Universidade Federal da Paraíba
Centro de Ciências Exatas e da Natureza
Programa de Pós-Graduação em Matemática
Doutorado em Matemática

Unimodular multilinear forms on sequence spaces and summability principles

por

Lisiane Rezende dos Santos

João Pessoa - PB
Agosto de 2020

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Unimodular multilinear forms on sequence spaces and summability principles

por

Lisiane Rezende dos Santos [†]

sob orientação do

Prof. Dr. Nacib Gurgel Albuquerque

Tese apresentada ao Corpo Docente do Programa de Pós-Graduação em Matemática da UFPB, como requisito parcial para obtenção do título de Doutor em Matemática.

João Pessoa - PB

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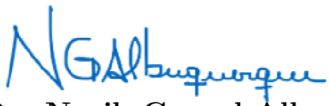
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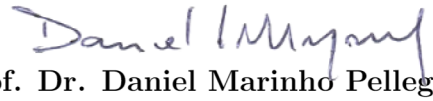
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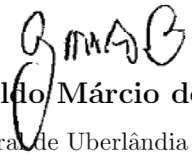
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Arthur Schopenhauer

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Dedicatória

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Resumo

Este trabalho está dividido em duas partes. Inicialmente investigamos a existência de formas unimodulares (complexas ou reais) com coeficientes de módulo 1 com norma *relativamente* pequena em espaços ℓ_p^n. Melhoramos a estimativa superior da norma sup da desigualdade de Kahane–Salem–Zygmund para formas multilineares: dados inteiros $m, n_1, \ldots, n_m \geq 1$ e $(p_1, \ldots, p_m) \in [1, +\infty]^m$, existe uma aplicação m-linear $A : \ell_{p_1}^{n_1} \times \cdots \times \ell_{p_m}^{n_m} \to \mathbb{K}$ com $\|A\| \leq B_m \cdot \left(\sum_{k=1}^m n_k\right)^{1-\frac{1}{\gamma}} \cdot \prod_{k=1}^m n_k^{\max\left(\frac{1}{\gamma}-\frac{1}{p_k},0\right)}$, onde $\gamma := \min\{2, \max\{p_k : p_k \leq 2\}\}$ e $B_m > 0$ é constante. Esta estimativa é usada para respondermos, de forma definitiva, o comportamento assintótico do ínfimo das normas de formas unimodulares em ℓ_p^n quando $p \geq 2$. A segunda parte trata de temas de somabilidade de operadores multilineares. Investigamos uma recente técnica introduzida por Pellegrino *et al.* [56] para obter um princípio de regularidade em espaços de sequências e, como aplicação, aprimoramos um resultado de inclusão para operadores somantes. Em seguida, no Capítulo 3, tratamos de uma noção geral de somabilidade (operadores Λ–somantes) que unifica as noções de somabilidade múltipla e absoluta para operadores multilineares, onde fornecemos um resultado geral de inclusão que engloba os correspondentes resultados de cada classe. Aplicações à teoria da desigualdade multilinear de Hardy–Littlewood são obtidas em ambos contextos.

Palavras-chave: Desigualdade de Kahane–Salem–Zygmund; formas unimodulares; estimativas assintóticas; operadores somantes; Teorema de Inclusão; Princípio de Regularidade; Desigualdade de Hardy–Littlewood.

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Abstract

This work is divided into two parts. Initially, we investigate the existence of unimodular (complex or real) forms with *relatively* small norms on ℓ_p^n spaces. We improve the sup norm upper estimate in Kahane–Salem–Zygmund inequality for multilinear forms: given positive $m, n_1, \ldots, n_m \geq 1$ and $(p_1, \ldots, p_m) \in [1, +\infty]^m$, there exists an m-linear map $A : \ell_{p_1}^{n_1} \times \cdots \times \ell_{p_m}^{n_m} \to \mathbb{K}$ with $\|A\| \leq B_m \cdot \left(\sum_{k=1}^m n_k\right)^{1-\frac{1}{\gamma}} \cdot \prod_{k=1}^m n_k^{\max\left(\frac{1}{\gamma}-\frac{1}{p_k}, 0\right)}$, where $\gamma := \min\{2, \max\{p_k : p_k \leq 2\}\}$ and $B_m > 0$ is a constant. This norm estimate is used to offer a definitive answer to the asymptotic behavior from infimum of norms of unimodular forms on ℓ_p^n when $p \geq 2$. The second part concerns the summability of operators and related subjects. We apply a recent technique introduced by Pellegrino *et al.* [56] to obtain an improved regularity principle on sequence spaces and an inclusion theorem for summing operators. Next, we deal with a general summability notion (Λ–summing operators, see Chapter 3) that unifies the multiple and absolutely summing operators. A general inclusion theorem that encompasses the correspondent result on each class is provided. We also obtain applications to the multilinear Hardy–Littlewood inequality in both contexts.

Keywords: Kahane–Salem–Zygmund inequality; asymptotic estimates; unimodular forms; summing operators; inclusion theorems; regularity principle; Hardy–Littlewood inequality.

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Introduction

Part I: Asymptotic estimates for unimodular multilinear forms with small norms on sequence spaces

The Kahane–Salem–Zygmund inequality is a seminal result regarding the maximum modulus of a random homogeneous polynomial (or multilinear form) in several scalar (real or complex) variables. In the classical work *"Some random series of functions"* by J. P. Kahane (see [45, Theorem 4, Chapter 6]), it is presented a version on the unit polydisk $\mathbb{D}^n = B_{\ell_\infty^n}$ and, later, by using more accessible techniques, H. P. Boas [28, Theorem 4] obtained an extension to the unit ball $B_{\ell_p^n}$ (of the n-dimensional Banach space ℓ_p^n, $1 \leq p \leq \infty$). As mentioned by Boas, the purpose of the inequality, in the multilinear context, is *"to obtain a multilinear unimodular form, that is, a multilinear form with coefficients with modulus 1, on $(\ell_p^n)^m$ into $\mathbb{K}$ (the real or complex scalar field) with small supremum norm."* The precise statement is as follows.

For positive integers m, n and $p \in [1, \infty]$ and a certain choice of signs $\varepsilon_{\mathbf{j}} = \pm 1$, $\mathbf{j} = (j_1, \ldots, j_m)$, let us consider an m-linear unimodular form $A : \left(\ell_p^n\right)^m \to \mathbb{K}$ defined by

$$A\left(x^{(1)}, \ldots, x^{(m)}\right) := \sum_{j_1, \ldots, j_m = 1}^{n} \varepsilon_{\mathbf{j}} x_{j_1}^{(1)} \cdots x_{j_m}^{(m)}.$$

If all signs are 1, we have the supremum norm $\|A\| = n^m$.

Problem 1. *How small can $\|A\|$ be by choosing suitable signs $+1$ and -1?*

A nontrivial solution to this problem is provided by the Kahane–Salem–Zygmund

inequality. Probabilistic tools are used in order to obtain this result. We present the multilinear version obtained by Boas next.

Multilinear Kahane-Salem-Zygmund inequality. *Let $p \in [1, +\infty]$, and consider integers $n, m \geq 1$. Then there exists a symmetric multilinear map $A : \left(\ell_p^n\right)^m \to \mathbb{C}$ of the form*

$$A\left(z_1, \ldots, z_m\right) = \sum_{j_1, \ldots, j_m = 1}^{n} \pm z_{j_1}^1 \cdots z_{j_m}^m,$$

such that the supremum of $\left|A\left(z_1, \ldots, z_m\right)\right|$ when every n-vector $z_k \in \ell_p^n$ lies in the unit ball of ℓ_p^n is at most

$$\sqrt{32m \log\left(6m\right)} \cdot \begin{cases} n^{\frac{1}{2}} \left(m!\right)^{\left(1-\frac{1}{p}\right)}, & \text{if } 1 \leq p < 2; \\ n^{\frac{1}{2}+\left(\frac{1}{2}-\frac{1}{p}\right)m}, & \text{if } 2 \leq p \leq \infty. \end{cases}$$

The recent literature contains many different extensions and variants of this result, regarding independent Bernoulli variables, polynomial and multilinear versions on Reinhardt domains, and Bernoulli polynomials on Banach spaces (see [50]). Deep applications are found in several fields of modern analysis: multidimensional Bohr radius, Bohnenblust–Hille and Hardy–Littlewood inequalities, unconditionality in spaces of homogeneous polynomials on Banach spaces, Boolean functions, Dirichlet series, and classical combinatorial games (see, for instance, [1, 12, 13, 18, 20, 21, 29, 34–36, 45, 56, 57]). In the following, we discuss in details two important applications connected with the inequality we profoundly worked on in this text.

Application 1: Asymptotic estimates for unimodular forms

The initial motivation for the investigation of asymptotic estimates for unimodular forms, besides its intrinsic interest, is a combinatorial game called the *"unbalancing lights problem"* or *"Gale–Berlekamp switching game"* (we refer the reader to the classical work of Alon and Spencer [10]): given an array $\left(a_{ij}\right)_{n \times n}$ of lights, each either on ($a_{ij} = 1$) or off ($a_{ij} = -1$), let us suppose that for each row and each column there is a switch so that if the switch is pulled ($x_i = -1$ for row i and $y_j = -1$ for column j) all of the lights in that line are switched, on to off or off to on. The problem asks to maximize the difference between the number of lights in opposite states (on and off) in the given array.

A generalization of the problem to higher dimensions can be described as follows. Given $p \in [2,\infty]$, m,n positive integers and scalars $a_{i_1\dots i_d} \in \mathbb{K}$ with $|a_{i_1\dots i_m}| = 1$, $1 \leq i_1,\dots,i_m \leq n$, one needs to provide *lower bounds* for

$$g_{d,n}^{\mathbb{K}}(p) := \max \left| \sum_{i_1,\dots,i_m=1}^{n} a_{i_1\dots i_m} x_{i_1}^{(1)} \cdots x_{i_m}^{(m)} \right|,$$

where the maximum is taken over scalars $x_j^{(k)}$ such that $\left\| \left(x_j^{(k)} \right)_{j=1}^{n} \right\|_p = 1, \quad k = 1,\dots,d.$

Notice that $g_{m,n}^{\mathbb{K}}(p) = \|U\|$, where $U : \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{K}$ is the unimodular d-linear form

$$U\left(x^{(1)},\dots,x^{(m)}\right) := \sum_{i_1,\dots,i_m=1}^{n} a_{i_1\dots i_m} x_{i_1}^{(1)} \cdots x_{i_m}^{(m)}.$$

Also notice that the original problem is the case $p = \infty$ and $m = 2$. Recently, Araújo and Pellegrino provided an efficient answer to this problem.

Theorem 1 ([12])**.** *Let $p \in [2,\infty]$ and $m \geq 2$ be a positive integer. Then, for all n,*

$$g_{m,n}^{\mathbb{K}}(p) \geq \frac{1}{1,3m^{0,365}} \cdot n^{\frac{m+1}{2}-\frac{m}{p}}.$$

Moreover, the exponent $\frac{m+1}{2} - \frac{m}{p}$ is sharp.

This motivates the study of the following inferior asymptotic estimates: *does there exist a constant $B_m \in (0,\infty)$ such that*

$$\inf_{A\in\mathcal{U}} \frac{\|A\|}{n^{\frac{m+1}{2}-\frac{m}{p}}} \geq B_m,$$

where $\mathcal{U}$ consists of the family of all unimodular m-linear forms $\ell_p^n \times \cdots \times \ell_p^n \to \mathbb{K}$? Due to Littlewood's (ℓ_1,ℓ_2)-inequalities, the answer to this question is positive when $p \geq 2$. An analogous question (the superior asymptotic estimate) can also be stated: does there exist a constant $C_m \in (0,\infty)$ such that

$$\inf_{A\in\mathcal{U}} \frac{\|A\|}{n^{\frac{m+1}{2}-\frac{m}{p}}} \leq C_m?$$

The main ingredient to solve these problems is the Kahane-Salem-Zygmund inequality. Naturally, one may ask about the general case of the asymptotic behavior of the unimod-

ular forms $U : \ell_{p_1}^{n_1} \times \cdots \times \ell_{p_m}^{n_m} \to \mathbb{K}$ with small norms.

Problem 2. *Is there a function $f(n_1, \ldots, n_m)$ and C_m, B_m satisfying*

$$0 < B_m \leq \inf_{A \in \mathcal{U}} \frac{\|A\|}{f(n_1, \ldots, n_m)} \leq C_m < \infty$$

for all $n_1, \ldots, n_m \geq 1$?

This function f describes the asymptotic growth of such multilinear forms A with small norms. As we will see in Chapter 1, we provided the following solution, when $p_1, \ldots, p_m \geq 2$ (for the f, see Theorem 1.11).

$$f(n_1, \ldots, n_m) = \left(n_1^{1/2} + \cdots + n_m^{1/2}\right) \prod_{j=1}^{m} n_j^{\frac{1}{2} - \frac{1}{p_j}}.$$

In order to do this, we revisit the Kahane–Salem–Zygmund multilinear inequality and, using interpolation and duality, we improve the norm estimate when $p \in [1, 2]$ (see Theorem 1.7). Moreover, we will see that the asymptotic behavior previously discussed implies that the Kahane–Salem–Zygmund inequality we obtain is sharp.

Application 2: Bohnenblust–Hille and Hardy–Littlewood inequalities

The Bohnenblust–Hille and Hardy–Littlewood inequalities are, in short, estimates on multilinear forms on ℓ_p spaces. These results originated in 1930, with Littelwood's 4/3 inequality for bilinear forms on $c_0 \times c_0$ [47]. In the following year, Bohnenblust and Hille [30] extended that result to multilinear forms. In 1934, Hardy and Littlewood [43] provided a version of Littelwood's 4/3 inequality on ℓ_p spaces (for $1 \leq p < \infty$). In 1981, Praciano–Pereira [61] extended that result to multilinear forms on ℓ_p spaces (for $1 \leq p < \infty$) and, more recently, several authors have been investigating the subject. In 2016, Dimant and Sevilla–Peris [38] deeply analysed the case $p \in (m, 2m]$ of the Hardy–Littlewood inequality for m-linear forms.

The following theorem encompasses all the aforementioned results in a simple and clear inequality and, also, provides applications to the contractivity of Bohnenblust–Hille and Hardy–Littlewood constants (see [1]).

Theorem 2. *Let $1 \leq d \leq m$ and $m_1, \ldots, m_d \geq 1$ be positive integers such that $m_1 + \cdots + m_d = m$ and also let $\mathbf{p} := (p_1, \ldots, p_m) \in (1, \infty]^m$.*

(1) If $0 \leq \left|\frac{1}{\mathbf{p}}\right| \leq \frac{1}{2}$ and $\mathbf{q} := (q_1, \ldots, q_d) \in \left[\left(1 - \left|\frac{1}{\mathbf{p}}\right|\right)^{-1}, 2\right]^d$ then, for every continuous m–linear form $T : X_{p_1} \times \cdots \times X_{p_m} \to \mathbb{K}$, there exists a constant $C_{d,\mathbf{p}} \geq 1$ such that

$$\left(\sum_{j_1=1}^{\infty}\left(\cdots\left(\sum_{j_d=1}^{\infty}\left|T\left(e_{j_1}^{m_1}, \ldots, e_{j_d}^{m_d}\right)\right|^{q_d}\right)^{\frac{q_{d-1}}{q_d}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} \leq C_{d,\mathbf{p}}^{\mathbb{K}}\|T\|$$

if, and only if, $\left|\frac{1}{\mathbf{q}}\right| \leq \frac{d+1}{2} - \left|\frac{1}{\mathbf{p}}\right|$. In other words, the exponents are optimal.

(2) If $\frac{1}{2} \leq \left|\frac{1}{\mathbf{p}}\right| < 1$ then, for every continuous m–linear form $T : X_{p_1} \times \cdots \times X_{p_m} \to \mathbb{K}$, there exists a constant $D_{d,\mathbf{p}} \geq 1$ such that

$$\left(\sum_{j_1,\ldots,j_d=1}^{\infty}\left|T\left(e_{j_1}^{m_1}, \ldots, e_{j_d}^{m_d}\right)\right|^{\frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|}}\right)^{1-\left|\frac{1}{\mathbf{p}}\right|} \leq D_{d,\mathbf{p}}^{\mathbb{K}}\|T\|. \tag{1}$$

Moreover, the exponent in (1) is optimal.

Here, for $1 \leq p < \infty$, let $X_p := \ell_p := \ell_p(\mathbb{K})$ and $X_\infty := c_0 = \{(x_n)_{n=1}^{\infty} \subset \mathbb{K} : \lim x_n = 0\}$. If $m_1, \ldots, m_d$ are positive integers such that $m_1 + \cdots + m_d = m$, then $(e_{j_1}^{m_1}, \ldots, e_{j_d}^{m_d})$ represents the m-tuple

$$\left(e_{j_1}, \overset{m_1 \text{ times}}{\ldots}, e_{j_1}, \ldots, e_{j_d}, \overset{m_d \text{ times}}{\ldots}, e_{j_d}\right).$$

The optimality of the exponents in the previous result is obtained by a clever and suitable use of the Kahane–Salem–Zygmund inequality. In Chapter 1, the generalization and improved version we obtained (Theorem 1.7) is used to provide variants of Theorem 2 (see Theorem 1.18 and Theorem 1.20).

Part II: Anisotropic Regularity Principle on sequence spaces and a unified summability principle

We can say that the theory of absolutely summing operators has its foundations on two main concepts: absolutely and unconditionally summable sequences. Given a sequence $(x_j)_{j\in\mathbb{N}}$ in a normed vector space E, we say that it is *absolutely summable* if

$\sum_{j=1}^{\infty} \|x_j\| < \infty$ and *unconditionally* summable if $\sum_{j=1}^{\infty} x_{\sigma(j)}$ converges for any permutation $\sigma : \mathbb{N} \to \mathbb{N}$.

A result due to Dirichlet [39] guarantees that a scalar sequence is absolutely summable if, and only if, it is unconditionally summable. However, A. Dvoretzky and C. A. Rogers [40] proved that every infinite dimensional Banach space has an unconditionally summable sequence which fails to be absolutely summable. This is the starting point to an abundant theory of summing operators.

Considering E and F Banach spaces, a bounded linear operator $u : E \to F$ is said to be *absolutely summing* if it sends absolutely summable sequences into unconditionally summable sequences. This concept was introduced by A. Grothendieck, in the 1950's, in which occasion he called this type of operator "application semi-intégrales à droite". Later, A. Pietsch [60] introduced the concept of absolutely p-summing linear operators and B. Mitiagin and A. Pelczyński [51] expanded this concept by introducing the absolutely $(r;p)$-summing linear operators. In 1968 J. Lindenstrauss's and A. Pełczyński's seminal paper [46] re-wrote Grothendieck's Résumé in a more comprehensible form, thus helping bring the subject into the spotlight.

The theory of multiple summing multilinear mappings was introduced by M. Matos in [49] and, independently, by F. Bombal, D. Pérez–García and I. Villanueva in [31,58]. This class is one of the most useful and fruitful multilinear generalizations of the summability concept for multilinear operators. Let $1 \leq p_1, \ldots, p_m \leq q < \infty$. A bounded m-linear operator $T : E_1 \times \cdots \times E_m \to F$ is multiple $(q; p_1, \ldots, p_m)$-summing if there exists $C > 0$ such that

$$\left(\sum_{j_1,\ldots,j_m=1}^{\infty} \left\| T(x_{j_1}^{(1)}, \ldots, x_{j_m}^{(m)}) \right\|^q \right)^{\frac{1}{q}} \leq C \prod_{k=1}^{m} \left\| (x_j^{(k)})_{j=1}^{\infty} \right\|_{w,p_k},$$

for every $(x_j^{(k)})_{j=1}^{\infty} \in \ell_{p_k}^w(E_k)$, $k = 1, \ldots, m$. The class of all multiple $(q; p_1, \ldots, p_m)$-summing operators from $E_1 \times \cdots \times E_m$ to F will be denoted by $\Pi_{(q;p_1,\ldots,p_m)}^m(E_1, \ldots, E_m; F)$. This theory has important connections with the Bohnenblust–Hille and Hardy-Littlewood inequalities and its applications in applied sciences.

It is in this environment that this part of the work is set. In Chapter 2 we explore the ideas from the Regularity Principle by Pellegrino, Santos, Serrano and Teixeira [56] to prove an Anisotropic Regularity Principle on sequence spaces (see Theorem 2.2). Besides that, we use the same techniques and arguments to get an anisotropic inclusion result

for summing operators, generalizing recent approaches of [56] and Bayart [17]. It is interesting to note that a sharper version of the Hardy–Littlewood inequalities for m-linear forms, not encompassed by several recent attempts (for instance, [16]), is provided as an application. In fact, we show that under the same hypothesis of the Theorem of Dimant and Sevilla–Peris (see Theorem 2.8, Chapter 2) we have

$$\left(\sum_{j_1=1}^{n}\left(\cdots\left(\sum_{j_m=1}^{n}\left\|T\left(e_{j_1},\ldots,e_{j_m}\right)\right\|^{s_m}\right)^{\frac{s_{m-1}}{s_m}}\cdots\right)^{\frac{s_1}{s_2}}\right)^{\frac{1}{s_1}}\leq\left(\sqrt{2}\right)^{m-1}\|T\|,$$

with

$$s_1=\frac{p}{p-m},\ldots,s_m=\frac{2mp}{mp+p-2m},$$

which is quite an improvement of theorem 2.8. Despite the huge advance recently obtained in this direction in [16], our results and techniques were not encompassed by those techniques.

Besides generalizing several important/interesting results on summing operators, Theorem 2 motivated the authors of [1] to give the following unifying notion of summing multilinear operators (the essence of the notion of *partially* multiple summing operators was first sketched in [54, Definition 2.2.1]). In an attempt to unify most of the different multilinear approaches, the notion of Λ-summability arose naturally (see [19] and the references therein). The following definition is more general and encompasses anisotropic sums.

Definition 1. *Let $E_1,\ldots,E_m,F$ be Banach spaces, m be a positive integer, $(\mathbf{r};\mathbf{p}):=(r_1,\ldots,r_m;p_1,\ldots,p_m)\in[1,\infty)^{2m}$ and $\Lambda\subset\mathbb{N}^m$ be a set of indexes. A multilinear operator $T:E_1\times\cdots\times E_m\to F$ is Λ-$(\mathbf{r};\mathbf{p})$-summing if there exists a constant $C>0$ such that for all $x^k\in\ell_{p_k}^w\left(E_k\right)$, $k=1,\ldots,m$,*

$$\left(\sum_{j_1=1}^{\infty}\left(\cdots\left(\sum_{j_m=1}^{\infty}\left\|T\left(x_{j_1}^1,\ldots,x_{j_m}^m\right)\mathbf{1}_\Lambda(j_1,\ldots,j_m)\right\|^{r_m}\right)^{\frac{r_{m-1}}{r_m}}\cdots\right)^{\frac{r_1}{r_2}}\right)^{\frac{1}{r_1}}\leq C\cdot\prod_{k=1}^{m}\left\|x^k\right\|_{w,p_k},\tag{2}$$

where $\mathbf{1}_\Lambda$ is the characteristic function of Λ. We represent the class of all Λ-$(\mathbf{r};\mathbf{p})$-summing multilinear operators from $E_1\times\cdots\times E_m$ to F by $\Pi_{(\mathbf{r};\mathbf{p})}^{\Lambda}(E_1,\ldots,E_m;F)$. When

$r_1 = \cdots = r_m = r$ *and* $p_1 = \cdots = p_m = p$*, we will represent* $\Pi^{\Lambda}_{(\mathbf{r};\mathbf{p})}$ *by* $\Pi^{\Lambda}_{(r;p)}$.

When $\Lambda = \Lambda_{\text{as}} := \{(j, \ldots, j) : j \in \mathbb{N}\}$, Definition 3.1 recovers the notion of $(r;\mathbf{p})$-absolutely summing operators, whose class is denoted by $\Pi^{\text{as}}_{(r;\mathbf{p})}$; as for $\Lambda = \mathbb{N}^m$ we get the notion of $(\mathbf{r};\mathbf{p})$-multiple summing operators, and its class is represented by $\Pi^{m}_{(\mathbf{r};\mathbf{p})}$.

As far as this theory is concerned, there is a vast literature investigating inclusion results in cases where $\Lambda = \Lambda_{\text{as}}$ or $\Lambda = \mathbb{N}^m$ (see, [7, 18, 37, 56]). Accordingly, the main contribution of Chapter 3 is an Inclusion Theorem for the case in which the set Λ is formed by blocks. The set Λ is called a *block* if

$$\Lambda = \{(j_1, \overset{m_1 \text{ times}}{\ldots}, j_1, \ldots, j_d, \overset{m_d \text{ times}}{\ldots}, j_d) : j_1, \ldots, j_d \in \mathbb{N}\},$$

where $1 \leq m_1, \ldots, m_d \leq m$ are fixed positive integers such that $m_1 + \cdots + m_d = m$. Provided that Λ is a block, we shall prove the inclusion

$$\Pi^{\Lambda}_{(r;\mathbf{p})} \subset \Pi^{\Lambda}_{(\mathbf{s};\mathbf{q})},$$

for suitable values of $s_1, \ldots, s_m$. As a consequence, we present applications concerning Hardy–Littlewood's multilinear inequalities. We get better exponents than Theorem 2 item (2) and we recover Theorem 2.6.

Preliminaries and Notation

Throughout this work, E and F shall stand for Banach spaces over the scalar field $\mathbb{K}$ of real or complex numbers. Here, ℓ_p^n stands for the n-dimensional Euclidean space $\mathbb{K}^n$ with the ℓ_p-norm, $p \in [1,\infty]$. The topological dual of E and its closed unit ball are denoted by E^* and B_{E^*}, respectively. As usual, the open ball with center x and radius r is denoted by $B(x,r)$, its closure by $\overline{B}(x,r)$ and the center and radius are omitted when we deal with the open and closed unit balls. The cardinal of a set $\mathcal{I}$ is denoted by $\operatorname{card}\mathcal{I}$. Moreover, we use the usual p^* to denote the conjugate of $p \in [1,\infty]$, i.e., $\frac{1}{p} + \frac{1}{p^*} = 1$. We consider the spaces $\ell_p(E)$ and $\ell_p^w(E)$ of *strongly* and *weakly* p-summable sequences on E, respectively. Recall that $\ell_p(E)$ comprises all sequences $(x_j)_{j\in\mathbb{N}} \in E^{\mathbb{N}}$ such that the sequence of scalars $(\|x_j\|)_{j\in\mathbb{N}}$ belongs to ℓ_p; whereas $\ell_p^w(E)$ is the space of all sequences $(x_j)_{j\in\mathbb{N}} \in E^{\mathbb{N}}$ such that $(\varphi(x_j))_{j\in\mathbb{N}} \in \ell_p$ for all $\varphi \in E^*$. The following expressions

$$\|(x_j)_{j\in\mathbb{N}}\|_p := \left\|(\|x_j\|_E)_{j\in\mathbb{N}}\right\|_{\ell_p} \qquad \text{and} \qquad \|(x_j)_{j\in\mathbb{N}}\|_{w,p} := \sup_{\|\varphi\|\leq 1} \left\|(\varphi(x_j))_{j\in\mathbb{N}}\right\|_{\ell_p}$$

define complete norms on $\ell_p(E)$ and $\ell_p^w(E)$ respectively.

For a positive integer m, $\mathbf{p}$ stands for a multiple exponent $\mathbf{p} := (p_1,\ldots,p_m) \in (0,\infty)^m$ and given $A \subset \{1,\ldots,m\}$ we set

$$\left|\frac{1}{\mathbf{p}}\right|_{j\in A} := \sum_{j\in A} \frac{1}{p_j}.$$

Also, for $1 \leq k \leq m$, we define $|1/\mathbf{p}|_{j\geq k} := |1/\mathbf{p}|_{j\in\{k,\ldots,m\}}$; we simply write $|1/\mathbf{p}|$ instead

of $|1/\mathbf{p}|_{j\geq 1}$, and

$$\|T\| := \sup_{\|x_1\|,\ldots,\|x_m\|\leq 1} \|T(x_1,\ldots,x_m)\|$$

for any continuous m-linear mapping $T : E_1 \times \cdots \times E_m \to F$. Moreover, for $1 \leq p < \infty$, let $X_p := \ell_p := \ell_p(\mathbb{K})$ and $X_\infty := c_0 = \{(x_n)_{n=1}^\infty \subset \mathbb{K} : \lim x_n = 0\}$. If $m_1,\ldots,m_d$ are positive integers such that $m_1+\cdots+m_d = m$, then $(e_{j_1}^{m_1},\ldots,e_{j_d}^{m_d})$ represents the m-tuple

$$(e_{j_1},\overset{m_1 \text{ times}}{\ldots},e_{j_1},\ldots,e_{j_d},\overset{m_d \text{ times}}{\ldots},e_{j_d}),$$

where the e_j's are the canonical vectors of the sequence space $\mathbb{K}^{\mathbb{N}}$. For a non-empty subset $D \subset \mathbb{N}$ we denote the set of multi–indices $\mathbf{j} = (j_1,\ldots,j_m)$, with each $j_k \in D$, by

$$\mathcal{M}(m,D) := \{\mathbf{j} = (j_1,\ldots,j_m) \in \mathbb{N}^m;\ j_k \in D,\ k = 1,\ldots,m\} = D^m.$$

We also denote

$$\mathcal{M}(m,n) := \mathcal{M}(m,\{1,2,\ldots,n\}).$$

Let $\mathbf{p} := (p_1,\ldots,p_m) \in [1,\infty]^m$. Using the definitions of mixed norm $L_{\mathbf{p}}$ spaces from [23], the mixed norm sequence space

$$\ell_{\mathbf{p}}(E) := \ell_{p_1}\left(\ell_{p_2}\left(\cdots\left(\ell_{p_m}\left(E\right)\right)\cdots\right)\right)$$

consists of all index vector valued matrices $\mathbf{x} := (x_{\mathbf{j}})_{\mathbf{j}\in\mathbb{N}^m}$ with finite $\mathbf{p}$-norm. Notice that each norm $\|\cdot\|_{p_k}$ is taken over the index j_k and that each index j_k is related to the $\|\cdot\|_{p_k}$ norm. For instance, when $\mathbf{p} \in [1,\infty)^m$ a vector matrix $\mathbf{x}$ belongs to $\ell_{\mathbf{p}}(E)$ if, and only if,

$$\|\mathbf{x}\|_{\mathbf{p}} := \left(\sum_{j_1=1}^\infty\left(\sum_{j_2=1}^\infty\left(\cdots\left(\sum_{j_{m-1}=1}^\infty\left(\sum_{j_m=1}^\infty \|x_{\mathbf{j}}\|^{p_m}\right)^{\frac{p_{m-1}}{p_m}}\right)^{\frac{p_{m-2}}{p_{m-1}}}\cdots\right)^{\frac{p_2}{p_3}}\right)^{\frac{p_1}{p_2}}\right)^{\frac{1}{p_1}} < \infty.$$

Hölder's inequality for mixed ℓ_p spaces

The following general version of Hölder's inequality was presented in the classical paper [23] on mixed norms in L_p spaces. We shall now work with $L_p(\mathbb{N}) = \ell_p$, since it is

the case we are interested in. The result below seems to have been first observed by A. Benedek and R. Panzone (see [23]):

Hölder's inequality for mixed $\ell_{\mathbf{p}}$ spaces. *Let m, n, N be positive integers, $\mathbf{r} \in (0,\infty)^m$ and $\mathbf{q}(1), \dots, \mathbf{q}(N) \in (0,\infty]^m$ be such that*

$$\tfrac{1}{r_j} = \tfrac{1}{q_j(1)} + \cdots + \tfrac{1}{q_j(N)}, \quad j \in \{1, 2, \dots, m\}$$

and also let $\mathbf{a}_k := (a_{\mathbf{i}}^k)_{\mathbf{i}\in\mathcal{M}(m,n)}$, $k = 1, \dots, N$ be scalar matrices. Then

$$\left\| \prod_{k=1}^{N} \mathbf{a}_k \right\|_{\mathbf{r}} \leq \prod_{k=1}^{N} \|\mathbf{a}_k\|_{\mathbf{q}(k)} .$$

In particular, if each $\mathbf{q}(k) \in [1,\infty)^m$, we have

$$\left(\sum_{i_1=1}^{n} \left(\dots \left(\sum_{i_m=1}^{n} |a_{\mathbf{i}}^1 a_{\mathbf{i}}^2 \dots a_{\mathbf{i}}^N|^{r_m} \right)^{\frac{r_{m-1}}{r_m}} \dots \right)^{\frac{r_1}{r_2}} \right)^{\frac{1}{r_1}}$$
$$\leq \prod_{k=1}^{N} \left[\left(\sum_{i_1=1}^{n} \left(\dots \left(\sum_{i_m=1}^{n} |a_{\mathbf{i}}^k|^{q_m(k)} \right)^{\frac{q_{m-1}(k)}{q_m(k)}} \dots \right)^{\frac{q_1(k)}{q_2(k)}} \right)^{\frac{1}{q_1(k)}} \right] .$$

A consequence of the previous Hölder's inequality is the interpolative approach summarized as follows.

Hölder's interpolative inequality for multiple exponents. *Let m, n, N be positive integers and $\mathbf{r}, \mathbf{q}(1), \dots, \mathbf{q}(N) \in (0,\infty]^m$ and $\theta_1, \dots, \theta_N \in [0,1]$ be such that $\theta_1 + \cdots + \theta_N = 1$ and*

$$\tfrac{1}{r_j} = \tfrac{\theta_1}{q_j(1)} + \cdots + \tfrac{\theta_N}{q_j(N)}, \quad \text{for all } j = 1, \dots, m.$$

Then, for all scalar matrices $\mathbf{a} = (a_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,n)}$ we have

$$\|\mathbf{a}\|_{\mathbf{r}} \leq \prod_{k=1}^{N} \|\mathbf{a}\|_{\mathbf{q}(k)}^{\theta_k} .$$

In particular, if each $\mathbf{q}(k) \in [1,\infty)^m$*, the previous inequality means that*

$$\left(\sum_{i_1=1}^{n}\left(\cdots\left(\sum_{i_m=1}^{n}|a_{\mathbf{i}}|^{r_m}\right)^{\frac{r_{m-1}}{r_m}}\cdots\right)^{\frac{r_1}{r_2}}\right)^{\frac{1}{r_1}}$$
$$\leq \prod_{k=1}^{N}\left[\left(\sum_{i_1=1}^{n}\left(\cdots\left(\sum_{i_m=1}^{n}|a_{\mathbf{i}}|^{q_m(k)}\right)^{\frac{q_{m-1}(k)}{q_m(k)}}\cdots\right)^{\frac{q_1(k)}{q_2(k)}}\right)^{\frac{1}{q_1(k)}}\right]^{\theta_k}.$$

The main result of Chapter 2 has a general status, but basic facts are used along its proof. We will now present the necessary ingredients to prove it.

Inclusions on ℓ_p spaces

The following result provides us with the necessary conditions for inclusions in the ℓ_p's spaces [37, Proposition 5.5.2]:

Inclusion on ℓ_p spaces. *For* $q \geq p > 0$*,*

$$\left(\sum_{i=1}^{n} a_i^q\right)^{1/q} \leq \left(\sum_{i=1}^{n} a_i^p\right)^{1/p},$$

for all positives integers $n, a_1, \ldots, a_n$*.*

That is, the inclusion $i : \ell_p \hookrightarrow \ell_q$ *is continuous,* $\|i\| \leq 1$ *and* $\|\cdot\|_{\ell_q} \leq \|\cdot\|_{\ell_p}$*, whenever* $q \geq p \geq 1$*.*

Minkowski's inequality

The following result is a corollary of one of the many versions of Minkowski's inequality (see [41, Corollary 5.4.2]):

Minkowski's inequality. *For any* $0 < p \leq q < \infty$ *and for any scalar matrix* $(a_{ij})_{i,j\in\mathbb{N}} \in \ell_p(\ell_q)$*,*

$$\left(\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}|a_{ij}|^p\right)^{q/p}\right)^{1/q} \leq \left(\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty}|a_{ij}|^q\right)^{p/q}\right)^{1/p},$$

Linear Inclusion Theorem

The following is the definition of absolutely $(r;p)$-summing linear operators:

Definition 2. *For $r,p\geq 1$, a linear operator $T:E\to F$ is said to be absolutely $(r;p)$-summing if there exists a constant $C>0$ such that*

$$\left(\sum_{j=1}^{\infty}\|T(x_j)\|^r\right)^{\frac{1}{r}}\leq C\left\|(x_j)_{j\in\mathbb{N}}\right\|_{w,p}$$

for any weakly p-summable *vector sequence* $(x_j)_{j\in\mathbb{N}}\in\ell_p^w(E)$.

The class $\Pi_{(r;p)}$ of all absolutely $(r;p)$-summing operators is a Banach space, with the norm $(\pi_{(r;p)}(\cdot))$ defined by the infimum taken over the constants $C>0$ satisfying the inequality above. Notice that an absolutely summing operator maps a weakly summable sequence into a strongly summable sequence.

For linear operators, it is folklore that p-summability implies q-summability whenever $1\leq p\leq q$. More generally, although basic, the following is the classical linear inclusion theorem [37, p. 198], which is the starting point of these types of results.

Linear Inclusion Theorem. *If $s\geq r$, $q\geq p$ and $\frac{1}{p}-\frac{1}{r}\leq\frac{1}{q}-\frac{1}{s}$, then every absolutely $(r;p)$-summing linear operator is absolutely $(s;q)$-summing.*

Generalized Hardy–Littlewood inequality

Recently an extended version of the Hardy–Littlewood inequality for $0\leq|1/\mathbf{p}|\leq 1/2$ was presented in [4].

Theorem 3. *Let be $\mathbf{p}\in[1,\infty]^m$ such that $0\leq|1/\mathbf{p}|\leq 1/2$. Let also $\mathbf{q}:=(q_1,\ldots,q_m)\in\left[(1-|1/\mathbf{p}|)^{-1},2\right]^m$. The following are equivalent:*

(1) There is a (optimal) constant $D^{\mathbb{K}}_{m,\mathbf{p},\mathbf{q}}\geq 1$ such that

$$\left(\sum_{j_1=1}^{\infty}\left(\cdots\left(\sum_{j_m=1}^{\infty}|A(e_{j_1},\ldots,e_{j_m})|^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}}\leq D^{\mathbb{K}}_{m,\mathbf{p},\mathbf{q}}\|A\|$$

for all continuous m-linear forms $A : X_{p_1} \times \cdots \times X_{p_m} \to \mathbb{K}$.

(2) $\left|\frac{1}{\mathbf{q}}\right| \leq \frac{m+1}{2} - \left|\frac{1}{\mathbf{p}}\right|$.

The Riesz–Thorin Interpolation Theorem

Let (Ω, Σ, μ) be a measure space where μ is a positive measure and $0 < p < \infty$. Two functions are said to be equivalent if they coincide up to a μ–null set. We define $\mathcal{L}_p(\Omega, \Sigma, \mu)$ to be the collection of those (real- or complex-valued) measurable functions for which

$$\|f\|_{L_p} = \left(\int_\Omega |f|^p \, d\mu\right)^{\frac{1}{p}} < \infty.$$

If $f = g$ almost everywhere, then $\left(\int_\Omega |f-g|^p \, d\mu\right)^{\frac{1}{p}} = 0$ and $\left(\int_\Omega |f|^p \, d\mu\right)^{\frac{1}{p}} = \left(\int_\Omega |g|^p \, d\mu\right)^{\frac{1}{p}}$. We therefore identify functions which are equal almost everywhere, and denote the resulting space by $L_p = L_p(\Omega, \Sigma, \mu)$. If $1 \leq p < \infty$ then $(L_p, \|\cdot\|_p)$ is a Banach space, with

$$\|f\|_p = \left(\int_\Omega |f|^p \, d\mu\right)^{\frac{1}{p}}.$$

In the case $p = \infty$, L_p consists of all μ–measurable and essentially bounded functions, and

$$\|f\|_\infty = ess\sup|f| < \infty,$$

where

$$ess\sup f = \left\{t : \lambda_{\|f\|}(t) = \mu(\|f\| > t) = 0\right\}.$$

In the future, we will write $ess\sup|f|$ simply as $\sup|f|$.

The next result guarantees that, if $0 < p_0 < p < p_1 \leq \infty$ then

$$L_{p_0} \cap L_{p_1} \subset L_p \subset L_{p_0} + L_{p_1}.$$

Theorem 4. [41, Theorem 5.5.1]

(i) (Lyapunov's inequality) Suppose that $0 < p_0 < p_1 < \infty$ *and that* $0 < \theta < 1$. *Let* $p = (1-\theta)p_0 + \theta p_1$. *If* $f \in L_{p_0} \cap L_{p_1}$ *then* $f \in L_p$ *and* $\|f\|_p^p \leq \|f\|_{p_0}^{(1-\theta)p_0} \|f\|_{p_1}^{\theta p_1}$.

(ii) (Littlewood's inequality) Suppose that $0 < p_0 < p_1 < \infty$ *and that* $0 < \theta < 1$. *Define* p *by* $1/p = (1-\theta)/p_0 + \theta/p_1$. *If* $f \in L_{p_0} \cap L_{p_1}$ *then* $f \in L_p$ *and* $\|f\|_p^p \leq \|f\|_{p_0}^{(1-\theta)} \|f\|_{p_1}^{\theta}$.

(iii) Suppose that $0 < p_0 < p_1 \leq \infty$ *and that* $0 < \theta < 1$. *Define* p *by* $1/p = (1-\theta)/p_0 + \theta/p_1$. *If* $f \in L_p$ *then there exist functions* $g \in L_{p_0}$ *and* $h \in L_{p_1}$ *such that* $f = g + h$ *and* $\|g\|_{p_0}^{(1-\theta)} \|h\|_{p_1}^{\theta} \leq \|f\|_p^p$.

Theorem 5 is an adaptation of [26, Theorem 1.1.1], and for its proof we need the following results.

Proposition 1. *(See* [44, Theorem 7.8]*). Suppose that* (Ω, Σ, μ) *is a measure space and* $1 \leq p \leq \infty$. *Then the simple functions that belong to* L_p *are dense in* L_p.

Lemma 1. *(See* [27, Lemma 2.3]*, see also* [26, Lemma 1.1.2]*). (Hadamard three line lemma) Let* $F : \{z \in \mathbb{C} : 0 \leq Rez \leq 1\} \to \mathbb{C}$ *be bounded and continuous on the closed strip* $0 \leq Rez \leq 1$ *and analytic on the open strip* $0 < Rez < 1$. *For* $0 \leq \theta \leq 1$ *let* $N_\theta := \sup_{t \in \mathbb{R}} |F(\theta + it)|$. *Then we have*

$$N_\theta \leq N_0^{1-\theta} N_1^{\theta}.$$

For convenience, the following result is called Riesz–Thorin Interpolation Theorem.

Theorem 5. [26, Exercise 13] *Let* $p_1^{(0)}, \ldots, p_d^{(0)}, p_1^{(1)}, \ldots, p_d^{(1)}, q_1, q_0 \in [1, \infty]$, $0 \leq \theta \leq 1$, (Ω, Σ, μ) *and* (Ψ, Υ, ν) *be measure spaces. Define* $p_1, \ldots, p_d, q$ *by*

$$\frac{1}{p_k} = \frac{1-\theta}{p_k^{(0)}} + \frac{\theta}{p_k^{(1)}} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

with $k = 1, \ldots, d$. *Suppose* T *is a bounded multilinear mapping*

$$T : \left(L_{p_1^{(0)}} + L_{p_1^{(1)}}\right) \times \cdots \times \left(L_{p_d^{(0)}} + L_{p_d^{(1)}}\right) \to \left(L_{q_0} + L_{q_1}\right).$$

with

$$\|T(f_1, \ldots, f_d)\|_{q_0} \leq M_0 \cdot \prod_{k=1}^{d} \|f_k\|_{L_{p_k^{(0)}}} \text{ for } (f_1, \ldots, f_d) \in L_{p_1^{(0)}} \times \cdots \times L_{p_d^{(0)}}$$

and

$$\|T(f_1, \ldots, f_d)\|_{q_1} \leq M_1 \cdot \prod_{k=1}^{d} \|f_k\|_{L_{p_k^{(1)}}} \text{ for } (f_1, \ldots, f_d) \in L_{p_1^{(1)}} \times \cdots \times L_{p_d^{(1)}}.$$

Then

$$T : L_{p_1} \times \cdots \times L_{p_d} \to L_q$$

with norm M is such that

$$M \leq M_0^{1-\theta} M_1^{\theta}. \tag{3}$$

Proof. Since the simple functions are dense in L_p-spaces for all $1 \leq p \leq \infty$ it is sufficient to show inequality (3) for this class of functions in Ω. Indeed, suppose (3) holds for simple functions and let $f_k \in L_{p_k}$. Then, for each $k = 1, \ldots, d$, there exists a sequence of simple functions $(f_k^n)_{n\in\mathbb{N}}$ such that $f_k^n \longrightarrow f_k$. The multilinearity of T implies that

$$\begin{aligned} T(f_1, \ldots, f_d) &= T(f_1 - f_1^n, f_2, \ldots, f_d) + T(f_1^n, f_2 - f_2^n, f_3 \ldots, f_d) + \cdots \\ &+ T\left(f_1^n, \ldots, f_{d-1}^n, f_d - f_d^n\right) + T(f_1^n, \ldots, f_d^n). \end{aligned}$$

Thus

$$\begin{aligned} \|T(f_1, \ldots, f_d)\|_q &= \|T(f_1 - f_1^n, f_2, \ldots, f_d)\|_q + \|T(f_1^n, f_2 - f_2^n, f_3 \ldots, f_d)\|_q + \cdots \\ \cdots &+ \left\|T\left(f_1^n, \ldots, f_{d-1}^n, f_d - f_d^n\right)\right\|_q + \|T(f_1^n, \ldots, f_d^n)\|_q \\ &\leq M\Bigg(\|f_1 - f_1^n\|_{p_1} \prod_{k=2}^{d} \|f_k\|_{p_k} + \|f_1^n\|_{p_1} \|f_2 - f_2^n\|_{p_2} \prod_{k=3}^{d} \|f_k\|_{p_k} + \cdots \\ \cdots &+ \prod_{k=1}^{d-1} \|f_k^n\|_{p_k} \|f_d - f_d^n\|_{p_d} \Bigg) + M_0^{1-\theta} M_1^{\theta} \prod_{k=1}^{d} \|f_k^n\|_{p_k} \\ &\leq M\Bigg(\|f_1 - f_1^n\|_{p_1} \prod_{k=2}^{d} \|f_k\|_{p_k} + \|f_1^n\|_{p_1} \|f_2 - f_2^n\|_{p_2} \prod_{k=3}^{d} \|f_k\|_{p_k} + \cdots \\ \cdots &+ \prod_{k=1}^{d-1} \|f_k^n\|_{p_k} \|f_d - f_d^n\|_{p_d} \Bigg) + M_0^{1-\theta} M_1^{\theta} \Bigg(\|f_1^n - f_1\|_{p_1} \prod_{k=2}^{d} \|f_k^n\|_{p_k} + \\ &+ \|f_1\|_{p_1} \|f_2^n - f_2\|_{p_2} \prod_{k=3}^{d} \|f_k^n\|_{p_k} + \|f_1\|_{p_1} \|f_2\|_{p_2} \|f_3^n - f_3\|_{p_3} \prod_{k=4}^{d} \|f_k^n\|_{p_k} + \\ &+ \cdots + \prod_{k=1}^{d-1} \|f_k\|_{p_k} \|f_d^n - f_d\|_{p_d} + \prod_{k=1}^{d} \|f_k\|_{p_k} \Bigg). \end{aligned}$$

Therefore, passing to the limit, we obtain the result.

Now recall that

$$\|h\|_q = \sup\left\{\left|\int_\Psi hg\,d\nu\right| : \|g\|_{q'} = 1\right\},$$

thus

$$\begin{aligned} M &= \sup\left\{\|T(f_1,\dots,f_d)\|_q : \|f_1\|_{p_1} = \cdots \|f_d\|_{p_d} = 1\right\} \\ &= \sup\left\{\left|\int_\Psi T(f_1,\dots,f_d)\,g\,d\nu\right| : \|f_1\|_{p_1} = \cdots \|f_d\|_{p_d} = \|g\|_{q'} = 1\right\}. \end{aligned}$$

Let $f_1,\dots,f_d$ be simple functions on Ω and write

$$f_k = \sum_{l=1}^{n_k} a_l^{(k)} \mathcal{X}_{A_l^{(k)}} = \sum_{l=1}^{n_k} \left|a_l^{(k)}\right| e^{i\theta_l^{(k)}} \mathcal{X}_{A_l^{(k)}},$$

where $a_1^{(k)},\dots,a_{n_k}^{(k)} \in \mathbb{C}$ and $A_1^{(k)},\dots,A_{n_k}^{(k)}$ are pairwise disjoint measurable sets in Ω with positive finite measure, for each $k = 1,\dots,d$. Now let g be a simple function on Ψ and again we write

$$g = \sum_{j=1}^{n} b_j \mathcal{X}_{B_j} = \sum_{j=1}^{n} |b_j| e^{i\alpha_j} \mathcal{X}_{B_j}$$

where $b_1,\dots,b_n \in \mathbb{C}$ and $B_1,\dots,B_n$ are pairwise disjoint measurable sets in Ψ with positive finite measure.

For $z \in S := \{z \in \mathbb{C} : 0 \le Re z \le 1\}$ let $p_k(z)$ and $q'(z)$ be defined as

$$\tfrac{1}{p_k(z)} := \tfrac{1-z}{p_i^{(0)}} + \tfrac{z}{p_i^{(1)}} \quad \text{and} \quad \tfrac{1}{q'(z)} := \tfrac{1-z}{q_0'} + \tfrac{z}{q_1'},$$

for $k = 1,\dots,d$. Note that $p_k(0) = p_0$, $p_k(\theta) = p_k$ and $p_k(1) = p_1$ as well as $q'(0) = q_0'$, $q'(\theta) = q'$ and $q'(1) = q_1'$. If $p_k = \infty$ let $f_z^k := f_k$, and if $q = 1$ let $g_z := g$ for $z \in S$. Otherwise define, using $p_k(z)$ and $q'(z)$, new functions by

$$f_z^{(k)}(x) := \sum_{l=1}^{n_k} \left|a_l^{(k)}\right|^{\frac{p_k}{p_k(z)}} e^{i\theta_l^{(k)}} \mathcal{X}_{A_l^{(k)}} \quad \text{and} \quad g_z(x) := \sum_{j=1}^{n} |b_j|^{\frac{q'}{q'(z)}} e^{i\alpha_j} \mathcal{X}_{B_j}.$$

It follows that $f_z^{(k)} \in L_{p_k^{(j)}}$ and $g_z \in L_{q_j'}$ and hence $T\left(f_z^{(1)},\dots,f_z^{(d)}\right) \in L_{q_j}$, $j = 0, 1$. Finally, we define

$$F(z) := \int_\Psi T\left(f_z^{(1)},\dots,f_z^{(d)}\right) g_z\,d\nu.$$

For the sake of clarity we assume $1 \le p_k < \infty$ and $q \ne 1$. Therefore by definition of the

functions f_z^k and g_z,

$$F(z) = \sum_{j_1=1}^{n_1} \dots \sum_{j_d=1}^{n_d} \sum_{j=1}^{n} \left|a_{j_1}^{(1)}\right|^{\frac{p_1}{p_1(z)}} \cdots \left|a_{j_d}^{(d)}\right|^{\frac{p_d}{p_d(z)}} \times |b_j|^{\frac{q'}{q'(z)}} e^{i\left(\theta_{j_1}^{(1)}+\cdots+\theta_{j_d}^{(d)}+\alpha_j\right)} \int_{B_j} T\left(\mathcal{X}_{A_{j_1}^{(1)}}, \dots, \mathcal{X}_{A_{j_d}^{(d)}}\right) \mathcal{X}_{B_j} d\nu.$$

This shows that F is a linear combination of terms of the form β^z with $\beta > 0$. Moreover, for all functions β^z with $z \in S$ we have

$$|\beta^z| = \left|e^{z \log \beta}\right| = \left|e^{Rez \log \beta}\right| = \left|\beta^{Rez}\right| = \beta^{Rez} \leq \max(1,\, \beta).$$

So F is analytic, bounded and continuous on S and therefore satisfies the assumptions of Lemma 1. It implies that

$$\sup_{t\in\mathbb{R}} |F(\theta + it)| \leq \left(\sup_{t\in\mathbb{R}} |F(it)|\right)^{1-\theta} \cdot \left(\sup_{t\in\mathbb{R}} |F(1+it)|\right)^{\theta}. \tag{4}$$

Next we estimate $|F(it)|$ and $|F(1+iy)|$. By Hölder's inequality we have

$$\begin{aligned} |F(it)| &= \left|\int_\Psi T\left(f_{it}^{(1)}, \dots, f_{it}^{(d)}\right) g_{it}\, d\nu\right| \leq \int_\Psi \left|T\left(f_{it}^{(1)}, \dots, f_{it}^{(d)}\right) g_{it}\right| d\nu \\ &\leq \left\|T\left(f_{it}^{(1)}, \dots, f_{it}^{(d)}\right)\right\|_{q_0} \cdot \|g_{it}\|_{q_0'} \leq M_0 \prod_{k=1}^{d} \left\|f_{it}^{(k)}\right\|_{p_k^{(0)}} \cdot \|g_{it}\|_{q_0'}, \end{aligned}$$

and

$$\begin{aligned} |F(1+it)| &\leq \int_\Psi \left|T\left(f_{1+it}^{(1)}, \dots, f_{1+it}^{(d)}\right) g_{1+it}\right| d\nu \leq \left\|T\left(f_{1+it}^{(1)}, \dots, f_{1+it}^{(d)}\right)\right\|_{q_1} \cdot \|g_{1+it}\|_{q_1'} \\ &\leq M_1 \prod_{k=1}^{d} \left\|f_{it}^{(k)}\right\|_{p_k^{(1)}} \cdot \|g_{it}\|_{q_1'}. \end{aligned}$$

Furthermore, by definition we have

$$\begin{aligned} \left\|f_{it}^{(k)}\right\|_{p_k^{(0)}}^{p_k^{(0)}} &= \sum_{l=1}^{n_k} \left|\left|a_l^{(k)}\right|^{\frac{p_k}{p_k(it)}} e^{i\theta_l^{(k)}}\right|^{p_k^{(0)}} \mathcal{X}_{A_l^{(k)}} = \sum_{l=1}^{n_k} \left|a_l^{(k)}\right|^{Re\left(\frac{p_k}{p_k(it)}\right)\cdot p_k^{(0)}} \mathcal{X}_{A_l^{(k)}} \\ &= \sum_{l=1}^{n_k} \left|a_l^{(k)}\right|^{\frac{p_k}{p_k^{(0)}}\cdot p_k^{(0)}} \mathcal{X}_{A_l^{(k)}} = \sum_{l=1}^{n_k} \left|a_l^{(k)}\right|^{p_k} \mathcal{X}_{A_l^{(k)}} = \|f_k\|_{p_k}^{p_k} = 1, \end{aligned}$$

for each $k = 1, \ldots, d$. Analogously we obtain $\|g_{it}\|_{q_0'} = \left\|f_{1+it}^{(k)}\right\|_{p_k^{(1)}} = \|g_{1+it}\|_{q_1'} = 1$. So

$$\sup_{t\in\mathbb{R}} |F(it)| \leq M_0 \quad \text{and} \quad \sup_{t\in\mathbb{R}} |F(1+it)| \leq M_1.$$

Now note that, for every $k = 1, \ldots, d$, $f_\theta^{(k)} = f_k$ and $g_\theta = g$. Thus

$$F(\theta) = \int_\Psi T\left(f_1, \ldots, f_d\right) g \, d\nu.$$

Therefore, by (4) we have

$$\left| \int_V T\left(f_1, \ldots, f_d\right) g \, d\nu \right| \leq M_0^{1-\theta} M_1^\theta.$$

□

Part I

Asymptotic estimates for unimodular multilinear forms with small norms on sequence spaces

Chapter 1

Unimodular multilinear forms and applications

In this chapter we investigate the asymptotic behavior of multilinear forms on sequence spaces $A : \ell_{p_1}^{n_1} \times \cdots \times \ell_{p_m}^{n_m} \to \mathbb{K}$ with coefficients of modulus 1 (unimodular forms for short). In order to do this, an extension of the Kahane–Salem–Zygmund multilinear inequality is obtained. The main results presented here came from the paper [8], whose main references are [10, 12, 28].The results of this chapter are present in

> [8] *Asymptotic estimates for unimodular multilinear forms with small norms on sequence spaces, Bulletin of the Brazilian Mathematical Society, New Series (2019).*

The so-called classical Kahane–Salem–Zygmund multilinear inequality provides a unimodular multilinear form mapping $\ell_p^n \times \cdots \times \ell_p^n$ into $\mathbb{K}$ (see [28]) with relatively small supremum norm. This result is a fundamental tool in modern analysis with a broad range of applications involving, for instance, the Bohr radius, and Bohnenblust–Hille and Hardy–Littlewood multilinear inequalities (see, e.g., [1, 13, 18, 20, 21, 29, 34–36, 45, 56, 57]).

Here, the sections are organized as follows: in Section 1.1 we investigate the asymptotic estimates of unimodular forms, and for that we will need the Kahane–Salem–Zygmund inequality, so in Section 1.2 we revisit the Kahane–Salem–Zygmund multilinear inequality. The proof of this inequality is an adaptation of the original proof (see [28] and [5, Lemma 3.1]). Later in this section we use interpolation and duality to improve the norm estimate when $p \in [1, 2]$. Finally, in Section 1.3, it is proved that the Kahane–Salem–Zygmund

inequality we obtained is sharp when $p_1, \dots, p_m \in [2, \infty]$. In Section 1.4 some applications are presented.

1.1 Asymptotic estimates of unimodular forms

In this section we investigate the asymptotic growth of

$$\inf_{A \in \mathcal{U}} \|A\|$$

where $\mathcal{U}$ comprises all unimodular multilinear forms $A : \ell_{p_1}^{n_1} \times \cdots \times \ell_{p_m}^{n_m} \to \mathbb{K}$. One of the main tools is the Kahane–Salem–Zygmund inequality which gives us the following: For $p_1, \dots, p_m \geq 2$ and $\mathbb{K}$ the real or complex scalar field, there exists an m-linear form $A : \ell_{p_1}^{n} \times \cdots \times \ell_{p_m}^{n} \to \mathbb{K}$ of the form

$$A\left(x^1, \dots, x^m\right) = \sum_{j_1, \dots, j_m = 1}^{n} \varepsilon_{\mathbf{j}} x_{j_1}^1 \cdots x_{j_m}^m,$$

with $\varepsilon_{\mathbf{j}} \in \{-1, 1\}$, $\mathbf{j} := (j_1, \dots, j_m)$, such that

$$\|A\| \leq C_m \cdot n^{\frac{m+1}{2} - \sum_{j=1}^{m} \frac{1}{p_j}},$$

where $C_m := (m!)^{1 - \max\left\{\frac{1}{2}, \frac{1}{\max\{p_1, \dots, p_m\}}\right\}} \sqrt{32m \log(6m)}$. On the other hand, if we consider Theorem 3, for the case of multilinear forms with coefficients of modulus 1, it is possible to verify that there is a constant $B_m > 0$ such that

$$\|A\| \geq B_m \cdot n^{\frac{m+1}{2} - \sum_{j=1}^{m} \frac{1}{p_j}}$$

for all unimodular m-linear forms $A : \ell_{p_1}^{n} \times \cdots \times \ell_{p_m}^{n} \to \mathbb{K}$ with $p_1, \dots, p_m \geq 2$. Thus we conclude that the exponent is sharp and we can summarize this sharpness by writing

$$0 < B_m \leq \inf_{A \in \mathcal{U}} \frac{\|A\|}{n^{\frac{m+1}{2} - \sum_{j=1}^{m} \frac{1}{p_j}}} \leq C_m < \infty, \tag{1.1}$$

where $p_1, \dots, p_m \geq 2$ and the infimum ranges over all unimodular m-linear forms $A : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$. From (1.1) we get that

$$0 < B_m \leq \inf_{A \in \mathcal{U}} \frac{\|A\|}{n^s} \leq C_m < \infty \Leftrightarrow s = \frac{m+1}{2} - \sum_{j=1}^{m} \frac{1}{p_j}. \tag{1.2}$$

Indeed, suppose that there exists s_1 such that

$$0 < B_m^{'} \leq \inf_{A \in \mathcal{U}} \frac{\|A\|}{n^{s_1}} \leq C_m^{'} < \infty.$$

In this case we should have that

$$n^{s_1 - s} \leq C_m / B_m^{'} \text{ and } n^{s - s_1} \leq C_m^{'} / B_m.$$

This concludes the statement.

Problem 1.1. *What about the asymptotic behavior of* $\inf_{A \in \mathcal{U}} \|A\|$*, where the infimum is estimated over all unimodular* m*-linear forms* $A : \ell_{p_1}^{n_1} \times \cdots \times \ell_{p_m}^{n_m} \to \mathbb{K}$*?*

The following simple lemmata show that the answer to Problem 1.1 is not so immediate.

Lemma 1.2. *If there are* $s_1, \dots, s_m \in \mathbb{R}$ *such that there exist* $C_m, B_m \in (0, \infty)$ *with*

$$B_m \leq \inf_{A \in \mathcal{U}} \frac{\|A\|}{n_1^{s_1} \cdots n_m^{s_m}} \leq C_m$$

for all $n_1, \dots, n_m \geq 1$*, then* $s_1, \dots, s_m$ *are unique.*

Proof. To prove that s_1 is unique it suffices to consider $n_2 = \cdots = n_m = 1$ and so on. □

Lemma 1.3. *There are no* $s_1, \dots, s_m \in \mathbb{R}$ *such that there exist* $C_m, B_m \in (0, \infty)$ *and*

$$B_m \leq \inf_{A \in \mathcal{U}} \frac{\|A\|}{n_1^{s_1} \cdots n_m^{s_m}} \leq C_m$$

for all $n_1, \dots, n_m \geq 1$.

Proof. Considering $n_2 = \cdots = n_m = 1$ we conclude that $s_1 \geq 1 - \frac{1}{p_1}$. In fact, every

unimodular form is of the type

$$A(x^1,\dots,x^m)=\left(\sum_{j=1}^{n_1}\pm x_j^1\right)x_1^2\cdots x_1^m$$

and

$$\|A\|=n_1^{1-\frac{1}{p_1}}.$$

By the previous lemma, we have $s_1=1-\frac{1}{p_1}$. Analogously,

$$s_2=1-\frac{1}{p_2},\dots,s_m=1-\frac{1}{p_m}.$$

Now, considering $n_1=\cdots=n_m=n$, supposing we would have

$$B_m\leq\inf_{A\in\mathcal{U}}\frac{\|A\|}{n^{m-\sum_{j=1}^d\frac{1}{p_j}}}\leq C_m$$

contradicts (1.2). □

An answer to Problem 1.1 is a function $f(n_1,\dots,n_m)$ so that there exist $C_m,B_m\in(0,\infty)$ such that

$$B_m\leq\inf_{A\in\mathcal{U}}\frac{\|A\|}{f(n_1,\dots,n_m)}\leq C_m$$

for all $n_1,\dots,n_m\geq 1$. This function f describes the asymptotic growth of such multilinear forms A with small norms. We shall show (see Theorem 1.11) that, for $p_1,\dots,p_m\geq 2$, f can be chosen as

$$\left(n_1^{1/2}+\cdots+n_m^{1/2}\right)\prod_{j=1}^m n_j^{\frac{1}{2}-\frac{1}{p_j}}.$$

In order to do this, we prove a version of the Kahane-Salem-Zygmund multilinear inequality on the space $\ell_{p_1}^{n_1}\times\cdots\times\ell_{p_m}^{n_m}$ with sup norm having a better upper bound estimate when dealing with some p_k between 1 and 2 (see Theorem 1.7). Moreover, applications concerning the Hardy–Littlewood inequality are provided.

1.2 Kahane–Salem–Zygmund inequality revisited

The main result in this section is Theorem 1.7, a version of the Kahane-Salem-Zygmund multilinear inequality on the space $\ell_{p_1}^{n_1}\times\cdots\times\ell_{p_m}^{n_m}$ with sup norm having a

better upper bound estimate when dealing with some p_k between 1 and 2. Two basic tools are needed: a useful upper bound for the probability of a non-negative function to be greater than or equal to some positive constant (Lemma 1.4); and a covering argument (Lemma 1.5). The techniques follow along the lines of [28]. We present the sketch of the proof for the sake of completeness.

Lemma 1.4. *(Chebyshev–Markov Inequality* [22, Lemma 20.1]*) Let (Ω, Σ, μ) be a measure space, f be a measurable function on Ω, and $g : \mathbb{R} \to \mathbb{R}$ be a non-negative function non-decreasing on the range of f and $g(t) > 0$ if $t > 0$. Then, for every positive real number α,*

$$\mu\left([f \geq \alpha]\right) \leq \frac{1}{g(\alpha)} \int_{\Omega} g \circ f \, d\mu.$$

In particular,

$$\mu\left([|f| \geq \alpha]\right) \leq \frac{1}{\alpha^p} \int_{\Omega} |f|^p \, d\mu$$

holds for every real number $p > 0$.

Lemma 1.5. *(Covering argument* [28, p. 333]*) Let r be a positive real number. Then the open unit ball B of ℓ_p^n can be covered by a collection of open ℓ_p^n-balls of radius r, with the number of balls in the collection not exceeding $\left(1 + 2r^{-1}\right)^{2n}$, and the centers of the balls lying in the closed unit ball $\overline{B}$ of ℓ_p^n.*

The version of the Kahane–Salem–Zygmund inequality on $\ell_{p_1}^{n_1} \times \cdots \times \ell_{p_m}^{n_m}$ is presented below.

Proposition 1.6. *Let $m, n_1, \ldots, n_m \geq 1$ be positive integers and $p_1, \ldots, p_m \in [1, \infty]$. Then there exist signs $\varepsilon_{\mathbf{j}} = \pm 1$ and an m-linear map $A : \ell_{p_1}^{n_1} \times \cdots \times \ell_{p_m}^{n_m} \to \mathbb{K}$ of the form*

$$A\left(z^1, \ldots, z^m\right) = \sum_{j_1=1}^{n_1} \cdots \sum_{j_m=1}^{n_m} \varepsilon_{\mathbf{j}} z_{j_1}^1 \cdots z_{j_m}^m,$$

such that

$$\|A\| \leq C_m \cdot \prod_{k=1}^{m} n_k^{\max\left\{\frac{1}{2} - \frac{1}{p_k}, 0\right\}} \cdot \left(\sum_{k=1}^{m} n_k\right)^{\frac{1}{2}},$$

with $C_m = 8(m!)^{1-\max\left\{\frac{1}{2}, \frac{1}{p}\right\}} \sqrt{\log(1+4m)}$ and $p := \max\{p_1, \ldots, p_m\}$.

Proof. The argument consists on a probabilistic estimate combined with the covering lemma. Initially we establish the following notation: z denotes an m-tuple $z^1, \ldots, z^m$

with $z^k \in \ell^{n_k}_{p_k}$ and $k = 1, \ldots, m$, $\mathbf{j} := (j_1, \ldots, j_m)$ where each j_k is between 1 and n_k, sums and products run over all such m-tuples of integers. A prime symbol " $'$ " on a sum or product means that it is restricted to m-tuples of integers that are arranged in non-decreasing order, and $\mathbf{j} \sim \mathbf{k}$ indicates that the m-tuples $\mathbf{j}$ and $\mathbf{k}$ are permutations of each other. The symbol $z_{\mathbf{j}}$ is shorthand for the monomial $z^1_{j_1} \cdots z^m_{j_m}$. Thus, for a fixed a multilinear map like the one in the statement, it can be written in the form

$$A(z) = \sum_{\mathbf{k}}{}' \left(\pm \sum_{\mathbf{j} \sim \mathbf{k}} z_{\mathbf{j}} \right),$$

and in this expression all of the plus and minus signs are independent of each other. Let us fix a point $z \in \overline{B}^{n_1}_{p_1} \times \cdots \times \overline{B}^{n_m}_{p_m}$. For each m-tuple of positive integers $\mathbf{k} := (k_1, \ldots, k_m)$ in non-decreasing order, let us choose a Rademacher function $r_{\mathbf{k}}$, and consider the random sum

$$A(t, z) = \sum_{\mathbf{k}}{}' \left(r_{\mathbf{k}}(t) \sum_{\mathbf{j} \sim \mathbf{k}} z_{\mathbf{j}} \right),$$

where t lies in the interval $I := [0, 1]$. The immediate goal is to obtain an upper estimate on the probability that this sum has large modulus. Let λ be an arbitrary positive number (whose value we will specify afterwards in terms of m, $n_1, \ldots, n_m$ and p). Invoking the independence of the Rademacher functions, we may compute the expectation (that is, the integral with respect to t) of the exponential of the real part of $\lambda A(t, z)$ by computing the product over non-decreasing m-tuples $\mathbf{k}$ of the expectation of $e^{\left(\lambda r_{\mathbf{k}}(t) \sum_{\mathbf{j} \sim \mathbf{k}} z_{\mathbf{j}}\right)}$. More precisely:

$$\operatorname{Re}\left[\lambda A(t, z)\right] = \sum_{\mathbf{k}}{}' \left(\lambda r_{\mathbf{k}}(t) \sum_{\mathbf{j} \sim \mathbf{k}} \operatorname{Re} z_{\mathbf{j}} \right)$$

which implies

$$e^{\operatorname{Re}[\lambda A(t,z)]} = \prod_{\mathbf{k}}{}' e^{\left(\lambda r_{\mathbf{k}}(t) \sum_{\mathbf{j} \sim \mathbf{k}} \operatorname{Re} z_{\mathbf{j}}\right)}.$$

Denoting $\alpha_{\mathbf{k}} := \lambda \sum_{\mathbf{j} \sim \mathbf{k}} \operatorname{Re} z_{\mathbf{j}}$, using the independence and orthogonality properties of Rademacher functions, and

$$\int_I e^{\beta r_i(t)}\, dt = \int_I \left(\sum_{m=0}^{+\infty} \frac{\beta^m r_i^m(t)}{m!} \right) dt = \sum_{m=0}^{+\infty} \frac{\beta^{2m}}{(2m)!} = \cosh(\beta),$$

we get

$$\begin{aligned}\int_I e^{\mathrm{Re}[\lambda A(t,z)]}\,dt &= \int_I \prod_{\mathbf{k}}{}' e^{\alpha_{\mathbf{k}} r_{\mathbf{k}}(t)}\,dt \\ &= \prod_{\mathbf{k}}{}' \int_I e^{\alpha_{\mathbf{k}} r_{\mathbf{k}}(t)}\,dt \\ &= \prod_{\mathbf{k}}{}' \cosh\left(\lambda \sum_{\mathbf{j}\sim\mathbf{k}} \mathrm{Re}\, z_{\mathbf{j}}\right).\end{aligned}$$

In view of the inequality

$$\cosh x \le e^{\frac{x^2}{2}} \qquad \text{for all } x \in \mathbb{R}$$

(which follows from $k!\cdot 2^k \le k!\cdot(k+1)\cdots(2k) = (2k)!$ and series comparison), we get an upper bound for the expectation of $\mathrm{Re}\,[\lambda A\,(t,z)]$:

$$\int_I e^{\mathrm{Re}[\lambda A(t,z)]}\,dt \le \prod_{\mathbf{k}}{}' e^{\frac{1}{2}\left(\lambda \sum_{\mathbf{j}\sim\mathbf{k}} \mathrm{Re}\, z_{\mathbf{j}}\right)^2} = e^{\frac{1}{2}\lambda^2 \sum_{\mathbf{k}}'\left(\sum_{\mathbf{j}\sim\mathbf{k}} \mathrm{Re}\ z_{\mathbf{j}}\right)^2}. \tag{1.3}$$

Let $p := \max\{p_1,\ldots,p_m\}$, $m(t) := \min\{t,2\}$ and $M(t) := \max\{t,2\}$, for a real positive t. By Hölder's inequality,

$$\begin{aligned}\left(\sum_{\mathbf{j}\sim\mathbf{k}} \mathrm{Re}\ z_{\mathbf{j}}\right)^2 &= \left|\sum_{\mathbf{j}\sim\mathbf{k}} \mathrm{Re}\ z_{\mathbf{j}}\right|^2 \\ &\le \left(\sum_{\mathbf{j}\sim\mathbf{k}} |\mathrm{Re}\ z_{\mathbf{j}}|\right)^2 \\ &\le \left(\sum_{\mathbf{j}\sim\mathbf{k}} |z_{\mathbf{j}}|\right)^2 \\ &\le (m!)^{2\left(1-\frac{1}{m(p)}\right)} \cdot \left(\sum_{\mathbf{j}\sim\mathbf{k}} |z_{\mathbf{j}}|^{m(p)}\right)^{\frac{2}{m(p)}},\end{aligned} \tag{1.4}$$

where Hölder's inequality was applied in the last step. The exponent $2/m(p)$ is equal to 1 when $p \in [2,\infty]$. When $p \in [1,2]$, it is equal to $2/p$ and belongs to $[1,2]$, so replacing it by 1 can only increase the right-hand side, since $\sum_{\mathbf{j}\sim\mathbf{k}} |z_{\mathbf{j}}|^p \le 1$, which is obtained from the following argument: each n_k-vector z^k lies in $B_{p_k}^{n_k}$, so $\sum_{j=1}^{n_k} \left|z_j^k\right|^{p_k} \le 1$ and by norm

inequality on ℓ_r,

$$\begin{aligned}\sum_{\mathbf{j}\sim\mathbf{k}}|z_{\mathbf{j}}|^p &= \sum_{\mathbf{j}=(j_1,\dots,j_m)\sim\mathbf{k}}\left|z_{j_1}^1\cdots z_{j_m}^m\right|^p\\ &\le \sum_{j_1=1}^{n_1}\cdots\sum_{j_m=1}^{n_m}\left|z_{j_1}^1\cdots z_{j_m}^m\right|^p\\ &= \prod_{k=1}^{m}\left(\sum_{j_k=1}^{n_k}\left|z_{j_k}^k\right|^p\right)\\ &= \left(\|z^1\|_{\ell_p^{n_1}}\cdots\|z^m\|_{\ell_p^{n_m}}\right)^p\\ &\le \left(\|z^1\|_{\ell_{p_1}^{n_1}}\cdots\|z^m\|_{\ell_{p_m}^{n_m}}\right)^p\\ &\le 1. \end{aligned}\tag{1.5}$$

Inequalities (1.4) and (1.5) lead us to conclude, for both cases $1\le p\le 2$ and $2\le p\le\infty$, that

$$\begin{aligned}\sum_{\mathbf{k}}'\left(\sum_{\mathbf{j}\sim\mathbf{k}}\operatorname{Re}\,z_{\mathbf{j}}\right)^2 &\le (m!)^{2\left(1-\frac{1}{m(p)}\right)}\cdot\sum_{\mathbf{k}}'\left(\sum_{\mathbf{j}\sim\mathbf{k}}|z_{\mathbf{j}}|^{m(p)}\right)^{\frac{2}{m(p)}}\\ &\le (m!)^{2\left(1-\frac{1}{m(p)}\right)}\cdot\sum_{\mathbf{k}}'\sum_{\mathbf{j}\sim\mathbf{k}}|z_{\mathbf{j}}|^{m(p)}\\ &= (m!)^{2\left(1-\frac{1}{m(p)}\right)}\cdot\left(\sum_{j_1=1}^{n_1}\cdots\sum_{j_m=1}^{n_m}\left|z_{j_1}^1\cdots z_{j_m}^m\right|^{m(p)}\right)\\ &= (m!)^{2\left(1-\frac{1}{m(p)}\right)}\cdot\left(\sum_{j_1=1}^{n_1}\left|z_{j_1}^1\right|^{m(p)}\right)\cdots\left(\sum_{j_m=1}^{n_m}\left|z_{j_m}^m\right|^{m(p)}\right)\\ &= (m!)^{2\left(1-\frac{1}{m(p)}\right)}\|z^1\|_{\ell_{m(p)}^{n_1}}^{m(p)}\cdots\|z^m\|_{\ell_{m(p)}^{n_m}}^{m(p)}.\end{aligned}$$

Using the following facts and choice for exponents:

- $m(p_k)\le m(p)$ and $\left|z_j^k\right|\le\|z^k\|_{\ell_{p_k}^{n_k}}\le 1$ imply $\left|z_j^k\right|^{m(p)}\le\left|z_j^k\right|^{m(p_k)}$,
- $m(p_k)\cdot M(p_k)/2 = p_k$ and
- Hölder's inequality with exponents $\frac{M(p_k)}{2}$ and $\left(1-\frac{2}{M(p_k)}\right)^{-1}$,

we increase the upper bound:

$$\begin{aligned}
\|z^k\|_{\ell^{n_k}_{m(p)}}^{m(p)} &= \sum_{j=1}^{n_k} \left|z_j^k\right|^{m(p)} \\
&\leq \sum_{j=1}^{n_k} \left|z_j^k\right|^{m(p_k)} \\
&\leq {n_k}^{1-\frac{2}{M(p_k)}} \cdot \left(\sum_{j=1}^{n_k} \left|z_j^k\right|^{\frac{m(p_k)M(p_k)}{2}} \right)^{\frac{2}{M(p_k)}} \\
&= {n_k}^{1-\frac{2}{M(p_k)}} \cdot \left(\sum_{j=1}^{n_k} \left|z_j^k\right|^{p_k} \right)^{\frac{2}{M(p_k)}} \\
&= {n_k}^{1-\frac{2}{M(p_k)}} \cdot \|z^k\|_{\ell^{n_k}_{p_k}}^{\frac{2p_k}{M(p_k)}} \\
&\leq {n_k}^{1-\frac{2}{M(p_k)}}.
\end{aligned}$$

Therefore, (1.3) is bounded above by

$$\begin{aligned}
\int_I e^{\lambda \mathrm{Re}[A(t,z)]}\, dt &\leq e^{\frac{1}{2}\lambda^2 \sum_{\mathbf{k}}' \left(\sum_{\mathbf{j}\sim\mathbf{k}} \mathrm{Re}\ z_{\mathbf{j}}\right)^2} \\
&\leq \exp\left[\frac{1}{2}\lambda^2 \, (m!)^{2\left(1-\frac{1}{m(p)}\right)} \cdot {n_1}^{2\left(\frac{1}{2}-\frac{1}{M(p_1)}\right)} \cdots {n_m}^{2\left(\frac{1}{2}-\frac{1}{M(p_m)}\right)} \right].
\end{aligned}$$

Let R be an arbitrary positive real number (whose value will be specified later in terms of $m, n_1, \ldots, n_m$ and p). Using the previous estimates and applying Lemma 1.4 with $g(t) := e^t$ and

$$\mathcal{A} := \{t \in I \,:\, \mathrm{Re}\,[A\,(t,z)] \geq R\} = \{t \in I \,:\, \lambda \mathrm{Re}\,[A\,(t,z)] \geq \lambda R\}\,,$$

we get:

$$\begin{aligned}
\mu\,(\mathcal{A}) &\leq \frac{1}{e^{\lambda R}} \int_I e^{\lambda \mathrm{Re}[A(t,z)]}\, dt \\
&\leq \exp\left[-R\lambda + \frac{1}{2}\lambda^2 \, (m!)^{2\left(1-\frac{1}{m(p)}\right)} \cdot {n_1}^{2\left(\frac{1}{2}-\frac{1}{M(p_1)}\right)} \cdots {n_m}^{2\left(\frac{1}{2}-\frac{1}{M(p_m)}\right)} \right]. \qquad (1.6)
\end{aligned}$$

By symmetric reasoning, the same estimate is obtained for the measure of the points $t \in I$ such that $\mathrm{Re}\,[A\,(t,z)]$ is less or equal to $-R$, which is the same set of points that

satisfy $(-\lambda)\operatorname{Re}[A(t,z)] \geq \lambda R$. Since

$$[|\operatorname{Re}[A(t,z)]| \geq R] = [\operatorname{Re}[A(t,z)] \geq R] \cup [\operatorname{Re}[A(t,z)] \leq -R],$$

this set has measure at most twice the bound (1.6). The same argument applies to the imaginary part of $A(t,z)$. In view of the inequality $|w| \leq \sqrt{2}\max\{|\operatorname{Re} w|, |\operatorname{Im} w|\}$, $|A(t,z)| \geq \sqrt{2}R$ implies $|\operatorname{Re}[A(t,z)]| \geq R$ or $|\operatorname{Im}[A(t,z)]| \geq R$. Thus

$$\left[|A(t,z)| \geq \sqrt{2}R\right] \subset [|\operatorname{Re}[A(t,z)]| \geq R] \cup [|\operatorname{Im}[A(t,z)]| \geq R].$$

Therefore, the probability that $|A(t,z)|$ exceeds $\sqrt{2}R$ is at most

$$4 \cdot \exp\left[-R\lambda + \frac{1}{2}\lambda^2 (m!)^{2\left(1-\frac{1}{m(p)}\right)} \cdot {n_1}^{2\left(\frac{1}{2}-\frac{1}{M(p_1)}\right)} \cdots {n_m}^{2\left(\frac{1}{2}-\frac{1}{M(p_m)}\right)}\right].$$

This probabilistic estimate holds for an arbitrary but fixed z. The second part of the proof uses Lemma 1.5 and a simple Lipschitz estimate for $A(t,z)$. Suppose that $z, w \in \mathbb{K}^{n_1} \times \cdots \times \mathbb{K}^{n_m}$ are such that each one of the n_k-vector components z^k and w^k lie in $\overline{B}_{p_k}^{n_k}$ and $\left\|z^k - w^k\right\|_{\ell_{p_k}^{n_k}} \leq \varepsilon$ for every $k = 1, \ldots, m$. The multilinearity of A implies that

$$\begin{aligned} A\left(t, z^1, \ldots, z^m\right) &= A\left(t, z^1 - w^1, z^2, \ldots, z^m\right) \\ &\quad + A\left(t, w^1, z^2 - w^2, z^3 \ldots, z^m\right) \\ &\quad + \cdots \\ &\quad + A\left(t, w^1, \ldots, w^{m-1}, z^m - w^m\right) \\ &\quad + A\left(t, w^1, \ldots, w^m\right). \end{aligned}$$

Consequently, the boundedness of A guarantees that

$$|A(t,z) - A(t,w)| \leq \varepsilon \cdot m \cdot \sup_{\mathbf{z} \in \overline{B}_{p_1}^{n_1} \times \cdots \times \overline{B}_{p_m}^{n_m}} |A(t,\mathbf{z})|.$$

Now let us fix an arbitrary point $z \in B_{p_1}^{n_1} \times \cdots \times B_{p_m}^{n_m}$. Taking $r = 1/2m$, Lemma 1.5 assures that each open ball $B_{p_k}^{n_k}$ is covered by a collection of at most $(1+4m)^{2n_k}$ balls of radius $1/2m$ and with centers lying in the closed unit ball $\overline{B}_{p_k}^{n_k}$. Let $\mathcal{W} = \{w\}$ be the

collection of points $w = (w^1, \dots, w^m) \in \overline{B}_{p_1}^{n_1} \times \cdots \times \overline{B}_{p_m}^{n_m}$ with each component w^k having its center in some ball belonging to the cover of $B_{p_k}^{n_k}$, $k = 1, \dots, m$. Thus $\mathcal{W}$ does not exceed $(1+4m)^{2\sum_{k=1}^{m} n_k}$ points. Consequently, each n_k-vector component z^k of z must lie at a distance at most $1/2m$ from some w^k. Therefore, z, w fulfill the above Lipschitz property with $\varepsilon = 1/2m$ and

$$\begin{aligned} |A(t,z)| &\le |A(t,w)| + |A(t,z) - A(t,w)| \\ &\le \max_{w \in \mathcal{W}} |A(t,w)| + \frac{1}{2} \cdot \sup_{\mathbf{z} \in \overline{B}_{p_1}^{n_1} \times \cdots \times \overline{B}_{p_m}^{n_m}} |A(t,\mathbf{z})| . \end{aligned}$$

Since z is an arbitrary point, we get

$$\sup_{\mathbf{z} \in \overline{B}_{p_1}^{n_1} \times \cdots \times \overline{B}_{p_m}^{n_m}} |A(t,\mathbf{z})| \le 2 \cdot \max_{w \in \mathcal{W}} |A(t,w)| ,$$

which implies

$$\left[\sup_{\mathbf{z} \in \overline{B}_{p_1}^{n_1} \times \cdots \times \overline{B}_{p_m}^{n_m}} |A(t,\mathbf{z})| \ge 2\sqrt{2}R \right] \subset \bigcup_{w \in \mathcal{W}} \left[|A(t,w)| \ge \sqrt{2}R \right] .$$

Hence, applying the preceding probabilistic estimate to each point of the finite collection $\mathcal{W}$, we get that the probability $P_{\mathcal{S}}$ of the set

$$\mathcal{S} := \left[\sup_{\mathbf{z} \in \overline{B}_{p_1}^{n_1} \times \cdots \times \overline{B}_{p_m}^{n_m}} |A(t,\mathbf{z})| \ge 2\sqrt{2}R \right]$$

is at most

$$4\,(1+4m)^{2\sum_{k=1}^{m} n_k} \cdot \exp\left[-R\lambda + \frac{1}{2}\lambda^2 \,(m!)^{2\left(1-\frac{1}{m(p)}\right)} \cdot \prod_{k=1}^{m} {n_k}^{2\left(\frac{1}{2}-\frac{1}{M(p_k)}\right)} \right] .$$

Now taking the following values for the parameters R and λ,

$$\begin{aligned} R &:= \left(2\,(m!)^{2\left(1-\frac{1}{m(p)}\right)} \prod_{k=1}^{m} {n_k}^{2\left(\frac{1}{2}-\frac{1}{M(p_k)}\right)} \log\left(8\,(1+4m)^{2\sum_{k=1}^{m} n_k} \right) \right)^{\frac{1}{2}} , \\ \lambda &:= \frac{R}{(m!)^{2\left(1-\frac{1}{m(p)}\right)} \, {n_1}^{2\left(\frac{1}{2}-\frac{1}{M(p_1)}\right)} \cdots {n_m}^{2\left(\frac{1}{2}-\frac{1}{M(p_m)}\right)}} , \end{aligned}$$

we conclude that, with these choices, the probability $P_{\mathcal{S}}$ that the supremum of $|A(t,\cdot)|$ over $\overline{B}_{p_1}^{n_1} \times \cdots \times \overline{B}_{p_m}^{n_m}$ exceeds $2\sqrt{2}R$ is such that

$$P_{\mathcal{S}} \leq \frac{1}{2}.$$

This is achieved by looking for values for R and λ such that the previous probability is $1/2$, *i.e.*,

$$\exp\left(-R\lambda + \frac{\beta}{2}\lambda^2\right) = \frac{1}{2\alpha},$$

with $\alpha := 4\,(1+4m)^{2\sum_{k=1}^m n_k}$ and $\beta := (m!)^{2\left(1-\frac{1}{m(p)}\right)} \cdot \prod_{k=1}^m {n_k}^{2\left(\frac{1}{2}-\frac{1}{M(p_k)}\right)}$. This is equivalent to solve the following equation with respect to λ

$$\frac{\beta}{2}\lambda^2 - R\lambda + \log(2\alpha) = 0,$$

which provides, imposing the condition $R^2 - 2\beta\log(2\alpha) = 0$, the unique solution $\lambda = R/\beta$, and these are precisely the values presented previously.

Therefore, there exists a particular value $t_0 \in [0,1]$ such that the supremum of $|A(t_0,\cdot)|$ over $\overline{B}_{p_1}^{n_1} \times \cdots \times \overline{B}_{p_m}^{n_m}$ is no more than

$$2\sqrt{2}R = \left(16\,(m!)^{2\left(1-\frac{1}{m(p)}\right)} \prod_{k=1}^m {n_k}^{2\left(\frac{1}{2}-\frac{1}{M(p_k)}\right)} \log\left(8\,(1+4m)^{2\sum_{k=1}^m n_k}\right)\right)^{\frac{1}{2}}.$$

Moreover, notice that,

$$\begin{aligned}\left(\log\left(8\,(1+4m)^{2\sum_{k=1}^m n_k}\right)\right)^{\frac{1}{2}} &\leq \sqrt{2}\cdot\max\left\{\log 8,\ 2\left(\sum_{k=1}^m n_k\right)\log(1+4m)\right\}^{\frac{1}{2}} \\ &= 2\cdot\left(\sum_{k=1}^m n_k\right)^{\frac{1}{2}} \cdot \sqrt{\log(1+4m)}.\end{aligned}$$

The values of the Rademacher functions at the particular value t_0 produce the pattern of signs $\varepsilon_{\mathbf{j}} := r_{\mathbf{j}}(t_0)$ stated. □

Borrowing an argument from [18], we improve the estimate of the norm of A when dealing with some p_k between 1 and 2. We will fix the following notation for a constant

involved in the results of this section:

$$C_m := 8(m!)^{1-\max\left\{\frac{1}{2},\frac{1}{p}\right\}}\sqrt{\log(1+4m)},$$

with $p := \max\{p_1, \ldots, p_m\}$.

Theorem 1.7. *Let $m, n_1, \ldots, n_m \geq 1$ be positive integers and $p_1, \ldots, p_m \in [1, \infty]$. Then there exist signs $\varepsilon_{\mathbf{j}} = \pm 1$ and an m-linear map $A : \ell_{p_1}^{n_1} \times \cdots \times \ell_{p_m}^{n_m} \to \mathbb{K}$ of the form*

$$A\left(z^1, \ldots, z^m\right) = \sum_{j_1=1}^{n_1} \cdots \sum_{j_m=1}^{n_m} \varepsilon_{\mathbf{j}} z_{j_1}^1 \cdots z_{j_m}^m,$$

such that

$$\|A\| \leq (C_m)^{2\left(1-\frac{1}{\gamma}\right)} \cdot \left(\sum_{k=1}^{m} n_k\right)^{1-\frac{1}{\gamma}} \cdot \prod_{k=1}^{m} n_k^{\max\left\{\frac{1}{\gamma}-\frac{1}{p_k},0\right\}},$$

with $\gamma := \min\left\{2, \max\{p_k : p_k \leq 2\}\right\}$.

Proof. If $p_k \geq 2$ for all $k = 1, \cdots, m$, then the estimate above coincides with the estimate of Proposition 1.6. For the sake of clarity, we shall assume that $\{k \in \{1, \cdots, m\} : p_k \geq 2\} = \{1, \ldots, d\}$ and, for $p > 0$, define

$$\beta(p) := \begin{cases} 1 - \frac{1}{p}, \text{ if } p \geq 2; \\ 0, \text{ otherwise.} \end{cases}$$

The general form of the Kahane–Salem–Zygmund inequality guarantees the existence of an m-linear map $A : \ell_{p_1}^{n_1} \times \cdots \times \ell_{p_d}^{n_d} \times \ell_2^{n_{d+1}} \times \cdots \times \ell_2^{n_m} \to \mathbb{K}$ such that

$$\begin{aligned} \|A\|_{\mathcal{L}\left(\ell_{p_1}^{n_1} \times \cdots \times \ell_{p_m}^{n_m} \times \ell_2^{n_{m+1}} \times \cdots \times \ell_2^{n_m}; \mathbb{K}\right)} &\leq C_m \cdot \left(\sum_{k=1}^{m} n_k\right)^{\frac{1}{2}} \cdot \prod_{k=1}^{d} n_k^{\frac{1}{2}-\frac{1}{p_k}} \\ &= C_m \cdot \left(\sum_{k=1}^{m} n_k\right)^{\frac{1}{2}} \cdot \prod_{k=1}^{m} n_k^{\max\left\{\frac{1}{2}-\frac{1}{p_k},0\right\}}. \end{aligned}$$

The restriction of this map to $\ell_{p_1}^{n_1} \times \cdots \times \ell_{p_d}^{n_d} \times \ell_1^{n_{d+1}} \times \cdots \times \ell_1^{n_m}$ has norm not greater than

$$\prod_{k=1}^{d} n_k^{1/p_k^*} = \prod_{k=1}^{m} n_k^{\beta(p_k)}.$$

Since $1 \leq \gamma \leq 2$, there exists $\theta \in (0,1)$ such that $\frac{1}{\gamma} = \frac{1-\theta}{1} + \frac{\theta}{2}$. Note that $\theta = \frac{2}{\gamma^*}$. Now write $\frac{1}{p_k} = \frac{1-\theta}{p_k} + \frac{\theta}{p_k}$ for all $k \in \{1, \cdots, d\}$. By using Theorem 5,

$$\begin{aligned}
&\|A\|_{\mathcal{L}\left(\ell_{p_1}^{n_1} \times \cdots \times \ell_{p_d}^{n_d} \times \ell_{\gamma}^{n_{d+1}} \times \cdots \times \ell_{\gamma}^{n_m}; \mathbb{K}\right)} \\
&\leq \|A\|^{1-\theta}_{\mathcal{L}\left(\ell_{p_1}^{n_1} \times \cdots \times \ell_{p_d}^{n_d} \times \ell_{1}^{n_{d+1}} \times \cdots \times \ell_{1}^{n_m}; \mathbb{K}\right)} \cdot \|A\|^{\theta}_{\mathcal{L}\left(\ell_{p_1}^{n_1} \times \cdots \times \ell_{p_d}^{n_d} \times \ell_{2}^{n_{d+1}} \times \cdots \times \ell_{2}^{n_m}; \mathbb{K}\right)} \\
&\leq \prod_{k=1}^{m} n_k^{(1-\theta)\beta(p_k)} \left[C_m \cdot \left(\sum_{k=1}^{m} n_k \right)^{\frac{1}{2}} \cdot \prod_{k=1}^{m} n_k^{\max\left\{\frac{1}{2} - \frac{1}{p_k}, 0\right\}} \right]^{\theta} \\
&= (C_m)^{\theta} \cdot \left(\sum_{k=1}^{m} n_k \right)^{\frac{\theta}{2}} \cdot \prod_{k=1}^{m} n_k^{(1-\theta)\beta(p_k) + \max\left\{\frac{\theta}{\gamma} - \frac{\theta}{p_k}, 0\right\}} \\
&= (C_m)^{\frac{2}{\gamma^*}} \cdot \left(\sum_{k=1}^{m} n_k \right)^{\frac{1}{\gamma^*}} \cdot \prod_{k=1}^{m} n_k^{\max\left\{\frac{1}{\gamma} - \frac{1}{p_k}, 0\right\}}.
\end{aligned}$$

For each $k \notin \{1, \cdots, d\}$ the unit ball of $\ell_{p_k}^{n_k}$ lies in the unit ball of $\ell_{\gamma}^{n_k}$, so

$$\begin{aligned}
\|A\|_{\mathcal{L}\left(\ell_{p_1}^{n_1} \times \cdots \times \ell_{p_m}^{n_m}; \mathbb{K}\right)} &\leq \|A\|_{\mathcal{L}\left(\ell_{p_1}^{n_1} \times \cdots \times \ell_{p_d}^{n_d} \times \ell_{\gamma}^{n_{d+1}} \times \cdots \times \ell_{\gamma}^{n_m}; \mathbb{K}\right)} \\
&\leq (C_m)^{\frac{2}{\gamma^*}} \cdot \left(\sum_{k=1}^{m} n_k \right)^{\frac{1}{\gamma^*}} \cdot \prod_{k=1}^{m} n_k^{\max\left\{\frac{1}{\gamma} - \frac{1}{p_k}, 0\right\}}.
\end{aligned}$$

□

In order to clarify the result above, we illustrate the simpler case $n_1 = \cdots = n_m = n$.

Corollary 1.8. *The norm of the m-linear form A in Theorem 1.7, with $n_1 = \cdots = n_m = n$, has the following estimate:*

$$\|A\| \leq C \cdot n^{1 - \frac{1}{\gamma} + \sum_{k=1}^{m} \max\left\{\frac{1}{\gamma} - \frac{1}{p_k}, 0\right\}},$$

with $C = \left(m \cdot C_m^2\right)^{1 - \frac{1}{\gamma}}$ and $\gamma := \min\left\{2, \max\{p_k : p_k \leq 2\}\right\}$.

The polynomial and vector valued versions of the previous result read as follows.

Corollary 1.9. *Let $p \in [1, \infty]$ and consider integers $n, m \geq 1$. Then there exist signs $\varepsilon_{\mathbf{j}} = \pm 1$ and a homogeneous polynomial P of degree m in the variable $z \in \ell_p^n$ of the form*

$$\sum_{|\alpha|=m} \varepsilon_{\mathbf{j}} \binom{d}{\alpha} z^{\alpha}$$

such that

$$\|P\| \leq (C_m)^{2\left(1-\frac{1}{\gamma}\right)} \cdot n^{1-\frac{1}{\gamma}+m\cdot\max\left\{\frac{1}{2}-\frac{1}{p},0\right\}},$$

with $\gamma := \min\{p, 2\}$.

Corollary 1.10. *Let* $m, n_1, \ldots, n_{m+1} \geq 1$, $s, p_1, \ldots, p_m \in [1, \infty]$. *Then there exist signs* $\varepsilon_{\mathbf{j}} = \pm 1$ *and an* m*-linear map* $A : \ell_{p_1}^{n_1} \times \cdots \times \ell_{p_m}^{n_m} \to \ell_s^{n_{m+1}}$ *of the form*

$$A\left(z^1, \ldots, z^m\right) = \sum_{j_1=1}^{n_1} \cdots \sum_{j_m=1}^{n_m} \sum_{j_{m+1}=1}^{n_{m+1}} \varepsilon_{\mathbf{j}} z_{j_1}^1 \cdots z_{j_m}^m e_{j_{m+1}},$$

such that

$$\|A\| \leq (C_m)^{2\left(1-\frac{1}{\gamma}\right)} \cdot \left(\sum_{k=1}^{m+1} n_k\right)^{1-\frac{1}{\gamma}} \cdot \prod_{k=1}^{m+1} n_k^{\max\left\{\frac{1}{\gamma}-\frac{1}{p_k},0\right\}},$$

with $p_{m+1} := s^*$ *and* $\gamma := \min\left\{2, \max\{p_k : p_k \leq 2\}\right\}$.

Proof. Initially note that the map

$$T \in \mathcal{L}\left(\ell_{p_1}^{n_1} \times \cdots \times \ell_{p_m}^{n_m} \times \ell_{s^*}^{n_{m+1}}; \mathbb{K}\right) \longrightarrow \overline{T} \in \mathcal{L}\left(\ell_{p_1}^{n_1} \times \cdots \times \ell_{p_m}^{n_m}; \ell_s^{n_{m+1}}\right)$$

given by

$$\begin{aligned} &\left(z \mapsto \sum_{j_1=1}^{n_1} \cdots \sum_{j_m=1}^{n_m} \sum_{j_{m+1}=1}^{n_{m+1}} a_{i_1,\ldots,i_m}^{i_{m+1}} z_{j_1}^1 \cdots z_{j_m}^m z_{j_{m+1}}^{m+1}\right) \\ \mapsto &\left(z \mapsto \sum_{j_1=1}^{n_1} \cdots \sum_{j_m=1}^{n_m} \sum_{j_{m+1}=1}^{n_{m+1}} a_{i_1,\ldots,i_m}^{i_{m+1}} z_{j_1}^1 \cdots z_{j_m}^m e_{j_{m+1}}\right) \end{aligned}$$

is an isometric isomorphism. Indeed, for any $z = (z^1, \ldots, z^m) \in \ell_{p_1}^{n_1} \times \cdots \times \ell_{p_m}^{n_m}$,

$$\begin{aligned} \overline{T}(z) &= \sum_{j_1=1}^{n_1} \cdots \sum_{j_m=1}^{n_m} \sum_{j_{m+1}=1}^{n_{m+1}} a_{i_1,\ldots,i_m}^{i_{m+1}} z_{j_1}^1 \cdots z_{j_m}^m e_{j_{m+1}} \\ &= \sum_{j_1=1}^{n_1} \cdots \sum_{j_m=1}^{n_m} a_{i_1,\ldots,i_m}^1 z_{j_1}^1 \cdots z_{j_m}^m e_1 + \cdots \\ &\quad + \sum_{j_1=1}^{n_1} \cdots \sum_{j_m=1}^{n_m} a_{i_1,\ldots,i_m}^{n_{m+1}} z_{j_1}^1 \cdots z_{j_m}^m e_{n_{m+1}} \\ &= \left(\sum_{j_1=1}^{n_1} \cdots \sum_{j_m=1}^{n_m} a_{i_1,\ldots,i_m}^k z_{j_1}^1 \cdots z_{j_m}^m\right)_{k=1}^{n_{m+1}}, \end{aligned}$$

then

$$\|\overline{T}(z)\|_{\ell_s^{n_{m+1}}} = \left(\sum_{k=1}^{n_{m+1}} \left|\sum_{j_1=1}^{n_1} \cdots \sum_{j_m=1}^{n_m} a_{i_1,\ldots,i_m}^k z_{j_1}^1 \cdots z_{j_m}^m\right|^s\right)^{\frac{1}{s}}.$$

Using Hölder's inequality we have

$$\begin{aligned}
\left|T(z, z^{m+1})\right| &= \left|\sum_{j_1=1}^{n_1} \cdots \sum_{j_m=1}^{n_m} \sum_{j_{m+1}=1}^{n_{m+1}} a_{i_1,\ldots,i_m}^{i_{m+1}} z_{j_1}^1 \cdots z_{j_m}^m z_{j_{m+1}}^{m+1}\right| \\
&= \left|\sum_{j_1=1}^{n_1} \cdots \sum_{j_m=1}^{n_m} a_{i_1,\ldots,i_m}^1 z_{j_1}^1 \cdots z_{j_m}^m z_1^{m+1} + \cdots \right. \\
&\quad \left. + \sum_{j_1=1}^{n_1} \cdots \sum_{j_m=1}^{n_m} a_{i_1,\ldots,i_m}^{n_{m+1}} z_{j_1}^1 \cdots z_{j_m}^m z_{n_{m+1}}^{m+1}\right| \\
&= \left|\sum_{k=1}^{n_{m+1}} \left(\sum_{j_1=1}^{n_1} \cdots \sum_{j_m=1}^{n_m} a_{i_1,\ldots,i_m}^k z_{j_1}^1 \cdots z_{j_m}^m\right) z_k^{m+1}\right| \\
&\leq \sum_{k=1}^{n_{m+1}} \left|\left(\sum_{j_1=1}^{n_1} \cdots \sum_{j_m=1}^{n_m} a_{i_1,\ldots,i_m}^k z_{j_1}^1 \cdots z_{j_m}^m\right) z_k^{m+1}\right| \\
&\leq \left(\sum_{k=1}^{n_{m+1}} \left|\sum_{j_1=1}^{n_1} \cdots \sum_{j_m=1}^{n_m} a_{i_1,\ldots,i_m}^k z_{j_1}^1 \cdots z_{j_m}^m\right|^s\right)^{\frac{1}{s}} \cdot \left(\sum_{k=1}^{n_{m+1}} \left|z_k^{m+1}\right|^{s^*}\right)^{\frac{1}{s^*}} \\
&= \|\overline{T}(z)\|_{\ell_s^{n_{m+1}}} \cdot \|z^{m+1}\|_{\ell_{s^*}^{n_{m+1}}} \\
&\leq \|\overline{T}\|_{\mathcal{L}\left(\ell_{p_1}^{n_1} \times \cdots \times \ell_{p_m}^{n_m}; \ell_s^{n_{m+1}}\right)} \cdot \|z^1\|_{\ell_{p_1}^{n_1}} \cdots \|z^m\|_{\ell_{p_m}^{n_m}} \cdot \|z^{m+1}\|_{\ell_{s^*}^{n_{m+1}}}.
\end{aligned}$$

Thus,

$$\|T\|_{\mathcal{L}\left(\ell_{p_1}^{n_1} \times \cdots \times \ell_{p_m}^{n_m} \times \ell_{s^*}^{n_{m+1}}; \mathbb{K}\right)} \leq \|\overline{T}\|_{\mathcal{L}\left(\ell_{p_1}^{n_1} \times \cdots \times \ell_{p_m}^{n_m}; \ell_s^{n_{m+1}}\right)}.$$

On the other hand, arguing by duality, for each $z = (z^1, \ldots, z^m) \in \ell_{p_1}^{n_1} \times \cdots \times \ell_{p_m}^{n_m}$ there is $z^{m+1} = (z_1^{m+1}, \ldots, z_{n_{m+1}}^{m+1}) \in \ell_{s^*}^{n_{m+1}}$, with $\|z^{m+1}\|_{\ell_{s^*}^{n_{m+1}}} = 1$, such that

$$\begin{aligned}
\|\overline{T}(z)\|_{\ell_s^{n_{m+1}}} &= \sum_{k=1}^{n_{m+1}} \left(\sum_{j_1=1}^{n_1} \cdots \sum_{j_m=1}^{n_m} a_{i_1,\ldots,i_m}^k z_{j_1}^1 \cdots z_{j_m}^m\right) z_k^{m+1} \\
&= T(z, z^{m+1}) \\
&\leq \|T\|_{\mathcal{L}\left(\ell_{p_1}^{n_1} \times \cdots \times \ell_{p_m}^{n_m} \times \ell_{s^*}^{n_{m+1}}; \mathbb{K}\right)} \cdot \|z^1\|_{\ell_{p_1}^{n_1}} \cdots \|z^m\|_{\ell_{p_m}^{n_m}} \cdot \|z^{m+1}\|_{\ell_{s^*}^{n_{m+1}}} \\
&= \|T\|_{\mathcal{L}\left(\ell_{p_1}^{n_1} \times \cdots \times \ell_{p_m}^{n_m} \times \ell_{s^*}^{n_{m+1}}; \mathbb{K}\right)} \cdot \|z^1\|_{\ell_{p_1}^{n_1}} \cdots \|z^m\|_{\ell_{p_m}^{n_m}},
\end{aligned}$$

and so,

$$\|\overline{T}\|_{\mathcal{L}\left(\ell_{p_1}^{n_1}\times\cdots\times\ell_{p_m}^{n_m};\ell_s^{n_{m+1}}\right)} \leq \|T\|_{\mathcal{L}\left(\ell_{p_1}^{n_1}\times\cdots\times\ell_{p_m}^{n_m}\times\ell_{s^*}^{n_{m+1}};\mathbb{K}\right)},$$

that is, T and $\overline{T}$ have the same norm.

Now, from Theorem 1.6, let us consider the $(m+1)$-linear form $A_0 : \ell_{p_1}^{n_1}\times\cdots\times\ell_{p_m}^{n_m}\times\ell_{s^*}^{n_{m+1}} \to \mathbb{K}$ given by

$$A_0(z^1,\ldots,z^{m+1}) := \sum_{j_1=1}^{n_1}\cdots\sum_{j_m=1}^{n_m}\sum_{j_{m+1}=1}^{n_{m+1}} \pm z_{j_1}^1\cdots z_{j_m}^m z_{j_{m+1}}^{m+1}.$$

Using the isometric map previously described, we obtain an operator $\overline{A_0} : \ell_{p_1}^{n_1}\times\cdots\times\ell_{p_m}^{n_m} \to \ell_s^{n_{m+1}}$ defined by

$$\overline{A_0}(z^1,\ldots,z^m) = \sum_{j_1=1}^{n_1}\cdots\sum_{j_m=1}^{n_m}\sum_{j_{m+1}=1}^{n_{m+1}} \pm z_{j_1}^1\cdots z_{j_m}^m e_{j_{m+1}},$$

which fulfills $\|\overline{A_0}\| = \|A_0\|$. Let us take $A := \overline{A_0}$. Therefore, by using Theorem 1.7,

$$\|A\| \leq (C_m)^{\frac{2}{\gamma^*}}\cdot\left(\sum_{k=1}^{m+1} n_k\right)^{\frac{1}{\gamma^*}}\cdot\prod_{k=1}^{m+1} n_k^{\max\left\{\frac{1}{\gamma}-\frac{1}{p_k},0\right\}}.$$

□

1.3 Asymptotic optimality

The following result shows that the Kahane–Salem–Zygmund inequality that was obtained in Theorem 1.7 is sharp when $p_1,\ldots,p_m \in [2,\infty]$.

Theorem 1.11. *Let $p_1,\ldots,p_m \in [2,\infty]$. Then*

$$\frac{1}{m\cdot 2^{\frac{m-1}{2}}} \leq \inf \frac{\|A\|}{\left(n_1^{1/2}+\cdots+n_m^{1/2}\right)\prod_{j=1}^m n_j^{\frac{1}{2}-\frac{1}{p_j}}} \leq C_m, \tag{1.7}$$

with $C_m := 8(m!)^{1-\max\left\{\frac{1}{2},\frac{1}{\max\{p_1,\ldots,p_m\}}\right\}}\sqrt{\log(1+4m)}$, for all unimodular m-linear forms $A : \ell_{p_1}^{n_1}\times\cdots\times\ell_{p_m}^{n_m} \to \mathbb{K}$.

Proof. Littlewood's (ℓ_1, ℓ_2)-inequalities (see [5, 6]) tell us that

$$\sum_{i_1=1}^{n_1}\left(\sum_{i_2,\ldots,i_m=1}^{n_2,\ldots,n_m}\left|T\left(e_{i_1,\ldots,}e_{i_m}\right)\right|^2\right)^{1/2} \leq \left(\sqrt{2}\right)^{m-1}\|T\|,$$

$$\vdots$$

$$\left(\sum_{i_1,\ldots,i_{m-1}=1}^{n_1,\ldots,n_{m-1}}\left(\sum_{i_m=1}^{n_m}\left|T\left(e_{i_1,\ldots,}e_{i_m}\right)\right|\right)^2\right)^{1/2} \leq \left(\sqrt{2}\right)^{m-1}\|T\|,$$

for all $T : \ell_\infty^{n_1} \times \cdots \times \ell_\infty^{n_m} \to \mathbb{K}$. Thus

$$n_1 n_2^{1/2} ... n_m^{1/2} \leq \left(\sqrt{2}\right)^{m-1}\|T\|,$$

$$\vdots$$

$$n_1^{1/2} ... n_{m-1}^{1/2} n_m \leq \left(\sqrt{2}\right)^{m-1}\|T\|,$$

for all unimodular $T : \ell_\infty^{n_1} \times \cdots \times \ell_\infty^{n_m} \to \mathbb{K}$. Thus

$$\|T\| \geq \frac{1}{m\left(\sqrt{2}\right)^{m-1}}\left(n_1^{1/2} + \cdots + n_m^{1/2}\right)\prod_{j=1}^{m} n_j^{\frac{1}{2}}$$

for all unimodular $T : \ell_\infty^{n_1} \times \cdots \times \ell_\infty^{n_m} \to \mathbb{K}$. Now consider a unimodular $S : \ell_{p_1}^{n_1} \times \cdots \times \ell_{p_m}^{n_m} \to \mathbb{K}$. Let us denote by S_∞ the same operator but now with domain $\ell_\infty^{n_1} \times \cdots \times \ell_\infty^{n_m}$. We have

$$\|S_\infty\| \geq \frac{1}{m\left(\sqrt{2}\right)^{m-1}}\left(n_1^{1/2} + \cdots + n_m^{1/2}\right)\prod_{j=1}^{m} n_j^{\frac{1}{2}},$$

hence there exist $x^{(1)}, ..., x^{(m)}$ with $\left|x_j^{(k)}\right| = 1$ for all k, j such that

$$\left|S_\infty\left(x^{(1)}, ..., x^{(m)}\right)\right| \geq \frac{1}{m\left(\sqrt{2}\right)^{m-1}}\left(n_1^{1/2} + \cdots + n_m^{1/2}\right)\prod_{j=1}^{m} n_j^{\frac{1}{2}}.$$

Consider

$$y^{(k)} = \frac{1}{n_k^{1/p_k}} x^{(k)}$$

for all k. Since each $y^{(k)}$ belongs to the closed unit ball of $\ell_{p_k}^{n_k}$ we have

$$\|S\| \geq \left|S\left(y^{(1)}, ..., y^{(m)}\right)\right|$$

$$
\begin{aligned}
&= \frac{1}{n_1^{1/p_1}\cdots n_m^{1/p_m}} \left|S\left(x^{(1)},...,x^{(m)}\right)\right| \\
&= \frac{1}{n_1^{1/p_1}\cdots n_m^{1/p_m}} \left|S_\infty\left(x^{(1)},...,x^{(m)}\right)\right| \\
&\geq \frac{1}{n_1^{1/p_1}\cdots n_m^{1/p_m}} \frac{1}{m\left(\sqrt{2}\right)^{m-1}} \left(n_1^{1/2}+\cdots+n_m^{1/2}\right) \prod_{j=1}^m n_j^{\frac{1}{2}} \\
&= \frac{1}{m\left(\sqrt{2}\right)^{m-1}} \left(n_1^{1/2}+\cdots+n_m^{1/2}\right) \prod_{j=1}^m n_j^{\frac{1}{2}-\frac{1}{p_j}}.
\end{aligned}
$$

We thus conclude that

$$
\frac{1}{m\left(\sqrt{2}\right)^{m-1}} \leq \inf \frac{\|A\|}{\left(n_1^{1/2}+\cdots+n_m^{1/2}\right) \prod\limits_{j=1}^m n_j^{\frac{1}{2}-\frac{1}{p_j}}}.
$$

By Theorem 1.7, for all $p_1,\ldots,p_m \in [2,\infty]$ there is a constant $C_m > 0$ and a multilinear form $T_0 : \ell_{p_1}^{n_1} \times \cdots \times \ell_{p_m}^{n_m} \to \mathbb{K}$ such that

$$
\|T_0\| \leq C_m \left(n_1^{1/2}+\cdots+n_m^{1/2}\right) \prod_{j=1}^m n_j^{\frac{1}{2}-\frac{1}{p_j}}.
$$

We thus conclude that

$$
\frac{1}{m\left(\sqrt{2}\right)^{m-1}} \leq \inf \frac{\|A\|}{\left(n_1^{1/2}+\cdots+n_m^{1/2}\right) \prod\limits_{j=1}^m n_j^{\frac{1}{2}-\frac{1}{p_j}}} \leq C_m.
$$

□

Remark 1.12 (Case $1 \leq p \leq 2$)**.** *The search for the optimal constants and exponents involved are natural and relevant topics for further investigation in this framework. The determination of the unknown optimal exponents for the case $p \in (1,2)$ of Theorem 1.7 relies on an open problem on the interpolation of certain multilinear forms (see [12]): it is well-known that every m-linear form on $\ell_p^n \times \cdots \times \ell_p^n$ is multiple $(r,1)$-summing, for*

$$
r = \frac{mp}{m+p-1}, \qquad \text{for } p = 1 \text{ or } p = 2.
$$

What about intermediate results for $p \in (1,2)$? If an interpolative approach could tell us that every m-linear operator from $\ell_p^n \times \cdots \times \ell_p^n$ is multiple $(mp/(m+p-1),1)$-summing,

a simple computation as in the proof of Theorem 1.11 would tell us that the exponents in Theorem 1.7, with $p_1 = \cdots = p_m = p \in (1,2)$ *and* $n_1 = \cdots = n_m = n$*, would be in fact admissible exponents and would give us the estimate for the asymptotic behavior pursued in Problem 1.1 (see also (1.1)):*

$$s \geq 1 - \frac{1}{p}.$$

Even in the linear case, similar problems remain open. Based on the discussion presented above, we conjecture the following optimal result that would also complete the solution to Problem 1.1.

Conjecture 1.13. *Let* $p_1, \ldots, p_m \in [1, \infty]$*. There exist* $B_m, C_m > 0$ *such that*

$$B_m \leq \inf \frac{\|A\|}{\left(\sum_{k=1}^m n_k^{1-\frac{1}{\gamma}}\right) \cdot \prod_{k=1}^m n_k^{\max\left\{\frac{1}{\gamma}-\frac{1}{p_k}, 0\right\}}} \leq C_m,$$

with $\gamma := \min\{2, \max\{p_k : p_k \leq 2\}\}$*, for all unimodular* m*-linear forms* $A : \ell_{p_1}^{n_1} \times \cdots \times \ell_{p_m}^{n_m} \to \mathbb{K}$ *and the exponents involved are sharp.*

Notice that Theorem 1.11 provides the following asymptotic growth of $\inf \|A\|$ with respect to $n_1, \ldots, n_m$:

$$\frac{1}{m \cdot 2^{\frac{m-1}{2}}} \leq \inf \frac{\|A\|}{\left(\sum_{j=1}^m n_j^{1/2}\right) \prod_{j=1}^m n_j^{\frac{1}{2}-\frac{1}{p_j}}} \leq 8(m!)^{1-\max\left\{\frac{1}{2}, \frac{1}{p}\right\}} \sqrt{\log(1+4m)},$$

where $p := \max\{p_1, \ldots, p_m\}$. Clearly there is big gap between the lower and upper estimates. Some interesting problems for further investigation are presented next.

Problem 1.14. *Is there a better lower bound to replace*

$$m^{-1} 2^{\frac{1-m}{2}}?$$

Problem 1.15. *Is there a better upper bound to replace*

$$8(m!)^{1-\max\left\{\frac{1}{2}, \frac{1}{\max\{p_1, \ldots, p_m\}}\right\}} \sqrt{\log(1+4m)}?$$

Problem 1.16. *In the complex case, what about unimodular forms with complex coeffi-*

cients with modulus 1, *instead of just* 1 *and* -1*? It is obvious that* (1.7) *holds. What about the lower and upper bounds?*

1.4 Applications to classical inequalities

The Kahane–Salem–Zygmund inequality is often used to prove the optimality of exponents in the Hardy–Littlewood inequality. The next results follow in this direction. The fact that the multilinear form provided in Theorem 1.7 is defined on $\ell_{p_1}^{n_1} \times \cdots \times \ell_{p_m}^{n_m}$ for arbitrary finite dimensions $n_1, \cdots, n_m$ has a crucial role. Throughout this section, X_p stands for ℓ_p if $1 \leq p < \infty$ and $X_\infty := c_0$.

1.4.1 On a result of Santos and Velanga

From now on, the symbol $e_j^{m_j}$ stands for $(e_j, \overset{m_j times}{\ldots}, e_j)$, where e_j is the j-th canonical vector. The following result, due to Santos and Velanga [62, Theorem 1.2], is a complete classification of the Bohnenblust–Hille multilinear type inequality.

Theorem 1.17. *Let* $1 \leq d \leq m$ *and* $m_1, \ldots, m_d$ *be positive integers such that* $m_1 + \cdots + m_d = m$*. Also let* $\boldsymbol{\rho} := (\rho_1, \ldots, \rho_d) \in (0, \infty)^d$*. The following assertions are equivalent:*

(I) There is a constant $B_{d,\boldsymbol{\rho}}^{\mathbb{K}} \geq 1$ *such that, for all* m*-linear forms* $T : c_0 \times \cdots \times c_0 \to \mathbb{K}$*,*

$$\left(\sum_{j_1=1}^{+\infty} \left(\cdots \left(\sum_{j_d=1}^{+\infty} \left| T\left(e_{j_1}^{m_1}, \ldots, e_{j_d}^{m_d}\right) \right|^{\rho_d} \right)^{\frac{\rho_{d-1}}{\rho_d}} \cdots \right)^{\frac{\rho_1}{\rho_2}} \right)^{\frac{1}{\rho_1}} \leq B_{d,\boldsymbol{\rho}}^{\mathbb{K}} \|T\|.$$

(II) For all $\mathcal{I} \subset \{1, \ldots, k\}$*,* $\sum_{j \in \mathcal{I}} \frac{1}{\rho_j} \leq \frac{card\,\mathcal{I}+1}{2}$*.*

The main result of this section shows that (I)⇒(II) is valid in the general setting of the Hardy–Littlewood inequality, which deals with both c_0 or ℓ_p spaces. We show that Theorem 1.18 provides an alternative straightforward proof of (I)⇒(II) of Santos and Velanga.

Theorem 1.18. *Let* $1 \leq d \leq m$ *and* $m_1, \ldots, m_d$ *be positive integers such that* $m_1 + \cdots + m_d = m$*. Also let*

$$\mathbf{p} := \left(\mathbf{p}^1, \ldots, \mathbf{p}^d\right) \in [1, \infty]^m, \quad \mathbf{p}^j := \left(p_1^j, \ldots, p_{m_j}^j\right) \in (1, \infty]^{m_j},$$

for $j = 1, \dots, d$ *and* $\rho := (\rho_1, \dots, \rho_d) \in (0, \infty)^d$. *If there is a constant* $C^{\mathbb{K}}_{d,\boldsymbol{\rho},\mathbf{p}} \geq 1$ *such that*

$$\left(\sum_{j_1=1}^{\infty} \left(\cdots \left(\sum_{j_d=1}^{\infty} \left| T\left(e_{j_1}^{m_1}, \dots, e_{j_d}^{m_d}\right) \right|^{\rho_d} \right)^{\frac{\rho_{d-1}}{\rho_d}} \cdots \right)^{\frac{\rho_1}{\rho_2}} \right)^{\frac{1}{\rho_1}} \leq C^{\mathbb{K}}_{d,\boldsymbol{\rho},\mathbf{p}} \|T\|, \tag{1.8}$$

for all m*-linear forms* $T : X_{p_1^1} \times \cdots \times X_{p_{m_1}^1} \times \cdots \times X_{p_1^d} \times \cdots X_{p_{m_d}^d} \to \mathbb{K}$, *then*

$$\sum_{j \in \mathcal{I}} \frac{1}{\rho_j} \leq \frac{card\mathcal{I} + 1}{2} - \sum_{j \in \mathcal{I}} \left| \frac{1}{\mathbf{p}^j} \right|,$$

for all $\mathcal{I} \subset \{1, \dots, d\}$.

Proof. Let us suppose (1.8) holds. Since $\mathcal{L}\left(\widehat{\otimes}^{\pi}_{1 \leq j \leq m_1} X_{p_j^1}, \cdots, \widehat{\otimes}^{\pi}_{1 \leq j \leq m_d} X_{p_j^d}; \mathbb{K}\right)$ and $\mathcal{L}\left(X_{p_1^1}, \cdots, X_{p_{m_1}^1}, \cdots, X_{p_1^d}, \cdots, X_{p_{m_d}^d}; \mathbb{K}\right)$ are isometric, we get an operator

$$\widehat{T} : \widehat{\otimes}^{\pi}_{1 \leq j \leq m_1} X_{p_j^1} \times \cdots \times \widehat{\otimes}^{\pi}_{1 \leq j \leq m_d} X_{p_j^d} \to \mathbb{K},$$

that satisfies

$$\widehat{T}\left(\otimes_{1 \leq j \leq m_1} x_j^1, \cdots, \otimes_{1 \leq j \leq m_d} x_j^d\right) = T\left(x_1^1, \cdots, x_{m_1}^1, \cdots, x_1^d, \cdots, x_{m_d}^d\right).$$

Then,

$$\widehat{T}\left(\otimes_{m_1} e_{j_1}, \cdots, \otimes_{m_d} e_{j_d}\right) = T\left(e_{j_1}^{m_1}, \cdots, e_{j_d}^{m_d}\right)$$

and $\|\widehat{T}\| = \|T\|$. Using [14, Theorem 1.3] it is possible to prove that it is sufficient to deal with k-linear forms $A : X_{r_1} \times \cdots \times X_{r_d} \to \mathbb{K}$ with $\frac{1}{r_j} = \left| \frac{1}{\mathbf{p}^j} \right|$, $j = 1, \dots, d$, defined by

$$A\left(y^1, \cdots, y^d\right) := \widehat{T}\left(u_1(y^1), \cdots, u_d(y^d)\right),$$

which is bounded and fulfills $\|A\| \leq \|\widehat{T}\|$. Therefore,

$$\left(\sum_{j_1=1}^{n_1} \left(\cdots \left(\sum_{j_m=1}^{n_d} |A(e_{j_1}, \dots, e_{j_d})|^{\rho_d} \right)^{\frac{\rho_{d-1}}{\rho_d}} \cdots \right)^{\frac{\rho_1}{\rho_2}} \right)^{\frac{1}{\rho_1}} \leq C^{\mathbb{K}}_{d,\boldsymbol{\rho},\mathbf{p}} \|A\|,$$

for all positive integers $n_1, \dots, n_d$ and all k-linear forms $A : X_{r_1} \times \cdots \times X_{r_d} \to \mathbb{K}$, with $\frac{1}{r_j} = \left|\frac{1}{\mathbf{p}^j}\right|$, $j = 1, \dots, d$. Using the map from Theorem 1.7, we conclude that, for all positive integers $n_1, \dots, n_d$,

$$n_1^{\frac{1}{\rho_1}} \cdots n_d^{\frac{1}{\rho_d}} \leq C_m \cdot n_1^{\frac{1}{2}-\frac{1}{r_1}} \cdots n_d^{\frac{1}{2}-\frac{1}{r_d}} \cdot (n_1 + \cdots + n_d)^{\frac{1}{2}}.$$

Let $\mathcal{I} \subset \{1, \dots, d\}$. Taking $n_j = n$ for $j \in \mathcal{I}$ and $n_j = 1$ for $j \in \mathcal{I}^c := \{1, \dots, d\} \setminus \mathcal{I}$,

$$n^{\left[\sum_{j\in\mathcal{I}} \frac{1}{\rho_j} - \frac{\operatorname{card}\mathcal{I}+1}{2} + \sum_{j\in\mathcal{I}} \frac{1}{r_j}\right]} \leq C_m \cdot \left(\operatorname{card}\mathcal{I} + \operatorname{card}\mathcal{I}^c \cdot \frac{1}{n}\right)^{\frac{1}{2}},$$

so we conclude that

$$\sum_{j\in\mathcal{I}} \frac{1}{\rho_j} \leq \frac{\operatorname{card}\mathcal{I}+1}{2} - \sum_{j\in\mathcal{I}} \left|\frac{1}{\mathbf{p}^j}\right|.$$

□

1.4.2 Dimension blow-up estimates

We turn our attention to the order in which the estimates blow–up with respect to the dimension in the cases when the exponents $(\rho_1, \dots, \rho_m)$ do not fulfill the range required in the Hardy–Littlewood inequality (see [1]). In this framework, there is no constant $C^{\mathbb{K}}_{m,\boldsymbol{\rho},\mathbf{p}} \geq 1$, independent of the dimensions $n_1, \dots, n_m$, for which the inequality

$$\left(\sum_{j_1=1}^{n_1} \left(\cdots \left(\sum_{j_m=1}^{n_m} |T(e_{j_1}, \dots, e_{j_m})|^{\rho_m}\right)^{\frac{\rho_{m-1}}{\rho_m}} \cdots\right)^{\frac{\rho_1}{\rho_2}}\right)^{\frac{1}{\rho_1}} \leq C^{\mathbb{K}}_{m,\boldsymbol{\rho},\mathbf{p}} \|T\|,$$

is universally satisfied for all m-linear forms $T : X_{p_1} \times \cdots \times X_{p_m} \to \mathbb{K}$. Recent papers present results along these lines [6, 56]. The next result follows this fashion.

Theorem 1.19. [6, Theorem 5] *Let $m \geq 2$ be an integer. If $\left|\frac{1}{\mathbf{p}}\right| \leq \frac{1}{2}$ and $\mathbf{r} \in [1, 2]^m$, then there is a constant $C^{\mathbb{K}}_{m,\mathbf{r},\mathbf{p}} \geq 1$ (not depending on n) such that*

$$\left(\sum_{i_1=1}^{n} \left(\cdots \left(\sum_{i_m=1}^{n} |T(e_{i_1}, \dots, e_{i_m})|^{r_m}\right)^{\frac{r_{m-1}}{r_m}} \cdots\right)^{\frac{r_1}{r_2}}\right)^{\frac{1}{r_1}}$$

$$\leq C^{\mathbb{K}}_{m,\mathbf{r},\mathbf{p}} \cdot n^{\max\left\{\left|\frac{1}{\mathbf{r}}\right|-\frac{m+1}{2}+\left|\frac{1}{\mathbf{p}}\right|,0\right\}}\|T\|,$$

for all m-linear forms $T : \ell^n_{p_1} \times \cdots \times \ell^n_{p_m} \to \mathbb{K}$ and all positive integers n. Moreover, the exponent $\max\left\{\left|\frac{1}{\mathbf{r}}\right| - \frac{m+1}{2} + \left|\frac{1}{\mathbf{p}}\right|, 0\right\}$ *is optimal.*

Our goal is to improve this result, specifying the dependence arisen when dealing with distinct dimensions $n_1, \ldots, n_m$. More precisely, the result obtained reads as follows.

Theorem 1.20. *Let $m \geq 2$ be an integer, $\mathbf{p} := (p_1, \ldots, p_m) \in [1, \infty]^m$ be such that $\left|\frac{1}{\mathbf{p}}\right| \leq \frac{1}{2}$ and also let $s_k, \rho_k \in (0, \infty)$, for $k = 1, \ldots, m$. Suppose there exists $D^{\mathbb{K}}_{m,\boldsymbol{\rho},\mathbf{p}} \geq 1$ such that*

$$\left(\sum_{j_1=1}^{n_1}\left(\ldots\left(\sum_{j_m=1}^{n_m}|T(e_{j_1},\ldots,e_{j_m})|^{\rho_m}\right)^{\frac{\rho_{m-1}}{\rho_m}}\ldots\right)^{\frac{\rho_1}{\rho_2}}\right)^{\frac{1}{\rho_1}} \leq D^{\mathbb{K}}_{m,\boldsymbol{\rho},\mathbf{p}} n_1^{s_1}\cdots n_m^{s_m}\|T\|,$$

for all bounded m-linear forms $T : \ell^{n_1}_{p_1} \times \cdots \times \ell^{n_m}_{p_m} \to \mathbb{K}$ and any positive integers $n_1, \ldots, n_m$. Then for all $\mathcal{I} \subset \{1, \ldots, m\}$,

$$\sum_{j\in\mathcal{I}} s_j \geq \max\left\{0, \sum_{j\in\mathcal{I}}\frac{1}{\rho_j} - \frac{card\mathcal{I}+1}{2} + \sum_{j\in\mathcal{I}}\frac{1}{p_j}\right\}.$$

Proof. Let us consider the m-linear form $A : \ell^{n_1}_{p_1} \times \cdots \times \ell^{n_m}_{p_m} \to \mathbb{K}$ given by Theorem 1.7. Since $|A(e_{j_1}, \ldots, e_{j_m})| = 1$ for any $j_1, \ldots, j_m$, we have

$$\left(\sum_{j_1=1}^{n_1}\left(\ldots\left(\sum_{j_m=1}^{n_m}|A(e_{j_1},\ldots,e_{j_m})|^{\rho_m}\right)^{\frac{\rho_{m-1}}{\rho_m}}\ldots\right)^{\frac{\rho_1}{\rho_2}}\right)^{\frac{1}{\rho_1}} = n_1^{\frac{1}{\rho_1}}\cdots n_m^{\frac{1}{\rho_m}}.$$

By Theorem 1.7,

$$\begin{aligned} n_1^{\frac{1}{\rho_1}}\cdots n_m^{\frac{1}{\rho_m}} &\leq D^{\mathbb{K}}_{m,\boldsymbol{\rho},\mathbf{p}} n_1^{s_1}\cdots n_m^{s_m}\|A\| \\ &\leq D^{\mathbb{K}}_{m,\boldsymbol{\rho},\mathbf{p}} n_1^{s_1+\frac{1}{2}-\frac{1}{p_1}}\cdots n_m^{s_m+\frac{1}{2}-\frac{1}{p_m}}(n_1+\cdots+n_m)^{\frac{1}{2}} \end{aligned}$$

and so

$$\frac{n_1^{\frac{1}{\rho_1}-s_1+\frac{1}{p_1}-\frac{1}{2}}\cdots n_m^{\frac{1}{\rho_m}-s_m+\frac{1}{p_m}-\frac{1}{2}}}{(n_1+\cdots+n_m)^{\frac{1}{2}}} \leq D^{\mathbb{K}}_{m,\boldsymbol{\rho},\mathbf{p}}.$$

Let $\mathcal{I} \subset \{1, \dots, m\}$, $n_j = n$ for each $j \in \mathcal{I}$ and $n_j = 1$ for $j \in \mathcal{I}^c := \{1, \dots, m\} \setminus \mathcal{I}$, thus

$$\lim_{n\to\infty} \left[\frac{1}{\left(\operatorname{card}\mathcal{I} + \frac{\operatorname{card}\mathcal{I}^c}{n}\right)^{\frac{1}{2}}} \cdot n^{\sum_{j\in\mathcal{I}} \frac{1}{\rho_j} - \sum_{j\in\mathcal{I}} s_j - \frac{\operatorname{card}\mathcal{I}+1}{2} + \sum_{j\in\mathcal{I}} \frac{1}{p_j}} \right] \leq D^{\mathbb{K}}_{m,\boldsymbol{\rho},\mathbf{p}},$$

and therefore,

$$\sum_{j\in\mathcal{I}} \frac{1}{\rho_j} - \sum_{j\in\mathcal{I}} s_j - \frac{\operatorname{card}\mathcal{I}+1}{2} + \sum_{j\in\mathcal{I}} \frac{1}{p_j} \leq 0.$$

□

Part II

Anisotropic regularity principle on sequence spaces and a unified summability principle

Chapter 2

Anisotropic regularity principle on sequence spaces

This chapter deals with a *Regularity Principle* on sequence spaces. Introduced by D. Pellegrino, J. Santos, D. Serrano–Rodríguez and E. Teixeira in [56], this kind of result is important due to its capacity for providing applications in the theory of summing operators, such as inclusion theorems, and results involving the classical inequalities of Bohnenblust–Hille and Hardy–Littlewood. We present here results from

> [7] *Anisotropic regularity principle in sequence spaces and applications, Communications in Contemporary Mathematics (2018).*

This chapter is organized as follows. The first section presents an anisotropic regularity principle for sequence spaces and next, in Section 2.2, we provide the main concepts from the theory, which are used in these sections, and an anisotropic inclusion theorem for summing operators is obtained. The final section uses the results obtained to prove Hardy–Littlewood inequalities for multilinear operators.

2.1 Anisotropic regularity principle

The investigation of regularity results in this setting was initiated in [55] and expanded in [56]. In this section we present an extension of these results.

Let $m \geq 2$ and also Z_1, V, $W_1, \ldots, W_m$ be arbitrary non-empty sets and $Z_2, \ldots, Z_m$

be vector spaces. Let also

$$R_k : Z_k \times W_k \to [0,\infty) \quad \text{and} \quad S : Z_1 \times \cdots \times Z_m \times V \to [0,\infty)$$

be functions satisfying, with $k = 1, \ldots, m$,

$$R_k(\lambda z, w) = \lambda R_k(z, w)$$

and

$$S(z_1, \ldots, z_{j-1}, \lambda z_j, z_{j+1}, \ldots, z_m, \nu) = \lambda S(z_1, \ldots, z_{j-1}, z_j, z_{j+1}, \ldots, z_m, \nu)$$

for all scalars $\lambda \geq 0$, $z, z_k \in Z_k$, $w \in W_k$ and $\nu \in V$ with $k \in \{1, \ldots, m\}$ and $j \in \{2, \ldots, m\}$. We shall work with each $p_k \geq 1$ and also assume that

$$\sup_{w \in W_k} \left(\sum_{j=1}^{n_k} R_k\left(z_j^k, w\right)^{p_k} \right)^{\frac{1}{p_k}} < \infty,$$

for all $z_k \in Z_k$ and $n_k \in \mathbb{N}$, with $k = 1, \ldots, m$. The following theorem will be an important tool in this section and provides a useful regularity principle for anisotropic summability of sequences (see [56, Theorem 2.3]).

Theorem 2.1. *Let $r_1, r_2, p_1, p_2 \geq 1$ and $s_1 \geq r_1$ and $q_1 \geq p_1$. Assume that there exists a constant $C > 0$ such that*

$$\sup_{\nu \in V} \left(\sum_{i=1}^{n_1} \left(\sum_{j=1}^{n_2} S\left(z_i^1, z_j^2, \nu\right)^{r_2} \right)^{\frac{r_1}{r_2}} \right)^{\frac{1}{r_1}} \leq C \cdot \sup_{w \in W_1} \left(\sum_{i=1}^{n_1} R_1\left(z_i^1, w\right)^{p_1} \right)^{\frac{1}{p_1}} \cdot \sup_{w \in W_2} \left(\sum_{j=1}^{n_2} R_2\left(z_j^2, w\right)^{p_2} \right)^{\frac{1}{p_2}},$$

for all z_i^1, z_j^2 and all $n_1, n_2 \in \mathbb{N}$. Then

$$\sup_{\nu \in V} \left(\sum_{i=1}^{n_1} \left(\sum_{j=1}^{n_2} S\left(z_i^1, z_j^2, \nu\right)^{r_2} \right)^{\frac{s_1}{r_2}} \right)^{\frac{1}{s_1}} \leq C \cdot \sup_{w \in W_1} \left(\sum_{i=1}^{n_1} R_1\left(z_i^1, w\right)^{q_1} \right)^{\frac{1}{q_1}} \cdot \sup_{w \in W_2} \left(\sum_{j=1}^{n_2} R_2\left(z_j^2, w\right)^{p_2} \right)^{\frac{1}{p_2}},$$

for all z_i^1, z_j^2 and all $n_1, n_2 \in \mathbb{N}$, with

$$\frac{1}{p_1} - \frac{1}{r_1} \leq \frac{1}{s_1} - \frac{1}{q_1}.$$

Despite the abstract context, the proof of the theorem below is similar to the proof of

Theorem 2.6.

Theorem 2.2. *[Anisotropic Regularity Principle]Let m be a positive integer, $r \geq 1$, $\mathbf{s}, \mathbf{p}, \mathbf{q} \in [1,\infty)^m$ be such that $q_k \geq p_k$, for $k = 1, \ldots, m$ and*

$$\frac{1}{r} - \left|\frac{1}{\mathbf{p}}\right| + \left|\frac{1}{\mathbf{q}}\right| > 0.$$

Assume that there exists a constant $C > 0$ such that

$$\sup_{\nu \in V} \left(\sum_{j_1=1}^{n_1} \cdots \sum_{j_m=1}^{n_m} S\left(z_{j_1}^1, \ldots, z_{j_m}^m, \nu\right)^r \right)^{\frac{1}{r}} \leq C \cdot \prod_{k=1}^{m} \sup_{w \in W_k} \left(\sum_{j=1}^{n_k} R_k\left(z_j^k, w\right)^{p_k} \right)^{\frac{1}{p_k}},$$

for all $z_j^{(k)} \in Z_k$ and $n_k \in \mathbb{N}$ with $k = 1, \ldots, m$. Then

$$\sup_{\nu \in V} \left(\sum_{j_1=1}^{n_1} \left(\cdots \left(\sum_{j_m=1}^{n_m} S\left(z_{j_1}^1, \ldots, z_{j_m}^m, \nu\right)^{s_m} \right)^{\frac{s_{m-1}}{s_m}} \cdots \right)^{\frac{s_1}{s_2}} \right)^{\frac{1}{s_1}} \leq C \cdot \prod_{k=1}^{m} \sup_{w \in W_k} \left(\sum_{j=1}^{n_k} R_k\left(z_j^k, w\right)^{q_k} \right)^{\frac{1}{q_k}},$$

for all $z_j^{(k)} \in Z_k$ and $n_k \in \mathbb{N}$, $k = 1, \ldots, m$, with

$$\frac{1}{s_k} - \left|\frac{1}{\mathbf{q}}\right|_{j \geq k} = \frac{1}{r} - \left|\frac{1}{\mathbf{p}}\right|_{j \geq k}, \quad k \in \{1, \ldots, m\}.$$

Proof. We will proceed by induction on m. The initial bilinear case is a straightforward application of Theorem 2.1. The ideas used are revealed in the case $m = 3$, thus we discuss it in detail. Assume that there exists a constant $C > 0$ such that

$$\sup_{\nu \in V} \left(\sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} \sum_{j_3=1}^{n_3} S\left(z_{j_1}^1, z_{j_2}^2, z_{j_3}^3, \nu\right)^r \right)^{\frac{1}{r}} \leq C \cdot \prod_{k=1}^{3} \sup_{w \in W_k} \left(\sum_{j=1}^{n_k} R_k\left(z_j^k, w\right)^{p_k} \right)^{\frac{1}{p_k}}, \tag{2.1}$$

for all $z_j^{(k)} \in Z_k$ and $n_k \in \mathbb{N}$ with $k = 1, 2, 3$. Let $(z_j^1)_{j \in \mathbb{N}} \in Z_1^{\mathbb{N}}$ be fixed. Define

$$S_3(z, w, \nu) = \left(\sum_{j_1=1}^{n_1} S\left(z_{j_1}^1, w, z, \nu\right)^r \right)^{\frac{1}{r}}.$$

By (2.1) we obtain, for all $z_j^{(k)} \in Z_k$ with $k = 2, 3$,

$$\sup_{\nu\in V}\left(\sum_{j_3=1}^{n_3}\left(\sum_{j_2=1}^{n_2}S_3(z_{j_3}^3,z_{j_2}^2,\nu)^r\right)^{\frac{1}{r}\cdot r}\right)^{\frac{1}{r}}\le C_3\cdot\sup_{w\in W_3}\left(\sum_{j=1}^{n_3}R_3\left(z_j^3,w\right)^{p_3}\right)^{\frac{1}{p_3}}\cdot\sup_{w\in W_2}\left(\sum_{j=1}^{n_2}R_2\left(z_j^2,w\right)^{p_2}\right)^{\frac{1}{p_2}},$$

with $C_3=C\cdot\sup_{w\in W_1}\left(\sum_{j=1}^{n_1}R_1\left(z_j^1,w\right)^{p_1}\right)^{\frac{1}{p_1}}$. By Theorem 2.1,

$$\sup_{\nu\in V}\left(\sum_{j_3=1}^{n_3}\left(\sum_{j_2=1}^{n_2}S_3(z_{j_3}^3,z_{j_2}^2,\nu)^r\right)^{\frac{1}{r}\cdot s_3}\right)^{\frac{1}{s_3}}\le C_3\cdot\sup_{w\in W_3}\left(\sum_{j=1}^{n_3}R_3\left(z_j^3,w\right)^{q_3}\right)^{\frac{1}{q_3}}\cdot\sup_{w\in W_2}\left(\sum_{j=1}^{n_2}R_2\left(z_j^2,w\right)^{p_2}\right)^{\frac{1}{p_2}},$$

with $q_3\ge p_3,\ s_3\ge r$ such that

$$\frac{1}{p_3}-\frac{1}{r}\le\frac{1}{q_3}-\frac{1}{s_3}.$$

Let us take $\frac{1}{s_3}=\frac{1}{r}-\frac{1}{p_3}+\frac{1}{q_3}>0$. Accordingly,

$$\begin{aligned}
&\sup_{\nu\in V}\left(\sum_{j_2=1}^{n_2}\left(\sum_{j_1=1}^{n_1}\left(\sum_{j_3=1}^{n_3}S\left(z_{j_1}^1,z_{j_2}^2,z_{j_3}^3,\nu\right)^{s_3}\right)^{\frac{s_3}{s_3}}\right)^{\frac{s_3}{s_3}}\right)^{\frac{1}{s_3}}\\
&=\sup_{\nu\in V}\left(\sum_{j_3=1}^{n_3}\left(\sum_{j_1=1}^{n_1}\left(\sum_{j_2=1}^{n_2}S\left(z_{j_1}^1,z_{j_2}^2,z_{j_3}^3,\nu\right)^{s_3}\right)^{\frac{s_3}{s_3}}\right)^{\frac{s_3}{s_3}}\right)^{\frac{1}{s_3}}\\
&\le\sup_{\nu\in V}\left(\sum_{j_3=1}^{\infty}\left(\sum_{j_1=1}^{n_1}\sum_{j_2=1}^{n_2}S\left(z_{j_1}^1,z_{j_2}^2,z_{j_3}^3,\nu\right)^{r}\right)^{\frac{s_3}{r}}\right)^{\frac{1}{s_3}}\\
&\le C\cdot\sup_{w\in W_3}\left(\sum_{j=1}^{n_3}R_3\left(z_j^3,w\right)^{q_3}\right)^{\frac{1}{q_3}}\cdot\prod_{k=1}^{2}\sup_{w\in W_k}\left(\sum_{j=1}^{n_k}R_k\left(z_j^k,w\right)^{p_k}\right)^{\frac{1}{p_k}}.
\end{aligned}\tag{2.2}$$

We fix $(z_j^3)_{j\in\mathbb{N}}\in Z_3^{\mathbb{N}}$ and define

$$S_2(z,w,\nu)=\left(\sum_{j_3=1}^{n_3}S\left(w,z,z_{j_3}^3,\nu\right)^{s_3}\right)^{\frac{1}{s_3}}.$$

Thus (2.2) leads us to

$$\sup_{\nu\in V}\left(\sum_{j_2=1}^{n_2}\left(\sum_{j_1=1}^{n_1}S_2(z_{j_2}^2,z_{j_1}^1,\nu)^{s_3}\right)^{\frac{1}{s_3}\cdot s_3}\right)^{\frac{1}{s_3}}\leq C_2\cdot\sup_{w\in W_2}\left(\sum_{j=1}^{n_2}R_2\left(z_j^2,w\right)^{p_2}\right)^{\frac{1}{p_2}}\cdot\sup_{w\in W_1}\left(\sum_{j=1}^{n_1}R_1\left(z_j^1,w\right)^{p_1}\right)^{\frac{1}{p_1}},$$

for all $z_j^{(k)}\in Z_k$ with $k=1,2$ and $C_2=C\cdot\sup_{w\in W_3}\left(\sum_{j=1}^{n_1}R_3\left(z_j^3,w\right)^{q_3}\right)^{\frac{1}{q_3}}$. Using Theorem 2.1 we get

$$\sup_{\nu\in V}\left(\sum_{j_2=1}^{n_2}\left(\sum_{j_1=1}^{n_1}S_2(z_{j_2}^2,z_{j_1}^1,\nu)^{s_3}\right)^{\frac{1}{s_3}\cdot s_2}\right)^{\frac{1}{s_2}}\leq C_2\cdot\sup_{w\in W_2}\left(\sum_{j=1}^{n_2}R_2\left(z_j^2,w\right)^{q_2}\right)^{\frac{1}{q_2}}\cdot\sup_{w\in W_1}\left(\sum_{j=1}^{n_1}R_1\left(z_j^1,w\right)^{p_1}\right)^{\frac{1}{p_1}},$$

with $q_2\geq p_2$, $s_2\geq s_3$ and $\frac{1}{p_2}-\frac{1}{s_3}\leq\frac{1}{q_2}-\frac{1}{s_2}$. Taking

$$\frac{1}{s_2}=\frac{1}{s_3}-\frac{1}{p_2}+\frac{1}{q_2}=\frac{1}{r}-\frac{1}{p_2}-\frac{1}{p_3}+\frac{1}{q_2}+\frac{1}{q_3},$$

since $s_3\leq s_2$, we have

$$\begin{aligned}\left(\sum_{j_1=1}^{n_1}\left(\sum_{j_2=1}^{n_2}\left(\sum_{j_3=1}^{n_2}S\left(z_{j_1}^1,z_{j_2}^2,z_{j_3}^3,\nu\right)^{s_3}\right)^{\frac{s_2}{s_3}}\right)^{\frac{s_2}{s_2}}\right)^{\frac{1}{s_2}}&=\left(\sum_{j_2=1}^{n_2}\left(\sum_{j_1=1}^{n_1}\left(\sum_{j_3=1}^{n_3}S\left(z_{j_1}^1,z_{j_2}^2,z_{j_3}^3,\nu\right)^{s_3}\right)^{\frac{s_2}{s_3}}\right)^{\frac{s_2}{s_2}}\right)^{\frac{1}{s_2}}\\&\leq\left(\sum_{j_2=1}^{n_2}\left(\sum_{j_1=1}^{n_1}\left(\sum_{j_3=1}^{n_3}S\left(z_{j_1}^1,z_{j_2}^2,z_{j_3}^3,\nu\right)^{s_3}\right)^{\frac{s_3}{s_3}}\right)^{\frac{s_2}{s_3}}\right)^{\frac{1}{s_2}}\\&\leq C\cdot\sup_{w\in W_1}\left(\sum_{j=1}^{n_1}R_1\left(z_j^1,w\right)^{p_1}\right)^{\frac{1}{p_1}}\\&\quad\times\prod_{k=2}^{3}\sup_{w\in W_k}\left(\sum_{j=1}^{n_k}R_k\left(z_j^k,w\right)^{q_k}\right)^{\frac{1}{q_k}}.\end{aligned}\tag{2.3}$$

Now let us fix $(z_j^3)_{j\in\mathbb{N}}\in Z_3^{\mathbb{N}}$ and define

$$S_1(w,z,\nu)=\left(\sum_{j_3=1}^{n_3}S\left(w,z,z_{j_3}^3,\nu\right)^{s_3}\right)^{\frac{1}{s_3}}.$$

Thus, by inequality (2.3)

$$\sup_{\nu\in V}\left(\sum_{j_1=1}^{n_1}\left(\sum_{j_2=1}^{n_2}S_2(z_{j_1}^1,z_{j_2}^2,\nu)^{s_2}\right)^{\frac{1}{s_2}\cdot s_2}\right)^{\frac{1}{s_2}}\leq C_1\cdot\sup_{w\in W_1}\left(\sum_{j=1}^{n_1}R_1\left(z_j^1,w\right)^{p_1}\right)^{\frac{1}{p_1}}\cdot\sup_{w\in W_1}\left(\sum_{j=1}^{n_2}R_2\left(z_j^2,w\right)^{q_2}\right)^{\frac{1}{q_2}},$$

for all $z_j^{(k)} \in Z_k$ with $k = 1, 2$ and $C_1 = C \cdot \sup_{w \in W_3} \left(\sum_{j=1}^{n_3} R_3 \left(z_j^3, w \right)^{q_3} \right)^{\frac{1}{q_3}}$. By Theorem 2.1, we get

$$\sup_{\nu \in V} \left(\sum_{j_1=1}^{n_1} \left(\sum_{j_2=1}^{n_2} S_1(z_{j_1}^1, z_{j_2}^2, \nu)^{s_2} \right)^{\frac{1}{s_2} \cdot s_1} \right)^{\frac{1}{s_1}} \leq C_1 \cdot \sup_{w \in W_1} \left(\sum_{j=1}^{n_1} R_1 \left(z_j^1, w \right)^{q_1} \right)^{\frac{1}{q_1}} \cdot \sup_{w \in W_2} \left(\sum_{j=1}^{n_2} R_2 \left(z_j^2, w \right)^{q_2} \right)^{\frac{1}{q_2}},$$

with $q_1 \geq p_1$ and $s_1 \geq s_2$ such that

$$\frac{1}{p_1} - \frac{1}{s_2} \leq \frac{1}{q_1} - \frac{1}{s_1}.$$

By choosing $\frac{1}{s_1} = \frac{1}{s_2} - \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{r} - \sum_{k=1}^{3} \frac{1}{p_k} + \sum_{k=1}^{3} \frac{1}{q_k} > 0$, we have

$$\sup_{\nu \in V} \left(\sum_{j_1=1}^{n_1} \left(\sum_{j_2=1}^{n_2} \left(\sum_{j_3=1}^{n_2} S \left(z_{j_1}^1, z_{j_2}^2, z_{j_3}^3, \nu \right)^{s_3} \right)^{\frac{s_2}{s_3}} \right)^{\frac{s_1}{s_2}} \right)^{\frac{1}{s_1}} \leq C \cdot \prod_{k=1}^{3} \sup_{w \in W_k} \left(\sum_{j=1}^{n_k} R_k \left(z_j^k, w \right)^{q_k} \right)^{\frac{1}{q_k}},$$

for all $z_j^{(k)} \in Z_k$ and $n_k \in \mathbb{N}$ with $k = 1, 2, 3$. Now we will conclude the induction argument. Let us suppose the result is true for $m - 1$ and assume that there exists a constant $C > 0$ such that

$$\sup_{\nu \in V} \left(\sum_{j_1=1}^{n_1} \cdots \sum_{j_m=1}^{n_m} S \left(z_{j_1}^1, \ldots, z_{j_m}^m, \nu \right)^r \right)^{\frac{1}{r}} \leq C \cdot \prod_{k=1}^{m} \sup_{w \in W_k} \left(\sum_{j=1}^{n_k} R_k \left(z_j^k, w \right)^{p_k} \right)^{\frac{1}{p_k}},$$

for all $z_j^{(k)} \in Z_k$ and $n_k \in \mathbb{N}$ with $k = 1, \ldots, m$. For a fixed $z_j^1 \in Z_1$, define

$$F(z_2, \ldots, z_m, \nu) := \left(\sum_{j_1=1}^{n_1} S \left(z_{j_1}^1, z_2, \ldots, z_m, \nu \right)^r \right)^{\frac{1}{r}}.$$

By the induction hypothesis,

$$\sup_{\nu \in V} \left(\sum_{j_2=1}^{n_2} \left(\cdots \left(\sum_{j_m=1}^{n_m} F \left(z_{j_2}^2, \ldots, z_{j_m}^m, \nu \right)^{s_m} \right)^{\frac{s_{m-1}}{s_m}} \cdots \right)^{\frac{s_2}{s_3}} \right)^{\frac{1}{s_2}} \leq C' \cdot \prod_{k=2}^{m} \sup_{w \in W_k} \left(\sum_{j=1}^{n_k} R_k \left(z_j^k, w \right)^{q_k} \right)^{\frac{1}{q_k}},$$

for all $z_j^{(k)} \in Z_k$ and $n_k \in \mathbb{N}$, $k = 2, \ldots, m$ with $C' = C \cdot \sup_{w \in W_1} \left(\sum_{j=1}^{n_1} R_1 \left(z_j^1, w \right)^{p_1} \right)^{\frac{1}{p_1}}$.

Consequently, by norm inequality on ℓ_p and the Minkowski inequality,

$$\begin{aligned}
&\left(\sum_{j_1=1}^{n_1}\left(\sum_{j_2=1}^{n_2}\left(\dots\left(\sum_{j_m=1}^{n_m} S\left(z_{j_1}^1,\dots,z_{j_m}^m,\nu\right)^{s_m}\right)^{\frac{s_{m-1}}{s_m}}\dots\right)^{\frac{s_2}{s_3}}\right)^{\frac{s_2}{s_2}}\right)^{\frac{1}{s_2}} \\
&\leq \left(\sum_{j_2=1}^{n_2}\left(\dots\left(\sum_{j_m=1}^{n_m}\left(\sum_{j_1=1}^{n_1} S\left(z_{j_1}^1,\dots,z_{j_m}^m,\nu\right)^{r}\right)^{\frac{s_m}{r}}\right)^{\frac{s_{m-1}}{s_m}}\dots\right)^{\frac{s_2}{s_3}}\right)^{\frac{1}{s_2}} \\
&\leq C\sup_{w\in W_1}\left(\sum_{j=1}^{n_1} R_1\left(z_j^1,w\right)^{p_1}\right)^{\frac{1}{p_1}}\prod_{k=2}^{m}\sup_{w\in W_k}\left(\sum_{j=1}^{n_k} R_k\left(z_j^k,w\right)^{q_k}\right)^{\frac{1}{q_k}}, \qquad (2.4)
\end{aligned}$$

with $r \leq s_m \leq \cdots \leq s_2$ and

$$\frac{1}{s_2} = \frac{1}{r} - \sum_{k=2}^{m}\frac{1}{p_k} + \sum_{k=2}^{m}\frac{1}{q_k}.$$

Fixing $z_j^k \in Z_k$, $k = 3,\dots,m$ and defining

$$Q(w,z,\nu) = \left(\sum_{j_3=1}^{n_3}\left(\dots\left(\sum_{j_m=1}^{n_m} S\left(w,z,z_{j_3}^3,\dots,z_{j_m}^m,\nu\right)^{s_m}\right)^{\frac{s_{m-1}}{s_m}}\dots\right)^{\frac{s_3}{s_4}}\right)^{\frac{1}{s_3}},$$

we have, by applying Theorem 2.1 in inequality (2.4), with $q_1 \geq p_1$ and $s_1 \geq s_2$ that

$$\frac{1}{p_1} - \frac{1}{s_2} \leq \frac{1}{q_1} - \frac{1}{s_1},$$

and

$$\left(\sum_{j_1=1}^{n_1}\left(\sum_{j_2=1}^{n_2} Q(z_{j_1}^1, z_{j_2}^2,\nu)^{s_2}\right)^{\frac{s_1}{s_2}}\right)^{\frac{1}{s_1}} \leq C\cdot\sup_{w\in W_1}\left(\sum_{j=1}^{n_1} R_1\left(z_j^1,w\right)^{q_1}\right)^{\frac{1}{q_1}}\cdot\sup_{w\in W_2}\left(\sum_{j=1}^{n_2} R_2\left(z_j^2,w\right)^{q_2}\right)^{\frac{1}{q_2}}.$$

Take $\frac{1}{s_1} = \frac{1}{s_2} - \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{r} - \left|\frac{1}{\mathbf{p}}\right| + \left|\frac{1}{\mathbf{q}}\right| > 0$, and thus

$$\left(\sum_{j_1=1}^{n_1}\left(\dots\left(\sum_{j_m=1}^{n_m} S\left(z_{j_1}^1,\dots,z_{j_m}^m,\nu\right)^{s_m}\right)^{\frac{s_{m-1}}{s_m}}\dots\right)^{\frac{s_1}{s_2}}\right)^{\frac{1}{s_1}} \leq C\prod_{k=1}^{m}\sup_{w\in W_k}\left(\sum_{j=1}^{n_k} R_k\left(z_j^k,w\right)^{q_k}\right)^{\frac{1}{q_k}}.$$

This concludes the proof. □

2.2 Anisotropic inclusion theorem for multilinear summing operators

The concept of absolutely summing operators was introduced by A. Grothendieck [42]. Later, A. Pietsch [60] introduced the concept of absolutely p-summing linear operators and B. Mitiagin and A. Pelczyński [51] broadened this concept and defined the absolutely $(r;p)$-summing linear operators. Afterwards, the investigation of summing operators was directed to the multilinear framework and several different lines of investigation emerged. Before introducing these notions, for the sake of clarity we recall basic summability concepts about multilinear operators.

Definition 2.3. [11, Definition 4.2] *For $\mathbf{r},\mathbf{p}\in[1,\infty)^m$ and $r\in[1,\infty)$, a multilinear operator $T:E_1\times\cdots\times E_m\to F$ is said to be*

(I) multiple $(\mathbf{r};\mathbf{p})$-summing *if there exists a constant $C>0$ such that for all sequences $x^k:=(x_j^k)_{j\in\mathbb{N}}\in\ell_{p_k}^w(E_k)$, $k=1,\ldots,m$,*

$$\left\|(Tx_{\mathbf{j}})_{\mathbf{j}\in\mathbb{N}^m}\right\|_{\mathbf{r}}:=\left(\sum_{j_1=1}^{\infty}\left(\cdots\left(\sum_{j_m=1}^{\infty}\|Tx_{\mathbf{j}}\|^{r_m}\right)^{\frac{r_{m-1}}{r_m}}\cdots\right)^{\frac{r_1}{r_2}}\right)^{\frac{1}{r_1}}\leq C\prod_{k=1}^{m}\|x^k\|_{w,p_k},\tag{2.5}$$

where $Tx_{\mathbf{j}}:=T\left(x_{j_1}^1,\ldots,x_{j_m}^m\right)$.

(II) absolutely $(r;\mathbf{p})$-summing *if it has the same property (2.5) with the norm on the left hand side replaced by*

$$\left\|\left(T(x_j^1,\ldots,x_j^m)\right)_{j\in\mathbb{N}}\right\|_r:=\left(\sum_{j=1}^{\infty}\left\|T\left(x_j^1,\ldots,x_j^m\right)\right\|^r\right)^{\frac{1}{r}}.$$

Both classes $\Pi_{(\mathbf{r};\mathbf{p})}^m$ of all multiple $(\mathbf{r};\mathbf{p})$-summing multilinear operators and $\Pi_{(r;\mathbf{p})}^{\mathrm{as}}$ of all absolutely $(r;\mathbf{p})$-summing multilinear operators are Banach spaces with the norms defined by the infimum taken over the constants $C>0$ in (2.5). The multiple and absolutely summing norms are denoted by $\pi_{(\mathbf{r};\mathbf{p})}^m(\cdot)$ and $\pi_{(r;\mathbf{p})}^{\mathrm{as}}(\cdot)$, respectively. When $r_1=\cdots=r_m=$

r we simply write $(r;\mathbf{p})$ instead of $(\mathbf{r};\mathbf{p})$; similarly, if $p_1 = \cdots = p_m = p$, we just write $(\mathbf{r};p)$.

Throughout the development of the theory of summing operators, inclusion theorems were revealed as challenging problems and indispensable tools (see, for instance, [58]). An important inclusion result was obtained recently in [56, Proposition 3.4], where the authors proved the following multilinear inclusion result.

Theorem 2.4 (Pellegrino, Santos, Serrano and Teixeira)**.** *Let m be a positive integer, $r,p,q \in [1,\infty)$ be such that $q \geq p$ and*

$$\frac{1}{r} - \frac{m}{p} + \frac{m}{q} > 0.$$

Then

$$\Pi^m_{(r;p)}(E_1,\ldots,E_m;F) \subset \Pi^m_{(s;q)}(E_1,\ldots,E_m;F),$$

for any Banach spaces $E_1,\ldots,E_m,F$, where

$$\frac{1}{s} - \frac{m}{q} = \frac{1}{r} - \frac{m}{p},$$

and the inclusion operator has norm 1.

Soon after, independently and with a different technique, F. Bayart in [17, Theorem 1.2] obtained a more general version.

Theorem 2.5 (Bayart)**.** *Let m be a positive integer, $r,s \in [1,\infty)$, $\mathbf{p},\mathbf{q} \in [1,\infty)^m$ be such that $q_k \geq p_k$, $k = 1,\ldots,m$ and*

$$\frac{1}{r} - \left|\frac{1}{\mathbf{p}}\right| + \left|\frac{1}{\mathbf{q}}\right| > 0.$$

Then

$$\Pi^m_{(r;\mathbf{p})}(E_1,\ldots,E_m;F) \subset \Pi^m_{(s;\mathbf{q})}(E_1,\ldots,E_m;F),$$

for any Banach spaces $E_1,\ldots,E_m,F$, where

$$\frac{1}{s} - \left|\frac{1}{\mathbf{q}}\right| = \frac{1}{r} - \left|\frac{1}{\mathbf{p}}\right|.$$

We provide an inclusion theorem that encompasses and improves the previous results in the anisotropic sense. The techniques are inspired by [56] and are contained in the proof of the forthcoming Regularity Principle in Section 2.1 (see Theorem 2.2). The inclusion result obtained reads as follows.

Theorem 2.6. *Let m be a positive integer, $r \geq 1$, $\mathbf{s}, \mathbf{p}, \mathbf{q} \in [1, \infty)^m$ be such that $q_k \geq p_k$, for $k = 1, \ldots, m$, and*

$$\frac{1}{r} - \left|\frac{1}{\mathbf{p}}\right| + \left|\frac{1}{\mathbf{q}}\right| > 0.$$

Then

$$\Pi^m_{(r;\mathbf{p})}\left(E_1, \ldots, E_m; F\right) \subset \Pi^m_{(\mathbf{s};\mathbf{q})}\left(E_1, \ldots, E_m; F\right),$$

for any Banach spaces $E_1, \ldots, E_m, F$, where

$$\frac{1}{s_k} - \left|\frac{1}{\mathbf{q}}\right|_{j \geq k} = \frac{1}{r} - \left|\frac{1}{\mathbf{p}}\right|_{j \geq k},$$

for each $k \in \{1, \ldots, m\}$, and the inclusion operator has norm 1.

Proof. The argument is inspired by the Regularity Principle of [56, Theorem 2.1]. We will proceed by induction on m. The initial bilinear case is a straightforward application of the classical inclusion of absolutely summing linear operators and norm inequality on ℓ_r spaces. The ideas used are revealed in the case $m = 3$; thus we discuss it in detail. Let $T \in \Pi^3_{(r;\mathbf{p})}\left(E_1, E_2, E_3; F\right)$. Then there exists a constant $C > 0$ such that

$$\left(\sum_{j_3=1}^{\infty}\left(\sum_{j_1,j_2=1}^{\infty}\left\|T\left(x_{\mathbf{j}}\right)\right\|^r\right)^{\frac{1}{r}\cdot r}\right)^{\frac{1}{r}} \leq C\prod_{k=1}^{3}\|x^k\|_{w,p_k},$$

for all sequences $x^k = (x^k_j)_{j\in\mathbb{N}} \in \ell^w_{p_k}(E_k)$, $k = 1, 2, 3$. Let $x^k \in \ell^w_{p_k}(E_k)$ with $k = 1, 2$ be fixed. Define $v_3 : E_3 \longrightarrow \ell_{(r,r)}(F)$ by

$$v_3(x_3) = \left(T\left(x^1_{j_1}, x^2_{j_2}, x_3\right)\right)_{j_1,j_2\in\mathbb{N}},$$

for all $x_3 \in E_3$. The linear inclusion [37, Theorem 10.4] tells us that $v_3 \in \Pi_{(s_3;q_3)}\left(E_3; \ell_{(r,r)}(F)\right)$

with $q_3 \geq p_3$ and $s_3 \geq r$ such that

$$\frac{1}{p_3} - \frac{1}{r} = \frac{1}{q_3} - \frac{1}{s_3}.$$

Applying norm inequality on ℓ_p with the exponents $r \leq s_3$ and using the fact that v_3 is $(s_3; q_3)$-summing, we obtain

$$\left(\sum_{j_2=1}^{\infty}\left(\sum_{j_1=1}^{\infty}\left(\sum_{j_3=1}^{\infty}\left\|T\left(x_{\mathbf{j}}\right)\right\|^{s_3}\right)^{\frac{s_3}{s_3}}\right)^{\frac{s_3}{s_3}}\right)^{\frac{1}{s_3}} \leq C\left\|x^3\right\|_{w,q_3}\prod_{k=1}^{2}\left\|x^k\right\|_{w,p_k}.$$

Fixing $x^1 \in \ell_{p_1}^w(E_1)$, $x^3 \in \ell_{q_3}^w(E_3)$, defining $v_2 : E_2 \longrightarrow \ell_{(s_3,s_3)}(F)$ by

$$v_2(x_2) = \left(T\left(x_{j_1}^1, x_2, x_{j_3}^3\right)\right)_{j_1,j_3\in\mathbb{N}},$$

and using the linear inclusion theorem, we obtain that $v_2 \in \Pi_{(s_2;q_2)}\left(E_2; \ell_{(s_3,s_3)}(F)\right)$ with

$$\frac{1}{s_2} = \frac{1}{r} - \left(\frac{1}{p_2} + \frac{1}{p_3}\right) + \left(\frac{1}{q_2} + \frac{1}{q_3}\right).$$

By norm inequality on ℓ_p with $s_3 \leq s_2$ and the previous estimate obtained from the fact that v_2 is $(s_2; q_2)$-summing, it follows that

$$\left(\sum_{j_1=1}^{\infty}\left(\sum_{j_2=1}^{\infty}\left(\sum_{j_3=1}^{\infty}\left\|T\left(x_{\mathbf{j}}\right)\right\|^{s_3}\right)^{\frac{s_2}{s_3}}\right)^{\frac{s_2}{s_2}}\right)^{\frac{1}{s_2}} \leq C\left\|x^1\right\|_{w,p_1}\prod_{k=2}^{3}\left\|x^k\right\|_{w,q_k}.$$

Now let us fix $x^k \in \ell_{q_k}^w(E_k)$ with $k = 2, 3$ and let us define, for all $x_1 \in E_1$,

$$v_1(x_1) = \left(T\left(x_1, x_{j_2}^2, x_{j_3}^3\right)\right)_{j_2,j_3\in\mathbb{N}}.$$

Proceeding analogously, we obtain that $T \in \Pi_{(\mathbf{s};\mathbf{q})}^3\left(E_1, E_2, E_3; F\right)$, with

$$\frac{1}{s_1} = \frac{1}{r} - \sum_{k=1}^{3}\frac{1}{p_k} + \sum_{k=1}^{3}\frac{1}{q_k} > 0.$$

Now we shall conclude the proof by an induction argument. Let us suppose the result

is true for $m-1$ and let $T \in \Pi^m_{(r;\mathbf{p})}(E_1,\dots,E_m;F)$. For a fixed $x^1 \in \ell^w_{p_1}(E_1)$, $v : E_2 \times \cdots \times E_m \to \ell_r(F)$, given by

$$v(x_2,\dots,x_m) := \left(T(x^1_{j_1},x_2,\dots,x_m)\right)_{j_1\in\mathbb{N}},$$

belongs to $\Pi^{m-1}_{(r;p_2,\dots,p_m)}(E_2,\dots,E_m;\ell_r(F))$. By the induction hypothesis,

$$\left(\sum_{j_2=1}^{\infty}\left(\sum_{j_3=1}^{\infty}\left(\dots\left(\sum_{j_m=1}^{\infty}\left\|v\left(x^2_{j_2},\dots,x^m_{j_m}\right)\right\|_r^{s_m}\right)^{\frac{s_{m-1}}{s_m}}\dots\right)^{\frac{s_3}{s_4}}\right)^{\frac{s_2}{s_3}}\right)^{\frac{1}{s_2}} \le C'\prod_{k=2}^{m}\left\|x^k\right\|_{w,q_k},$$

for all sequences $x^k \in \ell^w_{p_k}(E_k)$, $k=2,\dots m$ with $C' = C\left\|x^1\right\|_{w,p_1}$. Consequently, by norm inequality on ℓ_p and the Minkowski inequality,

$$\left(\sum_{j_1=1}^{\infty}\left(\sum_{j_2=1}^{\infty}\left(\dots\left(\sum_{j_m=1}^{\infty}\left\|T\left(x_{\mathbf{j}}\right)\right\|^{s_m}\right)^{\frac{s_{m-1}}{s_m}}\dots\right)^{\frac{s_2}{s_3}}\right)^{\frac{s_2}{s_2}}\right)^{\frac{1}{s_2}} \le C'\prod_{k=2}^{m}\left\|x^k\right\|_{w,q_k} \tag{2.6}$$

with $r \le s_m \le \cdots \le s_2$ and

$$\frac{1}{s_2} = \frac{1}{r} - \sum_{k=2}^{m}\frac{1}{p_k} + \sum_{k=2}^{m}\frac{1}{q_k}.$$

Fixing $x^k \in \ell^w_{q_k}(E_k)$, $k=2,\dots,m$ and defining, for all $x_1 \in E_1$,

$$u(x_1) = \left(T\left(x_1, x^2_{j_2}\dots,x^m_{j_m}\right)\right)_{j_2,\dots,j_m\in\mathbb{N}}$$

we have that $u \in \Pi_{(s_2;p_1)}\left(E_1;\ell_{(s_2,\dots,s_m)}(F)\right)$. Applying the classical linear inclusion on (2.6), with $q_1 \ge p_1$ and $s_1 \ge s_2$ such that

$$\frac{1}{p_1} - \frac{1}{s_2} = \frac{1}{q_1} - \frac{1}{s_1},$$

we get $u \in \Pi_{(s_1;q_1)}\left(E_1;\ell_{(s_2,\ldots,s_m)}(F)\right)$. Since $\frac{1}{s_1} = \frac{1}{r} - \left|\frac{1}{\mathbf{p}}\right| + \left|\frac{1}{\mathbf{q}}\right| > 0$ we have

$$\left(\sum_{j_1=1}^{\infty}\left(\ldots\left(\sum_{j_m=1}^{\infty}\left\|T\left(x_{\mathbf{j}}\right)\right\|^{s_m}\right)^{\frac{s_{m-1}}{s_m}}\ldots\right)^{\frac{s_1}{s_2}}\right)^{\frac{1}{s_1}} \leq C\prod_{k=1}^{m}\left\|x^k\right\|_{w,q_k}.$$

Therefore, $T \in \Pi^m_{(\mathbf{s};\mathbf{q})}\left(E_1,\ldots,E_m;F\right)$. Also note that the inclusion operator has norm 1, so the constant C is preserved. This concludes the proof. □

It is important to highlight the difference between, 2.5 and 2.6. Under the hypotheses of Theorem 2.6, by using the usual inclusion of ℓ_p spaces and Theorem 2.5 with $\frac{1}{s_1} := \frac{1}{r} - \left|\frac{1}{\mathbf{p}}\right| + \left|\frac{1}{\mathbf{q}}\right|$ one may get that

$$\Pi^m_{(\mathbf{r};\mathbf{p})} \subset \Pi^m_{(s_1;\mathbf{q})}.$$

But if $r \leq s_m \leq \cdots \leq s_1$, then

$$\Pi^m_{(\mathbf{s};\mathbf{q})} \subset \Pi^m_{(s_1;\mathbf{q})}.$$

Nevertheless, we may not conclude by Theorem 2.5 that

$$\Pi^m_{(r;\mathbf{p})} \subset \Pi^m_{(\mathbf{s};\mathbf{q})}.$$

However this is provided by Theorem 2.6. Let us illustrate with a numerical example: let $r := 3$, $\mathbf{s} := (5,3)$, $\mathbf{p} := (3,2)$ and $\mathbf{q} := (5,2)$. From Theorem 2.5 we have

$$\Pi^2_{(3;\mathbf{p})} \subset \Pi^2_{(5;\mathbf{q})},$$

while Theorem 2.6 provides

$$\Pi^2_{(3;\mathbf{p})} \subset \Pi^2_{(5,3;\mathbf{q})}.$$

The same can also be done for $m = 3$: let $r := 2$, $\mathbf{s} := (6,3,2)$, $\mathbf{p} := (2,2,1)$ and $\mathbf{q} := (3,3,1)$. Then

$$\Pi^3_{(2;\mathbf{p})} \subset \Pi^3_{(6,3,2;\mathbf{q})} \subset \Pi^3_{(6;\mathbf{q})},$$

where the first inclusion is assured by Theorem 2.6.

2.3 Applications

The Hardy–Littlewood inequalities have been deeply investigated in recent years (see, for instance, [5, 13, 16, 35, 36, 38, 48, 52, 56, 57]).

Theorem 2.7 (Albuquerque, Bayart, Pellegrino, Seoane [5], 2014). *Let $\mathbf{p}, \mathbf{s} \in [1,\infty]^m$ be such that $|1/\mathbf{p}| \leq 1/2$ and $\mathbf{s} \in \left[(1-|1/\mathbf{p}|)^{-1}, 2\right]^m$. There is a constant $B^{\mathbb{K}}_{m,\mathbf{p},\mathbf{s}} \geq 1$ such that*

$$\left(\sum_{j_1=1}^{n}\left(\cdots\left(\sum_{j_m=1}^{n}|A(e_{j_1},\ldots,e_{j_m})|^{s_m}\right)^{\frac{s_{m-1}}{s_m}}\cdots\right)^{\frac{s_1}{s_2}}\right)^{\frac{1}{s_1}} \leq B^{\mathbb{K}}_{m,\mathbf{p},\mathbf{s}}\|A\|,$$

for all positive integers n and every m-linear form $A : \ell^n_{p_1} \times \cdots \times \ell^n_{p_m} \to \mathbb{K}$ if, and only if,

$$\left|\frac{1}{\mathbf{s}}\right| \leq \frac{m+1}{2} - \left|\frac{1}{\mathbf{p}}\right|.$$

Theorem 2.8. *[Dimant, Sevilla-Peris [38], 2016] Let $\mathbf{p} \in [1,\infty]^m$ be such that $1/2 \leq |1/\mathbf{p}| < 1$. There exists a constant $D^{\mathbb{K}}_{m,\mathbf{p}} \geq 1$ such that*

$$\left(\sum_{j_1,\ldots,j_m=1}^{n}|A(e_{j_1},\ldots,e_{j_m})|^{\frac{1}{1-|1/\mathbf{p}|}}\right)^{1-|1/\mathbf{p}|} \leq D^{\mathbb{K}}_{m,\mathbf{p}}\|A\|, \tag{2.7}$$

for all positive integers n and every m-linear form $A : \ell^n_{p_1} \times \cdots \times \ell^n_{p_m} \to \mathbb{K}$. Moreover, the exponent is optimal.

The above exponent $\frac{1}{1-|1/\mathbf{p}|}$ is optimal, but not in the anisotropic sense. The following recent result is a kind of anisotropic extension.

Theorem 2.9 (Aron, Núñez, Pellegrino, Serrano [16], 2017). *Let $m \geq 2$, $s_1,\ldots,s_m > 0$, and $1 < p_m \leq 2 < p_1,\ldots,p_{m-1}$, with $|1/\mathbf{p}| < 1$. There is a constant $C^{\mathbb{K}}_{m,\mathbf{p},\mathbf{s}} \geq 1$ such that*

$$\left(\sum_{j_1=1}^{n}\left(\cdots\left(\sum_{j_m=1}^{n}|A(e_{j_1},\ldots,e_{j_m})|^{s_m}\right)^{\frac{s_{m-1}}{s_m}}\cdots\right)^{\frac{s_1}{s_2}}\right)^{\frac{1}{s_1}} \leq C^{\mathbb{K}}_{m,\mathbf{p},\mathbf{s}}\|A\|,$$

for all positive integers n and every m-linear form $A : \ell^n_{p_1} \times \cdots \times \ell^n_{p_m} \to \mathbb{K}$ if, and only if,

$$s_k \geq \left[1 - \left|\frac{1}{\mathbf{p}}\right|_{j\geq k}\right]^{-1}, \quad \text{for } k = 1,\ldots,m.$$

Recently, Cavalcante [35] has shown that (2.7) is a consequence of the inclusion result for multiple summing operators due to Pellegrino *et al.* [56] combined with Theorem 2.7. The standard isometries between $\mathcal{L}(\ell_p, E)$ and $\ell_{p^*}^w(E)$, for $1 < p \leq \infty$, allow us to read the previous Theorems 2.7, 2.8, 2.9 as coincidence results (see [37]). The key point is to begin with the coincidence below, obtained by revisiting Theorem 2.7 as a coincidence result with $s_1 = \cdots = s_m = 2$ and $p_1 = \cdots = p_m = 2m$,

$$\Pi^m_{(2;(2m)^*)}(E_1, \ldots, E_m; \mathbb{K}) = \mathcal{L}(E_1, \ldots, E_m; \mathbb{K}), \tag{2.8}$$

for all Banach spaces $E_1, \ldots, E_m$. We shall combine these isometries with the inclusion Theorem 2.6 to obtain refined inclusions and coincidences results.

Theorem 2.10. *Let m be a positive integer and $\mathbf{s}, \mathbf{p} \in [1, \infty]^m$. If $|1/\mathbf{p}| < 1$ and $p_1, \ldots, p_m \leq 2m$, then*

$$\Pi^m_{(2;(2m)^*)}(E_1, \ldots, E_m; \mathbb{K}) \subset \Pi^m_{(\mathbf{s};p_1^*,\ldots,p_m^*)}(E_1, \ldots, E_m; \mathbb{K}),$$

for any Banach spaces $E_1, \ldots, E_m$, where

$$s_k = \left[\frac{1}{2} + \frac{m-k+1}{2m} - \left|\frac{1}{\mathbf{p}}\right|_{j\geq k}\right]^{-1} \qquad \text{for } k = 1, \ldots, m.$$

Proof. Since each $p_k^* \geq (2m)^*$, because by hypothesis $p_k \leq 2m$, and

$$\frac{1}{2} - m \cdot \frac{1}{(2m)^*} + \left|\frac{1}{\mathbf{p}^*}\right| = \frac{1}{2} - m \cdot \left(1 - \frac{1}{2m}\right) + m - \left|\frac{1}{\mathbf{p}}\right| = 1 - \left|\frac{1}{\mathbf{p}}\right| > 0,$$

applying Theorem 2.6, the stated inclusion is obtained with

$$\frac{1}{s_k} = \frac{1}{2} - (m-k+1) \cdot \left(1 - \frac{1}{2m}\right) + (m-k+1) - \left|\frac{1}{\mathbf{p}}\right|_{j\geq k},$$

for each $k = 1, \ldots, m$. □

Corollary 2.11. *If $|1/\mathbf{p}| < 1$ and $p_1, \ldots, p_m \leq 2m$, then*

$$\Pi^m_{(\mathbf{s};p_1^*,\ldots,p_m^*)}(E_1, \ldots, E_m; \mathbb{K}) = \mathcal{L}(E_1, \ldots, E_m; \mathbb{K}),$$

for any Banach spaces $E_1, \ldots, E_m$ *and*

$$s_k = \left[\frac{1}{2} + \frac{m-k+1}{2m} - \left|\frac{1}{\mathbf{p}}\right|_{j\geq k}\right]^{-1} \qquad \text{for } k = 1, \ldots, m.$$

Proof. Applying Theorem 2.10 and coincidence (2.8) we get the result. □

Bringing Theorem 2.10 to the context of sequence spaces, the announced anisotropic result will be achieved. The main result of this section reads as follows.

Corollary 2.12. *Let* m *be a positive integer and* $\mathbf{p} \in [1,\infty)^m$ *be such that* $|1/\mathbf{p}| < 1$ *and* $p_1, \ldots, p_m \leq 2m$. *There is a constant* $D^{\mathbb{K}}_{m,\mathbf{p},\mathbf{s}} \geq 1$ *such that*

$$\left(\sum_{j_1=1}^{n_1}\left(\cdots\left(\sum_{j_m=1}^{n_m}|A(e_{j_1},\ldots,e_{j_m})|^{s_m}\right)^{\frac{s_{m-1}}{s_m}}\cdots\right)^{\frac{s_1}{s_2}}\right)^{\frac{1}{s_1}} \leq D^{\mathbb{K}}_{m,\mathbf{p},\mathbf{s}}\cdot\|A\|, \tag{2.9}$$

for all positive integers $n_1, \ldots n_m$ *and all* m*-linear forms* $A : \ell^{n_1}_{p_1} \times \cdots \times \ell^{n_m}_{p_m} \to \mathbb{K}$, *where*

$$s_k = \left[\frac{1}{2} + \frac{m-k+1}{2m} - \left|\frac{1}{\mathbf{p}}\right|_{j\geq k}\right]^{-1}, \qquad \text{for } k = 1, \ldots, m.$$

Proof. By the previous corollary,

$$\Pi^m_{(\mathbf{s};p_1^*,\ldots,p_m^*)}\left(\ell^{n_1}_{p_1},\ldots,\ell^{n_m}_{p_m};\mathbb{K}\right) = \mathcal{L}\left(\ell^{n_1}_{p_1},\ldots,\ell^{n_m}_{p_m};\mathbb{K}\right),$$

with

$$s_k = \left[\frac{1}{2} + \frac{m-k+1}{2m} - \left|\frac{1}{\mathbf{p}}\right|_{j\geq k}\right]^{-1}, \qquad \text{for } k = 1, \ldots, m.$$

Since $\|\cdot\| \leq \pi_{(\boldsymbol{s};\boldsymbol{p^*})}(\cdot)$ (see [11, Prposition 5.3]) the Open Mapping Theorem ensures that there exists a constant $D^{\mathbb{K}}_{m,\mathbf{p},\mathbf{s}} > 0$ such that

$$\pi_{(\boldsymbol{s};\boldsymbol{p^*})}(\cdot) \leq D^{\mathbb{K}}_{m,\mathbf{p},\mathbf{s}}\cdot\|\cdot\|.$$

Now let us consider $A \in \mathcal{L}\left(\ell_{p_1}^{n_1}, \ldots, \ell_{p_m}^{n_m}; \mathbb{K}\right)$. By Corollary 2.11,

$$\left(\sum_{j_1=1}^{n_1}\left(\cdots\left(\sum_{j_m=1}^{n_m}\left|A\left(e_{j_1},\ldots,e_{j_m}\right)\right|^{s_m}\right)^{\frac{s_{m-1}}{s_m}}\cdots\right)^{\frac{s_1}{s_2}}\right)^{\frac{1}{s_1}} \leq \pi_{(\boldsymbol{s};\boldsymbol{p}^*)}(A)\cdot\prod_{k=1}^{m}\left\|e^k\right\|_{w,p_k^*}$$

$$= \pi_{(\boldsymbol{s};\boldsymbol{p}^*)}(A)$$

$$\leq D_{m,\mathbf{p},\mathbf{s}}^{\mathbb{K}}\cdot\|A\|$$

for all positive integers $n_1,\ldots n_m$ and all m-linear forms $A:\ell_{p_1}^{n_1}\times\cdots\times\ell_{p_m}^{n_m}\to\mathbb{K}$. □

In order to clarify the new result, we illustrate a simpler case: when dealing with $m<p_1=\cdots=p_m=p\leq 2m$, we get (2.9) with exponents

$$s_k=\left[\frac{1}{2}-(m-k+1)\cdot\left(\frac{1}{p}-\frac{1}{2m}\right)\right]^{-1},\qquad \text{for } k=1,\ldots,m,$$

that is,

$$s_1=\frac{p}{p-m},\ldots,s_m=\frac{2mp}{mp+p-2m}.$$

It is obvious that the above exponents are better than the estimates of Theorem 2.8 that provides

$$s_1=\cdots=s_m=\frac{p}{p-m}.$$

The following example is illustrative:

Example 2.13. *Suppose $m=3$ and $p=4$. By Theorem 2.8 we know that (2.9) holds with*

$$s_k=4 \text{ for } k=1,2,3,$$

whereas by Corollary 2.12 we have

$$s_1=4,\quad s_2=3\quad \text{and}\quad s_3=12/5.$$

Chapter 3

Summing operators and a unified summability principle

We investigate Λ-summing operators, a concept that recovers the absolute and multiple summability notions. A general summability principle plays the main role: an inclusion theorem that unifies the related result for absolutely and multiple summing operators. This chapter discusses the manuscript

[3] *A Summability Principle and Applications, preprint (2018).*

We begin this chapter with some definitions, results and preliminary notations needed for the development of the subsequent sections. In Section 3.2 we present an inclusion theorem that improves, recovers and unifies results of [7, 17, 53, 56]. The final Section 3.3 provides applications to classical inequalities of multilinear operators.

3.1 Λ-Summing operators: a unified concept

In an attempt to unify most of the different multilinear approaches, the notion of Λ-summability arose naturally (see [19] and the references therein).

The following definition, kindly suggested by D. Pellegrino, is more general, encompassing anisotropic sums:

Definition 3.1. *Let $E_1, \ldots, E_m, F$ be Banach spaces, m be a positive integer, $(\mathbf{r};\mathbf{p}) := (r_1, \ldots, r_m; p_1, \ldots, p_m) \in [1,\infty)^{2m}$ and $\Lambda \subset \mathbb{N}^m$ be a set of multi–indexes. A multilinear*

operator $T : E_1 \times \cdots \times E_m \to F$ *is* Λ-$(\mathbf{r};\mathbf{p})$*-summing if there exists a constant* $C > 0$ *such that for all* $x^k \in \ell_{p_k}^w(E_k)$, $k = 1, \ldots, m$,

$$\left(\sum_{j_1=1}^{\infty} \left(\cdots \left(\sum_{j_m=1}^{\infty} \left\| T\left(x_{j_1}^1, \ldots, x_{j_m}^m\right) \mathbf{1}_\Lambda(j_1, \ldots, j_m) \right\|^{r_m} \right)^{\frac{r_{m-1}}{r_m}} \cdots \right)^{\frac{r_1}{r_2}} \right)^{\frac{1}{r_1}}$$
$$\leq C \cdot \prod_{k=1}^{m} \left\| x^k \right\|_{w,p_k}, \tag{3.1}$$

where $\mathbf{1}_\Lambda$ *is the characteristic function of* Λ. *We represent the class of all* Λ-$(\mathbf{r};\mathbf{p})$*-summing multilinear operators from* $E_1 \times \cdots \times E_m$ *to* F *by* $\Pi^{\Lambda}_{(\mathbf{r};\mathbf{p})}(E_1, \ldots, E_m; F)$. *When* $r_1 = \cdots = r_m = r$ *and* $p_1 = \cdots = p_m = p$, *we will represent* $\Pi^{\Lambda}_{(\mathbf{r};\mathbf{p})}$ *by* $\Pi^{\Lambda}_{(r;p)}$.

When $\Lambda = \Lambda_{\text{as}} := \{(j, \ldots, j) : j \in \mathbb{N}\}$, Definition 3.1 recovers the notion (see Definition 2.3, Chapter 2) of $(r;\mathbf{p})$-absolutely summing operators, denoted by $\Pi^{\text{as}}_{(r;\mathbf{p})}$. When $\Lambda = \mathbb{N}^m$, we get the notion of $(\mathbf{r};\mathbf{p})$-multiple summing operators, denoted by $\Pi^{m}_{(\mathbf{r};\mathbf{p})}$.

In the general setting, the case on which Λ is an arbitrary subset of $\mathbb{N}^m$, it is difficult to deal with the strong mixed norm on the left hand side of (3.1). Recently, Botelho and Nogueira investigated in depth this general setting (we refer the reader to [33]). Nevertheless, there are particular cases for Λ in which the notions of absolutely and multiple summing operators are unified at once: when Λ is a *block type set* (Definition 3.2). Next we introduce a simple and fundamental notation we need to elucidate these cases.

For vector spaces $V_1, \ldots, V_m$ and an element $v_k \in V_k$ for some $k \in \{1, \ldots, m\}$, the symbol $v_k \cdot e_k$ is defined as

$$v_k \cdot e_k := (0, \ldots, 0, v_k, 0, \ldots, 0) \in V_1 \times \cdots \times V_m,$$

that is, the element in $V_1 \times \cdots \times V_m$ with v_k in the k-th coordinate and 0 elsewhere. For positive integers m, d with $1 \leq d \leq m$ and $\mathcal{I} = \{I_1, \ldots, I_d\}$ a partition formed by non-empty disjoints subsets of $\{1, \ldots, m\}$ such that $\cup_{k=1}^{d} I_k = \{1, \ldots, m\}$, the expression

$$\sum_{k=1}^{d} \sum_{j \in I_k} v_{j_k} \cdot e_j$$

will be very useful throughout this Chapter. The following example clarifies this notation. For instance, for

$$m = 5, d = 3, \mathcal{I} = \{I_1, I_2, I_3\} \text{ with } I_1 = \{1, 3\}, I_2 = \{2, 4\}, I_3 = \{5\},$$

we have

$$\begin{aligned}
\sum_{k=1}^{3} \sum_{j \in I_k} e_{j_k} \cdot e_j &= \sum_{j \in I_1} e_{j_1} \cdot e_j + \sum_{j \in I_2} e_{j_2} \cdot e_j + \sum_{j \in I_3} e_{j_3} \cdot e_j \\
&= e_{j_1} \cdot e_1 + e_{j_1} \cdot e_3 + \sum_{j \in I_2} e_{j_2} \cdot e_j + \sum_{j \in I_3} e_{j_3} \cdot e_j \\
&= (e_{j_1}, 0, 0, 0, 0) + (0, 0, e_{j_1}, 0, 0) + \sum_{j \in I_2} e_{j_2} \cdot e_j + \sum_{j \in I_3} e_{j_3} \cdot e_j \\
&= (e_{j_1}, e_{j_2}, e_{j_1}, e_{j_2}, e_{j_3}).
\end{aligned}$$

Definition 3.2. *Let m, d be positive integers with $1 \leq d \leq m$. If*

$$\Lambda = \Lambda_b := \left\{ \sum_{k=1}^{d} \sum_{j \in I_k} j_k \cdot e_j \ : \ j_1, \ldots, j_d \in \mathbb{N} \right\} \subseteq \mathbb{N}^m,$$

where $\mathcal{I} = \{I_1, \ldots, I_d\}$ is a partition of non-empty disjoint subsets of $\{1, \ldots, m\}$, we say that the set of multi–indexes $\Lambda = \Lambda_b$ is a block set of $\mathcal{I}$-type. *In this situation, we use the notation*

$$\Pi^{\mathcal{I}}_{(\mathbf{r};\mathbf{p})} := \Pi^{\Lambda_b}_{(\mathbf{r};\mathbf{p})}.$$

Surely, the simple standard case for a *block type set* is

$$\Lambda = \left\{ (j_1, \overset{m_1 \text{ times}}{\ldots}, j_1, \ldots, j_d, \overset{m_d \text{ times}}{\ldots}, j_d) : j_1, \ldots, j_d \in \mathbb{N} \right\},$$

where $1 \leq m_1, \ldots, m_d \leq m$ are fixed positive integers such that $m_1 + \cdots + m_d = m$. Notice that, by considering partitions $\mathcal{I}_{\text{as}} := \{\{1, \ldots, m\}\}$ and $\mathcal{I}_m := \{\{1\}, \ldots, \{m\}\}$, the classes of absolute and multiple summing operators are recovered:

$$\Pi^{\text{as}}_{(r;\mathbf{p})} = \Pi^{\mathcal{I}_{\text{as}}}_{(r;\mathbf{p})} \quad \text{and} \quad \Pi^{m}_{(r;\mathbf{p})} = \Pi^{\mathcal{I}_m}_{(r;\mathbf{p})}.$$

Clearly, if Λ is a block set of $\mathcal{I}$-type, then

$$\begin{aligned}&\left\|(Tx_{\mathbf{j}})_{\mathbf{j}\in\Lambda}\right\|_{\ell_{\mathbf{r}}(F)}\\&=\left(\sum_{j_1=1}^{\infty}\left(\cdots\left(\sum_{j_d=1}^{\infty}\left\|T\left(\sum_{k=1}^{d}\sum_{j\in I_k}x_{j_n}^{(j)}\cdot e_j\right)\right\|_F^{r_d}\right)^{\frac{r_{d-1}}{r_d}}\cdots\right)^{\frac{r_1}{r_2}}\right)^{\frac{1}{r_1}}.\end{aligned}$$

The Λ-$(\mathbf{r};\mathbf{p})$-summing operators share standard properties with the classical summing operators. Next, a list of these properties is presented.

Proposition 3.3. *Let $E_1,\dots,E_m,F$ be Banach spaces, m be a positive integer, $(\mathbf{r};\mathbf{p}):=(r_1,\dots,r_m;p_1,\dots,p_m)\in[1,\infty)^{2m}$ and $\Lambda\subset\mathbb{N}^m$ be a set of multi–indexes.*

(1) Then $\Pi^{\Lambda}_{(\mathbf{r};\mathbf{p})}(E_1,\dots,E_m;F)$ is a subspace of $\mathcal{L}(E_1,\dots,E_m;F)$. Moreover, the infimum taken over all possible constants $C>0$ satisfying (3.1), i.e.,

$$\pi^{\Lambda}_{\mathbf{r};\mathbf{p}}(T):=\inf\left\{C>0\ ;\ \ (3.1)\text{ holds for all }x^k\in\ell^w_{p_k}(E_k),k=1,...,m\right\},$$

defines a norm on $\Pi^{\Lambda}_{(\mathbf{r};\mathbf{p})}(E_1,\dots,E_m;F)$, which is denoted by $\pi^{\Lambda}_{\mathbf{r};\mathbf{p}}(\cdot)$.

(2) If $T\in\Pi^{\Lambda}_{(\mathbf{r};\mathbf{p})}(E_1,\dots,E_m;F)$, then $\|T\|\le\pi^{\Lambda}_{\mathbf{r};\mathbf{p}}(T)$.

(3) Then $(\Pi^{\Lambda}_{(\mathbf{r};\mathbf{p})}(E_1,\dots,E_m;F),\pi^{\Lambda}_{\mathbf{r};\mathbf{p}}(\cdot))$ is a Banach space.

As expected, in order to avoid situations in which the class $\Pi^{\Lambda}_{(\mathbf{r};\mathbf{p})}$ is trivial, the exponents must fulfill a condition that mixes the classical cases related to absolutely and multiple summability.

Proposition 3.4. *Let $E_1,\dots,E_m,F$ be Banach spaces and let m,d be positive integers with $1\le d\le m$, and $(\mathbf{r},\mathbf{p}):=(r_1,\dots,r_d;p_1,\dots,p_m)\in[1,\infty)^{m+d}$. Let also $\mathcal{I}=\{I_1,\dots,I_d\}$ be a partition of $\{1,\dots,m\}$. There exists $k\in\{1,\dots,d\}$ such that $1/r_k>\sum_{j\in I_k}1/p_j$, if and only if*

$$\Pi^{\mathcal{I}}_{(\mathbf{r};\mathbf{p})}(E_1,\dots,E_m;F)=\{0\}.$$

3.2 Blocks inclusion theorem

As mentioned in Chapter 2, inclusion theorems play an important role in the literature. There is a vast literature investigating inclusion theorems in the framework of absolutely and multiple summing multiple operators [1, 7, 11, 17, 19, 20, 24, 32, 35, 37, 42, 43, 46, 47, 56, 59, 60]. The main contribution of the present chapter is an inclusion theorem for the case in which the set Λ is formed by *blocks*: we shall prove the inclusion

$$\Pi^{\Lambda}_{(r;\mathbf{p})} \subset \Pi^{\Lambda}_{(\mathbf{s};\mathbf{q})},$$

for suitable values of $s_1, \ldots, s_m$ (Theorem 3.6). At the end of this Chapter (see Section 3.3, Theorem 3.9) we apply our main result to the investigation of Hardy–Littlewood inequalities for multilinear forms.

In the multilinear framework, for the notion of absolutely summing operators (when Λ is the diagonal) the linear inclusion theorem is extended as follows (see [49]):

Theorem 3.5. *Let m be a positive integer, $1 \leq r \leq s < \infty$ and $\mathbf{p}, \mathbf{q} \in [1, \infty)^m$ be such that $q_k \geq p_k$, for $k = 1, \ldots, m$. Then*

$$\Pi^{\mathrm{as}}_{(r;\mathbf{p})}(E_1, \ldots, E_m; F) \subset \Pi^{\mathrm{as}}_{(s;\mathbf{q})}(E_1, \ldots, E_m; F),$$

for any Banach spaces $E_1, \ldots, E_m, F$, where

$$\frac{1}{s} - \left|\frac{1}{\mathbf{q}}\right| = \frac{1}{r} - \left|\frac{1}{\mathbf{p}}\right|,$$

and the inclusion operator has norm 1.

For multiple summing operators, inclusion theorems are more subtle (see, for instance, [32, 59]). Recently, this subject was investigated by several authors and using different techniques (see [7, Theorem 3], [17, Theorem 1.2], [53, Theorem 3] and [56, Proposition 3.3]). Our main result recovers all the aforementioned results (see Chapter 2, Theorem 2.6):

Theorem 3.6. *Let $1 \leq d \leq m$ be positive integers and $r \geq 1$, $\mathbf{p}, \mathbf{q} \in [1, \infty)^m$. Let also $\mathcal{I} = \{I_1, \ldots, I_d\}$ be a partition of $\{1, \ldots, m\}$ and suppose that Λ is a block-set of $\mathcal{I}$-type.*

Then

$$\Pi^{\mathcal{I}}_{(r;\mathbf{p})}\left(E_1,\ldots,E_m;F\right)\subset\Pi^{\mathcal{I}}_{(\mathbf{s};\mathbf{q})}\left(E_1,\ldots,E_m;F\right),$$

for any Banach spaces $E_1,\ldots,E_m,F$*, where*

$$\frac{1}{s_k}-\left|\frac{1}{\mathbf{q}}\right|_{j\in\bigcup_{i=k}^{d}I_i}=\frac{1}{r}-\left|\frac{1}{\mathbf{p}}\right|_{j\in\bigcup_{i=k}^{d}I_i},\ k=1,\ldots,d,$$

whenever $q_j\geq p_j,\ j=1,\ldots,m$ *and*

$$\frac{1}{r}-\left|\frac{1}{\mathbf{p}}\right|+\left|\frac{1}{\mathbf{q}}\right|>0,$$

or $q_1>p_1,\ q_j\geq p_j,\ j=2,\ldots,m$ *and*

$$\frac{1}{r}-\left|\frac{1}{\mathbf{p}}\right|+\left|\frac{1}{\mathbf{q}}\right|=0.$$

Moreover, the inclusion operator has norm 1*.*

Proof. Let us consider the two cases separately:

(A) $q_j\geq p_j$ for all $j=1,\ldots,m$ and

$$\frac{1}{r}-\left|\frac{1}{\mathbf{p}}\right|+\left|\frac{1}{\mathbf{q}}\right|>0.$$

(B) $q_1>p_1$ and $q_j\geq p_j$ for all $j=2,\ldots,m$ and

$$\frac{1}{r}-\left|\frac{1}{\mathbf{p}}\right|+\left|\frac{1}{\mathbf{q}}\right|=0.$$

We begin by proving (A). We proceed by induction on d. The bilinear case is a straightforward application of Theorems 3.5 and [7, Theorem 3]. Let us suppose the result is true for $d-1$. Let $T\in\Pi^{\mathcal{I}}_{(r;\mathbf{p})}\left(E_1,\ldots,E_m;F\right)$, i.e.,

$$\left(\sum_{j_1=1}^{\infty}\left(\ldots\left(\sum_{j_d=1}^{\infty}\left\|T\left(\sum_{k=1}^{d}\sum_{j\in J_k}x^j_{j_k}e_j\right)\right\|^r\right)^{\frac{r}{r}}\ldots\right)^{\frac{r}{r}}\right)^{\frac{1}{r}}$$

$$= \left(\sum_{j_1,\dots,j_d=1}^{\infty} \left\| T\left(\sum_{k=1}^{d} \sum_{j\in J_k} x_{j_k}^j e_j \right) \right\|^r \right)^{\frac{1}{r}} \leq C \cdot \prod_{k=1}^{m} \left\| x^k \right\|_{w,p_k},$$

for all $x^k \in \ell_{p_k}^w(E_k)$. Without loss of generality, suppose that $I_1 = \{1,\dots,l\}$. Thus, $\mathcal{J} := \{I_2,\dots,I_d\}$ is a partition of $\{l+1,\dots,m\}$. For fixed $x^k \in \ell_{p_k}^w(E_k), k = 1,\dots,l$, let us define $w : E_{l+1} \times \cdots \times E_m \to \ell_r(F)$ by

$$w(x_{l+1},\dots,x_m) := \left(T(x_j^1,\dots,x_j^l,x_{l+1},\dots,x_m)\right)_{j\in\mathbb{N}}.$$

Observe that w belongs to $\Pi^{\mathcal{J}}_{(r;p_{l+1},\dots,p_m)}(E_{l+1},\dots,E_m;\ell_r(F))$. Consequently, by the induction hypothesis, norm inequality on ℓ_p and the Minkowski inequality,

$$\begin{aligned}
&\left(\sum_{j_1=1}^{\infty} \left(\sum_{j_2=1}^{\infty} \left(\dots \left(\sum_{j_d=1}^{\infty} \left\| T\left(\sum_{k=1}^{d} \sum_{j\in I_k} x_{j_k}^j e_j \right) \right\|^{s_d} \right)^{\frac{s_{d-1}}{s_d}} \dots \right)^{\frac{s_2}{s_3}} \right)^{\frac{s_2}{s_2}} \right)^{\frac{1}{s_2}} \\
&\leq \left(\sum_{j_2=1}^{\infty} \left(\dots \left(\sum_{j_d=1}^{\infty} \left(\sum_{j_1=1}^{\infty} \left\| T\left(\sum_{j\in I_1} x_{j_1}^j e_j + \sum_{k=2}^{d} \sum_{j\in I_k} x_{j_k}^j e_j \right) \right\|^r \right)^{\frac{s_d}{r}} \right)^{\frac{s_{d-1}}{s_d}} \dots \right)^{\frac{s_2}{s_3}} \right)^{\frac{1}{s_2}} \\
&= \left(\sum_{j_2=1}^{\infty} \left(\dots \left(\sum_{j_d=1}^{\infty} \left\| w\left(\sum_{k=2}^{d} \sum_{j\in I_k} x_{j_k}^j e_j \right) \right\|_r^{s_d} \right)^{\frac{s_{d-1}}{s_d}} \dots \right)^{\frac{s_2}{s_3}} \right)^{\frac{1}{s_2}} \\
&\leq C \cdot \prod_{k=1}^{l} \left\| x^k \right\|_{w,p_k} \cdot \prod_{k=l+1}^{m} \left\| x^k \right\|_{w,q_k},
\end{aligned}$$

with $r \leq s_d \leq \cdots \leq s_2$ and

$$\frac{1}{s_k} = \frac{1}{r} - \left| \frac{1}{\mathbf{p}} \right|_{j\in\bigcup_{i=k}^{d} I_i} + \left| \frac{1}{\mathbf{q}} \right|_{j\in\bigcup_{i=k}^{d} I_i}, \quad \text{for each} \quad k = 2,\dots,d.$$

Now for fixed $x^k \in \ell_{p_k}^w(E_k), k = l+1,\dots,m$, let us define, for all $(x_1,\dots,x_l) \in E_1 \times \cdots \times E_l$,

$$\psi(x_1,\dots,x_l) := \left(T\left(\sum_{j\in I_1} x_j e_j + \sum_{k=2}^{d} \sum_{j\in I_k} x_{j_k}^j e_j \right) \right)_{j_2,\dots,j_d\in\mathbb{N}}.$$

Observe that

$$\left(\sum_{j_1=1}^{\infty}\left\|\psi\left(x_{j_1}^1,\ldots,x_{j_1}^l\right)\right\|_{\ell_{(s_2,\ldots,s_d)}}^{s_2}\right)^{\frac{1}{s_2}}\leq C_1\cdot\prod_{k=1}^{l}\left\|x^k\right\|_{w,p_k},$$

where $C_1=C\cdot\prod_{k=l+1}^{m}\left\|x^k\right\|_{w,q_k}$, i.e., $\psi\in\Pi^{\mathrm{as}}_{(s_2;p_1,\ldots,p_l)}\left(E_1,\ldots,E_l;\ell_{s_2,\ldots,s_d}(F)\right)$. From Theorem 3.5 we conclude that $\psi\in\Pi^{as}_{(s_1;q_1,\ldots,q_l)}\left(E_1,\ldots,E_l;\ell_{s_2,\ldots,s_d}(F)\right)$, with

$$\frac{1}{s_1}=\frac{1}{s_2}-\sum_{j\in I_1}\frac{1}{p_j}+\sum_{j\in I_1}\frac{1}{q_j}=\frac{1}{r}-\left|\frac{1}{\mathbf{p}}\right|_{j\in\bigcup_{i=1}^{d}I_i}+\left|\frac{1}{\mathbf{q}}\right|_{j\in\bigcup_{i=1}^{d}I_i}.$$

Now we deal with case (B). Let $T\in\Pi^{\mathcal{I}}_{(r;\mathbf{p})}\left(E_1,\ldots,E_m;F\right)$, from $q_1>p_1$ it follows that

$$\frac{1}{r}-\left|\frac{1}{\mathbf{p}}\right|_{\geq 2}+\left|\frac{1}{\mathbf{q}}\right|_{\geq 2}>0.$$

For the sake of simplicity, we suppose that $I_1=\{1\}$. Then $\mathcal{K}:=\{I_2,\ldots,I_d\}$ is a partition of $\{2,\ldots,m\}$. For fixed $x^1\in\ell_{p_1}^w(E_1)$, define $\beta:E_2\times\cdots\times E_m\to\ell_r(F)$ by

$$\beta(x_2,\ldots,x_m):=\left(T(x_j^1,x_2,\ldots,x_m)\right)_{j\in\mathbb{N}}.$$

Observe that $\beta\in\Pi^{\mathcal{K}}_{(r;p_2,\ldots,p_m)}\left(E_2,\ldots,E_m;\ell_r(F)\right)$ and, consequently, by Theorem 3.6, norm inequality on ℓ_p and Minkowski's inequality,

$$\begin{aligned}
&\left(\sum_{j_1=1}^{\infty}\left(\sum_{j_2=1}^{\infty}\left(\ldots\left(\sum_{j_d=1}^{\infty}\left\|T\left(\sum_{k=1}^{d}\sum_{j\in I_k}x_{j_k}^j e_j\right)\right\|^{s_d}\right)^{\frac{s_{d-1}}{s_d}}\ldots\right)^{\frac{s_2}{s_3}}\right)^{\frac{s_2}{s_2}}\right)^{\frac{1}{s_2}}\\
&\leq\left(\sum_{j_2=1}^{\infty}\left(\ldots\left(\sum_{j_d=1}^{\infty}\left(\sum_{j_1=1}^{\infty}\left\|T\left(x_{j_1}^1e_1+\sum_{k=2}^{d}\sum_{j\in I_k}x_{j_k}^j e_j\right)\right\|^{r}\right)^{\frac{s_d}{r}}\right)^{\frac{s_{d-1}}{s_d}}\ldots\right)^{\frac{s_2}{s_3}}\right)^{\frac{1}{s_2}}\\
&=\left(\sum_{j_2=1}^{\infty}\left(\ldots\left(\sum_{j_d=1}^{\infty}\left\|\beta\left(\sum_{k=2}^{d}\sum_{j\in I_k}x_{j_k}^j e_j\right)\right\|_r^{s_d}\right)^{\frac{s_{d-1}}{s_d}}\ldots\right)^{\frac{s_2}{s_3}}\right)^{\frac{1}{s_2}}\\
&\leq C\cdot\left\|x^1\right\|_{w,p_1}\cdot\prod_{k=2}^{m}\left\|x^k\right\|_{w,q_k},
\end{aligned}$$

with $r \leq s_d \leq \cdots \leq s_2$ and

$$\frac{1}{s_k} = \frac{1}{r} - \left|\frac{1}{\mathbf{p}}\right|_{j\in\bigcup_{i=k}^{d} I_i} + \left|\frac{1}{\mathbf{q}}\right|_{j\in\bigcup_{i=k}^{d} I_i}, \quad \text{for each} \quad k = 2,\ldots,d.$$

Now we fix $x^k \in \ell_{p_k}^w(E_k), k = 2,\ldots,m$, and define, for all $x \in E_1$,

$$\xi(x) := \left(T\left(x\,e_1 + \sum_{k=2}^{d}\sum_{j\in I_k} x_{j_k}^j e_j\right)\right)_{j_2,\ldots,j_d\in\mathbb{N}}.$$

Observe that

$$\left(\sum_{j_1=1}^{\infty} \left\|\xi\left(x_{j_1}^1\right)\right\|_{\ell_{(s_2,\ldots,s_d)}}^{s_2}\right)^{\frac{1}{s_2}} \leq C_1 \cdot \left\|x^1\right\|_{w,p_1},$$

where $C_1 = C\cdot\prod_{k=2}^{m}\left\|x^k\right\|_{w,q_k}$, i.e., $\xi \in \Pi_{(s_2;p_1)}\left(E_1;\ell_{s_2,\ldots,s_d}(F)\right)$. By the classical inclusion theorem [37] we conclude that $\xi \in \Pi_{(s_1;q_1)}\left(E_1;\ell_{s_2,\ldots,s_d}(F)\right)$, with

$$\frac{1}{s_1} = \frac{1}{s_2} - \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{r} - \left|\frac{1}{\mathbf{p}}\right|_{j\in\bigcup_{i=1}^{d} I_i} + \left|\frac{1}{\mathbf{q}}\right|_{j\in\bigcup_{i=1}^{d} I_i}.$$

This concludes the proof. □

3.3 Applications

In 1934, Hardy and Littlewood [43] generalized Littlewood's 4/3 inequality to bilinear maps defined on $\ell_p \times \ell_q$. In 1981, Praciano-Pereira [61] extended the Hardy–Littlewood inequalities to m-linear forms on $\ell_{p_1}\times\cdots\times\ell_{p_m}$ for $0 \leq \frac{1}{p_1}+\cdots+\frac{1}{p_m} \leq \frac{1}{2}$ and just recently Dimant and Sevilla-Peris [38] extended the Hardy–Littlewood inequalities for $\frac{1}{2} \leq \frac{1}{p_1}+\cdots+\frac{1}{p_m} < 1$. In this section, we will deal with the Hardy–Littlewood/Dimant–Sevilla-Peris inequalities.

Theorem 3.7 (Hardy–Littlewood/Dimant–Sevilla-Peris [38,43]). *Let $\mathbf{p} \in [1,\infty)^m$ be such that $\frac{1}{2} \leq \left|\frac{1}{\mathbf{p}}\right| < 1$. Then, for every continuous m–linear form $T : X_{p_1}\times\cdots\times X_{p_m} \to \mathbb{K}$, there exists a constant $C_{m,\mathbf{p}}^{\mathbb{K}} \geq 1$ such that*

$$\left(\sum_{i_1,\ldots,i_m=1}^{\infty} \left|T\left(e_{i_1},\ldots,e_{i_m}\right)\right|^{\frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|}}\right)^{1-\left|\frac{1}{\mathbf{p}}\right|} \leq C_{m,\mathbf{p}}^{\mathbb{K}}\left\|T\right\|$$

and the exponent is optimal.

In [1, Theorem 2.4] the following version with blocks of the Hardy–Littlewood/Dimant–Sevilla-Peris inequalities was proven.

Theorem 3.8. *Let $\mathbf{p} \in [1,\infty)^m$ and let $1 \leq d \leq m$, $m_1, \ldots, m_d$ be positive integers such that $m_1 + \cdots + m_d = m$. If $\frac{1}{2} \leq \left|\frac{1}{\mathbf{p}}\right| < 1$, then, for every continuous m–linear form $T : X_{p_1} \times \cdots \times X_{p_m} \to \mathbb{K}$, we have*

$$\left(\sum_{j_1,\ldots,j_d=1}^{\infty} \left| T\left(e_{j_1}^{m_1}, \ldots, e_{j_d}^{m_d}\right)\right|^{\frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|}} \right)^{1-\left|\frac{1}{\mathbf{p}}\right|} \leq C_{k,\mathbf{p}}^{\mathbb{K}} \|T\|$$

for some constant $C_{k,\mathbf{p}}^{\mathbb{K}} \geq 1$. Moreover, the exponent is optimal.

Although the exponents are sharp in both of the above theorems, it is already known that these results can be improved if we deal with anisotropic exponents (see [2, Theorem 3.4 and Theorem 3.5] and [7, Corollary 2]). Our inclusion theorem has an immediate application in the investigation of the Hardy–Littlewood inequalities for multilinear forms; more precisely, we use the inclusion theorem 3.6 to improve the above results for $p_1, \ldots, p_m \leq 2m$.

Theorem 3.9. *Let m be a positive integer, $\mathcal{I} = \{I_1, \ldots, I_d\}$ be a partition of $\{1, \ldots, m\}$ and $\mathbf{p} \in (1, 2m]^m$ such that $|1/\mathbf{p}| < 1$. Then, there exists a constant $D_{m,\mathbf{p},\mathbf{s}}^{\mathbb{K}} \geq 1$ such that*

$$\left(\sum_{j_1=1}^{n} \left(\cdots \left(\sum_{j_d=1}^{n} \left| A\left(\sum_{k=1}^{d} \sum_{j \in I_k} e_{i_k} \cdot e_j \right) \right|^{s_d} \right)^{\frac{s_{d-1}}{s_d}} \cdots \right)^{\frac{s_1}{s_2}} \right)^{\frac{1}{s_1}} \leq D_{m,\mathbf{p},\mathbf{s}}^{\mathbb{K}} \|A\|, \tag{3.2}$$

for all positive integer n and all m-linear forms $A : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$, with

$$s_k = \left[\frac{1}{2} - \left| \frac{1}{\mathbf{p}} \right|_{j \in \bigcup_{i=k}^{d} I_i} + \frac{1}{2m} \cdot \sum_{i=k}^{d} |I_i| \right]^{-1}, \quad \text{for } k = 1, \ldots, d.$$

Proof. Let $r := 2 = (1 - |1/\mathbf{2m}|)^{-1}$. Thus

$$\frac{1}{2} - \left| \frac{1}{(\mathbf{2m})^*} \right| + \left| \frac{1}{\mathbf{p}^*} \right| = \frac{1}{2} - m + \frac{1}{2} + m - \left| \frac{1}{\mathbf{p}} \right| = 1 - \left| \frac{1}{\mathbf{p}} \right| > 0$$

and, from Theorem 3.6, we have

$$\Pi^{\mathcal{I}}_{(2;(2m)^*,\dots,(2m)^*)}(E_1,\dots,E_m;\mathbb{K}) \subset \Pi^{\mathcal{I}}_{\left(\mathbf{s};p_1^*,\dots,p_m^*\right)}(E_1,\dots,E_m;\mathbb{K}),$$

for any Banach spaces $E_1,\dots,E_m$, with

$$\begin{aligned} s_k &= \left[\frac{1}{2} - \left|\frac{1}{(\mathbf{2m})^*}\right|_{j\in\bigcup_{i=k}^d I_i} + \left|\frac{1}{\mathbf{p}^*}\right|_{j\in\bigcup_{i=k}^d I_i}\right]^{-1} \\ &= \left[\frac{1}{2} - \sum_{i=k}^d |I_i| + \frac{1}{2m}\cdot\sum_{i=k}^d |I_i| + \sum_{i=k}^d |I_i| - \left|\frac{1}{\mathbf{p}}\right|_{j\in\bigcup_{i=k}^d I_i}\right]^{-1} \\ &= \left[\frac{1}{2} - \left|\frac{1}{\mathbf{p}}\right|_{j\in\bigcup_{i=k}^d I_i} + \frac{1}{2m}\cdot\sum_{i=k}^d |I_i|\right]^{-1}, \quad \text{for each } k=1,\dots,d. \end{aligned}$$

From Theorem 3.8 we have

$$\Pi^{\mathcal{I}}_{(2;(2m)^*,\dots,(2m)^*)}(E_1,\dots,E_m;\mathbb{K}) = \mathcal{L}(E_1,\dots,E_m;\mathbb{K}),$$

and thus

$$\Pi^{\mathcal{I}}_{\left(\mathbf{s};p_1^*,\dots,p_m^*\right)}(E_1,\dots,E_m;\mathbb{K}) = \mathcal{L}(E_1,\dots,E_m;\mathbb{K}),$$

for all Banach spaces $E_1,\dots,E_m$. Using the standard isometries between $\mathcal{L}(X_p,E)$ and $\ell^w_{p^*}(E)$, for $1<p\leq\infty$, the proof is completed. □

In order to clarify the new result, we illustrate with the following simple case that provides better exponents than the estimates of Theorem 3.8.

Corollary 3.10. *Let $1\leq d\leq m$ and let $m_1,\dots,m_d$ be positive integers such that $m_1+\cdots+m_d=m$. If $m<p\leq 2m$, then*

$$\left(\sum_{j_1=1}^{\infty}\left(\cdots\left(\sum_{j_d=1}^{\infty}\left|A\left(e_{j_1}^{m_1},\dots,e_{j_d}^{m_d}\right)\right|^{s_d}\right)^{\frac{s_{d-1}}{s_d}}\cdots\right)^{\frac{s_1}{s_2}}\right)^{\frac{1}{s_1}} \leq D^{\mathbb{K}}_{m,\mathbf{p},\mathbf{s}}\|A\|,$$

for all m-linear forms $A:\ell_p^n\times\cdots\times\ell_p^n\to\mathbb{K}$, with

$$s_k = \left[\frac{1}{2} - (m_k+\cdots+m_d)\cdot\left(\frac{1}{p}-\frac{1}{2m}\right)\right]^{-1}, \quad \textit{for} \quad k=1,\dots,d.$$

Notice that the exponents provided are

$$s_1 = \frac{p}{p-m}, \ldots, s_d = \frac{2mp}{mp + pm_d - 2mm_d},$$

while Theorem 3.8 provides the exponents

$$s_1 = \cdots = s_d = \frac{p}{p-m}.$$

Corollary 3.10 recovers [7, Corollary 2] and, when $d = 1$, it recovers a result of [63].

The following example illustrates the improvement obtained:

Example 3.11. *Suppose* $m = 4, d = 2, p = 6, I_1 = \{1, 2\}$ *and* $I_2 = \{3, 4\}$. *By Theorem 3.8 we know that* (3.2) *holds with*

$$s_k = 3 \quad \textit{for} \quad k = 1, 2,$$

whereas by Theorem 3.9 we have

$$s_1 = 3 \quad \textit{and} \quad s_2 = 12/5.$$

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