



Universidade Federal da Paraíba
Centro de Ciências Exatas e da Natureza
Departamento de Física

Programa de Pós-Graduação Stricto Sensu em Física

**UNVEILING THE UNIVERSAL CLASS
CATEGORIZATION OF
ALTLAND-ZIRNBAUER IN
MESOSCOPIC SYSTEMS: PROBING
ENTANGLEMENT AND
CONDUCTANCE FLUCTUATIONS**

Rafael Alves Batista

Tese de Doutorado
João Pessoa
August 8, 2023

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Trabalho apresentado ao Programa de Pós-Graduação Stricto Sensu em Física do Departamento de Física da Universidade Federal da Paraíba como requisito parcial para obtenção do grau de Doutor em Física.

Orientador: *Jorge Gabriel Gomes de Souza Ramos*

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Ata da Sessão Pública da Defesa de tese de Doutorado do aluno Rafael Alves Batista, candidato ao Título de Doutor em Física na Área de Concentração Física da Matéria Condensada.

Aos dezoito dias do mês de julho do ano de dois mil e vinte e três, às 09:00, virtualmente, através do link <https://meet.google.com/bvi-wdlhv-bvb>, reuniram-se os membros da Banca Examinadora constituída para avaliar a tese de Doutorado, na área de Física da Matéria Condensada, de **Rafael Alves Batista**. A banca foi composta pelos(as) professores(as) doutores(as): Jorge Gabriel Gomes de Souza Ramos (PPGF/UFPB), orientador e presidente da banca examinadora, Dionísio Bazeia Filho (PPGF/UFPB), João Antônio Plascak (PPGF/UFPB), Caio Henrique Lewenkopf (UFF) e Tarcísio Marciano da Rocha Filho (UNB). Dando início aos trabalhos, o Prof. Jorge Gabriel Gomes de Souza Ramos comunicou aos presentes a finalidade da reunião. A seguir, passou a palavra para o candidato para que o mesmo fizesse, oralmente, a exposição da pesquisa de tese intitulada "*Unveiling the universal class categorization of Altland-Zirnbauer in mesoscopic systems: probing entanglement and conductance fluctuations*". Concluída a exposição, o candidato foi arguido pela Banca Examinadora, que emitiu o parecer "**aprovado**". Assim sendo, deve a Universidade Federal da Paraíba expedir o respectivo diploma de Doutor em Física na forma da lei. E para constar, Danilo Wilson Lemos Menezes, Técnico em Assuntos Educacionais, redigiu a presente ata que vai assinada pelos membros da Banca Examinadora. João Pessoa, Paraíba, **18 de julho de 2023**.

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*I dedicate this work to God, who is the beginning.
To my family, for being the foundation of my life.*

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*“Be like a vine that produces grapes without looking for anything in
return.”*

—MARCUS AURELIUS

Resumo

Neste trabalho, exploramos o uso das dez classes de simetria de Altland-Zirnbauer para descrever o transporte quântico em sistemas mesoscópicos, focando em pontos quânticos caóticos. Utilizamos simulações numéricas baseadas na Teoria de Matrizes Aleatórias (RMT) para investigar propriedades de transporte, como condutância e potência do ruído de disparo, além de quantificadores de emaranhamento, como concorrência, emaranhamento de formação e desigualdade de Bell. Iniciamos apresentando o sistema de estudo, o bilhar quântico caótico, e derivamos a fórmula de Mahaux-Weindenmüller para a matriz de espalhamento. Assim, introduzimos os principais observáveis do transporte quântico: a condutância e o ruído de disparo. Em seguida, exploramos as simetrias fundamentais presentes em sistemas mesoscópicos, como a simetria de reversão temporal, rotação de spin, quiralidade e partícula-buraco, analisando como essas simetrias impõem restrições ao hamiltoniano e à matriz de espalhamento. Em seguida, apresentamos resultados numéricos que destacam a influência dessas simetrias nas propriedades de transporte e emaranhamento. Realizamos uma análise estatística da condutância e do ruído de disparo para bilhares de Schrödinger, Dirac e Andreev, revelando o papel das simetrias quirais e de partícula-buraco nas flutuações universais da condutância. Além disso, investigamos o emaranhamento quântico em bilhares quânticos caóticos, observando violações da desigualdade de Bell em todos os casos, com dependências específicas em relação ao número de ressonâncias e transparência da barreira. Destacamos que é mais fácil violar a desigualdade de Bell nos graus de liberdade quiral/sub-rede e partículas-buraco do que nos graus de liberdade orbitais. Por fim, evidenciamos um comportamento único e intrigante: no bilhar de Andreev, a realização do emaranhamento sempre resulta na produção de pares de Bell no limite de canal único, independentemente da presença da simetria de reversão temporal e das taxas de tunelamento. Esses pares emaranhados são formados por modos de Majorana em interfaces de diferentes fases topológicas, sugerindo sua utilidade como ferramenta para produção e controle de emaranhamento, bem como para aplicações em computação quântica.

Palavras-chave: Classificação de Altland-Zirnbauer; Teoria de Matrizes Aleatórias; Picos de Condutância; Potência do Ruído de Disparo; Emaranhamento Quântico; Desigualdade de Bell.

Abstract

In this thesis, we present the ten classes of Altland-Zirnbauer symmetry and how they are used to describe universal quantum transport in mesoscopic systems, particularly chaotic quantum dots. We perform numerical simulations using Random Matrix Theory (RMT) to explore transport properties such as conductance and shot noise power, as well as entanglement quantifiers including concurrence, entanglement formation, and Bell inequality. We begin by introducing the description of our system, the chaotic quantum billiard. Then, we derive the Mahaux-Weidenmüller formula for the scattering matrix, and consequently, the conductance and shot noise power. We then present the possible fundamental symmetries present in mesoscopic systems, such as time-reversal symmetry, spin rotation, chirality, and particle-hole symmetry. In this way, we explore how these symmetries impose constraints on the Hamiltonian and, consequently, the scattering matrix. Finally, we present numerical results showing the influence of these fundamental symmetries on transport and entanglement properties. We provide a statistical analysis of the conductance and shot noise power for Schrödinger, Dirac, and Andreev billiards. Specifically, we observe that chiral and particle-hole symmetries play a role in the universal conductance fluctuations. In addition to transport properties, we perform numerical simulations of quantum entanglement in chaotic quantum billiards. We observe violations of the Bell inequality for all billiards and analyze their dependencies on the number of resonances and barrier transparency. We demonstrate that it is easier to violate the Bell inequality in chiral/sublattice and particle-hole degrees of freedom than in orbital degrees of freedom. Moreover, our results exhibit a unique and peculiar behavior: the realization of entanglement in the Andreev billiard always results in the production of Bell pairs in the single-channel limit, regardless of the presence of time-reversal symmetry and tunneling rates. The entangled pairs are formed by Majorana modes in an interface of different topological phases and may serve as a useful tool for entanglement generation and control, as well as quantum computing.

Keywords: Tenfold way; Altland-Zirnbauer classification; Random-Matrix Theory; Conductance Peaks; Shot Noise Power; Quantum Entanglement; Bell's inequality.

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CHAPTER 1

INTRODUCTION

1.1 Mesoscopic Physics

The last decades witnessed intense research activity to miniaturize electronic devices. Over the past 60 years, the progress made in computers, smartphones, and communication has influenced technological evolution, and consequently our lives. Enthusiastic about the miniaturization of electronic devices, Gordon Moore (co-founder of Fairchild Semiconductor and Intel) observed that the number of transistors in a dense integrated circuit (IC) doubles about every two years. This empirical relationship became known as *Moore's Law* in the semiconductor industry. However, although this intense production of smaller structures is beneficial for the technological development, this process faces a remarkable challenge in the physical realm: quantum effects.

Mesoscopic physics is an area of condensed matter physics that deals with small conductors that are intermediate between the quantum world of microscopic systems and the classical world of macroscopic systems. In this regime, the system consists of a large number of atoms, but their behavior is governed by quantum mechanics. Thus, the small conductors are much larger than microscopic objects like atoms, but not large enough to be described by classical physics. In Fig.(1.1) we present our mesoscopic system of interest: the quantum dot. In parts a), b), and c) we see a schematic view, scanning electron micrographs of the few-electron single-dot and double-dot devices, respectively. In Fig.(1.1 a)), the light gray area is a two-dimensional electron gas (2DEG) fabricated from heterostructures of GaAs and AlGaAs by a technique called molecular-beam epitaxy. The application of metal gate electrodes, represented by dark gray in the figure, leads to depleted regions (white area) that confine the 2DEG. An electric current is introduced by applying a potential difference across the ohmic contacts.

In a ballistic mesoscopic system, the device dimension L must be comparable to or smaller than the electronic mean free path ℓ_e , which is the average distance that an electron travels before it is scattered by an impurity. Usually, the electrons travel in a conductor with Fermi energy E_F and effective mass m^* , so their de Broglie wavelength is the Fermi wavelength $\lambda_F = h/\sqrt{2m^*E_F}$. If the device dimension is comparable to λ_F , the electrons will show a typical wave characteristic, and their kinetic energy will be discretized, fact known as *size quantization*. Due to the wave behavior, each electron is associated with a wave function $\Psi = \Psi_0 e^{i\phi}$, with ϕ as the phase of the wave function of amplitude Ψ_0 and can present diffraction depending on the obstacle. Nevertheless, diffraction effects disappear as temperature increases due to the production of incoherent scattering with the phonon bath. In the case of low temperatures, the device can also destroy the coherence of the electrons due to inelastic scatterings with phonons, other electrons, or magnetic impurities. Therefore, we call a system mesoscopic when the elec-

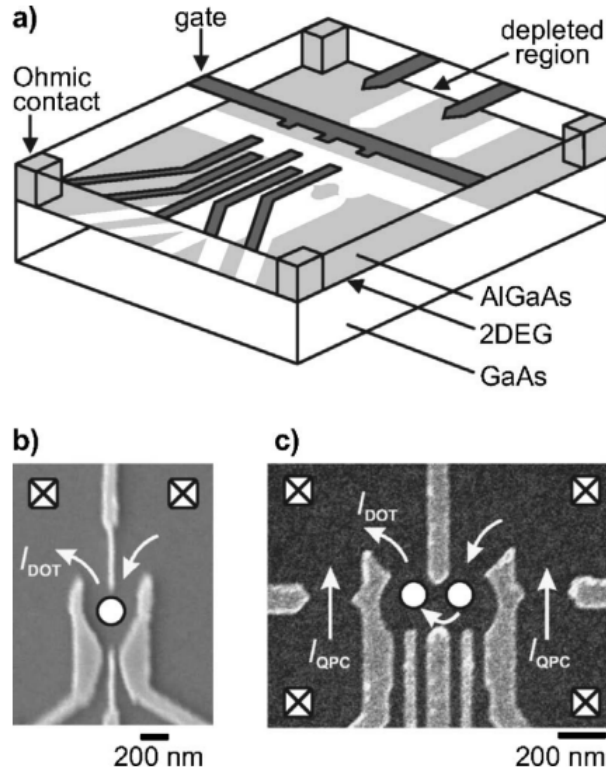


Figure 1.1 Quantum dot device defined by metal surface electrodes. Figure taken from article [1].

tron's phase coherence length L_ϕ (the typical distance the electron travels without losing phase coherence) is larger than or comparable to the system's size L ($L \leq L_\phi$). Another interesting property of the mesoscopic systems is the charging effects. In a conventional device, the capacitance of the device is so large that the energy needed to charge it with a single electron is negligibly small. However, in a mesoscopic system, the capacitances may be sufficiently small, such that single-electron charging effects may become observable. Finally, another important feature of mesoscopic physics is spin dephasing length ℓ_s . Summarizing, the ℓ_s is the typical distance the electron's spin travels without losing its spin phase coherence. In a nanostructure in which the dimension is smaller than the spin dephasing length ℓ_s , the conductance can depend on the spin polarization of the injected current. This fact provides relevant importance of spin effects in technological applications, such as spintronics, magnetic storage devices, and even spin qubit quantum computers. This entire discussion of conventional and mesoscopic devices is summarized in Figure 1.2.

1.2 Conductance

The property which quantifies the electric current passing through a conventional or mesoscopic device is called conductance. In a conventional device, such as a classical wire, the conductance is given by $G = \sigma W / L$, where W and L are the areas and the length of the conductor.

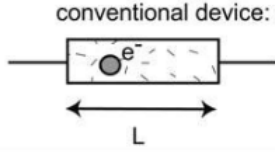
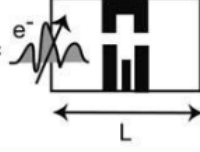
 conventional device:	 mesoscopic device:
$L \gg l_e$ diffusive	$L \lesssim l_e$ ballistic
$L \gg l_\phi$ incoherent	$L \lesssim l_\phi$ phase coherent
$L \gg \lambda_F$ no size quantization	$L \lesssim \lambda_F$ size quantization
$e^2/C < k_B \Theta$ no single electron charging	$e^2/C \gtrsim k_B \Theta$ single electron charging effects
$L \gg l_s$ no spin effects	$L \lesssim l_s$ spin effects

Figure 1.2 Properties of conventional and mesoscopic devices. Figure taken from book [2].

However, in a mesoscopic system, conductance is a property directly related to the probability of an electron going from one lead to the other due to the electron's wave-particle duality. Due to the miniaturization process, the size quantization plays a fundamental role in a ballistic quantum dot: the conductance is discretized and is given by integers of a quantized unit called the quantum of conductance $G_0 = 2e^2/h$, where e , h and the factor 2 are the elementary charges, Planck constant, and a factor due to the spin degeneracy, respectively.

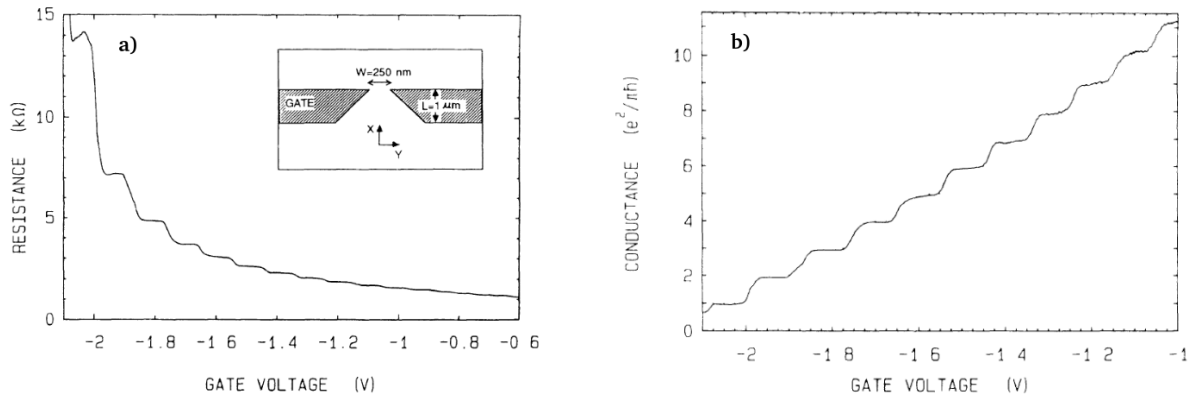


Figure 1.3 Resistance a) and Conductance b) of a constriction, also known as quantum point contact (QPC). Figures taken and modified from reference [3].

In Fig.(1.3), we see experimental results [3] which show discrete changes in the resistance and conductance as a function of the gate voltage V . In a quantum point contact, as depicted in insert figure in Fig.(1.3 a)), the conductance is given by the Landauer formula

$$G = G_0 \times \sum_{i=1}^N T_i, \quad (1.1)$$

where T_i and N are the transmission eigenvalues and the number of open channels, respectively. The transmission eigenvalues T_i represent the probability of an electron going from an open channel of a lead to another lead. The number of open channels N arises from the size quantization of the lead, and this represents the amount of kinetic energy available to the electron. As a size quantization fact, the N tends to increase as the lead's width W becomes larger. If the lead's width W becomes much larger than the Fermi wavelength λ_F (30 nm in GaAs/AlGaAs), then the discrete increases in conductance disappear and the conductance starts to depend linearly on the width $G \propto W$, reaching the classic limit.

Another notable phenomenon in mesoscopic devices is universal conductance fluctuations (UCF). Experimental results have shown that the transport properties of small conductors depend on external parameters, such as applied bias voltage and magnetic field, in a random way [4, 5, 6, 7, 8]. Nevertheless, the statistical properties of these random signals of conductance have well-defined values, such as probability distribution, mean, variance, root mean square (RMS) and etc. As an example of conductance fluctuations, the magnetoconductance in three different systems is depicted in Fig.(1.4). In the first part (a), the magnetoconductance was measured in a polycrystalline gold film ring with 38nm thick having a resistivity $\rho \simeq 5\mu\Omega \cdot cm$ at $T = 4K$ [4]. In the second part (b), the magnetoconductance was obtained for quasi-1D metal oxide-silicon field-effect transistors (MOSFET) in the lengths and widths ranges of $0.04 - 1.0\mu m$ [6]. The last part (c) is a numerical simulation of the magnetoresistance which consists of a nearest-neighbor tight-binding model on an infinite two-dimensional strip containing a finite disordered region of N sites in length and M sites in width [7].

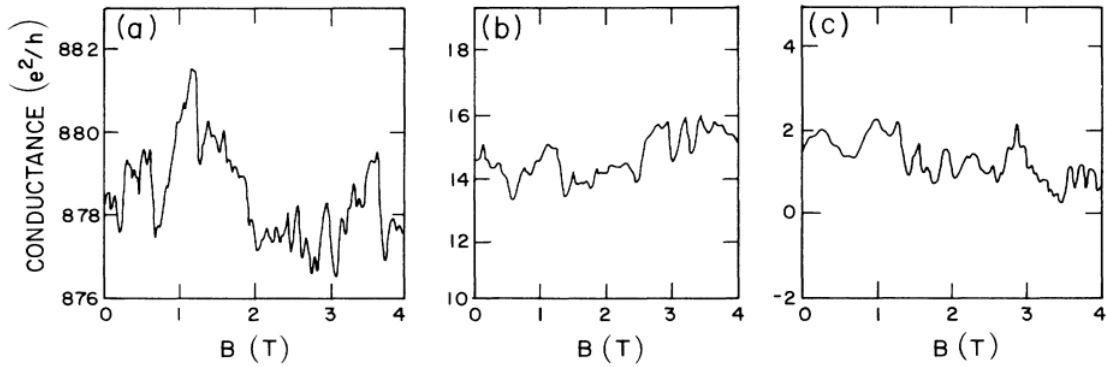


Figure 1.4 Magnetoconductance in three different systems. (a) $g(B)$ in 0.8- μm -diam gold ring from Ref. [4, 5]. (b) $g(B)$ for a quasi-1D silicon MOSFET from Ref. [6]. (c) Numerical calculation of $g(B)$ performed by Ref. [7]. Figure taken from reference [8].

The UCF mechanism is reasoned by wave characteristics of quantum mechanics. By presenting phase-coherent effects such as destructive or constructive interference, the fluctuations are manifestations of the wave superposition of the quantum regime. As electrons are treated as waves, fluctuations arise from interference between them after passing through the conductor along different trajectories. This fact leads to different phase shifts which will interfere between them, yielding fluctuations. Although such fluctuations are imperceptible in the classical

regime, their root mean square (RMS) value of conductance can be estimated [9]

$$\text{rms}[G] = \sqrt{\text{var}(G)} = \sqrt{\langle G^2 \rangle - \langle G \rangle^2} \sim \frac{e^2}{h}. \quad (1.2)$$

The root mean square does not depend on the number of open channels, thus the fluctuations are on the order of the conductance average ($\sim e^2/h$) in the quantum regime. However, as the number of open channels increases, that root mean square becomes infinitely smaller than the conductance average, making the fluctuations imperceptible in the classical regime measurements.

1.3 Graphene

Advances in condensed matter physics and nanotechnology have allowed the control and production of graphene monolayers [10, 11, 12]. Graphene is made out of carbon atoms arranged on a honeycomb structure made out of hexagons, as depicted in Fig 1.5. Each carbon atom of the lattice has six electrons: two electrons are in the inner shell $1s$, three electrons are in the 3 in-plane covalent bonds in the sp^2 configuration ($2s^2, 2p_x, 2p_y$), and a single electron occupying the p_z orbital perpendicular to the plane. The physics of graphene, including electron transport, is associated with the two-dimensional fluid formed by those p_z electrons.

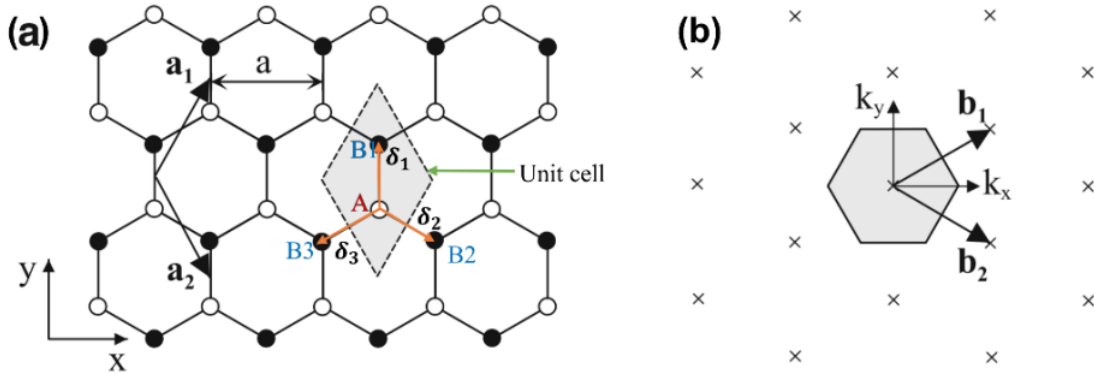


Figure 1.5 Graphene Structure. a) Honeycomb structure of monolayer graphene; white (black) circles represent carbon atoms on A (B) sites. The unit cell, drawn in (a), is the smallest portion of the structure that shows the pattern of the entire lattice. b) The reciprocal lattice of monolayer graphene, that is entirely broken into Brillouin zones (shaded hexagon). Figure taken from [13].

The electrons move over two carbon atoms sub-lattices A and B, represented by white and black circles in Fig.(1.5 (a)), respectively. A particular sub-lattice A (B) is entirely spanned by the basis vector:

$$\mathbf{a}_1 = \frac{a}{2}(\mathbf{e}_x + \sqrt{3}\mathbf{e}_y), \quad \mathbf{a}_2 = \frac{a}{2}(\mathbf{e}_x - \sqrt{3}\mathbf{e}_y), \quad (1.3)$$

where $a = \sqrt{3}a_0 \approx \sqrt{3} \times 1.42 = 2.46 \text{ \AA}$ is the lattice constant. The vectors δ_α connect any

A-site to its three B-type nearest neighbors, and are defined by:

$$\delta_1 = \frac{a}{\sqrt{3}}\mathbf{e}_y, \quad \delta_{2,3} = \frac{a}{2}(\pm\mathbf{e}_x - \frac{1}{\sqrt{3}}\mathbf{e}_y). \quad (1.4)$$

The set of vectors $\mathbf{a}_{1,2}$ and $\delta_{1,2,3}$ are sufficient to describe the graphene in the real lattice. In Fig.(1.5 (b)), the primitive reciprocal lattice vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ are given by

$$\mathbf{b}_1 = \frac{2\pi}{a}\mathbf{e}_x + \frac{2\pi}{\sqrt{3}a}\mathbf{e}_y, \quad \mathbf{b}_2 = \frac{2\pi}{a}\mathbf{e}_x - \frac{2\pi}{\sqrt{3}a}\mathbf{e}_y, \quad (1.5)$$

satisfying the condition $\mathbf{a}_i \cdot \mathbf{b}_j = 2\pi\delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker delta.

In this thesis, we are interested in electronic transport properties in so-called Dirac billiards. Efficient in describing condensed matter systems whose quasiparticles obey the relativistic Dirac equation, the Dirac billiards are quantum dot devices made entirely from graphene [14], as depicted in Fig.(1.6). The peculiar structure of graphene, discussed above, supports a curious aspect: its low-level excitations are massless, chiral, and Dirac fermions. This fact is confirmed by analysing its band structure.

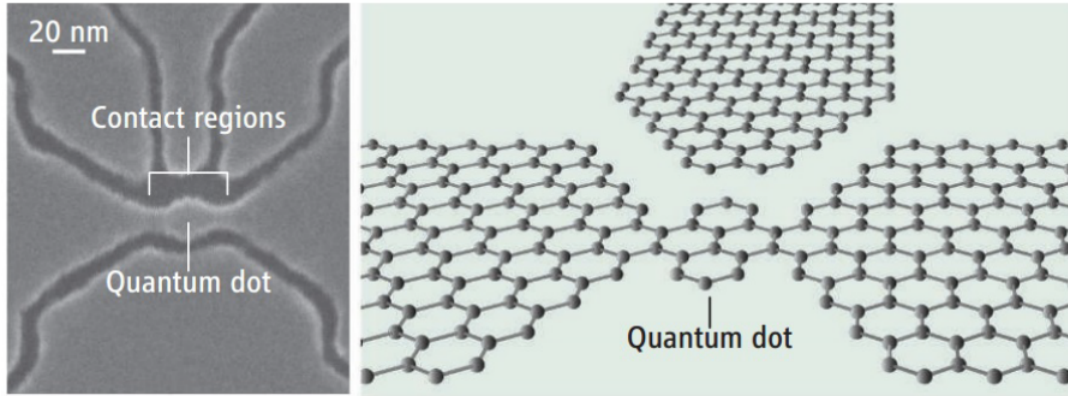


Figure 1.6 Quantum dot made out of graphene. Figure taken from [15].

Firstly revealed by [16], the honeycomb lattice of graphene has an unusual band structure, shown in Fig.(1.7). The quasiparticle transport in graphene is divided into two bands: electrons filling states in the conduction band ($E > 0$) and hole excitations (empty states) in the valence band ($E < 0$).

Near each corner of the Brillouin zone ($\delta\mathbf{k}a \ll 1$), the energy has the following linear dispersion relation

$$|E| = \hbar v |\delta\mathbf{k}|, \quad (1.6)$$

where $\delta\mathbf{k} \equiv \mathbf{k} - \mathbf{K}$ is the displacement from the corner at wave vector $\mathbf{K}$ and v is the velocity of quasiparticles. The velocity can be calculated by $v \equiv \frac{1}{2}\sqrt{3}\tau a/\hbar \approx 10^6 \text{ m/s}$, where $a = 0.246 \text{ nm}$ is the lattice constant and $\tau \approx 3 \text{ eV}$ is the nearest-neighbor hopping energy. In the points $\mathbf{K}$ and $\mathbf{K}'$, the conduction and valence band touch each other at $E = 0$; this fact indicates that graphene has zero band gap, and it is therefore usually described as a zero-gap semiconductor. Moreover,

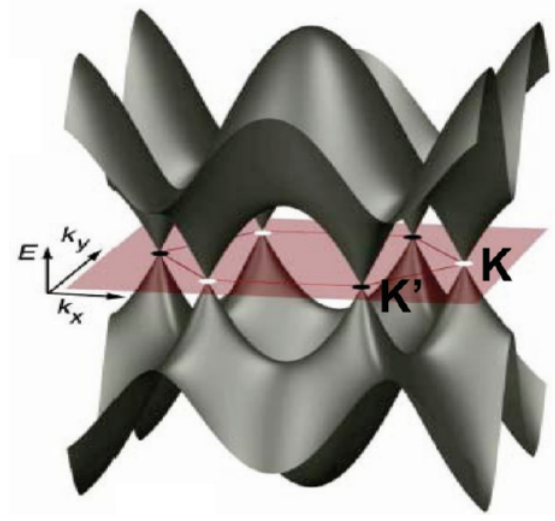


Figure 1.7 The band structure of graphene $E(k_x, k_y)$. The red hexagon is the first Brillouin zone (see Fig.1.5 (b)). The black and white corner points of the hexagon are called Dirac points. Figure taken from [17].

due to the two sublattice structures (A and B) and the linear dispersion relation, the system has the relativistic Dirac-Hamiltonian form [18]

$$H\Psi = v_F(\hbar\mathbf{k}) \cdot \boldsymbol{\sigma}\Psi = E\Psi \quad \rightarrow \quad -iv_F\hbar \begin{pmatrix} 0 & \partial_x - i\partial_y \\ \partial_x + i\partial_y & 0 \end{pmatrix} \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix} = E \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix}, \quad (1.7)$$

where the off-diagonal structure of Hamiltonian reveals a coupling between Ψ_A and Ψ_B but not to itself. Then, the two wave-function amplitudes Ψ_A and Ψ_B are coupled by the nearest-neighbor hopping τ in the form

$$E\Psi_A = \tau \sum_{\text{neighbors}} \Psi_B, \quad E\Psi_B = \tau \sum_{\text{neighbors}} \Psi_A. \quad (1.8)$$

The operator $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, described by Pauli matrices σ_i , acts on the sublattice degree of freedom and it mimics the operator from the real electron spin. On the other hand, another degree of freedom arises due to the hexagon structure of the first Brillouin zone: the valley degree of freedom. The Eq. (1.7) describes states with wave vector $\mathbf{k}$ in the valley centered at $\mathbf{K} = (4\pi/3a)\hat{\mathbf{x}}$ (corner of the Brillouin zone). However, the valley at the opposite corner ($\mathbf{K}' = -\mathbf{K}$) provides independent set of states whose amplitudes Ψ'_A and Ψ'_B satisfy the same Dirac Eq. (1.7) with $p_x \rightarrow -p_x$. Thus, the spinor $\Psi = (\Psi_A, \Psi_B, -\Psi_B, \Psi_A)$ obeys the valley-isotropic representation

$$\begin{pmatrix} v\mathbf{p} \cdot \boldsymbol{\sigma} & 0 \\ 0 & v\mathbf{p} \cdot \boldsymbol{\sigma} \end{pmatrix} \Psi = E\Psi. \quad (1.9)$$

The Eq. (1.9) can be written in the compact form

$$H = v\mathbf{p} \cdot \boldsymbol{\sigma} \otimes \tau_0, \quad (1.10)$$

where $\tau = (\tau_1, \tau_2, \tau_3)$, described by Pauli matrices τ_i , acts on valley degree of freedom and τ_0 is the 2×2 unit matrix.

Quasi-particles in graphene are called "chiral" because their direction of the motion and the direction of the pseudospin are the same. The current operator, defined by

$$\mathbf{j} = v\boldsymbol{\sigma} \otimes \tau_0, \quad (1.11)$$

shows that if an electron moves in the x or y direction, then it has a pseudospin pointing in the x or y direction.

1.4 Andreev Reflection

In the last decades, several normal-superconductor hybrid structures have been fabricated and exploited. In the present work, we focus on the transport properties of devices with normal-metal (N) and superconductor (S) interfaces called Andreev billiards. Fig.(1.8) presents a quantum dot manufactured in a high-mobility InAs/AlSb heterostructure and contacted by four superconducting Nb electrodes. Usually, the electron dynamics in a normal conductor, 2DEG GaAs/AlGaAs heterostructure, for example, has the quantum mechanical description given by the Schrödinger equation. However, the presence of a superconducting segment in contact with the normal metal plays a fundamental role in the equation that governs the system dynamics, especially the energy spectrum of the billiard.

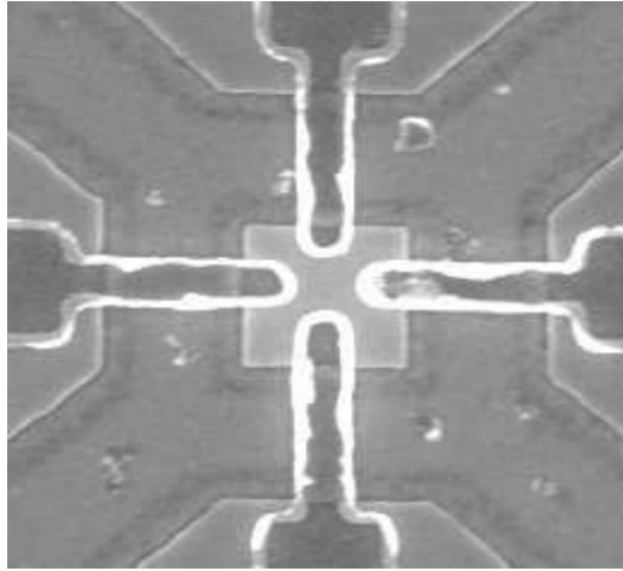


Figure 1.8 Quantum dot (central square) contacted by four superconducting electrodes. The central square has dimensions $500 \text{ nm} \times 500 \text{ nm}$. Figure taken from [19].

The contact of a superconducting with a non-superconducting material leads to the proximity effects whose microscopic model description in terms of single electron processes is known as *Andreev Reflection*. As depicted in Fig.(1.9), [20] discovered that an electron in the Fermi

level of a normal metal retraces its path when it collides with a superconductor, as if time is reversed. Moreover, the particle's charge is reversed, i.e., the negatively charged electron (e) is converted into a positively charged hole (h). In a usual reflection, the particle conserves charge but does not conserve momentum. Conversely, in an Andreev reflection, the particle conserves momentum but does not conserve charge: the electron is converted as a hole with the same momentum and opposite velocity.

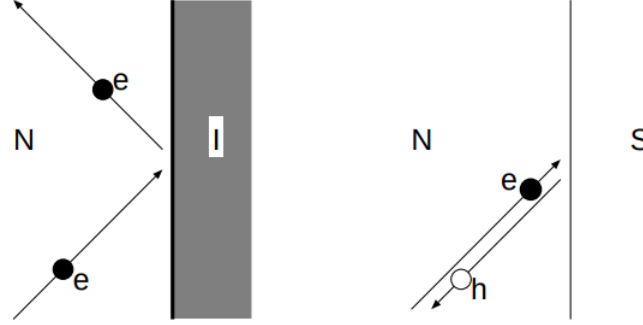


Figure 1.9 Normal reflection by an insulator (I) and Andreev reflection by a superconductor (S) in the left and right panel, respectively. Figure taken from [21].

The mechanism behind this peculiar property is associated with the superconducting nature of the Cooper pairs. An incident electron, coming from the metal, forms a Cooper pair with another when it reaches the superconducting condensate, leaving a hole with the opposite velocity in the metal. This phenomenon converts a dissipative electrical current in the normal metal into a dissipationless supercurrent in the superconductor.

Although Cooper pairs tend to condense into the same ground quantum state, they are not the only ones responsible for the unusual properties of superconductivity. Beyond the pair formation in the ground quantum state, superconductors host the fermionic excitations Ψ called Bogoliubov quasi-particles [22]. The excited state Ψ is a coherent superposition of the electron and hole states ψ_e and ψ_h , respectively, and its $\pm 2e$ charge fluctuations are absorbed by Cooper pairs of the superconducting condensate. These fluctuations conserve modulo $2e$ of the Bogoliubov quasiparticle charge, revealing a curious symmetry in nature called symmetry of charge conjugation or *particle-hole symmetry*. In the first quantization, the equation that governs the dynamics of Bogoliubov quasiparticles is given by the mean-field Hamiltonian

$$H\Psi(\mathbf{r},t) = -i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t), \quad H = \begin{pmatrix} H_0 - E_F & -i\sigma_y\Delta \\ i\sigma_y\Delta^* & E_F - H_0^* \end{pmatrix}, \quad (1.12)$$

$$\Psi = (\psi_e, \psi_h) = (\psi_{e,\uparrow}, \psi_{e,\downarrow}, \psi_{h,\uparrow}, \psi_{h,\downarrow}).$$

This equation is known as the Bogoliubov-De Gennes (BdG) equation. The presence of particle-hole symmetry is incorporated into the block structure of Eq. (1.12). The upper-left block $H_0 - E_F$ of H acts on the electron component ψ_e , while the lower-right block $E_F - H_0^*$ acts on the hole component ψ_h . Electrons and holes with opposite spin bands $\uparrow, \downarrow$ are coupled by the off-diagonal blocks through the pair potential Δ . If H_0 is spin-independent, then the Hamilto-

nian H decouples into the two blocks

$$H_{\pm} = \begin{pmatrix} H_0 - E_F & \pm\Delta \\ \pm\Delta^* & E_F - H_0^* \end{pmatrix}, \quad (1.13)$$

acting separately on $(\psi_{e,\downarrow}, \psi_{h,\uparrow})$ and $(\psi_{e,\uparrow}, \psi_{h,\downarrow})$. Therefore, the Bogoliubov-De Gennes (BdG) equations [23] take the form

$$\begin{aligned} H_{BG} \begin{pmatrix} u \\ v \end{pmatrix} &= E \begin{pmatrix} u \\ v \end{pmatrix}, \\ H_{BG} &= \begin{pmatrix} H & \Delta(\mathbf{r}) \\ \Delta^*(\mathbf{r}) & -H^* \end{pmatrix}. \end{aligned} \quad (1.14)$$

Each eigenfunction of $\Psi = (u, v) = (\psi_{e,\downarrow}, \psi_{h,\uparrow})$ of H_{BG} at energy $E > 0$ has a copy $i\tau_y \mathcal{K}\Psi = (\psi_{h,\uparrow}^*, -\psi_{e,\downarrow}^*)$ at $-E$, where τ_y and $\mathcal{K}$ are the y -component of Pauli matrix acting on the particle-hole sector and the operator of complex conjugation, respectively. Hence, for each Hamiltonian eigenvalue E positive, there is a symmetrical eigenvalue $-E$ around zero.

1.5 Random Matrix Theory

A powerful technique to determine the spectrum of a chaotic system is the random-matrix theory (RMT). Initially addressed in nuclear physics, the universal conductance fluctuations were owing to the chaotic resonant scattering in heavy nuclei, which led Wigner to discover the random nature of Hamiltonians [24]. Nuclei are many-body systems, and chaos manifests in ways different from those of few-degrees-of-freedom systems. To explain the fluctuations and peaks in the cross-section of nuclear reactions, Bohr proposed his compound-nucleus model depicted in Fig.(1.10).

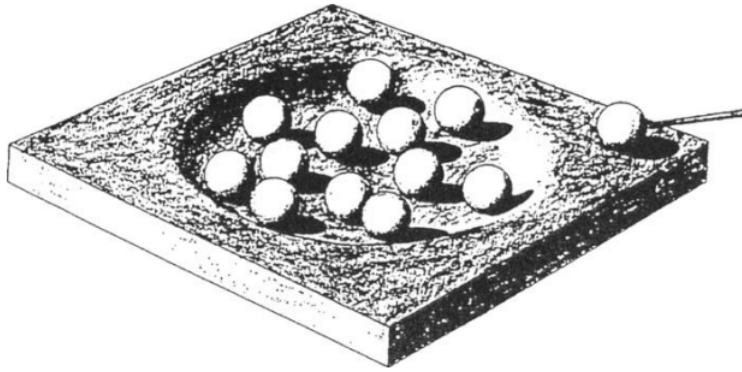


Figure 1.10 Bohr's wooden toy model of the compound nucleus. This Figure is taken from [25].

An incident ball strikes billiard balls in a shallow basin, dividing its original kinetic energy among all the balls. Considering both basin and balls are perfectly smooth and elastic, the collisions would continue until the kinetic energy happens again to be concentrated upon a ball

close to the edge. After numerous collisions, this ball would receive enough kinetic energy to quit the basin, leaving the others with insufficient kinetic energy to escape.

Bohr's idea that the nucleus is a chaotic, complex, and strongly interacting system motivated Wigner to introduce a random matrices approach. This approach characterizes spectra by their fluctuation properties: *nearest-neighbors spacing* (NNS) distribution. The statistical theory of spectra considers an ensemble of random matrices, instead of the actual nuclear Hamiltonian (which was unknown in the 1950s). So this approach does not aim to calculate and compare individual spectra with data. Instead, it determines the joint probability distribution of the eigenvalues and calculates spectral fluctuation measures, such as the NNS distribution, as averages over the ensemble. Due to its statistical aspect, the random-matrix theory is a phenomenological theory that does not refer to an underlying system-specific Hamiltonian. Further, the RMT developed by Wigner and Dyson classifies systems by their symmetry properties.

Wigner and Dyson considered an ensemble of $N \times N$ Hermitian matrices $\mathcal{H}$, with probability distribution

$$P(\mathcal{H}) = c \exp[-\beta \text{Tr} V(\mathcal{H})], \quad (1.15)$$

where c and β are a normalization constant and a symmetry index, respectively. In the case of $V(\mathcal{H}) \propto \mathcal{H}^2$, the ensemble is called Gaussian, with independent distributed matrix elements (because $\text{Tr} \mathcal{H}^2 = \text{Tr} \mathcal{H} \mathcal{H}^\dagger = \sum_{ij} |\mathcal{H}_{ij}|^2$). As we deal with mesoscopic systems constituted by many resonances $N \rightarrow \infty$, the spectral correlations become largely independent of V . This is the called *universality* of spectral correlations, where it stays away from the edge of the spectrum.

For the Wigner-Dyson Gaussian ensemble, the β index in Eq. (1.15) counts the number of degrees of freedom in the matrix elements of $\mathcal{H}$ and it is directly related to the symmetry classes. The matrix elements can be real, complex, or real quaternion numbers, corresponding to $\beta = 1, 2$ or 4 , respectively. These indexes are related to the presence/absence of time reversal (TRS) and spin-rotation (SRS) symmetries in the three following ensembles

- **Gaussian Orthogonal Ensemble (GOE):** The index $\beta = 1$, classified by Cartan's symmetric spaces as "AI", represents a system that preserves time-reversal (TRS) and spins rotational symmetries (SRS). In these systems, the time-reversal operator $\mathcal{T}$ satisfies $\mathcal{T}^2 = +1$, and the Hamiltonian of this system has to be a real-symmetric matrix.
- **Gaussian Unitary Ensemble (GUE):** The $\beta = 2$, classified by Cartan's symmetric spaces as "A", indicates systems with time-reversal symmetry broken by a magnetic field or magnetic impurities in the quantum dot. When the TRS is broken, it does not matter if SRS is preserved or broken. In this case, each element of the Hamiltonian is a complex number. As the time-reversal symmetry is broken in these systems, we indicate its absence as $\mathcal{T}^2 = \times$.
- **Gaussian Symplectic Ensemble (GSE):** The $\beta = 4$, classified by Cartan's symmetric spaces as "AII", represents a system with SRS broken but TRS preserved. Mathematically, the Hamiltonian is written in the quaternionic form $H = a \otimes 1 + i(b \otimes \sigma_x + c \otimes \sigma_y + d \otimes \sigma_z)$. In these systems, the spin-orbit coupling arises due to the relativistic effects between the apparent magnetic field seen from the electron perspective and the magnetic moment of the electron associated with its intrinsic spin. The time-reversal operator satisfies $\mathcal{T}^2 = -1$.

Therefore, each β index has its respective Hamiltonian transformation $\mathcal{H} \rightarrow U\mathcal{H}U^{-1}$, where U is orthogonal, unitary or symplectic for $\beta = 1, 2$ or 4 , respectively.

Beyond the Wigner-Dyson ensemble, new symmetry classes were discovered by [26, 27] known as chiral symmetry. Initially addressed in quantum chromodynamics (QCD), it was realized that the hadronic world at low energies is governed by chiral symmetry and its respective spontaneous breaking. The quarks' dynamic inside the hadron is described by the off-diagonal Hamiltonian called as Dirac operator $\mathcal{D}$

$$\mathcal{D} = \begin{pmatrix} 0 & iW \\ iW^\dagger & 0 \end{pmatrix}, \quad (1.16)$$

where W couples the right-handed and the left-handed quarks. The Dirac operator $\mathcal{D}$ satisfies the additional chiral symmetry

$$\{\gamma_5, \mathcal{D}\} = 0 \quad \rightarrow \quad \mathcal{D} = -\gamma_5 \mathcal{D} \gamma_5, \quad (1.17)$$

where $\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4 = \text{diag}(1, 1, -1, -1)$ and γ_μ are the Euclidean gamma matrices with $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. This additional constraint expresses that right-handed ψ^R and left-handed ψ^L quarks can be rotated independently $\gamma_5 \psi^{R/L} = \pm \psi^{R/L}$. Moreover, the eigenvalue equation for $\mathcal{D}$ is

$$\mathcal{D} \psi_n = i\lambda_n \psi_n, \quad (1.18)$$

with eigenvalues $\pm i\lambda_n$ occurring in pairs and eigenfunctions ψ_n and $\gamma_5 \psi_n$.

Analogous to the Wigner-Dyson ensemble, the matrix elements of W are either real [$\beta = 1$, chiral Gaussian Orthogonal Ensemble (chGOE)], complex [$\beta = 2$, chiral Gaussian Unitary Ensemble (chGUE)], or quaternion real [$\beta = 4$, chiral Gaussian Symplectic Ensemble (chGSE)]. In addition to the time reversal operator $\mathcal{T}$ mentioned in the Wigner-Dyson ensemble, the chiral ensemble depends on the charge conjugation operator $\mathcal{C}$ in a particular way: the value of $\mathcal{C}^2$ must be the same as $\mathcal{T}^2$. Physically, these three ensembles are used for the following cases

- **Chiral Gaussian Orthogonal Ensemble (chGOE):** The index $\beta = 1$, classified by Cartan's symmetric spaces as "BDI", represents a system that preserves time-reversal (TRS) and spins rotational symmetries (SRS). The matrix elements of W have to be real in this system. Both time-reversal $\mathcal{T}$ and the charge conjugation $\mathcal{C}$ operators satisfy $\mathcal{T}^2 = +1$ and $\mathcal{C}^2 = +1$, respectively.
- **Chiral Gaussian Unitary Ensemble (chGUE):** The $\beta = 2$, classified by Cartan's symmetric spaces as "AIII", indicates systems with time-reversal symmetry broken by a magnetic field or magnetic impurities in the quantum dot. When the TRS is broken, it does not matter if SRS is preserved or broken. Mathematically, each matrix element of W is represented by a complex number. In addition, both time-reversal and particle-hole symmetries are not present in these systems ($\mathcal{T}^2 = \times$, $\mathcal{C}^2 = \times$). However, the chiral symmetry, given by the product $\mathcal{C}\mathcal{T}$, is present in the form $H\mathcal{C}\mathcal{T} = -\mathcal{C}\mathcal{T}H$.
- **Chiral Gaussian Symplectic Ensemble (chGSE):** The $\beta = 4$, classified by Cartan's symmetric spaces as "CII", represents a system with SRS broken but TRS preserved.

Mathematically, the W matrix is written in the quaternionic form $W = a \otimes 1 + i(b \otimes \sigma_x + c \otimes \sigma_y + d \otimes \sigma_z)$. In these systems, the spin-orbit coupling arises due to the relativistic effects between the apparent magnetic field seen from the electron perspective and the magnetic moment of the electron associated with its intrinsic spin. In chGSE, both time-reversal and charge conjugation operators satisfy $\mathcal{T}^2 = -1$ and $\mathcal{C}^2 = -1$, respectively.

All these three ensembles have the additional validity of the chiral symmetry, given by Eq.(1.17), presented in quark's dynamic inside the hadron. Although the random matrix theory for chiral systems was initially introduced in quantum chromodynamics, chirality also naturally emerges in graphene due to the carbon honeycomb lattice [28].

Posteriorly, [29] considered four new classes of symmetries by describing mesoscopic normal-metal–superconducting (NS) systems. Similar Wigner-Dyson, the universality classes are distinguished by their behavior under time reversal and rotation of the electron's spin. However, those classes arise due to the effects of particle-hole conversion at the NS interface, a phenomenon known as *Andreev reflection*. It allows an electron to be converted into a hole in the normal metal region, thereby adding a Cooper pair to (or removing it) from the superconducting condensate. The formalism used to describe this peculiar behavior is given by the Boboliubov de Gennes (BdG) Hamiltonian

$$\hat{H} = \sum_{\alpha\beta}^{2N} \left(h_{\alpha\beta} c_{\alpha}^{\dagger} c_{\beta} + \frac{1}{2} \Delta_{\alpha\beta} c_{\alpha}^{\dagger} c_{\beta}^{\dagger} + \frac{1}{2} \Delta_{\alpha\beta}^* c_{\beta} c_{\alpha} \right), \quad (1.19)$$

where α, β are multi-indexes that combine the quasiparticles' orbital and spin quantum numbers. They range from 1 to $2N$, where N is the number of orbital states, and the factor 2 appears due to spin degeneracy. The usual creation and annihilation operators $c_{\alpha}^{\dagger}$ and c_{β} obey the canonical anticommutation relations $c_{\alpha}^{\dagger} c_{\beta} + c_{\beta}^{\dagger} c_{\alpha} = \delta_{\alpha\beta}$. Therefore, the Hamiltonian can be written in the form

$$\hat{H} = \frac{1}{2} \begin{pmatrix} \mathbf{c}^{\dagger} & \mathbf{c} \end{pmatrix} \mathcal{H} \begin{pmatrix} \mathbf{c} \\ \mathbf{c}^{\dagger} \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} h & \Delta \\ -\Delta^* & -h^T \end{pmatrix}. \quad (1.20)$$

Hermiticity requires that h and Δ be hermitian and antisymmetric, respectively

$$\begin{aligned} h = h^{\dagger} &\rightarrow h_{\alpha\beta} = h_{\beta\alpha}^*, \\ \Delta = -\Delta^T &\rightarrow \Delta_{\alpha\beta} = -\Delta_{\beta\alpha}. \end{aligned} \quad (1.21)$$

Even though hermiticity plays a fundamental role in the spectrum of the system, a set of hermitian matrices does not close under commutation, whereas anti-hermitian ones $X := i\mathcal{H}$ do. The particle-hole symmetry is written in the form

$$H = -\mathcal{C}H\mathcal{C}^{-1}, \quad (1.22)$$

where the $\mathcal{C}$ is the charge conjugation operator. Therefore, the particle-hole symmetry, together with the hermiticity conditions Eq. (1.21), can be written in terms of X

$$\begin{aligned} -X^{\dagger} &= X = -\Sigma_x X^T \Sigma_x, \\ \Sigma_x &= \begin{pmatrix} 0 & 1_{2N} \\ 1_{2N} & 0 \end{pmatrix} = \sigma_x \otimes 1_{2N}. \end{aligned} \quad (1.23)$$

Our goal is to impose fundamental symmetries, such as time-reversal and spin rotation symmetries, in the matrix $\mathcal{H}$ with dimension $4N \times 4N$. The time-reversal and spin-rotation combined generate four symmetry classes, classified by Cartan's symmetric spaces as D , $DIII$, C , and CI .

- **Symmetry Class D:** This symmetry class represents a system with neither time-reversal (TRS) nor spin rotational invariance (SRS). Thus, the matrices h and Δ have no additional symmetry properties beyond the particle-hole symmetry given by Eq.(1.23). In this symmetry class, the charge conjugation and time-reversal symmetry operators $\mathcal{C}$ and $\mathcal{T}$ satisfy $\mathcal{C}^2 = +1$ and $\mathcal{T}^2 = \times$, respectively.
- **Symmetry Class C:** This symmetry class corresponds to systems without time-reversal symmetry but with spin-rotation invariance. The anti-hermitian matrix $X = i\mathcal{H}$ is decomposed into two commuting sub-blocks. One corresponds to spin-up particles and spin-down holes, and the other to spin-down particles and spin-up holes. For a physical description, it is sufficient to focus on one of them, spin-up particles and spin-down holes $X_\uparrow$ for example

$$X_\uparrow = \begin{pmatrix} a & b \\ c & -a^T \end{pmatrix}. \quad (1.24)$$

The condition Eq.(1.23), the antihermiticity of X and the commutation of X with the generators of spin rotations $J_k = (E_{pp} \otimes \sigma_k - E_{hh} \otimes \sigma_k^T) \otimes 1_N$ can be summarized by the equation

$$-X^\dagger = X = -\Sigma_y X^T \Sigma_y, \quad \Sigma_y = \sigma_y \otimes 1_N. \quad (1.25)$$

The peculiar structure of Eq.(1.25) is associated with charge conjugation operator $\mathcal{C}^2 = -1$. Since the time-reversal symmetry is broken, then $\mathcal{T}^2 = \times$.

- **Symmetry Class DIII:** We now consider systems with time-reversal symmetry but without spin-rotation invariance. Beyond the particle-hole symmetry imposed by the condition Eq.(1.23), the time-reversal operator $\mathcal{T}$ satisfies $\mathcal{T}^2 = -1$ and acts on the BdG-Hamiltonian of these systems by

$$\mathcal{T} : \mathcal{H} \mapsto \tau \mathcal{H}^* \tau^{-1}, \quad (1.26)$$

where $\tau = 1_2 \otimes i\sigma_y \otimes 1_N$. Thus, using $X = i\mathcal{H}$, a time-reversal invariant system is subject to the additional constraint $X = \tau X^T \tau^{-1}$. The charge conjugation operator $\mathcal{C}$ satisfies $\mathcal{C}^2 = +1$ in this class.

- **Symmetry Class CI:** Finally, the symmetry class CI corresponds to systems with both time-reversal and spin-rotation symmetries. Recall the spin-rotation invariance in symmetry class C causes the BdG-Hamiltonian $X = i\mathcal{H}_\uparrow$ to obey the relations $-X^\dagger = X = -\Sigma_y X^T \Sigma_y$. The additional time-reversal symmetry acts on the single spin Hamiltonian $X = i\mathcal{H}_\uparrow$ in the simplified form $\mathcal{T} : X \mapsto X^T$. Beyond the condition Eq.(1.25), time-reversal symmetry imposes the Hamiltonian X to be symmetric $X = X^T$. In this symmetry class, the charge conjugation and time-reversal symmetry operators $\mathcal{C}$ and $\mathcal{T}$ satisfy $\mathcal{C}^2 = -1$ and $\mathcal{T}^2 = +1$, respectively.

The tenfold-way Altland-Zirnbauer classification is summarized in Table.1.1. The $\mathcal{T}^2$ in the Table.1.1 represents the symmetry of the Hamiltonian, $\mathcal{T}H = H\mathcal{T}$, under time-reversal operation. In a similar way, the $\mathcal{C}^2$ stands for the symmetry of the Hamiltonian, $\mathcal{C}H = -H\mathcal{C}$, under charge conjugation operation. The anticommutation of the Hamiltonian with the new operator $\mathcal{C}\mathcal{T}$, which is the product of charge conjugation and time-reversal operators, provides a special symmetry $H\mathcal{C}\mathcal{T} = -\mathcal{C}\mathcal{T}H$, known as chiral symmetry of the Hamiltonian.

Symmetry class		$\mathcal{T}^2 =$	$\mathcal{C}^2 =$	$H\mathcal{C}\mathcal{T} = -\mathcal{C}\mathcal{T}H, (\mathcal{C}\mathcal{T})^2 =$
A	Unitary	$\times$	$\times$	$\times$
AI	Orthogonal	$+1$	$\times$	$\times$
AII	Symplectic	-1	$\times$	$\times$
$AIII_v$ ($v \in \mathbb{N}$)	Chiral unitary	$\times$	$\times$	$+1$
BDI_v ($v \in \mathbb{N}$)	Chiral orthogonal	$+1$	$+1$	$+1$
CII_v ($v \in \mathbb{N}$)	Chiral symplectic	-1	-1	$+1$
C		$\times$	-1	$\times$
CI		$+1$	-1	-1
D_v ($v = 0, 1$)		$\times$	$+1$	$\times$
$DIII_v$ ($v = 0, 1$)		-1	$+1$	-1

Table 1.1 The tenfold-way classification of the Altland-Zirnbauer Hamiltonian ensembles.

1.6 Shot-Noise and Quantum Entanglement

1.6.1 Shot-Noise

One of the founding fathers of mesoscopic physics Rolf Landauer once said: *The noise is the signal*. While physicists initially aimed to reduce current noise in mesoscopic systems, Rolf Landauer's insight emerged: he recognized that within current fluctuations lies information beyond the scope of time-averaged values. In general, noise is the fluctuation in time of the electrical current or beam intensity in electronic or photonic systems, respectively. These fluctuations can provide information that would not be possible to obtain by performing a simple average over time of the electrical current or beam intensity. Mesoscopic systems present unavoidable sources of noise, which are thermal (Nyquist-Johnson noise) and shot noise [30]. The former is associated with fluctuations in the number of occupations in the states due to thermal agitation [31], and the latter is a consequence of the quantization of the charge [32]. The shot noise become one of the most important topics studied in mesoscopic systems due to its importance in the detection of fractionally charged quasiparticles in fractional quantum Hall (QH) systems, which led to the Nobel Prize in Physics in 1998 for the discovery of the fractional QH effect [33, 34].

Introducing the shot-noise, let V be a bias voltage applied to a conductor, for example, a resistor, as shown in Fig.(1.11). The current I passes through a conductor and reaches a time-resolved high-precision ammeter. In a quantum mechanical description, the current operator

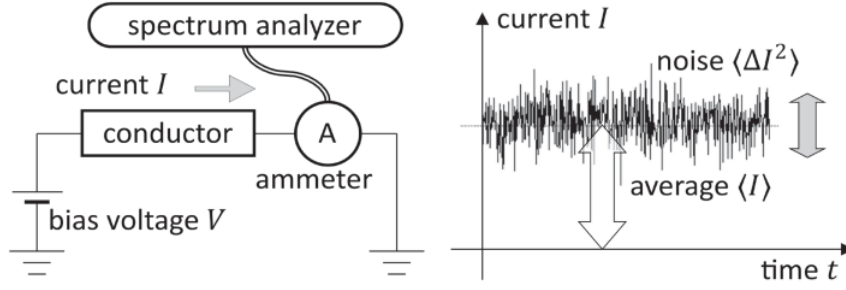


Figure 1.11 The left panel shows the current and noise-measurement setup. The right panel shows the measurement of the current I . The shot noise is a dispersion measure of the current, given by $\langle \Delta I^2 \rangle$, and is obtained using a spectrum analyzer. This Figure is taken from [35].

$\hat{I}(t)$ has a dependence on time with time-averaged value $\langle \hat{I}(t) \rangle$ and its fluctuation $\Delta \hat{I}(t) \equiv \hat{I}(t) - \langle \hat{I}(t) \rangle$. We can define the second-order current-current correlation function as

$$C(t, t') \equiv \langle \Delta \hat{I}(t) \Delta \hat{I}(t') \rangle = \langle \hat{I}(t) \hat{I}(t') \rangle - \langle \hat{I}(t) \rangle \langle \hat{I}(t') \rangle. \quad (1.27)$$

Let us consider the Fourier transform of the correlation function $C(t, t')$ over the measurement time interval $-\tau/2 \leq t \leq \tau/2$:

$$S(\omega) \equiv \lim_{\tau \rightarrow \infty} \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} dt \int_{-\tau/2}^{\tau/2} dt' C(t, t') e^{i\omega(t-t')}. \quad (1.28)$$

We define the quantity $S(\omega)$ as the current-noise power spectral density (PSD) or, in abbreviated form, shot noise power. Our goal in defining the shot noise power is the same as the conductance, given by Eq.(1.1): we want to describe the quantum transport properties, such as conductance and shot noise power, in terms of transmission probabilities T_i ($n = 1, 2, 3 \dots$). For this purpose, the shot noise power S for low-frequencies is described as

$$S = \frac{2e^2}{\pi\hbar} \sum_i T_i(1 - T_i) |eV| = 2e | \langle I \rangle | F, \quad (1.29)$$

where F is called as Fano factor and it represents a proportion between the noise and the conductance

$$F = \frac{\sum_i T_i(1 - T_i)}{\sum_i T_i}. \quad (1.30)$$

Both shot noise power S and Fano factor F are expressed as the sum of noise contributions from N independent open channels, similar to the case of conductance G . In addition, the shot-noise power S is given as the product of the current $| \langle I \rangle |$ and the effective charge of a charge carrier e . Thus, the effective charge can be evaluated by measuring both the current and shot noise.

Beyond the measurements of effective charge, current fluctuations correlations can be a useful tool for measuring quantum entanglement in mesoscopic systems. Nonequilibrium transport and shot noise make quantum entanglement reproducible and measurable in chaotic quantum billiards.

1.6.2 Quantum Entanglement

One of the cornerstone resources to make quantum information processing viable in electronic systems, quantum entanglement is an effect of quantum mechanics with no classical correspondence. If two or more particles are entangled, then their non-local correlations cannot be acquired by concepts of classical mechanics. In 1935, Einstein *et al.* [36] questioned whether the description of physical reality by quantum mechanics could be complete. Unlike a classical system, a quantum system can be in a superposition of possible states, a phenomenon known as quantum superposition. However, when submitted to a measurement, the system collapses into one of the possible states that form the superposition. Then, the objective reality of the system is accessible only when a measurement is performed, destroying the superposition state. Nevertheless, Einstein argued that nature had an objective reality, which did not depend on measurement. This perspective on nature, defended by Einstein, became known as the *EPR paradox*, along with his famous quote: *God does not play dice*.

The discussion about the EPR paradox was adapted by Bohm [37] in the following way. Let us consider an entangled particle pairs source, as depicted in Fig.(1.12), which could be the photons' polarization or electrons' spins. In the case of the electron, both Alice (A) and Bob (B) receive electrons to perform measurements of their spin, such measurements performed are indicated by J_A and J_B in Fig.(1.12), respectively. The quantum state $|\Psi\rangle$ that describes the whole system is written in the form:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}[|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B], \quad (1.31)$$

where $|\uparrow\rangle$ and $|\downarrow\rangle$ stand for spin up and down projections on the z -axis, respectively. The state $|\Psi\rangle$ shows that if Alice measures spin-up projection, then Bob would obtain a spin-down projection and vice-versa.

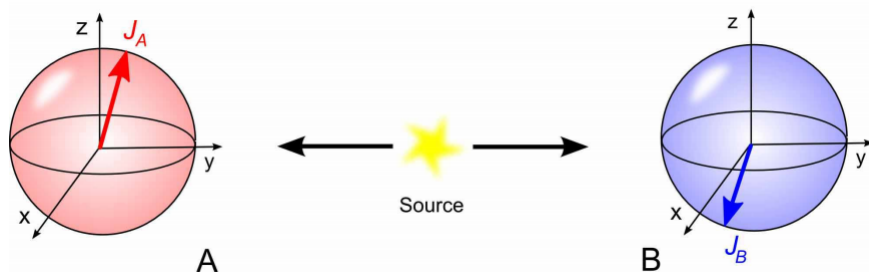


Figure 1.12 Experiment proposed by Bohm for test EPR paradox, a source emits a pair of entangled particles that Alice and Bob will measure their spin's projection with observers J_A and J_B , respectively. The figure was taken from [38].

As mentioned before, Einstein argued that nature had an objective reality that did not depend on measurement. From the Bohm experimental perspective, the electron's spins would already be determined even before the measurement. On the other hand, the Copenhagen interpretation claims that electrons' spins were in a quantum superposition. In addition, the act

of measuring causes a collapse of the quantum state $|\Psi\rangle$, providing an objective reality for electrons' spins.

The fact that Einstein differs from the ideas of the Copenhagen interpretation is associated with some concepts of relativity. The description of the quantum state $|\Psi\rangle$ over space-time is not explicit, so if Alice measures spin-up, for example, then she knows that Bob will obtain spin-down. It can happen even if their measurements are instantaneous and their distance is sufficiently long, avoiding any causal relation between them. Ironically, Einstein called this phenomenon as *Spooky action at a distance* [39]. For Einstein, the correlation between the measurements would be intermediate by a physical entity, such as an electromagnetic field; however, such fields are limited by the speed of light c .

In 1952, Bohm [40] proposed a theory that aims to describe statistical correlations through variables that we could not obtain physically, whose variables are called hidden variables. The goal is to recover the statistical inference provided by quantum mechanics experiments. However, in 1964, J. Bell [41] shows that a hidden-variable theory was incompatible with quantum mechanics prediction through his inequality, known as Bell's inequality. He shows that quantum mechanics can violate his inequality, conversely of any hidden-variable theory. Such violations reveal the presence of quantum entanglement in the system.

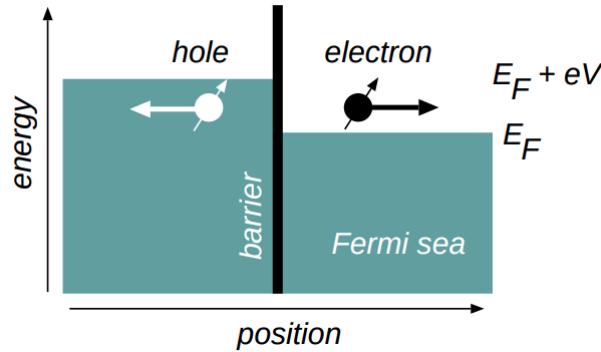


Figure 1.13 Entanglement production in a tunneling process. The voltage difference V creates a particle-hole entangled state due to the scattering in a tunnel barrier separating two metals. The figure was taken from [42].

In mesoscopic systems, entanglement arises due to the particles' interactions and the resulting nonlinearities of their dynamics [42]. In a semiconductor, a voltage difference V , applied between two metals separated by a tunnel barrier, can entangle a particle-hole pair at the same energy. In Fig.(1.13), the tunneling of the electron (e) entangles its respective spin with the spin of a hole (h) left behind. Then, the quantum state $|\psi\rangle$ created from tunneling is given by

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow_h \uparrow_e\rangle - |\downarrow_h \downarrow_e\rangle). \quad (1.32)$$

Then, the Bell inequality becomes a test to distinguish quantum from classical correlations by comparing the correlation along different spin directions.

A variety of entanglement implementations can be found in mesoscopic systems. First, it was found *spin-entangled Bell pair* by spatially separated electron-hole pair created in tunneling events [43]. The entanglement of the spin degree of freedom requires ferromagnetic contacts to measure the spin-resolved correlator needed for Bell inequality tests [44, 45, 46]. In addition to the spin degree of freedom, it is possible to entangle spatial/orbital degree of freedom [47]. The orbital entanglers refer to different spatial degrees of freedom. In this thesis, these spatial degrees of freedom are the energy levels that arise due to the confinement in quantum dots' leads.

Beyond the orbital and spin, the relativistic quantum mechanics and superconductivity enable the entanglement of other degrees of freedom. First, the entanglement of the sub-lattice degree of freedom arises due to the spin-orbit coupling between the spin and sublattice pseudospin of massless Dirac electrons in graphene [48]. Then, the relativistic nature of Dirac's equation provides this new chiral/sub-lattice degree of freedom and its respective entanglement can be obtained experimentally [49].

Finally, we present the application of entanglement in normal-superconductor hybrid structures. Although superconducting circuits already provide violations of Bell's inequality [50], we are interested in the entanglement of systems that support Majorana Modes in superconducting quantum dots [51]. In these systems, the particle-hole symmetry is present and plays a fundamental role in entanglement. In special, experimental setups demonstrate that the particle-hole symmetry enables entangling particles with holes [52]. Therefore, these systems could be a building block for topological quantum computing [53, 54].

1.7 Thesis Framework

Next, in the second chapter, we explore the description of a chaotic quantum billiard. We discuss about the parts of chaotic quantum billiard, such as the chaotic cavity, the leads, and quantum point contacts. Moreover, we describe the whole system by an auto-adjoint extension of quantum states from the waveguides and the scattering cavity. Then, we are able to obtain the scattering matrix and, consequently, quantum transport properties, such as conductance and shot noise power.

In the third chapter, we will explain possible fundamental symmetries present in mesoscopic systems, such as time-reversal, spin rotation, particle-hole, and chiral/sub-lattice symmetries. Moreover, we will classify the Hamiltonian according to the fundamental symmetries combinations and their respective symmetric spaces.

In the fourth chapter, we implement those fundamental symmetries in the Hamiltonian of chaotic ballistic billiards. In special, we investigate the constraints over the Hamiltonian and Scattering matrices' entries due to the presence of symmetries. In this way, we assign the Wigner-Dyson, Chiral, and Altland-Zirnbauer superconducting ensembles to the Schrödinger, Dirac, and Andreev billiards, respectively.

After this, we present the quantum entanglement and Bell's inequality in the fifth chapter. In this part, we show the importance of the violation of Bell's inequality as a test of non-local realism correlations. In addition, we approach how quantum entanglement is obtained in mesoscopic systems. We introduce entanglement quantities, such as concurrence and entanglement of formation, and explore their relation with the violation of Bell's inequality.

In the sixth chapter, we present the results of this thesis. We present numerical simulations using the Random-Matrix Theory (RMT) to investigate transport properties, such as conductance and shot-noise power, as well as entanglement quantifiers, such as concurrence, the entanglement of formation and Bell's inequality for Schrödinger, Dirac, and Andreev billiards. Lastly, we explore the universal conductance fluctuations (UCF) by performing numerical simulations of the conductance as a function of the energy. We investigate the relation between the density of conductance peaks and the length of the correlation for the Schrödinger, Dirac, and Andreev billiards.

In the seventh chapter, we present the conclusions of our numerical results. Initially, we discuss the importance of UCF and how quantum interference correction can be an efficient way to analyze it properly. Next, we approach the entanglement production in all quantum billiards and how the chiral/sub-lattice and particle-hole degrees of freedom influence it. In special, the superconducting quantum dots can be a useful tool for entanglement production due to the Majorana Zero Modes. Finally, we explore the density of conductance peaks in Schrödinger, Dirac, and Andreev billiards. We also analyze the correlation function of conductance and related its correlation length with the density of conductance peaks.

Chaotic Quantum Billiards

2.1 Description of a chaotic quantum billiard

In mesoscopic systems, the quantum transport is performed in a small-scale region called the scattering cavity, shown in Fig.(2.1) as the dot, which is connected with electrons conduction regions called leads or waveguides. The interface between the scattering cavity and the leads is called quantum point contact [55, 56]. Confinement of electrons generates energy quantization; therefore, the scattering cavity presents discrete allowed energies for electronic transport called *resonances*. In chaotic ballistic billiards, both wave functions and resonances are extremely complex to be obtained. However, these billiards present resonances whose statistical behavior is well-defined and can be described by the Random-Matrix Theory (RMT).

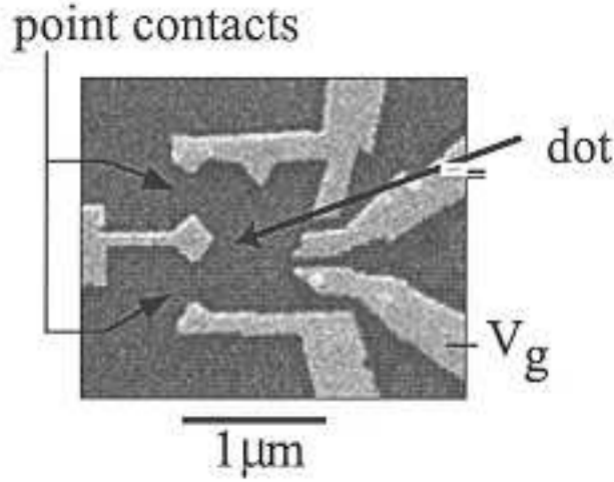


Figure 2.1 Electronic micrograph of a quantum dot, the dark region is a two-dimensional electron gas (2DEG) formed by a heterogeneous structure of GaAs/AlGaAs. The grey regions are electrostatic potentials produced by metallic gates. The Figure was taken from [57].

About the waveguides, the wave functions are always combinations of plane waves. For simplicity, the waveguide is a semi-infinite wire along the x -direction with squared cross-section format in y and z axes, as depicted in Fig.(2.2). The confinement in y and z axes generates energy quantization, which we call *propagating modes*. In these terminals, there will be a set of states denoted by $|n, E\rangle$, where E represents the energy of the state and n is the transverse modes and also can index any degree of freedom, such as sublattice, spin and etc. For a given energy E , we consider N open channels that satisfy the following orthogonality

condition

$$\langle n, E_1 | m, E_2 \rangle = \delta_{mn} \delta(E_1 - E_2), \text{ where } n, m = 1 \dots N. \quad (2.1)$$

Similarly, the scattering cavity has a discrete set of states $\{|\mu\rangle\}$, where $\mu = 1 \dots M$ represents the M resonances.

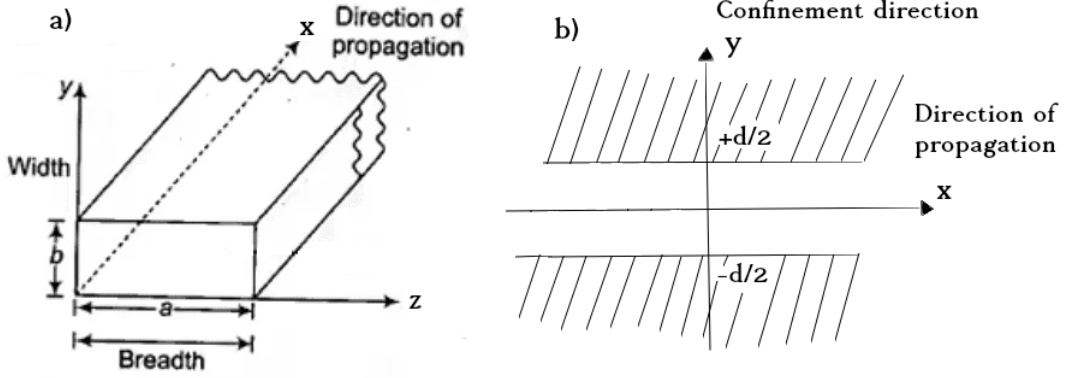


Figure 2.2 Representation of the waveguide in three (a) and two (b) dimensions.

Assuming the existence of propagating modes in waveguides and resonances in scattering cavity, we describe the quantum transport in chaotic quantum billiard by the following approach [58]. Initially, we consider the Hamiltonian of the whole system without coupling between the waveguide and the scattering cavity as follows

$$H_0 = \sum_{\mu, \nu=1}^M (H_{ri})_{\mu\nu} |\mu\rangle \langle \nu| + \sum_{n=1}^N \int dE |n, E\rangle E \langle n, E|, \quad (2.2)$$

where the first and second terms represent the energy of resonant states in the scattering region and the open channels in the waveguide, respectively. The coupling between the resonant states and the open channels is written as

$$V = \sum_{\mu\nu} \left(|\mu\rangle \int dE W_{\mu n} \langle n, E| + \text{h.c} \right). \quad (2.3)$$

Thus, the total Hamiltonian is given by $H = H_0 + V$. The Hamiltonian H acts over the Hilbert space $\mathcal{H}$, i.e $H : \mathcal{H} \rightarrow \mathcal{H}$, where the inner product is well-defined and the space is normed with the following Hamiltonian property

$$\langle H\Phi^{(1)} | \Phi^{(2)} \rangle = \langle \Phi^{(1)} | H^\dagger \Phi^{(2)} \rangle, \quad \forall \Phi^{(1)}, \Phi^{(2)} \in \mathcal{H}, \quad (2.4)$$

where $H^\dagger$ is the adjoint operator.

The coefficient $W_{\mu n}$ describes the coupling in quantum point contact between the resonant states and open channels. The flow of electrons is controllable through the width L and length A , depicted in Fig.(2.3 a)).

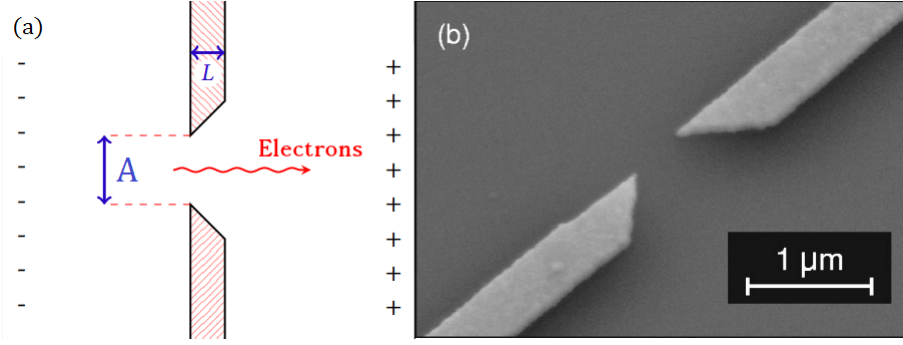


Figure 2.3 (a) Representation of quantum point contact [59]. (b) Quantum point contact captured by electronic micrography [60].

Before reaching the quantum point contact, the dynamic of electrons in the waveguide is described by the Schrödinger equation

$$-\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi(x, y) = \frac{(\hbar k)^2}{2m} \psi(x, y), \quad (2.5)$$

where m is the electron mass and $\hbar k$ is its momentum. Considering the two dimension waveguide, shown in Fig.(2.2 b)), the state $\psi(x, y)$ assumes free particle solution in x -axis and confined behavior in y -axis, that is $\psi(x, y = \pm d/2) = 0$. The normalized free particle solution for the x -axis is

$$\psi_n(x) = \frac{1}{\sqrt{2\pi\hbar^2 k_n/m}} [A_n e^{-ik_n x} + B_n e^{ik_n x}], \quad (2.6)$$

while the confined solution in y -axis is

$$\phi_n(y) = \sqrt{\frac{2}{d}} \sin \left[\left(\frac{n\pi}{d} \right) \left(y + \frac{d}{2} \right) \right], \quad n = 1, \dots, M. \quad (2.7)$$

Thus, the general solution $\Psi(x, y) = \psi(x)\phi(y)$ has total energy given by

$$\frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 k_x^2}{2m} + \frac{\hbar^2 k_y^2}{2m}. \quad (2.8)$$

Due to the confinement in y -axis, the wavenumber is discretized $k_y = (n\pi/d)$. Therefore, k_x is written as

$$k_{xn} = \left[k^2 - \left(\frac{n\pi}{d} \right)^2 \right]^{1/2}. \quad (2.9)$$

Once we are interested in quantum transport phenomena, the k_{xn} must be real, avoiding exponential decay of probabilities in the waveguides. Thus, as the maximum value of n is N , quantum transport imposes

$$\frac{kd}{\pi} \geq N, \quad (2.10)$$

thus the number of open channels is related to the incident electron wavenumber k and the lead width d . Therefore, it is possible to determine the number of open channels N in the lead by adjusting the incident electron energy E_k , and consequently the wavenumber k , and its width d .

Defined the wave functions in the leads and assumed the existence of M resonances in the scattering cavity, we can express the state of the whole system Φ as an auto-adjoint extension of both waveguides $\Psi \in L^2(\mathbb{R}^+, \mathbb{C}^N)$ and scattering cavity $u \in \mathbb{C}^M$ Hilbert spaces cavity

$$\Phi = (u, \Psi)^T, \text{ where } \Phi \in L^2(\mathbb{R}^+, \mathbb{C}^N) \oplus \mathbb{C}^M. \quad (2.11)$$

The Hamiltonian operator is written as

$$\hat{H} = \begin{pmatrix} \hat{H}_{ri} & \hat{W}^{(1)} \\ \hat{W}^{(2)} & \hat{H}_c \end{pmatrix}, \quad (2.12)$$

where $\hat{H}_{ri}$ represents the Hamiltonian of the scattering cavity, whereas the $\hat{H}_c$ represents the Hamiltonian of the leads

$$\hat{H}_c = -\frac{\hbar^2}{2m} \text{diag} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \dots, \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \quad (2.13)$$

and the $\hat{W}$'s represent the coupling contacts. Admitting that the coupling is local [61, 62], we set $\hat{W}^{(2)} = 0$ and

$$\hat{W}^{(1)}\Psi = \int_{-d/2}^{d/2} dy W(y) \Psi(0, y) = \hat{\omega} \psi(0), \quad (2.14)$$

where $\Psi(0, y) = \phi_n(y) \psi(0)$ and entries of $\hat{\omega}$ as

$$\omega_{\mu n} = \int_{-d/2}^{d/2} dy W_{\mu n}(y) \phi_n(y). \quad (2.15)$$

Hence, the Hamiltonian $\hat{H}$ acts over the extended Hilbert space state Φ in the form

$$\hat{H}\Phi = (\hat{H}_{ri}u + \hat{\omega}\psi(0), \hat{H}_c\Psi)^T, \quad (2.16)$$

where the inner product is

$$\langle \Phi_1 | \Phi_2 \rangle = u_1^\dagger u_2 + \langle \Psi_1 | \Psi_2 \rangle, \quad \langle \Psi_1 | \Psi_2 \rangle = \int_{-d/2}^{d/2} dy \int_0^\infty dx \Psi_1^\dagger \Psi_2. \quad (2.17)$$

From now on, we consider the hermiticity condition of Hamiltonian. First, we determine that the energy spectrum of the scattering region is made out of real eigenvalues, i.e., we impose that the Hamiltonian H_{ri} is hermitian. Hence, we obtain

$$\begin{aligned} \langle \hat{H}\Phi_1 | \Phi_2 \rangle - \langle \Phi_1 | \hat{H}\Phi_2 \rangle &= \psi_1^\dagger(0) \hat{\omega}^\dagger u_2 + \langle \hat{H}_c \Psi_1 | \Psi_2 \rangle \\ &\quad - u_1^\dagger \hat{\omega} \psi_2(0) - \langle \Psi_1 | \hat{H}_c \Psi_2 \rangle. \end{aligned} \quad (2.18)$$

Performing an integration by parts on $\langle \hat{H}_c \Psi_1 | \Psi_2 \rangle$ and $\langle \Psi_1 | \hat{H}_c \Psi_2 \rangle$, we have

$$\begin{aligned} \langle \hat{H} \Phi_1 | \Phi_2 \rangle - \langle \Phi_1 | \hat{H} \Phi_2 \rangle = & \psi_1^\dagger(0) \hat{\omega}^\dagger u_2 + \frac{\hbar^2}{2m} \left(\frac{\partial \psi^\dagger}{\partial x} \right) \Big|_{x=0} \psi_2(0) \\ & - u_1^\dagger \hat{\omega} \psi_2(0) - \frac{\hbar^2}{2m} \psi_1^\dagger(0) \left(\frac{\partial \psi_2}{\partial x} \right) \Big|_{x=0}. \end{aligned} \quad (2.19)$$

We observe that the Hamiltonian $\hat{H}$ is not self-adjoint. In Ref.[62] is considered constraints of the wavefunction around $x = 0$ that turns the Hamiltonian $\hat{H}$ self-adjoint. For our proposal, let us consider the contour condition

$$\hat{\omega}^\dagger u = \frac{\hbar^2}{2m} \left(\frac{\partial \psi}{\partial x} \right) \Big|_{x=0}. \quad (2.20)$$

An immediate consequence is the following eigenvalue equation

$$\hat{H} \Phi = (\hat{H}_{ri} u + \hat{\omega} \psi(0), \hat{H}_c \Psi)^T = E(u, \Psi)^T, \quad (2.21)$$

where the first component satisfies

$$u = (E - \hat{H}_{ri})^{-1} \hat{\omega} \psi(0). \quad (2.22)$$

Substituting Eq.(2.22) in Eq.(2.20), we have

$$\hat{\omega}^\dagger (E - \hat{H}_{ri})^{-1} \hat{\omega} \psi(0) = \frac{\hbar^2}{2m} \left(\frac{\partial \psi}{\partial x} \right) \Big|_{x=0}. \quad (2.23)$$

Using $\psi(x)$ from Eq.(2.6) in the above one and defining

$$\hat{W} = \sqrt{\frac{2m}{\pi \hbar^2}} \hat{\omega} k^{-1/2}, \quad (2.24)$$

where k é given by $k \equiv \text{diag}(k_1, k_2, \dots, k_N)$. Basically, the physical meaning of the operator $\hat{W}$ is to couple the resonant states in the scattering cavity with the propagating modes in the waveguides. This operator satisfies the special condition

$$\hat{W}_l^\dagger \hat{W}_{l'} = \frac{M \Delta}{\pi^2} \hat{\omega}_{ll'} \delta_{ll'}. \quad (2.25)$$

This condition claims that direct processes are not allowed, i.e processes whose electrons transmit from one waveguide to another without passing through the scattering region. The M and Δ represent the number of resonances and the average of the nearest neighbor distribution of Hamiltonian H_{ri} eigenvalues, also called the mean level spacing, respectively.

Proceeding forward, we can express the scattering coefficients B 's in terms of incident coefficients A 's from Eq.(2.6)

$$B = 1 - i\pi \hat{W}^\dagger (E - \hat{H}_{ri})^{-1} \hat{W} 1 + i\pi \hat{W}^\dagger (E - \hat{H}_{ri})^{-1} \hat{W} A, \quad (2.26)$$

obtaining then the scattering matrix S by the relation $B = SA$. The scattering matrix S can be written in the form $S = (1 - iF)(1 + iF)^{-1}$, where $F = \pi\hat{W}^\dagger(E - \hat{H}_{ri})^{-1}\hat{W}$. Therefore, following the appendix A, the final form of S matrix is given by

$$S = 1 - 2i\pi\hat{W}^\dagger \frac{1}{E - \hat{H}_{ri} + i\pi\hat{W}\hat{W}^\dagger} \hat{W}. \quad (2.27)$$

This equation is known as the Mahaux-Weidenmüller formula. All relevant observables are described in terms of the transmission eigenvalues of the S -matrix. The Mahaux-Weidenmüller formula connects quantum transport properties with the details of the energy spectrum in the chaotic quantum billiard.

2.2 Scattering Matrix and Conductance

Once obtained the scattering matrix, we express the amplitudes of probabilities of scattering waves through the amplitudes of probabilities from incident waves. For example, let us consider the case when there are two waveguides, at the left and the right of the scattering region

$$\begin{aligned} \Psi(x_L, y_L, z_L) &= \sum_n \frac{1}{\sqrt{2\pi\hbar v_n}} \phi_n(y_L, z_L) \left[a_{Ln} e^{ik_x^n x_L} + b_{Ln} e^{-ik_x^n x_L} \right], \\ \Psi(x_R, y_R, z_R) &= \sum_m \frac{1}{\sqrt{2\pi\hbar v_m}} \phi_m(y_R, z_R) \left[a_{Rm} e^{-ik_x^m x_R} + b_{Rm} e^{ik_x^m x_R} \right]. \end{aligned} \quad (2.28)$$

The coefficients a_{Ln} , a_{Rm} are the amplitudes of incident particles, whereas b_{Ln} , b_{Rm} are the amplitudes of the waves transmitted through or reflected back from the scattering region. These coefficients are not independent, the amplitude of the wave reflected from the obstacle linearly depends on the amplitudes of incoming waves in all the channels

$$b_{\alpha l} = \sum_{\beta=L,R} \sum_{l'} s_{\alpha l, \beta l'} a_{\beta l'}, \quad \alpha = L, R, \quad l = n, m. \quad (2.29)$$

The proportionality coefficients are combined into a $(N_L + N_R) \times (N_L + N_R)$ scattering matrix S , given by the Mahaux-Weidenmüller formula Eq.(2.27). It has the following block structure

$$S = \begin{pmatrix} S_{LL} & S_{LR} \\ S_{RL} & S_{RR} \end{pmatrix} \equiv \begin{pmatrix} \hat{r} & \hat{t}' \\ \hat{t} & \hat{r}' \end{pmatrix}. \quad (2.30)$$

The reflection matrix $\hat{r}$ has dimension $N_L \times N_L$ and describes the reflection of the waves coming from the left. For example, $|r_{nn'}|^2$ is the probability of an electron coming from the left in the transverse channel n' being reflected in the channel n . The $N_R \times N_L$ transmission matrix $\hat{t}$ is responsible for the transmission through the scattering region. For example, $|t_{mn}|^2$ is the probability of an electron coming from the left in the transverse channel n being transmitted in the channel m at the right. The $N_R \times N_R$ reflection matrix $\hat{r}'$ describes the reflection of electrons coming from the right. Finally, $N_L \times N_R$ transmission matrix $\hat{t}'$ describes the transmission of electrons coming from the right to the left waveguide.

Since we are dealing with probabilities, any scattering matrix satisfies the unitarity condition $S^\dagger S = \hat{1}$ due to the conservation of probability. The diagonal elements of $S^\dagger S$ is given by

$$(S^\dagger S)_{nn} = \sum_{n'} |r_{nn'}|^2 + \sum_m |t_{mn}|^2 = 1, \quad (2.31)$$

where it represents the total probability of an electron in channel n being either reflected or transmitted to any channel. The sum $R_n = \sum_{n'} |r_{nn'}|^2$ represents the probability of being reflected to channel n from all possible starting channels n' . The sum $T_n = \sum_m |t_{mn}|^2$ determines the probability of an electron being transmitted from the n channel at the left to all possible channels m at the right. The relation between R_n and T_n is ensured by the unitarity condition

$$1 - (\hat{r}^\dagger \hat{r})_{nn} = 1 - \sum_{n'} |r_{nn'}|^2 = 1 - R_n = T_n = \sum_m |t_{mn}|^2 = (\hat{t}^\dagger \hat{t})_{nn}. \quad (2.32)$$

The sum of the transmission eigenvalues T_n represents the conduction of electrons from the left to the right lead. Therefore, the conductance of electrons G is defined by

$$G = G_0 \text{Tr} t t^\dagger = G_0 \sum_n T_n, \quad (2.33)$$

where the G_0 is the quantum of conductance. The Eq.(2.33) is known as the (two-terminal) *Landauer formula* [63]. Remember here that we are dealing with the case of zero temperature in this work. The relation between the conductance, current, and voltage is given by

$$\langle I \rangle = GV, \quad G = \frac{2e^2}{h} \sum_n T_n, \quad (2.34)$$

where factor 2 represents the spin degeneracy at zero magnetic fields. In some experimental setups, the electric current through a circuit is expressed in terms of the current-voltage I-V curve. In this case, the slope of the current in relation to the voltage dI/dV is referred to as differential conductance G .

As mentioned in the introduction, a notable phenomenon in mesoscopic devices is universal conductance fluctuations (UCF) reasoned by wave characteristics of quantum mechanics. These fluctuations are a remarkable signature of chaos in the quantum scattering processes. The study of fluctuations is, therefore, a powerful source of information on the detailed structure of a statistical system. Initially addressed in nuclear physics, correlation functions are introduced for the cross section (analogous to conductance) to study these fluctuations on a systematic basis [64]. In Fig.(2.4 a)), the differential cross section was measured for the light-ion reaction $p + {}^{35}\text{Cl} \rightarrow \alpha + {}^{32}\text{S}$ as a function of the incident proton energy [65]. To properly analyse the fluctuations, Erickson calculated the cross-section correlation functions [64]

$$C(\Delta E) = \langle (\sigma(E + \Delta E) - \langle \sigma \rangle)(\sigma(E) - \langle \sigma \rangle) \rangle = \langle \sigma(E + \Delta E) \sigma(E) \rangle - \langle \sigma \rangle^2. \quad (2.35)$$

In Fig.(2.4 b)), the autocorrelation function of the cross-section in Fig.(2.4 a)) is fitted by a Lorentzian function with width γ

$$C(\Delta E) = \frac{1}{1 + (\Delta E / \gamma)^2}, \quad (2.36)$$

whose width corresponds to the lifetime $\hbar/\gamma$ of the excited compound nucleus ^{36}Ar . The decay rate $\gamma_c/\hbar$ of a ^{36}Ar resonance level in a specific channel c is given by the frequency of attempted decays times the transmission coefficient $\tau_H^{-1}T_c$, where $\tau_H = \hbar/\Delta$ provides the residence time of the particle inside the chaotic cavity (known as Heisenberg time). The total width, $\gamma = \sum_c \gamma_c$, depends on mean level spacing Δ of nuclei resonances and the number of open channels N

$$\gamma = \frac{\Delta}{2\pi} \sum_c^N T_c. \quad (2.37)$$

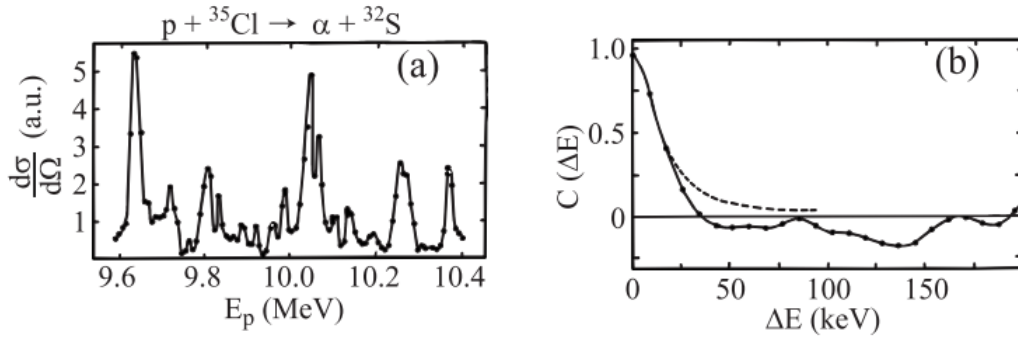


Figure 2.4 Ericson fluctuations in the compound nucleus. (a) The differential cross section $d\sigma/d\Omega$ at $\theta = 170^\circ$ for the reaction $p + {}^{35}\text{Cl} \rightarrow \alpha + {}^{32}\text{S}$ vs the incoming proton energy E_p . (b) The energy autocorrelation function $C(\Delta E)$ (solid line) of the cross-section in panel (a) [66], the dashed line is a Lorentzian fit. The figure was taken and modified from [67].

Similar to the differential cross section for compound nucleus reactions, the conductance G , measured in an open chaotic dot as a function of the electron's Fermi momentum k_F , also presents fluctuations, as depicted in Fig.(2.5). Although the average conductance increases as k_F increases, the magnitude of the fluctuations remains constant ($\sim e^2/h$), as shown in Fig.(2.5 a)). In part Fig.(2.5 b)), the power spectrum $S(f_V)$ of the panel a) follows an exponential shape in a log-linear scale. The conductance autocorrelation function of G can be obtained by applying the inverse Fourier transform, where the conjugate variable of the frequency f_V is the gate voltage (equivalent to the Fermi energy). In this case, the conductance G autocorrelation function corresponds to a Lorentzian shape

$$C(\Delta E) = \langle G(E + \Delta E)G(E) \rangle - \langle G \rangle^2 = \frac{1}{1 + (\Delta E/\gamma)^2}. \quad (2.38)$$

Even though correlation functions provide a study of fluctuations systematically, all statistical properties, such as mean, variance, root mean square, and correlation functions, require performing averages over an ensemble. This task becomes arduous depending on the number of experimental parameters that the setup makes available. For this purpose, inspired by methods of the theory of stochastic process, [68] were able to extract useful statistical information from the analysis of a single cross-section curve in nuclear physics. This method relates the average number of maxima (or minima) in cross-section per unit energy range $\langle \rho_\epsilon \rangle$ with the

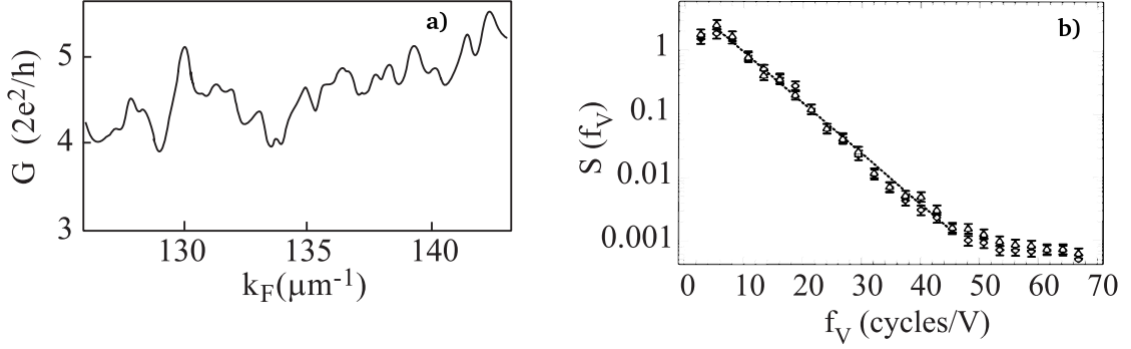


Figure 2.5 Conductance fluctuations in an open chaotic quantum dot. a) The conductance vs the incoming electron's Fermi momentum k_F . b) The power spectrum $S(f_V)$ of the conductance in panel (a), the dashed line is a linear fit in a log-linear scale. The figure was taken and modified from [67].

auto-correlation length γ of a single realization. The relation between the number of maxima and the auto-correlation length is given by

$$\langle \rho_\varepsilon \rangle = \alpha / \gamma, \quad (2.39)$$

where α assumes the value 0.5.

Analogous to nuclear physics, proposals to associate the average number of extreme points in the dimensionless conductance, T , as a function of an arbitrary external parameter Z with its respective auto-correlation length γ_Z for quantum dots have emerged [69]. In mesoscopic physics, the parameter Z can be associated with an applied gate voltage causing deformation in the quantum dot, an external magnetic field B , the Fermi energy E_F , etc. Basically, in the case where the external parameter Z is the electron energy ε and for many channels $N \gg 1$, the correlation function is a Lorentzian and gives

$$\langle \rho_\varepsilon \rangle = \frac{\sqrt{3}}{\pi\gamma} \approx \frac{0.55}{\gamma}. \quad (2.40)$$

Therefore, the analysis of $\langle \rho_Z \rangle$ has become a method that brings both theoretical and experimental simplifications for nuclear and mesoscopic physics. It provides the autocorrelation length only by doing a maximum count of a single realization, avoiding massive data storage. In this thesis, we show numerical simulations which recover results available in the literature about universal conductance fluctuations and conductance correlation functions. Moreover, we analyse the influence of the tunneling barriers on correlation functions and the relation between maxima density and autocorrelation length.

2.3 Shot Noise

Together with the conductance, another important property for quantum transport in mesoscopic devices is the shot noise power. As mentioned in the introduction, the shot noise is

described as the correlation function of current fluctuations $\Delta\hat{I}(t) = \hat{I}(t) - \langle\hat{I}(t)\rangle$

$$C(t, t') \equiv \langle\Delta\hat{I}(t)\Delta\hat{I}(t')\rangle = \langle\hat{I}(t)\hat{I}(t')\rangle - \langle\hat{I}(t)\rangle\langle\hat{I}(t')\rangle. \quad (2.41)$$

First, we need to describe the current operator $\hat{I}$ as a function of the field operators in the leads, then we introduce the Hamiltonian of the electrons in the left lead in the second quantization

$$H = \sum_k \hbar v_F (k - k_F) a_{L,k}^\dagger a_{L,k} + \sum_k (-\hbar v_F) (k + k_F) b_{L,k}^\dagger b_{L,k}, \quad (2.42)$$

where $a_{L,k}$ and $b_{L,k}$ represent the incident and reflected electrons in the left lead. Hence, the current operator in the left (L) lead is

$$\hat{I}(x) = \frac{\hbar e}{2im} \left(\hat{\psi}^\dagger(x) \frac{d\hat{\psi}(x)}{dx} - \frac{d\hat{\psi}^\dagger(x)}{dx} \hat{\psi}(x) \right), \quad (2.43)$$

where $\hat{\psi}(x) = \sum_k (1/\sqrt{L}) \exp(ikx) c_k$ is the field operator, L is the length of the lead and c_k stands for both income a_k and outcome b_k propagating modes. The current operator in the left lead is written as

$$\hat{I}(x) = \frac{ev_F}{L} \sum_{k,k'} (a_{L,k}^\dagger a_{L,k'} - b_{L,k}^\dagger b_{L,k'}) e^{i(k'-k)x}. \quad (2.44)$$

In the interface between the lead and the chaotic scattering cavity ($x = 0$), the current operator becomes

$$\hat{I}(x=0) = \frac{ev_F}{L} \sum_{k,k'} (a_{L,k}^\dagger a_{L,k'} - b_{L,k}^\dagger b_{L,k'}). \quad (2.45)$$

Therefore, we express the current operator written in terms of the scattering matrix blocks

$$\hat{I}_L = \frac{ev_F}{L} \sum_{\alpha=L,R} \sum_{\beta=L,R} \sum_{k,k'} a_{\alpha,k}^\dagger A_L^{\alpha\beta}(k, k') a_{\beta,k'}, \quad (2.46)$$

$$A_L^{\alpha\beta}(k, k') = \delta_{L,\alpha} \delta_{L,\beta} - [s^{L\alpha}(k)]^* s^{L\beta}(k'). \quad (2.47)$$

Since we are interested in calculating the correlation function of current fluctuations in time, we define the time evolution of the current operator in the Heisenberg representation

$$\hat{I}_L(t) = \exp\left(\frac{iHt}{\hbar}\right) \hat{I}_L \exp\left(-\frac{iHt}{\hbar}\right). \quad (2.48)$$

Although the current-current correlation function is calculated by Eq.(2.41), our main goal is to obtain the current-noise power spectral density (PSD) $S(\omega)$, given by

$$S(\omega) \equiv \lim_{\tau \rightarrow \infty} \int_{-\tau_2}^{\tau_2} dt \int_{-\tau_2}^{\tau_2} dt' C(t, t') e^{i\omega(t-t')}. \quad (2.49)$$

If the Hamiltonian is time-independent, $C(t, t')$ depends only on the time difference $\Delta t = t - t'$ and $C(t, t')$ equals zero for large Δt . Thus, the noise becomes

$$\begin{aligned} S(\omega) &= \lim_{\tau \rightarrow \infty} \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} dt' \int_{-\infty}^{\infty} d(\Delta t) C(\Delta t) e^{i\omega \Delta t}, \\ &= 2 \int_{-\infty}^{\infty} d(\Delta t) C(\Delta t) e^{i\omega \Delta t}. \end{aligned} \quad (2.50)$$

In this thesis, we focus on the noise at low frequencies $S(0)$, then the zero-frequency noise is expressed as

$$S \equiv S(0) = 2 \int_{-\infty}^{\infty} dt \left[\langle \hat{I}_L(t) \hat{I}_L(0) \rangle - \langle \hat{I}_L(t) \rangle \langle \hat{I}_L(0) \rangle \right]. \quad (2.51)$$

By using the Heisenberg representation of current given by Eq.(2.48) into Eq.(2.50), the zero-frequency noise becomes

$$\begin{aligned} S &= 2 \int_{-\infty}^{\infty} dt \left(\frac{ev_F}{L} \right)^2 \sum_{k, k', k'', k'''} \sum_{\alpha, \beta, \alpha', \beta'} \\ &A_L^{\alpha\beta}(k, k') A_L^{\alpha'\beta'}(k'', k''') \left[\langle a_{\alpha, k}^\dagger a_{\beta, k'} a_{\alpha', k''}^\dagger a_{\beta', k'''} \rangle \right. \\ &\left. - \langle a_{\alpha, k}^\dagger a_{\beta, k'} \rangle \langle a_{\alpha', k''}^\dagger a_{\beta', k'''} \rangle \right] \exp \left[\frac{i(\epsilon_k - \epsilon_{k'})t}{\hbar} \right]. \end{aligned} \quad (2.52)$$

For finite temperature, the statistical average of $a_{\alpha, k}^\dagger a_{\beta, k'}$ is given by

$$\langle a_{\alpha, k}^\dagger a_{\beta, k'} \rangle = \delta_{\alpha\beta} \delta_{k, k'} f_\alpha(k), \quad (2.53)$$

where $f_\alpha(k)$ is the Fermi-Dirac distribution function in the lead α

$$f_\alpha(k) \equiv \frac{1}{\exp[(\epsilon_k - \mu_\alpha)/k_B T_\alpha] + 1}, \quad (2.54)$$

with k_B being the Boltzmann constant and μ_α, T_α stand for the chemical potential and the temperature of the reservoir coupled in the lead α , respectively. Wick's theorem lead to

$$\langle a_{\alpha, k}^\dagger a_{\beta, k'} a_{\alpha', k''}^\dagger a_{\beta', k'''} \rangle - \langle a_{\alpha, k}^\dagger a_{\beta, k'} \rangle \langle a_{\alpha', k''}^\dagger a_{\beta', k'''} \rangle \quad (2.55)$$

$$= \langle a_{\alpha, k}^\dagger a_{\beta', k'''} \rangle \langle a_{\beta, k'} a_{\alpha', k''}^\dagger \rangle \quad (2.56)$$

$$= \delta_{\alpha, \beta'} \delta_{k, k'''} \delta_{\beta, \alpha'} \delta_{k', k''} f_\alpha(k) [1 - f_\beta(k')]. \quad (2.57)$$

The term $[1 - f_\beta(k')]$ in the shot noise power shows the exchange term in the statistical nature of Fermi particles. By replacing the summation of the wavenumbers by integration over the energy, we obtain

$$\begin{aligned} S &= \frac{2e^2}{(2\pi\hbar)^2} \int d\epsilon \int d\epsilon' \int_{-\infty}^{\infty} dt \\ &\sum_{\alpha, \beta} A_L^{\alpha\beta}(\epsilon, \epsilon') A_L^{\beta\alpha}(\epsilon', \epsilon) f_\alpha(\epsilon) [1 - f_\beta(\epsilon')] \exp \left[\frac{i(\epsilon - \epsilon')t}{\hbar} \right]. \end{aligned} \quad (2.58)$$

Using the Dirac's delta definition $\int_{-\infty}^{\infty} dt e^{i(\varepsilon - \varepsilon')t/\hbar} = 2\pi\hbar\delta(\varepsilon - \varepsilon')$, we obtain the general formula for the current noise

$$S = \frac{e^2}{\pi\hbar} \int d\varepsilon \sum_{\alpha,\beta} A_L^{\alpha\beta}(\varepsilon, \varepsilon) A_L^{\beta\alpha}(\varepsilon, \varepsilon) f_{\alpha}(\varepsilon) [1 - f_{\beta}(\varepsilon)]. \quad (2.59)$$

This equation includes both thermal and shot noise formulas. First, the thermal noise can be obtained by considering that the system is in equilibrium, or $|eV| \ll k_B T_{\alpha}$, where the thermal noise dominates over the shot noise. In this case, $eV = 0$,

$$f(\varepsilon)[1 - f(\varepsilon)] = k_B T_{\alpha} \left(-\frac{\partial f}{\partial \varepsilon} \right), \quad (2.60)$$

then

$$\begin{aligned} S &= \frac{e^2 k_B T_{\alpha}}{\pi\hbar} \int d\varepsilon \sum_{\alpha,\beta} A_L^{\alpha\beta}(\varepsilon, \varepsilon) A_L^{\beta\alpha}(\varepsilon, \varepsilon) \left(-\frac{df}{d\varepsilon} \right) \\ &= \frac{2e^2 k_B T_{\alpha}}{\pi\hbar} \int d\varepsilon T(\varepsilon) \left(-\frac{df}{d\varepsilon} \right), \end{aligned} \quad (2.61)$$

where $T(\varepsilon)$ is the probability transmission as a function of the electron's energy for a finite temperature system. The remaining integral is the conductance for finite temperature and, therefore, the thermal-noise formula is written as

$$S = 4k_B T_{\alpha} G. \quad (2.62)$$

For the zero temperature regime, we consider $eV \neq 0$ and $f_L(\varepsilon) = \Theta(-\varepsilon + eV)$ and $f_R(\varepsilon) = \Theta(-\varepsilon)$ with Θ being the Heavside step function

$$\Theta(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases} \quad (2.63)$$

When $eV > 0$, $f_{\alpha}(\varepsilon)[1 - f_{\beta}(\varepsilon)] \neq 0$ holds only for $\alpha = L, \beta = R$ and $0 < \varepsilon < eV$. Hence, the zero-frequency noise S becomes

$$S = \frac{e^2}{\pi\hbar} A_L^{LR}(\varepsilon, \varepsilon) A_L^{RL}(\varepsilon, \varepsilon) \times |eV|. \quad (2.64)$$

Considering that the energy dependence of T is negligibly small, then

$$A_L^{LR} A_L^{RL} = |t|^2 |r|^2 = |t|^2 (1 - |t|^2) = T(1 - T). \quad (2.65)$$

Therefore, the shot noise for one channel case is given by

$$S = \frac{e^2}{\pi\hbar} T(1 - T) |eV|. \quad (2.66)$$

The general formula for the shot noise power S is described as

$$S = \frac{2e^2}{\pi\hbar} \sum_n T_n(1 - T_n)|eV|, \quad (2.67)$$

where the factor 2 represents the spin degeneracy at zero magnetic fields. In this thesis, we are interested in calculating the shot noise power in chaotic quantum billiards, which is the current fluctuations due to the discrete nature of charge carriers. Sometimes, shot noise is referred to as partition noise since it appears in systems where the current is partitioned into transmitted and reflected parts.

Both conductance and shot noise power are important quantities measured in mesoscopic systems. Our interest is to explore the influence of fundamental symmetries on these properties. Therefore, in the next chapter, we will present the possible fundamental symmetries present in electronic mesoscopic systems, in particular the chaotic quantum billiard. In special, we explore antiunitary symmetries, such as time reversal and charge conjugation symmetries, and how it affects the energy spectrum of the Hamiltonian. Finally, we explore the unitary symmetry called chiral and its respective consequences on the energy spectrum.

Time Reversal and Unitary Symmetries

3.1 Time Reversal Symmetry in Classical Systems

Time reversal or, more accurately, reversal of motion is a property that some systems exhibit when their physical description becomes invariant in the face of movement inversion. For example, let's suppose a classical trajectory of a particle subject to a force field, as shown in Fig.(3.1). At $t = 0$, let the particle stop and reverse its motion, that is $\vec{p}|_{t=0} \rightarrow -\vec{p}|_{t=0}$. The particle traverses backward along the same trajectory, and it will be hard to say whether it goes forward or backward in time.

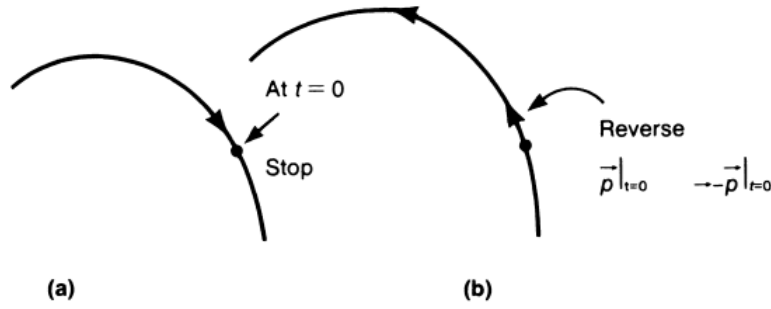


Figure 3.1 a) The classical trajectory stops at $t = 0$. b) Reversed trajectory $\vec{p}|_{t=0} \rightarrow -\vec{p}|_{t=0}$. The figure was taken from [70].

Mathematically, if $\vec{x}(t)$ is a solution to Newton's equation

$$m \frac{d^2 \vec{x}}{dt^2} = -\nabla \vec{V}(x), \quad (3.1)$$

then $\vec{x}(-t)$ is a possible solution in the same force field derivable from the potential V and the system becomes invariant under the transformation

$$t \rightarrow -t, \quad \vec{x} \rightarrow \vec{x}, \quad \vec{p} \rightarrow -\vec{p}, \quad (3.2)$$

making the Hamiltonian an even function of all moments $H(\vec{x}, \vec{p}) = H(\vec{x}, -\vec{p})$. In this case, we say that the Hamiltonian is invariant under time reversal.

In a different situation, the motion of a charged particle in an external magnetic field is not invariant under conventional time reversal. Without a magnetic field, we could not see the difference between an electron beam moving from the left to the right and conversely, as in the

case of Fig.(3.1). However, in the presence of a uniform magnetic field, as shown in Fig.(3.2), the time-reversal invariance of the charged particle is broken. Thus, now it is possible to say whether a beam goes to the left or the right.

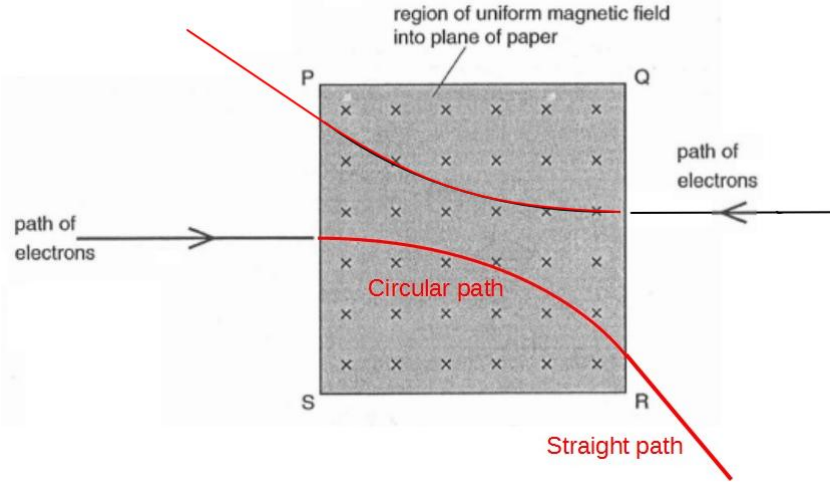


Figure 3.2 The electron trajectories in a region with a uniform magnetic field.

To obtain a quite symmetrical situation when a uniform magnetic field is applied, from a microscopic point of view, we should reverse the current that causes the magnetic field $\vec{B}$

$$\vec{E} \rightarrow \vec{E}, \quad \vec{B} \rightarrow -\vec{B}, \quad \rho \rightarrow \rho, \quad \vec{j} \rightarrow -\vec{j}, \quad \vec{v} \rightarrow -\vec{v}. \quad (3.3)$$

In this way, both Maxwell equations

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho, \quad \vec{\nabla} \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} = \frac{4\pi\vec{j}}{c}, \quad \vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}, \quad (3.4)$$

and Lorentz force equation

$$\vec{F} = e[\vec{E} + (1/c)\vec{v} \times \vec{B}], \quad (3.5)$$

would be invariant under both time reversal conditions $t \rightarrow -t$ and Eq.(3.3).

Hamiltonian systems with no time-reversal invariance, such as the charged particle in an external magnetic field, must not be confused with dissipative systems. According to Liouville's theorem, all Hamiltonian motions conserve phase-space volumes, except systems in contact with dissipative entities where such phase-space volumes contract in time. Systems with and without time-reversal invariance never attracted much attention in classical physics, however, this invariance has a significant difference in quantum systems.

3.2 Time Reversal Symmetry in Quantum Systems

3.2.1 Spinless Systems

From now on, we describe the time-reversal and other antiunitary symmetries by following the Ref.[71]. In particular, we are interested in how time-reversal symmetry can be expressed

in quantum systems. The time-reversal invariance plays an important role in the description of a quantum system. Our starting point will be spinless systems governed by the Schrödinger equation

$$i\hbar \frac{\partial \psi(\vec{x}, t)}{\partial t} = H\psi(\vec{x}, t). \quad (3.6)$$

We say that a system is a time-reversal invariant if, for any given solution $\psi(\vec{x}, t)$, there is another one, $\psi'(\vec{x}, t')$, with $t' = -t$ and ψ' uniquely related to ψ . The conventional Hamiltonian for a spinless particle assumes the form

$$H(\vec{x}, \vec{p}) = \frac{\vec{p}^2}{2m} + V(\vec{x}), \quad V(\vec{x}) = V^*(\vec{x}). \quad (3.7)$$

Thus, the time reversal invariance is obtained when

$$\begin{aligned} t &\rightarrow -t, \quad \vec{x} \rightarrow \vec{x}, \quad \vec{p} \rightarrow -\vec{p}, \\ \psi(\vec{x}) &\rightarrow \psi^*(\vec{x}) = K\psi(x). \end{aligned} \quad (3.8)$$

Hence, the time-reversal symmetry claims that if $\psi(\vec{x}, t)$ solves Eq.(3.6), then $\psi'(\vec{x}, t) = K\psi(\vec{x}, -t)$ also does. Therefore, for a spinless system, the time-reversal operator $\mathcal{T}$ is given by the complex conjugation $\mathcal{T} = K$, which satisfies

$$K^2 = 1, \quad (3.9)$$

i.e, it equals its inverse, $K = K^{-1}$. The time-reversal operator $\mathcal{T}$ satisfies both anti-linearity and anti-unitarity conditions

$$\begin{cases} K(c_1\psi_1(\vec{x}) + c_2\psi_2(\vec{x})) = c_1^*K\psi_1(\vec{x}) + c_2^*K\psi_2(\vec{x}), \\ \langle K\psi | K\phi \rangle = \langle \psi | \phi \rangle^* = \langle \phi | \psi \rangle, \end{cases} \quad (3.10)$$

where the latter comes from the requirement of conservation probabilities under complex conjugation transformation. In general, any antiunitary operator θ can be written as

$$\theta = UK, \quad (3.11)$$

where U is a unitary operator and K is the complex-conjugate operator which acts on any coefficient that multiplies a ket. The antiunitary operator, given by Eq.(3.11), is a useful tool for any state representation. In our case, time-reversal operators $\mathcal{T}$ are antiunitary and have the simple form $\mathcal{T} = K$ for spinless particles.

The time-reversal operator $\mathcal{T}$ commutes with the Hamiltonian H in a time-reversal invariant spinless system

$$[H, \mathcal{T}] = 0, \quad \mathcal{T}^2 = +1. \quad (3.12)$$

In these systems, the Hamiltonian H can always be given a real matrix representation and such a representation can be found without diagonalizing H . To see it, let's start by constructing a set of mutually orthogonal vectors ψ_v , such that

$$\psi_1 = a_1\phi_1 + \mathcal{T}a_1\phi_1, \quad (3.13)$$

where ψ_1 is $\mathcal{T}$ invariant, i.e, $\mathcal{T}\psi_1 = \psi_1$, with ϕ_1 and a_1 being a vector and complex number, respectively. Thus, with respect to a $\mathcal{T}$ invariant basis, the Hamiltonian $H = \mathcal{T}H\mathcal{T}$ is real

$$\begin{aligned} H_{\mu\nu} &= \langle \psi_\mu | H \psi_\nu \rangle = \langle \mathcal{T}\psi_\mu | \mathcal{T}H\psi_\nu \rangle^* \\ &= \langle \psi_\mu | \mathcal{T}H\mathcal{T}^2\psi_\nu \rangle^* = \langle \psi_\mu | \mathcal{T}H\mathcal{T}\psi_\nu \rangle^* = H_{\mu\nu}^*. \end{aligned} \quad (3.14)$$

Therefore, the Hamiltonians are generically real matrices.

Our focus lies in Hamiltonian transformations that uphold both Hermiticity and eigenvalue preservation. Such transformations are referred to as canonical transformations, and they are encompassed by the Lie group of real orthogonal matrices, such that $OO^T = 1$, applicable to real Hamiltonians. The requirement of preserving eigenvalues and Hermiticity provides a group of orthogonal transformations that transforms a real matrix H into another real matrix H' in the form $H \mapsto H' = OH O^T$. In special, a $\mathcal{T}$ -invariant Hamiltonian ($\mathcal{T}^2 = +1$) can be diagonalized by the group $SO(N)$, whose elements are unit-determinant orthogonal matrices $\det(O) = +1$. These Hamiltonians belong to the *orthogonal symmetry class O*.

3.2.2 Spin-1/2 Systems

As we mentioned before, the time-reversal operator $\mathcal{T}$ must be antiunitary. This condition is ensured due to the explicit factor i in Schrödinger's equation and the invariance of the overlap modulus $|\langle \psi | \phi \rangle|$. Another requirement over the time-reversal operator $\mathcal{T}$ is the reproducibility of the same wave function, at least to within a phase factor, when $\mathcal{T}$ is applied twice

$$\mathcal{T}^2 |\psi\rangle = \alpha |\psi\rangle, \quad (3.15)$$

where $|\alpha| = 1$. This requirement imposes that $\alpha^2 = 1$, or

$$\mathcal{T}^2 = \pm 1. \quad (3.16)$$

As discussed in the previous section, the positive sign express the conventional time reversal for spinless particles. However, the new case $\mathcal{T}^2 = -1$ represents systems with spin-1/2 particles. For these systems, the time-reversal operator $\mathcal{T}$ acts on orbital angular momentum or spin in the form

$$\mathcal{T}\vec{J}\mathcal{T}^{-1} = -\vec{J}, \text{ where } \vec{J} = \vec{L} \text{ or } \vec{S}. \quad (3.17)$$

The conventional time-reversal operator $\mathcal{T}$ for spin-1/2 particles is given by

$$\mathcal{T} = i\sigma_y K = e^{i\pi\sigma_y/2} K, \quad (3.18)$$

where the complex conjugation K acts on both wave functions described over the position space and the spinor representations. Unlike the spinless particles, the $\mathcal{T}^2$ squares to -1 for spin-1/2 systems. Indeed, $\mathcal{T}^2 = (i\sigma_y K)(i\sigma_y K) = (i\sigma_y)^2 = -1$.

In a system with a half-integer total spin, Kramers' degeneracy theorem states that for every energy eigenstate of a system whose Hamiltonian is invariant under the time reversal operator $\mathcal{T}$, there exists a time-reversed eigenstate with identical energy, i.e., let H be a Hamiltonian invariant under a time reversal $\mathcal{T}$

$$[H, \mathcal{T}] = 0, \quad (3.19)$$

if ψ is an eigenfunction with eigenvalue E , so is $\mathcal{T}\psi$. For spinless systems, where $\mathcal{T}^2 = +1$, we have the trivial case $\mathcal{T}\psi = \psi$ without loss of generality. However, for Spin-1/2 systems, where $\mathcal{T}^2 = -1$, ψ and $\mathcal{T}\psi$ are orthogonal,

$$\langle \psi | \mathcal{T}\psi \rangle = \langle \mathcal{T}\psi | \mathcal{T}^2\psi \rangle^* = -\langle \mathcal{T}\psi | \psi \rangle^* = -\langle \psi | \mathcal{T}\psi \rangle = 0, \quad (3.20)$$

and therefore all eigenvalues of H are doubly degenerate, this is the known Kramers' degeneracy theorem. Thus, the dimension of the Hilbert space must, if finite, be even. Moreover, if the total-spin quantum number is s , then a half-integer total spin system has an even number of $2s + 1$ eigenstates.

The time-reversal operator $\mathcal{T} = ZK = i\sigma_y K$ provides a Hamiltonian h_{mn} with the following block structure

$$h_{mn} = h_{mn}^{(0)} 1 + i\mathbf{h}_{mn} \cdot \boldsymbol{\sigma}, \quad (3.21)$$

where all the amplitudes $h_{mn}^{(\mu)}$ are real, $h_{mn}^{(\mu)} = h_{mn}^{(\mu)*}$. In addition, the hermiticity condition of the Hamiltonian H requires that the amplitudes $h_{mn}^{(\mu)}$ satisfy

$$\begin{cases} h_{mn}^{(0)} = h_{nm}^{(0)}, \\ h_{mn}^{(k)} = -h_{nm}^{(k)}, \quad k = 1, 2, 3. \end{cases} \quad (3.22)$$

From now on, we are interested in finding canonical transformations that preserve this Hamiltonian structure. In particular, our goal is to find a subgroup of unitary matrices that preserve the form $\mathcal{T} = ZK$ of the time-reversal operator. A desired unitary basis transformation S must obey the following condition

$$\mathcal{T} = S\mathcal{T}S^{-1} = SZKS^{-1} = SZS^T K \quad \longrightarrow \quad SZS^T = Z. \quad (3.23)$$

The requirement $SZS^T = Z$ defines the Lie group $Sp(2N)$. Therefore, if $\mathcal{T}^2 = -1$ then the $2N \times 2N$ Hamiltonian is $\mathcal{T}$ -invariant under symplectic transformations $Sp(2N)$. This class of Hamiltonians is known as *symplectic symmetry class S*.

3.2.3 Systems Without Time-Reversal Invariance

For all quantum closed systems, the Hamiltonians are represented by Hermitian matrices. The Hamiltonian matrices unrestricted by antiunitary symmetries are in general complex. We are interested in Hamiltonian transformations $H' = AHA^{-1}$ that preserve both Hermiticity and eigenvalues. These transformations are called canonical transformations and satisfy the following condition

$$(AHA^{-1})^\dagger = AHA^{-1} \Leftrightarrow [H, A^\dagger A] = 0. \quad (3.24)$$

Considering that H is a real diagonal in eigen-representation and the requirement of preservation of normalization of vectors, then $A^\dagger A = 1$ must be the unit matrix. The class of canonical transformations of Hamiltonians which are unrestricted by antiunitary symmetries is thus constituted by unitary matrices $U(N)$. The set of complex Hermitian Hamiltonians is called by the *Unitary symmetry class U*.

The orthogonal, unitary and symplectic symmetry classes constitute systems where the electron's dynamic follows the Schrödinger equation. However, new symmetry classes were discovered associated with systems that support relativistic and superconducting physics through the Dirac and Bogoliubov De-Gennes equations, respectively. In the following part, we explore the symmetry classes beyond Dyson's Threefold way.

3.3 Beyond Dyson's Threefold Way

At the end of the 20th century, new symmetry classes were discovered in quantum chromodynamics [27] and normal-superconductor hybrid mesoscopic devices [72, 29, 73]. In particular, these classes presented a universal behavior not predicted by Dyson's threefold way. Martin Zirnbauer, along with A. Altland, claimed that seven further symmetry classes would be necessary to explain the universal behavior of both low-energy Dirac spectrum in chromodynamics and low-energy excitations in disordered superconductors. This approach rests on the assumption of effective single-Fermion theories. Next, we will address the emergence of these symmetry classes in mesoscopic systems.

3.3.1 Normal-Superconducting Hybrid Systems

Observed by Martin Zirnbauer and Alexander Altland, four new universality classes emerge from normal-superconducting hybrid structures, such as a normal quantum dot with a superconductor material attached at the boundary. As mentioned in the introduction, the Andreev reflection plays a fundamental role in the chaotic dynamic of the electron inside these types of billiards. When an electron from a normal conductor enters the superconductor, it couples with another electron and forms a Cooper-pair inside the superconductor and, consequently, leaves a hole with opposite velocity in the normal conductor region. Therefore, an electron with energy ε is scattered into a hole of energy $-\varepsilon$ with phase $\pi/2 - \phi$, where ϕ is the phase of superconducting order parameter at the interface. The quasiparticles inside a chaotic normal-conductor billiard attached by a superconductor material can be described by a mean-field approximation which yields an effective single-particle theory. This approximation is given by the second-quantized Bardeen-Cooper-Schrieffer (BCS) Hamiltonian [29]

$$\hat{H} = \sum_{\alpha\beta} (h_{\alpha\beta} c_{\alpha}^{\dagger} c_{\beta} + \frac{1}{2} \Delta_{\alpha\beta} c_{\alpha}^{\dagger} c_{\beta}^{\dagger} + \frac{1}{2} \Delta_{\alpha\beta}^{*} c_{\alpha} c_{\beta}), \quad (3.25)$$

where the creation and annihilation operators $c_{\alpha}^{\dagger}$ and c_{α} satisfy the known Fermions anticommutation algebra

$$c_{\alpha} c_{\beta}^{\dagger} + c_{\beta}^{\dagger} c_{\alpha} = \delta_{\alpha\beta}. \quad (3.26)$$

In a matrix form, the Hamiltonian H is written as

$$\hat{H} = \frac{1}{2} \begin{pmatrix} c^{\dagger} & c \end{pmatrix} \begin{pmatrix} h & \Delta \\ -\Delta^{*} & -h^T \end{pmatrix} \begin{pmatrix} c \\ c^{\dagger} \end{pmatrix} + \text{const}, \quad (3.27)$$

where the coefficients of the Hamiltonian H are determined uniquely by the $4N \times 4N$ -matrix $\mathcal{H}$

$$\mathcal{H} = \begin{pmatrix} h & \Delta \\ -\Delta^* & -h^T \end{pmatrix}, \quad (3.28)$$

where h , Δ and $-h^T$ represent the particle Hamiltonian sector, the coupling between particles and holes and the hole Hamiltonian sector, respectively. The $4N \times 4N$ Hamiltonian $\mathcal{H}$ is the Bogolyubov-deGennes (BdG) Hamiltonian. The system is constituted by N orbital states, where each orbital state has two spin states and the additional particle-hole degree of freedom. Both Hermiticity and Fermi statistics, represented respectively as

$$h_{\alpha\beta} = h_{\beta\alpha}^*, \quad \Delta_{\alpha\beta} = -\Delta_{\beta\alpha}, \quad (3.29)$$

take the following Hamiltonian form

$$\mathcal{H} = -\Sigma_x \mathcal{H}^* \Sigma_x, \quad \Sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (3.30)$$

Under this Hamiltonian constraint, if ψ is an eigenstate of $\mathcal{H}$ with eigenvalue ω , then $\Sigma_x \psi^*$ is also an eigenstate but with eigenvalue $-\omega$. Thus, we see a symmetry of the spectrum around $E = 0$. The invariance of the Hamiltonian under the interchange $c \leftrightarrow c^\dagger$ of creation and annihilation operators reveals a supposed "symmetry" associated with the particle-hole degree of freedom. From Eq.(3.30), the antiunitary charge conjugation operator $\mathcal{C} = \Sigma_x K$ can be defined such that

$$\mathcal{C} = \Sigma_x K, \quad \mathcal{C}^2 = 1, \quad \mathcal{C} \mathcal{H} \mathcal{C}^{-1} = -\mathcal{H}, \quad \mathcal{C} \mathcal{H} + \mathcal{H} \mathcal{C} = 0 \quad \rightarrow \quad \{\mathcal{H}, \mathcal{C}\} = 0. \quad (3.31)$$

Even though the BdG Hamiltonian $\mathcal{H}$ is not symmetric under $\mathcal{C}$ -operation in the sense that they do not commute, the particle-hole symmetry is characterized as a remarkable antiunitary symmetry where $\mathcal{H}$ and $\mathcal{C}$ anti-commute. Since $\mathcal{H}$ does not close under commutation, the imaginary Hamiltonian $X := i\mathcal{H}$ does

$$\begin{aligned} \mathcal{C} X \mathcal{C}^{-1} &= (\Sigma_x K)(i\mathcal{H})(\Sigma_x K)^{-1} = \Sigma_x \underbrace{[K(i\mathcal{H})K^{-1}]}_{-i\mathcal{H}^*} \Sigma_x^{-1} \\ &= -i \underbrace{\Sigma_x \mathcal{H}^* \Sigma_x^{-1}}_{-\mathcal{H}} = i\mathcal{H} = X \quad \rightarrow \quad [\mathcal{C}, X] = 0, \end{aligned} \quad (3.32)$$

where the spectrum of these imaginary hermitian matrices is symmetric about zero. In particular, we also obtain imaginary BdG Hamiltonian by conjugating it with the unitary $4N \times 4N$ matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \quad (3.33)$$

where we can verify

$$\mathcal{H}_U = U \mathcal{H} U^{-1} = -\mathcal{H}_U^* = -\mathcal{H}_U^T. \quad (3.34)$$

This imaginary Hamiltonian $\mathcal{H}_U$ is diagonalized by an $SO(4N)$ matrix g , where $g\mathcal{H}_U g^{-1} = \text{diag}(\omega_1, \omega_2, \dots, \omega_{2N}, -\omega_1, -\omega_2, \dots, -\omega_{2N})$. Consequently, the gU diagonalizes the BdH Hamiltonian $\mathcal{H}$.

Symmetry class D. This symmetry class is constituted by Hamiltonians with neither time-reversal symmetry nor spin-rotation invariance. Hence, the Hamiltonian components h and Δ have no symmetry properties beyond the hermicity of h and skew-symmetry of Δ . In terms of $X = i\mathcal{H}$, the particle-hole symmetry is presented in the form

$$\begin{aligned} -X^\dagger &= X = -\Sigma_x X^T \Sigma_x, \\ \Sigma_x &= \begin{pmatrix} 0 & 1_{2N} \\ 1_{2N} & 0 \end{pmatrix} = \sigma_x \otimes 1_{2N}. \end{aligned} \quad (3.35)$$

The X satisfies some Lie algebra, i.e., if two matrices X, Y satisfy the Eq.(3.35), then so does their commutator $[X, Y]$. The set of real anti-Hermitian matrices $X_U = i\mathcal{H}_U = X_U^*$ forms the Lie algebra $so(4N)$. In addition, the algebra formed by the $X = i\mathcal{H}$ is isomorphic to that formed by the $X_U = i\mathcal{H}_U = X_U^*$, the $so(4N)$. This algebra is named " D " due to the $so(4N) \equiv D_{2N}$ Cartan's classification.

Symmetry class DIII. This symmetry class considers systems with time-reversal symmetry but without spin-rotation invariance. The time reversal invariance is given by $[\mathcal{T}, \mathcal{H}] = 0$ where, for an effective single-electron theory, the time reversal operator assumes the form

$$\mathcal{T} = i\sigma_y K, \quad \mathcal{T}^2 = -1, \quad (3.36)$$

with σ_y being the Pauli matrix acting on spin space. More precisely, the reversal operator $\mathcal{T}$ is written in the form

$$\mathcal{T} = (1_2 \otimes i\sigma_y \otimes 1_N) K \equiv \tau K, \quad (3.37)$$

where the first identity 1_2 expresses the particle-hole sector, the second operator $i\sigma_y$ acts on the spin space and, lastly, the identity 1_N is associated with the orbital space. The time reversal invariance of the Hamiltonian is given by

$$\mathcal{H} = \tau \mathcal{H}^* \tau^{-1}, \quad \tau = \begin{pmatrix} i\sigma_y & 0 \\ 0 & i\sigma_y \end{pmatrix}. \quad (3.38)$$

As we have seen, any Hamiltonian invariant under a time reversal $\mathcal{T}$, with $\mathcal{T}^2 = -1$, has the spectrum characterized by the Kramers' degeneracy.

Similar the class D , we consider the antiHermitian matrices $X = i\mathcal{H} = -X^\dagger$ to characterize the class of Hamiltonians. The reason we consider them is that these Hamiltonians form a closed algebra under commutation. For the class D , which satisfies the Eq.(3.35), we have seen that the algebra is isomorphic to $so(4N)$. However, for the class $DIII$, the particle-hole symmetry restriction over X and the additional time reversal property given by Eq.(3.38) translate the following constraints

$$\begin{aligned} -X^\dagger &= X = -\Sigma_x X^T \Sigma_x, \quad \text{and} \\ X &= \tau X^T \tau^{-1}. \end{aligned} \quad (3.39)$$

We denote the set of solutions $X = +\tau X^T \tau^{-1}$ in $so(4N)$ as the set $\mathcal{P}$. Indeed, for two members X_1 and $X_2 \in \mathcal{P}$, their commutator does not obey, at least to a minus signal, the condition

$$[X_1, X_2] = \tau[X_1^T, X_2^T]\tau^{-1} = -\tau[X_1, X_2]^T \tau^{-1}. \quad (3.40)$$

The space $\mathcal{P}$ is given by the tangent space of the quotient $SO(4N)/U(2N)$ of the corresponding Lie groups. This quotient is the symmetric space labeled as *DIII* by Cartan and corresponds to the symmetry class of time reversal invariant BdG Hamiltonians when the spin rotational symmetry is broken.

Symmetry class C. Next we consider systems with isotropy in spin space but without time-reversal symmetry. In these systems, the generators of spin rotations J_k ($k = x, y, z$) commute with the Hamiltonian, $[J_k, \mathcal{H}] = 0$. In the second quantization, the angular momentum is written in the form $J_k = (\hbar/2) \sum_{qss'} c_{qs}^\dagger \sigma_{kss'} c_{qs}$ where σ_k , $k = x, y, z$ are the Pauli matrices. In matrix representation, the angular momentum is a matrix $4N \times 4N$

$$J_k = \frac{\hbar}{4} \begin{pmatrix} \sigma_k & 0 \\ 0 & -\sigma_k^T \end{pmatrix} \otimes 1_N. \quad (3.41)$$

For the antiHermitian representation $X = i\mathcal{H}$, the commutation $[X, J_k] = 0$ imposes the following restrictions to the $2N \times 2N$ block representation of X

$$\begin{aligned} X_{pp} &= ih = 1_2 \otimes a, & X_{ph} &= i\Delta = i\sigma_y \otimes b, \\ X_{hp} &= -i\Delta^* = -\sigma_y \otimes c, & X_{hh} &= -ih^T = -1_2 \otimes a^T. \end{aligned} \quad (3.42)$$

Thus, the antiHermitian representation X can be written in terms of $N \times N$ blocks

$$X = \begin{pmatrix} a & 0 & 0 & b \\ 0 & a & -b & 0 \\ 0 & -c & -a^T & 0 \\ c & 0 & 0 & -a^T \end{pmatrix}, \quad (3.43)$$

where X can be decomposed into two subblocks. The outer subblock

$$X_\uparrow = \begin{pmatrix} a & b \\ c & -a^T \end{pmatrix} \quad (3.44)$$

corresponds to spin-up particles and spin-down holes. The inner subblock $X_\downarrow$ corresponds to spin-down particles and spin-up holes and relates to the $X_\uparrow$ by $b \rightarrow -b$ and $c \rightarrow -c$. Hence, it suffices to focus on $X_\uparrow$. Since the condition $X = -\Sigma_x X^T \Sigma_x$ makes $X_{ph} = -X_{ph}^T$, with $X_{ph} = i\sigma_y \otimes b$, then b from $X_\uparrow$ has to be symmetric $b = b^T$. Similarly, the block c from $X_\uparrow$ follows the constraint $c = c^T$. Moreover, the antiHermiticity requires $a = -a^\dagger$ and $c = -b^\dagger$. All these constraints are summarized by

$$-X_\uparrow^\dagger = X_\uparrow = -\Sigma_y X_\uparrow^T \Sigma_y, \quad \Sigma_y = \sigma_y \otimes 1_N. \quad (3.45)$$

This defines the Lie algebra $sp(2N)$. For an $S \in Sp(2N)$ such that $SXS^T = X$, the constraint $X = \Sigma_y X^T \Sigma_y^T$ preserves the structure

$$X = \Sigma_y (SXS^T)^T \Sigma_y^T = \Sigma_y SX^T S^T \Sigma_y^T = (\Sigma_y S) X^T (\Sigma_y S)^T. \quad (3.46)$$

The canonical transformations for this symmetry class are matrices from $Sp(2N)$, and such matrices can diagonalize the Hamiltonian $X_\uparrow$. In Cartan's notation, the present symmetry class is called C .

The charge conjugation operator can be refined for the subblock $\mathcal{H}_\uparrow = -iX_\uparrow$. For BdG Hamiltonians, the anticommutation $\mathcal{H}\mathcal{C} + \mathcal{C}\mathcal{H} = 0$ is guaranteed. In the same way, the subblock $\mathcal{H}_\uparrow$ anticommutes with the new charge conjugation operator $\mathcal{C}_\uparrow = \Sigma_y K$ and $\mathcal{C}_\uparrow^2 = -1$. The same definition can be made for the spin-down subblock, therefore the total charge conjugation $\mathcal{C} = \mathcal{C}_\uparrow \otimes \mathcal{C}_\downarrow$ squares to minus one $\mathcal{C}^2 = -1$ for class C .

Symmetry class CI. The last class of BdG Hamiltonians represents systems with invariance under both spin rotation and time reversal. The invariant spin-rotation representation

$$X = \begin{pmatrix} 1_2 \otimes a & i\sigma_y \otimes b \\ -i\sigma_y \otimes c & -1_2 \otimes a^T \end{pmatrix} \quad (3.47)$$

with the additional time reversal restriction $X = \tau X^T \tau^{-1}$, where $\tau = 1_2 \otimes i\sigma_y \otimes 1_N$, entails that X is a symmetric matrix $X = X^T$. Similar the total Hamiltonian X , the two commuting subblocks are also symmetric matrices, $X_{\uparrow,\downarrow} = X_{\uparrow,\downarrow}^T$. Hence, the present symmetry class is formed by the set $\mathcal{P}$ of symmetric matrices in $sp(2N)$, and it is considered as the tangent space of the coset space $Sp(2N)/U(N)$ which is the type CI à la Cartan. Thus, the present symmetry class is labeled as CI . Similar the class C , the antiunitary charge operator $\mathcal{C}$ also squares to -1 , i.e. , $\mathcal{C}^2 = -1$.

3.3.2 Systems with Chiral Symmetry

Now we will deal with chiral symmetry in mesoscopic systems. As pointed out by Beenakker [74], there is a correspondence between the valley degree of freedom in graphene and the chirality of neutrinos. In particular, the electrons in graphene can be called "chiral" because their direction of motion is tied to the direction of the pseudo-spin, which corresponds to the sublattice degree of freedom. The current operator $\vec{j}$ can be written in the form $\vec{j} = v\vec{\sigma} \otimes \tau_0$, where v is the velocity of the electron in graphene, $\vec{\sigma}$ is called as pseudo-spin Pauli matrices acting on sublattice degree of freedom and τ_0 is the unit 2×2 matrix acting on valley degree of freedom. Near the Dirac points, the two-dimensional graphene structure allows the Hamiltonian $v\vec{p} \cdot \vec{\sigma}$ to be written in an off-diagonal sub-blocks form. In these Dirac points, the chiral symmetry is recognized by its block representation

$$H = \begin{pmatrix} 0 & D \\ D^\dagger & 0 \end{pmatrix}. \quad (3.48)$$

This particular structure appears in quantum physics of relativistic electrons and is called the Dirac operator. This block representation supports a "symmetry" of the form

$$H = -\mathcal{P}H\mathcal{P}^{-1}, \quad \mathcal{P}\mathcal{P}^\dagger = 1, \quad \mathcal{P}^2 = 1. \quad (3.49)$$

We say "symmetry" because no conserved quantity comes with it since $\mathcal{P}$ anti commutes with H . However, this anticommutativity implies an energy spectrum symmetric about zero, similar

to particle-hole symmetry. The main difference between chiral and particle-hole symmetry is that the latter is described by an antiunitary charge conjugation operator $\mathcal{C}$ and the former by a unitary operator $\mathcal{P}$. In particular, choosing an appropriate basis, the unitary operator $\mathcal{P}$ of the chiral symmetry can be identified by the Pauli matrix in the z direction, that is $\mathcal{P} = \sigma_z$ [75]. The unitary operator $\mathcal{P}$ is known as chiral symmetry operator.

Both unitary $\mathcal{P}$ and antiunitary $\mathcal{C}$ operators lead to a symmetric energy spectrum. Let us suppose an eigenstate $|\nu\rangle$ of the Hamiltonian H , i.e $H|\nu\rangle = E|\nu\rangle$. Hence, the symmetric energy spectrum of the Hamiltonian can be viewed by the following eigenvalue equations

$$\begin{aligned} H\mathcal{P}|\nu\rangle &= -E\mathcal{P}|\nu\rangle, \\ H\mathcal{C}|\nu\rangle &= -E\mathcal{C}|\nu\rangle. \end{aligned} \quad (3.50)$$

Therefore, the Hamiltonian operator H anti-commutes with either a unitary or an anti-unitary symmetry operator, $\{\mathcal{P}, H\} = 0$ or $\{\mathcal{C}, H\} = 0$. In a time-reversal invariant system, if $\mathcal{P}$ is a unitary spectral mirror symmetry, then exists an anti-unitary spectral symmetry operator $\mathcal{C} \equiv \mathcal{P}\mathcal{T}$ that anti-commutes with the Hamiltonian. In the same way, a system with both types of spectral mirror symmetries preserves the time-reversal symmetry with respect to $\mathcal{T} \equiv \mathcal{P}\mathcal{C}$.

Summarizing, the Table.(3.1) presents the tenfold-way classification of the Altland-Zirnbauer Hamiltonian ensembles with their respective symmetric spaces.

Symmetry class		$\mathcal{T}^2 =$	$\mathcal{C}^2 =$	$(\mathcal{C}\mathcal{T})^2 =$	Symmetric space
A	Unitary	$\times$	$\times$	$\times$	$U(N)$
AI	Orthogonal	$+1$	$\times$	$\times$	$U(N)/O(N)$
AII	Symplectic	-1	$\times$	$\times$	$U(2N)/Sp(N)$
$AIII_\nu$ ($\nu \in \mathbb{N}$)	Chiral unitary	$\times$	$\times$	$+1$	$U(2N + \nu)/U(N + \nu)U(N)$
BDI_ν ($\nu \in \mathbb{N}$)	Chiral orthogonal	$+1$	$+1$	$+1$	$SO(2N + \nu)/SO(N + \nu)SO(N)$
CII_ν ($\nu \in \mathbb{N}$)	Chiral symplectic	-1	-1	$+1$	$Sp(2N + \nu)/Sp(N + \nu)Sp(N)$
C		$\times$	-1	$\times$	$Sp(2N)$
CI		$+1$	-1	-1	$Sp(2N)/U(N)$
D_ν ($\nu = 0, 1$)		$\times$	$+1$	$\times$	$SO(4N + \nu)$
$DIII_\nu$ ($\nu = 0, 1$)		-1	$+1$	-1	$SO(4N + 2\nu)/U(2N + \nu)$

Table 3.1 The tenfold-way classification of the Altland-Zirnbauer Hamiltonian ensembles and their respective symmetric spaces.

Random Matrix Theory in Quantum Transport

4.1 Random Matrix Theory

The goal of this chapter is to investigate the role of symmetries in quantum transport properties. For this purpose, we implement the time-reversal and unitary symmetries (chapter 3) in our physical system, the chaotic quantum billiard (chapter 2). This chapter assigns the scattering cavity Hamiltonian simply by H . In this way, the Hamiltonian of the scattering cavity in Chapter 2 will be modeled by the Random-Matrix Theory (RMT). Interested in analytical statistical properties of scattering resonances observed in nuclear reactions, Wigner [76] introduced a set of Hamiltonians, known as *ensemble*, whose elements share spectral features imposed by the underlying symmetry, such as time-reversal and spin-rotation symmetries. However, the success of RMT is due to its application in the quantum transport of chaotic systems, in special mesoscopic devices. In our case, the RMT explains the statistical properties of the energy spectrum of a scattering cavity with irregular shapes. By considering the chaotic behavior of electrons in these billiards, each realization of the irregular shape would provide a random Hamiltonian whose eigenvalues have distribution, average, variance and etc, well-defined.

For a random-matrix approach, the Hamiltonian operator is represented by an $M \times M$ Hermitian matrix $H = H^\dagger$ with the probability distribution

$$P(H) \propto \exp\left[-\frac{c}{M} \text{Tr} H^2\right],$$

$$c = \frac{\pi^2 \beta_E}{8 \delta_0^2} \times \begin{cases} 2 & \text{in the classes A, AI, AII,} \\ 1 & \text{in other classes,} \end{cases} \quad (4.1)$$

where δ_0 is the mean level spacing of H in the bulk of the spectrum and $\beta_E \in \{1, 2, 4\}$ describes the strength of the level repulsion. As we deal with mesoscopic systems constituted by many resonances $M \rightarrow \infty$, the spectral correlations stay away from the edge of the spectrum. Thus, the *universality* of spectral correlations arise, making mesoscopic phenomena independent of sample size or the degree of disorder within certain limits.

In the table (4.1), each symmetry class has its respective β_E index. Both α_E and β_E are exponents that express the repulsion of neighboring eigenvalues of H . The integer ν counts the number of d_E -fold degenerate, topologically protected zero-modes (Majorana in class D, BDI, and DIII).

	D	BDI	DIII	C	CI	CII	A	AI	AII	AIII
$H\mathcal{C} = -\mathcal{C}H, \mathcal{C}^2 =$	+1	+1	+1	-1	-1	-1	$\times$	$\times$	$\times$	$\times$
$H\mathcal{T} = \mathcal{T}H, \mathcal{T}^2 =$	$\times$	+1	-1	$\times$	+1	-1	$\times$	+1	-1	$\times$
$H\mathcal{C}\mathcal{T} = -\mathcal{C}\mathcal{T}H$	$\times$	$\checkmark$	$\checkmark$	$\times$	$\checkmark$	$\checkmark$	$\times$	$\times$	$\times$	$\checkmark$
ν	0,1	0,1,2,...	0,1	0	0	0,1,2,...	0	0	0	0,1,2,...
d_E	1	1	2	1	1	2	1	1	2	1
α_E	0	0	1	2	1	3	0	0	0	1
β_E	2	1	4	2	1	4	2	1	4	2

Table 4.1 The tenfold-way classification of the Altland-Zirnbauer Hamiltonian ensembles. Both α_E and β_E are repulsion exponents of the neighboring eigenvalues of H . The ν is a topological number associated with protected zero-modes.

4.2 Schrödinger billiard

The Schrödinger billiard is constituted by the A, AI, and AII symmetry classes of the Wigner-Dyson Ensemble. In this billiard, the Hamiltonian has the following probability distribution:

$$P(H) \propto \exp\left(-\frac{c}{M} \text{Tr} H^2\right), \quad c = \frac{\pi^2 \beta_E}{4\delta_0^2}, \quad (4.2)$$

$\beta_E \in \{1, 2, 4\}$ describes the strength of the level repulsion for the orthogonal, unitary and symplectic classes, respectively. From now on, we are interested in implementing the statistical properties of the random matrix theory for the Wigner-Dyson ensemble. We start by considering the Gaussian distribution of the Hamiltonian

$$P(H) \propto \exp\left(-\frac{2\pi^2\beta}{8\delta_0^2 M} \text{Tr} H^2\right) = \exp\left(-\frac{\beta M}{4\lambda^2} \text{Tr} H^2\right), \quad (4.3)$$

where λ is a constant of proportionality between the number of resonances M and the mean level spacing δ_0 , with $\lambda = M\delta_0/\pi \approx 0.5$ in our simulations. From the Gaussian distribution of the Hamiltonian $P(H)$, the trace $\text{Tr} H^2$ can be written in the form

$$\begin{aligned} \text{Tr} H^2 &= \text{Tr} |H|^2 = \sum_{ij} |H_{ij}|^2 = \sum_i |H_{ii}|^2 + \sum_{i \neq j} |H_{ij}|^2 \\ &= \sum_i |H_{ii}|^2 + 2 \sum_{i < j} |H_{ij}|^2, \end{aligned} \quad (4.4)$$

thus, there are diagonal (H_{ii}) and off-diagonal (H_{ij}) contributions to the trace. This decomposition of the trace allows us to express the statistical independence for each entry of the

Hamiltonian H_{ij}

$$\begin{aligned}
P(H) &\propto \exp\left(-\frac{\beta M}{4\lambda^2}\left[\sum_i |H_{ii}|^2 + 2\sum_{i<j} |H_{ij}|^2\right]\right), \\
&= \exp\left(-\frac{\beta M}{4\lambda^2}\sum_i |H_{ii}|^2\right) \exp\left(-\frac{\beta M}{4\lambda^2} 2\sum_{i<j} |H_{ij}|^2\right), \\
&= \prod_i \exp\left(-\frac{\beta M}{4\lambda^2}|H_{ii}|^2\right) \prod_{i<j} \exp\left(-\frac{\beta M}{4\lambda^2}(2|H_{ij}|^2)\right), \\
&= \prod_{ij} P(H_{ij}).
\end{aligned} \tag{4.5}$$

As mentioned, each entry H_{ij} is statistically independent and the Gaussian distribution has different weights for diagonal and off-diagonal elements of the Hamiltonian

$$\begin{cases} P(H_{ii}) \propto \exp\left(-\frac{\beta M}{4\lambda^2}|H_{ii}|^2\right), & i = j \\ P(H_{ij}) \propto \exp\left(-\frac{\beta M}{4\lambda^2} 2|H_{ij}|^2\right), & i < j. \end{cases} \tag{4.6}$$

Once we have explored the probability distribution of the Hamiltonian entries for classes of the Wigner-Dyson ensemble, we will discuss how symmetry implies constraints on the Hamiltonian inputs.

4.2.1 Orthogonal Class - AI

In the orthogonal class, also known as GOE (Gaussian Orthogonal Ensemble), the system preserves both time-reversal (TRS) and spin rotational symmetries (SRS). For the class AI, the table (4.1) shows the time-reversal operator $\mathcal{T}$ squares to +1, so it is simply the conjugation operator $\mathcal{T} = \mathcal{K}$ such that $\mathcal{T}^2 = +1$. Therefore, the Hamiltonian of this system satisfies the following condition

$$\begin{aligned} \mathcal{T} = \mathcal{K}, \quad (\mathcal{T}^2 = +1) &\rightarrow \mathcal{T}H\mathcal{T}^{-1} = H^* = H, \\ &H = H^* = H^T, \end{aligned} \tag{4.7}$$

then H must be real and symmetric.

Now we are concerned with the statistical properties of the Hamiltonian H for the orthogonal class. Following the Eq.(4.6), the probability distribution is divided into two parts: diagonal and off-diagonal. The diagonal part provides

$$\begin{aligned} P(H_{ii}) &\propto \exp\left(-\frac{\beta M}{4\lambda^2}|H_{ii}|^2\right) = \exp\left(-\frac{|H_{ii}|^2}{2(2\lambda^2/\beta M)}\right), \text{ then} \\ \langle |H_{ii}|^2 \rangle &= \frac{2\lambda^2}{\beta M} = \frac{2\lambda^2}{M}. \end{aligned} \tag{4.8}$$

On the other hand, the off-diagonal gives us

$$P(H_{ij}) \propto \exp\left(-\frac{\beta M}{4\lambda^2} 2|H_{ij}|^2\right) = \exp\left(-\frac{|H_{ij}|^2}{2(\lambda^2/\beta M)}\right), \text{ then} \quad (4.9)$$

$$\langle |H_{ij}|^2 \rangle = \frac{\lambda^2}{\beta M} = \frac{\lambda^2}{M}.$$

Therefore, considering a $M \times M$ random matrix A with each entry following $A_{ij} \sim \mathcal{N}(0, 1)$, a real and symmetric Hamiltonian H can be constructed by splitting the matrix A into the following parts

$$\text{diag}(A) = \begin{pmatrix} A_{11} & 0 & 0 & \dots & 0 & 0 \\ 0 & A_{22} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & A_{M-1,M-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & A_{MM} \end{pmatrix}, \quad C = \begin{pmatrix} 0 & A_{12} & A_{13} & \dots & A_{1M} \\ 0 & 0 & A_{23} & \dots & A_{2M} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & A_{M-1,M} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad (4.10)$$

such that

$$H = \sqrt{\frac{2\lambda^2}{M}} \text{diag}(A) + \sqrt{\frac{\lambda^2}{M}} (C + C^T). \quad (4.11)$$

In fact, the factors that multiply the matrices $\text{diag}(A)$ and C represent the standard deviation σ of the Hamiltonian entries. The second term of the Hamiltonian imposes the symmetric matrix condition $H = H^T$.

4.2.2 Unitary Class - A

In the unitary class, also known as GUE (Gaussian Unitary Ensemble), the time-reversal symmetry (TRS) is broken. This happens when a magnetic field applied in the quantum dot breaks the time-reversal symmetry. For the Wigner-Dyson ensembles, the spin rotation symmetry does not play a role when time-reversal symmetry is broken. Thus, the spin degree of freedom would add a mere degeneracy that can be ruled out. In the unitary class, each element of the Hamiltonian is a complex number

$$H = H^{(0)} + iH^{(1)}, \text{ where } H^{(0)}, H^{(1)} \text{ are real matrices.} \quad (4.12)$$

Since, the Hamiltonian H is Hermitian $H = H^\dagger$, then $H^{(0)}$ and $H^{(1)}$ must be symmetric and anti-symmetric, respectively

$$\begin{cases} H^{(0)} = H^{(0)T}, \\ H^{(1)} = -H^{(1)T}. \end{cases} \quad (4.13)$$

Thus, the symmetric and antisymmetric parts of the Hamiltonian H were described above by Eq.(4.13).

Now we implement the statistical properties of the Hamiltonian H for the unitary class. In the table (4.1), the unitary class, also classified by A, is indicated by the $\beta = 2$. Similar to the

orthogonal case, the probability distribution of the diagonal elements provides us

$$P(H_{ii}) \propto \exp\left(-\frac{\beta M}{4\lambda^2}|H_{ii}|^2\right). \quad (4.14)$$

However, we remember that $H = H^{(0)} + iH^{(1)}$, with $H^{(1)}$ being an antisymmetric matrix. Then, its contribution for diagonal elements H_{ii} is null. Therefore,

$$\begin{aligned} P(H_{ii}) &\propto \exp\left(-\frac{\beta M}{4\lambda^2}|H_{ii}|^2\right) = \exp\left(-\frac{[H^{(0)}]_{ii}^2}{2(2\lambda^2/\beta M)}\right), \text{ then} \\ \langle |H_{ii}|^2 \rangle &= \langle [H^{(0)}]_{ii}^2 \rangle = \frac{2\lambda^2}{\beta M} = \frac{\lambda^2}{M}. \end{aligned} \quad (4.15)$$

On the other hand, the off-diagonal elements gives us

$$\begin{aligned} P(H_{ij}) &\propto \exp\left(-\frac{\beta M}{4\lambda^2} 2|H_{ij}|^2\right) = \exp\left(-\frac{|H_{ij}|^2}{2(\lambda^2/\beta M)}\right), \\ &= \exp\left(-\frac{[H^{(0)}]_{ij}^2}{2(\lambda^2/\beta M)}\right) \exp\left(-\frac{[H^{(1)}]_{ij}^2}{2(\lambda^2/\beta M)}\right), \text{ then} \\ \langle |H_{ij}|^2 \rangle &= \langle [H^{(0)}]_{ij}^2 \rangle + \langle [H^{(1)}]_{ij}^2 \rangle, \\ &= \frac{\lambda^2}{\beta M} + \frac{\lambda^2}{\beta M} = \frac{\lambda^2}{M}. \end{aligned} \quad (4.16)$$

Similar to the orthogonal Hamiltonian of Eq.(4.11), we write the standard deviation as the factor of the random matrices. For example, consider again a random matrix A with each entry following $A_{ij} \sim \mathcal{N}(0, 1)$. The symmetric part H^0 of the Hamiltonian can be obtained in terms of $\text{diag}(A)$ and C , defined in Eq.(4.10), by

$$H^0 = \sqrt{\frac{\lambda^2}{2M}}(\sqrt{2}\text{diag}(A) + C + C^T), \quad (4.17)$$

where the variance of the symmetric part is given by

$$\langle [H^0]^2 \rangle = \begin{cases} \frac{\lambda^2}{M}, & i = j, \\ \frac{\lambda^2}{2M}, & i \neq j. \end{cases} \quad (4.18)$$

Observe that for $i = j$, the variance of H^0 is equal to the variance of the total Hamiltonian H , i.e $\text{var}([H^{(0)}]_{ij}) = \text{var}(H_{ij})$. This happens because H^1 is antisymmetric with null diagonal elements. On the other hand for $i \neq j$, the variance of H^0 is half of the total Hamiltonian H ; the remaining part is contained in H^1 .

The antisymmetric part H^1 is obtained by defining B

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ A_{21} & 0 & 0 & 0 & 0 \\ A_{31} & A_{32} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{M1} & A_{M2} & \dots & A_{M,M-1} & 0 \end{pmatrix}, \quad (4.19)$$

where $A_{ij} \sim \mathcal{N}(0, 1)$ and

$$H^1 = \sqrt{\frac{\lambda^2}{2M}}(B - B^T), \text{ with } \langle [H^1]^2 \rangle = \begin{cases} 0, & i = j, \\ \frac{\lambda^2}{2M}, & i \neq j. \end{cases} \quad (4.20)$$

Therefore, $H = H^0 + iH^1$ with both H^0 and H^1 from Eq.(4.17) and Eq.(4.20), respectively.

4.2.3 Symplectic Class - AII

In the symplectic class, also known as GSE (Gaussian Symplectic Ensemble), the system has SRS broken but the TRS is preserved. In these systems, the spin degree of freedom emerges due to the Rashba spin-orbit coupling. As indicated by the table (4.1), the time-reversal operator squares to -1. It indicates that the time-reversal symmetry, $\mathcal{T}H\mathcal{T}^{-1} = H$, is assured by a time-reversal operator $\mathcal{T} = i\sigma_y\mathcal{K}$, where $\sigma_y = 1_M \otimes \sigma_y$ acts in the spin-degree of freedom. Therefore,

$$\begin{aligned} \mathcal{T} &= i\sigma_y\mathcal{K}, \quad (\mathcal{T}^2 = -1) \quad \rightarrow \quad \mathcal{T}H\mathcal{T}^{-1} = \sigma_y H^* \sigma_y = H, \\ H &= \sigma_y H^* \sigma_y \quad \rightarrow \quad H = H_0 \otimes 1 + i(H_1 \otimes \sigma_x + H_2 \otimes \sigma_y + H_3 \otimes \sigma_z), \end{aligned} \quad (4.21)$$

then the Hamiltonian H has a quaternionic structure, where H_0 is real symmetric and H_1, H_2, H_3 are real antisymmetric matrices

$$\begin{cases} H_0 = H_0^T, \\ H_i = -H_i^T, \text{ for } i = 1, 2, 3. \end{cases} \quad (4.22)$$

Now we are concerned with the statistical aspects of the Hamiltonian H for the symplectic class. In the table (4.1), the symplectic class, also classified by AII, is indicated by the $\beta = 4$. Similar to the unitary case, the only contribution of the diagonal elements comes from the matrix H_0 , then

$$\begin{aligned} P(H_{ii}) &\propto \exp\left(-\frac{\beta M}{4\lambda^2}|H_{ii}|^2\right) = \exp\left(-\frac{[H^{(0)}]_{ii}^2}{2(2\lambda^2/\beta M)}\right), \\ \langle [H^{(0)}]_{ij}^2 \rangle &= \frac{2\lambda^2}{\beta M} = \frac{\lambda^2}{2M}. \end{aligned} \quad (4.23)$$

Considering the quaternionic structure of the Hamiltonian and its respective modulus

$$|H_{ij}|^2 = [H^0]_{ij}^2 + [H^1]_{ij}^2 + [H^2]_{ij}^2 + [H^3]_{ij}^2, \quad (4.24)$$

so the off-diagonal elements provide the following probability distribution and variance

$$\begin{aligned} P(H_{ij}) &\propto \exp\left(-\frac{\beta M}{2\lambda^2}|H_{ij}|^2\right) = \prod_{k=0}^3 \exp\left(-\frac{[H^{(k)}]_{ij}^2}{2(\lambda^2/\beta M)}\right), \\ \langle [H^{(k)}]_{ij}^2 \rangle &= \frac{\lambda^2}{\beta M} = \frac{\lambda^2}{4M}. \end{aligned} \quad (4.25)$$

The symmetric part H^0 of the Hamiltonian can be obtained in terms of $\text{diag}(A)$ and C , defined in Eq.(4.10), by

$$H^0 = \sqrt{\frac{\lambda^2}{4M}} \left(\sqrt{2} \text{diag}(A) + (C + C^T) \right), \quad (4.26)$$

where

$$\begin{cases} \langle [H^0]_{ij}^2 \rangle = \lambda^2/2M, & i = j, \\ \langle [H^0]_{ij}^2 \rangle = \lambda^2/4M, & i \neq j. \end{cases} \quad (4.27)$$

The antisymmetric matrices H^1 , H^2 and H^3 are obtained by defining matrices E_k , such that

$$E_k = \begin{pmatrix} 0 & [Z^{(k)}]_{12} & [Z^{(k)}]_{13} & \dots & [Z^{(k)}]_{1M} \\ 0 & 0 & [Z^{(k)}]_{23} & \dots & [Z^{(k)}]_{2M} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & [Z^{(k)}]_{M-1,M} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \text{ for } k \in \{1, 2, 3\}, \quad (4.28)$$

and $[Z^{(k)}]_{ij} \sim \mathcal{N}(0, 1)$ an random variable normally distributed. Therefore, the k -part of the Hamiltonian H^k can be written as

$$H^k = \sqrt{\frac{\lambda^2}{4M}} (E_k - E_k^T), \text{ for } k \in \{1, 2, 3\}. \quad (4.29)$$

Therefore, $H = H_0 \otimes 1 + i(H_1 \otimes \sigma_x + H_2 \otimes \sigma_y + H_3 \otimes \sigma_z)$ with H^0 and H^k given by Eq.(4.26) and Eq.(4.29), respectively.

4.3 Dirac Billiard

The Dirac billiard is constituted by the BDI, AIII and CII symmetry classes of the Chiral Ensemble. The BDI, AIII and CII are also known as Chiral Gaussian Orthogonal (ChGOE), Unitary (ChGUE) and Symplectic (ChGSE) Ensembles, respectively. In the Dirac billiards, the probability distribution for the Hamiltonian H is given by

$$P(H) \propto \exp \left[-\frac{\pi^2 \beta}{8 M \delta_0^2} \text{Tr} H^2 \right], \quad (4.30)$$

where $\beta = 1, 2$, and 4 for ChGOE, ChGUE and ChGSE, respectively.

All previously cited ensembles have in common the chiral/sub-lattice symmetry, which is presented in the form [75]

$$\{\sigma_z, H\} = 0 \rightarrow H = -\sigma_z H \sigma_z, \quad \sigma_z = \sigma_z \otimes 1_M. \quad (4.31)$$

Thus, together with the Hermiticity condition $H = H^\dagger$, the $2M \times 2M$ Hamiltonian is presented as off-diagonal block representation

$$H = \begin{pmatrix} 0 & D \\ D^\dagger & 0 \end{pmatrix}. \quad (4.32)$$

In this way, the statistical properties of the random matrix theory will be implemented in the block D . So, the probability distribution Eq.(4.30) for the block D is

$$P(D) \propto \exp\left(-\frac{\beta M}{2\lambda^2} \text{Tr} DD^\dagger\right), \quad (4.33)$$

where $\lambda = 2M\delta_0/\pi$. The trace $\text{Tr} DD^\dagger$ can be written as

$$\begin{aligned} \text{Tr} DD^\dagger &= \sum_i (DD^\dagger)_{ii} = \sum_i \left[\sum_j D_{ij} D_{ij}^* \right], \\ &= \sum_i |D_{ii}|^2 + \sum_{i \neq j} |D_{ij}|^2. \end{aligned} \quad (4.34)$$

Thus, the probability distribution for the block D is

$$P(D) \propto \prod_i \exp\left(-\frac{\beta M}{2\lambda^2} |D_{ii}|^2\right) \prod_{i \neq j} \exp\left(-\frac{\beta M}{2\lambda^2} |D_{ij}|^2\right). \quad (4.35)$$

Each entry D_{ij} is statistically independent and follows the Gaussian distribution

$$P(D_{ij}) \propto \exp\left(-\frac{\beta M}{2\lambda^2} |D_{ij}|^2\right), \quad \forall i, j. \quad (4.36)$$

This probability distribution is valid for all symmetry classes of the Chiral ensemble. We are interested in how symmetry affects the Hamiltonian inputs.

4.3.1 Chiral Orthogonal Class - BDI

Similar to the Schrödinger billiard, the Chiral Orthogonal class, also known as ChGOE (Gaussian Orthogonal Ensemble) and index $\beta = 1$, represents chiral systems that preserve both time-reversal (TRS) and spin rotational symmetries (SRS). For class BDI, the table (4.1) shows the time-reversal operator $\mathcal{T}$ squares to $+1$, so it is simply the conjugation operator $\mathcal{T} = \mathcal{K}$ such that $\mathcal{T}^2 = +1$. Therefore, the Hamiltonian of this system satisfies the following condition

$$\begin{aligned} \mathcal{T} = \mathcal{K}, \quad (\mathcal{T}^2 = +1) \quad \rightarrow \quad \mathcal{T} H \mathcal{T}^{-1} &= H^* = H, \\ H &= H^* = H^T, \end{aligned} \quad (4.37)$$

then H must be real and symmetric. The only condition over the block D is it must be a real matrix $D = D^*$.

Since the probability distribution of any entry D_{ij} is given by (4.36), we have the variance $\text{var}(D_{ij}) = \lambda^2/M$ for $\beta = 1$, then

$$\langle |D_{ij}|^2 \rangle = \frac{\lambda^2}{M} \rightarrow D_{ij} \propto \frac{\lambda}{\sqrt{M}}. \quad (4.38)$$

In this way, the block D can be constructed by using a $M \times M$ random matrix A with each entry following $A_{ij} \sim \mathcal{N}(0, 1)$. Thus, the block D is given by

$$D = \frac{\lambda}{\sqrt{M}} A, \quad (4.39)$$

and, consequently, the Hamiltonian can be obtained by Eq.(4.32).

4.3.2 Chiral Unitary Class - AIII

The chiral unitary class $\beta = 2$, also known as ChGUE (Chiral Gaussian Unitary Ensemble), represents chiral systems with time-reversal symmetry broken. This ensemble models those chiral chaotic systems where a strong enough magnetic field is applied. The only symmetry preserved in this ensemble is the chiral/sub-lattice symmetry Eq.(4.31). Therefore, each element of the block D , and consequently the Hamiltonian H , is a complex number

$$D_{ij} = D_{ij}^{(0)} + iD_{ij}^{(1)}, \text{ where } D^{(0)}, D^{(1)} \text{ are real matrices.} \quad (4.40)$$

Since any entry D_{ij} is a complex number, then the modulus $|D_{ij}|^2$ is written as $|D_{ij}|^2 = [D^{(0)}]_{ij}^2 + [D^{(1)}]_{ij}^2$. Then,

$$\begin{aligned} \langle |D_{ij}|^2 \rangle &= \langle [D^{(0)}]_{ij}^2 \rangle + \langle [D^{(1)}]_{ij}^2 \rangle, \\ &= 2 \frac{\lambda^2}{\beta M} = \frac{\lambda^2}{M}. \end{aligned} \quad (4.41)$$

Therefore, the constants of proportionality are determined

$$D_{ij} \propto \frac{\lambda}{\sqrt{M}} \rightarrow \begin{cases} D_{ij}^{(0)} \propto \lambda / \sqrt{2M}, \\ D_{ij}^{(1)} \propto \lambda / \sqrt{2M}. \end{cases} \quad (4.42)$$

In this way, the block D can be constructed by using two $M \times M$ random matrix A, B with each entry following $A_{ij}, B_{ij} \sim \mathcal{N}(0, 1)$. Thus, the block D is given by

$$D = \frac{\lambda}{\sqrt{2M}}(A + iB). \quad (4.43)$$

4.3.3 Chiral Symplectic Class - CII

The chiral symplectic class, also known as ChGSE (Chiral Gaussian Symplectic Ensemble), represents systems with SRS broken but the TRS preserved. In these systems, the spin degeneracy is presented as an additional degree of freedom, beyond the chiral/sub-lattice one. Similar to the Schrödinger billiard case, the time-reversal operator $\mathcal{T}$ is given by $\mathcal{T} = i\sigma_y \mathcal{K}$ and the Hamiltonian has the same quaternionic structure. Thus, the block D can be written as

$$D = D^{(0)} \otimes 1_2 + i \sum_{j=1}^3 D^{(j)} \otimes \sigma_j, \text{ with } D^k \text{ being real matrices, } \forall k. \quad (4.44)$$

Then, the Hamiltonian satisfies the TRS constraint for SRS broken systems $H = \sigma_y H^* \sigma_y$.

The modulus of any entry D_{ij} is written as $|D_{ij}|^2 = [D^{(0)}]_{ij}^2 + [D^{(1)}]_{ij}^2 + [D^{(2)}]_{ij}^2 + [D^{(3)}]_{ij}^2$, then the probability distribution $P(D_{ij})$ reads

$$P(D_{ij}) \propto \exp\left(-\frac{\beta M}{2\lambda^2} |D_{ij}|^2\right) = \prod_{k=0}^3 \exp\left(-\frac{[D^{(k)}]_{ij}^2}{2(\lambda^2/\beta M)}\right), \quad (4.45)$$

with $\text{var}([D^{(k)}]_{ij}) = \lambda^2/4M$.

In this way, the block D can be constructed by using four $M \times M$ random matrix Z^k , where $k = 0, 1, 2, 3$, with each entry following $[Z^k]_{ij} \sim \mathcal{N}(0, 1)$, such that

$$D = \frac{\lambda}{\sqrt{4M}} \left(Z^0 \otimes 1_2 + i \sum_{k=1}^3 Z^k \otimes \sigma_k \right), \quad (4.46)$$

with $\text{var}(D_{ij}) = \lambda^2/M$.

4.4 Andreev Billiard

The Andreev billiard is constituted by the D, DIII, C and CI symmetry classes of the Altland-Zirnbauer Ensemble. In the Andreev billiards, the probability distribution for the Hamiltonian H is given by

$$P(H) \propto \exp \left[-\frac{\pi^2 \beta}{8 M \delta_0^2} \text{Tr} H^2 \right], \quad (4.47)$$

where $\beta = 2, 4, 2, 1$ for the D, DIII, C and CI symmetry classes (see table 4.1), respectively. From now on, we are interested in implementing the statistical properties of the random matrix theory for the Altland-Zirnbauer Ensemble. Moreover, we will connect these statistical properties with the symmetry aspects for each symmetry class. For that, we will use the probability distributions presented in Ref.[77] to implement the statistical aspects of each Altland-Zirnbauer class.

In normal-superconducting hybrid structures, the particle-hole (or charge-conjugation) symmetry is a property of the mean-field theory of superconductivity, with the Hamiltonian H given by [75]

$$\begin{aligned} \mathcal{H} &= \hat{\Psi}^\dagger H \hat{\Psi}, \quad \hat{\Psi} = (\hat{\psi}, \hat{\psi}^\dagger) = (\hat{\psi}_\uparrow, \hat{\psi}_\downarrow, \hat{\psi}_\uparrow^\dagger, \hat{\psi}_\downarrow^\dagger), \\ H &= \begin{pmatrix} H_0 - E_F & -i\sigma_y \Delta \\ i\sigma_y \Delta^* & E_F - H_0^* \end{pmatrix}, \end{aligned} \quad (4.48)$$

and the second quantization operator $\hat{\Psi}$ is a four-component Nambu spinor. In the particle-hole representation, the equation that governs the dynamics of Bogoliubov particles is

$$H\Psi(\vec{r}, t) = i\hbar \frac{\partial \Psi(\vec{r}, t)}{\partial t}, \quad \Psi = (\psi_e, \psi_h) = (\psi_{e\uparrow}, \psi_{e\downarrow}, \psi_{h\uparrow}, \psi_{h\downarrow}), \quad (4.49)$$

where satisfies the particle-hole symmetry

$$H = -\mathcal{C}H\mathcal{C}^{-1} = -\tau_x H^* \tau_x, \quad \mathcal{C} = \tau_x \mathcal{K}. \quad (4.50)$$

4.4.1 Class - D

We start by considering systems with the least degree of symmetry, i.e. systems with neither time-reversal symmetry nor spin-rotation invariance. We have introduced the Bogoliubov-de

Genes (BdG) Hamiltonian in the particle-hole representation. However, the Majorana representation can be obtained by the unitary transformation

$$\begin{aligned} H &\mapsto \Omega H \Omega^\dagger, \quad \Omega = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \\ \Psi &\mapsto \Omega \Psi = \sqrt{\frac{1}{2}} \begin{pmatrix} \psi_e + \psi_h \\ i\psi_e - i\psi_h \end{pmatrix}. \end{aligned} \quad (4.51)$$

Thus, the particle-hole symmetry in the Majorana representation becomes simply

$$\mathcal{C} = \mathcal{K}, \quad H = -H^*. \quad (4.52)$$

This can be viewed, alternatively, in the table (4.1). The particle-hole symmetry reads

$$\begin{aligned} \mathcal{C}^2 = +1 &\rightarrow \mathcal{C} = \mathcal{K}, \text{ then} \\ H = -\mathcal{C}H\mathcal{C}^{-1} &\rightarrow H = -H^*. \end{aligned} \quad (4.53)$$

Therefore, H can be written in terms of an antisymmetric matrix A , such that $H = iA$. The equation that governs Bogoliubov quasiparticles becomes a real wave equation,

$$A\Psi(\vec{r}, t) = \hbar \frac{\partial \Psi(\vec{r}, t)}{\partial t}, \quad (4.54)$$

and the corresponding field operator is self-conjugate, $\Psi^\dagger(\vec{r}, t) = \Psi(\vec{r}, t)$. For this reason, we say that the Bogoliubov quasiparticle is a Majorana fermion.

In the Gaussian ensemble, the upper-diagonal matrix elements A_{nm} ($n < m$) all have identical and independent distributions,

$$P(\{A_{nm}\}) \propto \prod_{1=n<m}^M \exp\left(-\frac{\pi^2 A_{nm}^2}{2M\delta_0^2}\right) = \prod_{1=n<m}^M \exp\left(-\frac{A_{nm}^2}{2(\lambda^2/M)}\right), \quad (4.55)$$

where $\lambda = M^2 \delta_0^2 / \pi^2$. Then, A_{nm} follows a Gaussian distribution with mean $\langle A_{nm} \rangle = 0$ and variance $\langle A_{nm}^2 \rangle = \lambda^2 / M$. Therefore, the Hamiltonian for the class D can be constructed by

$$H = \sqrt{\frac{\lambda^2}{M}} (C - C^T), \quad (4.56)$$

where C is given by Eq.(4.10).

4.4.2 Class - DIII

The class DIII represents systems with time-reversal symmetry but without spin-rotation invariance. Following the table (4.1), the charge-conjugation and time-reversal operators satisfy $\mathcal{C}^2 = +1$ and $\mathcal{T}^2 = -1$, respectively. Then

$$\begin{cases} H = -\mathcal{C}H\mathcal{C}^{-1}, & \mathcal{C}^2 = +1 \rightarrow \mathcal{C} = \mathcal{K}, \\ H = \mathcal{T}H\mathcal{T}^{-1}, & \mathcal{T}^2 = -1 \rightarrow \mathcal{T} = i\sigma_y \mathcal{K}. \end{cases} \longrightarrow \begin{cases} H = -H^*, \\ H = \sigma_y H^* \sigma_y. \end{cases} \quad (4.57)$$

The Hamiltonian, which satisfies all these conditions, must have a quaternionic structure given by:

$$H = i(H_1 \otimes \sigma_x + H_3 \otimes \sigma_z), \quad \begin{cases} H_1 = -H_1^T, \\ H_3 = -H_3^T, \end{cases} \quad (4.58)$$

and satisfies the anticommutation relation $\{\sigma_y, H\} = 0$. In the Gaussian ensemble, the Hamiltonian of the class DIII ($\beta = 4$) has the following probability distribution

$$\begin{aligned} P(\{H_{nm}\}) &\propto \prod_{1=n < m}^M \exp\left(-\frac{\pi^2}{M\delta_0^2}([H^1]_{nm}^2 + [H^3]_{nm}^2)\right), \\ &= \prod_{1=n < m}^M \exp\left(-\frac{[H^1]_{nm}^2}{2(\lambda^2/2M)}\right) \exp\left(-\frac{[H^3]_{nm}^2}{2(\lambda^2/2M)}\right). \end{aligned} \quad (4.59)$$

Thus, both H_1 and H_3 can be written as

$$\begin{cases} H_1 = \sqrt{\frac{\lambda^2}{2M}}(B - B^T), \\ H_3 = \sqrt{\frac{\lambda^2}{2M}}(C - C^T), \end{cases} \quad (4.60)$$

where C and B are given by Eq.(4.10) and Eq.(4.19), respectively.

4.4.3 Class - C

The symmetry class C stands for systems without time-reversal symmetry but with spin-rotation invariance. The table 4.1 shows us that charge-conjugation operator squares to -1, $\mathcal{C}^2 = -1$, then the charge-conjugation operator is given by $\mathcal{C} = i\tau_y \mathcal{K}$, where τ_y is the y -Pauli matrix acting on the particle-hole degree of freedom. The particle-hole symmetry reads

$$\begin{aligned} H &= -\mathcal{C}H\mathcal{C}^{-1}, \text{ with } \mathcal{C} = i\tau_y \mathcal{K}, \\ H &= -\tau_y H^* \tau_y, \end{aligned} \quad (4.61)$$

where the Hamiltonian H can be written in terms of a quaternion anti-Hermitian matrix Q

$$H = iQ, \quad Q = 1_2 \otimes H_0 + i(\tau_x \otimes \Delta_1 + \tau_y \otimes \Delta_2 + \tau_z \otimes H_3). \quad (4.62)$$

The anti-hermiticity of Q requires that

$$\begin{cases} H_0 = -H_0^T, \\ X = X^T, \text{ where } X \in \{\Delta_1, \Delta_2, H_3\}. \end{cases} \quad (4.63)$$

Thus, the Gaussian ensemble implies the following distribution for the independent matrix elements

$$\begin{aligned} P(\{Q_{nm}\}) &\propto \prod_{n=1}^M \exp\left(-\frac{\pi^2}{4M\delta_0^2}([\Delta_1]_{nn}^2 + [\Delta_2]_{nn}^2 + [H_3]_{nn}^2)\right) \\ &\times \prod_{1=n < m}^M \exp\left(-\frac{\pi^2}{2M\delta_0^2}([H_0]_{nm}^2 + [\Delta_1]_{nm}^2 + [\Delta_2]_{nm}^2 + [H_3]_{nm}^2)\right). \end{aligned} \quad (4.64)$$

Considering the matrices E_k of Eq.(4.28) and both $\text{diag}(A)$ and C from Eq.(4.10), we have

$$\begin{aligned} H_0 &= \sqrt{\frac{\lambda^2}{4M}}(C - C^T), \\ X &= \sqrt{\frac{\lambda^2}{4M}}\left(\frac{2}{\sqrt{3}}\text{diag}(A_k) + E_k + E_k^T\right), \quad X \in \{\Delta_1, \Delta_2, H_3\}, \end{aligned} \quad (4.65)$$

with $k \in \{1, 2, 3\}$ for Δ_1, Δ_2 and H_3 , respectively. Hence, the variance for each part of the Hamiltonian is

$$\begin{cases} \text{var}(X) = (\lambda^2/3M), & \text{var}(H_0) = 0, & i = j, \\ \text{var}(H_0) = \text{var}(X) = \lambda^2/4M, & i \neq j. \end{cases} \quad (4.66)$$

4.4.4 Class - CI

Finally, we turn to systems with both time-reversal symmetry and spin-rotation invariance. In the class CI, both time-reversal and charge-conjugation operators satisfy $\mathcal{C}^2 = -1$ and $\mathcal{T}^2 = +1$, respectively. Then,

$$\begin{cases} H = -\mathcal{C}H\mathcal{C}^{-1}, & \mathcal{C}^2 = -1 \rightarrow \mathcal{C} = i\tau_y\mathcal{K}, \\ H = \mathcal{T}H\mathcal{T}^{-1}, & \mathcal{T}^2 = +1 \rightarrow \mathcal{T} = \mathcal{K}. \end{cases} \longrightarrow \begin{cases} H = -\tau_y H^* \tau_y, \\ H = H^*. \end{cases} \quad (4.67)$$

Hence, time-reversal and particle-hole symmetries provide

$$\begin{cases} H = -\tau_y H^* \tau_y, \\ H = H^* \end{cases} \longrightarrow H = -\tau_y H \tau_y, \quad (4.68)$$

so the Hamiltonian satisfies the anticommutation relation $\{\tau_y, H\} = 0$. The proper Hamiltonian H to describe the symmetries conditions of class CI is

$$H = \tau_x \otimes \Delta_1 + \tau_z \otimes H_3, \quad \begin{cases} \Delta_1 = \Delta_1^T, \\ H_3 = H_3^T. \end{cases} \quad (4.69)$$

with Δ_1 and H_3 being symmetric matrices. The probability distribution of the independent matrix elements in the Gaussian ensemble is

$$\begin{aligned} P(\{H_{nm}\}) &\propto \prod_{n=1}^M \exp\left(-\frac{\pi^2}{2M\delta_0^2}([\Delta_1]_{nn}^2 + [H_3]_{nn}^2)\right) \\ &\times \prod_{1=n < m}^M \exp\left(-\frac{\pi^2}{M\delta_0^2}([\Delta_1]_{nm}^2 + [H_3]_{nm}^2)\right) \\ &= \prod_{n=1}^M \exp\left(-\frac{1}{2(\lambda^2/M)}([\Delta_1]_{nn}^2 + [H_3]_{nn}^2)\right) \\ &\times \prod_{1=n < m}^M \exp\left(-\frac{1}{2(\lambda^2/2M)}([\Delta_1]_{nm}^2 + [H_3]_{nm}^2)\right) \end{aligned} \quad (4.70)$$

So, given two random matrices E_1 and E_2 of Eq.(4.28), we obtain Δ_1 and H_3 by

$$\begin{aligned}\Delta_1 &= \sqrt{\frac{\lambda^2}{2M}}(\sqrt{2}\text{diag}(A_1) + E_1 + E_1^T), \\ H_3 &= \sqrt{\frac{\lambda^2}{2M}}(\sqrt{2}\text{diag}(A_2) + E_2 + E_2^T).\end{aligned}\tag{4.71}$$

Hence, the variance for each part $X \in \{\Delta_1, H_3\}$ of the Hamiltonian is

$$\text{var}(X) = \begin{cases} \lambda^2/M, & i = j \\ \lambda^2/2M, & i \neq j. \end{cases}\tag{4.72}$$

4.5 Symmetries and The Scattering Matrix

The approach of random-matrix theory also applies to the scattering matrix S for chaotic mesoscopic systems. In particular, chaotic scattering was a concept that originates from classical mechanics, referring to the exponential sensitivity of a trajectory to a slight change in the initial condition [78]. However, in quantum mechanics, we consider an ensemble of scattering matrices S produced by slight deformations of the scattering potential [79].

The solution of quantum transport relies on the determination of the scattering matrix S in terms of incident and outgoing wave amplitudes $\psi_n^{\text{in}}, \psi_n^{\text{out}}, n = 1, 2, \dots, N$, through of the following equation

$$\psi_n^{\text{out}}(E) = \sum_{m=1}^N S_{nm}(E) \psi_m^{\text{in}}(E).\tag{4.73}$$

The scattering is elastic, so both $\psi_n^{\text{in}}, \psi_n^{\text{out}}$ are at the same energy E . It is also conservative, then

$$\sum_n |\psi_n^{\text{out}}|^2 = \sum_n |\psi_n^{\text{in}}|^2.\tag{4.74}$$

Thus, in general, the S -matrix is a unitary $N \times N$ matrix, i.e $S \in U(N)$

$$S^{-1}(E) = S^\dagger(E) \leftrightarrow \sum_k S_{nk}(E) S_{mmk}^*(E) = \delta_{nm}.\tag{4.75}$$

The connection between the quantum scattering matrix S and the Hamiltonian $\hat{H}$ of the chaotic billiard can be achieved by using the Mahaux-Weindemüller formula

$$S(E) = 1 - 2i\pi \hat{W}^\dagger \frac{1}{E - \hat{H} + i\pi \hat{W} \hat{W}^\dagger} \hat{W}, \quad S = \begin{bmatrix} r & t' \\ t & r' \end{bmatrix},\tag{4.76}$$

where the matrix $\hat{W}$ is a $M \times N$ matrix and represents the coupling between the N open channels in the leads and the M resonances of the chaotic billiard. The blocks r and t (r' and t') represent reflection and transmission from particles coming from the left (right) lead, respectively. We

emphasize here that transmission processes without the transition through the resonant cavity are forbidden. Thus, this condition imposes the orthogonality constraint equation [58, 80]

$$W_\alpha W_\beta^\dagger = \gamma_\alpha \frac{M\delta_0}{\pi^2} \delta_{\alpha,\beta}, \quad (4.77)$$

where δ_0 is the mean energy level spacing of the resonant cavity, M is the number of resonances and γ_α is the diagonal matrix containing the tunneling rates $\Gamma_{\alpha,a} \in [0, 1]$ of the a channel supported by the lead α through the relation $\Gamma_{\alpha,a} = \text{sech}^2[-\ln(\gamma_{\alpha a})/2]$.

The symmetries play a fundamental role in the scattering matrix. In the table (4.2), we present how the symmetry implies constraints in the entries of the S matrix. For this purpose, we will discuss the connection between the Hamiltonian symmetries and the constraint of the scattering matrix S for each class of symmetry.

	D	BDI	DIII	C	CI	CII	A	AI	AII	AIII
$H\mathcal{C} = -\mathcal{C}H, \mathcal{C}^2 =$	+1	+1	+1	-1	-1	-1	$\times$	$\times$	$\times$	$\times$
$H\mathcal{T} = \mathcal{T}H, \mathcal{T}^2 =$	$\times$	+1	-1	$\times$	+1	-1	$\times$	+1	-1	$\times$
$H\mathcal{C}\mathcal{T} = -\mathcal{C}\mathcal{T}H$	$\times$	$\checkmark$	$\checkmark$	$\times$	$\checkmark$	$\checkmark$	$\times$	$\times$	$\times$	$\checkmark$
$S^{-1} =$	S^T (orthogonal)			$\tau_y S^T \tau_y$ (symplectic)			$S^\dagger$ (unitary)			
$S^T =$	$\times$	$+S$	$-S$	$\times$	$+S$	$-S$	$\times$	$+S$	$-S$	S^*
d_T	1	1	2	2	2	2	1	1	2	1
α_T	-1	-1	-1	2	1	-1	0	0	0	-1
β_T	1	1	2	4	2	4	2	1	4	2

Table 4.2 The tenfold-way classification of the Altland-Zirnbauer Hamiltonian ensembles. Both α_T and β_T are repulsion exponents of the neighboring eigenvalues of the transmission matrix $tt^\dagger$. The d_T represents the eigenvalues degeneracy of the matrix $tt^\dagger$.

• Orthogonal Class - AI

In the previous section, we show that the Hamiltonian H must be real and symmetric for the orthogonal class, $H = H^* = H^T$. This constraint imposes the following constraint on the transposition of matrix S^T

$$\begin{aligned}
 S^T(E) &= 1 - 2i\pi W^T \frac{1}{(E - \hat{H} + i\pi \hat{W} \hat{W}^T)^T} W, \\
 &= 1 - 2i\pi W^T \frac{1}{E^T - \hat{H}^T + i\pi \hat{W} \hat{W}^T} W, \\
 &= 1 - 2i\pi W^T \frac{1}{(E - \hat{H} + i\pi \hat{W} \hat{W}^T)} W, \\
 &= S(E) \quad \rightarrow \quad S^T(E) = +S(E).
 \end{aligned} \quad (4.78)$$

Thus, the scattering matrix S must be symmetric for the orthogonal class, $S^T(E) = +S(E)$. This constraint imposed by time-versal symmetry restricts S to a subset of $U(N)$.

The symmetric scattering matrix $S = S^T$ applies to electrons when their spin is conserved by the scattering potential. This ensemble of scattering matrix is known as Circular Orthogonal Ensemble (COE) because it refers to the fact that unitary symmetric matrices form the coset $U(N)/O(N)$ of the orthogonal group $O(N)$.

- **Unitary Class - A**

The unitary class, characterized by the absence of any fundamental symmetry, refers to a uniform distribution of the ensemble in the unitary group,

$$P(S) = \text{const}, \quad S \in U(N). \quad (4.79)$$

There is no any constraint in S -matrix beyond its unitarity $S^{-1} = S^\dagger$.

- **Symplectic Class - AII**

In the presence of spin-orbit coupling the constraint of time-reversal symmetry reads

$$H = \sigma_y H^* \sigma_y, \quad (4.80)$$

where $\sigma_y = 1_M \otimes \sigma_y$ is the Pauli matrix acting on the M resonances of the the chaotic billiard. Then, the symmetry condition on the S matrix is given by

$$\begin{aligned} S(E) &= 1 - 2i\pi W^T \frac{1}{(E - \hat{H} + i\pi W W^T)} W, \\ &= 1 - 2i\pi W^T \frac{1}{(E - \sigma_y H^* \sigma_y + i\pi W W^T)} W, \\ &= 1 - 2i\pi W^T \sigma_y \frac{1}{(E - H^* + i\pi \sigma_y W W^T \sigma_y)} \sigma_y W. \end{aligned} \quad (4.81)$$

From now on, we define the Pauli matrix acting on the open channels in the leads as $\Sigma_y = 1_N \otimes \sigma_y$, where σ_y is the usual 2×2 Pauli matrix. Moreover, since the Hamiltonian is Hermitian and symmetric ($H^* = H^T$), we obtain

$$\begin{aligned} S(E) &= \Sigma_y \left\{ 1 - 2i\pi (\Sigma_y W^T \sigma_y) \frac{1}{[E - H^T + i\pi (\sigma_y W \Sigma_y) (\Sigma_y W^T \sigma_y)]^T} (\sigma_y W \Sigma_y) \right\} \Sigma_y, \\ &= \Sigma_y \left\{ 1 - 2i\pi (\sigma_y W \Sigma_y)^T \frac{1}{[E - H + i\pi (\sigma_y W \Sigma_y) (\sigma_y W \Sigma_y)^T]} (\sigma_y W \Sigma_y) \right\} \Sigma_y. \end{aligned} \quad (4.82)$$

By imposing the constraint on the coupling $W \mapsto \sigma_y W \Sigma_y$, we conclude that

$$S(E) = \Sigma_y S^T(E) \Sigma_y, \quad \text{with } W = \sigma_y W \Sigma_y, \quad (4.83)$$

it means that the scattering matrix is found to be self-dual $S = \Sigma_y S^T \Sigma_y$. Moreover, in Ref.[81] is shown that the scattering matrix S is antisymmetric for an appropriate basis change

$$S^T = -S. \quad (4.84)$$

This is called the Circular Symplectic Ensemble (CSE) because S is in the coset $\in U(N)/Sp(N)$ of the unitary symplectic group $Sp(N)$.

- **Chiral Orthogonal Class - BDI**

In this part, we start to consider the chiral ensembles. First, the chiral orthogonal class (BDI) satisfies two conditions

$$\begin{cases} H &= -\sigma_z H \sigma_z, \quad \sigma_z = \sigma_z \otimes 1_M, \\ H &= \mathcal{T} H \mathcal{T}^{-1} = H^*, \quad \mathcal{T} = \mathcal{K}, \end{cases} \quad (4.85)$$

where the first and second conditions are the chiral/sub-lattice and time-reversal symmetries, respectively. The chiral/sub-lattice symmetry implies the following constraint of the S matrix

$$\begin{aligned} S(E) &= 1 - 2i\pi W^T \frac{1}{E - H + i\pi W W^T} W, \\ &= 1 - 2i\pi W^T \frac{1}{E + \sigma_z H \sigma_z + i\pi W W^T} W, \\ &= 1 - 2i\pi W^T \sigma_z \frac{1}{E + H + i\pi \sigma_z W W^T \sigma_z} \sigma_z W. \end{aligned} \quad (4.86)$$

Now we define the Pauli matrix acting on the open channels in the leads as $\Sigma_z = 1_N \otimes \sigma_z$. Moreover, since the Hamiltonian is Hermitian $H = H^\dagger$, we obtain

$$\begin{aligned} S(E) &= \Sigma_z \left\{ 1 - 2i\pi (\Sigma_z W^T \sigma_z) \frac{1}{[E + H^\dagger + i\pi (\sigma_z W \Sigma_z) (\Sigma_z W^T \sigma_z)]} (\sigma_z W \Sigma_z) \right\} \Sigma_z, \\ &= \Sigma_z \left\{ 1 - 2i\pi (\Sigma_z W^T \sigma_z) \frac{-1}{[(-E) - H + i\pi (\sigma_z W \Sigma_z) (\sigma_z W \Sigma_z)^T]^\dagger} (\sigma_z W \Sigma_z) \right\} \Sigma_z, \\ &= \Sigma_z \left\{ 1 - 2i\pi (\Sigma_z W^T \sigma_z) \frac{1}{[(-E) - H + i\pi (\sigma_z W \Sigma_z) (\sigma_z W \Sigma_z)^T]} (\sigma_z W \Sigma_z) \right\}^\dagger \Sigma_z, \\ &= \Sigma_z S^\dagger(-E) \Sigma_z, \quad \rightarrow \quad S(E) = \Sigma_z S^\dagger(-E) \Sigma_z. \end{aligned} \quad (4.87)$$

By imposing the constraint on the coupling $W \mapsto \sigma_z W \Sigma_z$, we obtain the chiral/sub-lattice symmetry for the S -matrix

$$S(E) = \Sigma_z S^\dagger(-E) \Sigma_z, \quad \text{with } W = \sigma_z W \Sigma_z, \quad \Sigma_z = 1_N \otimes \sigma_z. \quad (4.88)$$

Beyond the chiral/sub-lattice condition, the orthogonal chiral class also has the following constraint $H = H^*$ due to the time-reversal symmetry, which makes the matrix S symmetric, $S = S^T$.

- **Chiral Unitary Class - AIII**

The chiral unitary class (AIII) only satisfies the chiral-sublattice symmetry, $H = -\sigma_z H \sigma_z$. Thus, the only condition for the S -matrix is that given by Eq.(4.88).

- **Chiral Symplectic Class - CII**

The chiral symplectic class is constituted by the symmetries conditions

$$\begin{cases} H = -\sigma_z H \sigma_z, \\ H = \mathcal{T} H \mathcal{T}^{-1} = \sigma_y H^* \sigma_y, \end{cases} \quad (4.89)$$

where $\sigma_z = \sigma_z \otimes 1_M \otimes 1_2$ and $\sigma_y = 1_2 \otimes 1_M \otimes \sigma_y$ are the Pauli matrix acting on chiral/sub-lattice and spin degrees of freedom, respectively. Beyond the chiral/sub-lattice symmetry for the S -matrix given by Eq.(4.88), the time-reversal symmetry for systems with SRS broken provides the additional constraint

$$S(E) = \Sigma_y S^T(E) \Sigma_y, \quad \Sigma_y = 1_N \otimes 1_2 \otimes \sigma_y, \quad (4.90)$$

where makes S an antisymmetric matrix for an appropriated change of basis, $S = -S^T$.

• **Class - D**

For the class D of the Altland-Zirnbauer classification, the Hamiltonian is imaginary and antisymmetric, $H = -H^*$. Thus, the scattering matrix has the following constraint

$$\begin{aligned} S(E) &= 1 - 2i\pi W^T \frac{1}{E + H^* + i\pi W W^\dagger} W, \\ &= \left\{ 1 - 2i\pi W^T \frac{1}{[(-E) - H + i\pi W W^T]} W \right\}^*, \\ &= S^*(-E). \end{aligned} \quad (4.91)$$

Therefore, the S -matrix satisfies $S(E) = S^*(-E)$ for the class D.

• **Class - DIII**

In the presence of spin-orbit coupling, the constraint of time-reversal symmetry for systems that preserve particle-hole symmetry is

$$H = -\sigma_y H \sigma_y, \quad \sigma_y = 1_M \otimes \sigma_y. \quad (4.92)$$

Thus, the scattering matrix S has the following constraint

$$\begin{aligned} S(E) &= 1 - 2i\pi W^T \frac{1}{E - H + i\pi W W^T} W, \\ &= 1 - 2i\pi W^T \frac{1}{E + \sigma_y H \sigma_y + i\pi W W^T} W, \\ &= 1 - 2i\pi W^T \sigma_y \frac{1}{(E + H - i\pi \sigma_y W W^T \sigma_y)^\dagger} \sigma_y W, \\ &= \Sigma_y \left\{ 1 - 2i\pi (\sigma_y W \Sigma_y)^T \frac{1}{[(-E) - H + i\pi (\sigma_y W \Sigma_y)(\sigma_y W \Sigma_y)^T]} (\sigma_y W \Sigma_y) \right\}^\dagger \Sigma_y, \\ &= \Sigma_y S^\dagger(-E) \Sigma_y, \quad \rightarrow \quad S(E) = \Sigma_y S^\dagger(-E) \Sigma_y, \end{aligned} \quad (4.93)$$

where $\Sigma_y = 1_N \otimes \sigma_y$. Therefore, the scattering matrix S satisfies $S(E) = \Sigma_y S^\dagger(-E) \Sigma_y$ for the class DIII.

- **Class - C**

In class C of the Altland-Zirnbauer classification, the Hamiltonian is given by

$$H = iQ, \quad Q = 1_2 \otimes H_0 + i(\tau_x \otimes \Delta_1 + \tau_y \otimes \Delta_2 + \tau_z \otimes H_3), \quad (4.94)$$

where τ is the Pauli matrix acting on the particle-hole degree of freedom of the Hamiltonian resonances. The particle-hole symmetry reads

$$H = -\tau_y H^* \tau_y, \quad \tau_y = \tau_y \otimes 1_M. \quad (4.95)$$

Thus, the scattering matrix S has the following constraint

$$\begin{aligned} S(E) &= 1 - 2i\pi W^T \frac{1}{[E + \tau_y H^* \tau_y + i\pi W W^T]} W, \\ &= 1 - 2i\pi W^T \tau_y \frac{1}{[E + H^* + i\pi \tau_y W W^T \tau_y]} \tau_y W, \\ &= \left\{ 1 + 2i\pi W^T \tau_y \frac{1}{[E + H - i\pi \tau_y W W^T \tau_y]} \tau_y W \right\}^*, \\ &= \tau_y \left\{ 1 - 2i\pi (\tau_y W^T \tau_y) \frac{1}{[-E - H + i\pi (\tau_y W \tau_y)(\tau_y W^T \tau_y)]} (\tau_y W \tau_y) \right\}^* \tau_y, \\ &= \tau_y S^*(-E) \tau_y, \quad \rightarrow \quad S(E) = \tau_y S^*(-E) \tau_y, \end{aligned} \quad (4.96)$$

where $\tau_y = 1_N \otimes \tau_y$ is the Pauli matrix acting on the particle-hole degree of freedom of the open channels in the leads. Therefore, the particle-hole symmetry implies the scattering matrix constraint

$$S(E) = \tau_y S^*(-E) \tau_y \quad (4.97)$$

for class C.

- **Class - CI**

Class CI is constituted by the following symmetry constraints of the Hamiltonian

$$\begin{cases} H = -\tau_y H^* \tau_y, \\ H = H^*. \end{cases} \quad (4.98)$$

The first condition provides the same scattering matrix constraint $S(E) = \tau_y S^*(-E) \tau_y$ of class C. However, the additional time-reversal symmetry makes the scattering matrix symmetric, $S = S^T$.

In this chapter, we present the influence of fundamental symmetries on both Hamiltonian H and Scattering matrix S . We also show how the random matrix theory (RMT) can be implemented under the impositions of symmetries on the Hamiltonian entries. The goal of this thesis is to explore the influence of those fundamental symmetries on universal conductance fluctuations (UCF) and entanglement production. Therefore, in the next section, we present the theoretical background of quantum entanglement and how it can be measured in mesoscopic systems.

Quantum Entanglement

5.1 EPR paradox and Bell's Inequality

One of the cornerstone resources to make quantum information processing viable in electronic systems, quantum entanglement is an effect of quantum mechanics with no classical correspondence. In 1935, Einstein *et al.* [36] questioned whether the description of physical reality by quantum mechanics could be complete. In 1952, Bohm [40] proposed a theory that aims to describe statistical correlations through variables that we could not obtain physically, whose variables are called hidden variables. This theory was known as the Hidden-variable theory. The EPR paradox was better adapted by the following description [37]. Consider a pair of half-integer spin particles in a singlet state where each particle is moving in the opposite direction of the other. Spin measurements can be performed using magnets, as in the Stern-Gerlach experiment [82]. Using a unitary vector $\vec{a}$ that represents the magnetic field direction, we obtain the spin projections $\vec{\sigma}_1 \cdot \vec{a}$ and $\vec{\sigma}_2 \cdot \vec{a}$ of the particles 1 and 2 in the direction $\vec{a}$, respectively. For a maximally entangled pair, also known as Bell pair, if the measurement component $\vec{\sigma}_1 \cdot \vec{a}$ is equal $+1$, then $\vec{\sigma}_2 \cdot \vec{a}$ provides -1 and vice-versa. At first, measurements on each particle can be carried out over a sufficiently long distance, avoiding causality between events. Let us consider a hidden variable λ , the result A and B of measurements $\vec{\sigma}_1 \cdot \vec{a}$ and $\vec{\sigma}_2 \cdot \vec{b}$, respectively, are

$$A(\vec{a}, \lambda) = \pm 1, B(\vec{b}, \lambda) = \pm 1. \quad (5.1)$$

The assumption of a hidden variable λ means that the particle spins are already predetermined before the measurement. This hypothesis is known as *reality*. Moreover, observe from Eq.(5.1) that the result A and B do not depend on magnetic field directions $\vec{b}$ and $\vec{a}$, respectively. Therefore, a measurement in one particle does not interfere with the measurement of the other particle and vice-versa. This hypothesis is known as *locality*. We are interested in the statistical aspects of a realistic local theory. Given a hidden variable λ , the probability distribution $\rho(\lambda)$ provides the following correlation for both random variables $A(\vec{a}, \lambda)$ and $B(\vec{b}, \lambda)$

$$P(\vec{a}, \vec{b}) = \int d\lambda \rho(\lambda) A(\vec{a}, \lambda) B(\vec{b}, \lambda). \quad (5.2)$$

In the quantum mechanical theory, this correlation for a maximally entangled singlet state is given by

$$\langle \vec{\sigma}_1 \cdot \vec{a} \vec{\sigma}_2 \cdot \vec{b} \rangle = -\vec{a} \cdot \vec{b}. \quad (5.3)$$

In general, John Bell [83] showed that any hidden-variable theory does not reproduce the quantum mechanic's results. Mathematically, Eq.(5.2) would never reproduce all results of Eq.(5.3).

Therefore, a real and local theory would never provide all the statistical aspects of quantum mechanics. We will present this disagreement with the following argument. Let us consider a probability distribution $\rho(\lambda)$ such that

$$\int d\lambda \rho(\lambda) = 1. \quad (5.4)$$

Let us also consider

$$\overline{P}(\vec{a}, \vec{b}) \text{ and } \overline{-\vec{a} \cdot \vec{b}} \quad (5.5)$$

as independent averaging of $P(\vec{a}, \vec{b})$ and $-\vec{a} \cdot \vec{b}$ over vectors $\vec{a}'$ and $\vec{b}'$ within specified small angles of $\vec{a}$ and $\vec{b}$. We assume that the absolute value of the difference between $\overline{P}(\vec{a}, \vec{b})$ and $\overline{-\vec{a} \cdot \vec{b}}$ is smaller than an arbitrary ε

$$|\overline{P}(\vec{a}, \vec{b}) + \overline{\vec{a} \cdot \vec{b}}| \leq \varepsilon. \quad (5.6)$$

Our goal is to show that given a δ such that

$$|\overline{\vec{a} \cdot \vec{b}} - \vec{a} \cdot \vec{b}| \leq \delta, \quad (5.7)$$

it is not possible to find an ε small enough. In the appendix B, we find the following inequality

$$4(\varepsilon + \delta) \geq \sqrt{2} - 1. \quad (5.8)$$

It means that, for any δ , we can not find an ε small enough. Therefore, Eq.(5.3) can not be represented by Eq.(5.2).

Even though Bell provided his famous inequality to show that any hidden-variable theory does not reproduce all the quantum mechanic's predictions, Clauser *et al.* [84] develop an inequality from correlations that would facilitate experimental purposes, known as CHSH inequality. Consider the quantity $\mathcal{S}_{CHSH}$

$$\mathcal{S}_{CHSH} = A(\lambda, \vec{a})B(\lambda, \vec{b}) - A(\lambda, \vec{a})B(\lambda, \vec{b}') + A(\lambda, \vec{a}')B(\lambda, \vec{b}) + A(\lambda, \vec{a}')B(\lambda, \vec{b}'), \quad (5.9)$$

that can also be written in the form

$$\mathcal{S}_{CHSH} = A(\lambda, \vec{a})[B(\lambda, \vec{b}) - B(\lambda, \vec{b}')] + A(\lambda, \vec{a}')[B(\lambda, \vec{b}) + B(\lambda, \vec{b}')]. \quad (5.10)$$

Since both A and B result in ± 1 , then:

$$\mathcal{S}_{CHSH}(\lambda, \vec{a}, \vec{a}', \vec{b}, \vec{b}') = \pm 2. \quad (5.11)$$

Performing a average of $\mathcal{S}_{CHSH}$ in relation to the hidden-variable λ , we obtain

$$-2 \leq \int d\lambda \rho(\lambda) \mathcal{S}_{CHSH}(\lambda, \vec{a}, \vec{a}', \vec{b}, \vec{b}') \leq 2. \quad (5.12)$$

We can write the CHSH parameter $\mathcal{S}_{CHSH}$ in terms of the correlations $P(\vec{a}, \vec{b})$ of Eq.(5.2)

$$\mathcal{S}_{CHSH}(\vec{a}, \vec{a}', \vec{b}, \vec{b}') = P(\vec{a}, \vec{b}) - P(\vec{a}, \vec{b}') + P(\vec{a}', \vec{b}) + P(\vec{a}', \vec{b}'), \quad (5.13)$$

where $|\mathcal{S}_{CHSH}| \leq 2$. Therefore, the inequality Eq.(5.13) holds for any hidden-variable theory.

Quantum mechanics shows that Bell pairs violate the inequality Eq.(5.13). A fermionic Bell pair is given by the singlet state

$$|\Psi\rangle = \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}. \quad (5.14)$$

In the appendix C, we obtain $|\mathcal{S}_{CHSH}| = 2\sqrt{2} > 2$, then violating the inequality Eq.(5.13). Therefore, real localism is incompatible with quantum mechanics predictions.

5.2 Quantum Entanglement in Mesoscopic Systems

We are interested in how the symmetries affect the entanglement production in mesoscopic systems, in special the chaotic quantum billiard. Consider the entanglement of two electrons in a system constituted by a chaotic scattering region connected by two leads at the left and right [42, 43]. We assume 2 independent open channels in each lead. Incident electrons from the left in a range eV above the Fermi energy E_F have the form

$$|\Psi_{\text{in}}\rangle = \prod_{0 < \varepsilon < eV} a_{\text{in},1}^\dagger(\varepsilon) a_{\text{in},2}^\dagger(\varepsilon) |0\rangle. \quad (5.15)$$

The operator $a_{\text{in},i}^\dagger(\varepsilon)$ indicates a fermion occupying the i th channel incident from the left at the energy ε above the Fermi level, whereas $b_{\text{in},i}^\dagger(\varepsilon)$ stands for the right lead. In matrix notation, the income state is

$$|\Psi_{\text{in},i}\rangle = \prod_{\varepsilon} \begin{pmatrix} a_{\text{in}}^\dagger & b_{\text{in}}^\dagger \end{pmatrix} \begin{pmatrix} \frac{1}{2}\sigma_y & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_{\text{in}}^\dagger \\ b_{\text{in}}^\dagger \end{pmatrix} |0\rangle. \quad (5.16)$$

The S -matrix relates the income-outcome states, $|\Psi_{\text{out}}\rangle = S|\Psi_{\text{in}}\rangle$, in the form

$$\begin{pmatrix} a_{\text{out}} \\ b_{\text{out}} \end{pmatrix} = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \begin{pmatrix} a_{\text{in}} \\ b_{\text{in}} \end{pmatrix}. \quad (5.17)$$

Since each lead has two open channels, the scattering matrix is a unitary 4×4 matrix, where r, r', t, t' are submatrices with dimension 2×2 . These submatrices describe the reflection and transmission of incident states from the left or the right. The outcome state $|\Psi_{\text{out}}\rangle$ can be obtained by substituting Eq.(5.17) into Eq.(5.16)

$$|\Psi_{\text{out}}\rangle = \prod_{\varepsilon} (a_{\text{out}}^\dagger r \sigma_y t^T b_{\text{out}}^\dagger + [r \sigma_y r^T]_{12} a_{\text{out},1}^\dagger a_{\text{out},2}^\dagger + [t \sigma_y t^T]_{12} b_{\text{out},1}^\dagger b_{\text{out},2}^\dagger) |0\rangle. \quad (5.18)$$

The first term represents one particle in each lead, whereas the second and the third stand for two particles in the left and right leads, respectively. Therefore, the output state can be written in terms of a new vacuum state $a_{\text{out},1}^\dagger a_{\text{out},2}^\dagger |0\rangle$ in the form

$$|\Psi_{\text{out}}\rangle = \prod_{\varepsilon} (\sqrt{w} |\Phi\rangle + \sqrt{1-w} |0\rangle), \quad (5.19)$$

where $|\Phi\rangle$ is the entangled state

$$|\Phi\rangle = w^{-1/2} c_{\text{out},i}^\dagger \gamma b_{\text{out}}^\dagger |0\rangle, \quad \gamma = \sigma_y r \sigma_y t^T. \quad (5.20)$$

The $c_{\text{out},i} = a_{\text{out},i}^\dagger$ is the hole operator at the left of the chaotic scattering region. Then, the state $|\Psi\rangle$ is a superposition of the vacuum state, $|0\rangle$, and a particle-hole state, $|\Phi\rangle$, with weight $w = \text{Tr } \gamma \gamma^\dagger$.

From now on, we are interested in quantifying the entanglement of the state $|\Phi\rangle$. For this purpose, Wootters [85] defined the quantity concurrence C as the degree of state separability. In our case, Beenakker [43] calculated the concurrence in mesoscopic systems, in special the chaotic quantum billiard. Consider a state of interest $|\psi\rangle$, such that

$$|\psi\rangle = \sum_{i=1}^2 \sum_{j=1}^2 \gamma_{ij} |i\rangle_A |j\rangle_B, \quad (5.21)$$

where i and j represent the open channels in the leads. We calculate the density operator of the left lead ρ_A , for example, by taking the partial trace of the density operator $|\psi\rangle \langle \psi|$

$$\begin{aligned} \rho_A &= \text{Tr}_B |\psi\rangle \langle \psi| = \text{Tr}_B \left\{ \left[\sum_{i=1}^2 \sum_{j=1}^2 \gamma_{ij} |i\rangle_A |j\rangle_B \right] \left[\sum_{i'=1}^2 \sum_{j'=1}^2 \gamma_{j'i'}^* \langle i'|_A \langle j'|_B \right] \right\} \\ &= \text{Tr}_B \left\{ \sum_{i,i',j,j'=1}^2 \gamma_{ij} \gamma_{j'i'}^* |i\rangle_A \langle i'|_A \otimes |j\rangle_B \langle j'|_B \right\} \\ &= \sum_{i,i'=1}^2 \left[\sum_{j=1}^2 \gamma_{ij} \gamma_{ji'}^* \right] |i\rangle_A \langle i'|_A \\ &= \sum_{i,i'=1}^2 (\gamma \gamma^\dagger)_{ii'} |i\rangle_A \langle i'|_A. \end{aligned} \quad (5.22)$$

Wootters also defined the entanglement of formation that connects the concept of entanglement with Shannon's information [86]. The Shannon information for quantum theory is given by [87]

$$\begin{aligned} \mathcal{E} &= -\text{Tr}_A \rho_A \log_2 \rho_A \\ &= -\text{Tr}_A \left\{ \sum_{i,i'=1}^2 (\gamma \gamma^\dagger)_{ii'} |i\rangle_A \langle i'|_A \log_2 \sum_{k,k'=1}^2 (\gamma \gamma^\dagger)_{kk'} |k\rangle_A \langle k'|_A \right\}. \end{aligned} \quad (5.23)$$

To simplify the calculations, we will diagonalize the matrix $\gamma \gamma^\dagger$

$$\gamma \gamma^\dagger = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} \begin{pmatrix} \gamma_{11}^* & \gamma_{12}^* \\ \gamma_{21}^* & \gamma_{22}^* \end{pmatrix} = \begin{pmatrix} |\gamma_{11}|^2 + |\gamma_{12}|^2 & \gamma_{11} \gamma_{21}^* + \gamma_{12} \gamma_{22}^* \\ \gamma_{21} \gamma_{11}^* + \gamma_{22} \gamma_{12}^* & |\gamma_{21}|^2 + |\gamma_{22}|^2 \end{pmatrix}, \quad (5.24)$$

where we obtain, for a normalized state $\text{Tr } \gamma \gamma^\dagger = 1$, the eigenvalues $\lambda_\pm$

$$\lambda_\pm = \frac{1 \pm \sqrt{1 - 4 \det \gamma \gamma^\dagger}}{2}. \quad (5.25)$$

The diagonalized state ρ_A is written as

$$\rho_A = \frac{1 + \sqrt{1 - 4 \det \gamma \gamma^\dagger}}{2} |\lambda_+\rangle \langle \lambda_+| + \left[1 - \left(\frac{1 + \sqrt{1 - 4 \det \gamma \gamma^\dagger}}{2} \right) \right] |\lambda_-\rangle \langle \lambda_-|, \quad (5.26)$$

and, therefore, the entanglement of formation reads

$$\begin{aligned} \mathcal{E} = & - \left(\frac{1 + \sqrt{1 - 4 \det \gamma \gamma^\dagger}}{2} \right) \log_2 \left(\frac{1 + \sqrt{1 - 4 \det \gamma \gamma^\dagger}}{2} \right) \\ & - \left[1 - \left(\frac{1 + \sqrt{1 - 4 \det \gamma \gamma^\dagger}}{2} \right) \right] \log_2 \left[1 - \left(\frac{1 + \sqrt{1 - 4 \det \gamma \gamma^\dagger}}{2} \right) \right]. \end{aligned} \quad (5.27)$$

In a simple way, the entanglement of formation $\mathcal{E}$ is given by

$$\begin{aligned} \mathcal{E} = & -x \log_2 x - (1-x) \log_2 (1-x), \\ x = & \frac{1 + \sqrt{1 - 4 \det \gamma \gamma^\dagger}}{2}. \end{aligned} \quad (5.28)$$

The physical meaning of $\mathcal{E}$ is the cost of the resources needed to prepare a given entangled state [88]. The degree of entanglement is quantified by the concurrence [43, 85]. For an unnormalized state, the concurrence is obtained by

$$C = 2 \sqrt{\det \gamma \gamma^\dagger / \text{Tr} \gamma \gamma^\dagger}. \quad (5.29)$$

Since Eq.(5.28) stands for the normalized state, the concurrence is given by $C = \det \gamma \gamma^\dagger$. Therefore, the entanglement of formation is expressed in terms of the concurrence C .

Now, we can connect the concepts of entanglement with quantum scattering by describing the concurrence in terms of probability transmission eigenvalues T_1 and T_2 of the matrix $t^\dagger t$. First, the weight w is written as

$$\begin{aligned} w = \text{Tr} \gamma \gamma^\dagger &= \text{Tr} (r \sigma_y t^T) (r \sigma_y t^T)^\dagger = \text{Tr} \{ (r \sigma_y t^T) t^* \sigma_y r^\dagger \}, \\ &= \text{Tr} \{ r^\dagger r \sigma_y t^T t^* \sigma_y \} = \text{Tr} \{ (1 - t^\dagger t) \sigma_y (t^\dagger t)^T \sigma_y \}. \end{aligned} \quad (5.30)$$

Supposing that $t^\dagger t$ is diagonalizable, i.e

$$t^\dagger t = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \quad (5.31)$$

where T_1 and T_2 are probability transmission eigenvalues of electrons in channels 1 and 2 of the left lead tunneling to the right one, respectively. Then, we obtain the weight w

$$w = T_1 + T_2 - 2T_1 T_2, \quad (5.32)$$

whereas the $\det \gamma \gamma^\dagger$ reads

$$\begin{aligned} \det \gamma \gamma^\dagger &= \det \left[(r \sigma_y t^T) (r \sigma_y t^T)^\dagger \right] = \det (r \sigma_y t^T t^* \sigma_y r^\dagger), \\ &= \det (r^\dagger r \sigma_y t^T t^* \sigma_y) = \det \{ (1 - t^\dagger t) \sigma_y (t^\dagger t)^T \sigma_y \}, \\ &= T_1 (1 - T_1) T_2 (1 - T_2). \end{aligned} \quad (5.33)$$

Hence, the concurrence C is obtained by

$$C = \frac{2\sqrt{T_1(1-T_1)T_2(1-T_2)}}{T_1 + T_2 - 2T_1T_2}. \quad (5.34)$$

This is the general formula of the concurrence, written in terms of the transmission eigenvalues T_1 and T_2 . In the opaque regime, where the barrier's transparency Γ tends to 0 ($\Gamma \rightarrow 0$), the transmission eigenvalues T_1 and T_2 become very small $T_1, T_2 \ll 1$ and the concurrence

$$C \approx \frac{2\sqrt{T_1T_2}}{T_1 + T_2}. \quad (5.35)$$

As we mentioned before, the concurrence and entanglement of formation quantify the degree of separability of a given state and the cost of the resources needed to prepare this given entangled state, respectively. However, another way to explore the entanglement of quantum states is through the violations of Bell's inequality. In mesoscopic physics, the Bell's parameter S_{CHSH} is usually defined in terms of the current noises or correlators [42, 43]

$$N_{ij} = \int_{-\infty}^{\infty} dt \overline{\delta I_{L,i}(t) \delta I_{R,j}(0)}, \quad (5.36)$$

where N_{ij} is the correlation between the current fluctuations of the i th channel in the left (L) lead and the j th channel in the right (R) lead. In low temperatures ($kT \ll eV$), the correlator for low frequencies and opaque regime $\Gamma \rightarrow 0$ is given by

$$N_{ij} = (-e^3 V / h) |(rt^\dagger)_{ij}|^2. \quad (5.37)$$

In Appendix D, we present the relationship between these correlators and the shot noise power, which allows us to write the ratio between the noises

$$E = \frac{\text{Tr} \sigma_z r t^\dagger \sigma_z t r^\dagger}{\text{Tr} r^\dagger r t^\dagger t} = \frac{N_{11} + N_{22} - N_{12} - N_{21}}{N_{11} + N_{12} + N_{21} + N_{22}}. \quad (5.38)$$

By mixing the channels locally in the left and right leads, the reflection and transmission matrices r and t are transformed as $r \rightarrow U_L r$, $t \rightarrow U_R t$ with unitary 2×2 matrices U_L, U_R , such that the relation between the noises becomes

$$E(U_L, U_R) = \frac{\text{Tr} U_L^\dagger \sigma_z U_L r t^\dagger U_R^\dagger \sigma_z U_R t r^\dagger}{\text{Tr} r^\dagger r t^\dagger t}. \quad (5.39)$$

Then, the Bell-CHSH parameter (Clauser-Horne-Shimony-Holt) is given by [84]

$$S_{\text{CHSH}} = E(U_L, U_R) + E(U_L', U_R) + E(U_L, U_R') - E(U_L', U_R'). \quad (5.40)$$

If $|S_{\text{CHSH}}| > 2$ for some set of unitary matrices U_L, U_R, U_L', U_R' we say the state is entangled. The Ref [89] shows the maximum violation of Bell's inequality

$$S_{\text{CHSH}, \text{max}} = 2[1 + 4T_1 T_2 (T_1 + T_2)^{-2}]^{1/2} > 2. \quad (5.41)$$

In tunneling approximation regime $T_1, T_2 \ll 1$, the concurrence is given by $C \approx 2\sqrt{T_1 T_2}/(T_1 + T_2)$. Therefore, we can obtain the relation $\mathcal{S}_{\text{CHSH, max}} = 2(1 + C^2)^{1/2}$ between the concurrence and the maximal violation of the CHSH inequality [90].

Dephasing can be investigated in the channels by introducing random phase factors in each edge channel.

$$U_L \rightarrow U_L \begin{pmatrix} e^{i\phi_1} & 0 \\ 0 & e^{i\phi_2} \end{pmatrix}, \quad U_R \rightarrow U_R \begin{pmatrix} e^{i\psi_1} & 0 \\ 0 & e^{i\psi_2} \end{pmatrix}. \quad (5.42)$$

Averaging $E(U_L, U_R)$ over the random phases uniformly in $(0, 2\pi)$, we find

$$\begin{aligned} \mathcal{S}_{\text{Dephase}} &= \frac{2|\text{Tr} \sigma_z r t^\dagger \sigma_z t r^\dagger|}{\text{Tr} r^\dagger r t^\dagger t} \\ &= \frac{2|N_{11} + N_{22} - N_{12} - N_{21}|}{N_{11} + N_{22} + N_{12} + N_{21}} \leq 2. \end{aligned} \quad (5.43)$$

Hence, for strong dephasing, there is no violation of the Bell inequality $|\mathcal{S}| \leq 2$.

Without the tunneling approximation assumption, i.e, for any Γ , we can calculate the maximal violation of the CHSH inequality using the correlator

$$N_{\mathbf{ab}} = \frac{[\text{Tr}(\mathbf{a} \cdot \boldsymbol{\sigma}) r r^\dagger][\text{Tr}(\mathbf{b} \cdot \boldsymbol{\sigma}) t t^\dagger] - \text{Tr}(\mathbf{a} \cdot \boldsymbol{\sigma}) r t^\dagger (\mathbf{b} \cdot \boldsymbol{\sigma}) t r^\dagger}{(\text{Tr} r r^\dagger)(\text{Tr} t t^\dagger) - \text{Tr} r t^\dagger t r^\dagger}, \quad (5.44)$$

and an appropriate base to measure the violation of CHSH inequality of the degrees of freedom

$$\begin{cases} U_{\mathbf{a}}^\dagger \sigma_z U_{\mathbf{a}} = \mathbf{a} \cdot \boldsymbol{\sigma}, \\ U_{\mathbf{b}}^\dagger \sigma_z U_{\mathbf{b}} = \mathbf{b} \cdot \boldsymbol{\sigma}. \end{cases} \quad (5.45)$$

We can use the correlator $N_{\mathbf{ab}}$ to obtain the Bell-CHSH parameter

$$\mathcal{S}_{\text{CHSH}} = |N_{\mathbf{ab}} + N_{\mathbf{ab}'} + N_{\mathbf{a}'\mathbf{b}} - N_{\mathbf{a}'\mathbf{b}'}| \leq 2, \quad (5.46)$$

and verify the relation between the concurrence and the maximal violation of the CHSH inequality beyond the tunneling regime [91],

$$\mathcal{S}_{\text{CHSH, max}} = 2\sqrt{1 + C^2}. \quad (5.47)$$

CHAPTER 6

Results

6.1 Numerical Simulation Method

The results of this thesis come from numerical simulation. In general, the method used consists in generating random matrices to simulate the Hamiltonian H of the chaotic billiard. The geometry of the billiard edge can be changed through electrostatic potentials applied by gates. In our case, each edge setting provides a random matrix and, therefore, we analyze the statistical properties of an ensemble of random Hamiltonians. In Chapter 4, we explore how the Hamiltonians are constructed for all 10 classes of Altland-Zirnbauer classifications. In our system, the billiard consists of a central cavity connected by two leads, shown in Fig.(6.1). Each lead has N_i open channels connected by its respective electrochemical potential μ_i , where $i = 1, 2$. For the sake of simplicity, in this thesis, we will only consider the case of symmetric leads $N_1 = N_2 = N$.

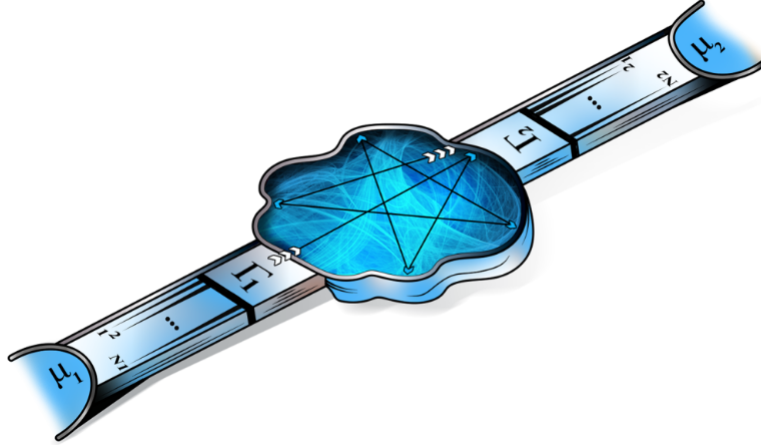


Figure 6.1 Quantum billiard setup. The central cavity is connected by two leads. Each lead has N_i open channels connected by its respective electrochemical potential μ_i . The tunneling rate between the leads and the chaotic central cavity is represented by Γ_i .

The tunneling rate between the leads and the chaotic central cavity is represented by Γ_i . Experimentally, the interface between the leads and the chaotic cavity is known as quantum point contact (QPC) [55, 92, 56, 93]. In our case, the $\Gamma_i \in [0, 1]$, where $\Gamma \rightarrow 0$ and $\Gamma = 1$ represent the opaque and transparent barrier regimes, respectively. The tunneling rate is responsible for modulating the coupling between the open channels of the leads and the resonances of the chaotic cavity through the coupling matrix W .

For the Schrödinger billiard, the $M \times (N_1 + N_2)$ coupling matrix W is given by

$$\begin{aligned} [W^1]_{ij} &= \frac{(1 - \sqrt{1 - \Gamma})}{\sqrt{\Gamma}} \sqrt{\frac{2\lambda}{\pi(M+1)}} \sin\left(\frac{ij\pi}{M+1}\right), \\ [W^2]_{ij} &= \frac{(1 - \sqrt{1 - \Gamma})}{\sqrt{\Gamma}} \sqrt{\frac{2\lambda}{\pi(M+1)}} \sin\left(\frac{i(j+N_1)\pi}{M+1}\right), \end{aligned} \quad (6.1)$$

with $i = 1, \dots, M$ and $j = 1, \dots, N$,

and $W = [W^1, W^2]$. For the symplectic class, breaking the spin rotation symmetry doubles the matrix dimension W due to the spin degree of freedom, i.e. $W \mapsto W \otimes 1_2$. Thus, the additional symmetry $W = \sigma_y W \Sigma_y$ must be satisfied.

For the Dirac billiard, the chiral/sub-lattice degree of freedom doubles the dimension of the coupling matrix W . In Chapter 4, we presented that the coupling matrix W follows the symmetry condition

$$W = \sigma_z W \Sigma_z, \quad (6.2)$$

where $\sigma_z = \sigma_z \otimes 1_M$ and $\Sigma_z = 1_N \otimes \sigma_z$. Then, for the chiral ensemble, we construct the matrices

$$\begin{aligned} [W^1]_{ij} &= \frac{(1 - \sqrt{1 - \Gamma})}{\sqrt{\Gamma}} \sqrt{\frac{2\lambda}{\pi(M+1)}} \sin\left(\frac{ij\pi}{M+1}\right), \\ [W^2]_{ij} &= \frac{(1 - \sqrt{1 - \Gamma})}{\sqrt{\Gamma}} \sqrt{\frac{2\lambda}{\pi(M+1)}} \sin\left(\frac{i(j+2N_1)\pi}{M+1}\right), \end{aligned} \quad (6.3)$$

with $i = 1, \dots, M$ and $j = 1, \dots, 2N$,

and we impose the symmetry constraint by setting W^i in the form

$$W^i = \begin{pmatrix} [W^i]_1 & 0 & [W^i]_3 & 0 & \dots & [W^i]_{2N-1} & 0 \\ 0 & -[W^i]_2 & 0 & -[W^i]_4 & \dots & 0 & -[W^i]_{2N} \end{pmatrix}, \quad (6.4)$$

where the lower index k of $[W^i]_k$ represents the k -column of the matrices in Eq.(6.3). In a single-channel case $N = 1$, for example, the chiral symmetry implies the block structure of W

$$W^i = \begin{pmatrix} [W^i]_1 & 0 \\ 0 & -[W^i]_2 \end{pmatrix}, \quad W^i = \sigma_z W^i \Sigma_z, \quad (6.5)$$

the index $i = 1, 2$ represent the first and second column of the matrices W^i in Eq.(6.3), respectively. Similar to the Schrödinger billiard case, the W is obtained by $W = [W^1, W^2]$.

For the Andreev billiard, we consider the coupling matrix W in the form [94]

$$\begin{aligned} W_{nm} &= w_n \delta_{nm}, \quad 1 \leq n \leq M, \quad 1 \leq m \leq 2N, \\ |w_n|^2 &= \frac{M \delta_0}{\pi^2 \Gamma_n} (2 - \Gamma_n - 2\sqrt{1 - \Gamma_n}). \end{aligned} \quad (6.6)$$

Then, we construct W^1 and W^2 such that

$$\begin{aligned} W^1 &= \frac{(1 - \sqrt{1 - \Gamma})}{\sqrt{\Gamma}} \sqrt{\frac{\lambda}{\pi}} \delta_{i,j}, \\ W^2 &= \frac{(1 - \sqrt{1 - \Gamma})}{\sqrt{\Gamma}} \sqrt{\frac{\lambda}{\pi}} \delta_{(i+N_1),j}, \end{aligned} \quad (6.7)$$

with $i = 1, \dots, M$ and $j = 1, \dots, N$,

and $W = [W^1, W^2]$ for the class D. For the class DIII, the additional symmetry $W = \sigma_y W \Sigma_y$ doubles the matrix dimension due to the spin degree of freedom, then the coupling matrix becomes $W \mapsto W \otimes 1_2$. For both classes C and CI, the symmetry constraint $W = \tau_y W \tau_y$ doubles the matrix dimension due to the Bogoliubov quasiparticle degree of freedom, then $W \mapsto W \otimes 1_2$.

From now on, we can obtain the Mahaux-Weindemuller scattering matrix

$$S = 1 - 2i\pi W^T \frac{1}{(E - H + i\pi W W^T)} W, \quad S = \begin{bmatrix} r & t' \\ t & r' \end{bmatrix}, \quad (6.8)$$

and, consequently, physical observables such as conductance, shot noise, concurrence and entanglement of formation, and Bell's inequality. When plotting the simulation, we will also show the exact results obtained by the diagrammatic method [95] or supersymmetric nonlinear sigma model [94, 96]. Specifically, by simulating both conductance and shot noise power, we can effectively explore the thermodynamic uncertainty relations in mesoscopic devices [97], which goes beyond the scope of this thesis.

6.2 Conductance

The conductance quantifies the electric current passing through the mesoscopic device. In our case, the conductance is given by the Landauer formula

$$G = G_0 \text{Tr} t t^\dagger = d G_0 \times \sum_{i=1}^N T_i, \quad (6.9)$$

where $G_0 = 2e^2/h$ is the quantum of conductance, the d represents the spin and/or particle-hole degeneracy and $\{T_i\}_{i=1}^N$ are the eigenvalues of the transmission matrix $t t^\dagger$.

• Schrödinger Billiard - SB

In this thesis, we perform the Monte Carlo method to generate 10^6 random Hamiltonians with dimensions 100×100 and calculate their respective conductance. Fig.(6.2) is shown the distribution of conductance for the Schrödinger billiard with different numbers of open channels N in the ideal case (transparent barrier $\Gamma = 1$), recovering some results of literature [98, 99, 100]. The orthogonal (AI), unitary (A), and symplectic (AII) symmetry classes are indicated by the index $\beta = 1, 2$ and 4, respectively. The distribution of $\min(N_1, N_2)$ nonzero transmission eigenvalues is [21]

$$P(\{T_n\}) \propto \prod_{n < m} |T_n - T_m|^\beta \prod_k T_k^{\frac{1}{2}\beta(|N_2 - N_1| + 1 - 2/\beta)}. \quad (6.10)$$

In particular, our simulations agree with the single-channel ballistic point contacts $N_1 = N_2 = N = 1$ probability distribution of the conductance [99, 100]

$$P(g) \propto g^{-1+\beta/2}, \quad g = G/G_0. \quad (6.11)$$

As the number of channels increases, the distribution of conductance becomes a normal distribution $\mathcal{N}(\mu, \sigma)$. The regime with many open channels $N \gg 1$ is known as the semi-classical regime, where we expect a correspondence with classical physics. As the number of channels increases, quantum properties, such as quantum coherence, become imperceptible to the transport of electrons in the classical limit. This reveals a universal behavior of the system in the sense that the distribution of conductance will follow the normal distribution $\mathcal{N}(\mu, \sigma)$ when $N \rightarrow \infty$, with μ and σ depending on the number of open channels. We say that this is the semi-classical regime, where small amounts of channels ($N \approx 5$) are needed to achieve it. In the context of a single-channel (quantum regime), the conductance distribution deviates significantly from Gaussian, yet when considering the sum of conductance across numerous open channels, a Gaussian distribution emerges. This phenomenon is a direct outcome of the Central Limit Theorem (CLT).

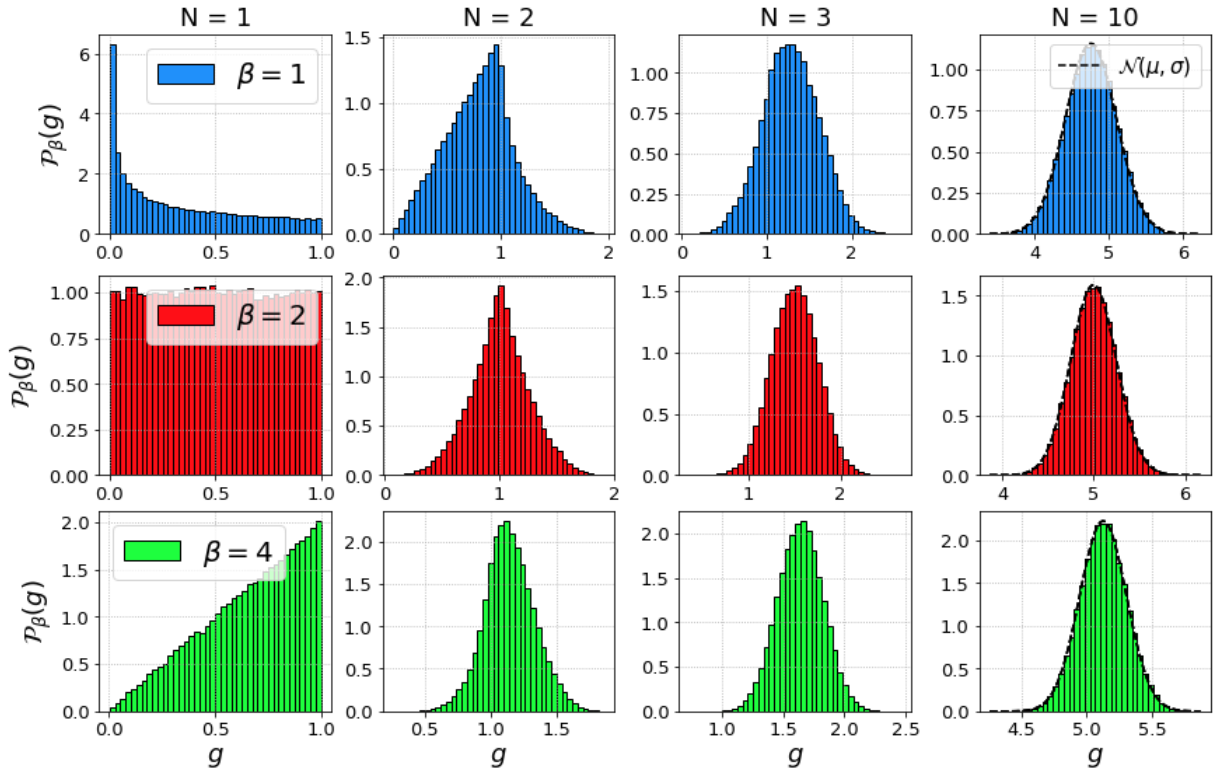


Figure 6.2 Conductance distribution for the Schrödinger billiard with different numbers of open channels N in the ideal case (transparent barrier regime $\Gamma = 1$). We consider $M = 100$ resonances and 10^6 random Hamiltonians.

The mean and variance of the conductance distribution are shown in Fig.(6.3). Our simula-

tions agree with the analytical results [99] for the average conductance $\langle g \rangle$

$$\langle g \rangle = \frac{N_1 N_2}{N_1 + N_2 - 1 + 2/\beta}. \quad (6.12)$$

The average of conductance is proportional to the number of open channels, i.e. $\langle g \rangle \propto N$. However, we can observe a small difference in the average conductance for different symmetry classes. This difference is associated with the underlying symmetry of the system and is called *quantum interference correction* (QIC). The error bars in Fig.(6.3) are smaller in size than the points associated with the values, thus we neglect the error bar in our simulations due to a large number of realizations.

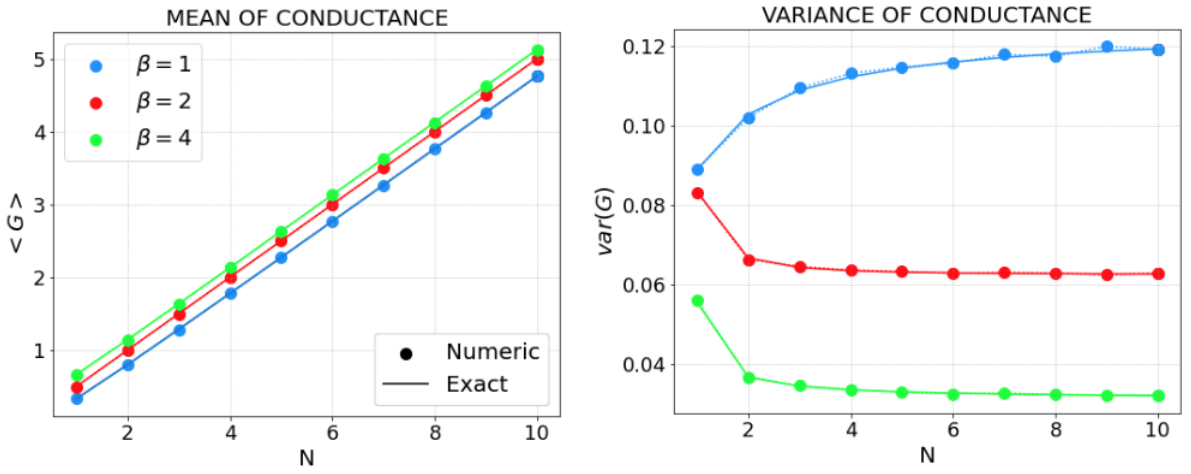


Figure 6.3 Mean $\langle g \rangle$ and variance $\text{var}(g)$ of conductance for the Schrödinger billiard with different numbers of open channels N in the ideal case (transparent barrier regime $\Gamma = 1$).

Our simulations agree with analytical results [101] for the variance of conductance

$$\text{var}(g) = \frac{2N_1 N_2 (N_1 - 1 + 2/\beta)(N_2 - 1 + 2/\beta)}{\beta(N_T - 1 + 4/\beta)(N_T - 2 + 2/\beta)(N_T - 1 + 2/\beta)^2}, \quad N_T = N_1 + N_2. \quad (6.13)$$

In special, the variance remains constant as N increases. Thus, the magnitude of the conductance fluctuations, $\text{var}(g)$, becomes independent of the size of the system for chaotic billiards [102]. This fact is known as universal conductance fluctuations (UCF). In the semi-classical regime, the variance of conductance for the orthogonal, unitary, and symplectic classes is $1/8$, $1/16$, and $1/32$, respectively.

Until now we have discussed the statistical properties of conductance for the Schrödinger billiard in the ideal case (the transparent barrier $\Gamma = 1$). However, the barrier plays a fundamental role in the quantum transport properties and, in particular, the conductance distribution. Fig.(6.4) shows the effect of tunneling on the conductance distribution in the extreme quantum regime $N = 1$. First, we see the conductance distribution tends to focus on 0 for all symmetry classes. As $\Gamma \rightarrow 0$, the barrier becomes bigger causing the particle tunneling probability, and

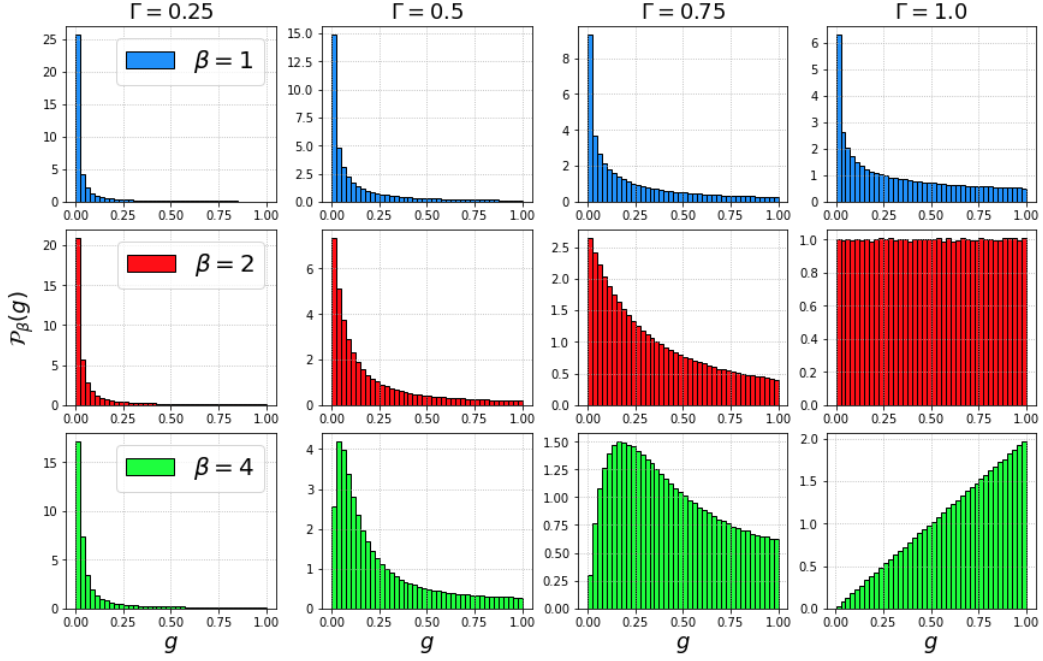


Figure 6.4 Conductance distribution with different values of the barrier transparency Γ in the extreme quantum regime $N = 1$. We consider $M = 100$ resonances and 10^6 random Hamiltonians.

consequently the conductance, to decrease considerably. Fig.(6.5) shows our numerical simulations of the average of conductance as a function of the barrier transparency Γ for different numbers of open channels N in the leads.

For the semi-classical regime, our result agrees with the analytical formula of the conductance average as a function of the barrier transparency Γ for symmetric leads [103, 95]

$$\langle g \rangle = \frac{1}{2}N\Gamma + \frac{1}{4}(1 - 2/\beta)\Gamma. \quad (6.14)$$

Therefore, the conductance average depends linearly on the barrier transparency Γ for the Schrödinger billiard.

We also present the variance of conductance as a function of the barrier transparency Γ in Fig.(6.6). In the semi-classical regime, the analytical result was obtained by [103, 95]

$$\text{var}(g) = \frac{1}{8\beta} \left[1 + (1 - \Gamma)^2 \right]. \quad (6.15)$$

Although there is an agreement between the analytical result and the simulation for the ideal case, both results differ in the opaque limit. This divergence can be a limitation of the numerical method used due to the singular values for the conductance in the opaque regime.

We have mentioned the quantum interference correction (QIC) in a brief way. However, the QIC strongly affects the conduction properties. In Schrödinger billiard (SB), the QIC δg is obtained by subtracting the semi-classical term, $\langle g \rangle_{\text{semi-classical}}$, of the orthogonal and symplectic conductance averages

$$\delta g_{\beta=1 \text{ or } 4} = \langle g \rangle_{\beta=1 \text{ or } 4} - \langle g \rangle_{\text{semi-classical}}. \quad (6.16)$$

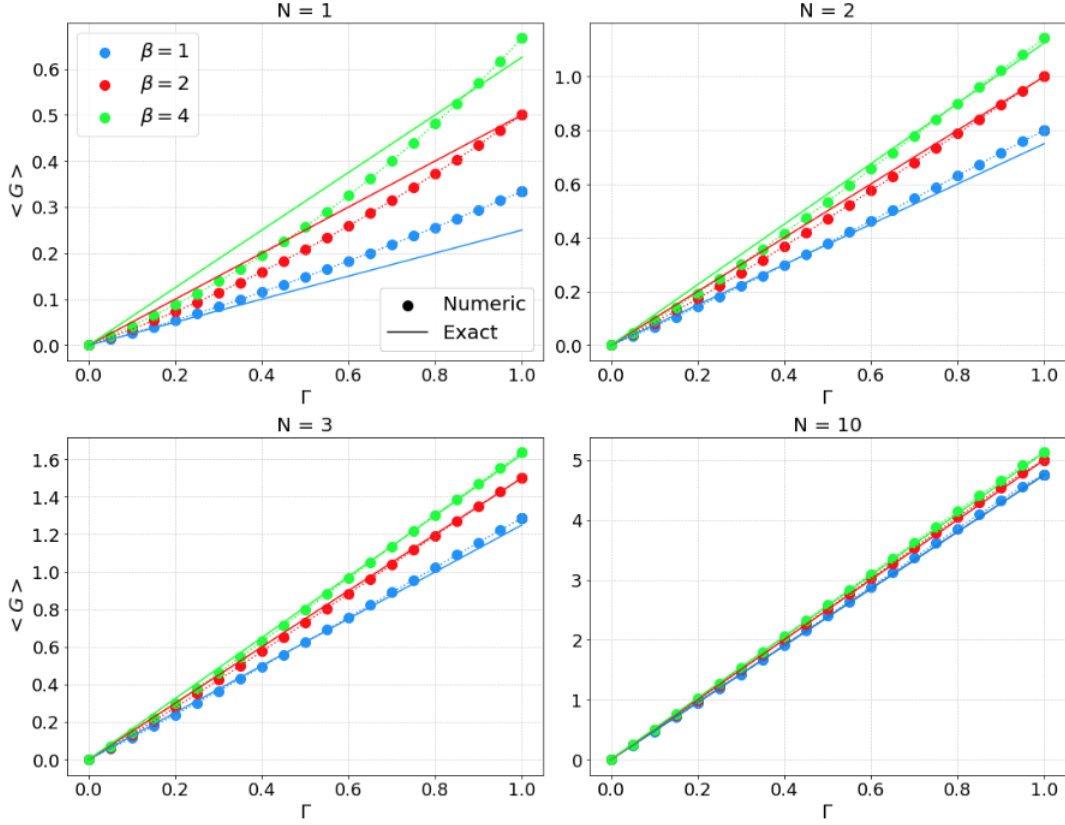


Figure 6.5 Average of conductance as a function of the barrier transparency Γ for different numbers of open channels N in the leads. We consider $M = 100$ resonances and 10^6 random Hamiltonians.

The semi-classical term in the ideal case is given by $\langle g \rangle_{\text{semi-classical}} = N_1 N_2 / (N_1 + N_2)$, where $\langle g \rangle_{\text{semi-classical}} = N/2$ for symmetrical leads $N_1 = N_2 = N$. For a non-ideal case, the semi-classical term is given by $\langle g \rangle_{\text{semi-classical}} = N\Gamma/2$. The QIC of conductance for the transparent barrier $\Gamma = 1$ case is given by

$$\delta g = \left(1 - \frac{2}{\beta}\right) \frac{1}{4}. \quad (6.17)$$

In Fig.(6.7), we present our numerical results of the QIC for the Schrödinger billiard, which agrees with the analytical result in the ideal case $\Gamma = 1$ (left panel). The $\delta g_{\beta=1}$ and $\delta g_{\beta=4}$ present negative and positive QIC for the conductance, respectively. Since the QIC is negative for $\beta = 1$, the conduction of electrons in the orthogonal class becomes less than in the unitary symmetry case. Quantum interference in systems that preserve both time-reversal and spin rotation symmetries makes electrons more localized, thus reducing the value of the conductance. This phenomenon is known as *weak localization* [104]. In the opposite way, the QIC is positive for $\beta = 4$, then the conduction in the symplectic class becomes larger than the unitary case. Therefore, the quantum interference in systems that breaks SRS but preserve TRS makes electrons less localized, increasing the value of the conductance. This phenomenon is known as *weak antilocalization* [105].

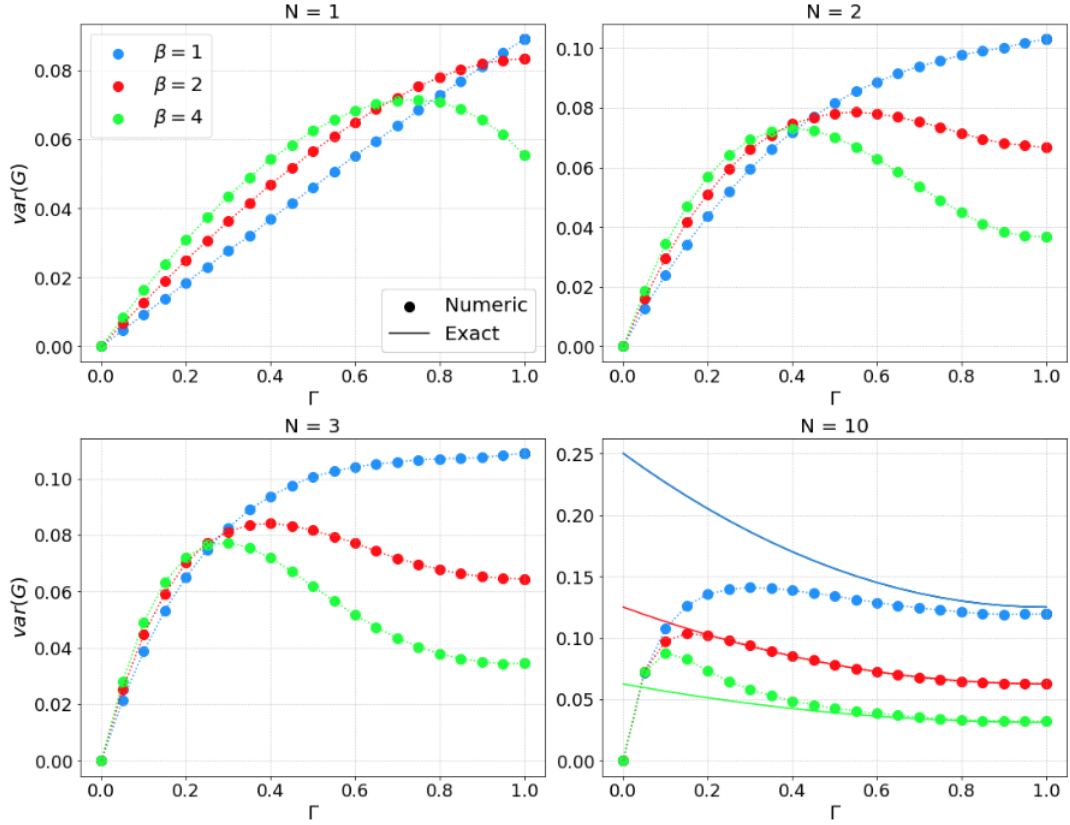


Figure 6.6 Variance of conductance as a function of the barrier transparency Γ for different numbers of open channels N in the leads.

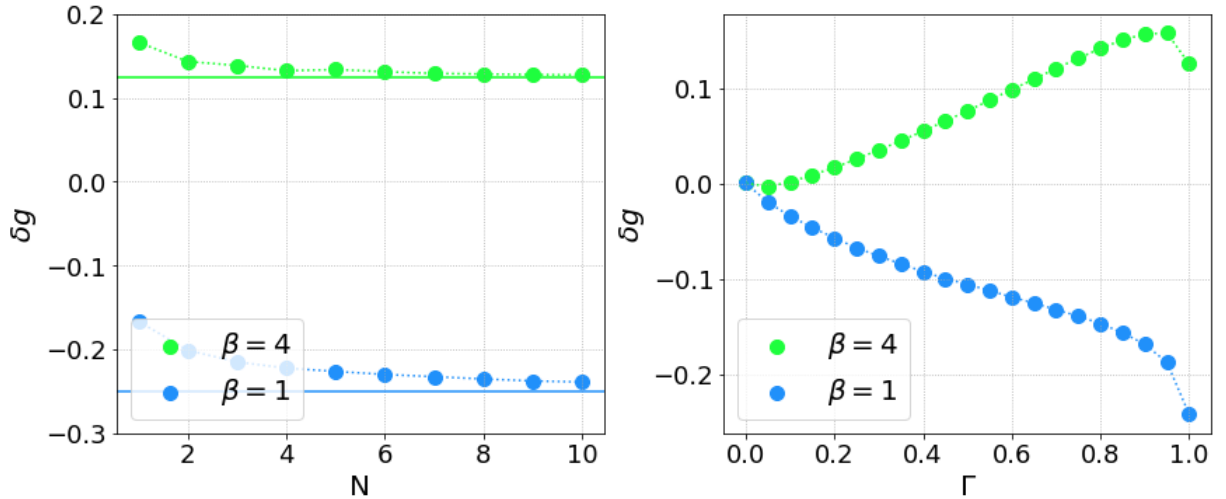


Figure 6.7 Quantum Interference Correction (QIC) of conductance for the Schrödinger billiard as a function of the number of open channels N (left panel) and as a function of the transparency barrier Γ in the semi-classical regime (right panel). The solid lines represent the analytical result Eq.(6.17).

• Dirac Billiard - DB

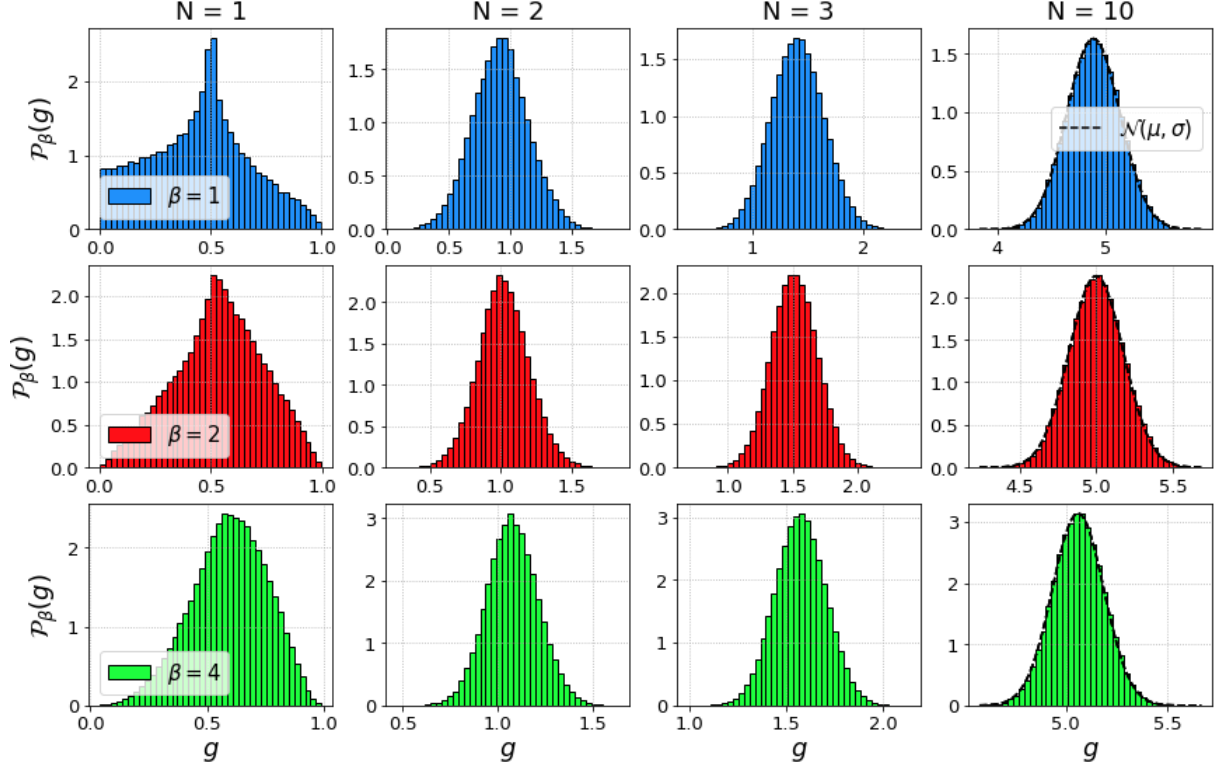


Figure 6.8 Conductance distribution for the Dirac billiard with different numbers of open channels N in the ideal case (transparent barrier regime $\Gamma = 1$). We consider $M = 2 \times 100$ resonances and 10^6 random Hamiltonians.

We perform the simulation with 10^6 random Hamiltonians with dimensions 200×200 and calculate their respective conductance. Dirac billiards presents a challenge because the matrix dimensions double due to the chiral/sub-lattice degree of freedom. The chiral orthogonal (BDI), chiral unitary (AIII) and chiral symplectic (CII) symmetry classes are indicated by the index $\beta = 1, 2$ and 4 , respectively. In Fig.(6.8), we plot the distribution of conductance with different numbers of open channels N in the ideal case. Ref.[106] present numerical simulations of the conductance distribution for the extreme quantum regime $N = 1$, where our results are in agreement with the literature. Ref.[75] shows the distribution of transmission probabilities T_1, T_2 for the class BDI (see Fig. 28 of [75] and compare with $P_{\beta=1}(G)$ for the $N=1$ in Fig.(6.8))

$$P_{BDI}(T_1, T_2) \propto (T_1 T_2)^{-1/2} (1 - T_1)^{-1/2} (1 - T_2)^{-1/2} \sqrt{1 - \min(T_1, T_2)}. \quad (6.18)$$

Similar to the Schrödinger billiard, the conductance distribution converges to the normal distribution $\mathcal{N}(\mu, \sigma)$ as N increases, reaching the semi-classical regime. Again, this is a consequence of the Central Limit Theorem (CLT).

The mean and variance of the conductance distribution are shown in Fig.(6.9). Our simulations agree with the analytical results for both average and variance of conductance [106].

First, the conductance average $\langle g \rangle$ is given

$$\langle g \rangle = \frac{4\beta N_1 N_2 N_T}{(\beta N_T + 1)(2N_T - 1)}, \quad (6.19)$$

where $g = G/G_0$ with $G_0 = 2e^2/h$ and $N_T = N_1 + N_2$. In a similar way, the analytical result for the variance of conductance $\text{var}(g)$ [106]

$$\text{var}(g) = \begin{cases} \frac{16N_1 N_2 N_T (3 + 2N_T^3 + 4N_1 N_2 N_T^2 - 4N_T - 4N_1 N_2 - 5N_1^2 - 5N_2^2)}{(2N_T - 3)(2N_T - 1)^2 (N_T + 3)(N_T + 1)^2 (2N_T + 1)} & (\beta = 1), \\ \frac{8N_1 N_2 (3 + 16N_1 N_2 N_T^2 - 6N_T^2 - 6N_1^2 - 6N_2^2)}{(2N_T - 3)(2N_T + 3)(2N_T + 1)^2 (2N_T + 1)^2} & (\beta = 2), \\ \frac{32N_1 N_2 N_T (3 - 16N_1 N_2 + 8N_T - 20N_1^2 - 20N_2^2 - 16N_T^3 + 64N_1 N_2 N_T^2)}{(4N_T + 3)(4N_T + 1)^2 (2N_T - 3)(2N_T - 1)^2 (4N_T - 1)} & (\beta = 4). \end{cases} \quad (6.20)$$

Analogous to the Schrödinger billiard, the variance remains constant as N increases. Thus, it reveals the universal conductance fluctuations in chiral billiards. Comparing the variance of Schrödinger and Dirac billiards, the variance of conductance for the Schrödinger billiard is twice the Dirac one. This fact is just a matter of normalization due to the chiral degree of freedom, then the $\text{var}(g)_{DB} \approx \text{var}(g)_{SB}$ when we consider that one open channel $N = 1$ has two chiral degrees of freedom.

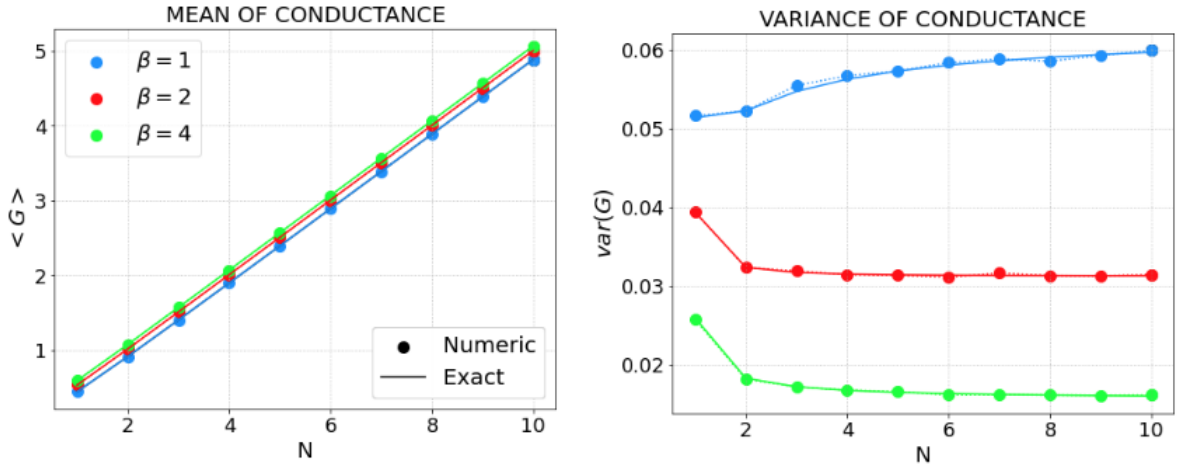


Figure 6.9 Mean $\langle g \rangle$ and variance $\text{var}(g)$ of conductance for the Dirac billiard with different numbers of open channels N in the ideal case (transparent barrier regime $\Gamma = 1$).

Until now we have discussed the conductance for the ideal Dirac billiard, i.e., the transparent barrier case $\Gamma = 1$. Now we will explore non-ideal Dirac billiards, where the barrier transparency is arbitrary $\Gamma \in [0, 1]$. In this regime, analytical results are challenging and remain open problems. Fig.(6.10) shows the effect of tunneling on the conductance distribution in the extreme quantum regime $N = 1$ for the Dirac billiard. Similar to the Schrödinger case, the conductance distribution tends to focus on 0 as Γ goes to the opaque regime. However, the distributions for the chiral orthogonal and chiral unitary present cusps in 0.5 when Γ increases, this may be associated with the dependence on Majorana zero-modes (see the dependence of conductance distribution on the number ν of Majorana zero-modes in Fig.28 of the Ref.[75]).

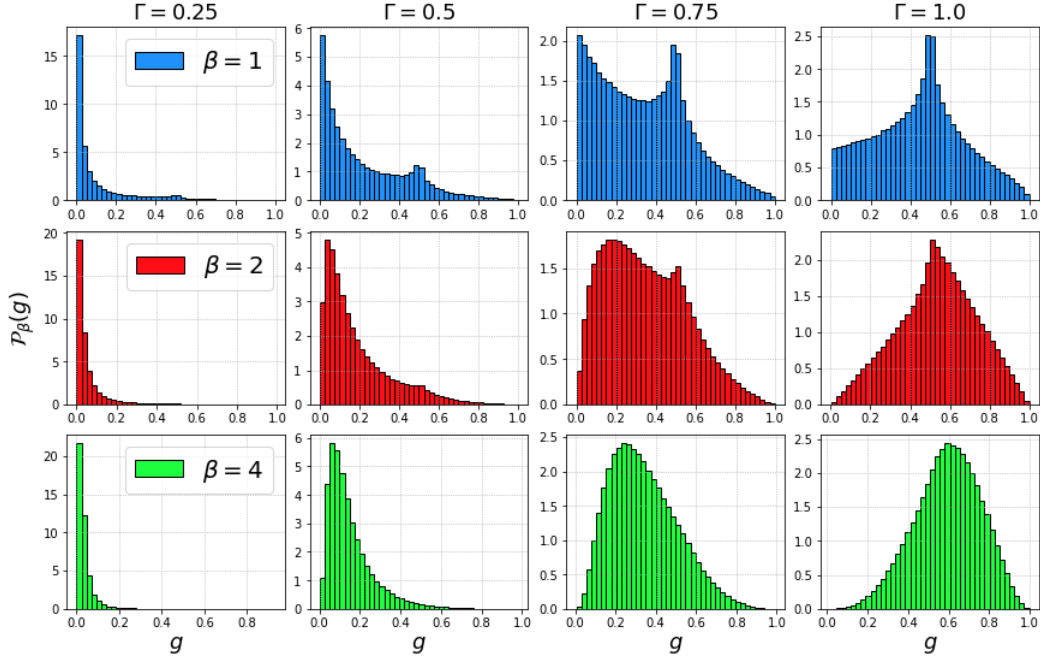


Figure 6.10 Conductance distribution with different values of the barrier transparency Γ in the extreme quantum regime $N = 1$ for the Dirac billiard. We consider $M = 2 \times 100$ resonances and 10^6 random Hamiltonians.

In Fig.(6.11), we present our numerical simulations of the conductance average $\langle G \rangle$ as a function of the barrier transparency Γ for different numbers of open channels N in the leads. Similar to the Schrödinger billiard, the conductance average tends to depend linearly on the barrier transparency Γ for the semi-classical regime. However, even for the semi-classical regime, the dependence on statistical properties of conductance, such as average and variance, remains without an analytical solution. Therefore, there is a strong motivation for the numerical investigations and the obtention of general results in Dirac billiards.

In Fig.(6.12), we present our results of the conductance variance $\text{var}(g)$ as a function of the barrier transparency Γ for different numbers of open channels N in the leads. Although the dependence of variance on the transparency barrier for the Dirac billiard is similar to the Schrödinger case (see Fig.(6.6) for the semi-classical regime $N = 10$), the lack of analytical results makes us unable to carry out conclusions. However, we can see an approximate linear behavior of the average conductance for the semi-classical regime, similar to the Schrödinger case.

We also investigate the QIC for the conductance in Dirac billiards. The QIC in Dirac billiards is obtained [107] by subtracting the semi-classical term, $\langle g \rangle_{\text{semi-classical}} = N\Gamma/2$, of the orthogonal and symplectic conductance averages, then

$$\delta g = \frac{1}{2} \left(1 - \frac{2}{\beta} \right) \frac{1}{4}, \quad (6.21)$$

where the factor $1/2$ is associated with the degeneracy of the chiral/sub-lattice degree of freedom. The left panel of Fig.(6.13) shows that our numerical results agree with the analytical

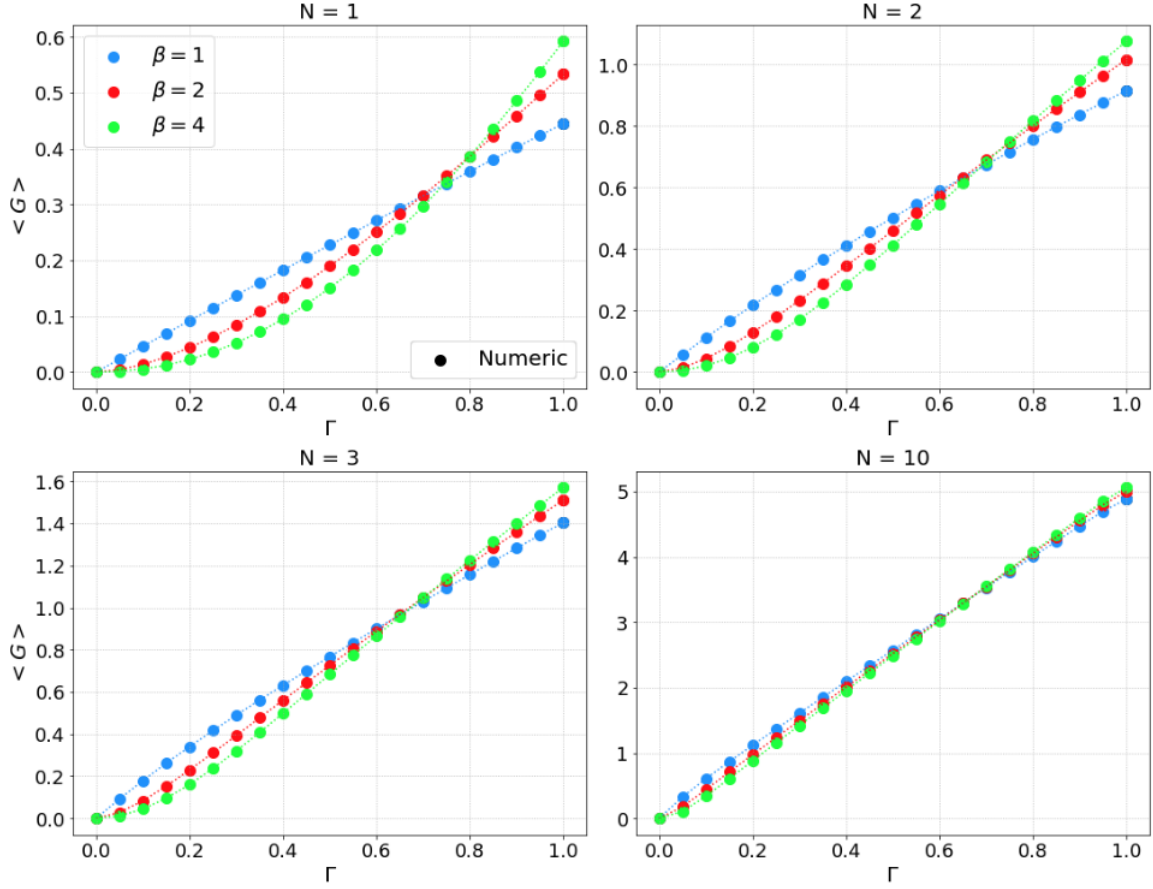


Figure 6.11 Conductance average as a function of the barrier transparency Γ for different numbers of open channels N in the leads for Dirac billiard.

one. The right panel shows a counterintuitive property present in the quantum transport of systems governed by the Dirac equation. The Klein paradox [108] refers to the amplification of the tunneling probability in relativistic systems if the potential barrier is increased. Depending on the height of the barrier, an incident electron with enough energy can create a pair of particle-antiparticle when it hits the barrier. Hence, free antiparticle currents can be detected in the barrier region. This unusual property can be seen as we decrease the barrier transparency Γ . The QIC for the chiral orthogonal class $\beta = 1$ goes from weak localization ($\delta g < 0$) to a weak antilocalization ($\delta g > 0$) regime as the Γ decreases. Conversely, the QIC for the chiral symplectic class $\beta = 4$ goes from weak antilocalization ($\delta g > 0$) to a weak localization ($\delta g < 0$) regime as Γ decreases. Thus, the barrier transparency Γ can change the system from a weak localized to a weak anti-localized regime and vice-versa. Therefore, these transitions manifest behavior that may be associated with Klein's tunneling.

• Andreev Billiard - AB

For the Andreev billiard, we perform a simulation with 10^6 random Hamiltonians and calculate the conductance for each realization. The Andreev billiard is constituted by the sym-

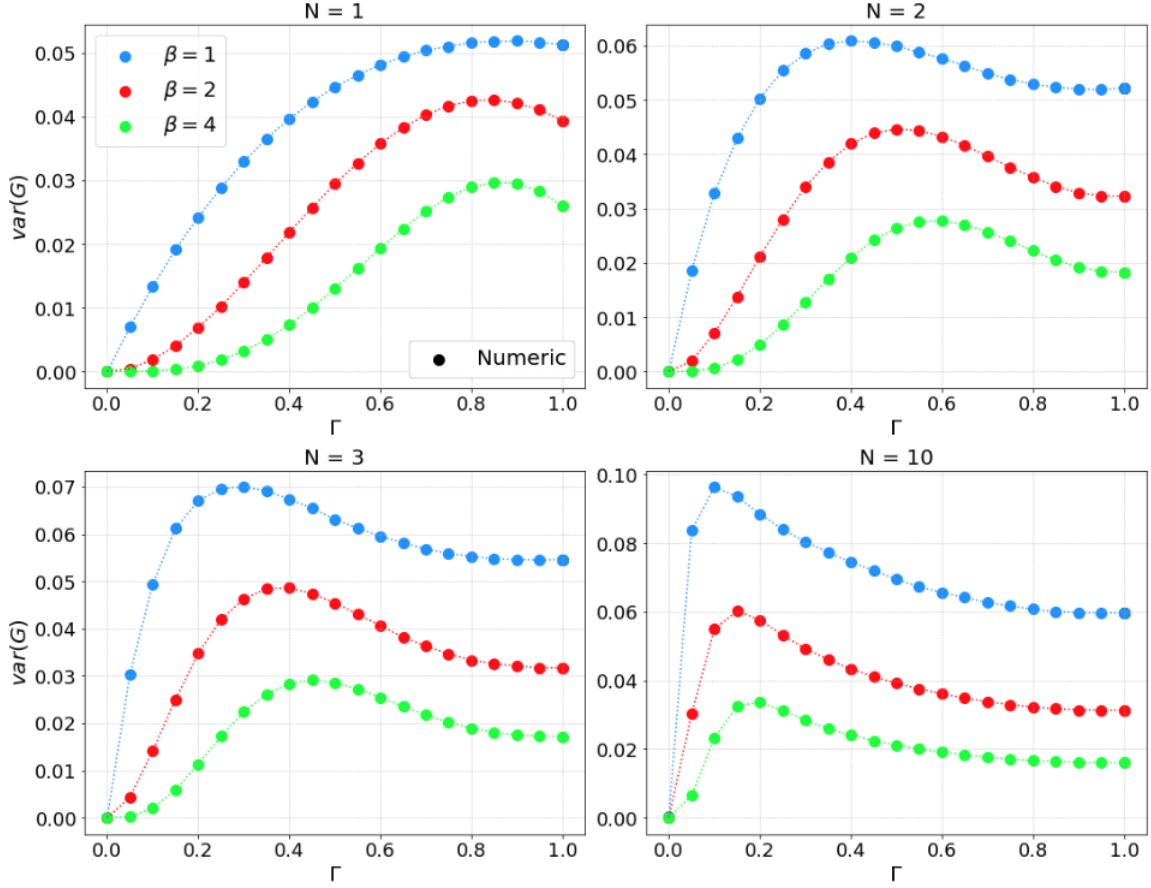


Figure 6.12 Variance of conductance as a function of the barrier transparency Γ for different numbers of open channels N in the leads for Dirac billiard.

metry classes D, DIII, C and CI. Then, depending on the symmetry class, Andreev billiards presents a challenge because the matrix dimensions double due to spin and/or particle-hole degree of freedom. For class D, we use random Hamiltonians with dimensions 100×100 . For class DIII, the Hamiltonians have dimensions 200×200 due to the spin degree of freedom. Both classes C and CI were generated by random Hamiltonians with dimension 200×200 due to the Bogoliubov quasiparticle degree of freedom.

Fig.(6.14) is depicted the distribution of thermal conductance for all symmetry classes of the Andreev billiard with different numbers of open channels N in the ideal case (transparent barrier $\Gamma = 1$), recovering some results of literature [51]. Here we say thermal transport, instead of electrical transport, because the Bogoliubov quasiparticles that are transmitted through a superconducting quantum dot carry a definite amount of energy rather than a definite amount of charge. In particular, the analytical result of the conductance for the extreme quantum regime $N = 1$ is given by [51]

$$P(g) \propto g^{-1+\beta/2}(1-g)^{\gamma/2}, \quad 0 < g < 1, \quad g = G/dG_0, \quad (6.22)$$

where $\beta = \{1, 2, 4, 2\}$ and $\gamma = \{-1, -1, 2, 1\}$ for the classes D, DIII, C and CI, respectively.

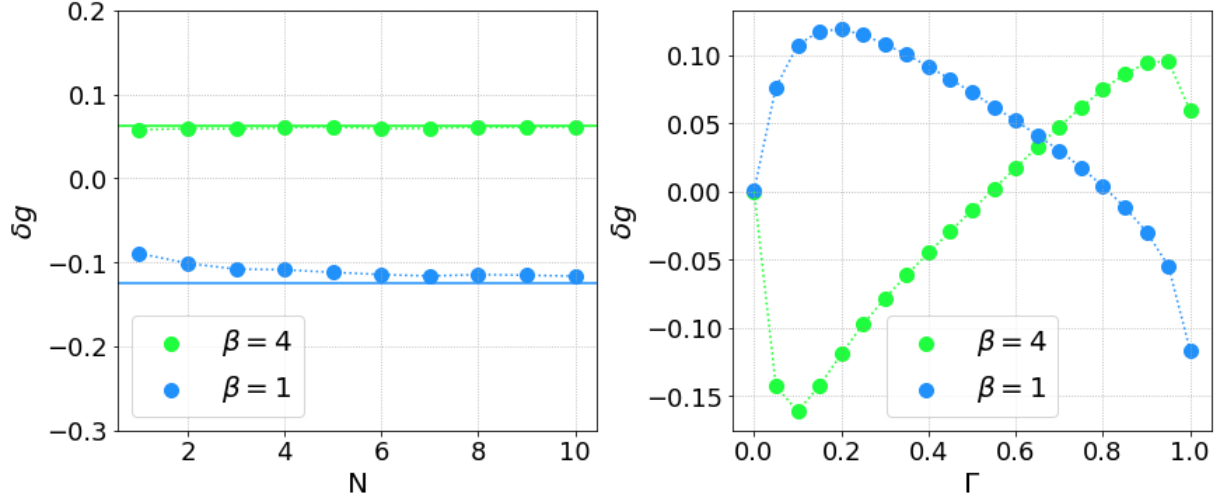


Figure 6.13 Quantum Interference Correction (QIC) of conductance for the Dirac billiard as a function of the number of open channels N (left panel) and as a function of the transparency barrier Γ in the semi-classical regime (right panel). The solid lines represent the analytical result Eq.(6.21).

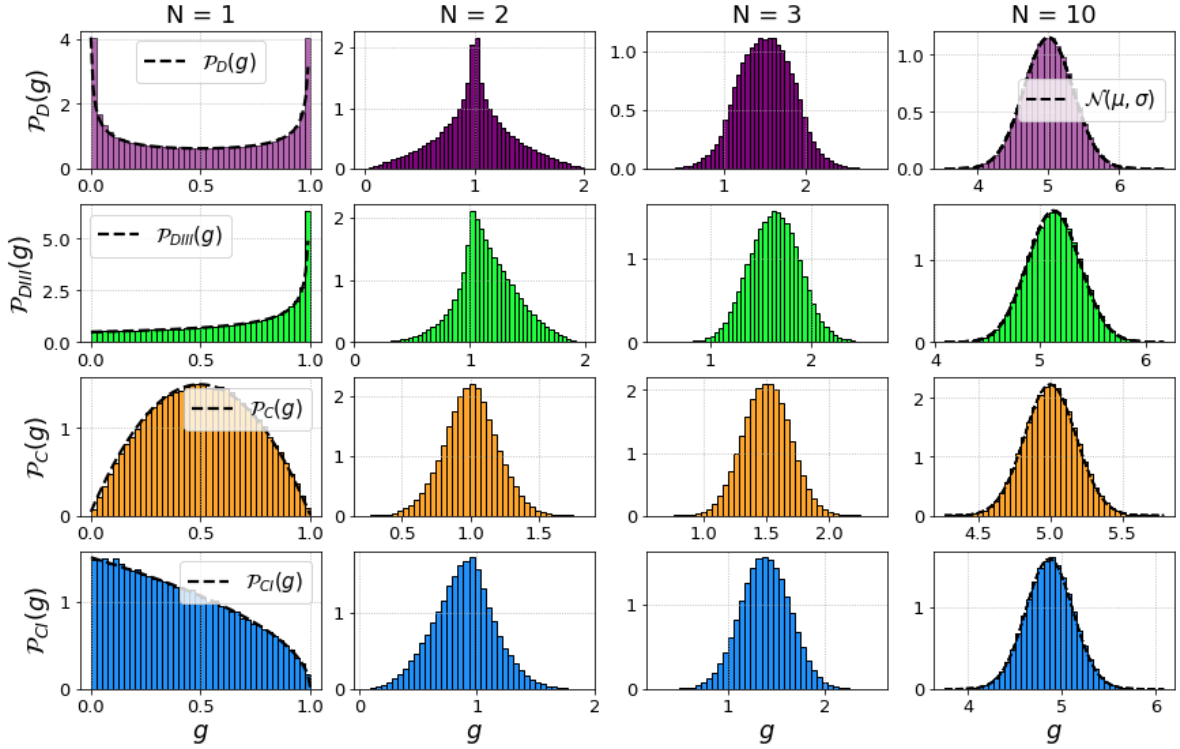


Figure 6.14 Conductance distribution for the Andreev billiard with different numbers of open channels N in the ideal case (transparent barrier regime $\Gamma = 1$). The dashed line represents the analytical result.

As the number of channels N increases, the distribution of conductance becomes a normal distribution $\mathcal{N}(\mu, \sigma)$. Empirically, we witness the manifestation of the Central Limit Theo-

rem's convergence. Therefore, the semi-classical regime implies the universal behavior of the conductance distribution.

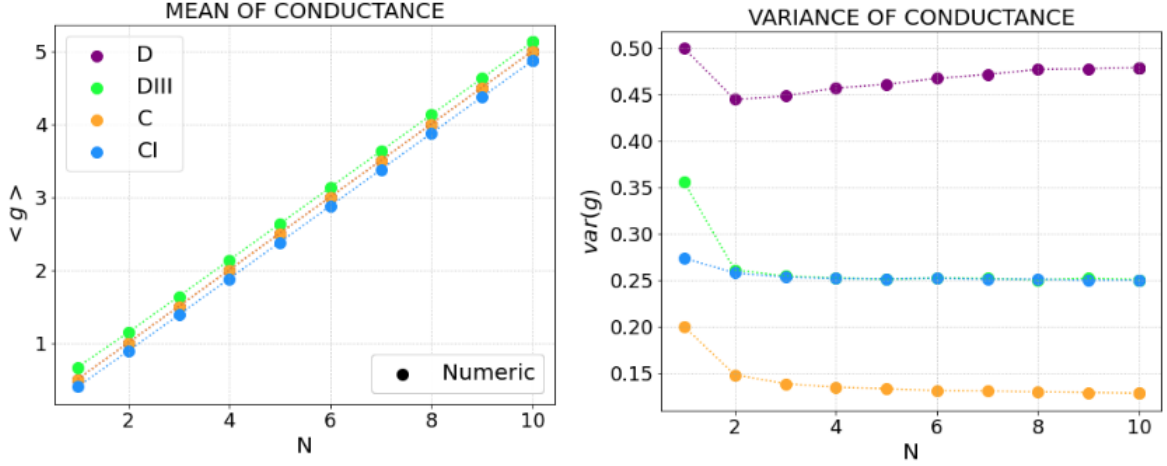


Figure 6.15 Mean $\langle g \rangle$ and variance $\text{var}(g)$ of conductance for the Andreev billiard with different numbers of open channels N in the ideal case (transparent barrier regime $\Gamma = 1$).

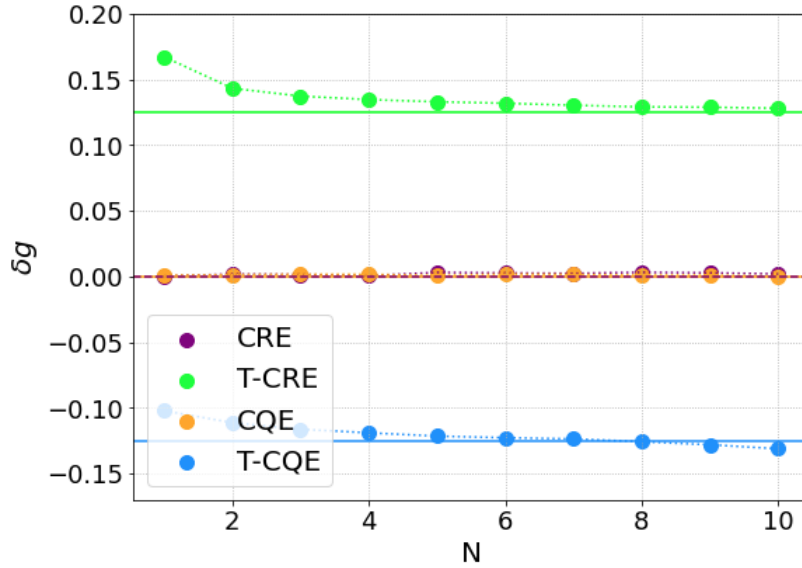


Figure 6.16 Quantum Interference Correction (QIC) of conductance as a function of the number of open channels N for the Andreev billiard. The solid line represents the analytical RMT prediction for the semi-classical regime [51].

The mean and variance of the conductance distribution are shown in Fig.(6.15). Our simulations agree with the analytical results [51] for the average conductance $\langle g \rangle = \langle G/dG_0 \rangle =$

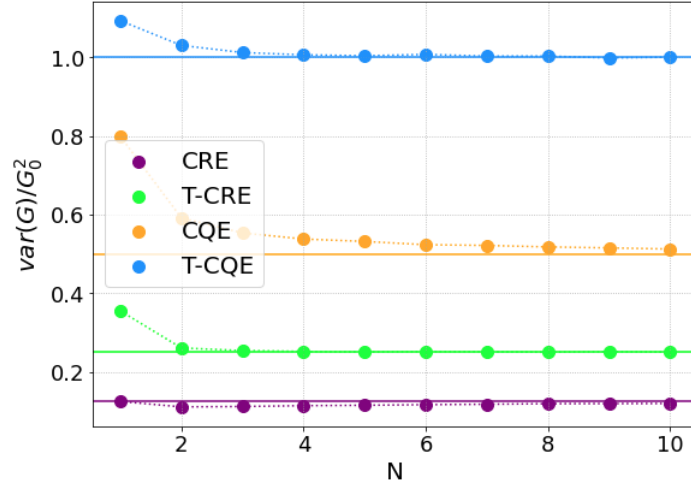


Figure 6.17 Variance of conductance $\text{var}(G)/G_0^2$ for the Andreev billiard with different numbers of open channels N in the ideal case (transparent barrier regime $\Gamma = 1$). The solid line represents the analytical RMT prediction for the semi-classical regime [51].

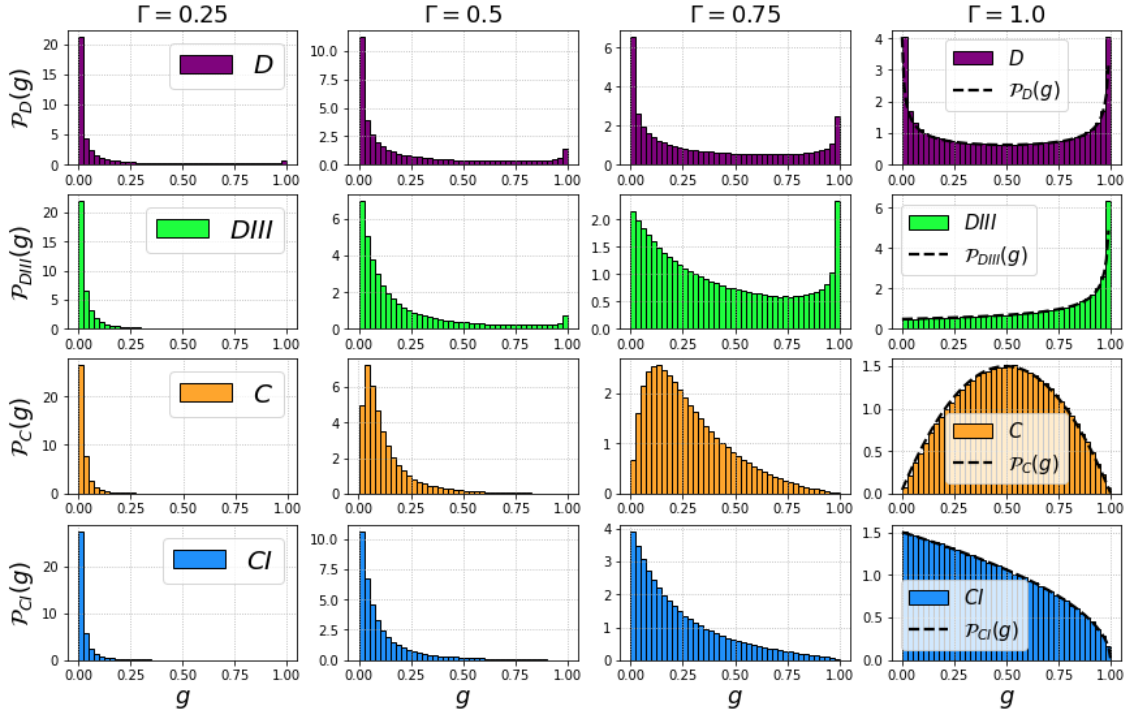


Figure 6.18 Conductance distribution with different values of the barrier transparency Γ in the extreme quantum regime $N = 1$ for the Andreev billiard.

$g_0 + \delta g$, where

$$\begin{aligned} g_0 &= \frac{N_1 N_2}{N_1 + N_2}, \\ \delta g &= \frac{1}{\beta} (\beta - 2 - \gamma) \frac{N_1 N_2}{(N_1 + N_2)^2}. \end{aligned} \quad (6.23)$$

Filling in the values β and γ for the δg , we obtain the quantum interference correction for the symmetry classes

$$\delta g = \begin{cases} 0, & \text{in the } D \text{ and } C, \\ -1/8, & \text{in the } CI, \\ 1/8, & \text{in the } DIII. \end{cases} \quad (6.24)$$

Our results also agree with the variance of conductance $\langle g \rangle$, where the analytical form [51] is

$$\text{var } g = \frac{2(N_1 N_2)^2}{(N_1 + N_2)^4} = \begin{cases} 1/2, & \text{for the class } D, \\ 1/4, & \text{for the class } DIII, \\ 1/8, & \text{for the class } C, \\ 1/4, & \text{for the class } CI. \end{cases} \quad (6.25)$$

Fig.(6.16) shows the result of quantum interference correction by our simulations. The CRE, T-CRE, CQE and T-CQE stand for D, DIII, C and CI, respectively. The CRE and CQE refer to the circular real ensemble and circular quaternion ensemble, whereas the presence of time-reversal symmetry is indicated by the prefix T in T-CRE and T-CQE.

In Ref.[51], the variance of thermal conductance is given by $\text{var}(G) = G_0^2/p$, where $p = 8, 4, 2, 1$ for the symmetry classes D (CRE), DIII (T-CRE), CQE (C) and T-CQE (CI), respectively. In Fig.(6.17), we present our numerical result of the variance of conductance for all classes of symmetry. We see the agreement between our numerical results with the analytical ones.

Until now we have discussed the conductance for the ideal Andreev billiard, i.e., the transparent barrier case $\Gamma = 1$. Some statistical properties of conductance, such as distribution, average and variance, are well-known in the literature for this regime. However, the lack of both numerical and analytics results for non-ideal Andreev billiards motivates us to explore the quantum transport for an arbitrary transparency barrier Γ . Therefore, the results of conductance in the presence of the barrier for Andreev's billiards are new to the literature. Fig.(6.18) shows the effect of tunneling on the conductance distribution in the extreme quantum regime $N = 1$ for the Andreev billiard. Similar to the Schrödinger and Dirac cases, the conductance distribution tends to focus on 0 as Γ goes to the opaque regime ($\Gamma \rightarrow 0$). When the system tends to the ideal regime $\Gamma \rightarrow 1$, the conductance distribution converges to the known analytical results Eq.(6.22) presented in [51].

In Fig.(6.19), we present our numerical simulations of the conductance average $\langle g \rangle$ as a function of the barrier transparency Γ for different numbers of open channels N in the leads. For the semi-classical regime $N = 10$, the tendency of conductance average to converge to linear behavior as a function of Γ is smaller in the Andreev than the Schrödinger and Dirac billiards.

In Fig.(6.20), we present the variance of conductance $\text{var } g$ as a function of the barrier transparency Γ for different numbers of open channels N in the leads, we recover the ideal case Eq.(6.25) when $\Gamma \rightarrow 1$. We conclude that the variance $\text{var } g$ for classes that preserve TRS (DIII and CI) are equal when the transparency of the barrier is large enough $\Gamma > 0.5$ in the semi-classical regime.

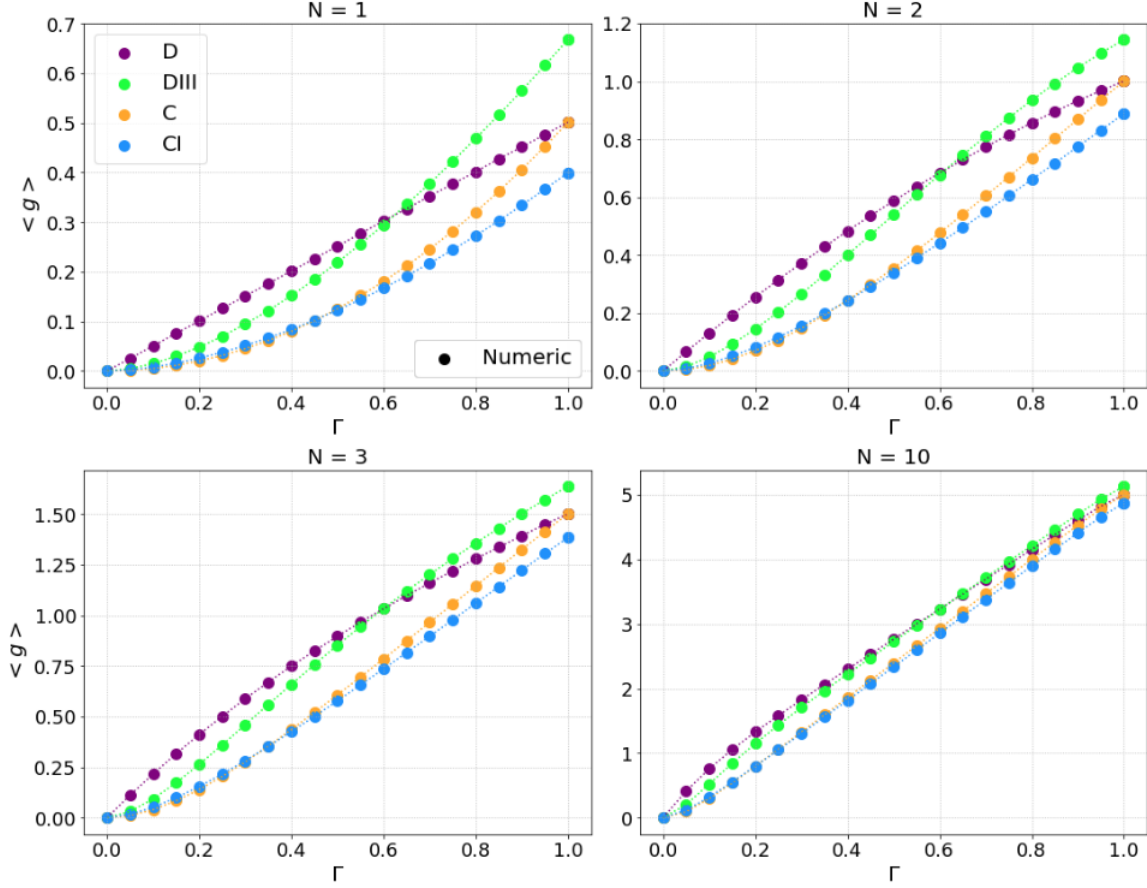


Figure 6.19 Conductance average as a function of the barrier transparency Γ for different numbers of open channels N in the leads for Andreev billiard.

Finally, Fig.(6.21) shows the quantum interference correction of conductance for the Andreev billiard as a function of the barrier transparency Γ . The QIC is obtained by subtracting the semi-classical term, $\langle g \rangle_{\text{semi-classical}} = N\Gamma/2$, of the ensemble's conductance averages. Thus, we obtain the QIC for the conductance by

$$\delta g_{\text{Class X}} = \langle g \rangle_{\text{Class X}} - \langle g \rangle_{\text{semi-classical}}, \quad (6.26)$$

where X can be the class D (CRE), DIII (T-CRE), C (CQE) and CI (T-CQE). Similar to the Schrödinger and Dirac billiards, the breaking of the time-reversal symmetry implies an absence of the quantum interference term for the ideal case $\Gamma = 1$. Therefore, $\delta g = 0$ in both classes D and C in this regime. However, as the barrier increases, i.e. $\Gamma \rightarrow 0$, the presence of QIC even for those D and C symmetry classes is evident. Weak localization (Weak-antilocalization) is presented in classes C and CI (D and DIII). Nevertheless, transitions from WAL ($\delta g > 0$) to WL ($\delta g < 0$) can be seen in classes DIII and C as the barrier transparency Γ decreases, respectively.

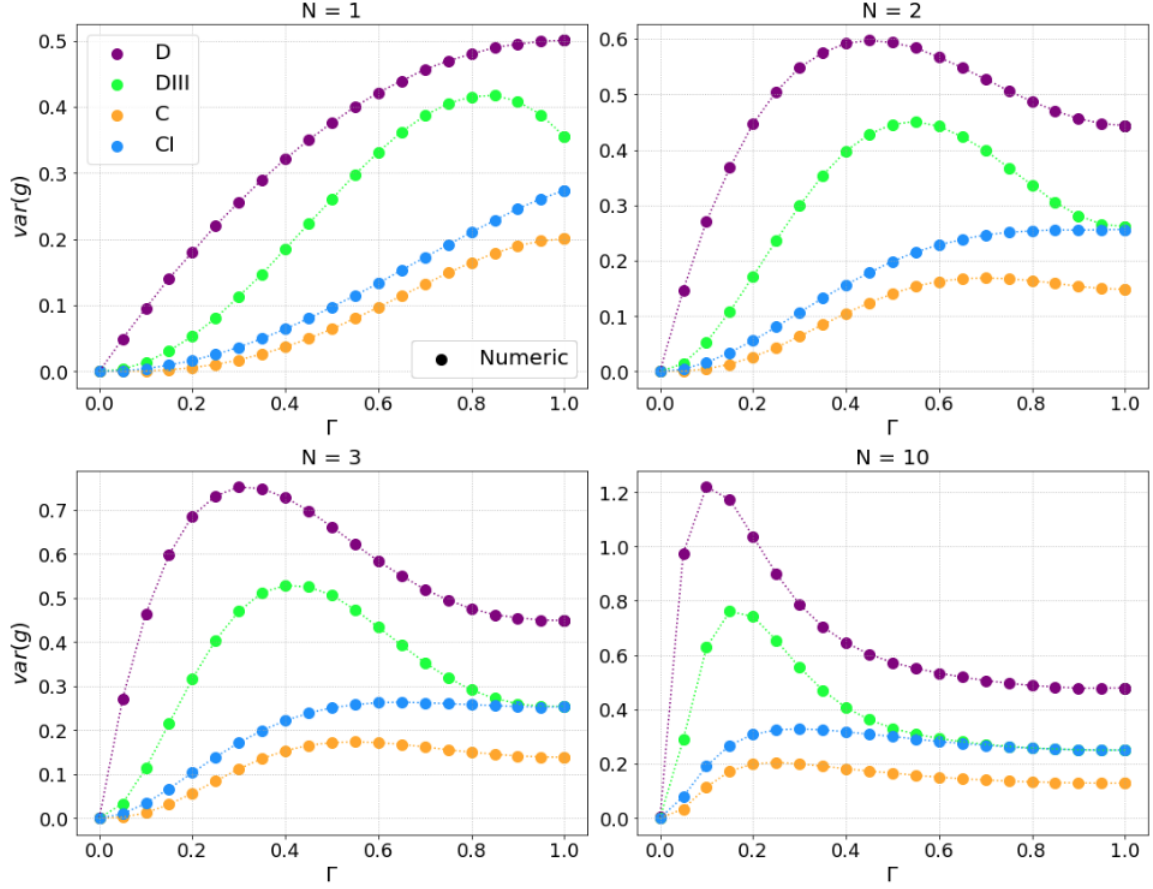


Figure 6.20 Variance of conductance as a function of the barrier transparency Γ for different numbers of open channels N in the leads for Andreev billiard.

6.3 Shot Noise Power

Shot noise quantifies the correlation of charge fluctuations of the current in time [30], $I(t)$. In our case, this correlation can be obtained by defining the second-order current-current correlation function

$$C(t, t') \equiv \langle \Delta \hat{I}(t) \Delta \hat{I}(t') \rangle = \langle \hat{I}(t) \hat{I}(t') \rangle - \langle \hat{I}(t) \rangle \langle \hat{I}(t') \rangle. \quad (6.27)$$

However, in mesoscopic systems, we are interested in the Fourier transform of $C(t, t')$

$$S(\omega) \equiv \lim_{\tau \rightarrow \infty} \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} dt \int_{-\tau/2}^{\tau/2} dt' C(t, t') e^{i\omega(t-t')}, \quad (6.28)$$

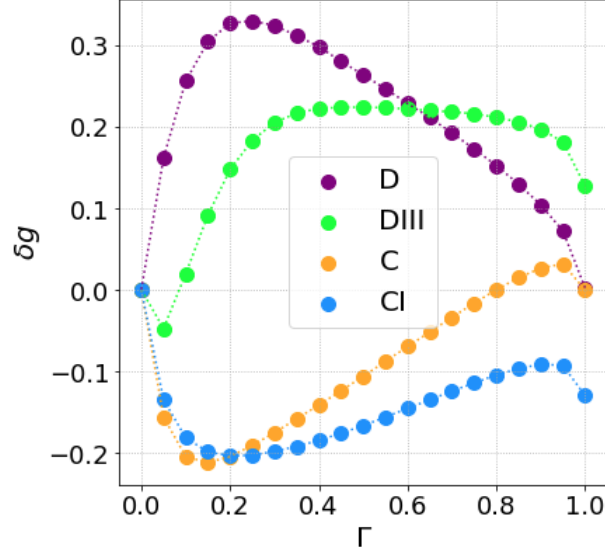


Figure 6.21 Quantum Interference Correction (QIC) of conductance for the Andreev billiard as a function of the barrier transparency Γ in the semi-classical regime.

which $S(\omega)$ is known as current-noise power spectral density (PSD) or shot noise power. In mesoscopic systems, shot noise power S for low-frequencies is described by

$$S = \frac{2e^2}{\pi\hbar} |eV| \text{Tr} r r^\dagger t t^\dagger \rightarrow S = S_0 \sum_i T_i (1 - T_i), \quad (6.29)$$

$$s = \sum_i T_i (1 - T_i), \quad s = S/S_0,$$

with $S_0 = 2e^2|eV|/\pi\hbar$ and the factor $1 - T_i$ being the reduction of noise due to the Pauli principle. Analogous to the conductance, the shot-noise power depends on the eigenvalues of the transmission matrix $tt^\dagger$, $\{T_i\}_{i=1}^N$. Next, we present the statistical properties of shot-noise power such as distribution, average and variances. Moreover, we explore the tunneling effect on this quantum transport property.

• Schrödinger Billiard - SB

We perform the Monte Carlo method to generate 10^6 random Hamiltonians with dimensions 100×100 and calculate their respective shot-noise power. Fig.(6.22) is shown the distribution of shot-noise power for the Schrödinger billiard with different numbers of open channels N in the ideal case (transparent barrier $\Gamma = 1$). Similar to conductance, the distribution of shot noise power becomes a normal distribution $\mathcal{N}(\mu, \sigma)$ as the number of channels increases (semi-classical regime), revealing a universal behavior. As a result, the accumulation of shot noise power across multiple open channels gives rise to a Gaussian distribution, thereby revealing the underlying implication of the Central Limit Theorem (CLT).

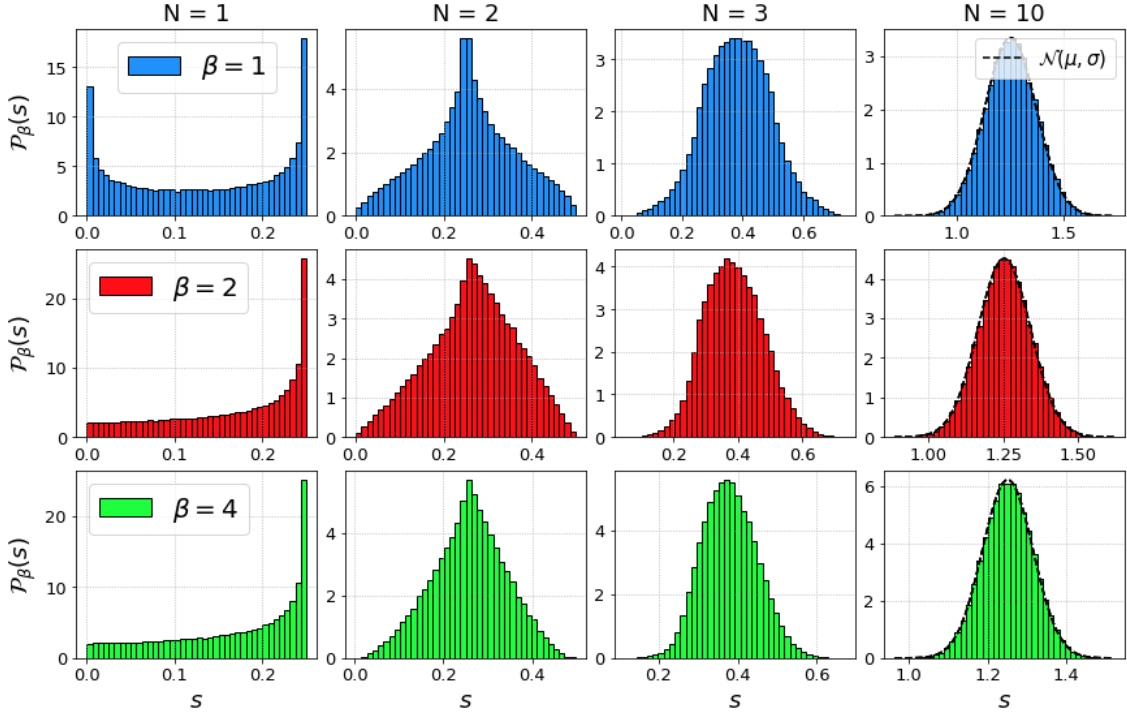


Figure 6.22 Shot-noise power distribution for the Schrödinger billiard with different numbers of open channels N in the ideal case (transparent barrier regime $\Gamma = 1$).

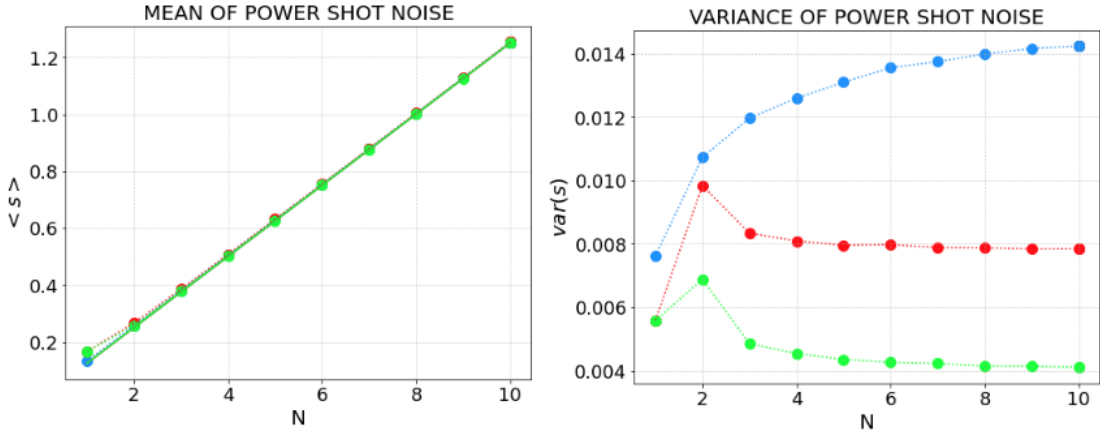


Figure 6.23 Mean $\langle s \rangle$ and variance $\text{var}(s)$ of shot noise power for the Schrödinger billiard with different numbers of open channels N in the ideal case (transparent barrier regime $\Gamma = 1$).

The mean and variance of the conductance distribution are shown in Fig.(6.23). Our simulations agree with the analytical results [109] for the average shot noise power $\langle s \rangle$

$$\langle s \rangle = \frac{N_1^2 N_2^2}{(N_1 + N_2)^3} - \left(1 - \frac{2}{\beta}\right) \frac{N_1 N_2 (N_1 - N_2)^2}{(N_1 + N_2)^4} + \mathcal{O}(N^{-1}). \quad (6.30)$$

Since we treat only symmetric leads, the average shot noise power is the same for all symmetry

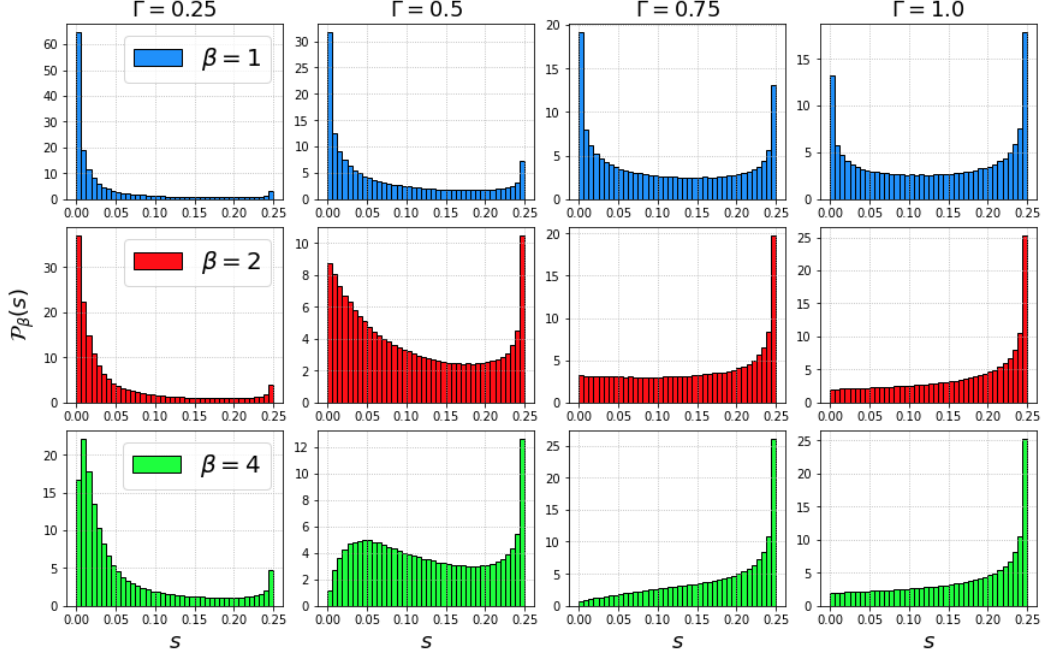


Figure 6.24 Shot noise power distribution with different values of the barrier transparency Γ in the extreme quantum regime $N = 1$.

classes of the Wigner-Dyson ensemble. The average of shot noise power is proportional to the number of open channels, $s \propto N$.

The lack of analytical results makes it impossible for us to compare the formula of $\text{var}(s)$ with our numerical results. However, we conclude that the variance for the shot noise power remains constant in the semi-classical regime $N \gg 1$. Thus, the magnitude of the shot noise fluctuations, $\text{var}(s)$, also presents universal fluctuations.

From now on, we present the statistical properties of shot noise power for the non-ideal Schödinger billiards. Analytical results of the shot noise power in the presence of a barrier are a challenge for mesoscopic physics not yet explored, thus motivating us to perform numerical simulations. Fig.(6.24) shows the effect of tunneling on the shot noise power distribution in the extreme quantum regime $N = 1$. As the barrier increases $\Gamma \rightarrow 0$, the transmission eigenvalues $\{T_i\}_{i=1}^N$ tend to zero. Therefore, both conductance and shot noise power tend to focus on 0 for all symmetry classes. For $\Gamma = 1$, we recover the distribution of shot noise power for the ideal case (see Fig.(6.22)).

In Fig.(6.25) we present the average of shot noise power $\langle s \rangle$ as a function of the barrier transparency Γ for different numbers of open channels N in the leads. Analytical results were performed to obtain the shot noise power average for an arbitrary tunneling transparency Γ case [109, 110, 111]. In particular, our results agree with the semi-classical regime

$$\langle s \rangle = \frac{N\Gamma}{8}(2 - \Gamma). \quad (6.31)$$

Unlike the conductance, the shot noise power has a quadratic dependence on the barrier transparency Γ . This dependence is related to the factor $1 - T_i$ of shot noise power definition

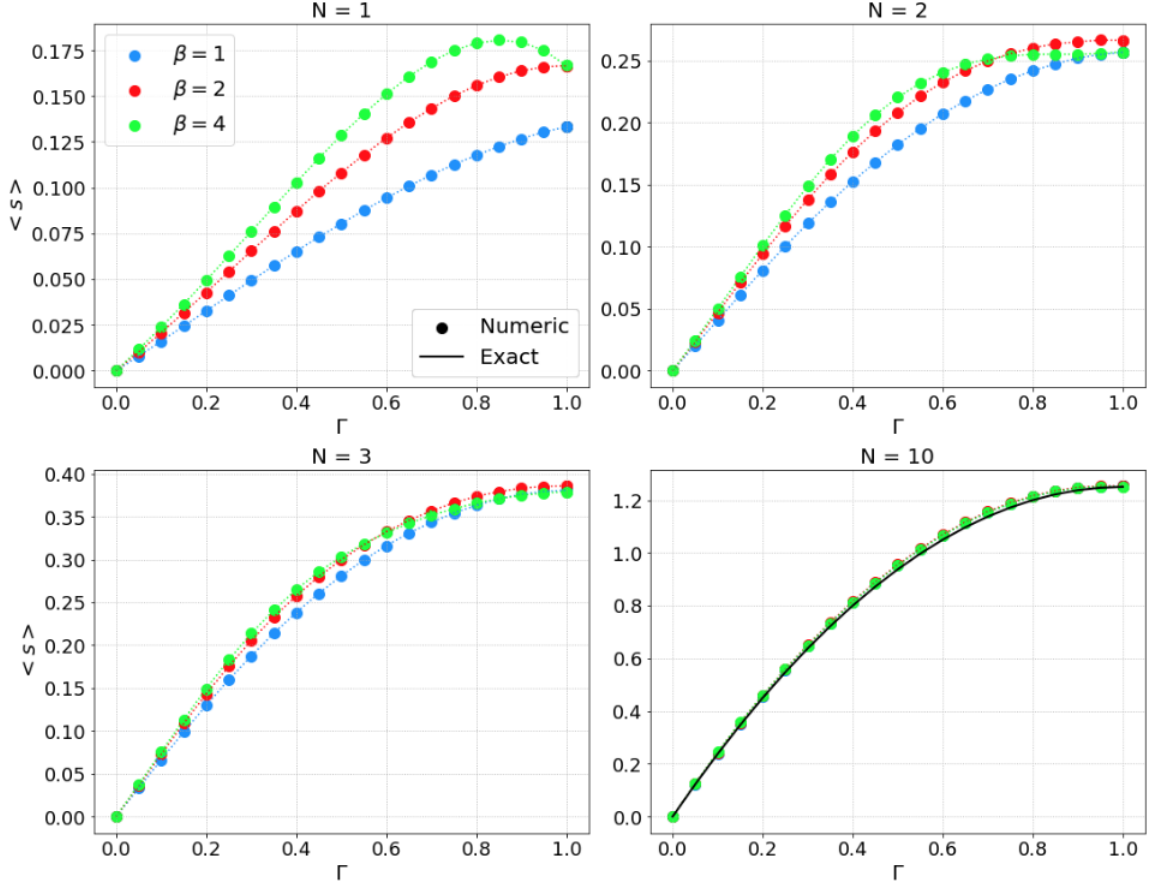


Figure 6.25 Shot noise power average as a function of the barrier transparency Γ for different numbers of open channels N in the leads for Schrödinger billiard. The black solid line represents the analytical result given by Eq.(6.31).

Eq.(6.29) due to the Pauli principle.

In Fig.(6.26), we present our results of the variance for the shot noise power $\text{var}(s)$ as a function of the barrier transparency Γ for different numbers of open channels N in the leads. Since the distributions tend to focus on 0 when $\Gamma \rightarrow 0$, the variance also tends to 0 for all symmetry classes. Moreover, the $\text{var}(s)$ converges to fixed values when $\Gamma = 1$ in the semi-classical regime, presenting a universal behavior of the shot noise power fluctuations. The lack of analytical results makes us unable to carry out further conclusions.

Finally, Fig.(6.27) shows the quantum interference correction (QIC) for the shot noise power in Schrödinger billiard as a function of the number of open channels N (left panel) and as a function of the barrier transparency Γ (right panel). In Schrödinger billiard (SB), the QIC of the shot noise power in transparent barrier regime ($\Gamma = 1$) is obtained by subtracting the semi-classical term,

$$\langle s \rangle_{\text{semi-classical}} = \frac{N_1^2 N_2^2}{(N_1 + N_2)^3}, \text{ for } \Gamma = 1, \quad (6.32)$$

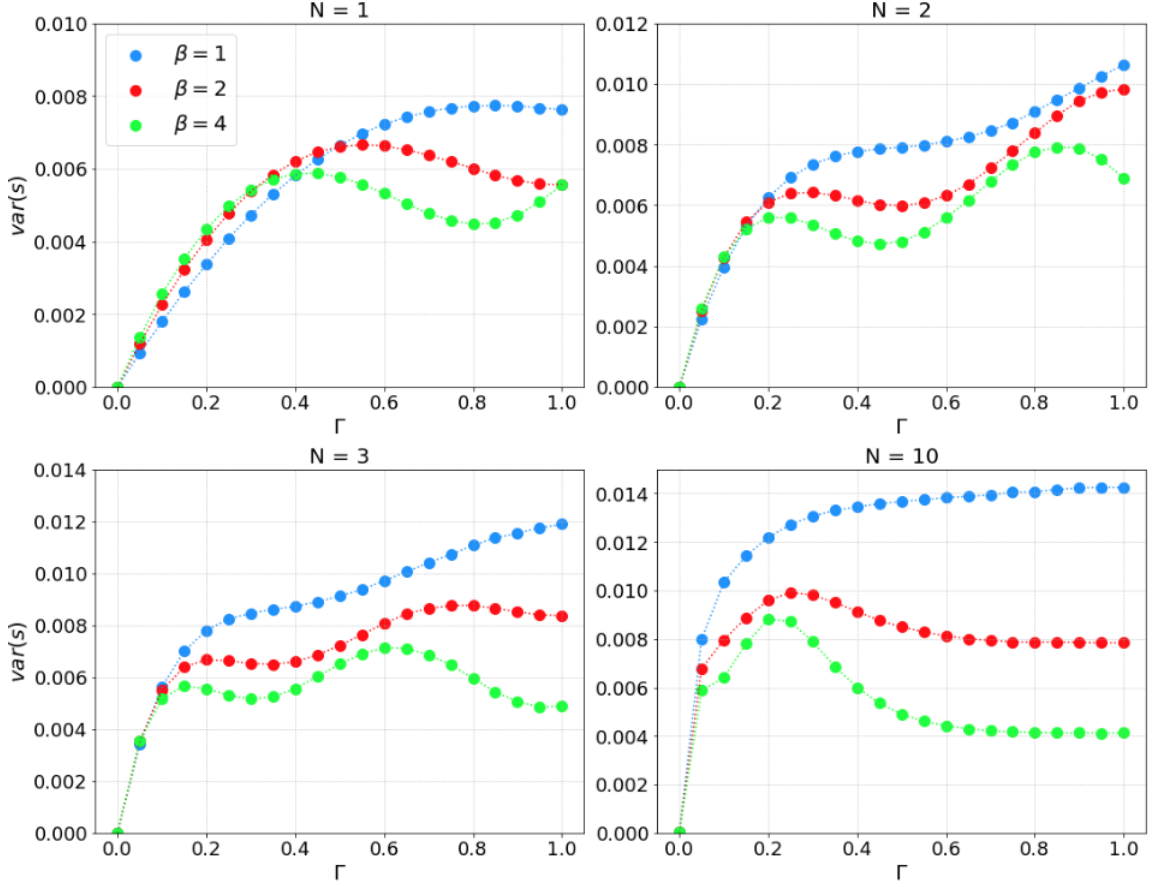


Figure 6.26 Variance of shot noise power as a function of the barrier transparency Γ for different numbers of open channels N in the leads.

of the orthogonal and symplectic shot noise power averages $\langle s \rangle_{\beta=1}$ and $\langle s \rangle_{\beta=4}$, respectively

$$\delta s = \langle s \rangle_{\beta} - \langle s \rangle_{\text{semi-classical}}. \quad (6.33)$$

For symmetrical leads and the ideal transparent barrier, the QIC of the shot noise power is null for both classes orthogonal and symplectic.

For non ideal billiards with arbitrary transparency Γ , the semi-classical term $\langle s \rangle_{\text{semi-classical}}$ is given by Eq.(6.31). Thus, the QIC of shot noise power, δs , is shown in the right panel of Fig(6.27). Unlike conductance, the QIC of shot noise power is always positive for both classes.

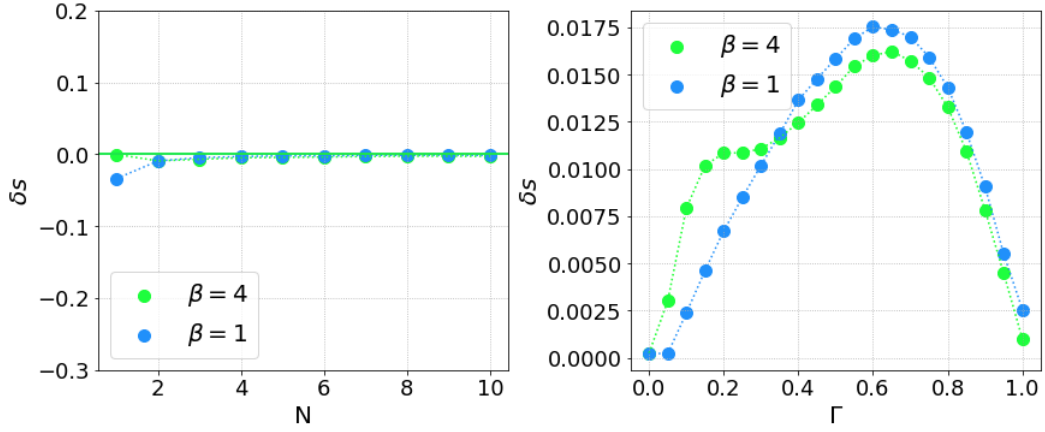


Figure 6.27 Quantum Interference Correction (QIC) of shot noise power for the Schrödinger billiard as a function of the number of open channels N (left panel) and as a function of the transparency barrier Γ in the semi-classical regime (right panel).

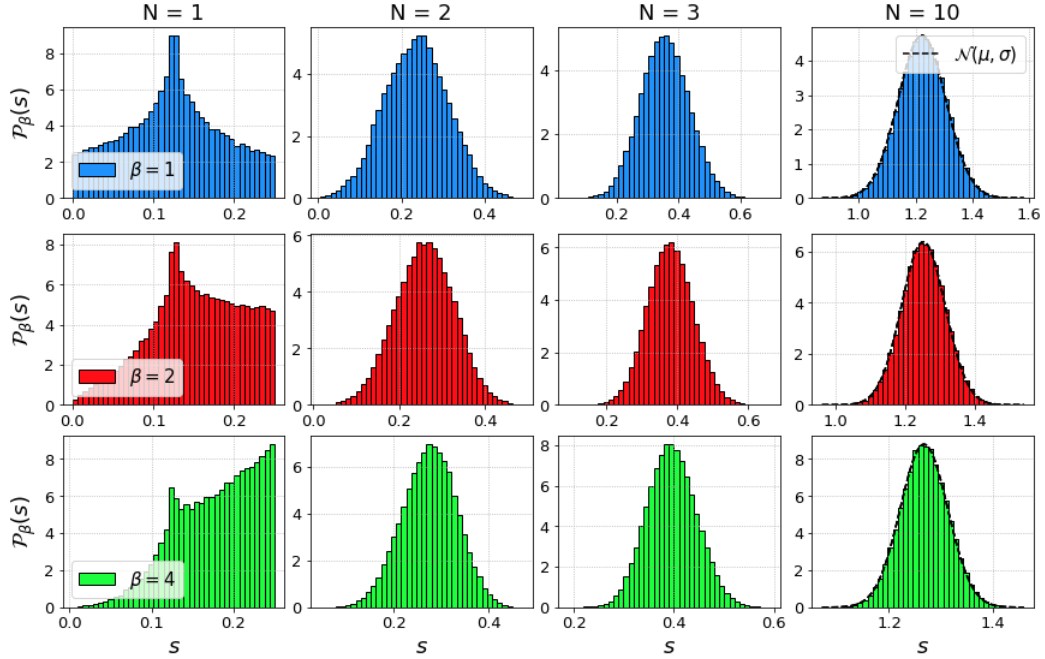


Figure 6.28 Shot-noise power distribution for the Dirac billiard with different numbers of open channels N in the ideal case (transparent barrier regime $\Gamma = 1$).

• Dirac Billiard - DB

We generate 10^6 random Hamiltonians with dimensions 200×200 and calculate their respective shot noise power. Fig.(6.28) shows the distribution of shot noise power with different numbers of open channels N in the ideal case $\Gamma = 1$. Again, the distribution of shot noise power becomes a normal distribution $\mathcal{N}(\mu, \sigma)$ as $N \gg 1$ in the semi-classical regime, showing a universal behavior. Again, this is a consequence of the Central Limit Theorem (CLT).

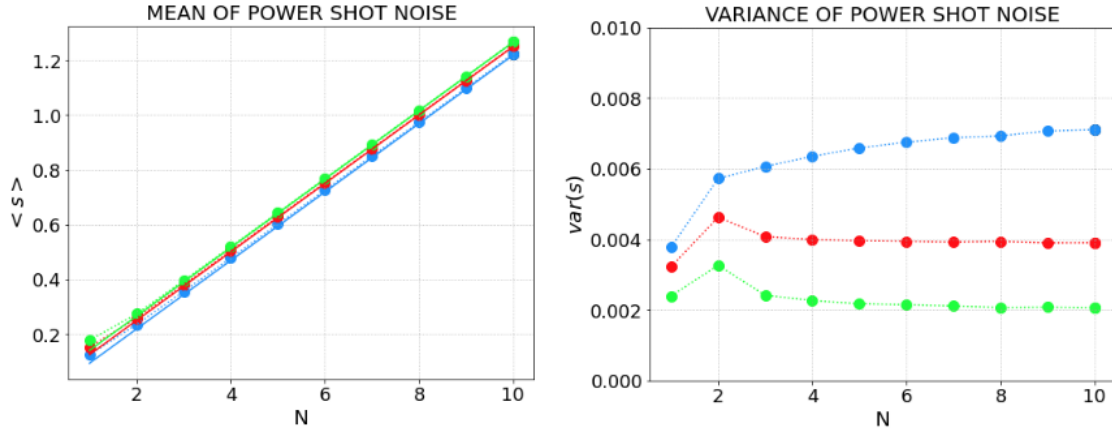


Figure 6.29 Mean $\langle s \rangle$ and variance $\text{var}(s)$ of shot noise power for the Dirac billiard with different numbers of open channels N in the ideal case (transparent barrier regime $\Gamma = 1$).

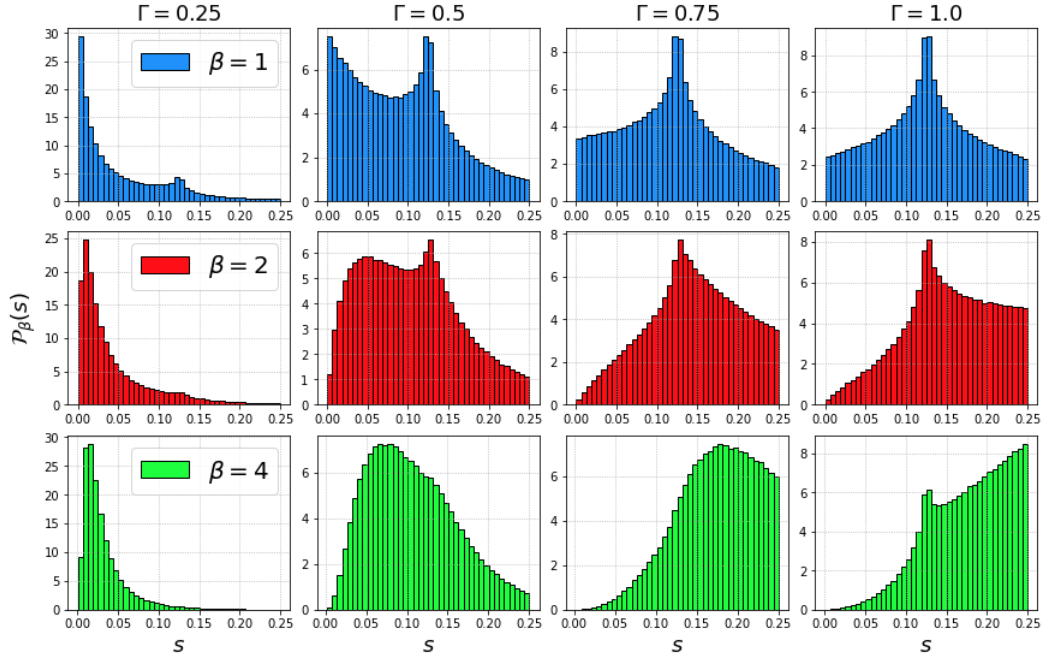


Figure 6.30 Shot noise power distribution with different values of the barrier transparency Γ in the extreme quantum regime $N = 1$ for the Dirac billiard.

The mean and variance of shot noise power distribution are shown in Fig.(6.29). Analytical results for the average shot noise power were obtained by [107] for all symmetry classes

$$\langle s \rangle = 2 \frac{N_1^2 N_2^2}{N_T^3} + \left(\frac{2}{\beta} - 1 \right) \frac{N_1 N_2 (N_1 - N_2)^2}{N_T^4} - 2 \left(\frac{2}{\beta} - 1 \right) \frac{N_1^2 N_2^2}{N_T^4} + \mathcal{O}(N_T^{-1}). \quad (6.34)$$

Observe that the analytical result for the average shot noise power $\langle s \rangle$ has the main term, $2N_1^2 N_2^2 / N_T^3$, and higher-order contributions. Unlike the Schrödinger billiard, Dirac billiards

present higher-order contributions for symmetric leads, $2((2/\beta) - 1)N_1^2 N_2^2 / N_T^4$. Thus, we can differentiate the average shot noise power for different symmetry classes in Dirac billiards, unlike Schrödinger billiards (see Fig.(6.23)).

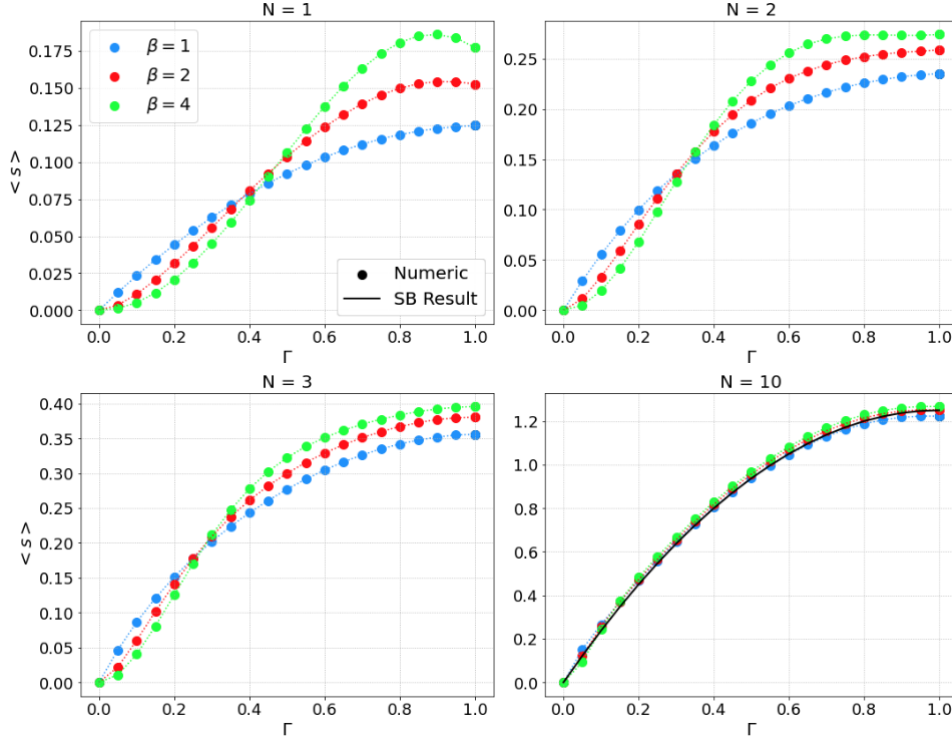


Figure 6.31 Average of shot noise power as a function of the barrier transparency Γ with different numbers of open channels N in the leads for Dirac billiard. The black solid line represents the analytical result for the Schrödinger Billiard, given by Eq.(6.31).

The variance for the shot noise power $\text{var}(s)$ remains constant in the semi-classical regime $N \gg 1$ (see Fig.(6.29)). Therefore, the magnitude of the shot noise fluctuations presents universal fluctuations. The lack of analytical results makes it impossible for us to draw further conclusions.

Now we present the statistical properties of shot noise power for the non-ideal Dirac billiards. Analytical results of the shot noise power in the presence of a barrier are a challenge for mesoscopic physics not yet explored, thus motivating us to perform numerical simulations. In Fig.(6.30), we present shot noise power distribution with different values of the barrier transparency Γ in the extreme quantum regime $N = 1$ for the Dirac billiard. As the barrier increases $\Gamma \rightarrow 0$, the distributions tend to focus on 0 for all symmetry classes. Moreover, a particular feature of Dirac billiards is that cusps in shot noise power distribution appear as Γ increases. For $\Gamma = 1$, we recover the distribution of shot noise power for the ideal case (see Fig.(6.28)).

In Fig.(6.31) we present the average of shot noise power $\langle s \rangle$ as a function of the barrier transparency Γ for different numbers of open channels N in the leads. Compared to the Schrödinger billiard (see Fig.(6.25)), the average of shot noise power presents the same behavior of Eq.(6.31) in the Dirac billiard. However, the presence of the QIC for the shot noise power

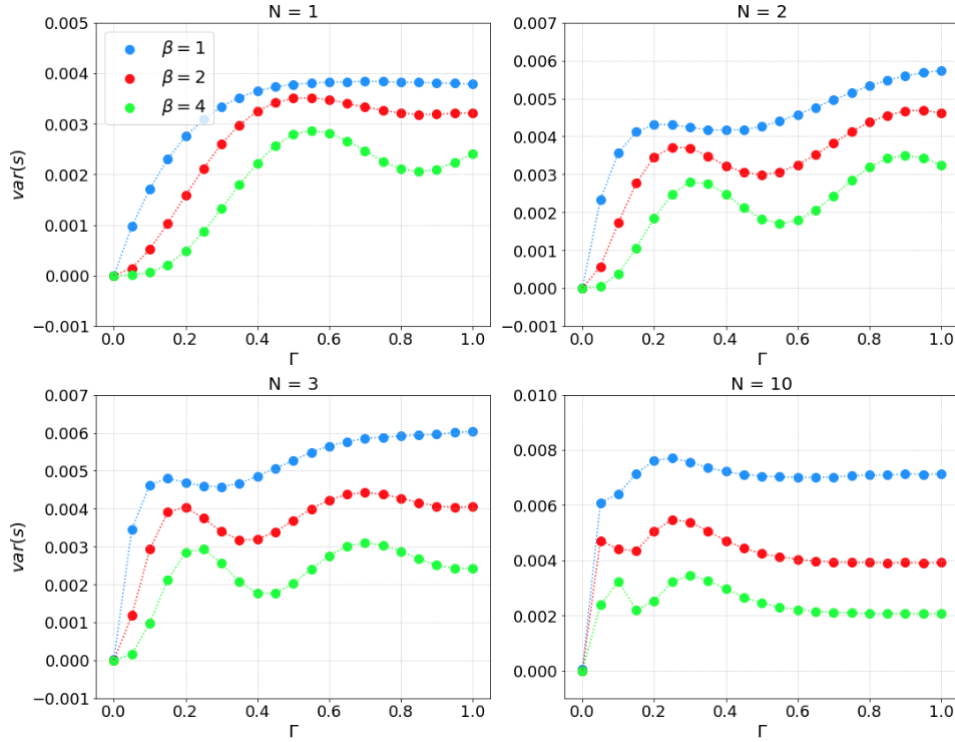


Figure 6.32 Variance of shot noise power as a function of the barrier transparency Γ with different numbers of open channels N in the leads for the Dirac billiard.

in the Dirac billiard is evident even in the semi-classical regime.

The variance of the shot noise power $\text{var}(s)$ as a function of the barrier transparency Γ is depicted in Fig.(6.32) for different numbers of open channels N in the leads. The variance tends to focus on 0 in the opaque regime $\Gamma \rightarrow 0$. Moreover, the $\text{var}(s)$ converges to fixed values for large enough Γ values in the semi-classical regime, presenting a universal behavior of the shot noise power fluctuations. The lack of analytical results makes us unable to carry out further conclusions.

We have mentioned the presence of the QIC in the average shot noise power part. Here we present the QIC for the shot noise power in Fig.(6.33) as a function of the number of open channels N . The exact result for the QIC of shot noise power was obtained by [107]

$$\delta_s = -\frac{1}{2} \left(\frac{2}{\beta} - 1 \right) \frac{N_1^2 N_2^2}{N_T^4}, \quad (6.35)$$

and for symmetrical leads ($N_1 = N_2 = N$) we obtain

$$\delta_s = \frac{1}{2} \left(1 - \frac{2}{\beta} \right) \frac{1}{16}. \quad (6.36)$$

The factor $1/2$ is associated with the degeneracy of the chiral/sub-lattice degree of freedom. Fig.(6.33) shows the QIC as a function of the number of open channels N (left panel) and as a

function of the transparency barrier Γ in the semi-classical regime (right panel). Observe that our numerical results converge to the Eq.(6.36) in the ideal case. For the non-ideal billiard, the result shows the transition from WL to WAL (from WAL to WL) for the orthogonal (symplectic) class. Depending on the symmetry class, the tunneling barrier can suppress or amplify shot noise power.

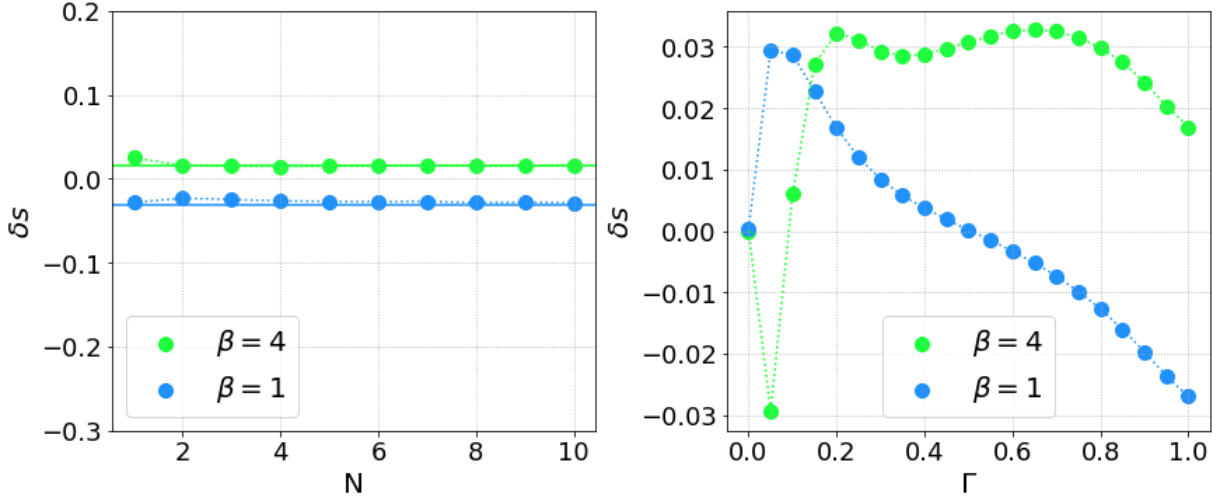


Figure 6.33 Quantum Interference Correction (QIC) of shot noise power for the Dirac billiard as a function of the number of open channels N (left panel) and as a function of the transparency barrier Γ in the semi-classical regime (right panel). The solid lines in the left panel represent the analytical result Eq.(6.36).

• Andreev Billiard

So far, the study of shot noise power distribution for Andreev billiards is not found in the literature. Thus, numerical simulations are relevant as a first approach to the subject. Therefore, we perform the simulation of 10^6 random Hamiltonians and calculate the shot noise power for each realization. Similar to the conductance, we use random Hamiltonians with dimensions 100×100 for class D and dimensions 200×200 for the others (DIII, C and CI) due to the spin/particle-hole degrees of freedom. In Fig(6.34) is depicted the distribution of thermal shot noise power for all symmetry classes of the Andreev billiard with different numbers of open channels N in the ideal case (transparent barrier $\Gamma = 1$). As the number of open channels increases $N \gg 1$, the shot noise power distribution becomes normal $\mathcal{N}(\mu, \sigma)$ for all symmetry classes. Again, we witness the manifestation of the Central Limit Theorem's (CLT) convergence.

The mean and variance of shot noise power distribution are shown in Fig.(6.35). In general, the average of shot noise power is proportional to the number of open channels, $s \propto N$. The lack of analytical results makes it impossible for us to compare our numerical simulation with other techniques. However, we observe that the presence/absence of spin rotational symmetry is a determinant for the average shot noise power, whereas the time-reversal symmetry is for the

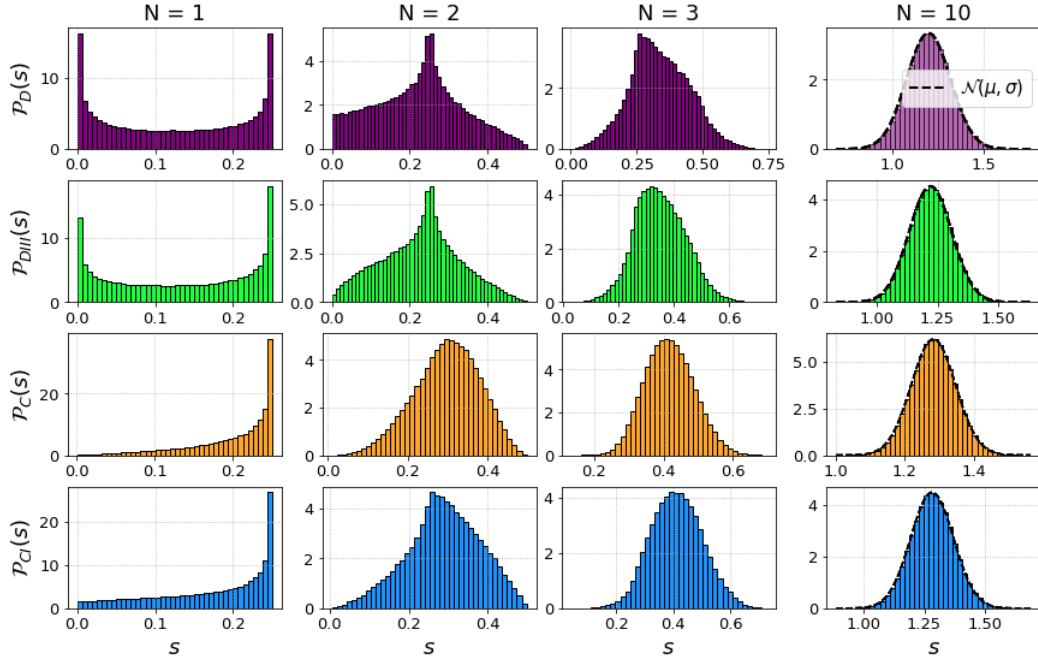


Figure 6.34 Shot-noise power distribution for the Andreev billiard with different numbers of open channels N in the ideal case (transparent barrier regime $\Gamma = 1$).

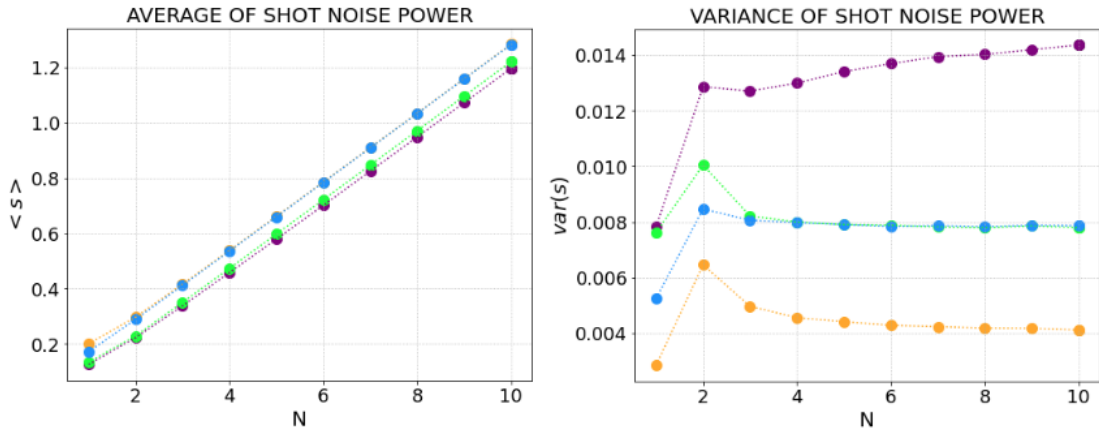


Figure 6.35 Average $\langle s \rangle$ and variance $\text{var}(s)$ of shot noise power for the Andreev billiard with different numbers of open channels N in the ideal case (transparent barrier regime $\Gamma = 1$).

variance. First, the average of shot noise power for classes C (blue) and CI (orange) are equal, where both classes preserve spin rotational symmetry. Then, the average shot noise power for these symmetry classes is determined by the presence of SRS.

For the variance of shot noise power, we observe that classes CI and DIII present the same variance in the semi-classical regime, where both classes preserve time-reversal symmetry. Then, the variance of the shot noise power for these symmetry classes is determined by the presence of TRS. Moreover, the variance for all symmetry classes remains constant as N in-

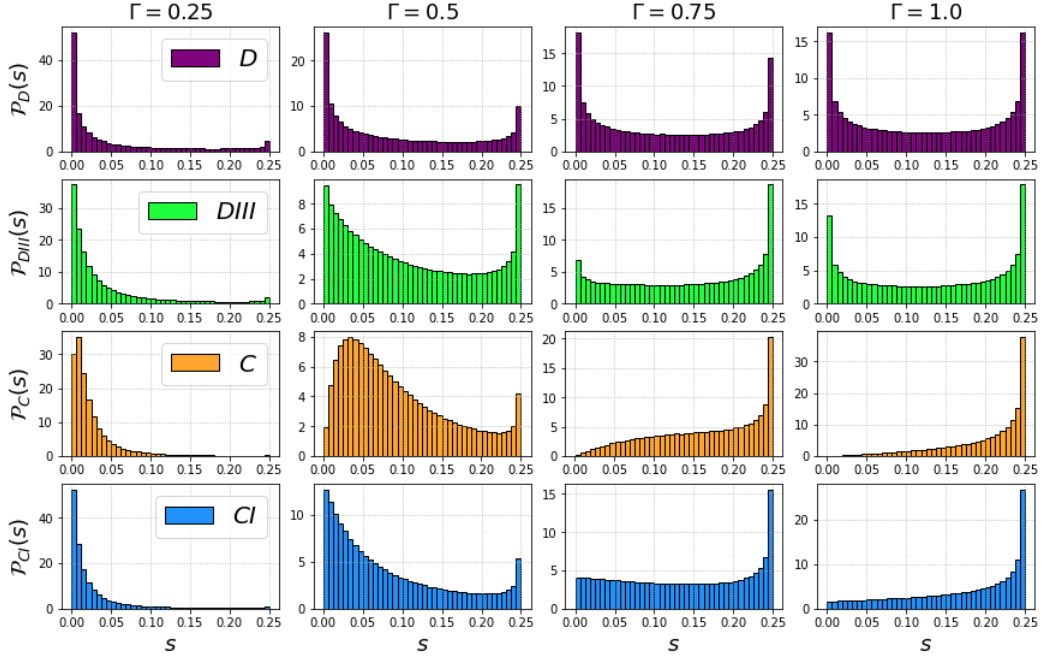


Figure 6.36 Shot noise power distribution with different values of the barrier transparency Γ in the extreme quantum regime $N = 1$.

creases, indicating universal fluctuations of shot noise power even for normal-superconducting heterostructures.

Now we present the statistical properties of shot noise power for the non-ideal Andreev billiards in Fig(6.36). Again, analytical results of the shot noise power distribution in the presence of a barrier are a challenge for mesoscopic physics not yet explored, thus motivating us to perform numerical simulations. The transmission probabilities $\{T_i\}_{i=1}^N$ converge to 0 in the opaque regime, then the shot noise power distributions tend to focus on 0 for all symmetry classes. Moreover, we recover the distribution of shot noise power for the ideal case $\Gamma = 1$ (see Fig.(6.34)).

In Fig(6.37), we present the average of shot noise power $\langle s \rangle$ as a function of the barrier transparency Γ for different numbers of open channels N in the leads. The average of shot noise power presents a similar behavior to the Schrödinger Billiard in the semi-classical regime ($N = 10$), given by Eq.(6.31). However, the presence of the QIC for the shot noise power is evident in Andreev billiards.

In Fig.(6.38), we present our results of the variance for the shot noise power $\text{var}(s)$ as a function of the barrier transparency Γ for different numbers of open channels N in the leads. Since the distributions tend to focus on 0 when $\Gamma \rightarrow 0$, the variance also tends to 0 for all symmetry classes. Moreover, the $\text{var}(s)$ converges to fixed values when $\Gamma = 1$ in the semi-classical regime ($N = 10$), presenting a universal behavior of the shot noise power fluctuations. The lack of analytical results makes us unable to carry out further conclusions.

The QIC of the shot noise power for Andreev billiards is depicted in Fig.(6.39). The QIC is obtained by subtracting the semi-classical term of the ensemble's shot noise power averages.

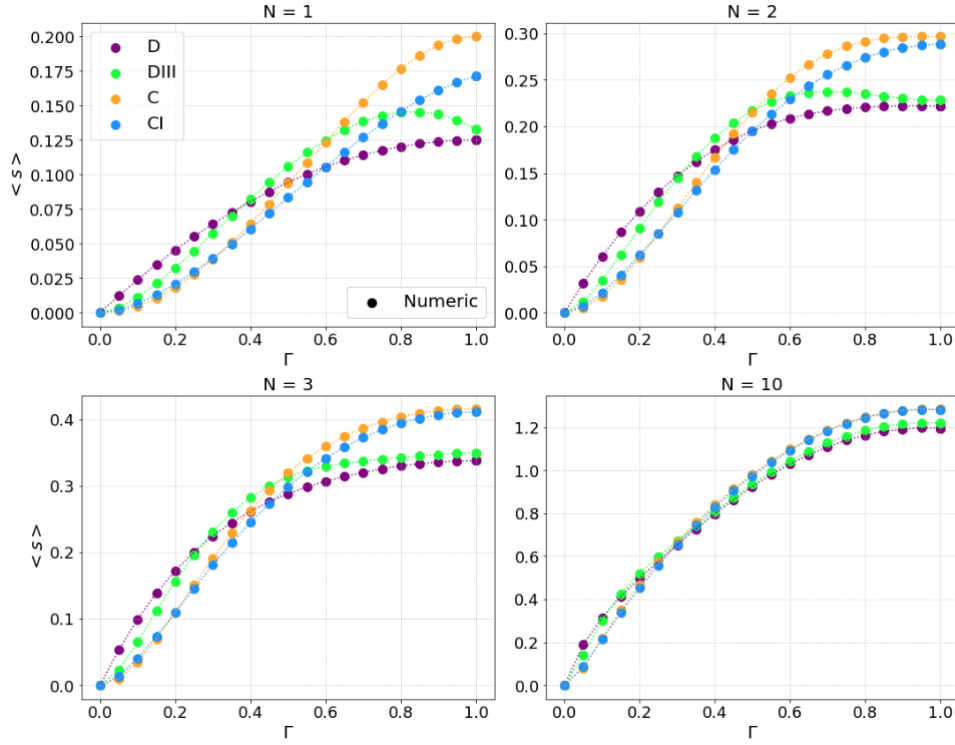


Figure 6.37 Average of shot noise power as a function of the barrier transparency Γ with different numbers of open channels N in the leads for Andreev billiard. The average of shot noise power presents a similar behavior to the Schrödinger Billiard in the semi-classical regime, given by Eq.(6.31).

In this case, we consider the semi-classical term, $\langle s \rangle_{\text{semi-classical}} = N/8$ that is the Eq.(6.31) for the ideal case $\Gamma = 1$. Thus, we obtain the QIC for the shot noise power in the left panel of Fig.(6.39) by

$$\delta s_{\text{Class X}} = \langle s \rangle_{\text{Class X}} - \langle s \rangle_{\text{semi-classical}}, \quad (6.37)$$

where X can be the class D (CRE), DIII (T-CRE), C (CQE) and CI (T-CQE). Analytical results of the QIC are not yet found in the literature but, by numerical methods, we found the following QIC values

$$\delta s = \begin{cases} 1/32, & \text{for classes C and CI,} \\ -1/16, & \text{class D,} \\ -1/32, & \text{class DIII.} \end{cases} \quad (6.38)$$

For the right panel, we consider the semi-classical term $\langle s \rangle_{\text{semi-classical}}$ given by Eq.(6.31). Then, we calculate the QIC for the shot noise power in the right panel of Fig.(6.39) by Eq.(6.37). We observe the transitions from WL to WAL (from WAL to WL) for classes D and DIII (for classes C and CI) as the barrier transparency Γ decreases. Since the spin rotation symmetry is preserved (broken) in classes C and CI (D and DIII), we observe that the invariance of spin rotation plays a fundamental role in the shot noise power.

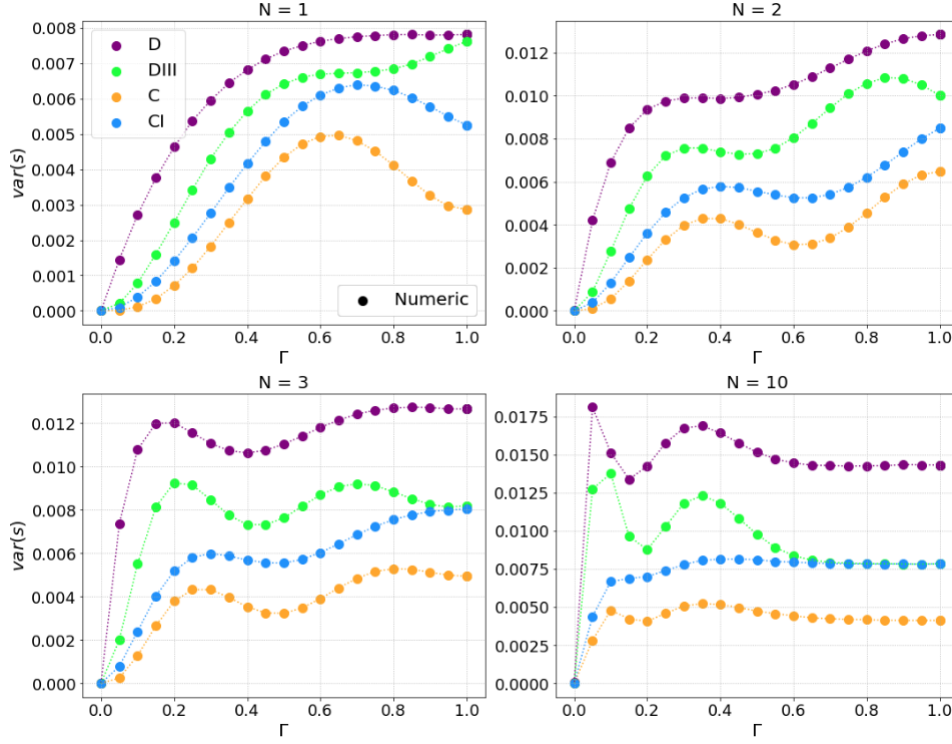


Figure 6.38 Variance of shot noise power as a function of the barrier transparency Γ with different numbers of open channels N in the leads for the Andreev billiard.

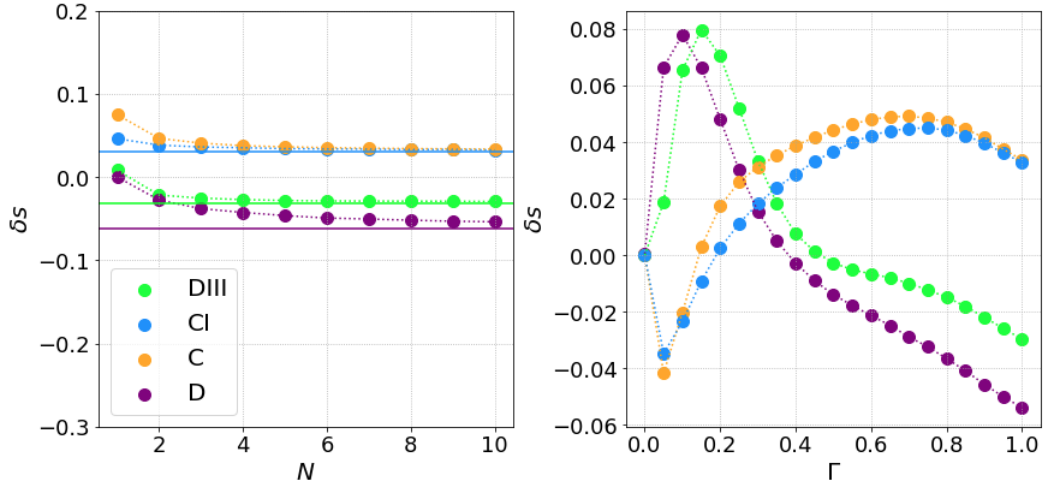


Figure 6.39 Quantum Interference Correction (QIC) of shot noise power for the Dirac billiard as a function of the number of open channels N (left panel) and as a function of the transparency barrier Γ in the semi-classical regime (right panel). The solid lines in the left panel represent Eq.(6.38).

6.4 Concurrence and Entanglement of Formation

The concurrence quantifies the degree of entanglement of electrons scattered by a chaotic billiard. In our case, the conductance is given by

$$C = 2 \frac{\sqrt{(1-T_1)T_1(1-T_2)T_2}}{T_1 + T_2 - 2T_1T_2}, \quad (6.39)$$

where the interval of possible values of concurrence is $0 \leq C \leq 1$ with 0 and 1 representing the state without any entanglement and the state with maximal entanglement (Bell pair), respectively. Through the concurrence C , we also define the entanglement of formation $\mathcal{E}$,

$$\mathcal{E}(C) = h\left(\frac{1 + \sqrt{1 - C^2}}{2}\right), \quad (6.40)$$

with

$$h(x) = -x \log_2(x) - (1 - x) \log_2(1 - x). \quad (6.41)$$

The physical meaning of $\mathcal{E}$ is the cost of the resources needed to prepare a given entangled state.

In this part, we are interested in how time-reversal symmetry affects the production of entanglement in chaotic billiards. Thus, the spin degeneracy of the channels is disregarded for simplicity [112]. In this way, we analyse only the orthogonal and unitary for the Schrödinger billiard. For the Dirac billiard, we investigate the chiral orthogonal and chiral unitary. Finally, we address the classes C and CI for the Andreev billiards.

• Schrödinger Billiard - SB

We perform numerical simulation for the Schrödinger Billiard with 10^6 realization of $\hat{H}$ to describe a chaotic cavity with 100 resonances and $N = 2$ open channels in each lead. We also consider the barrier's transparency value Γ as a parameter. We obtain the concurrence for orthogonal and unitary ensembles in the transparent barrier regime, shown in Fig.(6.40). These results are compatible with those shown in Ref.[113, 114]. The concurrence distribution of the orthogonal ensemble ($\beta = 1$) is written in the analytical form as [113]

$$P_1(C) = \frac{2}{(1 + C)^2}. \quad (6.42)$$

The mean value and variance of the concurrence distribution for GOE shown in Fig.(6.40) are

$$\begin{aligned} \langle C \rangle_1 &= 0.3860, \\ \text{var}(C)_1 &= 0.0781, \end{aligned} \quad (6.43)$$

and for the entanglement of formation distribution, we obtain

$$\begin{aligned} \langle \mathcal{E} \rangle_1 &= 0.285, \\ \text{var}(\mathcal{E})_1 &= 0.078. \end{aligned} \quad (6.44)$$

In the case of GUE, as we can see in Fig.(6.40), the concurrence distribution converges to the analytical form [113]:

$$\begin{aligned} P_2(C) = \frac{2C}{(1 - C^2)^3} &\left[3(2 + 3C^2) \operatorname{arctanh}(\sqrt{1 - C^2}) \right. \\ &\left. - (11 + 4C^2) \sqrt{1 - C^2} \right]. \end{aligned} \quad (6.45)$$

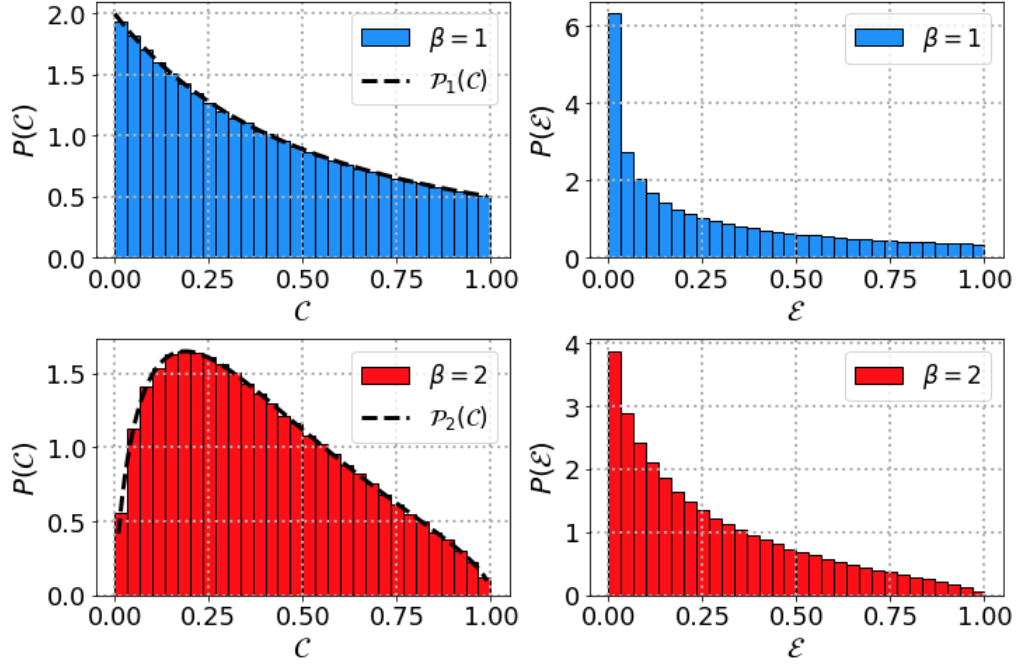


Figure 6.40 Distribution of concurrence and entanglement of formation for the Schrödinger billiard if time-reversal symmetry (TRS) is preserved (orthogonal - blue) or broken (unitary - red). The dashed lines represent the analytical result Eq.(6.42) and Eq.(6.45) for the orthogonal and unitary ensembles, respectively.

The mean value and variance of $P_2(C)$ are, respectively,

$$\begin{aligned}\langle C \rangle_2 &= 0.3876, \\ \text{var}(C)_2 &= 0.0564.\end{aligned}\tag{6.46}$$

The mean value and variance of the entanglement of formation distribution for the GUE are

$$\begin{aligned}\langle \mathcal{E} \rangle_2 &= 0.2732, \\ \text{var}(\mathcal{E})_2 &= 0.056,\end{aligned}\tag{6.47}$$

which are results in great accordance with those obtained by the references [112, 113].

In addition, we obtain the quantum interference correction (QIC) of concurrence and entanglement of formation by subtracting the classical term (unitary ensemble) of the orthogonal ensemble:

$$\begin{aligned}\langle C \rangle_{QIC} &= \langle C \rangle_1 - \langle C \rangle_2 = -0.0016, \\ \langle \mathcal{E} \rangle_{QIC} &= \langle \mathcal{E} \rangle_1 - \langle \mathcal{E} \rangle_2 = 0.0118.\end{aligned}\tag{6.48}$$

Now we consider a non-ideal chaotic billiard with an arbitrary transparency barrier $\Gamma \in [0, 1]$. The scatter of the histogram of concurrence and entanglement of formation with Γ as a parameter appears in Fig.(6.41), these results are in great accord with the Ref.[114].

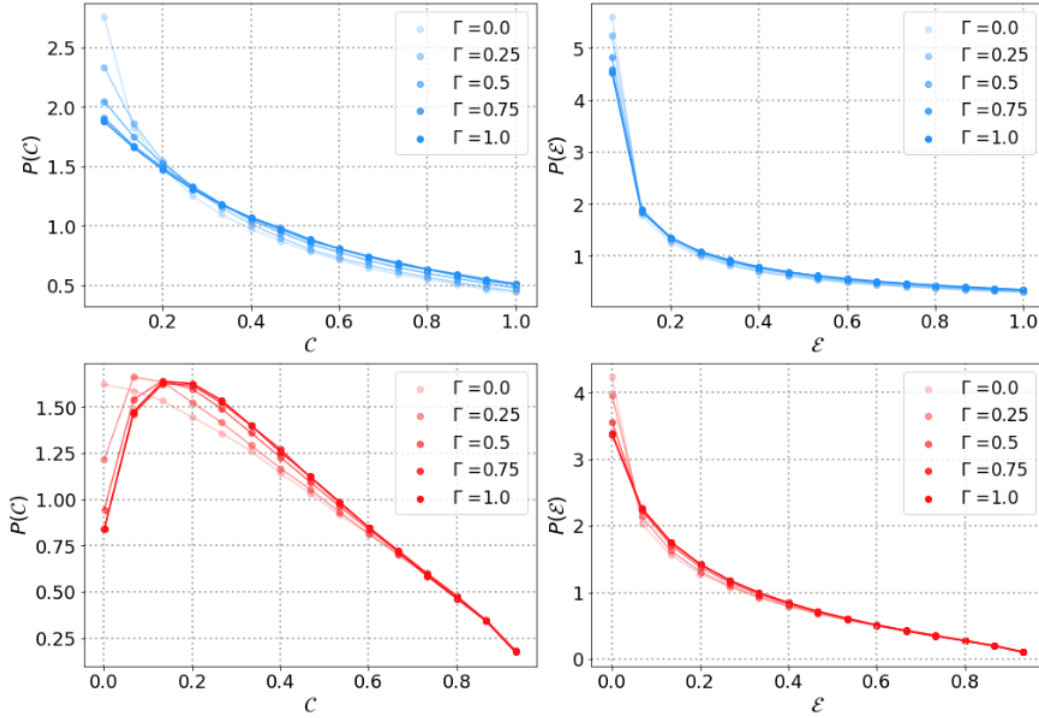


Figure 6.41 Scatter of the distribution of concurrence and entanglement of formation for the Schrödinger billiard taking Γ as a parameter.

We obtain the first novel numerical result regarding the dependence of the mean and variance of concurrence and entanglement on Γ , shown in Fig.(6.42). In Ref.[112] was pointed out that the effect of broken TRS on the average concurrence is unusually small, less than 1%. However, the difference between the mean of concurrence for GOE and GUE increases as the tunneling decreases. Therefore, for concurrence, quantum interference phenomena became more relevant in the tunneling regime ($\Gamma \rightarrow 0$). Moreover, the mean value of the entanglement of formation exhibits a transition from a depletion of the QIC term in the tunneling regime ($\Gamma \rightarrow 0$) to an amplification in the transparent barrier regime ($\Gamma = 1$).

To properly analyze the quantum interference term, we plot the QIC of concurrence and entanglement of formation as a function of Γ in Fig.(6.43). The QIC increases as Γ increases in both cases. As we can see, there is a linear relation between the quantum interference correction of both concurrence and entanglement. The linear regression provides the following linear equations

$$\begin{aligned} C^{\text{QIC}}(\Gamma) &= -0.0238 + 0.0232 \Gamma, \quad 0 \leq \Gamma \leq 1, \\ \mathcal{E}^{\text{QIC}}(\Gamma) &= -0.0110 + 0.0246 \Gamma, \quad 0 \leq \Gamma \leq 1. \end{aligned} \quad (6.49)$$

This linear behavior of the QIC for both concurrence and entanglement is unique to the Schrödinger billiard. We will see in the next sections the behavior of the QIC of concurrence and entanglement for the Dirac and Andreev billiards.

• Dirac Billiard - DB

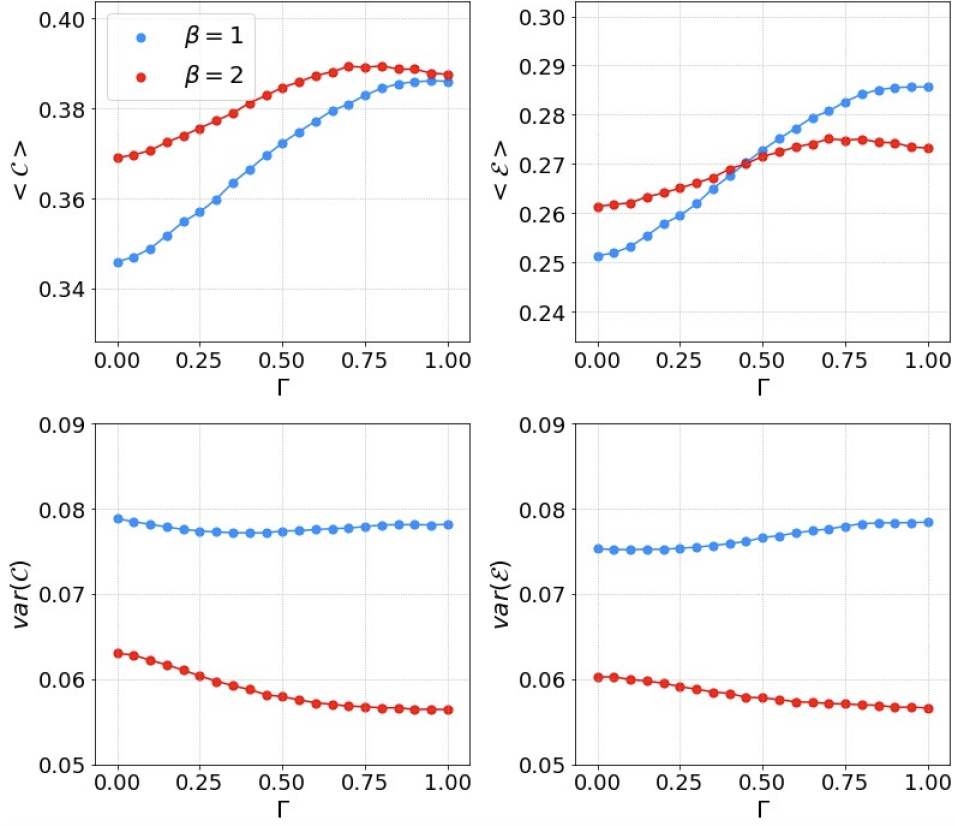


Figure 6.42 Mean and variance of concurrence and entanglement of formation for the Schrödinger billiard vs barrier's transparency Γ .

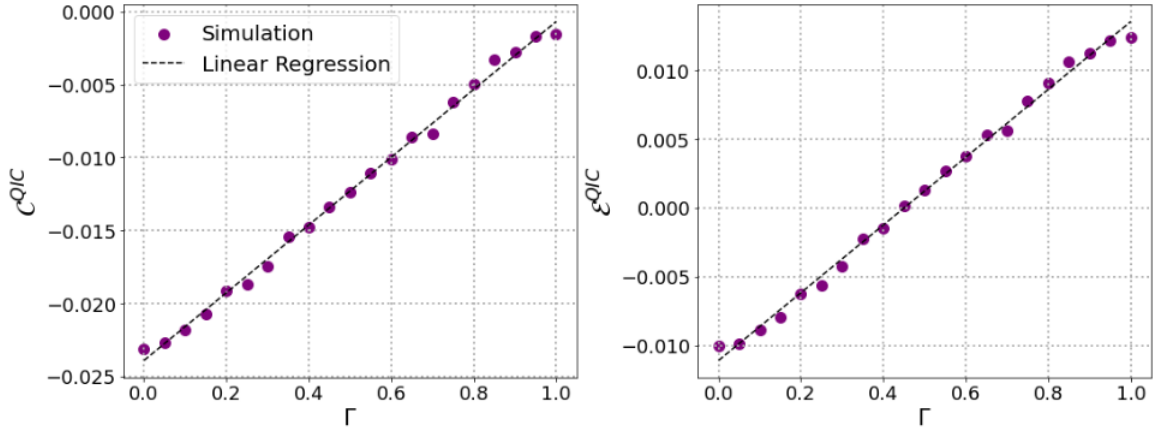


Figure 6.43 The quantum interference correction of concurrence and entanglement in Schrödinger billiard as a function of Γ .

To obtain the distribution of concurrence and entanglement of formation for the non-ideal Dirac billiard, we develop a numerical simulation similar to the one in the previous section. Additionally, we recover the results for the ideal case ($\Gamma = 1$) of the Refs.[115, 106]. We model

the billiard through 10^6 realization of $\hat{H}$ with 100 resonances, i.e, $\hat{H}$ with dimensions 200×200 . The Hamiltonian satisfies the following anti-commutation relation [27, 116]

$$H = -\sigma_z H \sigma_z, \quad \sigma_z = \begin{bmatrix} \mathbf{1}_M & 0 \\ 0 & -\mathbf{1}_M \end{bmatrix}. \quad (6.50)$$

The H matrix has a double dimension ($2M \times 2M$) due to the two sublattice degrees of freedom, and the number of open channels in each lead is $N = 1$ (high quantum regime). To simulate non-ideal Dirac billiards, the computational power required is greater than the case of Schrödinger billiards given that the scattering matrix model encompasses the inversion of $2M \times 2M$ matrices, where M is the number of resonances without the chiral/sublattice degree of freedom.

For the chiral orthogonal and unitary ensembles, we obtain their respective distributions of concurrence and entanglement shown in Fig.(6.44). The concurrence distribution of the chiral

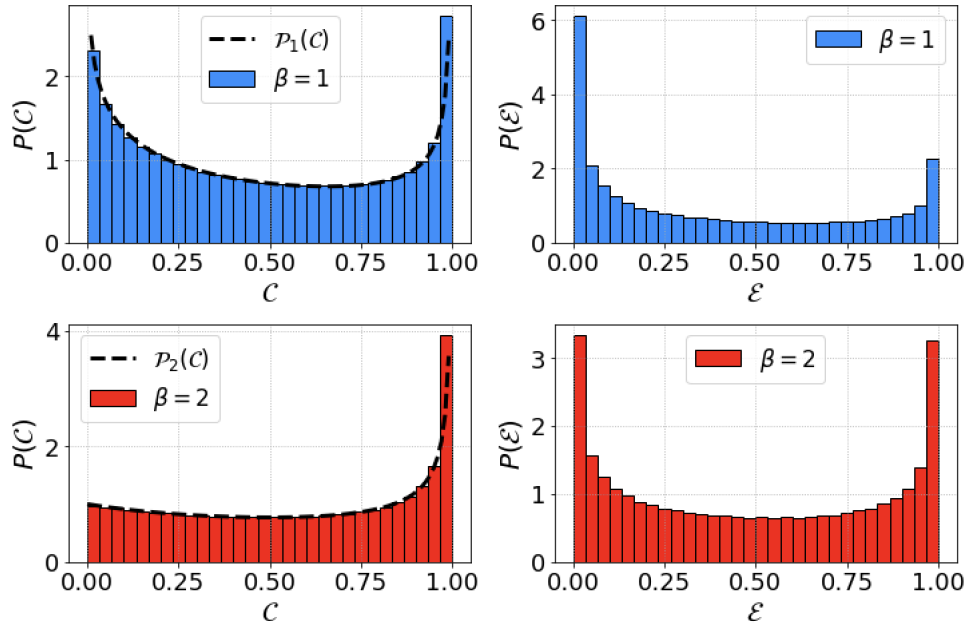


Figure 6.44 Distribution of concurrence and entanglement of formation for the Dirac billiard when time-reversal symmetry (TRS) is preserved (chiral orthogonal - blue) and broken (chiral unitary - red).

orthogonal ensemble (TRS preserved, $\beta = 1$) is written in the analytical form as [115]

$$P_1(C) = \frac{1}{2} \frac{1}{(1-C^2)} \left\{ \frac{\sqrt{2}}{2} \sqrt{1 + \frac{\sqrt{1-C^2}}{1-C^2}} \right. \\ \left. \times \operatorname{arccoth} \left[\sqrt{1 + \frac{C^2}{2(1-C^2)(1 + \frac{\sqrt{1-C^2}}{1-C^2})}} \right] - 1 \right\}. \quad (6.51)$$

We obtain the mean value and variance of concurrence for the chiral orthogonal case, which is

$$\begin{aligned} \langle C \rangle_1 &= 0.4682, \\ \operatorname{var}(C)_1 &= 0.1115. \end{aligned} \quad (6.52)$$

In the same case, we obtain the mean value and variance for the entanglement of formation,

$$\begin{aligned}\langle \mathcal{E} \rangle_1 &= 0.381, \\ \text{var}(\mathcal{E})_1 &= 0.122.\end{aligned}\tag{6.53}$$

The distribution of concurrence for the chiral unitary ensemble (TRS broken, $\beta = 2$) is written in the analytical form as

$$P_2(C) = \frac{\sqrt{1-C^2}}{(1+C)(1-C^2)}.\tag{6.54}$$

For the chiral unitary ensemble, the mean value and variance of the distribution of concurrence are

$$\begin{aligned}\langle C \rangle_2 &= 0.5710, \\ \text{var}(C)_2 &= 0.1035.\end{aligned}\tag{6.55}$$

In the case of the entanglement of formation, we obtain the following results

$$\begin{aligned}\langle \mathcal{E} \rangle_2 &= 0.4866, \\ \text{var}(\mathcal{E})_2 &= 0.1223.\end{aligned}\tag{6.56}$$

Once we have obtained the average of concurrence and entanglement distribution, we can calculate the QIC

$$\begin{aligned}\langle C \rangle_{QIC} &= \langle C \rangle_1 - \langle C \rangle_2 = -0.102, \\ \langle \mathcal{E} \rangle_{QIC} &= \langle \mathcal{E} \rangle_1 - \langle \mathcal{E} \rangle_2 = -0.105.\end{aligned}\tag{6.57}$$

Now we consider a non-ideal chaotic billiard with an arbitrary transparency barrier $\Gamma \in [0, 1]$. The scatter histogram of concurrence and entanglement of formation with Γ as a parameter is shown in Fig.(6.45).

We investigate in detail the dependence of the mean and variance of concurrence and entanglement of formation on Γ for Dirac billiard, as shown in Fig.(6.46). Compared with Schrödinger billiard in Fig.(6.42), the mean of concurrence and entanglement of formation in the Dirac billiard does not present transition from depletion to amplification of quantum interference term as Schrödinger billiard. It means that chirality protects against this transition.

In Fig.(6.47), we plot the quantum interference correction as a function of Γ . The decrease in tunneling rates produces an amplification of the quantum interference term, an exotic behavior of the Dirac billiard called the Klein paradox. Thus, in mesoscopic physics systems, the Klein paradox is manifested by the inversion of quantum anti-localization due to the time-reversal symmetry breaking. This inversion of the QIC can be found in other transport observables, such as conductance [28, 117]. We perform a quadratic regression to fit the data obtained by simulation in Fig. 6.47, which provides quadratic Eq. (6.58). Therefore, unlike the Schrödinger billiard, which follows a linear fit, the manifestation of Klein tunneling in the Dirac billiard gives a quadratic fit to the quantum interference correction for both concurrence and entanglement of formation.

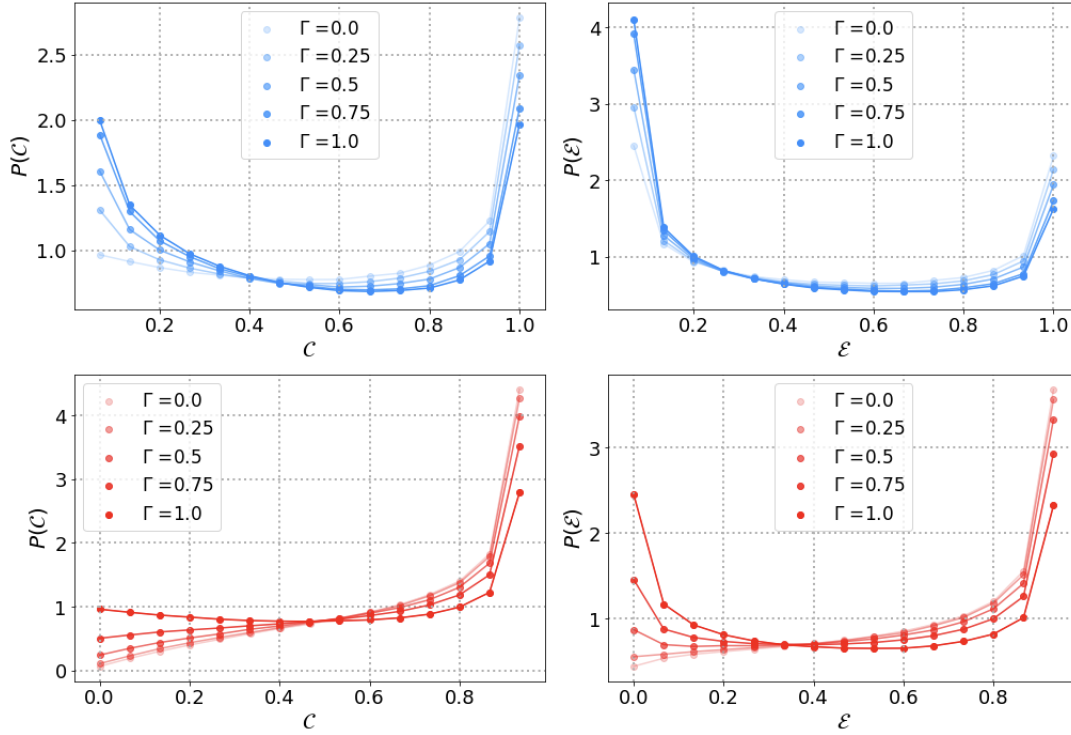


Figure 6.45 Scatter of the distribution of concurrence and entanglement of formation for the Dirac billiard taking Γ as a parameter.

$$\begin{aligned} C^{\text{QIC}}(\Gamma) &= -0.1590 - 0.0837 \Gamma + 0.1368 \Gamma^2, \quad 0 \leq \Gamma \leq 1, \\ \mathcal{E}^{\text{QIC}}(\Gamma) &= -0.1694 - 0.0791 \Gamma + 0.1420 \Gamma^2, \quad 0 \leq \Gamma \leq 1. \end{aligned} \quad (6.58)$$

	Ensembles	Chaotic Billiard	TRS	$\langle C \rangle$	$ \langle C \rangle_{\text{QIC}} $	$\langle \mathcal{E} \rangle$	$ \langle \mathcal{E} \rangle_{\text{QIC}} $
$\Gamma = 1$	Wigner-Dyson	Schrödinger	YES	0.3860	0.0016	0.285	0.0118
			NO	0.3876		0.2732	
	Chiral	Dirac	YES	0.4682	0.102	0.381	0.105
			NO	0.5710		0.4866	
$\Gamma \rightarrow 0$	Wigner-Dyson	Schrödinger	YES	0.3459	0.0231	0.2513	0.01
			NO	0.3690		0.2613	
	Chiral	Dirac	YES	0.5706	0.1616	0.4862	0.1719
			NO	0.7322		0.6581	

Table 6.1 A table containing the average and quantum interference correction of concurrence and entanglement for Schrödinger and Dirac billiards if time-reversal symmetry (TRS) is preserved or broken ($\Gamma = 1$ and $\Gamma \rightarrow 0$).

In table 6.1, we show the average, variance, and QIC of Schrödinger and Dirac billiards if time-reversal symmetry (TRS) is preserved or broken for both ideal ($\Gamma = 1$) and tunneling

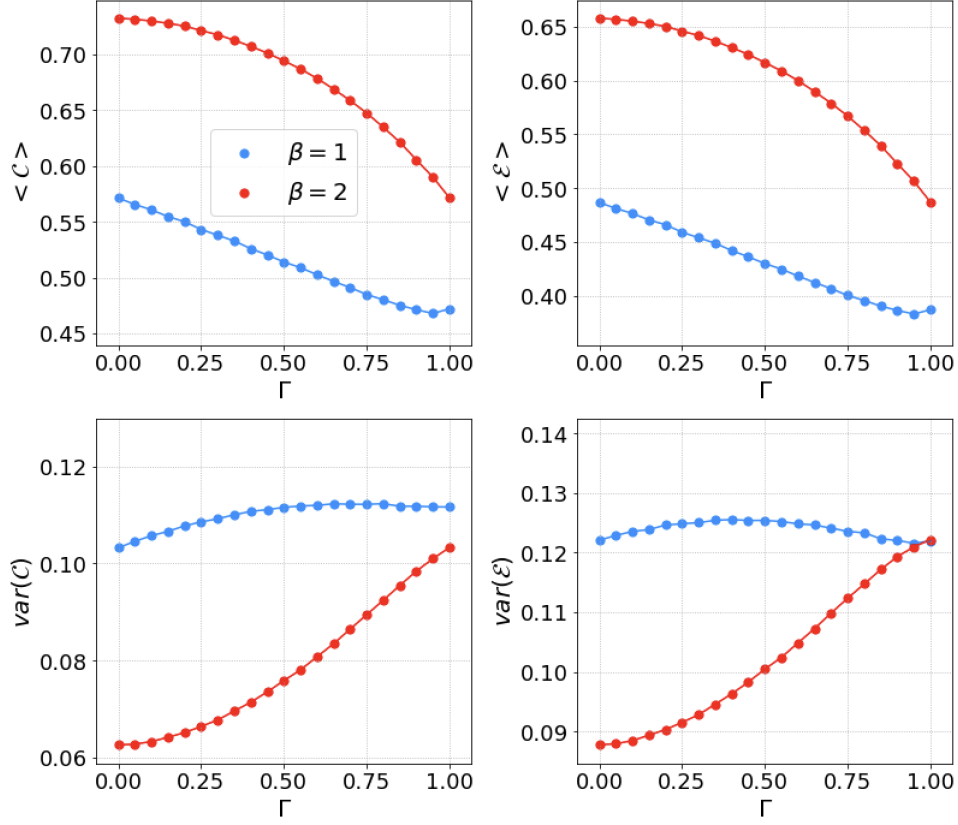


Figure 6.46 Mean and variance of concurrence and entanglement of formation for the Dirac billiard vs barrier's transparency Γ .

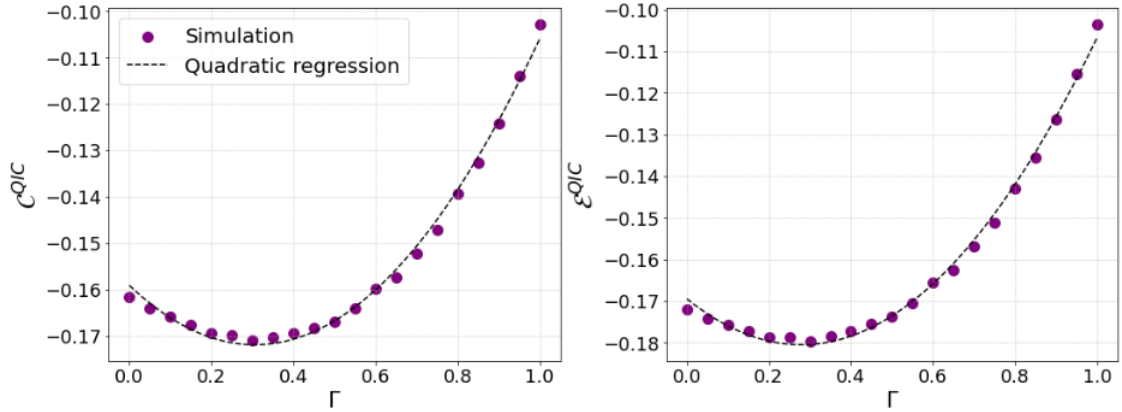


Figure 6.47 The quantum interference correction of concurrence and entanglement in Dirac billiard as a function of Γ .

($\Gamma \rightarrow 0$) regimes. As the central result in table 6.1, we can see that the quantum interference correction of concurrence $\langle C \rangle_{QIC}$ and entanglement of formation $\langle \mathcal{E} \rangle_{QIC}$ in Dirac billiard is the order of magnitude ten times larger than Schrödinger billiard for both regimes $\Gamma = 1$ and

$\Gamma \rightarrow 0$. The contribution of quantum effects to the entanglement of formation becomes more evident in the chiral degrees of freedom than in orbital ones.

• **Andreev Billiard - AB**

We obtain the distribution of concurrence and entanglement of formation for both classes C and CI, shown in Fig.(6.48), where we recover the results for the ideal case ($\Gamma = 1$) of the Ref.[118]. Since concurrence is indicated by C , we use the alternative nomenclature CQE and T-CQE for the classes C and CI, respectively. We model the billiard through 10^6 realization of $\hat{H}$ with $M = 100$ resonances, i.e, $\hat{H}$ with dimensions 200×200 . The H matrix doubles the dimension ($2M \times 2M$) due to the two electron-hole degrees of freedom, and the number of open channels in each lead is $N = 2$. In this case, the transmission eigenvalues T_1 and T_2 are double degenerated. Therefore, we use the two different transmissions T_1 and T_2 eigenvalues ($T_1 \neq T_2$) to calculate the concurrence by Eq.(6.39).

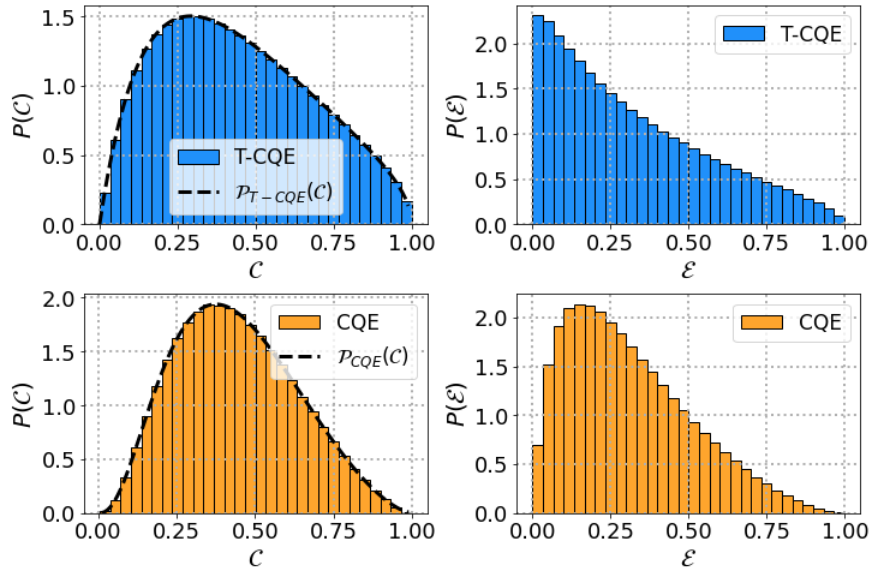


Figure 6.48 Distribution of concurrence and entanglement of formation for the number of open channels $N = 2$ when time-reversal symmetry (TRS) is preserved (T-CQE - blue) and broken (CQE - orange). The dashed line represents the analytical expression for the concurrence, given by Eq.6.59 and Eq.6.60 for the classes T-CQE and CQE, respectively.

The analytical form of the concurrence distribution for class T-CQE and CQE (TRS preserved and broken, respectively) were obtained by Ref.[118]

$$P_{T-CQE}(C) = 15 \frac{C(1-C)}{\sqrt{1-C^2}(1+C^3)}, \quad (6.59)$$

$$\begin{aligned}
P_{\text{CQE}}(C) = & \frac{5}{8} \frac{C^3}{(1-C^2)^6} \left[2 \operatorname{arctanh} \sqrt{1-C^2} \right. \\
& \times (560 + 5880C^2 + 7350C^4 + 1225C^6) \\
& - (2904 + 16264C^2 + 10350C^4 + 512C^6) \\
& \left. \times \sqrt{1-C^2} \right].
\end{aligned} \tag{6.60}$$

Therefore, our numerical results agree with the analytical formula obtained [118]. In table 6.2, we show the average, variance, and QIC for the number of open channels $N = 2$ if time-reversal symmetry (TRS) is preserved or broken (CI and C classes, respectively) in the ideal tunneling regime $\Gamma = 1$ which appropriately converges with the results of the literature.

Symmetry class	$\langle C \rangle$	$ \langle C \rangle_{\text{QIC}} $	$\text{var}(C)$	$\langle \mathcal{E} \rangle$	$ \langle \mathcal{E} \rangle_{\text{QIC}} $	$\text{var}(\mathcal{E})$
T-CQE	0.438	0.0019	0.0558	0.3207	0.0098	0.0601
CQE	0.440		0.0358	0.3109		0.0388

Table 6.2 A table containing the average, variance, and quantum interference correction of concurrence and entanglement for Andreev billiard if time-reversal symmetry (TRS) is preserved or broken ($\Gamma = 1$).

As new results, we investigate the dependence of entanglement production on tunneling rate Γ . In Fig.(6.49), we show the scatter histogram of concurrence and entanglement of formation with Γ as a parameter. As the histogram remains unchanged with different values of Γ , the entanglement production on the particle-hole degree of freedom is not affected by the tunneling rate.

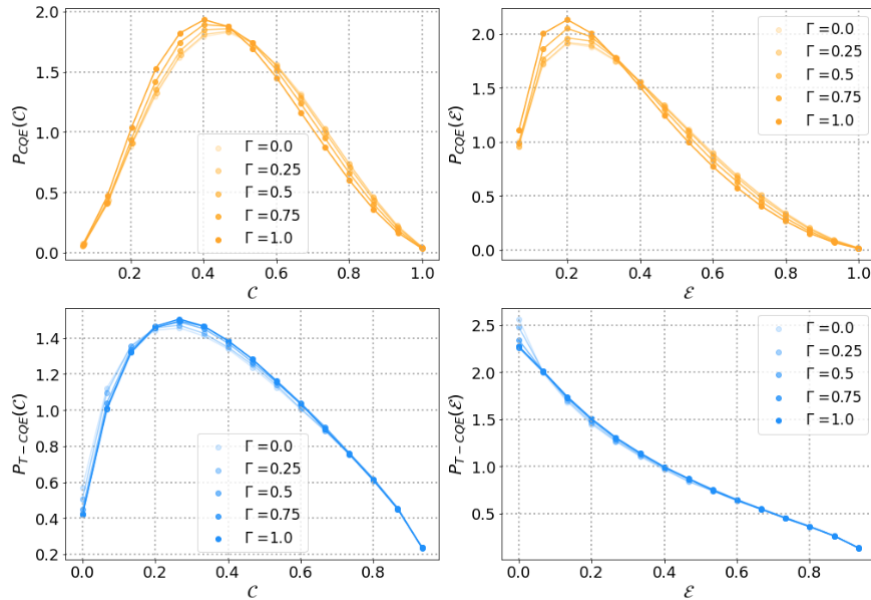


Figure 6.49 Scatter of the distribution of concurrence and entanglement of formation for the Andreev billiard taking the tunneling rate Γ as a parameter.

The dependence of mean and variance of concurrence and entanglement on Γ is depicted in Fig.(6.50). The change in concurrence averages from the opaque to the transparent barrier regime is 4.5% and 2.3% for the class C (CQE) and CI (T-CQE), respectively, so it is an unimpressive variation. Similar to the Schrödinger billiard, the average entanglement of formation presents a transition from depletion to amplification as the barrier transparency Γ increases. The barrier transparency does not change significantly the variance of both concurrence and entanglement of formation.

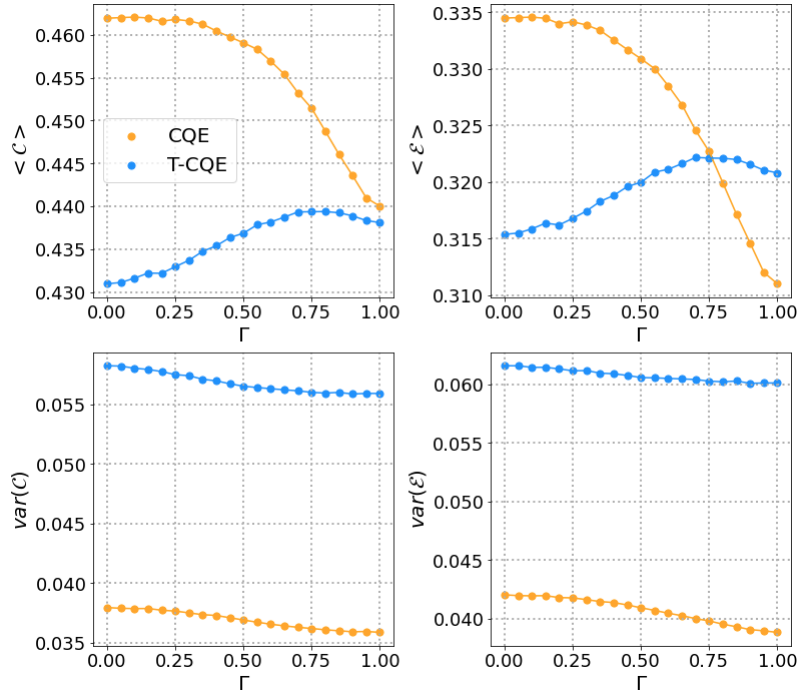


Figure 6.50 Mean and variance of concurrence and entanglement of formation for the Andreev billiard vs barrier's transparency Γ .

In Fig.(6.51), we plot the quantum interference correction (QIC) as a function of Γ . Both QIC of concurrence and entanglement of formation increase as the tunneling rate Γ decreases. This amplification of the quantum interference term is an exotic behavior of the Andreev billiard, known as the Klein paradox. Thus, in mesoscopic physics systems, the Klein paradox is manifested by the inversion of quantum anti-localization due to the time-reversal symmetry breaking. We perform a quadratic regression to fit the data obtained by simulation in Fig.(6.51), which provides the quadratic functions (6.61).

$$\begin{aligned} C^{\text{QIC}}(\Gamma) &= 0.0315 - 0.0068 \Gamma - 0.0247 \Gamma^2, \quad 0 \leq \Gamma \leq 1, \\ \mathcal{E}^{\text{QIC}}(\Gamma) &= 0.0196 - 0.0047 \Gamma - 0.0267 \Gamma^2, \quad 0 \leq \Gamma \leq 1. \end{aligned} \quad (6.61)$$

Therefore, analogous to the Dirac billiard, Andreev billiards also present a quadratic behavior of quantum interference correction for both concurrence and entanglement of formation.

We also obtain the distribution of concurrence and entanglement of formation for both classes C and CI in the single-channel limit $N = 1$, shown in Fig.(6.52). The concurrence

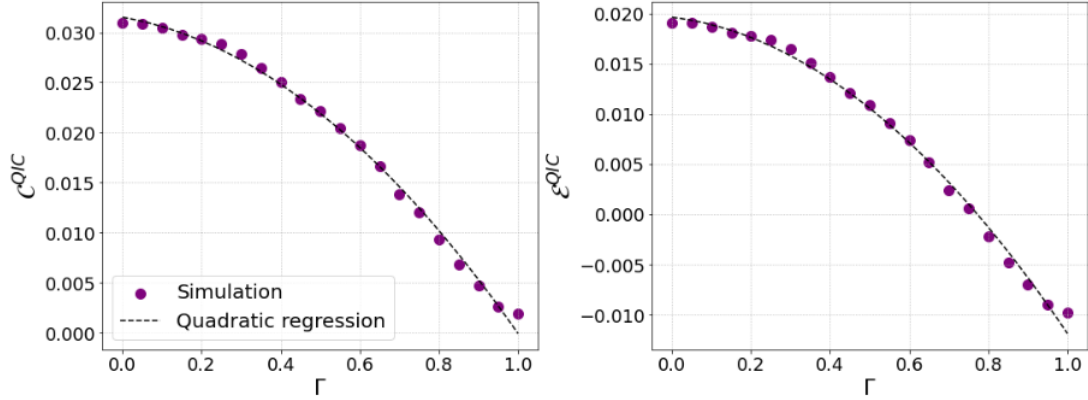


Figure 6.51 The quantum interference correction of concurrence and entanglement in Andreev billiard as a function of Γ .

C always results in 1 without statistical fluctuations. For $N = 1$ open channel in each lead, the particle-hole symmetry (Eq.(1.23)) doubles the dimension of the Hamiltonian H and, consequently, the scattering matrix S (with dimension 4×4). Therefore, the transmission matrix product $tt^\dagger$ (with dimension 2×2) produces twofold degenerate transmission eigenvalue $\tau_1 = \tau_2$ for all realizations of hamiltonian H . This equality is a consequence of particle-hole symmetry, so it does not depend on the realization, tunneling rate Γ and presence/absence of time-reversal symmetry.

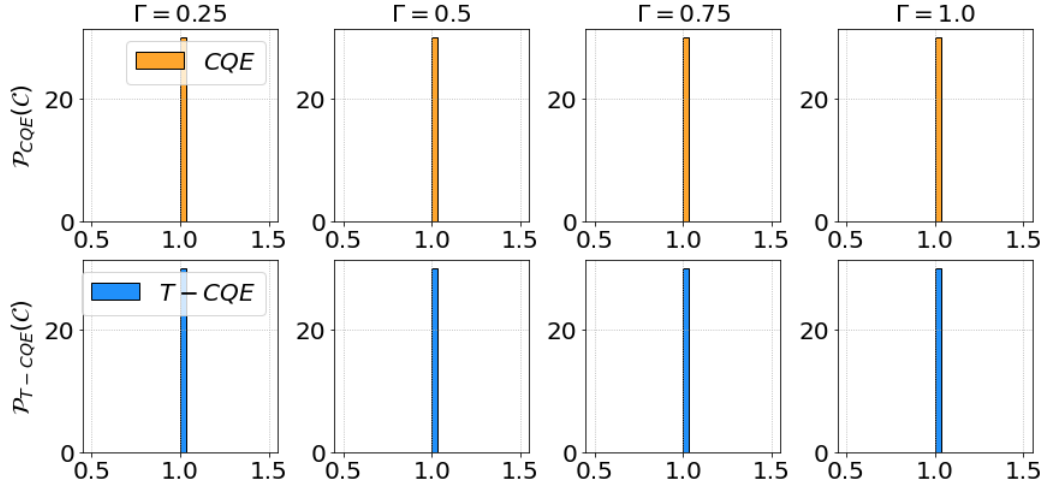


Figure 6.52 Distribution of concurrence and entanglement of formation of single-channel limit when time-reversal symmetry (TRS) is preserved (CI - blue) and broken (C - orange). The concurrence $C = 1$ reveals the production of Bell's pair on Majorana modes and helical Majorana modes in classes C and CI, respectively.

As the Beenakker formalism [112] calculates the entanglement between particles from different leads, our result shows that scatterings for Altland-Zirnbauer classes C (CRE) and CI (T-CRE) always produce Bell pairs. As previously mentioned [51], the entangled pairs are

formed by Majorana and helical Majorana modes at the interface between different topological phases, and they are supported by the symmetry classes C and CI, respectively. Then, the Majorana modes can be a useful tool for the production and control of quantum entanglement and computing. Therefore, the particle-hole symmetry provides an ideal physical system to create qubits for quantum computing [52].

The concurrence and entanglement of formation are properties that quantify the entanglement degree of the system. More peculiarly, the former quantities exhibit the signal of the Majorana modes in mesoscopic billiards, as we demonstrated. However, further questions about signatures of quantum noise in the non-classical correlation remain open. Considering this, we study, in the next section, the violations of Bell's inequality using the shot-noise power, another crucial transport observable.

6.5 Bell's Inequality

In this part, we are interested in Bell's parameter S_{CHSH} . For an arbitrary tunneling regime $\Gamma \in [0, 1]$, we calculate the maximal violation of the CHSH inequality using the correlator [119, 42]

$$C_{\mathbf{a}\mathbf{b}} = \frac{[\text{Tr}(\mathbf{a} \cdot \boldsymbol{\sigma})rr^\dagger][\text{Tr}(\mathbf{b} \cdot \boldsymbol{\sigma})tt^\dagger] - \text{Tr}(\mathbf{a} \cdot \boldsymbol{\sigma})rt^\dagger(\mathbf{b} \cdot \boldsymbol{\sigma})tr^\dagger}{(\text{Tr}rr^\dagger)(\text{Tr}tt^\dagger) - \text{Tr}rt^\dagger tr^\dagger}, \quad (6.62)$$

with an appropriate base to measure the violation of CHSH inequality of the degrees of freedom

$$\begin{cases} U_{\mathbf{a}}^\dagger \boldsymbol{\sigma}_z U_{\mathbf{a}} = \mathbf{a} \cdot \boldsymbol{\sigma}, \\ U_{\mathbf{b}}^\dagger \boldsymbol{\sigma}_z U_{\mathbf{b}} = \mathbf{b} \cdot \boldsymbol{\sigma}. \end{cases} \quad (6.63)$$

We use the correlator $C_{\mathbf{a}\mathbf{b}}$ to obtain the Bell-CHSH parameter

$$S_{\text{CHSH}} = |C_{\mathbf{a}\mathbf{b}} + C_{\mathbf{a}\mathbf{b}'} + C_{\mathbf{a}'\mathbf{b}} - C_{\mathbf{a}'\mathbf{b}'}| \leq 2, \quad (6.64)$$

and verify the relation between the concurrence C and the maximal violation of the CHSH inequality beyond the tunneling regime,

$$S_{\text{CHSH}, \max} = 2\sqrt{1 + C^2}. \quad (6.65)$$

Dephasing can also be investigated in the channels by introducing random phase factors in each edge channel.

$$U_L \rightarrow U_L \begin{pmatrix} e^{i\phi_1} & 0 \\ 0 & e^{i\phi_2} \end{pmatrix}, \quad U_R \rightarrow U_R \begin{pmatrix} e^{i\psi_1} & 0 \\ 0 & e^{i\psi_2} \end{pmatrix}. \quad (6.66)$$

Averaging $E(U_L, U_R)$ over the random phases uniformly in $(0, 2\pi)$, we find

$$\begin{aligned} S_{\text{Dephase}} &= \frac{2|\text{Tr} \boldsymbol{\sigma}_z r t^\dagger \boldsymbol{\sigma}_z t r^\dagger|}{\text{Tr} r^\dagger r t^\dagger t} \\ &= \frac{2|C_{11} + C_{22} - C_{12} - C_{21}|}{C_{11} + C_{22} + C_{12} + C_{21}} \leq 2. \end{aligned} \quad (6.67)$$

Hence, for sufficiently strong dephasing, there is no violation of the Bell inequality $|S| \leq 2$.

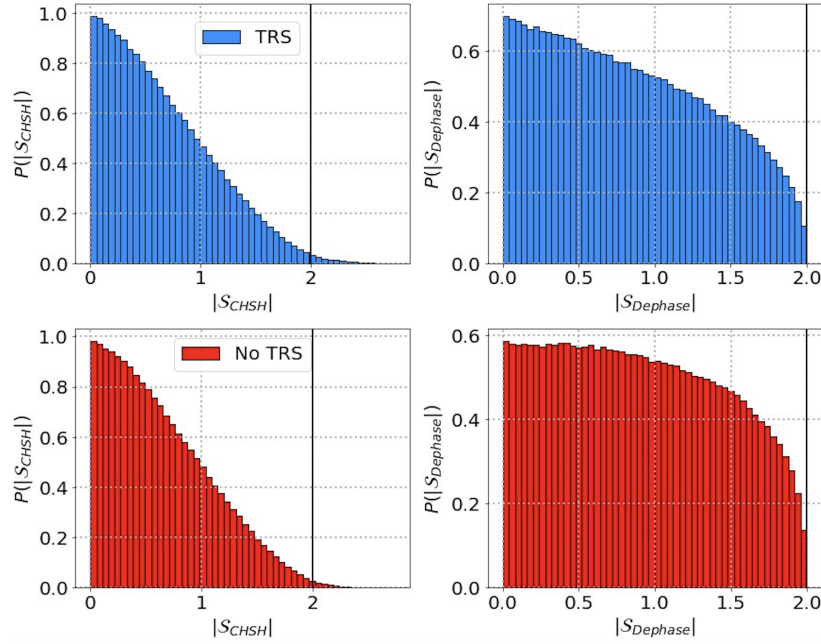


Figure 6.53 Distribution of Bell's Parameter with S_{Dephase} and without S_{CHSH} dephasing for Schrödinger billiard ($\Gamma = 1$).

• Schrödinger Billiard - SB

We perform a numerical simulation to obtain Bell's parameters for the Schrödinger billiard to detect violations. We perform 10^7 realizations in the regime without barrier $\Gamma = 1$. The number of open channels in each lead is $N = 2$, where we define the set of random unitary matrices $U_L, U_R, U_{L'},$ and $U_{R'}$ in each realization of Eq.(6.63). After that, we calculate the correlator C_{ab} given by Eq.(6.62) and finally obtain S_{CHSH} . The distribution of the Bell parameter for a quantum billiard with 100 resonances in both cases, with and without TRS, is shown in Fig.(6.53). We found 6834 (0,683%) and 4773 (0,477%) violations in GOE and GUE, respectively. We also implemented the dephasing in the channels by introducing random phase factors in each edge channel by Eq.(6.67). When the dephasing is applied, no violations of the Bell inequality occur.

The distribution of Bell's parameter in the tunneling regime ($\Gamma \rightarrow 0$) is shown in Fig.(6.54), where $\Gamma = 10^{-4}$. The quantity of violations remains the same when compared with the no-barrier case. We found 6048 (0,604%) and 4817 (0,481%) violations in GOE and GUE, respectively. There is no significant difference in comparing the number of violations between the opaque $\Gamma \rightarrow 0$ and transparent $\Gamma = 1$ regimes.

• Dirac Billiard - DB

Furthermore, we perform numerical simulations of Bell's parameters for the Dirac billiard, with 10^7 realizations in the regime without barrier $\Gamma = 1$. We set the number of open channels in each lead as $N = 1$. Unlike the Schrödinger billiard, each open channel in the Dirac billiard

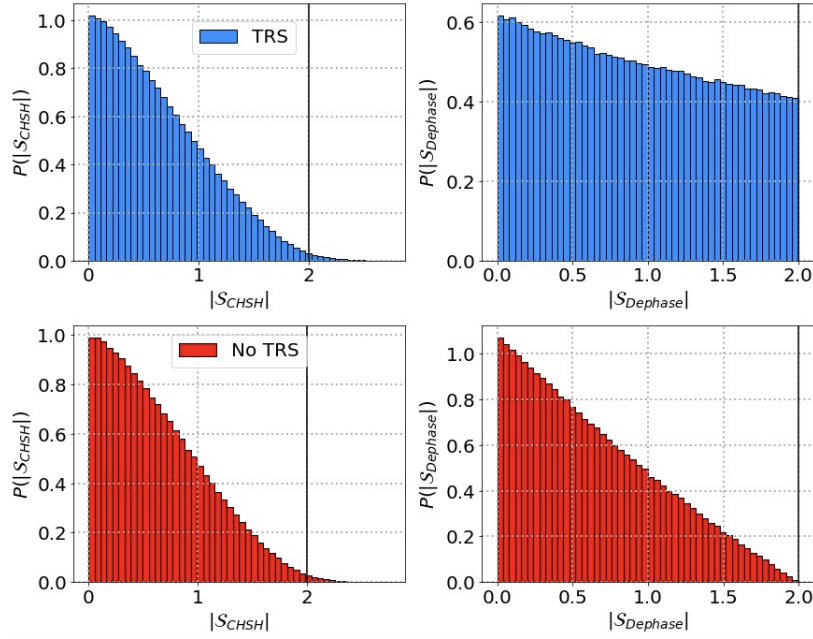


Figure 6.54 Distribution of Bell's Parameter with $\mathcal{S}_{\text{Dephase}}$ and without $\mathcal{S}_{\text{CHSH}}$ dephasing for Schrödinger billiard ($\Gamma \rightarrow 0$).

has two sub-channels related to the sublattice degrees of freedom. Thus, we are interested in the violations of Bell's inequality of these degrees of freedom. For the Dirac billiard, we obtain the distribution of Bell's parameter in the regime without barrier ($\Gamma = 1$) shown in Fig.(6.55), we found 15091 (1,509%) and 20222 (2,022%) violations in chGOE and chGUE, respectively. These results demonstrate a significant difference between Schrödinger and Dirac billiards. The violations in the orthogonal ensemble increase from 0,683% (Schrödinger) to 1,509% (Dirac), and the unitary ensemble, increases from 0,477% (Schrödinger) to 2,022% (Dirac). Therefore, it is easier to detect violations in chiral/sublattices degrees of freedom of the Dirac billiard than orbital degrees of freedom of the Schrödinger billiard.

For the tunneling regime case ($\Gamma \rightarrow 0$), we obtain the distribution of Bell's parameter, shown in Fig.(6.56). We found 20180 (2,018%) and 30602 (3,060%) violations for the chGOE and chGUE, respectively. Again there is a significant difference between Schrödinger and Dirac billiards. The violations in the orthogonal ensemble increase from 0,604% (Schrödinger) to 2,018% (Dirac), and in the unitary ensemble, increases from 0,481% (Schrödinger) to 3,060% (Dirac). In short, we can see in Table 6.3 that there are more violations of Bell's inequality in the Dirac billiard than in the Schrödinger billiard.

In Fig.(6.56), we also present the Bell parameter when applying a dephase in the channels. We observe a peculiar behavior of the parameter $|\mathcal{S}_{\text{Dephase}}|$ in tunneling regime $\Gamma \rightarrow 0$. To analyze it properly, we simulate the correlators C_{ij} of Eq.(6.67). In the Fig.(6.57) we see the boxplot of correlators $C_{ij} = |(rt^\dagger)_{ij}|^2$ that show us $C_{11}, C_{22} \rightarrow 0$ in the tunneling regime, then $\mathcal{S}_{\text{Dephase}}$ of Eq.(6.67) is equal to 2. It means there is no correlation between current fluctuations and the same chiral degree of freedom from different leads in this regime.

We also count violations of Bell inequality as a function of resonances for the Schrödinger

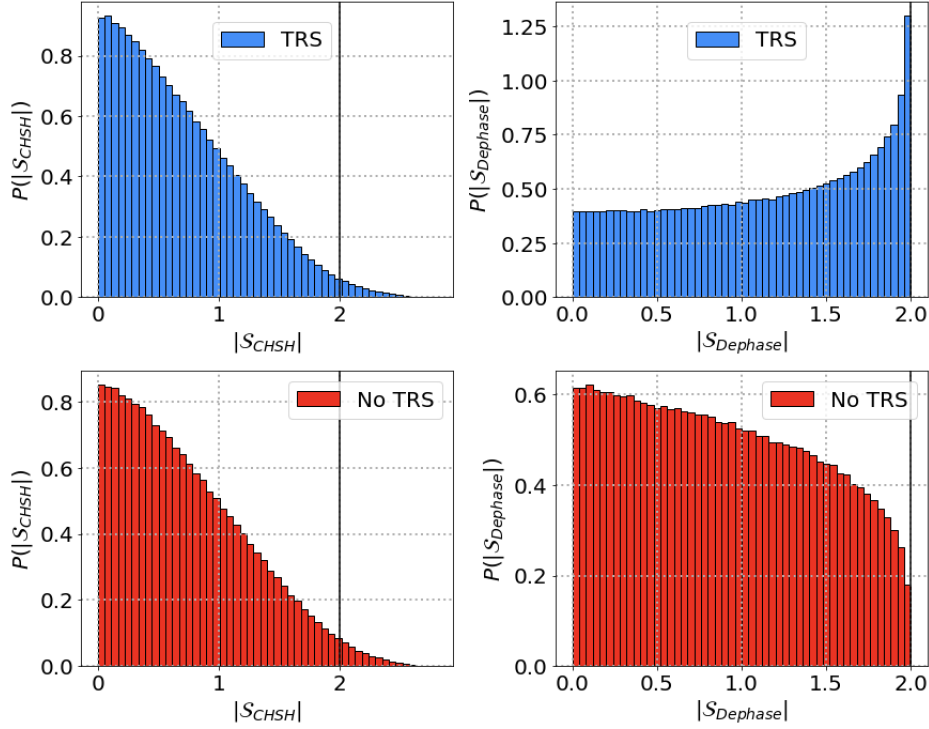


Figure 6.55 Distribution of Bell's Parameter with S_{Dephase} and without S_{CHSH} dephasing for Dirac billiard ($\Gamma = 1$).

Ensembles	Chaotic Billiard	TRS	Violations ($\Gamma = 1$)	Violations ($\Gamma \rightarrow 0$)
Wigner-Dyson	Schrödinger	YES	6843 (0.683%)	6048 (0.604%)
		NO	4773 (0.477%)	4817 (0.481%)
Chiral	Dirac	YES	15091 (1.509%)	20180 (2.018%)
		NO	20222 (2.022%)	30602 (3.060%)

Table 6.3 Table containing the violations for Schrödinger and Dirac billiards if time-reversal symmetry (TRS) is preserved or broken ($\Gamma = 1$, $\Gamma \rightarrow 0$).

and Dirac billiards, where we define $\Gamma = 1$. Initially, we vary the number of resonances from 50 to 300, shown in Fig.(6.58). The number of violations remains constant as a function of the resonances, regardless of the billiard. Thus, under the RMT validity, the chaotic billiard size does not play a role in Bell's inequality violations.

Since the number of violations remains constant with the number of resonances, we set the number of resonances equal to 100 and count violations of Bell inequality as a function of the barriers' transparency Γ , shown in the Fig.(6.59). The cases $\Gamma \rightarrow 0$ and $\Gamma = 1$ represent the tunneling approximation regime and beyond the tunneling approximation, respectively. The violations remain constant in the Schrödinger billiard and slightly decrease in the Dirac billiard.

In addition, we realize a numerical simulation for the Bell's Parameter with a fixed base of measurement instead of using random unitary matrices U_L , U_R , $U_{L'}$ and $U_{R'}$. In Fig.(6.60) is

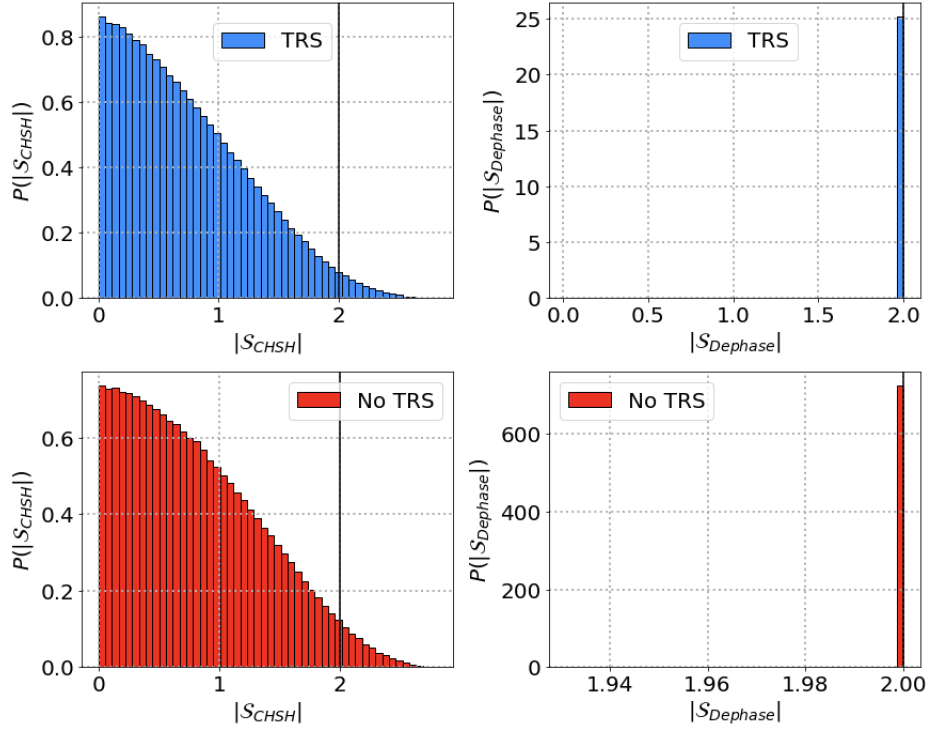


Figure 6.56 Distribution of Bell's Parameter with and without dephasing for Dirac billiard ($\Gamma \rightarrow 0$).

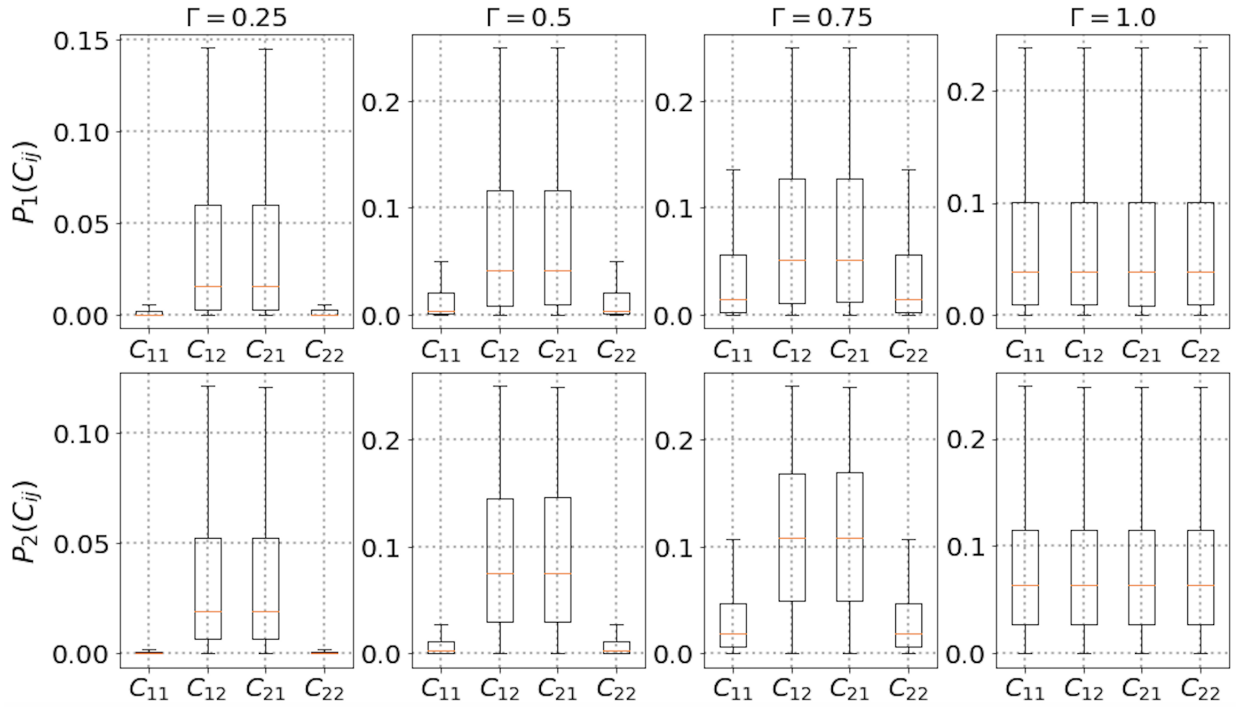


Figure 6.57 Boxplot of correlators for $\beta = 1, 2$ in the Dirac Billiard.

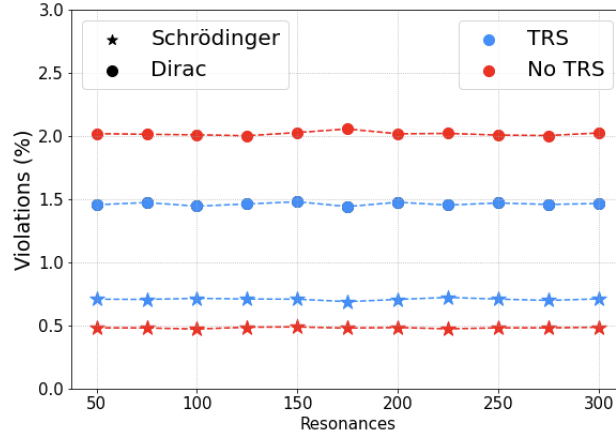


Figure 6.58 Violations as a function of resonances for the Schrödinger and Dirac billiards.

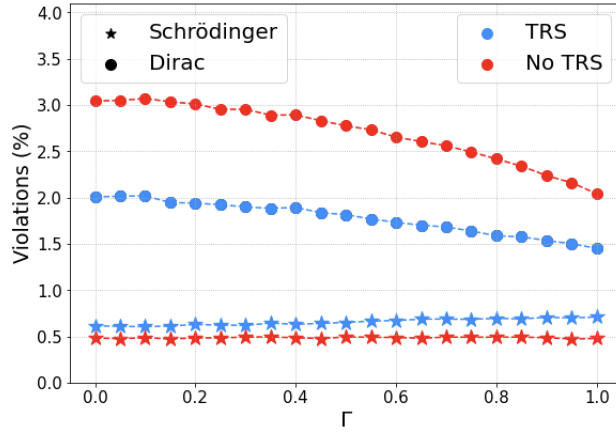


Figure 6.59 Violations as function of Γ for the Schrödinger and Dirac billiards.

shown the distribution of Bell's Parameter $\mathcal{S}_{\text{CHSH}}$ for the Schrödinger billiard. To obtain Bell's parameter, we use the following measurement base

$$\begin{cases} \mathbf{a} = \hat{k}, & \mathbf{b} = -\frac{1}{\sqrt{2}}(\hat{i} + \hat{k}), \\ \mathbf{a}' = \hat{i}, & \mathbf{b}' = -\frac{1}{\sqrt{2}}(\hat{i} - \hat{k}), \end{cases} \quad (6.68)$$

where we use $\mathbf{a}$, $\mathbf{a}'$, $\mathbf{b}$ and $\mathbf{b}'$ in the Eq.(6.62). In the Fig.(6.60) is depicted the distribution of Bell's Parameter $\mathcal{S}_{\text{CHSH}}$ for Dirac billiard in the base of Eq.(6.68) rotated by $\pi/4$ in y-axis $R_y(\pi/4)$.

The Fig.(6.61) shows the relation between the maximal violation of Bell's inequality $\mathcal{S}_{\text{max}}$ in the intervals of concurrences [120]. For that 4×10^7 realization in the regime without barrier $\Gamma = 1$ was performed. The yellow scatter is obtained by calculating the maximal violation according to the concurrence C given by Eq.(6.65). In contrast, the blue and red scatters are obtained by picking up the maximal violation of Bell's inequality of a specific range in the concurrence of orthogonal and unitary ensembles, respectively. Therefore, chirality does not

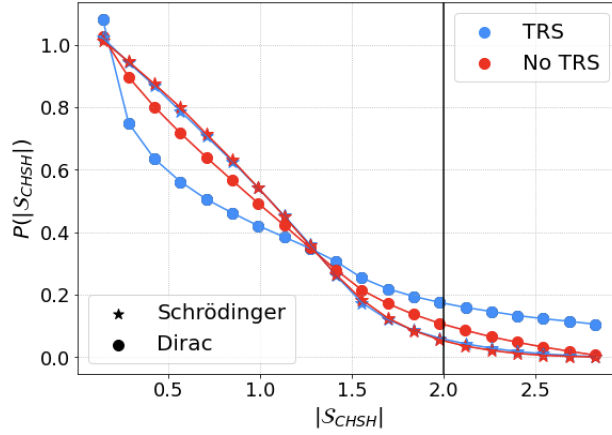


Figure 6.60 Distribution of Bell's parameter for Schrödinger and Dirac billiard with a fixed measurement base. The base chosen for the Schrödinger billiard is given by Eq.(6.68). For the Dirac billiard is the basis Eq.(6.68) rotated by $\pi/4$ in y-axis $R_y(\pi/4)$ ($\Gamma = 1$).

show a significant difference between the maximal violation of Bell's inequality versus the concurrence.

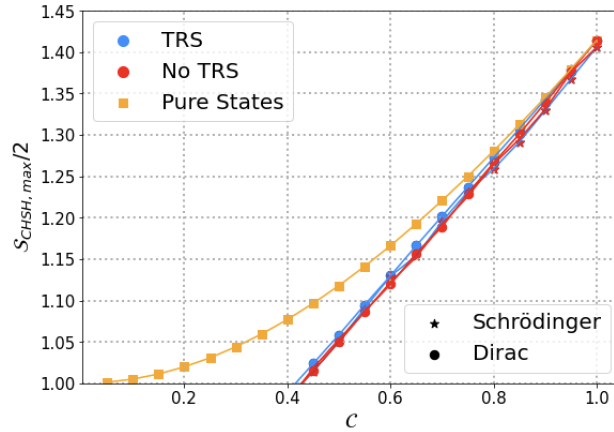


Figure 6.61 Maximal violation as a function of concurrence for the Schrödinger and Dirac billiards.

• Andreev Billiard - AB

We perform numerical simulations of Bell's parameters for the Andreev billiard, with 10^7 realizations in the regime without barrier $\Gamma = 1$. We set the number of open channels in each lead as $N = 1$. Each open channel has two sub-channels related to the electron-hole degree of freedom. Thus, we are interested in the violations of Bell's inequality of these degrees of freedom. We obtain the distribution of Bell's parameter in the regime without barrier ($\Gamma = 1$) shown in Fig.(6.62), we found 70956 (7,095%) and 71179 (7,117%) violations in CI and C classes, respectively. These results demonstrate an insignificant difference of the CHSH parameter $\mathcal{S}_{CHSH}$ under the presence/absence of TRS. In addition, we investigate the Bell parameter when

a dephase is applied $\mathcal{S}_{\text{Dephase}}$. Independent of the presence/absence of TRS, violations are not present in this case.

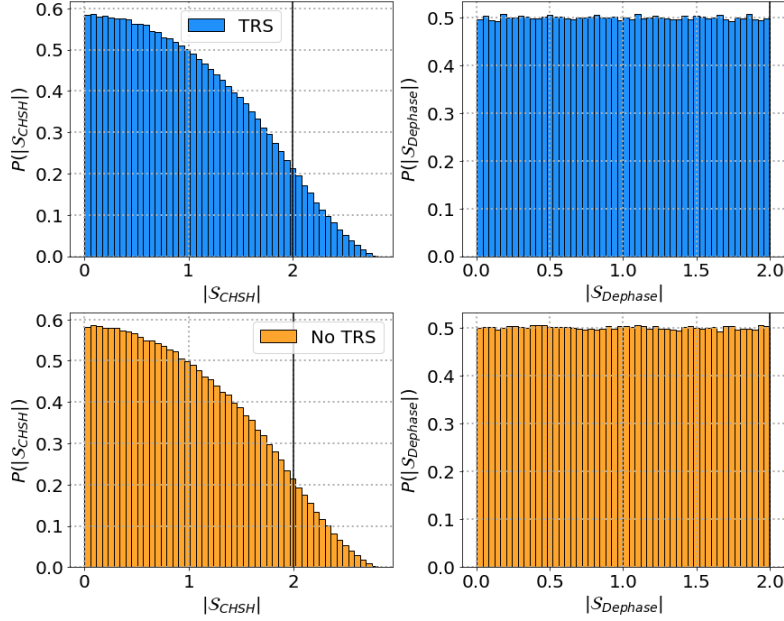


Figure 6.62 Distribution of Bell's Parameter with $\mathcal{S}_{\text{Dephase}}$ and without $\mathcal{S}_{\text{CHSH}}$ dephasing for Andreev billiard ($\Gamma = 1$).

For the tunneling regime case ($\Gamma \rightarrow 0$), we obtain the distribution of Bell's parameter, shown in Fig.(6.63). We found 70829 (7,082%) and 71055 (7,105%) violations for the CI and C classes, respectively. Therefore, there is no significant difference between opaque and transparent barrier regimes.

We also count violations of Bell inequality as a function of resonances for the Andreev billiard, where we define $\Gamma = 1$. Initially, we vary the number of resonances from 50 to 300, shown in Fig.(6.64). The number of violations remains constant at around 7,1% as a function of the resonances, regardless of the presence of time-reversal symmetry. Thus, under the RMT validity, the chaotic billiard size does not play a role in Bell's inequality violations.

Since the number of violations remains constant with the number of resonances, we set the number of resonances equal to 100 and count violations of Bell inequality as a function of the barriers' transparency Γ , shown in the Fig.(6.65). The violations remain constant at around 7,1% for any tunneling rate regime.

In addition, we realize a numerical simulation for the Bell's Parameter with a fixed base of measurement instead of using random unitary matrices U_L , U_R , $U_{L'}$ and $U_{R'}$. In Fig. 6.60 is shown the distribution of Bell's Parameter $\mathcal{S}_{\text{CHSH}}$ for the Andreev billiard. To obtain Bell's parameter, we use the following measurement base

$$\begin{cases} \mathbf{a} = \hat{k}, & \mathbf{b} = -\frac{1}{\sqrt{2}}(\hat{i} + \hat{k}), \\ \mathbf{a}' = \hat{i}, & \mathbf{b}' = -\frac{1}{\sqrt{2}}(\hat{i} - \hat{k}), \end{cases} \quad (6.69)$$

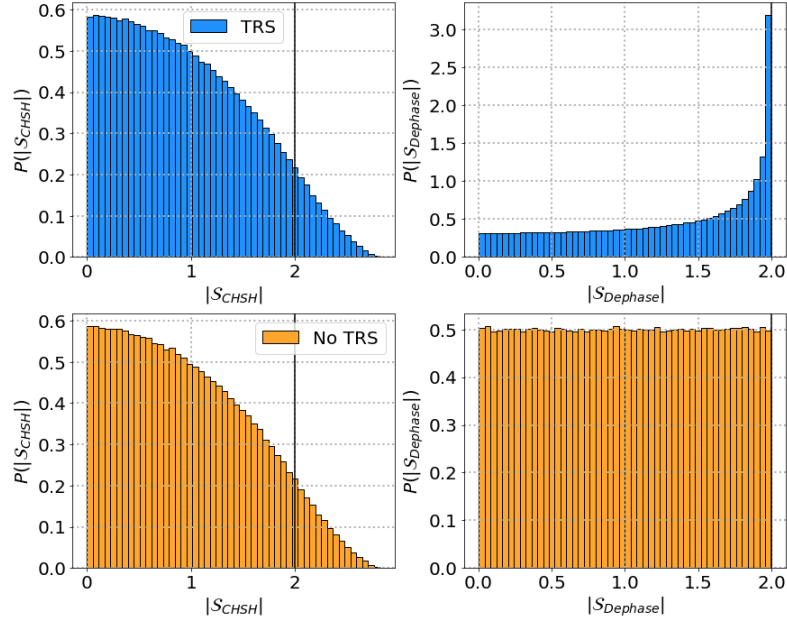


Figure 6.63 Distribution of Bell's Parameter with S_{Dephase} and without S_{CHSH} dephasing for Andreev billiard ($\Gamma \rightarrow 0$).

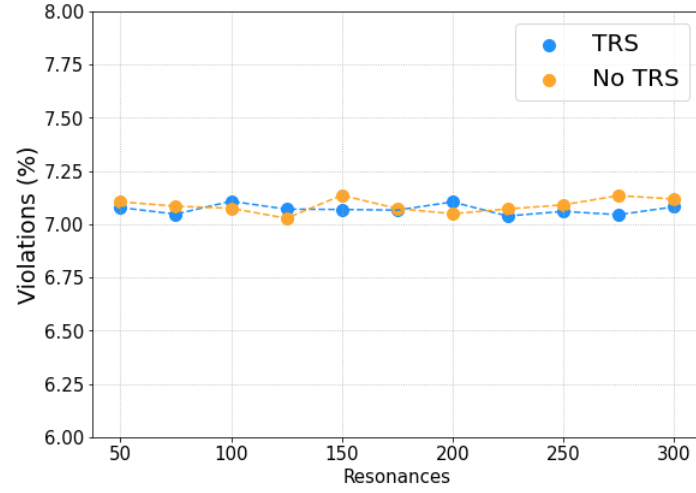


Figure 6.64 Violations as a function of resonances for the Schrödinger and Andreev billiards.

where we use $\mathbf{a}$, $\mathbf{a}'$, $\mathbf{b}$ and $\mathbf{b}'$ in the Eq.(6.62). In the Fig.(6.66) is depicted the distribution of Bell's Parameter S_{CHSH} for Andreev billiard in the base given by Eq.(6.69).

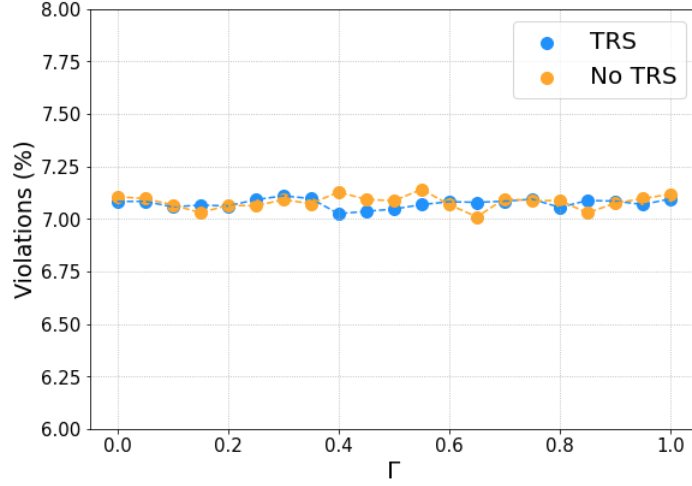


Figure 6.65 Violations as function of Γ for the Schrödinger and Andreev billiards.

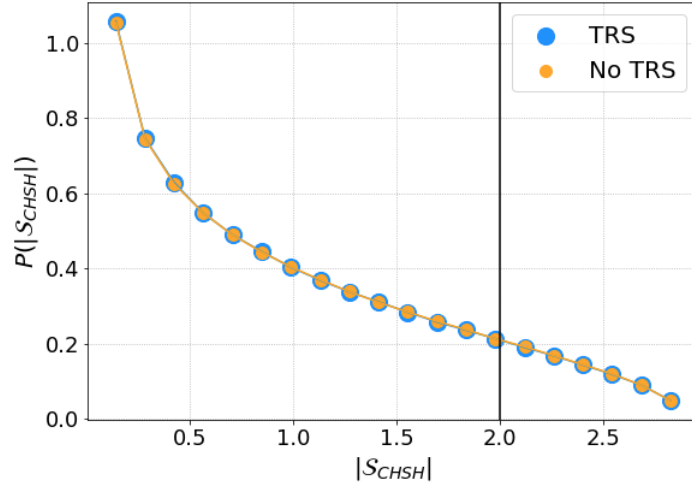


Figure 6.66 Scatter of the distribution of Bell's parameter for Andreev billiard with a fixed measurement base. The base chosen is given by Eq.(6.68) with $\Gamma = 1$.

6.6 Density of Maxima for the Conductance

Universal conductance fluctuations (UCF) are a remarkable signature of chaos in the quantum scattering processes [94, 80, 121, 71, 9, 101, 122]. Initially addressed in nuclear physics, these fluctuations were owing the chaotic resonant scattering in heavy nuclei [25], which led Wigner to discover the random nature of Hamiltonians [123]. Due to the precise experimental control, artificial mesoscopic chaotic quantum dots (QDs) has become one of the most discussed physical systems for UCF studies [2]. In such systems, the conductance, described by the Landauer formula, exhibits random fluctuations as an external parameter Z , such as a magnetic field B or the electronic energy ε . In order to analyze these fluctuations, correlation function $C(\delta Z) = \langle G(Z)G(\delta Z) \rangle - \langle G(Z) \rangle^2$ is usually defined to measure the degree of coher-

ence in fully chaotic systems, in which is quantified by the correlation width length γ_Z . In nuclear physics, Ericson [64] obtained a Lorentzian shape for the cross-section correlation as a function of the energy variation

$$\frac{C(\delta\epsilon)}{\text{var}(g)} = [1 + (\delta\epsilon/\gamma_\epsilon)^2]^{-1}. \quad (6.70)$$

Furthermore, Brink and Stephen [68] were able to access the correlation width length by analyzing the density of maxima in a single cross-section curve, avoiding the laborious experimental data obtention associated with calculating the correlation function.

On the other hand, random matrix theory (RMT) [94, 25, 123] can explain an observed UCF and its respective correlation function (CF). According to the RMT, the electronic transport properties does depend only on fundamental symmetries, that were initially characterized by Wigner and Dyson [124, 125, 21] in three ensembles: circular orthogonal ensemble (COE) ($\beta = 1$), circular unitary ensemble (CUE) ($\beta = 2$) and circular symplectic ensemble (CSE) ($\beta = 4$). It was shown that the CF amplitude (variance) of a UCF depends only on the Dyson symmetry index β and that its shape in Dyson ensembles can be Lorentzian [126, 102].

The tunneling barriers also play a leading role in UCFs. The effect of the tunneling barrier on the density of maxima is well established [127] and the CF obtained in the limit that the number of open channels N is large and tunneling probabilities $\Gamma \rightarrow 0$ is given by [128]

$$\frac{C(\delta\epsilon)}{\text{var}(g)} = \frac{1 - (\delta\epsilon/\gamma)^2}{[1 + (\delta\epsilon/\gamma)^2]^2}, \quad (6.71)$$

which is a strongly non-Lorentzian shape for the conductance CF. All these theoretical developments were primarily concerned with the CFs and density of maxima for the Wigner-Dyson class of symmetry. In this thesis, we are interested in universal conductance fluctuations (UCF) beyond the Wigner-Dyson classes, such as the symmetry classes that emerge from chiral (Dirac billiard) and normal-superconductor hybrid heterostructures (Andreev billiard).

In chaotic mesoscopic physics, RMT also predicts the existence of universality beyond the Wigner-Dyson classes previously mentioned [129, 130, 131]. The two sublattices in the honeycomb graphene structure imply the emergence of chiral symmetry, which is presented in some special systems known as topological insulators [132, 129, 133] and the phenomenological counter-part, the relativistic chaos [134, 135]. Nevertheless, the signature of chirality disappears subtly as the number of open channels increases, turning experimental detection a notorious challenge. Superconductivity also plays a fundamental role in mesoscopic systems. The particle-hole degree of freedom presents peculiar features in universal conductance fluctuations that will be addressed in this section.

In the present work, we explore how chiral and particle-hole degrees of freedom affect the shape of CFs and density of maxima for both quantum regimes, which is achieved for a single open channel, and for many open channels, known as the semi-classical regimes. Therefore, the analysis of conductance correlations and the density of maxima for Dirac and Andreev billiards is a new branch of research that seeks to determine the relationship between symmetries (chiral and particle-hole) and correlations/the density of maxima of conductance.

- **Schrödinger Billiard - SB**

We perform numerical simulations to obtain the dimensionless conductance, represented here as G , as a function of the energy ε for the orthogonal class (GOE) of the Schrödinger billiard. We extend our analysis for both semi-classical and quantum regimes with arbitrary barrier values, as represented in Fig.(6.67).

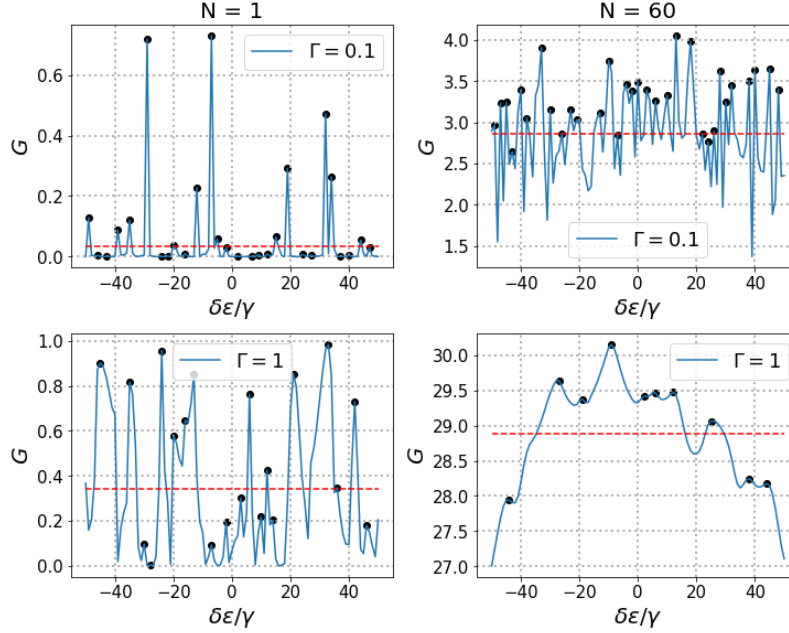


Figure 6.67 Typical dimensionless conductance as a function of energy for quantum ($N = 1$) and semi-classical regime ($N = 60$) with different barrier values Γ . The red dashed line represents the RMT prediction for $\langle T \rangle$.

We obtain the correlation function (CF) of those signals in Fig.(6.67) for N perfectly coupled modes with $\Gamma = 1$, depicted in Fig.(6.68). The ensemble average is taken over 30000 realizations of the H matrices with $M = 200$. The transmission auto-correlation length γ is given by the Weisskopf estimate $\gamma = (N_1 + N_2)\Delta/2\pi = N\Delta/\pi$, where Δ represents the mean resonance spacing. We present the agreement between the Lorentzian distribution and the CF for the semi-classical regime, which is a known result by literature. As a new result, the quantum regime presents a Lorentzian shape with a higher auto-correlation length.

In the Fig.(6.69), we see the product of the density max average $\langle \rho_\varepsilon \rangle$ with the auto-correlation length γ in terms of the number of channels N . The value converges to 0.55 as N increases, agreeing with the Gaussian process prediction [69]

$$\langle \rho_\varepsilon \rangle = \frac{\sqrt{3}}{\pi\gamma} \approx \frac{0.55}{\gamma}. \quad (6.72)$$

We also perform the CF of those signals for both quantum ($N = 1$) and semi-classical regime ($N = 30$) with different barrier values Γ , depicted in Fig.(6.70). As we previously mentioned, the CF in the quantum regime follows the Lorentzian distribution for any value of the barrier's

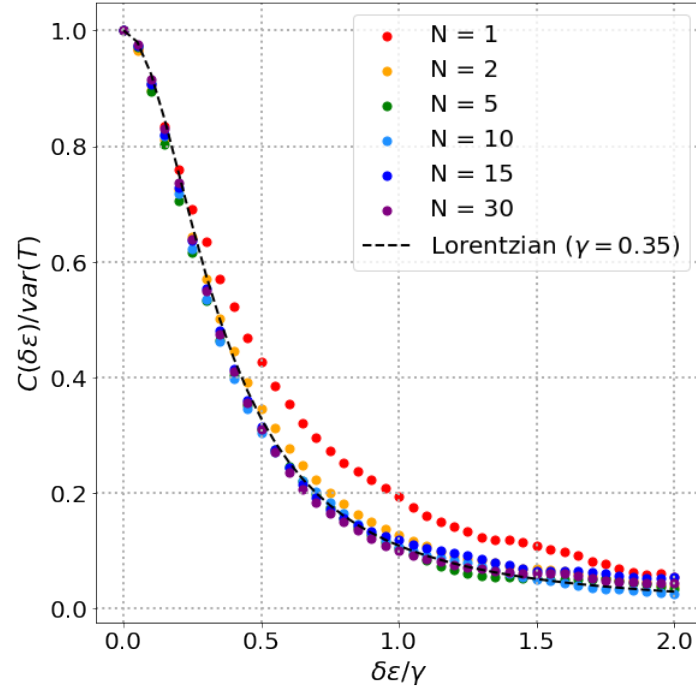


Figure 6.68 CF of the dimensionless conductance as a function of energy for different numbers of open channels N with $\Gamma = 1$.

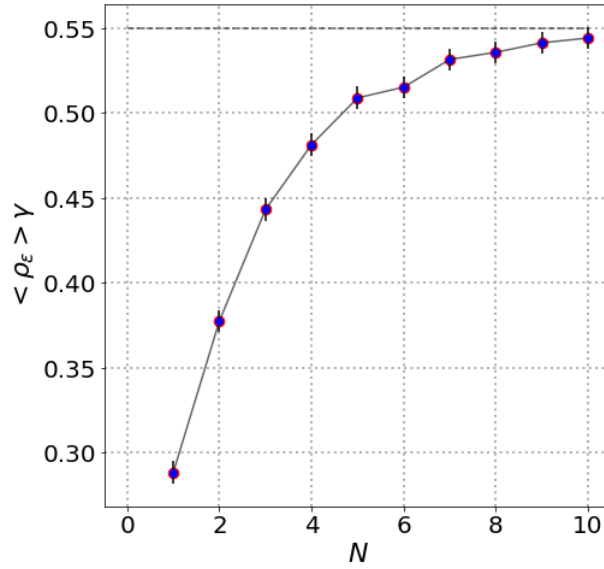


Figure 6.69 Product of the density max average ρ_ϵ with the auto-correlation length γ in terms of the number of channel N . The value converges to 0.55 as N increases, agreeing with the Gaussian process prediction [69].

transparency, in particular for the transparent barrier case $\Gamma = 1$. In a semi-classical regime, the

CF satisfies the general form [128]

$$\frac{C(\delta\epsilon)}{1/(8\beta)} = \frac{3\Gamma(2-\Gamma)-2}{1+(\delta\epsilon/\gamma)^2} + \frac{4[1+\Gamma(\Gamma-2)]}{[1+(\delta\epsilon/\gamma)^2]^2}. \quad (6.73)$$

In the ideal case, $\Gamma = 1$, we recover the usual Lorentzian shape. However, in the opaque limit ($\Gamma \rightarrow 0$), there is a manifestation of anti-correlation present in different physical systems, as quantum dots [136] and in the nuclear reaction $p + {}^{35}\text{Cl} \rightarrow \alpha + {}^{32}\text{S}$ (Fig. 6 of Ref.[67]).

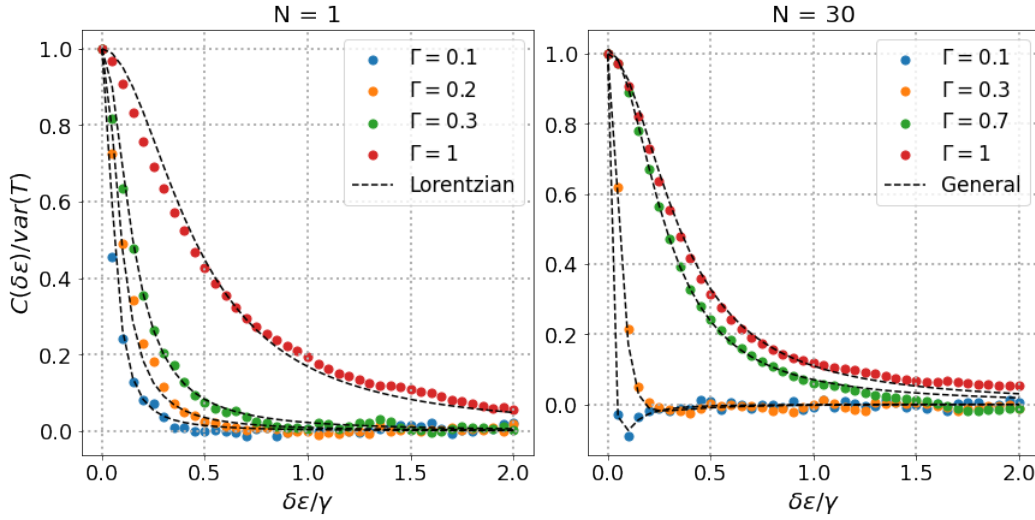


Figure 6.70 CF of the conductance as a function of energy for quantum ($N = 1$) and semi-classical regime ($N = 30$) with different barrier values Γ . The General in the right panel indicates the correlation function given by Eq.(6.73).

The Fig.(6.71) shows a new numerical result that agrees with the analytical result obtained for the semi-classical regime available in the literature, which is the product of the density max average $\langle\rho_\epsilon\rangle$ with the auto-correlation length γ in terms of the barrier value Γ . In a semi-classical regime, we present the convergence between our simulations and analytical [128] results

$$\langle\rho_\epsilon\rangle\gamma = \frac{3}{\pi}\sqrt{\frac{9\Gamma^2 - 18\Gamma + 10}{5\Gamma^2 - 10\Gamma + 6}}. \quad (6.74)$$

• Dirac Billiard - DB

We also obtain numerically the dimensionless conductance as a function of the energy for the chiral orthogonal class (ChGOE) of the Dirac billiard, extending the analysis for the semi-classical and quantum regimes with arbitrary barrier values, depicted in Fig. 6.72. The Hamiltonian of the Dirac billiard satisfies the following the anti-commutation relation

$$H = -\sigma_z H \sigma_z. \quad (6.75)$$

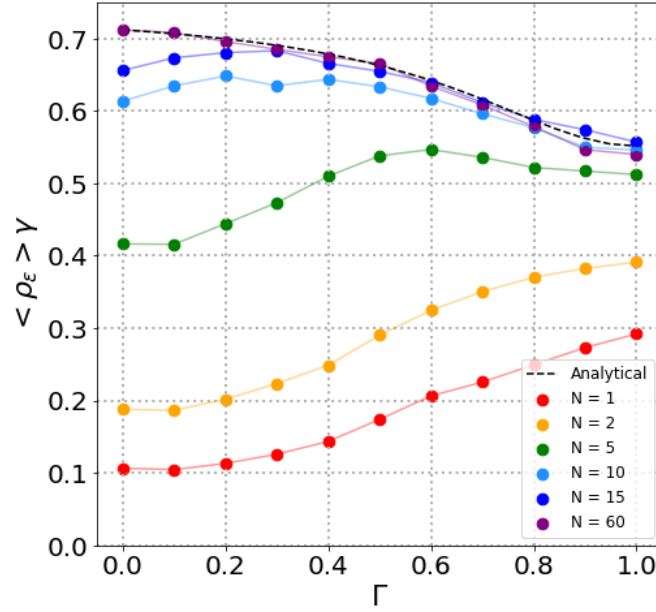


Figure 6.71 Product of the density max average ρ_ε with the auto-correlation length γ in terms of the barrier value Γ . In the semi-classical regime, our numerical result agrees with the analytical one [128], represented by the Eq.(6.74).

Since the matrices' dimensions double due to the two sublattices' degrees of freedom, the Dirac billiards simulation requires more computational power than Schrödinger one. Therefore, we treat $N = 30$ as the semi-classical regime in this case. For the scattering matrix, the Eq. 6.75 translates into [116]

$$S(\varepsilon) = \sigma_z S^\dagger(-\varepsilon) \sigma_z. \quad (6.76)$$

In our case, due to the sub-lattice and the time-reversal symmetries are present, the dimensionless conductance $G = \text{Tr}(t^\dagger t)$ is an even function of the energy ε , as we can see in Fig. 6.72. For a case of comparison between billiards, we plot a typical realization of the dimensionless conductance for Schrödinger and for the positive part $\delta\varepsilon \geq 0$ of Dirac billiard in Fig 6.73, with $\delta\varepsilon_{SB}$ and $\delta\varepsilon_{DB}$ representing its respective energy scales for quantum and semi-classical regimes.

Firstly, we analyze the CF of dimensionless conductance signals in the range $\delta\varepsilon/\gamma \geq 0$, depicted in Fig. 6.74. The ensemble average is taken over 30000 realizations of the H matrices with $M = 200$. The CF in Dirac billiards converges to the Lorentzian shape for any number of open channels N .

In the Fig. 6.75, we see the product of the density max average $\langle \rho_\varepsilon \rangle$ with the auto-correlation length γ in terms of the number of channels N . Unlike the Schrödinger billiard described by Eq. 6.72, the value converges to 0.6 as N increases, revealing an amplification in the density of maxima for the chiral class of symmetry

$$\langle \rho_\varepsilon \rangle \approx \frac{0.6}{\gamma}. \quad (6.77)$$

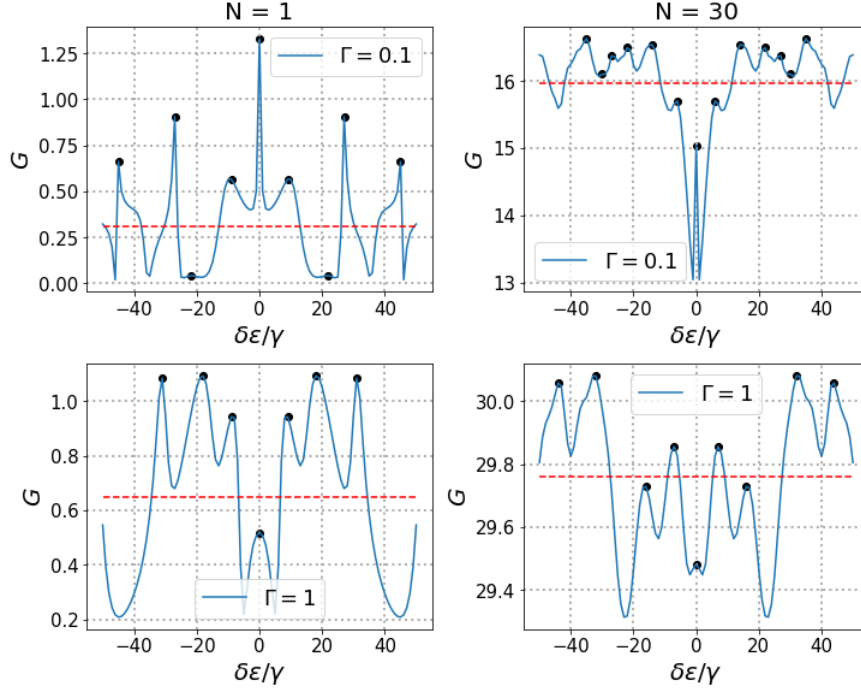


Figure 6.72 Typical realization of the dimensionless conductance as a function of energy for quantum ($N = 1$) and semi-classical regime ($N = 30$) with different barrier values Γ for the Dirac billiard.

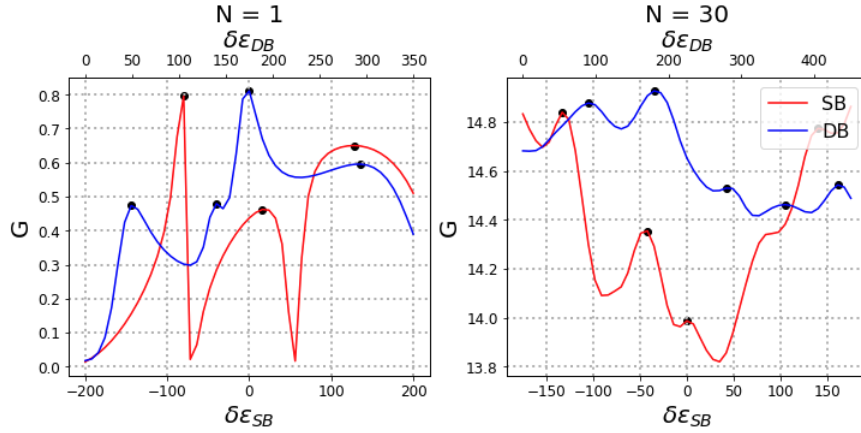


Figure 6.73 Typical realization of the dimensionless conductance as a function of energy for Schrodinger (SB) and Dirac (DB) billiards with different energy scales for quantum ($N = 1$) and semi-classical regimes ($N = 30$).

We also calculated the CF for quantum ($N = 1$) and semi-classical regime ($N = 30$) with different barrier values Γ , depicted in Fig. 6.76. In the semi-classical regime, Schrödinger (Fig. 6.70) and Dirac (Fig. 6.76) billiard have the same behavior, given by the Eq. 6.73. In particular, they present anti-correlation in the opaque limit ($\Gamma \rightarrow 0$) and the usual Lorentzian shape in the ideal case ($\Gamma = 1$). However, in the quantum regime, Wigner-Dyson class of symmetry preserves the CF Lorentzian shape for any value of barrier transparency, whereas

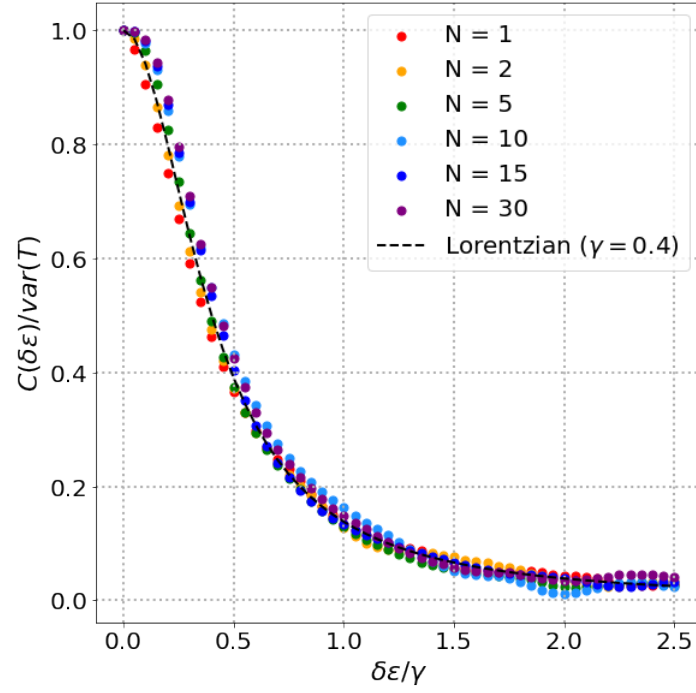


Figure 6.74 CF of the dimensionless conductance as a function of energy ε for the ideal barrier case ($\Gamma = 1$).

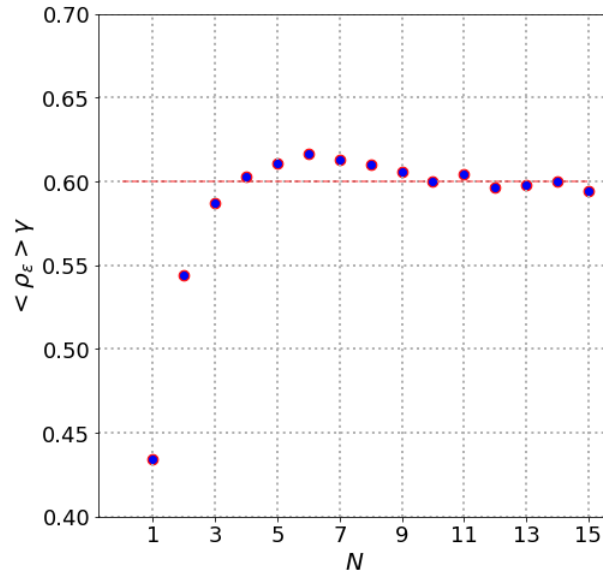


Figure 6.75 Product of the density max average ρ_ε with the auto-correlation length γ in terms of the number of channel N . The value converges to 0.6 as N increases.

chirality presents the anti-correlation behavior in opaque limit.

In the Fig. 6.77, we see the product of the density max average $\langle \rho_\varepsilon \rangle$ with the auto-correlation length γ in terms of the number of the barrier value Γ . The curves for $N \geq 5$

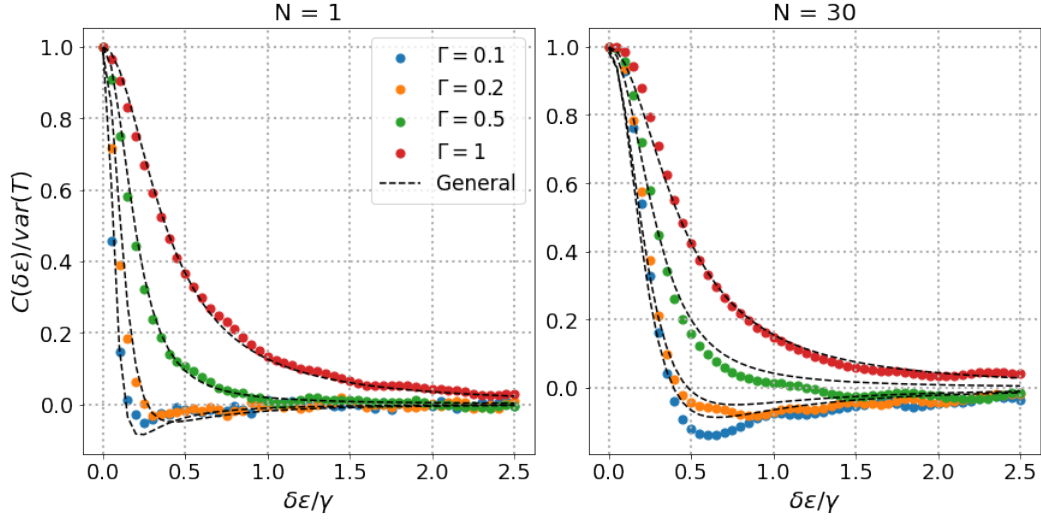


Figure 6.76 CF of the dimensionless conductance as a function of energy for quantum ($N = 1$) and semi-classical regime ($N = 30$) with different transparency barrier values Γ for the Dirac billiard. The General indicates the correlation function given by Eq.(6.73).

converge to 0.6 for $\Gamma = 1$. Moreover, we observe that Dirac billiard also follows the Eq. 6.74 in a semi-classical regime, however it presents an amplification in the density of maxima

$$\langle \rho_\varepsilon \rangle_{DB}(\Gamma) = \langle \rho_\varepsilon \rangle_{SB}(\Gamma) + 0.05 \quad (6.78)$$

for any value of Γ .

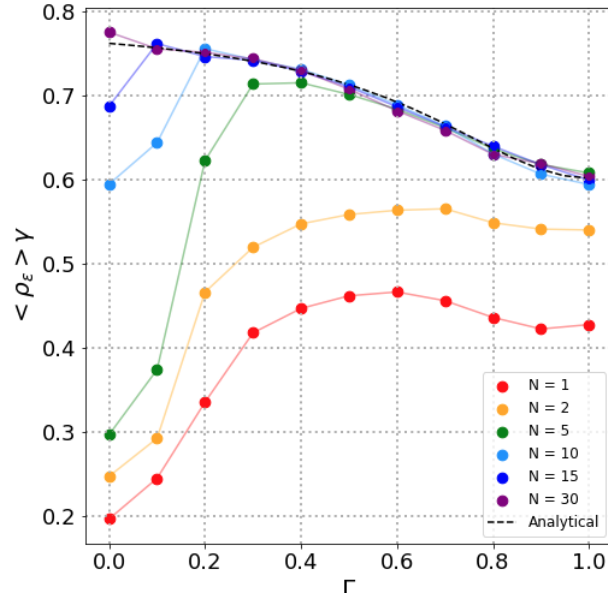


Figure 6.77 Product of the density max average ρ_ε with the auto-correlation length γ in terms of the value of tunneling barrier Γ .

• Andreev Billiard - AB

We also obtain numerically the dimensionless conductance as a function of the energy for the class CI of the Andreev billiard, extending the analysis for the semi-classical and quantum regimes with arbitrary barrier values, depicted in Fig.(6.78). The Hamiltonian of the Andreev billiard satisfies the following the anti-commutation relation

$$H = -\tau_y H \tau_y. \quad (6.79)$$

Since the matrices' dimensions double due to the particle-hole degree of freedom, the Andreev billiards simulation requires more computational power than Schrödinger one. Therefore, we treat $N = 30$ as the semi-classical regime in this case. For the scattering matrix, the Eq.(6.79) translates into

$$S(E) = \tau_y S^*(-E) \tau_y. \quad (6.80)$$

In our case, due to the particle-hole and the time-reversal symmetries are present, the dimensionless conductance $G = \text{Tr}(t^\dagger t)$ is an even function of the energy ε , as we can see in Fig.(6.78). For a case of comparison between billiards, we plot a typical realization of the dimensionless conductance for Schrödinger and for the positive part $\delta\varepsilon \geq 0$ of Andreev billiard in Fig.(6.79), with $\delta\varepsilon_{SB}$ and $\delta\varepsilon_{AB}$ representing its respective energy scales for quantum and semi-classical regimes.

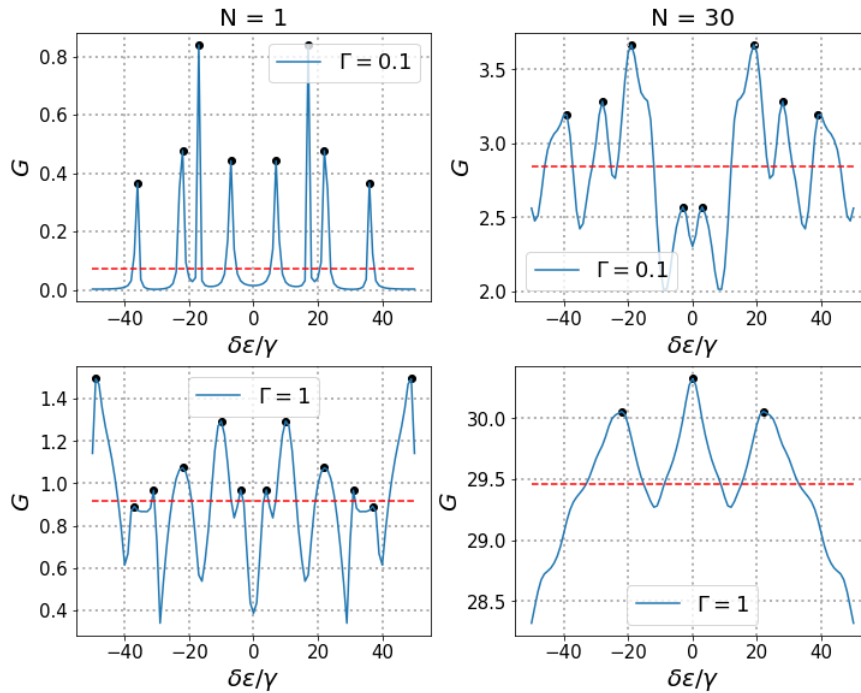


Figure 6.78 Typical realization of the dimensionless conductance as a function of energy for quantum ($N = 1$) and semi-classical regime ($N = 30$) with different barrier values Γ for the Andreev billiard.

We analyze the CF of dimensionless conductance signals in the range $\delta\varepsilon/\gamma \geq 0$, depicted in Fig.(6.80). The ensemble average is taken over 30000 realizations of the H matrices with

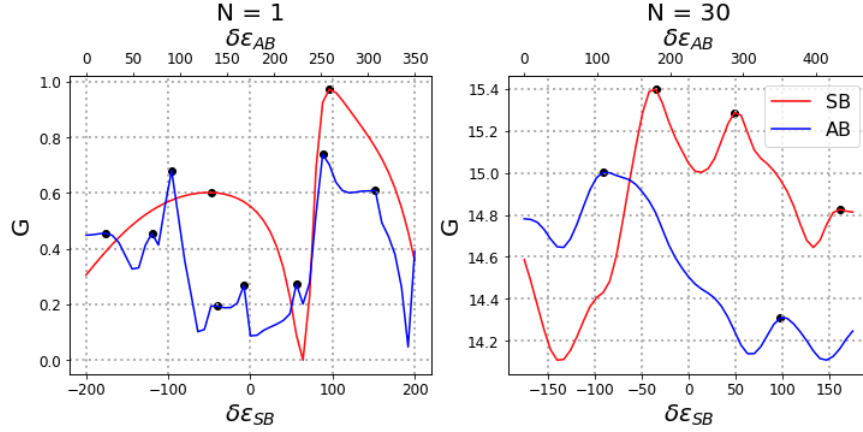


Figure 6.79 Typical realization of the dimensionless conductance as a function of energy for Schrodinger (SB) and Andreev (AB) billiards with different energy scales for quantum ($N = 1$) and semi-classical regimes ($N = 30$).

$M = 200$. The CF in Andreev billiards converges to the Lorentzian shape for any number of open channels N .

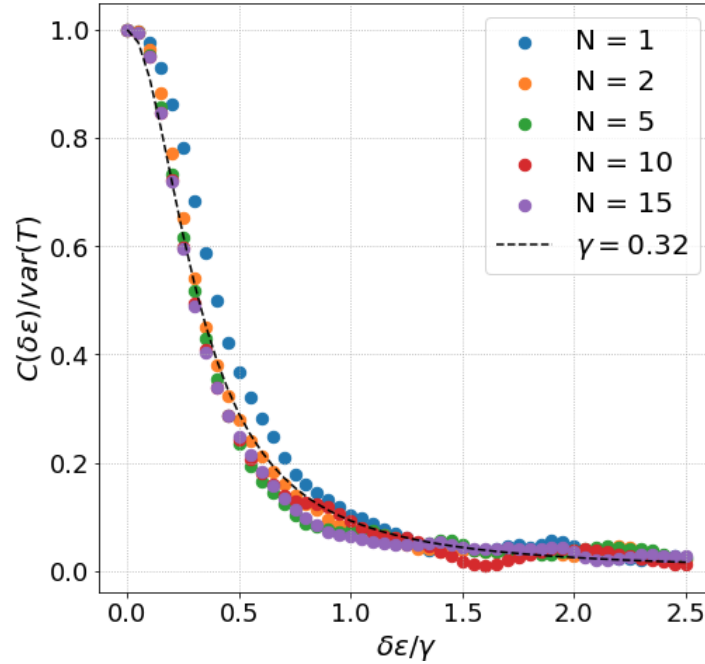


Figure 6.80 CF of the dimensionless conductance as a function of energy ϵ for the ideal barrier case ($\Gamma = 1$).

In the Fig.(6.81), we see the product of the density max average $\langle\rho_\epsilon\rangle$ with the auto-correlation length γ in terms of the number of channels N . Unlike the Schrödinger billiard described by Eq.(6.72), the value converges to 0.6 as N increases, revealing an amplification in the density

of maxima for the particle-hole class of symmetry

$$\langle \rho_\varepsilon \rangle \approx \frac{0.6}{\gamma}. \quad (6.81)$$

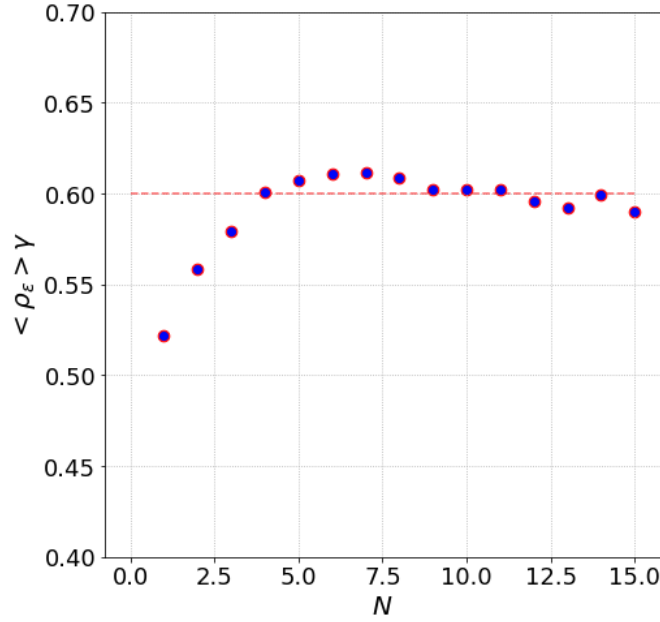


Figure 6.81 Product of the density max average ρ_ε with the auto-correlation length γ in terms of the number of channel N . The value converges to 0.6 as N increases.

We also calculated the CF for quantum ($N = 1$) and semi-classical regime ($N = 30$) with different barrier values Γ , depicted in Fig. 6.82. In the semi-classical regime, Schrödinger (Fig. 6.70) and Andreev (Fig. 6.82) billiard have the same behavior, given by the Eq.(6.73). In particular, they present anti-correlation in the opaque limit ($\Gamma \rightarrow 0$) and the usual Lorentzian shape in the ideal case ($\Gamma = 1$). In the quantum regime, the correlation follows the Lorentzian shape for the transparent barrier $\Gamma = 1$. Unlike the Dirac billiards, the Andreev billiards does not present anti-correlation in the opaque limit in quantum regime.

In the Fig.(6.83), we see the product of the density max average $\langle \rho_\varepsilon \rangle$ with the auto-correlation length γ in terms of the number of the barrier value Γ . The curves for $N \geq 5$ converge to 0.6 for $\Gamma = 1$. Moreover, we observe that Andreev billiard also follows the Eq.(6.74) in a semi-classical regime, however it presents an amplification in the density of maxima

$$\langle \rho_\varepsilon \rangle_{AB}(\Gamma) = \langle \rho_\varepsilon \rangle_{SB}(\Gamma) + 0.05 \quad (6.82)$$

for any value of Γ .

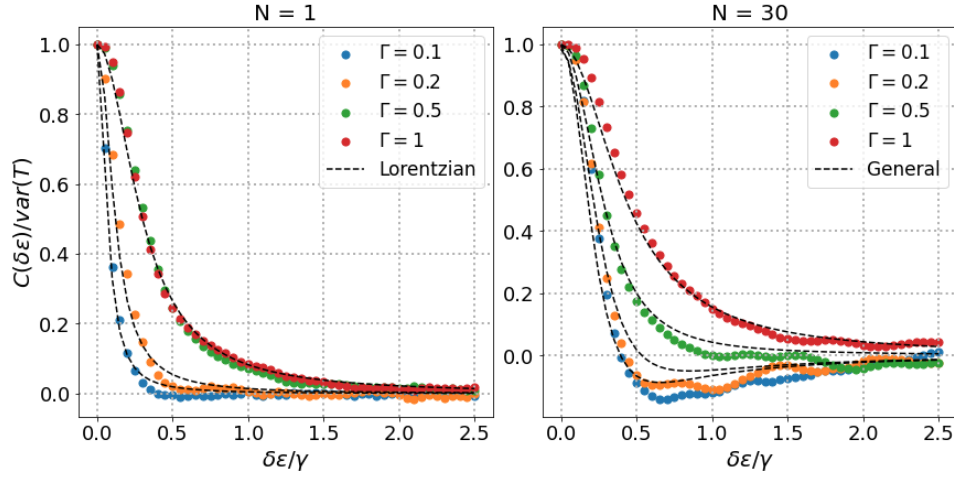


Figure 6.82 CF of the dimensionless conductance as a function of energy for quantum ($N = 1$) and semi-classical regime ($N = 30$) with different transparency barrier values Γ for the Andreev billiard. The General in the right panel indicates the correlation function given by Eq.(6.73).

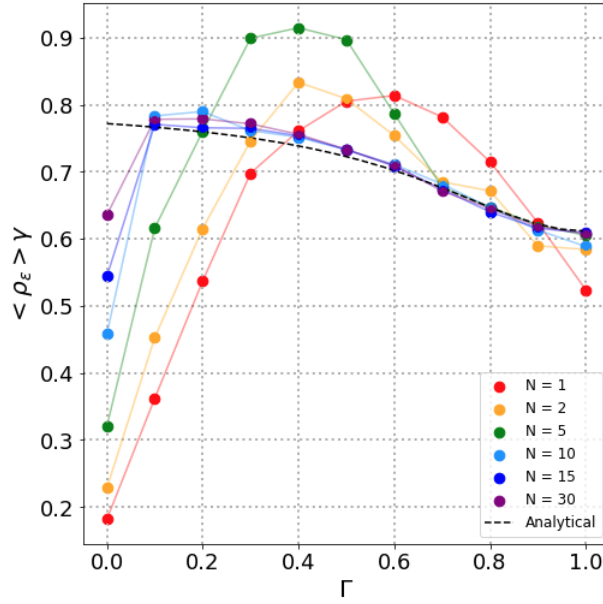


Figure 6.83 Product of the density max average ρ_ϵ with the auto-correlation length γ in terms of the value of tunneling barrier Γ .

Conclusion

7.1 Conclusion

In this work, we perform a numerical investigation of transport and entanglement properties in chaotic quantum billiards. By using the Random-Matrix Theory (RMT), we explore transport properties, such as conductance and shot-noise power, as well as entanglement quantifiers, such as concurrence, the entanglement of formation, and the violation of Bell's inequality. First, for all billiards (Schrödinger, Dirac, and Andreev), we recover the known distribution of conductance in the quantum regime and also demonstrate that it converges to the normal distribution in the semi-classical regime. We show that the conductance average behaves linearly in relation to the number of channels. Moreover, the conductance variance becomes constant as the number of channels increases, indicating a universal conductance fluctuations (UCF) behavior. To analyse properly the UCF, we obtain the quantum interference correction (QIC) for conductance in relation to the number of channels and the barrier transparency. Unlike the Schrödinger billiard, both Dirac and Andreev present transitions between weak localization and weak-antilocalization regimes as the barrier transparency increases. Then, the QIC of conductance for these billiards presents this peculiar behavior due to the chiral/particle-hole degrees of freedom.

For the shot noise power, we also obtain its distribution in the quantum regime and show that it converges to the normal distribution in the semi-classical regime. Similar to the conductance, the average of shot noise power behaves linearly in relation to the number of channels and its variance becomes constant as the number of channels increases, indicating a universal conductance fluctuations (UCF) behavior. Analysing the UCF, we obtain the quantum interference correction (QIC) for the shot noise power in relation to the number of channels and the barrier transparency. Unlike the Schrödinger billiard, the QIC for the shot noise power is nonzero in the ideal case (transparent barrier) for Dirac and Andreev billiards. Then, the chiral and particle-hole symmetries play a fundamental role in the QIC for the shot noise power. In the presence of a barrier, the QIC of shot noise power presents transitions between weak localization and weak antilocalization regimes as the barrier transparency increases.

About the entanglement quantifiers, we examine the concurrence, entanglement of formation, and the Bell parameter for Schrödinger and Dirac chaotic billiards. We calculate the concurrence and entanglement of formation for both billiards and compare the influence of orbital and chiral/sub-lattices degree of freedom on the entanglement production. In particular, when we analyze the effects of the tunneling barrier on concurrence for non-ideal Schrödinger and Dirac billiards, we observe that the quantum interference terms follow a linear and quadratic pattern, respectively. This particular quadratic behavior on the entanglement production of

Dirac billiard is a consequence of Klein tunneling in chiral/sub-lattices degree of freedom.

Moreover, we present violations of the Bell inequality in those chaotic billiards, whose entanglement arises due to the chaos in the billiard without requiring interactions. We show that the Dirac billiard produces more violations of Bell's inequality than the Schrödinger billiard, which means that it is easier to entangle chiral/sub-lattices degrees of freedom of the Dirac billiard than the orbital degrees of freedom of the Schrödinger billiard. The investigation of dephasing in the channels presents no violations in Bell's parameter for all tunneling regimes for both billiards. We also explore the production of entanglement with the number of resonances, where the number of Bell's inequality violations remains unchanged. Thus, under the random matrix theory validity, the size of the chaotic billiard does not play a role in Bell's inequality violations. Considering the dependence of Bell's parameter on barrier transparency, the number of violations remains constant for the Schrödinger billiard. In the Dirac billiard, the violations slightly increase as the barrier increases. Thus the sub-lattice/chiral degrees of freedom are more sensitive to the presence of the barrier for the Bell's parameter observable.

We also investigate the entanglement and the Bell parameter for Andreev billiard. We calculate the concurrence and entanglement of formation and compare the influence of the electron-hole degree of freedom on the entanglement production. We analyze the effects of the tunneling barrier on concurrence and entanglement of formation, where we observe that both concurrence and entanglement of formation are perdurable on tunneling effects. In special, we show that the tunneling effect does not affect the production of pair Bell for single-channel limit, which is made up of Majorana Modes. Therefore, the chiral and helical modes in respective symmetry classes C and CI are maximally entangled and perdurable under tunneling effects and the absence/presence of time reversal symmetry. Then, the Majorana modes can be a useful tool for the production and control of quantum entanglement and computing. Therefore, the particle-hole symmetry provides an ideal physical system to create qubits for quantum computing.

We present violations of the Bell inequality in Andreev billiards, whose entanglement arises due to the chaos in the billiard without requiring interactions. We show that the number of Bell's inequality violations is not affected by time-reversal symmetry and tunneling effects. The investigation of dephasing in the channels presents no violations in Bell's parameter for all tunneling regimes. We also explore Bell's violation with the number of resonances, where the number of Bell's inequality violations remains unchanged. Thus, under the random matrix theory validity, the size of the chaotic billiard does not play a role in Bell's inequality violations. Considering the dependence of Bell's parameter on barrier transparency, the number of violations also remains constant. Therefore, the number of Bell's inequality violations on Majorana and helical Majorana modes is not affected by the size of the chaotic billiard, time-reversal symmetry, and tunneling effects.

Finally, we also explore the universal conductance fluctuations (UCF) for Schrödinger, Dirac, and Andreev billiards. We extend the analysis of dimensionless conductance maxima density as a function of the energy for the chiral (ChGOE) and particle-hole (CI) classes of symmetry. For the Schrödinger billiard, we recover known results of the correlation function for Wigner-Dyson class and, as a new result, we obtain the relation between density maxima average and the auto-correlation length, also its respective analytical semi-classical behavior,

given by Eq.(6.74). For the Dirac billiard, we show numerically that the correlation function also satisfies the general form, given by Eq.(6.73), in both quantum and semi-classical regimes. Moreover, given an auto-correlation length, $\gamma = 1$ for example, the density max average in chiral ($\langle \rho_\epsilon \rangle \approx 0.6$) is higher than in Wigner-Dyson ($\langle \rho_\epsilon \rangle \approx 0.55$) class of symmetry for the ideal barrier case. This shows us that chirality exhibits a larger number of maxima than the Wigner-Dyson symmetry class. Finally, we also obtain the relation between the density maxima average and the auto-correlation length for the chiral class of symmetry. In a semi-classical regime, the product follows the same behavior of Wigner-Dyson class of symmetry, given by Eq.(6.74). However, there is an amplification of 0.05 for any value transparency barrier value, which is a signature of the emergence of chirality in the density of maxima of the dimensionless conductance.

Similar to the chiral case, the Andreev billiards also present a correlation function that satisfies the general form Eq.(6.73) in the semi-classical regime. However, the Andreev billiards does not present anti-correlation in the quantum regime. Moreover, the Andreev billiards also produces a density max average $\langle \rho_\epsilon \rangle \approx 0.6$ as in the chiral case. Therefore, both chiral and particle-hole degrees of freedom result in a slightly larger number of maxima than the Wigner-Dyson symmetry class. Finally, the product between the density maxima average and the auto-correlation length follows the same behavior of Wigner-Dyson class of symmetry with an amplification of 0.05 as in the chiral case. Therefore, both Dirac and Andreev billiards present produce a few more maxima than in the Schrödinger case due to the presence of chiral and particle-hole symmetries, respectively.

7.2 Perspective

A fascinating avenue for future exploration is the transition between integrable and chaotic quantum dots. By employing advanced numerical simulations and experimental setups, this research can investigate the mechanisms underlying this transition, uncovering novel quantum phenomena. Moreover, this approach can reveal how chaotic semi-classical dynamical systems can be described in terms of quantum theory. This research area is known as quantum chaos.

The role of spin-orbit coupling as a phase transition parameter between the GOE and GSE ensembles presents an exciting opportunity for further investigation. Exploring how spin interacts with quantum systems can lead to insights into fundamental quantum behavior, such as transport property transitions. Experimental setups to manipulate and measure spin-orbit interactions offers potential pathways for developing new quantum devices. Hence, employing numerical simulations holds the potential to pave a path toward devising experimental configurations for controlling spin-orbit coupling.

Detecting the signal of chirality is another potential investigation. Because of the challenges involved in studying hadronic systems, observing chirality in experiments becomes an intricate undertaking. However, mesoscopic systems are ideal for simulating chirality through sublattice symmetry. The signal of chirality could help us to understand more about how certain materials work at a fundamental level. Moreover, researchers aim to find a way to spot these chiral signals among other effects because it could offer a powerful tool for computers or super-accurate sensors.

The study of a few resonances cavity and billiard geometry dependence is another relevant topic for future exploration. Observing how small cavities vibrate with different shapes and sizes could reveal exotic patterns and provide insights into quantum chaos. Understanding these patterns could present more about how waves behave inside the chaotic cavity and would be possible to investigate the validity of the random-matrix theory. The comprehension could drive the creation of compact structures proficient in efficient energy storage or expediting signal transmission in devices.

Scattering Matrix Deduction

The S matrix in the form

$$S = (1 - iF)(1 + iF)^{-1}, \text{ where } F = \pi \hat{W}^\dagger (E - H_{ri})^{-1} \hat{W}, \quad (\text{A.1})$$

can be written in an alternative way

$$\begin{aligned} S &= (1 - iF)(1 + iF)^{-1} = (1 + iF - iF - iF)(1 + iF)^{-1}, \\ &= (1 + iF)(1 + iF)^{-1} - 2iF(1 + iF)^{-1} = 1 - 2iF(1 + iF)^{-1}. \end{aligned} \quad (\text{A.2})$$

Now, we substitute $F = \pi \hat{W}^\dagger (E - H_{ri})^{-1} \hat{W}$ into the S

$$\begin{aligned} S &= 1 - 2i\pi \hat{W}^\dagger (E - \hat{H}_{ri})^{-1} \hat{W} (1 + i\pi \hat{W}^\dagger (E - \hat{H}_{ri})^{-1} \hat{W})^{-1}, \\ &= 1 - 2i(\pi \hat{W}^\dagger (E - \hat{H}_{ri})^{-1} \hat{W}) \sum_{l=0}^{\infty} (i\pi \hat{W}^\dagger (E - \hat{H}_{ri})^{-1} \hat{W})^l, \\ &= 1 - 2 \sum_{l=0}^{\infty} (i\pi \hat{W}^\dagger (E - \hat{H}_{ri})^{-1} \hat{W})^{l+1}, \\ &= 1 - 2 \left[i\pi \hat{W}^\dagger (E - \hat{H}_{ri})^{-1} \hat{W} + (i\pi)^2 \hat{W}^\dagger (E - \hat{H}_{ri})^{-1} \hat{W} \hat{W}^\dagger (E - \hat{H}_{ri})^{-1} \hat{W} + \dots \right], \\ &= 1 - 2i\pi \hat{W}^\dagger \left[1 + i\pi (E - \hat{H}_{ri})^{-1} \hat{W} \hat{W}^\dagger + \dots \right] (E - \hat{H}_{ri})^{-1} \hat{W}, \\ &= 1 - 2i\pi \hat{W}^\dagger \left[\sum_{l=0}^{\infty} (i\pi (E - \hat{H}_{ri})^{-1} \hat{W} \hat{W}^\dagger)^l \right] (E - \hat{H}_{ri})^{-1} \hat{W}, \\ &= 1 - 2\pi i \hat{W}^\dagger \left[1 + i\pi (E - \hat{H}_{ri})^{-1} \hat{W} \hat{W}^\dagger \right]^{-1} (E - \hat{H}_{ri})^{-1} \hat{W}, \\ &= 1 - 2\pi i \hat{W}^\dagger ((E - \hat{H}_{ri}) + i\pi \hat{W} \hat{W}^\dagger)^{-1} \hat{W}, \\ S &= 1 - 2i\pi \hat{W}^\dagger (E - \hat{H}_{ri} + i\pi \hat{W} \hat{W}^\dagger)^{-1} \hat{W}, \end{aligned} \quad (\text{A.3})$$

where we use in the first and sixth steps the definition of geometric series and their convergence.

APPENDIX B

Bell's Inequality

First, we assume that the absolute difference between $\bar{P}(\vec{a}, \vec{b})$ and $-\vec{a} \cdot \vec{b}$ is smaller than an arbitrary ε

$$|\bar{P}(\vec{a}, \vec{b}) + \vec{a} \cdot \vec{b}| \leq \varepsilon. \quad (\text{B.1})$$

Now, given a δ such that

$$|\vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{b}| \leq \delta, \quad (\text{B.2})$$

we show that it is not possible to find an ε small enough in the Eq.(B.1). We know that

$$\begin{aligned} |\bar{P}(\vec{a}, \vec{b}) + \vec{a} \cdot \vec{b}| &\leq |\bar{P}(\vec{a}, \vec{b}) + \vec{a} \cdot \vec{b}| + |\vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{b}| \\ &\leq \varepsilon + \delta. \end{aligned} \quad (\text{B.3})$$

and $\bar{P}(\vec{a}, \vec{b})$ is written in the form

$$\bar{P}(\vec{a}, \vec{b}) = \int d\lambda \bar{A}(\vec{a}, \lambda) \bar{B}(\vec{b}, \lambda), \text{ where } |\bar{A}(\vec{a}, \lambda)| \leq 1 \text{ and } |\bar{B}(\vec{b}, \lambda)| \leq 1. \quad (\text{B.4})$$

Setting $\vec{a} = \vec{b}$, we have the Eq.(B.3):

$$|\int \rho(\lambda) (\bar{A}(\vec{b}, \lambda) \bar{B}(\vec{b}, \lambda) + 1) d\lambda| \leq \varepsilon + \delta. \quad (\text{B.5})$$

On the other hand,

$$\begin{aligned} |\bar{P}(\vec{a}, \vec{b}) - \bar{P}(\vec{a}, \vec{c})| &= \int d\lambda \rho(\lambda) [\bar{A}(\vec{a}, \lambda) \bar{B}(\vec{b}, \lambda) - \bar{A}(\vec{a}, \lambda) \bar{B}(\vec{c}, \lambda)] \\ &= \int d\lambda \rho(\lambda) \bar{A}(\vec{a}, \lambda) \bar{B}(\vec{b}, \lambda) [1 + \bar{A}(\vec{b}, \lambda) \bar{B}(\vec{c}, \lambda)] \\ &\quad - \int d\lambda \rho(\lambda) \bar{A}(\vec{a}, \lambda) \bar{B}(\vec{c}, \lambda) [1 + \bar{A}(\vec{b}, \lambda) \bar{B}(\vec{b}, \lambda)]. \end{aligned} \quad (\text{B.6})$$

Since $|\bar{A}(\vec{a}, \lambda)|$ and $|\bar{B}(\vec{b}, \lambda)|$ are less than 1, then

$$\begin{aligned} |\bar{P}(\vec{a}, \vec{b}) - \bar{P}(\vec{a}, \vec{c})| &\leq \int d\lambda \rho(\lambda) + \int d\lambda \rho(\lambda) \bar{A}(\vec{b}, \lambda) \bar{B}(\vec{c}, \lambda) \\ &\quad + |\int \rho(\lambda) (\bar{A}(\vec{b}, \lambda) \bar{B}(\vec{b}, \lambda) + 1) d\lambda|. \end{aligned} \quad (\text{B.7})$$

By Eq.B.5 and the definition $\bar{P}(a, b)$, we obtain

$$|\bar{P}(\vec{a}, \vec{b}) - \bar{P}(\vec{a}, \vec{c})| \leq 1 + \bar{P}(\vec{b}, \vec{c}) + (\varepsilon + \delta). \quad (\text{B.8})$$

On the one hand,

$$\begin{aligned} |\bar{P}(\vec{a}, \vec{b}) - \bar{P}(\vec{a}, \vec{c})| &\geq |\bar{P}(\vec{a}, \vec{b}) + \vec{a} \cdot \vec{c}| - |\bar{P}(\vec{a}, \vec{c}) + \vec{a} \cdot \vec{c}| \\ &\geq |\bar{P}(\vec{a}, \vec{b}) + \vec{a} \cdot \vec{c}| - (\varepsilon + \delta). \end{aligned} \quad (\text{B.9})$$

On the other hand,

$$\begin{aligned} |\bar{P}(\vec{a}, \vec{b}) + \vec{a} \cdot \vec{c}| &= |\bar{P}(\vec{a}, \vec{b}) + \vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}| \\ &= |\vec{a} \cdot \vec{c} - \vec{a} \cdot \vec{b} - (-\bar{P}(\vec{a}, \vec{b}) - \vec{a} \cdot \vec{b})| \\ &\geq |\vec{a} \cdot \vec{c} - \vec{a} \cdot \vec{b}| - |\bar{P}(\vec{a}, \vec{b}) + \vec{a} \cdot \vec{b}| \\ &\geq |\vec{a} \cdot \vec{c} - \vec{a} \cdot \vec{b}| - (\varepsilon + \delta). \end{aligned} \quad (\text{B.10})$$

Substituting Eq.(B.10) into Eq.(B.9), we have

$$|\bar{P}(\vec{a}, \vec{b}) - \bar{P}(\vec{a}, \vec{c})| \geq |\vec{a} \cdot \vec{c} - \vec{a} \cdot \vec{b}| - 2(\varepsilon + \delta). \quad (\text{B.11})$$

Now, Substituting Eq.(B.8) into Eq.(B.11), we obtain

$$|\vec{a} \cdot \vec{c} - \vec{a} \cdot \vec{b}| - 2(\varepsilon + \delta) \leq 1 + \bar{P}(\vec{b}, \vec{c}) + (\varepsilon + \delta). \quad (\text{B.12})$$

Observe that

$$\begin{aligned} \varepsilon + \delta &\geq |\bar{P}(\vec{b}, \vec{c}) + \vec{b} \cdot \vec{c}| = |\bar{P}(\vec{b}, \vec{c}) - (-\vec{b} \cdot \vec{c})| \geq |\bar{P}(\vec{b}, \vec{c})| - |(-\vec{b} \cdot \vec{c})| \\ &= \bar{P}(\vec{b}, \vec{c}) - \vec{b} \cdot \vec{c}. \end{aligned} \quad (\text{B.13})$$

Therefore,

$$\bar{P}(\vec{b}, \vec{c}) \leq \vec{b} \cdot \vec{c} + (\varepsilon + \delta). \quad (\text{B.14})$$

Substituting Eq.(B.12) into Eq.(B.14), we obtain

$$|\vec{a} \cdot \vec{c} - \vec{a} \cdot \vec{b}| - 2(\varepsilon + \delta) \leq 1 - \vec{b} \cdot \vec{c} + 2(\varepsilon + \delta), \quad (\text{B.15})$$

or better,

$$4(\varepsilon + \delta) \geq |\vec{a} \cdot \vec{c} - \vec{a} \cdot \vec{b}| + \vec{b} \cdot \vec{c} - 1. \quad (\text{B.16})$$

Now assuming that $\vec{a} \cdot \vec{c} = 0$, $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \frac{1}{\sqrt{2}}$, for example, we have

$$4(\varepsilon + \delta) \geq \sqrt{2} - 1. \quad (\text{B.17})$$

Therefore, given a sufficiently small δ , we cannot obtain an arbitrarily small ε .

APPENDIX C

Violation of Bell's Inequality

Let A_0, A_1 and B_0, B_1 be Alice and Bob's observables, respectively, we can define the Bell's parameter $\mathcal{S}_{CHSH}$ as

$$\hat{\mathcal{S}}_{CHSH} = A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1. \quad (\text{C.1})$$

The average of these observables over a quantum state $|\psi\rangle$ is given by

$$\langle A_0 \otimes B_0 \rangle_{|\psi\rangle} = \langle \psi | A_0 \otimes B_0 | \psi \rangle, \quad (\text{C.2})$$

then the Bell's parameter is obtained by

$$\mathcal{S}_{CHSH} = \langle \mathcal{S}_{CHSH} \rangle = \langle A_0 \otimes B_0 \rangle + \langle A_0 \otimes B_1 \rangle + \langle A_1 \otimes B_0 \rangle - \langle A_1 \otimes B_1 \rangle. \quad (\text{C.3})$$

Suppose that Alice and Bob share the entangled state $|\psi\rangle$ given by

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}[|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle] = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}. \quad (\text{C.4})$$

If Alice and Bob perform the measurements $A_0 = \sigma_1, A_1 = \sigma_3$ and $B_0 = \frac{\sigma_1 + \sigma_3}{\sqrt{2}}, B_1 = \frac{\sigma_1 - \sigma_3}{\sqrt{2}}$, respectively, the first term becomes

$$\begin{aligned} A_0 \otimes B_0 &= \sigma_1 \otimes \left(\frac{\sigma_1 + \sigma_3}{\sqrt{2}} \right), \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}, \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (\text{C.5})$$

Then,

$$\langle \psi | A_0 \otimes B_0 | \psi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & -1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = -\frac{1}{\sqrt{2}} \quad (\text{C.6})$$

Similarly, we can obtain the other terms:

$$\begin{aligned}
 \langle \psi | A_0 \otimes B_1 | \psi \rangle &= -\frac{1}{\sqrt{2}} \\
 \langle \psi | A_1 \otimes B_0 | \psi \rangle &= -\frac{1}{\sqrt{2}} \\
 \langle \psi | A_1 \otimes B_1 | \psi \rangle &= \frac{1}{\sqrt{2}}
 \end{aligned} \tag{C.7}$$

and, finally, $\langle S_{CHSH} \rangle$ becomes

$$\begin{aligned}
 \langle S_{CHSH} \rangle &= \langle A_0 \otimes B_0 \rangle + \langle A_0 \otimes B_1 \rangle + \langle A_1 \otimes B_0 \rangle - \langle A_1 \otimes B_1 \rangle \\
 &= \left(-\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{\sqrt{2}}\right) = -2\sqrt{2}
 \end{aligned} \tag{C.8}$$

Therefore, we found the maximum violation of Bell's inequality

$$|\langle S_{CHSH} \rangle| = 2\sqrt{2} > 2. \tag{C.9}$$

APPENDIX D

Correlator and Noise Relationship

The dimensionless shot noise power $s = \text{Tr } r^\dagger r t^\dagger t$ can be written in the following way

$$\begin{aligned}
 \text{Tr } r^\dagger r t^\dagger t &= \text{Tr} \left(t r^\dagger r t^\dagger \right) = \text{Tr} \left((r t^\dagger)^\dagger (r t^\dagger) \right) \\
 &= \text{Tr} \left(A^\dagger A \right) = \sum_{i=1}^2 (A^\dagger A)_{ii}, \text{ onde } A = r t^\dagger. \\
 &= \sum_{i=1}^2 \sum_{k=1}^2 A_{ik}^\dagger A_{ki} = \sum_{i=1}^2 \sum_{k=1}^2 A_{ki}^* A_{ki} \\
 &= \sum_{i,k=1}^2 |A_{ki}|^2 = \sum_{i,k=1}^2 |(r t^\dagger)_{ki}|^2 = \sum_{i,k=1}^2 N_{ik} \\
 &= N_{11} + N_{12} + N_{21} + N_{22}.
 \end{aligned} \tag{D.1}$$

Similarly, the noise correlator $\text{Tr } \sigma_z r t^\dagger \sigma_z t r^\dagger$ is written as

$$\begin{aligned}
 \text{Tr } \sigma_z r t^\dagger \sigma_z t r^\dagger &= \text{Tr} \{ \sigma_z r t^\dagger (r t^\dagger \sigma_z)^\dagger \} \\
 &= \text{Tr} \{ (\sigma_z A) (A \sigma_z)^\dagger \} = \sum_{i=1}^2 \left[(\sigma_z A) (A \sigma_z)^\dagger \right]_{ii} \\
 &= \sum_{i=1}^2 \left\{ \sum_{k=1}^2 (\sigma_z A)_{ik} (A \sigma_z)_{ki}^\dagger \right\} \\
 &= \text{Tr} \begin{pmatrix} |A_{11}|^2 - |A_{12}|^2 & A_{11} A_{21}^* - A_{12} A_{22}^* \\ -A_{21} A_{11}^* + A_{22} A_{12}^* & -|A_{21}|^2 - |A_{22}|^2 \end{pmatrix} \\
 &= |A_{11}|^2 + |A_{22}|^2 - |A_{12}|^2 - |A_{21}|^2 \\
 &= |(r t^\dagger)_{11}|^2 + |(r t^\dagger)_{22}|^2 - |(r t^\dagger)_{12}|^2 - |(r t^\dagger)_{21}|^2 \\
 &= N_{11} + N_{22} - N_{12} - N_{21}.
 \end{aligned} \tag{D.2}$$

Then, the ratio between the noise correlator and the dimensionless shot noise power

$$E = \frac{\text{Tr } \sigma_z r t^\dagger \sigma_z t r^\dagger}{\text{Tr } r^\dagger r t^\dagger t} = \frac{N_{11} + N_{22} - N_{12} - N_{21}}{N_{11} + N_{12} + N_{21} + N_{22}}. \tag{D.3}$$

provides a useful tool to obtain a violation of Bell's inequality.

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