**Universidade Federal da Paraíba**
**Programa de Pós-Graduação em Matemática**
**Doutorado em Matemática**

# Limite de escala para um sistema de partículas com várias leis de conservação

**por**

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**Março/2022**

# Limite de escala para um sistema de partículas com várias leis de conservação

por

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sob orientação do

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e coorientação da

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Tese apresentada ao Corpo Docente do Programa de Pós-Graduação em Matemática - UFPB, como requisito parcial para obtenção do título de Doutor em Matemática.

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**Março/2022**

†Este trabalho contou com apoio financeiro da Capes

UNIVERSIDADE FEDERAL DA PARAÍBA
CENTRO DE CIÊNCIAS EXATAS E DA NATUREZA
CAMPUS I – DEPARTAMENTO DE MATEMÁTICA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

**ATA DE DEFESA DE DOUTORADO JUNTO AO PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA DA UNIVERSIDADE FEDERAL DA PARAÍBA, REALIZADA NO DIA 31 DE MARÇO DE 2022.**

Ao trigésimo primeiro dia de março de dois mil e vinte e dois, às 10:00 horas, por meio da plataforma virtual *Google Meet*, através do link: https://meet.google.com/ssv-juvy-zgj, em conformidade com o parágrafo único do Art. 80 da Resolução CONSEPE nº 79/2013, que regulamenta a defesa de trabalho final por videoconferência, seguindo os mesmos preceitos da defesa presencial, foi aberta a sessão pública de Defesa de tese intitulada “**Limite de escala para um sistema de partículas com várias leis de conservação**”, da aluna **Oslenne Nogueira de Araújo**, que havia cumprido, anteriormente, todos os requisitos para a obtenção do grau de Doutora em Matemática, sob a orientação do Prof. Dr. Alexandre de Bustamante Simas. A Banca Examinadora, indicada pelo Colegiado do Programa de Pós-Graduação em Matemática, foi composta pelos professores: Alexandre de Bustamante Simas (Orientador), Ana Patrícia Carvalho Gonçalves (Coorientadora – IST/Universidade de Lisboa), Felipe Wallison Chaves Silva (UFPB), Adriana Neumann de Oliveira (UFRGS), Milton Jara (IMPA) e Fábio Júlio da Silva Valentim (UFES). O professor Alexandre de Bustamante Simas, em virtude da sua condição de orientador, presidiu os trabalhos e, depois das formalidades de apresentação, convidou a aluna a discorrer sobre o conteúdo da tese. Concluída a explanação, a candidata foi arguida pela banca examinadora que, em seguida, sem a presença da aluna, finalizando os trabalhos, reuniu-se para deliberar tendo concedido a menção: **APROVADA.** E, para constar, foi lavrada a presente ata que será assinada pelos membros da Banca Examinadora.

João Pessoa, 31 de março de 2022.

Alexandre de Bustamante Simas ____________________

Ana Patrícia Carvalho Gonçalves ____________________

Felipe Wallison Chaves Silva ____________________

Adriana Neumann de Oliveira ____________________

Fábio Júlio da Silva Valentim ____________________

Milton Jara ____________________

**Catalogação na publicação**
**Seção de Catalogação e Classificação**

A663l Araújo, Oslenne Nogueira de.
Limite de escala para um sistema de partículas com várias leis de conservação / Oslenne Nogueira de Araújo. - João Pessoa, 2022.
143 f. : il.

Orientação: Alexandre de Bustamante Simas.
Coorientação: Ana Patrícia Carvalho Gonçalves.
Tese (Doutorado) - UFPB/CCEN.

1. Método da entropia. 2. Limite hidrodinâmico. 3. Flutuações no equilíbrio. I. Simas, Alexandre de Bustamante. II. Gonçalves, Ana Patrícia Carvalho. III. Título.

UFPB/BC CDU 51-7(043)

Elaborado por RUSTON SAMMEVILLE ALEXANDRE MARQUES DA SILVA - CRB-15/0386

# Resumo

Esta tese está dividida em duas partes. Na primeira parte obtemos o limite hidrodinâmico para um sistema de partículas com velocidades em contato com reservatórios. Teremos um parâmetro $\theta$, que regula a intensidade dos reservatórios e obteremos um sistema de equações diferenciais parciais com diferentes condições de bordo dependendo do parâmetro $\theta$. Nosso objetivo é analisar o impacto da força dos reservatórios (mudando o valor de $\theta$) no comportamento macroscópico do sistema. O limite hidrodinâmico deste modelo no caso $\theta = 0$ foi provado em [3].

Na segunda parte desta tese obtemos as flutuações no equilíbrio para o mesmo modelo com bordos periódicos.

**Palavras-chave:** Método da Entropia; Limite Hidrodinâmico; Flutuações no Equilíbrio;

# Abstract

This Ph.D. thesis consists of two parts. In the first part, we discuss the hydrodynamic limit of the weakly asymmetric exclusion process with collision among particles having different velocities and in contact with stochastic reservoirs. We will have a parameter $\theta$ and a system of partial differential equations with boundary conditions that change depending on this parameter $\theta$. We aim to analyze the impact of the reservoirs (change the value of $\theta$) on the macroscopic behavior of the system. The hydrodynamic limit of this model in the case $\theta = 0$ was proved in [3].

In the second part of this work, we obtain the equilibrium fluctuation for the same model with periodic boundary conditions.

**Keywords:** Entropy Method; Hydrodynamic Limit; Equilibrium Fluctuations;

# Agradecimentos

Pela generosidade e por ter me acolhido tão bem longe de casa, agradeço à minha coorientadora, Profa. Patrícia, com quem pude aprender muito além da matemática.

Ao meu orientador, Prof. Alexandre, pela confiança e ensinamentos.

Ao meu esposo, Damião Júnio, pelo incentivo, carinho, paciência e por ser sempre o meu porto seguro quando mais preciso.

Aos membros da banca examinadora, Profa. Adriana Neumann, Prof. Fábio J. Valentim, Prof. Felipe Wallison, Prof. Milton Jara e Prof. Flank Bezerra, pela disponibilidade e pelas contribuições.

Aos professores e funcionários do Programa de Pós-graduação em Matemática da UFPB pelo suporte.

Ao orgão financiador, Capes, pelo apoio financeiro.

E a todos que de alguma forma contribuiram para a realização desse trabalho, deixo aqui meu sincero agradecimento.

*"If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is."*

*John Von Neumann*

Dedicado a

*Damião Júnio.*

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# Introduction

Interacting particle systems is an area of probability devoted to the mathematical analysis of random models (stochastic process) that arise from statistical physics, biology, and many other fields of science. Interacting particle systems were introduced in the 1970s by Spitzer [14]. A classic problem in this field is to derive macroscopic laws of the thermodynamic quantities of a physical system, considering a microscopic dynamics which is composed of particles that move according to some prescribed stochastic law. These macroscopic laws are governed by Partial Differential Equations (PDEs) or stochastic PDEs, depending if one is looking at the convergence to the mean, or fluctuations around that mean. Convergence to the mean is a scaling limit, called the hydrodynamic limit. This limit will be the solution of a partial differential equation, called the hydrodynamic equation, and with it we can understand how the temporal evolution of the spatial density of particles is, see [1].

To make the reading as pleasant as possible, we will informally describe the model, which we will work with in Chapter 1 for $d = 1$(see Figure 1 and 2). Let the set of possible velocities $\mathcal{V}$, be a finite subset of $\mathbb{R}$, and for $x \in \mathbb{R}$. Moreover, fix a velocity $v \in \mathcal{V}$, at any given time, each site of $\{1, \ldots, N-1\}$ is either empty or occupied by one particle with velocity $v$. In $\{1, \ldots, N-1\}$, each particle attempts to jump to one of its neighbors with the same velocity, with a weakly asymmetric rate. To prevent the occurrence of more than one particle per site with the same velocity $v$, we introduce an exclusion rule that suppresses each jump to an already occupied site, with the fixed velocity $v$. The boundary dynamics is given by the following birth and death process at the sites 1 or $N-1$. A particle is inserted into the system with rate $\frac{\alpha_v}{N^\theta}$ at site 1 if the site is empty, while if the site 1 is occupied a particle is removed from the system with rate $\frac{1-\alpha_v}{N^\theta}$. On the other hand, at site $N-1$ a particle is inserted into the system, with rate $\frac{\beta_v}{N^\theta}$, if the site is empty, while a particle is removed at $N-1$ if the site is occupied, with rate $\frac{1-\beta_v}{N^\theta}$. Superposed to this dynamics, there is a collision process that exchanges velocities of particles in the same site in a way that the moment is conserved.

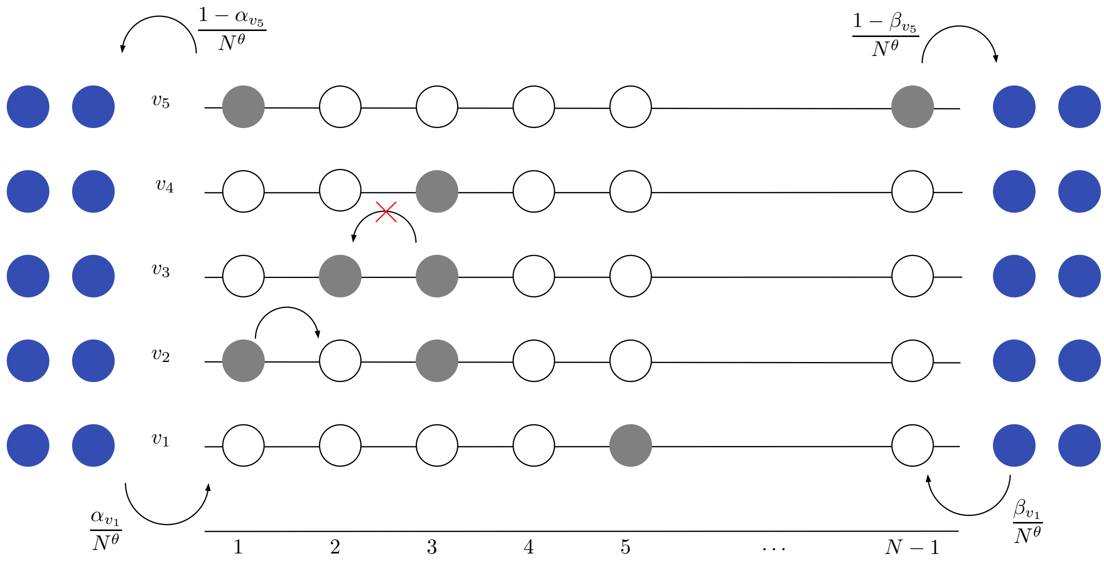


Figure 1: Illustration of the dynamics

We draw some pictures to illustrate the dynamics that we will work on in the following sections. In Figure 1, we have an illustration of the dynamics, the particles at the bulk are colored in gray, and the particles at the two reservoirs are colored in blue. Note that if a particle at site $x$ with velocity $v$, attempts to jump to an already occupied site $y$ with velocity $v$, the jump is not allowed. In this case the particle does not move, see for example in Figure 1, the particle at site 3 with velocity $v_3$ is not allowed to jump to site 2 with velocity $v_3$. On the other hand, if the destination site is empty the jump is performed, see for example in Figure 1, the particle at site 1 with velocity $v_2$ is allowed to jump to site 2 with velocity $v_2$. Let us suppose that the clock associated to the left-most reservoir rings, since there exist no particle at site 1 with velocity $v_1$, a particle can be injected into the system at the site 1 with velocity $v_1$ with rate $\frac{\alpha_{v_1}}{N^\theta}$. Also, if the clock associated to the site 1 with velocity $v_5$ rings, the particle leaves the system at rate $\frac{1-\alpha_{v_5}}{N^\theta}$ (See Figure 1). Analogously, suppose that the clock associated to the right-most reservoir rings, since there exist no particle at site $N-1$ with velocity $v_1$, a particle can be injected into the system at the site $N-1$ with velocity $v_1$ with rate $\frac{\beta_{v_1}}{N^\theta}$. Also, if the clock associated to the site $N-1$ with velocity

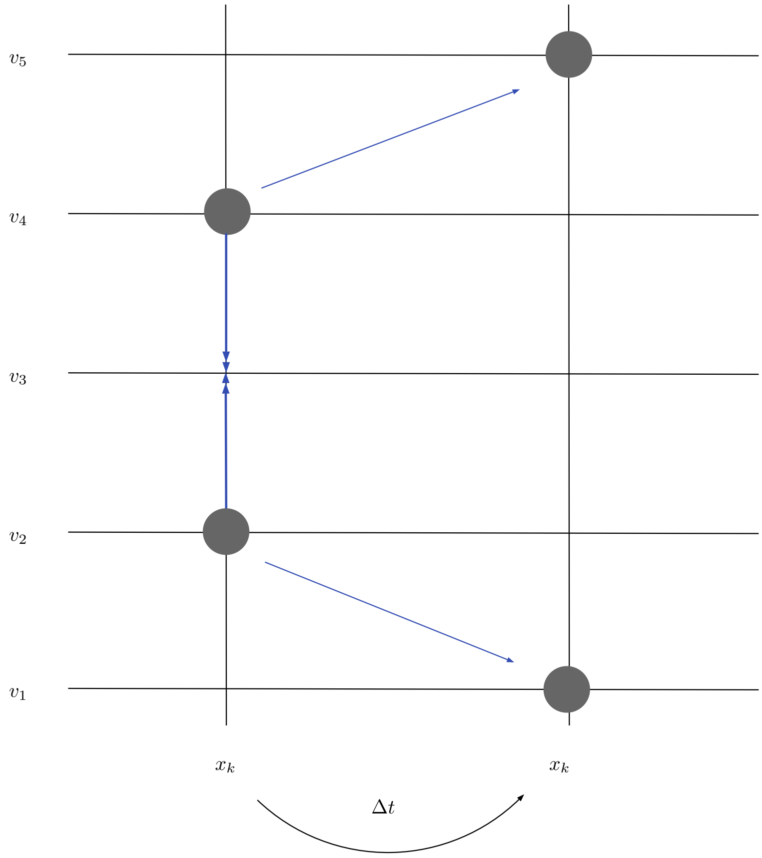


Figure 2: Illustration of the collision dynamics

$v_5$ rings, the particle leaves the system at rate $\frac{1-\beta_{v_5}}{N^\theta}$ (See Figure 1).

Now let us suppose that the clock associated to the site $x_k$ rings (see Figure 2). We have that two particles at $x_k$, with velocities $v_2$ and $v_4$ collide at rate one and produce two particles at the same site, with velocities $v_1$ and $v_5$, with $v_2+v_4=v_1+v_5$.

In Chapter 1 our goal is to show that the system can be described by a hydrodynamic equation: fix a macroscopic time interval $[0,T]$, and consider the dynamical behavior of the empirical density and momentum over such an interval. The law of large numbers for the empirical density and momentum, which is called hydrodynamic limit, and in the context of the diffusive scaling limit here considered, is given by a system of parabolic evolution equations which is called hydrodynamic equations.

In Chapter 2 we study the equilibrium fluctuations for the model that we presented in Chapter 1 with periodic boundary conditions, which can be viewed as a central limit theorem for the empirical distribution of particles when the system starts from an equilibrium measure. The purpose of this work is to study the density fluctuation field of this system as $N \to \infty$. We prove that the density field converges weakly to a generalized Ornstein-Uhlenbeck process.

These notes are organized as follows: In Chapter 1, Sections 1.1 to 1.4, we establish the notation adopted in this work and state some useful results. In Section 1.5, we state the main theorem of Chapter 1; the proof of the theorem is postponed to Section 1.10. In Section 1.7, we prove the Replacement Lemmas for the hydrodynamic limits. In Section 1.12, we prove uniqueness of weak solutions of the hydrodynamic equations, which are also needed for the hydrodynamic limits. In Chapter 2, we describe in details the model, in Sections 2.1, 2.2 and 2.3, that we study. Then, in Section 2.4, we start the analysis of the equilibrium fluctuations for this model, introducing the fluctuation field and we state our main results. We present the Boltzmann-Gibbs principle in Section 2.6; and in Section 2.9, we prove tightness of the density fluctuation field. In the Appendices, we present some technical results that are needed along with the proofs.

# Frequently Used Notation

- $C^{m,n}([0,T], D^d)$ is the space of continuous functions with $m$ continuous derivatives in time $t \in [0,T]$ and $n$ continuous derivatives in the space $D^d$;
- $\lfloor r \rfloor$ denote the integer part of $r$;
- $\mathcal{L}_N^{ex}$ is the generator of the exclusion part of the dynamics;
- $\mathcal{L}_N^{c}$ is the generator of the collision part of the dynamics;
- $\mathcal{L}_N^{b}$ is the generator of the boundary part of the dynamics;
- $\mathcal{V}$ is the set of velocities;
- $D_N^d = S_N \times \mathbb{T}_N^{d-1}$;
- $\mathcal{D}([0,T], \mathcal{M}_+ \times \mathcal{M}^d)$ is the set of right continuous functions with left limits taking values in $\mathcal{M}_+ \times \mathcal{M}^d$;
- $(\mathbb{Q}_N)_{N\geq 1}$ is a sequence of probability measures defined on $\mathcal{D}([0,T], \mathcal{M}_+ \times \mathcal{M}^d)$;
- $Q$ is the set of all collisions which preserve momentum;
- $\mathscr{H}^1(D^d)$ the Sobolev space of measurable functions in $L^2(D^d)$ that have generalized derivatives in $L^2(D^d)$;
- $\Omega_T = (0,T) \times D^d$;
- $C_c^\infty(\Omega_T)$ stands for the set of infinitely differentiable functions (with respect to time and space) with compact support contained in $\Omega_T$;
- $C_0^\infty(\Omega_T)$ stands for the space of infinitely differentiable functions vanishing at the boundary of $D^d$.

# Hydrodynamic Limit

## 1.1 Notation and Results

We start by establishing the notation to be used throughout this chapter. Let $\mathbb{T}_N^d = \{0, \ldots, N-1\}^d = (\mathbb{Z}/N\mathbb{Z})^d$ be the $d$-dimensional discrete torus, and denote by $D_N^d$ the set $S_N \times \mathbb{T}_N^{d-1}$, which will henceforth be called by *bulk*, where $S_N = \{1, \ldots, N-1\}$. Further, denote the $d$-dimensional torus by $\mathbb{T}^d = [0,1)^d = (\mathbb{R}/\mathbb{Z})^d$, and let $D^d = [0,1] \times \mathbb{T}^{d-1}$. Moreover, let $\mathcal{V} \subset \mathbb{R}^d$ be a finite set of velocities $v = (v_1, \ldots, v_d)$. Assume that $\mathcal{V}$ is invariant under reflections and permutations of the coordinates, i.e.,

$$(v_1, \ldots, v_{i-1}, -v_i, v_{i+1}, \ldots, v_d) \text{ and } (v_{\sigma(1)}, \ldots, v_{\sigma(d)})$$

belong to $\mathcal{V}$ for all $1 \leq i \leq d$, and all permutations $\sigma$ of $\{1, \ldots, d\}$, provided $(v_1, \ldots, v_d)$ belongs to $\mathcal{V}$.

At each site of $D_N^d$, at most one particle with a certain velocity is allowed. We also denote by $\eta(x,v) \in \{0,1\}$ the number of particles with fixed velocity $v \in \mathcal{V}$ at site $x \in D_N^d$; by $\eta_x = \{\eta(x,v);\, v \in \mathcal{V}\}$ the number of particles in each velocity $v$ at site $x$; and a configuration by $\eta = \{\eta_x;\, x \in D_N^d\}$. The set of particle configurations is $X_N = (\{0,1\}^{\mathcal{V}})^{D_N^d}$.

On the interior of the domain, the dynamics consist of two parts:

(i) each particle in the system evolves according to the nearest neighbor weakly asymmetric random walk with exclusion among particles with the same velocity,

(ii) binary collisions between particles with different velocities.

Let $p(x,v)$ be an irreducible transition probability with finite range, and mean velocity

$v$, i.e.,

$$\sum_{x\in\mathbb{Z}^d} xp(x,v) = v.$$

The jump law and the waiting times are chosen so that the jump rate from site $x$, with velocity $v$, to site $x+y$, with the same velocity $v$, is given by

$$P_N(y,v) = \frac{1}{2}\sum_{j=1}^{d}(\delta_{y,e_j} + \delta_{y,-e_j}) + \frac{1}{N}p(y,v), \tag{1.1}$$

where $\delta_{x,y}$ stands for the Kronecker delta, which is equal to one if $x=y$ and 0 otherwise, and $\{e_1,\dots,e_d\}$ is the canonical basis in $\mathbb{R}^d$.

## 1.2 Infinitesimal Generator

In this section, we describe the model that we are going to consider in these thesis. Our main interest is to analyze the stochastic lattice gas model given by the generator $\mathcal{L}_N$, which is the superposition of the Glauber dynamics with the collision and exclusion dynamics:

$$\mathcal{L}_N = N^2\{\mathcal{L}_N^b + \mathcal{L}_N^c + \mathcal{L}_N^{ex}\}, \tag{1.2}$$

where $\mathcal{L}_N^b$ denotes the generator of the Glauber dynamics, modeling insertion or removal of particles, $\mathcal{L}_N^c$ denotes the generator that models the collision part of the dynamics and lastly, $\mathcal{L}_N^{ex}$ models the exclusion part of the dynamics. Note that in (1.2) time has been speeded up diffusively due to the factor $N^2$.

Let $f: X_N \to \mathbb{R}$. The generator of the exclusion part of the dynamics, $\mathcal{L}_N^{ex}$, is given by

$$(\mathcal{L}_N^{ex}f)(\eta) = \sum_{v\in\mathcal{V}}\sum_{x,z\in D_N^d} \eta(x,v)(1-\eta(z,v))P_N(z-x,v)[f(\eta^{x,z,v}) - f(\eta)],$$

where

$$\eta^{x,y,v}(z,w) = \begin{cases} \eta(y,v) \text{ if } w=v \text{ and } z=x, \\ \eta(x,v) \text{ if } w=v \text{ and } z=y, \\ \eta(z,w) \text{ otherwise.} \end{cases}$$

Because the definition of $P_N$, in (1.1), we can use the decomposition

$$\mathcal{L}_N^{ex} = \mathcal{L}_N^{ex,1} + \mathcal{L}_N^{ex,2},$$

where

$$(\mathcal{L}_N^{ex,1} f)(\eta) = \frac{1}{2} \sum_{v \in \mathcal{V}} \sum_{\substack{x,z \in D_N^d \\ |z-x|=1}} \eta(x,v)(1-\eta(z,v))[f(\eta^{x,z,v}) - f(\eta)],$$

and

$$(\mathcal{L}_N^{ex,2} f)(\eta) = \frac{1}{N} \sum_{v \in \mathcal{V}} \sum_{x,z \in D_N^d} \eta(x,v)(1-\eta(z,v)) p(z-x,v)[f(\eta^{x,z,v}) - f(\eta)].$$

The generator of the collision part of the dynamics, $\mathcal{L}_N^c$, is given by

$$(\mathcal{L}_N^c f)(\eta) = \sum_{y \in D_N^d} \sum_{q \in Q} p_c(y,q,\eta)[f(\eta^{y,q}) - f(\eta)],$$

where $Q$ is the set of all collisions which preserve momentum:

$$Q = \{q = (v,w,v',w') \in \mathcal{V}^4 : v + w = v' + w'\}.$$

The rate $p_c(y,q,\eta)$ is given by

$$p_c(y,q,\eta) = \eta(y,v)\eta(y,w)[1-\eta(y,v')][1-\eta(y,w')],$$

and for $q = (v_0, v_1, v_2, v_3)$, the configuration $\eta^{y,q}$ after the collision is defined as

$$\eta^{y,q}(z,u) = \begin{cases} \eta(y, v_{j+2}) \text{ if } z = y \text{ and } u = v_j \text{ for some } 0 \le j \le 3, \\ \eta(z,u) \text{ otherwise,} \end{cases}$$

where the index of $v_{j+2}$ should be taken modulo 4.

Particles of velocities $v$ and $w$ at the same site collide at rate one and produce two particles of velocities $v'$ and $w'$ at the same site and $v + w = v' + w'$.

Finally, the generator of the Glauber dynamics is given by

$$(\mathcal{L}_N^b f)(\eta) = \sum_{\substack{x\in D_N^d\\ x_1=1}} \sum_{v\in\mathcal{V}} \Big(\frac{\alpha_v(\tilde{x}/N)}{N^\theta}[1-\eta(x,v)] + \frac{1-\alpha_v(\tilde{x}/N)}{N^\theta}[\eta(x,v)]\Big)[f(\sigma^{x,v}\eta)-f(\eta)]$$

$$+ \sum_{\substack{x\in D_N^d\\ x_1=N-1}} \sum_{v\in\mathcal{V}} \Big(\frac{\beta_v(\tilde{x}/N)}{N^\theta}[1-\eta(x,v)] + \frac{1-\beta_v(\tilde{x}/N)}{N^\theta}[\eta(x,v)]\Big)[f(\sigma^{x,v}\eta)-f(\eta)],$$

where $\tilde{x} = (x_2, \ldots, x_d)$, $x = (x_1, \tilde{x})$ and

$$\sigma^{x,v}\eta(y,w) = \begin{cases} 1-\eta(x,v), & \text{if } w = v \text{ and } y = x, \\ \eta(y,w), & \text{otherwise} \end{cases}$$

for every $v \in \mathcal{V}$, $\alpha_v$, $\beta_v \in C^2(\mathbb{T}^{d-1})$ and $\theta \geq 0$. We also assume that, for every $v \in \mathcal{V}$, the functions $\alpha_v$ and $\beta_v$, have images belonging to some compact subset of $(0,1)$, which means that $\alpha_v(\cdot)$ and $\beta_v(\cdot)$ are bounded away from 0 and 1. The functions $\alpha_v(\cdot)$ and $\beta_v(\cdot)$, which affect the birth and death rates at the boundary, represent the density of the reservoirs.

In the text, sometimes it will be more convenient to write

$$\begin{aligned} r_x(\eta,\alpha) &= \alpha_v(\tilde{x}/N)(1-\eta(x,v)) + (1-\alpha_v(\tilde{x}/N))\eta(x,v), \\ r_x(\eta,\beta) &= \beta_v(\tilde{x}/N)(1-\eta(x,v)) + (1-\beta_v(\tilde{x}/N))\eta(x,v). \end{aligned} \tag{1.3}$$

Let $\{\eta(t),\, t \geq 0\}$ be the Markov process with generator $\mathcal{L}_N$ and denote by $\{S_t^N,\, t \geq 0\}$ the semigroup associated to $\mathcal{L}_N$.

Let $\mathcal{D}(\mathbb{R}_+, X_N)$ be the set of right continuous functions with left limits taking values in $X_N$ endowed with the Skorohod topology. For a probability measure $\mu$ on $X_N$, denote by $\mathbb{P}_\mu$ the measure on the path space $\mathcal{D}(\mathbb{R}_+, X_N)$ induced by $\{\eta(t) : t \geq 0\}$ and the initial measure $\mu$. The expectation with respect to $\mathbb{P}_\mu$ is denoted by $\mathbb{E}_\mu$.

## 1.3 Mass and Momentum

For each configuration $\xi \in \{0,1\}^{\mathcal{V}}$, denote by $I_0(\xi)$ the mass of $\xi$ and by $I_k(\xi)$, $k = 1, \dots, d$, the momentum of $\xi$, i.e.,

$$I_0(\xi) = \sum_{v \in \mathcal{V}} \xi(v), \quad I_k(\xi) = \sum_{v \in \mathcal{V}} v_k \xi(v).$$

Set $I(\xi) := (I_0(\xi), \dots, I_d(\xi))$. Assume that the set of velocities is chosen in such a way that the unique conserved quantities by the random walk dynamics described above are the mass and the momentum: $\sum_{x \in D_N^d} I(\eta_x)$.

Two examples of sets of velocities satisfying these conditions can be found in [7], one of the models is the following. Denote by $\mathcal{E} = \{e = \pm e_i \text{ for some } i = 1, \dots, d\}$, let $\mathcal{V} = \mathcal{E}$, with this choice, the only possible collisions are those $q = (v, w, v', w')$ such that $v + w = 0$ and $v' + w' = 0$.

For each chemical potential $\lambda = (\lambda_0, \dots, \lambda_d) \in \mathbb{R}^{d+1}$, denote by $m_\lambda$ the probability measure on $\{0,1\}^{\mathcal{V}}$ given by

$$m_\lambda(\xi) = \frac{1}{Z(\lambda)} \exp\{\lambda \cdot I(\xi)\}, \tag{1.4}$$

where $Z(\lambda)$ is a normalizing constant. Note that $m_\lambda$ is a product measure on $\{0,1\}^{\mathcal{V}}$, i.e., the variables $\{\xi(v) : v \in \mathcal{V}\}$ are independent under $m_\lambda$.

Denote by $\mu_\lambda^N$ the product measure on $X_N$, with marginals given by

$$\mu_\lambda^N\{\eta : \eta(x, \cdot) = \xi\} = m_\lambda(\xi),$$

for each $\xi \in \{0,1\}^{\mathcal{V}}$ and $x \in D_N^d$. Note that $\{\eta(x,v) : x \in D_N^d, v \in \mathcal{V}\}$ are independent variables under $\mu_\lambda^N$, and that the measure $\mu_\lambda^N$ is invariant for the exclusion process with periodic boundary conditions, in this case the generator is given by $\mathcal{L}_N = N^2\{\mathcal{L}_N^c + \mathcal{L}_N^{ex}\}$. The expectation under $\mu_\lambda^N$ of the mass and momentum are, respectively, given by

$$\rho(\lambda) := E_{\mu_\lambda^N}[I_0(\eta_x)] = \sum_{v \in \mathcal{V}} \theta_v(\lambda),$$

$$\varrho_k(\lambda) := E_{\mu_\lambda^N}[I_k(\eta_x)] = \sum_{v \in \mathcal{V}} v_k \theta_v(\lambda).$$

In the last formula, $\theta_v(\lambda)$ denotes the expected value of the density of particles with velocity $v$ under $m_\lambda$:

$$\theta_v(\lambda) := E_{m_\lambda}[\xi(v)] = \frac{\exp\left\{\lambda_0 + \sum_{k=1}^{d} \lambda_k v_k\right\}}{1 + \exp\left\{\lambda_0 + \sum_{k=1}^{d} \lambda_k v_k\right\}}. \tag{1.5}$$

Denote by $(\rho, \varrho)(\lambda) := (\rho(\lambda), \varrho_1(\lambda), \ldots, \varrho_d(\lambda))$ the map that associates the chemical potential to the vector of density and momentum. It is possible to prove that $(\rho, \varrho)$ is a diffeomorphism onto $\mathfrak{U} \subset \mathbb{R}^{d+1}$, the interior of the convex envelope of $\{I(\xi), \xi \in \{0,1\}^{\mathcal{V}}\}$. Denote by $\Lambda = (\Lambda_0, \ldots, \Lambda_d) : \mathfrak{U} \to \mathbb{R}^{d+1}$ the inverse of $(\rho, \varrho)$. This correspondence allows one to parameterize the invariant states by the density and momentum: for each $(\rho, \varrho) \in \mathfrak{U}$, we have a product measure $\nu^N_{\rho,\varrho} = \mu^N_{\Lambda(\rho,\varrho)}$ on $X_N$.

## 1.4 Hydrodynamic Equations

From now on, we fix a finite time horizon $[0, T]$. We denote by $C^{m,n}([0,T] \times D^d)$ the set of functions defined on $[0,T] \times D^d$ that are $m$ times differentiable on the first variable, $n$ times differentiable on the second variable and have continuous derivatives. For a function $G := G(t,u) \in C^{m,n}([0,T] \times D^d)$, we denote by $\partial_t G$ its derivative with respect to the time variable $t$ and by $\partial_{u_i} G$ its derivative with respect to the space variable $u_i$, with $i = 1, \ldots, d$. For simplicity of notation, we set $\Delta G := \sum_{i=1}^{d} \frac{\partial^2 G}{\partial {u_i}^2}$ and $\nabla G$ represents the generalized gradient of the function $G$. Finally, $C_0^{m,n}([0,T] \times D^d)$ is the set of functions $G \in C^{m,n}([0,T] \times D^d)$ such that for any time $t$ the function $G_t$ vanishes at the boundary, that is, $G_t(0, \tilde{u}) = G_t(1, \tilde{u}) = 0$, where we denote $u \in D^d$ by $(u_1, \tilde{u})$, with $\tilde{u} \in \mathbb{T}^{d-1}$.

Let $(B, \|\cdot\|_B)$ be a separable Banach space. We denote by $L^2([0,T], B)$ the Banach space of measurable functions $U : [0,T] \to B$ for which

$$\|U\|^2_{L^2([0,T],B)} = \int_0^T \|U_t\|^2_B \, dt < \infty.$$

Moreover, we denote by $\mathscr{H}^1(D^d)$ the Sobolev space of measurable functions in $L^2(D^d)$ that have generalized derivatives in $L^2(D^d)$.

Now that we have introduced all the notation and the spaces of functions that we will use, we can define the system of partial differential equations and the respective notions of weak solutions which are involved in the hydrodynamic limit of this model, under different boundary coditions.

### 1.4.1 Dirichlet boundary conditions

For $x = (x_1, \tilde{x}) \in \{0,1\} \times \mathbb{T}^{d-1}$, consider

$$d(x) = \begin{cases} \sum_{v\in\mathcal{V}} (\alpha_v(\tilde{x}), v_1\alpha_v(\tilde{x}), \ldots, v_d\alpha_v(\tilde{x})) & \text{if } x_1 = 0, \\ \sum_{v\in\mathcal{V}} (\beta_v(\tilde{x}), v_1\beta_v(\tilde{x}), \ldots, v_d\beta_v(\tilde{x})) & \text{if } x_1 = 1. \end{cases} \tag{1.6}$$

**Definition 1.** *Fix a measurable density profile $\rho_0 : D^d \to \mathbb{R}_+$, and a measurable momentum profile $\varrho_0 : D^d \to \mathbb{R}^d$. We say that $(\rho, \varrho) : [0,T] \times D^d \to \mathbb{R}_+ \times \mathbb{R}^d$ is a weak solution of the system of parabolic partial differential equations*

$$\begin{cases} \partial_t(\rho, \varrho) + \sum_{v\in\mathcal{V}} \tilde{v}[v \cdot \nabla\chi(\theta_v(\Lambda(\rho, \varrho)))] = \frac{1}{2}\Delta(\rho, \varrho), \\ (\rho, \varrho)(0, \cdot) = (\rho_0, \varrho_0)(\cdot) \quad \textit{and} \quad (\rho, \varrho)(t, x) = d(x),\ x \in \{0,1\} \times \mathbb{T}^{d-1}, \end{cases} \tag{1.7}$$

*where $\chi(r) = r(1-r)$ is the static compressibility and for each velocity $v = (v_1, \ldots, v_d)$, we define $\tilde{v} = (1, v_1, \ldots, v_d)$, if the following two conditions hold:*

*(i)* $(\rho, \varrho) \in L^2([0,T], \mathscr{H}^1(D^d))$;

*(ii)* $(\rho, \varrho)$ *satisfies the weak formulation:*

$$\begin{aligned} &\int_{D^d} (\rho, \varrho)(T, u) G(T, u)\, du - \int_{D^d} (\rho_0, \varrho_0)(u) G(0, u)\, du = \\ &\int_0^T dt \int_{D^d} du \left\{ (\rho, \varrho)(t, u)\partial_t G(t, u) + \frac{1}{2}(\rho, \varrho)(t, u) \sum_{i=1}^d \frac{\partial^2 G}{\partial u_i^2}(t, u) \right\} \\ &-\frac{1}{2}\int_0^T\int_{\{1\}\times\mathbb{T}^{d-1}} d(1, \tilde{u})\, \frac{\partial G}{\partial u_1}(t, 1, \tilde{u})\, dS\, dt + \frac{1}{2}\int_0^T\int_{\{0\}\times\mathbb{T}^{d-1}} d(0, \tilde{u})\, \frac{\partial G}{\partial u_1}(t, 0, \tilde{u})\, dS\, dt \\ &+ \int_0^T\int_{D^d} \sum_{v\in\mathcal{V}} \tilde{v} \cdot \chi(\theta_v(\Lambda(\rho, \varrho))) \sum_{i=1}^d v_i \frac{\partial G}{\partial u_i}(t, u)\, du\, dt \end{aligned} \tag{1.8}$$

*for all $t \in [0,T]$ and any function $G : [0,T] \times D^d \to \mathbb{R}^{d+1}$ in $C_0^{1,2}([0,T] \times D^d)$.*

### 1.4.2 Robin boundary conditions

**Definition 2.** *Fix a measurable density profile $\rho_0 : D^d \to \mathbb{R}_+$, and a measurable momentum profile $\varrho_0 : D^d \to \mathbb{R}^d$. We say that $(\rho, \varrho) : [0,T] \times D^d \to \mathbb{R}_+ \times \mathbb{R}^d$ is a weak solution of the system of parabolic partial differential equations*

$$\begin{cases} \partial_t(\rho,\varrho) + \sum_{v\in\mathcal{V}} \tilde{v}[v \cdot \nabla\chi(\theta_v(\Lambda(\rho,\varrho)))] = \frac{1}{2}\Delta(\rho,\varrho), \\ \frac{\partial(\rho,\varrho)}{\partial u_1}(t,0,\tilde{u}) - 2\sum_{v\in\mathcal{V}} \tilde{v}[v \cdot \chi(\theta_v(\Lambda(\rho,\varrho)))] = (\rho,\varrho)(t,0,\tilde{u}) - \sum_{v\in\mathcal{V}} v_k\alpha_v(\tilde{u}), \\ \frac{\partial(\rho,\varrho)}{\partial u_1}(t,1,\tilde{u}) - 2\sum_{v\in\mathcal{V}} \tilde{v}[v \cdot \chi(\theta_v(\Lambda(\rho,\varrho)))] = \sum_{v\in\mathcal{V}} v_k\beta_v(\tilde{u}) - (\rho,\varrho)(t,1,\tilde{u}), \quad t \in (0,T], \\ (\rho,\varrho)(0,\cdot) = (\rho_0,\varrho_0)(\cdot) \end{cases} \tag{1.9}$$

*where $\chi(r) = r(1-r)$ is the static compressibility of the system and for each velocity $v = (v_1, \ldots, v_d)$, we define $\tilde{v} = (1, v_1, \ldots, v_d)$, if the following two conditions hold:*

*(i)* $(\rho,\varrho) \in L^2([0,T], \mathscr{H}^1(D^d))$;

*(ii)* $(\rho,\varrho)$ *satisfies the weak formulation:*

$$\begin{aligned} &\int_{D^d} (\rho,\varrho)(T,u)G(T,u)\,du - \int_{D^d} (\rho,\varrho)(0,u)G(0,u)\,du = \\ &\int_0^T\int_{D^d} (\rho,\varrho)(t,u)\partial_t G(t,u)\,du\,dt \\ &+ \int_0^T\int_{D^d} \sum_{v\in\mathcal{V}} \tilde{v}\cdot\chi(\theta_v(\Lambda(\rho,\varrho))) \sum_{i=1}^d v_i \frac{\partial G}{\partial u_i}(t,u)\,du\,dt \\ &+ \frac{1}{2}\int_0^T\int_{\{1\}\times\mathbb{T}^{d-1}} [\sum_{v\in\mathcal{V}} v_k\beta_v(\tilde{u}) - (\rho,\varrho)(t,1,\tilde{u})]G(t,1,\tilde{u})\,dS\,dt \\ &- \frac{1}{2}\int_0^T\int_{\{0\}\times\mathbb{T}^{d-1}} [(\rho,\varrho)(t,0,\tilde{u}) - \sum_{v\in\mathcal{V}} v_k\alpha_v(\tilde{u})]G(t,0,\tilde{u})\,dS\,dt \\ &- \frac{1}{2}\int_0^T\int_{\{1\}\times\mathbb{T}^{d-1}} (\rho,\varrho)(t,1,\tilde{u})\frac{\partial G}{\partial u_1}(t,1,\tilde{u})\,dS\,dt \\ &+ \frac{1}{2}\int_0^T\int_{\{0\}\times\mathbb{T}^{d-1}} (\rho,\varrho)(t,0,\tilde{u})\frac{\partial G}{\partial u_1}(t,0,\tilde{u})\,dS\,dt \\ &+ \frac{1}{2}\int_0^T\int_{D^d} (\rho,\varrho)(t,u)\sum_{i=1}^d \frac{\partial^2 G}{\partial u_i^2}(t,u)\,du\,dt \end{aligned} \tag{1.10}$$

*for all $t \in [0,T]$ and any function $G : [0,T] \times D^d \to \mathbb{R}^{d+1}$ in $C^{1,2}([0,T] \times D^d)$.*

### 1.4.3 Neumann boundary conditions

**Definition 3.** *Fix a measurable density profile $\rho_0 : D^d \to \mathbb{R}_+$, and a measurable momentum profile $\varrho_0 : D^d \to \mathbb{R}^d$. We say that $(\rho, \varrho) : [0,T] \times D^d \to \mathbb{R}_+ \times \mathbb{R}^d$ is a weak solution of the system of parabolic partial differential equations*

$$\begin{cases} \partial_t(\rho,\varrho) + \sum_{v\in\mathcal{V}} \tilde{v}[v \cdot \nabla\chi(\theta_v(\Lambda(\rho,\varrho)))] = \frac{1}{2}\Delta(\rho,\varrho), \\ \frac{\partial(\rho,\varrho)}{\partial u_1}(t,0,\tilde{u}) - 2\sum_{v\in\mathcal{V}} \tilde{v}[v \cdot \chi(\theta_v(\Lambda(\rho,\varrho)))] = 0, \\ \frac{\partial(\rho,\varrho)}{\partial u_1}(t,1,\tilde{u}) - 2\sum_{v\in\mathcal{V}} \tilde{v}[v \cdot \chi(\theta_v(\Lambda(\rho,\varrho)))] = 0, \quad t \in (0,T], \\ (\rho,\varrho)(0,\cdot) = (\rho_0,\varrho_0)(\cdot) \end{cases} \tag{1.11}$$

*where $\chi(r) = r(1-r)$ is the static compressibility of the system and for each velocity $v = (v_1, \ldots, v_d)$, we define $\tilde{v} = (1, v_1, \ldots, v_d)$, if the following two conditions hold:*

*(i)* $(\rho,\varrho) \in L^2([0,T], \mathscr{H}^1(D^d))$;

*(ii)* $(\rho,\varrho)$ *satisfies the weak formulation:*

$$\begin{aligned} &\int_{D^d} (\rho,\varrho)(T,u)G(T,u)\,du - \int_{D^d} (\rho,\varrho)(0,u)G(0,u)\,du = \\ &\int_0^T\int_{D^d} (\rho,\varrho)(t,u)\partial_t G(t,u)\,du\,dt \\ &+ \int_0^T\int_{D^d} \sum_{v\in\mathcal{V}} \tilde{v} \cdot \chi(\theta_v(\Lambda(\rho,\varrho))) \sum_{i=1}^d v_i \frac{\partial G}{\partial u_i}(t,u)\,du\,dt \\ &-\frac{1}{2}\int_0^T\int_{\{1\}\times\mathbb{T}^{d-1}} (\rho,\varrho)(t,1,\tilde{u})\frac{\partial G}{\partial u_1}(t,1,\tilde{u})\,dS\,dt \\ &+\frac{1}{2}\int_0^T\int_{\{0\}\times\mathbb{T}^{d-1}} (\rho,\varrho)(t,0,\tilde{u})\frac{\partial G}{\partial u_1}(t,0,\tilde{u})\,dS\,dt \\ &+\frac{1}{2}\int_0^T\int_{D^d} (\rho,\varrho)(t,u)\sum_{i=1}^d \frac{\partial^2 G}{\partial u_i^2}(t,u)\,du\,dt \end{aligned} \tag{1.12}$$

*for all $t \in [0,T]$ and any function $G : [0,T] \times D^d \to \mathbb{R}^{d+1}$ in $C^{1,2}([0,T] \times D^d)$.*

**Remark 1.** *We obtain the weak formulation of the system of partial differential equations with one of the above boundary conditions by multiplying both sides of the identity*

$$\partial_t(\rho, \varrho) + \sum_{v\in\mathcal{V}} \tilde{v}[v \cdot \nabla\chi(\theta_v(\Lambda(\rho, \varrho)))] = \frac{1}{2}\Delta(\rho, \varrho)^{1} \tag{1.13}$$

*by a test function $G$, then integrating both in space and time and finally, performing two formal integration by parts in space and one in time. Finally, applying the respective boundary conditions, we obtain the corresponding integral equations. For more details, see Appendix A.1.*

## 1.5 Hydrodynamic Limit for the Boundary Driven Process

Let $\mathcal{M}_+$ be the space of finite positive measures on $D^d$ endowed with the weak topology, and let $\mathcal{M}$ be the space of bounded variation signed measures on $D^d$ endowed with the weak topology. Let $\mathcal{M}_+ \times \mathcal{M}^d$ be the cartesian product of these spaces endowed with the product topology, which is metrizable.

Recall that the conserved quantities are the mass and momentum presented in Section 1.3. For $k = 0, \ldots, d$, denote by $\pi_t^{k,N}$ the empirical measure associated to the $k$-th conserved quantity:

$$\pi_t^{k,N}(du) = \frac{1}{N^d} \sum_{x\in D_N^d} I_k(\eta_x(t))\delta_{x/N}(du), \tag{1.14}$$

where $\delta_u(du)$ stands for the Dirac measure supported on $u \in [0,1]^d$. We denote by $\langle \pi_t^{k,N}, G\rangle$ the integral of a test function $G$ with respect to the empirical measure $\pi_t^{k,N}$, and let $\langle f, g\rangle_\nu$ be the inner product in $L^2(\nu)$ of $f$ and $g$:

$$\langle f, g\rangle_\nu = \int f g \, d\nu.$$

Let $\mathcal{D}([0,T], \mathcal{M}_+ \times \mathcal{M}^d)$ be the set of right continuous functions with left limits taking values on $\mathcal{M}_+ \times \mathcal{M}^d$ endowed with the Skorohod topology. We consider the sequence $(\mathbb{Q}_N)_N$ of probability measures on $\mathcal{D}([0,T], \mathcal{M}_+ \times \mathcal{M}^d)$ that corresponds to the Markov process $\pi_t^N = (\pi_t^{0,N}, \ldots, \pi_t^{d,N})$ starting from $\mu^N$.

[1] Remember that in equation (1.13) for each velocity $v = (v_1, \ldots, v_d) \in \mathcal{V}$, we define $\tilde{v} = (1, v_1, \ldots, v_d)$

At this point we need to fix initial measurable profiles $\rho_0 : D^d \to \mathbb{R}_+$ and $\varrho_0 : D^d \to \mathbb{R}^d$, where $\varrho_0 = (\varrho_{0,1}, \dots, \varrho_{0,d})$ and an initial distribution $(\mu^N)_N$ on $X_N$.

Before introducing the main result, we establish some definitions.

**Definition 4.** *We say that $(\rho, \varrho)$ has finite energy if its components belong to $L^2([0,T], \mathscr{H}^1(D^d))$, i.e., $\nabla\rho$ and $\nabla\varrho_k$ are measurable functions and*

$$\int_0^T ds \left( \int_{D^d} \|\nabla\rho(s,u)\|^2 du \right) < \infty, \quad \int_0^T ds \left( \int_{D^d} \|\nabla\varrho_k(s,u)\|^2 du \right) < \infty,$$

*for $k = 1, \dots, d$.*

**Definition 5.** *We say that a sequence of probability measures $(\mu^N)_N$ on $X_N$ is associated to the density profile $\rho_0$ and to the momentum profile $\varrho_0$, if, for every continuous function $G : D^d \to \mathbb{R}$ and for every $\delta > 0$,*

$$\lim_{N\to\infty} \mu^N \left[ \eta : \left| \frac{1}{N^d} \sum_{x \in D_N^d} G\left(\tfrac{x}{N}\right) I_0(\eta_x) - \int_{D^d} G(u)\rho_0(u) du \right| > \delta \right] = 0,$$

*and for every $1 \le k \le d$,*

$$\lim_{N\to\infty} \mu^N \left[ \eta : \left| \frac{1}{N^d} \sum_{x \in D_N^d} G\left(\tfrac{x}{N}\right) I_k(\eta_x) - \int_{D^d} G(u)\varrho_{0,k}(u) du \right| > \delta \right] = 0.$$

**Theorem 1.** *Let $\rho_0$ and $\varrho_0$ be measurable functions, also let $(\mu^N)_N$ be a sequence of probability measures on $X_N$ associated to the profile $(\rho_0, \varrho_0)$. Then, for every $t \in [0,T]$, for every continuous function $G : D^d \to \mathbb{R}$, and for every $\delta > 0$,*

$$\lim_{N\to\infty} \mathbb{P}_{\mu^N} \left[ \left| \frac{1}{N^d} \sum_{x \in D_N^d} G\left(\tfrac{x}{N}\right) I_0(\eta_x(t)) - \int_{D^d} G(u)\rho(t,u)\, du \right| > \delta \right] = 0,$$

*and for $1 \le k \le d$,*

$$\lim_{N\to\infty} \mathbb{P}_{\mu^N} \left[ \left| \frac{1}{N^d} \sum_{x \in D_N^d} G\left(\tfrac{x}{N}\right) I_k(\eta_x(t)) - \int_{D^d} G(u)\varrho_k(t,u)\, du \right| > \delta \right] = 0,$$

*where $(\rho, \varrho)$ has finite energy (see Definition 4) and it is the unique weak solution of:*

- (1.7) *as given in Definition 1, if $0 \le \theta < 1$;*
- (1.9) *as given in Definition 2, if $\theta = 1$;*
- (1.11) *as given in Definition 3, if $\theta > 1$.*

**Remark 2.** *We split the proof of Theorem 1 into*

*(i) proof of tightness of the sequence* $(\mathbb{Q}_N)_N$*,*

*(ii) characterization of the unique limiting point* $\mathbb{Q}^*$ *of the sequence.*

*These two results, together, imply the weak convergence of* $(\mathbb{Q}_N)_N$ *to* $\mathbb{Q}^*$ *as* $N \to \infty$*.*

## 1.6 Heuristics for Hydrodynamic Equations

We need to introduce a function $\kappa$, which is going to be described later in Remark 3, to be able to obtain some entropy estimates that are essential to the proof of the hydrodynamic limit. We then consider $\nu_\kappa^N$ as the product measure on $X_N$ with marginals given by

$$\nu_\kappa^N\{\eta : \eta(x,\cdot) = \xi\} = m_{\Lambda(\kappa(x))}(\xi), \tag{1.15}$$

where $m_\lambda(\cdot)$ was defined in (1.4).

Next, we give the main ideas which are behind the identification of limit points as a weak solution of the system of parabolic partial differential equations given before, but we only present the heuristic argument.

We fix a function $H : [0,T] \times D^d \to \mathbb{R}^{d+1}$ which is continuously differentiable in time and twice continuously differentiable in space. By Dynkin's formula, see for example in [1, Appendix A1,Lemma 5.1],

$$M_t^{N,k}(H) = \langle \pi_t^{k,N}, H\rangle - \langle \pi_0^{k,N}, H\rangle - \int_0^t (\mathcal{L}_N + \partial_s)\langle \pi_s^{k,N}, H\rangle \, ds \tag{1.16}$$

is a martingale with respect to the natural filtration $\mathcal{F}_t = \sigma(\eta(s), s \leq t)$. We can rewrite

$$M_t^{N,k}(H) = \langle \pi_t^{k,N}, H\rangle - \langle \pi_0^{k,N}, H\rangle - \int_0^t \partial_s\langle \pi_s^{k,N}, H\rangle \, ds - \int_0^t N^2\mathcal{L}_N^c\langle \pi_s^{k,N}, H\rangle \, ds$$

$$- \int_0^t N^2\mathcal{L}_N^{ex,1}\langle \pi_s^{k,N}, H\rangle \, ds - \int_0^t N^2\mathcal{L}_N^{ex,2}\langle \pi_s^{k,N}, H\rangle \, ds - \int_0^t N^2\mathcal{L}_N^{b}\langle \pi_s^{k,N}, H\rangle \, ds.$$

Therefore, from (A.4), (A.6), (A.8) and (A.10), we have that

$$
\begin{aligned}
M_t^{N,k}(H) \; &= \langle \pi_t^{k,N}, H\rangle - \langle \pi_0^{k,N}, H\rangle - \int_0^t \langle \pi_s^{k,N}, \partial_s H\rangle \, ds - \frac{1}{2}\int_0^t \langle \pi_s^{k,N}, \Delta_N H\rangle \, ds \\
&+ \frac{1}{2}\int_0^t \frac{1}{N^{d-1}} \sum_{\substack{x\in D_N^d \\ x_1=N-1}} I_k(\eta_x(s)) \left\{ N\left[ H\left(\tfrac{N}{N}, \tfrac{\tilde{x}}{N}\right) - H\left(\tfrac{N-1}{N}, \tfrac{\tilde{x}}{N}\right)\right]\right\} ds \\
&- \frac{1}{2}\int_0^t \frac{1}{N^{d-1}} \sum_{\substack{x\in D_N^d \\ x_1=1}} I_k(\eta_x(s)) \left\{ -N\left[ H\left(\tfrac{0}{N}, \tfrac{\tilde{x}}{N}\right) - H\left(\tfrac{1}{N}, \tfrac{\tilde{x}}{N}\right)\right]\right\} ds \\
&- \int_0^t \frac{1}{N^d} \sum_{j=1}^d \sum_{x\in D_N^d} (\partial_{u_j}^N H)\Big(\tfrac{x}{N}\Big) \tau_x W_{j,k}^{N,\eta_s} \, ds \\
&- \int_0^t \frac{N^{1-\theta}}{N^d} \sum_{\substack{x\in D_N^d \\ x_1=1}} \sum_{v\in\mathcal{V}} v_k \, NH\left(\tfrac{1}{N}, \tfrac{\tilde{x}}{N}\right) [\alpha_v(\tfrac{\tilde{x}}{N}) - \eta_{sN^2}(1, \tilde{x}, v)] \, ds \\
&- \int_0^t \frac{N^{1-\theta}}{N^d} \sum_{\substack{x\in D_N^d \\ x_1=N-1}} \sum_{v\in\mathcal{V}} v_k \, NH\left(\tfrac{N-1}{N}, \tfrac{\tilde{x}}{N}\right) [\beta_v(\tfrac{\tilde{x}}{N}) - \eta_{sN^2}(N-1, \tilde{x}, v)] \, ds
\end{aligned}
\tag{1.17}
$$

where $\tau_x$ stands for the translation by $x$ on the state space $X_N$ so that $(\tau_x\eta)(y,v) = \eta(x+y,v)$ for all $x,y\in\mathbb{Z}^d, v\in\mathcal{V}$, and $W_{j,k}^{N,\eta_s}$ is given by:

$$
W_{j,k}^{N,\eta_s} = \sum_{v\in\mathcal{V}} v_k \sum_{z\in\mathbb{Z}^d} p(z,v) z_j \eta_s(0,v)[1-\eta_s(z,v)],
$$

where $v_0 = 1$. Since $p(\cdot, v)$ is of finite range,

$$
E_{\mu_\lambda^N}\left[W_{j,k}^{N,\eta_{sN^2}}\right] = \sum_{v\in\mathcal{V}} v_k v_j \chi(\theta_v(\lambda)),
$$

where $\chi(r) = r(1-r)$ as mentioned previously in Section 1.4.

### 1.6.1 The case $\theta \in [0,1)$

In this regime, we consider the test function $H \in C_0^{1,2}([0,T]\times D^d)$. Then, we can subtract $H\left(\frac{0}{N}, \frac{\tilde{x}}{N}\right)$ (resp. $H\left(\frac{N}{N}, \frac{\tilde{x}}{N}\right)$) in the eighth term (resp. the ninth term) at the right-hand side of (1.17) and then, making a Taylor expansion of $H$, we get that

$$
\begin{aligned}
M_t^{N,k}(H) \;&= \langle \pi_t^{k,N}, H\rangle - \langle \pi_0^{k,N}, H\rangle - \int_0^t \langle \pi_s^{k,N}, \partial_s H\rangle\, ds - \frac{1}{2}\int_0^t \langle \pi_s^{k,N}, \Delta_N H\rangle\, ds \\
&+ \frac{1}{2}\int_0^t \langle \pi_s^{k,N,b_{N-1}}, \partial_{u_1}^{N,+} H\rangle\, ds - \frac{1}{2}\int_0^t \langle \pi_s^{k,N,b_1}, \partial_{u_1}^{N,-} H\rangle\, ds \\
&- \int_0^t \frac{1}{N^d}\sum_{j=1}^d \sum_{x\in D_N^d} (\partial_{u_j}^N H)\Big(\tfrac{x}{N}\Big)\tau_x W_{j,k}^{N,\eta_{sN^2}}\, ds \\
&- \int_0^t \frac{N^{1-\theta}}{N^d} \sum_{\substack{x\in D_N^d\\ x_1=1}} \sum_{v\in\mathcal{V}} v_k\, \partial_{x_1}^{N,-} H\left(\tfrac{1}{N},\tfrac{\tilde{x}}{N}\right)\left[\alpha_v(\tfrac{\tilde{x}}{N}) - \eta_{sN^2}(1,\tilde{x},v)\right] ds \\
&+ \int_0^t \frac{N^{1-\theta}}{N^d} \sum_{\substack{x\in D_N^d\\ x_1=N-1}} \sum_{v\in\mathcal{V}} v_k\, \partial_{x_1}^{N,+} H\left(\tfrac{N-1}{N},\tfrac{\tilde{x}}{N}\right)\left[\beta_v(\tfrac{\tilde{x}}{N}) - \eta_{sN^2}(N-1,\tilde{x},v)\right] ds
\end{aligned}
$$

plus a term that vanishes, as $N\to\infty$.

Above

$$\partial_{x_1}^{N,-} H\left(\tfrac{1}{N},\tfrac{\tilde{x}}{N}\right) = -N\left[H\left(\tfrac{0}{N},\tfrac{\tilde{x}}{N}\right) - H\left(\tfrac{1}{N},\tfrac{\tilde{x}}{N}\right)\right]$$

and

$$\partial_{x_1}^{N,+} H\left(\tfrac{N-1}{N},\tfrac{\tilde{x}}{N}\right) = N\left[H\left(\tfrac{N}{N},\tfrac{\tilde{x}}{N}\right) - H\left(\tfrac{N-1}{N},\tfrac{\tilde{x}}{N}\right)\right].$$

As a consequence of Lemma 3, we can replace $\eta_{sN^2}(1,\tilde{x},v)$ by $\alpha_v\left(\tfrac{\tilde{x}}{N}\right)$ and $\eta_{sN^2}(N-1,\tilde{x},v)$ by $\beta_v\left(\tfrac{\tilde{x}}{N}\right)$. Then, we have

$$
\begin{aligned}
M_t^{N,k}(H) \;&= \langle \pi_t^{k,N}, H\rangle - \langle \pi_0^{k,N}, H\rangle - \int_0^t \langle \pi_s^{k,N}, \partial_s H\rangle\, ds - \frac{1}{2}\int_0^t \langle \pi_s^{k,N}, \Delta_N H\rangle\, ds \\
&+ \frac{1}{2}\int_0^t \langle \pi_s^{k,N,b_{N-1}}, \partial_{u_1}^{N,+} H\rangle\, ds - \frac{1}{2}\int_0^t \langle \pi_s^{k,N,b_1}, \partial_{u_1}^{N,-} H\rangle\, ds \\
&- \int_0^t \frac{1}{N^d}\sum_{j=1}^d \sum_{x\in D_N^d} (\partial_{u_j}^N H)\Big(\tfrac{x}{N}\Big)\tau_x W_{j,k}^{N,\eta_{sN^2}}\, ds
\end{aligned}
$$

plus a term that vanishes, as $N\to\infty$.

Taking the expectation with respect to $\nu_\kappa^N$ in the expression above we get

$$
\begin{aligned}
0 \quad &= \frac{1}{N^d} \sum_{x \in D_N^d} H\left(\tfrac{x}{N}\right) E_{\nu_\kappa^N}[I_k(\eta_x(t)) - I_k(\eta_x(0))] - \int_0^t \frac{1}{N^d} \sum_{x \in D_N^d} \partial_s H\left(\tfrac{x}{N}\right) E_{\nu_\kappa^N}[I_k(\eta_x)]\, ds \\
&- \frac{1}{2} \int_0^t \frac{1}{N^d} \sum_{x \in D_N^d} E_{\nu_\kappa^N}[I_k(\eta_x(s))] \Delta_N H\Big(\tfrac{x}{N}\Big)\, ds \\
&+ \frac{1}{2} \int_0^t \frac{1}{N^{d-1}} \sum_{\substack{x \in D_N^d \\ x_1 = N-1}} E_{\nu_\kappa^N}[I_k(\eta_x(s))] \partial_{u_1}^{N,+} H\Big(\tfrac{N-1}{N}, \tfrac{\tilde{x}}{N}\Big)\, ds \\
&- \frac{1}{2} \int_0^t \frac{1}{N^{d-1}} \sum_{\substack{x \in D_N^d \\ x_1 = 1}} E_{\nu_\kappa^N}[I_k(\eta_x(s))] \partial_{u_1}^{N,-} H\Big(\tfrac{1}{N}, \tfrac{\tilde{x}}{N}\Big)\, ds \\
&- \int_0^t \frac{1}{N^d} \sum_{j=1}^d \sum_{x \in D_N^d} (\partial_{u_j}^N H)\Big(\tfrac{x}{N}\Big) \sum_{v \in \mathcal{V}} v_k v_j \chi(\theta_v(\kappa)) ds
\end{aligned}
$$

plus a term that vanishes, as $N \to \infty$.

Note that, the restriction $\theta \geq 0$ comes from the fact that the errors, which arise from the Taylor expansion in H, have to vanish as $N \to \infty$ and the restriction $\theta < 1$ comes from the replacement of the occupation variables $\eta_{sN^2}(1, \tilde{x}, v)$ and $\eta_{sN^2}(N - 1, \tilde{x}, v)$ by $\alpha_v(\tilde{x})$ and $\beta_v(\tilde{x})$, respectively, see Lemma 3. At this point compare the previous expression with the weak formulation given in (1.8).

### 1.6.2 The case $\theta = 1$

In this regime, we consider the test function $H \in C^{1,2}([0,T] \times D^d)$ . We get

$$
\begin{aligned}
M_t^{N,k}(H) \; &= \langle \pi_t^{k,N}, H\rangle - \langle \pi_0^{k,N}, H\rangle - \int_0^t \langle \pi_s^{k,N}, \partial_s H\rangle \, ds - \frac{1}{2}\int_0^t \langle \pi_s^{k,N}, \Delta_N H\rangle \, ds \\
&+ \frac{1}{2}\int_0^t \langle \pi_s^{k,N,b_{N-1}}, \partial_{u_1}^{N,+} H\rangle \, ds - \frac{1}{2}\int_0^t \langle \pi_s^{k,N,b_1}, \partial_{u_1}^{N,-} H\rangle \, ds \\
&- \int_0^t \frac{1}{N^d} \sum_{j=1}^{d} \sum_{x \in D_N^d} (\partial_{u_j}^N H)\Big(\tfrac{x}{N}\Big) \tau_x W_{j,k}^{N,\eta_{sN^2}} \, ds \\
&- \int_0^t \frac{N^{2-\theta}}{N^d} \sum_{\substack{x \in D_N^d \\ x_1 = 1}} \sum_{v \in \mathcal{V}} v_k \, H\left(\tfrac{1}{N}, \tfrac{\tilde{x}}{N}\right) \left[\alpha_v\left(\tfrac{\tilde{x}}{N}\right) - \eta_{sN^2}(1, \tilde{x}, v)\right] ds \\
&+ \int_0^t \frac{N^{2-\theta}}{N^d} \sum_{\substack{x \in D_N^d \\ x_1 = N-1}} \sum_{v \in \mathcal{V}} v_k \, H\left(\tfrac{N-1}{N}, \tfrac{\tilde{x}}{N}\right) \left[\eta_{sN^2}(N-1, \tilde{x}, v) - \beta_v\left(\tfrac{\tilde{x}}{N}\right)\right] ds.
\end{aligned}
$$

For this regime, the replacement done before, due to Lemma 3, is no longer valid. Nevertheless, we can replace the integral in time of $\eta_s(1, \tilde{x}, v)$ (resp. $\eta_s(N-1, \tilde{x}, v)$) by integral in time of the average in a box around $(1, \tilde{x}, v)$ (resp. $(N-1, \tilde{x}, v)$):

$$
\overrightarrow{\eta}_{sN^2}^{\varepsilon N}(1, \tilde{x}, v) := \frac{1}{\varepsilon N} \sum_{x_1=1}^{1+\varepsilon N} \eta_{sN^2}(x_1, \tilde{x}, v), \qquad \overleftarrow{\eta}_{sN^2}^{\varepsilon N}(N-1, \tilde{x}, v) := \frac{1}{\varepsilon N} \sum_{x_1=N-1}^{N-1-\varepsilon N} \eta_{sN^2}(x_1, \tilde{x}, v).
$$

Here we note that the sum above goes from 1 to $1+\lfloor \varepsilon N \rfloor$, but for sake of simplicity, we write $1+\varepsilon N$. By noting that

$$
\overrightarrow{\eta}_{sN^2}^{\varepsilon N}(1, \tilde{x}, v) \sim (\rho, \varrho)_s(0), \quad (\text{ resp. } \overleftarrow{\eta}_{sN^2}^{\varepsilon N}(N-1, \tilde{x}, v) \sim (\rho, \varrho)_s(1)),
$$

in some sense, that we will be clear in the Section 1.11, for more information see [18].

We obtain that

$$
\begin{aligned}
0 \; &= \frac{1}{N^d}\sum_{x\in D_N^d} H\left(\tfrac{x}{N}\right) E_{\nu_\kappa^N}[I_k(\eta_x(t)) - I_k(\eta_x(0))] - \frac{1}{N^d}\sum_{x\in D_N^d} \partial_s H\left(\tfrac{x}{N}\right) E_{\nu_\kappa^N}[I_k(\eta_x)] \\
&-\frac{1}{2}\int_0^t \frac{1}{N^d}\sum_{x\in D_N^d} E_{\nu_\kappa^N}[I_k(\eta_{sN^2}(x))]\Delta_N H\Big(\tfrac{x}{N}\Big)\, ds \\
&+\frac{1}{2}\int_0^t \frac{1}{N^{d-1}}\sum_{\substack{x\in D_N^d \\ x_1=N-1}} E_{\nu_\kappa^N}[I_k(\eta_{sN^2}(x))]\partial_{u_1}^{N,+} H\Big(\tfrac{N-1}{N},\tfrac{\tilde{x}}{N}\Big)\, ds \\
&-\frac{1}{2}\int_0^t \frac{1}{N^{d-1}}\sum_{\substack{x\in D_N^d \\ x_1=1}} E_{\nu_\kappa^N}[I_k(\eta_{sN^2}(x))]\partial_{u_1}^{N,-} H\Big(\tfrac{1}{N},\tfrac{\tilde{x}}{N}\Big)\, ds \\
&-\int_0^t \frac{1}{N^d}\sum_{j=1}^d\sum_{x\in D_N^d} (\partial_{u_j}^N H)\Big(\tfrac{x}{N}\Big)\sum_{v\in\mathcal{V}} v_k v_j \chi(\theta_v(\kappa)) ds \\
&-\int_0^t \frac{N^{2-\theta}}{N^d}\sum_{\substack{x\in D_N^d \\ x_1=1}} H\Big(\tfrac{1}{N},\tfrac{\tilde{x}}{N}\Big) E_{\nu_\kappa^N}\Big[\sum_{v\in\mathcal{V}} v_k\alpha_v(\tfrac{\tilde{x}}{N}) - I_k(\eta_{sN^2}((1,\tilde{x},v))\Big] \\
&+\int_0^t \frac{N^{2-\theta}}{N^d}\sum_{\substack{x\in D_N^d \\ x_1=N-1}} H\Big(\tfrac{N-1}{N},\tfrac{\tilde{x}}{N}\Big) E_{\nu_\kappa^N}\Big[I_k(\eta_{sN^2}(N-1,\tilde{x},v)) - \sum_{v\in\mathcal{V}} v_k\beta_v(\tfrac{\tilde{x}}{N})\Big].
\end{aligned}
$$

At this point, compare the previous expression with the weak formulation given in (1.10).

### 1.6.3 The case $\theta > 1$

This regime is quite similar to the previous one. We consider again an arbitrary function $H \in C^{1,2}([0,T]\times D^d)$. Note that the last two terms at the right-hand side of (1.16) vanish as $N \to \infty$ since $\theta > 1$. Then, repeating the same arguments as in the previous subsection, we obtain

$$
\begin{aligned}
0 \; &= \frac{1}{N^d} \sum_{x \in D_N^d} H\left(\tfrac{x}{N}\right) E_{\nu_\kappa^N}[I_k(\eta_x(t)) - I_k(\eta_x(0))] - \frac{1}{N^d} \sum_{x \in D_N^d} \partial_s H\left(\tfrac{x}{N}\right) E_{\nu_\kappa^N}[I_k(\eta_x)] \\
&-\frac{1}{2}\int_0^t \frac{1}{N^d} \sum_{x \in D_N^d} E_{\nu_\kappa^N}[I_k(\eta_x(s))]\Delta_N H\left(\tfrac{x}{N}\right) ds \\
&+\frac{1}{2}\int_0^t \frac{1}{N^{d-1}} \sum_{\substack{x \in D_N^d \\ x_1 = N-1}} E_{\nu_\kappa^N}[I_k(\eta_{sN^2}(x))]\partial_{u_1}^{N,+} H\left(\tfrac{N-1}{N}, \tfrac{\tilde{x}}{N}\right) ds \\
&-\frac{1}{2}\int_0^t \frac{1}{N^{d-1}} \sum_{\substack{x \in D_N^d \\ x_1 = 1}} E_{\nu_\kappa^N}[I_k(\eta_{sN^2}(x))]\partial_{u_1}^{N,-} H\left(\tfrac{1}{N}, \tfrac{\tilde{x}}{N}\right) ds \\
&-\int_0^t \frac{1}{N^d} \sum_{j=1}^d \sum_{x \in D_N^d} (\partial_{u_j}^N H)\left(\tfrac{x}{N}\right) \sum_{v \in \mathcal{V}} v_k v_j \chi(\theta_v(\kappa)) ds.
\end{aligned}
$$

Again compare with the weak formulation given in (1.12).

## 1.7 Replacement Lemmas

### 1.7.1 Estimates on Dirichlet forms

The Dirichlet form $\langle \mathcal{L}_N \sqrt{f}, \sqrt{f} \rangle_{\nu_\kappa^N}$ does not always have a closed form. In this section, we compare the Dirichlet form with the closed form defined below, for each function $f : X_N \to \mathbb{R}$,

$$
D_{\nu_\kappa^N}(\sqrt{f}) = D_{\nu_\kappa^N}^{ex}(\sqrt{f}) + D_{\nu_\kappa^N}^{c}(\sqrt{f}) + D_{\nu_\kappa^N}^{b}(\sqrt{f}),
$$

and

$$
D_{\nu_\kappa^N}^{ex}(\sqrt{f}) = \sum_{v \in \mathcal{V}} \sum_{x \in D_N^d} \sum_{z \in D_N^d} c_{(x,z,v)}(\eta) \int [\sqrt{f(\eta^{x,z,v})} - \sqrt{f(\eta)}]^2 d\nu_\kappa^N,
$$

where $c_{(x,z,v)}(\eta) = \eta(x,v)(1-\eta(z,v))P_N(z-x,v)$,

$$
D_{\nu_\kappa^N}^{c}(\sqrt{f}) = \sum_{q \in \mathcal{Q}} \sum_{x \in D_N^d} \int p_c(x,q,\eta)[\sqrt{f(\eta^{x,q})} - \sqrt{f(\eta)}]^2 d\nu_\kappa^N
$$

and

$$\begin{aligned} D^b_{\nu^N_\kappa}(\sqrt{f}) \; &= \sum_{v\in\mathcal{V}} \sum_{x\in\{1\}\times\mathbb{T}^{d-1}_N} \int \frac{r_x(\eta,\alpha)}{N^\theta}[\sqrt{f(\sigma^{x,v}\eta)} - \sqrt{f(\eta)}]^2 d\nu^N_\kappa \\ &+ \sum_{v\in\mathcal{V}} \sum_{x\in\{N-1\}\times\mathbb{T}^{d-1}_N} \int \frac{r_x(\eta,\beta)}{N^\theta}[\sqrt{f(\sigma^{x,v}\eta)} - \sqrt{f(\eta)}]^2 d\nu^N_\kappa, \end{aligned}$$

where $r_x(\eta,\alpha)$ and $r_x(\eta,\beta)$ was defined in (1.3).

In order to prove the next proposition, we need some intermediate results. For that purpose, we recall from [4] the following lemmas:

**Lemma 1.** *Let $T : \eta \in \Omega_N \to T(\eta) \in \Omega_N$ be a transformation in the configuration space and $c : \eta \in \Omega_N \to c(\eta)$ be a positive local function. Let $f$ be a density with respect to a probability measure $\nu^N_\kappa$ on $\Omega_N$. Then, we have that*

$$\begin{aligned} &\langle c(\eta)[\sqrt{f(T(\eta))} - \sqrt{f(\eta)}], \sqrt{f(\eta)}\rangle_{\nu^N_\kappa} \le -\frac{1}{4}\int c(\eta)(\sqrt{f(T(\eta))} - \sqrt{f(\eta)})^2\, d\nu^N_\kappa \\ &+ \frac{1}{16}\int \frac{1}{c(\eta)}\left[c(\eta) - c(T(\eta))\frac{\nu^N_\kappa(T(\eta))}{\nu^N_\kappa(\eta)}\right]^2 (\sqrt{f(T(\eta))} + \sqrt{f(\eta)})^2\, d\nu^N_\kappa. \end{aligned} \tag{1.18}$$

**Lemma 2.** *Let $f$ be a density with respect to a probability measure $\nu^N_\kappa$ on $\Omega_N$. Then, we have that*

$$\sup_{x\neq y} \int f(\eta^{x,y,v}) d\nu^N_\kappa \le C$$

$$\sup_{x} \int f(\eta^{x,q}) d\nu^N_\kappa \le C$$

$$\sup_{x} \int f(\sigma^{x,v}\eta) d\nu^N_\kappa \le C.$$

**Remark 3.** *For each $v\in\mathcal{V}$, consider the functions $\kappa^v_k : D^d \to (0,1)$, for $k = 0,\dots,d$. We will have two situations for the function $\kappa = \sum_{v\in\mathcal{V}}(\kappa^v_0, v_1\kappa^v_1,\dots,v_d\kappa^v_d)$:*

- *When $\theta\in[0,1)$, we will assume that $\kappa^v_k$ are smooth functions, for $k=0,\dots,d$, such that the restriction of $\kappa$ to $\{0\}\times\mathbb{T}^{d-1}$ equals to the vector valued function $d(0,\tilde{x})$ defined in (1.6), and that the restricion of $\kappa$ to $\{1\}\times\mathbb{T}^{d-1}$ equals to vector valued function $d(1,\tilde{x})$, also defined in (1.6);*
- *when $\theta\ge 1$, we will assume that $\kappa$ is a constant function.*

As a consequence of Lemmas 1 and 2, we conclude that

**Corollary 1.** *Let $\kappa$ be a function as in Remark 3. Let $f : X_N \to \mathbb{R}$ be a density with respect to the measure $\nu_\kappa^N$, which was mentioned previously in (1.15). Then,*

*(i) if $\kappa$ is a constant function, then*

$$N^2\langle \mathcal{L}_N^{ex}\sqrt{f}, \sqrt{f}\rangle_{\nu_\kappa^N} = -\frac{N^2}{2} D^{ex}_{\nu_\kappa^N}(\sqrt{f});$$

*(ii) if $\kappa$ is a smooth function, then*

$$N^2\langle \mathcal{L}_N^{ex}\sqrt{f}, \sqrt{f}\rangle_{\nu_\kappa^N} = -\frac{N^2}{4} D^{ex}_{\nu_\kappa^N}(\sqrt{f}) + \mathcal{E}_N(\kappa)$$

*with $|\mathcal{E}_N(\kappa)| \leq CN^d$.*

*Proof.* By writing $N^2\langle \mathcal{L}_N^{ex}\sqrt{f}, \sqrt{f}\rangle_{\nu_\kappa^N}$ as its half plus its half and summing and subtracting the term needed to complete the square, we have that

$$\begin{aligned}
&N^2\langle \mathcal{L}_N^{ex}\sqrt{f}, \sqrt{f}\rangle_{\nu_\kappa^N}\\
&= \frac{N^2}{2}\int \sum_{v\in\mathcal{V}}\sum_{x\in D_N^d}\sum_{z\in D_N^d} c_{(x,z,v)}(\eta)[\sqrt{f(\eta^{x,z,v})} - \sqrt{f(\eta)}]\sqrt{f(\eta)}\, d\nu_\kappa^N\\
&+\frac{N^2}{2}\int \sum_{v\in\mathcal{V}}\sum_{x\in D_N^d}\sum_{z\in D_N^d} c_{(x,z,v)}(\eta)[\sqrt{f(\eta^{x,z,v})} - \sqrt{f(\eta)}]\sqrt{f(\eta)}\, d\nu_\kappa^N\\
&+\frac{N^2}{2}\int \sum_{v\in\mathcal{V}}\sum_{x\in D_N^d}\sum_{z\in D_N^d} c_{(x,z,v)}(\eta)[\sqrt{f(\eta^{x,z,v})} - \sqrt{f(\eta)}]\sqrt{f(\eta^{x,z,v})}\, d\nu_\kappa^N\\
&-\frac{N^2}{2}\int \sum_{v\in\mathcal{V}}\sum_{x\in D_N^d}\sum_{z\in D_N^d} c_{(x,z,v)}(\eta)[\sqrt{f(\eta^{x,z,v})} - \sqrt{f(\eta)}]\sqrt{f(\eta^{x,z,v})}\, d\nu_\kappa^N,
\end{aligned}$$

where $c_{(x,z,v)}(\eta) = \eta(x,v)(1-\eta(z,v))P_N(z-x,v)$. Putting together the first and fourth terms and doing a change of variables in the second term, we obtain that the last display equals to

$$\begin{aligned}
&-\frac{N^2}{2}\int \sum_{v\in\mathcal{V}}\sum_{x\in D_N^d}\sum_{z\in D_N^d} c_{(x,z,v)}(\eta)[\sqrt{f(\eta^{x,z,v})} - \sqrt{f(\eta)}]^2\, d\nu_\kappa^N\\
&+\frac{N^2}{2}\int \sum_{v\in\mathcal{V}}\sum_{x\in D_N^d}\sum_{z\in D_N^d} c_{(x,z,v)}(\eta)[\sqrt{f(\eta^{x,z,v})} - \sqrt{f(\eta)}]\sqrt{f(\eta^{x,z,v})}\, d\nu_\kappa^N\\
&-\frac{N^2}{2}\int \sum_{v\in\mathcal{V}}\sum_{x\in D_N^d}\sum_{z\in D_N^d} c_{(z,x,v)}(\eta)\tfrac{\nu_\kappa^N(\eta^{x,z,v})}{\nu_\kappa^N(\eta)}[\sqrt{f(\eta^{x,z,v})} - \sqrt{f(\eta)}][\sqrt{f(\eta^{x,z,v})}]d\nu_\kappa^N.
\end{aligned}$$

Last display equals to

$$-\frac{N^2}{2}\int\sum_{v\in\mathcal{V}}\sum_{x\in D_N^d}\sum_{z\in D_N^d} c_{(x,z,v)}(\eta)[\sqrt{f(\eta^{x,z,v})}-\sqrt{f(\eta)}]^2\,d\nu_\kappa^N$$
$$+\frac{N^2}{2}\int\sum_{v\in\mathcal{V}}\sum_{x\in D_N^d}\sum_{z\in D_N^d} c_{(x,z,v)}(\eta)\left(1-\tfrac{\nu_\kappa^N(\eta^{x,z,v})}{\nu_\kappa^N(\eta)}\right)\sqrt{f(\eta^{x,z,v})}[\sqrt{f(\eta^{x,z,v})}-\sqrt{f(\eta)}]\,d\nu_\kappa^N.$$

Hence, $N^2\langle\mathcal{L}_N^{ex}\sqrt{f},\sqrt{f}\rangle_{\nu_\kappa^N}=-\dfrac{N^2}{2}D^{ex}_{\nu_\kappa^N}(\sqrt{f})+g_N(\kappa)$, where

$$g_N(\kappa)=\frac{N^2}{2}\int\sum_{v\in\mathcal{V}}\sum_{x\in D_N^d}\sum_{z\in D_N^d} c_{(x,z,v)}(\eta)\Big(1-\tfrac{\nu_\kappa^N(\eta^{x,z,v})}{\nu_\kappa^N(\eta)}\Big)\sqrt{f(\eta^{x,z,v})}[\sqrt{f(\eta^{x,z,v})}-\sqrt{f(\eta)}]d\nu_\kappa^N.$$

To handle $g_N(\kappa)$, we start by observing that if we set

$$\gamma_{x,v}=\frac{\theta_v(\Lambda(\kappa(x)))}{1-\theta_v(\Lambda(\kappa(x)))},\tag{1.19}$$

then

$$\left|1-\frac{\nu_\kappa^N(\eta^{x,z,v})}{\nu_\kappa^N(\eta)}\right|=\left|1-\frac{\gamma_{z,v}}{\gamma_{x,v}}\right|.\tag{1.20}$$

Thus, if $\kappa$ is constant, then $g_N(\kappa)=0$.

On the other hand, if $\kappa$ is not constant, we need to redo the analysis of $g_N(\kappa)$. Applying Young's inequality, (that is $ab\le\frac{1}{2}a^2+\frac{1}{2}b^2$), in $g_N(\kappa)$, with

$$a=\frac{N}{\sqrt{2}}\sqrt{\eta(x,v)(1-\eta(z,v)P_N(z-x,v))}[\sqrt{f(\eta^{x,z,v})}-\sqrt{f(\eta)}]$$

and

$$b=\frac{N}{\sqrt{2}}\sqrt{\eta(x,v)(1-\eta(z,v)P_N(z-x,v))}\sqrt{f(\eta^{x,z,v})}\left(1-\tfrac{\nu_\kappa^N(\eta^{x,z,v})}{\nu_\kappa^N(\eta)}\right),$$

we can bound $g_N(\kappa)$ from above by

$$\frac{N^2}{4}\int\sum_{v\in\mathcal{V}}\sum_{x\in D_N^d}\sum_{z\in D_N^d}\eta(x,v)(1-\eta(z,v))P_N(z-x,v)[\sqrt{f(\eta^{x,z,v})}-\sqrt{f(\eta)}]^2\,d\nu_\kappa^N$$
$$+\frac{N^2}{4}\int\sum_{v\in\mathcal{V}}\sum_{x\in D_N^d}\sum_{z\in D_N^d}\eta(x,v)(1-\eta(z,v))P_N(z-x,v)\left(1-\tfrac{\nu_\kappa^N(\eta^{x,z,v})}{\nu_\kappa^N(\eta)}\right)^2 f(\eta^{x,z,v})\,d\nu_\kappa^N.$$

Thus $|g_N(\kappa)|\le\frac{N^2}{4}D^{ex}_{\nu_\kappa^N}(\sqrt{f})+\mathcal{E}_N(\kappa)$, where

$$\mathcal{E}_N(\kappa):=\frac{N^2}{4}\int\sum_{v\in\mathcal{V}}\sum_{x\in D_N^d}\sum_{z\in D_N^d} c_{(x,z,v)}(\eta)\left(1-\tfrac{\nu_\kappa^N(\eta^{x,z,v})}{\nu_\kappa^N(\eta)}\right)^2 f(\eta^{x,z,v})\,d\nu_\kappa^N.$$

Doing again the change of variables $\eta^{x,z,v} = \xi$, we obtain

$$\mathcal{E}_N(\kappa) = \frac{N^2}{4} \int \sum_{v\in\mathcal{V}} \sum_{x\in D_N^d} \sum_{z\in D_N^d} c_{(z,x,v)}(\eta) \left|1 - \frac{\nu_\kappa^N(\eta)}{\nu_\kappa^N(\eta^{x,z,v})}\right|^2 \frac{\nu_\kappa^N(\eta^{x,z,v})}{\nu_\kappa^N(\eta)} f(\eta)\, d\nu_\kappa^N.$$

Now, observe that

$$\left|1 - \frac{\nu_\kappa^N(\eta)}{\nu_\kappa^N(\eta^{x,z,v})}\right| = \left(1 - \frac{\gamma_{x,v}}{\gamma_{z,v}}\right) \le \tilde{c}\|\gamma'\|_\infty \frac{1}{N}, \tag{1.21}$$

since $\gamma$ is bounded away from zero, see (1.19), and

$$\left|\frac{\nu_\kappa^N(\eta^{x,z,v})}{\nu_\kappa^N(\eta)}\right| \le C.$$

Also note that $f$ is a density with respect to $\nu_\kappa^N$, therefore,

$$|\mathcal{E}_N(\kappa)| \le CN^d.$$

This finishes the proof of Corollary 1. □

**Corollary 2.** *If $\kappa$ is the function defined in Remark 3 and let $f$ be a density with respect to the measure $\nu_\kappa^N$, which was mentioned previously in (1.15), we have that*

$$N^2\langle \mathcal{L}_N^c \sqrt{f}, \sqrt{f}\rangle_{\nu_\kappa^N} = -\frac{N^2}{2} D^c_{\nu_\kappa^N}(\sqrt{f}). \tag{1.22}$$

*Proof.* Let $q = (v, w, v', w')$ and $\tilde{q} = (v', w', v, w)$. Note that

$$\begin{aligned} p_c(y, q, \xi^{y,q}) &= \xi^{y,q}(y,v)\xi^{y,q}(y,w)(1 - \xi^{y,q}(y,v'))(1 - \xi^{y,q}(y,w')) \\ &= \xi(y,v')\xi(y,w')(1-\xi(y,v))(1-\xi(y,w)) \\ &= p_c(y,\tilde{q},\xi). \end{aligned}$$

By writing the term at the left-hand side of (1.22) as its half, plus its half and summing and subtracting the term needed to complete the square as appears in $D^c_{\nu_\kappa^N}(\sqrt{f})$, we have that

$$
\begin{aligned}
N^2\langle \mathcal{L}_N^c\sqrt{f},\sqrt{f}\rangle_{\nu_\kappa^N} &= \frac{N^2}{2}\int \sum_{y\in D_N^d}\sum_{q\in Q} p_c(y,q,\eta)[\sqrt{f(\eta^{y,q})}-\sqrt{f(\eta)}]\sqrt{f(\eta)}\, d\nu_\kappa^N \\
&+\frac{N^2}{2}\int \sum_{y\in D_N^d}\sum_{q\in Q} p_c(y,q,\eta)[\sqrt{f(\eta^{y,q})}-\sqrt{f(\eta)}]\sqrt{f(\eta)}\, d\nu_\kappa^N \\
&+\frac{N^2}{2}\int \sum_{y\in D_N^d}\sum_{q\in Q} p_c(y,q,\eta)[\sqrt{f(\eta^{y,q})}-\sqrt{f(\eta)}]\sqrt{f(\eta^{y,q})}\, d\nu_\kappa^N \\
&-\frac{N^2}{2}\int \sum_{y\in D_N^d}\sum_{q\in Q} p_c(y,q,\eta)[\sqrt{f(\eta^{y,q})}-\sqrt{f(\eta)}]\sqrt{f(\eta^{y,q})}\, d\nu_\kappa^N .
\end{aligned}
\tag{1.23}
$$

Using a change of variables

$$
\begin{aligned}
N^2\langle \mathcal{L}_N^c\sqrt{f},\sqrt{f}\rangle_{\nu_\kappa^N} &= -\frac{N^2}{2}\int \sum_{y\in D_N^d}\sum_{q\in Q} p_c(y,q,\eta)[\sqrt{f(\eta^{y,q})}-\sqrt{f(\eta)}]^2\, d\nu_\kappa^N \\
&+\frac{N^2}{2}\int \sum_{y\in D_N^d}\sum_{q\in Q} p_c(y,q,\eta)\left[\sqrt{f(\eta^{y,q})}-\sqrt{f(\eta)}\right]\sqrt{f(\eta^{y,q})}\, d\nu_\kappa^N \\
&-\frac{N^2}{2}\int \sum_{y\in D_N^d}\sum_{q\in Q} p_c(y,q,\eta)\left[\sqrt{f(\eta^{y,q})}-\sqrt{f(\eta)}\right]\sqrt{f(\eta^{y,q})}\tfrac{\nu_\kappa^N(\eta^{y,q})}{\nu_\kappa^N(\eta)}\, d\nu_\kappa^N .
\end{aligned}
$$

Putting together the second and third terms in last display we get

$$
\begin{aligned}
&N^2\langle \mathcal{L}_N^c\sqrt{f},\sqrt{f}\rangle_{\nu_\kappa^N} \\
&= -\frac{N^2}{2}\int \sum_{y\in D_N^d}\sum_{q\in Q} p_c(y,q,\eta)[\sqrt{f(\eta^{y,q})}-\sqrt{f(\eta)}]^2\, d\nu_\kappa^N \\
&+\frac{N^2}{2}\int \sum_{y\in D_N^d}\sum_{q\in Q} p_c(y,q,\eta)\left[\sqrt{f(\eta^{y,q})}-\sqrt{f(\eta)}\right]\sqrt{f(\eta^{y,q})}\left[1-\tfrac{\nu_\kappa^N(\eta^{y,q})}{\nu_\kappa^N(\eta)}\right] d\nu_\kappa^N .
\end{aligned}
$$

Since $v+w=v'+w'$, we observe that

$$
\frac{\nu_\kappa^N(\eta^{y,q})}{\nu_\kappa^N(\eta)} = \frac{\gamma_{y,v'}\gamma_{y,w'}}{\gamma_{y,v}\gamma_{y,w}} = 1,
$$

therefore,

$$
N^2\langle \mathcal{L}_N^c\sqrt{f},\sqrt{f}\rangle_{\nu_\kappa^N} = -\frac{N^2}{2} D^c_{\nu_\kappa^N}(\sqrt{f}).
$$

This finishes the proof of Corollary 2. □

**Corollary 3.** *If $\kappa$ is the function defined in Remark 3 and let $f$ be a density with* *respect to the measure $\nu_\kappa^N$, which was mentioned previously in* (1.15)*, we have that*

$$N^2\langle \mathcal{L}_N^{b,\alpha}\sqrt{f},\sqrt{f}\rangle_{\nu_\kappa^N} = -\frac{N^2}{4}D^{b,\alpha}_{\nu_\kappa^N}(\sqrt{f}) + \mathcal{E}_N^\alpha(\kappa), \tag{1.24}$$

*with*

$$|\mathcal{E}_N^\alpha(\kappa)| \leq \frac{CN^{d+1}}{N^\theta}\left|m_{\Lambda(\kappa(x/N))} - \alpha_v\left(\tfrac{\tilde{x}}{N}\right)\right|.$$

*Analogously,*

$$N^2\langle \mathcal{L}_N^{b,\beta}\sqrt{f},\sqrt{f}\rangle_{\nu_\kappa^N} = -\frac{N^2}{4}D^{b,\beta}_{\nu_\kappa^N}(\sqrt{f}) + \mathcal{E}_N^\beta(\kappa), \tag{1.25}$$

*with*

$$|\mathcal{E}_N^\beta(\kappa)| \leq \frac{CN^{d+1}}{N^\theta}\left|m_{\Lambda(\kappa(x/N))} - \beta_v\left(\tfrac{\tilde{x}}{N}\right)\right|.$$

*Proof.* We present the proof for the left boundary since the other case is analogous. Splitting the integral on the left-hand side of (1.24) into the integral over the sets $A_0 = \{\eta : \eta((1,\tilde{x}),v) = 0\}$ and $A_1 = \{\eta : \eta((1,\tilde{x}),v) = 1\}$, we obtain

$$\begin{aligned}
&N^2\langle \mathcal{L}_N^{b,\alpha}\sqrt{f},\sqrt{f}\rangle_{\nu_\kappa^N} = \\
&\frac{N^2}{N^\theta}\int_{A_0}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}}\alpha_v\left(\tfrac{\tilde{x}}{N}\right)[\sqrt{f(\sigma^{x,v}\eta)} - \sqrt{f(\eta)}]\sqrt{f(\eta)}\,d\nu_\kappa^N \\
&+\frac{N^2}{N^\theta}\int_{A_1}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}}\left(1-\alpha_v\left(\tfrac{\tilde{x}}{N}\right)\right)[\sqrt{f(\sigma^{x,v}\eta)} - \sqrt{f(\eta)}]\sqrt{f(\eta)}\,d\nu_\kappa^N \\
&=\frac{N^2}{N^\theta}\int_{A_0}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}}\alpha_v\left(\tfrac{\tilde{x}}{N}\right)\sqrt{f(\sigma^{x,v}\eta)}\sqrt{f(\eta)}\,d\nu_\kappa^N \\
&+\frac{N^2}{N^\theta}\int_{A_1}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}}\left(1-\alpha_v\left(\tfrac{\tilde{x}}{N}\right)\right)\sqrt{f(\sigma^{x,v}\eta)}\sqrt{f(\eta)}\,d\nu_\kappa^N \\
&-\frac{N^2}{N^\theta}\int_{A_0}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}}\alpha_v\left(\tfrac{\tilde{x}}{N}\right)[\sqrt{f(\eta)}]^2\,d\nu_\kappa^N \\
&-\frac{N^2}{N^\theta}\int_{A_1}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}}\left(1-\alpha_v\left(\tfrac{\tilde{x}}{N}\right)\right)[\sqrt{f(\eta)}]^2\,d\nu_\kappa^N.
\end{aligned}$$

Last display can be rewritten as

$$
\begin{aligned}
&N^2\langle \mathcal{L}_N^{b,\alpha}\sqrt{f},\sqrt{f}\rangle_{\nu_\kappa^N} = \\
&= \frac{N^2}{N^\theta}\int_{A_0}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}}\alpha_v\left(\tfrac{\tilde{x}}{N}\right)\left[\sqrt{f(\sigma^{x,v}\eta)}\sqrt{f(\eta)}-\tfrac{1}{2}[\sqrt{f(\eta)}]^2\right]d\nu_\kappa^N \\
&+\frac{N^2}{N^\theta}\int_{A_1}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}}\left(1-\alpha_v\left(\tfrac{\tilde{x}}{N}\right)\right)\left[\sqrt{f(\sigma^{x,v}\eta)}\sqrt{f(\eta)}-\tfrac{1}{2}[\sqrt{f(\eta)}]^2\right]d\nu_\kappa^N \\
&-\frac{N^2}{2N^\theta}\int_{A_0}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}}\alpha_v\left(\tfrac{\tilde{x}}{N}\right)[\sqrt{f(\eta)}]^2\,d\nu_\kappa^N \\
&-\frac{N^2}{2N^\theta}\int_{A_1}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}}\left(1-\alpha_v\left(\tfrac{\tilde{x}}{N}\right)\right)[\sqrt{f(\eta)}]^2\,d\nu_\kappa^N.
\end{aligned}
$$

Summing and subtracting the term needed to complete the square, we obtain

$$
\begin{aligned}
N^2\langle \mathcal{L}_N^{b,\alpha}\sqrt{f},\sqrt{f}\rangle_{\nu_\kappa^N} &= -\frac{N^2}{2N^\theta}\int_{A_0}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}}\alpha_v\left(\tfrac{\tilde{x}}{N}\right)\left[\sqrt{f(\sigma^{x,v}\eta)}-\sqrt{f(\eta)}\right]^2 d\nu_\kappa^N \\
&-\frac{N^2}{2N^\theta}\int_{A_1}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}}\left(1-\alpha_v\left(\tfrac{\tilde{x}}{N}\right)\right)\left[\sqrt{f(\sigma^{x,v}\eta)}-\sqrt{f(\eta)}\right]^2 d\nu_\kappa^N \\
&+\frac{N^2}{2N^\theta}\int_{A_0}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}}\alpha_v\left(\tfrac{\tilde{x}}{N}\right)[\sqrt{f(\sigma^{x,v}\eta)}]^2\,d\nu_\kappa^N \\
&+\frac{N^2}{2N^\theta}\int_{A_1}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}}\left(1-\alpha_v\left(\tfrac{\tilde{x}}{N}\right)\right)[\sqrt{f(\sigma^{x,v}\eta)}]^2\,d\nu_\kappa^N \\
&-\frac{N^2}{2N^\theta}\int_{A_0}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}}\alpha_v\left(\tfrac{\tilde{x}}{N}\right)[\sqrt{f(\eta)}]^2\,d\nu_\kappa^N \\
&-\frac{N^2}{2N^\theta}\int_{A_1}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}}\left(1-\alpha_v\left(\tfrac{\tilde{x}}{N}\right)\right)[\sqrt{f(\eta)}]^2\,d\nu_\kappa^N.
\end{aligned}
$$

Using a change of variables on the last two terms above, we obtain

$$
\begin{aligned}
&N^2\langle \mathcal{L}_N^{b,\alpha}\sqrt{f},\sqrt{f}\rangle_{\nu_\kappa^N} = \\
&= -\frac{N^2}{2N^\theta}\int_{A_0}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}}\alpha_v\left(\tfrac{\tilde{x}}{N}\right)\left[\sqrt{f(\sigma^{x,v}\eta)}-\sqrt{f(\eta)}\right]^2 d\nu_\kappa^N \\
&-\frac{N^2}{2N^\theta}\int_{A_1}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}}\left(1-\alpha_v\left(\tfrac{\tilde{x}}{N}\right)\right)\left[\sqrt{f(\sigma^{x,v}\eta)}-\sqrt{f(\eta)}\right]^2 d\nu_\kappa^N \\
&+\frac{N^2}{2N^\theta}\int_{A_0}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}}\alpha_v\left(\tfrac{\tilde{x}}{N}\right)[\sqrt{f(\sigma^{x,v}\eta)}]^2 d\nu_\kappa^N \\
&+\frac{N^2}{2N^\theta}\int_{A_1}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}}\left(1-\alpha_v\left(\tfrac{\tilde{x}}{N}\right)\right)[\sqrt{f(\sigma^{x,v}\eta)}]^2 d\nu_\kappa^N \\
&-\frac{N^2}{2N^\theta}\int_{A_1}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}}\alpha_v\left(\tfrac{\tilde{x}}{N}\right)\frac{1-m_{\Lambda(\kappa(x/N))}}{m_{\Lambda(\kappa(x/N))}}[\sqrt{f(\sigma^{x,v}\eta)}]^2 d\nu_\kappa^N \\
&-\frac{N^2}{2N^\theta}\int_{A_0}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}}\left(1-\alpha_v\left(\tfrac{\tilde{x}}{N}\right)\right)\frac{m_{\Lambda(\kappa(x/N))}}{1-m_{\Lambda(\kappa(x/N))}}[\sqrt{f(\sigma^{x,v}\eta)}]^2 d\nu_\kappa^N .
\end{aligned}
\tag{1.26}
$$

For a general function $\kappa(\cdot)$, we can rewrite (1.26) and we have that $N^2\langle \mathcal{L}_N^{b,\alpha}\sqrt{f},\sqrt{f}\rangle_{\nu_\kappa^N}$ is equal to $-\frac{N^2}{2}D^{b,\alpha}_{\nu_\kappa^N}(\sqrt{f})$ plus

$$
\begin{aligned}
&-\frac{N^2}{2N^\theta}\int_{A_1}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}}\alpha_v\left(\tfrac{\tilde{x}}{N}\right)\left[\frac{1-m_{\Lambda(\kappa(x/N))}}{m_{\Lambda(\kappa(x/N))}}-\frac{1-\alpha_v(\frac{\tilde{x}}{N})}{\alpha_v(\frac{\tilde{x}}{N})}\right][\sqrt{f(\sigma^{x,v}\eta)}]^2 d\nu_\kappa^N \\
&-\frac{N^2}{2N^\theta}\int_{A_0}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}}\left(1-\alpha_v\left(\tfrac{\tilde{x}}{N}\right)\right)\left[\frac{m_{\Lambda(\kappa(x/N))}}{1-m_{\Lambda(\kappa(x/N))}}\frac{\alpha_v(\frac{\tilde{x}}{N})}{1-\alpha_v(\frac{\tilde{x}}{N})}\right][\sqrt{f(\sigma^{x,v}\eta)}]^2 d\nu_\kappa^N .
\end{aligned}
$$

The expression above is bounded by

$$
\frac{CN^{d+1}}{N^\theta}\left|m_{\Lambda(\kappa(x/N))}-\alpha_v\left(\tfrac{\tilde{x}}{N}\right)\right| .
$$

□

**Remark 4.** *In order to prove the Replacement Lemma, see Proposition 1, we need some intermediate results. Observe that, if $H(\mu^N|\nu_\kappa^N)$ is the relative entropy of the measure $\mu^N$ with respect to $\nu_\kappa^N$, see (1.15), then there exists a constant $C_\kappa$ such that*

$H(\mu^N|\nu_\kappa^N) \le C_\kappa N^d$. *To prove it, note that by the definition of the entropy,*

$$H(\mu^N|\nu_\kappa^N) = \int \log\left(\frac{\mu^N(\eta)}{\nu_\kappa^N(\eta)}\right)\mu^N(\eta) \le \int \log\left(\frac{1}{\nu_\kappa^N(\eta)}\right)\mu^N(\eta).$$

*Since the measure* $\nu_\kappa^N$ *is a product measure with marginal given by*

$$\nu_\kappa^N\{\eta : \eta(x,\cdot) = \xi\} = m_{\Lambda(\kappa(x))}(\xi),$$

*where* $m_\lambda(\cdot)$ *was defined in* (1.4)*, we obtain that the last display is bounded from above by*

$$\begin{aligned}
&\int \log\left(\frac{1}{\inf\limits_{x\in D^d}(m_{\Lambda(\kappa(x))}) \wedge (1 - m_{\Lambda(\kappa(x))})}\right)^{N^d} \mu^N(\eta)\\
&= N^d \log\left(\frac{1}{\inf\limits_{x\in D^d}(m_{\Lambda(\kappa(x))}) \wedge (1 - m_{\Lambda(\kappa(x))})}\right)\\
&= C_\kappa N^d.
\end{aligned}$$

*Since the function* $\kappa_k^v$*, defined in Remark 3, is continuous, the image of each* $\kappa_k^v$ *is a compact set bounded away from* 0 *and* 1*. Hence, from the definition of the measure* $m$*, we have that* $m_{\Lambda(\kappa(x/N))} > 0$ *and* $m_{\Lambda(\kappa(x/N))} < 1$*. The constant* $C_\kappa = \log\left(\dfrac{1}{\inf\limits_{x\in D^d} m_{\Lambda(\kappa(x/N))} \wedge (1 - m_{\Lambda(\kappa(x/N))})}\right)$.

### 1.7.2 Replacement Lemma for the Boundary

Fix $k = 0, \dots, d$, a continuous function $G : [0,T] \times \mathbb{T}^{d-1} \to \mathbb{R}^{d+1}$, and consider the quantities

$$V_k^{1,l}(\eta_s, \alpha, G) = \frac{1}{N^{d-1}} \sum_{\tilde{x} \in \mathbb{T}^{d-1}} G_k(s, \tilde{x}/N) \left( I_k(\eta_{(1,\tilde{x})}(s)) - \sum_{v \in \mathcal{V}} v_k \alpha_v(\tilde{x}/N) \right),$$

$$V_k^{1,r}(\eta_s, \beta, G) = \frac{1}{N^{d-1}} \sum_{\tilde{x} \in \mathbb{T}^{d-1}} G_k(s, \tilde{x}/N) \left( I_k(\eta_{(N-1,\tilde{x})}(s)) - \sum_{v \in \mathcal{V}} v_k \beta_v(\tilde{x}/N) \right),$$

$$V_k^{2,l}(\eta_s, \alpha, G) = \frac{1}{N^{d-1}} \sum_{\tilde{x} \in \mathbb{T}^{d-1}} G_k(s, \tilde{x}) \left( I_k(\eta_{(1,\tilde{x})}(s)) - \frac{1}{\lfloor N\epsilon \rfloor} \sum_{x_1=1}^{\lfloor N\epsilon \rfloor + 1} I_k(\eta_{(x_1,\tilde{x})}(s)) \right),$$

$$V_k^{2,r}(\eta_s, \beta, G) = \frac{1}{N^{d-1}} \sum_{\tilde{x} \in \mathbb{T}^{d-1}} G_k(s, \tilde{x}) \left( I_k(\eta_{(N-1,\tilde{x})}(s)) - \frac{1}{\lfloor N\epsilon \rfloor} \sum_{x_1=N-1-\lfloor N\epsilon \rfloor}^{N-1} I_k(\eta_{(x_1,\tilde{x})}(s)) \right),$$

where $s \in [0,T]$, and $0 \le k \le d$ consider that $G_k$ are the components of function $G$. The main result of this section is the following proposition:

**Proposition 1** (Replacement Lemma for the boundary)**.** *For each $0 \le t \le T$, $0 \le k \le d$, and $G : [0,T] \times D^d \to \mathbb{R}$ continuous,*

$$\limsup_{N \to \infty} E_{\mu^N} \left[ \left| \int_0^t ds\, V_k^{j,\vartheta}(\eta_s, \zeta, G) \right| \right] = 0,$$

*where $j = \{1,2\}$, and*

$$\vartheta = \begin{cases} l, & \text{if} \quad \zeta = \alpha, \\ r, & \text{if} \quad \zeta = \beta. \end{cases}$$

*Proof.* By the entropy inequality and Jensen's inequality for any $A > 0$ the expectation in the statement of the proposition is bounded from above by

$$\frac{H(\mu^N|\nu_\kappa^N)}{AN^d} + \frac{1}{AN^d} \log E_{\nu_\kappa^N} \left[ \exp \left\{ \left| \int_0^t ds\, AN^d V_k^{j,\vartheta}(\eta_s, \zeta, G) \right| \right\} \right]. \tag{1.27}$$

By Remark 4, the left-most term is bounded by $\frac{C_\kappa}{A}$, so we only need to show that the right-most term vanishes as $N \to \infty$. Since $e^{|x|} \le e^x + e^{-x}$ and

$$\limsup_{N \to \infty} N^{-d} \log\{a_N + b_N\} \le \max\{\limsup_{N \to \infty} N^{-d} \log(a_N), \limsup_{N \to \infty} N^{-d} \log(b_N)\},$$

we may remove the absolute value from the expression (1.27). By Feynman-Kac formula, see for instance [1], we obtain that (1.27) is bounded from above by

$$\frac{C_\kappa}{A} + t\sup_f \left\{ \int V_k^{j,\vartheta}(\eta,\zeta,G) f(\eta)\, d\nu_\kappa^N + \frac{\langle \mathcal{L}_N \sqrt{f}, \sqrt{f}\rangle_{\nu_\kappa^N}}{AN^{d-2}} \right\}.$$

In this formula the supremum is taken over all densities $f$ with respect to $\nu_\kappa^N$. The proof follows from an application of the auxiliary lemmas given below.

**Lemma 3.** *For every $0 \le t \le T$, $0 \le k \le d$, and every continuous function $G : [0,T] \times \mathbb{T}^{d-1} \to \mathbb{R}^{d+1}$,*

$$\langle V_k^{1,\vartheta}(\eta,\zeta,G), f(\eta)\rangle_{\nu_\kappa^N} \le CBN^\theta + \frac{C'}{B} D^b_{\nu_\kappa^N}(\sqrt{f}) + K$$

*where we have the following cases*

$$\vartheta = \begin{cases} l, & \text{if} \quad \zeta = \alpha, \\ r, & \text{if} \quad \zeta = \beta. \end{cases}$$

*Proof.* We prove for $\vartheta = l$, since for $\vartheta = r$ the proof is entirely analogous. Note that $G$ is continuous and its domain is a compact set, hence, we may prove the above result without $G$. Note that

$$\begin{aligned} &\frac{1}{N^{d-1}} \sum_{\tilde{x} \in \mathbb{T}_N^{d-1}} \int f(\eta) \left[ I_k(\eta_{(1,\tilde{x})}) - \sum_{v \in \mathcal{V}} v_k \alpha_v(\tilde{x}/N) \right] d\nu_\kappa^N \\ &= \frac{1}{N^{d-1}} \sum_{\tilde{x} \in \mathbb{T}_N^{d-1}} \sum_{v \in \mathcal{V}} v_k \int f(\eta) [\eta(1,\tilde{x},v) - \alpha_v(\tilde{x}/N)]\, d\nu_\kappa^N. \end{aligned}$$

By summing and subtracting an appropriate term, last term is equal to

$$\begin{aligned} &\frac{1}{2} \left| \frac{1}{N^{d-1}} \sum_{\tilde{x} \in \mathbb{T}_N^{d-1}} \sum_{v \in \mathcal{V}} v_k \int [f(\eta) - f(\sigma^{x,v}\eta)][\eta(1,\tilde{x},v) - \alpha_v(\tilde{x}/N)]\, d\nu_\kappa^N \right| \\ &+ \frac{1}{2} \left| \frac{1}{N^{d-1}} \sum_{\tilde{x} \in \mathbb{T}_N^{d-1}} \sum_{v \in \mathcal{V}} v_k \int [f(\eta) + f(\sigma^{x,v}\eta)][\eta(1,\tilde{x},v) - \alpha_v(\tilde{x}/N)]\, d\nu_\kappa^N \right|. \end{aligned} \tag{1.28}$$

Applying Young's inequality on the first term of last display, we can bound it from

above by

$$
\begin{aligned}
&\frac{B}{4}\left|\frac{N^\theta}{N^{d-1}}\sum_{\tilde{x}\in\mathbb{T}_N^{d-1}}\sum_{v\in\mathcal{V}}v_k\int[\sqrt{f(\eta)}+\sqrt{f(\sigma^{x,v}\eta)}]^2\frac{[\eta(1,\tilde{x},v)-\alpha_v(\tilde{x}/N)]^2}{r_x(\eta,\alpha)}\,d\nu_\kappa^N\right| \\
&+\frac{1}{4B}\left|\frac{1}{N^{d-1}}\sum_{\tilde{x}\in\mathbb{T}_N^{d-1}}\sum_{v\in\mathcal{V}}v_k\int[\sqrt{f(\eta)}-\sqrt{f(\sigma^{x,v}\eta)}]^2\frac{r_x(\eta,\alpha)}{N^\theta}\,d\nu_\kappa^N\right|
\end{aligned}
\tag{1.29}
$$

where $r_x(\eta,\alpha)=\alpha_v(\tilde{x}/N)[1-\eta(x,v)]+[1-\alpha_v(\tilde{x}/N)]\eta(x,v)$ and this holds for any $B>0$. Since

$$[\eta(1,\tilde{x},v)-\alpha_v(\tilde{x}/N)]^2\le 1$$

and $r_x(\eta,\alpha)\le 1$, we obtain that (1.29) is bounded from above by

$$BN^\theta C+\frac{C'}{B}D^{b,\alpha}_{\nu_\kappa^N}(\sqrt{f}),$$

with $C$ and $C'$ constants.

Now, we analyze the second term on (1.28). Note that

$$
\begin{aligned}
&\frac{1}{2}\left|\frac{1}{N^{d-1}}\sum_{\tilde{x}\in\mathbb{T}_N^{d-1}}\sum_{v\in\mathcal{V}}v_k\int[f(\eta)+f(\sigma^{x,v}\eta)][\eta(1,\tilde{x},v)-\alpha_v(\tilde{x}/N)]\,d\nu_\kappa^N\right| \\
&=\frac{1}{2}\left|\frac{1}{N^{d-1}}\sum_{\tilde{x}\in\mathbb{T}_N^{d-1}}\sum_{v\in\mathcal{V}}v_k\int f(\eta)[\eta(1,\tilde{x},v)-\alpha_v(\tilde{x}/N)]\,d\nu_\kappa^N\right| \\
&+\frac{1}{2}\left|\frac{1}{N^{d-1}}\sum_{\tilde{x}\in\mathbb{T}_N^{d-1}}\sum_{v\in\mathcal{V}}v_k\int f(\sigma^{x,v}\eta)[\eta(1,\tilde{x},v)-\alpha_v(\tilde{x}/N)]\,d\nu_\kappa^N\right|.
\end{aligned}
$$

Using that $[\eta(1,\tilde{x},v)-\alpha_v(\tilde{x}/N)]\le 1$, we obtain that the first term above is bounded by a constant $K_1$. After a change of variables on the second term above, we obtain that it is also bounded by a constant $K_2$. Therefore,

$$\langle V_k^{1,l}(\eta,\alpha,G),f(\eta)\rangle_{\nu_\kappa^N}\le K_1BN^\theta+\frac{C'}{B}D^{b,\alpha}_{\nu_\kappa^N}(\sqrt{f})+K_1+K_2.$$

□

**Lemma 4.** *For every $0\le t\le T$, $0\le k\le d$, and every continuous function $G:[0,T]\times\mathbb{T}^{d-1}\to\mathbb{R}^{d+1}$,*

$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\langle V_k^{2,\vartheta}(\eta,\zeta,G),f(\eta)\rangle_{\nu_\kappa^N}=0,$$

*where we have the following cases*

$$\vartheta = \begin{cases} l, & \textit{if} \quad \zeta = \alpha, \\ r, & \textit{if} \quad \zeta = \beta. \end{cases}$$

*Proof.* First of all, note that since $G$ is continuous and its domain $[0,T]\times D^d$ is compact, it is enough to prove the result without $G$. We will only prove for $\vartheta = l$, since for $\vartheta = r$ the proof is entirely analogous. Observe that

$$\begin{aligned}
&\langle V_k^{2,l}(\eta,\zeta,G), f(\eta)\rangle_{\nu_\kappa^N} \\
&= \frac{1}{N^{d-1}} \sum_{\tilde{x}\in\mathbb{T}_N^{d-1}} \int f(\eta) \left[ I_k(\eta_{(1,\tilde{x})}) - \frac{1}{N\varepsilon} \sum_{x_1=1+1}^{N\varepsilon+1} I_k(\eta_{(x_1,\tilde{x})}) \right] d\nu_\kappa^N \\
&= \frac{1}{N^{d-1}} \sum_{\tilde{x}\in\mathbb{T}_N^{d-1}} \sum_{v\in\mathcal{V}} v_k \int f(\eta) \left[ \eta(1,\tilde{x},v) - \frac{1}{N\varepsilon} \sum_{x_1=1+1}^{N\varepsilon+1} \eta(x_1,\tilde{x},v) \right] d\nu_\kappa^N \\
&= \frac{1}{N^{d-1}} \sum_{\tilde{x}\in\mathbb{T}_N^{d-1}} \sum_{v\in\mathcal{V}} v_k \int f(\eta) \left[ \frac{1}{N\varepsilon} \sum_{x_1=1+1}^{N\varepsilon+1} \{\eta(1,\tilde{x},v) - \eta(x_1,\tilde{x},v)\} \right] d\nu_\kappa^N.
\end{aligned}$$

By writing the term $\frac{1}{N\varepsilon} \sum_{x_1=1+1}^{N\varepsilon+1} \{\eta(1,\tilde{x},v) - \eta(x_1,\tilde{x},v)\}$ as a telescopic sum, we obtain that the last term is equal to

$$\frac{1}{N^{d-1}} \sum_{\tilde{x}\in\mathbb{T}_N^{d-1}} \sum_{v\in\mathcal{V}} v_k \int f(\eta) \left[ \frac{1}{N\varepsilon} \sum_{x_1=1+1}^{N\varepsilon+1} \sum_{y=1}^{x_1-1} \{\eta(y,\tilde{x},v) - \eta(y+1,\tilde{x},v)\} \right] d\nu_\kappa^N.$$

Writing this sum as twice its half, performing change of variables [2], we obtain that the last display is equal to

$$+\frac{1}{N^{d-1}} \sum_{\tilde{x}\in\mathbb{T}_N^{d-1}} \sum_{v\in\mathcal{V}} v_k \tfrac{1}{2N\varepsilon} \sum_{x_1=1+1}^{N\varepsilon+1} \sum_{y=1}^{x_1-1} \int \left[ f(\eta) - f(\eta^{y,y+1,v}) \right] (\eta(y,\tilde{x},v) - \eta(y+1,\tilde{x},v))\, d\nu_\kappa^N. \tag{1.30}$$

Rewriting $[f(\eta) - f(\eta^{y,y+1,v})]$ as $[\sqrt{f(\eta)} - \sqrt{f(\eta^{y,y+1,v})}][\sqrt{f(\eta)} + \sqrt{f(\eta^{y,y+1,v})}]$ and using Young's inequality, for all $B > 0$, we obtain that (1.30) is bounded from above

[2] In this case, we will assume that $\kappa$ is constant

by

$$\frac{1}{N^{d-1}}\sum_{\tilde{x}\in\mathbb{T}_N^{d-1}}\sum_{v\in\mathcal{V}}v_k\frac{B}{2N\varepsilon}\sum_{x_1=1+1}^{N\varepsilon+1}\sum_{y=1}^{x_1-1}\int\left[\sqrt{f(\eta)}-\sqrt{f(\eta^{y,y+1,v})}\right]^2 d\nu_\kappa^N$$

$$+\frac{1}{N^{d-1}}\sum_{\tilde{x}\in\mathbb{T}_N^{d-1}}\sum_{v\in\mathcal{V}}v_k\frac{1}{2BN\varepsilon}\sum_{x_1=1+1}^{N\varepsilon+1}\sum_{y=1}^{x_1-1}\int\left[\sqrt{f(\eta)}+\sqrt{f(\eta^{y,y+1,v})}\right]^2(\eta(y,\tilde{x},v)-\eta(y+1,\tilde{x},v))^2\,d\nu_\kappa^N.$$

Using that $f$ is a density for $\nu_\kappa^N$, the second term in last display is bounded by $\frac{CN\varepsilon}{B}$. Letting the sum in $y$ run from 1 to $N-1$, the first term in last display is bounded by $BD^{ex}_{\nu_\kappa^N}(\sqrt{f})$. By Corollary 1, since $\kappa$ is a constant function, we obtain that

$$D^{ex}_{\nu_\kappa^N}(\sqrt{f})=-\langle\mathcal{L}_N^{ex}\sqrt{f},\sqrt{f}\rangle_{\nu_\kappa^N}.$$

Since $0\le D^{c}_{\nu_\kappa^N}(\sqrt{f})=-\langle\mathcal{L}_N^{c}\sqrt{f},\sqrt{f}\rangle_{\nu_\kappa^N}$ and $0\le D^{b}_{\nu_\kappa^N}(\sqrt{f})$, using Corollary 3, we have that $D^{b,\alpha}_{\nu_\kappa^N}(\sqrt{f})=-\langle\mathcal{L}_N^{b,\alpha}\sqrt{f},\sqrt{f}\rangle_{\nu_\kappa^N}$. Therefore,

$$D^{ex}_{\nu_\kappa^N}(\sqrt{f})\le -B\langle\mathcal{L}_N\sqrt{f},\sqrt{f}\rangle_{\nu_\kappa^N}.$$

□

### 1.7.3 Replacement Lemma at the bulk

Before we state the replacement lemma that will allow us to prove that the limit points $\mathbb{Q}^*$ are concentrated on weak solutions of the system of partial differential equations (1.13), we introduce some notations. Fix $L\ge 1$ and a configuration $\eta$. Let $\mathbf{I}^L(x,\eta)=(I_0^L(x),\dots,I_d^L(x))$ be the average of the conserved quantities in a cube of length $L$ centered at $x$:

$$\mathbf{I}^L(x,\eta)=\frac{1}{|\Lambda_L|}\sum_{z\in x+\Lambda_L}\mathbf{I}(\eta_z),$$

where, $\Lambda_L=\{-L,\dots,L\}^d$ and $|\Lambda_L|=(2L+1)^d$ is the discrete volume of the box $\Lambda_L$. Let $\mathfrak{B}_L$ be the set of all possible values of $\mathbf{I}^L(0,\eta)$ for $\eta\in(\{0,1\}^{\mathcal{V}})^{\Lambda_L}$, that is,

$$\mathfrak{B}_L=\{\mathbf{I}^L(0,\eta);\,\eta\in(\{0,1\}^{\mathcal{V}})^{\Lambda_L}\}.$$

Note that $\mathfrak{B}_L$ is a finite subset of the convex envelope of $\{\mathbf{I}(\xi) : \xi \in \{0,1\}^{\mathcal{V}}\}$. The set of configurations $(\{0,1\}^{\mathcal{V}})^{\Lambda_L}$ splits into invariant subsets: for $\mathbf{i}$ in $\mathfrak{B}_L$, let

$$\mathcal{H}_L(\mathbf{i}) := \{\eta \in (\{0,1\}^{\mathcal{V}})^{\Lambda_L} : \mathbf{I}^L(0) = \mathbf{i}\}.$$

For each $\mathbf{i}$ in $\mathfrak{B}_L$, define the canonical measure $\nu_{\Lambda_L,\mathbf{i}}$ as the uniform probability measure on $\mathcal{H}_L(\mathbf{i})$. Note that for every $\lambda$ in $\mathbb{R}^{d+1}$

$$\nu_{\Lambda_L,\mathbf{i}}(\cdot) = \mu_\lambda^{\Lambda_L}(\cdot \,|\, \mathbf{I}^L(0) = \mathbf{i}).$$

Let $\langle g\,;\, f\rangle_\mu$ stand for the covariance of $g$ and $f$ with respect to $\mu$, i.e.,

$$\langle g\,;\, f\rangle_\mu = E_\mu[fg] - E_\mu[f]E_\mu[g].$$

**Proposition 2.** *[Equivalence of ensembles] Fix $\ell, L$, the cubes $\Lambda_\ell \subset \Lambda_L$, for each $\boldsymbol{i} \in \mathfrak{B}_L$, denote by $\nu^\ell$ the projection of the canonical measure $\nu_{\Lambda_L,\boldsymbol{i}}$ on $\Lambda_\ell$ and by $\mu^\ell$ the projection of the grand canonical measure $\mu^L_{\Lambda(\boldsymbol{i})}$ on $\Lambda_\ell$. There exists a finite constant $C(\ell,\mathcal{V})$, depending only on $\ell$ and $\mathcal{V}$, such that*

$$|E_{\mu^\ell}[f] - E_{\nu^\ell}[f]| \leq \frac{C(\ell,\mathcal{V})}{|\Lambda_L|}\langle f; f\rangle_{\mu^\ell}^{1/2}$$

*for every function $f : (\{0,1\}^{\mathcal{V}})^{\Lambda_\ell} \mapsto \mathbb{R}$.*

The proof of Proposition 2 can be found in [6].

**Lemma 5** (Replacement lemma)**.** *For all $\delta > 0$, $1 \leq j \leq d$, $0 \leq k \leq d$:*

$$\limsup_{\epsilon\to 0}\limsup_{N\to\infty} \mathbb{P}_{\mu^N}\left[\int_0^T \frac{1}{N^d}\sum_{x\in D_N^d} \tau_x V_{\epsilon N}^{j,k}(\eta(s))\,ds \geq \delta\right] = 0,$$

*where*

$$V_\ell^{j,k}(\eta) = \Big|\frac{1}{(2l+1)^d}\sum_{y\in\Lambda_\ell}\sum_{v\in\mathcal{V}} v_k \sum_{z\in\mathbb{Z}^d} p(z,v) z_j \tau_y[\eta(0,v)(1-\eta(z,v))] \\ - \sum_{v\in\mathcal{V}} v_j v_k \chi(\theta_v(\Lambda(\boldsymbol{I}^\ell(0))))\Big|.$$

Note that $V_{\epsilon N}^{j,k}$ is well-defined for large $N$ since $p(\cdot, v)$ is of finite range. We now observe that Corolaries 1 and 2 permit us to prove the previous replacement lemma for the boundary driven exclusion process by using the process without the boundary part of the generator. For the proof of Lemma 5, see [3, Lemma 3.7].

## 1.8 Tightness

In this section, we show that the sequence of probability measures $(\mathbb{Q}_N)_N$, defined in Section 1.5, is tight in the Skorohod space $\mathcal{D}([0,T],\, \mathcal{M}_+ \times \mathcal{M}^d)$. In order to do that, we invoke the Aldous' criterion, which says that:

**Lemma 6.** *A sequence $(\mathbb{Q}_N)_{N\geq 1}$ of probability measures defined on $\mathcal{D}([0,T], \mathcal{M}_+ \times \mathcal{M}^d)$ is tight if, and only if, these two conditions hold:*

*a. For every $t \in [0,T]$ and every $\varepsilon > 0$, there exists $K_\varepsilon^t \subset \mathcal{M}_+ \times \mathcal{M}^d$ compact, such that*

$$\sup_{N\geq 1} \mathbb{Q}_N\big(\pi_t^{k,N} \notin K_\varepsilon^t\big) \leq \varepsilon,$$

*b. For every $\varepsilon > 0$*

$$\lim_{\gamma\to 0} \limsup_{N\to\infty} \sup_{\substack{\tau\in\mathfrak{T} \\ \overline{\tau}\leq\gamma}} \mathbb{Q}_N\big(\, d(\pi_{\tau+\overline{\tau}}^{k,N}, \pi_\tau^{k,N}) > \varepsilon\,\big) = 0,$$

*where $\mathfrak{T}$ denotes the set of stopping times with respect to the canonical filtration, bounded by $T$ and $d$ is the metric in the space $\mathcal{M}_+ \times \mathcal{M}^d$. We assume that all the stopping times are bounded by $T$, thus, $\tau + \overline{\tau}$ should be understood as $(\tau + \overline{\tau}) \wedge T$.*

By [1, Chapter 4, Proposition 1.7] it is enough to show that for every function $H$ in a dense subset of $C(D^d)$, with respect to the uniform topology, the sequence of measures, that corresponds to the real processes $\langle \pi_t^{k,N}, H\rangle$, is tight. In our setting, condition $a.$ above translates by saying that:

$$\lim_{A\to+\infty} \lim_{N\to+\infty} \mathbb{P}_{\mu^N}\Big(|\langle \pi_t^{k,N}, H\rangle| > A\Big) = 0. \tag{1.31}$$

This is a consequence of Chebychev's inequality and the fact that for the exclusion type dynamics, the number of particles per site is at most one for each fixed velocity. So, it remains to show condition $b$. In this context and since we are considering the real processes $\langle \pi_t^{k,N}, H\rangle$, the distance $d$ above is the usual distance in $\mathbb{R}$. Then, we must show that for all $\varepsilon > 0$ and any function $H$ in a dense subset of $C(D^d)$, with respect to the uniform topology, the following holds:

$$\lim_{\delta\to 0}\limsup_{N\to+\infty}\sup_{\substack{\tau\in\mathfrak{T}\\ \overline{\tau}\leq\delta}}\mathbb{P}_{\mu^N}\Big(\eta:|\langle\pi^{k,N}_{\tau+\overline{\tau}},H\rangle-\langle\pi^{k,N}_{\tau},H\rangle|>\varepsilon\Big)=0. \tag{1.32}$$

Recall that it is enough to prove the assertion for functions $H$ in a dense subset of $C(D^d)$ with respect to the uniform topology. We will split the proof into two cases:

### 1.8.1 The case $\theta \geq 1$

Recall from (1.16) that, $M^{N,k}_t(H)$ is a martingale with respect to the natural filtration $(\mathcal{F}_t)_{t\geq 0}$. Then

$$\mathbb{P}_{\mu^N}\Big(\eta:\Big|\langle\pi^{k,N}_{\tau+\overline{\tau}},H\rangle-\langle\pi^{k,N}_{\tau},H\rangle|>\varepsilon\Big)$$

$$=\mathbb{P}_{\mu^N}\Big(\eta:|M^{N,k}_{\tau}(H)-M^{N,k}_{\tau+\overline{\tau}}(H)+\int_{\tau}^{\tau+\overline{\tau}}\mathcal{L}_N\langle\pi^{k,N}_s,H\rangle\,ds\Big|>\varepsilon\Big)$$

$$\leq\mathbb{P}_{\mu^N}\Big(\eta:|M^{N,k}_{\tau}(H)-M^{N,k}_{\tau+\overline{\tau}}(H)\Big|>\frac{\varepsilon}{2}\Big)+\mathbb{P}_{\mu^N}\Big(\eta:\Big|\int_{\tau}^{\tau+\overline{\tau}}\mathcal{L}_N\langle\pi^{k,N}_s,H\rangle\,ds\Big|>\frac{\varepsilon}{2}\Big).$$

Applying Chebychev's inequality (resp. Markov's inequality) in the first (resp. second) term on the right-hand side of last inequality, we can bound the previous expression from above by

$$\frac{2}{\varepsilon^2}\mathbb{E}_{\mu^N}\Big[\Big(M^{N,k}_{\tau}(H)-M^{N,k}_{\tau+\overline{\tau}}(H)\Big)^2\Big]+\frac{2}{\varepsilon}\mathbb{E}_{\mu^N}\Big[\Big|\int_{\tau}^{\tau+\overline{\tau}}\mathcal{L}_N\langle\pi^{k,N}_s,H\rangle\,ds\Big|\Big].$$

Therefore, in order to prove (1.32) it is enough to show that

$$\lim_{\delta\to 0}\limsup_{N\to+\infty}\sup_{\substack{\tau\in\mathfrak{T}\\ \overline{\tau}\leq\delta}}\mathbb{E}_{\mu^N}\Big[\Big|\int_{\tau}^{\tau+\overline{\tau}}\mathcal{L}_N\langle\pi^{k,N}_s,H\rangle\,ds\Big|\Big]=0, \tag{1.33}$$

and

$$\lim_{\delta\to 0}\limsup_{N\to+\infty}\sup_{\substack{\tau\in\mathfrak{T}\\ \overline{\tau}\leq\delta}}\mathbb{E}_{\mu^N}\Big[\Big(M^{N,k}_{\tau}(H)-M^{N,k}_{\tau+\overline{\tau}}(H)\Big)^2\Big]=0. \tag{1.34}$$

Let us start by proving (1.33). Given a test function $H\in C^2(D^d)$, we will show that there exists a constant $C$ such that $\mathcal{L}_N\langle\pi^{k,N}_s,H\rangle\leq C$ for any $s\leq T$. For that purpose, we use the computations of Appendix A.0.1, where we derived the expression of $\mathcal{L}_N\langle \pi_s^{k,N}, H\rangle$. Note that,

$$|\mathcal{L}_N\langle \pi_s^{k,N}, H\rangle| \leq |N^2\mathcal{L}_N^{ex,1}\langle \pi_s^{k,N}, H\rangle| + |N^2\mathcal{L}_N^{ex,2}\langle \pi_s^{k,N}, H\rangle| + |N^2\mathcal{L}_N^{c}\langle \pi_s^{k,N}, H\rangle| + |N^2\mathcal{L}_N^{b}\langle \pi_s^{k,N}, H\rangle|.$$

Let us bound this separately. Note that,

$$\begin{aligned}
|N^2\mathcal{L}_N^{ex,1}\langle \pi_s^{k,N}, H\rangle| &\leq |\langle \pi_s^{k,N}, \frac{1}{2}\Delta_N H\rangle| + \Big|\frac{N}{2N^d}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}} v_k \eta(1,\tilde{x},v)\partial_{u_1}^{N,+}H(0,\tilde{x})\Big| \\
&+\Big|\frac{N}{2N^d}\sum_{\substack{x\in D_N^d\\ x_1=N-1}}\sum_{v\in\mathcal{V}} v_k \eta(N-1,\tilde{x},v)\partial_{u_1}^{N,-}H(1,\tilde{x})\Big| \\
&\leq \frac{1}{2}\|H''\|_\infty + \frac{CNN^{d-1}}{2N^d}\|H'\|_\infty + \frac{CNN^{d-1}}{2N^d}\|H'\|_\infty \\
&= \frac{1}{2}\|H''\|_\infty + C\|H'\|_\infty,
\end{aligned} \tag{1.35}$$

since $|\eta_{sN^2}(x,v)| \leq 1$ for all $s \in [0,T]$, for each $v \in \mathcal{V}$ fixed and since $H \in C^2(D^d)$. Similarly,

$$\begin{aligned}
|N^2\mathcal{L}_N^{b}\langle \pi_s^{k,N}, H\rangle| &\leq \Big|\frac{N^2}{N^d N^\theta}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}} v_k H\left(\tfrac{1}{N},\tfrac{\tilde{x}}{N}\right)[\alpha_v - \eta(1,\tilde{x},v)]\Big| \\
&+\Big|\frac{N^2}{N^d N^\theta}\sum_{\substack{x\in D_N^d\\ x_1=N-1}}\sum_{v\in\mathcal{V}} v_k H\left(\tfrac{N-1}{N},\tfrac{\tilde{x}}{N}\right)[\beta_v - \eta(N-1,\tilde{x},v)]\Big| \\
&\leq \frac{CN^2N^{d-1}}{N^dN^\theta}\|H\|_\infty + \frac{CN^2N^{d-1}}{N^dN^\theta}\|H\|_\infty \\
&= 2CN^{1-\theta}\|H\|_\infty.
\end{aligned} \tag{1.36}$$

Also,

$$
\begin{aligned}
&|N^2\mathcal{L}_N^{ex,2}\langle \pi_s^{k,N}, H\rangle| \\
&\leq \Big|\frac{1}{N^d}\sum_{j=1}^{d}\sum_{x\in D_N^d}(\partial_{u_j}^N H)\left(\tfrac{x}{N}\right)\sum_{v\in\mathcal{V}} v_k \sum_{z\in\mathbb{Z}^d} p(z,v) z_j \eta(x,v)(1-\eta(x+z,v))\Big| \\
&\leq \frac{\tilde{C}N^d\|H'\|_\infty}{N^d} \\
&= \tilde{C}\|H'\|_\infty.
\end{aligned} \tag{1.37}
$$

By Lemma 18,

$$|N^2\mathcal{L}_N^c\langle \pi_s^{k,N}, H\rangle| = 0. \tag{1.38}$$

Therefore, by (1.35), (1.36), (1.37) and (1.38), we have that

$$|\mathcal{L}_N\langle \pi_s^{k,N}, H\rangle| \leq C.$$

This proves (1.33) for $\theta \geq 1$.

Now we will prove (1.34). Applying Dynkin's formula, we have that

$$(M_t^{N,k}(H))^2 - \int_0^t \mathcal{L}_N[\langle \pi_s^{k,N}, H\rangle]^2 - 2\langle \pi_s^{k,N}, H\rangle \mathcal{L}_N\langle \pi_s^{k,N}, H\rangle ds$$

is a martingale with respect to the natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$. By Lemma 20, we have that

$$
\begin{aligned}
&N^2\mathcal{L}_N^{ex,1}\langle \pi_s^{k,N}, H\rangle^2 - 2\langle \pi_s^{k,N}, H\rangle N^2\mathcal{L}_N^{ex,1}\langle \pi_s^{k,N}, H\rangle \\
&= \frac{1}{2N^{2d}}\sum_{v\in\mathcal{V}}\sum_{x\in D_N^d}\sum_{j=1}^{d} v_k^2\left[\eta(x,v)-\eta(x+e_j,v))\right]^2[\partial_{u_j}^N H\left(\tfrac{x}{N}\right)]^2
\end{aligned}
$$

and by using the fact that $|\eta_{sN^2}(x,v)| \leq 1$ for all $s \in [0,t]$ and fixed $v$, the last expression is bounded from above by $\frac{C}{N^d}\|H'\|_\infty$. We have from Lemma 21 that

$$
\begin{aligned}
&N^2\mathcal{L}_N^{ex,2}\langle \pi_s^{k,N}, H\rangle^2 - 2\langle \pi_s^{k,N}, H\rangle N^2\mathcal{L}_N^{ex,2}\langle \pi_s^{k,N}, H\rangle \\
&= \frac{1}{N^{2d+1}}\sum_{v\in\mathcal{V}}\sum_{x\in D_N^d}\sum_{w\in\mathbb{Z}^d} v_k^2\,\eta(x,v)(1-\eta(x+w,v))p(w,v)w_j^2[\partial_{u_j}^N H\left(\tfrac{x}{N}\right)]^2
\end{aligned}
$$

and by using the fact that $|\eta_{sN^2}(x,v)| \leq 1$ for all $s \in [0,t]$ last expression is bounded from above by $\frac{\tilde{C}}{N^{d+1}}\|H'\|_\infty$. Additionally, from Lemma 22 we have

$$
\begin{aligned}
&N^2\mathcal{L}_N^b\langle \pi_s^{k,N}, H\rangle^2 - 2\langle \pi_s^{k,N}, H\rangle N^2\mathcal{L}_N^b\langle \pi_s^{k,N}, H\rangle \\
&= \frac{N^2}{N^{2d}} \sum_{\substack{x\in D_N^d\\ x_1=1}} \sum_{v\in\mathcal{V}} \left[\frac{\alpha_v(\frac{\tilde{x}}{N})(1-\eta(x,v)) + (1-\alpha_v(\frac{\tilde{x}}{N}))\eta(x,v)}{N^\theta}\right] v_k^2\, H^2\left(\tfrac{x}{N}\right) \\
&+\frac{N^2}{N^{2d}} \sum_{\substack{x\in D_N^d\\ x_1=1}} \sum_{v\in\mathcal{V}} \left[\frac{\beta_v(\frac{\tilde{x}}{N})(1-\eta(x,v)) + (1-\beta_v(\frac{\tilde{x}}{N}))\eta(x,v)}{N^\theta}\right] v_k^2\, H^2\left(\tfrac{x}{N}\right)
\end{aligned}
\tag{1.39}
$$

and by using the fact that $|\eta_{sN^2}(x,v)| \leq 1$ for all $s \in [0,t]$ last expression is bounded from above by $\frac{N^{1-\theta}\overline{C}}{N^d}\|H\|_\infty$, where $\overline{C}$ comes from the fact that the set $\mathcal{V}$ is finite. This finishes the proof of tightness in the case $\theta \geq 1$, since $C^2(D^d)$ is a subset dense of $C(D^d)$ with respect to uniform topology.

### 1.8.2 The case $\theta \in [0, 1)$

If we try to apply the same strategy used for $\theta \geq 1$ we will run into trouble when trying to control the modulus of continuity of $\int_0^t N^2\mathcal{L}_N^b\langle \pi_s^{k,N}, H\rangle\, ds$, because the expression in (1.36) can explode when $N \to \infty$. We will prove (1.32) first for functions $H \in C_c^2(D^d)$ and then we can extend it, by a $L^1$ approximation procedure which is explained below, to functions $H \in C^1(D^d)$. We can see in this case that

$$
\begin{aligned}
|N^2\mathcal{L}_N^{ex,1}\langle \pi_s^{k,N}, H\rangle| &\leq |\langle \pi_s^{k,N}, \frac{1}{2}\Delta_N H\rangle| + \Big|\frac{N}{2N^d} \sum_{\substack{x\in D_N^d\\ x_1=1}} \sum_{v\in\mathcal{V}} v_k \eta(1,\tilde{x},v)\partial_{u_1}^{N,+}H(0,\tilde{x})\Big| \\
&+\Big|\frac{N}{2N^d} \sum_{\substack{x\in D_N^d\\ x_1=N-1}} \sum_{v\in\mathcal{V}} v_k \eta(N-1,\tilde{x},v)\partial_{u_1}^{N,-}H(1,\tilde{x})\Big| \\
&\leq \frac{1}{2}\|H''\|_\infty.
\end{aligned}
$$

Also,

$$
\begin{aligned}
&|N^2\mathcal{L}_N^b\langle\pi_s^{k,N},H\rangle| \\
&\leq \Big|\frac{N^2}{N^dN^\theta}\sum_{\substack{x\in D_N^d\\x_1=1}}\sum_{v\in\mathcal{V}} v_k H\left(\tfrac{1}{N},\tfrac{\tilde{x}}{N}\right)[\alpha_v(\tilde{x})-\eta(1,\tilde{x},v)]\Big| \\
&+\Big|\frac{N^2}{N^dN^\theta}\sum_{\substack{x\in D_N^d\\x_1=N-1}}\sum_{v\in\mathcal{V}} v_k H\left(\tfrac{N-1}{N},\tfrac{\tilde{x}}{N}\right)[\beta_v(\tilde{x})-\eta(N-1,\tilde{x},v)]\Big| \\
&\leq \Big|\frac{N}{N^dN^\theta}\sum_{\substack{x\in D_N^d\\x_1=1}}\sum_{v\in\mathcal{V}} v_k\, N\left[H\left(\tfrac{1}{N},\tfrac{\tilde{x}}{N}\right)-H\left(\tfrac{0}{N},\tfrac{\tilde{x}}{N}\right)\right][\alpha_v(\tilde{x})-\eta(1,\tilde{x},v)]\Big| \\
&+\Big|\frac{N}{N^dN^\theta}\sum_{\substack{x\in D_N^d\\x_1=N-1}}\sum_{v\in\mathcal{V}} v_k\, N\left[H\left(\tfrac{N-1}{N},\tfrac{\tilde{x}}{N}\right)-H\left(\tfrac{N}{N},\tfrac{\tilde{x}}{N}\right)\right][\beta_v(\tilde{x})-\eta(N-1,\tilde{x},v)]\Big| \\
&\leq \Big|\frac{N}{N^dN^\theta}\sum_{\substack{x\in D_N^d\\x_1=1}}\sum_{v\in\mathcal{V}} v_k\, \partial_{u_1}^{N,+}H\left(\tfrac{0}{N},\tfrac{\tilde{x}}{N}\right)[\alpha_v(\tilde{x})-\eta(1,\tilde{x},v)]\Big| \\
&+\Big|\frac{N}{N^dN^\theta}\sum_{\substack{x\in D_N^d\\x_1=N-1}}\sum_{v\in\mathcal{V}} v_k\, \partial_{u_1}^{N,-}H\left(\tfrac{N}{N},\tfrac{\tilde{x}}{N}\right)[\beta_v(\tilde{x})-\eta(N-1,\tilde{x},v)]\Big| \\
&=0
\end{aligned}
$$

since $H\in C_c^2(D^d)$. This finishes the proof of (1.33) for any $\theta\geq 0$.

To prove (1.34) for $\theta\in[0,1)$, we use the same computations we did for the case $\theta\geq 1$. Since $H\in C_c^2(D^d)$, the equation (1.39) is equal to zero, but as mentioned before, we need to extend this result to functions in $C^1(D^d)$. To accomplish that, we take a function $H\in C^1(D^d)\subset L^1(D^d)$, and we take a sequence of functions $(H_j)_{j\geq 0}\in C_c^2(D^d)$ converging to $H$, with respect to the $L^1$-norm, as $j\to\infty$. Now since the probability in (1.32) is less or equal than

$$
\mathbb{P}_{\mu^N}\left(\eta:|\langle\pi_{\tau+\bar{\tau}}^{k,N},H_j\rangle-\langle\pi_\tau^{k,N},H_j\rangle|>\frac{\varepsilon}{2}\right)
$$

$$
+\mathbb{P}_{\mu^N}\left(\eta:|\langle\pi_{\tau+\bar{\tau}}^{k,N},(H-H_j)\rangle-\langle\pi_\tau^{k,N},(H-H_j)\rangle|>\frac{\varepsilon}{2}\right)
$$

and since $H_j$ has compact support, from the computation above, it remains only to check that the last probability vanishes as $N\to\infty$ and then $j\to\infty$. For that purpose, we use the fact that

$$|\langle \pi^{k,N}_{\tau+\bar{\tau}}, (H-H_j)\rangle - \langle \pi^{k,N}_{\tau}, (H-H_j)\rangle| \leq \frac{C}{N^d}\sum_{x\in D^d_N} |(H-H_j)\left(\tfrac{x}{N}\right)|$$

and we use the estimate

$$\begin{aligned}
&\frac{1}{N^d}\sum_{x\in D^d_N} |(H-H_j)\left(\tfrac{x}{N}\right)| \\
&\leq \sum_{x\in D^d_N}\int_{\frac{x}{N}}^{\frac{x+1}{N}} |(H-H_j)\left(\tfrac{x}{N}\right) - (H-H_j)(u)|\,du + \int_{D^d} |(H-H_j)(u)|\,du \\
&\leq \frac{1}{N^d}\|(H-H_j)'\|_\infty + \int_{D^d} |(H-H_j)(u)|\,du.
\end{aligned}$$

The result follows by first taking $N\to\infty$ and then $j\to\infty$.

## 1.9 Energy Estimates

We will now define some quantities in order to prove that each component of the vector solution belongs, in fact, to $\mathscr{H}^1([0,T]\times D^d)$.

Let the energy $\mathscr{E} : \mathcal{D}([0,T],\mathcal{M}) \to [0,\infty]$ be given by

$$\mathscr{E}(\pi) = \sum_{i=1}^d \mathscr{E}_i(\pi),$$

with

$$\mathscr{E}_i(\pi) = \sup_{G\in C^\infty_c(\Omega_T)} \left\{ 2\int_0^T dt\,\langle \pi_t, \partial_{u_i} G_t\rangle - \int_0^T dt \int_{D^d} du\, G(t,u)^2 \right\},$$

where $\Omega_T = (0,T)\times D^d$ and $C^\infty_c(\Omega_T)$ stands for the set of infinitely differentiable functions (with respect to time and space) with compact support contained in $\Omega_T$. For any $G\in C^\infty_c(\Omega_T)$, $1\leq i\leq d$ and $C\geq 0$, let the functional $\mathscr{E}^G_{i,C} : \mathcal{D}([0,T],\mathcal{M}) \to \mathbb{R}$ be given by

$$\mathscr{E}^G_{i,C}(\pi) = \int_0^T ds\,\langle \pi_s, \partial_{u_i} G_s\rangle - C\int_0^T ds\int_{D^d} du\, G(s,u)^2.$$

Note that

$$\sup_{G\in C^\infty_c(\Omega_T)} \{\mathscr{E}^G_{i,C}\} = \frac{\mathscr{E}_i(\pi)}{4C}. \tag{1.40}$$

It is well-known that $\mathscr{E}(\pi)$ is finite if, and only if, $\pi$ has a generalized gradient,

$\nabla\pi = (\partial_{u_1}\pi, \ldots, \partial_{u_d}\pi)$, which is a measurable function and

$$\tilde{\mathscr{E}}(\pi) = \int_0^T ds \int_{D^d} du \, \|\nabla\pi_t(u)\|^2 < \infty,$$

in which case, $\mathscr{E}(\pi) = \tilde{\mathscr{E}}(\pi)$. Recall from Section 1.8 that the sequence $(\mathbb{Q}_N)_N$ is tight. We have that:

**Proposition 3.** *Let $\mathbb{Q}^*$ be any limit point of the sequence of measures $(\mathbb{Q}_N)_N$. Then,*

$$E_{\mathbb{Q}^*}\left[\int_0^T ds \left(\int_{D^d} \|\nabla\rho(s,u)\|^2 \, du\right)\right] < \infty$$

*and*

$$E_{\mathbb{Q}^*}\left[\int_0^T ds \left(\int_{D^d} \|\nabla\varrho_k(s,u)\|^2 \, du\right)\right] < \infty,$$

*for $k = 1, \ldots, d$.*

The proof follows from the next lemma and Riesz Representation Theorem.

**Lemma 7.** *For all $\theta \geq 0$, there is a positive constant $C > 0$ such that*

$$E_{\mathbb{Q}^*}\left[\sup_G \left\{\int_0^T \int_{D^d} \partial_{u_i} G(s,u)\varrho_k(s,u) du ds - C\int_0^T ds \int_{D^d} du G(s,u)^2\right\}\right] < \infty,$$

*for $k = 0, 1, \ldots, d$, where the supremum is carried over all the functions $G \in C^\infty(\Omega_T)$ and $\varrho_0 = \rho$.*

*Proof.* Let $\{G^m : m \geq 1\}$ be a sequence of functions in $C_c^\infty(\Omega_T)$ (the space of infinitely differentiable functions with compact support). Thus, it is sufficient to prove that, for every $r \geq 1$,

$$E_{\mathbb{Q}^*}\left[\max_{1\leq m\leq r}\left\{\mathscr{E}_{i,C}^{G^m}(\pi^{k,N})\right\}\right] \leq \tilde{C}, \tag{1.41}$$

for some constant $\tilde{C} > 0$, independent of $r$. The expression on the left-hand side of (1.41) is equal to

$$\lim_{N\to\infty} E_{\mu^N}\left[\max_{1\leq m\leq r}\left\{\int_0^T \langle \partial_{u_i} G^m(s,u), \pi_s^{k,N}\rangle ds - C\int_0^T ds \int_{D^d} du G^m(s,u)^2\right\}\right]. \tag{1.42}$$

By the relative entropy bound (see Remark 4), Jensen's inequality and $\exp\{\max_{1\leq j\leq k} a_j\} \leq$

$\sum_{1\le j\le k} \exp a_j$, the expectation in (1.42) is bounded from above by

$$\frac{H(\mu^N|\nu_\kappa^N)}{N^d}$$

$$+\frac{1}{N^d}\log\sum_{1\le m\le r} E_{\nu_\kappa^N}\left[\exp\left\{\int_0^T N\langle\partial_{u_i}G^m(s,u),\pi_s^{k,N}\rangle ds - C\int_0^T ds\int_{D^d} duG^m(s,u)^2\right\}\right],$$

where the profile $\kappa$ is the same used in Section 1.7.

We can bound the first term in the sum above by $C_\kappa$. It is enough to show, for a fixed function $G$, that

$$\limsup_{N\to\infty}\frac{1}{N^d}\log E_{\nu_\kappa^N}\left[\exp\left\{\int_0^T N\langle\partial_{u_i}G(s,u),\pi_s^{k,N}\rangle ds - C\int_0^T ds\int_{D^d} duG(s,u)^2\right\}\right]\le\tilde c$$

for some constant $\tilde c$ independent of $G$. Then the result follows from the next lemma and the definition of the empirical measure.

**Lemma 8.** *There exists a constant $C_0 = C_0(\kappa) > 0$, such that for every $i = 1,\dots,d$ every $k = 0,\dots,d$ and every function $G \in C_c^\infty(\Omega_T)$*

$$\limsup_{N\to\infty}\frac{1}{N^d}\log E_{\nu_\kappa^N}\left[\exp\{N^d\mathscr{E}_{i,C_0}^G(\pi^{k,N})\}\right]\le C_0.$$

*Proof.* Writing $\partial_{u_i}G_s\left(\frac{x}{N}\right) = N\left[G_s\left(\frac{x+e_i}{N}\right) - G_s\left(\frac{x}{N}\right)\right] + O(N^{-1})$ and summing by parts (the compact support of $G$ takes care of the boundary term), by applying the Feynman-Kac formula and using the same arguments as in the proof of Lemma 1, we have that

$$\frac{1}{N^d}\log E_{\nu_\kappa^N}\left[\exp\left\{N\int_0^T ds\sum_{x\in D_N^d}(I_k(\eta_x(s)) - I_k(\eta_{x-e_i}(s)))G\left(s,\tfrac{x}{N}\right)\right\}\right]$$

is bounded from above by

$$\frac{1}{N^d}\int_0^T \lambda_s^N\, ds,$$

where $\lambda_s^N$ is equal to

$$\sup_f\left\{\left\langle N\sum_{x\in D_N^d}((I_k(\eta_x(s)) - I_k(\eta_{x-e_i}(s)))G\left(s,\tfrac{x}{N}\right),f\right\rangle_{\nu_\kappa^N} + N^2\langle\mathcal{L}_N\sqrt{f},\sqrt{f}\rangle_{\nu_\kappa^N}\right\},\tag{1.43}$$

where the supremum is taken over all densities $f$ with respect to $\nu_\kappa^N$. By Corollaries 1, 2 and 3, for a constant function $\kappa$, the expression inside brackets is bounded from

above by

$$-\frac{N^2}{2}D_{\nu_\kappa^N}(\sqrt{f})+\sum_{x\in D_N^d}\left\{NG\left(s,\tfrac{x}{N}\right)\int[I_k(\eta_x(s))-I_k(\eta_{x-e_i}(s))]f(\eta)d\nu_\kappa^N\right\}.$$

We now rewrite the term inside the brackets as

$$\sum_{v\in\mathcal{V}}v_k\sum_{x\in D_N^d}\left\{\int NG\left(s,\tfrac{x}{N}\right)[\eta(x,v)-\eta(x-e_i,v)]f(\eta)d\nu_\kappa^N\right\}. \tag{1.44}$$

After a simple computation, we may rewrite the terms inside the brackets of the above expression as

$$\begin{aligned}
&NG\left(s,\tfrac{x}{N}\right)\int[\eta(x,v)-\eta(x-e_i,v)]f(\eta)d\nu_\kappa^N\\
&=NG\left(s,\tfrac{x}{N}\right)\int\eta(x,v)f(\eta)d\nu_\kappa^N\\
&-NG\left(s,\tfrac{x}{N}\right)\int\eta(x,v)f(\eta^{x-e_i,x,v})\tfrac{\nu_\kappa^N(\eta^{x,x-e_i,v})}{\nu_\kappa^N(\eta)}d\nu_\kappa^N\\
&=NG\left(s,\tfrac{x}{N}\right)\int\eta(x,v)[f(\eta)-f(\eta^{x-e_i,x,v})]d\nu_\kappa^N.
\end{aligned}$$

By using $f(\eta)-f(\eta^{x-e_i,x,v})=[\sqrt{f(\eta)}-\sqrt{f(\eta^{x-e_i,x,v})}][\sqrt{f(\eta)}+\sqrt{f(\eta^{x-e_i,x,v})}]$ and applying Young's inequality, the above expression is bounded from above by

$$\begin{aligned}
&\frac{N^2}{2}\int[\sqrt{f(\eta^{x-e_i,x,v})}-\sqrt{f(\eta)}]^2d\nu_\kappa^N\\
&+2G\left(s,\tfrac{x}{N}\right)^2\int\eta(x,v)(\sqrt{f(\eta)}+\sqrt{f(\eta^{x-e_i,x,v})})^2d\nu_\kappa^N.
\end{aligned}$$

Using the above estimate, we have that (1.44) is clearly bounded by $\frac{N^2}{2}D_{\nu_\kappa^N}(\sqrt{f})+CG\left(s,\frac{x}{N}\right)^2$, where $C$ is a positive constant. Thus, letting $C_0=C$, the statement of the lemma holds. Now we will analyze (1.43) for a general function $\kappa$. By Corollaries 1, 2 and 3, the expression inside brackets is bounded from above by

$$CN^d-\frac{N^2}{4}D_{\nu_\kappa^N}(\sqrt{f})+\sum_{x\in D_N^d}\left\{NG\left(s,\tfrac{x}{N}\right)\int[I_k(\eta_x(s))-I_k(\eta_{x-e_i}(s))]f(\eta)d\nu_\kappa^N\right\}.$$

We will analyze the term inside brackets above

$$\sum_{x\in D_N^d}\sum_{v\in\mathcal{V}}v_k\left\{NG\left(s,\tfrac{x}{N}\right)\int[\eta(x,v)-\eta(x-e_i,v)]f(\eta)d\nu_\kappa^N\right\}. \tag{1.45}$$

Now rewrite the term inside the brackets as

$$NG\left(s,\tfrac{x}{N}\right)\int[\eta(x,v)-\eta(x-e_i,v)]f(\eta)d\nu_\kappa^N$$

$$=NG\left(s,\tfrac{x}{N}\right)\int\eta(x,v)f(\eta)d\nu_\kappa^N$$

$$-NG\left(s,\tfrac{x}{N}\right)\int\eta(x,v)f(\eta^{x-e_i,x,v})\tfrac{\nu_\kappa^N(\eta^{x,x-e_i,v})}{\nu_\kappa^N(\eta)}d\nu_\kappa^N$$

$$=NG\left(s,\tfrac{x}{N}\right)\int\eta(x,v)[f(\eta)-f(\eta^{x-e_i,x,v})]d\nu_\kappa^N$$

$$+G\left(s,\tfrac{x}{N}\right)\int\eta(x,v)f(\eta^{x-e_i,x,v})N\left[1-\tfrac{\nu_\kappa^N(\eta^{x,x-e_i,v})}{\nu_\kappa^N(\eta)}\right]d\nu_\kappa^N.$$

Since $f(\eta)-f(\eta^{x-e_i,x,v})=[\sqrt{f(\eta)}-\sqrt{f(\eta^{x-e_i,x,v})}][\sqrt{f(\eta)}+\sqrt{f(\eta^{x-e_i,x,v})}]$ and applying Young's inequality, the expression is bounded from above by

$$N^2\int\frac{1}{2}[\sqrt{f(\eta^{x-e_i,x,v})}-\sqrt{f(\eta)}]^2d\nu_\kappa^N$$

$$+2G\left(s,\tfrac{x}{N}\right)^2\int\eta(x,v)(\sqrt{f(\eta)}+\sqrt{f(\eta^{x-e_i,x,v})})^2d\nu_\kappa^N$$

$$+G\left(s,\tfrac{x}{N}\right)^2\int f(\eta^{x-e_i,x,v})d\nu_\kappa^N$$

$$+\frac{1}{4}\int\eta(x,v)f(\eta^{x-e_i,x,v})\left[N\left(1-\tfrac{\nu_\kappa^N(\eta^{x,x-e_i,v})}{\nu_\kappa^N(\eta)}\right)\right]^2d\nu_\kappa^N.$$

Using the above estimate, (1.45) is clearly bounded by $C_1+C_1G\left(s,\tfrac{x}{N}\right)^2$, by some positive constant $C_1=C_1(\kappa)$, using the estimate (1.21) and the fact that $f$ is a density with respect to $\nu_\kappa^N$. Thus, letting $C_0=C+C_1$, the statement of the lemma follows. □

*Proof of Proposition 3.* Let $\{G_m : 1\le m\le r\}$ be a sequence of functions in $C_c^\infty(\Omega_T)$ (the space of infinitely differentiable functions with compact support) and $1\le i\le d$, and $0\le k\le d$, be integers. By the entropy inequality, see Remark 4, there exists a constant $C_\kappa>0$ such that

$$E_{\mu^N}\left[\max_{1\le m\le r}\left\{\mathscr{E}_{i,C_0}^{G_m}(\pi^{k,N})\right\}\right]\le C_\kappa+\frac{1}{N^d}\log E_{\nu_\kappa^N}\left[\exp\left\{N^d\max_{1\le m\le r}\{\mathscr{E}_{i,C_0}^{G_m}(\pi^{k,N})\}\right\}\right].$$

Therefore, using Lemma 8 together with the elementary inequalities

$$\limsup_{N\to\infty} N^{-d}\log(a_N+b_N) \le \limsup_{N\to\infty}\max\left\{\limsup_{N\to\infty} N^{-d}\log(a_N), \limsup_{N\to\infty} N^{-d}\log(b_N)\right\}$$

and

$$\exp\{\max\{x_1,\dots,x_n\}\} \le \exp(x_1)+\dots+\exp(x_n)$$

we set that

$$\begin{aligned} E_{\mathbb{Q}^*}\left[\max_{1\le m\le r}\left\{\mathscr{E}_{i,C_0}^{G_m}(\pi^{k,N})\right\}\right] &= \lim_{N\to\infty} E_{\mu^N}\left[\max_{1\le m\le r}\left\{\mathscr{E}_{i,C_0}^{G_m}(\pi^{k,N})\right\}\right] \\ &\le C_\kappa + C_0. \end{aligned}$$

Using this, the equation (1.40) and the monotone convergence Theorem, we obtain the desired result. □

## 1.10 Proof of Theorem 1

Since there is at most one particle per site we have, by a standard argument, that all limit points $\mathbb{Q}^*$ of $(\mathbb{Q}_N)_N$ are concentrated on an absolutely continuous measures with respect to the Lebesgue measure. For more details, see [1]. Thus,

$$\mathbb{Q}^*\{\pi;\, \pi^k(du) = \varrho_k(u)du, \text{ for all } 0\le k\le d\} = 1,$$

where $\pi^k$ denotes the $k$-th component of $\pi$ and $\varrho_0 = \rho$.

We consider the martingale

$$M_t^{N,k}(H) = \langle \pi_t^{k,N}, H\rangle - \langle \pi_0^{k,N}, H\rangle - \int_0^t \mathcal{L}_N\langle \pi_s^{k,N}, H\rangle\, ds$$

which can be rewritten explicity as

$$\begin{aligned} M_t^{N,k}(H) &= \langle \pi_t^{k,N}, H\rangle - \langle \pi_0^{k,N}, H\rangle - \int_0^t N^2\mathcal{L}_N^{ex,1}\langle \pi_s^{k,N}, H\rangle\, ds \\ &- \int_0^t N^2\mathcal{L}_N^{ex,2}\langle \pi_s^{k,N}, H\rangle\, ds - \int_0^t N^2\mathcal{L}_N^{b}\langle \pi_s^{k,N}, H\rangle\, ds - \int_0^t N^2\mathcal{L}_N^{c}\langle \pi_s^{k,N}, H\rangle\, ds. \end{aligned}$$

By equations (A.4), (A.6), (A.8) and (A.10), we have that

$$
\begin{aligned}
M_t^{N,k}(H) &= \langle \pi_t^{k,N}, H\rangle - \langle \pi_0^{k,N}, H\rangle - \frac{1}{2N^d}\sum_{x\in D_N^d}\sum_{v\in\mathcal{V}} v_k\,\eta(x,v)\Delta_N H\left(\tfrac{x}{N}\right) \\
&+\frac{N^2}{2N^d}\sum_{\substack{x\in D_N^d\\ x_1=N-1}}\sum_{v\in\mathcal{V}} v_k\,\left[H\left(\tfrac{N-1}{N},\tfrac{\tilde{x}}{N}\right) - H\left(\tfrac{N}{N},\tfrac{\tilde{x}}{N}\right)\right]\eta(N-1,\tilde{x},v) \\
&+\frac{N^2}{2N^d}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}} v_k\,\left[H\left(\tfrac{1}{N},\tfrac{\tilde{x}}{N}\right) - H\left(\tfrac{0}{N},\tfrac{\tilde{x}}{N}\right)\right]\eta(1,\tilde{x},v) \\
&-\frac{1}{N^d}\sum_{j=1}^{d}\sum_{x\in D_N^d}(\partial_{u_j}^N H)\left(\frac{x}{N}\right)\tau_x W_{j,k}^{N,\eta_s} \\
&-\frac{N^2}{N^dN^\theta}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}} v_k H\left(\frac{x}{N}\right)\left[\alpha_v\left(\tfrac{\tilde{x}}{N}\right) - \eta(1,\tilde{x},v)\right] \\
&-\frac{N^2}{N^dN^\theta}\sum_{\substack{x\in D_N^d\\ x_1=N-1}}\sum_{v\in\mathcal{V}} v_k H\left(\tfrac{x}{N}\right)\left[\beta_v\left(\tfrac{\tilde{x}}{N}\right) - \eta(N-1,\tilde{x},v)\right]
\end{aligned}
$$

where $(\tau_x\eta)(z,v) = \eta(x+z,v)$ and $W_{j,k}^{N,\eta_s} = \sum_{v\in\mathcal{V}} v_k \sum_{z\in\mathbb{Z}^d} p(z,v)z_j\eta_s(0,v)(1-\eta_s(z,v))$.

We then apply the Replacement Lemma to rewrite the martingale in terms of the empirical measure. Further, we apply Lemma 1 (replacement lemma for the boundary) to obtain that all limit points satisfy the integral identity in the definition of the corresponding weak solution.

Using the previous computations and the tightness of the sequence of measures $(\mathbb{Q}_N)_N$, we conclude that all limit points are concentrated on weak solutions of

$$
\partial_t(\rho,\varrho) + \sum_{v\in\mathcal{V}}\tilde{v}[v\cdot\nabla\chi(\theta_v(\Lambda(\rho,\varrho)))] = \frac{1}{2}\Delta(\rho,\varrho)
$$

with boundary conditions depending on $\theta$. The uniqueness of weak solutions of the above equation implies that there is at most one limit point. Moreover, by Proposition 3, each limit point of $(\mathbb{Q}_N)_N$ is concentrated on a vector of measures with finite energy, that is: whose components have a density with respect to the Lebesgue measure that belongs to the Sobolev space $\mathscr{H}^1(D^d)$. This completes the proof of Theorem 1. □

## 1.11 Characterization of the limit points

This section deals with the characterization of the limit points in the three regime of $\theta \geq 0$.

### 1.11.1 Characterization of the limit points for $\theta \in [0, 1)$

Now we look at the limit points of the sequence $\{\mathbb{Q}_N\}_{N\geq 1}$.

**Proposition 4.** *If $\mathbb{Q}^*$ is a limit point of $\{\mathbb{Q}_N\}_{N\geq 1}$, then*

$$\mathbb{Q}^*\Bigg[\pi_\cdot : \int_{D^d} (\rho,\varrho)(t,x)G(t,x)\,dx - \int_{D^d} (\rho,\varrho)(0,x)G(0,x)\,dx$$

$$-\int_0^t \int_{D^d} \sum_{v\in\mathcal{V}} \tilde{v}\cdot\chi(\theta_v(\Lambda(\rho,\varrho))) \sum_{i=1}^d v_i \frac{\partial G}{\partial x_i}(r,x)\,dx\,dr$$

$$+\frac{1}{2}\int_0^t \int_{\{1\}\times\mathbb{T}^{d-1}} d(1,\tilde{x})\frac{\partial G}{\partial x_1}(r,1,\tilde{x})\,dS\,dr - \frac{1}{2}\int_0^t \int_{\{0\}\times\mathbb{T}^{d-1}} d(0,\tilde{x})\frac{\partial G}{\partial x_1}(r,0,\tilde{x})\,dS\,dr$$

$$-\int_0^t \int_{D^d} (\rho,\varrho)(r,x)\left(\partial_r G(r,x) + \frac{1}{2}\Delta G\right)\,dx\,dr = 0\Bigg] = 1,$$

*for all $t \in [0,T]$, $\forall G \in C_0^{1,2}([0,T]\times D^d)$.*

*Proof.* It is enough to verify that, for $\delta > 0$ and $G \in C_0^{1,2}([0,T]\times D^d)$ fixed,

$$\mathbb{Q}^*\Bigg[\pi_\cdot : \sup_{0\leq t\leq T}\Bigg|\int_{D^d} (\rho,\varrho)(t,x)G(t,x)\,dx - \int_{D^d} (\rho,\varrho)(0,x)G(0,x)\,dx$$

$$-\int_0^t \int_{D^d} \sum_{v\in\mathcal{V}} \tilde{v}\cdot\chi(\theta_v(\Lambda(\rho,\varrho))) \sum_{i=1}^d v_i \frac{\partial G}{\partial x_i}(r,x)\,du\,dr$$

$$+\frac{1}{2}\int_0^t \int_{\{1\}\times\mathbb{T}^{d-1}} d(1,\tilde{x})\frac{\partial G}{\partial x_1}(r,1,\tilde{x})\,dS\,dr - \frac{1}{2}\int_0^t \int_{\{0\}\times\mathbb{T}^{d-1}} d(0,\tilde{x})\frac{\partial G}{\partial x_1}(r,0,\tilde{x})\,dS\,dr$$

$$-\int_0^t \int_{D^d} (\rho,\varrho)(r,x)\left(\partial_r G(r,x) + \frac{1}{2}\Delta G\right)\,dx\,dr\Bigg| > \delta\Bigg] = 0.$$

Since the set considered above is an open set, we can use the Portmanteau's Theorem directly and bound the last probability by

$$\liminf_{N\to\infty} \mathbb{Q}_N \Bigg[\pi. : \sup_{0\le t\le T} \Big|\langle \pi_t^k, G_t\rangle - \langle \pi_0^k, G_0\rangle$$

$$- \int_0^t \int_{D^d} \sum_{v\in\mathcal{V}} \tilde{v}\cdot \chi(\theta_v(\Lambda(\rho,\varrho))) \sum_{i=1}^d v_i \frac{\partial G}{\partial x_i}(r,x)\, dx\, dr$$

$$+\frac{1}{2}\int_0^t \int_{\{1\}\times\mathbb{T}^{d-1}} d(1,\tilde{x}) \frac{\partial G}{\partial x_1}(r,1,\tilde{x})\, dS\, dr - \frac{1}{2}\int_0^t \int_{\{0\}\times\mathbb{T}^{d-1}} d(0,\tilde{x}) \frac{\partial G}{\partial x_1}(r,0,\tilde{x})\, dS\, dr$$

$$- \int_0^t \Big\langle \pi_r^k, \Big(\partial_r G(r,x) + \frac{1}{2}\Delta G\Big)\Big\rangle\, dx\, dr\Big| > \delta\Bigg] = 0.$$

Summing and subtracting $\int_0^t N^2 \mathcal{L}_N \langle \pi_r^{k,N}, G_r\rangle\, dr$ in the expression above, we can bound it by the sum of

$$\liminf_{N\to\infty} \mathbb{Q}_N \Big[\sup_{0\le t\le T} |M_t^{N,k}(G)| > \frac{\delta}{2}\Big]$$

and

$$\limsup_{N\to\infty} \mathbb{P}_{\mu^N} \Bigg[\eta. : \sup_{0\le t\le T} \Big|\int_0^t N^2 \mathcal{L}_N \langle \pi_r^{k,N}, G_r\rangle\, dr - \frac{1}{2}\int_0^t \langle \pi_r^{k,N}, \Delta G\rangle\, dr$$

$$- \int_0^t \int_{D^d} \sum_{v\in\mathcal{V}} \tilde{v}\cdot \chi(\theta_v(\Lambda(\rho,\varrho))) \sum_{i=1}^d v_i \frac{\partial G}{\partial x_i}(r,x)\, dx\, dr$$

$$+ \frac{1}{2}\int_0^t \int_{\{1\}\times\mathbb{T}^{d-1}} d(1,\tilde{u}) \frac{\partial G}{\partial u_1}(r,1,\tilde{x})\, dS\, dr - \frac{1}{2}\int_0^t \int_{\{0\}\times\mathbb{T}^{d-1}} d(0,\tilde{x}) \frac{\partial G}{\partial u_1}(r,0,\tilde{u})\, dS\, dr\Big| > \frac{\delta}{2}\Bigg],$$

where $M_r^{N,k}(G)$ was defined in (1.17) and $\pi_r^{k,N}$ is the empirical measure defined in (1.14). Now, let us bound the expression inside the probability above by the sum of the following terms

$$\sup_{0\le t\le T} \Big|\int_0^t \big[\frac{1}{2N^d} \sum_{x\in D_N^d} I_k(\eta_x(r))\Delta_N G\left(\tfrac{x}{N}\right) - \frac{1}{2N^d} \sum_{x\in D_N^d} I_k(\eta_x(r))\Delta G\left(\tfrac{x}{N}\right)\big]\, dr\Big|,$$

$$\sup_{0\le t\le T} \Big|\int_0^t \frac{1}{2N^{d-1}} \sum_{\substack{x\in D_N^d \\ x_1 = N-1}} I_k(\eta_{(N-1,\tilde{x})}(r)) \left\{\partial_{x_1}^{N,+} G\left(\tfrac{N-1}{N}, \tfrac{\tilde{x}}{N}\right) - \partial_{x_1} G_r(1,\tilde{x})\right\}\, dr\Big|,$$

$$\sup_{0\le t\le T} \Big|\frac{1}{2}\int_0^t \partial_{x_1} G_r(1,\tilde{x}) \left[I_k(\eta_{(N-1,\tilde{x})}(r)) - d(1,\tilde{x})\right]\, dr\Big|,$$

$$\sup_{0\le t\le T}\left|\int_0^t \frac{1}{2N^{d-1}}\sum_{\substack{x\in D_N^d\\ x_1=1}} I_k(\eta_{(1,\tilde{x})}(r))\left\{\partial_{x_1}^{N,-}G\left(\tfrac{1}{N},\tfrac{\tilde{x}}{N}\right)-\partial_{x_1}G_r(0,\tilde{x})\right\}\,dr\right|,$$

$$\sup_{0\le t\le T}\left|\frac{1}{2}\int_0^t \partial_{x_1}G_r(0,\tilde{x})\left[I_k(\eta_{(1,\tilde{x})}(r))-d(0,\tilde{x})\right]\,dr\right|,$$

$$\sup_{0\le t\le T}\left|\int_0^t \frac{N^{1-\theta}}{N^d}\sum_{\substack{x\in D_N^d\\ x_1=1}} \partial_{x_1}^{N,-}G\left(\tfrac{1}{N},\tfrac{\tilde{x}}{N}\right)\left[\sum_{v\in\mathcal{V}}\alpha_v(\tfrac{\tilde{x}}{N})-I_k(\eta_{(1,\tilde{x})})\right]\,dr\right|,$$

$$\sup_{0\le t\le T}\left|\int_0^t \frac{N^{1-\theta}}{N^d}\sum_{\substack{x\in D_N^d\\ x_1=N-1}} \partial_{x_1}^{N,+}G\left(\tfrac{N-1}{N},\tfrac{\tilde{x}}{N}\right)\left[\sum_{v\in\mathcal{V}}\beta_v(\tfrac{\tilde{x}}{N})-I_k(\eta_{(N-1,\tilde{x})})\right]\,dr\right|$$

and

$$\sup_{0\le t\le T}\left|\int_0^t\left[\frac{1}{N^d}\sum_{j=1}^d\sum_{x\in D_N^d}(\partial_{x_j}^N G)(\tfrac{x}{N})\tau_x W_{j,k}-\int_{D^d}\sum_{v\in\mathcal{V}}\tilde{v}\cdot\chi(\theta_v(\Lambda(\rho,\varrho)))\sum_{i=1}^d v_i\frac{\partial G}{\partial x_i}\,dx\right]dr\right|$$

Since $G\in C_0^{1,2}([0,T]\times D^d)$ and using the Replacement Lemmas it is easy to see that terms above converges to zero, as $N\to\infty$. This concludes the proof. □

### 1.11.2 Characterization of limit points for $\theta=1$

Now we look at the limit points of the sequence $\{\mathbb{Q}_N\}_{N\ge1}$.

**Proposition 5.** *If $\mathbb{Q}^*$ is a limit point of $\{\mathbb{Q}_N\}_{N\geq 1}$, then*

$$\mathbb{Q}^* \Bigg[\pi. : \int_{D^d} (\rho,\varrho)(t,x)G(t,x)\,dx - \int_{D^d} (\rho,\varrho)(0,x)G(0,x)\,dx$$

$$- \int_0^t \int_{D^d} \sum_{v\in\mathcal{V}} \tilde{v}\cdot\chi(\theta_v(\Lambda(\rho,\varrho))) \sum_{i=1}^d v_i \frac{\partial G}{\partial x_i}(r,x)\,dx\,dr$$

$$+\frac{1}{2}\int_0^t \int_{\{1\}\times\mathbb{T}^{d-1}} (\rho,\varrho)(r,1,\tilde{x})\frac{\partial G}{\partial x_1}(r,1,\tilde{x})\,dS\,dr$$

$$-\frac{1}{2}\int_0^t \int_{\{0\}\times\mathbb{T}^{d-1}} (\rho,\varrho)(r,0,\tilde{x})\frac{\partial G}{\partial x_1}(r,0,\tilde{x})\,dS\,dr$$

$$-\frac{1}{2}\int_0^t \int_{\{1\}\times\mathbb{T}^{d-1}} \left[\sum_{v\in\mathcal{V}} v_k\beta_v(\tilde{x}) - (\rho,\varrho)(r,1,\tilde{x})\right] G(r,1,\tilde{x})\,dS\,dr$$

$$-\frac{1}{2}\int_0^t \int_{\{0\}\times\mathbb{T}^{d-1}} \left[\sum_{v\in\mathcal{V}} v_k\alpha_v(\tilde{x}) - (\rho,\varrho)(r,0,\tilde{x})\right] G(r,0,\tilde{x})\,dS\,dr$$

$$- \int_0^t \int_{D^d} (\rho,\varrho)(r,x)\left(\partial_r G(r,x) + \frac{1}{2}\Delta G\right)\,dx\,dr = 0\Bigg] = 1,$$

*for all $t \in [0,T]$, $\forall G \in C^{1,2}([0,T]\times D^d)$.*

*Proof.* It is enough to verify that, for $\delta > 0$ and $G \in C^{1,2}([0,T]\times D^d)$ fixed,

$$\mathbb{Q}^* \Bigg[\pi. : \sup_{0\leq t\leq T} \Bigg|\int_{D^d} (\rho,\varrho)(t,x)G(t,x)\,dx - \int_{D^d} (\rho,\varrho)(0,x)G(0,x)\,dx$$

$$- \int_0^t \int_{D^d} \sum_{v\in\mathcal{V}} \tilde{v}\cdot\chi(\theta_v(\Lambda(\rho,\varrho))) \sum_{i=1}^d v_i \frac{\partial G}{\partial x_i}(r,x)\,dx\,dr$$

$$+\frac{1}{2}\int_0^t \int_{\{1\}\times\mathbb{T}^{d-1}} (\rho,\varrho)(r,1,\tilde{x})\frac{\partial G}{\partial x_1}(r,1,\tilde{x})\,dS\,dr$$

$$-\frac{1}{2}\int_0^t \int_{\{0\}\times\mathbb{T}^{d-1}} (\rho,\varrho)(r,0,\tilde{x})\frac{\partial G}{\partial x_1}(r,0,\tilde{x})\,dS\,dr$$

$$-\frac{1}{2}\int_0^t \int_{\{1\}\times\mathbb{T}^{d-1}} \left[\sum_{v\in\mathcal{V}} v_k\beta_v(\tilde{x}) - (\rho,\varrho)(r,1,\tilde{x})\right] G(r,1,\tilde{x})\,dS\,dr$$

$$-\frac{1}{2}\int_0^t \int_{\{0\}\times\mathbb{T}^{d-1}} \left[\sum_{v\in\mathcal{V}} v_k\alpha_v(\tilde{x}) - (\rho,\varrho)(r,0,\tilde{x})\right] G(r,0,\tilde{x})\,dS\,dr$$

$$- \int_0^t \int_{D^d} (\rho,\varrho)(r,x)\left(\partial_r G(r,x) + \frac{1}{2}\Delta G\right)\,dx\,dr\Bigg| > \delta\Bigg] = 0.$$

Rewrite the expression above as

$$
\begin{aligned}
&\mathbb{Q}^*\Bigg[\pi_\cdot : \sup_{0\le t\le T}\Bigg|\int_{D^d}(\rho,\varrho)(t,x)G(t,x)\,dx-\int_{D^d}(\rho,\varrho)(0,x)G(0,x)\,dx\\
&-\int_0^t\int_{D^d}\sum_{v\in\mathcal{V}}\tilde{v}\cdot\chi(\theta_v(\Lambda(\rho,\varrho)))\sum_{i=1}^d v_i\frac{\partial G}{\partial x_i}(r,x)\,dx\,dr\\
&+\frac{1}{2}\int_0^t\int_{\{1\}\times\mathbb{T}^{d-1}}(\rho,\varrho)(r,1,\tilde{x})\left[\frac{\partial G}{\partial x_1}(r,1,\tilde{x})+G(r,1,\tilde{x})\right]\,dS\,dr\\
&-\frac{1}{2}\int_0^t\int_{\{0\}\times\mathbb{T}^{d-1}}(\rho,\varrho)(r,0,\tilde{x})\left[\frac{\partial G}{\partial x_1}(r,0,\tilde{x})-G(r,0,\tilde{x})\right]\,dS\,dr\\
&-\frac{1}{2}\int_0^t\int_{\{1\}\times\mathbb{T}^{d-1}}G(r,1,\tilde{x})\sum_{v\in\mathcal{V}}v_k\beta_v(\tilde{x})\,dS\,dr\\
&-\frac{1}{2}\int_0^t\int_{\{0\}\times\mathbb{T}^{d-1}}G(r,0,\tilde{x})\sum_{v\in\mathcal{V}}v_k\alpha_v(\tilde{x})\,dS\,dr\\
&-\int_0^t\int_{D^d}(\rho,\varrho)(r,x)\left(\partial_r G(r,x)+\frac{1}{2}\Delta G\right)\,dx\,dr\Bigg|>\delta\Bigg]=0.
\end{aligned}
\tag{1.46}
$$

We would like to work with the probabilities $\mathbb{Q}_N$, as we did in the previous case, using Portmanteau's Theorem. Unfortunately, the set inside the above probability is not an open set in the Skorohod space. In order to avoid this problem, we fix $\varepsilon>0$ and we consider two approximations of the identity, for fixed $u_1\in[0,1]$ which are given on $w\in[0,1]$ by

$$\overleftarrow{\imath}_\varepsilon^{u_1}(w)=\frac{1}{\varepsilon}\mathbb{1}_{(u_1-\varepsilon,u_1]}(w)\ \text{ and }\ \overrightarrow{\imath}_\varepsilon^{u_1}(w)=\frac{1}{\varepsilon}\mathbb{1}_{[u_1,u_1+\varepsilon)}(w).$$

We use the notation

$$\langle\pi_r,\overleftarrow{\imath}_\varepsilon^{u_1}\rangle=\langle(\rho,\varrho)_r,\overleftarrow{\imath}_\varepsilon^{u_1}\rangle=\frac{1}{\varepsilon}\int_{u_1-\varepsilon}^{u_1}(\rho,\varrho)_r(w,\tilde{u})\,dw$$

and

$$\langle\pi_r,\overrightarrow{\imath}_\varepsilon^{u_1}\rangle=\langle(\rho,\varrho)_r,\overrightarrow{\imath}_\varepsilon^{u_1}\rangle=\frac{1}{\varepsilon}\int_{u_1}^{u_1+\varepsilon}(\rho,\varrho)_r(w,\tilde{u})\,dw.$$

By summing and subtracting proper terms, we bound the probability in (1.46) from

above by

$$
\begin{aligned}
&\mathbb{Q}^*\Bigg[\pi_\cdot : \sup_{0\le t\le T}\Bigg|\int_{D^d}(\rho,\varrho)(t,x)G(t,x)\,dx-\int_{D^d}(\rho,\varrho)(0,x)G(0,x)\,dx\\
&-\int_0^t\int_{D^d}\sum_{v\in\mathcal{V}}\tilde{v}\cdot\chi(\theta_v(\Lambda(\rho,\varrho)))\sum_{i=1}^{d}v_i\frac{\partial G}{\partial x_i}(r,x)\,dx\,dr\\
&+\frac{1}{2}\int_0^t\int_{\{1\}\times\mathbb{T}^{d-1}}\big[(\rho,\varrho)(r,1,\tilde{x})-\langle\pi_r,\overleftarrow{\imath}_{\varepsilon}^{1}\rangle\big]\left[\frac{\partial G}{\partial x_1}(r,1,\tilde{x})+G(r,1,\tilde{x})\right]dS\,dr\\
&+\frac{1}{2}\int_0^t\int_{\{1\}\times\mathbb{T}^{d-1}}\langle\pi_r,\overleftarrow{\imath}_{\varepsilon}^{1}\rangle\left[\frac{\partial G}{\partial x_1}(r,1,\tilde{x})+G(r,1,\tilde{x})\right]dS\,dr\\
&-\frac{1}{2}\int_0^t\int_{\{0\}\times\mathbb{T}^{d-1}}\big[(\rho,\varrho)(r,0,\tilde{x})-\langle\pi_r,\overrightarrow{\imath}_{\varepsilon}^{0}\rangle\big]\left[\frac{\partial G}{\partial x_1}(r,0,\tilde{x})-G(r,0,\tilde{x})\right]dS\,dr \qquad (1.47)\\
&-\frac{1}{2}\int_0^t\int_{\{0\}\times\mathbb{T}^{d-1}}\langle\pi_r,\overrightarrow{\imath}_{\varepsilon}^{0}\rangle\left[\frac{\partial G}{\partial x_1}(r,0,\tilde{x})-G(r,0,\tilde{x})\right]dS\,dr\\
&-\frac{1}{2}\int_0^t\int_{\{1\}\times\mathbb{T}^{d-1}}G(r,1,\tilde{x})\sum_{v\in\mathcal{V}}v_k\beta_v(\tilde{x})\,dS\,dr\\
&-\frac{1}{2}\int_0^t\int_{\{0\}\times\mathbb{T}^{d-1}}G(r,0,\tilde{x})\sum_{v\in\mathcal{V}}v_k\alpha_v(\tilde{x})\,dS\,dr\\
&-\int_0^t\int_{D^d}(\rho,\varrho)(r,x)\left(\partial_r G(r,x)+\frac{1}{2}\Delta G\right)dx\,dr\Bigg|>\tfrac{\delta}{5}\Bigg]=0.
\end{aligned}
$$

By Lebesgue's Differentiation Theorem, observe that, for almost $u_1\in[0,1]$,

$$\lim_{\varepsilon\to0}|(\rho,\varrho)(r,u_1,\tilde{x})-\langle\pi_r,\overleftarrow{\imath}_{\varepsilon}^{u_1}\rangle|=0$$

and

$$\lim_{\varepsilon\to0}|(\rho,\varrho)(r,u_1,\tilde{x})-\langle\pi_r,\overrightarrow{\imath}_{\varepsilon}^{u_1}\rangle|=0.$$

Since the functions $\overleftarrow{\imath}_{\varepsilon}^{u_1}$ and $\overleftarrow{\imath}_{\varepsilon}^{u_1}$ are not continuous, we cannot use Portmanteau's Theorem. However, we can approximate each one of these functions by continuous functions, in such a way that the error vanishes as $\varepsilon\to0$. Then, since the set inside the probability in (1.47) is an open set with respect to the Skorohod topology, we can use Portmanteau's Theorem and bound (1.47) from above by

$$
\begin{aligned}
&\liminf_{N\to\infty} \mathbb{Q}_N \Bigg[\pi. : \sup_{0\le t\le T} \Big|\langle \pi_t^k, G_t\rangle - \langle \pi_0^k, G_0\rangle \\
&- \int_0^t \int_{D^d} \sum_{v\in\mathcal{V}} \tilde{v}\cdot \chi(\theta_v(\Lambda(\rho,\varrho))) \sum_{i=1}^d v_i \frac{\partial G}{\partial x_i}(r,x)\, dx\, dr \\
&+\frac{1}{2}\int_0^t \int_{\{1\}\times\mathbb{T}^{d-1}} \overleftarrow{\eta}_r^{\varepsilon N}(N-1,\tilde{x},v)\left[\frac{\partial G}{\partial x_1}(r,1,\tilde{x}) + G(r,1,\tilde{x})\right] dS\, dr \\
&-\frac{1}{2}\int_0^t \int_{\{0\}\times\mathbb{T}^{d-1}} \overrightarrow{\eta}_r^{\varepsilon N}(1,\tilde{x},v)\left[\frac{\partial G}{\partial x_1}(r,0,\tilde{x}) - G(r,0,\tilde{x})\right] dS\, dr \\
&-\frac{1}{2}\int_0^t \int_{\{0\}\times\mathbb{T}^{d-1}} G(r,1,\tilde{x}) \sum_{v\in\mathcal{V}} v_k \beta_v(\tilde{x})\, dS\, dr \\
&-\frac{1}{2}\int_0^t \int_{\{0\}\times\mathbb{T}^{d-1}} G(r,0,\tilde{x}) \sum_{v\in\mathcal{V}} v_k \alpha_v(\tilde{x})\, dS\, dr \\
&- \int_0^t \int_{D^d} (\rho,\varrho)(r,x) \left(\partial_r G(r,x) + \frac{1}{2}\Delta G\right) dx\, dr\Big| > \tfrac{\delta}{8}\Bigg] = 0.
\end{aligned} \tag{1.48}
$$

Summing and subtracting $\int_0^t N^2 \mathcal{L}_N \langle \pi_s^{k,N}, G_s\rangle\, ds$ to the term inside the supremum in (1.48) from above by the sum of

$$
\mathbb{P}_{\mu^N}\left[\sup_{0\le t\le T} |M_t^{N,k}(G)| > \frac{\delta}{16}\right] \tag{1.49}
$$

and

$$
\begin{aligned}
&\mathbb{P}_{\mu^N}\Bigg[\sup_{0\le t\le T}\Bigg|\int_0^t N^2\mathcal{L}_N\langle \pi_s^{k,N}, G_s\rangle\, ds \\
&-\int_0^t\int_{D^d}\sum_{v\in\mathcal{V}}\tilde{v}\cdot\chi(\theta_v(\Lambda(\rho,\varrho)))\sum_{i=1}^d v_i\frac{\partial G}{\partial x_i}(r,x)\,dx\,dr \\
&+\frac{1}{2}\int_0^t\int_{\{1\}\times\mathbb{T}^{d-1}}\overleftarrow{\eta}_r^{\varepsilon N}(N-1,\tilde{x},v)\left[\frac{\partial G}{\partial x_1}(r,1,\tilde{x})+G(r,1,\tilde{x})\right]dS\,dr \\
&-\frac{1}{2}\int_0^t\int_{\{0\}\times\mathbb{T}^{d-1}}\overrightarrow{\eta}_r^{\varepsilon N}(1,\tilde{x},v)\left[\frac{\partial G}{\partial x_1}(r,0,\tilde{x})-G(r,0,\tilde{x})\right]dS\,dr \\
&-\frac{1}{2}\int_0^t\int_{\{0\}\times\mathbb{T}^{d-1}}G(r,1,\tilde{x})\sum_{v\in\mathcal{V}}v_k\beta_v(\tilde{x})\,dS\,dr \\
&-\frac{1}{2}\int_0^t\int_{\{0\}\times\mathbb{T}^{d-1}}G(r,0,\tilde{x})\sum_{v\in\mathcal{V}}v_k\alpha_v(\tilde{x})\,dS\,dr \\
&-\int_0^t\int_{D^d}(\rho,\varrho)(r,x)\left(\partial_r G(r,x)+\frac{1}{2}\Delta G\right)dx\,dr\Bigg|>\tfrac{\delta}{16}\Bigg]=0.
\end{aligned}
\tag{1.50}
$$

From Doob's inequality the term (1.49) vanishes as $N\to\infty$. We can bound (1.50) fom above by a sum of terms and doing the same argument from previous section, since $G\in C^{1,2}([0,T]\times D^d)$ and using the Replacement Lemmas it is easy to see that terms above converges to zero, as $N\to\infty$ and $\varepsilon\to 0$. This concludes the proof. □

### 1.11.3 Characterization of limit points for $\theta>1$

As in previous sections, we will look at the limit points of the sequence $\{\mathbb{Q}_N\}_{N\ge 1}$.

**Proposition 6.** *If $\mathbb{Q}^*$ is a limit point of $\{\mathbb{Q}_N\}_{N\geq 1}$, then it is true that*

$$\mathbb{Q}^*\Bigg[\pi_\cdot : \int_{D^d}(\rho,\varrho)(t,x)G(t,x)\,dx - \int_{D^d}(\rho,\varrho)(0,x)G(0,x)\,dx$$

$$-\int_0^t\int_{D^d}\sum_{v\in\mathcal{V}}\tilde{v}\cdot\chi(\theta_v(\Lambda(\rho,\varrho)))\sum_{i=1}^d v_i\frac{\partial G}{\partial x_i}(r,x)\,dx\,dr$$

$$+\frac{1}{2}\int_0^t\int_{\{1\}\times\mathbb{T}^{d-1}}(\rho,\varrho)(r,1,\tilde{x})\frac{\partial G}{\partial x_1}(r,1,\tilde{x})\,dS\,dr$$

$$-\frac{1}{2}\int_0^t\int_{\{0\}\times\mathbb{T}^{d-1}}(\rho,\varrho)(r,0,\tilde{x})\frac{\partial G}{\partial x_1}(r,0,\tilde{x})\,dS\,dr$$

$$-\int_0^t\int_{D^d}(\rho,\varrho)(r,x)\left(\partial_r G(r,x)+\frac{1}{2}\Delta G\right)\,dx\,dr = 0\Bigg] = 1,$$

*for all $t\in[0,T]$, $\forall G\in C^{1,2}([0,T]\times D^d)$.*

*Proof.* Following the same reasoning as in Proposition 4 and 5, it is enough to verify that, for $\delta>0$ and $G\in C^{1,2}([0,T]\times D^d)$ fixed, we have

$$\mathbb{Q}^*\Bigg[\pi_\cdot : \sup_{0\leq t\leq T}\Bigg|\int_{D^d}(\rho,\varrho)(t,x)G(t,x)\,dx - \int_{D^d}(\rho,\varrho)(0,x)G(0,x)\,dx$$

$$-\int_0^t\int_{D^d}\sum_{v\in\mathcal{V}}\tilde{v}\cdot\chi(\theta_v(\Lambda(\rho,\varrho)))\sum_{i=1}^d v_i\frac{\partial G}{\partial x_i}(r,x)\,dx\,dr$$

$$+\frac{1}{2}\int_0^t\int_{\{1\}\times\mathbb{T}^{d-1}}(\rho,\varrho)(r,1,\tilde{x})\frac{\partial G}{\partial x_1}(r,1,\tilde{x})\,dS\,dr$$

$$-\frac{1}{2}\int_0^t\int_{\{0\}\times\mathbb{T}^{d-1}}(\rho,\varrho)(r,0,\tilde{x})\frac{\partial G}{\partial x_1}(r,0,\tilde{x})\,dS\,dr$$

$$-\int_0^t\int_{D^d}(\rho,\varrho)(r,x)\left(\partial_r G(r,x)+\frac{1}{2}\Delta G\right)\,dx\,dr\Bigg| > \delta\Bigg] = 0.$$

We need to change the boundary terms $(\rho,\varrho)_r(0,\tilde{x})$ (resp. $(\rho,\varrho)_r(1,\tilde{x})$) by $\overrightarrow{\eta}_r^{\varepsilon N}(1,\tilde{x},v)$ (resp. $\overleftarrow{\eta}_r^{\varepsilon N}(N-1,\tilde{x},v)$). Then, we sum and subtract $\int_0^t N^2\mathcal{L}_N\langle\pi_r^{k,N},G\rangle\,dr$, it will be enough to analyze

$$
\begin{aligned}
&\limsup_{N\to\infty} \mathbb{P}_{\mu^N}\Bigg[\eta_\cdot : \sup_{0\le t\le T}\Bigg|\int_0^t N^2\mathcal{L}_N\langle \pi_r^{k,N}, G\rangle\,dr - \frac{1}{2}\int_0^t \langle \pi_r^{k,N}, \Delta G\rangle\,dr \\
&-\int_0^t\int_{D^d}\sum_{v\in\mathcal{V}}\tilde{v}\cdot\chi(\theta_v(\Lambda(\rho,\varrho)))\sum_{i=1}^d v_i\frac{\partial G}{\partial x_i}(r,x)\,dx\,dr \\
&+\frac{1}{2}\int_0^t\int_{\{1\}\times\mathbb{T}^{d-1}} \overleftarrow{\eta}_r^{\varepsilon N}(N-1,\tilde{x},v)\frac{\partial G}{\partial x_1}(r,1,\tilde{x})\,dS\,dr \\
&-\frac{1}{2}\int_0^t\int_{\{0\}\times\mathbb{T}^{d-1}} \overrightarrow{\eta}_r^{\varepsilon N}(1,\tilde{x},v)\frac{\partial G}{\partial x_1}(r,0,\tilde{x})\,dS\,dr\Bigg| > \delta\Bigg].
\end{aligned}
\tag{1.51}
$$

Doing the same as in the other cases and using that $G \in C^{1,2}([0,T]\times D^d)$ and $\theta > 1$, we just have to analyze the following, for all $\tilde{\delta}$ and $x_1 = \{1, N-1\}$

$$
\limsup_{N\to\infty} \mathbb{P}_{\mu^N}\left[\eta_\cdot : \sup_{0\le t\le T}\left|\int_0^t [\eta_s^{\varepsilon N}(x_1,\tilde{x}) - \eta_s(x_1,\tilde{x})]\frac{\partial G}{\partial x_1}\,dr\right| > \tilde{\delta}\right].
$$

Applying Replacement Lemma (4), we conclude that, taking limit when $\varepsilon \to 0$ the limit above goes to 0. This concludes the proof of this proposition. □

## 1.12 Uniqueness of weak solutions

To conclude the proof of the hydrodynamic limit, it remains to prove the uniqueness of weak solutions to (1.8), (1.10) and (1.12).

### 1.12.1 Uniqueness of weak solutions of (1.8)

Consider $(\rho^1,\varrho^1)$, $(\rho^2,\varrho^2)$ two weak solutions of (1.8) with the same initial condition and denote their difference by $(\overline{\rho},\overline{\varrho}) = (\rho^1-\rho^2, \varrho^1-\varrho^2)$. Let us define the set $\{\psi_z\}_z$ given by $\psi_z(u) = \sqrt{2}\sin(z\pi u)$ for $z \ge 1$ and $\psi_0(u) = 1$ which is an orthonormal basis of $L^2([0,1])$. Note that $(\overline{\rho},\overline{\varrho}) = (\overline{p}^0,\overline{p}^1,\dots,\overline{p}^d) = 0$ if, and only if, each component is equal to zero, which means that $\overline{p}^k = 0$ for $k = 0,\dots,d$. Let

$$
V_k(t) = \sum_{z\ge 0}\frac{1}{2a_z}\langle \overline{p}_t^k, \psi_z\rangle^2
$$

where $a_z = (z\pi)^2 + 1$. We claim that $V_k'(t) \le CV_k(t)$, where $C$ is a positive constant. Since $V_k(0) = 0$, $\forall\, k = 0,\dots,d$, from Gronwall's inequality we will conclude that

$V_k(t) \leq 0$, but since we know by definition that $V_k(t) \geq 0$, we are done. Now we need to show that the claim is true. Note that

$$V_k'(t) = \sum_{z\geq 0} \frac{1}{a_z} \langle \overline{p}_t^k, \psi_z \rangle \frac{d}{dt} \langle \overline{p}_t^k, \psi_z \rangle,$$

and from the integral formulation (1.8) we have that

$$\begin{aligned} \frac{d}{dt} \langle \overline{p}_t^k, \psi_z \rangle =& \langle \frac{d}{dt} \overline{p}_t^k, \psi_z \rangle + \langle \overline{p}_t^k, \frac{d}{dt} \psi_z \rangle \\ =& \frac{1}{2} \langle \overline{p}_t^k, \Delta \psi_z \rangle + \langle \chi(\theta_v(\Lambda(\rho_t^1, \varrho_t^1))) - \chi(\theta_v(\Lambda(\rho_t^2, \varrho_t^2))), \partial_u \psi_z \rangle. \end{aligned}$$

Since $\psi_z(u) = \sqrt{2} \sin(z\pi u)$ we have that $\partial_u \psi_z(u) = \sqrt{2} z\pi \cos(z\pi u)$ and $\Delta \psi_z(u) = -(z\pi)^2 \sqrt{2} \sin(z\pi u) = -(z\pi)^2 \psi_z$, then

$$V_k'(t) = \sum_{z\geq 0} \frac{-(z\pi)^2}{2a_z} \langle \overline{p}_t^k, \psi_z \rangle^2 + \sum_{z\geq 0} \frac{1}{a_z} \langle \overline{p}_t^k, \psi_z \rangle \langle \chi(\theta_v(\Lambda(\rho_t^1, \varrho_t^1))) - \chi(\theta_v(\Lambda(\rho_t^2, \varrho_t^2))), \partial_u \psi_z \rangle.$$

Using Young's inequality on the second term on the right-hand side of last identity, we bound that term from above by

$$\frac{1}{2A} \sum_{z\geq 0} \frac{1}{a_z} \langle \overline{p}_t^k, \psi_z \rangle^2 + \frac{A}{2} \sum_{z\geq 0} \frac{1}{a_z} \langle \chi(\theta_v(\Lambda(\rho_t^1, \varrho_t^1))) - \chi(\theta_v(\Lambda(\rho_t^2, \varrho_t^2))), \partial_u \psi_z \rangle^2, \forall A > 0.$$

Observe that $\partial_u \psi_z = z\pi \phi_z(u)$, with $\phi_z(u) = \sqrt{2} \cos(z\pi u)$ for $z \geq 1$ and $\phi_0(u) = 1$. Therefore, the second term at right-hand side in last display can be rewritten as

$$\begin{aligned} &\frac{A}{2} \sum_{z\geq 0} \frac{(z\pi)^2}{a_z} \langle \chi(\theta_v(\Lambda(\rho_t^1, \varrho_t^1))) - \chi(\theta_v(\Lambda(\rho_t^2, \varrho_t^2))), \phi_z \rangle^2 \\ \leq& \frac{A}{2} \sum_{z\geq 0} \langle \chi(\theta_v(\Lambda(\rho_t^1, \varrho_t^1))) - \chi(\theta_v(\Lambda(\rho_t^2, \varrho_t^2))), \phi_z \rangle^2 \end{aligned}$$

because of the choice for $a_z$. Observe that, since $\{\phi_z\}_z$ is an orthonormal basis of $L^2[0,1]$, we can rewritten the last display as

$$\frac{A}{2} \int_0^1 \left( \chi(\theta_v(\Lambda(\rho_t^1, \varrho_t^1))) - \chi(\theta_v(\Lambda(\rho_t^2, \varrho_t^2))) \right)^2 du.$$

Since $\chi(\theta_v(\Lambda(\rho_t, p_t)))$ is Lipschitz (see [16]), last display is bounded from above by $\frac{A}{2}\|\overline{\rho}_t\|_2^2$. Putting all this together, we conclude that

$$V_k'(t) \leq \sum_{z\geq 0} \left( \frac{-(z\pi)^2}{2a_z} + \frac{1}{2Aa_z} + \frac{A}{2} \right) \langle \overline{\rho}_t, \psi_z \rangle^2.$$

Taking $A = 1$, we get

$$V_k'(t) \leq \sum_{z\geq 0} \left( \frac{1}{2a_z} + \frac{1}{2} \right) \langle \overline{\rho}_t, \psi_z \rangle^2 = \frac{1+a_z}{2a_z} \langle \overline{\rho}_t, \psi_z \rangle^2 = C\, V_k(t).$$

And this concludes the proof of uniqueness of weak solutions for the problem (1.8).

### 1.12.2 Uniqueness of weak solutions of (1.12)

The proof above can be adapted to this case, as we describe now. Consider $(\rho^1, p^1)$ and $(\rho^2, p^2)$ two weak solutions of (1.12) with the same initial condition and denote by $(\overline{\rho}, \overline{p})$ their difference $(\overline{\rho}, \overline{p}) = (\rho^1 - \rho^2, p^1 - p^2)$. Now consider the set $\{\phi_z\}_z$ given by $\phi_z(u) = \sqrt{2}\cos(z\pi u)$ for $z \geq 1$ and $\phi_0(u) = 1$, which is an orthonormal basis of $L^2([0,1])$. Note that $(\overline{\rho}, \overline{p}) = (\overline{p}^0, \overline{p}^1, \ldots, \overline{p}^d) = 0$ if and only if each component is equal to zero, which means $\overline{p}^k = 0$ for $k = 0, \ldots, d$. Let

$$V_k(t) = \sum_{z\geq 0} \frac{1}{2a_z} \langle \overline{p}_t^k, \phi_z \rangle^2$$

where $a_z = (z\pi)^2 + 1$. We claim that $V_k'(t) \leq CV_k(t)$, where $C$ is a positive constant and since $V_k(0) = 0 \,\forall\, k = 0, \ldots, d$, from Gronwall's inequality we conclude that $V_k(t) \leq 0$, but we know by definition that $V_k(t) \geq 0$, and we are done. Now we need to show that the claim is true. Note that

$$V_k'(t) = \sum_{z\geq 0} \frac{1}{a_z} \langle \overline{p}_t^k, \phi_z \rangle \frac{d}{dt} \langle \overline{p}_t^k, \phi_z \rangle,$$

and from the integral formulation (1.12) we have that

$$
\begin{aligned}
\frac{d}{dt}\langle \overline{p}_t^k, \phi_z\rangle =& \langle \frac{d}{dt}\overline{p}_t^k, \phi_z\rangle + \langle \overline{p}_t^k, \frac{d}{dt}\phi_z\rangle \\
=& \frac{1}{2}\langle \overline{p}_t^k, \Delta\phi_z\rangle + \langle \chi(\theta_v(\Lambda(\rho_t^1, p_t^1))) - \chi(\theta_v(\Lambda(\rho_t^2, p_t^2))), \partial_u\phi_z\rangle \\
& - \frac{1}{2}\overline{p}_t^k(1)\partial_u\phi_z(1) + \frac{1}{2}\overline{p}_t^k(0)\partial_u\phi_z(0).
\end{aligned}
$$

Since $\partial_u\phi_z(0) = \partial_u\phi_z(1) = 0$, then

$$
\begin{aligned}
V_k'(t) \; &= \sum_{z\geq 0} -\frac{1}{2a_z}\langle \overline{p}_t^k, \phi_z\rangle\langle \overline{p}_t^k, \Delta\phi_z\rangle \\
&+ \sum_{z\geq 0} \frac{1}{a_z}\langle \overline{p}_t^k, \phi_z\rangle\langle \chi(\theta_v(\Lambda(\rho_t^1, p_t^1))) - \chi(\theta_v(\Lambda(\rho_t^2, p_t^2))), \partial_u\phi_z\rangle \\
&= \sum_{z\geq 0} -\frac{(z\pi)^2}{2a_z}\langle \overline{p}_t^k, \phi_z\rangle^2 + \sum_{z\geq 0} \frac{1}{a_z}\langle \overline{p}_t^k, \phi_z\rangle\langle \chi(\theta_v(\Lambda(\rho_t^1, p_t^1))) - \chi(\theta_v(\Lambda(\rho_t^2, p_t^2))), \partial_u\phi_z\rangle.
\end{aligned} \tag{1.52}
$$

Using Young's inequality in the second term of right-hand side of equation (1.52), this term is bounded from above by

$$
\frac{1}{2A}\sum_{z\geq 0}\frac{1}{a_z}\langle \overline{p}_t^k, \phi_z\rangle^2 + \frac{A}{2}\sum_{z\geq 0}\frac{1}{a_z}\langle \chi(\theta_v(\Lambda(\rho_t^1, \varrho_t^1))) - \chi(\theta_v(\Lambda(\rho_t^2, \varrho_t^2))), \partial_u\phi_z\rangle^2, \forall A > 0.
$$

Observe that $\partial_u\phi_z = -z\pi\psi_z(u)$, with $\psi_z(u) = \sqrt{2}\sin(z\pi u)$ for $z \geq 1$ and $\psi_0(u) = 1$. Therefore, the last term at right-hand side of last display can be rewritten as

$$
\begin{aligned}
&\frac{A}{2}\sum_{z\geq 0}\frac{(z\pi)^2}{a_z}\langle \chi(\theta_v(\Lambda(\rho_t^1, \varrho_t^1))) - \chi(\theta_v(\Lambda(\rho_t^2, \varrho_t^2))), \phi_z\rangle^2 \\
\leq\; &\frac{A}{2}\sum_{z\geq 0}\langle \chi(\theta_v(\Lambda(\rho_t^1, \varrho_t^1))) - \chi(\theta_v(\Lambda(\rho_t^2, \varrho_t^2))), \phi_z\rangle^2
\end{aligned}
$$

because of the choice for $a_z$. Observe that, since $\{\phi_z\}_z$ is an orthonormal basis of $L^2[0,1]$, we can rewritten the last display as

$$
\frac{A}{2}\int_0^1 \big(\chi(\theta_v(\Lambda(\rho_t^1, \varrho_t^1))) - \chi(\theta_v(\Lambda(\rho_t^2, \varrho_t^2)))\big)^2\, du.
$$

Since $\chi(\theta_v(\Lambda(\rho_t, p_t)))$ is Lipschitz (see [16]), we have the last display is bounded from above by $\frac{A}{2}\|\overline{\rho}_t\|_2^2$. Putting all this together we conclude that

$$V_k'(t) \leq \sum_{z\geq 0} \left( \frac{-(z\pi)^2}{2a_z} + \frac{1}{2Aa_z} + \frac{A}{2} \right) \langle \overline{p}_t^k, \psi_z \rangle^2.$$

Taking $A = 1$, then we get

$$V_k'(t) \leq \sum_{z\geq 0} \left( \frac{1}{2a_z} + \frac{1}{2} \right) \langle \overline{p}_t^k, \psi_z \rangle^2 = \frac{1 + a_z}{2a_z} \langle \overline{p}_t^k, \psi_z \rangle^2 = C\, V_k(t).$$

And this concludes the proof of uniqueness of weak solutions for the problem (1.12).

### 1.12.3 Uniqueness of weak solutions of (1.10)

We tried to adapt the same method used in the previous sections 1.12.1 and 1.12.2 for this case. For that, we use the linear combination of sin and cosine, which is an orthonormal basis of $L^2([0, 1])$. And follows the same as in the subsections 1.12.1 and 1.12.2. But the problem is that, when we derive this basis the result is no longer a basis.

Fortunately, we have an answer about that uniqueness for the case 1-dimensional. The proof is ipsis litteris as in [17].

# Equilibrium Fluctuations for a boundary driven stochastic lattice gas model with many conserved quantities

This work aims to study the equilibrium fluctuations of a weakly asymmetric exclusion process with collision among particles having different velocities with periodic boundary conditions. The reader can skip Sections 2.1, 2.2 and 2.3 since is similar to Sections 1.1, 1.2 and 1.3 from Chapter 1. Just make sure to keep in mind that now we have periodic boundary conditions.

## 2.1 Notation and Results

We start by fixing the notation to be used throughout this chapter. Let $\mathbb{T}_N^d = \{0, \ldots, N-1\}^d = (\mathbb{Z}/N\mathbb{Z})^d$ be the $d$-dimensional discrete torus. Moreover, let $\mathcal{V} \subset \mathbb{R}^d$ be a finite set of velocities $v = (v_1, \ldots, v_d)$. Assume that $\mathcal{V}$ is invariant under reflections and permutations of the coordinates, i.e.,

$$(v_1, \ldots, v_{i-1}, -v_i, v_{i+1}, \ldots, v_d) \text{ and } (v_{\sigma(1)}, \ldots, v_{\sigma(d)})$$

belong to $\mathcal{V}$ for all $1 \leq i \leq d$, and all permutations $\sigma$ of $\{1, \ldots, d\}$, provided $(v_1, \ldots, v_d)$ belongs to $\mathcal{V}$.

At each site of $\mathbb{T}_N^d$, at most one particle with a certain velocity is allowed. We also denote: the number of particles with velocity $v \in \mathcal{V}$ at $x \in \mathbb{T}_N^d$, by $\eta(x, v) \in \{0, 1\}$; the number of particles in each velocity $v$ at site $x$ by $\eta_x = \{\eta(x, v); v \in \mathcal{V}\}$; and a configuration by $\eta = \{\eta_x; x \in \mathbb{T}_N^d\}$. The set of particle configurations is $X_N = (\{0, 1\}^{\mathcal{V}})^{\mathbb{T}_N^d}$.

On the interior of the domain, the dynamics consist of two parts:

(i) each particle in the system evolves according to the nearest neighbor weakly asymmetric random walk with exclusion among particles with the same velocity,

(ii) binary collisions between particles with different velocities.

Let $p(x, v)$ be an irreducible transition probability with finite range, and mean velocity $v$, i.e.,

$$\sum_{x \in \mathbb{Z}^d} x p(x, v) = v.$$

The jump law and the waiting times are chosen so that the jump rate from site $x$ to site $x + y$ for a particle with velocity $v$ is given by

$$P_N(y, v) = \frac{1}{2} \sum_{j=1}^{d} (\delta_{y,e_j} + \delta_{y,-e_j}) + \frac{1}{N} p(y, v),$$

where $\delta_{x,y}$ stands for the Kronecker delta, which is equal to one if $x = y$ and 0 otherwise, and $\{e_1, \ldots, e_d\}$ is the canonical basis in $\mathbb{R}^d$.

## 2.2 Infinitesimal Generator

In this section, we describe the model that we are going to consider in this chapter. Our main interest is to analyze the stochastic lattice gas model given by the generator $\mathcal{L}_N$, which is the superposition of the collision and exclusion dynamics:

$$\mathcal{L}_N = N^2 \{\mathcal{L}_N^c + \mathcal{L}_N^{ex}\},\text{[1]} \tag{2.1}$$

where $\mathcal{L}_N^c$ denotes the generator that models the collision part of the dynamics and $\mathcal{L}_N^{ex}$ models the exclusion part of the dynamics.

Let $f : X_N \to \mathbb{R}$. The generator of the exclusion part of the dynamics, $\mathcal{L}_N^{ex}$, is given by

$$(\mathcal{L}_N^{ex} f)(\eta) = \sum_{v \in \mathcal{V}} \sum_{x, x+z \in \mathbb{T}_N^d} \eta(x, v)(1 - \eta(z, v)) P_N(z - x, v)[f(\eta^{x,z,v}) - f(\eta)]$$

[1] Note that in (2.1) time has been speeded up diffusively due to the factor $N^2$.

where

$$\eta^{x,y,v}(z,w) = \begin{cases} \eta(y,v) \text{ if } w = v \text{ and } z = x, \\ \eta(x,v) \text{ if } w = v \text{ and } z = y, \\ \eta(z,w) \text{ otherwise.} \end{cases}$$

We will often use the decomposition

$$\mathcal{L}_N^{ex} = \mathcal{L}_N^{ex,1} + \mathcal{L}_N^{ex,2},$$

where

$$(\mathcal{L}_N^{ex,1} f)(\eta) = \frac{1}{2} \sum_{v \in \mathcal{V}} \sum_{\substack{x, x+z \in \mathbb{T}_N^d \\ |z-x|=1}} \eta(x,v)(1-\eta(z,v))[f(\eta^{x,z,v}) - f(\eta)],$$

and

$$(\mathcal{L}_N^{ex,2} f)(\eta) = \frac{1}{N} \sum_{v \in \mathcal{V}} \sum_{x, x+z \in \mathbb{T}_N^d} \eta(x,v)(1-\eta(z,v))p(z-x,v)[f(\eta^{x,z,v}) - f(\eta)].$$

The generator of the collision part of the dynamics, $\mathcal{L}_N^c$, is given by

$$(\mathcal{L}_N^c f)(\eta) = \sum_{y \in \mathbb{T}_N^d} \sum_{q \in Q} p_c(y,q,\eta)[f(\eta^{y,q}) - f(\eta)],$$

where $Q$ is a set of all collisions which preserve momentum:

$$Q = \{q = (v,w,v',w') \in \mathcal{V}^4 : v + w = v' + w'\}.$$

The rate $p_c(y,q,\eta)$ is given by

$$p_c(y,q,\eta) = \eta(y,v)\eta(y,w)[1-\eta(y,v')][1-\eta(y,w')],$$

and for $q = (v_0, v_1, v_2, v_3)$, the configuration $\eta^{y,q}$ after the collision is defined as

$$\eta^{y,q}(z,u) = \begin{cases} \eta(y, v_{j+2}) \text{ if } z = y \text{ and } u = v_j \text{ for some } 0 \le j \le 3, \\ \eta(z,u) \text{ otherwise,} \end{cases}$$

where the index of $v_{j+2}$ should be taken modulo 4.

Particles of velocities $v$ and $w$ at the same site collide at rate one and produce two particles of velocities $v'$ and $w'$ at the same site.

Let $\{\eta(t),\, t \geq 0\}$ be the Markov process with generator $\mathcal{L}_N$ and denote by $\{S_t^N,\, t \geq 0\}$ the semigroup associated to $\mathcal{L}_N$.

Let $\mathcal{D}(\mathbb{R}_+, X_N)$ be the set of right continuous functions with left limits taking values in $X_N$. For a probability measure $\mu$ on $X_N$, denote by $\mathbb{P}_\mu$ the measure on the path space $\mathcal{D}(\mathbb{R}_+, X_N)$ induced by $\{\eta(t) : t \geq 0\}$ and the initial measure $\mu$. The expectation with respect to $\mathbb{P}_\mu$ is denoted by $\mathbb{E}_\mu$.

## 2.3 Mass and Momentum

For each configuration $\xi \in \{0,1\}^{\mathcal{V}}$, denote by $I_0(\xi)$ the mass of $\xi$ and by $I_k(\xi)$, $k = 1, \ldots, d$, the momentum of $\xi$, i.e.,

$$I_0(\xi) = \sum_{v \in \mathcal{V}} \xi(v), \quad I_k(\xi) = \sum_{v \in \mathcal{V}} v_k \xi(v).$$

Set $I(\xi) := (I_0(\xi), \ldots, I_d(\xi))$. Assume that the set of velocities is chosen in such a way that the unique conserved quantities by the random walk dynamics described above are the mass and the momentum: $\sum_{x \in \mathbb{T}_N^d} I(\eta_x)$. Two examples of sets of velocities satisfying these conditions can be found in [7].

For each chemical potential $\lambda = (\lambda_0, \ldots, \lambda_d) \in \mathbb{R}^{d+1}$, denote by $m_\lambda$ the probability measure on $\{0,1\}^{\mathcal{V}}$ given by

$$m_\lambda(\xi) = \frac{1}{Z(\lambda)} \exp\{\lambda \cdot I(\xi)\}, \tag{2.2}$$

where $Z(\lambda)$ is a normalizing constant. Note that $m_\lambda$ is a product measure on $\{0,1\}^{\mathcal{V}}$, i.e., that the variables $\{\xi(v) : v \in \mathcal{V}\}$ are independent under $m_\lambda$.

Denote by $\mu_\lambda^N$ the product measure on $X_N$, with marginals given by

$$\mu_\lambda^N\{\eta : \eta(x, \cdot) = \xi\} = m_\lambda(\xi),$$

for each $\xi \in \{0,1\}^{\mathcal{V}}$ and $x \in \mathbb{T}_N^d$. Note that $\{\eta(x,v) : x \in \mathbb{T}_N^d,\, v \in \mathcal{V}\}$ are independent variables under $\mu_\lambda^N$, and that the measure $\mu_\lambda^N$ is invariant for the exclusion process.

The expectation under $\mu_\lambda^N$ of the mass and momentum are, respectively, given by

$$\rho(\lambda) := E_{\mu_\lambda^N}[I_0(\eta_x)] = \sum_{v\in\mathcal{V}} \theta_v(\lambda),$$

$$\varrho_k(\lambda) := E_{\mu_\lambda^N}[I_k(\eta_x)] = \sum_{v\in\mathcal{V}} v_k\theta_v(\lambda).$$

In the last formula, $\theta_v(\lambda)$ denotes the expected value of the density of particles with velocity $v$ under $m_\lambda$:

$$\theta_v(\lambda) := E_{m_\lambda}[\xi(v)] = \frac{\exp\left\{\lambda_0 + \sum_{k=1}^{d}\lambda_k v_k\right\}}{1+\exp\left\{\lambda_0 + \sum_{k=1}^{d}\lambda_k v_k\right\}}.$$

Denote by $(\rho,\varrho)(\lambda) := (\rho(\lambda), \varrho_1(\lambda), \dots, \varrho_d(\lambda))$ the map that associates the chemical potential to the vector of density and momentum. It is possible to prove that $(\rho,\varrho)$ is a diffeomorphism onto $\mathfrak{U} \subset \mathbb{R}^{d+1}$, the interior of the convex envelope of $\{I(\xi),\, \xi \in \{0,1\}^{\mathcal{V}}\}$. Denote by $\Lambda = (\Lambda_0,\dots,\Lambda_d) : \mathfrak{U} \to \mathbb{R}^{d+1}$ the inverse of $(\rho,\varrho)$. This correspondence allows one to parameterize the invariant states by the density and momentum: for each $(\rho,\varrho)\in\mathfrak{U}$, we have a product measure $\nu_{\rho,\varrho}^N = \mu_{\Lambda(\rho,\varrho)}^N$ on $X_N$.

## 2.4 Density Fluctuations

In this section, we investigate the equilibrium fluctuations of $\pi^{k,N}$. We denote by $Y_\cdot^{N,k}$ the density fluctuation field associated to the $k$-th conserved quantity that acts on smooth functions $H$ as

$$Y_t^{N,k}(H) = N^{-\frac{d}{2}} \sum_{x\in\mathbb{T}_N^d} H\left(\tfrac{x}{N}\right)\left(I_k(\eta_{tN^2}(x)) - \rho^k\right), \tag{2.3}$$ [2]

where $\rho^0 = \rho$ and $\rho^k = \varrho_k$ for $k = 1,\dots,d$. The aim of this chapter is to prove that $Y_\cdot^{N,k}$ converges to a stationary Gaussian process with given space-time correlations.

To state the main theorem of this chapter we need to introduce some notation. Consider the lattice $\mathbb{Z}^d$ endowed with the lexicographical order. Let $h_0 \equiv 1$ and for

[2] Note the diffusive rescaling of time on the right-hand side of the (2.3).

each $z > 0$ (resp. $z < 0$), define

$$h_z(u) = \sqrt{2}\cos(2\pi z \cdot u) \text{ ( resp. } h_z(u) = \sqrt{2}\sin(2\pi z \cdot u)). \tag{2.4}$$

Here $\cdot$ denotes the inner product in $\mathbb{R}^d$. It is well known that the set $\{h_z, z \in \mathbb{Z}^d\}$ is an orthonormal basis of $L^2(\mathbb{T}^d)$: each function $f \in L^2(\mathbb{T}^d)$ can be written as

$$f = \sum_{z\in\mathbb{Z}^d} \langle f, h_z\rangle h_z.$$

In this formula and bellow $\langle\cdot,\cdot\rangle$ stands for the inner product of $L^2(\mathbb{T}^d)$.

Consider on $L^2(\mathbb{T}^d)$ the positive, symmetric linear operator $\mathscr{L} = (1 - \Delta)$. A simple computation shows that the functions $h_z$ are eigenvectors: $\mathscr{L}h_z = \gamma_z h_z$, where $\gamma_z = 1 + 4\pi^2\|z\|^2$. For a positive integer $p$, denote by $\mathscr{H}_p$ the Sobolev space of order $p$, which is the Hilbert space obtained as the completion of $C^\infty$ with respect to the inner product $\langle\cdot,\cdot\rangle_p$ defined by

$$\langle f, g\rangle_p = \langle f, \mathscr{L}^p g\rangle.$$

It is easy to check that $\mathscr{H}_p$ is the subspace of $L^2(\mathbb{T}^d)$ consisting of all functions $f$ such that

$$\sum_{z\in\mathbb{Z}^d} \langle f, h_z\rangle^2 \gamma_z^p < \infty.$$

In particular, if we denote $L^2(\mathbb{T}^d)$ by $\mathscr{H}_0$,

$$\mathscr{H}_0 \supset \mathscr{H}_1 \supset \mathscr{H}_2 \supset \cdots \tag{2.5}$$

Moreover, on $\mathscr{H}_p$ the inner product $\langle\cdot,\cdot\rangle_p$ can be expressed by

$$\langle f, g\rangle_p = \sum_{z\in\mathbb{Z}^d} \langle f, h_z\rangle\langle g, h_z\rangle \gamma_z^p.$$

For each positive integer $p$, denote by $\mathscr{H}_{-p}$ the dual of $\mathscr{H}_p$ relatively to the inner product $\langle\cdot,\cdot\rangle$. Note that $\mathscr{H}_{-p}$ can be obtained as the completion of $L^2(\mathbb{T}^d)$ with respect to the inner product obtained from the quadratic form $\langle f, f\rangle_{-p}$ defined by

$$\|f\|_{-p}^2 = \langle f, f\rangle_{-p} = \sup_{g\in\mathscr{H}_p} \left\{2\langle f, g\rangle - \langle g, g\rangle_p\right\}. \tag{2.6}$$

Furthermore, $\mathscr{H}_{-p}$ consists of all sequences $\{\langle f, h_z\rangle, z \in \mathbb{Z}^d\}$ such that

$$\sum_{z\in\mathbb{Z}^d} \langle f, h_z\rangle^2 \gamma_z^{-p} < \infty$$

and that the inner product $\langle f, g\rangle_{-p}$ of two functions $f, g \in \mathscr{H}_{-p}$ can be written as

$$\langle f, g\rangle_{-p} = \sum_{z\in\mathbb{Z}^d} \langle f, h_z\rangle\langle g, h_z\rangle \gamma_z^{-p}.$$

It follows also from the explicit characterization of $\mathscr{H}_{-p}$ and from (2.5) that

$$\cdots \subset \mathscr{H}_2 \subset \mathscr{H}_1 \subset \mathscr{H}_0 \subset \mathscr{H}_{-1} \subset \mathscr{H}_{-2} \subset \cdots$$

We shall consider the density fluctuation field $Y_t^{N,k}$ as taking values in the Sobolev space $\mathscr{H}_{-p}$ for some large enough $p$. Fix a time $T > 0$, a positive integer $p_0$ and denote by $\mathcal{D}([0,T], \mathscr{H}_{-p_0})$ (resp. $C([0,T], \mathscr{H}_{-p_0})$) the space of $\mathscr{H}_{-p_0}$ valued functions, that are right continuous with left limits (resp. continuous), endowed with the uniform weak topology: a sequence $\{Y_t^{k,j}\}_{j\geq 1}$ converges weakly to $Y_t^k$ uniformly in time, i.e., if for all $f \in \mathscr{H}_{p_0}$,

$$\lim_{j\to\infty} \sup_{0\leq t\leq T} \Big| \langle Y_t^{k,j}, f\rangle - \langle Y_t^k, f\rangle \Big| = 0.$$

Denote by $\mathbb{Q}_N$ the probability measure on $\mathcal{D}([0,T], \mathscr{H}_{-p_0})$ induced by the density fluctuation field $Y_\cdot^{N,k}$ introduced in (2.3) and the product measure $\nu_{\rho,\varrho}^N$, by $\mathbb{P}_N$ the probability measure on $\mathcal{D}([0,T], X_N)$ induced by the probability measure $\nu_{\rho,\varrho}^N$ and the Markov process $\eta_t$ speeded up by $N^2$ and denote by $\mathbb{E}_N$ the expectation with respect to $\mathbb{P}_N$. We denote by $\mathbb{Q}$ the limit point of $\mathbb{Q}_N$.

Fix $(\rho, \varrho)$. Based on [1, 19], we give here a characterization of the generalized Ornstein-Uhlenbeck process which is a solution of

$$dY_t = \frac{1}{2}\Delta Y_t - \sum_{v\in\mathcal{V}} \tilde{v}\langle \nabla Y_t \cdot v, \nabla F_v(\rho, \varrho)\rangle + \sqrt{Z} dW_t,$$

where $Z = \sum_{v\in\mathcal{V}} v_k^2\, \chi(\theta_v(\Lambda(\rho, p)))$, $F_v(\rho, \varrho) = \chi(\theta_v(\Lambda(\rho, \varrho)))$ and given $v = (v_1, \ldots, v_d) \in \mathcal{V}$ we denote by $\tilde{v} = (1, v_1, \ldots, v_d)$. We will see below that this process governs the equilibrium fluctuations of the density of particles of our model.

**Proposition 7.** *For each $(\rho, \varrho) \in \mathfrak{U}$ there exists an unique random $Y_\cdot$ taking values in* *the space* $C([0,T],\mathscr{H}_{-p_0})$ *such that:*

*(i) for every function* $H \in C([0,T],\mathscr{H}_{p_0})$*,* $M_t(H)$ *and* $N_t(H)$ *given by*

$$\begin{aligned} M_t(H) &= Y_t(H) - Y_0(H) - \int_0^t Y_s\left(\frac{1}{2}\Delta H\right) ds + \sum_{v\in\mathcal{V}} \tilde{v} \int_0^t \langle Y_s(v\cdot\nabla H), \nabla F_v(\rho,\varrho)\rangle ds, \\ N_t(H) &= (M_t(H))^2 - \|\mathscr{B}H\|_2^2 t, \end{aligned}$$

*are* $\mathcal{F}_t$*-martingales, where for each* $t\in[0,T]$*,* $\mathcal{F}_t = \sigma(Y_s(H);\, s\le t, H\in C([0,T],\mathscr{H}_{p_0}))$*. Above, for each velocity* $v=(v_1,\dots,v_d)\in\mathcal{V}$*, we define* $\tilde{v}=(1,v_1,\dots,v_d)$ *and* $(M_t(H))^2 = \big((M_t^0(H))^2,\dots,(M_t^d(H))^2\big)$*, also* $\mathscr{B}H = (\mathscr{B}_0 H,\dots,\mathscr{B}_d H)$ *with*

$$\mathscr{B}_k H = \sqrt{\sum_{v\in\mathcal{V}} v_k^2\, \chi(\theta_v(\Lambda(\rho,\varrho)))}\nabla H.$$

*(ii)* $Y_0$ *is a Gaussian field of mean zero and covariance given on* $H,G\in C([0,T],\mathscr{H}_{p_0})$ *by*

$$E_{\mathbb{Q}}[Y_0^k(H)Y_0^k(G)] = \chi(\alpha)\int_{\mathbb{T}^d} du H(u)G(u)$$

*Here* $\chi(\alpha)$ *stands for the static compressibility given by* $\chi(\alpha)=Var(\nu_\alpha,\eta(0,v))$*. Then, the sequence* $\{\mathbb{Q}_N\}_{N\ge1}$ *converges weakly to the probability measure* $\mathbb{Q}$*.*

**Theorem 2.** *Consider the Markov process* $\{\eta_{tN^2} : t\ge 0\}$ *starting from the invariant state* $\nu_{\rho,\varrho}$*. Then, the sequence of process* $\{Y_t^{N,k}\}_{N\ge1}$ *converges in ditribution, as* $N\to\infty$*, with respect to the Skorohod topology of* $\mathcal{D}([0,T],\mathscr{H}_{-p_0})$*, to* $Y_t\in C([0,T],\mathscr{H}_{-p_0})$*, the generalized Ornstein-Uhlenbeck process of characteristics* $\nabla,\Delta$ *which is the formal solution of the equation*

$$dY_t = \frac{1}{2}\Delta Y_t - \sum_{v\in\mathcal{V}} \tilde{v}\langle \nabla Y_t\cdot v, \nabla F_v(\rho,\varrho)\rangle + \sqrt{Z}dW_t.$$

## 2.5 Proof of Theorem 2

### 2.5.1 Martingale Problem

By Dynkin's formula, for a given function $H\in C([0,T],\mathscr{H}_{p_0})$

$$\begin{aligned} M_t^{N,k}(H) &= Y_t^{N,k}(H) - Y_0^{N,k}(H) - \int_0^t \Gamma_1^N(H)\, ds, \\ N_t^{N,k}(H) &= (M_t^{N,k}(H))^2 - \int_0^t \Gamma_2^N(H)\, ds, \end{aligned} \tag{2.7}$$

are martingales with respect to the filtration $\mathcal{G}_t := \sigma(\eta_s :\ s \le t)$, where

$$\Gamma_1^N(H) := (\partial_s + N^2\mathcal{L}_N)Y_s^{N,k}(H)$$

$$\Gamma_2^N(H) := N^2\mathcal{L}_N\left([Y_s^{N,k}(H)]^2\right) - 2Y_s^{N,k}(H)N^2\mathcal{L}_N[Y_s^{N,k}(H)].$$

By the computations of Appendix B, we obtain

$$\begin{aligned}\Gamma_1^N(H) &:= (\partial_s + N^2\mathcal{L}_N)Y_s^{N,k}(H)\\ &= \partial_s Y_s^{N,k}(H) + \frac{1}{2}Y_s^{N,k}[\Delta_N H\left(\tfrac{x}{N}\right)]\\ &-N^{-\frac{d}{2}}\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^{d}(\partial_{u_j}^N H)\left(\tfrac{x}{N}\right)[\tau_x W_{j,k}^{\eta_s} - \omega_k^{\rho,\varrho}],\end{aligned} \tag{2.8}$$

where

$$W_{j,k}^{\eta_s} := \sum_{v\in\mathcal{V}} v_k \sum_{z\in\mathbb{Z}^d} z_j p(z,v)\eta_s(0,v)[1-\eta_s(z,v)]$$

and

$$\omega_k^{\rho,\varrho} := E_{\nu_{\rho,\varrho}^N}[W_{j,k}^{\eta_s}] = \sum_{v\in\mathcal{V}} v_k v_j \chi(\theta_v(\Lambda(\rho,\varrho))).$$

Also by the computations of Appendix B,

$$\begin{aligned}&\Gamma_2^N(H) := N^2\mathcal{L}_N\left([Y_s^{N,k}(H)]^2\right) - 2Y_s^{N,k}(H)N^2\mathcal{L}_N[Y_s^{N,k}(H)]\\ &= \frac{1}{2N^d}\sum_{v\in\mathcal{V}}\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^{d} v_k^2\Big(\eta(x,v) - \eta(x+e_j,v)\Big)^2(\partial_j^N H\left(\tfrac{x}{N}\right))^2\\ &+\frac{1}{2N^{d+1}}\sum_{v\in\mathcal{V}} v_k^2 \sum_{x\in\mathbb{T}_N^d}\sum_{|w|\le R}\sum_{j=1}^{d}\eta(x,v)(1-\eta(x+w,v))p(w,v)\left(\partial_j^N H\left(\tfrac{x}{N}\right)\right)^2 w_j^2.\end{aligned} \tag{2.9}$$

[3]

The goal is to close the $M_t^{N,k}(H)$ in equation (2.7). Note that,

$$\begin{aligned}M_t^{N,k}(H) = Y_t^{N,k}(H) - Y_0^{N,k}(H) - \int_0^t \frac{1}{2N^{\frac{d}{2}}} \sum_{x\in\mathbb{T}_N^d} \Delta_N H\left(\tfrac{x}{N}\right)\left(I_k(\eta_x(s)) - \rho^k\right) ds \\ + \int_0^t \frac{1}{N^{\frac{d}{2}}} \sum_{x\in\mathbb{T}_N^d} \sum_{j=1}^{d} (\partial_j^N H)\left(\tfrac{x}{N}\right)\left[\tau_x W_{j,k}^{\eta_s} - \omega_k^{\rho,\varrho}\right] ds.\end{aligned} \tag{2.10}$$

By Theorem 3, the Boltzmann-Gibbs principle (see Section 2.6), where $F_v(\rho,\varrho) = \chi(\theta_v(\Lambda(\rho,\varrho)))$, we have that (2.10) is equal to

$$M_t^{N,k}(H) = Y_t^{N,k}(H) - Y_0^{N,k}(H) - \frac{1}{2}\int_0^t \frac{1}{N^{\frac{d}{2}}} \sum_{x\in\mathbb{T}_N^d} \Delta_N H\left(\tfrac{x}{N}\right)\left(I_k(\eta_x(s)) - \rho^k\right) ds$$

$$+ \int_0^t \frac{1}{N^{\frac{d}{2}}} \sum_{x\in\mathbb{T}_N^d} \sum_{j=1}^{d} (\partial_j^N H)\left(\tfrac{x}{N}\right) \sum_{v\in\mathcal{V}} v_k v_j \sum_{i=1}^{d} \partial_{\rho_i} F_v(\rho,\varrho)[I_i(\eta_x(s)) - \rho^i]\, ds,$$

rewrite the last equation as

$$\begin{aligned}& Y_t^{N,k}(H) - Y_0^{N,k}(H) - \int_0^t Y_s^{N,k}\left(\frac{1}{2}\Delta_N H\right) ds \\ & + \sum_{i=1}^{d} v_k \int_0^t Y_s^{N,i}\left(\sum_{v\in\mathcal{V}} \sum_{j=1}^{d} v_j \partial_{\rho_i} F_v(\rho,\varrho)\, \partial_j^N H\right) ds \\ & = Y_t^{N,k}(H) - Y_0^{N,k}(H) - \int_0^t Y_s^{N,k}\left(\frac{1}{2}\Delta_N H\right) ds \\ & + \sum_{v\in\mathcal{V}} v_k \sum_{i=0}^{d} \int_0^t Y_s^{N,i}\left(\sum_{j=1}^{d} v_j \partial_{\rho_i} F_v(\rho,\varrho)\, \partial_j^N H\right) ds \\ & = Y_t^{N,k}(H) - Y_0^{N,k}(H) - \int_0^t Y_s^{N,k}\left(\frac{1}{2}\Delta_N H\right) ds \\ & + \sum_{v\in\mathcal{V}} v_k \sum_{i=0}^{d} \int_0^t Y_s^{N,i}\left(\partial_{\rho_i} F_v(\rho,\varrho)\,(v\cdot\nabla H)\right) ds\end{aligned} \tag{2.11}$$

[3] $R$ is the range of $p$

$$= Y_t^{N,k}(H) - Y_0^{N,k}(H) - \int_0^t Y_s^{N,k}\left(\frac{1}{2}\Delta_N H\right) ds + \sum_{v\in\mathcal{V}} v_k \int_0^t \langle Y_s^N(v\cdot\nabla H), \nabla F_v(\rho,\varrho)\rangle\, ds$$

$$= Y_t^{N,k}(H) - Y_0^{N,k}(H) - \int_0^t Y_s^{N,k}\left(\frac{1}{2}\Delta_N H\right) ds + \sum_{v\in\mathcal{V}} \tilde{v} \int_0^t \langle Y_s^N(v\cdot\nabla H), \nabla F_v(\rho,\varrho)\rangle\, ds. \tag{2.12}$$

By the definition of $N_t^{N,k}$ in (2.7), using the computations of Appendix B, we obtain that

$$\begin{aligned} N_t^{N,k}(H) &= (M_t^{N,k}(H))^2 \\ &- \int_0^t \frac{1}{2N^d}\sum_{v\in\mathcal{V}}\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^d v_k^2(\eta_s(x,v)-\eta_s(x+e_j,v))^2(\partial_j^N H\left(\tfrac{x}{N}\right))^2 ds + R_t^{N,k}(H), \end{aligned}$$

where $R_t^{N,k}(H) = \dfrac{1}{2N^{d+1}}\displaystyle\sum_{v\in\mathcal{V}} v_k^2 \sum_{x\in\mathbb{T}_N^d}\sum_{|w|\le R}\sum_{j=1}^d \eta(x,v)(1-\eta(x+w,v))p(w,v)\left(\partial_j^N H\left(\tfrac{x}{N}\right)\right)^2 w_j^2$, is a martingale.

**Claim 1.** *Note that $R_t^{N,k}(H)$ vanishes as $N\to+\infty$ in $L^2(\nu_{\rho,\varrho})$.*

*Proof of the Claim 1: In fact, consider*

$$A_t^{N,k}(H) = \frac{1}{2N^d}\sum_{v\in\mathcal{V}} v_k^2 \sum_{x\in\mathbb{T}_N^d}\sum_{|w|\le R}\sum_{j=1}^d \eta(x,v)(1-\eta(x+w,v))p(w,v)\left(\partial_j^N H\left(\tfrac{x}{N}\right)\right)^2 w_j^2$$

*this implies that $R_t^{N,k}(H) = \frac{1}{N}A_t^{N,k}(H)$. We prove that $A_t^{N,k}(H)$ is bounded, which results in $R_t^{N,k}(H)$ vanishes as $N\to+\infty$ in $L^1(\nu_{\rho,\varrho})$. Hence, $R_t^{N,k}(H)$ vanishes as $N\to+\infty$ in $L^2(\nu_{\rho,\varrho})$. Note that*

$$\begin{aligned} A_t^{N,k}(H) &= \frac{1}{2N^d}\sum_{v\in\mathcal{V}} v_k^2 \sum_{x\in\mathbb{T}_N^d}\sum_{|w|\le R}\sum_{j=1}^d \eta(x,v)(1-\eta(x+w,v))p(w,v)\left(\partial_j^N H\left(\tfrac{x}{N}\right)\right)^2 w_j^2 \\ &\le \frac{1}{2N^d} C \sum_{x\in\mathbb{T}_N^d} (\partial_j^N H\left(\tfrac{x}{N}\right))^2, \end{aligned}$$

*since the set of velocities is finite and the the range of $p$ is finite. Observe that, last display converges to $\tilde{C}\displaystyle\int_{\mathbb{T}^d}\nabla H^2(x)\,dx$, as $N\to\infty$. Since $\tilde{C}\displaystyle\int_{\mathbb{T}^d}\nabla H^2(x)\,dx$ is bounded, then $A_t^{N,k}(H)$ is bounded. This proves the claim.*

**Claim 2.** *Since $\mathbb{E}_N\left[(\eta(x,v)-\eta(x+e_j,v))^2\right] = 2\chi(\theta_v(\Lambda(\rho,\varrho)))$, with $\chi(r) = r(1-r)$,* *then*

$$\mathbb{E}_N\Big[\Big(\int_0^t \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d} G\left(\tfrac{x}{N}\right)\big[\big(\eta(x,v)-\eta(x+e_j,v)\big)^2-2\chi(\theta_v(\Lambda(\rho,\varrho)))\big]ds\Big)^2\Big]\xrightarrow{N\to+\infty}0.$$

*Proof of the Claim 2: In fact, by Cauchy-Schwarz inequality, last expectation is bounded from above by*

$$\mathbb{E}_N\left[t\int_0^t\left(\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}G\left(\tfrac{x}{N}\right)\left[(\eta_s(x,v)-\eta_s(x+e_j,v))^2-2\chi(\theta_v(\Lambda(\rho,\varrho)))\right]ds\right)^2\right]$$

$$=\mathbb{E}_N\Big[\frac{t}{N^{2d}}\int_0^t\sum_{x,y\in\mathbb{T}_N^d}G\left(\tfrac{x}{N}\right)G\left(\tfrac{y}{N}\right)[(\eta_s(x,v)-\eta_s(x+e_j,v))^2-2\chi(\theta_v(\Lambda(\rho,\varrho)))]$$
$$\times[(\eta_s(y,v)-\eta_s(y+e_j,v))^2-2\chi(\theta_v(\Lambda(\rho,\varrho)))\,ds]\Big]$$

$$\leq t^2Var((\eta(x,v)-\eta(x+e_j,v))^2,\nu_{\rho,\varrho})\frac{3}{N^d}\langle G,G\rangle_{\mathbb{T}_N^d},$$

*which vanishes as* $N\to\infty$, *since* $y\in\{x-e_j,x,x+e_j\}$ *and* $Var((\eta(x,v)-\eta(x+e_j,v))^2,\nu_{\rho,\varrho})$ *does not depend on* $x$ *nor* $j$. *Therefore,*

$$\int_0^t\frac{1}{2N^d}\sum_{v\in\mathcal{V}}\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^d v_k^2(\eta_s(x,v)-\eta_s(x+e_j,v))^2(\partial_j^N H\left(\tfrac{x}{N}\right))^2ds$$
$$\xrightarrow{L^2}\int_0^t\sum_{v\in\mathcal{V}}\sum_{j=1}^d v_k^2\int_{\mathbb{T}^d}\chi(\theta_v(\Lambda(\rho,\varrho)))(\partial_jH(x))^2dx\,ds$$
$$=\int_0^t\sum_{j=1}^d\int_{\mathbb{T}^d}\left(\sum_{v\in\mathcal{V}}v_k^2\chi(\theta_v(\Lambda(\rho,\varrho)))\right)(\partial_jH(x))^2dx\,ds.$$

*Define*

$$\mathscr{B}_kH=\sqrt{\sum_{v\in\mathcal{V}}v_k^2\,\chi(\theta_v(\Lambda(\rho,\varrho)))}\nabla H\ \text{and}\ \mathscr{B}=(\mathscr{B}_0,\cdots,\mathscr{B}_d).\tag{2.13}$$

## 2.6 The Boltzmann-Gibbs Principle

*In this section we show that the martingales* $M_t^{N,k}$ *introduced in (2.7) can be expressed in terms of the fluctuation field* $Y_t^{N,k}$. *The Boltzmann-Gibbs principle is one of the main ingredients in the proof of the equilibrium fluctuations.*

**Theorem 3** (Boltzmann-Gibbs principle)**.** *For every continuous function* $G:\mathbb{T}^d\to\mathbb{R}$

*and every* $t \in [0,T]$, $1 \le j \le d$, $0 \le k \le d$,

$$\lim_{N\to\infty} \mathbb{E}_N \left[ \left( \int_0^t N^{-\frac{d}{2}} \sum_{x\in\mathbb{T}_N^d} G\left(\tfrac{x}{N}\right) \tau_x V_F^{j,k}(\eta_s) ds \right)^2 \right] = 0$$

*where*

$$V_F^{j,k}(\eta) = \sum_{v\in\mathcal{V}} v_k \left[ \sum_{z\in\mathbb{Z}^d} z_j p(z,v)\eta(0,v)(1-\eta(z,v)) - v_j F_v(\rho,\varrho) - v_j \sum_{i=0}^{d} \partial_{\rho_i} F_v(\rho,\varrho)[I_i(0)-\rho^i] \right]$$

*and* $F_v(\rho,\varrho) = \chi(\theta_v(\Lambda(\rho,\varrho)))$.

*Proof.* Fix a positive integer $l$ that shall increase to $\infty$ after $N$. For each $N$, we subdivide $\mathbb{T}_N^d$, the discrete $d$-dimensional torus, into non-overlapping cubes of linear size $l$. Denote them by $\{B_j\,;\, 1 \le j \le M^d\}$, where $M = \lfloor \frac{N}{l} \rfloor$: [4] for each $j$,

$$B_j = y_j + \{1,\ldots,l\}^d \text{ , for some } y_j \in \mathbb{T}_N^d \text{ and } B_i \cap B_j = \emptyset \text{ if } i \neq j.$$

Denote by $B_0 = \mathbb{T}_N^d \setminus \bigcup_{j=1}^{M^d} B_j$, By construction, the cardinality of $B_0$ is bounded by $dlN^{d-1}$. Once $p(\cdot,\cdot)$ is the probability transition function, which has finite range for each $v \in \mathcal{V}$ and $\mathcal{V}$ is finite. We will denote by $s_p$ be the maximus range of $p$, that is $s_p = \max\{|x| : \exists v \in \mathcal{V}; p(x,v) > 0\}$. Let $\Lambda_{s_p}$ be the smallest cube centered at origin that contains the support of $p$. Denote by $B_i^0$ the interior of the cube $B_i$, i.e., the sites $x$ in $B_i$ that are at a distance at least $s_p$ from the boundary:

$$B_i^0 = \{x \in B_i\,; \quad d(x, \mathbb{T}_N^d \setminus B_i) > s_p\}.$$

Note that $\forall x \in B_i^0$, $\tau_x V_F^{j,k}(\eta)$ is measurable with respect to $\sigma(\eta(z);\, z \in B_i)$. In particular, since $\nu_{\rho,\varrho}^N$ is product measure and $B_i \cap B_j = \emptyset$, the $\sigma$-algebra $\sigma(\eta(z);\, z \in B_i)$ and $\sigma(\eta(z);\, z \in B_j)$ are independent $i \neq j$. Then, $\tau_x V_F^{i,k}(\eta)$ is independent of $\tau_y V_F^{j,k}(\eta)$ if $x \in B_i^0$ and $y \in B_j^0$, $i \neq j$.

Let

$$B^0 = \bigcup_{i=1}^{M^d} B_i^0 \quad \text{ and } \quad B^1 = \mathbb{T}_N^d \setminus B^0.$$

By construction, the cardinality of $B^1$ is bounded by $dN^d\,(c(p)l^{-1} + lN^{-1})$, for some constant $c(p) < \infty$ depending only on $p$.

[4] $\lfloor r \rfloor$ denote the integer part of $r$

With the notation we have just introduced, we have that

$$\begin{aligned} N^{-\frac{d}{2}} \sum_{x\in\mathbb{T}_N^d} G\left(\tfrac{x}{N}\right)\tau_x V_F^{j,k}(\eta) &= N^{-\frac{d}{2}} \sum_{x\in B^1} G\left(\tfrac{x}{N}\right)\tau_x V_F^{j,k}(\eta) \\ &+ N^{-\frac{d}{2}} \sum_{i=1}^{M^d}\sum_{x\in B_i^0} [G\left(\tfrac{x}{N}\right) - G\left(\tfrac{y_i}{N}\right)]\tau_x V_F^{j,k}(\eta) \\ &+ N^{-\frac{d}{2}} \sum_{i=1}^{M^d} G\left(\tfrac{y_i}{N}\right) \sum_{x\in B_i^0} \tau_x V_F^{j,k}(\eta), \end{aligned}$$

where $y_i$ is a point in $B_i$.

We claim that the expected value of the $L^2$ norm of the time integral of the first two expressions on the right-hand side, vanishes as $N \uparrow +\infty$ and then $l \uparrow +\infty$. The first step is to prove that

$$\lim_{l\to\infty}\lim_{N\to\infty} \mathbb{E}_N\left[\left(\int_0^t N^{-\frac{d}{2}} \sum_{x\in B^1} G\left(\tfrac{x}{N}\right)\tau_x V_F^{j,k}(\eta_s)ds\right)^2\right] = 0. \tag{2.14}$$

By Cauchy-Schwarz inequality and invariance of $\nu_{\rho,\varrho}^N$ last expectation is bounded from above by

$$\begin{aligned} &\mathbb{E}_N\left[\left(\int_0^t N^{-\frac{d}{2}} \sum_{x\in B^1} G\left(\tfrac{x}{N}\right)\tau_x V_F^{j,k}(\eta_s)ds\right)^2\right] \\ &\le t\int_0^t \mathbb{E}_N\left[\left(N^{-\frac{d}{2}} \sum_{x\in B^1} G\left(\tfrac{x}{N}\right)\tau_x V_F^{j,k}(\eta_s)\right)^2\right] ds \\ &= t^2\, E_{\nu_{\rho,\varrho}^N}\left[\left(N^{-\frac{d}{2}} \sum_{x\in B^1} G\left(\tfrac{x}{N}\right)\tau_x V_F^{j,k}(\eta)\right)^2\right]. \end{aligned}$$

Note that $E_{\nu_{\rho,\varrho}^N}[V_F^{j,k}] = 0$. Furthermore, if $x, y \in \mathbb{T}_N^d$ such that $\|x-y\| > 2s_p$, then $E_{\nu_{\rho,\varrho}^N}[\tau_x V_F^{j,k}(\eta)\cdot\tau_y V_F^{j,k}(\eta)] = 0$. Therefore, the last expression is bounded by

$$t^2 N^{-d} \sum_{\substack{x,y\in B^1\\ \|x-y\|\le 2s_p}} G\left(\tfrac{x}{N}\right) G\left(\tfrac{y}{N}\right)\, E_{\nu_{\rho,\varrho}^N}\left[\tau_x V_F^{j,k}\tau_y V_F^{j,k}\right].$$

Note that, for each $(\rho,\varrho)$ fixed, $V_F^{j,k}(\eta)$ is bounded, thus, $V_F^{j,k}(\cdot) \in L^2(\nu_{\rho,\varrho}^N)$. Since $G$ is continuous, $G\left(\frac{x}{N}\right)$ and $G\left(\frac{y}{N}\right)$ are uniformly bounded on $\mathbb{T}_N^d$. We obtain that the last expression is bounded from above by

$$\begin{aligned} & t^2\, N^{-d} C\, |B^1|\, (4s_p)^d \\ & \leq t^2\, N^{-d} C\, \left( dN^d C(p) l^{-1} + l N^{-1} \right)\, (4s_p)^d \\ & \leq \frac{t^2\, C\, d\, C(p)\, (4s_p)^d}{l} + \frac{t^2\, l\, C\, (4s_p)^d}{N^{d+1}} \longrightarrow 0 \end{aligned} \tag{2.15}$$

when $N \uparrow +\infty$ and then $l \uparrow +\infty$.

Applying the same arguments, for the second term on the right-hand side of equation (2.14), since $G$ is continuous on $\mathbb{T}^d$, which is compact then $G$ is uniformly continuous in $\mathbb{T}^d$. Denote by

$$A_{l,N} = \sup_{\|x-y\| \leq \frac{l}{N}} |G(x) - G(y)|^2.$$

For each $l$ fixed, we have that $\lim\limits_{N \to +\infty} A_{l,N} = 0$. Therefore,

$$\begin{aligned} & \mathbb{E}_N \left[ \left( \int_0^t N^{-\frac{d}{2}} \sum_{i=1}^{M^d} \sum_{x \in B_i^0} [G\left(\tfrac{x}{N}\right) - G\left(\tfrac{y_i}{N}\right)] \tau_x V_F^{j,k}(\eta_s) ds \right)^2 \right] \\ & \leq t^2\, E_{\nu_{\rho,\varrho}^N} \left[ N^{-d} \sum_{i,m=1}^{M^d} \sum_{\substack{x \in B_i^0 \\ z \in B_m^0}} \left(G\left(\tfrac{x}{N}\right) - G\left(\tfrac{y_i}{N}\right)\right) \left(G\left(\tfrac{z}{N}\right) - G\left(\tfrac{y_m}{N}\right)\right) [\tau_x V_F^{j,k} \tau_z V_F^{j,k}] \right]. \end{aligned}$$

If $m \neq i$, $\tau_x V_F^{j,k}$ is independent of $\tau_z V_F^{j,k}$ and $E_{\nu_{\rho,\varrho}^N}[V_F^{j,k}(\eta)] = 0$. Then, the last expression is bounded from above by

$$t^2\, E_{\nu_{\rho,\varrho}^N} \left[ N^{-d} \sum_{i=1}^{M^d} \sum_{x,z \in B_i^0} \left(G\left(\tfrac{x}{N}\right) - G\left(\tfrac{y_i}{N}\right)\right) \left(G\left(\tfrac{z}{N}\right) - G\left(\tfrac{y_i}{N}\right)\right) [\tau_x V_F^{j,k} \tau_z V_F^{j,k}] \right]$$

the same argument above can be applied when $\|x - z\| > 2s_p$. Therefore, (2.15) is bounded from above by

$$t^2\, E_{\nu_{\rho,\varrho}^N}\left[ N^{-d}\sum_{i=1}^{M^d}\sum_{\substack{x,z\in B_i^0\\ \|x-z\|\le 2s_p}} \left(G\left(\tfrac{x}{N}\right)-G\left(\tfrac{y_i}{N}\right)\right)\left(G\left(\tfrac{z}{N}\right)-G\left(\tfrac{y_i}{N}\right)\right)\left[\tau_x V_F^{j,k}\tau_z V_F^{j,k}\right]\right]$$

$$\le t^2\, E_{\nu_{\rho,\varrho}^N}\left[ N^{-d}\sum_{i=1}^{M^d}\sum_{\substack{x,z\in B_i^0\\ \|x-z\|\le 2s_p}} A_{l,N}\, \tau_x V_F^{j,k}\tau_z V_F^{j,k}\right]$$

for each $(\rho,\varrho)$ fixed, $V_F^{j,k}$ is bounded uniformly on $\eta$. So, the last display is bounded from above by

$$\begin{aligned}
& t^2\, A_{l,N}\, C\, M^d\, |B_1|\, (4s_p)^d\, N^{-d}\\
\le\ & t^2\, A_{l,N}\, C\, M^d\, l^d\, (4s_p)^d\, N^{-d}\\
\le\ & t^2\, A_{l,N}\, C\, \frac{N^d}{l^d}\, l^d\, (4s_p)^d\, N^{-d}\\
=\ & t^2\, A_{l,N}\, C\, (4s_p)^d
\end{aligned}$$

and the expression above vanishes as $N\to+\infty$.

In order to conclude the proof it remains to show that

$$\lim_{l\to\infty}\lim_{N\to\infty}\mathbb{E}_N\left[\left(\int_0^t N^{-\frac{d}{2}}\sum_{i=1}^{M^d} G\left(\tfrac{y_i}{N}\right)\sum_{x\in B_i^0}\tau_x V_F^{j,k}(\eta)ds\right)^2\right]=0.$$

For each $1\le i\le M^d$, denote by $\xi_i$ the configuration $\{\eta(x) : x\in B_i\}$, and by $\mathcal{L}_{B_i}$ the generator $\mathcal{L}_N$ restricted to the cube $B_i$,

$$\mathcal{L}_{B_i}=\mathcal{L}_{B_i}^{ex}+\mathcal{L}_{B_i}^{c},$$

where

$$(\mathcal{L}_{B_i}^{ex}f)(\eta)=\sum_{v\in\mathcal{V}}\sum_{x,z\in B_i}\eta(x,v)(1-\eta(z,v))P_N(z-x,v)[f(\eta^{x,z,v})-f(\eta)]$$

and

$$(\mathcal{L}_{B_i}^{c}f)(\eta)=\sum_{y\in B_i}\sum_{q\in Q}p_c(y,q,\eta)[f(\eta^{y,q})-f(\eta)].$$

Consider a $L^2(\nu_{\rho,\varrho}^N)$ cylinder function $f$ measurable with respect to the $\sigma$-algebra $\sigma(\eta(x),\, x\in B_1)$ and denote by $f_i$ the translation of $f$, that makes $f_i$ measurable with respect to the $\sigma$-algebra $\sigma(\eta(x),\, x\in B_i)$. By definition of the generator $\mathcal{L}_{B_i}$, $\mathcal{L}_{B_i}f$ is also measurable with respect to $\sigma(\eta(x),\, x\in B_i)$. By [1, Appendix A1, Proposition 6.1], for $t>0$

$$\mathbb{E}_N\left[\left(\int_0^t N^{-\frac{d}{2}}\sum_{i=1}^{M^d} G\left(\tfrac{y_i}{N}\right)\mathcal{L}_{B_i}f_i(\xi_i(s))\right)^2 ds\right]$$
$$\leq 20t\sup_{h\in L^2(\nu^N_{\rho,\varrho})}\left[2\int V^N_{G,f}(\eta)h(\eta)\nu^N_{\rho,\varrho}(d\eta)-N^2\langle h,(-\mathcal{L}_N h)\rangle_{\nu^N_{\rho,\varrho}}\right],\text{[5]}$$

where $V^N_{G,f}(\eta):=N^{-\frac{d}{2}}\sum_{i=1}^{M^d} G\left(\tfrac{y_i}{N}\right)\mathcal{L}_{B_i}f_i(\xi_i)$.

**Claim 3.** *We claim that*

$$\langle\mathcal{L}^c_{B_i}f,h\rangle_{\nu^N_{\rho,\varrho}}=\langle f,\mathcal{L}^c_{B_i}h\rangle_{\nu^N_{\rho,\varrho}}$$

*this means that, the collision generator is a symmetric operator.*

*Proof of the claim 3:* First of all, note that $\eta^{y,q}=\xi$ implying that $\xi^{y,\tilde{q}}=(\eta^{y,q})^{y,\tilde{q}}=\eta$ where $q=(v_0,v_1,v_2,v_3),\tilde{q}=(v_2,v_3,v_0,v_1)$ and $\dfrac{d\nu^N_{\rho,\varrho}(\xi)}{d\nu^N_{\rho,\varrho}(\eta)}=1$. From this we obtain that

$$\begin{aligned}
\langle\mathcal{L}^c_{B_i}f,h\rangle_{\nu^N_{\rho,\varrho}} &= \int(\mathcal{L}^c_{B_i}f)(\eta)h(\eta)d\nu^N_{\rho,\varrho}\\
&= \int\sum_{y\in B_i}\sum_{q\in Q}p_c(y,q,\eta)[f(\eta^{y,q})-f(\eta)]h(\eta)d\nu^N_{\rho,\varrho}\\
&= \sum_{y\in B_i}\sum_{q\in Q}\int p_c(y,q,\eta)f(\eta^{y,q})h(\eta)d\nu^N_{\rho,\varrho}-\sum_{y\in B_i}\sum_{q\in Q}\int p_c(y,q,\eta)f(\eta)h(\eta)d\nu^N_{\rho,\varrho}.
\end{aligned}$$

Performing a change of variables $\eta=\xi^{y,\tilde{q}}$, the last display can be rewritten as

$$\sum_{y\in B_i}\sum_{q\in Q}\int p_c(y,q,\xi^{y,\tilde{q}})f(\xi)h(\xi^{y,\tilde{q}})d\nu^N_{\rho,\varrho}-\sum_{y\in B_i}\sum_{q\in Q}\int p_c(y,q,\xi)f(\xi)h(\xi)d\nu^N_{\rho,\varrho}. \tag{2.16}$$

Note that

$$\begin{aligned}
p_c(y,q,\xi^{y,\tilde{q}}) &= \xi^{y,\tilde{q}}(y,v_0)\xi^{y,\tilde{q}}(y,v_1)[1-\xi^{y,\tilde{q}}(y,v_2)][1-\xi^{y,\tilde{q}}(y,v_3)]\\
&= \xi(y,v_2)\xi(y,v_3)[1-\xi(y,v_0)][1-\xi(y,v_1)]\\
&= p_c(y,\tilde{q},\xi).
\end{aligned}$$

[5] In the formula $\langle\cdot,\cdot\rangle_{\nu^N_{\rho,\varrho}}$ denotes the inner product in $L^2(\nu^N_{\rho,\varrho})$

Therefore,

$$
\begin{aligned}
(2.16) \quad &= \sum_{y\in B_i}\sum_{q\in Q}\int p_c(y,\tilde{q},\xi)f(\xi)h(\xi^{y,\tilde{q}})d\nu^N_{\rho,\varrho} - \sum_{y\in B_i}\sum_{q\in Q}\int p_c(y,q,\xi)f(\xi)h(\xi)d\nu^N_{\rho,\varrho}\\
&= \sum_{y\in B_i}\sum_{q\in Q}\int p_c(y,q,\xi)f(\xi)h(\xi^{y,q})d\nu^N_{\rho,\varrho} - \sum_{y\in B_i}\sum_{q\in Q}\int p_c(y,q,\xi)f(\xi)h(\xi)d\nu^N_{\rho,\varrho}\\
&= \sum_{y\in B_i}\sum_{q\in Q}\int p_c(y,q,\xi)f(\xi)[h(\xi^{y,q})-h(\xi)]d\nu^N_{\rho,\varrho}\\
&= \langle f, \mathcal{L}^c_{B_i}h\rangle_{\nu^N_{\rho,\varrho}}
\end{aligned}
$$

and this finishes the proof of the claim. However, since

$$\mathcal{L}_{B_i} = \mathcal{L}^{ex,1}_{B_i} + \mathcal{L}^{ex,2}_{B_i} + \mathcal{L}^c_{B_i},$$

we will need the following result.

**Proposition 8.** *For all $f_i \in L^2(\nu^N_{\rho,\varrho})$*

$$\lim_{l\to+\infty}\lim_{N\to+\infty}\mathbb{E}_N\left[\left(\int_0^t N^{-\frac{d}{2}}\sum_{i=1}^{M^d} G\left(\tfrac{y_i}{N}\right)\mathcal{L}^{ex,2}_{B_i}f_i(\xi_i(s))\right)^2 ds\right] = 0.$$

*Proof of the Proposition 8: Using Cauchy-Schwarz inequality and Tonelli's theorem, we obtain*

$$
\begin{aligned}
&\mathbb{E}_N\left[\left(\int_0^t N^{-\frac{d}{2}}\sum_{i=1}^{M^d} G\left(\tfrac{y_i}{N}\right)\mathcal{L}^{ex,2}_{B_i}f_i(\xi_i(s))\right)^2 ds\right]\\
&\le \mathbb{E}_N\left[t\int_0^t\left(N^{-\frac{d}{2}}\sum_{i=1}^{M^d} G\left(\tfrac{y_i}{N}\right)\mathcal{L}^{ex,2}_{B_i}f_i(\xi_i(s))\right)^2 ds\right] \qquad (2.17)\\
&= t^2 E_{\nu^N_{\rho,\varrho}}\left[N^{-d}\left(\sum_{i=1}^{M^d} G\left(\tfrac{y_i}{N}\right)\mathcal{L}^{ex,2}_{B_i}f_i(\xi_i(s)\right)^2\right].
\end{aligned}
$$

*Observe that*

$$E_{\nu^N_{\rho,\varrho}}[\mathcal{L}^{ex,2}_{B_i}f_i] = 0$$

*and* $\mathcal{L}_{B_i} f$ *is independent of* $\mathcal{L}_{B_j} f$*, if* $i \neq j$*, as*

$$(\mathcal{L}^{ex,2}_{B_i} f)(\eta) = \frac{1}{N} \sum_{v \in \mathcal{V}} \sum_{x,z \in B_i} \eta(x,v)[1-\eta(z,v)]p(z-x,v)[f(\eta^{x,z,v}) - f(\eta)].$$

*Therefore, we have that* (2.17) *is bounded from above by*

$$\frac{t^2}{N^d} \|G\|^2_\infty \, M^d \, E_{\nu^N_{\rho,\varrho}}\big[\big(\mathcal{L}^{ex,2}_{B_1} f_1(\xi_1)\big)^2\big]. \tag{2.18}$$

*Now let us estimate* $E_{\nu^N_{\rho,\varrho}}\big[\big(\mathcal{L}^{ex,2}_{B_1} f\big)^2\big]$*. Let* $s_p$ *be the range of* $p$*, i.e.,* $p(x-z,v) \leq \mathbb{1}_{\{\|x-z\| \leq s_p\}}$*. We have that*

$$\begin{aligned}
&E_{\nu^N_{\rho,\varrho}}\left[\left(\mathcal{L}^{ex,2}_{B_1} f\right)^2\right] \\
&= E_{\nu^N_{\rho,\varrho}}\left[\left(\frac{1}{N} \sum_{v \in \mathcal{V}} \sum_{x,z \in B_1} \eta(x,v)[1-\eta(z,v)]p(z-x,v)[f(\eta^{x,z,v}) - f(\eta)]\right)^2\right] \\
&\leq \frac{1}{N^2} E_{\nu^N_{\rho,\varrho}}\left[\left(\sum_{v \in \mathcal{V}} \sum_{x,z \in B_1} \mathbb{1}_{\{\|x-z\| \leq s_p\}}[f(\eta^{x,z,v}) - f(\eta)]\right)^2\right] \\
&\leq \frac{2}{N^2} E_{\nu^N_{\rho,\varrho}}\left[\left(\sum_{v \in \mathcal{V}} \sum_{x,z \in B_1} \mathbb{1}_{\{\|x-z\| \leq s_p\}} f(\eta^{x,z,v})\right)^2\right] \\
&+ \frac{2}{N^2} E_{\nu^N_{\rho,\varrho}}\left[\left(\sum_{v \in \mathcal{V}} \sum_{x,z \in B_1} \mathbb{1}_{\{\|x-z\| \leq s_p\}} f(\eta)\right)^2\right].
\end{aligned}$$

*Denote by*

$$A = \{(v,x,z) \,;\, v \in \mathcal{V},\ x,z \in B_1 \text{ with } \|x-z\| \leq s_p\}.$$

*Consequently, the last display is bounded from above by*

$$\frac{2}{N^2} E_{\nu^N_{\rho,\varrho}}\left[\sum_{v \in \mathcal{V}} \sum_{x,z \in B_1} \mathbb{1}_{\{\|x-z\| \leq s_p\}}[f(\eta^{x,z,v})]^2\right] + \frac{2}{N^2} \,|A|^2 \, \|f\|^2_{L^2(\nu^N_{\rho,\varrho})}.$$

*Doing the change of variables* $\eta^{x,z,v} \mapsto \eta$*, we obtain*

$$\frac{2}{N^2} \,|A|\, E_{\nu^N_{\rho,\varrho}}\left[\sum_{v \in \mathcal{V}} \sum_{x,z \in B_1} \mathbb{1}_{\{\|x-z\| \leq s_p\}}[f(\eta)]^2 \frac{\gamma_{x,v}}{\gamma_{z,v}}\right] + \frac{2}{N^2} \,|A|^2 \, \|f\|^2_{L^2(\nu^N_{\rho,\varrho})}. \tag{2.19}$$

*Consider*

$$\begin{array}{rccc}
g \;:\; & (\mathbb{T}^d)^2 & \longrightarrow & \mathbb{R} \\
& (x,z) & \longmapsto & \dfrac{\gamma_{x,v}}{\gamma_{z,v}},
\end{array}$$

*and note that $g \in C^\infty$, and since $(\mathbb{T}^d)^2$ is compact, we conclude that $g$ is bounded. Consequently,*

$$(2.19) \leq \frac{2}{N^2}|A|^2 \|f\|^2_{L^2(\nu^N_{\rho,\varrho})} \|g\|_\infty + \frac{2}{N^2}|A|^2 \|f\|^2_{L^2(\nu^N_{\rho,\varrho})}$$

$$= \frac{2}{N^2}|A|^2 \|f\|^2_{L^2(\nu^N_{\rho,\varrho})} (\|g\|_\infty + 1)\,.$$

*Therefore, using this estimate in (2.18), we obtain that*

$$\begin{aligned}
&\mathbb{E}_N\left[\left(\int_0^t N^{-\frac{d}{2}} \sum_{i=1}^{M^d} G\left(\tfrac{y_i}{N}\right) \mathcal{L}^{ex,2}_{B_i} f_i(\xi_i(s))\right)^2 ds\right] \\
&\leq \frac{t^2}{N^d}\|G\|^2_\infty M^d\, 2\,(\|g\|_\infty + 1)\frac{|A|^2}{N^2}\|f\|^2_{L^2(\nu^N_{\rho,\varrho})} \\
&\leq \frac{C}{N^d}\frac{|A(l)|^2}{N^2}\frac{N^d}{l^d} \\
&= \frac{C}{l^d}\frac{|A|^2}{N^2} \longrightarrow 0,
\end{aligned}$$

*for fixed $l$ and $N \to +\infty$. This finishes the prof of Proposition 8.*

*Set*

$$\mathcal{L}^{sym}_{B_i} := \mathcal{L}^{ex,1}_{B_i} + \mathcal{L}^{c}_{B_i}. \tag{2.20}$$

*Thus, by Proposition 8, in order to show that*

$$\lim_{l\to+\infty}\lim_{N\to+\infty}\mathbb{E}_N\left[\left(\int_0^t N^{-\frac{d}{2}} \sum_{i=1}^{M^d} G\left(\tfrac{y_i}{N}\right) \mathcal{L}_{B_i} f_i(\xi_i(s))\right)^2 ds\right] = 0,$$

*it is enough to show that*

$$\lim_{l\to+\infty}\lim_{N\to+\infty}\mathbb{E}_N\left[\left(\int_0^t N^{-\frac{d}{2}} \sum_{i=1}^{M^d} G\left(\tfrac{y_i}{N}\right) \mathcal{L}^{sym}_{B_i} f_i(\xi_i(s))\right)^2 ds\right] = 0.$$

*By [1, Appendix A1, Proposition 6.1], we have*

$$\begin{aligned}
&\lim_{l\to+\infty}\lim_{N\to+\infty}\mathbb{E}_N\left[\left(\int_0^t N^{-\frac{d}{2}} \sum_{i=1}^{M^d} G\left(\tfrac{y_i}{N}\right) \mathcal{L}^{sym}_{B_i} f_i(\xi_i(s))\right)^2 ds\right] \\
&\leq 20t\langle V^{N,sym}_{G,f}, (-N^2\mathcal{L}^{sym}_N)^{-1} V^{N,sym}_{G,f}\rangle_{\nu^N_{\rho,\varrho}}
\end{aligned}$$

*where* $V^{N,sym}_{G,f} := N^{-\frac{d}{2}} \sum_{i=1}^{M^d} G\left(\frac{y_i}{N}\right) \mathcal{L}^{sym}_{B_i} f_i(\xi_i)$.

*Note that*

$$\langle \quad \cdot \quad , (-N^2 \mathcal{L}^{sym}_N)^{-1} \quad \cdot \quad \rangle_{\nu^N_{\rho,\varrho}}$$

*induces a norm on* $\mathscr{H}_{-1}$. *Therefore, we obtain*

$$20t \langle V^{N,sym}_{G,f}, (-N^2 \mathcal{L}^{sym}_N)^{-1} V^{N,sym}_{G,f} \rangle_{\nu^N_{\rho,\varrho}} =$$

$$20t \sup_h \left\{ 2 \int V^{N,sym}_{G,f}(\eta) h(\eta)\, d\nu^N_{\rho,\varrho} - N^2 \langle h, -\mathcal{L}^{sym}_N h \rangle_{\nu^N_{\rho,\varrho}} \right\},$$

*where the supremum is taken over all the functions* $h$ *in* $L^2(\nu^N_{\rho,\varrho})$. *Observe that*

$$2 \int V^{N,sym}_{G,f}(\eta) h(\eta)\, d\nu^N_{\rho,\varrho}$$

$$= 2\, N^{-\frac{d}{2}} \sum_{i=1}^{M^d} G\left(\tfrac{y_i}{N}\right) \int \mathcal{L}^{sym}_{B_i} f_i(\xi_i) h(\eta) d\nu^N_{\rho,\varrho}$$

$$\leq 2\, N^{-\frac{d}{2}} \sum_{i=1}^{M^d} G\left(\tfrac{y_i}{N}\right) \left[ \frac{1}{2\gamma_i} \langle -\mathcal{L}^{sym}_{B_i} f_i, f_i \rangle_{\nu^N_{\rho,\varrho}} + \frac{\gamma_i}{2} \langle -\mathcal{L}^{sym}_{B_i} h, h \rangle_{\nu^N_{\rho,\varrho}} \right]$$

*choosing* $\gamma_i = N^{2+\frac{d}{2}} |G\left(\frac{y_i}{N}\right)|^{-1} \mathbb{1}_{\left\{G\left(\frac{y_i}{N}\right) \neq 0\right\}}$, *we obtain that last display is bounded from above by*

$$2\, N^{-\frac{d}{2}} \sum_{i=1}^{M^d} G\left(\tfrac{y_i}{N}\right) \left[ \frac{|G\left(\frac{y_i}{N}\right)|}{2N^{2+\frac{d}{2}}} \langle -\mathcal{L}^{sym}_{B_i} f_i, f_i \rangle_{\nu^N_{\rho,\varrho}} + \frac{N^{2+\frac{d}{2}}}{2|G\left(\frac{y_i}{N}\right)|} \mathbb{1}_{\left\{G\left(\frac{y_i}{N}\right) \neq 0\right\}} \langle -\mathcal{L}^{sym}_{B_i} h, h \rangle_{\nu^N_{\rho,\varrho}} \right].$$

*Since* $\langle -\mathcal{L}^{sym}_{B_i} h, h \rangle \geq 0$, *we have that*

$$\leq \sum_{i=1}^{M^d} \frac{sgn(G\left(\frac{y_i}{N}\right))\, G\left(\frac{y_i}{N}\right)^2}{N^{d+2}} \langle -\mathcal{L}^{sym}_{B_i} f_i, f_i \rangle_{\nu^N_{\rho,\varrho}} + N^2 \sum_{i=1}^{M^d} sgn(G\left(\tfrac{y_i}{N}\right)) \langle -\mathcal{L}^{sym}_{B_i} h, h \rangle_{\nu^N_{\rho,\varrho}}$$

$$\leq \frac{\|G\|^2_\infty M^d}{N^{d+2}} \langle -\mathcal{L}^{sym}_{B_1} f_1, f_1 \rangle_{\nu^N_{\rho,\varrho}} + N^2 \sum_{i=1}^{M^d} \langle -\mathcal{L}^{sym}_{B_i} h, h \rangle_{\nu^N_{\rho,\varrho}}$$

$$\leq \frac{\|G\|^2_\infty M^d}{N^{d+2}} \langle -\mathcal{L}^{sym}_{B_1} f_1, f_1 \rangle_{\nu^N_{\rho,\varrho}} + N^2 \langle -\mathcal{L}^{sym}_N h, h \rangle_{\nu^N_{\rho,\varrho}}.$$

*Therefore,*

$$
\begin{aligned}
&20t \sup_h \left\{ 2 \int V_{G,f}^{N,sym}(\eta) h(\eta)\, d\nu_{\rho,\varrho}^N - N^2 \langle h, -\mathcal{L}_N^{sym} h \rangle_{\nu_{\rho,\varrho}^N} \right\} \\
&\le \frac{20t\, \|G\|_\infty^2\, M^d}{N^{d+2}} \langle -\mathcal{L}_{B_1}^{sym} f_1, f_1 \rangle_{\nu_{\rho,\varrho}^N} \\
&= \frac{20t\, \|G\|_\infty^2\, N^d}{l^d\, N^{d+2}} \langle -\mathcal{L}_{B_1}^{sym} f_1, f_1 \rangle_{\nu_{\rho,\varrho}^N}
\end{aligned}
$$

*which vanishes as* $N \to +\infty$.

*Observe that, since* $(x+y)^2 \le 2x^2 + 2y^2$, *we have that*

$$
\begin{aligned}
&\mathbb{E}_N \left[ \left( \int_0^t N^{-\frac{d}{2}} \sum_{i=1}^{M^d} G\left(\tfrac{y_i}{N}\right) \sum_{x \in B_i^0} \tau_x V_F^{j,k}(\eta) ds \right)^2 \right] \le \\
&2\, \mathbb{E}_N \left[ \left( \int_0^t N^{-\frac{d}{2}} \sum_{i=1}^{M^d} G\left(\tfrac{y_i}{N}\right) \Big( \sum_{x \in B_i^0} \tau_x V_F^{j,k}(\eta) - \mathcal{L}_{B_i} f_i(\xi_i) \Big) ds \right)^2 \right] + \\
&+2\, \mathbb{E}_N \left[ \left( \int_0^t N^{-\frac{d}{2}} \sum_{i=1}^{M^d} G\left(\tfrac{y_i}{N}\right) \mathcal{L}_{B_i} f_i(\xi_i) ds \right)^2 \right].
\end{aligned}
\tag{2.21}
$$

**Claim 4.** *By the inequality* (2.21) *to prove*

$$
\lim_{l \to +\infty} \lim_{N \to +\infty} \mathbb{E}_N \left[ \left( \int_0^t N^{-\frac{d}{2}} \sum_{i=1}^{M^d} G\left(\tfrac{y_i}{N}\right) \sum_{x \in B_i^0} \tau_x V_F^{j,k}(\eta) ds \right)^2 \right] = 0.
$$

*It is enought to show that*

$$
\lim_{l \to +\infty} \inf_f \lim_{N \to +\infty} \mathbb{E}_N \left[ \left( \int_0^t N^{-\frac{d}{2}} \sum_{i=1}^{M^d} G\left(\tfrac{y_i}{N}\right) \Big( \sum_{x \in B_i^0} \tau_x V_F^{j,k}(\eta) - \mathcal{L}_{B_i} f_i(\xi_i) \Big) ds \right)^2 \right] = 0.
$$

*where the infimum is taken over all the functions* $f$ *in* $L^2(\nu_{\rho,\varrho}^N)$ *measurable with respect to* $\sigma(\eta(x), x \in B_1)$ *and* $f_i$ *stands for the translation of* $f$ *that makes it measurable with respect to* $\sigma(\eta(x), x \in B_i)$.

*Proof of the claim 4: In fact, using Proposition 8, we obtain that*

$$\mathbb{E}_N\left[\left(\int_0^t N^{-\frac{d}{2}}\sum_{i=1}^{M^d} G\left(\tfrac{y_i}{N}\right)\Big(\sum_{x\in B_i^0}\tau_x V_F^{j,k}(\eta)-\mathcal{L}_{B_i}f_i(\xi_i)\Big)ds\right)^2\right]$$

$$\leq 2\,\mathbb{E}_N\left[\left(\int_0^t N^{-\frac{d}{2}}\sum_{i=1}^{M^d} G\left(\tfrac{y_i}{N}\right)\Big(\sum_{x\in B_i^0}\tau_x V_F^{j,k}(\eta)-\mathcal{L}^{sym}_{B_i}f_i(\xi_i)\Big)ds\right)^2\right]$$

$$+2\,\mathbb{E}_N\left[\left(\int_0^t N^{-\frac{d}{2}}\sum_{i=1}^{M^d} G\left(\tfrac{y_i}{N}\right)\mathcal{L}^{ex,2}_{B_i}f_i(\xi_i)ds\right)^2\right].$$

*Finally, to prove the claim it is enough show that*

$$\mathbb{E}_N\left[\left(\int_0^t N^{-\frac{d}{2}}\sum_{i=1}^{M^d} G\left(\tfrac{y_i}{N}\right)\Big(\sum_{x\in B_i^0}\tau_x V_F^{j,k}(\eta)-\mathcal{L}^{sym}_{B_i}f_i(\xi_i)\Big)ds\right)^2\right]\longrightarrow 0,$$

*when $N\to\infty$. From the Cauchy-Schwarz inequality and Fubini's theorem, we set*

$$\begin{aligned}
&\mathbb{E}_N\left[\left(\int_0^t N^{-\frac{d}{2}}\sum_{i=1}^{M^d} G\left(\tfrac{y_i}{N}\right)\Big(\sum_{x\in B_i^0}\tau_x V_F^{j,k}(\eta)-\mathcal{L}^{sym}_{B_i}f_i(\xi_i)\Big)ds\right)^2\right]\\
&\leq t\int_0^t N^{-d}\,\mathbb{E}_N\left[\left(\sum_{i=1}^{M^d} G\left(\tfrac{y_i}{N}\right)\Big(\sum_{x\in B_i^0}\tau_x V_F^{j,k}(\eta)-\mathcal{L}^{sym}_{B_i}f_i(\xi_i)\Big)\right)^2\right]ds \qquad (2.22)\\
&= t^2N^{-d}\,E_{\nu^N_{\rho,\varrho}}\left[\left(\sum_{i=1}^{M^d} G\left(\tfrac{y_i}{N}\right)\Big(\sum_{x\in B_i^0}\tau_x V_F^{j,k}(\eta)-\mathcal{L}^{sym}_{B_i}f_i(\xi_i)\Big)\right)^2\right]
\end{aligned}$$

*since the support of $\tau_x V_F^{j,k}(\eta)-\mathcal{L}^{sym}_{B_i}f_i$ and $\tau_y V_F^{j,k}(\eta)-\mathcal{L}^{sym}_{B_\ell}f_\ell$ are disjoints for $x\in B_i^0, y\in B_\ell^0$, $i\neq\ell$. Last display is equal to*

$$\begin{aligned}
&t^2N^{-d}\,E_{\nu^N_{\rho,\varrho}}\left[\sum_{i=1}^{M^d}\Big(G\left(\tfrac{y_i}{N}\right)\Big)^2\Big(\sum_{x\in B_i^0}\tau_x V_F^{j,k}(\eta)-\mathcal{L}^{sym}_{B_i}f_i(\xi_i)\Big)^2\right]\\
&\leq t^2N^{-d}\|G\|^2_\infty M^d\,E_{\nu^N_{\rho,\varrho}}\left[\Big(\sum_{x\in B_1^0}\tau_x V_F^{j,k}(\eta)-\mathcal{L}^{sym}_{B_1}f_1(\xi_1)\Big)^2\right] \qquad (2.23)\\
&\leq t^2\,\|G\|^2_\infty\, l^{-d}\,E_{\nu^N_{\rho,\varrho}}\left[\Big(\sum_{x\in B_1^0}\tau_x V_F^{j,k}(\eta)-\mathcal{L}^{sym}_{B_1}f_1(\xi_1)\Big)^2\right].
\end{aligned}$$

*Denote by* $R(\mathcal{L}^{sym}_{B_1})$ *the range of the generator* $\mathcal{L}^{sym}_{B_1}$ *in* $L^2(\nu^{B_1}_{\rho,\varrho})$ *and* $R^{\perp}(\mathcal{L}^{sym}_{B_1})$ *the orthogonal space to* $R(\mathcal{L}^{sym}_{B_1})$*. Fix a measurable function* $h \in \sigma\big(\eta(x)\,;\, x \in B_1\big)$*. The formula*

$$\inf_{f\in L^2(\nu^{B_1}_{\rho,\varrho})} E_{\nu^{B_1}_{\rho,\varrho}}\left[(h-\mathcal{L}^{sym}_{B_1} f)^2\right]$$

*corresponds to the projection of* $h$ *on* $R^{\perp}(\mathcal{L}^{sym}_{B_1})$*. Consider*

$$\mathscr{V}_{B_1} = \{I^{B_1}(0,\eta)\,;\eta \in (\{0,1\}^{\mathcal{V}})^{B_1}\}$$

*where* $I^{B_1}(x) = \frac{1}{|\Lambda_{B_1}|}\sum_{z\in\Lambda_{B_1}} I(\eta_z)$*. The set* $\mathscr{V}_{B_1}$ *is the set of all the possible values of* $I^{B_1}(y_1)$*. Denote*

$$\mathcal{H}_{B_1}(i) = \{\eta \in (\{0,1\}^{\mathcal{V}})^{B_1};\, I^{B_1}(y_1) = i\}$$

$$\nu_{B_1,i}(\cdot) = \nu^{B_1}_{\rho,\varrho}(\cdot\,|I^{B_1}=i)$$

$$\mathcal{M}^0_{B_1,i} = \{f \in L^2(\nu_{B_1,i}); E_{\nu_{B_1,i}}[f]=0\}.$$

*Note that* $\mathcal{M}^0_{B_1,i}$ *has codimension 1 and* $R(\mathcal{L}^{sym}_{B_1})$ *is a subset of* $\mathcal{M}^0_{B_1,i}$*. Since* $\nu^{B_1}_{\rho,\varrho}$ *is invariant for the dynamics generated by* $\mathcal{L}^{sym}_{B_1}$*, and since the conserved quantities by the dynamics are the mass and the momentum,* $\nu_{B_1,i}$ *is invariant by dynamics generated by* $\mathcal{L}^{sym}_{B_1}$*. On the other hand, the kernel of* $\mathcal{L}^{sym}_{B_1}$ *reduces to the constant functions since* $\mathcal{L}^{sym}_{B_1} f = 0$ *implies that* $\langle f, (-\mathcal{L}^{sym}_{B_1})f\rangle_{\nu_{B_1,i}} = 0$ *that in turn forces* $f$ *to be constant.*[6] *Consequently,*

$$\dim\ker\mathcal{L}^{sym}_{B_1} = 1.$$

*And thus* $R(\mathcal{L}^{sym}_{B_1})$ *has codimension 1 because* $R(\mathcal{L}^{sym}_{B_1}) \subset \mathcal{M}^0_{B_1,i}$*. Since* $\mathcal{M}^0_{B_1,i}$ *has codimension 1, follows that* $R(\mathcal{L}^{sym}_{B_1}) = \mathcal{M}^0_{B_1,i}$*. Observe that* $\mathcal{M}^0_{B_1,i}$ *is the space of orthogonal functions to constant functions in* $L^2(\nu_{B_1,i})$*, i.e,*

$$f \perp 1 \Leftrightarrow \langle f,1\rangle_{L^2(\nu_{B_1,i})} = 0 \Leftrightarrow E_{\nu_{B_1,i}}[f]=0.$$

*Therefore,* $[\mathcal{M}^0_{B_1,i}]^{\perp}$ *= constant functions in* $L^2(\nu_{B_1,i})$*. Thus,* $R(\mathcal{L}_{B_1})^{\perp}$ *= constant functions in* $L^2(\nu_{B_1,i})$*.* $R(\mathcal{L}_{B_1})^{\perp}$ *consists of all functions that depends on the configuration* $\eta$ *only through its the vector mass and momentum* $i$*. In particular, the infimum*

[6] In Appendix B, we prove that $\langle -\mathcal{L}^c_N f, f\rangle_{\nu^N_{\rho,\varrho}}$ is nonnegative. Using this, we have that $\langle -\mathcal{L}^{ex,1}_N f, f\rangle_{\nu^N_{\rho,\varrho}} + \langle f, -\mathcal{L}^c f\rangle_{\nu^N_{\rho,\varrho}} = 0$ implies $\langle -\mathcal{L}^{ex,1}_N f, f\rangle_{\nu^N_{\rho,\varrho}} = 0$ for $\nu$-almost every $\eta$, $f(\eta) = f(\eta^{x,z,v})$ for all $x, z \in \mathbb{T}^d_N$ and $v \in \mathcal{V}$. Thus, $f$ is almost surely constant.

*over all $f \in \nu_{\rho,\varrho}^{B_1}$ of the expression (2.22) is equal to* [7]

$$\begin{aligned} &\inf_{f\in L^2(\nu_{\rho,\varrho}^{B_1})} t^2 l^{-d}\|G\|_\infty^2 E_{\nu_{\rho,\varrho}^{B_1}}\Big[\big(\sum_{x\in B_1^0}\tau_x V_F^{j,k} - \mathcal{L}_{B_1}^{sym} f_1\big)^2\Big] \\ &= \inf_{g\in R(\mathcal{L}_{B_1}^{sym})} t^2 l^{-d}\|G\|_\infty^2 E_{\nu_{\rho,\varrho}^{B_1}}\Big[\big(\sum_{x\in B_1^0}\tau_x V_F^{j,k} - g\big)^2\Big] \\ &= t^2 l^{-d}\|G\|_\infty^2 E_{\nu_{\rho,\varrho}^{B_1}}\Big[\Big(E_{\nu_{\rho,\varrho}^{B_1}}[\sum_{x\in B_1^0}\tau_x V_F^{j,k}\,|\, I^{B_1}(y_1)]\Big)^2\Big]. \end{aligned} \tag{2.24}$$

*Note that $I^{B_1}(y_1)$ is the vector average of mass and momentum on $B_1$. For $x \in B_1^0$, $\tau_x V_F^{j,k}$ depends only on $B_1$, and since $\nu_{\rho,\varrho}^{B_1}$ is homogeneous, we have that*

$$E_{\nu_{\rho,\varrho}^{B_1}}\Big[\tau_x\Psi | I^{B_1}(y_1)\Big]$$

*does not depend on $x$, above $\Psi := \sum_{v\in\mathcal{V}} v_k \sum_{z\in\mathbb{Z}^d} z_j\, p(z,v)\eta(0,v)(1-\eta(z,v))$. Define,*

$$\tilde\Psi_l(I^{B_1}(y_1)) := E_{\nu_{\rho,\varrho}^{B_1}}[\tau_x\Psi | I^{B_1}(y_1)]$$

*and $\tilde\Psi(\rho,\varrho) = E_{\nu_{\rho,\varrho}}[\Psi]$. We can rewrite (2.24) as*

$$t^2\frac{|B_1^0|^2}{l^d}\|G\|_\infty^2 E_{\nu_{\rho,\varrho}^{B_1}}\Big[\Big(\tilde\Psi_l\big(I^{B_1}(y_1)\big) - \tilde\Psi(\rho,\varrho) - \sum_{i=0}^d \frac{\partial\tilde\Psi}{\partial\rho_i}(\rho,\varrho)\big(I_i^{B_1}(y_1) - \rho^i\big)\Big)^2\Big]. \tag{2.25}$$

*Denote by $\ell$ the range of the function $\Psi$. By construction, denote by $\Lambda_\ell = \{-\ell,\dots,\ell\}^d$, and note that $\Lambda_\ell \subset B_1$. Using Equivalence of ensembles (see **??**), we have that*

$$|\tilde\Psi_l(I^{B_1}(y_1)) - \tilde\Psi(I^{B_1}(y_1))| \le \frac{C(\ell,\nu)}{|B_1|}\langle\Psi,\Psi\rangle^{\frac12}_{\mu^\ell_{I^{B_1}(y_1)}}. \tag{2.26}$$

*We can bound*

$$|\tilde\Psi_l(I^{B_1}(y_1)) - \tilde\Psi(I^{B_1}(y_1))| \le \frac{C(\ell,\nu)}{|B_1|}C(\Psi).$$

*Therefore, (2.25) is bounded from above by*

$$2t^2\frac{|B_1^0|^2}{l^d}\|G\|_\infty^2 E_{\nu_{\rho,\varrho}^{B_1}}\Big[\Big(\tilde\Psi_l\big(I^{B_1}(y_1)\big) - \tilde\Psi(I^{B_1}(y_1))\Big)^2\Big] \tag{2.27}$$

$$+2t^2\frac{|B_1^0|^2}{l^d}\|G\|_\infty^2 E_{\nu_{\rho,\varrho}^{B_1}}\Big[\Big(\tilde\Psi\big(I^{B_1}(y_1)\big) - \tilde\Psi(\rho,\varrho) - \sum_{j=0}^d \frac{\partial\tilde\Psi}{\partial\rho_i}(\rho,\varrho)\big(I_j^{B_1}(y_1) - \rho_j\big)\Big)^2\Big]. \tag{2.28}$$

[7] Let $X$ be a Hilbert space and $A \subset X$ a closed subset. Then, $\inf_{g\in A}\|f-g\|^2 = \|f_{A^\perp}\|^2$.

*From* (2.26)*, we can see there exists a constant* $C(\ell,\nu,\Psi)$ *such that* (2.27) *is bounded from above by*

$$2t^2\frac{|B_1^0|^2}{l^d}\,\|G\|_\infty^2\,\frac{C(\ell,\nu,\Psi)}{l^{2d}}$$

$$=\frac{2t^2\|G\|_\infty^2 C(\ell,\nu,\Psi)}{l^d}\xrightarrow{l\to\infty}0.$$

*Now, let us estimate* (2.28)*. In order to do that define*

$$\mathsf{K}\big((\rho,\varrho),I_j^{B_1}(y_1)\big):=\big|\tilde\Psi(I^{B_1}(y_1))-\tilde\Psi(\rho,\varrho)-\sum_{j=0}^{d}\frac{\partial\tilde\Psi}{\partial\rho_j}(\rho,\varrho)(I_j^{B_1}(y_1)-\rho^j)\big|^2.$$

*Note that* $\mathsf{K}((\rho,\varrho),I_j^{B_1}(y_1))$ *is bounded, and observe that for* $(\rho,\varrho)$ *fixed,* $\frac{\partial\tilde\Psi}{\partial\rho_j}(\rho,\varrho)$ *is bounded since the set of velocities is finite.*

**Remark 5.** *By the classical large deviations theorem for Bernoulli, if* $0<\varepsilon<\frac{1}{4}d((\rho,\varrho),\partial\mathscr{V})$ *there exists constants* $C(\varepsilon)>0$ *and* $m(\varepsilon)>0$ *such that*

$$P(\|I^{B_1}(y_1)-\rho\|>\varepsilon)\le C(\varepsilon)\exp\{-l^d m(\varepsilon)\}. \tag{2.29}$$

*We can split* (2.28) *when* $\mathbb{1}_{\{\|I^{B_1}(y_1)-\rho\|>\varepsilon\}}$ *and* $\mathbb{1}_{\{\|I^{B_1}(y_1)-\rho\|\le\varepsilon\}}$*. Using Remark* 5*, we obtain that* (2.28) *vanishes as* $\ell\to\infty$

$$2t^2\frac{|B_1^0|^2}{l^d}\|G\|_\infty^2 E_{\nu_{\rho,\varrho}^{B_1}}\left[\left(\mathsf{K}(\rho,\varrho),I_j^{B_1}(y_1)\right)\mathbb{1}_{\{\|I^{B_1}(y_1)-\rho\|>\varepsilon\}}\right]\longrightarrow 0 \tag{2.30}$$

*when* $l\longrightarrow+\infty$.

*To finish the proof is enough to show that*

$$2t^2\frac{|B_1^0|^2}{l^d}\|G\|_\infty^2 E_{\nu_{\rho,\varrho}^{B_1}}\left[\left(\mathsf{K}(\rho,\varrho),I_j^{B_1}(y_1)\right)\mathbb{1}_{\{\|I^{B_1}(y_1)-\rho\|\le\varepsilon\}}\right]\xrightarrow{l\to+\infty}0.$$

*Using Taylor's expansion up to second order on* $\tilde\Psi$ *and the fact that* $\tilde\Psi\in C^\infty$ *in the compact ball* $\overline{B\big((\rho,\varrho),\varepsilon\big)}$*, there exists a constant* $C(\varepsilon,\tilde\Psi(\rho,\varrho))$ *such that the last display becomes bounded from above by*

$$2t^2\frac{|B_1^0|^2}{l^d}\|G\|_\infty^2 C(d,\varepsilon,\tilde\Psi(\rho,\varrho))\sum_{j=0}^{d}E_{\nu_{\rho,\varrho}^{B_1}}\left[\left(I_j^{B_1}(y_1)-\rho_j\right)^4\right]$$

$$\le 2t^2\frac{|B_1^0|^2}{l^d}\|G\|_\infty^2\tilde C(d,\varepsilon,\tilde\Psi(\rho,\varrho))l^{-2d}$$

$$\le 2t^2\frac{l^{2d}}{l^d}\|G\|_\infty^2\tilde C(d,\varepsilon,\tilde\Psi(\rho,\varrho))l^{-2d}$$

*and vanishes as $l \longrightarrow +\infty$. This concludes the proof of the Boltzmann-Gibbs principle.* □

## 2.7 Convergence at initial time

*For $t \geq 0$, let $\mathcal{F}_t$ be the $\sigma$-algebra on $\mathcal{D}([0,T], \mathscr{H}_{-p_0})$ generated by $Y_s(H)$ for $s \leq t$ and $H$ in $C^\infty(\mathbb{T}^d)$ and set $\mathcal{F} = \sigma\left(\cup_{t\geq 0}\mathcal{F}_t\right)$.*

**Lemma 9.** *For every continuous function $H : \mathbb{T}^d \to \mathbb{R}$ and every $t > 0$,*

$$\lim_{N\to+\infty} \log \mathbb{E}_N \left[\exp\{iw \cdot Y_t(H)\}\right] = -w^T \frac{\mathrm{X}(\rho,\varrho)}{2} w\langle H, H\rangle,$$

*where $\mathrm{X}(\rho,\varrho) = [v_j v_k \chi(\theta_v(\Lambda(\rho,\varrho)))]_{j,k}$ is a $(d+1)\times(d+1)$ matrix, with $(j,k)$-th entry is given by $[v_j v_k \chi(\theta_v(\Lambda(\rho,\varrho)))]$.*

*Proof.* Since $\nu^N_{\rho,\varrho}$ is an invariant product measure,

$$\begin{aligned}
&\log \mathbb{E}_N \left[\exp\{iw \cdot Y_t(H)\}\right] \\
&= \log \mathbb{E}_N \left[\exp\{i \sum_{j=0}^{d} w_j Y_t^j(H)\}\right] \\
&= \log \mathbb{E}_{\nu^N_{\rho,\varrho}} \left[\exp\left\{ i \sum_{j=0}^{d} w_j \sum_{x\in\mathbb{T}^d_N} N^{-\frac{d}{2}} H\left(\tfrac{x}{N}\right)\left(I^j(\eta_x(t)) - \rho_j\right)\right\}\right] \\
&= \sum_{x\in\mathbb{T}^d_N} \log \mathbb{E}_{\nu^N_{\rho,\varrho}} \left[\exp\left\{ i \sum_{j=0}^{d} w_j N^{-\frac{d}{2}} H\left(\tfrac{x}{N}\right)\left(I^j(\eta_x(t)) - \rho_j\right)\right\}\right] \\
&= \sum_{x\in\mathbb{T}^d_N} \log \mathbb{E}_{\nu^N_{\rho,\varrho}} \left[\exp\left\{ N^{-\frac{d}{2}} H\left(\tfrac{x}{N}\right) i \sum_{j=0}^{d} w_j \left(I^j(\eta_x(t)) - \rho_j\right)\right\}\right] \\
&= \sum_{x\in\mathbb{T}^d_N} -\frac{N^{-d}}{2} H^2\left(\tfrac{x}{N}\right)\, s\left(\sum_{j=0}^{d} w_j I^j(\eta_x(t))\right) + O(N^{-\frac{3d}{2}}) \\
&= \sum_{x\in\mathbb{T}^d_N} -\frac{H^2\left(\tfrac{x}{N}\right)}{2N^d} \sum_{j=0}^{d}\sum_{k=0}^{d} w_j w_k\, cov\left(I^j(\eta_x), I^k(\eta_x)\right) + O(N^{-\frac{3d}{2}}) \\
&= \sum_{x\in\mathbb{T}^d_N} -\frac{H^2\left(\tfrac{x}{N}\right)}{2N^d} \sum_{j=0}^{d}\sum_{k=0}^{d} w_j w_k v_j v_k \chi(\theta_v(\Lambda(\rho,\varrho))) + O(N^{-\frac{3d}{2}}).
\end{aligned}$$

Since

$$
\begin{aligned}
&cov\left(I^j(\eta_x), I^k(\eta_x)\right) = \\
&= E\left[\left(\sum_{v\in\mathcal{V}} v_k\eta(x,v) - E(\sum_{v\in\mathcal{V}} v_k\eta(x,v))\right)\left(\sum_{\tilde{v}\in\mathcal{V}} \tilde{v}_k\eta(x,\tilde{v}) - E(\sum_{\tilde{v}\in\mathcal{V}} \tilde{v}_k\eta(x,\tilde{v}))\right)\right] \\
&= \sum_{v\in\mathcal{V}} v_j v_k \chi(\theta_v(\Lambda(\rho,\varrho))),
\end{aligned}
$$

then, we have that

$$
\lim_{N\to+\infty} \log \mathbb{E}_N\left[\exp\{iw\cdot Y_t(H)\}\right] = -w^T\frac{\mathrm{X}(\rho,\varrho)}{2}w\langle H,H\rangle.
$$

□

**Corollary 4.** *Restricted to $\mathcal{F}_0$, $\mathbb{Q}$ is a Gaussian field with covariance given by*

$$
E_{\mathbb{Q}}[Y_0^j(G)Y_0^k(H)] = \frac{v_j v_k\chi(\theta_v(\Lambda(\rho,\varrho)))}{2}\langle H,G\rangle.
$$

*Proof.* Fix a positive integer $n, \theta\in\mathbb{R}^n$ and $H_1,\dots,H_n$ in $\mathscr{H}_{p_0}$. Since $Y_0^j, Y_0^k$ are linear, and since, by assumption, $\mathbb{Q}_N$ converges weakly to $\mathbb{Q}$, by the previous lemma,

$$
\begin{aligned}
&\log \mathbb{E}_{\mathbb{Q}}\left[\exp\left\{i\sum_{k=0}^{d} w_k\sum_{j=1}^{n}\theta_j Y_0^k(H_j)\right\}\right] \\
&= \log \mathbb{E}_{\mathbb{Q}}\left[\exp\left\{i\sum_{k=0}^{d} w_k Y_0^k(\sum_{j=1}^{n}\theta_j H_j)\right\}\right] \\
&= -w^T\frac{\mathrm{X}(\rho,\varrho)}{2}w\left\langle\sum_{j=1}^{n}\theta_j H_j, \sum_{j=1}^{n}\theta_j H_j\right\rangle.
\end{aligned}
\tag{2.31}
$$

□

## 2.8 Proof of Proposition 7

*Before proving Proposition 7, we recall the following results.*

**Proposition 9.** *Let $\{M_t^n;\ t\in[0,T]\}_{n\in\mathbb{N}}$ be a sequence of martingales converging in distribution to some process $\{M_t;\ t\in[0,T]\}$ as $n\to\infty$. If the sequence of random variables $\{M_t : t\in[0,T], n\in\mathbb{N}\}$ is uniformly integrable, then $\{M_t;\ t\in[0,T]\}$ is a martingale.*

*Also the criterion of uniformly integrability:*

**Proposition 10.** *If $U_\alpha$ is a sequence of random variables integrables and*

$$\sup_\alpha \|U_\alpha\|_{L^p} < +\infty,$$

*for some $p > 1$. Then, $\{U_\alpha\}$ is uniformly integrable.*

*Proof.* Note that $\int |U_\alpha \cdot \mathbb{1}_{A_n}| \, dP \longrightarrow 0$ as $n \to +\infty$. Since by Hölder's inequality, we have that

$$\begin{aligned}\int |U_\alpha \cdot \mathbb{1}_{A_n}| \, dP &\leq \|U_\alpha\|_{L^p} \cdot \|\mathbb{1}_{A_n}\|_{L^q} \\ &= \|U_\alpha\|_{L^p} \cdot (P(A_n))^{\frac{1}{q}} \\ &\leq M \cdot (P(A_n))^{\frac{1}{q}}\end{aligned}$$

due to $\sup_\alpha \|U_\alpha\| \leq M$. □

*We will use Propositions 9 and 10 to prove that $M_t(H)$ is a martingale. By Proposition 9, we need to show that $M_t(H)$ is uniformly integrable. To prove that, we will use the Proposition 10, with $p = 2$, then, we need to show that*

$$\sup_{N,t} E[(M_t(H))^2] < +\infty.$$

*Note that $\{M_t^{N,k}(H)\}$ is uniformly integrable:*

$$\mathbb{E}[(M_t^{N,k}(H))^2] =$$

$$\mathbb{E}\left[N_t^{N,k}(H) + \int_0^t \frac{1}{2N^d} \sum_{v\in\mathcal{V}} \sum_{x\in\mathbb{T}_N^d} \sum_{j=1}^d v_k^2(\eta_s(x,v) - \eta_s(x+e_j,v))^2 (\partial_j^N H\left(\tfrac{x}{N}\right))^2 ds\right]$$

$$= \mathbb{E}\Big[N_t^{N,k}(H)\Big] + \mathbb{E}\Big[\int_0^t \frac{1}{2N^d} \sum_{v\in\mathcal{V}} \sum_{x\in\mathbb{T}_N^d} \sum_{j=1}^d v_k^2(\eta_s(x,v) - \eta_s(x+e_j,v))^2 (\partial_j^N H\left(\tfrac{x}{N}\right))^2 ds\Big]$$

$$= \frac{1}{2N^d} \int_0^t \sum_{v\in\mathcal{V}} \sum_{x\in\mathbb{T}_N^d} \sum_{j=1}^d v_k^2 \mathbb{E}\Big[(\eta_s(x,v) - \eta_s(x+e_j,v))^2\Big] (\partial_j^N H\left(\tfrac{x}{N}\right))^2 ds$$

$$\leq T\langle \nabla H, \nabla H\rangle \sum_{v\in\mathcal{V}} v_k^2\, \chi(\theta_v(\Lambda(\rho,\varrho))),$$

*which is bounded. Therefore, $\{M_t^{N,k}(H)\}$ is uniformly integrable.*

**Remark 6.** *If $M_t^N$ converges weakly to $M_t$ in Skorohod topology then from Skorohod representation Theorem, we have that exist*

$$W_t^N \overset{d}{=} M_t^N \quad \textit{and} \quad W_t \overset{d}{=} M_t$$

*such that*

$$W_t^N \to W_t \quad \textit{almost surely.}$$

*Since uniformly integrable depends only on the distribution, we have that* $\{W_t^N\}$ *is uniformly integrable. Then,*

$$W_t^N \xrightarrow{L^1} W_t.$$[8]

*Since* $W_t^N$ *is a martingale with respect to the natural filtration* $\mathcal{G}_t = \sigma\{\eta(s);\, s \leq t\}$, *we have that for any* $A \in \mathcal{G}_s$, $s \leq t$

$$E[W_t^N \mathbb{1}_A] = E[W_s^N \mathbb{1}_A].$$

*Since* $\mathbb{1}_A$ *is bounded and* $W_t^N \xrightarrow{L^1} W_t$, *then*

$$E[W_t^N \mathbb{1}_A] \to E[W_t \mathbb{1}_A].$$

*On the other hand,*

$$E[W_t^N \mathbb{1}_A] = E[W_s^N \mathbb{1}_A] \to E[W_s \mathbb{1}_A].$$

*This implies that*

$$E[W_t | \mathcal{G}_s] = W_s,$$

*which means that* $W$ *is a martingale. Since* $W$ *and* $M$ *have the same distribution,* $M$ *is a martingale. In fact, for any* $A \in \mathcal{G}_s$, *we have*

$$E[M_s \mathbb{1}_A] = E[W_s \mathbb{1}_A] = E[W_t \mathbb{1}_A] = E[M_t \mathbb{1}_A],$$

*which implies that* $E[M_t|\mathcal{G}_s] = M_s$.

## 2.9 Tightness

*We prove in this section that the sequence of probability measures* $(\mathbb{Q}_N)_{N\geq 1}$ *is tight and all limit points are concentrated on continuous paths. We first review some aspects of the uniform weak topology on* $\mathcal{D}([0,T], \mathscr{H}_{-p})$ *introduced in the beginning of the chapter. Throughout this section* $p$ *stands for a positive integer satisfying*

$$p > 2 + \frac{d}{2}. \tag{2.32}$$

*For* $\delta > 0$ *and a path* $Y$ *in* $\mathcal{D}([0,T], \mathscr{H}_{-p})$ *define the modulus of continuity* $w_\delta(Y)$ *by*

$$w_\delta(Y) = \sup_{\substack{|s-t|\leq\delta \\ 0\leq s,t\leq T}} \|Y_t - Y_s\|_{-p}.$$

[8] This result follows from Vitali's convergence Theorem

*To check the definition of $\|\cdot\|_{-p}$ see 2.6.*

*The first result provides sufficient conditions for a subset to be weakly relatively compact.*

**Lemma 10.** *A subset $A$ of $\mathcal{D}([0,T],\mathscr{H}_{-p})$ is relatively compact for the uniform weak topology if*

*(i)* $\sup_{Y\in A}\sup_{0\leq t\leq T}\|Y_t\|_{-p}<\infty$

*(ii)* $\lim_{\delta\to 0}\sup_{Y\in A} w_\delta(Y)=0.$

*From this lemma we deduce a criterion for tightness of a sequence of probability measures $P_N$ defined on $\mathcal{D}([0,T],\mathscr{H}_{-p})$.*

**Lemma 11.** *A sequence $\{P_N,\ N\geq 1\}$ of probability measures defined on $\mathcal{D}([0,T],\mathscr{H}_{-p})$ is tight if for every $0\leq t\leq T$,*

$$\lim_{A\to\infty}\limsup_{N\to\infty} P_N\left[\sup_{0\leq t\leq T}\|Y_t\|_{-p}>A\right]=0$$

*and*

$$\lim_{\delta\to 0}\limsup_{N\to\infty} P_N[w_\delta(Y)\geq\varepsilon]=0$$

*for every $\varepsilon>0$.*

*We have now all elements to prove tightness of the sequence $(\mathbb{Q}_N)_N$ introduced in the beginning of the chapter.*

**Proposition 11.** *The sequence of probability measures $\mathbb{Q}_N$ is tight. Moreover, all limit points are concentrated on continuous paths.*

*The proof of this proposition is divided in several lemmas. We start with a key estimate. For each $z\in\mathbb{Z}^d$, denote by $M_t^{z,k}$ and $N_t^{z,k}$ the martingales introduced before with $M_t^{z,k}=M_t^{N,k}(h_z)$ and $N_t^{z,k}=N_t^{N,k}(h_z)$, where $h_z$ was introduced in (2.4). To keep notation simple let*

$$\Gamma_1^{z,k}(s)=\frac{1}{2N^{\frac{d}{2}}}\sum_{x\in\mathbb{T}_N^d}\Delta_N h_z\left(\tfrac{x}{N}\right)\left(I_k(\eta_x(s))-\rho^k\right)-\frac{1}{N^{\frac{d}{2}}}\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^{d}(\partial_{u_j}^N h_z)\left(\tfrac{x}{N}\right)\left[\tau_x W_{j,k}^{\eta_s}-\omega_k^{\rho,\varrho}\right] \tag{2.33}$$

*where*

$$W_{j,k}^{\eta_s}=\sum_{v\in\mathcal{V}}v_k\sum_{z\in\mathbb{Z}^d}z_j p(z,v)\eta_s(0,v)[1-\eta_s(z,v)],$$

*and*

$$\omega_k^{\rho,\varrho} = E_{\nu_{\rho,\varrho}^N}[W_{j,k}^s] = \sum_{v\in\mathcal{V}} v_k v_j \chi(\theta_v(\Lambda(\rho,\varrho))).$$

*Also, let*

$$\Gamma_2^{z,k}(s) = \frac{1}{2N^d}\sum_{v\in\mathcal{V}}\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^{d} v_k^2\left(\eta_s(x,v)-\eta_s(x+e_j,v)\right)^2(\partial_j^N h_z\left(\tfrac{x}{N}\right))^2. \tag{2.34}$$

*Therefore*

$$M_t^{z,k} = Y_t^{N,k}(h_z) - Y_0^{k,N}(h_z) - \int_0^t \Gamma_1^{z,k}(s)ds \tag{2.35}$$

*and*

$$N_t^{z,k} = (M_t^{z,k})^2 - \int_0^t \Gamma_2^{z,k}(s)ds. \tag{2.36}$$

**Lemma 12.** *There exists a finite constant $C(\rho,\varrho,v,T)$ such that $\forall z\in\mathbb{Z}^d$,*

$$\limsup_{N\to\infty}\mathbb{E}_N[\sup_{0\le t\le T}|Y_t^{N,k}(h_z)|^2]$$

$$\le C(\rho,\varrho,v,T)\left\{\langle h_z,h_z\rangle + \langle\nabla h_z,\nabla h_z\rangle + \langle\Delta h_z,\Delta h_z\rangle\right\}.$$

*Proof.* Rewrite $Y_t^{N,k}(h_z)$ as $M_t^{z,k} + Y_0^{k,N}(h_z) + \int_0^t \Gamma_1^{z,k}(s)ds$. Note that,

$$\begin{aligned}\left|Y_t^{N,k}(h_z)\right| &\le |M_t^{z,k}| + |Y_0^{k,N}(h_z)| + \left|\int_0^t \Gamma_1^{z,k}(s)ds\right| \\ &\le |M_t^{z,k}| + |Y_0^{k,N}(h_z)| + \int_0^t |\Gamma_1^{z,k}(s)|ds.\end{aligned} \tag{2.37}$$

Consequently

$$\left|Y_t^{N,k}(h_z)\right|^2 \le 2^3\left\{|M_t^{z,k}|^2 + |Y_0^{k,N}(h_z)|^2 + \left(\int_0^t \Gamma_1^{z,k}(s)ds\right)^2\right\}.$$

Now we compute each term separately. Then,

$$\begin{aligned}
(I) := \mathbb{E}_N \left[ \sup_{0\le t\le T} |Y_0^{k,N}(h_z)|^2 \right] &= \mathbb{E}_N[\,|Y_0^{k,N}(h_z)|^2] \\
&= \mathbb{E}_N\left[ \left( \frac{1}{N^{\frac{d}{2}}} \sum_{x\in\mathbb{T}_N^d} h_z\left(\tfrac{x}{N}\right)\left(I_k(\eta_x(0,v)) - \rho^k\right) \right)^2 \right] \\
&= \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} (h_z\left(\tfrac{x}{N}\right))^2 \mathbb{E}_N[(I_k(\eta_x(0,v)) - \rho^k)^2] \\
&= \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} (h_z\left(\tfrac{x}{N}\right))^2\, var(I_k(\eta_x(0,v)), \nu_{\rho,\varrho}),
\end{aligned}$$

where $var(I_k(\eta), \nu_{\rho,\varrho}) = \sum_{v\in\mathcal{V}} v_k^2\, var(\eta(0,v)) = \sum_{v\in\mathcal{V}} v_k^2\, \chi(\theta_v(\Lambda(\rho,\varrho)))$.

Since $h_z$ is continuous,

$$\mathbb{E}_N[(Y_0^{N,k}(h_z))^2] \xrightarrow{N\to+\infty} var(I_k(\eta_x(0,v)), \nu_{\rho,\varrho})\langle h_z, h_z\rangle$$

On the other hand, since $M_t^{z,k}$ is a martingale, by Doob's inequality

$$(II) := \mathbb{E}_N\left[ \sup_{0\le t\le T} |M_t^{z,k}|^2 \right] \le 4\mathbb{E}_N\left[ |M_T^{z,k}|^2 \right]. \tag{2.38}$$

By definition of the martingale $N^{z,k}$, we have

$$\mathbb{E}_N[\, N_t^{z,k}\,] = \mathbb{E}_N[\, N_0^{z,k}\,] = 0 \quad \forall\, t.$$

Consequently,

$$0 = \mathbb{E}_N[\, N_T^{z,k}\,] = \mathbb{E}_N[(M_T^{z,k})^2] - \mathbb{E}_N\left[ \int_0^T \Gamma_2^{z,k}(s)ds \right].$$

The right-hand side of (2.38) is equal to

$$
\begin{aligned}
&\mathbb{E}_N[(M_T^{z,k})^2] = \mathbb{E}_N\left[\int_0^T \Gamma_2^{z,k}(s)ds\right] \\
&= \mathbb{E}_N\left[\int_0^T \frac{1}{2N^d}\sum_{v\in\mathcal{V}}\sum_{j=1}^d\sum_{x\in\mathbb{T}_N^d} v_k^2(\eta_s(x,v)-\eta_s(x+e_j,v))^2(\partial_j^N h_z\left(\tfrac{x}{N}\right))^2 ds\right] \\
&= \frac{1}{2N^d}\sum_{v\in\mathcal{V}}\sum_{j=1}^d\sum_{x\in\mathbb{T}_N^d} v_k^2(\partial_j^N h_z\left(\tfrac{x}{N}\right))^2 \int_0^T \mathbb{E}_N\left[(\eta_s(x,v)-\eta_s(x+e_j,v))^2\right] ds \\
&= \frac{1}{N^d}\left(T\sum_{v\in\mathcal{V}} v_k^2\,\chi(\theta_v(\Lambda(\rho,\varrho)))\right)\sum_{j=1}^d\sum_{x\in\mathbb{T}_N^d}(\partial_j^N h_z\left(\tfrac{x}{N}\right))^2
\end{aligned}
\tag{2.39}
$$

and the last display converges to

$$
\begin{aligned}
&\left(T\sum_{v\in\mathcal{V}} v_k^2\,\chi(\theta_v(\Lambda(\rho,\varrho)))\right)\sum_{j=1}^d\int_{\mathbb{T}^d}(\partial_j h_z\left(\tfrac{x}{N}\right))^2 ds \\
&= \left(T\sum_{v\in\mathcal{V}} v_k^2\,\chi(\theta_v(\Lambda(\rho,\varrho)))\right)\langle\nabla h_z,\nabla h_z\rangle
\end{aligned}
\tag{2.40}
$$

as $N\to\infty$.

**Remark 7.** *Note that*

$$
\begin{aligned}
&E_N\left[(\eta_s(x,v)-\eta_s(x+e_j,v))^2\right] \\
&= E_N\left[(\eta_s^2(x,v)-2\eta_s(x,v)\eta_s(x+e_j,v)+\eta_s^2(x+e_j,v)\right] \\
&= E_N\left[(\eta_s(x,v)-2\eta_s(x,v)\eta_s(x+e_j,v)+\eta_s(x+e_j,v)\right] \\
&= \theta_v(\Lambda(\rho,\varrho))-2\theta_v(\Lambda(\rho,\varrho))^2+\theta_v(\Lambda(\rho,\varrho)) \\
&= 2\chi(\theta_v(\Lambda(\rho,\varrho))).
\end{aligned}
$$

To finish, it remains to bound the other term, namely,

$$
(III) := \mathbb{E}_N\left[\sup_{0\le t\le T}\left(\int_0^t \Gamma_1^{z,k}(s)\,ds\right)^2\right].
$$

Observe that, using Cauchy-Schwarz inequality

$$
\begin{aligned}
&\mathbb{E}_N\left[\sup_{0\le t\le T}\left(\int_0^t \Gamma_1^{z,k}(s)\,ds\right)^2\right] \le \mathbb{E}_N\left[T\int_0^T \left(\Gamma_1^{z,k}(s)\right)^2 ds\right]\\
&= T\int_0^T \mathbb{E}_N\left[\left(\Gamma_1^{z,k}(s)\right)^2\right] ds\\
&\le T\int_0^T \mathbb{E}_N\left[\left(\frac{1}{2N^{\frac{d}{2}}}\sum_{x\in\mathbb{T}_N^d}\Delta_N h_z\left(\tfrac{x}{N}\right)\left(I_k(\eta_x(s))-\rho^k\right)\right.\right.\\
&\left.\left.-\frac{1}{N^{\frac{d}{2}}}\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^d(\partial_{u_j}^N h_z)\left(\tfrac{x}{N}\right)[\tau_x W_{j,k}^{\eta_s}-\omega_k^{\rho,\varrho}]\right)^2\right] ds.
\end{aligned}
\tag{2.41}
$$

Since $(a-b)^2\le 2a^2+2b^2$, last display is bounded from above by

$$
\begin{aligned}
&\le 2T\int_0^T\left\{\mathbb{E}_N\left[\frac{1}{4N^d}\sum_{x\in\mathbb{T}_N^d}\Delta_N^2 h_z\left(\tfrac{x}{N}\right)\left(I_k(\eta_x(s))-\rho^k\right)^2\right]+\right.\\
&\left.+\,\mathbb{E}_N\left[\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^d(\partial_{u_j}^N h_z)^2\left(\tfrac{x}{N}\right)(\tau_x W_{j,k}^{\eta_s}-\omega_k^{\rho,\varrho})^2\right]\right\} ds\\
&= \frac{T}{2}\int_0^T\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\Delta_N^2 h_z\left(\tfrac{x}{N}\right)\mathbb{E}_N\left[\left(I_k(\eta_x(s))-\rho^k\right)^2\right] ds+\\
&+2T\int_0^T\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^d(\partial_{u_j}^N h_z)^2\left(\tfrac{x}{N}\right)\mathbb{E}_N\left[(\tau_x W_{j,k}^{\eta_s}-\omega_k^{\rho,\varrho})^2\right]\}\ ds\\
&= \frac{T}{2}\int_0^T\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\Delta_N^2 h_z\left(\tfrac{x}{N}\right)\ var(I_k(\eta),\nu_{\rho,\varrho})\,ds+\\
&+2T\int_0^T\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^d(\partial_{u_j}^N h_z)^2\left(\tfrac{x}{N}\right)\ var(\tau_x W_{j,k}^{s},\nu_{\rho,\varrho})\,ds
\end{aligned}
\tag{2.42}
$$

and this converges to

$$
\frac{T^2}{2}\,var(I_k(\eta),\nu_{\rho,\varrho})\,\langle\Delta h_z,\Delta h_z\rangle+2T^2\,var(\tau_x W_{j,k}^s,\nu_{\rho,\varrho})\,\langle\nabla h_z,\nabla h_z\rangle
$$

when $N\to+\infty$. □

**Corollary 5.** *For each* $p>2+\frac{d}{2}$

*(i)* $\limsup_{N\to+\infty} \mathbb{E}_N \left[ \sup_{0\le t\le T} \|Y_t\|_{-p}^2 \right] < \infty$

*(ii)* $\lim_{n\to+\infty} \limsup_{N\to+\infty} \mathbb{E}_N \left[ \sup_{0\le t\le T} \sum_{\|z\|\ge n} (Y_t(h_z))^2 \gamma_z^{-p} \right] = 0.$

*Proof.* Recall the definition of $\mathscr{H}_p$ and the inner product $\langle\cdot,\cdot\rangle_p$. We have

$$\|Y_t^{N,k}\|_{-p}^2 = \sum_{z\in\mathbb{Z}^d} (Y_t^{N,k}(h_z))^2 \gamma_z^{-p}.$$

The expect in first expression can be estimates as

$$\begin{aligned} &\mathbb{E}_N \left[ \sup_{0\le t\le T} \|Y_t^{N,k}\|_{-p}^2 \right] = \mathbb{E}_N \left[ \sup_{0\le t\le T} \sum_{z\in\mathbb{Z}^d} (Y_t^{N,k}(h_z))^2 \gamma_z^{-p} \right] \\ &\le \sum_{z\in\mathbb{Z}^d} \gamma_z^{-p}\, \mathbb{E}_N \left[ \sup_{0\le t\le T} (Y_t^{N,k}(h_z))^2 \right]. \end{aligned} \tag{2.43}$$

By the previous lemma, we have

$$\begin{aligned} &\limsup_{N\to+\infty} \mathbb{E}_N \left[ \sup_{0\le t\le T} \|Y_t^{N,k}\|_{-p}^2 \right] \\ &\le \limsup_{N\to+\infty} \sum_{z\in\mathbb{Z}^d} \gamma_z^{-p}\, \mathbb{E}_N \left[ \sup_{0\le t\le T} (Y_t^{N,k}(h_z))^2 \right] \\ &= \sum_{z\in\mathbb{Z}^d} \gamma_z^{-p} \limsup_{N\to+\infty} \mathbb{E}_N \left[ \sup_{0\le t\le T} (Y_t^{N,k}(h_z))^2 \right] \\ &\le \sum_{z\in\mathbb{Z}^d} \gamma_z^{-p}\, C(\rho,\varrho,v,T) \left\{ \langle h_z, h_z\rangle + \langle \nabla h_z, \nabla h_z\rangle + \langle \Delta h_z, \Delta h_z\rangle \right\} \\ &= C(\rho,\varrho,v,T) \sum_{z\in\mathbb{Z}^d} \gamma_z^{-p} \left\{ 1 + \langle \nabla h_z, \nabla h_z\rangle + \langle \Delta h_z, \Delta h_z\rangle \right\} \\ &= C(\rho,\varrho,v,T) \sum_{z\in\mathbb{Z}^d} \frac{1}{(1+4\pi^2\|z\|^2)^p} \left\{ 1 + \langle \nabla h_z, \nabla h_z\rangle + \langle \Delta h_z, \Delta h_z\rangle \right\} \\ &= C(\rho,\varrho,v,T) \sum_{z\in\mathbb{Z}^d} \frac{1}{[1+(2\pi\|z\|)^2]^p} \left\{ 1 + (2\pi\|z\|)^2 + (2\pi\|z\|)^4 \right\} \\ &\le C(\rho,\varrho,v,T) \sum_{z\in\mathbb{Z}^d} \frac{[1+(2\pi\|z\|)^2]^2}{[1+(2\pi\|z\|)^2]^p} \\ &= C(\rho,\varrho,v,T) \sum_{z\in\mathbb{Z}^d} \frac{1}{[1+(2\pi\|z\|)^2]^{p-2}} \end{aligned}$$

which is finite as long as $2(p-2)-d>0 \iff p>2+\frac{d}{2}$. This proves the first statement. The second one follows by the same argument. □

*It follows from Lemma 12 and Corollary 5 that, in order to prove that the sequence* $(\mathbb{Q}_N)_N$ *is tight, we only have to show that for every* $\varepsilon>0$,

$$\lim_{\delta\to 0}\limsup_{N\to+\infty}\mathbb{P}_N[\omega_\delta(Y)>\varepsilon]=0.$$

*In view of part (ii) of the previous corollary, this result follows from the following lemma:*

**Lemma 13.** *For every positive integer* $n$ *and every* $\varepsilon>0$,

$$\lim_{\delta\to 0}\limsup_{N\to+\infty}\mathbb{P}_N\left[\sup_{\substack{|s-t|\le\delta\\0\le s,t\le T}}\sum_{\|z\|\le n}(Y_t(h_z)-Y_s(h_z))^2\gamma_z^{-p}>\varepsilon\right]=0.$$

*Proof.* To prove this lemma it is enough to show that

$$\lim_{\delta\to 0}\limsup_{N\to+\infty}\mathbb{P}_N\left[\sup_{\substack{|s-t|\le\delta\\0\le s,t\le T}}(Y_t(h_z)-Y_s(h_z))^2>\varepsilon\right]=0.$$

for every $z\in\mathbb{Z}^d$and $\varepsilon>0$.

Fix $z\in\mathbb{Z}^d$ and recall the definition of $M_t^{z,k}$. Since

$$Y_t^{N,k}(h_z)=Y_0^{N,k}(h_z)+M_t^{N,k}+\int_0^t\Gamma_1^{z,k}(s)\,ds$$

the lemma follows from the next two results.

**Lemma 14.** *Fix a function* $G\in C^2(\mathbb{T}^d)$*. For every* $\varepsilon>0$,

$$\lim_{\delta\to 0}\limsup_{N\to+\infty}\mathbb{P}_N\left[\sup_{\substack{|s-t|\le\delta\\0\le s,t\le T}}|M_t^{N,k}(G)-M_s^{k,N}(G)|>\varepsilon\right]=0.$$

*Proof.* Denote by $\omega'_\delta(M^{k,N}(G))$ the modified modulus of continuity defined as

$$\omega'_\delta(M^{k,N}(G))=\inf_{t_i}\max_{0\le i\le r}\sup_{t_i\le s<t\le t_{i+1}}|M_t^{N,k}(G)-M_s^{k,N}(G)|$$

where the infimum is taken over all partitions of $[0,T]$ such that

$$\begin{cases}0=t_0<t_1<\dots t_r=T\\ t_{i+1}-t_i>\delta \text{ with } 0\le i<r.\end{cases}$$

Since

$$
\begin{aligned}
&\sup_t |M_t^{N,k}(G) - M_{t^-}^{k,N}(G)| \\
&= \sup_t |Y_t^{N,k}(G) - Y_{t^-}^{N,k}(G)| \\
&= \sup_t |N^{-\frac{d}{2}} \sum_{x\in\mathbb{T}_N^d} G\left(\tfrac{x}{N}\right)(I_k(\eta_x(t)) - \rho^k) - N^{-\frac{d}{2}} \sum_{x\in\mathbb{T}_N^d} G\left(\tfrac{x}{N}\right)(I_k(\eta_x(t^-)) - \rho^k)| \\
&= \sup_t |N^{-\frac{d}{2}} \sum_{x\in\mathbb{T}_N^d} G\left(\tfrac{x}{N}\right)[I_k(\eta_x(t)) - I_k(\eta_x(t^-))]|.
\end{aligned}
\tag{2.44}
$$

Since the process is a Markovian process, each particle waits for ring of random clocks exponentially distributed and independent. Consequently, there exists $\{x^*, x^*+e_j\} \in \mathbb{T}_N^d$ such that $(I_k(\eta_x(t)) - I_k(\eta_x(t^-))) = 0$ for every $x \in \mathbb{T}_N^d$, with $x \notin \{x^*, x^*+e_j\}$.

Besides that, if $x \in \{x^*, x^*+e_j\}$

$$
I_k(\eta_x(t)) - I_k(\eta_x(t^-)) \;=\; \sum_{v\in\mathcal{V}} v_k(\eta_t(x,v) - \eta_{t^-}(x,v))
$$

and there exists $v^* \in \mathcal{V}$ such that

$$
\sum_{v\in\mathcal{V}} v_k(\eta_t(x,v) - \eta_{t^-}(x,v)) = (\eta_t(x,v^*) - \eta_{t^-}(x,v^*))v_k^* = \pm v_k^*.
$$

Therefore,

$$
N^{-\frac{d}{2}} \sum_{x\in\mathbb{T}_N^d} G\left(\tfrac{x}{N}\right)[I_k(\eta_x(t)) - I_k(\eta_x(t^-))] = v_k^*\, N^{-\frac{d}{2}}(G\left(\tfrac{x}{N}\right) - G\left(\tfrac{x+e_j}{N}\right)).
$$

From this we get

$$
\begin{aligned}
(2.44) = \;& \sup_t |N^{-\frac{d}{2}}\, v_k^*[G\left(\tfrac{x}{N}\right) - G\left(\tfrac{x+e_j}{N}\right)| \\
= \;& v_k^*\, N^{-\frac{d}{2}}|G\left(\tfrac{x}{N}\right) - G\left(\tfrac{x+e_j}{N}\right)| \\
\leq \;& v_k^*\, N^{-\frac{d}{2}}\tfrac{1}{N}G'\left(\tfrac{\tilde{x}}{N}\right) \text{ with } \tilde{x} \in (x, x+e_j) \\
\leq \;& \sup_{w\in\mathbb{T}^d} G'(w)\, N^{-\frac{d}{2}-1} \\
= \;& N^{-(1+\frac{d}{2})}\, C(G).
\end{aligned}
\tag{2.45}
$$

Besides,

$$
\omega_\delta(M^G) \leq 2\omega'_\delta(M^G) + \sup_t |M_t^G - M_{t^-}^G|
$$

in order to prove the lemma we just need to show that, for every $\varepsilon > 0$

$$
\lim_{\delta\to 0} \limsup_{N\to\infty} \mathbb{P}_N\left[\omega'_\delta(M^{G,k}) > \varepsilon\right] = 0.
$$

By Aldous' criterion, see for example [1, Chapter 4, Proposition 1.6], it is enough to check that for every $\varepsilon > 0$:

$$\lim_{\delta\to 0}\limsup_{N\to+\infty}\sup_{\substack{\tau\in\mathfrak{T}_T\\ 0\le\theta\le\delta}}\mathbb{P}_N\left[|M^{k,N}_{\tau+\theta}(G)-M^{k,N}_{\tau}(G)|>\varepsilon\right]=0,$$

where $\mathfrak{T}_T$ stands for all the stopping times bounded by $T$. By Chebychev inequality, the last probability is less than or equal to

$$\frac{1}{\varepsilon^2}\mathbb{E}_N\left[(M^{k,N}_{\tau+\theta}(G)-M^{k,N}_{\tau}(G))^2\right]=\frac{1}{\varepsilon^2}\mathbb{E}_N\left[(M^{k,N}_{\tau+\theta}(G))^2-(M^{k,N}_{\tau}(G))^2\right]$$

because $M^{N,k}_t(G)$ is a martingale and $\tau$ a bounded stopping time. By (2.36) this expression is bounded from above by

$$\frac{1}{\varepsilon^2}\mathbb{E}_N\left[\int_0^\delta\Gamma^{G,k}_2(r)\,dr\right].$$

because $\nu^N_{\rho,\varrho}$ is invariant, $\tau$ a stopping time and $\theta$ is bounded from above by *delta*. The limit as $N\to\infty$ of this last expression is less than or equal to $\delta\varepsilon^{-2}C\|\nabla G\|_2^2$, this concludes the proof of the lemma. □

**Lemma 15.** *Fix a function $G\in C^2(\mathbb{T}^d)$. For every $\varepsilon>0$,*

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\mathbb{P}_N\Big[\sup_{\substack{0\le s,t\le T\\|s-t|\le\delta}}\Big|\int_s^t\Gamma^{G,k}_1(r)\,dr\Big|>\varepsilon\Big]=0.$$

*Proof.* By using the expression of $\Gamma^{G,k}_1(r)$, see (2.33), we obtain that

$$\begin{aligned}&\mathbb{P}_N\Big[\sup_{\substack{0\le s,t\le T\\|s-t|\le\delta}}\Big|\int_s^t\Gamma^{G,k}_1(r)\,dr\Big|>\varepsilon\Big]\\&=\mathbb{P}_N\left[\sup_{\substack{0\le s,t\le T\\|s-t|\le\delta}}\left|\int_s^t\frac{1}{2N^{\frac{d}{2}}}\sum_{x\in\mathbb{T}^d_N}\Delta_N G\left(\tfrac{x}{N}\right)\left(I_k(\eta_x(r))-\rho^k\right)\right.\right.\\&\left.\left.-\frac{1}{N^{\frac{d}{2}}}\sum_{x\in\mathbb{T}^d_N}\sum_{j=1}^d(\partial^N_{u_j}G)\left(\tfrac{x}{N}\right)[\tau_xW^r_{j,k}-\omega^{\rho,\varrho}_k]\,dr\right|>\varepsilon\right].\end{aligned}\tag{2.46}$$

Observe that the expression above is bounded by

$$\begin{aligned}&\mathbb{P}_N\Big[\sup_{\substack{0\le s,t\le T\\|s-t|\le\delta}}\Big|\int_s^t \frac{1}{2N^{\frac{d}{2}}}\sum_{x\in\mathbb{T}_N^d}\Delta_N G\left(\tfrac{x}{N}\right)\left(I_k(\eta_x(r))-\rho^k\right)\,dr\Big|>\frac{\varepsilon}{2}\Big]\\&+\mathbb{P}_N\Big[\sup_{\substack{0\le s,t\le T\\|s-t|\le\delta}}\Big|\frac{1}{N^{\frac{d}{2}}}\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^d(\partial_{u_j}^N G)\left(\tfrac{x}{N}\right)[\tau_x W_{j,k}^r-\omega_k^{\rho,\varrho}]\,dr\Big|>\frac{\varepsilon}{2}\Big].\end{aligned}\tag{2.47}$$

We will compute each term above separately. For the first term, using Chebychev inequality, we obtain

$$\begin{aligned}&\mathbb{P}_N\Big[\sup_{\substack{0\le s,t\le T\\|s-t|\le\delta}}\Big|\int_s^t \frac{1}{2N^{\frac{d}{2}}}\sum_{x\in\mathbb{T}_N^d}\Delta_N G\left(\tfrac{x}{N}\right)\left(I_k(\eta_x(r))-\rho^k\right)\,dr\Big|>\frac{\varepsilon}{2}\Big]\\&\le\frac{4}{\varepsilon^2}\mathbb{E}_N\Big[\Big(\sup_{\substack{0\le s,t\le T\\|s-t|\le\delta}}\Big|\int_s^t \frac{1}{2N^{\frac{d}{2}}}\sum_{x\in\mathbb{T}_N^d}\Delta_N G\left(\tfrac{x}{N}\right)\left(I_k(\eta_x(r))-\rho^k\right)\,dr\Big|\Big)^2\Big]\\&=\frac{4}{\varepsilon^2}\mathbb{E}_N\Big[\sup_{\substack{0\le s,t\le T\\|s-t|\le\delta}}\Big|\int_s^t \frac{1}{2N^{\frac{d}{2}}}\sum_{x\in\mathbb{T}_N^d}\Delta_N G\left(\tfrac{x}{N}\right)\left(I_k(\eta_x(r))-\rho^k\right)\,dr\Big|^2\Big]\\&\le\frac{4}{\varepsilon^2}\mathbb{E}_N\Big[\sup_{\substack{0\le s,t\le T\\|s-t|\le\delta}}|t-s|\int_s^t \frac{1}{4N^d}\Big[\sum_{x\in\mathbb{T}_N^d}\Delta_N G\left(\tfrac{x}{N}\right)\left(I_k(\eta_x(r))-\rho^k\right)\Big]^2\,dr\Big]\end{aligned}\tag{2.48}$$

since $|t-s|\le\delta$ and $0\le s,t\le T$ we can bound (2.48) from above by

$$\begin{aligned}&\le\frac{\delta}{N^d\varepsilon^2}\mathbb{E}_N\Big[\int_0^T\Big(\sum_{x\in\mathbb{T}_N^d}\Delta_N G\left(\tfrac{x}{N}\right)\left(I_k(\eta_x(r))-\rho^k\right)\Big)^2dr\Big]\\&=\frac{\delta}{N^d\varepsilon^2}\int_0^T\mathbb{E}_N\Big[\Big(\sum_{x\in\mathbb{T}_N^d}\Delta_N G\left(\tfrac{x}{N}\right)\left(I_k(\eta_x(r))-\rho^k\right)\Big)^2\Big]dr\\&=\frac{\delta}{N^d\varepsilon^2}\int_0^T\sum_{x\in\mathbb{T}_N^d}(\Delta_N G)^2\left(\tfrac{x}{N}\right)\mathbb{E}_N\Big[\left(I_k(\eta_x(r))-\rho^k\right)^2\Big]dr\\&=\frac{T\delta}{N^d\varepsilon^2}\sum_{x\in\mathbb{T}_N^d}(\Delta_N G)^2\left(\tfrac{x}{N}\right)\mathbb{E}_N\Big[\left(I_k(\eta_x(r))-\rho^k\right)^2\Big]\\&=\frac{T\delta}{N^d\varepsilon^2}\,var(I_k(\eta_x(r)),\nu_{\rho,\varrho})\sum_{x\in\mathbb{T}_N^d}(\Delta_N G)^2\left(\tfrac{x}{N}\right)\end{aligned}\tag{2.49}$$

as $N\to\infty$, this converges to

$$\frac{T\delta}{\varepsilon^2}\,var(I_k(\eta_x(r)),\nu_{\rho,\varrho})\int_{\mathbb{T}^d}(\Delta G)^2\,dx$$

since $\dfrac{T}{\varepsilon^2}\, var(I_k(\eta_x(r)), \nu_{\rho,\varrho}) \displaystyle\int_{\mathbb{T}^d} (\Delta G)^2\, dx$ is bounded, when $\delta \to 0$, we obtain that first term of (2.47) goes to zero.

On the other hand, we have that second term of (2.47) is bounded from above by

$$
\begin{aligned}
&\mathbb{P}_N\Big[ \sup_{\substack{0\le s,t\le T\\ |s-t|\le\delta}} \Big| \frac{1}{N^{\frac{d}{2}}} \sum_{x\in\mathbb{T}_N^d} \sum_{j=1}^{d} (\partial_{u_j}^N G)\left(\tfrac{x}{N}\right) [\tau_x W_{j,k}^r - \omega_k^{\rho,\varrho}]\, dr \Big| > \frac{\varepsilon}{2} \Big] \\
&\le \frac{4}{\varepsilon^2} \mathbb{E}_N\Big[ \Big( \sup_{\substack{0\le s,t\le T\\ |s-t|\le\delta}} \Big| \int_s^t \frac{1}{N^{\frac{d}{2}}} \sum_{x\in\mathbb{T}_N^d} \sum_{j=1}^{d} (\partial_{u_j}^N G)\left(\tfrac{x}{N}\right) (\tau_x W_{j,k}^r - \omega_k^{\rho,\varrho})\, dr \Big| \Big)^2 \Big] \\
&= \frac{4}{\varepsilon^2} \mathbb{E}_N\Big[ \sup_{\substack{0\le s,t\le T\\ |s-t|\le\delta}} \Big| \int_s^t \frac{1}{N^{\frac{d}{2}}} \sum_{x\in\mathbb{T}_N^d} \sum_{j=1}^{d} (\partial_{u_j}^N G)\left(\tfrac{x}{N}\right) (\tau_x W_{j,k}^r - \omega_k^{\rho,\varrho})\, dr \Big|^2 \Big] \\
&\le \frac{4}{\varepsilon^2} \mathbb{E}_N\Big[ \sup_{\substack{0\le s,t\le T\\ |s-t|\le\delta}} |t-s| \int_s^t \frac{1}{N^d} \Big[ \sum_{x\in\mathbb{T}_N^d} \sum_{j=1}^{d} (\partial_{u_j}^N G)\left(\tfrac{x}{N}\right) (\tau_x W_{j,k}^r - \omega_k^{\rho,\varrho}) \Big]^2 dr \Big] \\
&\le \frac{4\delta}{N^d \varepsilon^2} \mathbb{E}_N\Big[ \sup_{\substack{0\le s,t\le T\\ |s-t|\le\delta}} \int_0^T \Big[ \sum_{x\in\mathbb{T}_N^d} \sum_{j=1}^{d} (\partial_{u_j}^N G)\left(\tfrac{x}{N}\right) (\tau_x W_{j,k}^r - \omega_k^{\rho,\varrho}) \Big]^2 dr \Big] \qquad (2.50) \\
&= \frac{4T\delta}{N^d \varepsilon^2} \Big\{ \mathbb{E}_N\Big[ \sum_{x\in\mathbb{T}_N^d} \sum_{j=1}^{d} (\partial_{u_j}^N G)^2 \left(\tfrac{x}{N}\right) (\tau_x W_{j,k}^r - \omega_k^{\rho,\varrho})^2 \Big] \\
&+ \mathbb{E}_N\Big[ \sum_{x\neq y} \sum_{j=1}^{d} (\partial_{u_j}^N G)^2 \left(\tfrac{x}{N}\right) (\tau_x W_{j,k}^r - \omega_k^{\rho,\varrho}) (\partial_{u_j}^N G)^2 \left(\tfrac{y}{N}\right) (\tau_y W_{j,k}^r - \omega_k^{\rho,\varrho}) \Big] \Big\} \\
&= \frac{4T\delta}{N^d \varepsilon^2} \sum_{x\in\mathbb{T}_N^d} \sum_{j=1}^{d} (\partial_{u_j}^N G)^2 \left(\tfrac{x}{N}\right) \mathbb{E}_N\Big[ (\tau_x W_{j,k}^r - \omega_k^{\rho,\varrho})^2 \Big] \\
&= \frac{4T\delta}{N^d \varepsilon^2} \sum_{x\in\mathbb{T}_N^d} \sum_{j=1}^{d} (\partial_{u_j}^N G)^2 \left(\tfrac{x}{N}\right)\, var(W_{j,k}^r, \nu_{\rho,\varrho})
\end{aligned}
$$

and when $N \to \infty$ this goes to

$$
\frac{4T\delta}{\varepsilon^2}\, var(W_{j,k}^r, \nu_{\rho,\varrho}) \int_{\mathbb{T}^d} \nabla G^2 dx.
$$

Since $\dfrac{4T}{\varepsilon^2}\, var(W_{j,k}^r, \nu_{\rho,\varrho}) \displaystyle\int_{\mathbb{T}^d} (\nabla G)^2\, dx$ is bounded, when $\delta \to 0$ we obtain that second term of (2.47) goes to zero. This concludes the proof of tightness. □

# Appendix A

*In this section, we establish some technical results that are needed in order to prove the hydrodynamic limit for the model discussed in the previous sections.*

### A.0.1 Computation of $\mathcal{L}_N\langle \pi_t^{k,N}, H\rangle$

*Recall that the conserved quantities are the mass and the momentum. For* $k = 0, \dots, d$, *denote by* $\pi_t^{k,N}$ *the empirical measure associated to the* $k$*-th conserved quantity:*

$$\pi_t^{k,N} = \frac{1}{N^d} \sum_{x \in D_N^d} I_k(\eta_x(t))\delta_{x/N}, \tag{A.1}$$

*where* $\delta_u$ *stands for the Dirac measure supported on* $u$*. We denote by* $\langle \pi_t^{k,N}, H\rangle$ *the integral of a test function* $H$ *with respect to the empirical measure* $\pi_t^{k,N}$.

*Further, denote by* $\pi_t^{k,N,b_1}$ *and* $\pi_t^{k,N,b_{N-1}}$ *the empirical measures associated to the* $k$*-th thermodynamic quantity restricted to the boundary:*

$$\pi_t^{k,N,b_i} = \frac{1}{N^{d-1}} \sum_{\substack{x \in D_N^d \\ x_1 = i}} I_k(\eta_x(t))\delta_{x/N},$$

*for* $i = 1, N-1$.

*Let* $\mathcal{L}_N := N^2\{\mathcal{L}_N^{ex,1} + \mathcal{L}_N^{ex,2} + \mathcal{L}_N^{c} + \mathcal{L}_N^{b}\}$ *and* $\pi_t^{k,N} = \frac{1}{N^d} \sum_{x \in D_N^d} I_k(\eta_x(t))\delta_{\frac{x}{N}}$*. Let us compute the action of the generator of the empirical measure. We do this separately to make the presentation easier to follow.*

**Lemma 16.** *Recall the definition of the empirical measure that was defined in (A.1).*

*Let $H$ be a test function, we obtain that*

$$
\begin{aligned}
N^2\mathcal{L}_N^{ex,1}\langle \pi_t^{k,N}, H\rangle &= \frac{1}{2N^d}\sum_{x\in D_N^d}\sum_{v\in\mathcal{V}} v_k\,\eta(x,v)\Delta_N H\left(\tfrac{x}{N}\right)\\
&+\frac{N^2}{2N^d}\sum_{\substack{x\in D_N^d\\ x_1=N-1}}\sum_{v\in\mathcal{V}} v_k\,\left[H\left(\tfrac{N-1}{N},\tfrac{\tilde{x}}{N}\right)-H\left(\tfrac{N}{N},\tfrac{\tilde{x}}{N}\right)\right]\eta(N-1,\tilde{x},v)\\
&+\frac{N^2}{2N^d}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}} v_k\,\left[H\left(\tfrac{1}{N},\tfrac{\tilde{x}}{N}\right)-H\left(\tfrac{0}{N},\tfrac{\tilde{x}}{N}\right)\right]\eta(1,\tilde{x},v).
\end{aligned}
\tag{A.2}
$$

*Proof.* Since the operator is linear, we just need to compute $\mathcal{L}_N^{ex,1}(\eta(x,v))$. For $f(\eta)=\eta(x,v)$ with $x_1\neq\{1,N-1\}$ note that

$$
\begin{aligned}
(\mathcal{L}_N^{ex,1}f)(\eta) = \frac{1}{2}\sum_{w\in\mathcal{V}}\sum_{y\in D_N^d}\sum_{j=1}^{d}&[\eta(y,w)(1-\eta(y+e_j,w))+\eta(y+e_j,w)(1-\eta(y,w))]\\
&\times[f(\eta^{y,y+e_j,w})-f(\eta)]\\
+\left[\eta(y,w)(1-\eta(y-e_j,w))+\eta(y-e_j,w)(1-\eta(y,w))[f(\eta^{y-e_j,y,w})-f(\eta)]\right].
\end{aligned}
$$

We have that

$$\eta^{y,y+e_j,w}(x,w)=\eta(y,v) \text{ if } x=y+e_j \text{ and } w=v$$

and

$$\eta^{y,y+e_j,w}(x,w)=\eta(y+e_j,v) \text{ if } x=y \text{ and } w=v$$

if $w\neq v \Longrightarrow \eta^{y,y+e_j,w}(x,v)=\eta(x,v)$. Hence,

$$
\begin{aligned}
(\mathcal{L}_N^{ex,1}f)(\eta) &= \frac{1}{2}\sum_{j=1}^{d}[(\eta(x+e_j,v)(1-\eta(x,v)))-(\eta(x,v)(1-\eta(x+e_j,v)))]\\
&+[(\eta(x,v)(1-\eta(x-e_j,v)))-(\eta(x-e_j,v)(1-\eta(x,v)))]\\
&=\frac{1}{2}\sum_{j=1}^{d}\eta(x+e_j,v)-\eta(x,v)+\eta(x-e_j,v)-\eta(x,v)\\
&=\frac{1}{2}\sum_{j=1}^{d}\eta(x+e_j,v)+\eta(x-e_j,v)-2\eta(x,v).
\end{aligned}
$$

Now consider $f(\eta)=\eta(x,v)$ for $x_1=1$, we have that

$$(\mathcal{L}_N^{ex,1}f)(\eta)=\frac{1}{2}\left(\eta(2,\tilde{x},v)-\eta(1,\tilde{x},v)\right)$$

and for $x_1=N-1$

$$(\mathcal{L}_N^{ex,1}f)(\eta)=\frac{1}{2}\left(\eta(N-2,\tilde{x},v)-\eta(N-1,\tilde{x},v)\right).$$

Therefore,

$$\begin{aligned}
N^2\mathcal{L}_N^{ex,1}\langle\pi_t^{k,N},H\rangle &= \frac{N^2}{2N^d}\sum_{\substack{x\in D_N^d\\ x_1\neq\{1,N-1\}}}\sum_{v\in\mathcal{V}}\sum_{j=1}^{d} v_k H\left(\tfrac{x}{N}\right)\left[\eta(x+e_j,v)+\eta(x-e_j,v)-2\eta(x,v)\right]\\
&+\frac{N^2}{2N^d}\sum_{\substack{x\in D_N^d\\ x_1=N-1}}\sum_{v\in\mathcal{V}} v_k H\left(\tfrac{N-1}{N},\tfrac{\tilde{x}}{N}\right)\left(\eta(N-2,\tilde{x},v)-\eta(N-1,\tilde{x},v)\right)\\
&+\frac{N^2}{2N^d}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}} v_k H\left(\tfrac{1}{N},\tfrac{\tilde{x}}{N}\right)\left(\eta(2,\tilde{x},v)-\eta(1,\tilde{x},v)\right),
\end{aligned}$$

where $\tilde{x}=(x_2,\ldots,x_d)$. Grouping the terms, we have that

$$\begin{aligned}
N^2\mathcal{L}_N^{ex,1}\langle\pi_t^{k,N},H\rangle &= \frac{N^2}{2N^d}\sum_{\substack{x\in D_N^d\\ x_1\neq\{1,N-1\}}}\sum_{v\in\mathcal{V}}\sum_{j=1}^{d} v_k\,\eta(x,v)\left[H\left(\tfrac{x+e_j}{N}\right)+H\left(\tfrac{x-e_j}{N}\right)-2H\left(\tfrac{x}{N}\right)\right]\\
&+\frac{N^2}{2N^d}\sum_{\substack{x\in D_N^d\\ x_1=N-1}}\sum_{v\in\mathcal{V}} v_k\left[H\left(\tfrac{N-2}{N},\tfrac{\tilde{x}}{N}\right)-H\left(\tfrac{N-1}{N},\tfrac{\tilde{x}}{N}\right)\right]\eta(N-1,\tilde{x},v)\\
&+\frac{N^2}{2N^d}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}} v_k\left[H\left(\tfrac{2}{N},\tfrac{\tilde{x}}{N}\right)-H\left(\tfrac{1}{N},\tfrac{\tilde{x}}{N}\right)\right]\eta(1,\tilde{x},v).
\end{aligned}\tag{A.3}$$

To force the appearance of the discrete Laplacian, we will add and subtract the expression below with $i=\{1,N-1\}$

$$\frac{N^2}{2N^d}\sum_{\substack{x\in D_N^d\\ x_1=i}}\sum_{v\in\mathcal{V}} v_k\,\eta(x,v)\left[H\left(\tfrac{x+e_1}{N}\right)+H\left(\tfrac{x-e_1}{N}\right)-2H\left(\tfrac{x}{N}\right)\right].$$

Then we get

$$N^2\mathcal{L}_N^{ex,1}\langle \pi_t^{k,N}, H\rangle$$

$$= \frac{N^2}{2N^d}\sum_{x\in D_N^d}\sum_{v\in\mathcal{V}}\sum_{j=1}^{d} v_k\,\eta(x,v)\left[H\left(\tfrac{x+e_j}{N}\right)+H\left(\tfrac{x-e_j}{N}\right)-2H\left(\tfrac{x}{N}\right)\right]$$

$$+\frac{N^2}{2N^d}\sum_{\substack{x\in D_N^d\\ x_1=N-1}}\sum_{v\in\mathcal{V}} v_k\left[H\left(\tfrac{N-2}{N},\tfrac{\tilde{x}}{N}\right)-H\left(\tfrac{N-1}{N},\tfrac{\tilde{x}}{N}\right)\right]\eta(N-1,\tilde{x},v)$$

$$-\frac{N^2}{2N^d}\sum_{\substack{x\in D_N^d\\ x_1=N-1}}\sum_{v\in\mathcal{V}} v_k\eta(N-1,\tilde{x},v)\left[H\left(\tfrac{N}{N},\tfrac{\tilde{x}}{N}\right)+H\left(\tfrac{N-2}{N},\tfrac{\tilde{x}}{N}\right)-2H\left(\tfrac{N-1}{N},\tfrac{\tilde{x}}{N}\right)\right]$$

$$+\frac{N^2}{2N^d}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}} v_k\left[H\left(\tfrac{2}{N},\tfrac{\tilde{x}}{N}\right)-H\left(\tfrac{1}{N},\tfrac{\tilde{x}}{N}\right)\right]\eta(1,\tilde{x},v)$$

$$-\frac{N^2}{2N^d}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}} v_k\,\eta(1,\tilde{x},v)\left[H\left(\tfrac{2}{N},\tfrac{\tilde{x}}{N}\right)+H\left(\tfrac{0}{N},\tfrac{\tilde{x}}{N}\right)-2H\left(\tfrac{1}{N},\tfrac{\tilde{x}}{N}\right)\right],$$

Which gives

$$\begin{aligned}N^2\mathcal{L}_N^{ex,1}\langle \pi_t^{k,N}, H\rangle &= \frac{1}{2N^d}\sum_{x\in D_N^d}\sum_{v\in\mathcal{V}} v_k\,\eta(x,v)\Delta_N H\left(\tfrac{x}{N}\right)+\\ &+\frac{N^2}{2N^d}\sum_{\substack{x\in D_N^d\\ x_1=N-1}}\sum_{v\in\mathcal{V}} v_k\left[H\left(\tfrac{N-1}{N},\tfrac{\tilde{x}}{N}\right)-H\left(\tfrac{N}{N},\tfrac{\tilde{x}}{N}\right)\right]\eta(N-1,\tilde{x},v)+\\ &+\frac{N^2}{2N^d}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}} v_k\left[H\left(\tfrac{1}{N},\tfrac{\tilde{x}}{N}\right)-H\left(\tfrac{0}{N},\tfrac{\tilde{x}}{N}\right)\right]\eta(1,\tilde{x},v).\end{aligned} \tag{A.4}$$

□

**Lemma 17.** *Recall the definition of the empirical measure that was defined in (A.1). Let $H$ be a test function, we obtain that*

$$\begin{aligned}N^2\mathcal{L}_N^{b}\langle \pi_t^{k,N}, H\rangle &= \frac{N^2}{N^dN^\theta}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}} v_kH\left(\tfrac{x}{N}\right)\left[\alpha_v\left(\frac{\tilde{x}}{N}\right)-\eta(1,\tilde{x},v)\right]\\ &+\frac{N^2}{N^dN^\theta}\sum_{\substack{x\in D_N^d\\ x_1=N-1}}\sum_{v\in\mathcal{V}} v_kH\left(\tfrac{x}{N}\right)\left[\beta_v\left(\frac{\tilde{x}}{N}\right)-\eta(N-1,\tilde{x},v)\right].\end{aligned} \tag{A.5}$$

*Proof.* Observe that

$$N^2\mathcal{L}_N^b\langle \pi_t^{k,N}, H\rangle = \frac{N^2}{N^d}\sum_{x\in D_N^d}\sum_{v\in\mathcal{V}} v_k H\left(\tfrac{x}{N}\right)\mathcal{L}_N^b(\eta(x,v))$$

Since the operator is linear, we just need to compute $\mathcal{L}_N^b(\eta(x,v))$, for $f(\eta)=\eta(x,v)$. We have that

$$(\mathcal{L}_N^b f)(\eta) = \sum_{\substack{z\in D_N^d\\ z_1=1}}\sum_{w\in\mathcal{V}}\frac{1}{N^\theta}\left[\alpha_v\left(\tfrac{\tilde{z}}{N}\right)(1-\eta(1,\tilde{z},w))+\eta(1,\tilde{z},w)(1-\alpha_v\left(\tfrac{\tilde{z}}{N}\right))\right]$$

$$\times[f(\sigma^{z,w}\eta)-f(\eta)]$$

$$+\sum_{\substack{z\in D_N^d\\ z_1=N-1}}\sum_{w\in\mathcal{V}}\frac{1}{N^\theta}\left[\beta_v\left(\tfrac{\tilde{z}}{N}\right)(1-\eta(N-1,\tilde{z},w))+\eta(N-1,\tilde{z},w)(1-\beta_v\left(\tfrac{\tilde{z}}{N}\right))\right]$$

$$\times[f(\sigma^{z,w}\eta)-f(\eta)]$$

where

$$\sigma^{z,w}\eta(x,v) = \begin{cases} 1-\eta(x,v), & \text{if } w=v \text{ and } x=z,\\ \eta(x,v), & \text{otherwise.}\end{cases}$$

Note that for $w\neq v$ and $x\neq z$ we have that $f(\sigma^{z,w}\eta)-f(\eta)$ vanishes, then

$$(\mathcal{L}_N^b f)(\eta) = \left[\alpha_v\left(\tfrac{\tilde{x}}{N}\right)\frac{(1-\eta(1,\tilde{x},v))}{N^\theta}+\frac{\eta(1,\tilde{x},v)}{N^\theta}(1-\alpha_v\left(\tfrac{\tilde{x}}{N}\right))[1-2\eta(1,\tilde{x},v)]\right]$$

$$+\left[\beta_v\left(\tfrac{\tilde{x}}{N}\right)\frac{(1-\eta(N-1,\tilde{x},v))}{N^\theta}+\frac{\eta(N-1,\tilde{x},v)}{N^\theta}(1-\beta_v\left(\tfrac{\tilde{x}}{N}\right))[1-2\eta(N-1,\tilde{x},v)]\right]$$

$$=\alpha_v\left(\tfrac{\tilde{x}}{N}\right)\frac{(1-\eta(1,\tilde{x},v))}{N^\theta}-\frac{\eta(1,\tilde{x},v)}{N^\theta}(1-\alpha_v\left(\tfrac{\tilde{x}}{N}\right))$$

$$+\beta_v\left(\tfrac{\tilde{x}}{N}\right)\frac{(1-\eta(N-1,\tilde{x},v))}{N^\theta}-\frac{\eta(N-1,\tilde{x},v)}{N^\theta}(1-\beta_v\left(\tfrac{\tilde{x}}{N}\right))$$

$$=\frac{1}{N^\theta}\left[\alpha_v\left(\tfrac{\tilde{x}}{N}\right)-\eta(1,\tilde{x},v)\right]+\frac{1}{N^\theta}\left[\beta_v\left(\tfrac{\tilde{x}}{N}\right)-\eta(N-1,\tilde{x},v)\right].$$

Therefore, we have that

$$\begin{aligned}
N^2\mathcal{L}_N^b\langle \pi_t^{k,N}, H\rangle &= \frac{N^2}{N^d}\sum_{x\in D_N^d}\sum_{v\in\mathcal{V}} v_k H\left(\tfrac{x}{N}\right)\mathcal{L}_N^b(\eta(x,v)) \\
&= \frac{N^2}{N^d}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}} v_k H\left(\tfrac{x}{N}\right)\frac{1}{N^\theta}[\alpha_v\left(\tfrac{\tilde{x}}{N}\right) - \eta(1,\tilde{x},v)] \\
&+ \frac{N^2}{N^d}\sum_{\substack{x\in D_N^d\\ x_1=N-1}}\sum_{v\in\mathcal{V}} v_k H\left(\tfrac{x}{N}\right)\frac{1}{N^\theta}[\beta_v\left(\tfrac{\tilde{x}}{N}\right) - \eta(N-1,\tilde{x},v)] \\
&= \frac{N^2}{N^dN^\theta}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}} v_k H\left(\tfrac{x}{N}\right)[\alpha_v\left(\tfrac{\tilde{x}}{N}\right) - \eta(1,\tilde{x},v)] \\
&+ \frac{N^2}{N^dN^\theta}\sum_{\substack{x\in D_N^d\\ x_1=N-1}}\sum_{v\in\mathcal{V}} v_k H\left(\tfrac{x}{N}\right)[\beta_v\left(\tfrac{\tilde{x}}{N}\right) - \eta(N-1,\tilde{x},v)].
\end{aligned} \tag{A.6}$$

□

**Lemma 18.** *Recall the definition of the empirical measure that was defined in (A.1). Let $H$ be a test function, we obtain that*

$$N^2\mathcal{L}_N^c\langle \pi_t^{k,N}, H\rangle = 0. \tag{A.7}$$

*Proof.* Observe that

$$N^2\mathcal{L}_N^c\langle \pi_t^{k,N}, H\rangle = \frac{N^2}{N^d}\sum_{x\in D_N^d}\sum_{v\in\mathcal{V}} v_k H\left(\tfrac{x}{N}\right)\mathcal{L}_N^c(\eta(x,v)).$$

Since the operator is linear, we just need to compute $\mathcal{L}_N^c(\eta(x,v))$ For $f(\eta) = \eta(x,v)$, we have that

$$
\begin{aligned}
&(\mathcal{L}_N^c f)(\eta)\\
&= \sum_{y\in D_N^d}\sum_{q\in Q} p_c(y,q,\eta)[f(\eta^{y,q})-f(\eta)]\\
&= \sum_{q\in Q} p_c(x,q,\eta)[\eta(x,v_{j+2})-\eta(x,v)]\\
&= \eta(x,v_0)\eta(x,v_1)(1-\eta(x,v_2))(1-\eta(x,v_3))\\
&\times[\eta(x,v_2)-\eta(x,v_0)+\eta(x,v_3)-\eta(x,v_1)+\eta(x,v_0)-\eta(x,v_2)+\eta(x,v_1)-\eta(x,v_3)]\\
&= 0.
\end{aligned}
$$

Therefore,

$$N^2\mathcal{L}_N^c\langle \pi_t^{k,N}, H\rangle = 0. \tag{A.8}$$

□

**Lemma 19.** *Recall the definition of the empirical measure that was defined in* (A.1). *Let $H$ be a test function, we obtain that*

$$N^2\mathcal{L}_N^{ex,2}\langle \pi_t^{k,N}, H\rangle = \frac{1}{N^d}\sum_{j=1}^{d}\sum_{x\in D_N^d}(\partial_{u_j}^N H)\left(\tfrac{x}{N}\right)\tau_x^1 W_{j,k}^{N,s} \tag{A.9}$$

*where* $(\tau_x^1\eta)(z,v)=\eta(x+z,v)$ *and* $W_{j,k}^{N,s} = \sum_{v\in\mathcal{V}} v_k \sum_{z\in\mathbb{Z}^d} p(z,v)z_j\eta_s(0,v)(1-\eta_s(z,v))$.

*Proof.* Observe that

$$N^2\mathcal{L}_N^{ex,2}\langle \pi_t^{k,N}, H\rangle = \frac{N^2}{N^d}\sum_{x\in D_N^d}\sum_{v\in\mathcal{V}} v_k H\left(\tfrac{x}{N}\right)\mathcal{L}_N^{ex,2}(\eta(x,v))$$

Since the operator is linear, we just need to compute $\mathcal{L}_N^{ex,2}(\eta(x,v))$, for $f(\eta)=\eta(x,v)$, we have that

$$
\begin{aligned}
(\mathcal{L}_N^{ex,2} f)(\eta) &= \frac{1}{N} \sum_{w \in \mathcal{V}} \sum_{y,z \in D_N^d} \eta(y,w)(1-\eta(z,w))p(z-y,w)[f(\eta^{y,z,w}) - f(\eta)] \\
&= \frac{1}{N} \sum_{y \in D_N^d} \eta(y,v)(1-\eta(x,v))p(x-y,v)[\eta(y,v) - \eta(x,v)] \\
&+ \frac{1}{N} \sum_{z \in D_N^d} \eta(x,v)(1-\eta(z,v))p(z-x,v)[\eta(z,v) - \eta(x,v)] \\
&= \frac{1}{N} \sum_{y \in D_N^d} \eta(y,v)(1-\eta(x,v))p(x-y,v)[\eta(y,v) - \eta(x,v)] \\
&+ \frac{1}{N} \sum_{y \in D_N^d} \eta(x,v)(1-\eta(y,v))p(y-x,v)[\eta(y,v) - \eta(x,v)] \\
&= \frac{1}{N} \sum_{y \in D_N^d} [\eta(y,v)(1-\eta(x,v))p(x-y,v)] - [\eta(x,v)(1-\eta(y,v))p(y-x,v)].
\end{aligned}
$$

Then

$$
N^2 \mathcal{L}_N^{ex,2} \langle \pi_t^{k,N}, H \rangle
$$

$$
\begin{aligned}
&= \frac{N^2}{N^{d+1}} \sum_{x,y \in D_N^d} \sum_{v \in \mathcal{V}} v_k H\left(\tfrac{x}{N}\right) [\eta(y,v)(1-\eta(x,v))p(x-y,v)] - [\eta(x,v)(1-\eta(y,v))p(y-x,v)] \\
&= -\frac{N^2}{N^{d+1}} \sum_{x,y \in D_N^d} \sum_{v \in \mathcal{V}} v_k H\left(\tfrac{x}{N}\right) [\eta(x,v)(1-\eta(y,v))p(y-x,v)] \\
&+ \frac{N^2}{N^{d+1}} \sum_{x,y \in D_N^d} \sum_{v \in \mathcal{V}} v_k H\left(\tfrac{x}{N}\right) [\eta(y,v)(1-\eta(x,v))p(x-y,v)]
\end{aligned}
$$

and we can change $x$ by $y$ in the second term, to get that

$$
\begin{aligned}
&N^2\mathcal{L}_N^{ex,2}\langle \pi_t^{k,N}, H\rangle = \\
&-\frac{N^2}{N^{d+1}}\sum_{x,y\in D_N^d}\sum_{v\in\mathcal{V}} v_k H\left(\tfrac{x}{N}\right)[\eta(x,v)(1-\eta(y,v))p(y-x,v)] \\
&+\frac{N^2}{N^{d+1}}\sum_{x,y\in D_N^d}\sum_{v\in\mathcal{V}} v_k H\left(\tfrac{y}{N}\right)[\eta(x,v)(1-\eta(y,v))p(y-x,v)] \\
&= -\frac{N^2}{N^{d+1}}\sum_{x\in D_N^d}\sum_{v\in\mathcal{V}}\sum_{z\in\mathbb{Z}^d} v_k\, H\left(\tfrac{x}{N}\right)[\eta(x,v)(1-\eta(x+z,v))p(z,v)] \\
&+\frac{N^2}{N^{d+1}}\sum_{x\in D_N^d}\sum_{v\in\mathcal{V}}\sum_{z\in\mathbb{Z}^d} v_k\, H\left(\tfrac{x+z}{N}\right)[\eta(x,v)(1-\eta(x+z,v))p(z,v)] \\
&= \frac{N^2}{N^{d+1}}\sum_{x\in D_N^d}\sum_{v\in\mathcal{V}}\sum_{z\in\mathbb{Z}^d} v_k\,[\eta(x,v)(1-\eta(x+z,v))p(z,v)]\left[H\left(\tfrac{x+z}{N}\right)-H\left(\tfrac{x}{N}\right)\right] \\
&= \frac{1}{N^d}\sum_{x\in D_N^d}\sum_{v\in\mathcal{V}}\sum_{z\in\mathbb{Z}^d} v_k\,[\eta(x,v)(1-\eta(x+z,v))p(z,v)]\sum_{j=1}^{d}(\partial_{u_j}^N H)\left(\tfrac{x}{N}\right)z_j \\
&= \frac{1}{N^d}\sum_{j=1}^{d}\sum_{x\in D_N^d}(\partial_{u_j}^N H)\left(\tfrac{x}{N}\right)\sum_{v\in\mathcal{V}} v_k \sum_{z\in\mathbb{Z}^d} p(z,v)z_j\eta(x,v)(1-\eta(x+z,v)) \\
&= \frac{1}{N^d}\sum_{j=1}^{d}\sum_{x\in D_N^d}(\partial_{u_j}^N H)\left(\tfrac{x}{N}\right)\tau_x^1 W_{j,k}^{N,s}.
\end{aligned}
\tag{A.10}
$$

□

### A.0.2 Computation of $\mathcal{L}_N\langle \pi_t^{k,N}, H\rangle^2 - 2\langle \pi_t^{k,N}, H\rangle\mathcal{L}_N\langle \pi_t^{k,N}, H\rangle$

*In order to simplify the presentation, we split the generator and calculate separately each term.*

**Remark 8.** *For $\pi_t^{k,N}$ the empirical measure associated to the k-th thermodynamic* *quantity introduced in* (A.1). *Then*

$$
\begin{aligned}
&\langle \pi_t^{k,N}, H\rangle I^{x,z,v} - \langle \pi_t^{k,N}, H\rangle \\
&= \frac{1}{N^d}\sum_{y\in D_N^d} I_k(\eta_y^{x,z,v}(t))H\left(\tfrac{y}{N}\right) - \frac{1}{N^d}\sum_{y\in D_N^d} I_k(\eta_y(t))H\left(\tfrac{y}{N}\right)\\
&= \frac{1}{N^d}\sum_{y\in D_N^d}\sum_{w\in\mathcal{V}} w_k\eta^{x,z,v}(y,w)H\left(\tfrac{y}{N}\right) - \frac{1}{N^d}\sum_{y\in D_N^d}\sum_{w\in\mathcal{V}} w_k\eta(y,w)H\left(\tfrac{y}{N}\right)\\
&= \frac{1}{N^d}\sum_{y\in D_N^d}\sum_{w\in\mathcal{V}} w_k H\left(\tfrac{y}{N}\right)\left[\eta^{x,z,v}(y,w) - \eta(y,w)\right].
\end{aligned}
$$

*Taking* $v = w$, $y = x$ *and* $v = w$, $y = z$,

$$
\begin{aligned}
&= \frac{1}{N^d}v_k H\left(\tfrac{x}{N}\right)\left[\eta(z,v) - \eta(x,v)\right] + \frac{1}{N^d}v_k H\left(\tfrac{z}{N}\right)\left[\eta(x,v) - \eta(z,v)\right]\\
&= \frac{1}{N^d}v_k\left[\eta(z,v) - \eta(x,v)\right]\left[H\left(\tfrac{z}{N}\right) - H\left(\tfrac{x}{N}\right)\right].
\end{aligned}
\tag{A.11}
$$

**Lemma 20.** *For* $\pi_t^{k,N}$ *the empirical measure associated to the* $k$*-th thermodynamic quantity introduced in* (A.1) *it holds,*

$$
\begin{aligned}
&N^2\mathcal{L}_N^{ex,1}\langle \pi_t^{k,N}, H\rangle^2 - 2\langle \pi_t^{k,N}, H\rangle N^2\mathcal{L}_N^{ex,1}\langle \pi_t^{k,N}, H\rangle\\
&= \frac{1}{2N^{2d}}\sum_{v\in\mathcal{V}}\sum_{x\in D_N^d}\sum_{j=1}^{d} v_k^2\left[\eta(x,v) - \eta(x+e_j,v))\right]^2\left[\partial_{u_j}^N H\left(\tfrac{x}{N}\right)\right]^2.
\end{aligned}
\tag{A.12}
$$

*Proof.* Note that

$$N^2\mathcal{L}_N^{ex,1}\langle \pi_t^{k,N},H\rangle^2 - 2\langle \pi_t^{k,N},H\rangle N^2\mathcal{L}_N^{ex,1}\langle \pi_t^{k,N},H\rangle$$

$$= \frac{N^2}{2}\sum_{v\in\mathcal{V}}\sum_{x,z\in D_N^d}\eta(x,v)(1-\eta(z,v))[\langle \pi_t^{k,N},H\rangle^2 I^{x,z,v} - \langle \pi_t^{k,N},H\rangle^2]$$

$$-2\langle \pi_t^{k,N},H\rangle\frac{N^2}{2}\sum_{v\in\mathcal{V}}\sum_{x,z}\eta(x,v)(1-\eta(z,v))[\langle \pi_t^{k,N},H\rangle I^{x,z,v} - \langle \pi_t^{k,N},H\rangle]$$

$$= \frac{N^2}{2}\sum_{v\in\mathcal{V}}\sum_{x,z\in D_N^d}\eta(x,v)(1-\eta(z,v))[\langle \pi_t^{k,N},H\rangle^2 I^{x,z,v} - \langle \pi_t^{k,N},H\rangle^2$$

$$-2\langle \pi_t^{k,N},H\rangle\langle \pi_t^{k,N},H\rangle I^{x,z,v} + 2\langle \pi_t^{k,N},H\rangle^2]$$

$$= \frac{N^2}{2}\sum_{v\in\mathcal{V}}\sum_{x,z\in D_N^d}\eta(x,v)(1-\eta(z,v))[\langle \pi_t^{k,N},H\rangle I^{x,z,v} - \langle \pi_t^{k,N},H\rangle]^2$$

$$= \frac{N^2}{2}\sum_{v\in\mathcal{V}}\sum_{x,z\in D_N^d}\eta(x,v)(1-\eta(z,v))\left[\frac{1}{N^d}v_k[\eta(z,v)-\eta(x,v)][H\left(\tfrac{z}{N}\right) - H\left(\tfrac{x}{N}\right)]\right]^2$$

$$= \frac{N^2}{2N^{2d}}\sum_{v\in\mathcal{V}}\sum_{x,z\in D_N^d}v_k^2\,\eta(x,v)(1-\eta(z,v))[\eta(z,v)-\eta(x,v)]^2[H\left(\tfrac{z}{N}\right) - H\left(\tfrac{x}{N}\right)]^2$$

$$= \frac{N^2}{2N^{2d}}\sum_{v\in\mathcal{V}}\sum_{x,z\in D_N^d}v_k^2\,\eta(x,v)(1-\eta(z,v))[H\left(\tfrac{z}{N}\right) - H\left(\tfrac{x}{N}\right)]^2$$

$$= \frac{N^2}{2N^{2d}}\sum_{v\in\mathcal{V}}\sum_{x\in D_N^d}\sum_{j=1}^{d}v_k^2\,\Big\{\eta(x,v)(1-\eta(x+e_j,v))[H\left(\tfrac{x+e_j}{N}\right) - H\left(\tfrac{x}{N}\right)]^2$$

$$+\,\eta(x,v)(1-\eta(x-e_j,v))\left[H\left(\tfrac{x-e_j}{N}\right) - H\left(\tfrac{x}{N}\right)\right]^2\Big\}$$

$$= \frac{N^2}{2N^{2d}}\sum_{v\in\mathcal{V}}\sum_{x\in D_N^d}\sum_{j=1}^{d}v_k^2\,[\eta(x,v)(1-\eta(x+e_j,v)) + \eta(x+e_j,v)(1-\eta(x,v))]$$

$$\times[H\left(\tfrac{x+e_j}{N}\right) - H\left(\tfrac{x}{N}\right)]^2$$

$$= \frac{N^2}{2N^{2d}}\sum_{v\in\mathcal{V}}\sum_{x\in D_N^d}\sum_{j=1}^{d}v_k^2\,[\eta(x,v)-\eta(x+e_j,v))]^2\,[H\left(\tfrac{x+e_j}{N}\right) - H\left(\tfrac{x}{N}\right)]^2$$

$$= \frac{1}{2N^{2d}}\sum_{v\in\mathcal{V}}\sum_{x\in D_N^d}\sum_{j=1}^{d}v_k^2\,[\eta(x,v)-\eta(x+e_j,v))]^2\,[\partial_{u_j}^N H\left(\tfrac{x}{N}\right)]^2.$$

□

**Lemma 21.** *For $\pi_t^{k,N}$ the empirical measure associated to the k-th thermodynamic quantity introduced in (A.1) it holds*

$$\begin{aligned}&N^2\mathcal{L}_N^{ex,2}\langle\pi_t^{k,N},H\rangle^2-2\langle\pi_t^{k,N},H\rangle N^2\mathcal{L}_N^{ex,2}\langle\pi_t^{k,N},H\rangle\\&=\frac{1}{N^{2d+1}}\sum_{v\in\mathcal{V}}\sum_{x\in D_N^d}\sum_{w\in\mathbb{Z}^d}v_k^2\,\eta(x,v)(1-\eta(x+w,v))p(w,v)w_j^2[\partial_{u_j}^N H\left(\tfrac{x}{N}\right)]^2.\end{aligned}\tag{A.13}$$

*Proof.* Note that

$$\begin{aligned}&N^2\mathcal{L}_N^{ex,2}\langle\pi_t^{k,N},H\rangle^2-2\langle\pi_t^{k,N},H\rangle N^2\mathcal{L}_N^{ex,2}\langle\pi_t^{k,N},H\rangle\\
&=N^2\frac{1}{N}\sum_{v\in\mathcal{V}}\sum_{x,z\in D_N^d}\eta(x,v)(1-\eta(z,v))p(z-x,v)[\langle\pi_t^{k,N},H\rangle^2I^{x,z,v}-\langle\pi_t^{k,N},H\rangle^2]\\
&-2\langle\pi_t^{k,N},H\rangle N^2\frac{1}{N}\sum_{v\in\mathcal{V}}\sum_{x,z\in D_N^d}\eta(x,v)(1-\eta(z,v))p(z-x,v)[\langle\pi_t^{k,N},H\rangle I^{x,z,v}-\langle\pi_t^{k,N},H\rangle]\\
&=N^2\frac{1}{N}\sum_{v\in\mathcal{V}}\sum_{x,z\in D_N^d}\eta(x,v)(1-\eta(z,v))p(z-x,v)[\langle\pi_t^{k,N},H\rangle^2I^{x,z,v}-\langle\pi_t^{k,N},H\rangle^2\\
&-2\langle\pi_t^{k,N},H\rangle\langle\pi_t^{k,N},H\rangle I^{x,z,v}+2\langle\pi_t^{k,N},H\rangle^2]\\
&=N^2\frac{1}{N}\sum_{v\in\mathcal{V}}\sum_{x,z\in D_N^d}\eta(x,v)(1-\eta(z,v))p(z-x,v)[\langle\pi_t^{k,N},H\rangle I^{x,z,v}-\langle\pi_t^{k,N},H\rangle]^2\\
&=\frac{N^2}{N}\sum_{v\in\mathcal{V}}\sum_{x,z\in D_N^d}\eta(x,v)(1-\eta(z,v))p(z-x,v)[\frac{1}{N^d}v_k\left[\eta(z,v)-\eta(x,v)\right]\left[H\left(\tfrac{z}{N}\right)-H\left(\tfrac{x}{N}\right)\right]]^2\\
&=\frac{N^2}{N^{2d+1}}\sum_{v\in\mathcal{V}}\sum_{x,z\in D_N^d}v_k^2\,\eta(x,v)(1-\eta(z,v))p(z-x,v)[\eta(z,v)-\eta(x,v)]^2[H\left(\tfrac{z}{N}\right)-H\left(\tfrac{x}{N}\right)]^2\\
&=\frac{N^2}{N^{2d+1}}\sum_{v\in\mathcal{V}}\sum_{x,z\in D_N^d}v_k^2\,\eta(x,v)(1-\eta(z,v))p(z-x,v)[H\left(\tfrac{z}{N}\right)-H\left(\tfrac{x}{N}\right)]^2\\
&=\frac{N^2}{N^{2d+1}}\sum_{v\in\mathcal{V}}\sum_{x\in D_N^d}\sum_{w\in\mathbb{Z}^d}v_k^2\,\eta(x,v)(1-\eta(x+w,v))p(w,v)[H(x+w/N)-H\left(\tfrac{x}{N}\right)]^2\\
&=\frac{1}{N^{2d+1}}\sum_{v\in\mathcal{V}}\sum_{x\in D_N^d}\sum_{w\in\mathbb{Z}^d}v_k^2\,\eta(x,v)(1-\eta(x+w,v))p(w,v)w_j^2[\partial_{u_j}^N H\left(\tfrac{x}{N}\right)]^2.\end{aligned}$$

□

**Remark 9.** *Note that*

$$
\begin{aligned}
&\langle \pi_t^{k,N}, H\rangle I^{x,v} - \langle \pi_t^{k,N}, H\rangle \\
&= \frac{1}{N^d}\sum_{y\in D_N^d} I_k(\eta_y^{x,v}(t))H\left(\tfrac{y}{N}\right) - \frac{1}{N^d}\sum_{y\in D_N^d} I_k(\eta_y(t))H\left(\tfrac{y}{N}\right)\\
&= \frac{1}{N^d}\sum_{y\in D_N^d}\sum_{w\in\mathcal{V}} w_k\eta^{x,v}(y,w)H\left(\tfrac{y}{N}\right) - \frac{1}{N^d}\sum_{y\in D_N^d}\sum_{w\in\mathcal{V}} w_k\eta(y,w)H\left(\tfrac{y}{N}\right)\\
&= \frac{1}{N^d}\sum_{y\in D_N^d}\sum_{w\in\mathcal{V}} w_k H\left(\tfrac{y}{N}\right)\left[\eta^{x,v}(y,w)-\eta(y,w)\right].
\end{aligned}
$$

*Taking $v = w$ and $y = x$ last expression is equal to*

$$
\begin{aligned}
&= \frac{1}{N^d}\left[v_k H\left(\tfrac{x}{N}\right)(1-\eta(x,v)) - v_k H\left(\tfrac{x}{N}\right)\eta(x,v)\right]\\
&= \frac{1}{N^d} v_k\left[1-2\eta(x,v)\right]H\left(\tfrac{x}{N}\right).
\end{aligned}
\tag{A.14}
$$

**Lemma 22.** *For $\pi_t^{k,N}$ the empirical measure associated to the k-th thermodynamic quantity introduced in* (A.1) *it holds*

$$
\begin{aligned}
&N^2\mathcal{L}_N^b\langle \pi_t^{k,N}, H\rangle^2 - 2\langle \pi_t^{k,N}, H\rangle N^2\mathcal{L}_N^b\langle \pi_t^{k,N}, H\rangle\\
&= \frac{N^2}{N^{2d}}\sum_{\substack{x\in D_N^d\\ x_1=1}}\sum_{v\in\mathcal{V}}\left[\frac{\alpha_v(\frac{\tilde{x}}{N})(1-\eta(x,v)) + (1-\alpha_v(\frac{\tilde{x}}{N}))\eta(x,v)}{N^\theta}\right] v_k^2\, H^2\left(\tfrac{x}{N}\right)\\
&+\frac{N^2}{N^{2d}}\sum_{\substack{x\in D_N^d\\ x_1=N-1}}\sum_{v\in\mathcal{V}}\left[\frac{\beta_v(\frac{\tilde{x}}{N})(1-\eta(x,v)) + (1-\beta_v(\frac{\tilde{x}}{N}))\eta(x,v)}{N^\theta}\right] v_k^2\, H^2\left(\tfrac{x}{N}\right).
\end{aligned}
\tag{A.15}
$$

*Proof.* In order to make the presentation easier, we denote by

$$
r_\alpha(\eta) = \left[\frac{\alpha_v(\frac{\tilde{x}}{N})(1-\eta(x,v)) + (1-\alpha_v(\frac{\tilde{x}}{N}))\eta(x,v)}{N^\theta}\right]
$$

and computing (A.15) for $x_1 = 1$, we obtain

$$
\begin{aligned}
& N^2\mathcal{L}_N^b\langle \pi_t^{k,N}, H\rangle^2 - 2\langle \pi_t^{k,N}, H\rangle N^2\mathcal{L}_N^b\langle \pi_t^{k,N}, H\rangle \\
&= N^2 \sum_{\substack{x\in D_N^d\\ x_1=1}} \sum_{v\in\mathcal{V}} r_\alpha(\eta)[\langle \pi_t^{k,N}, H\rangle^2 I^{x,v} - \langle \pi_t^{k,N}, H\rangle^2] \\
& -2\langle \pi_t^{k,N}, H\rangle N^2 \sum_{\substack{x\in D_N^d\\ x_1=1}} \sum_{v\in\mathcal{V}} r_\alpha(\eta)[\langle \pi_t^{k,N}, H\rangle I^{x,v} - \langle \pi_t^{k,N}, H\rangle] \\
&= \frac{N^2}{N^{2d}} \sum_{\substack{x\in D_N^d\\ x_1=1}} \sum_{v\in\mathcal{V}} r_\alpha(\eta) v_k^2\, H^2\left(\tfrac{x}{N}\right)(1-2\eta(x,v))^2 \\
&= \frac{N^2}{N^{2d}} \sum_{\substack{x\in D_N^d\\ x_1=1}} \sum_{v\in\mathcal{V}} r_\alpha(\eta) v_k^2\, H^2\left(\tfrac{x}{N}\right).
\end{aligned}
$$

For $x_1 = N-1$

$$
\begin{aligned}
& N^2\mathcal{L}_N^b\langle \pi_t^{k,N}, H\rangle^2 - 2\langle \pi_t^{k,N}, H\rangle N^2\mathcal{L}_N^b\langle \pi_t^{k,N}, H\rangle \\
&= \frac{N^2}{N^{2d}} \sum_{\substack{x\in D_N^d\\ x_1=N-1}} \sum_{v\in\mathcal{V}} \left[\frac{\beta_v(\frac{\tilde{x}}{N})(1-\eta(x,v)) + (1-\beta_v(\frac{\tilde{x}}{N}))\eta(x,v)}{N^\theta}\right] v_k^2\, H^2\left(\tfrac{x}{N}\right).
\end{aligned}
$$

□

*Therefore, by Lemmas 20, 21, and 22. We can establish the following proposition.*

**Proposition 12.** *For $\pi_t^{k,N}$ the empirical measure associated to the $k$-th thermodynamic quantity introduced in (A.1), it holds*

$$
\begin{aligned}
& \mathcal{L}_N\langle \pi_t^{k,N}, H\rangle^2 - 2\langle \pi_t^{k,N}, H\rangle \mathcal{L}_N\langle \pi_t^{k,N}, H\rangle \\
&= \frac{1}{2N^{2d}} \sum_{v\in\mathcal{V}} \sum_{x\in D_N^d} \sum_{j=1}^{d} v_k^2\, [\eta(x,v) - \eta(x+e_j,v))]^2\, [\partial_{u_j}^N H\left(\tfrac{x}{N}\right)]^2 \\
& + \frac{1}{N^{2d+1}} \sum_{v\in\mathcal{V}} \sum_{x\in D_N^d} \sum_{w\in\mathbb{Z}^d} v_k^2\, \eta(x,v)(1-\eta(x+w,v)) p(w,v) w_j^2 [\partial_{u_j}^N H\left(\tfrac{x}{N}\right)]^2 \\
& + \frac{N^2}{N^{2d}} \sum_{\substack{x\in D_N^d\\ x_1=1}} \sum_{v\in\mathcal{V}} \left[\frac{\alpha_v(\frac{\tilde{x}}{N})(1-\eta(x,v)) + (1-\alpha_v(\frac{\tilde{x}}{N}))\eta(x,v)}{N^\theta}\right] v_k^2\, H^2\left(\tfrac{x}{N}\right) \\
& + \frac{N^2}{N^{2d}} \sum_{\substack{x\in D_N^d\\ x_1=N-1}} \sum_{v\in\mathcal{V}} \left[\frac{\beta_v(\frac{\tilde{x}}{N})(1-\eta(x,v)) + (1-\beta_v(\frac{\tilde{x}}{N}))\eta(x,v)}{N^\theta}\right] v_k^2\, H^2\left(\tfrac{x}{N}\right).
\end{aligned}
\tag{A.16}
$$

## A.1 Deriving the weak formulation

*Note that the weak formulation of the system of PDEs can be obtained in the following way. Take a test function* $G \in C^{1,2}([0,T] \times D^d)$ *and multiply both sides of the equality*

$$\partial_t(\rho, \varrho) + \sum_{v \in \mathcal{V}} \tilde{v}[v \cdot \nabla \chi(\theta_v(\Lambda(\rho, \varrho)))] = \frac{1}{2}\Delta(\rho, \varrho) \tag{A.17}$$

*by* $G$ *and then integrate both sides in time and space to get*

$$\int_0^T \int_{D^d} \partial_t(\rho, \varrho) G(t,u)\, du\, dt + \int_0^T \int_{D^d} \sum_{v \in \mathcal{V}} \tilde{v}[v \cdot \nabla \chi(\theta_v(\Lambda(\rho, \varrho)))] G(t,u)\, du\, dt$$
$$= \frac{1}{2} \int_0^T \int_{D^d} \Delta(\rho, \varrho) G(t,u)\, du\, dt.$$

*Computing each term separately, we perform integration by parts in the time integral and we get to*

$$\int_0^T \int_{D^d} \partial_t(\rho, \varrho) G(t,u)\, du\, dt = \int_{D^d} (\rho, \varrho)(T,u) G(T,u)\, du - \int_{D^d} (\rho, \varrho)(0,u) G(0,u)\, du$$
$$- \int_0^T \int_{D^d} (\rho, \varrho)(t,u) \partial_t G(t,u)\, du\, dt.$$

*Let* $dS$ *be the Lebesgue measure on* $\mathbb{T}^{d-1}$*,*

$$\begin{aligned}
&\int_0^T \int_{D^d} \sum_{v \in \mathcal{V}} \tilde{v}[v \cdot \nabla \chi(\theta_v(\Lambda(\rho, \varrho)))] G(t,u)\, du\, dt = \\
&\int_0^T \int_{\{1\} \times \mathbb{T}^{d-1}} \sum_{v \in \mathcal{V}} \tilde{v}[v \cdot \chi(\theta_v(\Lambda(\rho, \varrho)))] G(t,u)\, dS\, dt \\
&- \int_0^T \int_{\{0\} \times \mathbb{T}^{d-1}} \sum_{v \in \mathcal{V}} \tilde{v}[v \cdot \chi(\theta_v(\Lambda(\rho, \varrho)))] G(t,u)\, dS\, dt \\
&- \int_0^T \int_{D^d} \sum_{v \in \mathcal{V}} \tilde{v} \cdot \chi(\theta_v(\Lambda(\rho, \varrho))) \sum_{i=1}^{d} v_i \frac{\partial G}{\partial u_i}(t,u)\, du\, dt.
\end{aligned} \tag{A.18}$$

*Doing integration by parts twice in the spacial integral*

$$\frac{1}{2}\int_0^T\int_{D^d}\Delta(\rho,\varrho)G(t,u)\,du\,dt=\frac{1}{2}\int_0^T\int_{\{1\}\times\mathbb{T}^{d-1}}\frac{\partial(\rho,\varrho)}{\partial u_1}(t,u)G(t,u)\,dS\,dt$$

$$-\frac{1}{2}\int_0^T\int_{\{0\}\times\mathbb{T}^{d-1}}\frac{\partial(\rho,\varrho)}{\partial u_1}(t,u)G(t,u)\,dS\,dt-\frac{1}{2}\int_0^T\int_{\{1\}\times\mathbb{T}^{d-1}}(\rho,\varrho)(t,u)\frac{\partial G}{\partial u_1}(t,u)\,dS\,dt$$

$$+\frac{1}{2}\int_0^T\int_{\{0\}\times\mathbb{T}^{d-1}}(\rho,\varrho)(t,u)\frac{\partial G}{\partial u_1}(t,u)\,dS\,dt+\frac{1}{2}\int_0^T\int_{D^d}(\rho,\varrho)(t,u)\sum_{i=1}^{d}\frac{\partial^2 G}{\partial u_i^2}(t,u)\,du\,dt.$$

*Putting together the last identities, we obtain*

$$\int_{D^d}(\rho,\varrho)(T,u)G(T,u)\,du-\int_{D^d}(\rho,\varrho)(0,u)G(0,u)\,du=$$

$$\int_0^T\int_{D^d}(\rho,\varrho)(t,u)\partial_t G(t,u)\,du\,dt-\int_0^T\int_{\{1\}\times\mathbb{T}^{d-1}}\sum_{v\in\mathcal{V}}\tilde{v}[v\cdot\chi(\theta_v(\Lambda(\rho,\varrho)))]G(t,u)\,dS\,dt$$

$$+\int_0^T\int_{\{0\}\times\mathbb{T}^{d-1}}\sum_{v\in\mathcal{V}}\tilde{v}[v\cdot\chi(\theta_v(\Lambda(\rho,\varrho)))]G(t,u)\,dS\,dt$$

$$+\int_0^T\int_{D^d}\sum_{v\in\mathcal{V}}\tilde{v}\cdot\chi(\theta_v(\Lambda(\rho,\varrho)))\sum_{i=1}^{d}v_i\frac{\partial G}{\partial u_i}(t,u)\,du\,dt$$

$$+\frac{1}{2}\int_0^T\int_{\{1\}\times\mathbb{T}^{d-1}}\frac{\partial(\rho,\varrho)}{\partial u_1}(t,u)G(t,u)\,dS\,dt-\frac{1}{2}\int_0^T\int_{\{0\}\times\mathbb{T}^{d-1}}\frac{\partial(\rho,\varrho)}{\partial u_1}(t,u)G(t,u)\,dS\,dt$$

$$-\frac{1}{2}\int_0^T\int_{\{1\}\times\mathbb{T}^{d-1}}(\rho,\varrho)(t,u)\frac{\partial G}{\partial u_1}(t,u)\,dS\,dt+\frac{1}{2}\int_0^T\int_{\{0\}\times\mathbb{T}^{d-1}}(\rho,\varrho)(t,u)\frac{\partial G}{\partial u_1}(t,u)\,dS\,dt$$

$$+\frac{1}{2}\int_0^T\int_{D^d}(\rho,\varrho)(t,u)\sum_{i=1}^{d}\frac{\partial^2 G}{\partial u_i^2}(t,u)\,du\,dt. \tag{A.19}$$

# Appendix B

*In this appendix, we establish some technical results that are needed in order to prove the equilibrium fluctuations for the model discussed in the previous sections.*

### B.0.1 Computations of $\mathcal{L}_N[Y_t^{N,k}(H)]$

*We will compute separately, in order to simplify the presentation. Recall that*

$$Y_t^{N,k}(H) = N^{-\frac{d}{2}} \sum_{x \in D_N^d} H\left(\tfrac{x}{N}\right)(I_k(\eta_x(t)) - \rho^k). \tag{B.1}$$

**Lemma 23.** *For a test function $H$, we obtain that*

$$N^2 \mathcal{L}_N^{ex,1}[Y_t^{N,k}(H)] = \frac{1}{2} Y_t^{N,k}[\Delta_N H\left(\tfrac{x}{N}\right)]. \tag{B.2}$$

*Proof.*

$$N^2\mathcal{L}_N^{ex,1}[Y_t^{N,k}(H)] = N^2\mathcal{L}_N^{ex,1}\left[N^{-\frac{d}{2}}\sum_{x\in\mathbb{T}_N^d} H\left(\tfrac{x}{N}\right)\left(I_k(\eta_x(t))-\rho^k\right)\right]$$

$$= N^{-\frac{d}{2}+2}\sum_{x\in\mathbb{T}_N^d} H\left(\tfrac{x}{N}\right)\mathcal{L}_N^{ex,1}\left(I_k(\eta_x(t))\right)$$

$$= N^{-\frac{d}{2}+2}\sum_{x\in\mathbb{T}_N^d} H\left(\tfrac{x}{N}\right)\frac{1}{2}\sum_{j=1}^{d}[I_k(\eta_{x+e_j}(t))+I_k(\eta_{x-e_j}(t))-2I_k(\eta_x(t))]$$

$$= \tfrac{1}{2}N^{-\frac{d}{2}+2}\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^{d}\left[H\left(\tfrac{x}{N}\right)I_k(\eta_{x+e_j}(t))+H\left(\tfrac{x}{N}\right)I_k(\eta_{x-e_j}(t))-2H\left(\tfrac{x}{N}\right)I_k(\eta_x(t))\right]$$

$$= \tfrac{1}{2}N^{-\frac{d}{2}+2}\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^{d}\left[H\left(\tfrac{x-e_j}{N}\right)I_k(\eta_x(t))+H\left(\tfrac{x+e_j}{N}\right)I_k(\eta_x(t))-2H\left(\tfrac{x}{N}\right)I_k(\eta_x(t))\right]$$

$$= \frac{1}{2}N^{-\frac{d}{2}+2}\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^{d}\left[H\left(\tfrac{x-e_j}{N}\right)+H\left(\tfrac{x+e_j}{N}\right)-2H\left(\tfrac{x}{N}\right)\right]I_k(\eta_x(t))$$

$$= \frac{1}{2}N^{-\frac{d}{2}+2}\sum_{x\in\mathbb{T}_N^d}\Delta_N H\left(\tfrac{x}{N}\right)I_k(\eta_x(t))$$

$$= \frac{1}{2}N^{-\frac{d}{2}+2}\sum_{x\in\mathbb{T}_N^d}\Delta_N H\left(\tfrac{x}{N}\right)\left[I_k(\eta_x(t))-\rho^k\right]$$

$$= \frac{1}{2}Y_t^{N,k}[\Delta_N H\left(\tfrac{x}{N}\right)].$$

□

**Lemma 24.** *For a test function $H$, we obtain that*

$$N^2\mathcal{L}_N^{ex,2}[Y_t^{N,k}(H)] = -N^{-\frac{d}{2}}\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^{d}(\partial_{u_j}^N H)\left(\tfrac{x}{N}\right)[\tau_x W_{j,k}^t-\omega_k^{\rho,\varrho}], \tag{B.3}$$

*where*

$$W_{j,k}^t := \sum_{v\in\mathcal{V}} v_k \sum_{z\in\mathbb{Z}^d} z_j p(z,v)\eta_t(0,v)[1-\eta_t(z,v)]$$

*and*

$$\omega_k^{\rho,\varrho} := E_{\nu_{\rho,\varrho}^N}[W_{j,k}^t] = \sum_{v\in\mathcal{V}} v_k v_j \chi(\theta_v(\Lambda(\rho,\varrho))).$$

*Proof.*

$$N^2\mathcal{L}_N^{ex,2}[Y_t^{N,k}(H)] = N^2\mathcal{L}_N^{ex,2}\left[N^{-\frac{d}{2}}\sum_{x\in\mathbb{T}_N^d} H\left(\tfrac{x}{N}\right)\big(I_k(\eta_x(t)) - \rho^k\big)\right]$$

$$= N^{-\frac{d}{2}+2}\sum_{x\in\mathbb{T}_N^d} H\left(\tfrac{x}{N}\right)\mathcal{L}_N^{ex,2}\big(I_k(\eta_x(t)\big)$$

$$= -N^{-\frac{d}{2}}\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^{d}(\partial_{u_j}^N H)\left(\tfrac{x}{N}\right)\tau_x W_{j,k}^t$$

$$= -N^{-\frac{d}{2}}\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^{d}(\partial_{u_j}^N H)\left(\tfrac{x}{N}\right)[\tau_x W_{j,k}^t - \omega_k^{\rho,\varrho}]$$

$$= \sum_{j=1}^{d}\nabla F^{j,k}(\rho,\varrho)Y_t^{N,k}[(\partial_{u_j}^N H)\left(\tfrac{x}{N}\right)].$$

□

**Lemma 25.** *For a test function $H$, we obtain that*

$$N^2\mathcal{L}_N^c[Y_t^{N,k}(H)] = 0. \tag{B.4}$$

*Proof.*

$$N^2\mathcal{L}_N^c[Y_t^{N,k}(H)] = N^2\mathcal{L}_N^c\left[N^{-\frac{d}{2}}\sum_{x\in\mathbb{T}_N^d} H\left(\tfrac{x}{N}\right)\big(I_k(\eta_x(t)) - \rho^k\big)\right]$$

$$= N^{-\frac{d}{2}+2}\sum_{x\in\mathbb{T}_N^d} H\left(\tfrac{x}{N}\right)\mathcal{L}_N^c\big(I_k(\eta_x(t)\big)$$

$$= 0.$$

□

### B.0.2 Computations of $\mathcal{L}_N\Big([Y_t^{N,k}(H)]^2\Big) - 2Y_t^{N,k}(H)\mathcal{L}_N[Y_t^{N,k}(H)]$

*We will compute each term separately, in order to simplify the presentation.*

**Remark 10.** *Note that*

$$Y_t^{N,k}(H)(I^{x,z,v}) - Y_t^{N,k}(H) = N^{-\frac{d}{2}}v_k[H\left(\tfrac{x}{N}\right) - H\left(\tfrac{z}{N}\right)][\eta(x,v) - \eta(z,v)]. \tag{B.5}$$

*Indeed,*

$$
\begin{aligned}
& Y_t^{N,k}(H)(I^{x,z,v}) - Y_t^{N,k}(H) \\
&= N^{-\frac{d}{2}} \sum_{y \in \mathbb{T}_N^d} H\left(\tfrac{y}{N}\right)[I_k(\eta^{x,z,v}(y)) - \rho^k] - N^{-\frac{d}{2}} \sum_{y \in \mathbb{T}_N^d} H\left(\tfrac{y}{N}\right)[I_k(\eta_y(t)) - \rho^k] \\
&= N^{-\frac{d}{2}} \sum_{y \in \mathbb{T}_N^d} H\left(\tfrac{y}{N}\right)[I_k(\eta^{x,z,v}(y)) - I_k(\eta_y(t))] \\
&= N^{-\frac{d}{2}} \sum_{y \in \mathbb{T}_N^d} H\left(\tfrac{y}{N}\right)[\sum_{v \in \mathcal{V}} v_k \eta^{x,z,v}(y) - \sum_{v \in \mathcal{V}} v_k \eta(y)].
\end{aligned}
$$

*If $y = x$ and $y = z$, because when $y \neq x, z$ the equation above vanishes. Therefore, the last display is equal to*

$$
\begin{aligned}
& N^{-\frac{d}{2}}[H\left(\tfrac{x}{N}\right) v_k \eta(z,v) - H\left(\tfrac{x}{N}\right) v_k \eta(x,v) + H\left(\tfrac{z}{N}\right) v_k \eta(x,v) - H\left(\tfrac{z}{N}\right) v_k \eta(z,v)] \\
&= N^{-\frac{d}{2}} v_k [H\left(\tfrac{x}{N}\right) - H\left(\tfrac{z}{N}\right)][\eta(x,v) - \eta(z,v)].
\end{aligned}
\tag{B.6}
$$

*Using the notation introduced in Remark 10, we obtain that*

$$
N^2 \mathcal{L}_N^{ex,1}[Y_t^{N,k}(H)] = \frac{N^2}{2} \sum_{v \in \mathcal{V}} \sum_{x,z \in \mathbb{T}_N^d} \eta(x,v)[1 - \eta(z,v)]\left[Y_t^{N,k}(H)(I^{x,z,v}) - (Y_t^{N,k}(H))\right].
\tag{B.7}
$$

*Now we will prove the following lemmas:*

**Lemma 26.** *Let $H$ be a test function. We have that*

$$
\begin{aligned}
& N^2 \mathcal{L}_N^{ex,1}[(Y_t^{N,k}(H))^2] - 2 Y_t^{N,k}(H) N^2 \mathcal{L}_N^{ex,1}[Y_t^{N,k}(H)] \\
&= \frac{1}{2N^d} \sum_{v \in \mathcal{V}} \sum_{x \in \mathbb{T}_N^d} \sum_{j=1}^{d} v_k^2 \Big(\eta(x,v) - \eta(x+e_j,v)\Big)^2 (\partial_{u_j}^N H\left(\tfrac{x}{N}\right))^2.
\end{aligned}
\tag{B.8}
$$

*Proof.* Using the notation we introduced above, we have that

$$N^2\mathcal{L}_N^{ex,1}[(Y_t^{N,k}(H))^2] - 2Y_t^{N,k}(H)N^2\mathcal{L}_N^{ex,1}[Y_t^{N,k}(H)]$$

$$= \frac{N^2}{2}\sum_{v\in\mathcal{V}}\sum_{\substack{x,z\in\mathbb{T}_N^d\\ ||x-z||=1}} \eta(x,v)[1-\eta(z,v)]\big[(Y_t^{N,k}(H))^2(I^{x,z,v}) - (Y_t^{N,k}(H))^2\big]$$
$$-2Y_t^{N,k}(H)\frac{N^2}{2}\sum_{v\in\mathcal{V}}\sum_{\substack{x,z\in\mathbb{T}_N^d\\ ||x-z||=1}} \eta(x,v)[1-\eta(z,v)]\big[(Y_t^{N,k}(H))(I^{x,z,v}) - (Y_t^{N,k}(H))\big]$$
$$= \frac{N^2}{2}\sum_{v\in\mathcal{V}}\sum_{\substack{x,z\in\mathbb{T}_N^d\\ ||x-z||=1}} \eta(x,v)[1-\eta(z,v)]\big[\big((Y_t^{N,k}(H))(I^{x,z,v}) - (Y_t^{N,k}(H))\big)^2\big].$$

Now using the Remark 10 last expression is equal to

$$\tfrac{N^2}{2}\sum_{v\in\mathcal{V}}\sum_{\substack{x,z\in\mathbb{T}_N^d\\ ||x-z||=1}} \eta(x,v)[1-\eta(z,v)]\left[N^{-\frac{d}{2}}v_k(H\left(\tfrac{x}{N}\right) - H\left(\tfrac{z}{N}\right))(\eta(x,v)-\eta(z,v))\right]^2$$
$$= \frac{N^2}{2}N^{-d}\sum_{v\in\mathcal{V}}\sum_{\substack{x,z\in\mathbb{T}_N^d\\ ||x-z||=1}} \eta(x,v)[1-\eta(z,v)]v_k^2(H\left(\tfrac{x}{N}\right) - H\left(\tfrac{z}{N}\right))^2(\eta(x,v)-\eta(z,v))^2. \tag{B.9}$$

**Remark 11.** *Note that*

$$\eta(x,v)(1-\eta(z,v))(\eta(z,v)-\eta(x,v))^2$$

$$= (\eta(x,v)-\eta(x,v)\eta(z,v))[\eta(z,v)^2 - 2\eta(z,v)\eta(x,v) + \eta(x,v)^2]$$

$$= \eta(x,v)\eta(z,v) - 2\eta(z,v)\eta(x,v) + \eta(x,v) - \eta(x,v)\eta(z,v)$$

$$+2\eta(x,v)\eta(z,v) - \eta(x,v)\eta(z,v)$$

$$= \eta(x,v)[1-\eta(z,v)].$$

By this remark, we can rewrite (B.9) as

$$= \frac{N^2}{2N^d}\sum_{v\in\mathcal{V}}\sum_{\substack{x,z\in\mathbb{T}_N^d\\ ||x-z||=1}} \eta(x,v)[1-\eta(z,v)]v_k^2(H\left(\tfrac{x}{N}\right) - H\left(\tfrac{z}{N}\right))^2.$$

Since $\|x-z\|=1$, we obtain

$$\frac{N^2}{2N^d}\sum_{v\in\mathcal{V}}\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^{d}v_k^2\Big[\eta(x,v)(1-\eta(x+e_j,v))(H\left(\tfrac{x}{N}\right)-H\left(\tfrac{x+e_j}{N}\right))^2$$
$$+\eta(x,v)(1-\eta(x-e_j,v))(H\left(\tfrac{x-e_j}{N}\right)-H\left(\tfrac{x}{N}\right))^2\Big]$$

$$=\frac{N^2}{2N^d}\sum_{v\in\mathcal{V}}\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^{d}v_k^2\Big[\eta(x,v)(1-\eta(x+e_j,v))(H\left(\tfrac{x}{N}\right)-H\left(\tfrac{x+e_j}{N}\right))^2$$
$$+\eta(x+e_j,v)(1-\eta(x,v))(H\left(\tfrac{x}{N}\right)-H\left(\tfrac{x+e_j}{N}\right))^2\Big] \tag{B.10}$$

$$=\frac{N^2}{2N^d}\sum_{v\in\mathcal{V}}\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^{d}v_k^2\left[\eta(x,v)(1-\eta(x+e_j,v))+\eta(x+e_j,v)(1-\eta(x,v))\right]$$
$$\times(H\left(\tfrac{x}{N}\right)-H\left(\tfrac{x+e_j}{N}\right))^2$$

$$=\frac{N^2}{2N^d}\sum_{v\in\mathcal{V}}\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^{d}v_k^2\left(\eta(x,v)-\eta(x+e_j,v)\right)^2(H\left(\tfrac{x}{N}\right)-H\left(\tfrac{x+e_j}{N}\right))^2$$
$$=\frac{1}{2N^d}\sum_{v\in\mathcal{V}}\sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^{d}v_k^2\left(\eta(x,v)-\eta(x+e_j,v)\right)^2(\partial_{u_j}^N H\left(\tfrac{x}{N}\right))^2. \tag{B.11}$$

□

**Lemma 27.** *For a test function $H$, we have that*

$$N^2\mathcal{L}_N^{ex,2}[(Y_t^{N,k}(H))^2]-2Y_t^{N,k}(H)N^2\mathcal{L}_N^{ex,2}[Y_t^{N,k}(H)]$$

$$=\frac{1}{2N^{d+1}}\sum_{v\in\mathcal{V}}v_k^2\sum_{x\in\mathbb{T}_N^d}\sum_{|w|\le R}\sum_{j=1}^{d}\eta(x,v)(1-\eta(x+w,v))p(w,v)\left(\partial_j^N H\left(\tfrac{x}{N}\right)\right)^2 w_j^2. \tag{B.12}$$

*Proof.* Using Remark 10, we can rewrite

$$N^2\mathcal{L}_N^{ex,2}[(Y_t^{N,k}(H))^2] - 2Y_t^{N,k}(H)N^2\mathcal{L}_N^{ex,2}[Y_t^{N,k}(H)]$$

$$= \frac{N^2}{2}\left\{\frac{1}{N}\sum_{v\in\mathcal{V}}\sum_{x,z\in\mathbb{T}_N^d}\eta(x,v)(1-\eta(z,v))p(z-x,v)\Big[Y_t^{N,k}(H)(I^{x,z,v}) - Y_t^{N,k}(H)\Big]^2\right\}$$
$$= \frac{N^2}{2N^{d+1}}\sum_{v\in\mathcal{V}}\sum_{x,z\in\mathbb{T}_N^d}\eta(x,v)(1-\eta(z,v))p(z-x,v)v_k^2\Big(\eta(x,v)-\eta(z,v)\Big)^2(H\left(\tfrac{x}{N}\right) - H\left(\tfrac{z}{N}\right))^2$$
$$= \frac{N^2}{2N^{d+1}}\sum_{v\in\mathcal{V}}\sum_{x,z\in\mathbb{T}_N^d} v_k^2\,\eta(x,v)(1-\eta(z,v))p(z-x,v)(H\left(\tfrac{x}{N}\right) - H\left(\tfrac{z}{N}\right))^2.$$

Note that $p(w,v) = 0$ for $|w| > R$, where $R$ is the range of $p$. Therefore, writing $z - x = w$ with $|w| \le R$, we obtain

$$\frac{N^2}{2N^{d+1}}\sum_{v\in\mathcal{V}}v_k^2\sum_{x,w\in\mathbb{T}_N^d}\eta(x,v)(1-\eta(x+w,v))p(w,v)\left(H(x+w/N) - H\left(\tfrac{x}{N}\right)\right)^2$$
$$= \frac{1}{2N^{d+1}}\sum_{v\in\mathcal{V}}v_k^2\sum_{x,w\in\mathbb{T}_N^d}\eta(x,v)(1-\eta(x+w,v))p(w,v)\left(\sum_{j=1}^{d}\partial_j^N H\left(\tfrac{x}{N}\right)w_j\right)^2$$
$$= \frac{1}{2N^{d+1}}\sum_{v\in\mathcal{V}}v_k^2\sum_{x,w\in\mathbb{T}_N^d}\sum_{j=1}^{d}\eta(x,v)(1-\eta(x+w,v))p(w,v)\left(\partial_j^N H\left(\tfrac{x}{N}\right)\right)^2 w_j^2$$
$$= \frac{1}{2N^{d+1}}\sum_{v\in\mathcal{V}}v_k^2\sum_{x\in\mathbb{T}_N^d}\sum_{|w|\le R}\sum_{j=1}^{d}\eta(x,v)(1-\eta(x+w,v))p(w,v)\left(\partial_j^N H\left(\tfrac{x}{N}\right)\right)^2 w_j^2.$$

□

**Lemma 28.** *For a test function $H$, we obtain that*

$$N^2\mathcal{L}_N^c\Big([Y_t^{N,k}(H)]^2\Big) = 0. \tag{B.13}$$

*Proof.*

$$
\begin{aligned}
&N^2\mathcal{L}_N^c\Big([Y_t^{N,k}(H)]^2\Big) = N^2\mathcal{L}_N^c\left[\left(N^{-\frac{d}{2}}\sum_{x\in\mathbb{T}_N^d} H\left(\tfrac{x}{N}\right)\left(I_k(\eta_x(t))-\rho^k\right)\right)^2\right]\\
&= N^{-d+2}\mathcal{L}_N^c\left[\sum_{x,y\in\mathbb{T}_N^d} H\left(\tfrac{x}{N}\right)H\left(\tfrac{y}{N}\right)\left(I_k(\eta_x(t))-\rho^k\right)\left(I_k(\eta_y(t))-\rho^k\right)\right]\\
&= N^{-d+2}\sum_{x,y\in\mathbb{T}_N^d} H\left(\tfrac{x}{N}\right)H\left(\tfrac{y}{N}\right)\,\mathcal{L}_N^c\left[I_k(\eta_x(t))I_k(\eta_y(t)-\rho^k I_k(\eta_x(t)-\rho^k I_k(\eta_y(t)+(\rho^k)^2\right]\\
&= N^{-d+2}\sum_{x,y\in\mathbb{T}_N^d} H\left(\tfrac{x}{N}\right)H\left(\tfrac{y}{N}\right)\,\mathcal{L}_N^c\left[I_k(\eta_x(t))I_k(\eta_y(t)\right],
\end{aligned}
\tag{B.14}
$$

note that $\mathcal{L}_N^c[(\rho_k)^2]=0$ and $\mathcal{L}_N^c[I_k(\eta_x(t))]=0$ because momentum is preserved.

**Claim 5.** *To conclude the computations, let us prove that for every real function $f$ if $I_k(\eta_x)=I_k(\eta_x^{z,q})$ it holds*

$$\mathcal{L}_N^c[f(I_k(\eta))]=0.$$

This claim close the case when $x=y$, in the last equality of (B.14).

*Proof of the claim.* In fact, if $v,w\in\mathcal{V}$ are such that

$$\begin{cases}\eta(z,v')=\eta(z,w')=0\\ \eta(z,v)=\eta(z,w)=1,\end{cases}$$

then, for $q=(v,w,v',w')\in Q$. We have

$$\begin{cases}\eta^{y,q}(z,v)=\eta^{y,q}(z,w)=0\\ \eta^{y,q}(z,v')=\eta^{y,q}(z,w')=1.\end{cases}$$

If $z\neq x$ we obtain $I_k(\eta_x^{z,q})=I_k(\eta_x)$.

On the other hand, if $z=x$,

$$
\begin{aligned}
I_k(\eta_x^{x,q}) &= \sum_{v\in\mathcal{V}} v_k\eta^{x,q}(x,v)\\
&= \sum_{\tilde v\in\tilde{\mathcal{V}}^*}\tilde v_k+v'_k+w'_k\\
&= \sum_{\tilde v\in\tilde{\mathcal{V}}^*}\tilde v_k+v_k+w_k
\end{aligned}
$$

where $\tilde{\mathcal{V}}=\{\tilde v\in\mathcal{V};\ \eta(x,\tilde v)=1\}$ and $\tilde{\mathcal{V}}^*=\tilde v\backslash\{v,w,v',w'\}$. Therefore the last display results to be equal to

$$\sum_{\tilde{v}\in\tilde{\mathcal{V}}} \tilde{v}_k = I_k(\eta_x). \tag{B.15}$$

From the Claim 5, we can conclude that

$$\mathcal{L}_N^c([I_k(\eta_x)]^2) = 0.$$

**Claim 6.** *To finish the proof of Lemma 28, we need to show that*

$$I_k(\eta_x)I_k(\eta_y) = I_k(\eta_x^{z,q})I_k(\eta_y^{z,q}).$$

*Proof of the claim.* In fact, observe that

$$\text{if } z \notin \{x,y\} \Longrightarrow \begin{cases} I_k(\eta_x^{z,q}) = I_k(\eta_x) \\ I_k(\eta_y^{z,q}) = I_k(\eta_y) \end{cases}$$

consequently,

$$I_k(\eta_x)I_k(\eta_y) = I_k(\eta_x^{z,q})I_k(\eta_y^{z,q}).$$

On the other hand, if

$$z = x \quad \Longrightarrow \begin{cases} I_k(\eta_x^{z,q}) = I_k(\eta_x^{x,q}) = I_k(\eta_x) \\ I_k(\eta_y^{z,q}) = I_k(\eta_y^{x,q}) = I_k(\eta_y) \end{cases}$$

we also have that

$$I_k(\eta_x)I_k(\eta_y) = I_k(\eta_x^{z,q})I_k(\eta_y^{z,q}).$$

If

$$z = y \quad \Longrightarrow \begin{cases} I_k(\eta_x^{z,q}) = I_k(\eta_x^{y,q}) = I_k(\eta_x) \\ I_k(\eta_y^{z,q}) = I_k(\eta_y^{y,q}) = I_k(\eta_y) \end{cases}$$

we obtain

$$I_k(\eta_x)I_k(\eta_y) = I_k(\eta_x^{z,q})I_k(\eta_y^{z,q}),$$

and this proves the claim.

From the Claims 5 and 6, we have that

$$N^2\mathcal{L}_N^c\Big([Y_t^{N,k}(H)]^2\Big) = 0.$$

this finishes the proof of the Lemma 28.

## B.1 Some extra results

**Remark 12.** *By Taylor expansion, we have that*

$$\mathbb{E}_N\left[\left(\int_0^t Y_s^{N,k}\left(\frac{1}{2}\Delta_N H\right)ds - \int_0^t Y_s^{N,k}\left(\frac{1}{2}\Delta H\right)ds\right)^2\right] \longrightarrow 0,$$

*as $N \to +\infty$. Also*

$$\begin{aligned}\mathbb{E}_N\Bigg[\Bigg(&\sum_{v\in\mathcal{V}} v_k \sum_{i=0}^{d}\int_0^t Y_s^{N,i}(\sum_{j=1}^{d} v_j \partial_{\rho_i} F_v(\rho,\varrho)\partial_{u_j}^N H)\,ds\\ -&\sum_{v\in\mathcal{V}} v_k \sum_{i=0}^{d}\int_0^t Y_s^{N,i}(\sum_{j=1}^{d} v_j \partial_{\rho_i} F_v(\rho,\varrho)\partial_{u_j} H)\,ds\Bigg)^2\Bigg] \longrightarrow 0,\end{aligned}$$

*as $N \to +\infty$.*

□

**Remark 13.** *Note that*

$$\langle -\mathcal{L}_N^c f, f\rangle_{\nu_{\rho,\varrho}^N} \geq 0. \tag{B.16}$$

*Proof.* By writing the term at the left-hand side of (B.16) as its half plus its half, we have that

$$\begin{aligned}\langle -\mathcal{L}_N^c f, f\rangle_{\nu_\kappa^N} &= -\frac{1}{2}\int \sum_{y\in D_N^d}\sum_{q\in Q} p_c(y,q,\eta)[f(\eta^{y,q}) - f(\eta)]f(\eta)\,d\nu_\kappa^N\\ &-\frac{1}{2}\int \sum_{y\in D_N^d}\sum_{q\in Q} p_c(y,q,\eta)[f(\eta^{y,q}) - f(\eta)]f(\eta)\,d\nu_\kappa^N\end{aligned}$$

Performing a change of variables in one of the terms, last display is equal to

$$\begin{aligned}&-\frac{1}{2}\int \sum_{y\in D_N^d}\sum_{q\in Q} p_c(y,q,\eta)[f(\eta^{y,q}) - f(\eta)]f(\eta)\,d\nu_\kappa^N\\ &+\frac{1}{2}\int \sum_{y\in D_N^d}\sum_{q\in Q} p_c(y,q,\eta)[f(\eta^{y,q}) - f(\eta)]f(\eta^{y,q})\,d\nu_\kappa^N\\ &=\frac{1}{2}\int \sum_{y\in D_N^d}\sum_{q\in Q} p_c(y,q,\eta)[f(\eta^{y,q}) - f(\eta)]^2\,d\nu_\kappa^N\end{aligned}$$

this implies that $\langle -\mathcal{L}_N^c f, f\rangle_{\nu_{\rho,\varrho}^N} \geq 0$. □

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