \lambda(r). \tag{2.7}$$

Using the expression for the line element above, we can specify every component of the metric tensor, which is represented matricially in the following way:

$$g_{\mu\nu} = \begin{bmatrix} e^{\nu} & 0 & 0 & 0 \\ 0 & -e^{\lambda} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 sin^2\theta \end{bmatrix}, \tag{2.8}$$

its inverse, on the other hand, is the contravariant metric tensor, that can be represented matricially as:

$$g^{\mu\nu} = \begin{bmatrix} e^{-\nu} & 0 & 0 & 0 \\ 0 & -e^{-\lambda} & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2 sin^2\theta} \end{bmatrix}, \tag{2.9}$$

Having calculated $g_{\mu\nu}$ and $g^{\mu\nu}$, we are able to demonstrate that the only non-null Christoffel symbols are:

$$
\begin{aligned}
\Gamma^0_{01} &= \Gamma^0_{10} = \frac{1}{2}\nu', \\
\Gamma^2_{21} &= \Gamma^2_{12} = \frac{1}{r}, \\
\Gamma^3_{13} &= \Gamma^3_{31} = \frac{1}{r}, \\
\Gamma^3_{23} &= \Gamma^3_{32} = cotg\theta, \\
\Gamma^1_{00} &= \frac{1}{2}e^{\nu-\lambda}\nu', \\
\Gamma^1_{11} &= \frac{1}{2}\lambda', \\
\Gamma^1_{22} &= -re^{-\lambda}, \\
\Gamma^1_{33} &= -rsin^2\theta e^{-\lambda}, \\
\Gamma^2_{33} &= -sin\theta cos\theta,
\end{aligned}
\tag{2.10}
$$

where the apostrophe means differentiation on the r variable.

With these Christoffel symbols, we can determine the non-null Ricci curvature tensor components:

$$
R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\mu\rho} + \Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\rho\sigma} - \Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\rho\sigma},
$$

which in this context are:

$$
\begin{aligned}
R_{00} &= -e^{(\nu-\lambda)}\left[\frac{\nu''}{2} + \left(\frac{\nu'}{2}\right)^2 - \frac{\nu'\lambda'}{4} + \frac{\nu'}{r}\right], \\
R_{11} &= \left[\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \left(\frac{\nu'}{2}\right)^2 - \frac{\lambda'}{r}\right], \\
R_{22} &= e^{-\lambda}\left[1 + \frac{1}{2}r(\nu' - \lambda')\right] - 1, \\
R_{33} &= R_{22}sin^2\theta,
\end{aligned}
\tag{2.11}
$$

and finally, with all this information gathered, we can calculate every component of the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$:

$$
\begin{aligned}
G_{00} &= R_{00} - \frac{1}{2}g_{00}R = e^{\nu}\left[e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) - \frac{1}{r^2}\right], \\
G_{11} &= R_{11} - \frac{1}{2}g_{11}R = -\left[\frac{1}{r^2} + \frac{\nu'}{r}\right] + \frac{e^{\lambda}}{r^2}, \\
G_{22} &= R_{22} - \frac{1}{2}g_{22}R = -\frac{1}{2}r^2 e^{-\lambda}\left[\nu'' + \frac{(\nu')^2}{2} - \frac{\nu'\lambda'}{2} + \frac{\nu' - \lambda'}{r}\right], \\
G_{33} &= R_{33} - \frac{1}{2}g_{33}R = \left[R_{22} - \frac{1}{2}g_{22}R\right] sin^2\theta,
\end{aligned} \tag{2.12}
$$

which specifies the first member of Einstein's field equations:

$$
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -T_{\mu\nu}.
$$

With the first member of Einstein's field equation completely explicited in terms of functions ν and λ, there is nothing left to do until the energy-momentum tensor $T_{\mu\nu}$ be specified. For Schwarzschild's black hole solution, this energy-momentum tensor, in the region outside the gravitating body, vanishes, which means that in this region $T_{\mu\nu} = 0$ [39, 40]. Thus, after some algebra we find the well known line-element:

$$
ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \frac{1}{\left(1 - \frac{2m}{r}\right)}dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{2.13}
$$

This is the Schwarzschild metric [39, 40]. The parameter which was introduced initially as an integration constant is the Newtonian mass measured at large distance.

It is worth calling attention to the fact that this result has a stranger meaning, namely, that the spherically symmetric solution to the vacuum Einstein equations is necessarily static, and this solution is the Schwarzschild one, which depends only on the total mass and nothing else.

However, for the more complicated scenarios in which we are going to work in at this chapter, such as the Reissner-Nordstrom's charged black hole[68, 69], or the Reissner-Nordstrom's charged black hole with cloud of strings at the spherically symmetric Letelier spacetime [57], we will need first to determine their energy-momentum tensor, and then solve their Einstein's equation for the metric in order to completely specify their solutions. We leave the entire next two sections for these specific tasks

2.1.1 Energy-Momentum Tensor

In 1918 Reissner-Nordstrom [68][69], and in 1979 Letelier [57], developed different black hole models with the purpose of generalising the current theoretic formulations of black holes for more elaborated scenarios. Reissner-Nordstrom's black hole model had introducted a black hole in which the liquid charge was non-null and which was surrounded by a eletromagnetic field. Meanwhile, Letelier, through a similar procedure to the one developed for the description of a cloud of particles approached by a incoherent perfect fluid invariant under reparametization, developed a black hole model surrounded by a cloud of strings.

In order to incorporate Letelier's and Reissner-Nordstrom's models in our description, we should reproduce the obtaining of the energy-momentum tensors for the eletromagnetic field, and for the cloud of strings, and then finally unify both of them in an unique formulation. We are going to treat each model individually in the following sections.

Energy-Momentum Tensor for a charged black hole

We know the fact that, from the electromagnetic theory, the energy-momentum tensor at the external region to a eletric source is not zero. In fact, its symmetric form is given by:

$$T_{\mu\nu} = 2\left(F_{\mu\rho}F^{\rho}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right), \tag{2.14}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{2.15}$$

is the electromagnetic tensor that describes the electromagnetic field in the neighbourhood of any charged source; and A_μ is the four-vector electromagnetic potential.

Since we are describing here in this context a source (the black hole) with spherical symmetry, and assuming that this configuration is also static, then, we must have that:

$$A_2 = A_3 = 0; A_0 = A_0(r,t), A_1 = A_1(r,t). \tag{2.16}$$

Thus, due to this potential, the most general components of the Maxwell tensor $F_{\mu\nu}$ in the description of the eletromagnetic field must have the form

$$F_{01} = -F_{10} = \alpha(r,t), \tag{2.17}$$

$$F_{\mu\nu} = 0, \text{ for other components}, \tag{2.18}$$

which means that we are not considering the magnetic field.

In view of the metric given by Eq. (2.8), we can also find the non-null components of this tensor in its contravariant form and in its mixed form through the following calculations:

$$F^0_1 = g^{00}F_{10} = -e^{-\nu}\alpha(r,t), \tag{2.19}$$

$$F^1_0 = g^{11}F_{01} = -e^{-\lambda}\alpha(r,t), \tag{2.20}$$

$$F^{01} = g^{00}F^1_0 = -e^{-(\lambda+\nu)}\alpha(r,t), \tag{2.21}$$

$$F^{10} = g^{11}F^0_1 = -e^{-(\nu+\lambda)}\alpha(r,t). \tag{2.22}$$

From the results above, the scalar $F_{\mu\rho}F^{\rho\mu}$ can be calculated as follows:

$$F_{\mu\rho}F^{\rho\mu} = F_{\mu 0}F^{\mu 0} + F_{\mu 1}F^{\mu 1} + F_{\mu 2}F^{\mu 2} + F_{\mu 3}F^{\mu 3}. \tag{2.23}$$

The last two terms are mutually anulated, remaining:

$$F_{\mu\rho}F^{\rho\mu} = F_{\mu0}F^{\mu0} + F_{\mu1}F^{\mu1} = F_{10}F^{10} + F_{01}F^{01} = -e^{-(\nu+\lambda)}(\alpha(r,t))^2 - e^{-(\nu+\lambda)}[\alpha(r,t)]^2, \quad (2.24)$$

and therefore

$$F_{\mu\rho}F^{\rho\mu} = -2e^{-(\nu+\lambda)}[\alpha(r,t)]^2. \quad (2.25)$$

Then, we can obtain the expression for the energy-momentum tensor by replacing the result obtained above, for the scalar $F_{\mu\rho}F^{\rho\mu}$, into the equation (2.14). The result found is:

$$T_{00}^{\text{EM}} = 2\left[F_{01}F_0^1 + \frac{1}{4}g_{00}2e^{-(\nu+\lambda)}(\alpha(r,t))^2\right],$$
$$T_{00}^{\text{EM}} = -e^{-\lambda}[\alpha(r,t)]^2; \quad (2.26)$$

$$T_{11}^{\text{EM}} = 2\left[F_{10}F_1^0 + \frac{1}{4}g_{11}2e^{-(\nu+\lambda)}(\alpha(r,t))^2\right],$$
$$T_{11}^{\text{EM}} = e^{-\nu}[\alpha(r,t)]^2; \quad (2.27)$$

$$T_{22}^{\text{EM}} = 2\left[-\frac{1}{4}g_{22}2e^{-(\nu+\lambda)}(\alpha(r,t))^2\right],$$
$$T_{22}^{\text{EM}} = -r^2e^{-(\nu+\lambda)}[\alpha(r,t)]^2; \quad (2.28)$$

$$T_{33}^{\text{EM}} = 2\left[-\frac{1}{4}g_{33}2e^{-(\nu+\lambda)}(\alpha(r,t))^2\right],$$
$$T_{33}^{\text{EM}} = -r^2e^{-(\nu+\lambda)}[\alpha(r,t)]^2 sen^2\theta = T_{22}sen^2\theta, \quad (2.29)$$

where the superscript "EM" refers to the electromagnetic field.

It remains for us to determine the function $\alpha(r,t)$ in order to completely explicit the energy-momentum tensor, for the electromagnetic field produced by a static and spherically symmetric electric source, in terms of the functions $\nu(r)$, $\lambda(r)$ and of the variable r. For this purpose, we should solve the Maxwell equations at the neighbourhood of the electric source, where $J^\mu = 0$. The covariant form of Maxwell equations is:

$$\nabla_\mu F^{\mu\nu} = 0. \tag{2.30}$$

Expanding the four-divergency above, we get:

$$\nabla_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + \Gamma^\mu_{\mu\alpha} F^{\alpha\nu} + \Gamma^\nu_{\mu\alpha} F^{\mu\alpha} = 0. \tag{2.31}$$

but $\Gamma^\nu_{\mu\alpha}$ is symmetric on the indexes μ and α of the double sum, while $F^{\mu\alpha}$ is anti symmetric in those. Therefore:

$$\Gamma^\nu_{\mu\alpha} F^{\mu\alpha} = 0. \tag{2.32}$$

So,

$$\nabla_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + \Gamma^\mu_{\mu\alpha} F^{\alpha\nu} = 0. \tag{2.33}$$

It happens that due to the form of the Christoffel connections, and due to a mathematical identity that can be found in tensor algebra books [70], we should have that:

$$\Gamma^\mu_{\mu\alpha} = \frac{1}{2} g^{\mu\mu} \frac{\partial g_{\mu\mu}}{\partial x^\alpha} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} \sqrt{-g}, \tag{2.34}$$

where g is the metric determinant. Accordingly:

$$\nabla_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + \left(\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} \sqrt{-g} \right) F^{\alpha\nu} = 0, \tag{2.35}$$

that can be written in a more convenient way as:

$$\nabla_\mu F^{\mu\nu} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\partial_\mu F^{\mu\nu} + (\partial_\mu\sqrt{-g})F^{\alpha\nu}\right) = 0, \tag{2.36}$$

which by Leibniz's rule is:

$$\nabla_\mu F^{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}F^{\mu\nu}\right) = 0, \tag{2.37}$$

In view of the metric tensor (2.8), it is also possible to calculate the metric determinant as:

$$g = -e^{(\nu+\lambda)}r^4 sin^2\theta, \tag{2.38}$$

and therefore:

$$\sqrt{-g} = \sqrt{e^{(\nu+\lambda)}r^4 sin^2\theta}, \tag{2.39}$$

Nevertheless, for the majority of metrics describing spherically symmetric black holes, we can verify that:

$$\nu = -\lambda. \tag{2.40}$$

Here in this context it is not different and this result is verified in the next section. We will use it beforehand here, what leads us to:

$$\sqrt{-g} = r^2 sin\theta. \tag{2.41}$$

With this result, the Maxwell equations can finally be written as:

$$\nabla_\mu F^{\mu\nu} = \frac{1}{r^2 sin\theta}\left[\partial_0(r^2 sin\theta F^{0\nu}) + \partial_1(r^2 sin\theta F^{1\nu})\right] = 0, \tag{2.42}$$

which leads us to the following equations:

$$\nabla_\mu F^{\mu 0} = \frac{1}{r^2 sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 sin\theta \alpha(r,t)\right)\right] = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\alpha(r,t)\right) = 0, \tag{2.43}$$

$$\nabla_\mu F^{\mu 1} = \frac{1}{r^2 sin\theta}\left[-\frac{\partial}{\partial t}\left(r^2 sin\theta \alpha(r,t)\right)\right] = -\frac{\partial\alpha}{\partial t}(r,t) = 0. \tag{2.44}$$

From this second equation we can conclude that:

$$\alpha(r,t) = \alpha(r).$$

Meanwhile the first one, which is the generalized Gauss Law, can be integrated over the whole volume containing the eletric charge e, resulting in:

$$\alpha(r) = \frac{e}{r^2}. \tag{2.45}$$

Substituting this result into the equations (2.26)-(2.29), we can finally present the explicit form of the energy-momentum tensor associated with the electromagnetic field produced by a static and spherically symmetric charged black hole. The non-null energy-momentum tensor components are listed below:

$$\begin{aligned}
T_{00}^{\text{EM}} &= -\frac{e^2}{r^4}\exp\left(-\lambda\right); \\
T_{11}^{\text{EM}} &= \frac{e^2}{r^4}\exp\left(-\nu\right); \\
T_{22}^{\text{EM}} &= -r^2\exp\left[-(\nu+\lambda)\right]\frac{e^2}{r^4}; \\
T_{33}^{\text{EM}} &= sin^2\theta T_{22}^{\text{Charge}}.
\end{aligned} \tag{2.46}$$

We can use these results to show that the trace of $T_{\mu\nu}$ is identically zero, as expected for the electromagnetic field in four dimensions.

Energy-Momentum Tensor for a black hole with cloud of strings

The action which describes a string immersed in the spacetime according to the the formulation proposed by Letelier [57] is:

$$s = \int L d\lambda^0 d\lambda^1, \tag{2.47}$$

$$L = M\sqrt{-\gamma}, \tag{2.48}$$

where L is the lagrangian density, M is a dimensionless constant which features the string and

$$\gamma = \det \gamma_{AB}, \tag{2.49}$$

$$\gamma_{AB} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda^A} \frac{\partial x^\nu}{\partial \lambda^B}, \tag{2.50}$$

where $x^\mu = x^\mu(\lambda^A)$ describes the string world surface and $\lambda^A = (\lambda^0, \lambda^1)$ are the parameters of this world surface, with λ^0 being a timelike parameter and λ^1 a spacelike parameter.

Associeted with this string world surface, we have the bivector:

$$\Sigma^{\mu\nu} = \epsilon^{AB} \frac{\partial x^\mu}{\partial \lambda^A} \frac{\partial x^\nu}{\partial \lambda^B}, \tag{2.51}$$

where $\epsilon^{\alpha\beta}$ is the 2-dimensional Levi-Civita symbol, normalized as follows: $\epsilon^{01} = -\epsilon^{10} = 1$.

Combining the equation above, the equations (2.49), (2.50) and the equation (2.48) for the lagrangian density, we can write this last one as follows:

$$L = M \left(-\frac{1}{2} \Sigma^{\mu\nu} \Sigma_{\mu\nu} \right)^{\frac{1}{2}}, \tag{2.52}$$

and therefore, the energy-momentum tensor for a single string is:

$$t^{\mu\nu} \equiv 2 \frac{\partial}{\partial g_{\mu\nu}} L = \frac{M}{\sqrt{-\gamma}} \Sigma^{\mu\beta} \Sigma^{\nu}_{\beta}. \tag{2.53}$$

Similarly to a cloud of particles [57], we will now consider the world surfaces for a cloud of strings as described by: $X^\mu = X^\mu(\lambda^A, \zeta, \eta)$, where ζ and η are the parameters which label a specific world surface and λ^A are the parameters which describe the evolution of this world surface specifically. A cloud of strings is also featured by a proper density ρ_{cs}. The energy-momentum tensor for a cloud of strings in the most general case is accordingly:

$$T^{\mu\nu^{\mathrm{CS}}} = \frac{\rho_{cs}}{\sqrt{-\gamma}} \Sigma^{\mu\beta}\Sigma^{\nu}_{\beta}, \tag{2.54}$$

where the superscript "CS" refers to the cloud of strings.

However, the spacetime symmetry considered in this framework narrows the density ρ_{cs} and the bivector $\Sigma_{\mu\nu}$ to be functions of r only. So, considering this symmetry, and the following invariance on reparametrization:

$$\lambda^0 \to \lambda^{0*} = \lambda^{0*}(\lambda^A, \zeta, \eta),$$
$$\lambda^1 \to \lambda^{1*} = \lambda^{1*}(\lambda^A, \zeta, \eta),$$

we concluded through an algebric development made by Letelier, that the bivector $\Sigma_{\mu\nu}$ is constrained to have only Σ^{01} and Σ^{10} as its non-null components. This result also allows us to obtain the solution for the general conservation equation[57]:

$$\nabla_\mu(\rho\Sigma^{\mu\nu}) = 0, \tag{2.55}$$

as:

$$\Sigma^{01} = \frac{b}{\rho_{cs} r^2} e^{\frac{-(\lambda+\nu)}{2}}, \tag{2.56}$$

where b is an integration constant which will be associated to the presence of the cloud of strings. We also notice that the gauge invariant density $(-\gamma)^{\frac{1}{2}}\rho_{cs}$ has got the value:

$$(-\gamma)^{\frac{1}{2}}\rho_{cs} = \frac{b}{r^2}, \tag{2.57}$$

and therefore, b is a positive constant.

From Eq. (2.56) and (2.8) for the metric tensor, we can calculate the non-null bivector components in its mixed form, which are listed below:

$$\Sigma_0^1 = g_{0\mu}\Sigma^{\mu 1} = g_{00}\Sigma^{01} = \frac{b}{\rho_{cs} r^2} e^{\frac{(\nu-\lambda)}{2}}, \tag{2.58}$$

$$\Sigma_1^0 = g_{1\mu}\Sigma^{\mu 0} = g_{11}\Sigma^{10} = \frac{b}{\rho_{cs} r^2} e^{\frac{(\lambda-\nu)}{2}}. \tag{2.59}$$

Having got all this information, we can explicit the non-null components of the energy-momentum tensor from Eq. (2.54), which in its contravariant form are:

$$T^{00^{\text{CS}}} = \frac{\rho_{cs}}{\sqrt{-\gamma}}\Sigma^{01}\Sigma_1^0 = \frac{b}{r^2}e^{-\nu}, \tag{2.60}$$

$$T^{11^{\text{CS}}} = \frac{\rho_{cs}}{\sqrt{-\gamma}}\Sigma^{10}\Sigma_0^1 = -\frac{b}{r^2}e^{-\lambda}. \tag{2.61}$$

And multiplying by the metric, we can get the indices down:

$$T_{\mu\nu}^{\text{CS}} = g_{\mu\alpha}g_{\nu\beta}T^{\alpha\beta^{\text{CS}}}. \tag{2.62}$$

Finally, expliciting the non-null components of the energy-momentum tensor, associated with the cloud of strings, in its covariant form, we get:

$$T_{00}^{\text{CS}} = g_{00}g_{00}T^{00^{\text{CS}}} = \frac{b}{r^2}e^{\nu}, \tag{2.63}$$

$$T_{11}^{\text{CS}} = g_{11}g_{11}T^{11^{\text{CS}}} = -\frac{b}{r^2}e^{\lambda}. \tag{2.64}$$

2.1.2 Line element for a static and charged black hole with cloud of strings

We have that for a static charged black hole with cloud of strings, the total energy-momentum tensor $T_{\mu\nu}$ must be a combination of the contributions from each of its sources considered separatly, the contribution from the electromagnetic field $T^{EM}_{\mu\nu}$ and the contribution from the cloud of strings $T^{\mathrm{CS}}_{\mu\nu}$, so:

$$T_{\mu\nu} = T^{\mathrm{CS}}_{\mu\nu} + T^{\mathrm{EM}}_{\mu\nu}, \tag{2.65}$$

and its non-null components are:

$$T_{00} = \frac{b}{r^2}e^{\nu} - \frac{Q^2}{r^4}e^{-\lambda}; \tag{2.66}$$

$$T_{11} = -\frac{b}{r^2}e^{\nu} + \frac{Q^2}{r^4}e^{-\nu}; \tag{2.67}$$

$$T_{22} = -r^2 e^{-(\nu+\lambda)}\frac{Q^2}{r^4}; \tag{2.68}$$

$$T_{33} = sin^2\theta T_{22}, \tag{2.69}$$

where $Q = e$ is the charge.

Thus, the Einstein's field equations read as follows:

$$e^{\nu}\left[e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) - \frac{1}{r^2}\right] = -e^{\nu}\left[\frac{b}{r^2} - \frac{Q^2}{r^4}e^{-(\lambda+\nu)}\right]; \tag{2.70}$$

$$-\left[\frac{1}{r^2} + \frac{\nu'}{r}\right] + \frac{e^{\lambda}}{r^2} = -e^{\lambda}\left[-\frac{b}{r^2} + \frac{Q^2}{r^4}e^{-(\nu+\lambda)}\right]; \tag{2.71}$$

$$-\frac{1}{2}r^2 e^{-\lambda}\left[\nu'' + \frac{(\nu')^2}{2} - \frac{\nu'\lambda'}{2} + \frac{\nu' - \lambda'}{r}\right] = -\frac{r^2}{2}\left[-2e^{-(\nu+\lambda)}\frac{Q^2}{r^4}\right]. \tag{2.72}$$

The $(3-3)$ component of Einstein's field equations leads us to the same result obtained by $(2-2)$ component. Summing the equation (2.70) multiplied by $e^{-\nu}$, with (2.71) multiplied by $e^{-\lambda}$, we get:

$$e^{-\lambda}\left(\frac{\nu' + \lambda'}{r}\right) = 0,$$

that leads us to:

$$\nu' = -\lambda',$$

and, therefore:

$$\nu = -\lambda, \tag{2.73}$$

since ν and λ are parameters and can be reparameterized in a convenient way in order to eliminate the integration constant. This result proves Eq. (2.40), that we used in the previous section.

Following a procedure similar to the one propoused by Kiselev [71], we define, without lost of generality, the parameters $\nu(r)$ and $\lambda(r)$ in terms of a function $f(r)$ (still to be obtained) through the following expression:

$$\nu = -\lambda = ln(1 + f). \tag{2.74}$$

Inserting the result of Eq. (2.73) into the equation (2.70) or equation (2.71), we get that:

$$e^{\nu}\left[\frac{1}{r^2} + \frac{\nu'}{r}\right] - \frac{1}{r^2} = -\frac{b}{r^2} + \frac{Q^2}{r^4},$$

which together with (2.74) and through a relatively simple algebric development, leads us to:

$$\frac{1}{r^2}(f + rf') = -\frac{b}{r^2} + \frac{Q^2}{r^4}. \tag{2.75}$$

On the other hand, using (2.74) into (2.72), we get:

$$f'' + 2\frac{f'}{r} = 2\frac{Q^2}{r^4}. \tag{2.76}$$

Summing Eq. (2.75) with Eq. (2.76), we finally get, after multiplying the whole equation by r^2, the following second order ordinary differential equation for $f(r)$:

$$r^2 f'' + 3rf' + f = -b + \frac{Q^2}{r^2}. \tag{2.77}$$

This equation can easily be solved by the standart differential calculus procedures. We then admit that its general solution is a sum of a particular solution for the whole equation, with the general solution for the associated homogeneous differential equation, i.e.:

$$f(r) = f_h(r) + f_p(r). \tag{2.78}$$

The general solution for the associated homogeneous differential equation

$$r^2 f'' + 3rf' + f = 0,$$

is obtained in a very simple way, and it is given by:

$$f_h(r) = -\frac{r_q}{r}, \tag{2.79}$$

where r_q is an integration constant.

On the other hand, the particular solution for the whole equation demands more mathematical development. To begin the procedure, we suppose it has the form:

$$f_p(r) = c_0 + c_1 r + c_2 r^2 + \frac{c_{-1}}{r} + \frac{c_{-2}}{r^2}, \tag{2.80}$$

Substituting this particular solution into the equation (2.77), we identify the constants above as:

$$
\begin{aligned}
c_0 &= -b, \\
c_1 &= 0, \\
c_2 &= 0, \\
c_{-2} &= Q^2.
\end{aligned} \tag{2.81}
$$

The c_{-1} constant is arbitrary and can be conveniently absorbed by the homogeneous solution, which contains a term of the same order in the variable r. We then identify the resulting constant at the order (-1) term as the Schwarzschild radius $2m$.

Thus, we finally obtain that the particular solution for Eq. (2.77) is:

$$f_p(r) = -b + \frac{Q^2}{r^2}, \tag{2.82}$$

and therefore, by the equation (2.78), the general solution for equation (2.77) is:

$$f(r) = -\frac{2m}{r} - b + \frac{Q^2}{r^2}. \tag{2.83}$$

Finally, the parameters $\nu(r)$ and $\lambda(r)$ are:

$$\nu = -\lambda = ln\left(1 - b - \frac{2m}{r} + \frac{Q^2}{r^2}\right). \tag{2.84}$$

Defining the function $h(r)$ as:

$$h(r) = e^{\nu(r)}, \tag{2.85}$$

we finally obtain that the line element at the neighbourhood of a static and charged black hole with cloud of string is:

$$ds^2 = h(r)dt^2 - \frac{1}{h(r)}dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2), \tag{2.86}$$

with

$$h(r) = 1 - b - \frac{2m}{r} + \frac{Q^2}{r^2}. \tag{2.87}$$

The constants of integration, m and $Q = e$ are identified with the mass and electric charge of the black hole.

The metric given by equation (2.86) will be called Reissner-Nordstrom black hole with a cloud of strings metric. It is asymptotically Minkowskian as $r \to \infty$. Observe that the gravitational field is affected by the presence of the cloud of strings.

Consider, now, the metric coefficient

$$h(r) = 1 - b - \frac{2m}{r} + \frac{Q^2}{r^2}.$$

The real roots of $h(r)$ exist if and only if

$$m^2 \geq (1-b)Q^2,$$

and those occur at $r = r_\pm$, where

$$r_\pm = \frac{m \pm \sqrt{m^2 - (1-b)Q^2}}{1-b},$$

and thus, this result is influenced by the cloud of strings. This means that the event horizons are affected by the presence of the cloud of strings.

2.2 Rotating Axially-Symmetric Black Holes

In this section, we will investigate the procedure that allows us to obtain the rotating black hole from its corresponding static configuration. This procedure was created by Newman and Janis to obtain the Kerr solution, which describes an uncharged rotating gravitational body, from the Schwarzschild solution, which as we have seen describes an uncharged static black hole. It was also used to obtain the Kerr-Newman solution, for a charged rotating black hole, from the static and charged one, i.e., Reissner-Nordstrom black hole. Our main goal in this section is to gain familiarity with this process, as well as with Newman-Penrose formalism, in order to apply it for obtaining the Kerr-Newman black hole with a cloud of strings.

In order to discuss this approach, we start by introducting the idea of tetrad or frame formalism, and after that, the idea of a null-tetrad, which its systematic use is the basis of the Newman-Penrose formalism [72].

Tetrads

We start by considering four 4-vector fields $\left\{e_{(a)}\right\}$, with $(a) = 0, 1, 2, 3$, such that their scalar products define:

$$g_{(a)(b)} = g(e_{(a)}, e_{(b)}), \tag{2.88}$$

where $g_{(a)(b)}$ is a constant matrix.

We will naturaly have:

$$e_{(a)} = e^{\alpha}_{(a)} \frac{\partial}{\partial x^{\alpha}}, \tag{2.89}$$

where $\left\{\frac{\partial}{\partial x^{\alpha}}\right\}$ for $\alpha = 0, 1, 2, 3$ is the coordinate basis, and $e^{\alpha}_{(a)}$ are the components of $\left\{e_{(a)}\right\}$ in this basis.

Assuming that this new basis $\left\{e_{(a)}\right\}$ defined in a specific point P of our manifold is linearly independent, that is to say that:

$$c_0 e_{(0)} + c_1 e_{(1)} + c_2 e_{(2)} + c_3 e_{(3)} \Leftrightarrow c_0 = c_1 = c_2 = c_3 = 0; \tag{2.90}$$

we will have that the matrix $e^{\alpha}_{(a)}$ is inversible, so we can write:

$$\frac{\partial}{\partial x^\alpha} = e^{(a)}_\alpha e_{(a)}, \tag{2.91}$$

where

$$e^{(a)}_\rho e^\alpha_{(a)} = \delta^\alpha_\rho. \tag{2.92}$$

With this result, and using the bilinearity of the metric, we can rewrite Eq. (2.88) as:

$$\begin{aligned} g_{(a)(b)} &= g(e_{(a)}, e_{(b)}) = g\left(e^\alpha_{(a)}\frac{\partial}{\partial x^\alpha}, e^\beta_{(b)}\frac{\partial}{\partial x^\beta}\right) = e^\alpha_{(a)}e^\beta_{(b)} g\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right), \\ g_{(a)(b)} &= e^\alpha_{(a)}e^\beta_{(b)} g_{\alpha\beta}. \end{aligned} \tag{2.93}$$

Multiplying the equation above by $e^{(a)}_\rho$, we get:

$$e_{\rho(b)} = g_{(a)(b)} e^{(a)}_\rho. \tag{2.94}$$

Similarly, if we define $g^{(a)(b)}$ as the inverse matrix of $g_{(a)(b)}$, we can show that:

$$e^{(b)}_\rho = g^{(a)(b)} e_{\rho(a)}. \tag{2.95}$$

Multiplying these two equations by the contravariant metric $g^{\rho\lambda}$, we also show that:

$$e^\lambda_{(b)} = g_{(a)(b)} e^{\lambda(a)}, \tag{2.96}$$

$$e^{\lambda(a)} = g^{(a)(b)} e^\lambda_{(b)}. \tag{2.97}$$

So, we have just proved that the tetrad indices are raised and lowered by the matrices $g^{(a)(b)}$ and $g_{(a)(b)}$, respectively. Thus, relatively to the tetrad indices, the matrices $g^{(a)(b)}$ and $g_{(a)(b)}$ have the same role of the metric.

Multiplying equation (2.93) by $e^{(a)}_\rho$ and $e^{(b)}_\gamma$, we also show that:

$$g_{\rho\gamma} = e^{(a)}_{\rho} e^{(b)}_{\gamma} g_{(a)(b)} \tag{2.98}$$

As an example of a particular set of tetrad, or reference frame, we can select a specific set of tetrad vector fields, defined by: $\left\{e_{(a)}\right\} = \left\{e_{(0)}, e_{(1)}, e_{(2)}, e_{(3)}\right\} = \{\mathbf{v}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ in a point P of the manifold, such that: $e_{(0)}$ is a timelike vector:

$$g(e_{(0)}, e_{(0)}) > 0,$$

and $e_{(i)}$ for $i = 1, 2, 3$ are spacelike vectors:

$$g(e_{(i)}, e_{(i)}) < 0,$$

and this particular set of vectors $\left\{e_{(a)}\right\}$ with $a = 0, 1, 2, 3$ is orthogonal:

$$g(e_{(a)}, e_{(b)}) = 0,$$

for $a \neq b$.

We can also normalize these vectors in order to get:

$$\begin{aligned} g(e_{(0)}, e_{(0)}) &= 1, \\ g(e_{(i)}, e_{(i)}) &= -1, \end{aligned}$$

According to these definitions, the quantities $\left\{e_{(a)}\right\}$ are such that:

$$g_{(a)(b)} = g(e_{(a)}, e_{(b)}) = \eta_{(a)(b)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{2.99}$$

In other words, the metric at this specific point P of the manifold and in this specific frame rate $\left\{e_{(a)}\right\} = \left\{e_{(0)}, e_{(1)}, e_{(2)}, e_{(3)}\right\} = \{\mathbf{v}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is the Minkowski metric.

For this frame rate $\left\{e_{(a)}\right\}$, we can take the particular geometric interpretation of the tetrad as the specific basis of the tangent space in P $\{\mathbf{T_P}\}$, associeted with the coordinate system $(\bar{x}^{(a)})$, such that the *local transformation* of the coordinate system $(x^{(a)})$, associated with the coordinate basis $\left\{\frac{\partial}{\partial x^\alpha}\right\}$, to these *local coordinates* $(\bar{x}^{(a)})$, transforms the metric $g_{\alpha\beta}$ of the manifold to the Minkowski metric $\eta_{(a)(b)}$ in P.

The coordinate system $(\bar{x}^{(a)})$ will be called local coordinates, as we have said, while the reference frame defined by $\left\{e_{(a)}\right\} = \left\{e_{(0)}, e_{(1)}, e_{(2)}, e_{(3)}\right\} = \{\mathbf{v}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ will be called local Lorentz frame. Also, the observers associated with this frame are called local Lorentz observers. These concepts of local Lorentz frame and local Lorentz observers will be very useful in the next chapter, since the spacetime manifold of general relativity has the following important property: it is *locally Lorentzian*, i.e., for every point P of the manifold, it is always possible to find a local coordinate system such that the the metric assumes the Minkowski form (2.99).

Finally, we can also add the important information that there exists an arbitrarity in the choice of the tetrad basis $\left\{e_{(a)}\right\}$. The expressions:

$$\begin{aligned} g_{\alpha\beta} &= e_\alpha^{(a)} e_\beta^{(b)} g_{(a)(b)}, \\ g_{(a)(b)} &= e_{(a)}^\alpha e_{(b)}^\beta g_{\alpha\beta}, \end{aligned} \tag{2.100}$$

remain unchanged by a local Lorentz transformation $L_b^a(P)$ in $e_\alpha^{(a)}$, i.e., given $e_\alpha^{(a)}$ defined in P, we have that:

$$e_\alpha^{(a)}(P) \mapsto \bar{e}_\alpha^{(a)}(P) = L_b^a(P) e_\alpha^{(b)}(P), \tag{2.101}$$

with $L_b^a(P)$ such that:

$$L^a_c(P)\eta_{(a)(b)}L^b_d(P) = \eta_{(c)(d)}, \tag{2.102}$$

keeps the expressions (2.100) above unchanged.

Null Tetrad: Newman-Penrose formalism

We now define the tetrad as $\{e_{(a)}\} = \{e_{(0)}, e_{(1)}, e_{(2)}, e_{(3)}\} = \{\mathbf{l}, \mathbf{n}, \mathbf{m}, \bar{\mathbf{m}}\}$, where:

$$\begin{aligned}
e_{(0)} &= \mathbf{l} = \frac{1}{\sqrt{2}}(\mathbf{v} + \mathbf{i}), \\
e_{(1)} &= \mathbf{n} = \frac{1}{\sqrt{2}}(\mathbf{v} - \mathbf{i}), \\
e_{(2)} &= \mathbf{m} = \frac{1}{\sqrt{2}}(\mathbf{j} + i\mathbf{k}), \\
e_{(3)} &= \bar{\mathbf{m}} = \frac{1}{\sqrt{2}}(\mathbf{j} - i\mathbf{k}),
\end{aligned} \tag{2.103}$$

in which case $\mathbf{l}$,$\mathbf{n}$,$\mathbf{m}$ and $\bar{\mathbf{m}}$ are *null vectors*; that is:

$$g(e_{(a)}, e_{(a)}) = 0, \quad \text{for } a = 0, 1, 2, 3. \tag{2.104}$$

Furthermore, these vectors satisfy the following normalization condition:

$$\begin{aligned}
g(e_{(0)}, e_{(1)}) = g(e_{(1)}, e_{(0)}) = 1, \\
g(e_{(2)}, e_{(3)}) = g(e_{(3)}, e_{(2)}) = -1,
\end{aligned} \tag{2.105}$$

with the others scalar products between these vectors equal to zero.

It defines the following *null-tetrad* frame metric:

$$g_{(a)(b)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \tag{2.106}$$

Thus, if we remind ourselves of Eq. (2.91), we decompose the metric $g_{\alpha\beta}$ using equations (2.98), (2.103) and (2.106), into the products of the null tetrad vectors' components according to:

$$g_{\alpha\beta} = l_\alpha n_\beta + l_\beta n_\alpha - m_\alpha \bar{m}_\beta - m_\beta \bar{m}_\alpha. \tag{2.107}$$

The contravariant form of this equation is:

$$g^{\alpha\beta} = l^\alpha n^\beta + l^\beta n^\alpha - m^\alpha \bar{m}^\beta - m^\beta \bar{m}^\alpha. \tag{2.108}$$

2.2.1 Rotating Uncharged Black Hole - Kerr Metric

As it was said in the beginning of this section, the procedure of Newman and Janis [72], which allows us to obtain a rotating black hole, has as starting point its associated static black hole, which plays the role of a seed. Thus, for the case of a rotating uncharged black hole, also called Kerr black hole, we must start from the static uncharged black hole, called Schwarzschild black hole:

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \frac{1}{\left(1 - \frac{2m}{r}\right)} dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{2.109}$$

This solution in advanced Eddington-Finkelstein coordinates [73] is described by the following non-zero components of the contravariant metric tensor $g^{\mu\nu}$:

$$g^{01} = -1, \quad g^{11} = -\left(1 - \tfrac{2m}{r}\right),$$
$$g^{22} = -\frac{1}{r^2}, \qquad g^{33} = -\tfrac{1}{r^2 sin^2\theta}. \tag{2.110}$$

It is straight forward to show, using equation (2.108), that these metric components can be written in terms of the following null tetrad components:

$$\begin{aligned} l^a &= (0,1,0,0), \\ n^a &= \left(-1, -\frac{1}{2}\left(1 - \frac{2m}{r}\right), 0, 0\right), \\ m^a &= \frac{1}{r\sqrt{2}}\left(0,0,1,\frac{i}{sin\theta}\right), \\ \bar{m}^a &= (m^a)^*. \end{aligned} \tag{2.111}$$

Here we come to the most important steps of the Newman and Janis procedure, they consist in:

- First, we allow the coordinate r to take complex values.

  So the tetrad can be rewritten in the following way:

$$\begin{aligned} l^a &= (0,1,0,0), \\ n^a &= \left(-1, -\frac{1}{2}\left(1 - m(r^{-1} + \bar{r}^{-1})\right), 0, 0\right), \\ m^a &= \frac{1}{\bar{r}\sqrt{2}}\left(0,0,1,\frac{i}{sin\theta}\right), \\ \bar{m}^a &= \frac{1}{r\sqrt{2}}\left(0,0,1,-\frac{i}{sin\theta}\right), \end{aligned} \tag{2.112}$$

- Next, we formally perform the complex coordinate transformations:

$$
\begin{aligned}
v &\rightarrow v' = v + iacos\theta, \\
r &\rightarrow r' = r + iacos\theta, \\
\theta &\rightarrow \theta', \\
\phi &\rightarrow \bar{\phi},
\end{aligned} \tag{2.113}
$$

on the null tetrad.

- Finally, we require v' and r' to be real again, and then we obtain the following tetrad:

$$
\begin{aligned}
l^a &= (0, 1, 0, 0), \\
n^a &= \left(-1, -\frac{1}{2}\left(1 - \frac{2mr'}{r'^2 + a^2cos^2\theta}\right), 0, 0\right), \\
m^a &= \frac{1}{\sqrt{2}(r' + iacos\theta)}\left(-iasin\theta, -iasin\theta, 1, \frac{i}{sin\theta}\right), \\
\bar{m}^a &= \frac{1}{\sqrt{2}(r' - iacos\theta)}\left(iasin\theta, iasin\theta, 1, -\frac{i}{sin\theta}\right),
\end{aligned} \tag{2.114}
$$

and this is the final form of the null-tetrad for Kerr's metric in advanced Eddington-Finkelstein coordinates, obtained here by the Newman and Janis procedure, which is described above in three steps.

Using now equation (2.108), we get the promised Kerr metric in its contravariant form, which gives rise to the following line element in the advanced Eddington-Finkelstein coordinates system:

$$
\begin{aligned}
ds^2 = \; & \left(1 - \tfrac{2mr}{\rho^2}\right) dv^2 - 2dvdr + \tfrac{2mr}{\rho^2}(2asin^2\theta)dvd\bar{\phi} \\
& +2asin^2\theta drd\bar{\phi} - \rho^2 d\theta^2 \\
& - \left((r^2 + a^2)sin^2\theta + \tfrac{2mr}{\rho^2}(a^2sin^4\theta)\right) d\bar{\phi}^2,
\end{aligned} \tag{2.115}
$$

where $\rho^2 = r^2 + a^2cos^2\theta$.

To obtain the analogue of the Schwarzschild form, we carry out the coordinate transformation from the old Eddington-Finkelstein coordinates system $(v, r, \theta, \bar{\phi})$ to the usual coordinates system (t, r, θ, ϕ). In this case, the transformation of the coordinate differentials is made by:

$$\begin{aligned} dv &= dt + \frac{2mr + \Delta_r^k}{\Delta_r^k} dr, \\ d\bar{\phi} &= d\phi + \frac{a}{\Delta_r^k} dr, \end{aligned} \tag{2.116}$$

where $\Delta_r^k = r^2 + a^2 - 2mr$ and the index k stands for "Kerr". The coordinates r and θ remain unchanged. This leads to the Kerr's solution form in the so called Boyer-Lindquist coordinates, which reads as:

$$\begin{aligned} ds^2 = \; & \tfrac{\Delta_r^k}{\rho^2}(dt - asin^2\theta d\phi)^2 - \tfrac{sin^2\theta}{\rho^2}[(r^2 + a^2)d\phi - adt]^2 \\ & -\tfrac{\rho^2}{\Delta_r^k} dr^2 - \rho^2 d\theta^2. \end{aligned} \tag{2.117}$$

This is the line element for the Kerr solution that we aimed to obtain.

2.2.2 Rotating Charged Black Hole with Cloud of Strings

Similarly to the previous case above, in order to obtain the metric for the charged rotating black hole with a cloud of strings, we need to start from the metric of the static charged black hole with cloud of strings, obtained in the section 2.1.2, which is given by:

$$ds^2 = h(r)dt^2 - \frac{1}{h(r)}dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2), \tag{2.118}$$

with

$$h(r) = 1 - b - \frac{2m}{r} + \frac{e^2}{r^2}. \tag{2.119}$$

Having this information, we will only need to repeat the Newman and Janis procedure described previously, but now with this metric. As result, we will obtain our aimed solution for the charged rotating black hole with cloud of strings.

In order to do so, firstly, we rewrite this metric into the Eddington-Finkelstein coordinates, by means of the transformation:

$$dv = dt - \frac{dr}{h(r)}, \tag{2.120}$$

and, thus, we get

$$ds^2 = h(r)dv^2 - dvdr - r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{2.121}$$

Now, we determine the null tetrad basis which describes this metric as:

$$\begin{aligned} l^a &= (0,1,0,0), \\ n^a &= \left(-1, -\frac{h(r)}{2}, 0, 0\right), \\ m^a &= \frac{1}{r\sqrt{2}}\left(0,0,1,\frac{i}{sin\theta}\right), \\ \bar{m}^a &= (m^a)^*. \end{aligned} \tag{2.122}$$

Next, we allow the coordinate r to take complex values, and then we formally perform the complex coordinate transformations:

$$\begin{aligned} v &\rightarrow v' = v + iacos\theta, \\ r &\rightarrow r' = r + iacos\theta, \\ \theta &\rightarrow \theta', \\ \phi &\rightarrow \bar{\phi}, \end{aligned} \tag{2.123}$$

on the null tetrad.

We also use the changes $h(r) \to H(r,\theta)$, and $r \to \rho$. Thus, the null tetrad is rewritten in the following form:

$$\begin{aligned} l^a &= (0,1,0,0), \\ n^a &= \left(-1, -\frac{H(r,\theta)}{2}, 0, 0\right), \\ m^a &= \frac{1}{\sqrt{2}(r' + iacos\theta)}\left(-iasin\theta, -iasin\theta, 1, \frac{i}{sin\theta}\right), \\ \bar{m}^a &= \frac{1}{\sqrt{2}(r' - iacos\theta)}\left(iasin\theta, iasin\theta, 1, -\frac{i}{sin\theta}\right), \end{aligned} \tag{2.124}$$

where

$$H(r,\theta) = \frac{r^2 h(r) + a^2 \cos^2\theta}{\rho^2}, \tag{2.125}$$

This null tetrad above is the right null tetrad for the charged rotating black hole with cloud of strings in the Eddington-Finkelstein coordinates system. However, we are willing to obtain the metric for this black hole in the Boyer-Lindquist coordinates. Therefore, we use the following transformations:

$$dv = dt + \lambda(r)dr, \quad d\bar{\phi} = d\phi + \chi(r)dr, \tag{2.126}$$

where

$$\lambda(r) = \frac{r^2 + a^2}{r^2 h(r) + a^2}, \quad \chi(r) = \frac{a}{r^2 h(r) + a^2}, \tag{2.127}$$

are chosen such that the non-diagonal components of the metric are null, excepting g_{03} and g_{30}. We can also write

$$\lambda(r) = \frac{r^2 + a^2}{\Delta_r}, \quad \chi(r) = \frac{a}{\Delta_r}, \tag{2.128}$$

where

$$\Delta_r = (r^2 + a^2) - br^2 - 2Mr + e^2, \tag{2.129}$$

Finally, using equation (2.108), we obtain the metric for the charged rotating black hole with cloud of strings, which its line element is written, in Boyer-Lindquist coordinates, as:

$$ds^2 = \frac{\Delta_r}{\rho^2}(dt - asin^2\theta d\phi)^2 - \frac{\rho^2}{\Delta_r}dr^2 - \rho^2 d\theta^2 - \frac{sin^2\theta}{\rho^2}(adt - (r^2 + a^2)d\phi)^2, \tag{2.130}$$

where $\rho^2 = r^2 + a^2cos^2\theta$.

The black hole horizons are determined by the condition $\Delta_r = 0$, or equivalently,

$$\Delta_r = (1 - b)\left[(r - r_+)(r - r_-)\right] = 0,$$

with r_+ and r_- being the event horizon and cauchy horizon, respectively, which are given by:

$$\begin{aligned} r_+ &= \frac{M}{1-b} + \frac{\sqrt{M^2 - (1-b)(a^2 + e^2)}}{1-b}, \\ r_- &= \frac{M}{1-b} - \frac{\sqrt{M^2 - (1-b)(a^2 + e^2)}}{1-b}. \end{aligned}$$

Note the role played by the cloud of strings on the horizons. As $0 < b < 1$, these horizons are enlarged and therefore all quantities written in terms of these horizons, as for example, in the thermodynamics of this black hole are affected as well.

The metric (2.130) above is the most important result for this second chapter. It is a solution to the Eistein field equations with an electromagnetic field and a cloud of strings as their sources. Our main objective for this dissertation is to write and solve the Dirac equation in this metric spacetime, i.e., at the neighbourhood of a charged rotating black hole with a cloud of strings. In order to do so, we would need, first of all, to obtain this metric. This initial goal, we can make sure that we have already accomplished.

In the next chapter, we shall discuss the generalization of Dirac equation for a curved spacetime, constructing a general relativistic Dirac equation. After that, in the fourth chapter, we will write this general relativistic Dirac equation for the specific spacetime scenario created by the metric (2.130). Finally, in the fifth chapter, we will discuss a theorem that allows us to obtain analytical solutions for this Dirac equation at the vicinity of some specific and important points, such as the event horizon, cauchy horizon, and the asymptotic limit at infinity.

Chapter Three

DIRAC EQUATION IN CURVED SPACETIMES

In this second chapter, our main goal is to obtain the general relativistic version of the Dirac equation by means of a particular prescription, which defines a spinorial connection for the covariant derivative of a spinor. This covariant derivative of a spinor (where covariant here stands for covariant under local Lorentz transformations), allows us to construct a general relativistic Dirac equation which is covariant under the group of local Lorentz transformations from special relativity (see Eq. (2.101) and Eq. (2.102)), and which is also covariant under the group of general coordinate transformations from general relativity.

In order to obtain and discuss this Dirac equation for curved spacetimes, we will firstly review, in the next section, the Dirac equation in Minkowsky spacetime. Then, when the concepts of the Dirac equation in special relativity are already established, we will introduce the procedure that generalizes this wave equation for the general relativistic context.

3.1 Dirac Equation in Special relativity

Dirac wanted a relativistic covariant wave equation of the Schroedinger form:

$$i\hbar \frac{\partial}{\partial t}\psi = \hat{H}\psi, \tag{3.1}$$

with positive definite probability density, and satisfying the energy-momentum relation:

$$E^2 = p^2c^2 + m_0^2c^4.$$

Since (3.1) is linear and first order in time derivative, it is natural to try to construct a hamiltonian that is also linear and first order in the spatial derivatives (equality of spatial and temporal coordinates). Hence, the desired equation must be of the form:

$$i\hbar \frac{\partial \psi}{\partial t} = \left[-i\hbar c \left(\hat{\alpha}_1 \frac{\partial}{\partial x^1} + \hat{\alpha}_2 \frac{\partial}{\partial x^2} + \hat{\alpha}_3 \frac{\partial}{\partial x^3}\right) + \hat{\beta} m_0 c^2\right]\psi = \hat{H}\psi. \tag{3.2}$$

The yet unknown coefficients $\hat{\alpha}_i$, for $i = 1, 2, 3$, and $\hat{\beta}$ cannot be simple numbers, otherwise equation (3.2) would not be form invariant with respect to simple spatial rotations [74].

We suspect that $\hat{\alpha}_i$, for $i = 1, 2, 3$, and $\hat{\beta}$ are matrices. Then, ψ cannot be a simple scalar but has to be a column vector

$$\psi = \begin{bmatrix} \psi_1(\overrightarrow{x}, t) \\ \psi_2(\overrightarrow{x}, t) \\ \vdots \\ \psi_N(\overrightarrow{x}, t) \end{bmatrix}, \tag{3.3}$$

from which a positive definite density of the form:

$$\rho(\mathbf{x}) = \psi^\dagger(\mathbf{x})\psi(\mathbf{x}) = \begin{bmatrix} \psi_1^*(\overrightarrow{x}, t) & \psi_2^*(\overrightarrow{x}, t) & \cdots & \psi_N^*(\overrightarrow{x}, t) \end{bmatrix} \cdot \begin{bmatrix} \psi_1(\overrightarrow{x}, t) \\ \psi_2(\overrightarrow{x}, t) \\ \vdots \\ \psi_N(\overrightarrow{x}, t) \end{bmatrix} = \sum_{i=1}^{N} \psi_i^* \psi_i, \tag{3.4}$$

can be constructed immediatly.

We still have to show that $\rho(\mathbf{x})$ is the temporal component of a four-current vector for which a continuity equation must exist. Only then is the probability interpretation of $\rho(\mathbf{x})$ ensured.

We shall call the wave function ψ of equation (3.3) spinors. The dimension N of the spinor is not yet known, but we will be able to decide this soon. The coefficients $\hat{\alpha}_i$, for $i = 1, 2, 3$, and $\hat{\beta}$ must be quadratic $N \times N$ matrices.

Thus, the Schroedinger-like equation (3.1), with (3.3), represents a system of N coupled first-order differential equations for the spinor components ψ_i, with $i = 1, 2, ..., N$. We can write each of these N coupled differential equations from the system of equations (3.2) in the following way:

$$i\hbar \frac{\partial}{\partial t}\psi_\sigma = \sum_{r=1}^{N} \left[-i\hbar c \left(\alpha_{1_{\sigma r}} \frac{\partial}{\partial x^1} + \alpha_{2_{\sigma r}} \frac{\partial}{\partial x^2} + \alpha_{3_{\sigma r}} \frac{\partial}{\partial x^3} \right) + \beta_{\sigma r} m_0 c^2 \right] \psi_r. \quad (3.5)$$

To continue, we demand the following natural properties:

- The correct energy-momentum relation for a relativistic free particle:

$$E^2 = p^2 c^2 + m_0^2 c^4. \quad (3.6)$$

- The continuity equation for the density (3.4):

$$\rho(\mathbf{x}) = \psi^\dagger(\mathbf{x})\psi(\mathbf{x}). \quad (3.7)$$

- And finally, the Lorentz covariance for equation (3.2).

In the next three sections, we are going to work in each one of these three requirements individually.

3.1.1 Energy-momentum relation requirement

To fulfill the first requirement of the energy-momentum relation (3.6), we require that every component ψ_σ of the spinor ψ satisfies the Klein-Gordon equation, that is to say:

$$-(\hbar c)^2 \left(\frac{\partial^2}{\partial (ct)^2} - \nabla^2 \right) \psi_\sigma = m_0^2 c^4 \psi_\sigma,$$

so

$$-\hbar^2 \frac{\partial^2}{\partial t^2} \psi_\sigma = (-\hbar^2 c^2 \nabla^2 + m_0^2 c^4) \psi_\sigma. \tag{3.8}$$

However, equation (3.2) may be written as:

$$i\hbar \frac{\partial}{\partial t} \psi = \frac{\hbar c}{i} \sum_{i=1}^{3} \hat{\alpha}_i \frac{\partial}{\partial x^i} \psi + \hat{\beta} m_0 c^2 \psi = \hat{H} \psi, \tag{3.9}$$

and applying $i\hbar \frac{\partial}{\partial t} = \hat{H}$ to both sides we get that for the σ-eth component of spinor ψ:

$$-\hbar^2 \frac{\partial^2}{\partial t^2} \psi_\sigma = -\hbar^2 c^2 \sum_{i,j=1}^{3} \left(\frac{\hat{\alpha}_i \hat{\alpha}_j - \hat{\alpha}_j \hat{\alpha}_i}{2} \right) \frac{\partial^2 \psi_\sigma}{\partial x^i \partial x^j} - i\hbar m_0 c^3 \sum_{i=1}^{3} (\hat{\alpha}_i \hat{\beta} + \hat{\beta} \hat{\alpha}_i) \frac{\partial \psi_\sigma}{\partial x^i} + \hat{\beta}^2 m_0^2 c^4 \psi_\sigma, \tag{3.10}$$

and comparing to equation (3.8), we must have the following requirements for matrices $\hat{\alpha}_i$, for $i = 1, 2, 3$, and $\hat{\beta}$:

$$\text{(i)} \quad \hat{\alpha}_i \hat{\alpha}_j + \hat{\alpha}_j \hat{\alpha}_i = 2I\delta_{ij}, \tag{3.11}$$

$$\text{(ii)} \quad \hat{\alpha}_i \hat{\beta} + \hat{\beta} \hat{\alpha}_i = 0, \tag{3.12}$$

$$\text{(iii)} \quad \hat{\alpha}_i^2 = \hat{\beta}^2 = I. \tag{3.13}$$

These anticommutation relations define an algebra for ψ matrices.

In order to establish hermeticity of hamiltonian $\hat{H}$, the matrices $\hat{\alpha}_i$, for $i = 1, 2, 3$, and $\hat{\beta}$ also have to be hermitian. Thus:

$$\hat{\alpha}_i^\dagger = \hat{\alpha}_i, \quad \hat{\beta}^\dagger = \hat{\beta}. \tag{3.14}$$

Therefore, the eigenvalues of the matrices $\hat{H}$, $\hat{\alpha}_i$, and $\hat{\beta}$ are real. Furthermore, according to equation (3.13) $\hat{\alpha}_i^2 = \hat{\beta}^2 = I$, so it follows that the eigenvalues of $\hat{\alpha}_i$ and $\hat{\beta}$ can only have the values ± 1.

Also, from equations (3.11), (3.12) and (3.13), we can deduce that the trace of each $\hat{\alpha}_i$ and $\hat{\beta}$ has to be zero [74]. However, since the eigenvalues of $\hat{\alpha}_i$ and $\hat{\beta}$ are all equal to ± 1, each matrix $\hat{\alpha}_i$ and $\hat{\beta}$ must have as many positive and negative eigenvalues in order to their trace be zero. Therefore N has to be an even number, that is to say that the dimension of the spinor and of the quadratic matrices $\hat{\alpha}_i$ and $\hat{\beta}$ has to be an even number.

The smallest dimension to be considered is $N = 2$, but it cannot be this number, because there exist only three anticommuting matrices in two dimmensions, namely the three Pauli matrices $\hat{\sigma}_i$. Therefore, the smallest dimmension for which the requirements (3.11), (3.12) and (3.13) can be fulfilled is $N = 4$. We will now study this case in more details and we indicate immediately one possible explicit representation of the Dirac matrices $(\hat{\beta}, \hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3)$ that will be:

$$\hat{\beta} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad \hat{\alpha}_1 = \begin{bmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{bmatrix}, \quad \text{for } i = 1, 2, 3; \tag{3.15}$$

where $\hat{\sigma}_i$ are the Pauli's 2×2 matrices and I is 2×2 unit matrix.

Indeed, we can easily check the validity of the requirements (3.11), (3.12) and (3.13) for this particular representation of the Dirac matrices.

Moreover, each set

$$\begin{aligned} \hat{\hat{\alpha}}_i &= \hat{U}\hat{\alpha}_i\hat{U}^{-1}; \\ \hat{\hat{\beta}} &= \hat{U}\hat{\beta}\hat{U}^{-1}, \end{aligned} \tag{3.16}$$

which is obtained from the original set $\hat{\alpha}_i$ and $\hat{\beta}$, by means of a unitary transformation $\hat{U}$, can be used equally as well as the one defined here in equation (3.15).

Physical results do not depend on the special choice of the Dirac matrices, what will allow us to eventually change our representation by mathematical convenience.

3.1.2 Continuity equation for the density requirement

The second requirement that the equation (3.2) needs to satisfy in order to be our relativistic wave equation of the Schoedinger type (3.1) is that the positive definite probability density (3.7), i.e.:

$$\rho(\mathbf{x}) = \psi^{\dagger}(\mathbf{x})\psi(\mathbf{x}),$$

be the temporal component of a four-current density which satisfies the equation of continuity $\partial_{\mu} J^{\mu} = 0$. We want to construct this four-current density and its equation of continuity. For that we take Eq. (3.2) and we multiply from the left by $\psi^{\dagger} = [\psi_1^* \ \psi_2^* \ \psi_3^* \ \psi_4^*]$, we get:

$$i\hbar\psi^{\dagger}\frac{\partial\psi}{\partial t} = \frac{\hbar c}{i}\sum_{i=1}^{3}\psi^{\dagger}\hat{\alpha}_i\frac{\partial\psi}{\partial x^i} + m_0 c^2\psi^{\dagger}\hat{\beta}\psi. \tag{3.17}$$

Moreover, we form the hermitian conjugate of equation (3.2) by taking the hermitian conjugate of the whole equation. After that, we multiply the result from the right by:

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix},$$

and we get:

$$-i\hbar\frac{\partial\psi^{\dagger}}{\partial t}\psi = -\frac{\hbar c}{i}\sum_{i=1}^{3}\frac{\partial\psi^{\dagger}}{\partial x^i}\hat{\alpha}_i\psi + m_0 c^2\psi^{\dagger}\hat{\beta}\psi. \tag{3.18}$$

Then we subtract (3.18) from (3.17) to get:

$$\frac{\partial}{\partial t}\left(\psi^{\dagger}\psi\right)+\sum_{k=1}^{3}\frac{\partial}{\partial x^{k}}\left(c\psi^{\dagger}\hat{\alpha}_{k}\psi\right)=0. \tag{3.19}$$

But this equation is basically:

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\overrightarrow{J}=0, \tag{3.20}$$

where $\rho=\psi^{\dagger}\psi$, and $J^{k}=c\psi^{\dagger}\hat{\alpha}_{k}\psi$.

The conservation law follows immediataly in the usual way, by integration of (3.20) over the volume V:

$$\int_{V}\frac{\partial\rho}{\partial t}dV=-\int_{V}\nabla\cdot\overrightarrow{J}dV,$$

or

$$\frac{\partial}{\partial t}\int_{V}\rho dV=-\oint_{S}\overrightarrow{J}\cdot d\overrightarrow{A}, \tag{3.21}$$

where V denotes a volume and S its surrounding surface.

Since ρ is positive definite and because of conservation law (3.21), we can accept the interpretation of ρ as a probability density. And accordingly, the three dimensional vector $\overrightarrow{J}$, where its components are given by $J^{k}=c\psi^{\dagger}\hat{\alpha}_{k}\psi$, is called probability density current, which allows us to interpret the matrices $\hat{\alpha}_{k}$ as operators related to the three dimensional velocity components.

Furthermore, $(c\rho,\overrightarrow{J})$ should form a four-current vector J^{μ}, such that the continuity equation (3.20) can be written as:

$$\partial_{\mu}J^{\mu}=0. \tag{3.22}$$

We can finally add here that according to special relativity, the components of four-current vector J^{μ} should transform from one inertial system into another inertial system by a Lorentz transformation.

$$J^{\mu} \mapsto J'^{\mu} = L^{\mu}_{\nu} J^{\nu}. \tag{3.23}$$

This point and in addition the form invariance (covariance) of the Dirac equation (3.2) with respect to Lorentz transformations have still to be shown before we can regard the Dirac equation (3.2) as an acceptable relativistic wave equation. We will work at this specific question in the next requirement.

3.1.3 Covariance under group of Lorentz transformations requirement

It will now be our task to find a relation between: the measurements of a observer A, and those of another observer B, which have been performed by both of them in their respective inertial system.

More precisely, we have to find the relation between:

$$\psi(\mathbf{x}),\ \text{from A} \quad \Leftrightarrow \quad \psi'(\mathbf{x}'),\ \text{from B} \tag{3.24}$$

So, for given $\psi(\mathbf{x})$ from A, a transformation (which we aim to obtain) must enable us to calculate $\psi'(\mathbf{x}')$ from B.

The requirement of Lorentz covariance now means that $\psi(\mathbf{x})$ in system A, as well as $\psi'(\mathbf{x}')$ in system B, must satisfy their respective Dirac equations, which have the same form for both systems of reference. This is precisely the relativity principle: only in this way do both inertial systems A and B become completely equivalent and indistinguible.

In the following considerations it is much more convenient to denote the Dirac equation in four-dimensional notation, in order to show the symmetry between the time coordinate ct and space coordinates x^i, with $i = 1, 2, 3$:

$$x^a = (ct, x^i). \tag{3.25}$$

where $a = 0, 1, 2, 3$.

We start that by the Dirac equation (3.2), which we can rewrite as:

$$\left[i\hbar \frac{\partial}{\partial t} + i\hbar c \left(\hat{\alpha}_1 \frac{\partial}{\partial x^1} + \hat{\alpha}_2 \frac{\partial}{\partial x^2} + \hat{\alpha}_3 \frac{\partial}{\partial x^3} \right) - \hat{\beta} m_0 c^2 \right] \psi = 0. \tag{3.26}$$

We multiply this equation by $\frac{1}{c}\hat{\beta}$ in order to get:

$$i\hbar \left[\hat{\beta} \frac{\partial}{\partial (ct)} + (\hat{\beta}\hat{\alpha}_1) \frac{\partial}{\partial x^1} + (\hat{\beta}\hat{\alpha}_2) \frac{\partial}{\partial x^2} + (\hat{\beta}\hat{\alpha}_3) \frac{\partial}{\partial x^3} \right] \psi - m_0 c \psi = 0. \tag{3.27}$$

With the definitions:

$$\hat{\gamma}^0 = \hat{\beta}, \tag{3.28}$$

$$\hat{\gamma}^i = \hat{\beta}\hat{\alpha}^i, \text{ for i=1,2,3.} \tag{3.29}$$

We get:

$$i\hbar \left[\hat{\gamma}^0 \frac{\partial}{\partial x^0} + \hat{\gamma}^1 \frac{\partial}{\partial x^1} + \hat{\gamma}^2 \frac{\partial}{\partial x^2} + \hat{\gamma}^3 \frac{\partial}{\partial x^3} \right] \psi - m_0 c \psi = 0. \tag{3.30}$$

From now on, we shall write the matrices $\hat{\gamma}^a$ without the hat "ˆ"; likewise, the four-vector position will be simply written as x. Thus, the Dirac equation (3.2) can be finally written as:

$$\left(i\hbar \gamma^a \frac{\partial}{\partial x^a} - m_0 c \right) \psi = 0, \tag{3.31}$$

where $\gamma^a = (\beta, \beta\alpha^i)$.

Furthermore, a more elegant formulation of the anticommutation relations (3.11), (3.12) and (3.13) is possible using the γ matrices defined by (3.28) and (3.29), and this formulation is:

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}I. \tag{3.32}$$

which is equivalent to all of those conditions (3.11), (3.12) and (3.13) at the same time.

We can also prove the important result that the gamma matrices γ^i, for $i = 1, 2, 3$, are unitary and anti-hermitian, i.e.:

$$(\gamma^i)^{-1} = (\gamma^i)^\dagger; \quad (\gamma^i)^\dagger = -\gamma^i. \tag{3.33}$$

While the γ^0 matrix is hermitian:

$$(\gamma^0)^\dagger = \gamma^0. \tag{3.34}$$

Finally, we can write down explicitly the γ^a matrices, defined by equations (3.28) and (3.29), as follows:

$$\gamma^0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}; \tag{3.35}$$

$$\gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}; \tag{3.36}$$

for $i = 1, 2, 3$.

In order to describe a particle with charge q interacting with an eletromagnetic field, we will need to introduce the eletromagnetic field potentials. We do so by using the minimal coupling:

$$p_a \mapsto p_a - \frac{q}{c}A_a \tag{3.37}$$

where $a = 0, 1, 2, 3$, and $p_a = i\hbar\frac{\partial}{\partial x^a}$. Then

$$\partial_a \mapsto \partial_a - \frac{q}{c}A_a. \tag{3.38}$$

Thus, the Dirac equation for a charged particle interacting with an eletromagnetic field described by the four-potential A_a is:

$$\left[i\hbar\gamma^a \left(\partial_a - \frac{q}{c}A_a\right) - m_0 c\right]\psi = 0, \tag{3.39}$$

where both ∂_a and A_a are four-vectors, hence their difference $\partial_a - \frac{q}{c}A_a$ is a four-vector as well.

Finally, we can now formalize the form invariance of the Dirac equation (3.39). As we have said at the beginning of this section, the form invariance, or covariance, of Dirac equation means two different things:

- There must be an explicit rule to enable observer B to calculate his $\psi'(x')$ if $\psi(x)$ of observer A is given. Hence $\psi'(x')$ of observer B describes the same physical state as $\psi(x)$ of observer A. We shall obtain this explicit transformation rule between $\psi'(x')$ of B and $\psi(x)$ of A in the "Lorentz Transformation for Spinors" subsection which comes after this one.

- According to the principle of relativity, which states that the physics is the same in every inertial system, $\psi'(x')$ of B must be a solution of the Dirac equation of observer B:

$$\left(i\hbar\gamma'^a \frac{\partial}{\partial x'^a} - m_0 c\right)\psi'(x') = 0. \tag{3.40}$$

While $\psi(x)$ of A must be a solution of the Dirac equation of observer A:

$$\left(i\hbar\gamma^a \frac{\partial}{\partial x^a} - m_0 c\right)\psi(x) = 0. \tag{3.41}$$

Additionally, γ'^a of equation (3.40) have to satisfy the anticommutation relation:

$$\gamma'^a\gamma'^b + \gamma'^b\gamma'^a = 2\eta^{ab}I, \tag{3.42}$$

as otherwise A and B could distinguish their inertial systems. And also, the conditions of hermiticity and anti hermiticity (3.33) and (3.34) must hold in all inertial systems as well.

Another evident requirement of the principle of relativity is that both observers see real energy eigenvaluers. So the hamiltonians $\hat{H}$ from observer A and $\hat{H}'$ from observer B must be hermitians. So

$$(\hat{H}')^\dagger = \hat{H}', \tag{3.43}$$

$$(\hat{H})^\dagger = \hat{H}. \tag{3.44}$$

We can prove this result if we rewrite (3.40) as:

$$i\hbar\gamma'^0\frac{\partial\psi'(x')}{\partial x'^0} = \left(-i\hbar\gamma'^k\frac{\partial}{\partial x'^k} + m_0 c\right)\psi'(x'), \tag{3.45}$$

or also:

$$i\hbar\frac{\partial\psi'(x')}{\partial x'^0} = \left(-i\hbar\gamma'^0\gamma'^k\frac{\partial}{\partial x'^k} + \gamma'^0 m_0 c\right)\psi'(x') = \hat{H}'\psi'(x'). \tag{3.46}$$

But that is the Dirac equation in the Schroedinger form, where the hamiltonian is:

$$\hat{H}' = \left(-i\hbar\gamma'^0\gamma'^k\frac{\partial}{\partial x'^k} + \gamma'^0 m_0 c\right). \tag{3.47}$$

In order for this hamiltonian to be hermitian, both γ'^0 and $\gamma'^0\gamma'^k$ must be hermitian. The first one is already guaranteed by equation (3.34), the second one, on the other hand, can be showed by taking its hermitian adjoint and using (3.42). Finally, it is demonstrated that both $\hat{H}'$ and $\hat{H}$, for B and A respectively, are hermitian, so their eigenvalues are real for both observers as required by the principle of relativity.

Furthermore, it can be shown by means of a rather long proof that all matrices γ'^a which satisfy:

$$\gamma'^a\gamma'^b + \gamma'^b\gamma'^a = 2\eta^{ab}I, \tag{3.48}$$

are identical up to an unitary transformation $\hat{U}$, that is:

$$\gamma'^a = \hat{U}^\dagger\gamma^a\hat{U}, \tag{3.49}$$

where $\hat{U}^\dagger = \hat{U}^{-1}$.

One of these sets $\{\gamma^a\}$ of four gamma matrices which satisfies (3.48) is the set of the Dirac gamma matrices from the standart representation (3.35) and (3.36), given by:

$$\{\gamma^a\} = \{\gamma^0, \gamma^i\} = \left\{ \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} ; \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix} \right\} \tag{3.50}$$

for $i = 1, 2, 3$.

However, we know that unitary transformations do not change the physics, so we can choose our set of gamma matrices $\{\gamma'^a\}$ associated with the Dirac equation (3.40) of observer B to be the same set of gamma matrices $\{\gamma^a\}$ as in the Lorentz system of observer A. So

$$\{\gamma'^a\} = \{\gamma^0, \gamma^i\} = \left\{ \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} ; \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix} \right\} \tag{3.51}$$

for $i = 1, 2, 3$.

Therefore, we shall no longer differentiate $\{\gamma'^a\}$ and $\{\gamma^a\}$, since we can always use:

$$\gamma'^a = \gamma^a. \tag{3.52}$$

Thus, the Dirac equation for the Lorentz system of the observer B (Eq. (3.40)) is now given by:

$$\left(i\hbar\gamma^a \frac{\partial}{\partial x'^a} - m_0 c\right)\psi'(x') = 0. \tag{3.53}$$

Lorentz Transformation for Spinors

Let A and B be two inertial systems such that A measures a state $\psi(x)$ while B measures the same state as $\psi'(x')$. We want to relate $\psi(x)$ for A with $\psi'(x')$ for B.

We know that the coordinates for the events measured by A and B are respectively: (x^a) and (x'^a), which are related by the Lorentz transformation for coordinates:

$$x'^a = L^a_b x^b, \tag{3.54}$$

which can be represented by:

$$\begin{aligned} x' &= \hat{L}x, \\ x &= \hat{L}^{-1}x'. \end{aligned} \tag{3.55}$$

This transformation is linear in the coordinates, and since the Dirac equation:

$$\left(i\hbar\gamma^a \frac{\partial}{\partial x^a} - m_0 c\right)\psi(x) = 0. \tag{3.56}$$

is linear too, we must expect that the transformation between $\psi(x)$ from A and $\psi'(x')$ from B is linear too.

Hence, this transformation must have the form:

$$\psi'(x') = \hat{S}(\hat{L})\psi(x), \tag{3.57}$$

or

$$\psi'(x') = \psi'(\hat{L}x) = \hat{S}(\hat{L})\psi(x) = \hat{S}(\hat{L})\psi(\hat{L}^{-1}x'), \tag{3.58}$$

where $\hat{L}$ denotes the matrix of Lorentz transformation, and $\hat{S}(\hat{L})$ is a 4×4 matrix which is function of the parameters of Lorentz transformation for coordinates L_b^a and which operates upon the four component spinor $\psi(x)$. Then

$$\hat{S}(\hat{L}) \equiv S_a^b(L_d^c). \tag{3.59}$$

Through $\hat{L}$, $\hat{S}$ depends on the relative velocities and spatial orientations of A and B.

The principle of relativity stating the invariance of physical laws for all inertial systems implies the existence of the inverse operator $\hat{S}^{-1}(\hat{L})$, that enables the observer A to construct his wave function $\psi(x)$ from the $\psi'(x')$ of observer B.

$$\psi(x) = \hat{S}^{-1}(\hat{L})\psi'(x'), \tag{3.60}$$

or,

$$\psi(x) = \psi(\hat{L}^{-1}x') = \hat{S}^{-1}(\hat{L})\psi'(x') = \hat{S}^{-1}(\hat{L})\psi'(\hat{L}x). \tag{3.61}$$

Then, we have:

$$\begin{aligned} \psi'(x') &= \psi'(\hat{L}x) = \hat{S}(\hat{L})\psi(x) = \hat{S}(\hat{L})\psi(\hat{L}^{-1}x'), \\ \psi(x) &= \psi(\hat{L}^{-1}x') = \hat{S}^{-1}(\hat{L})\psi'(x') = \hat{S}^{-1}(\hat{L})\psi'(\hat{L}x). \end{aligned} \tag{3.62}$$

However, from this first equation above we can also write:

$$\psi'(\hat{L}x) = \hat{S}(\hat{L})\psi(\hat{L}^{-1}\hat{L}x)$$

So,

$$\begin{aligned}\psi'(\hat{L}^{-1}x') &= \hat{S}(\hat{L}^{-1})\psi(\hat{L}\hat{L}^{-1}x'),\\ \psi'(x) &= \hat{S}(\hat{L}^{-1})\psi(\hat{L}x),\\ \psi(x) &= \hat{S}(\hat{L}^{-1})\psi'(x'). \end{aligned} \tag{3.63}$$

But comparing this result with the second equation of (3.62), we get that:

$$\hat{S}(\hat{L}^{-1}) = \hat{S}^{-1}(\hat{L}). \tag{3.64}$$

So, our aim is to construct $\hat{S}$ fulfilling all the conditions (3.62) and (3.64). In order to do that, we start from the Dirac equation of observer A:

$$\left(i\hbar\gamma^a \frac{\partial}{\partial x^a} - m_0 c\right)\psi(x) = 0,$$

and we use that:

$$\psi(x) = \hat{S}^{-1}(\hat{L})\psi'(x'),$$

so we get:

$$\left(i\hbar\gamma^a \frac{\partial}{\partial x^a} - m_0 c\right)\hat{S}^{-1}(\hat{L})\psi'(x') = 0,$$

or,

$$\left(i\hbar\gamma^a \hat{S}^{-1}(\hat{L})\frac{\partial}{\partial x^a} - m_0 c\hat{S}^{-1}(\hat{L})\right)\psi'(x') = 0.$$

Multiplying by $\hat{S}(\hat{L})$ from the left, and using that $\hat{S}(\hat{L})\hat{S}^{-1}(\hat{L}) = I$, we are led to:

$$\left(i\hbar\hat{S}(\hat{L})\gamma^a \hat{S}^{-1}(\hat{L})\frac{\partial}{\partial x^a} - m_0 c\right)\psi'(x') = 0.$$

But transforming $\frac{\partial}{\partial x^a}$ to the coordinates of the system B yields:

$$\frac{\partial}{\partial x^a} = L^b_a \frac{\partial}{\partial x'^b}. \tag{3.65}$$

Thus

$$\left(i\hbar \hat{S}(\hat{L}) \gamma^a \hat{S}^{-1}(\hat{L}) L^b_a \frac{\partial}{\partial x'^b} - m_0 c \right) \psi'(x') = 0,$$

or, finally:

$$\left(i\hbar \left(\hat{S}(\hat{L}) \gamma^a \hat{S}^{-1}(\hat{L}) L^b_a \right) \frac{\partial}{\partial x'^b} - m_0 c \right) \psi'(x') = 0. \tag{3.66}$$

However, comparing this equation with the Dirac equation for the Lorentz system of observer B, which is

$$\left(i\hbar \gamma^b \frac{\partial}{\partial x'^b} - m_0 c \right) \psi'(x') = 0,$$

we must have that $\hat{S}(\hat{L})$ satisfies:

$$\left(\hat{S}(\hat{L}) \gamma^a \hat{S}^{-1}(\hat{L}) L^b_a \right) = \gamma^b, \tag{3.67}$$

or equivalently

$$\hat{S}(\hat{L}) \gamma^a \hat{S}^{-1}(\hat{L}) = L^a_b \gamma^b. \tag{3.68}$$

This is the fundamental relation determining the operator $\hat{S}$, to find $\hat{S}$ means solving equations (3.68).

Once we have shown that there exists a solution $\hat{S}(\hat{L})$ for (3.68) and have found it, we will have proven the covariance of the Dirac equation!

We may already specify the definition of a spinor:

Definition: A wave function $\psi(x)$ is termed a four-component Lorentz spinor if it transforms according to

$$\psi'(x') = \hat{S}(\hat{L})\psi(x), \tag{3.69}$$

where $x' = \hat{L}x$, and $\hat{S}(\hat{L})$ satisfies the fundamental relation:

$$\hat{S}(\hat{L})\gamma^a \hat{S}^{-1}(\hat{L}) = L^a_b \gamma^b. \tag{3.70}$$

Such a four-component spinor is also frequently called bi-spinor, since it consists of two two-component spinors:

$$\psi = \begin{bmatrix} P_1 \\ Q_1 \end{bmatrix}, \text{ where } P_1 = \begin{bmatrix} \psi_0 \\ \psi_1 \end{bmatrix}, \text{ and } Q_1 = \begin{bmatrix} \psi_2 \\ \psi_3 \end{bmatrix}. \tag{3.71}$$

We will now discuss the procedure for obtaining the explicit form of the spinor transformation $\hat{S}(\hat{L})$, with $\hat{L}$ being some given specific Lorentz transformation.

In general, it is simplest to generate a continuous group transformation by constructing the group operator for infinitesimal transformations and then composing operators for finite transformations, such as rotations, translations, and more, by considering the application of these infinitesimal operators in series. Following the same pattern here, we will first construct the operator $\hat{S}(\hat{L})$ for infinitesimal proper Lorentz transformations, given by:

$$L^a_b = \delta^a_b + \Delta\omega^a_b, \tag{3.72}$$

where $\Delta\omega^{ab} = -\Delta\omega^{ba} = I^{ab}_n \Delta\omega$, with $\Delta\omega$ being the generalized infinitesimal angle of rotation, while I^{ab}_n is the unitary operator of rotation around the n-axis. We can see some examples of these infinitesimal Lorentz transformations in [74].

After that, we can finally construct $\hat{S}(\hat{L})$ for finite proper Lorentz transformations $\hat{L}$ by successive applications of the infinitesimal ones.

Let's start with $\hat{S}(\hat{L})$ for infinitesimal proper Lorentz transformations.

- **Construction of the $\hat{S}$ operator for infinitesimal Lorentz transformations**

We want to determine $\hat{S}(\hat{L})$, such that:

$$\psi'(x') = \hat{S}(\hat{L})\psi(x), \tag{3.73}$$

$$x'^a = L^a_b x^b, \tag{3.74}$$

where L^a_b describes an infinitesimal Lorentz transformation:

$$L^a_b = \delta^a_b + \Delta\omega^a_b = \delta^a_b + \Delta\omega(I_n)^a_b, \tag{3.75}$$

We can get that by expanding our operator $\hat{S}(\hat{L})$ in powers of $\Delta\omega^a_b$ and keeping only the linear terms of the infinitesimal generators. Then, we get:

$$\begin{aligned}\hat{S}(\hat{L}) &= \hat{S}(\hat{I} + \Delta\omega), \\ \hat{S}(\hat{L}) &= \hat{S}(\hat{I}) - \frac{i}{4}\hat{\sigma}_{ab}\Delta\omega^{ab},\end{aligned} \tag{3.76}$$

where the factor $-\frac{i}{4}$ was put for convinience, and the operator $\hat{\sigma}_{ab}$ is still unknown. This gives us:

$$\hat{S}(\hat{L}) = \hat{I} - \frac{i}{4}\hat{\sigma}_{ab}\Delta\omega^{ab}. \tag{3.77}$$

Since the inverse Lorentz transformation is made by changing $\Delta\omega \to -\Delta\omega$, we have that

$$\hat{S}^{-1}(\hat{L}) = \hat{I} + \frac{i}{4}\hat{\sigma}_{ab}\Delta\omega^{ab}. \tag{3.78}$$

As well as $\hat{I}$ and $\hat{S}(\hat{L})$, each of these coefficients $\hat{\sigma}_{ab}$ is a 4×4 matrix. By finding $\hat{\sigma}_{ab}$ we can determine the operator $\hat{S}(\hat{L})$.

So inserting these two expressions (3.77) and (3.78) as well as $L^a_b = \delta^a_b + \Delta\omega^a_b$ into the fundamental relation (3.70), we get after some algebra [74] that:

$$-2i\,(\delta^c_a\gamma_b - \delta^c_b\gamma_a) = [\hat{\sigma}_{ab}, \gamma^c]. \tag{3.79}$$

The problem of constructing $\hat{S}(\hat{L})$ that fulfills the fundamental relation (3.70) is now reduced to finding $\hat{\sigma}_{ab}$ which satisfies (3.79).

Since $\hat{\sigma}_{ab}$ is anti-symmetric in the indices a and b, it is natural to try an anti-symmetric product of two matrices γ^a:

$$\hat{\sigma}_{ab} = \frac{i}{2}[\gamma_a, \gamma_b], \tag{3.80}$$

and it can be shown that this form fulfills the requirement (3.79)[74].

In other words, for infinitesimal Lorentz transformations, the operator

$$\hat{S}(\hat{L}) = \hat{I} - \frac{i}{4}\hat{\sigma}_{ab}\Delta\omega^{ab}$$

fulfills the fundamental relation (3.70), when $\hat{\sigma}_{ab}$ is given by:

$$\hat{\sigma}_{ab} = \frac{i}{2}[\gamma_a, \gamma_b],$$

So finally, the operator $\hat{S}(\hat{L})$, for infinitesimal Lorentz transformations $L^a_b = \delta^a_b + \Delta\omega^a_b$, is given by:

$$\hat{S}(\hat{L}) = \hat{I} + \frac{1}{8}[\gamma_a, \gamma_b]\Delta\omega^{ab}. \tag{3.81}$$

The next step is to construct $\hat{S}(\hat{L})$ for finite Lorentz transformations by successive applications of the infinitesimal operators (3.81).

- **Construction of the $\hat{S}$ operator for finite Lorentz transformations**

Now we can construct the spinor transformation operator $\hat{S}(\hat{L})$ for a finite lorentz transformation;

$$\psi'(x') = \hat{S}(\hat{L})\psi(x), \tag{3.82}$$

$$x'^a = L^a_b x^b, \tag{3.83}$$

where now L^a_b is a finite Lorentz transformation.

We start from the operator for infinitesimal Lorentz transformations:

$$\hat{S}(\hat{L}) = \hat{I} - \frac{i}{4}\hat{\sigma}_{ab}\Delta\omega(I_n)^{ab}, \tag{3.84}$$

where $\Delta\omega$ is the infinitesimal rotation angle around a certain n-axis, $(I_n)^{ab}$ is the unit operator associated with this rotation, and $\hat{\sigma}_{ab}$ is given by:

$$\hat{\sigma}_{ab} = \frac{i}{2}[\gamma_a, \gamma_b], \tag{3.85}$$

and we apply this infinitesimal transformation N successive times, taking $N \to \infty$, in order to get:

$$\hat{S}(\hat{L}) = \lim_{N\to\infty}\left(\hat{I} - \frac{i}{4}\hat{\sigma}_{ab}\frac{\omega}{N}(I_n)^{ab}\right)^N. \tag{3.86}$$

However, this limit gives us:

$$\hat{S}(\hat{L}) = e^{-\frac{i}{4}\omega\hat{\sigma}_{ab}(I_n)^{ab}}. \tag{3.87}$$

Finally, this is the spinor transformation operator for a finite Lorentz transformation, and therefore:

$$\psi'(x') = e^{-\frac{i}{4}\omega\hat{\sigma}_{ab}(I_n)^{ab}}\psi(x). \tag{3.88}$$

By obtaining this operator $\hat{S}(\hat{L})$, which allows us to transform a spinor $\psi(x)$ from a certain Lorentz inertial frame into $\psi'(x')$ for any other Lorentz inertial frame, we have finally proved the Lorentz covariance of Dirac equation, which was the last requirement that this equation needed to fulfill in order to be our correct relativistic wave equation. With that, we finish this review of Dirac equation for special relativity and we can now discuss, in the following section, the procedure that generalizes this equation for curved spacetimes, allowing us to obtain a general relativistic Dirac equation.

3.2 Dirac Equation in Curved Spacetimes

As we previously saw in chapter 1, the Lorentzian character of general relativity spacetime always allows us to define local Lorentz frames, which consists of a local Lorentz system of coordinates $(\bar{x}^a)$ and the associated set of tetrads $\{e_{(a)}\}$, such that the metric assumes, locally, the Minkowski form. In other words, given any point P at the manifold, we can always choose a local Lorentz frame $(\bar{x}^a)$, associated with the tetrad basis $\{e_{(a)}\}$, in such a way that, locally at this point:

$$g\left(e_{(a)}, e_{(b)}\right) = \eta_{(a)(b)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{3.89}$$

where $e_{(a)} = e^{\alpha}_{(a)} \frac{\partial}{\partial x^{\alpha}}$.

Furthermore, associated with the basis $\{e_{(a)}\}$ of $\mathbf{T_P}$, we also have the local basis of one-forms $\{\theta^a\}$ of the dual space $\mathbf{T^*_P}$, such that:

$$\theta^a = e^{(a)}_{\alpha} dx^{\alpha}, \tag{3.90}$$

which allows the distance operator

$$g = g_{\alpha\beta} dx^{\alpha} dx^{\beta}, \tag{3.91}$$

to be written as:

$$g = \eta_{(a)(b)}\theta^a\theta^b. \tag{3.92}$$

This tetrad basis $\{e_{(a)}\}$ and this local basis of one-forms $\{\theta^a\}$ of $\mathbf{T_P}$ and $\mathbf{T_P^*}$, respectively, can always be chosen independently at every point P of spacetime.

We also saw in the section 1.3 that there exists an arbitrariness at this choice of the tetrad basis $\{e_{(a)}\}$, or equivalently, at the choice of the local basis $\{\theta^a\}$. The local expressions (3.89) and (3.92) remain unchanged by Local lorentz transformations:

$$\theta^a(P) \mapsto \bar{\theta}^a(P) = L^a_b(P)\theta^b(P), \tag{3.93}$$

with $L^a_b(P)$ such that

$$L^a_c(P)\eta_{(a)(b)}L^b_d(P) = \eta_{(c)(d)}, \tag{3.94}$$

It means that local Lorentz transformations don't change the Minkowski form of the metric (3.89) locally at the point P.

Thus, the tetrad basis $\{e_{(a)}\}$ and the local basis $\{\theta^a\}$ are defined with the arbitrariness of a local Lorentz transformation $L^a_b(P)$, which can be made independently at each and every point of the manifold.

Having in mind the Minkowskian structure of the spacetime, and the local Lorentz group of transformations $L^a_b(P)$, we can then transport, independently, for every point of the manifold, the whole theory of the Lorentz group representation. Accordingly, we can take at each point of the manifold, independently, a local Dirac spinorial structure!

So we define a Dirac spinor at the point P of a spacetime manifold, as a 4-component object:

$$\psi = \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{bmatrix}, \tag{3.95}$$

such that, under the group of local Lorentz transformations $\{L^a_b(P)\}$, it transforms as its correspondent in the flat spacetime (3.82):

$$\psi(x) \mapsto \psi'(x') = \hat{S}(\hat{L}(x))\psi(x), \tag{3.96}$$

where $\hat{S}(\hat{L}(x))$ is a 4×4 matrix which represents the local Lorentz transformation $L^a_b(x)$ for spinorial transformations.

We can also write for the correspondent hermitian conjugate:

$$\psi^\dagger(x) \mapsto \psi^{\dagger'}(x') = \psi^\dagger(x)\hat{S}^{-1}(\hat{L}(x)). \tag{3.97}$$

In terms of components, (3.96) and (3.97) are rewritten as:

$$\psi^a(x) \quad \mapsto \quad \psi'^a(x') = S^a_b(x)\psi^b(x), \tag{3.98}$$

$$\psi_a(x) \quad \mapsto \quad \psi'_a(x') = \psi_b(x)S^{b^{-1}}_a(x), \tag{3.99}$$

where:

$$S(x) \quad = \quad S(L(x)), \tag{3.100}$$

$$S^{-1}(x) \quad = \quad S^{-1}(L(x)). \tag{3.101}$$

On the other hand, under general coordinate transformations, i.e., under transformations:

$$x^\alpha \mapsto x'^\alpha = x'^\alpha(x), \tag{3.102}$$

the spinors transform as scalars:

$$\psi(x) \mapsto \psi'(x') = \psi(x). \tag{3.103}$$

So in this case, there is no relation between the group of general coordinate transformations over the entire manifold, and the group of local Lorentz transformations at a specific point P in the manifold, differently from the plane spacetime of the special relativity, where the group of lorentz transformations $\{L^a_b\}$ could construct a general representation of the group of general coordinate transformations over the entire manifold:

$$x'^a = L^a_b x^b. \tag{3.104}$$

Thus, in the case of a curved spacetime, the local Lorentzian structure exists independently at each and every point of the manifold, and the group of local Lorentz transformations $\{L^a_b(x)\}$ cannot construct a representation of the group of general coordinate transformations over the entire manifold.

Having understood these conceptual differences between the Dirac spinor for a flat spacetime, i.e., the Minkowski spacetime, and for a curved spacetime, we can now start the procedure which aims to obtain the general relativistic formulation of the Dirac equation. We start by defining the Dirac gamma matrices for curved spacetimes.

3.2.1 Dirac gamma matrices for curved spacetimes

Let the constant Dirac gamma matrices $\{\gamma^{(a)}\}$ be given by (3.50), where this set of matrices satisfies the relation:

$$\gamma^a \gamma^a + \gamma^b \gamma^a = 2\eta^{ab} I.$$

These matrices constitute a representation for the Clifford Algebra, at the basis of the Dirac spinors, associated to the Minkowski metric η^{ab}.

Using now the tetrad coefficients $(e^{\alpha}_{(a)})$, we can define, over the manifold, the matrices fields:

$$\gamma^{\mu}(x) = e^{\mu}_{(a)}(x)\gamma^{(a)}, \tag{3.105}$$

which in turn satisfy the following anti commutation relation:

$$\gamma^{\mu}(x)\gamma^{\nu}(x) + \gamma^{\nu}(x)\gamma^{\mu}(x) = 2g^{\mu\nu}(x)I, \tag{3.106}$$

for every point at the manifold.

These matrices fields $\gamma^{\mu}(x)$ constitute a Clifford Algebra associated to the manifold metric $g^{\mu\nu}(x)$.

Under the group of local Lorentz transformations, it is easy to show that the matrices fields $\gamma^{\mu}(x)$ transform as:

$$\gamma'^{\mu}(x) = S(x)\gamma^{\mu}(x)S^{-1}(x), \tag{3.107}$$

which in terms of the constant gamma matrices gives us the relation (3.70):

$$S(x)\gamma^{a}S^{-1}(x) = L^{a}_{b}\gamma^{b}. \tag{3.108}$$

On the other hand, under the group of general coordinate transformations:

$$x^{\alpha} \mapsto x'^{\alpha} = x'^{\alpha}(x), \tag{3.109}$$

the gamma matrices fields $\gamma^{\mu}(x)$ transform themselves as a contravariant 4-vector:

$$\gamma^{\mu}(x) \mapsto \gamma'^{\mu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\nu}}\gamma^{\nu}(x). \tag{3.110}$$

3.2.2 Covariant derivative of a spinor

We know that in order to generalize a special relativistic equation which contains derivatives for the general relativistic scenario, it is necessary to use the following usual prescription:

$$\partial_\alpha \mapsto \nabla_\alpha, \tag{3.111}$$

where ∇_α is a covariant derivative, i.e., a derivative which transforms itself as a covariant vector by general coordinate transformations. However, in this context here, we need our desired Dirac equation to be covariant under not only general coordinate transformations, but under local Lorentz transformations as well. So, in order to get that, we will need to define a covariant derivative of a spinor such that it transforms itself as a covariant vector under general coordinate transformations, and as a spinor under local Lorentz transformations.

We have previously seen that a spinor ψ^a transforms itself by the following rule for local Lorentz transformations for spinors:

$$\psi^a \mapsto \psi'^a(x') = (S(L(x)))^a_b\, \psi^b(x), \tag{3.112}$$

where $L(x)$ is a local Lorentz transformation.

The covariant derivative of a spinor:

$$\nabla_\alpha \psi^a(x)$$

is a derivative of a scalar in relation to general coordinate transformations

$$x^\alpha \mapsto x'^\alpha = x'^\alpha(x),$$

due to (3.103). But in order for this covariant derivative of a spinor to be also a spinor in relation to local Lorentz transformations $\{L^a_b(x)\}$, it should transform itself as:

$$\nabla_\alpha \psi^a \mapsto \nabla_\alpha \psi'^a(x') = (S(L(x)))^a_b\, \nabla_\alpha \psi^b(x),$$

under local Lorentz transformations.

In this way, the covariant derivative of a spinor will be a spinor as well. However, in order to obtain the transformation rule above, we should have by definition:

$$\nabla_\alpha \psi \equiv \partial_\alpha \psi - \Gamma_\alpha \psi, \tag{3.113}$$

i.e.;

$$\nabla_\alpha \equiv \partial_\alpha - \Gamma_\alpha, \tag{3.114}$$

where the Γ_α are called spinorial connections.

In relation to general coordinate transformations on the manifold, Γ_α should be transformed as:

$$\Gamma_\alpha = \frac{\partial x^\beta}{\partial x'^\alpha} \Gamma_\beta, \tag{3.115}$$

in order for the covariant derivative (3.114) to be transformed as a covariant vector under general coordinate transformations.

However, in relation to local Lorentz transformations, Γ_α should be transformed as:

$$\Gamma'_\alpha = S(L(x))\Gamma_\alpha S^{-1}(L(x)) + \left(S(L(x))\right)_{,\alpha} S^{-1}(L(x)), \tag{3.116}$$

in order for the covariant derivative of a spinor (3.114) to be transformed as a spinor as well. The last term in the equation (3.117) above is the non-homogeneous term of the similarity transformation.

Finally, the covariant derivative of a spinor $\nabla_\alpha \psi$ is also a spinor:

$$\nabla_\alpha \psi^a \mapsto \nabla_\alpha \psi'^a(x') = \left(S(L(x))\right)^a_b \nabla_\alpha \psi^b(x), \tag{3.117}$$

Generalizing this procedure for any quantity with both spinorial and tensorial indexes, i.e., with strictly defined spinorial transformation laws for local Lorentz transformations at the spinorial indexes, and with strictly defined tensorial transformation laws for general coordinate transformations at the tensorial indexes, we have the following generalization of the covariant derivative:

$$\nabla_\alpha F^{\mu}{}_{\nu}{}^{a}{}_{b} = \partial_\alpha F^{\mu}{}_{\nu}{}^{a}{}_{b} + \{{}^{\mu}_{\alpha\beta}\} F^{\beta}{}_{\nu}{}^{a}{}_{b} + \{{}^{\beta}_{\alpha\nu}\} F^{\mu}{}_{\beta}{}^{a}{}_{b} - (\Gamma_\alpha)^a_c F^{\mu}{}_{\nu}{}^{c}{}_{b} + (\Gamma_\alpha)^c_b F^{\mu}{}_{\nu}{}^{a}{}_{c}, \tag{3.118}$$

where the quantities $\{{}^{\mu}_{\alpha\beta}\}$ are the Christoffel connections, and the quantities $F^{\mu}{}_{\nu}{}^{a}{}_{b}$ are such that:

- under spinorial transformations, i.e., under local Lorentz transformations, they transform as:

$$F'^{\mu}{}_{\nu}{}^{a}{}_{b} = [S(L(x))]^a_c [S^{-1}(L(x))]^d_b F^{\mu}{}_{\nu}{}^{c}{}_{d}; \tag{3.119}$$

- and under tensorial transformations, i.e., under general coordinate transformations, they transform as:

$$F'^{\mu}{}_{\nu}{}^{a}{}_{b} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} F^{\alpha}{}_{\beta}{}^{a}{}_{b}. \tag{3.120}$$

However, for a riemannian manifold, we must have that :

$$\nabla_\alpha g_{\mu\nu} = 0, \tag{3.121}$$

which according to (3.106) yields:

$$\nabla_\alpha \left(\gamma_\mu(x)\gamma_\nu(x) + \gamma_\nu(x)\gamma_\mu(x)\right) = 0. \tag{3.122}$$

A necessary condition for (3.122) is:

$$\nabla_\alpha \left(\gamma_\mu(x)\right) = 0, \tag{3.123}$$

but using (3.118), it gives us:

$$\partial_\alpha(\gamma_\mu)^a_b - \left\{{\beta \atop \alpha\mu}\right\}(\gamma_\beta)^a_b - (\Gamma_\alpha)^a_c(\gamma_\mu)^c_b + (\Gamma_\alpha)^c_b(\gamma_\mu)^a_c = 0, \tag{3.124}$$

which can be solved for Γ_α, yielding:

$$\Gamma_\alpha = -\frac{1}{8}\left[\gamma^\mu(\partial_\alpha\gamma_\mu) - (\partial_\alpha\gamma_\mu)\gamma^\mu - \left\{{\beta \atop \alpha\mu}\right\}(\gamma^\mu\gamma_\beta - \gamma_\beta\gamma^\mu)\right]. \tag{3.125}$$

These spinorial affinities Γ_α above are known as Fock-Invanenko coefficients, and they can also be written in a more convinient way, by using equations: (3.105), for the general relativistic gamma matrices; and the following definition for the Ricci-rotation coefficients:

$$\gamma_{(a)(b)(c)} = -e_{\mu(a);\nu}e^\mu_{(b)}e^\nu_{(c)}; \tag{3.126}$$

so we get that the spinorial affinities Γ_α are given by:

$$\Gamma_\alpha = \frac{1}{8}\gamma_{(b)(a)(c)}e^{(c)}_\alpha[\gamma^a, \gamma^b]. \tag{3.127}$$

Furthermore, by the convention $C_\alpha \equiv C_{(a)}e^{(a)}_\alpha$, we have that:

$$\gamma_{(b)(a)(c)}e^{(c)}_\alpha \equiv \gamma_{(b)(a)\alpha}, \tag{3.128}$$

and therefore:

$$\Gamma_\alpha = \frac{1}{8}\gamma_{(b)(a)\alpha}[\gamma^a, \gamma^b]. \tag{3.129}$$

However, by equation (3.85), we finally have that the spinorial affinities Γ_α, such that the covariant derivative (3.133) be a spinor, are given by the expression:

$$\Gamma_\alpha = \frac{i}{4}\gamma_{(a)(b)\alpha}\sigma^{(a)(b)}. \tag{3.130}$$

Finally, with the concept of covariant derivative defined by (3.113), with Γ_α given by (3.130), which is covariant with relation to local Lorentz transformations and with relation to general coordinate transformations; the special relativistic Dirac equation (3.56) can finally be generalized for a riemannian manifold by means of the following prescription:

$$\partial_\mu \mapsto \nabla_\mu = \partial_\mu - \Gamma_\mu. \tag{3.131}$$

The new Dirac equation, covariant under the group of local Lorentz transformations and the group of general coordinate transformations, also called general relativistic Dirac equation, is written as follows:

$$(i\hbar\gamma^\mu(x)\nabla_\mu - mc)\,\psi(x) = (i\hbar\gamma^\mu(x)(\partial_\mu - \Gamma_\mu) - mc)\,\psi(x) = 0, \tag{3.132}$$

where,

$$\Gamma_\mu = \frac{i}{4}\gamma_{(a)(b)\mu}\sigma^{(a)(b)}.$$

With this equation (3.132) above, we have achieved the main objective of this entire chapter. The only thing left for us to do is to apply this equation for the Kerr-Newman black hole with cloud of strings spacetime, what we will do in the next chapter in order to obtain the radial Dirac equation for a spin $\frac{1}{2}$ particle at the neighbourhood of a charged rotating black hole with cloud of strings. After that, we will rewrite this radial equation as a generalized Heun equation, what will allow us to obtain its solution for some specific and important regions of spacetime.

Chapter Four

DIRAC EQUATION IN A CLASS OF BLACK HOLES WITH A CLOUD OF STRINGS

4.1 Introduction

In this chapter we study the behavior of fermions on a class of black hole space-times surrounded by a cloud of strings. We analyze the solutions of both the angular and radial parts of the Dirac equation in the Ker-Newman space-time surrounded by a cloud of strings, and in the particular cases, namely, Kerr surrounded by a cloud of strings, Reissner-Nordström and Schwarzschild, also surrounded by a cloud of strings. In these investigations, the solutions obtained by Kraniotis [2] are generalized in such a way to include the cloud of strings. Therefore, the motivation of the present chapter is to investigate the signature of a cloud of strings on the solutions of the Dirac equation in these space-times of different black holes, rotating and non-rotating, charged and uncharged, surrounded by a cloud of strings.

4.2 Dirac equation in the Kerr-Newman black hole with a cloud of strings

As we have seen in Chapter 2, the generalisation of the Kerr-Newman solution to contemplate the effect of the cloud of strings is described by the metric element which in Boyer-Lindquist (BL) coordinates is given by (in units where G = 1 and c = 1):

$$ds^2 = \frac{\Delta_r}{\rho^2}(dt - asin^2\theta d\phi)^2 - \frac{\rho^2}{\Delta_r}dr^2 - \rho^2 d\theta^2 - \frac{sin^2\theta}{\rho^2}(adt - (r^2 + a^2)d\phi)^2, \tag{4.1}$$

with

$$\Delta_r = (r^2 + a^2) - br^2 - 2Mr + e^2, \tag{4.2}$$

$$\rho^2 = r^2 + a^2 cos^2\theta \tag{4.3}$$

where a, b, M, e, denote the angular momentum per unit mass, the cloud of strings parameter, mass and electric charge of the black hole, respectively. This is accompanied by a non-zero electromagnetic field $F = dA$, where the vector potential is given by Eq. (5) of Kraniotis [2]:

$$A = -\frac{er}{\rho^2}(dt - asin^2\theta d\phi). \tag{4.4}$$

This metric can be described in terms of a local Newman-Penrose [72] null tetrad frame that is defined by:

$$l^{\mu} = \left(\frac{r^2+a^2}{\Delta_r}, 1, 0, \frac{a}{\Delta_r}\right), \tag{4.5}$$

$$n^{\mu} = \left(\frac{r^2+a^2}{2\rho^2}, -\frac{\Delta_r}{2\rho^2}, 0, \frac{a}{2\rho^2}\right), \tag{4.6}$$

$$m^{\mu} = \frac{1}{\sqrt{2}(r+iacos\theta)}\left(iasin\theta, 0, 1, \frac{i}{sin\theta}\right), \tag{4.7}$$

$$\bar{m}^{\mu} = \frac{1}{\sqrt{2}(r-iacos\theta)}\left(-iasin\theta, 0, 1, -\frac{i}{sin\theta}\right). \tag{4.8}$$

Thus the Kinnersley null tetrad coefficients, in the matrix form, for the Kerr-Newman spacetime with a cloud of strings, is given by:

$$e^{\mu}_{(a)} = \begin{bmatrix} \frac{r^2+a^2}{\Delta_r} & \frac{r^2+a^2}{2\rho^2} & \frac{iasin\theta}{\sqrt{2}(r+iacos\theta)} & \frac{-iasin\theta}{\sqrt{2}(r-iacos\theta)} \\ 1 & -\frac{\Delta_r}{2\rho^2} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}(r+iacos\theta)} & \frac{1}{\sqrt{2}(r-iacos\theta)} \\ \frac{a}{\Delta_r} & \frac{a}{2\rho^2} & \frac{i}{(r+iacos\theta)sin\theta\sqrt{2}} & \frac{-i}{(r-iacos\theta)sin\theta\sqrt{2}} \end{bmatrix}. \tag{4.9}$$

The inverse matrix, on the other hand, is given by:

$$e^{(a)}_{\mu} = (e^{\mu}_{(a)})^{-1} = \begin{bmatrix} \frac{\Delta_r}{2\rho^2} & \frac{1}{2} & 0 & \frac{-asin^2\theta\Delta_r}{2\rho^2} \\ 1 & -\frac{\rho^2}{\Delta_r} & 0 & -asin^2\theta \\ \frac{iasin\theta}{\sqrt{2}(r-iacos\theta)} & 0 & \frac{\rho^2}{\sqrt{2}(r-iacos\theta)} & -\frac{i(a^2+r^2)sin\theta}{\sqrt{2}(r-iacos\theta)} \\ \frac{-iasin\theta}{\sqrt{2}(r+iacos\theta)} & 0 & \frac{\rho^2}{\sqrt{2}(r+iacos\theta)} & \frac{i(a^2+r^2)sin\theta}{\sqrt{2}(r+iacos\theta)} \end{bmatrix}. \tag{4.10}$$

Also, we can (get down) the tetrad index (a) by means of the product with the Newman-Penrose local metric, to obtain the following useful matrix:

$$e_{\mu(b)} = \eta_{(b)(a)} e^{(a)}_{\mu} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\Delta_r}{2\rho^2} & \frac{1}{2} & 0 & \frac{-asin^2\theta\Delta_r}{2\rho^2} \\ 1 & -\frac{\rho^2}{\Delta_r} & 0 & -asin^2\theta \\ \frac{iasin\theta}{\sqrt{2}(r-iacos\theta)} & 0 & \frac{\rho^2}{\sqrt{2}(r-iacos\theta)} & -\frac{i(a^2+r^2)sin\theta}{\sqrt{2}(r-iacos\theta)} \\ \frac{-iasin\theta}{\sqrt{2}(r+iacos\theta)} & 0 & \frac{\rho^2}{\sqrt{2}(r+iacos\theta)} & \frac{i(a^2+r^2)sin\theta}{\sqrt{2}(r+iacos\theta)} \end{bmatrix},$$

$$e_{\mu(b)} = \begin{bmatrix} 1 & -\frac{\rho^2}{\Delta_r} & 0 & -asin^2\theta \\ \frac{\Delta_r}{2\rho^2} & \frac{1}{2} & 0 & \frac{-asin^2\theta\Delta_r}{2\rho^2} \\ \frac{iasin\theta}{\sqrt{2}(r+iacos\theta)} & 0 & -\frac{\rho^2}{\sqrt{2}(r+iacos\theta)} & -\frac{i(a^2+r^2)sin\theta}{\sqrt{2}(r+iacos\theta)} \\ \frac{-iasin\theta}{\sqrt{2}(r-iacos\theta)} & 0 & -\frac{\rho^2}{\sqrt{2}(r-iacos\theta)} & \frac{i(a^2+r^2)sin\theta}{\sqrt{2}(r-iacos\theta)} \end{bmatrix}. \quad (4.11)$$

4.2.1 Calculation of the Ricci-rotation coefficients in the Kerr-Newman spacetime with a cloud of strings

Using the matrices given in Eq. (4.9), (4.10) and (4.11) we can compute the Ricci-rotation coefficients:

$$\gamma_{(a)(b)(c)} = -e_{\mu(a);\nu} e^{\mu}_{(b)} e^{\nu}_{(c)}. \quad (4.12)$$

However, it is easier to calculate these coefficients by means of the λ-symbols [75], related to $\gamma_{(a)(b)(c)}$ by the following formula:

$$\gamma_{(a)(b)(c)} = \frac{1}{2}(\lambda_{(a)(b)(c)} + \lambda_{(c)(a)(b)} - \lambda_{(b)(c)(a)}), \quad (4.13)$$

where these λ-symbols are defined by the following equation:

$$\lambda_{(a)(b)(c)} = e_{\alpha(b),\beta}(e^{\alpha}_{(a)} e^{\beta}_{(c)} - e^{\beta}_{(a)} e^{\alpha}_{(c)}). \quad (4.14)$$

The formula (4.13) has the advantage that we only have to calculate ordinary derivatives of the components of the matrix in Eq(4.11).

The non-vanishing λ-symbols are computed to be:

$$\lambda_{102} = -\frac{a^2 sin\theta cos\theta}{\rho^2 \bar{\rho}}, \tag{4.15}$$

$$\lambda_{213} = -\frac{iacos\theta \Delta_r}{\rho^4}, \tag{4.16}$$

$$\lambda_{132} = -\frac{\Delta_r}{2\rho^2 \bar{\rho}}, \tag{4.17}$$

$$\lambda_{123} = -\frac{\Delta_r}{2\rho^2 \bar{\rho}*}, \tag{4.18}$$

$$\lambda_{023} = \frac{1}{\bar{\rho}*}, \tag{4.19}$$

$$\lambda_{203} = \frac{-2iacos\theta}{\rho^2}, \tag{4.20}$$

$$\lambda_{011} = -\frac{1}{2}\frac{d\Delta_r}{dr}\frac{1}{\rho^2} + r\frac{\Delta_r}{\rho^4}, \tag{4.21}$$

$$\lambda_{021} = \frac{\sqrt{2}iarsin\theta}{\rho^2 \bar{\rho}}, \tag{4.22}$$

$$\lambda_{223} = \frac{cos\theta}{sin\theta\sqrt{2}\bar{\rho}} + \frac{iasin\theta}{\sqrt{2}\bar{\rho}^2}, \tag{4.23}$$

$$\lambda_{130} = \frac{ira\sqrt{2}sin\theta}{\rho^2 \bar{\rho}*}, \tag{4.24}$$

$$\lambda_{301} = \frac{a^2\sqrt{2}sin\theta cos\theta}{\rho^2 \bar{\rho}*}, \tag{4.25}$$

$$\lambda_{332} = \frac{cos\theta}{sin\theta\sqrt{2}\bar{\rho}*} - \frac{iasin\theta}{\sqrt{2}\bar{\rho}*^2}, \tag{4.26}$$

where $\bar{\rho} = r + iacos\theta$ and $\rho^2 = \bar{\rho}\bar{\rho}*$.

Therefore, a calculation through the λ-symbols provides the following non-vanishing rotation coefficients for the Kerr-Newman spacetime with a cloud of strings [72]:

$$\pi = \gamma_{130} = \frac{1}{2}\frac{iasin\theta\sqrt{2}}{(\bar{\rho}*)^2}, \tag{4.27}$$

$$\beta = \frac{1}{2}(\gamma_{102} + \gamma_{232}) = \frac{cos\theta}{2\sqrt{2}sin\theta\bar{\rho}}, \tag{4.28}$$

$$\gamma = \frac{1}{2}(\gamma_{101} + \gamma_{231}) = \frac{1}{4\rho^2}\frac{d\Delta_r}{dr} - \frac{1}{2\rho^2\bar{\rho}*}\Delta_r, \tag{4.29}$$

$$\alpha = \frac{1}{2}(\gamma_{103} + \gamma_{233}) = \pi - \beta*, \tag{4.30}$$

$$\rho = \gamma_{203} = -\frac{1}{\bar{\rho}*}, \tag{4.31}$$

$$\mu = \gamma_{132} = -\frac{\Delta_r}{2\bar{\rho}*\rho^2}, \tag{4.32}$$

$$\tau = \gamma_{201} = -\frac{iasin\theta\sqrt{2}}{2\rho^2}, \tag{4.33}$$

$$\epsilon = \frac{1}{2}(\gamma_{100} + \gamma_{230}) = 0. \tag{4.34}$$

4.2.2 2-spinor form of the Dirac equation in the Kerr-Newman spacetime with a cloud of strings

The general relativistic Dirac equation is given by (in units where $\hbar = c = 1$):

$$(i\gamma^\mu(x)\nabla_\mu - m)\psi = 0, \tag{4.35}$$

where

$$\nabla_\mu = \partial_\mu - \Gamma_\mu, \tag{4.36}$$

$$\gamma^\mu = e^\mu_{(a)}(x)\gamma^{(a)}, \tag{4.37}$$

and $\gamma^{(a)}$ are the constant Dirac matrices.

In order for the Dirac equation above remain consistent with the 2-spinor form of the Dirac equation as considered by Chandrasekhar [17, 18, 2], we shall use the complex version of the Weyl (chiral) representation, which is defined in the following way [76]:

- The four component Dirac-spinor is written as:

$$\psi(x) = \begin{bmatrix} P^A(x) \\ \bar{Q}_{B'}(x) \end{bmatrix}, \tag{4.38}$$

where P^A and $\bar{Q}_{B'}$ are two dimensional complex vectors in C^2 space.

- The Dirac gamma matrices are defined by the complex version of the Weyl (chiral) representation as

$$\gamma^\mu = \begin{bmatrix} 0 & (\widetilde{\sigma}^\mu)^* \\ (\sigma^\mu)^* & 0 \end{bmatrix} \tag{4.39}$$

where $\widetilde{\sigma}^\mu$ and σ^μ are the Van der Waarden symbols, defined for curved spacetimes as:

$$\sigma^\mu = \sqrt{2} \begin{bmatrix} l^\mu & m^\mu \\ \bar{m}^\mu & n^\mu \end{bmatrix}, \tag{4.40}$$

$$\widetilde{\sigma}^\mu = \sqrt{2} \begin{bmatrix} n^\mu & -m^\mu \\ -\bar{m}^\mu & l^\mu \end{bmatrix}. \tag{4.41}$$

In accordance with Chandrasekhar[17], who showed that the Dirac equation is separable in the Kerr black hole space time, the following 2-spinor form of the Dirac equation can be obtained:

$$(\nabla_{AB'} + iqA_{AB'})P^A + i\mu_* \bar{Q}_{B'} = 0, \tag{4.42}$$

$$(\nabla_{AB'} - iqA_{AB'})Q^A + i\mu_* \bar{P}_{B'} = 0, \tag{4.43}$$

where $\nabla_{AB'} = \sigma^\mu_{AB'} \nabla_\mu$; $A_{AB'} = \sigma^\mu_{AB'} A_\mu$, with A_μ being the eletromagnetic vector potential given by Eq.(4.4); q is the charge or the coupling constant of the massive Dirac particle to the eletromagnetic vector field; and μ_* is the particle mass.

The components for $B' = 0, 1$ in $Eq.(4.42)$ lead to the following general relativistic Dirac equations in the Newman-Penrose formalism for the Kerr-Newman black hole with cloud of strings spacetime [17, 18, 2]:

$$
\begin{aligned}
(\delta + \beta - \tau + iqm^{\mu}A_{\mu})P^{(0)} + (D' - \gamma + \mu + iqn^{\mu}A_{\mu})P^{(1)} &= -i\mu_{*}\bar{Q}^{(0)'}, && (4.44)\\
(-D + \rho - \epsilon - iql^{\mu}A_{\mu})P^{(0)} + (-\delta' + \alpha - \pi - iq\bar{m}^{\mu}A_{\mu})P^{(1)} &= -i\mu_{*}\bar{Q}^{(1)'}, && (4.45)
\end{aligned}
$$

where

$$
\begin{aligned}
D = l^{\mu}\partial_{\mu} &= \left(\partial_r + \frac{1}{\Delta_r}\left((r^2+a^2)\partial_t + a\partial_{\phi}\right)\right), && (4.46)\\
D' = n^{\mu}\partial_{\mu} &= -\frac{\Delta_r}{2\rho^2}\left(\partial_r - \frac{(r^2+a^2)}{\Delta_r}\partial_t - \frac{a}{\Delta_r}\partial_{\phi}\right), && (4.47)\\
\delta = m^{\mu}\partial_{\mu} &= \frac{1}{\sqrt{2}\bar{\rho}}\left(\partial_{\theta} + i\left(asen\theta\partial_t + \frac{\partial_{\phi}}{sen\theta}\right)\right), && (4.48)\\
\delta' = \bar{m}^{\mu}\partial_{\mu} &= \frac{1}{\sqrt{2}\bar{\rho}*}\left(\partial_{\theta} - i\left(asen\theta\partial_t + \frac{\partial_{\phi}}{sen\theta}\right)\right), && (4.49)\\
iql^{\mu}A_{\mu} &= -\frac{iqer}{\Delta_r}, && (4.50)\\
iqn^{\mu}A_{\mu} &= -\frac{iqer}{2\rho^2}, && (4.51)\\
iqm^{\mu}A_{\mu} &= 0, && (4.52)\\
iq\bar{m}^{\mu}A_{\mu} &= 0, && (4.53)
\end{aligned}
$$

and the coefficients β,τ,γ,μ,ρ,ϵ,α and π are given in Eq.(4.27) $-$ (4.34).

These equations are formally similar to the ones obtained by Kraniotis [2], but with the function $\Delta(r)$ [2] redefined in order to consider the presence of the cloud of strings, which we are calling $\Delta_r(r)$.

4.3 Separation of the general relativistic Dirac equation in the Kerr-Newman space-time with cloud of strings

Assuming that the azimuthal and time-dependence of the components of the spinor are of the form $e^{i(m\phi-\omega t)}$, and also applying the following ansatz:

$$P^{(0)} = e^{i(m\phi-\omega t)}\frac{S^{(-)}(\theta)R^{(-)}(r)}{\sqrt{2}\bar{\rho}*}, \quad (4.54)$$

$$P^{(1)} = e^{i(m\phi-\omega t)}\frac{S^{(+)}(\theta)R^{(+)}(r)}{\sqrt{\Delta_r}}, \quad (4.55)$$

$$\bar{Q}^{(0)'} = -e^{i(m\phi-\omega t)}\frac{S^{(+)}(\theta)R^{(-)}(r)}{\sqrt{2}\bar{\rho}}, \quad (4.56)$$

$$\bar{Q}^{(1)'} = e^{i(m\phi-\omega t)}\frac{S^{(-)}(\theta)R^{(+)}(r)}{\sqrt{\Delta_r}}, \quad (4.57)$$

we get the following ordinary differential equations for the radial and angular parts of the general relativistic Dirac equation in the Kerr-Newman black hole with a cloud of strings spacetime:

$$\frac{dR^{(+)}}{dr}(r) + i\left(\frac{(\omega(r^2+a^2)-ma)+qer}{\Delta_r}\right)R^{(+)}(r) = \frac{(\lambda-i\mu r)}{\sqrt{\Delta_r}}R^{(-)}(r), \quad (4.58)$$

$$\frac{dR^{(-)}}{dr}(r) - i\left(\frac{(\omega(r^2+a^2)-ma)+qer}{\Delta_r}\right)R^{(-)}(r) = \frac{(\lambda+i\mu r)}{\sqrt{\Delta_r}}R^{(+)}(r), \quad (4.59)$$

$$\frac{dS^{(+)}}{d\theta}(\theta) + \left[\frac{m}{sin\theta} - \omega asin\theta + \frac{1}{2}cot\theta\right]S^{(+)}(\theta) = (-\lambda + \mu acos\theta)S^{(-)}(\theta), \quad (4.60)$$

$$\frac{dS^{(-)}}{d\theta}(\theta) - \left[\frac{m}{sin\theta} - \omega asin\theta - \frac{1}{2}cot\theta\right]S^{(-)}(\theta) = (\lambda + \mu acos\theta)S^{(+)}(\theta), \quad (4.61)$$

where λ is a separation constant and $\mu = \mu_*\sqrt{2}$.

Equations (4.58) and (4.59) determine the behaviour of the radial functions $R^{+}(r)$ and $R^{-}(r)$, while Eqs. (4.60) and (4.61) describe the angular part of the spinor according to the ansatz expressed in Eqs. (4.54) to (4.57).

Let us consider, firstly, the solution of the angular part of the Dirac equation given by Eqs. (4.60) and (4.61). Combining these two equations, we obtain a second order equation, which is known as Chandrasekhar-Page equation, and is given by:

$$\begin{aligned}&\left[\frac{1}{sin\theta}\frac{d}{d\theta}\left(sin\theta\frac{d}{d\theta}\right)\mp\frac{a\mu sin\theta}{\lambda\mp\mu acos\theta}\frac{d}{d\theta}+\left(\frac{1}{2}\mp a\omega cos\theta\right)^2\right.\\&-\left(\frac{m\pm\frac{1}{2}cos\theta}{sin\theta}\right)^2+2a\omega m-a^2\omega^2\\&\left.\mp\frac{a\mu\left(\frac{1}{2}cos\theta\mp a\omega sin^2\theta\pm m\right)}{\lambda\mp a\mu cos\theta}-a^2\mu cos^2\theta+\lambda^2-\frac{3}{4}\right]S^{(\pm)}=0.\end{aligned} \tag{4.62}$$

As we can clearly see, this differential equation, i.e., the angular part of general relativistic Dirac equation for Kerr-Newman black hole with cloud of strings is analogous to the one obtained for the case of Kerr-Newman without cloud of strings (Eq.(40) in [2]). Therefore, all results obtained in Kraniotis [2] for the angular part of general relativistic Dirac equation should be valid, which means that the solutions are given in terms of the Generalized Heun Functions, which will be analyzed in the next section. Only the radial part of the general relativistic Dirac equation is affected by the presence of the cloud of strings in the black hole, what we will see in section 4.5 .

4.4 Solution of the angular equation

In order to obtain the solution of the angular part of the Dirac equation, given by Eq. (4.62), we will perform a change of variable, by defining a new variable $x = cos\theta$. Then, considering the equation for $S^{(-)}(\theta) = S^{(-)}(arccosx)$, it was obtained the following result [2]:

$$\begin{aligned}&\left\{(1-x^2)\frac{d^2}{dx^2}-\left[\frac{a\mu(1-x^2)}{\lambda+a\mu x}+2x\right]\frac{d}{dx}+\left[\frac{a\mu\left(\frac{x}{2}-m\right)+a^2\mu\omega(1-x^2)}{\lambda+a\mu x}\right.\right.\\&\left.\left.-\frac{m^2-mx+\frac{1}{4}}{1-x^2}+a^2(\omega^2-\mu^2)x^2+a\omega x+2am\omega-a^2\omega^2-\frac{1}{4}+\lambda^2\right]\right\}S^{(-)}(x)=0.\end{aligned} \tag{4.63}$$

Note that Eq. (4.63) has singularities at $x = \pm 1$ and $-\frac{\lambda}{a\mu}$, which will be identified with a_1, a_2, a_3, namely, $(a_1, a_2, a_3) = (-1, 1, -\frac{\lambda}{a\mu})$. We can write Eq. (1) in a different form by using the new variable z, such that

$$z = \frac{x - a_1}{a_2 - a_1} = \frac{x+1}{2},$$

and mapping the singularity a_3 into z_3, which is given by

$$z_3 = \frac{a_3 - a_1}{a_2 - a_1} = \frac{-\frac{\lambda}{a\mu} + 1}{2},$$

equation (4.63) is transformed into the following equation:

$$\begin{aligned}&\left\{\frac{d^2}{dz^2} + \left[\frac{1}{z} + \frac{1}{z-1} - \frac{1}{z-z_3}\right]\frac{d}{dz} + 4a^2(\mu^2-\omega^2) + \frac{a^2(\mu^2-\omega^2)}{z-1} - \frac{a^2(\mu^2-\omega^2)}{z}\right.\\ &+\frac{1}{16}\frac{-4m^2-4m-1}{z^2} + \frac{1}{8}\frac{4m^2+1}{z-1} + \frac{1}{16}\frac{-4m^2+4m-1}{(z-1)^2} + \frac{1}{8}\frac{-4m^2-1}{z}\\ &+\frac{1}{4}\frac{8a\omega z_3^2 - 8a\omega z_3 + 2m - 2z_3 + 1}{z_3(z_3-1)(z-z_3)} + \frac{-2m+1}{(z-1)(-4+4z_3)} + \frac{1}{4}\frac{2m+1}{z_3 z}\\ &\left.+\frac{1}{4}\frac{4a^2\omega^2 - 8am\omega - 4a\omega - 4\lambda^2 + 1}{z-1} + \frac{1}{4}\frac{-4a^2\omega^2 + 8am\omega + 4a\omega + 4\lambda^2 - 1}{z}\right\} S(z) = 0,\end{aligned} \tag{4.64}$$

where $S(z) = S^{(-)}(2z-1)$.

Now, let us perform a transformation in order to reduce the power of the terms $\frac{1}{z^2}$ and $\frac{1}{(z-1)^2}$. This is achieved by transforming $S(z)$ into $u(z)$, such that

$$S(z) = e^{\alpha_1 z} z^{\alpha_2}(z-1)^{\alpha_3}(z-z_3)^{\alpha_4}u(z), \tag{4.65}$$

where α_i, with $(i = 1, 2, 3, 4)$, are given by[2]:

$$
\begin{aligned}
\alpha_1 &= \pm 2ia\sqrt{\mu^2-\omega^2},\\
\alpha_2 &= \frac{1}{2}|m+\frac{1}{2}|,\\
\alpha_3 &= \frac{1}{2}|m-\frac{1}{2}|,\\
\alpha_4 &= 0.
\end{aligned}
$$

The resulting equation for $u(z)$ can be written as

$$
\left[\frac{d^2}{dz^2}+\left(\frac{2\alpha_2+1}{z}+\frac{2\alpha_3+1}{z-1}-\frac{1}{z-z_3}\pm 2\alpha_1\right)\frac{d}{dz}+\sum_{i=1}^{3}\frac{C_i}{z-z_i}\right]u(z)=0, \tag{4.66}
$$

with

$$
\sum_{i=1}^{3} C_i = 0
$$

.

Equation (4.66) corresponds to a generalized Heun equation in a representation adopted by Schäfke and Schmidt [77]. The Heun equation is a second order equation with four regular singularities and represents a direct generalization of the hypergeometric equation. As particular cases of Heun equation, we can mention the Gauss hypergeometric, bessel, Legendre and Laguerre equations wich are very important equations in mathematical physics.

Now, let us consider $\mu = \omega$, and consider the representation adopted by Erdélyi [78], which can be written as:

$$
\frac{d^2u}{dz^2}+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\epsilon}{z-z_3}\right)\frac{du}{dz}+\frac{\alpha\beta z-q}{z(z-1)(z-z_3)}u=0, \tag{4.67}
$$

where the parameters $\alpha, \beta, \gamma, \delta, \epsilon, q$ and z_3 are complex numbers, in general, and arbitraries. They are related by:

$$
\gamma+\delta+\epsilon=\alpha+\beta+1.
$$

Equation (4.67) has four singularities, at $z = 0, 1, z_3$ and infinity. Their exponents are $\{0, 1-\gamma\}$, $\{0, 1-\delta\}$, $\{0, 1-\epsilon\}$ and $\{\alpha, \beta\}$, respectively. The sum of these exponents should be equal to 2.

Taking $\mu = \omega$, and then comparing Eqs. (4.66) and (4.67), we conclude that the coefficients C_i depend only on λ, a and μ. Therefore, there is no explicit dependence on the parameter that identifies the presence of the cloud of strings b. Otherwise, as was shown by Carter and McLenaghan [79], the constant λ is an eigenvalue of the square root of the total angular momentum operator, which in this case depends on the presence of the strings [58].

As a conclusion we can say that the solution of equation (4.67) is given by[80]:

$$u(z) = D_1 \mathrm{HeunG}(z_3, C_1, C_2, C_3, \alpha_2, \alpha_3; z) + D_2 z^{1-\alpha_2} \mathrm{HeunG}(z_3, C_1', C_2', C_3', \alpha_2', \alpha_3; z),$$

where C_1', C_2', C_3' and α_2' are appropriate combinations of C_1, C_2, C_3 and α_2.

It is worth to call attention to the fact that the solution given above does not depend explicitly on the parameter that codifies the presence of the cloud of strings, but it can depend implicitly on this parameter which enters in the definition of the eigenvalues of the angular momentum operator. These conclusions are the same for Kerr; Reissner-Nordstrom and Schwarzschild black holes with a cloud of strings.

Concerning the solutions for the specific cases of Reissner-Nordstrom and Schwarzschild black holes with a cloud of strings, which means that $a = 0$. In this case, the solutions are the same obtained by Dolan and Gair [20] and are given in terms of the 1/2 spin-weighted spherical harmonics [81], which have a relation with the Generalized Heun functions.

To close this section, let us remind that the solutions of the angular part of the Dirac equation for the class of black holes obtained from the Kerr-Newman black hole with a cloud of strings, by taking the electric charge equal to zero, the angular momentum equal to zero, or both quantities equal to zero, do not depend explicitly on the presence of the cloud of strings, but only implicitly through the eigenvalues of the total angular momentum operator of the fermionic particle. On the other hand, the radial part of the general relativistic Dirac equation is affected explicitly by the presence of the cloud of strings in the black hole, what we will discuss in details from the following section onwards.

4.5 Solution of the radial equation

In this section, we follow straight forwadly the development made by Kraniotis [2] from the formal point of view. The qualitative analyzes associated to the presence of a cloud of strings for all black hole configurations are given from the obtained results.

Combining the equations (4.58) and (4.59) in a similar way to what we have done for the angular equation, we obtain the following radial equation for the $R^{(-)}$ mode:

$$\begin{aligned} &\Delta_r \frac{d^2 R_{-\frac{1}{2}}(r)}{dr^2} + \left(\frac{1}{2}\frac{d\Delta_r}{dr} - \frac{i\mu\Delta_r}{\lambda + i\mu r}\right)\frac{dR_{-\frac{1}{2}}(r)}{dr} \\ &+ \left(K^2 + \frac{i}{2}K\frac{d\Delta_r}{dr}\right)\frac{1}{\Delta_r}R_{-\frac{1}{2}}(r) - \frac{\mu K}{\lambda + i\mu r}R_{-\frac{1}{2}}(r) \\ &+ \left(-2i\omega r - iqe - \lambda^2 - \mu^2 r^2\right)R_{-\frac{1}{2}}(r) = 0, \end{aligned} \tag{4.68}$$

where $K = K(r) \equiv (r^2 + a^2)\omega - ma + eqr$, and $R_{-\frac{1}{2}}(r) \equiv R^{(-)}(r)$.

However, we know that Δ_r is defined in Eq.(4.2), so if we differentiate that expression, we can rewrite (4.68) as follows:

$$\begin{aligned} &\frac{d^2 R_{-\frac{1}{2}}(r)}{dr^2} + \left(\frac{(1-b)r - M}{\Delta_r} - \frac{i\mu}{\lambda + i\mu r}\right)\frac{dR_{-\frac{1}{2}}(r)}{dr} \\ &+ \left[\frac{K^2 + iK((1-b)r - M)}{\Delta_r^2}\right]R_{-\frac{1}{2}}(r) - \frac{\mu K}{\Delta_r(\lambda + i\mu r)}R_{-\frac{1}{2}}(r) \\ &+ \frac{(-2i\omega r - iqe - \lambda^2 - \mu^2 r^2)}{\Delta_r}R_{-\frac{1}{2}}(r) = 0. \end{aligned} \tag{4.69}$$

In terms of its roots, Δ_r can be written as:

$$\Delta_r = (1-b)[(r - r_+)(r - r_-)], \tag{4.70}$$

where

$$r_{\pm} = \frac{M}{(1-b)} \pm \sqrt{\frac{M^2}{(1-b)^2} - \frac{e^2+a^2}{(1-b)}}, \tag{4.71}$$

so finally the radial equation Eq. (4.69) takes the form:

$$\begin{aligned} &\frac{d^2R_{-\frac{1}{2}}(r)}{dr^2} + \left(\frac{(1-b)r - M}{(1-b)[(r-r_+)(r-r_-)]} - \frac{i\mu}{\lambda + i\mu r}\right)\frac{dR_{-\frac{1}{2}}(r)}{dr} \\ &+ \left[\frac{K^2 + iK((1-b)r - M)}{[(1-b)((r-r_+)(r-r_-))]^2}\right]R_{-\frac{1}{2}}(r) - \frac{\mu K}{(1-b)[(r-r_+)(r-r_-)](\lambda + i\mu r)}R_{-\frac{1}{2}}(r) \\ &+ \frac{(-2i\omega r - iqe - \lambda^2 - \mu^2 r^2)}{(1-b)[(r-r_+)(r-r_-)]}R_{-\frac{1}{2}}(r) = 0. \end{aligned} \tag{4.72}$$

Now, we will apply the following transformation of variables to the equation (4.72) [80]:

$$z = \frac{r - r_-}{r_+ - r_-}, \tag{4.73}$$

which transforms the radii of the Cauchy horizon r_- and the event horizon r_+ to the points $z = 0$ and $z = 1$, respectively, and the singularity at $r_3 = \frac{i\lambda}{\mu}$ to the point $z_3 = \frac{r_3 - r_-}{r_+ - r_-}$.

With this transformation, the Eq. (4.72) becomes:

$$
\begin{aligned}
&\frac{d^2R(z)}{dz^2} + \left[\left(\frac{r_+ - \frac{M}{(1-b)}}{r_+ - r_-}\right)\frac{1}{z-1} - \left(\frac{r_- - \frac{M}{(1-b)}}{r_+ - r_-}\right)\frac{1}{z} - \frac{1}{z-z_3}\right]\frac{dR(z)}{dz} \\
&+ \frac{1}{z}\left(iq\frac{e}{(1-b)} + 2ir_-\frac{\omega}{(1-b)} + \frac{\lambda^2}{(1-b)}\right)R(z) + \frac{1}{z-1}\left(-iq\frac{e}{(1-b)} - 2ir_+\frac{\omega}{(1-b)} - \frac{\lambda^2}{(1-b)}\right)R(z) \\
&- \frac{r_+^2}{z-1}\frac{\mu^2}{(1-b)}R(z) + \frac{r_-^2}{z}\frac{\mu^2}{(1-b)}R(z) + \left(\frac{\omega^2}{(1-b)^2} - \frac{\mu^2}{(1-b)}\right)(r_+ - r_-)^2 R(z) \\
&- \frac{1}{z}\frac{i\frac{\mathcal{H}_-}{(1-b)}}{z_3(r_- - r_+)}R(z) + \frac{1}{z-1}\frac{i\frac{\mathcal{H}_+}{(1-b)}}{(z_3-1)(r_- - r_+)}R(z) \\
&- \frac{1}{z-z_3}\frac{\left[i\left(\frac{\omega}{(1-b)}(r_+-r_-)^2 z_3^2 + 2r_-(r_+-r_-)\frac{\omega}{(1-b)}z_3 + q(r_+-r_-)\frac{e}{(1-b)}z_3 + \frac{\mathcal{H}_-}{(1-b)}\right)\right]}{(r_- - r_+)z_3(z_3-1)}R(z) \\
&+ \frac{1}{(r_+-r_-)^2}\frac{1}{z}\Bigg[i\Bigg(a^2(r_++r_-)\frac{\omega}{(1-b)} - 2(a^2+r_+r_-)\frac{M}{(1-b)}\frac{\omega}{(1-b)} + r_-^2(3r_+-r_-)\frac{\omega}{(1-b)} \\
&+ \left(2a\frac{m}{(1-b)} - q(r_-+r_+)\frac{e}{(1-b)}\right)\frac{M}{(1-b)} - a(r_++r_-)\frac{m}{(1-b)} + 2qr_+r_-\frac{e}{(1-b)}\Bigg) \\
&+ 2qa(r_++r_-)\frac{e}{(1-b)}\left(a\frac{\omega}{(1-b)} - \frac{m}{(1-b)}\right) + 2q^2r_+r_-\frac{e^2}{(1-b)^2} + 2qr_-^2(3r_+-r_-)\frac{e}{(1-b)}\frac{\omega}{(1-b)} \\
&+4r_+r_-(a^2+r_-^2)\frac{\omega^2}{(1-b)^2} + 2a^2\frac{m^2}{(1-b)^2} - 4a(a^2+r_+r_-)\frac{\omega}{(1-b)}\frac{m}{(1-b)} + 2(a^4-r_-^4)\frac{\omega^2}{(1-b)^2}\Bigg] R(z) \\
&+ \frac{1}{(r_+-r_-)^2}\frac{1}{z-1}\Bigg[i\Bigg(-a^2(r_++r_-)\frac{\omega}{(1-b)} + 2(a^2+r_+r_-)\frac{M}{(1-b)}\frac{\omega}{(1-b)} + r_+^2(r_+-3r_-)\frac{\omega}{(1-b)} \\
&+q(r_-+r_+)\frac{e}{(1-b)}\frac{M}{(1-b)} + a\left(r_++r_- - 2\frac{M}{(1-b)}\right)\frac{m}{(1-b)} - 2qr_+r_-\frac{e}{(1-b)}\Bigg) \\
&- 2qa(r_++r_-)\frac{e}{(1-b)}\left(a\frac{\omega}{(1-b)}-\frac{m}{(1-b)}\right) - 2q^2r_+r_-\frac{e^2}{(1-b)^2} + 2qr_+^2(r_+-3r_-)\frac{e}{(1-b)}\frac{\omega}{(1-b)} \\
&-4r_+r_-(a^2+r_+^2)\frac{\omega^2}{(1-b)^2} - 2a^2\frac{m^2}{(1-b)^2} + 4a(a^2+r_+r_-)\frac{\omega}{(1-b)}\frac{m}{(1-b)} - 2(a^4-r_+^4)\frac{\omega^2}{(1-b)^2}\Bigg]R(z) \\
&+ \frac{1}{(r_+-r_-)^2}\frac{1}{z^2}\Bigg[i\Bigg((a^2+r_-^2)\left(r_- - \frac{M}{(1-b)}\right)\frac{\omega}{(1-b)} + \left(\frac{M}{(1-b)} - r_-\right)\left(a\frac{m}{(1-b)} - qr_-\frac{e}{(1-b)}\right)\Bigg) \\
&+ qr_-\left(2a\left(a\frac{\omega}{(1-b)} - \frac{m}{(1-b)}\right) + 2r_-^2\frac{\omega}{(1-b)}\right)\frac{e}{(1-b)} + q^2r_-^2\frac{e^2}{(1-b)^2} \\
&+2ar_-^2\left(a\frac{\omega}{(1-b)}-\frac{m}{(1-b)}\right)\frac{\omega}{(1-b)} + a^2\frac{m^2}{(1-b)^2} + (r_-^4+a^4)\frac{\omega^2}{(1-b)^2} - 2a^3\frac{m}{(1-b)}\frac{\omega}{(1-b)}\Bigg]R(z) \\
&+ \frac{1}{(r_+-r_-)^2}\frac{1}{(z-1)^2}\Bigg[i\Bigg((a^2+r_+^2)\left(r_+ - \frac{M}{(1-b)}\right)\frac{\omega}{(1-b)} \\
&+ \left(\frac{M}{(1-b)} - r_+\right)\left(a\frac{m}{(1-b)} - qr_+\frac{e}{(1-b)}\right)\Bigg) + qr_+\left(2a\left(a\frac{\omega}{(1-b)}-\frac{m}{(1-b)}\right) + 2r_+^2\frac{\omega}{(1-b)}\right)\frac{e}{(1-b)} \\
&+ q^2r_+^2\frac{e^2}{(1-b)^2} + 2ar_+^2\left(a\frac{\omega}{(1-b)}-\frac{m}{(1-b)}\right)\frac{\omega}{(1-b)} + a^2\frac{m^2}{(1-b)^2} + (r_+^4+a^4)\frac{\omega^2}{(1-b)^2} \\
&-2a^3\frac{m}{(1-b)}\frac{\omega}{(1-b)}\Bigg]R(z) = 0,
\end{aligned}
$$

(4.74)

where $R(z) \equiv R_{-\frac{1}{2}}(z)$ and

$$\mathcal{H}_{\pm} \equiv a^2\omega + eqr_{\pm} + \omega r_{\pm}^2 - am. \tag{4.75}$$

For convenience, we define the coefficients:

$$\alpha_{1,b} \equiv -\left(\frac{r_- - \frac{M}{(1-b)}}{r_+ - r_-}\right), \tag{4.76}$$

$$\alpha_{2,b} \equiv \left(\frac{r_+ - \frac{M}{(1-b)}}{r_+ - r_-}\right), \tag{4.77}$$

$$\begin{aligned}
\beta_{1,b} \equiv & \left(iq\frac{e}{(1-b)} + 2ir_-\frac{\omega}{(1-b)} + \frac{\lambda^2}{(1-b)}\right) + \frac{r_-^2\mu^2}{(1-b)} - \frac{i\frac{\mathcal{H}_-}{(1-b)}}{z_3(r_- - r_+)} \\
& + \frac{1}{(r_+ - r_-)^2}\left[i\left(a^2(r_+ + r_-)\frac{\omega}{(1-b)} - 2(a^2 + r_+r_-)\frac{M}{(1-b)}\frac{\omega}{(1-b)} + r_-^2(3r_+ - r_-)\frac{\omega}{(1-b)}\right)\right. \\
& \left. +i\left(\left(2a\frac{m}{(1-b)} - q(r_- + r_+)\frac{e}{(1-b)}\right)\frac{M}{(1-b)} - a(r_+ + r_-)\frac{m}{(1-b)} + 2qr_+r_-\frac{e}{(1-b)}\right)\right] \\
& + \frac{1}{(r_+ - r_-)^2}\left[2qa(r_+ + r_-)\frac{e}{(1-b)}\left(a\frac{\omega}{(1-b)} - \frac{m}{(1-b)}\right) + 2q^2r_+r_-\frac{e^2}{(1-b)^2}\right. \\
& + 2qr_-^2(3r_+ - r_-)\frac{e}{(1-b)}\frac{\omega}{(1-b)} + 4r_+r_-(a^2 + r_-^2)\frac{\omega^2}{(1-b)^2} + 2a^2\frac{m^2}{(1-b)^2} \\
& \left. -4a(a^2 + r_+r_-)\frac{\omega}{(1-b)}\frac{m}{(1-b)} + 2(a^4 - r_-^4)\frac{\omega^2}{(1-b)^2}\right];
\end{aligned} \tag{4.78}$$

$$
\begin{aligned}
\beta_{2,b} \equiv & \left(-iq\frac{e}{(1-b)} - 2ir_+\frac{\omega}{(1-b)} - \frac{\lambda^2}{(1-b)}\right) - \frac{r_+^2\mu^2}{(1-b)} + \frac{i\frac{\mathcal{H}_+}{(1-b)}}{(z_3-1)(r_- - r_+)} \\
& + \frac{1}{(r_+ - r_-)^2}\left[i\left(-a^2(r_+ + r_-)\frac{\omega}{(1-b)} + 2(a^2 + r_+r_-)\frac{M}{(1-b)}\frac{\omega}{(1-b)} + r_+^2(r_+ - 3r_-)\frac{\omega}{(1-b)}\right)\right. \\
& \left. + i\left(q(r_- + r_+)\frac{e}{(1-b)}\frac{M}{(1-b)} + a\left(r_+ + r_- - 2\frac{M}{(1-b)}\right)\frac{m}{(1-b)} - 2qr_+r_-\frac{e}{(1-b)}\right)\right] \\
& + \frac{1}{(r_+ - r_-)^2}\left[-2qa(r_+ + r_-)\frac{e}{(1-b)}\left(a\frac{\omega}{(1-b)} - \frac{m}{(1-b)}\right) - 2q^2r_+r_-\frac{e^2}{(1-b)^2}\right. \\
& + 2qr_+^2(r_+ - 3r_-)\frac{e}{(1-b)}\frac{\omega}{(1-b)} - 4r_+r_-(a^2 + r_+^2)\frac{\omega^2}{(1-b)^2} - 2a^2\frac{m^2}{(1-b)^2} \\
& \left. + 4a(a^2 + r_+r_-)\frac{\omega}{(1-b)}\frac{m}{(1-b)} - 2(a^4 - r_+^4)\frac{\omega^2}{(1-b)^2}\right];
\end{aligned}
\tag{4.79}
$$

$$
\beta_{3,b} \equiv -\frac{\left[i\left(\frac{\omega}{(1-b)}(r_+ - r_-)^2z_3^2 + 2r_-(r_+ - r_-)\frac{\omega}{(1-b)}z_3 + q(r_+ - r_-)\frac{e}{(1-b)}z_3 + \frac{\mathcal{H}_-}{(1-b)}\right)\right]}{(r_- - r_+)z_3(z_3-1)}; \tag{4.80}
$$

$$
\begin{aligned}
\eta_{1,b} \equiv & \frac{1}{(r_+ - r_-)^2}\left[i\left((a^2 + r_-^2)\left(r_- - \frac{M}{(1-b)}\right)\frac{\omega}{(1-b)} + \left(\frac{M}{(1-b)} - r_-\right)\left(a\frac{m}{(1-b)} - qr_-\frac{e}{(1-b)}\right)\right)\right. \\
& + qr_-\left(2a\left(a\frac{\omega}{(1-b)} - \frac{m}{(1-b)}\right) + 2r_-^2\frac{\omega}{(1-b)}\right)\frac{e}{(1-b)} + q^2r_-^2\frac{e^2}{(1-b)^2} \\
& + 2ar_-^2\left(a\frac{\omega}{(1-b)} - \frac{m}{(1-b)}\right)\frac{\omega}{(1-b)} + a^2\frac{m^2}{(1-b)^2} + (r_-^4 + a^4)\frac{\omega^2}{(1-b)^2} \\
& \left. -2a^3\frac{m}{(1-b)}\frac{\omega}{(1-b)}\right];
\end{aligned}
\tag{4.81}
$$

$$
\begin{aligned}
\eta_{2,b} \equiv & \frac{1}{(r_+ - r_-)^2}\left[i\left((a^2 + r_+^2)\left(r_+ - \frac{M}{(1-b)}\right)\frac{\omega}{(1-b)} + \left(\frac{M}{(1-b)} - r_+\right)\left(a\frac{m}{(1-b)} - qr_+\frac{e}{(1-b)}\right)\right)\right. \\
& + qr_+\left(2a\left(a\frac{\omega}{(1-b)} - \frac{m}{(1-b)}\right) + 2r_+^2\frac{\omega}{(1-b)}\right)\frac{e}{(1-b)} + q^2r_+^2\frac{e^2}{(1-b)^2} \\
& + 2ar_+^2\left(a\frac{\omega}{(1-b)} - \frac{m}{(1-b)}\right)\frac{\omega}{(1-b)} + a^2\frac{m^2}{(1-b)^2} + (r_+^4 + a^4)\frac{\omega^2}{(1-b)^2} \\
& \left. -2a^3\frac{m}{(1-b)}\frac{\omega}{(1-b)}\right];
\end{aligned}
\tag{4.82}
$$

$$\sigma_{0,b} \equiv \left(\frac{\omega^2}{(1-b)^2} - \frac{\mu^2}{(1-b)} \right) (r_+ - r_-)^2. \tag{4.83}$$

With the definitions (4.76) – (4.83), our radial equation (4.74), for the $R_{-\frac{1}{2}}(z)$ mode, can be written in a more compact way as:

$$R''(z) + \left(\sum_{i=1}^{2} \frac{\alpha_{i,b}}{(z-z_i)} - \frac{1}{z-z_3} \right) R'(z) + \left(\sum_{i=1}^{3} \frac{\beta_{i,b}}{(z-z_i)} + \sum_{i=1}^{2} \frac{\eta_{i,b}}{(z-z_i)^2} + \sigma_{0,b} \right) R(z) = 0, \tag{4.84}$$

where $z_1 = 0$ and $z_2 = 1$.

The indicial equation for the singularity at $z = z_1 = 0$ takes the form[80]:

$$\begin{aligned} F(s) &= s(s-1) + \alpha_{1,b} s + \eta_{1,b} \\ &= s^2 + \left(\frac{\frac{M}{(1-b)} - r_+}{r_+ - r_-} \right) s + \eta_{1,b}. \end{aligned} \tag{4.85}$$

Whose roots are:

$$\boldsymbol{\mu_1} \equiv s_{\pm}^{z=0} = \frac{-\left(\frac{\frac{M}{(1-b)} - r_+}{r_+ - r_-} \right) \pm \sqrt{\left(\frac{\frac{M}{(1-b)} - r_+}{r_+ - r_-} \right)^2 - 4\eta_{1,b}}}{2}. \tag{4.86}$$

Likewise, the indicial equation for the singularity at $z = z_2 = 1$ is [80]:

$$\begin{aligned} F(s) &= s(s-1) + \alpha_{2,b} s + \eta_{2,b} \\ &= s^2 + \left(\frac{r_- - \frac{M}{(1-b)}}{r_+ - r_-} \right) s + \eta_{2,b}, \end{aligned} \tag{4.87}$$

and its roots are:

$$\boldsymbol{\mu_2} \equiv s_{\pm}^{z=1} = \frac{\left(\frac{\frac{M}{(1-b)} - r_-}{r_+ - r_-} \right) \pm \sqrt{\left(\frac{\frac{M}{(1-b)} - r_-}{r_+ - r_-} \right)^2 - 4\eta_{2,b}}}{2}. \tag{4.88}$$

We apply now the F-homotopic transformation of $R(z)$ [80]:

$$R(z) = e^{\nu z} z^{\boldsymbol{\mu_1}} (z-1)^{\boldsymbol{\mu_2}} \bar{R}(z), \tag{4.89}$$

where $\nu = \pm i\sqrt{\frac{\omega^2}{(1-b)^2} - \frac{\mu^2}{(1-b)}}(r_+ - r_-)$.

Transformation (4.89) transforms the radial part of the Dirac equation in the curved spacetime of the Kerr-Newman black hole with cloud of strings (Eq.(4.84)) into a generalised Heun differential equation:

$$\left[\frac{d^2}{dz^2} + \left(\sum_{i=1}^{2} \frac{\alpha'_{i,b}}{(z-z_i)} - \frac{1}{z-z_3} + 2\nu\right)\frac{d}{dz} + \sum_{i=1}^{3} \frac{\beta'_{i,b}}{(z-z_i)}\right]\bar{R}(z) = 0, \tag{4.90}$$

where the coefficients in Eq. (4.90) are:

$$\alpha'_{1,b} = 2\boldsymbol{\mu_1} + \left(\frac{\frac{M}{(1-b)} - r_-}{r_+ - r_-}\right), \tag{4.91}$$

$$\alpha'_{2,b} = 2\boldsymbol{\mu_2} + \left(\frac{r_+ - \frac{M}{(1-b)}}{r_+ - r_-}\right), \tag{4.92}$$

$$\begin{aligned}\beta'_{1,b} =& \beta_{1,b} + 2i\sqrt{\frac{\omega^2}{(1-b)^2} - \frac{\mu^2}{(1-b)}}(r_+ - r_-)\boldsymbol{\mu_1} + \alpha_{1,b} i\sqrt{\frac{\omega^2}{(1-b)^2} - \frac{\mu^2}{(1-b)}}(r_+ - r_-) - 2\boldsymbol{\mu_1}\boldsymbol{\mu_2} \\ &- \alpha_{1,b}\boldsymbol{\mu_2} - \alpha_{2,b}\boldsymbol{\mu_1} + \frac{\boldsymbol{\mu_1}}{z_3},\end{aligned} \tag{4.93}$$

$$\begin{aligned}\beta'_{2,b} =& \beta_{2,b} + 2i\sqrt{\frac{\omega^2}{(1-b)^2} - \frac{\mu^2}{(1-b)}}(r_+ - r_-)\boldsymbol{\mu_2} + \alpha_{2,b} i\sqrt{\frac{\omega^2}{(1-b)^2} - \frac{\mu^2}{(1-b)}}(r_+ - r_-) + 2\boldsymbol{\mu_1}\boldsymbol{\mu_2} \\ &+ \alpha_{1,b}\boldsymbol{\mu_2} + \alpha_{2,b}\boldsymbol{\mu_1} - \frac{\boldsymbol{\mu_2}}{1-z_3},\end{aligned} \tag{4.94}$$

$$\beta'_{3,b} = \beta_{3,b} - i\sqrt{\frac{\omega^2}{(1-b)^2} - \frac{\mu^2}{(1-b)}}(r_+ - r_-) - \frac{\boldsymbol{\mu_1}}{z_3} + \frac{\boldsymbol{\mu_2}}{1-z_3}. \tag{4.95}$$

With some algebra, equation (4.90) can also be written in the general form of the generalised Heun differential equation as follows [80]:

$$\left[\frac{d^2}{dz^2}+\left(\frac{1-\mu_0}{z}+\frac{1-\mu_1}{(z-1)}+\frac{1-\mu_2}{(z-z_3)}+\alpha\right)\frac{d}{dz}+\frac{\beta_0+\beta_1 z+\beta_2 z^2}{z(z-1)(z-z_3)}\right]\bar{R}(z)=0, \tag{4.96}$$

where:

$$\mu_0 = 1-2\boldsymbol{\mu_1}-\left(\frac{\frac{M}{(1-b)}-r_-}{r_+-r_-}\right), \tag{4.97}$$

$$\mu_1 = 1-2\boldsymbol{\mu_2}-\left(\frac{r_+-\frac{M}{(1-b)}}{r_+-r_-}\right), \tag{4.98}$$

$$\mu_2 = 2, \tag{4.99}$$

$$\beta_0 = \beta'_{1,b}z_3, \tag{4.100}$$

$$\beta_1 = -\beta'_{1,b}(1+z_3)-\beta'_{2,b}z_3-\beta'_{3,b}, \tag{4.101}$$

$$\beta_2 = \beta'_{1,b}+\beta'_{2,b}+\beta'_{3,b}, \tag{4.102}$$

$$\alpha = 2\nu. \tag{4.103}$$

Therefore, the solutions for the radial part, given by Eq. (4.90), is written in terms of Generalized Heun Functions:

$$\begin{aligned}R_{-\frac{1}{2}}(z) =& e^{\alpha_{1,b}z}z^{\alpha_{2,b}}(z-1)^{\alpha_{3,b}}\left[\mathrm{HeunG}(z_3,\beta_{1,b},\beta_{2,b},\beta_{3,b},\alpha_{2,b},\alpha_{3,b};z)\right.\\ &\left.+\sigma_{0,b}z^{1-\alpha_{2,b}}\mathrm{HeunG}(z_3,\beta'_{1,b},\beta'_{2,b},\beta'_{3,b},\alpha'_{1,b},\alpha'_{2,b};z)\right].\end{aligned} \tag{4.104}$$

Let us close this section by examining the asymptotic behaviour of the solutions at infinity $r\to\infty$.

Our radial GHE (4.96) is a differential equation with three regular singularities at z_i, with $(i=1,2,3)$, and an irregular singularity at infinity. Using the result of [82] that the solution of such a differential equation can be expanded in a Maclaurin series given by:

$$\text{HeunG}(a,q;\alpha,\beta,\gamma,\delta;z)=\sum_{j=0}^{\infty}b_jz^j, \tag{4.105}$$

where b_0 can be assumed as equal to 1 and

$$a\gamma b_1 - qb_0 = 0, \tag{4.106}$$

$$x_j b_{j+1} - (Q_j+q)b_j + P_j b_{j-1} = 0\ ,\ j\geq 1. \tag{4.107}$$

with

$$P_j = (j-1+\alpha)(j-1+\beta), \tag{4.108}$$

$$Q_j = j[(j+1+\gamma)(1+a)+a\delta+\epsilon], \tag{4.109}$$

$$X_j = a(j+1)(j+\gamma), \tag{4.110}$$

where we are assuming the canonical form of the Heun equation which is given by

$$\frac{d^2u}{dz^2}+\left(\frac{\gamma}{z}+\frac{\delta}{(z-1)}+\frac{\epsilon}{(z-z_3)}\right)\frac{du}{dz}+\frac{\alpha\beta z-q}{z(z-1)(z-z_3)}u=0. \tag{4.111}$$

The parameters $\alpha,\beta,\gamma,\delta,\epsilon,q$ and z_3 obey the following relation:

$$\gamma+\delta+\epsilon=\alpha+\beta+1. \tag{4.112}$$

Thus, writting Eq. (4.96) in the canonical form given by Eq. (4.111), and following the discussions in Kraniotis' framework [2], we find that:

$$R_1^{\infty}(r)\sim e^{\pm i\sqrt{\frac{\omega^2}{(1-b)^2}-\frac{\mu^2}{(1-b)}}(r-r_-)}\left(\frac{r-r_-}{r_+-r_-}\right)^{-\left(\frac{C_1+C_2+C_3}{\pm 2i(r_+-r_-)\sqrt{\frac{\omega^2}{(1-b)^2}-\frac{\mu^2}{(1-b)}}}\right)}, \tag{4.113}$$

and

$$R_2^{\infty}(r) \sim e^{\mp i\sqrt{\frac{\omega^2}{(1-b)^2}-\frac{\mu^2}{(1-b)}}(r-r_-)} \left(\frac{r-r_-}{r_+-r_-}\right)^{\left(\frac{C_1+C_2+C_3}{\pm 2i(r_+-r_-)\sqrt{\frac{\omega^2}{(1-b)^2}-\frac{\mu^2}{(1-b)}}}\right)} . \tag{4.114}$$

Note that now, the solutions depend explicitly on the parameter that codifies the presence of the cloud of strings, and when $b = 0$ (absence of the cloud of strings) the results given by Kraniotis [2] are recovered, as expected.

As in the case of the angular solution, we can assume that there is no charge, or there is no angular momentum, or even, nor charge neither angular momentum, and obtain the solutions to Kerr, Reissner-Nordstrom and Schwarzschild black holes, all of them with a cloud of strings.

Chapter Five

ANALYTIC SOLUTIONS NEAR TO THE EVENT HORIZON

In chapter four we obtained the solutions of the angular and radial parts of the Dirac equation and showed that they are given in terms of the Generalized Heun Functions. We also discussed the asymptotic behaviour at $r \to \infty$.

In this chapter we will discuss some features of the Generalized Heun equation and discuss its behaviour near to the horizon $r = r_+$.

Using the representation adopted by Schäfke and Schmidt [77], the Generalized Heun equation reads as:

$$\left[\frac{d^2}{dz^2} + \left(\frac{1-\mu_0}{z} + \frac{1-\mu_1}{(z-1)} + \frac{1-\mu_2}{(z-c)} + \alpha\right)\frac{d}{dz} + \frac{\beta_0 + \beta_1 z + \beta_2 z^2}{z(z-1)(z-c)}\right] y(z) = 0, \tag{5.1}$$

where $c \in C - \{0, 1\}$, and $\alpha \neq 0$, and μ_j,β_j are complex parameters for $j = 0, 1, 2$.

From now on we will adopt the development made by Kraniotis [2] about the solution of Eq. (5.1). This equation (5.1) with three regular singular points and one irregular singular point at $0, 1, a$ and ∞, respectively, has been discussed in details in [77].

In this form, the exponents at the singularities $z = 0, 1, a$ are respectively $\{0, \mu_0\}$, $\{0, \mu_1\}$ and $\{0, \mu_2\}$.

Because of the simmetry of (5.1) in the parameters $\mu = (\mu_0, \mu_1, \mu_2)$ under certain index or F-Homotopic transformations, one allows the corfficient of the $y(z)$ in Eq. (5.1) to satisfy the form:

$$\frac{\beta_0 + \beta_1 z + \beta_2 z^2}{z(z-1)(z-c)} = \sum_{\sigma,\rho=0}^{2} \frac{1}{2}\left(\frac{1-\mu_\sigma}{z-z_\sigma}\right)\left(\frac{1-\mu_\rho}{z-z_\rho}\right) + \sum_{k=0}^{2} \frac{\frac{\alpha}{2}(1-\mu_k)+\lambda_k}{z-z_k}, \tag{5.2}$$

with $z_0 = 0$, $z_1 = 1$, $z_2 = c$, $\rho \neq \sigma$ and for some parameters $\lambda = (\lambda_0, \lambda_1, \lambda_2) \in C^3$.

We can see that, if we take:

$$c \equiv z_3; \tag{5.3}$$

$$\beta \equiv (\beta_0, \beta_1, \beta_2), \tag{5.4}$$

with $\beta_0, \beta_1, \beta_2$ given by equations (4.100) – (4.102);

$$\mu \equiv (\mu_0, \mu_1, \mu_2), \tag{5.5}$$

with μ_0, μ_1, μ_2 given by equations (4.97) – (4.99); and α given by Eq. (4.103), our radial equation (4.96) will be exactly equal to the equation (5.1),i.e., our equation (4.96) has the general form of the Generalised Heun Equation as we had previously stated. Furthermore, if we are able to find the analytical solutions for equation (5.1), we will have found the solutions for our Eq. (4.96) as well!

Although the Eq.(5.1) seems to be a very difficult equation to work in, his analytical solutions, at the vicinity of its singular regular points, can be found by means of the following theorem, proven in [77]:

Theorem 1: Let $c \in C - \{0, 1\}$ be fixed. Then, there exists a unique function $\eta = \eta(\cdot; c)$ holomorphic with respect to $(z, \mu, \alpha, \lambda) \in \{z \in C : |z| < min(1, |c|)\} \times C^7$, such that for each (μ, α, λ), $\eta(\cdot, \mu, \alpha, \lambda; c)$ is a solution of (5.1) satisfying $\eta(0, \mu, \alpha, \lambda; c) = \frac{1}{\Gamma(1-\mu_0)}$. The function η can be expanded in power series:

$$\eta(z,\mu,\alpha,\lambda;c) = \sum_{k=0}^{\infty} \frac{T_k(\mu,\alpha,\lambda;c)}{\Gamma(k+1-\mu_0)\Gamma(k+1)} z^k, \tag{5.6}$$

where the (unique) coefficients T_k are holomorphic in (μ,α,λ). In particular $T_0(\mu,\alpha,\lambda;c) = 1$.

Substituting our analytical solution (5.6), given by Theorem 1, into Eq.(5.1), we obtain recursion relations for the coefficients T_k:

$$T_1 = -\frac{\beta_0}{c} T_0, \tag{5.7}$$

$$T_2 = \left[\frac{2-\mu_0-\mu_2}{c} + (2-\mu_0-\mu_1) - \alpha - \frac{\beta_0}{c}\right] T_1 - \frac{\beta_1(1-\mu_0)}{c} T_0, \tag{5.8}$$

$$\begin{aligned} T_3 = &\left[2(3-\mu_0-\mu_1) + \frac{2}{c}(3-\mu_0-\mu_2) - 2\alpha - \frac{\beta_0}{c}\right] T_2 \\ &+ \left[-\frac{2}{c}(2-\mu_0)(3-\mu_0-\mu_1-\mu_2) + \frac{2(2-\mu_0)}{c}\alpha(c+1) - \frac{2(2-\mu_0)\beta_1}{c}\right] T_1 \\ &- \frac{2(2-\mu_0)(1-\mu_0)\beta_2}{c} T_0, \end{aligned} \tag{5.9}$$

...

which are summarised in the following four-term recurrence relation for T_k:

$$T_k = \varphi_1(k-1)T_{k-1} - \varphi_2(k-2)T_{k-2} + \varphi_3(k-3)T_{k-3}, \tag{5.10}$$

where $k \in N$, $T_{-1} = T_{-2} = 0$, and

$$\varphi_1(\xi) = \xi(\xi+1-\mu_0-\mu_1) + \frac{1}{c}\xi(\xi+1-\mu_0-\mu_2) - \alpha\xi - \frac{1}{c}\beta_0, \tag{5.11}$$

$$\varphi_2(\xi) = (\xi+1)(\xi+1-\mu_0)\left(\frac{1}{c}\xi(\xi+2-\mu_0-\mu_1-\mu_2) - \left(1+\frac{1}{c}\right)\alpha\xi + \frac{1}{c}\beta_1\right), \tag{5.12}$$

$$\varphi_3(\xi) = (\xi+1)(\xi+2)(\xi+1-\mu_0)(\xi+2-\mu_0)\left(-\frac{1}{a}\right)(a\xi+\beta_2). \quad (5.13)$$

The two following transformations demonstrate that all Floquet solutions at the vicinity of the regular singularities $z = 0$,$z = 1$ and $z = c$ can be constructed from the function η given in theorem 1. We will obtain two linear independent solutions at the vicinity of each regular singularity.

- The first set of transformations are the index transformations:

$$y(z) = z^{\sigma_0}(z-1)^{\sigma_1}(z-c)^{\sigma_2}\tilde{y}(z), \quad (5.14)$$

  with $\sigma_j \in \{0, \mu_j\}$ for $(j = 0, 1, 2)$.

  Considering that μ satisfies (5.2), a straithforward calculation [77] shows that (5.14) transforms (5.1) into an equation of the same type, but with $\tilde{\mu}_j = \mu_j - 2\sigma_j \in \{\mu_j, -\mu_j\}$ for $(j = 0, 1, 2)$, and $\tilde{\alpha} = \alpha$,$\tilde{\lambda} = \lambda$,$\tilde{c} = c$.

- The second set of transformations of interest is the class of linear transformations of the independent variable:

$$\tilde{z} = \epsilon z + \delta \quad (5.15)$$

  with $(\epsilon, \delta \in C)$ and $\epsilon \neq 0$, which map the simple singularities $\{0, 1, c\}$ into the simple singularities $\{0, 1, \tilde{c}\}$ and keep the irregular singularity at ∞ fixed.

  Table 1 contains six possible transformations (5.15) and yields all information about them.

Table 1					
	$\tilde{z}$	$\tilde{\mu}$	$\tilde{c}$	$\tilde{\alpha}$	$\tilde{\lambda}$
(a)	z	(μ_0, μ_1, μ_2)	c	α	λ
(b)	$1-z$	(μ_1, μ_0, μ_2)	$1-c$	$-\alpha$	$-\lambda$
(c)	$\frac{z}{c}$	(μ_0, μ_2, μ_1)	$\frac{1}{c}$	$c\alpha$	$c\lambda$
(d)	$\frac{1-z}{1-c}$	(μ_1, μ_2, μ_0)	$\frac{1}{1-c}$	$(c-1)\alpha$	$(c-1)\lambda$
(e)	$1-\frac{z}{c}$	(μ_2, μ_0, μ_1)	$1-\frac{1}{c}$	$-c\alpha$	$-c\lambda$
(f)	$\frac{c-z}{c-1}$	(μ_2, μ_1, μ_0)	$\frac{c}{c-1}$	$(1-c)\alpha$	$(1-c)\lambda$

Using (5.14) and Table 1, we can now define for each $j = 0, 1, 2$ a set of two linear independent Floquet solutions y_{j1} and y_{j2} for equation (5.1) at the vicinity of z_j in terms of the function η defined in (5.6) of theorem 1, by:

- for the vicinity of $z = 0$, i.e., for $|z| < min(1, |c|)$, we have the two L.I. solutions:

$$\bar{R}_{01}(z) = y_{01}(z, \mu, \alpha, \lambda) \equiv \eta(z, \mu_0, \mu_1, \mu_2, \alpha, \lambda; c), \tag{5.16}$$

$$\bar{R}_{02}(z) = y_{02}(z, \mu, \alpha, \lambda) \equiv z^{\mu_0}\eta(z, -\mu_0, \mu_1, \mu_2, \alpha, \lambda; c), \tag{5.17}$$

- for the vicinity of $z = 1$, i.e., for $|z-1| < min(1, |c-1|)$, we have the two L.I. solutions:

$$\bar{R}_{11}(z) = y_{11}(z, \mu, \alpha, \lambda) \equiv \eta(1-z, \mu_1, \mu_0, \mu_2, -\alpha, -\lambda; 1-c), \tag{5.18}$$

$$\bar{R}_{12}(z) = y_{12}(z, \mu, \alpha, \lambda) \equiv (1-z)^{\mu_1}\eta(1-z, -\mu_1, \mu_0, \mu_2, -\alpha, -\lambda; 1-c), \tag{5.19}$$

- for the vicinity of $z = c$, i.e., for $|z-c| < min(|c|, |1-c|)$, we have the two L.I. solutions:

$$\bar{R}_{21}(z) = y_{21}(z, \mu, \alpha, \lambda) \equiv \eta\left(1-\frac{z}{c}, \mu_2, \mu_0, \mu_1, -c\alpha, -c\lambda; 1-\frac{1}{c}\right), \tag{5.20}$$

$$\bar{R}_{22}(z) = y_{22}(z, \mu, \alpha, \lambda) \equiv \left(1-\frac{z}{c}\right)^{\mu_2}\eta\left(1-\frac{z}{c}, -\mu_2, \mu_0, \mu_1, -c\alpha, -c\lambda; 1-\frac{1}{c}\right), \tag{5.21}$$

where we have used the transformations from Table 1 (a), (b) and (e) respectively for each vicinity.

Equations (5.16) – (5.21) are the analytical solutions for our radial equation (4.96) at the vicinity of the three regular singularities $z = 0$,$z = 1$ and $z = c = z_3$, exactly what we aimed to obtain. In the next section, we are going to study these solutions near event horizon of our black hole.

5.1 The near event horizon analytic solutions $r \to r_+$

From theorem 1, and by means of the transformations (5.14) and (5.15),we can obtain the solution of our radial equation (4.96) near event horizon of our black hole, i.e., close to the point r_+.

In order to do that, in the first place, we need to change our independent variable r to a more convinient one, that is:

$$\zeta = \frac{r - r_+}{r_- - r_+}. \tag{5.22}$$

However, if we relate our new independent variable ζ, more convinient to study the radial equation (4.96) near the event horizon of the black hole, with z, we will find that:

$$\zeta = 1 - z, \tag{5.23}$$

but comparing that with the general class of linear transformations of the independet variable in Eq. (5.15), it will be clear that this transformation (5.22) or (5.23) is exactly the (b) transformation in Table 1, which produces the solutions at the vicinity of $z = 1$, or $r = r_+$, as we would expect. In its turn, these solutions are $\bar{R}_{11}(z)$ and $\bar{R}_{12}(z)$, which are given by Eq. (5.18) and (5.19).

By means of the formula (5.6), we can obtain the explicit form of these solutions, and they are read as:

$$\bar{R}_{11}(\zeta) = \eta(\zeta, \mu_1, \mu_0, \mu_2, -\alpha, -\lambda; 1-c) = \frac{1}{\Gamma(1-\mu_1)} + \frac{-\frac{\tilde{\beta}_0}{(1-c)}}{\Gamma(2-\mu_1)}\zeta$$
$$+ \left[\left(\frac{2-\mu_1-\mu_2}{1-c} + (2-\mu_1-\mu_0) + \alpha - \frac{\tilde{\beta}_0}{(1-c)}\right)T_1 - \frac{\tilde{\beta}_1(1-\mu_1)T_0}{1-c}\right]\frac{1}{2!\Gamma(3-\mu_1)}\zeta^2$$
$$+ ... \tag{5.24}$$

$$\bar{R}_{12}(\zeta) = \zeta^{\mu_1}\eta(\zeta, -\mu_1, \mu_0, \mu_2, -\alpha, -\lambda; 1-c) = \frac{\zeta^{\mu_1}}{\Gamma(1+\mu_1)} + \frac{-\frac{\tilde{\beta}_0}{(1-c)}}{\Gamma(2+\mu_1)}\zeta^{\mu_1+1}$$
$$+ \left[\left(\frac{2+\mu_1-\mu_2}{1-c} + (2+\mu_1-\mu_0) + \alpha - \frac{\tilde{\beta}_0}{(1-c)}\right)T_1 - \frac{\tilde{\beta}_1(1+\mu_1)T_0}{1-c}\right]\frac{1}{2!\Gamma(3+\mu_1)}\zeta^{\mu_1+2}$$
$$+ ... \tag{5.25}$$

However, from (4.89), we can write:

$$R(\zeta) = e^{\nu(1-\zeta)}(1-\zeta)^{\boldsymbol{\mu_1}}(-\zeta)^{\boldsymbol{\mu_2}}\bar{R}(\zeta). \tag{5.26}$$

Thus,

$$R_{11}(\zeta) = e^{\nu(1-\zeta)}(1-\zeta)^{\boldsymbol{\mu_1}}(-\zeta)^{\boldsymbol{\mu_2}}\bar{R}_{11}(\zeta), \tag{5.27}$$

$$R_{12}(\zeta) = e^{\nu(1-\zeta)}(1-\zeta)^{\boldsymbol{\mu_1}}(-\zeta)^{\boldsymbol{\mu_2}}\bar{R}_{12}(\zeta), \tag{5.28}$$

where $\nu = \pm i\sqrt{\frac{\omega^2}{(1-b)^2} - \frac{\mu^2}{(1-b)}}(r_+ - r_-)$.

Equations (5.24) and (5.25) are the analytic solutions for the radial part of Dirac equation in the vicinity of $\zeta = 0$, or $r = r_+$.

In terms of the original variable r, these solutions have the forms:

$$R_{11}(r) = e^{\pm i\sqrt{\frac{\omega^2}{(1-b)^2} - \frac{\mu^2}{(1-b)}}(r-r_-)}\left(\frac{r-r_-}{r_+-r_-}\right)^{\boldsymbol{\mu_1}}\left(\frac{r-r_+}{r_+-r_-}\right)^{\boldsymbol{\mu_2}}\bar{R}_{11}(r), \tag{5.29}$$

where $\bar{R}_{11}(r)$ is:

$$\begin{aligned}\bar{R}_{11}(r) =& \frac{1}{\Gamma(1-\mu_1)} + \frac{-\frac{\tilde{\beta}_0}{(1-c)}}{\Gamma(2-\mu_1)}\left(\frac{r-r_+}{r_--r_+}\right) \\ &+ \left[\left(\frac{2-\mu_1-\mu_2}{1-c} + (2-\mu_1-\mu_0) + \alpha - \frac{\tilde{\beta}_0}{(1-c)}\right)T_1 - \frac{\tilde{\beta}_1(1-\mu_1)T_0}{1-c}\right]\frac{1}{2!\Gamma(3-\mu_1)}\left(\frac{r-r_+}{r_--r_+}\right)^2 \\ &+ \ldots\end{aligned} \tag{5.30}$$

And

$$R_{12}(r) = e^{\pm i\sqrt{\frac{\omega^2}{(1-b)^2}-\frac{\mu^2}{(1-b)}}(r-r_-)}\left(\frac{r-r_-}{r_+-r_-}\right)^{\boldsymbol{\mu_1}}\left(\frac{r-r_+}{r_+-r_-}\right)^{\boldsymbol{\mu_2}}\bar{R}_{12}(r), \tag{5.31}$$

where $\bar{R}_{12}(r)$ is:

$$\begin{aligned}\bar{R}_{12}(r) =& \frac{\left(\frac{r-r_+}{r_--r_+}\right)^{\mu_1}}{\Gamma(1+\mu_1)} + \frac{-\frac{\tilde{\beta}_0}{(1-c)}}{\Gamma(2+\mu_1)}\left(\frac{r-r_+}{r_--r_+}\right)^{\mu_1+1} \\ &+ \left[\left(\frac{2+\mu_1-\mu_2}{1-c} + (2+\mu_1-\mu_0) + \alpha - \frac{\tilde{\beta}_0}{(1-c)}\right)T_1 - \frac{\tilde{\beta}_1(1+\mu_1)T_0}{1-c}\right]\frac{\left(\frac{r-r_+}{r_--r_+}\right)^{\mu_1+2}}{2!\Gamma(3+\mu_1)} \\ &+ \ldots\end{aligned} \tag{5.32}$$

Finally, (5.29) and (5.31) are the explicit forms of the analytical solutions of the radial part of the general relativistic Dirac equation near event horizon ($r = r_+$) of the Kerr-Newman black hole with cloud of strings, which we intended to obtain.

As a conclusion, the solutions of the radial part of the Dirac equation depend explicitly on the presence of the cloud of strings. Note that the exponential factor amplifies the values assumed by the solutions for large values of b, where ($0 < b < 1$), which means that the solutions are affected directly by the enlargement of the horizon. In other words, the values of the radial solutions close to the horizon, $r = r_+$, increases with the increasing of the horizon, which means that the presence of the cloud of strings affects the behaviour of the fermions placed in the background spacetime associated to a Kerr-Newman black hole with a cloud of strings.

Chapter Six

CONCLUSIONS

In this dissertation, we obtained the solution corresponding to charged and rotating black hole surrounded by a cloud of strings, which we are calling Kerr-Newman black hole with a cloud of strings, and we investigated the behaviour of $\frac{1}{2}$-spin particles when interacting with this black hole and the others which are obtained by taking the electric charge equal to zero, the angular momentum equal to zero or both quantities equal to zero.

We obtained the solutions of the Dirac equation which are given in terms of the Generalized Heun Function, both the angular and radial part. These equations appear not only in the case under consideration, but also in several others in different areas of physics. Concerning these solutions, which are well known in the literature in the context of Kerr-Newman, Kerr, Reissner-Nordstrom and Schwarzschild black holes, we reobtained them, and did a qualitative analyze in which concerns their dependence on the presence of the cloud of strings, and concluded that the radial solutions are strongly affected by the strings.

Future works can be done along this line, by considering the scattering process, quasinormal modes as well as the existence of Hawking radiation and superradiance in connection with the presence of a cloud of strings.

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