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Solutions for Critical Elliptic Systems on Compact Manifolds

by

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João Pessoa - PB
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under supervision of

Prof. Dr. Manassés Xavier de Souza

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Resumo

O trabalho apresentado nesta tese aborda resultados referentes à existência de soluções para três classes de sistemas elípticos fortemente acoplados em variedades Riemannianas compactas sem bordo. Nessas classes, estão envolvidas não linearidades acopladas com expoentes críticos no sentido das imersões de Sobolev e Hardy-Sobolev. A primeira e a segunda classe de problemas envolvem o operador Laplace-Beltrami sobre uma variedade e não linearidades com expoente crítico de Sobolev no primeiro caso e de Hardy-Sobolev no segundo. Na segunda classe, também consideramos potenciais do tipo Hardy. O terceiro problema envolve o operador p-Laplaciano e uma não linearidade com expoente crítico de Hardy-Sobolev. Dessa forma, em ambos os problemas, investigamos a falta de compacidade e como recuperá-la em algum nível de energia. Neste trabalho, a abordagem é realizada por meio de métodos variacionais.

Palavras-chave: Sistemas Elípticos de Segunda Ordem; Métodos Variacionais; Expoentes Críticos.

Abstract

The work presented in this thesis addresses results concerning the existence of solutions for three classes of strongly coupled elliptic systems on compact Riemannian manifolds without boundaries. In these classes, coupled nonlinearities with critical exponents in the sense of Sobolev and Hardy-Sobolev embeddings are involved. The first and second classes of problems involve the Laplace-Beltrami operator on a manifold and nonlinearities with a critical Sobolev exponent in the first case and Hardy-Sobolev exponent in the second case. In the second class, we also consider Hardy-type potentials. The third problem involves the p-Laplacian operator and a nonlinearity with a critical Hardy-Sobolev exponent. Thus, in both problems, we investigate the lack of compactness and how to recover it at some energy level. In this work, the approach is conducted through variational methods.

Keywords: Second Order Elliptical Systems; Variational Methods; Critical Exponents.

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"Não é apenas a questão, mas a maneira como você tenta resolvê-la."

Maryam Mirzakhani

Dedicatória

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Contents

Introduction

In this thesis, we study the existence of solutions for three classes of strongly coupled critical elliptic systems on compact Riemannian manifolds without boundary. Our primary objective is to investigate the influence of scalar curvature on the existence of solutions for these systems when Riemannian manifolds are involved, since, as far as we know, there is no research in this direction in the existing literature.

The study of coupled elliptic system has important applications in Mathematical Physics. They appear in the Hartree-Fock theory for Bose-Einstein double condensates, in fiber optics theory, in Langmuir wave theory in plasma physics and in the behavior of deep water waves and freak waves in the ocean. A general reference on such systems and their role in physics is duo to Ablowitz et al. [1].

Our study was motivated by some works: first, by paper due to Alves et al. [2], in which the authors proved results of the existence and non-existence of solutions to the following system of elliptic equations

$$\begin{cases} -\Delta u + au + bv = \dfrac{\alpha}{2^*} u|u|^{\alpha-2}|v|^{\beta} \ \text{ in } \ \Omega, \\ -\Delta v + bu + cv = \dfrac{\beta}{2^*} v|v|^{\beta-2}|u|^{\alpha} \ \text{ in } \ \Omega, \\ u = v = 0 \ \text{ on } \ \partial\Omega, \\ u, v > 0 \ \text{ in } \ \Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 3$, $a, b, c \in \mathbb{R}$, $2^* = \frac{2n}{n-2}$ is the critical Sobolev exponent and $\alpha, \beta > 1$. For this, the main point is to compare the value of $\alpha + \beta$ with the critical Sobolev exponent. Moreover, inspired by the paper of Brézis and Nirenberg [8], they also compare the two real eigenvalues of the matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \in M_{2\times 2}(\mathbb{R}),$$

denoted by μ_1 and μ_2 with λ_1 (the first eigenvalue of the Laplacian operator). For the reader interested in solutions for systems of this type, we would also like to refer, for instance [6, 10, 13, 25, 33, 34, 41]. Second, by the studies on the equation

$$-\Delta_{p,g}u + a(x)u = f(x)|u|^{2^*-2}u \quad \text{in} \quad M, \tag{1}$$

where (M, g) is a smooth closed Riemannian manifold of dimension n, $\Delta_{p,g}$ denotes the p-Laplace-Beltrami of M, f is a smooth function on M with $\max_M f > 0$ and a is a Hölder continuous function on M. This equation was studied, for example, by Druet in [15], who established existence results, for example, when $p = 2$, $n \geq 4$ and $-\Delta_g + a$ is a coercive operator, assuming a local condition, namely, if there exists a point x_0 in $\{x \in M : f(x) = \max_M f\}$ such that

$$a(x_0) < \frac{n-2}{4(n-1)}R_g(x_0) + \frac{(n-4)(n-2)}{8(n-1)}\frac{\Delta_g f(x_0)}{f(x_0)},$$

the existence of positive solution was established. In [48], Vétois also addressed the problem (1) with $p = 2$, $f \equiv 1$, $n \geq 4$ and $-\Delta_g + a$ coercive, under same local condition as above, the multiplicity of solutions is proved. Considering $a \equiv \frac{(n-2)}{4(n-1)}R_g$ and $p = 2$ in (1), then we have the prescribed scalar curvature equation

$$-\Delta_g u + \frac{(n-2)}{4(n-1)}R_g u = f(x)|u|^{2^*-2}u \quad \text{in} \quad M. \tag{2}$$

When f is constant, we get the well-known Yamabe equation, first considered by Yamabe [50], after by Trudinger [47], Aubin [3], Schoen [44] among others. In the form (2), it has been intensively studied, for example, by Kazdan and Warner [37], by Aubin and Hebey [5], Escobar and Schoen [21], Hebey and Vaugon [29]. Geometrically, the problem (2) is related to the existence of a conformal metric $\tilde{g}$ on M whose scalar curvature equals to $\frac{4(n-1)}{n-2}f$, in other words, given f a smooth function on M, if there is $\varphi \in C^\infty(M)$ with $\varphi > 0$ satisfying (2) then the conformal metric $\tilde{g} = \varphi^{\frac{4}{n-2}}g$ is such that $R_{\tilde{g}} \equiv \frac{4(n-1)}{(n-2)}f$. In addition to the works already highlighted, we would also like to mention Hebey [26], who considered the following elliptic system of equations

$$-\Delta_g u_i + \sum_{j=1}^{p} A_{ij}(x)u_j = u_i|u_i|^{2^*-2} \quad \text{in} \quad M, \quad i = 1, \cdots, p,$$

where $A = (A_{ij}) : M \to M_p$ is a smooth function, $p \in \mathbb{Z}$, $p \geq 1$, and $M_p^s(\mathbb{R})$ denotes the vector space of symmetric $p \times p$ real matrices. Assuming sufficient conditions on the matrix A related to the linear geometric potential $\frac{n-2}{4(n-1)} R_g$, the author studies the existence of minimizing solutions for this system, the existence of high-energy solutions, blow-up theory and its compactness properties.

In this present thesis, let (M, g) be a smooth closed Riemannian manifold of dimension n. Firstly, we are interested in the existence of solutions for the following elliptic system:

$$\begin{cases} -\Delta_g u + a(x)u + b(x)v = \dfrac{\alpha}{2^*} f(x) u|u|^{\alpha-2}|v|^{\beta} & \text{in } M, \\ -\Delta_g v + b(x)u + c(x)v = \dfrac{\beta}{2^*} f(x) v|v|^{\beta-2}|u|^{\alpha} & \text{in } M, \end{cases} \tag{3}$$

where Δ_g is the Laplace-Beltrami operator, a, b and c are functions Hölder continuous in M, f is a smooth function on M, and $\alpha > 1$, $\beta > 1$ are real constants satisfying $\alpha + \beta = 2^*$.

Subsequently, we will search for solutions to the Hardy-Sobolev type system:

$$\begin{cases} -\Delta_g u + \frac{\tilde{a}(x)}{\rho(x)^{\theta}} u + a(x)u + b(x)v = \dfrac{\alpha}{2^\star(s)} \frac{f(x)}{d_g(x,x_0)^s} u|u|^{\alpha-2}|v|^{\beta} & \text{in } M, \\ -\Delta_g v + \frac{\tilde{c}(x)}{\rho(x)^{\gamma}} v + b(x)u + c(x)v = \dfrac{\beta}{2^\star(s)} \frac{f(x)}{d_g(x,x_0)^s} v|v|^{\beta-2}|u|^{\alpha} & \text{in } M, \end{cases} \tag{4}$$

where Δ_g is the Laplace-Beltrami operator, d_g is the Riemannian distance, $a, b, c, \tilde{a}, \tilde{c} \in C^{0,\varrho}(M)$, for some $\varrho \in (0,1)$, $x_0 \in M$, $s \in [0,2)$ and $\theta, \gamma \in (0,2)$, ρ is a nonnegative continuous function such that $\rho(x) \approx d_g(x, x_0)$ for x near of x_0, $f \in C^\infty(M)$ with $f(x_0) = \max_M f > 0$ and $\alpha > 1$, $\beta > 1$ are real constants satisfying $\alpha + \beta = 2^\star(s)$, where $2^\star(s) = 2(n-s)/(n-2)$ is the critical Hardy-Sobolev exponent.

Finally, we investigate the existence of solutions for the generalized system:

$$\begin{cases} -\Delta_{p,g} u + a(x)|u|^{p-2}u + b(x)[(p-1)|u|^{p-2} + |v|^{p-2}]v = \frac{\alpha}{p^\star(s)} \dfrac{f(x)u|u|^{\alpha-2}|v|^{\beta}}{d_g(x,x_0)^s} & \text{in } M, \\ -\Delta_{p,g} v + b(x)[(p-1)|v|^{p-2} + |u|^{p-2}]u + c(x)|v|^{p-2}v = \frac{\beta}{p^\star(s)} \dfrac{f(x)v|v|^{\beta-2}|u|^{\alpha}}{d_g(x,x_0)^s} & \text{in } M, \end{cases} \tag{5}$$

where $\Delta_{p,g}$ is the p-Laplace-Beltrami operator, $p \in (1,n)$ with $p \neq 2$, $a, b, c \in C^{0,\varrho}(M)$ for some $\varrho \in (0,1)$ with $b \equiv 0$ when $1 < p < 2$, $x_0 \in M$, $s \in [0,p)$, $f \in C^\infty(M)$ with $f(x_0) = \max_M f > 0$ and $\alpha > 1$, $\beta > 1$ are real constants satisfying $\alpha + \beta = p^\star(s) = \frac{p(n-s)}{n-p}$.

The main difficulty in this type of problems is the lack of compactness. Indeed, if M is a compact Riemannian manifolds, we will denote by $H^{1,p}(M)$ the Sobolev space of all functions in $L^p(M)$ with one derivative (in the weak sense) in $L^p(M)$, when $p=2$ we simply write $H^1(M)$. As it is well known, the embeddings of Sobolev spaces, $H^1(M)$ in the Lesbegue space $L^{2^*}(M)$ and $H^{1,p}(M)$ $(1<p<n)$ in the weighted Lebesgue space $L^{p^\star(s)}(M, d_g(\cdot, x_0)^{-s})$ are continuous, but not compact (see [4, 11, 28, 42]).

In general, the use of variational methods to obtain non-trivial solutions to this type of problem requires appropriate estimates of the energy level or minimax level, due to lack of compactness. It is exactly in this step that we find the main geometric implications of the Riemannian manifold.

We shall work with the space $H^p := H^{1,p}(M) \times H^{1,p}(M)$ (we write $H = H^2$, when $p=2$) endowed with the norm

$$\|(u,v)\| = \left(\|u\|^p_{H^{1,p}} + \|v\|^p_{H^{1,p}}\right)^{1/p}.$$

We equip $H^{1,p}(M)$ with the standard $\|\cdot\|_{H^{1,p}}-$norm, that is, $\|u\|^p_{H^{1,p}} = \|\nabla u\|^p_p + \|u\|^p_p$, where $\|\cdot\|_q$ denotes the norm of the Lebesgue space $L^q(M)$, $q \geq 1$. The norm of $L^q(M) \times L^q(M)$ will be defined by $\|(u,v)\|_q = \left(\|u\|^q_q + \|v\|^q_q\right)^{1/q}$.

An important relation obtained by authors in [2] is the following:

$$\mathcal{S}_* = \left[\left(\frac{\alpha}{\beta}\right)^{\beta/2^*} + \left(\frac{\beta}{\alpha}\right)^{\alpha/2^*}\right] K_n^{-2}, \tag{6}$$

where $\alpha + \beta = 2^*$, K_n^{-2} is defined by

$$K_n^{-2} = \inf_{u \in H^1(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^n} |u|^{\alpha+\beta} dx\right)^{2/2^*}},$$

and $\mathcal{S}_*$ is defined by

$$\mathcal{S}_* = \inf_{(u,v)\in [H^1(\mathbb{R}^n)]^2\setminus\{0\}} \frac{\int_{\mathbb{R}^n} \left(|\nabla u|^2 + |\nabla v|^2\right) dx}{\left(\int_{\mathbb{R}^n} |u|^\alpha |v|^\beta dx\right)^{2/2^*}}. \tag{7}$$

It is known that K_n is the sharp constant for the embedding of $H^1(\mathbb{R}^n)$ into $L^{2^*}(\mathbb{R}^n)$.

In general, the relation (6) holds for $p \in (1,n)$ and $s \in [0,p)$, that is,

$$\mathcal{K}^{p,s}_{(\alpha,\beta)} = \kappa(\alpha,\beta) K(n,p,s), \tag{8}$$

where $\kappa(\alpha,\beta) := \left[\left(\frac{\alpha}{\beta}\right)^{\beta/p^\star(s)} + \left(\frac{\beta}{\alpha}\right)^{\alpha/p^\star(s)}\right]$ and $K(n,p,s)$ is the best Hardy-Sobolev constant defined by

$$K(n,p,s) = \inf_{u\in H^{1,p}(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n} |\nabla u|^p dx}{\left(\int_{\mathbb{R}^n} \frac{|u|^{p^\star(s)}}{|x|^s} dx\right)^{\frac{p}{p^\star(s)}}},$$

and $\mathcal{K}^{p,s}_{(\alpha,\beta)}$ is defined by

$$\mathcal{K}^{p,s}_{(\alpha,\beta)} = \inf_{(u,v)\in[H^{1,p}(\mathbb{R}^n)]^2\setminus\{0\}} \frac{\int_{\mathbb{R}^n} \left(|\nabla u|^p + |\nabla v|^p\right) dx}{\left(\int_{\mathbb{R}^n} \frac{|u|^\alpha |v|^\beta}{|x|^s} dx\right)^{\frac{p}{p^\star(s)}}}. \tag{9}$$

In our context, to deal with the lack of compactness, we will also establish Sobolev and Hardy-Sobolev inequalities, that is, when $p=2$ there is a positive constant B_0 such that

$$\left(\int_M |u|^\alpha |v|^\beta dv_g\right)^{2/2^*} \leq \mathcal{S}_*^{-1} \|(|\nabla u|, |\nabla v|)\|_2^2 + B_0 \|(u,v)\|_2^2 , \tag{10}$$

for all $(u,v)\in H$. Moreover, $(\mathcal{S}_*)^{-1}$ is the best constant such that (10) holds, thus this inequality is optimal. In general, given any $\varepsilon > 0$ there is a positive constant B_ε such that

$$\left(\int_M \frac{|u|^\alpha |v|^\beta}{d_g(x,x_0)^s} dv_g\right)^{\frac{p}{p^\star(s)}} \leq [(\mathcal{K}^{p,s}_{(\alpha,\beta)})^{-1} + \varepsilon] \|(|\nabla_g u|, |\nabla_g v|)\|_p^p + B_\varepsilon \|(u,v)\|_p^p , \tag{11}$$

for all $(u,v)\in H$. Here, $(\mathcal{K}^{p,s}_{(\alpha,\beta)})^{-1}$ is the best constant such that the inequality holds. When $p=2$ this inequality is optimal since it is true for $\varepsilon = 0$, while in the case $p\neq 2$, the inequality is not generally true for $\varepsilon = 0$. We prove these inequalities by applying the results obtained by Hebey and Vaugon [31], Jaber [35] and Chen and Liu [11]. We also need to prove a Brézis-Lieb lemma for the nonlinear term involved. More precisely, let $\ell \in L^\infty(M)$, if $u_m \rightharpoonup u$ and $v_m \rightharpoonup v$ in $H^{1,p}(M)$, then we have

$$\int_M \frac{\ell(x)|u_m|^\alpha |v_m|^\beta}{d_g(x,x_0)^s} dv_g = \int_M \frac{\ell(x)|u|^\alpha |v|^\beta}{d_g(x,x_0)^s} dv_g + \int_M \frac{\ell(x)|u_m - u|^\alpha |v_m - v|^\beta}{d_g(x,x_0)^s} dv_g + o_m(1),$$

where $o_m(1) \to 0$ as $m\to\infty$.

To achieve our goals, we have divided this work into three chapters. Next, we will describe them.

In Chapter 1, we are interested in problem (3). In this case, we assume some very general assumptions on the functions a, b, c and f that will allow us to obtain some existence results for this problem through variational methods. Precisely, we assume that the function f satisfies

$$\max_M f > 0, \tag{12}$$

and the functions a, b and c satisfy the following coercivity condition: there exists $C_0 > 0$ such that

$$\int_M (|\nabla_g u|^2 + |\nabla_g v|^2 + a(x)u^2 + 2b(x)uv + c(x)v^2) dv_g \geq C_0 \|(u,v)\|^2, \quad \forall\, (u,v) \in H. \tag{13}$$

In this context, our first result is the following:

Teorema 0.0.1 *Let (M, g) be a smooth closed Riemannian manifold of dimension $n \geq 3$. Let $\alpha, \beta > 1$ be two real numbers such that $\alpha + \beta = 2^*$, and let a, b and c be functions Hölder continuous in M, and $f \in C^\infty$, with a, b and c satisfying (13) and f satisfying (12), writing $h = \frac{\alpha}{2^*}a + \frac{2\sqrt{\alpha\beta}}{2^*}b + \frac{\beta}{2^*}c$. Let x_0 be some point in M such that $f(x_0) = \max_M f$. If, in addition, we assume that*

$$\begin{aligned} &(i)\ h(x_0) < \tfrac{n-2}{4(n-1)} R_g(x_0) + \tfrac{(n-4)(n-2)}{8(n-1)} \tfrac{\Delta_g f(x_0)}{f(x_0)}, \ \textit{if } n \geq 4, \\ &(ii)\ h(x_0) < \tfrac{1}{8} R_g(x_0) \ \textit{and } h \leq \tfrac{1}{8} R_g \ \textit{in } M, \ \textit{if } n = 3. \end{aligned} \tag{14}$$

Then, system (3) has a pair of nontrivial solutions.

As a consequence of Theorem 0.0.1, we have the following result.

Corollary 0.0.2 *Under the hypothesis of Theorem 0.0.1. If in addition $b \leq 0$ and*

$$\begin{aligned} &(i)\ \tfrac{\alpha}{2^*} a(x_0) + \tfrac{\beta}{2^*} c(x_0) < \tfrac{n-2}{4(n-1)} R_g(x_0) + \tfrac{(n-4)(n-2)}{8(n-1)} \tfrac{\Delta_g f(x_0)}{f(x_0)}, \ \textit{if } n \geq 4, \\ &(ii)\ \tfrac{\alpha}{2^*} a(x_0) + \tfrac{\beta}{2^*} c(x_0) < \tfrac{1}{8} R_g(x_0) \ \textit{and } \tfrac{\alpha}{2^*} a + \tfrac{\beta}{2^*} c \leq \tfrac{1}{8} R_g \ \textit{in } M, \ \textit{if } n = 3, \end{aligned} \tag{15}$$

then, (3) has a pair of positive solutions.

For the next results, consider the functional $E_h : H \to \mathbb{R}$ given by

$$E_h(u,v) = \int_M \left(|\nabla u|_g^2 + |\nabla v|_g^2 + au^2 + 2buv + cv^2 \right) dv_g \tag{16}$$

and let

$$S_{f,h}^{(\alpha,\beta)} = \inf \left\{ E_h(u,v) \ : \ u, v \in H^1(M) \text{ and } \int_M f(x)|u|^\alpha |v|^\beta dv_g = 1 \right\}. \tag{17}$$

In the next result we deal with the case where the functions a, b and c satisfy the condition:

$$\frac{\alpha}{2^*} a(x) + \frac{2\sqrt{\alpha\beta}}{2^*} b(x) + \frac{\beta}{2^*} c(x) \leq \frac{n-2}{4(n-1)} R_g(x), \ \forall x \in M. \tag{18}$$

We can state the following result.

Teorema 0.0.3 *Let (M,g) be a smooth closed Riemannian manifold of dimension $n \geq 3$. Let $\alpha, \beta > 1$ be two real numbers such that $\alpha + \beta = 2^*$, and let a, b and c be functions Hölder continuous in M, and $f \in C^\infty$, with a, b and c satisfying (13) and (18), and f satisfying (12). Let $x_0 \in M$ such that $f(x_0) = \max_M f$. If $S_{f,h}^{(\alpha,\beta)} < \frac{\mathcal{S}_*}{f(x_0)^{2/2^*}}$, where $\mathcal{S}_*$ is defined in (7). Then, system (3) has a pair of nontrivial solutions.*

Afterwards, we present sufficient conditions for the strict inequality

$$S_{f,h}^{(\alpha,\beta)} < \frac{\mathcal{S}_*}{f(x_0)^{2/2^*}}$$

to be true. To get this, inspired by [5, 21, 29], we denote by

$$\lambda_f(M,g) = \inf\left\{ \int_M (|\nabla u|_g^2 + \frac{n-2}{4(n-1)} R_g u^2) dv_g \;:\; \int_M f(x)|u|^{2^*} dv_g = 1 \right\}. \tag{19}$$

Recall that if $f \equiv 1$, $\lambda_f(M,g)$ is the Yamabe invariant of the manifold (M,g) and is usually denoted by $\lambda(M,g)$. In the particular case of the unit $n-$sphere $\mathbb{S}^n$ with the standard metric, denoted as $\lambda(\mathbb{S}^n)$, it is well known that when $\lambda_f(M,g) < \frac{\lambda(\mathbb{S}^n)}{(\max_M f)^{2/2^*}}$, there exists $\varphi \in C^\infty(M)$ with $\varphi > 0$ and $\int_M f\varphi^{2^*} dv_g = 1$ such that

$$-\Delta_g \varphi + \frac{n-2}{4(n-1)} R_g \varphi = \lambda_f(M,g) f \varphi^{\frac{n+2}{n-2}}, \tag{20}$$

with $\lambda_f(M,g) = \int_M \left(|\nabla\varphi|_g^2 + \frac{n-2}{4(n-1)} R_g \varphi^2 \right) dv_g$.

A special case is when we consider the unit $n-$sphere $\mathbb{S}^n$ with the standard metric g_0, where the scalar curvature is $R_{g_0} = n(n-1)$. Note that this case is included in Theorem 0.0.1 when we make the same assumptions. Therefore the following theorem is a case special of Theorem 0.0.3, when $M = \mathbb{S}^n/\Gamma$.

We state the following result.

Teorema 0.0.4 *Let Γ be a nontrivial finite group of isometries of $\mathbb{S}^n$ acting without fixed point on $\mathbb{S}^n$. Write $M = \mathbb{S}^n/\Gamma$, and let a, b, c and f be functions invariant under Γ and satisfying the same assumptions of Theorem 0.0.3. Then $S_{f,h}^{(\alpha,\beta)}(\mathbb{S}^n/\Gamma) < \frac{\mathcal{S}_*}{f(x_0)^{2/2^*}}$, and therefore, (3) has a pair of nontrivial solutions on $\mathbb{S}^n$.*

An interesting consequence is the following.

Corollary 0.0.5 *Suppose the same assumptions of Theorem 0.0.4. In addition, if we assume that $b \equiv 0$, $f \equiv 1$ and $a \equiv c \equiv \frac{n(n-2)}{4}$ then we get that $S^{(\alpha,\beta)}(\mathbb{S}^n) = \mathcal{S}_*$ and system (3) has infinitely many pair of positive solutions. Moreover, if (u,v) is* *a minimizer for* $S^{(\alpha,\beta)}(\mathbb{S}^n)$ *with* $u, v > 0$*, then up to rescaling* u *and* v *will have the following forms:*

$$u(x) = \xi_1(\rho_0 - \cos r)^{\frac{2-n}{2}} \text{ and } v(x) = \zeta_1(\rho_0 - \cos r)^{\frac{2-n}{2}} \tag{21}$$

where $\overline{x} \in \mathbb{S}^n$*,* $r = d_{g_0}(x, \overline{x})$*,* $\xi_1, \zeta_1 > 0$*,* $\rho_0 > 1$ *and* $\frac{\xi_1}{\zeta_1} = \left(\frac{\alpha}{\beta}\right)^{1/2}$*.*

The main results of this chapter, written together with my PhD advisor, are contained in the paper [43].

In chapter 2, inspired by [35], we investigate the coupled Hardy-Sobolev system (4), in which Hardy-type potentials are also involved. The interesting aspect of this study arises from the presence of singular terms, which adds complexity to the challenge of estimating the energy level. In this context, we assume that the functions $a, b, c, \tilde{a}, \tilde{c}$, and ρ satisfy:

($\mathcal{H}_1$) Coercivity condition, that is, there exists $C_0 > 0$ such that

$$\int_M \left(|\nabla_g u|^2 + |\nabla_g v|^2 + a(x)u^2 + 2buv + c(x)v^2 + \tfrac{\tilde{a}(x)}{\rho(x)^\theta}u^2 + \tfrac{\tilde{c}(x)}{\rho(x)^\gamma}v^2\right) dv_g \geq C_1\|(u,v)\|^2,$$

for all $(u, v) \in H$.

($\mathcal{H}_2$) The function ρ satisfies:

$$(i)\ \frac{\rho(x)}{d_g(x, x_0)} = 1 + O(d_g(x, x_0)^\mu),\ \forall x \in B_\delta(x_0);$$

$$(ii)\ \rho(x) > 0, \quad \forall x \in M \backslash B_\delta(x_0).$$

For some $\delta \in (0, i_g)$ (here i_g denotes the injectivity radius of (M, g)), where $\mu \in (0, 1)$.

Thus, our first result of this chapter can be stated as follows:

Teorema 0.0.6 *Let* (M, g) *be a smooth closed Riemannian manifold of dimension* $n \geq 3$*. Let* $a, b, c, \tilde{a}, \tilde{c}$ *and* ρ *be functions in* M *satisfying* ($\mathcal{H}_1$) *and* ($\mathcal{H}_2$)*, with* $\tilde{a}, \tilde{c} \in C^\infty(M)$*. Let* f *be smooth function such that* $f(x_0) = \max_M f > 0$*. In addition, assume that* $h := \frac{\alpha}{2^\star(s)}a + \frac{2\sqrt{\alpha\beta}}{2^\star(s)}b + \frac{\beta}{2^\star(s)}c$*,* $\tilde{a}$ *and* $\tilde{c}$ *satisfy:*

$$\begin{aligned} &(1)\ h(x_0) < \tfrac{(n-2)(6-s)}{12(2n-2-s)}R_g(x_0) + \tfrac{(n-2)(n-4)}{4(2n-2-s)}\tfrac{\Delta_g f(x_0)}{f(x_0)} \text{ and } \tilde{a}(x_0) = \tilde{c}(x_0) = 0, \text{ if } n \geq 4; \\ &(2) \text{ when } n = 3,\ h(x_0) < \tfrac{1}{8}R_g(x_0) \text{ and } h \leq \tfrac{1}{8}R_g \text{ in } M, \text{ or } h \equiv \tfrac{1}{8}R_g \text{ and } (M, g) \\ &\quad \text{is not conformally equivalent to the standard sphere } \mathbb{S}^3, \text{ and that } \tilde{a} \equiv \tilde{c} \equiv 0. \end{aligned} \tag{22}$$

Then, system (4) *has a pair of nontrivial weak solutions.*

An immediate consequence is the following:

Corollary 0.0.7 *Let (M,g) be a closed Riemannian manifold of dimension $n \geq 3$. We assume that $\tilde{a} \equiv \tilde{c} \equiv 0$. Let a, b, c and ρ be functions in M satisfying $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$. Let f be smooth function such that $f(x_0) = \max_M f > 0$. Suppose that $h := \frac{\alpha}{2^\star(s)}a + \frac{2\sqrt{\alpha\beta}}{2^\star(s)}b + \frac{\beta}{2^\star(s)}c$ satisfies:*

$$\begin{aligned} &(1)\ h(x_0) < \frac{(n-2)(6-s)}{12(2n-2-s)}R_g(x_0) + \frac{(n-2)(n-4)}{4(2n-2-s)}\frac{\Delta_g f(x_0)}{f(x_0)},\ \text{if } n \geq 4;\\ &(2)\ \text{when } n=3,\ h(x_0) < \tfrac{1}{8}R_g(x_0) \text{ and } h \leq \tfrac{1}{8}R_g \text{ in } M,\ \text{or } h \equiv \tfrac{1}{8}R_g \text{ and}\\ &\quad (M,g) \text{ is not conformally equivalent to the standard sphere } \mathbb{S}^3. \end{aligned} \tag{23}$$

Then, system (4) has a pair of nontrivial weak solutions.

For our second theorem, we assume only that the functions $\tilde{a}$ and $\tilde{c}$ are Hölder continuous. Now, we can state our second theorem of the chapter.

Teorema 0.0.8 *Let (M,g) be a closed Riemannian manifold of dimension $n \geq 3$. Let $a, b, c, \tilde{a}, \tilde{c}$ and ρ be functions in M satisfying $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, let f be smooth function such that $f(x_0) = \max_M f > 0$. Furthermore, suppose we are in one of the following cases:*

$$(1)\ \text{when } n \geq 4,\ \text{and} \begin{cases} \tilde{a}(x_0) < 0 & \text{if } \ \theta > \gamma,\\ \alpha\tilde{a}(x_0) + \beta\tilde{c}(x_0) < 0 & \text{if } \ \theta = \gamma,\\ \tilde{c}(x_0) < 0 & \text{if } \ \gamma > \theta, \end{cases} \tag{24}$$

$$(2)\ \text{when } n = 3,\ \text{and} \begin{cases} \tilde{a}(x_0) < 0 & \text{if } \ \theta > \gamma \geq 1,\\ \alpha\tilde{a}(x_0) + \beta\tilde{c}(x_0) < 0 & \text{if } \ \theta = \gamma \geq 1,\\ \tilde{c}(x_0) < 0 & \text{if } \ \gamma > \theta \geq 1. \end{cases} \tag{25}$$

Then, system (4) has a pair of nontrivial weak solutions.

Finally, in the Chapter 3, motivated by the works of Druet [15] and Chen and Liu [11]. In [15], the author considered equation (1) (as previously mentioned), while in [11], the authors studied a class of Hardy-Sobolev equation involving the p-Laplace operator. When $p \neq 2$, an interesting point in its existence results, is the condition: $2 < p^2 < n$ or $1 < p < \min\left\{\frac{n+2}{3}, 2\right\}$.

Here, we were able to improve the range to p, more specifically, when $2 < p \leq \frac{n+2}{3}$ or $1 < p < 2$ and $p \leq \sqrt{n}$, we establish existence results for (5). For this we assume that the functions a, b and c are Hölder continuous and f is a smooth function on M with

$\max_M f > 0$. In addition, these functions satisfy the following coercivity condition, that is, there exists $C_0 > 0$ such that

$$\int_M \left[|\nabla_g u|^p + |\nabla_g v|^p + a(x)|u|^p + b(x)uv(|u|^{p-2} + |v|^{p-2}) + c(x)|v|^p\right] dv_g \geq C_0\|(u,v)\|^p, \tag{26}$$

for all $(u, v) \in H^p$, where $b \equiv 0$ when $1 < p < 2$.

Our first result can be stated as follows:

Teorema 0.0.9 *Let (M, g) be a smooth closed Riemannian manifold of dimension $n \geq 5$, $p \in (2, n)$ and $s \in [0, p)$. Let a, b and c be functions Hölder continuous in M satisfying (26) and f smooth function on M such that $f(x_0) = \max_M f > 0$. In addition, assume that $2 < p \leq \frac{n+2}{3}$ and*

$$0 < R_g(x_0) + \frac{3(n + 2 - 3p)}{(3p - s)} \frac{\Delta f(x_0)}{f(x_0)}. \tag{27}$$

Then, system (5) has a pair of nontrivial weak solutions.

For our next theorem, we consider $b \equiv 0$, we write $h := \frac{\alpha}{p^\star(s)} a + \frac{\beta}{p^\star(s)} c$. We can state the following result.

Teorema 0.0.10 *Let (M, g) be a closed Riemannian manifold of dimension $n \geq 2$, $p \in (1, 2)$ and $s \in [0, p)$. Let a, b and c be functions Hölder continuous in M satisfying (26) and f a smooth function such that $f(x_0) = \max_M f > 0$. If in addition, we assume that $h(x_0) < 0$. Then, the system (5) has a pair of non-negative nontrivial weak solutions, when $n \geq 4$. The same conclusion holds when $n = 2, 3$, for $p \leq \sqrt{n}$.*

Concluding this introduction, we emphasize that the main tools used in this thesis are the following: "optimal" Sobolev and Hardy-Sobolev inequalities, Brézis-Lieb type lemma, minimization under constraint and Mountain Pass Theorem. In order not to resort to the Introduction and for the sake of the independence of the chapters, we will again present, in each chapter, the main results and the respective hypotheses.

Notation

- (M,g) denotes a smooth compact Riemannian manifold without boundary;
- i_g denotes the injectivity radius of (M,g);
- $B_\delta(x)$ denotes the geodesic ball centered in x with radius $\delta \in (0, i_g)$;
- $\det(g)$ is the determinant of the components of the metric g;
- $\mathbb{R}^n$ denotes the usual Euclidean space;
- $B_R(0)$ denotes an ball in $\mathbb{R}^n$ of radius R and center at the origin;
- $L^q(M) := \left\{u : M \to \mathbb{R} : u \text{ is measurable and } \int_M |u|^q dv_g < \infty\right\}$;
- $L^\infty(M) := \{u : M \to \mathbb{R} : u \text{ is measurable and } \sup_M |u(x)| < \infty\}$;
- $H^{1,p}(M)$ denotes the usual Sobolev space;
- $C^K(M)$ denotes the space of functions possessing continuous derivatives up to order k on M;
- $C^{k,\varrho}(M)$ denotes the Hölder space;
- $C_0^\infty(\Omega)$ denotes the space of infinitely differentiable functions whose support compact in Ω;
- $u_m \to u$ and $u_m \rightharpoonup u$ denote strongly and weak converge, respectively, in a normed space;
- $I_q^p := \displaystyle\int_0^\infty t^p(1+t)^{-q}dt.$

Chapter 1

On a Class of Strongly Coupled Critical Elliptic Systems

In this chapter, motivated by [2, 3, 10, 16, 21, 28, 26, 38, 44, 47, 50], we investigate the existence of solutions for a class of strongly coupled elliptic systems. The approach is variational, employing the Mountain Pass Theorem and minimization under constraint. One particularly intricate aspect is estimating the energy level (minimax level), especially when $n = 3$, as the argument is non-local, necessitating the use of an appropriate Green's function. The results of this chapter were published in the paper [43].

1.1 Introduction

Let (M, g) be a smooth closed Riemannian manifold of dimension $n \geq 3$. We are concerned with the existence of solutions of the following system:

$$\begin{cases} -\Delta_g u + a(x)u + b(x)v = \dfrac{\alpha}{2^*} f(x)u|u|^{\alpha-2}|v|^{\beta} & \text{in } M, \\ -\Delta_g v + b(x)u + c(x)v = \dfrac{\beta}{2^*} f(x)v|v|^{\beta-2}|u|^{\alpha} & \text{in } M, \end{cases} \tag{1.1}$$

where Δ_g is the Laplace-Beltrami operator, a, b and c are functions Hölder continuous in M, f is a smooth function, and $\alpha > 1$, $\beta > 1$ are real constants satisfying $\alpha + \beta = 2^*$, where $2^* = 2n/(n-2)$ is the critical Sobolev exponent.

The system (1.1) is strongly related to the equation

$$-\Delta_g u + a(x)u = f(x)|u|^{2^*-2}u \quad \text{in} \quad M,$$

which has been intensively studied (see for instance, Druet [15] and Vétois [48]), especially, when we have $a = \frac{n-2}{4(n-1)}R_g$, where we get the prescribe scalar curvature equation

$$-\Delta_g u + \frac{n-2}{4(n-1)}R_g u = f(x)|u|^{2^*-2}u, \quad \text{in} \quad M, \tag{1.2}$$

this problem is a generalization of the well-known Yamabe equation (when f is constant) whose positive solutions are such that the scalar curvature of the conformal metric $\tilde{g} = u^{2^*-2}g$ is equal to $\frac{4(n-1)}{n-2}f$ (for more details see Yamabe [50], Trudinger [47], Aubin [3], Schoen [44], Aubin and Hebey [5], Escobar and Schoen [21], Hebey and Vaugon [29]). The study of this equation both in the classical and prescribed form, together with the work of Alves et al. [2], inspired us in this investigation on the existence of solutions for (1.1).

Before presenting our main results, we need to introduce some notations and definitions. Throughout this work, we will denote by $H^1(M)$ the Sobolev space of all functions in $L^2(M)$ with one derivative (in the weak sense) in $L^2(M)$. We equip $H^1(M)$ with the standard $\|\cdot\|_{H^1}-$norm, that is, $\|u\|_{H^1}^2 = \|\nabla u\|_2^2 + \|u\|_2^2$, where $\|\cdot\|_q$ denotes the norm of the Lebesgue space $L^q(M)$, whenever $q \geq 1$. The norm of $L^q(M) \times L^q(M)$ will be defined by $\|(u,v)\|_q = \left(\|u\|_q^q + \|v\|_q\right)^{1/q}$.

We shall work with the space $H = H^1(M) \times H^1(M)$ endowed with the norm

$$\|(u,v)\| = \left(\|u\|_{H^1}^2 + \|v\|_{H^1}^2\right)^{1/2}.$$

In this context, we say that a pair of functions $(u,v) \in H$ is a weak solution of (1.1), if for all $(\varphi,\psi) \in H$, it holds

$$\begin{aligned}\int_M \left(\langle\nabla u, \nabla\varphi\rangle_g + \langle\nabla v, \nabla\psi\rangle_g + a(x)u\varphi + b(x)[u\psi + v\varphi] + c(x)v\psi\right) dv_g \\ = \int_M \frac{\alpha}{2^*}f(x)|u|^{\alpha-2}|v|^{\beta}u\varphi dv_g + \int_M \frac{\beta}{2^*}f(x)|v|^{\beta-2}|u|^{\alpha}v\psi dv_g.\end{aligned}$$

By elliptic regularity theory (for example, see Lee and Parker [38, Theorem 4.1]), any weak solution (u,v) of (1.1), is in $C^2 \times C^2$ when a, b and c are Hölder continuous, and is in $C^\infty \times C^\infty$ when a, b and c are smooth functions.

Recall the relation obtained by Alves et al. [2, Section 4] that we will use in this chapter is the following:

$$\mathcal{S}_* = \left[\left(\frac{\alpha}{\beta}\right)^{\beta/2^*} + \left(\frac{\beta}{\alpha}\right)^{\alpha/2^*} \right] K_n^{-2}, \tag{1.3}$$

where $\alpha + \beta = 2^*$, K_n^{-2} is defined by

$$K_n^{-2} = \inf_{u \in H^1(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^n} |u|^{\alpha+\beta} dx\right)^{2/2^*}},$$

and $\mathcal{S}_*$ is defined by

$$\mathcal{S}_* = \inf_{(u,v)\in[H^1(\mathbb{R}^n)]^2\setminus\{0\}} \frac{\int_{\mathbb{R}^n} \left(|\nabla u|^2 + |\nabla v|^2\right) dx}{\left(\int_{\mathbb{R}^n} |u|^{\alpha}|v|^{\beta} dx\right)^{2/2^*}}. \tag{1.4}$$

It is known that K_n is the sharp constant for the embedding of $H^1(\mathbb{R}^n)$ into $L^{2^*}(\mathbb{R}^n)$.

Throughout this chapter we assume some very general hypotheses on the functions a, b, c and f that will allow us to obtain some existence results for system (1.1) through variational methods. Precisely, we assume that the function f satisfies

$$\max_M f > 0 \tag{1.5}$$

and the functions a, b and c satisfy the following coercivity condition: there exists $C_0 > 0$ such that

$$\int_M (|\nabla_g u|^2 + |\nabla_g v|^2 + a(x)u^2 + 2b(x)uv + c(x)v^2) dv_g \geq C_0 \|(u,v)\|^2, \ \ \forall\, (u,v) \in H. \tag{1.6}$$

Our first result in this chapter can be stated as follows:

Teorema 1.1.1 *Let (M,g) be a closed Riemannian manifold of dimension $n \geq 3$. Let $\alpha, \beta > 1$ be two real numbers such that $\alpha + \beta = 2^*$, and let a, b and c be functions Hölder continuous in M, and $f \in C^\infty$, with a, b and c satisfying (1.6) and f satisfying (1.5), writing $h = \frac{\alpha}{2^*}a + \frac{2\sqrt{\alpha\beta}}{2^*}b + \frac{\beta}{2^*}c$. Let x_0 be some point in M such that $f(x_0) = \max_M f$. If, in addition, we assume that*

$$\begin{aligned} &(i)\ h(x_0) < \tfrac{n-2}{4(n-1)} R_g(x_0) + \tfrac{(n-4)(n-2)}{8(n-1)} \tfrac{\Delta_g f(x_0)}{f(x_0)}, \ \textit{if } n \geq 4, \\ &(ii)\ h(x_0) < \tfrac{1}{8} R_g(x_0) \ \textit{and}\ h \leq \tfrac{1}{8} R_g \ \textit{in } M, \ \textit{if } n = 3. \end{aligned} \tag{1.7}$$

Then, system (1.1) has a nontrivial solution.

Theorem 1.1.1 will be proved using the Mountain Pass Theorem without the Palais-Smale compactness condition. A delicate part is the estimating the minimax level in order to overcome the lack of compactness of the functional associated to system (1.1) caused by the critical growth of the nonlinearities. We achieve this objective following some ideas developed in [3, 4, 16]. Here we face some extra difficulties due to the strong coupling of the system.

As a consequence of Theorem 1.1.1, we prove the following results.

Corollary 1.1.2 *Suppose the same assumptions of Theorem 1.1.1. Let x_0 be some point in M such that $f(x_0) = \max_M f$. If, in addition, we assume $b \leq 0$ and*

$$\begin{aligned} &(i)\ \tfrac{\alpha}{2^*}a(x_0) + \tfrac{\beta}{2^*}c(x_0) < \tfrac{n-2}{4(n-1)}R_g(x_0) + \tfrac{(n-4)(n-2)}{8(n-1)}\tfrac{\Delta_g f(x_0)}{f(x_0)},\ \text{if } n \geq 4, \\ &(ii)\ \tfrac{\alpha}{2^*}a(x_0) + \tfrac{\beta}{2^*}c(x_0) < \tfrac{1}{8}R_g(x_0)\ \text{and}\ \tfrac{\alpha}{2^*}a + \tfrac{\beta}{2^*}c \leq \tfrac{1}{8}R_g\ \text{in } M,\ \text{if } n = 3. \end{aligned} \tag{1.8}$$

Then, system (1.1) has a pair of positive solutions.

Corollary 1.1.3 *Suppose the same assumptions of Theorem 1.1.1 and that f is constant and positive. Let x_0 be some point in M such that*

$$\begin{aligned} &(i)\ h(x_0) < \tfrac{n-2}{4(n-1)}R_g(x_0),\ \text{if } n \geq 4, \\ &(ii)\ h(x_0) < \tfrac{1}{8}R_g(x_0)\ \text{and}\ h \leq \tfrac{1}{8}R_g\ \text{in } M,\ \text{if } n = 3. \end{aligned} \tag{1.9}$$

Then, system (1.1) has a nontrivial solution.

For the next results, consider the functional $E_h : H \to \mathbb{R}$ given by

$$E_h(u, v) = \int_M \left(|\nabla u|_g^2 + |\nabla v|_g^2 + au^2 + 2buv + cv^2\right) dv_g \tag{1.10}$$

and let

$$S_{f,h}^{(\alpha,\beta)} = \inf\left\{E_h(u, v)\ :\ u, v \in H^1(M)\ \text{and}\ \int_M f(x)|u|^{\alpha}|v|^{\beta} dv_g = 1\right\}. \tag{1.11}$$

Define

$$\lambda_f(M, g) = \inf\left\{\int_M (|\nabla u|_g^2 + \frac{n-2}{4(n-1)}R_g u^2) dv_g\ :\ \int_M f(x)|u|^{2^*} dv_g = 1\right\}. \tag{1.12}$$

Remark 1.1.4 *When f is constant and equal to 1, $\lambda_f(M, g)$ is called of Yamabe invariant of the manifold (M, g), and is usually denoted by $\lambda(M, g)$. In the particular case of the unit $n-$sphere $\mathbb{S}^n$ with the standard metric is denoted by $\lambda(\mathbb{S}^n)$. It is well* *known that when* $\lambda_f(M,g) < \frac{\lambda(\mathbb{S}^n)}{(\max_M f)^{2/2^*}}$, *there exists* $\varphi \in C^\infty(M)$ *with* $\varphi > 0$ *and* $\int_M f\varphi^{2^*} dv_g = 1$ *such that*

$$-\Delta_g \varphi + \frac{n-2}{4(n-1)} R_g \varphi = \lambda_f(M,g) f \varphi^{\frac{n+2}{n-2}}, \tag{1.13}$$

with $\lambda_f(M,g) = \int_M \left(|\nabla\varphi|_g^2 + \frac{n-2}{4(n-1)} R_g \varphi^2 \right) dv_g$. *It is also known that* $\lambda(\mathbb{S}^n) = K_n^{-2}$, *with*

$$K_n = \sqrt{\frac{4}{n(n-2)\omega_n^{2/n}}},$$

where ω_n *is the volume of the unit* $n-$*sphere (see [3, 5, 4, 16]).*

In the next results we deal with the case where the functions a, b and c satisfy the condition:

$$\frac{\alpha}{2^*} a(x) + \frac{2\sqrt{\alpha\beta}}{2^*} b(x) + \frac{\beta}{2^*} c(x) \le \frac{n-2}{4(n-1)} R_g(x), \ \forall x \in M. \tag{1.14}$$

Remark 1.1.5 *The coercivity condition (1.6) and (1.14) imply that given* $\psi \in H^1(M)$ *and* $\xi, \zeta > 0$ *such that* $\left(\frac{\xi}{\zeta}\right)^2 = \frac{\alpha}{\beta}$, *then*

$$\begin{aligned} C_0(\xi^2+\zeta^2)\|\psi\|_{H^1}^2 &\le (\xi^2+\zeta^2)\|\nabla\psi\|_2^2 + \int_M (a\xi^2 + 2b\xi\zeta + c\zeta^2)\psi^2 dv_g \\ &= (\xi^2+\zeta^2)\left\{ \|\nabla\psi\|_2^2 + \int_M (\frac{\alpha}{2^*}a + \frac{2\sqrt{\alpha\beta}}{2^*}b + \frac{\beta}{2^*}c)\psi^2 dv_g \right\} \\ &\le (\xi^2+\zeta^2)\left\{ \|\nabla\psi\|_2^2 + \frac{n-2}{4(n-1)} \int_M R_g \psi^2 dv_g \right\}. \end{aligned}$$

Therefore, $-\Delta_g + \frac{n-2}{4(n-1)} R_g$ *is also coercive. In particular we are dealing with the case where the Yamabe invariant is positive.*

We can state the following result.

Teorema 1.1.6 *Let* (M, g) *be a closed Riemannian manifold of dimension* $n \ge 3$. *Let* $\alpha, \beta > 1$ *be two real numbers such that* $\alpha + \beta = 2^*$, *with* a, b *and* c *satisfying (1.6) and (1.14), and* f *satisfying (1.5). Let* x_0 *be some point in* M *such that* $f(x_0) = \max_M f$. *If* $S_{f,h}^{(\alpha,\beta)} < \frac{\mathcal{S}_*}{f(x_0)^{2/2^*}}$, *where* $\mathcal{S}_*$ *is defined in (1.4). Then, system (1.1) has a nontrivial solution.*

Complementing Theorem 1.1.6 and inspired by [21, 5, 30], we prove the following theorems:

Teorema 1.1.7 *Let (M,g) be a closed Riemannian manifold of dimension $n \geq 3$. Let a, b and c be functions Hölder continuous in M satisfying (1.6) and (1.14). Assume that $n \geq 6$ and M is not locally conformally flat. If at a point x_0 where $f(x_0) = \max_M f$ is such that the Weyl tensor is nonvanishing (that is, $|W_g(x_0)| \neq 0$). If we assume that*

(i) if $\Delta_g f(x_0) = 0$ when $n = 6$, or

(ii) if $\Delta_g f(x_0) = 0$ and $\left|\Delta_g^2 f(x_0)\right| / f(x_0)$ is small enough, when $n > 6$.

Then, $S_{f,h}^{(\alpha,\beta)} < \frac{\mathcal{S}_*}{f(x_0)^{2/2^*}}$. Consequently, system (1.1) has a nontrivial solution.

Teorema 1.1.8 *Let (M,g) be a closed Riemannian manifold of dimension $n \geq 3$. Let a, b and c be functions Hölder continuous in M satisfying (1.6) and (1.14). Assume that $n = 3, 4$ or 5, or M is locally conformally flat, when $n \geq 6$. Let $x_0 \in M$ be a point such that $f(x_0) = \max_M f > 0$. We have the following cases:*

(i) if $n = 3$ or, if $\Delta_g f(x_0) = 0$ when $n = 4, 5$;

(ii) if $\Delta_g f(x_0) = \Delta_g^2 f(x_0) = 0$, when $n = 6, 7$;

(iii) if $\Delta_g f(x_0) = \Delta_g^2 f(x_0) = 0$ and $\Delta_g^3 f(x_0) = 0$ or $|\nabla W_g(x_0)| = 0$, when $n = 8$.

Then, $S_{f,h}^{(\alpha,\beta)} < \frac{\mathcal{S}_*}{f(x_0)^{2/2^*}}$ unless M is conformal to the standard $\mathbb{S}^n$. Consequently, system (1.1) has a nontrivial solution. When $n > 8$ the same conclusion holds if $|\nabla W_g(x_0)| \neq 0$ and $\Delta_g^3 f(x_0) = 0$ or when $|\nabla W_g(x_0)| = 0$ if $|\nabla^2 W_g(x_0)| \neq 0$ and $\Delta_g^3 f(x_0) = \Delta_g^4 f(x_0) = 0$, or when all derivatives of W_g vanish at x_0 if $\Delta_g^m f(x_0) = 0$ for all $1 \leq m \leq \frac{n}{2} - 1$.

Corollary 1.1.9 *Suppose the same assumptions of Theorems 1.1.7 or 1.1.8. In addition, if $b \leq 0$ and the functions a and c satisfy*

$$\frac{\alpha}{2^*} a(x) + \frac{\beta}{2^*} c(x) \leq \frac{n-2}{4(n-1)} R_g(x), \ \forall x \in M. \tag{1.15}$$

Then, system (1.1) has a pair of positive solutions.

Corollary 1.1.10 *Suppose the same assumptions of Theorems 1.1.7 or 1.1.8. In addition, if we assume that $f \geq 0$, $b = 0$ and $a = c = \frac{n-2}{4(n-1)} R_g$. Then, system (1.1) has a nontrivial solution. Moreover, we have that*

$$S_{f,h}^{(\alpha,\beta)}(M) = \left[\left(\frac{\alpha}{\beta}\right)^{\beta/\alpha+\beta} + \left(\frac{\beta}{\alpha}\right)^{\alpha/\alpha+\beta} \right] \lambda_f(M,g). \tag{1.16}$$

Therefore, the pair $(\xi\varphi, \zeta\varphi)$ (up to rescaling) is solution for the system, for any positive solution $\varphi \in C^\infty$ of (1.13), where $\int_M f \varphi^{2^*} dv_g = 1$ and $\frac{\xi}{\zeta} = \left(\frac{\alpha}{\beta}\right)^{1/2}$.

A special case is when we consider the unit $n-$sphere $\mathbb{S}^n$ with the standard metric g_0, that is, the scalar curvature is $R_{g_0} = n(n-1)$. Note that this case is included in Theorem 1.1.1 when we assume the same hypotheses. Therefore the following theorem is a case special of Theorem 1.1.8, when $M = \mathbb{S}^n/\Gamma$, this result is inspired by [21].

Teorema 1.1.11 *Let Γ be a nontrivial finite group of isometries of $\mathbb{S}^n$ acting without fixed point on $\mathbb{S}^n$. Write $M = \mathbb{S}^n/\Gamma$, and let a, b, c and f be functions invariant under Γ and satisfying the same assumptions of Theorem 1.1.8. Then $S^{(\alpha,\beta)}_{f,h}(\mathbb{S}^n/\Gamma) < \frac{\mathcal{S}_*}{f(x_0)^{2/2^*}}$, and therefore, system (1.1) has a nontrivial solution on $\mathbb{S}^n$.*

Corollary 1.1.12 *Suppose the same assumptions of Theorem 1.1.11. In addition, if we assume that $b = 0$, $a = c = \frac{n(n-2)}{4}$ and $f \geq 0$. Then, system (1.1) has a nontrival solution. Moreover, we have*

$$S^{(\alpha,\beta)}_{f,h}(\mathbb{S}^n) = \left[\left(\frac{\alpha}{\beta}\right)^{\beta/\alpha+\beta} + \left(\frac{\beta}{\alpha}\right)^{\alpha/\alpha+\beta}\right] \lambda_f(\mathbb{S}^n, g_0). \tag{1.17}$$

Therefore, the pair $(\xi\varphi, \zeta\varphi)$ (up to rescaling) is solution for system (1.1), where $\varphi \in C^\infty$ is a positive solution of Eq. (1.13) on $\mathbb{S}^n$.

Corollary 1.1.13 *Suppose the same assumptions of Theorem 1.1.11. In addition, if we assume that $b = 0$, $f = 1$ and $a = c = \frac{n(n-2)}{4}$ then we get that $S^{(\alpha,\beta)}(\mathbb{S}^n) = \mathcal{S}_*$ and system (1.1) has infinitely many pair of positive solutions. Moreover, if (u, v) is a minimizer for $S^{(\alpha,\beta)}(\mathbb{S}^n)$ with $u, v > 0$, then up to rescaling u and v will have the following forms:*

$$u(x) = \xi_1(\rho_0 - \cos r)^{\frac{2-n}{2}} \text{ and } v(x) = \zeta_1(\rho_0 - \cos r)^{\frac{2-n}{2}} \tag{1.18}$$

where $\overline{x} \in \mathbb{S}^n$, $r = d_{g_0}(x, \overline{x})$, $\xi_1, \zeta_1 > 0$, $\rho_0 > 1$ and $\frac{\xi_1}{\zeta_1} = \left(\frac{\alpha}{\beta}\right)^{1/2}$.

Corollary 1.1.14 *Suppose the same assumptions of Theorem 1.1.11. In addition, if $b \leq 0$ and the functions a and c satisfy the following hypothesis*

$$\frac{\alpha}{2^*}a(x) + \frac{\beta}{2^*}c(x) \leq \frac{n(n-2)}{4}, \ \forall x \in M. \tag{1.19}$$

Then, system (1.1) has a pair of positive solutions on $\mathbb{S}^n$.

The chapter is organized as follows. In Sect. 1.2 we prove an essential Sobolev inequality to prove the main results. In Sect. 1.3 we prove Theorem 1.1.1 and its consequences. In Sect. 1.4 we prove Theorems 1.1.6, 1.1.7 and 1.1.8. We dedicate Sect. 1.5 for the case of the sphere $\mathbb{S}^n$.

1.2 Some preliminary results

In [31], Hebey and Vaugon have established that the best constant for the Sobolev inequality is K_n^2. Precisely, they proved that there is a positive constant B such that

$$\|u\|_{2^*}^2 \le K_n^2 \|\nabla u\|_2^2 + B\|u\|_2^2, \tag{1.20}$$

for all $u \in H^1(M)$. Moreover, if $\|u\|_{2^*}^2 \le K\|\nabla u\|_2^2 + C\|u\|_2^2$ for all $u \in H^1(M)$, where K and C are positive constants, then $K \ge K_n^2$.

Initially, we establish an inequality that will be used in the proof of the main results.

Lemma 1.2.1 *Let $\mathcal{S}_*$ be the constant defined in (1.4) when $\alpha + \beta = 2^*$. Then, there is a positive constant B_0 such that*

$$\left(\int_M |u|^\alpha |v|^\beta dv_g\right)^{2/2^*} \le \mathcal{S}_*^{-1} \|(|\nabla u|, |\nabla v|)\|_2^2 + B_0 \|(u,v)\|_2^2\,, \tag{1.21}$$

for all $(u,v) \in H$. Moreover, $(\mathcal{S}_*)^{-1}$ is the best constant such that the inequality holds.

Proof. Given $u, v \in H^1(M)$, since $\frac{\alpha}{2^*} + \frac{\beta}{2^*} = 1$, by Hölder's inequality,

$$\int_M |u|^\alpha |v|^\beta dv_g \le \left(\int_M |u|^{2^*} dv_g\right)^{\alpha/2^*} \left(\int_M |v|^{2^*} dv_g\right)^{\beta/2^*},$$

that is,

$$\left(\int_M |u|^\alpha |v|^\beta dv_g\right)^{2/2^*} \le \left(\|u\|_{2^*}^2\right)^{\alpha/2^*} \left(\|v\|_{2^*}^2\right)^{\beta/2^*}.$$

On the other hand, by Young's inequality,

$$\begin{aligned}
\left(\|u\|_{2^*}^2\right)^{\alpha/2^*} \left(\|v\|_{2^*}^2\right)^{\beta/2^*} &= \left(\varepsilon\|u\|_{2^*}^2\right)^{\alpha/2^*} \frac{\left(\|v\|_{2^*}^2\right)^{\beta/2^*}}{\varepsilon^{\alpha/2^*}} \\
&= \left(\varepsilon\|u\|_{2^*}^2\right)^{\alpha/2^*} \left(\|v\|_{2^*}^2 \varepsilon^{-\alpha/\beta}\right)^{\beta/2^*} \\
&\le \frac{\alpha}{2^*}\varepsilon\|u\|_{2^*}^2 + \frac{\beta}{2^*}\varepsilon^{-\alpha/\beta}\|v\|_{2^*}^2.
\end{aligned}$$

Choosing $\varepsilon = \left[\left(\frac{\alpha}{\beta}\right)^{\beta/2^*} + \left(\frac{\beta}{\alpha}\right)^{\alpha/2^*}\right]^{-1} \frac{2^*}{\alpha}$, by a straightforward calculation, we get

$$\frac{\alpha}{2^*}\varepsilon = \frac{\beta}{2^*}\varepsilon^{-\alpha/\beta} = \left[\left(\frac{\alpha}{\beta}\right)^{\beta/2^*} + \left(\frac{\beta}{\alpha}\right)^{\alpha/2^*}\right]^{-1},$$

and consequently,

$$\left(\int_M |u|^\alpha |v|^\beta dv_g\right)^{2/2^*} \le \left[\left(\frac{\alpha}{\beta}\right)^{\beta/2^*} + \left(\frac{\beta}{\alpha}\right)^{\alpha/2^*}\right]^{-1} \left(\|u\|_{2^*}^2 + \|v\|_{2^*}^2\right). \tag{1.22}$$

Using (1.22) and the Sobolev inequality (1.20), we can find $B > 0$ such that

$$\left(\int_M |u|^\alpha |v|^\beta dv_g\right)^{2/2^*} \leq \left[\left(\frac{\alpha}{\beta}\right)^{\beta/2^*} + \left(\frac{\beta}{\alpha}\right)^{\alpha/2^*}\right]^{-1} \left(K_n^2 \|(|\nabla u|, |\nabla v|)\|_2^2 + B\|(u,v)\|_2^2\right).$$

Therefore, we get that

$$\left(\int_M |u|^\alpha |v|^\beta dv_g\right)^{2/2^*} \leq \mathcal{S}_*^{-1} \|(|\nabla u|, |\nabla v|)\|_2^2 + B_0 \|(u,v)\|_2^2$$

for all $(u,v) \in H$, where $B_0 = B\left[\left(\frac{\alpha}{\beta}\right)^{\beta/2^*} + \left(\frac{\beta}{\alpha}\right)^{\alpha/2^*}\right]^{-1}$.

Finally, if S_0 is a positive constant such that

$$\left(\int_M |u|^\alpha |v|^\beta dv_g\right)^{2/2^*} \leq S_0 \|(|\nabla u|, |\nabla v|)\|_2^2 + B_1 \|(u,v)\|_2^2, \tag{1.23}$$

for all $(u,v) \in H$, where B_1 is some positive constant. We claim that $S_0 \geq \mathcal{S}_*^{-1}$. Indeed, given $\varphi \in H^1(M)$ and writing $u = \alpha^{1/2}\varphi$ and $v = \beta^{1/2}\varphi$, by (1.23) we have

$$\left(\alpha^{\alpha/2}\beta^{\beta/2}\right)^{2/2^*} \left(\int_M |\varphi|^{2^*} dv_g\right)^{2/2^*} \leq 2^* \left[S_0 \|\nabla\varphi\|_2^2 + B_1 \|\varphi\|_2^2\right],$$

which gives us

$$\begin{aligned}\left(\int_M |\varphi|^{2^*} dv_g\right)^{2/2^*} &\leq \frac{2^*}{\alpha^{\alpha/2^*}\beta^{\beta/2^*}} \left[S_0 \|\nabla\varphi\|_2^2 + B_1 \|\varphi\|_2^2\right] \\ &= \left[\frac{\alpha}{\alpha^{\alpha/2^*}\beta^{\beta/2^*}} + \frac{\beta}{\alpha^{\alpha/2^*}\beta^{\beta/2^*}}\right] \left(S_0 \|\nabla\varphi\|_2^2 + B_1 \|\varphi\|_2^2\right) \\ &= \left[\left(\frac{\alpha}{\beta}\right)^{\beta/2^*} + \left(\frac{\beta}{\alpha}\right)^{\alpha/2^*}\right] S_0 \|\nabla\varphi\|_2^2 + B_2 \|\varphi\|_2^2,\end{aligned}$$

for some $B_2 > 0$. Since K_n^{-2} is the best constant in the Sobolev embedding theorem (see [4, 31]), we reach that

$$\left[\left(\frac{\alpha}{\beta}\right)^{\beta/2^*} + \left(\frac{\beta}{\alpha}\right)^{\alpha/2^*}\right] S_0 \geq K_n^2,$$

and since $\mathcal{S}_* = \left[\left(\frac{\alpha}{\beta}\right)^{\beta/2^*} + \left(\frac{\beta}{\alpha}\right)^{\alpha/2^*}\right] K_n^{-2}$, we conclude the proof of the lemma. ∎

An immediate consequence of this result is the following inequality:

Corollary 1.2.2 *Let $C = \max\{\mathcal{S}_*^{-1}, B_0\}$, then we have*

$$\left(\int_M |u|^\alpha |v|^\beta dv_g\right)^{2/(\alpha+\beta)} \leq C\|(u,v)\|^2.$$

Another result that will be important later on is the following Brezis-Lieb type lemma.

Lemma 1.2.3 *Let* $u_m \rightharpoonup u$ *and* $v_m \rightharpoonup v$ *in* $H^1(M)$ *and let* $\ell \in L^\infty(M)$*. Then we have*

$$\int_M \ell(x)|u_m|^\alpha|v_m|^\beta dv_g = \int_M \ell(x)|u|^\alpha|v|^\beta dv_g + \int_M \ell(x)|u_m-u|^\alpha|v_m-v|^\beta dv_g + o_m(1),$$

where $o_m(1) \to 0$ *as* $m \to \infty$.

Proof. The proof is done in Lemma 3.2.5 in Chapter 3. ■

1.3 Proof of Theorem 1.1.1

We begin this section by introducing some notations and definitions. First, consider the functional $I : H \to \mathbb{R}$ associated to the system (1.1) given by

$$I(u,v) = \frac{1}{2}\int_M \left[|\nabla u|_g^2 + |\nabla v|_g^2 + a(x)u^2 + 2b(x)uv + c(x)v^2\right] dv_g - \frac{1}{2^*}\int_M f(x)|u|^\alpha|v|^\beta dv_g.$$

Since the functions a, b and c are Hölder continuous and f is a smooth function, we have that I is well defined and by standard arguments $I \in C^1(H,\mathbb{R})$ with

$$\begin{aligned} I'(u,v)\cdot(\varphi,\psi) = &\int_M \left(\langle \nabla u, \nabla\varphi\rangle_g + \langle\nabla v,\nabla\psi\rangle_g + a(x)u\varphi + b(x)[u\psi + v\varphi] + c(x)v\psi\right) dv_g \\ &- \int_M \left(\frac{\alpha}{2^*}f(x)|u|^{\alpha-2}|v|^\beta u\varphi + \frac{\beta}{2^*}f(x)|v|^{\beta-2}|u|^\alpha v\psi\right) dv_g. \end{aligned}$$

Hence, a critical point of I is a weak solution of system (1.1) and reciprocally. Moreover, it is easy to see that I satisfies the geometry of the Mountain Pass Theorem. Indeed, from Corollary 1.1.2 and by coercivity hypothesis (1.6) we have that

$$I(u,v) \geq \frac{C_0}{2}||(u,v)||^2 - \frac{C}{2^*}||(u,v)||^{2^*},$$

thus, there exist $R > 0$ (small enough) and $\rho > 0$ such that

$$I(u,v) \geq \rho \quad \text{whenever} \quad \|(u,v)\| = R. \tag{1.24}$$

Now, let $\varphi \in C_0^\infty(M)\backslash\{0\}$ such that $\int_M f(x)|\varphi|^{2^*}dv_g > 0$, for $t > 0$, note that:

$$\begin{aligned} I(t\varphi,t\varphi) &= \frac{t^2}{2}\int_M [2|\nabla_g\varphi|^2 + (a+2b+c)\varphi^2]dv_g - \frac{t^{2^*}}{2^*}\int_M f(x)|\varphi|^{2^*}dv_g \\ &\leq \frac{t^2}{2}||\varphi||_{H^1}^2 - \frac{t^{2^*}}{2^*}\int_M f(x)|\varphi|^{2^*}dv_g, \end{aligned}$$

thus, $\lim_{t\to\infty} I(t\varphi, t\varphi) = -\infty$ as $t \to \infty$. Therefore, there exists some $(\tilde{u}, \tilde{v}) \in H$ with $\|(\tilde{u}, \tilde{v})\| > R$ and such that $I(\tilde{u}, \tilde{v}) < 0$.

Now, for some pair $(\tilde{u}, \tilde{v})$ satisfying the second condition above, we consider the set $\Gamma = \{\gamma \in C([0,1], H) : \gamma(0) = 0 \text{ and } \gamma(1) = (\tilde{u}, \tilde{v})\}$, and so we can define the minimax level

$$\mathrm{c} := \inf_{\gamma\in\Gamma} \sup_{0\le t\le 1} I(\gamma(t)) \ge \rho. \tag{1.25}$$

Next, we will estimate the level c. This will be a very delicate result.

Lemma 1.3.1 *Suppose that (1.7) holds, then*

$$0 < \mathrm{c} < \frac{\mathcal{S}_*^{n/2}}{n f(x_0)^{(n-2)/2}}, \tag{1.26}$$

for some par $(\tilde{u}, \tilde{v}) \in H$, where c is defined in (1.25).

Proof. Initially, we will verify that there exists $(\overline{u}, \overline{v}) \in H$ such that

$$\mathrm{Q}(\overline{u}, \overline{v}) < \frac{\mathcal{S}_*}{(\max_M f)^{2/2^*}}, \tag{1.27}$$

where Q is defined by

$$\mathrm{Q}(u,v) := \frac{\int_M \left(|\nabla u|_g^2 + |\nabla v|_g^2\right) dv_g + \int_M \left(a(x)u^2 + 2b(x)uv + c(x)v^2\right) dv_g}{\left(\int_M f(x)|u|^\alpha |v|^\beta dv_g\right)^{2/2^*}}$$

for $(u, v) \in H$ with $\int_M f(x)|u|^\alpha|v|^\beta dv_g > 0$.

The proof will be done considering the cases $n \ge 4$ and $n = 3$.

Let $x_0 \in M$ be a point such that $f(x_0) = \max\{f(x) : x \in M\}$. We denote by $B_\delta(x_0)$ the geodesic ball of center x_0 and radius δ, with $\delta \in (0, i_g)$, where i_g is the injective radius of (M, g). We choose δ small enough if necessary such that $f(x) > 0$ on $B_{2\delta}(x_0)$. In normal coordinates we can write the following expansions

$$\begin{aligned} h(x)\eta(r)^2 &= h(x_0) + r^\theta O(1), \\ f(x)\eta(r)^{2^*} &= f(x_0) + \frac{1}{2}\partial_{ij} f(x_0) x^i x^j + r^3 O(1), \\ \int_{\mathbb{S}^{n-1}} \sqrt{\det(g)} d\sigma &= \omega_{n-1}\left(1 - \frac{R_g(x_0)}{6n} r^2 + r^4 O(1)\right), \end{aligned} \tag{1.28}$$

where $\det(g)$ is the determinant of the components of the metric g (the third expression can be seen in [27, Chapter 6, p. 283]) and $h = \frac{\alpha}{2^*}a + \frac{2\sqrt{\alpha\beta}}{2^*}b + \frac{\beta}{2^*}c$, with $\theta \in (0, 1)$ such that $h \in C^{0,\theta}(M)$, and $\eta \in C_0^\infty([-2\delta, 2\delta])$, with $\eta = 1$ in $[-\delta, \delta]$ and $0 \le \eta \le 1$.

Now, for $n \geq 4$ and $\epsilon > 0$, we consider the following family of functions

$$u_\epsilon(x) = \frac{\eta(d_g(x,x_0))}{(\epsilon + d_g(x,x_0)^2)^{(n-2)/2}}. \tag{1.29}$$

For $0 < p, q < \infty$, we put $I_q^p := \int_0^\infty t^p(1+t)^{-q}dt$, and then it holds that

$$\begin{aligned} &\frac{n-2}{n} I_n^{n/2} = I_n^{(n-2)/2} = \frac{\omega_n}{2^{n-1}\omega_{n-1}}, \\ &\frac{(n-2)^2}{2}\omega_{n-1} I_n^{n/2} = K_n^{-2}\left(\frac{n-2}{2n}\omega_{n-1} I_n^{n/2}\right)^{2/2^*}. \end{aligned} \tag{1.30}$$

When $n = 4$, from [3, 4], we get

$$\begin{aligned} &\int_M |\nabla u_\epsilon(x)|_g^2 dv_g = 2\omega_3\epsilon^{-1}\left(I_4^2 + \frac{1}{24}R_g(x_0)\epsilon\ln\epsilon + o(\epsilon\ln\epsilon)\right), \\ &\int_M h(x)u_\epsilon(x)^2 dv_g = -\frac{\omega_3}{2}h(x_0)\ln\epsilon + o(\ln\epsilon), \\ &\int_M f(x)u_\epsilon(x)^4 dv_g = \frac{\omega_3}{2}f(x_0)I_4^1\epsilon^{-2}\left(1 - \frac{1}{12}R_g(x_0)\epsilon + o(\epsilon)\right). \end{aligned} \tag{1.31}$$

Now considering $\xi, \zeta > 0$ such that $\frac{\xi}{\zeta} = \sqrt{\frac{\alpha}{\beta}}$, we obtain

$$\begin{aligned} \mathrm{Q}(\xi u_\epsilon, \zeta u_\epsilon) &= \frac{(\xi^2+\zeta^2)\int_M |\nabla u_\epsilon|_g^2 dv_g + \int_M \left[\xi^2 a(x) + 2\xi\zeta b(x) + \zeta^2 c(x)\right] u_\epsilon^2 dv_g}{(\xi^\alpha\zeta^\beta)^{2/2^*}\left(\int_M f(x)u_\epsilon^{2^*}dv_g\right)^{2/2^*}} \\ &= \frac{(\xi^2+\zeta^2)}{(\xi^\alpha\zeta^\beta)^{2/2^*}} \frac{\int_M |\nabla u_\epsilon|_g^2 dv_g + \int_M h(x)u_\epsilon^2 dv_g}{\left(\int_M f(x)u_\epsilon^{2^*}dv_g\right)^{2/2^*}}. \end{aligned}$$

Then, by (1.31) and (1.30), it follows that (for ϵ small enough):

$$\begin{aligned} \mathrm{Q}(\xi u_\epsilon, \zeta u_\epsilon) &= \frac{(\xi^2+\zeta^2)}{(\xi^\alpha\zeta^\beta)^{1/2}} \frac{\left[\omega_3 I_4^2 + \frac{\omega_3}{12}\epsilon\ln\epsilon\,(R_g(x_0) - 6h(x_0)) + o(\epsilon\ln\epsilon)\right]}{\left[\frac{\omega_3}{2}f(x_0)I_4^1(1 - \frac{1}{12}R_g(x)\epsilon + o(\epsilon))\right]^{1/2}} \\ &= \frac{\kappa(\alpha,\beta)}{f(x_0)^{1/2}} \frac{\left[2\omega_3 I_4^2 + \frac{\omega_3}{12}\epsilon\ln\epsilon\,(R_g(x_0) - 6h(x_0)) + o(\epsilon\ln\epsilon)\right]}{K_4^2 2\omega_3 I_4^2\left[1 - \frac{1}{24}R_g(x)\epsilon + o(\epsilon)\right]} \\ &= \frac{\kappa(\alpha,\beta)}{f(x_0)^{1/2}}\left\{K_4^{-2} + \frac{\frac{\omega_3}{12}\epsilon\ln\epsilon\,(R_g(x_0) - 6h(x_0)) + o(\epsilon\ln\epsilon) + \frac{1}{24}R_g(x_0)\epsilon + o(\epsilon)}{K_4^2 2\omega_3 I_4^2\left[1 - \frac{1}{24}R_g(x)\epsilon + o(\epsilon)\right]}\right\} \\ &= \frac{\kappa(\alpha,\beta)}{f(x_0)^{1/2}}\left\{K_4^{-2} + \frac{\frac{\omega_3}{12}\epsilon\ln\epsilon\,(R_g(x_0) - 6h(x_0)) + o(\epsilon\ln\epsilon)}{K_4^2 2\omega_3 I_4^2\left[1 - \frac{1}{24}R_g(x)\epsilon + o(\epsilon)\right]}\right\} \\ &= \frac{\kappa(\alpha,\beta)}{f(x_0)^{1/2}}\left\{K_4^{-2} + \frac{\frac{\omega_3}{12}\epsilon\ln\epsilon\,(R_g(x_0) - 6h(x_0))\left[1 - \epsilon O(1)\right] + \epsilon^2\ln(\epsilon)O(1) + o(\epsilon\ln\epsilon)}{K_4^2 2\omega_3 I_4^2\left[1 - \epsilon O(1)\right]}\right\} \\ &= \frac{\kappa(\alpha,\beta)}{f(x_0)^{1/2}}\left\{K_4^{-2} + \frac{K_4^{-2}}{24 I_4^2}\epsilon\ln\epsilon\,(R_g(x_0) - 6h(x_0)) + \frac{o(\epsilon\ln\epsilon)}{K_4^2 2\omega_3 I_4^2\left[1 - \frac{1}{24}R_g(x)\epsilon + o(\epsilon)\right]}\right\}, \end{aligned}$$

where $\kappa(\alpha,\beta) = \left[\left(\frac{\alpha}{\beta}\right)^{\beta/2^*} + \left(\frac{\beta}{\alpha}\right)^{\alpha/2^*}\right]$ and we use that $(1+t)^q = 1 + qt + \cdots + \frac{q(q-1)\dots(q-i+1)}{i!}t^i + \cdots$, for all $|t| < 1$, where $q \in \mathbb{R}$. Consequently, as $I_4^2 = \int_0^\infty \frac{t^2}{(1+t)^4}dt = \frac{1}{3}$ we reach

$$\mathrm{Q}(\xi u_\epsilon, \zeta u_\epsilon) \leq \frac{\mathcal{S}_*}{f(x_0)^{1/2}} + \frac{\mathcal{S}_*}{8f(x_0)^{1/2}}\epsilon \ln\epsilon\,(R_g(x_0) - 6h(x_0)) + o(\epsilon\ln\epsilon). \tag{1.32}$$

For $n > 4$, from [3, 4], we have

$$= \tfrac{(n-2)^2}{2} I_n^{n/2}\omega_{n-1}\epsilon^{(2-n)/2}\left(1 - \tfrac{n+2}{6n(n-4)}R_g(x_0)\epsilon + o(\epsilon)\right),$$

$$\int_M h(x)u_\epsilon^2 dv_g = \tfrac{2(n-2)(n-1)}{n(n-4)}\omega_{n-1}I_n^{n/2}h(x_0)\epsilon^{(4-n)/2} + o(\epsilon^{(4-n)/2}),$$

$$\int_M f(x)u_\epsilon^{2^*} dv_g = \tfrac{\omega_{n-1}}{2} f(x_0) I_n^{(n-2)/2}\epsilon^{-n/2}\left(1 - \tfrac{1}{2(n-2)}\left(-\frac{\Delta_g f(x_0)}{f(x_0)} + \frac{R_g(x_0)}{3}\right)\epsilon + o(\epsilon)\right).$$

Thus, similarly to what we did above, we find that

$$\begin{aligned}\mathrm{Q}(\xi u_\epsilon, \zeta u_\epsilon) \leq{}& \frac{\mathcal{S}_*}{f(x_0)^{2/2^*}} \\ &- \frac{\mathcal{S}_*}{(n-4)nf(x_0)^{2/2^*}}\left(\frac{(n-4)}{2}\frac{\Delta_g f(x_0)}{f(x_0)} + R_g(x_0) - \frac{4(n-1)}{(n-2)}h(x_0)\right)\epsilon + o(\epsilon).\end{aligned} \tag{1.33}$$

Now, we recall, by (1.7), that

$$h(x_0) < \frac{n-2}{4(n-1)}R_g(x_0) + \frac{(n-4)(n-2)}{8(n-1)}\frac{\Delta_g f(x_0)}{f(x_0)},$$

for $n \geq 4$. Then, by (1.32) and (1.33), it follows, for ϵ sufficient small, that (1.27) holds.

Now, we consider the case $n = 3$. As a, b and c satisfy the condition (1.6), it follows that $-\Delta_g + h$ is a coercive operator. Then we can consider $G_{x_0} : M\backslash\{x_0\} \longrightarrow \mathbb{R}$ the Green function of this operator, that is,

$$-\Delta_g G_{x_0} + hG_{x_0} = \delta_{x_0},$$

where δ_{x_0} is the Dirac mass at x_0. It is well known that for x close to x_0 we can write

$$G_{x_0}(x) = \frac{1}{\omega_2 d_g(x,x_0)} + m(x_0) + o(1).$$

Next, we will use Druet's idea [16]. By using the cut-off function η, we can write G_{x_0} as follows:

$$\omega_2 G_{x_0}(x) = \frac{\eta(d_g(x,x_0))}{d_g(x,x_0)} + w_h(x), \tag{1.34}$$

where $w_h \in C^\infty_{loc}(M\backslash\{x_0\})$. In $M\backslash B_\delta(x_0)$, we have

$$-\Delta_g w_h + h w_h = \Delta_g\left(\frac{\eta}{d_g(x,x_0)}\right) - h\frac{\eta}{d_g(x,x_0)}. \tag{1.35}$$

And, in $B_\delta(x_0)$, we write in normal coordinates

$$-\Delta_g w_h + h w_h = -\frac{\partial_r(\ln(\det(g)))}{2d_g(x,x_0)^2} - h\frac{1}{d_g(x,x_0)}. \tag{1.36}$$

In particular, we have that the right side of the above equation is in $L^s(M)$ for all $1 < s < 3$, so by standard elliptic theory, $w_h \in C^0(M) \cap H^1(M)$ and moreover $w_h(x_0) = \omega_2 m(x_0)$ (for more details see Druet [16]).

As we have assumed that $h \leq \frac{1}{8}R_g$ (see (1.7)), there exists $\overline{G}_{x_0}$ the Green function of $-\Delta_g + \frac{1}{8}R_g$, and as above we can write

$$\omega_2 \overline{G}_{x_0}(x) = \frac{\eta(d_g(x,x_0))}{d_g(x,x_0)} + \overline{w}(x). \tag{1.37}$$

Now, note that

$$\begin{aligned} -\Delta_g(\overline{w} - w_h) + \frac{1}{8}R_g(\overline{w} - w_h) &= -\omega_2\Delta_g(\overline{G}_{x_0} - G_{x_0}) + \omega_2\frac{1}{8}R_g(\overline{G}_{x_0} - G_{x_0}) \\ &= \left(h - \frac{1}{8}R_g\right)\omega_2 G_{x_0} \leq 0. \end{aligned}$$

Green's Formula and the hypothesis $h \leq \frac{1}{8}R_g$ (but not equal) gives us

$$(\overline{w} - w_h)(y) = \int_M \overline{G}_y(x)\left(h(x) - \frac{1}{8}R_g(x)\right)\omega_2 G_{x_0}(x)dv_g < 0, \tag{1.38}$$

so, $\overline{w}(y) < w_h(y)$, for all $y \in M$, in particular, as $\overline{w}(x_0) = \omega_2\overline{m}(x_0) \geq 0$ it follows that $w_h(x_0) > 0$ (here $\overline{m}(x_0)$ is given by the expansion of $\overline{G}_{x_0}$ in a neighborhood of x_0, and the positive mass theorem garantee that $\overline{m}(x_0) \geq 0$, see [44, 45].

For $\epsilon > 0$ and $x \in M$, we define the function

$$v_\epsilon(x) = \epsilon^{1/4}(u_\epsilon(x) + w_h(x)),$$

where u_ϵ is the test-function defined as (1.29).

As we did in case $n \geq 4$, we estimate $\mathrm{Q}(\xi v_\epsilon, \zeta v_\epsilon)$. For this we will estimate $\int_M(|\nabla v_\epsilon|_g^2 + hv_\epsilon^2)dg$ and $\int_M f(x)v_\epsilon^6 dv_g$. First, note that

$$\begin{aligned} \int_M(|\nabla v_\epsilon|_g^2 + hv_\epsilon^2)dg &= \int_M[v_\epsilon(-\Delta_g v_\epsilon) + hv_\epsilon^2]dv_g \\ &= \epsilon^{1/2}\int_M[U_\epsilon^2\eta(-\Delta_g\eta) - \eta\langle\nabla\eta, \nabla U_\epsilon^2\rangle_g + h\eta^2U_\epsilon^2]dv_g \\ &\quad + \epsilon^{1/2}\int_M \eta^2 U_\epsilon(-\Delta_g U_\epsilon)dv_g + \epsilon^{1/2}\int_M(-\Delta_g w_h + hw_h)(w_h + 2\eta U_\epsilon)dv_g, \end{aligned} \tag{1.39}$$

where $U_\epsilon(x) = \frac{1}{(\epsilon+d_g(x,x_0)^2)^{1/2}}$.

Note that we can write

$$U_\epsilon^2(x) = \frac{1}{d_g(x,x_0)^2} - \frac{\epsilon}{d_g(x,x_0)^2(\epsilon + d_g(x,x_0)^2)}. \tag{1.40}$$

With that we calculate:

$$\epsilon^{1/2}\int_M U_\epsilon^2 \eta(-\Delta_g \eta)dv_g = \epsilon^{1/2}\int_M \frac{\eta(-\Delta_g\eta)}{d_g(x,x_0)^2}dv_g + o(\epsilon^{1/2}), \tag{1.41}$$

$$\epsilon^{1/2}\int_M \eta\langle\nabla\eta, \nabla U_\epsilon^2\rangle_g dv_g = \epsilon^{1/2}\int_M \eta\left\langle\nabla\eta, \nabla\left(\frac{1}{d_g(x,x_0)^2}\right)\right\rangle_g dv_g + o(\epsilon^{1/2}), \tag{1.42}$$

$$\epsilon^{1/2}\int_M h\eta^2 U_\epsilon^2 dv_g = \epsilon^{1/2}\int_M h\frac{\eta^2}{d_g(x,x_0)^2}dv_g + o(\epsilon^{1/2}). \tag{1.43}$$

Now, as in normal coordinates the Laplacian of a radial function F can be written as follows $-\Delta_g F = \frac{1}{r^{n-1}\sqrt{\det(g)}}\partial_r(r^{n-1}\sqrt{\det(g)}\partial_r F)$, we have

$$-\Delta_g U_\epsilon = -\Delta U_\epsilon - \partial_r(\ln\sqrt{\det(g)})\partial_r U_\epsilon, \tag{1.44}$$

where $-\Delta$ is the Euclidean Laplacian. Since $-\Delta U_\epsilon = 3\epsilon U_\epsilon^5$, and using (1.40), we get that

$$\begin{aligned}\int_{B_\delta(x_0)} \eta^2 U_\epsilon(-\Delta_g U_\epsilon)dv_g &= \int_{B_\delta(0)} U_\epsilon(-\Delta U_\epsilon - \partial_r(\ln\sqrt{\det(g)})\partial_r U_\epsilon)\sqrt{\det(g)}dx \\ &= 3\epsilon\int_{B_\delta(0)} U_\epsilon^6 dx + O(\epsilon^{1/2}) + \int_{B_\delta(x_0)} \frac{\partial_r(\ln\det(g))\partial_r(U_\epsilon^2)}{4}dv_g \\ &= 3\epsilon^{-1/2}\omega_2\int_0^\infty \frac{s^2}{(1+s^2)^3}ds + \int_{B_\delta(x_0)}\frac{\partial_r(\ln\det(g))}{2d_g(x,x_0)^3}dv_g + O(\epsilon^{1/2}).\end{aligned}$$

So,

$$\int_{B_\delta(x_0)} \eta^2 U_\epsilon(-\Delta_g U_\epsilon)dv_g = \frac{3}{2}\omega_2 I_3^{1/2}\epsilon^{-1/2} + \int_{B_\delta(x_0)}\frac{\partial_r(\ln\det(g))}{2d_g(x,x_0)^3}dv_g + O(\epsilon^{1/2}). \tag{1.45}$$

Now, writing that

$$U_\epsilon(x) = \frac{1}{d_g(x,x_0)} - \frac{\epsilon}{d_g(x,x_0)(\epsilon+d_g(x,x_0)^2)^{1/2}[d_g(x,x_0) + (\epsilon+d_g(x,x_0)^2)^{1/2}]}, \tag{1.46}$$

we have

$$\int_{M\setminus B_\delta(x_0)} \eta^2 U_\epsilon(-\Delta_g U_\epsilon)dv_g = -\int_{M\setminus B_\delta(x_0)}\frac{\eta^2}{d_g(x,x_0)}\Delta_g\left(\frac{1}{d_g(x,x_0)}\right)dv_g + O(\epsilon^{1/2}). \tag{1.47}$$

So, we get that

$$\epsilon^{1/2}\int_M \eta^2 U_\epsilon(-\Delta_g U_\epsilon)dv_g = \frac{3}{2}\omega_2 I_3^{1/2} + \epsilon^{1/2}\int_{B_\delta(x_0)} \frac{\partial_r(\ln\det(g))}{2d_g(x,x_0)^3}dv_g \\ -\epsilon^{1/2}\int_{M\setminus B_\delta(x_0)} \frac{\eta^2}{d_g(x,x_0)}\Delta_g\left(\frac{1}{d_g(x,x_0)}\right)dv_g + o(\epsilon^{1/2}). \tag{1.48}$$

Finally, we calculate

$$\int_M(-\Delta_g w_h + hw_h)(w_h + 2\eta U_\epsilon)dv_g = \int_M(-\Delta_g w_h + hw_h)\left(w_h + \frac{2\eta}{d(x,x_0)}\right)dv_g \\ + \int_M(-\Delta_g w_h + hw_h) \\ \left(w_h + \frac{2\eta}{d_g(x,x_0)(\epsilon + d_g(x,x_0)^2)^{1/2}[d_g(x,x_0) + (\epsilon + d_g(x,x_0)^2)^{1/2}]}\right)dv_g,$$

first, by (1.34), we have

$$\int_M(-\Delta_g w_h + hw_h)\left(w_h + \frac{\eta}{d(x,x_0)}\right)dv_g = \int_M(-\Delta_g w_h + hw_h)\omega_2 G_{x_0}dv_g = \omega_2 w_h(x_0). \tag{1.49}$$

Second, we get from equations (1.35) and (1.36) that

$$\int_M(-\Delta_g w_h + hw_h)\left(\frac{\eta}{d(x,x_0)}\right)dv_g \\ = -\int_{B_\delta(x_0)}\left(\frac{\partial_r(\ln(\det(g)))}{2d_g(x,x_0)^2} + \frac{h}{d_g(x,x_0)}\right)\left(\frac{1}{d(x,x_0)}\right)dv_g \\ + \int_{M\setminus B_\delta(x_0)}\left(\Delta_g\left(\frac{\eta}{d_g(x,x_0)}\right) - \frac{h\eta}{d_g(x,x_0)}\right)\left(\frac{\eta}{d(x,x_0)}\right)dv_g,$$

so, we have

$$\int_M(-\Delta_g w_h + hw_h)\left(\frac{\eta}{d(x,x_0)}\right)dv_g = -\int_M \frac{h\eta^2}{d_g(x,x_0)^2} - \int_{B_\delta(x_0)}\left(\frac{\partial_r(\ln(\det(g)))}{2d_g(x,x_0)^3}\right)dv_g \\ + \int_{M\setminus B_\delta(x_0)}\left[\frac{\eta^2}{d_g(x,x_0)}\Delta_g\left(\frac{1}{d(x,x_0)}\right) + \frac{\eta}{d_g(x,x_0)^2}\Delta_g\eta\right]dv_g \\ + \int_{M\setminus B_\delta(x_0)}\eta\left\langle\nabla\eta,\nabla\left(\frac{1}{d_g(x,x_0)^2}\right)\right\rangle_g dv_g \tag{1.50}$$

Now, using the obtained in (1.41), (1.42), (1.43), (1.48) and (1.50) in the equation (1.39), gives us the following estimate

$$\int_M(|\nabla_g v_\epsilon|_g^2 + hv_\epsilon^2)dv_g = \frac{3}{2}\omega_2 I_3^{1/2} + \omega_2 w_h(x_0)\epsilon^{1/2} + o(\epsilon^{1/2}). \tag{1.51}$$

Now, we estimate $\int_M f(x)v_\epsilon^6 dv_g$.

$$\int_M f(x)v_\epsilon^6 dv_g = \epsilon^{3/2}\int_M f[u_\epsilon^6 + 6u_\epsilon^5 w_h + 15u_\epsilon^4 w_h^2 + 20u_\epsilon^3 w_h^3 + 15u_\epsilon^2 w_h^4 + 6u_\epsilon w_h^5 + w_h^6]dv_g \\ = \epsilon^{3/2}\int_M f[u_\epsilon^6 + 6u_\epsilon^5 w_h + 15u_\epsilon^4 w_h^2]dv_g + o(\epsilon^{1/2}). \tag{1.52}$$

Using the expansion (1.28) in normal coordinate,

$$\begin{aligned}\int_M f(x)u_\epsilon^6 dv_g =& f(x_0)\int_{B_\delta(x_0)} U_\epsilon^6 dx + O(1)\int_{B_\delta(x_0)} U_\epsilon^6 r^2 dx + \int_{B_{2\delta}(x_0)\setminus B_\delta(x_0)} f(x)u_\epsilon^6 dv_g \\ =& \frac{\omega_2}{2} f(x_0)\epsilon^{-3/2} I_3^{1/2} + O(\epsilon^{-1/2}) + O(1),\end{aligned}$$

so, we have

$$\int_M f(x)u_\epsilon^6 dv_g = \frac{\omega_2}{2} f(x_0)\epsilon^{-3/2} I_3^{1/2} + O(\epsilon^{-1/2}). \tag{1.53}$$

Similarly, we get

$$\int_M 6f(x)u_\epsilon^5 w_h dv_g = 3\omega_2 f(x_0)w_h(x_0)\epsilon^{-1} I_{5/2}^{1/2} + o(\epsilon^{-1/2}). \tag{1.54}$$

Also, we calculate

$$\int_M 15 f(x)u_\epsilon^4 w_h^2 dv_g = 15\omega_2 f(x_0)w_h(x_0)^2\epsilon^{-1/2} I_2^{1/2} + o(\epsilon^{-1/2}). \tag{1.55}$$

From what was obtained in (1.53)-(1.55) and the fact that $\omega_2 I_{5/2}^{1/2} = 2\int_{\mathbb{R}^n} U_1^5 dx = 2\int_{\mathbb{R}^n}(-\Delta U_1)dx = \frac{2}{3}\omega_2$, we have that

$$\int_M f(x)v_\epsilon^6 dv_g = \frac{\omega_2}{2} f(x_0) I_3^{1/2} + 2\omega_2 w_h(x_0)f(x_0)\epsilon^{1/2} + o(\epsilon^{1/2}). \tag{1.56}$$

Now, we can calculate $\mathrm{Q}(\xi v_\epsilon, \zeta v_\epsilon)$ for ϵ small enough, by the equations (1.51) and (1.56),

$$\begin{aligned}\frac{\int_M(|\nabla v_\epsilon|_g^2 + h v_\epsilon^2)dg}{\left(\int_M f(x)v_\epsilon^6 dv_g\right)^{1/3}} &= \frac{\frac{3}{2}\omega_2 I_3^{1/2} + \omega_2 w_h(x_0)\epsilon^{1/2} + o(\epsilon^{1/2})}{\left(\frac{\omega_2}{2} f(x_0)I_3^{1/2} + 2\omega_2 w_h(x_0)f(x_0)\epsilon^{1/2} + o(\epsilon^{1/2})\right)^{1/3}} \\ &= \frac{\frac{3}{2}\omega_2 I_3^{1/2} + \omega_2 w_h(x_0)\epsilon^{1/2} + o(\epsilon^{1/2})}{f(x_0)^{1/3}\left(\frac{\omega_2}{2} I_3^{1/2}\right)^{1/3}\left(1 + \frac{4w_h(x_0)}{3I_3^{1/2}}\epsilon^{1/2} + o(\epsilon^{1/2})\right)},\end{aligned}$$

as $I_3^{1/2} = \frac{1}{3}I_3^{3/2}$ and $\left(\frac{1}{6}\omega_2 I_3^{3/2}\right)^{1/3} = \frac{K_3^2}{2}\omega_2 I_3^{3/2}$ (see (1.30)), we get that

$$\frac{\int_M(|\nabla v_\epsilon|_g^2 + h v_\epsilon^2)dg}{\left(\int_M f(x)v_\epsilon^6 dv_g\right)^{1/3}} = \frac{K_3^{-2}}{f(x_0)^{1/3}} - \frac{\omega_2 w_h(x_0)\epsilon^{1/2}}{\frac{\omega_2}{2} I_3^{3/2} + 2\omega_2 w_h(x_0)\epsilon^{1/2} + o(\epsilon^{1/2})}.$$

As $w_h(x_0) > 0$, then

$$\mathrm{Q}(\xi v_\epsilon, \zeta v_\epsilon) < \left[\left(\frac{\alpha}{\beta}\right)^{\beta/6} + \left(\frac{\beta}{\alpha}\right)^{\alpha/6}\right]\frac{K_3^{-2}}{f(x_0)^{1/3}}. \tag{1.57}$$

Therefore, we obtain (1.27), when $n = 3$.

Now, in order to prove (1.26), we define for any $t > 0$ the following functional:

$$\Phi(t) = \begin{cases} I\left(t\xi u_\epsilon, t\zeta u_\epsilon\right), & \text{when } n \geq 4 \\ I\left(t\xi v_\epsilon, t\zeta v_\epsilon\right), & \text{when } n = 3 \end{cases}$$
$$= \begin{cases} \frac{t^2}{2} X_{u_\epsilon} - \frac{t^{2^*}}{2^*} Y_{u_\epsilon}, & \text{when } n \geq 4 \\ \frac{t^2}{2} X_{v_\epsilon} - \frac{t^{2^*}}{2^*} Y_{v_\epsilon}, & \text{when } n = 3, \end{cases}$$

where $X_u = (\xi^2+\zeta^2) \int_M \left[|\nabla u|_g^2 + (\xi^2 a + 2\xi\zeta b + \zeta^2 c)\, u^2\right] dv_g$ and $Y_u = \xi^\alpha \zeta^\beta \int_M f u^{2^*} dv_g$.

We want to find $t_0 > 0$ such that $\Phi'(t_0) = 0$, that is, such that $t_0 X - t_0^{2^*-1} Y = 0$. Hence,

$$t_0 = \begin{cases} \left(\dfrac{X_{u_\epsilon}}{Y_\epsilon}\right)^{(n-2)/4}, & \text{when } n \geq 4 \\ \left(\dfrac{X_{v_\epsilon}}{Y_{v_\epsilon}}\right)^{1/4}, & \text{when } n = 3. \end{cases}$$

Therefore, t_0 is the only critical point of Φ and since $\Phi(t) \to -\infty$ as $t \to \infty$, then t_0 is a maximum point for Φ.

Note that, by the above calculations, we get

$$\begin{aligned} \Phi(t_0) &= \begin{cases} \frac{1}{n}\left(\mathrm{Q}(\xi u_\epsilon(x), \zeta u_\epsilon(x))\right)^{n/2}, & \text{when } n \geq 4 \\ \frac{1}{3}\left(\mathrm{Q}(\xi v_\epsilon, \zeta v_\epsilon)\right)^{3/2}, & \text{when } n = 3 \end{cases} \\ &< \frac{\mathcal{S}_*^{n/2}}{n f(x_0)^{(n-2)/2}}. \end{aligned}$$

Choose $t_1 > t_0$ large such that $\Phi(t_1) < 0$ and write $\tilde{u} = t_1 \xi u_\epsilon(x)$ and $\tilde{v} = t_1 \zeta u_\epsilon(x)$ when $n \geq 4$ (and $\tilde{u} = t_1 \xi v_\epsilon$ and $\tilde{v} = t_1 \zeta v_\epsilon$ when $n = 3$). So,

$$\begin{aligned} 0 < \mathrm{c} = \inf_{\gamma\in\Gamma} \sup_{0\leqslant t\leq 1} I(\gamma(t)) &\leq \sup_{0\leqslant t\leq 1} I\left(tt_1\xi u_\epsilon(x), tt_1\zeta u_\epsilon(x)\right) \ (\text{we use } v_\epsilon \text{ if } n = 3) \\ &= \sup_{0<t\leq 1} \Phi(tt_1). \\ &\leq \Phi(t_0), \end{aligned}$$

which proves (1.26). This completes the proof. ■

We now have the tools for the proof of Theorem 1.1.1.

<u>*Proof of Theorem 1.1.1.*</u> By the General Minimax Principle [49, Theorem 2.8], there is a sequence $\{(u_m, v_m)\}$ in H such that

$$I(u_m, v_m) \to \mathrm{c} \quad \text{and} \quad I'(u_m, v_m) \to 0. \tag{1.58}$$

Now note that

$$I(u_m, v_m) - \frac{1}{2^*}I'(u_m,v_m)\cdot(u_m, v_m)$$
$$= \frac{1}{n}\int_M \left[|\nabla u_m|_g^2 + |\nabla v_m|_g^2 + a(x)u_m^2 + 2b(x)u_m v_m + c(x)v_m^2\right] dv_g.$$

Thus, by the coercivity hypothesis (1.6), we obtain that $\{(u_m, v_m)\}$ is bounded in H. Hence, there exists (u_0, v_0) in H such that, up to a subsequence,

$$\begin{aligned} &(u_m, v_m) \rightharpoonup (u_0, v_0) \ \text{ in } \ H;\\ &(u_m, v_m) \to (u_0, v_0) \ \text{ in } \ L^2(M)\times L^2(M);\\ &(u_m(x), v_m(x)) \to (u_0(x), v_0(x)) \ \text{ a.e in } \ M.\end{aligned} \tag{1.59}$$

It is easy to see that $f(x)|u_m|^{\alpha-2}u_m|v_m|^{\beta}$ is an uniformly bounded sequence in $L^{\frac{2^*}{2^*-1}}(M)$ and converges pointwisely to $f(x)|u_0|^{\alpha-2}u_0|v_0|^{\beta}$, from Lemma 4.8 in [36], we have

$$f(x)|u_m|^{\alpha-2}u_m|v_m|^{\beta} \rightharpoonup f(x)|u_0|^{\alpha-2}u_0|v_0|^{\beta} \ \text{ in } \ L^{\frac{2^*}{2^*-1}}(M). \tag{1.60}$$

Similarly we obtain the same for the sequence $f(x)|u_m|^{\alpha}v_m|v_m|^{\beta-2}$. As $I'(u_m, v_m)\cdot(\varphi,\psi) = o_m(1)$, for all $(\varphi,\psi)\in H$, by using (1.59), (1.60) and letting $m\to\infty$, we reach that $I'(u_0, v_0) = 0$, that is, (u_0, v_0) is a weak solution of (1.1).

The next step is to prove that $u_0 \neq 0$ and $v_0 \neq 0$.

First, let us see that $u_0 = 0$, if and only if, $v_0 = 0$. Indeed, if $u_0 = 0$, then $-\Delta_g v_0 + c(x)v_0 = 0$ in M. So by coercivity hypothesis (1.6), we have that $v_0 = 0$.

If $u_0 = 0$ and $v_0 = 0$, we write $\tau = \lim_{m\longrightarrow 0}\int_M \left(|\nabla u_m|_g^2 + |\nabla v_m|_g^2\right) dv_g$. Since $I'(u_m, v_m)\cdot(u_m, v_m) = o_m(1)$, then we get

$$\lim_{m\to\infty}\int_M f(x)|u_m|^{\alpha}|v_m|^{\beta}dv_g = \lim_{m\to\infty}\int_M \left(|\nabla u_m|_g^2 + |\nabla v_m|_g^2\right) dv_g = \tau.$$

On the other hand, since $I(u_m, v_m) = \mathrm{c} + o_m(1)$, then we get $\tau = n\mathrm{c}$.

Now, by Lemma 1.2.1, we know that there is a positive constant B_0 such that

$$\left(\int_M f(x)|u_m|^{\alpha}|v_m|^{\beta}dv_g\right)^{2/2^*} \le f(x_0)^{(n-2)/n}\left[\mathcal{S}_*^{-1}\left(\|(\nabla u_m, \nabla v_m)\|_2^2\right) + B_0\|(u_m, v_m)\|_2^2\right].$$

Thus, passing to the limit in the inequality above and using (1.59), we get $(n\mathrm{c})^{2/2^*} \le f(x_0)^{(n-2)/n}\mathcal{S}_*^{-1}n\mathrm{c}$. Hence,

$$\mathrm{c} \ge \frac{\mathcal{S}_*^{n/2}}{nf(x_0)^{(n-2)/2}}.$$

But, this contradicts the estimate obtained for the level c in Lemma 1.3.1. Therefore, $u_0 \neq 0$ and $v_0 \neq 0$. Thus, we conclude the proof of Theorem 1.1.1. ■

Proof of Corollary 1.1.2. Consider the functional $J : H \to \mathbb{R}$ defined by

$$J(u,v) = \frac{1}{2}\int_M \left[|\nabla u|_g^2 + |\nabla v|_g^2 + a(x)u^2 + 2b(x)uv + c(x)v^2\right] dv_g - \frac{1}{2^*}\int_M f(x)(u^+)^\alpha (v^+)^\beta dv_g.$$

This functional satisfies the same properties of I. Using the same test functions to estimate the minimax level and using the same steps as in the previous proof, one obtains that there exists $(u_0, v_0) \in H$ a nontrivial critical point of the functional J. Now, we will prove that u_0 and v_0 are positive solutions. First, we denote by $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$. Then, since $J'(u_0, v_0) \cdot (u_0^-, v_0^-) = 0$, we get

$$\begin{aligned} 0 =& \int_M \left[|\nabla u_0^-|_g^2 + |\nabla v_0^-|_g^2 + a(x)(u_0^-)^2 + b(x)[u_0 v_0^- + u_0^- v_0] + c(x)(v_0^-)^2\right] dv_g \\ &- \int_M f(x)\left[\alpha (u_0^+)^{\alpha-1}(v_0^+)^\beta u_0^+ u_0^- + \beta (u^+)^\alpha (v^+)^{\beta-1} v_0^+ v_0^-\right] dv_g \\ =& \int_M \left[|\nabla u_0^-|_g^2 + |\nabla v_0^-|_g^2 + a(x)(u_0^-)^2 + b(x)[u_0 v_0^- + u_0^- v_0] + c(x)(v_0^-)^2\right] dv_g \\ =& \int_M \left[|\nabla u_0^-|_g^2 + |\nabla v_0^-|_g^2 + a(x)(u_0^-)^2 + 2b(x)u_0^- v_0^- + c(x)(v_0^-)^2\right] dv_g \\ &+ \int_M b(x)[u_0^+ v_0^- + u_0^- v_0^+] dv_g. \end{aligned}$$

As $b \leq 0$ and $u_0^+ v_0^- + u_0^- v_0^+ \leq 0$, we deduce that

$$\int_M \left[|\nabla u_0^-|_g^2 + |\nabla v_0^-|_g^2 + a(x)(u_0^-)^2 + 2b(x)u_0^- v_0^- + c(x)(v_0^-)^2\right] dv_g \leq 0,$$

and consequently by (1.6), we reach $u_0^- = 0$ and $v_0^- = 0$. Therefore, $u_0 \geq 0$ and $v_0 \geq 0$. By elliptic regularity theory and maximum principle follows that $u_0 > 0$ and $v_0 > 0$. ■

1.4 Proof of Theorems 1.1.6, 1.1.7 and 1.1.8

In this section, we will study the case where the combination $h = \frac{\alpha}{2^*}a + \frac{2\sqrt{\alpha\beta}}{2^*}b + \frac{\beta}{2^*}c$ is less than or equal to $\frac{n-2}{4(n-1)}R_g$. We will begin by recalling some notations and definitions. Considering the functional $E_h : H \to \mathbb{R}$ given by

$$E_h(u,v) = \int_M \left(|\nabla u|_g^2 + |\nabla v|_g^2\right) dv_g + \int_M \left(au^2 + 2buv + cv^2\right) dv_g.$$

and constraint set $\Lambda_f^{\alpha,\beta} := \left\{(u,v) \in H \,:\, \int_M f(x)|u|^\alpha |v|^\beta dv_g = 1\right\}$.

Note that E_h is bounded from below on $\Lambda_f^{\alpha,\beta}$. Indeed, by the coercivity condition (1.6) and Corollary 1.2.2, we have

$$E_h(u,v) \geq C_0\|(u,v)\|^2 \geq C\left(\int_M |u|^\alpha|v|^\beta dv_g\right)^{2/2^*} \geq \frac{C}{f(x_0)^{2/2^*}},$$

for all $(u,v) \in \Lambda_f^{\alpha,\beta}$. Thus, we can consider

$$S_{f,h}^{(\alpha,\beta)} = \inf_{(u,v)\in\Lambda_f^{\alpha,\beta}} E_h(u,v). \tag{1.61}$$

If there exists $(u,v) \in \Lambda_f^{\alpha,\beta}$ which achieves the infimum $S_{f,h}^{(\alpha,\beta)}$, it turns out that (u,v) will be a weak solution of the following system

$$\begin{cases} -\Delta_g u + a(x)u + b(x)v &= S_{f,h}^{(\alpha,\beta)}\dfrac{2\alpha}{2^*}f(x)u|u|^{\alpha-2}|v|^\beta \text{ in } M, \\ -\Delta_g v + b(x)u + c(x)v &= S_{f,h}^{(\alpha,\beta)}\dfrac{2\beta}{2^*}f(x)v|v|^{\beta-2}|u|^\alpha \text{ in } M. \end{cases} \tag{1.62}$$

In order to achieve the existence result we need to recall some results due to Escobar-Schoen [21], Aubin-Hebey [5] and Hebey-Vaugon [30] for Prescribe scalar curvature problem, which prove that f is the scalar curvature of a conformal metric (see also [4]).

Before, let us remember that, when $\max_M f > 0$ it is known that $\lambda_f(M,g) \leq \frac{\lambda(\mathbb{S}^n)}{(\max_M f)^{2/2^*}}$, where $\lambda_f(M,g)$ is defined in (1.12), and if $\lambda_f(M,g) < \frac{\lambda(\mathbb{S}^n)}{(\max_M f)^{2/2^*}}$, then there is $\varphi \in C^\infty$ with $\varphi > 0$, $\int_M f(x)\varphi^{2^*}dv_g = 1$, and such that

$$\lambda_f(M,g) = \int_M \left(|\nabla\varphi|_g^2 + \frac{n-2}{4(n-1)}R_g\varphi^2\right)dv_g,$$

that is, φ is a positive solution of the equation $-\Delta_g u + \frac{n-2}{4(n-1)}R_g u = \lambda_f(M,g)fu^{2^*-1}$. Therefore, $\hat{g} = \varphi^{2^*-2}g$ is a conformal metric to g, where $f = R_{\hat{g}}$ is the scalar curvature of the metric $\hat{g}$, and moreover, $\lambda_f(M,\hat{g}) = \lambda_f(M,g)$.

Theorem A 1 (Escobar-Schoen [21]) *Let f be a C^∞ function with $\max_M f > 0$ on a compact riemannian manifold (M,g) not conformal to the sphere with the standard metric. Then if $n = 3$,*

$$\lambda_f(M,g) < \frac{\lambda(\mathbb{S}^n)}{(\max_M f)^{2/2^*}},$$

and consequently f is the scalar curvature of a conformal metric. The same conclusion holds for the locally conformally flat manifolds when $n \geq 4$ if at a point x_0 where f is maximal, all its derivatives up to order $n-2$ vanish.

Theorem B 1 (Aubin-Hebey [5]) *Assume that $n \geq 6$ and (M, g) is not locally conformally flat. Let f be a smooth function with $\max_M f > 0$. If at a point x_0 where $f(x_0) = \max_M f$ is such that the Weyl tensor is nonvanishing (that is, $|W_g(x_0)| \neq 0$) and $\Delta_g f(x_0) = 0$, then if $n = 6$,*

$$\lambda_f(M, g) < \frac{\lambda(\mathbb{S}^n)}{(\max_M f)^{2/2^*}},$$

and consequently f is the scalar curvature of a conformal metric. When $n > 6$ the same conclusion holds. If in addition $\left|\Delta_g^2 f(x_0)\right| / f(x_0)$ is small enough.

Theorem C 1 (Hebey-Vaugon [30]) *Let f be a C^∞ function satisfying $\max_M f > 0$ and $\Delta_g f(x_0) = 0$ at a point x_0 where f is maximum. Then*

$$\lambda_f(M, g) < \frac{\lambda(\mathbb{S}^n)}{(\max_M f)^{2/2^*}},$$

and consequently f is the scalar curvature of a conformal metric when $n = 4$ or 5, unless M is conformal to the standard $\mathbb{S}^n$. When $n \geq 6$ we suppose that $|W_g(x_0)| = 0$. The same conclusion holds if $\Delta_g^2 f(x_0) = 0$, when $n = 6$ or $n = 7$, and when $n = 8$ if in addition $\Delta_g^3 f(x_0) = 0$ or $|\nabla W_g(x_0)| \neq 0$. When $n > 8$ the same conclusion holds if $|\nabla W_g(x_0)| \neq 0$ and $\Delta_g^3 f(x_0) = 0$ or when $|\nabla W_g(x_0)| = 0$ if $|\nabla^2 W_g(x_0)| \neq 0$ and $\Delta_g^3 f(x_0) = \Delta_g^4 f(x_0) = 0$, or when all derivatives of W_g vanish at x_0 if $\Delta_g^m f(x_0) = 0$ for all $1 \leq m \leq \frac{n}{2} - 1$.

The next result is the first step to prove Theorems 1.1.7 and 1.1.8.

Lemma 1.4.1 *If $\lambda_f(M, g) < \frac{\lambda(\mathbb{S}^n)}{f(x_0)^{2/2^*}}$, then $S_{f,h}^{(\alpha,\beta)} < \dfrac{\mathcal{S}_*}{f(x_0)^{2/2^*}}$, where $\mathcal{S}_*$ is given in (1.4).*

Proof. Since $\lambda_f(M, g) < \frac{\lambda(\mathbb{S}^n)}{f(x_0)^{2/2^*}}$, from theorems A, B and C, there exists $\varphi \in C^\infty(M)$ with $\varphi > 0$, $\int_M f(x)\varphi^{2^*} dv_g = 1$ and such that

$$\lambda_f(M, g) = \int_M \left(|\nabla\varphi|_g^2 + \frac{n-2}{4(n-1)} R_g \varphi^2 \right) dv_g < \frac{\lambda(\mathbb{S}^n)}{f(x_0)^{2/2^*}}. \quad (1.63)$$

Now, consider the following pair of functions $(w_1, w_2) \in \Lambda_f^{\alpha,\beta}$, where $w_1 = \xi \left(\xi^\alpha \zeta^\beta\right)^{-1/2^*} \varphi$ and $w_2 = \zeta \left(\xi^\alpha \zeta^\beta\right)^{-1/2^*} \varphi$, with $\frac{\xi}{\zeta} = \sqrt{\frac{\alpha}{\beta}}$, thus

$$\begin{aligned} S_{f,h}^{(\alpha,\beta)} &\leq E_h(w_1, w_2) \\ &= \frac{(\xi^2 + \zeta^2) \int_M |\nabla\varphi|_g^2 dv_g + \int_M \left(\xi^2 a(x) + 2\xi\zeta b(x) + \zeta^2 c(x)\right) \varphi^2 dv_g}{\left(\xi^\alpha \zeta^\beta\right)^{2/2^*}} \\ &= \frac{(\xi^2 + \zeta^2)}{\left(\xi^\alpha \zeta^\beta\right)^{2/2^*}} \left\{ \int_M |\nabla\varphi|_g^2 dv_g + \int_M \left(\frac{\alpha}{2^*} a(x) + \frac{2\sqrt{\alpha\beta}}{2^*} b(x) + \frac{\beta}{2^*} c(x) \right) \varphi^2 dv_g \right\}. \end{aligned}$$

As $h = \frac{\alpha}{2^*}a + \frac{2\sqrt{\alpha\beta}}{2^*}b + \frac{\beta}{2^*}c \leq \frac{n-2}{4(n-1)}R_g$, it follows that

$$S_{f,h}^{(\alpha,\beta)} \leq \left[\left(\frac{\alpha}{\beta}\right)^{\beta/2^*} + \left(\frac{\beta}{\alpha}\right)^{\alpha/2^*}\right]\lambda_f(M,g). \tag{1.64}$$

Consequently,

$$S_{f,h}^{(\alpha,\beta)} < \left[\left(\frac{\alpha}{\beta}\right)^{\beta/2^*} + \left(\frac{\beta}{\alpha}\right)^{\alpha/2^*}\right]\frac{\lambda(\mathbb{S}^n)}{f(x_0)^{2/2^*}},$$

hence $S_{f,h}^{(\alpha,\beta)} < \frac{\mathcal{S}_*}{f(x_0)^{2/2^*}}$ as desired. Finishing the proof. ■

We will now prove the second auxiliary result of this section.

Lemma 1.4.2 *If $S_{f,h}^{(\alpha,\beta)} < \dfrac{\mathcal{S}_*}{f(x_0)^{2/2^*}}$, then there exists (u,v) in $\Lambda_f^{\alpha,\beta}$ such that $E_h(u,v) = S_{f,h}^{(\alpha,\beta)}$.*

Proof. Let $\{(u_m,v_m)\} \subset \Lambda_f^{\alpha,\beta}$ be a minimizing sequence for $S_{f,h}^{(\alpha,\beta)}$, that is,

$$E_h(u_m,v_m) = \|(|\nabla u_m|,|\nabla v_m|)\|_2^2 + \int_M (au_m^2 + 2bu_mv_m + cv_m^2)dv_g = S_{f,h}^{(\alpha,\beta)} + o_m(1), \tag{1.65}$$

where $o_m(1) \to 0$ as $m \to \infty$. By the coercivity hypothesis (1.6), it follows that $\{(u_m,v_m)\}$ is bounded in H. Thus, there exists (u,v) in H such that, up to a subsequence, $(u_m,v_m) \rightharpoonup (u,v)$ in H, $(u_m,v_m) \to (u,v)$ in $L^2(M)\times L^2(M)$, and $(u_m(x),v_m(x)) \to (u(x),v(x))$ a.e in M. From Lemma 1.2.1 and (1.65), we get

$$\begin{aligned}
1 &= \left(\int_M f(x)|u_m|^\alpha|v_m|^\beta dv_g\right)^{2/2^*} \\
&\leq f(x_0)^{2/2^*}\mathcal{S}_*^{-1}\|(|\nabla u_m|,|\nabla v_m|)\|_2^2 + f(x_0)^{2/2^*}B_0||(u_m,v_m)||_2^2 \\
&\leq f(x_0)^{2/2^*}\mathcal{S}_*^{-1}S_{f,h}^{(\alpha,\beta)} + f(x_0)^{2/2^*}B_0\|(u_m,v_m)\|_2^2 \\
&\quad - f(x_0)^{2/2^*}\mathcal{S}_*^{-1}\int_M (au_m^2 + 2bu_mv_m + cv_m^2)dv_g + o_m(1),
\end{aligned}$$

for some $B_0 > 0$. Letting $m \to \infty$, we obtain that

$$1 \leq f(x_0)^{2/2^*}\mathcal{S}_*^{-1}S_{f,h}^{(\alpha,\beta)} + f(x_0)^{2/2^*}B_0\|(u,v)\|_2^2 - f(x_0)^{2/2^*}\mathcal{S}_*^{-1}\int_M (au^2 + 2buv + cv^2)dv_g,$$

consequently, since $S_{f,h}^{(\alpha,\beta)} < \frac{\mathcal{S}_*}{f(x_0)^{2/2^*}}$, we get that

$$0 < 1 - f(x_0)^{2/2^*}\mathcal{S}_*^{-1}S_{f,h}^{(\alpha,\beta)} \leq f(x_0)^{2/2^*}B_0\|(u,v)\|_2^2 - f(x_0)^{2/2^*}\mathcal{S}_*^{-1}\int_M (au^2 + 2buv + cv^2)dv_g,$$

thus, we have that $u \neq 0$ or $v \neq 0$.

We claim that $u \neq 0$ and $v \neq 0$. Moreover, $(u, v) \in \Lambda_f^{\alpha,\beta}$ is a minimizing for $S_{f,h}^{(\alpha,\beta)}$. Indeed, rewriting (1.65), we have

$$E_h(u,v) + \|(|\nabla(u_m - u)|, |\nabla(v_m - v)|)\|_2^2 = S_{f,h}^{(\alpha,\beta)} + o_m(1). \tag{1.66}$$

On the other hand, since $1 = \int_M f(x)|u_m|^\alpha |v_m|^\beta dv_g$, Lemma 1.2.3 gives us

$$1 = \left(\int_M f(x)|u|^\alpha |v|^\beta dv_g + \int_M f(x)|u_m - u|^\alpha |v_m - v|^\beta dv_g + o_m(1) \right)^{2/2^*}. \tag{1.67}$$

Now, note that $\int_M f(x)|u|^\alpha |v|^\beta dv_g > 0$, otherwise, by (1.67) and Lemma 1.2.1, we would have

$$\begin{aligned} 1 &\leq \left(\int_M f(x)|u_m - u|^\alpha |v_m - v|^\beta dv_g \right)^{2/2^*} + o_m(1) \\ &\leq f(x_0)^{2/2^*} \mathcal{S}_*^{-1} \|(|\nabla_g(u_m - u)|, |\nabla_g(v_m - v)|)\|_2^2 + o_m(1), \end{aligned}$$

hence,

$$S_{f,h}^{(\alpha,\beta)} \leq f(x_0)^{2/2^*} \mathcal{S}_*^{-1} S_{f,h}^{(\alpha,\beta)} \|(|\nabla_g(u_m - u)|, |\nabla_g(v_m - v)|)\|_2^2 + o_m(1).$$

But, using the inequality above in (1.66), we get

$$\begin{aligned} E_h(u,v) &= S_{f,h}^{(\alpha,\beta)} - \|(|\nabla(u_m - u)|, |\nabla(v_m - v)|)\|_2^2 + o_m(1) \\ &\leq (f(x_0)^{2/2^*} \mathcal{S}_*^{-1} S_{f,h}^{(\alpha,\beta)} - 1) \|(|\nabla(u_m - u)|, |\nabla(v_m - v)|)\|_2^2 + o_m(1), \end{aligned}$$

again as $S_{f,h}^{(\alpha,\beta)} < \frac{\mathcal{S}_*}{f(x_0)^{2/2^*}}$, we reach that $E_h(u,v) \leq 0$, and so $u = v = 0$, which is a contradiction. Therefore, $\int_M f(x)|u|^\alpha |v|^\beta dv_g > 0$.

Now, returning to (1.67) we get

$$\begin{aligned} 1 &= \left(\int_M f(x)|u|^\alpha |v|^\beta dv_g + \int_M f(x)|u_m - u|^\alpha |v_m - v|^\beta dv_g + o_m(1) \right)^{2/2^*} \\ &\leq \left(\int_M f(x)|u|^\alpha |v|^\beta dv_g \right)^{2/2^*} + f(x_0)^{2/2^*} \left(\int_M |u_m - u|^\alpha |v_m - v|^\beta dv_g \right)^{2/2^*} + o_m(1) \\ &\leq \left(\int_M f(x)|u|^\alpha |v|^\beta dv_g \right)^{2/2^*} + f(x_0)^{2/2^*} \mathcal{S}_*^{-1} \|(|\nabla(u_m - u)|, |\nabla(v_m - v)|)\|_2^2 + o_m(1), \end{aligned}$$

as $S_{f,h}^{(\alpha,\beta)} > 0$, then

$$\begin{aligned} S_{f,h}^{(\alpha,\beta)} \leq\; & S_{f,h}^{(\alpha,\beta)} \left(\int_M f(x)|u|^\alpha |v|^\beta dv_g \right)^{2/2^*} \\ & + S_{f,h}^{(\alpha,\beta)} f(x_0)^{2/2^*} \mathcal{S}_*^{-1} \|(|\nabla(u_m - u)|, |\nabla(v_m - v)|)\|_2^2 + o_m(1). \end{aligned} \tag{1.68}$$

By using (1.66), it follows that

$$E_h(u,v) \leq S_{f,h}^{(\alpha,\beta)} \left(\int_M f(x)|u|^\alpha |v|^\beta dv_g \right)^{2/2^*}$$
$$+ (S_{f,h}^{(\alpha,\beta)} f(x_0)^{2/2^*} \mathcal{S}_*^{-1} - 1) \| (|\nabla_g (u_m - u)|, |\nabla_g (v_m - v)|) \|_2^2 + o_m(1).$$

Since $S_{f,h}^{(\alpha,\beta)} f(x_0)^{2/2^*} \mathcal{S}_*^{-1} - 1 < 0$, we have

$$E_h(u,v) \leq S_{f,h}^{(\alpha,\beta)} \left(\int_M f(x)|u|^\alpha |v|^\beta dv_g \right)^{2/2^*}.$$

The lower semicontinuity of E_h implies $E_h(u,v) \leq \liminf E_h(u_m, v_m) = S_{f,h}^{(\alpha,\beta)}$, and hence $0 < \tau = \int_M f(x)|u|^\alpha |v|^\beta dv_g \leq 1$, now writing $u_0 = \tau^{-1/2^*} u$ and $v_0 = \tau^{-1/2^*} v$, we have

$$E_h(u_0, v_0) = \frac{E_h(u,v)}{\left(\int_M f(x)|u|^\alpha |v|^\beta dv_g \right)^{2/2^*}} \leq S_{f,h}^{(\alpha,\beta)},$$

with $(u_0, v_0) \in \Lambda_f^{\alpha,\beta}$. By definition of $S_{f,h}^{(\alpha,\beta)}$ it follows that $E_h(u_0, v_0) = S_{f,h}^{(\alpha,\beta)}$, so we prove that $E_h(u,v) = S_{f,h}^{(\alpha,\beta)} \tau^{2/2^*}$.

Finally, we can check that $\tau = \int_M f(x)|u|^\alpha |v|^\beta dv_g = 1$, for this, we return to (1.66) and (1.68). Then

$$1 \leq \left(\int_M f(x)|u|^\alpha |v|^\beta dv_g \right)^{2/2^*} + f(x_0)^{2/2^*} \mathcal{S}_*^{-1} \left[S_{f,h}^{(\alpha,\beta)} - E_h(u,v) \right]$$
$$= \left(\int_M f(x)|u|^\alpha |v|^\beta dv_g \right)^{2/2^*} + f(x_0)^{2/2^*} \mathcal{S}_*^{-1} S_{f,h}^{(\alpha,\beta)} \left[1 - \left(\int_M f(x)|u|^\alpha |v|^\beta dv_g \right)^{2/2^*} \right].$$

Hence,

$$0 \leq - \left[1 - \left(\int_M f(x)|u|^\alpha |v|^\beta dv_g \right)^{2/2^*} \right]$$
$$+ f(x_0)^{2/2^*} \mathcal{S}_*^{-1} S_{f,h}^{(\alpha,\beta)} \left[1 - \left(\int_M f(x)|u|^\alpha |v|^\beta dv_g \right)^{2/2^*} \right]$$
$$= \left(-1 + f(x_0)^{2/2^*} \mathcal{S}_*^{-1} S_{f,h}^{(\alpha,\beta)} \right) \left[1 - \left(\int_M f(x)|u|^\alpha |v|^\beta dv_g \right)^{2/2^*} \right]$$

As $f(x_0)^{2/2^*} S_{f,h}^{(\alpha,\beta)} < \mathcal{S}_*$, then $\int_M f(x)|u|^\alpha |v|^\beta dv_g = 1$.

Consequenlty, we get that $(u,v) \in \Lambda_f^{\alpha,\beta}$, which proves that (u,v) is a minimizer for $S_{f,h}^{(\alpha,\beta)}$. ■

Now we can prove the main results of this section.

Proof of Theorem 1.1.6. Since $S_{f,h}^{(\alpha,\beta)} < \frac{\mathcal{S}_*}{f(x_0)^{2/2^*}}$, by Lemma 1.4.2 there exists $(u_0, v_0) \in \Lambda_f^{\alpha,\beta}$ such that $E_h(u_0, v_0) = S_{f,h}^{(\alpha,\beta)}$. Denote by $G(u,v) = \int_M f(x)|u|^\alpha |v|^\beta dv_g - 1$, where $(u,v) \in H$. Then, there is a Lagrange multiplier λ that satisfies

$$E_h'(u_0, v_0) \cdot (\varphi, \psi) - \lambda G'(u_0, v_0) \cdot (\varphi, \psi) = 0, \text{ for all } (\varphi, \psi) \in H. \tag{1.69}$$

Taking $\varphi = u_0$ and $\psi = v_0$ above, we have that $2E_h(u_0, v_0) = 2^*\lambda$, hence $\lambda = \frac{2}{2^*} S_{f,h}^{(\alpha,\beta)} > 0$. Therefore, by (1.69), we have that (u_0, v_0) is a weak solution of the following system

$$\begin{cases} -\Delta_g u + au + bv & = & S_{f,h}^{(\alpha,\beta)} \frac{2\alpha}{2^*} f(x) u |u|^{\alpha-2} |v|^\beta \text{ in } M, \\ -\Delta_g v + bu + cv & = & S_{f,h}^{(\alpha,\beta)} \frac{2\beta}{2^*} f(x) v |v|^{\beta-2} |u|^\alpha \text{ in } M. \end{cases} \tag{1.70}$$

It is easy to see that the pair $((2S_{f,h}^{(\alpha,\beta)})^{1/(2^*-2)} u_0, (2S_{f,h}^{(\alpha,\beta)})^{1/(2^*-2)} v_0)$ is a weak solution of system 1.1. This completes the proof. ∎

Proof of Theorems 1.1.7 and 1.1.8. From **Theorem B** and **Theorem C** together with Lemma 1.4.1, it follows that $S_{f,h}^{(\alpha,\beta)} < \frac{\mathcal{S}_*}{f(x_0)^{2/2^*}}$. Thus the proof follows similar to Theorem 1.1.6. ∎

Let us introduce some notations before of the proof of Corollary 1.1.9. Let

$$\Lambda_{f,+}^{\alpha,\beta} := \left\{ (u,v) \in H \; : \; \int_M f(x)(u^+)^\alpha (v^+)^\beta dv_g = 1 \right\} \text{ and } S_{f,h,+}^{(\alpha,\beta)} := \inf_{(u,v) \in \Lambda_{f,+}^{\alpha,\beta}} E_h(u,v).$$

Then, if $b \leq 0$ in M, it is easy to see that $E_h(|u|, |v|) \leq E_h(u,v)$, so if $(u,v) \in \Lambda_f^{\alpha,\beta}$ then $(|u|, |v|) \in \Lambda_{f,+}^{\alpha,\beta}$, and therefore, we deduce that $S_{f,h,+}^{(\alpha,\beta)} \leq S_{f,h}^{(\alpha,\beta)}$. Then, by Lemma 1.4.1, we have $S_{f,h,+}^{(\alpha,\beta)} < \frac{\mathcal{S}_*}{f(x_0)^{2/2^*}}$. Moreover, we claim that $S_{f,h}^{(\alpha,\beta)} > 0$, indeed,

$$\begin{aligned} E_h(u,v) \geq C_0 ||(u,v)||_H^2 &\geq C \left(\int_M |u|^\alpha |v|^\beta dv_g \right)^{2/2^*} \\ &\geq C \left(\int_M (u^+)^\alpha (v^+)^\beta dv_g \right)^{2/2^*} \geq \frac{C}{f(x_0)^{2/2^*}}, \end{aligned}$$

for all $(u,v) \in \Lambda_{f,+}^{\alpha,\beta}$.

Proof of Corollary 1.1.9. Let $\{(u_m, v_m)\} \subset \Lambda_{f,+}^{\alpha,\beta}$ be a minimizing sequence for $S_{f,h,+}^{(\alpha,\beta)}$. Arguing similarly to Lemma 1.4.2, we obtain a pair $(u,v) \in \Lambda_{f,+}^{\alpha,\beta}$ such that $E_h(u,v) = S_{f,h,+}^{(\alpha,\beta)}$, with $u \neq 0$ and $v \neq 0$, where $u_m \rightharpoonup u$ and $v_m \rightharpoonup v$ in $H^1(M)$. Now, we claim that

$u \geq 0$ and $v \geq 0$ in M. Indeed, if we consider $G_+(u,v) = \int_M f(x)(u^+)^\alpha (v^+)^\beta dv_g - 1$, there is a Lagrange multiplier λ such that

$$E'_h(u,v) \cdot (\varphi,\psi) - \lambda G'_+(u,v) \cdot (\varphi,\psi) = 0, \text{ for all } (\varphi,\psi) \in H. \tag{1.71}$$

Taking $\varphi = u^-$ and $\psi = v^-$ as test functions above, we have

$$2E_h(u^-,v^-) + 2\int_M b(u^+v^- + u^-v^+)dv_g = 0.$$

Since $b \leq 0$, it follows that $E_h(u^-,v^-) \leq 0$, hence $u^- = v^- = 0$. Thus, we conclude that $u \geq 0$ and $v \geq 0$. Considering $\varphi = u$ and $\psi = v$ as test functions in (1.71), we get $2E_h(u,v) = 2^*\lambda > 0$, and consequently $\lambda = \frac{2}{2^*} S^{(\alpha,\beta)}_{f,A,+} > 0$. Therefore, $((2S^{(\alpha,\beta)}_{f,h,+})^{1/(2^*-2)}u, (2S^{(\alpha,\beta)}_{f,h,+})^{1/(2^*-2)}v)$ is a weak positive solution of system (1.1), because the elliptic regularity theory gives us $u > 0$ and $v > 0$ in M. ■

Proof of Corollary 1.1.10. Here, we assume that $b = 0$, $a = c = \frac{(n-2)}{4(n-1)} R_g$ and $f \geq 0$. We claim that

$$S^{(\alpha,\beta)}_{f,h} = \left[\left(\frac{\alpha}{\beta} \right)^{\beta/2^*} + \left(\frac{\beta}{\alpha} \right)^{\alpha/2^*} \right] \lambda_f(M,g).$$

Indeed, from of the proof of Lemma 1.4.1 (see (1.64)), it is sufficient to prove that

$$S^{(\alpha,\beta)}_{f,h} \geq \left[\left(\frac{\alpha}{\beta} \right)^{\beta/2^*} + \left(\frac{\beta}{\alpha} \right)^{\alpha/2^*} \right] \lambda_f(M,g).$$

In order to achieve this goal, let $\{(u_m,v_m)\} \subset \Lambda^{\alpha,\beta}_f$ be a minimizing sequence for $S^{(\alpha,\beta)}_{f,h}$, that is,

$$\int_M \left(|\nabla_g u_m|^2_g + |\nabla_g v_m|^2_g + \frac{n-2}{4(n-1)} R_g \left(u_m^2 + v_m^2 \right) \right) dv_g = S^{(\alpha,\beta)}_{f,h} + o_m(1). \tag{1.72}$$

Define $w_m = t_m v_m$, where $t_m > 0$ is chosen so that

$$\int_M f(x)|u_m|^{2^*} dv_g = \int_M f(x)|w_m|^{2^*} dv_g.$$

By Young's inequality, we get that

$$\begin{aligned} t_m^\beta = \int_M f(x)|u_m|^\alpha |w_m|^\beta dv_g &\leq \frac{\alpha}{2^*}\int_M f(x)|u_m|^{2^*} dv_g + \frac{\beta}{2^*}\int_M f(x)|w_m|^{2^*} dv_g \\ &= \int_M f(x)|u_m|^{2^*} dv_g = \int_M f(x)|w_m|^{2^*} dv_g. \end{aligned} \tag{1.73}$$

Using (1.73) in (1.72), we have

$$\begin{aligned} S_{f,h}^{(\alpha,\beta)} + o_m(1) &= t_m^{2\beta/2^*} \frac{\int_M \left(|\nabla u_m|_g^2 + |\nabla v_m|_g^2\right) dv_g + \int_M \frac{n-2}{4(n-1)} R_g \left(u_m^2 + v_m^2\right) dv_g}{\left(\int_M f(x)|u_m|^\alpha |w_m|^\beta dv_g\right)^{2/2^*}} \\ &\geq t_m^{2\beta/2^*} \frac{\int_M \left(|\nabla u_m|_g^2 + \frac{n-2}{4(n-1)} R_g u_m^2\right) dv_g}{\left(\int_M f(x)|u_m|^{2^*} dv_g\right)^{2/2^*}} \\ &+ \quad t_m^{(2\beta/2^*)-2} \frac{\int_M \left(|\nabla w_m|_g^2 + \frac{n-2}{4(n-1)} R_g w_m^2\right) dv_g}{\left(\int_M f(x)|w_m|^{2^*} dv_g\right)^{\frac{2}{2^*}}} \\ &\geq (t_m^{2\beta/2^*} + t_m^{(2\beta/2^*)-2})\lambda_f(M,g). \end{aligned}$$

On the other hand, it is easy to see that $t^{2\beta/2^*} + t^{(2\beta/2^*)-2} \geq \left(\sqrt{\frac{\alpha}{\beta}}\right)^{2\beta/2^*} + \left(\sqrt{\frac{\beta}{\alpha}}\right)^{2\alpha/2^*}$, for all $t > 0$. Therefore,

$$S_{f,h}^{(\alpha,\beta)} \geq \left[\left(\frac{\alpha}{\beta}\right)^{\beta/2^*} + \left(\frac{\beta}{\alpha}\right)^{\alpha/2^*}\right] \lambda_f(M,g).$$

Thus, Corollary 1.1.10 follows by Lemma 1.4.2. ■

1.5 Case $\mathbb{S}^n$

Let $(\mathbb{S}^n, g_0)$ be the $n-$sphere, where g_0 is standard metric of $\mathbb{S}^n$. Due to the argument of Escobar and Schoen in [21] we can prove:

Lemma 1.5.1 *Let Γ be a nontrivial finite group of isometries of $\mathbb{S}^n$ acting without a fixed point on $\mathbb{S}^n$. Write $(M = \mathbb{S}^n/\Gamma, g)$, where g is the metric induced by $\pi : \mathbb{S}^n \longrightarrow \mathbb{S}^n/\Gamma$ covering map. Let $\overline{a}, \overline{b}, \overline{c}$ and $\overline{f}$ be functions in M satisfying the same assumptions of Theorem 1.1.8. Then we have that*

$$S_{\overline{f},\overline{h}}^{(\alpha,\beta)}(\mathbb{S}^n/\Gamma) < \frac{\mathcal{S}_*}{\overline{f}(x_0)^{2/2^*}}.$$

Proof. By hypotheses about Γ it is known that $M = \mathbb{S}^n/\Gamma$ is a compact Riemannian manifold locally conformally flat, which is not conformally diffeomorphic to $\mathbb{S}^n$. From **Theorem C**, we have $\lambda_{\overline{f}}(M,g) < \lambda(\mathbb{S}^n)/\overline{f}(x_0)^{2/2^*}$, and consequently

$$S_{\overline{f},\overline{h}}^{(\alpha,\beta)}(\mathbb{S}^n/\Gamma) < \frac{\mathcal{S}_*}{\overline{f}(x_0)^{2/2^*}}$$

as desired. ■

Proof of Theorem 1.1.11. By Lemma 1.4.2 and Lemma 1.5.1, it follows that there exists $(\overline{u}_0, \overline{v}_0) \in H$ weak solution of system (1.1) for $(M = \mathbb{S}^n/\Gamma, g)$. Since a, b, c and f are invariant under Γ (and recall that $\pi^* g = g_0$ and $\Delta_{g_0}(u \circ \pi) = (\Delta_g u) \circ \pi$, for $u \in C^2(M)$), then writing $u_0 = \overline{u}_0 \circ \pi$ and $v_0 = \overline{v}_0 \circ \pi$ we have that $(u_0, v_0) \in H^1(\mathbb{S}^n) \times H^1(\mathbb{S}^n)$ is a weak solution of the system

$$\begin{cases} -\Delta_{g_0} u + au + bv = \dfrac{\alpha}{2^*} f(x) u|u|^{\alpha-2}|v|^{\beta} & in \ \ \mathbb{S}^n, \\ -\Delta_{g_0} v + bu + cv = \dfrac{\beta}{2^*} f(x) v|v|^{\beta-2}|u|^{\alpha} & in \ \ \mathbb{S}^n, \end{cases} \tag{1.74}$$

which ends the proof of the theorem. ■

Proof of Corollary 1.1.13. As a consequence of the assumptions, from corollary 1.1.12 we immediately have that $S^{(\alpha,\beta)}(\mathbb{S}^n) = \mathcal{S}_*$. Let $\varphi \in C^\infty(\mathbb{S}^n)$ be a minimizer for $\lambda(\mathbb{S}^n)$, we can to see that $(\xi\varphi, \zeta\varphi)$ is a minimizer for $S^{(\alpha,\beta)}(\mathbb{S}^n)$. Indeed, notice that

$$\frac{\mathrm{Q}(\xi\varphi, \zeta\varphi)}{\left(\int_{\mathbb{S}^n} |\xi\varphi|^{\alpha}|\zeta\varphi|^{\beta} dv_{g_0}\right)^{2/2^*}} = \frac{(\xi^2 + \zeta^2)}{(\xi^\alpha \zeta^\beta)^{2/2^*}} \frac{\left(\|\nabla_{g_0}\varphi\|_2^2 + \frac{n(n-2)}{4}\|\varphi\|_2^2\right)}{\|\varphi\|_{2^*}^2} = \mathcal{S}_*. \tag{1.75}$$

So, $(\xi\varphi, \zeta\varphi)$ is a solution of the system

$$\begin{cases} -\Delta_{g_0} u + \frac{n(n-2)}{4} u & = & S^{(\alpha,\beta)}(\mathbb{S}^n) \dfrac{\alpha}{2^*} u|u|^{\alpha-2}|v|^{\beta} \ \text{ in } \ \mathbb{S}^n, \\ -\Delta_{g_0} v + \frac{n(n-2)}{4} v & = & S^{(\alpha,\beta)}(\mathbb{S}^n) \dfrac{\beta}{2^*} v|v|^{\beta-2}|u|^{\alpha} \ \text{ in } \ \mathbb{S}^n. \end{cases} \tag{1.76}$$

Hence the rescaling $((S^{(\alpha,\beta)}(\mathbb{S}^n))^{1/(2^*-2)}\xi\varphi, (S^{(\alpha,\beta)}(\mathbb{S}^n))^{1/(2^*-2)}\zeta\varphi)$ is solution of system (1.1). Therefore, we have infinite positive solutions for system (1.1), because for $x_0 \in \mathbb{S}^n$ fixed, and any $\rho > 1$, the functions

$$\varphi_{\rho,x_0}(x) = (\rho - \cos r)^{\frac{2-n}{2}} \tag{1.77}$$

are minimizer for $\lambda(\mathbb{S}^n)$, with $r = d_{g_0}(x, x_0)$ (for more details see Theorem 5.1 in [28]).

On the other hand, if (u, v) is a minimizer for $S^{(\alpha,\beta)}(\mathbb{S}^n)$, with $u, v \in C^\infty$, $u, v > 0$ and $\int_{\mathbb{S}^n} u^\alpha v^\beta dg_0 = 1$. Let $\sigma : \mathbb{S}^n \backslash \{P_N\} \longrightarrow \mathbb{R}^n$ be the stereographic projection, where P_N is the north pole of $\mathbb{S}^n$, since $(\sigma^{-1})^*(g_0) = U^{4/(n-2)} g_e$, where $U(y) = \left(\frac{2}{1+|y|^2}\right)^{(n-2)/2}$ and g_e is the Euclidian metric. So, we have

$$\begin{aligned} \mathcal{S}_* &= \int_{\mathbb{S}^n} [|\nabla u|_{g_0}^2 + |\nabla v|_{g_0}^2 + \frac{n(n-2)}{4}(u^2 + v^2)] dv_{g_0} \\ &= \int_{\mathbb{R}^n} [|\nabla[(u \circ \sigma^{-1})U]|^2 + |\nabla[(v \circ \sigma^{-1})U]|^2] dv_{g_e} \end{aligned} \tag{1.78}$$

and

$$\int_{\mathbb{S}^n} u^\alpha v^\beta dv_{g_0} = \int_{\mathbb{R}^n} [(u\circ\sigma^{-1})U]^\alpha [(v\circ\sigma^{-1})U]^\beta dv_{g_e} = 1. \tag{1.79}$$

Consequently, $(\overline{u},\overline{v})$ is a minimizer for $\mathcal{S}_*$, where $\overline{u} = (u\circ\sigma^{-1})U$ and $\overline{v} = (v\circ\sigma^{-1})U$, that is,

$$\begin{cases} -\Delta\overline{u} &= \mathcal{S}_*\dfrac{\alpha}{2^*}\overline{u}^{\alpha-1}\overline{v}^{\beta} \;\text{ in }\; \mathbb{R}^n,\\ -\Delta\overline{v} &= \mathcal{S}_*\dfrac{\beta}{2^*}\overline{u}^{\alpha}\overline{v}^{\beta-1} \;\text{ in }\; \mathbb{R}^n. \end{cases} \tag{1.80}$$

From Theorem 1.3 in [14], it follows that

$$\overline{u}(y) = \xi_1\left(\frac{\varepsilon_0}{\varepsilon_0^2+|y-y_0|^2}\right)^{(n-2)/2} \text{ and } \overline{v}(y) = \zeta_1\left(\frac{\varepsilon_0}{\varepsilon_0^2+|y-y_0|^2}\right)^{(n-2)/2}, \tag{1.81}$$

where $y_0\in\mathbb{R}^n$, $\varepsilon_0>0$ and $\xi_1,\zeta_1>0$ satisfying

$$\begin{aligned} \xi_1 n(n-2) &= \mathcal{S}_*\frac{\alpha}{2^*}\xi_1^{\alpha-1}\zeta_1^{\beta},\\ \zeta_1 n(n-2) &= \mathcal{S}_*\frac{\beta}{2^*}\xi_1^{\alpha}\zeta_1^{\beta-1},\\ \xi_1^{\alpha}\zeta_1^{\beta} &= \left[\frac{n(n-2)}{\lambda(\mathbb{S}^n)}\right]^{\frac{n}{2}}, \end{aligned}$$

so, a simple calculation gives us

$$\begin{aligned} \xi_1^2 &= \left(\frac{\alpha}{\beta}\right)^{\beta/2^*}\left[\frac{n(n-2)}{\lambda(\mathbb{S}^n)}\right]^{(n-2)/2},\\ \zeta_1^2 &= \left(\frac{\beta}{\alpha}\right)^{\alpha/2^*}\left[\frac{n(n-2)}{\lambda(\mathbb{S}^n)}\right]^{(n-2)/2},\\ \overline{u} &= \frac{\xi_1}{\zeta_1}\overline{v} = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}}\overline{v}. \end{aligned}$$

Therefore, by definition of $\overline{u}$ and $\overline{v}$ we get that $u = \frac{\xi_1}{\zeta_1}v$. Then $\xi_1^{-1}u$ is positive solution (up to a rescaling) of the equation $-\Delta_{g_0}w + \frac{n(n-2)}{4}w = w^{2^*-1}$ in $\mathbb{S}^n$. From Theorem 5.1 in [28] then up to a constant scale factor, u is of the following form, $u(x) = \xi_1(\rho_0-\cos r)^{\frac{2-n}{2}}$, so, $v(x) = \zeta_1(\rho_0-\cos r)^{\frac{2-n}{2}}$, where $r = d_{g_0}(x,x_0)$ and $\rho_0>1$. This complete the proof. ■

Chapter 2

On a Class of Hardy-Sobolev Elliptic Systems with Critical Exponents

This chapter is dedicated to the study of a class of critical Hardy-Sobolev systems. To this end, we establish a general existence theorem in which we employ minimization under constraint, along with an optimal Hardy-Sobolev inequality and a Brézis-Lieb type lemma with a singular weight. By estimating each term within the associated energy functional, we derive sufficient conditions for applying the general existence theorem. Here, our inspiration draws from [35, 42].

2.1 Introduction

Let (M, g) be a smooth closed Riemannian manifold of dimension $n \geq 3$. We are concerned with the existence of solutions of the following Hardy-Sobolev system:

$$\begin{cases} -\Delta_g u + \frac{\tilde{a}(x)}{\rho(x)^\theta} u + a(x)u + b(x)v = \dfrac{\alpha}{2^\star(s)} \frac{f(x)}{d_g(x,x_0)^s} u|u|^{\alpha-2}|v|^\beta & \text{in } M, \\ -\Delta_g v + \frac{\tilde{c}(x)}{\rho(x)^\gamma} v + b(x)u + c(x)v = \dfrac{\beta}{2^\star(s)} \frac{f(x)}{d_g(x,x_0)^s} v|v|^{\beta-2}|u|^\alpha & \text{in } M, \end{cases} \tag{2.1}$$

where Δ_g is the Laplace-Beltrami operator, $a, b, c, \tilde{a}, \tilde{c} \in C^{0,\varrho}(M)$, for some $\varrho \in (0, 1)$, $x_0 \in M$, $s \in [0, 2)$ and $\theta, \gamma \in (0, 2)$, ρ is a nonnegative continuous function such that $\rho(x) \approx d_g(x, x_0)$ for x near of x_0, $f \in C^\infty(M)$ with $f(x_0) = \max_M f > 0$ and $\alpha > 1$, $\beta > 1$ are real constants satisfying $\alpha + \beta = 2^\star(s)$, where $2^\star(s) = \frac{2(n-s)}{n-2}$ is the critical Hardy-Sobolev exponent.

Next, we would like to mention some works that are strongly related to the system we propose to study. We start with the Chapter 1 this thesis. The second is the paper due to Jaber [35], which the author considers elliptic Hardy-Sobolev equation

$$-\Delta_g u + a(x)u = \frac{|u|^{2^\star(s)-2}u}{d_g(x,x_0)^s} \quad \text{in} \quad M,$$

he proves an optimal Hardy-Sobolev inequality in the context of Riemannian manifolds and a existence result for this equation. Another paper that also motivated this study was the work of Madani [42], who considered the equation (Yamabe problem with singularities)

$$-\Delta_g u + a(x)u = \tilde{R}|u|^{2^\star(s)-2}u \quad \text{in} \quad M,$$

where $a \in L^q(M)$, with $q > \frac{n}{2}$ and $\tilde{R} \in \mathbb{R}$. The author proves an Hardy inequality on compact manifolds and existence result, in particular, when $a \equiv \frac{n-2}{4(n-1)}R_g$ with $R_g = R/d_g(\cdot,P)^\varrho$ for $P \in M$ a singular point for R_g of order $\varrho < 2$ and $R \in C^0(M)$. In addition, we would like to mention some works on coupled systems involving singular nonlinearities or singular potentials in the Euclidean domain, for example Huang and Kang [33], Long and Yang [41] and Mohammed and Yasmina [6].

In the present chapter, our interest is to examine the influence or not of the scalar curvature for the existence of solutions for the system (2.1) when we consider a compact Riemannian manifold without boundary.

Again, we will work in space $H = H^1(M) \times H^1(M)$ endowed with the norm

$$\|(u,v)\| = \left(\|u\|_{H^1}^2 + \|v\|_{H^1}^2\right)^{1/2}.$$

As it is know $2^\star(s) = \frac{2(n-s)}{n-2}$ is the critical exponet of the continuous embedding of $H^1(M)$ in the weighted Lebesgue space $L^q_{d_g,s}(M) := L^q(M, d_g(\cdot,x_0)^{-s})$ for $1 \le q \le 2^\star(s)$, which is a compact embedding when $1 \le q < 2^\star(s)$. Moreover, for any $\vartheta \in (0,2)$ the embedding $H^1(M) \hookrightarrow L^2(M,\rho^{-\vartheta})$ is compact (see for more details in [42, Theorem 1.3]). We equip $L^q_{d_s,s}(M)$ and $L^2(M,\rho^{-\vartheta})$ with the norm

$$||u||_{q,s} = \left(\int_M \frac{|u|^q}{d_g(x,x_0)^s} dv_g\right)^{\frac{1}{q}} \quad \text{and} \quad ||u||_{2,\rho^\vartheta} = \left(\int_M \frac{u^2}{\rho^\vartheta(x)} dv_g\right)^{\frac{1}{2}}.$$

In this context, we say that a pair of functions $(u,v) \in H$ is a weak solution of

(2.1), if for every $(\varphi,\psi)\in H$, it holds

$$\int_M (\langle\nabla_g u,\nabla_g\varphi\rangle + \langle\nabla_g v,\nabla_g\psi\rangle + \tfrac{\tilde{a}(x)}{\rho(x)^\theta}u\varphi + \tfrac{\tilde{c}(x)}{\rho(x)^\gamma}v\psi + au\varphi + b[u\psi+v\varphi] + cv\psi)dv_g$$
$$= \int_M \tfrac{\alpha}{2^\star(s)}\tfrac{f(x)}{d_g(x,x_0)^s}|u|^{\alpha-2}|v|^\beta u\varphi dv_g + \int_M \tfrac{\beta}{2^\star(s)}\tfrac{f(x)}{d_g(x,x_0)^s}|v|^{\beta-2}|u|^\alpha v\psi dv_g.$$

Recall of the relation (1.3) obtained by Alves et al. [2, Section 4] when $s=0$ from which borrowing the idea we obtain an general relation for $s\in[0,2)$ which is the following:

$$\mathcal{K}^s_{(\alpha,\beta)} = \left[\left(\frac{\alpha}{\beta}\right)^{\beta/2^\star(s)} + \left(\frac{\beta}{\alpha}\right)^{\alpha/2^\star(s)}\right]K(n,s)^{-1}, \tag{2.2}$$

where $K(n,s)$ is the best Hardy-Sobolev constant defined by

$$K(n,s)^{-1} = \inf_{u\in H^1(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n}|\nabla u|^2dx}{\left(\int_{\mathbb{R}^n}\frac{|u|^{2^\star(s)}}{|x|^s}dx\right)^{\frac{2}{2^\star(s)}}},$$

and $\mathcal{K}^s_{(\alpha,\beta)}$ is defined by

$$\mathcal{K}^s_{(\alpha,\beta)} = \inf_{(u,v)\in[H^1(\mathbb{R}^n)]^2\setminus\{0\}} \frac{\int_{\mathbb{R}^n}\left(|\nabla u|^2+|\nabla v|^2\right)dx}{\left(\int_{\mathbb{R}^n}\frac{|u|^\alpha|v|^\beta}{|x|^s}dx\right)^{\frac{2}{2^\star(s)}}}. \tag{2.3}$$

When $\alpha+\beta=2^\star(s)$.

Throughout this chapter, we make certain broad assumptions about the functions, which will enable us to derive existence results for the system (2.1) using variational methods. We assume that the functions $a,b,c,\tilde{a},\tilde{c}$, and ρ satisfy:

$(\mathcal{H}_1)$ Coercivity condition, that is, there exists $C_0>0$ such that

$$\int_M \left(|\nabla_g u|^2 + |\nabla_g v|^2 + a(x)u^2 + 2buv + c(x)v^2 + \tfrac{\tilde{a}(x)}{\rho(x)^\theta}u^2 + \tfrac{\tilde{c}(x)}{\rho(x)^\gamma}v^2\right)dv_g \geq C_1\|(u,v)\|^2,$$

for all $(u,v)\in H$.

$(\mathcal{H}_2)$ The function ρ satisfies:

$$(i)\ \frac{\rho(x)}{d_g(x,x_0)} = 1 + O(d_g(x,x_0)^\mu),\ \forall x\in B_\delta(x_0);$$
$$(ii)\ \rho(x)>0,\quad \forall x\in M\setminus B_\delta(x_0).$$

For some $\delta\in(0,i_g)$ (here i_g denotes the injectivity radius of (M,g)), where $\mu\in(0,1)$.

Thus, our first result in this chapter can be stated as follows:

Teorema 2.1.1 *Let (M,g) be a closed Riemannian manifold of dimension $n \geq 3$. Let $a, b, c, \tilde{a}, \tilde{c}$ and ρ be functions in M satisfying $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, with $\tilde{a}, \tilde{c} \in C^\infty(M)$. Let f be smooth function such that $f(x_0) = \max_M f > 0$. In addition if $h := \frac{\alpha}{2^\star(s)}a + \frac{2\sqrt{\alpha\beta}}{2^\star(s)}b + \frac{\beta}{2^\star(s)}c$, $\tilde{a}$ and $\tilde{c}$ satisfy:*

$$\begin{aligned} &(1)\ h(x_0) < \tfrac{(n-2)(6-s)}{12(2n-2-s)}R_g(x_0) + \tfrac{(n-2)(n-4)}{4(2n-2-s)}\tfrac{\Delta_g f(x_0)}{f(x_0)} \text{ and } \tilde{a}(x_0) = \tilde{c}(x_0) = 0, \text{ if } n \geq 4;\\ &(2)\ \text{when } n = 3,\ h(x_0) < \tfrac{1}{8}R_g(x_0) \text{ and } h \leq \tfrac{1}{8}R_g \text{ in } M, \text{ or } h \equiv \tfrac{1}{8}R_g \text{ and } (M,g)\\ &\quad \text{is not conformally equivalent to the standard sphere } \mathbb{S}^3, \text{ and that } \tilde{a} \equiv \tilde{c} \equiv 0. \end{aligned} \tag{2.4}$$

Then, system (2.1) has a pair of nontrivial weak solutions.

The first consequence of Theorem 2.1.1 is the following result.

Corollary 2.1.2 *Assuming the same assumptions of Theorem 2.1.1 with $\frac{\alpha}{2^\star(s)}a + \frac{\beta}{2^\star(s)}c$ instead of h. If in addition, we assume that $b \leq 0$ in M. Then, system (2.1) has a pair of non-negative nontrivial weak solutions.*

Another consequence is when we assume the same hypotheses of Theorem 2.1.1 with $\tilde{a} \equiv \tilde{c} \equiv 0$, that is, we obtain that the following system:

$$\begin{cases} -\Delta_g u + a(x)u + b(x)v = \dfrac{\alpha}{2^\star(s)} \frac{f(x)}{d_g(x,x_0)^s} u|u|^{\alpha-2}|v|^\beta & \text{in } M,\\ -\Delta_g v + b(x)u + c(x)v = \dfrac{\beta}{2^\star(s)} \frac{f(x)}{d_g(x,x_0)^s} v|v|^{\beta-2}|u|^\alpha & \text{in } M, \end{cases} \tag{2.5}$$

has a pair of nontrivial weak solutions. More precisely, we have the following result.

Corollary 2.1.3 *Let (M,g) be a closed Riemannian manifold of dimension $n \geq 3$. We assume that $\tilde{a} \equiv \tilde{c} \equiv 0$. Let a, b, c and ρ be functions in M satisfying $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$. Let f be smooth function such that $f(x_0) = \max_M f > 0$. In addition if $h := \frac{\alpha}{2^\star(s)}a + \frac{2\sqrt{\alpha\beta}}{2^\star(s)}b + \frac{\beta}{2^\star(s)}c$ satisfies:*

$$\begin{aligned} &(1)\ h(x_0) < \tfrac{(n-2)(6-s)}{12(2n-2-s)}R_g(x_0) + \tfrac{(n-2)(n-4)}{4(2n-2-s)}\tfrac{\Delta_g f(x_0)}{f(x_0)}, \text{ if } n \geq 4;\\ &(2)\ \text{when } n = 3,\ h(x_0) < \tfrac{1}{8}R_g(x_0) \text{ and } h \leq \tfrac{1}{8}R_g \text{ in } M, \text{ or } h \equiv \tfrac{1}{8}R_g \text{ and}\\ &\quad (M,g) \text{ is not conformally equivalent to the standard sphere } \mathbb{S}^3. \end{aligned} \tag{2.6}$$

Then, system (2.1) has a pair of nontrivial weak solutions.

For our second theorem, we just assume that the functions $\tilde{a}$ and $\tilde{c}$ are Hölder continuous, and that at least one of the three values $\tilde{a}(x_0)$, $\tilde{c}(x_0)$ or $\alpha\tilde{a}(x_0) + \beta\tilde{c}(x_0)$ is negative, such choice will depend on whether $(\theta - \gamma)$ is negative, positive or zero, respectively. Here, regarding the functions a, b and c, we only need to assume that they satisfy together with $\tilde{a}$ and $\tilde{c}$ the coercivity hypothesis $(\mathcal{H}_1)$.

Now, we can state our second theorem.

Teorema 2.1.4 *Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$. Let $a, b, c, \tilde{a}, \tilde{c}$ and ρ be functions in M satisfying $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, let f be smooth function such that $f(x_0) = \max_M f > 0$. Furthermore, suppose we are in one of the following cases:*

$$(1)\ \textit{when } n \geq 4, \textit{ and } \begin{cases} \tilde{a}(x_0) < 0 & \textit{if } \theta > \gamma, \\ \alpha\tilde{a}(x_0) + \beta\tilde{c}(x_0) < 0 & \textit{if } \theta = \gamma, \\ \tilde{c}(x_0) < 0 & \textit{if } \gamma > \theta, \end{cases} \tag{2.7}$$

$$(2)\ \textit{when } n = 3, \textit{ and } \begin{cases} \tilde{a}(x_0) < 0 & \textit{if } \theta > \gamma \geq 1, \\ \alpha\tilde{a}(x_0) + \beta\tilde{c}(x_0) < 0 & \textit{if } \theta = \gamma \geq 1, \\ \tilde{c}(x_0) < 0 & \textit{if } \gamma > \theta \geq 1. \end{cases} \tag{2.8}$$

Then, system (2.1) has a pair of nontrivial weak solutions.

Corollary 2.1.5 *With the same assumptions of Theorem 2.1.4. If we assume that the function $b \leq 0$ in M. Then, system (2.1) has a pair of non-negative nontrivial weak solution.*

Estimates of the terms in the functional associated with the system for an appropriate test function are a sensitive aspect in the proof of the main theorems. These estimates are essential to overcome the lack of compactness in these functionals caused by the critical growth of nonlinearities. We achieved this goal by adopting some ideas developed in [3, 16, 35]. In this context, we find additional challenges arising from the coupling of system components, in addition to Hardy-type potentials.

The chapter is organized as follows. In Sect. 2.2 we present an optimal Hardy-Sobolev type inequality, important to prove the main results. In Sect. 2.3 we prove a general existence theorem for system (2.1). In Sect. 2.4 we prove an auxiliary lemma and Theorem 2.1.1 and its consequences. In Sect. 2.5 we prove an auxiliary lemma and Theorem 2.1.4.

2.2 Preliminary

For the purpose of this chapter, in this section, we present some important results. First, as in [35] Jaber have established that the best constant for the Hardy-Sobolev inequality is $K(n,s)$. Precisely, he proved that there is a positive constant B such that

$$\left(\int_M \frac{|u|^{2^\star(s)}}{d_g(x,x_0)^s}dv_g\right)^{\frac{2}{2^\star(s)}} \leq K(n,s)\|\nabla u\|_2^2 + B\|u\|_2^2, \tag{2.9}$$

for all $u \in H^1(M)$.

Initially, we present an inequality that will be used in the proof of the main results.

Lemma 2.2.1 *Let $\mathcal{K}^s_{(\alpha,\beta)}$ be the constant defined in (2.3) when $\alpha+\beta = 2^\star(s)$. Then, there is a positive constant B_0 such that*

$$\left(\int_M \frac{|u|^\alpha|v|^\beta}{d_g(x,x_0)^s}dv_g\right)^{\frac{2}{2^\star(s)}} \leq (\mathcal{K}^s_{(\alpha,\beta)})^{-1}\|(|\nabla u|,|\nabla v|)\|_2^2 + B_0\|(u,v)\|_2^2\,, \tag{2.10}$$

for all $(u,v) \in H$. Moreover, $(\mathcal{K}^s_{(\alpha,\beta)})^{-1}$ is the best constant such that the inequality holds.

Proof. The proof is similar to Lemma 1.21 of Chapter 1. ■

An immediate consequence of this result is the following inequality.

Corollary 2.2.2 *Let $C = \max\{(\mathcal{K}^s_{(\alpha,\beta)})^{-1}, B_0\}$, then we have*

$$\left(\int_M \frac{|u|^\alpha|v|^\beta}{d_g(x,x_0)^s}dv_g\right)^{\frac{2}{2^\star(s)}} \leq C\|(u,v)\|^2.$$

Another result that will be important later on is the following Brezis-Lieb type lemma.

Lemma 2.2.3 *Let $u_m \rightharpoonup u$ and $v_m \rightharpoonup v$ in $H^1(M)$ and let $\ell \in L^\infty(M)$. Then we have*

$$\int_M \frac{\ell(x)|u_m|^\alpha|v_m|^\beta}{d_g(x,x_0)^s}dv_g = \int_M \frac{\ell(x)|u|^\alpha|v|^\beta}{d_g(x,x_0)^s}dv_g + \int_M \frac{\ell(x)|u_m-u|^\alpha|v_m-v|^\beta}{d_g(x,x_0)^s}dv_g + o_m(1),$$

where $o_m(1) \to 0$ as $m \to \infty$.

Proof. The proof is done in Lemma 3.2.5 in Chapter 3. ■

2.3 A General Existence Theorem

In the present section we prove a general existence result, for which we assume appropriate conditions.

Initially, consider the functional $E_h : H \to \mathbb{R}$ given by

$$E_h(u,v) = \int_M \left(|\nabla_g u|^2 + |\nabla_g v|^2 + a(x)u^2 + 2buv + c(x)v^2 + \frac{\tilde{a}(x)}{\rho(x)^\theta}u^2 + \frac{\tilde{c}(x)}{\rho(x)^\gamma}v^2 \right) dv_g,$$

and constraint set $\Lambda_{s,f}^{\alpha,\beta} := \left\{ (u,v) \in H \, : \, \int_M \frac{f(x)|u|^\alpha |v|^\beta}{d_g(x,x_0)^s} dv_g = 1 \right\}$.

Note that E_h is clearly well defined. Furthermore, it is also bounded from below on $\Lambda_s^{\alpha,\beta}$. Indeed, by the coercivity condition $(\mathcal{H}_1)$ and Corollary 2.2.2, we have

$$E_h(u,v) \geq C_0 \|(u,v)\|^2 \geq C \left(\int_M \frac{|u|^\alpha |v|^\beta}{d_g(x,x_0)^s} dv_g \right)^{\frac{2}{2^\star(s)}} \geq \frac{C}{f(x_0)^{\frac{2}{2^\star(s)}}} > 0,$$

for all $(u,v) \in \Lambda_{s,f}^{\alpha,\beta}$. Thus, we can consider

$$K_{s,f}^{(\alpha,\beta)}(M) = \inf_{(u,v)\in\Lambda_{s,f}^{\alpha,\beta}} E_h(u,v) > 0. \tag{2.11}$$

If there exists $(u,v) \in \Lambda_{s,f}^{\alpha,\beta}$ which achieves the infimum $K_{s,f}^{(\alpha,\beta)} (= K_{s,f}^{(\alpha,\beta)}(M))$, it turns out that (u,v) will be a weak solution of the following system

$$\begin{cases} -\Delta_g u + \frac{\tilde{a}(x)}{\rho(x)^\theta}u + a(x)u + b(x)v &= K_{s,f}^{(\alpha,\beta)} \frac{2\alpha}{2^\star(s)} \frac{u|u|^{\alpha-2}|v|^\beta}{d_g(x,x_0)^s} \quad \text{in} \quad M, \\ -\Delta_g v + \frac{\tilde{c}(x)}{\rho(x)^\gamma}v + b(x)u + c(x)v &= K_{s,f}^{(\alpha,\beta)} \frac{2\beta}{2^\star(s)} \frac{v|v|^{\beta-2}|u|^\alpha}{d_g(x,x_0)^s} \quad \text{in} \quad M. \end{cases} \tag{2.12}$$

Next, we will prove a general existence result.

Proposition 2.3.1 *If $f(x_0)^{\frac{2}{2^\star(s)}} K_{s,f}^{(\alpha,\beta)} < \mathcal{K}_{(\alpha,\beta)}^s$, then there exists (u,v) in $\Lambda_{s,f}^{\alpha,\beta}$ such that $E_h(u,v) = K_{s,f}^{(\alpha,\beta)}$.*

Proof. Let $\{(u_m,v_m)\} \subset \Lambda_{s,f}^{\alpha,\beta}$ be a minimizing sequence for $K_{s,f}^{(\alpha,\beta)}$, that is,

$$\begin{aligned} E_h(u_m,v_m) &= \|(|\nabla u_m|, |\nabla v_m|)\|_2^2 \\ &\quad + \int_M \left(\frac{\tilde{a}(x)}{\rho(x)^\theta}u_m^2 + \frac{\tilde{c}(x)}{\rho(x)^\gamma}v_m^2 + au_m^2 + 2bu_mv_m + cv_m^2 \right) dv_g \\ &= K_{s,f}^{(\alpha,\beta)} + o_m(1), \end{aligned} \tag{2.13}$$

where $o_m(1) \to 0$ as $m \to \infty$. By the coercivity hypothesis $(\mathcal{H}_1)$, it follows that $\{(u_m,v_m)\}$ is bounded in H. Thus, there exists (u,v) in H such that, up to a subsequence, $(u_m,v_m) \rightharpoonup (u,v)$ in H, $(u_m,v_m) \to (u,v)$ in $L^2(M) \times L^2(M)$, and

$(u_m(x), v_m(x)) \to (u(x), v(x))$ a.e in M. From Lemma 2.2.1 and (2.13), we get

$$
\begin{aligned}
1 &= \left(\int_M \frac{f(x)|u_m|^\alpha |v_m|^\beta}{d_g(x,x_0)^s} dv_g\right)^{\frac{2}{2^\star(s)}} \\
&\le f(x_0)^{\frac{2}{2^\star(s)}} \left[(\mathcal{K}^s_{(\alpha,\beta)})^{-1} \|(|\nabla u_m|, |\nabla v_m|)\|_2^2 + B_0||(u_m, v_m)||_2^2\right] \\
&\le f(x_0)^{\frac{2}{2^\star(s)}} (\mathcal{K}^s_{(\alpha,\beta)})^{-1} K^{(\alpha,\beta)}_{s,f} + f(x_0)^{\frac{2}{2^\star(s)}} B_0 \|(u_m, v_m)\|_2^2 \\
&\quad - f(x_0)^{\frac{2}{2^\star(s)}} (\mathcal{K}^s_{(\alpha,\beta)})^{-1} \int_M (au_m^2 + 2bu_m v_m + cv_m^2) dv_g \\
&\quad - f(x_0)^{\frac{2}{2^\star(s)}} (\mathcal{K}^s_{(\alpha,\beta)})^{-1} \int_M \left(\frac{\tilde{a}(x)}{\rho(x)^\theta} u_m^2 + \frac{\tilde{c}(x)}{\rho(x)^\gamma} v_m^2\right) dv_g + o_m(1).
\end{aligned}
$$

Letting $m \to \infty$, we obtain that

$$
\begin{aligned}
1 - \frac{f(x_0)^{\frac{2}{2^\star(s)}} K^{(\alpha,\beta)}_{s,f}}{\mathcal{K}^s_{(\alpha,\beta)}} &\le - f(x_0)^{\frac{2}{2^\star(s)}} (\mathcal{K}^s_{(\alpha,\beta)})^{-1} \int_M (au^2 + 2buv + cv^2) dv_g \\
&\quad - f(x_0)^{\frac{2}{2^\star(s)}} (\mathcal{K}^s_{(\alpha,\beta)})^{-1} \int_M \left(\tfrac{\tilde{a}(x)}{\rho(x)^\theta} u^2 + \tfrac{\tilde{c}(x)}{\rho(x)^\gamma} v^2\right) dv_g + f(x_0)^{\frac{2}{2^\star(s)}} B_0 \|(u, v)\|_2^2.
\end{aligned}
$$

Since $f(x_0)^{\frac{2}{2^\star(s)}} K^{(\alpha,\beta)}_{s,f} < \mathcal{K}^s_{(\alpha,\beta)}$, consequently, $0 < \|(u, v)\|_2^2$ this implies that $u \neq 0$ or $v \neq 0$.

We claim that $u \neq 0$ and $v \neq 0$. Moreover, $(u, v) \in \Lambda^{\alpha,\beta}_{s,f}$ is a minimizing for $K^{(\alpha,\beta)}_{s,f}$. Indeed, rewriting (2.13), we have

$$
E_h(u, v) + \|(|\nabla(u_m - u)|, |\nabla(v_m - v)|)\|_2^2 = K^{(\alpha,\beta)}_{s,f} + o_m(1). \tag{2.14}
$$

On the other hand, since $1 = \int_M \frac{f(x)}{d_g(x,x_0)^s} |u_m|^\alpha |v_m|^\beta dv_g$, Lemma 2.2.3 gives us

$$
1 = \left(\int_M \frac{f(x)|u|^\alpha |v|^\beta}{d_g(x,x_0)^s} dv_g + \int_M \frac{f(x)|u_m - u|^\alpha |v_m - v|^\beta}{d_g(x,x_0)^s} dv_g + o_m(1)\right)^{\frac{2}{2^\star(s)}}. \tag{2.15}
$$

Now, note that $\int_M \frac{f(x)}{d_g(x,x_0)^s} |u|^\alpha |v|^\beta dv_g > 0$, otherwise, by (2.15) and Lemma 2.2.1, we would have

$$
\begin{aligned}
1 &\le \left(\int_M \frac{f(x)}{d_g(x,x_0)^s} |u_m - u|^\alpha |v_m - v|^\beta dv_g\right)^{\frac{2}{2^\star(s)}} + o_m(1) \\
&\le f(x_0)^{\frac{2}{2^\star(s)}} (\mathcal{K}^s_{(\alpha,\beta)})^{-1} \|(|\nabla_g(u_m - u)|, |\nabla_g(v_m - v)|)\|_2^2 + o_m(1),
\end{aligned}
$$

hence,

$$
K^{(\alpha,\beta)}_{s,f} \le f(x_0)^{\frac{2}{2^\star(s)}} (\mathcal{K}^s_{(\alpha,\beta)})^{-1} K^{(\alpha,\beta)}_{s,f} \|(|\nabla_g(u_m - u)|, |\nabla_g(v_m - v)|)\|_2^2 + o_m(1).
$$

But, using the inequality above in (2.14), we get

$$
\begin{aligned}
E_h(u,v) &= K_{s,f}^{(\alpha,\beta)} - \|(|\nabla(u_m-u)|,|\nabla(v_m-v)|)\|_2^2 + o_m(1)\\
&\le (f(x_0)^{\frac{2}{2^\star(s)}}(\mathcal{K}^s_{(\alpha,\beta)})^{-1}K_{s,f}^{(\alpha,\beta)} - 1)\|(|\nabla(u_m-u)|,|\nabla(v_m-v)|)\|_2^2 + o_m(1),
\end{aligned}
$$

again as $f(x_0)^{\frac{2}{2^\star(s)}}K_{s,f}^{(\alpha,\beta)} < \mathcal{K}^s_{(\alpha,\beta)}$, we reach that $E_h(u,v)\le 0$, and so $u=v=0$, which is a contradiction. Therefore, $\int_M \frac{f(x)}{d_g(x,x_0)^s}|u|^\alpha|v|^\beta dv_g > 0$.

Now, returning to (2.15) we get

$$
\begin{aligned}
1 &= \left(\int_M \frac{f(x)}{d_g(x,x_0)^s}|u|^\alpha|v|^\beta dv_g + \int_M \frac{f(x)}{d_g(x,x_0)^s}|u_m-u|^\alpha|v_m-v|^\beta dv_g + o_m(1)\right)^{\frac{2}{2^\star(s)}}\\
&\le \left(\int_M \frac{f(x)|u|^\alpha|v|^\beta}{d_g(x,x_0)^s}dv_g\right)^{\frac{2}{2^\star(s)}} + f(x_0)^{\frac{2}{2^\star(s)}}\left(\int_M \frac{|u_m-u|^\alpha|v_m-v|^\beta}{d_g(x,x_0)^s}dv_g\right)^{\frac{2}{2^\star(s)}} + o_m(1)\\
&\le \left(\int_M \frac{f(x)|u|^\alpha|v|^\beta}{d_g(x,x_0)^s}dv_g\right)^{\frac{2}{2^\star(s)}}\\
&\quad + f(x_0)^{\frac{2}{2^\star(s)}}(\mathcal{K}^s_{(\alpha,\beta)})^{-1}\|(|\nabla(u_m-u)|,|\nabla(v_m-v)|)\|_2^2 + o_m(1),
\end{aligned}
$$

then we reach

$$
\begin{aligned}
K_{s,f}^{(\alpha,\beta)} \le\ & K_{s,f}^{(\alpha,\beta)}\left(\int_M \frac{f(x)}{d_g(x,x_0)^s}|u|^\alpha|v|^\beta dv_g\right)^{\frac{2}{2^\star(s)}}\\
&+ K_{s,f}^{(\alpha,\beta)} f(x_0)^{\frac{2}{2^\star(s)}}(\mathcal{K}^s_{(\alpha,\beta)})^{-1}\|(|\nabla(u_m-u)|,|\nabla(v_m-v)|)\|_2^2 + o_m(1).
\end{aligned}
\tag{2.16}
$$

By (2.14), it follows that

$$
\begin{aligned}
E_h(u,v) \le\ & K_{s,f}^{(\alpha,\beta)}\left(\int_M \frac{f(x)}{d_g(x,x_0)^s}|u|^\alpha|v|^\beta dv_g\right)^{\frac{2}{2^\star(s)}}\\
&+ (K_{s,f}^{(\alpha,\beta)} f(x_0)^{\frac{2}{2^\star(s)}}(\mathcal{K}^s_{(\alpha,\beta)})^{-1} - 1)\|(|\nabla_g(u_m-u)|,|\nabla_g(v_m-v)|)\|_2^2 + o_m(1).
\end{aligned}
$$

Since $f(x_0)^{\frac{2}{2^\star(s)}}K_{s,f}^{(\alpha,\beta)} < \mathcal{K}^s_{(\alpha,\beta)}$, then

$$
E_h(u,v) \le K_{s,f}^{(\alpha,\beta)}\left(\int_M \frac{f(x)}{d_g(x,x_0)^s}|u|^\alpha|v|^\beta dv_g\right)^{\frac{2}{2^\star(s)}}.
$$

The lower semicontinuity of E_h gives us $E_h(u,v) \le \liminf E_h(u_m,v_m) = K_{s,f}^{(\alpha,\beta)}$, and hence $0 < \int_M \frac{f(x)}{d_g(x,x_0)^s}|u|^\alpha|v|^\beta dv_g \le 1$, now writing $u_0 = \left(\int_M \frac{f(x)}{d_g(x,x_0)^s}|u|^\alpha|v|^\beta dv_g\right)^{-\frac{1}{2^\star(s)}} u$ and $v_0 = \left(\int_M \frac{f(x)}{d_g(x,x_0)^s}|u|^\alpha|v|^\beta dv_g\right)^{-\frac{1}{2^\star(s)}} v$, we have

$$
E_h(u_0,v_0) = \frac{E_h(u,v)}{\left(\int_M \frac{f(x)}{d_g(x,x_0)^s}|u|^\alpha|v|^\beta dv_g\right)^{\frac{2}{2^\star(s)}}} \le K_{s,f}^{(\alpha,\beta)},
$$

where $(u_0, v_0) \in \Lambda_{s,f}^{\alpha,\beta}$. By definition of $K_{s,f}^{(\alpha,\beta)}$ it follows that $E_h(u_0, v_0) = K_{s,f}^{(\alpha,\beta)}$, so we prove that

$$E_h(u,v) = K_{s,f}^{(\alpha,\beta)} \left(\int_M \frac{f(x)|u|^\alpha |v|^\beta}{d_g(x,x_0)^s} dv_g \right)^{\frac{2}{2^\star(s)}}.$$

Finally, we can check that $(u, v) \in \Lambda_{s,f}^{\alpha,\beta}$, for this, we return to (2.14) and (2.16). Then

$$\begin{aligned} 1 &\leq \left(\int_M \frac{f(x)|u|^\alpha |v|^\beta}{d_g(x,x_0)^s} dv_g \right)^{\frac{2}{2^\star(s)}} + f(x_0)^{\frac{2}{2^\star(s)}} (\mathcal{K}_{(\alpha,\beta)}^s)^{-1} \left[K_{s,f}^{(\alpha,\beta)} - E_h(u,v) \right] \\ &= \left(\int_M \frac{f(x)|u|^\alpha |v|^\beta}{d_g(x,x_0)^s} dv_g \right)^{\frac{2}{2^\star(s)}} \\ &\quad + f(x_0)^{\frac{2}{2^\star(s)}} (\mathcal{K}_{(\alpha,\beta)}^s)^{-1} K_{s,f}^{(\alpha,\beta)} \left[1 - \left(\int_M \frac{f(x)|u|^\alpha |v|^\beta}{d_g(x,x_0)^s} dv_g \right)^{\frac{2}{2^\star(s)}} \right]. \end{aligned}$$

Hence,

$$0 \leq \left(-1 + f(x_0)^{\frac{2}{2^\star(s)}} (\mathcal{K}_{(\alpha,\beta)}^s)^{-1} K_{s,f}^{(\alpha,\beta)} \right) \left[1 - \left(\int_M \frac{f(x)|u|^\alpha |v|^\beta}{d_g(x,x_0)^s} dv_g \right)^{\frac{2}{2^\star(s)}} \right].$$

As $f(x_0)^{\frac{2}{2^\star(s)}} (\mathcal{K}_{(\alpha,\beta)}^s)^{-1} K_{s,f}^{(\alpha,\beta)} < 1$, then $\int_M \frac{f(x)}{d_g(x,x_0)^s} |u|^\alpha |v|^\beta dv_g = 1$. Consequently, we get that $(u, v) \in \Lambda_{s,f}^{\alpha,\beta}$, which proves that (u, v) is a minimizer for $K_{s,f}^{(\alpha,\beta)}$. ∎

2.4 Proof of Theorem 2.1.1

In this section we will prove Theorems 2.1.1 and 2.1.4. First, we state an auxiliary lemma where we will show that under the assumptions of these theorems we will have $f(x_0)^{\frac{2}{2^\star(s)}} K_{s,f}^{(\alpha,\beta)} < \mathcal{K}_{(\alpha,\beta)}^s$.

For this, let $\delta \in (0, i_g/2)$ small enough such that $f > 0$ in $B_{2\delta}(x_0)$ (geodesic ball centered in x_0 and with radius 2δ). We consider the following cut-off function $\eta \in C_0^\infty([-2\delta, 2\delta])$, with $\eta = 1$ in $[-\delta, \delta]$, $0 \leq \eta \leq 1$ in $\mathbb{R}$, and we define the function

$$u_\epsilon(x) = \eta(d_g(x,x_0)) \left(\frac{\epsilon^{1-\frac{s}{2}}}{\epsilon^{2-s} + d_g(x,x_0)^{2-s}} \right)^{\frac{n-2}{2-s}}. \quad (2.17)$$

Remark 2.4.1 *As is known the function $\Phi(y) = (1 + |y|^{2-s})^{-\frac{n-2}{2-s}}$ (see [39]), with $y \in \mathbb{R}^n$ is an extremal for*

$$K(n,s)^{-1} = \inf_{u \in H^1(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^n} \frac{|u|^{2^\star(s)}}{|x|^s} dx \right)^{\frac{2}{2^\star(s)}}}.$$

Lemma 2.4.2 *Under the assumptions of Theorem 2.1.1 one get that*

$$f(x_0)^{\frac{2}{2^\star(s)}} K_{s,f}^{(\alpha,\beta)} < \mathcal{K}^s_{(\alpha,\beta)}.$$

Proof. In the proof of this estimate we use the ideas of [15] and [35]. According to the estimates obtained in [35, Section 3] for $n \geq 4$ and the test function u_ϵ we have

$$\int_M |\nabla_g u_\epsilon|^2 dv_g = \begin{cases} \int_{\mathbb{R}^n} |\nabla \Phi|^2 dy - \frac{\int_{\mathbb{R}^n} |y|^2 |\nabla \Phi|^2 dy}{6n} R_g(x_0)\epsilon^2 + o(\epsilon^2) \text{ if } n \geq 5, \\ \int_{\mathbb{R}^4} |\nabla \Phi|^2 dy - \frac{\omega_3}{6} R_g(x_0)\epsilon^2 \ln(\epsilon^{-1}) + O(\epsilon^2) \text{ if } n = 4, \end{cases} \tag{2.18}$$

and for $h \in C^0(M)$

$$\int_M h u_\epsilon^2 dv_g = \begin{cases} h(x_0)\epsilon^2 \int_{\mathbb{R}^n} \Phi^2 dy + o(\epsilon^2) \text{ if } n \geq 5, \\ h(x_0)\omega_3 \epsilon^2 \ln(\epsilon^{-1}) + O(\epsilon^2) \text{ if } n = 4. \end{cases} \tag{2.19}$$

Next, we want to estimate $\int_M \frac{\tilde{a}}{\rho^\theta} u_\epsilon^2 dv_g$ for $n \geq 4$. First, we have

$$\begin{aligned} \int_M \frac{\tilde{a}}{\rho^\theta} u_\epsilon^2 dv_g &= \int_{B_\delta(x_0)} \frac{\tilde{a}}{\rho^\theta} u_\epsilon^2 dv_g + \int_{M \setminus B_\delta(x_0)} \frac{\tilde{a}}{\rho^\theta} u_\epsilon^2 dv_g \\ &= \int_{B_\delta(x_0)} \frac{\tilde{a}}{\rho^\theta} u_\epsilon^2 dv_g + O(\epsilon^{n-2}). \end{aligned} \tag{2.20}$$

Since $\tilde{a}$ is smooth and by hypothesis $\tilde{a}(x_0) = 0$ we can write for $x \in B_\delta(x_0)$

$$\tilde{a}(x) = \partial_i \tilde{a}(x_0) x_i + \frac{1}{2} \partial_{ij} \tilde{a}(x_0) x_i x_j + O(r^3). \tag{2.21}$$

We can also write in normal coordinates the following expansion

$$\int_{\mathbb{S}^{n-1}} \sqrt{\det(g)} d\sigma = \omega_{n-1} \left(1 - \frac{R_g(x_0)}{6n} r^2 + O(1) r^4\right), \tag{2.22}$$

where $\sqrt{\det(g)}$ denote the determinant of the components of the metric g. So we get

$$\begin{aligned} \int_{B_\delta(x_0)} \frac{\tilde{a}}{\rho^\theta} u_\epsilon^2 dv_g &= \int_{B_\delta(0)} \frac{(\tilde{a} \circ \exp_{x_0})(y)}{(|y| + O(|y|^{1+\mu}))^\theta} ((u_\epsilon \circ \exp_{x_0})(y))^2 \sqrt{\det(g)} dy \\ &= \int_{B_\delta(0)} \frac{(\tilde{a} \circ \exp_{x_0})(y)}{|y|^\theta (1 + O(|y|^\mu))} ((u_\epsilon \circ \exp_{x_0})(y))^2 \sqrt{\det(g)} dy \\ &= \int_{B_\delta(0)} \frac{(\partial_i \tilde{a}(x_0) y_i + \frac{1}{2} \partial_{ij} \tilde{a}(x_0) y_i y_j + O(r^3))}{|y|^\theta (1 + O(|y|^\mu))} ((u_\epsilon \circ \exp_{x_0})(y))^2 \sqrt{\det(g)} dy \end{aligned}$$

noting that for each $i = 1, \ldots, n$ one has

$$\partial_i \tilde{a}(x_0) \int_{B_\delta(0)} \frac{y_i ((u_\epsilon \circ \exp_{x_0})(y))^2}{|y|^\theta (1 + O(|y|^\mu))} \sqrt{\det(g)} dy = 0,$$

so, we can rewrite

$$\int_{B_\delta(x_0)} \frac{\tilde{a}}{\rho^\theta} u_\epsilon^2 dv_g = \frac{1}{2}\partial_{ij}\tilde{a}(x_0)\int_{B_\delta(0)} \frac{y_i y_j((u_\epsilon \circ \exp_{x_0})(y))^2}{|y|^\theta\,(1+O(|y|^\mu))}\sqrt{\det(g)}dy + \int_{B_\delta(0)} \frac{O(r^3)((u_\epsilon \circ \exp_{x_0})(y))^2}{|y|^\theta\,(1+O(|y|^\mu))}\sqrt{\det(g)}dy,$$

again since

$$\int_{B_\delta(0)} \frac{y_i y_j((u_\epsilon \circ \exp_{x_0})(y))^2}{|y|^\theta\,(1+O(|y|^\mu))}\sqrt{\det(g)}dy = \frac{\omega_{n-1}}{n}\delta^{ij}\epsilon^{n-2}\int_0^\delta \frac{r^{n+1}(1+O(r^2))}{r^\theta\,(1+O(r^\mu))\left(\epsilon^{2-s}+|y|^{2-s}\right)^{\frac{2(n-2)}{2-s}}}dr$$

so we have

$$\int_{B_\delta(x_0)} \frac{\tilde{a}}{\rho^\theta} u_\epsilon^2 dv_g = \frac{1}{2n}\Delta_g\tilde{a}(x_0)\omega_{n-1}\epsilon^{n-2}\int_0^\delta \frac{r^{n+1}}{r^\theta\left(\epsilon^{2-s}+r^{2-s}\right)^{\frac{2(n-2)}{2-s}}}(1+O(r^2))dr + \int_0^\delta \frac{r^{n+1}O(r^\mu)}{r^\theta\,(1+O(r^\mu))\left(\epsilon^{2-s}+r^{2-s}\right)^{\frac{2(n-2)}{2-s}}}(1+O(r^2))dr.$$

Now let us calculate the first integral on the right hand side in separate cases: for $n \geq 6$ or $n = 5$ and $\theta > 1$, $n = 5$ and $\theta = 1$, $n = 5$ and $\theta < 1$, and $n = 4$. First, for $n \geq 6$ or $n = 5$ and $\theta > 1$ we have

$$\begin{aligned}\epsilon^{n-2}\int_0^\delta \frac{r^{n+1}}{r^\theta\left(\epsilon^{2-s}+r^{2-s}\right)^{\frac{2(n-2)}{2-s}}}dr &= \epsilon^{4-\theta}\int_0^{\frac{\delta}{\epsilon}} \frac{t^{n+1-\theta}}{\left(1+t^{2-s}\right)^{\frac{2(n-2)}{2-s}}}dt \\ &= O\left(\epsilon^{4-\theta}\int_0^\infty \frac{t^{n+1-\theta}}{\left(1+t^{2-s}\right)^{\frac{2(n-2)}{2-s}}}dt\right) \\ &= O(\epsilon^{4-\theta}).\end{aligned}$$

In the case, $n = 5$ and $\theta < 1$ we get

$$\begin{aligned}\epsilon^{n-2}\int_0^\delta \frac{r^{n+1}}{r^\theta\left(\epsilon^{2-s}+r^{2-s}\right)^{\frac{2(n-2)}{2-s}}}dr &= \epsilon^3\int_0^\delta \frac{r^6}{r^\theta\left(\epsilon^{2-s}+r^{2-s}\right)^{\frac{2(n-2)}{2-s}}}dr \\ &= O\left(\epsilon^3\int_0^\delta r^{-\theta}dr\right) \\ &= O(\epsilon^3\delta^{1-\theta}) \\ &= O(\epsilon^3).\end{aligned}$$

Now, in the case, $n = 5$ and $\theta = 1$, we reach (for ϵ small enough)

$$\begin{aligned}
\epsilon^{n-2}\int_0^{\delta}\frac{r^{n+1}}{r^{\theta}\left(\epsilon^{2-s}+r^{2-s}\right)^{\frac{2(n-2)}{2-s}}}dr &= \epsilon^3\int_0^{\frac{\delta}{\epsilon}}\frac{t^5}{(1+t^{2-s})^{\frac{6}{2-s}}}dt\\
&= \epsilon^3\int_0^{\delta}\frac{t^5}{(1+t^{2-s})^{\frac{6}{2-s}}}dt+\epsilon^3\int_{\delta}^{\frac{\delta}{\epsilon}}\frac{t^5}{(1+t^{2-s})^{\frac{6}{2-s}}}dt\\
&= O(\epsilon^3)+\epsilon^3\int_{\delta}^{\frac{\delta}{\epsilon}}\frac{1}{t}dt+\epsilon^3\int_{\delta}^{\frac{\delta}{\epsilon}}\left(\frac{t^5}{(1+t^{2-s})^{\frac{6}{2-s}}}-\frac{1}{t}\right)dt\\
&= O(\epsilon^3)+\epsilon^3\ln(\epsilon^{-1})+\epsilon^3\int_{\delta}^{\frac{\delta}{\epsilon}}t^5\left[\frac{1-\left(1+\frac{1}{t^{2-s}}\right)^{\frac{6}{2-s}}}{(1+t^{2-s})^{\frac{6}{2-s}}}\right]dt\\
&= O(\epsilon^3\ln(\epsilon^{-1}))+\epsilon^3 O\left(\int_{\delta}^{\frac{\delta}{\epsilon}}\frac{t^5}{t^{2-s}\left(1+t^{2-s}\right)^{\frac{6}{2-s}}}dt\right)\\
&= O(\epsilon^3\ln(\epsilon^{-1}))+\epsilon^3 O\left(\int_{\delta}^{\frac{\delta}{\epsilon}}t^{-3+s}dt\right)\\
&= O(\epsilon^3\ln(\epsilon^{-1}))+\epsilon^3 O(\epsilon^{2-s}).
\end{aligned}$$

Finally, in the case $n = 4$ we have

$$\begin{aligned}
\epsilon^{n-2}\int_0^{\delta}\frac{r^{n+1}}{r^{\theta}\left(\epsilon^{2-s}+r^{2-s}\right)^{\frac{2(n-2)}{2-s}}}dr &= \epsilon^2\int_0^{\delta}\frac{r^{5-\theta}}{(\epsilon^{2-s}+r^{2-s})^{\frac{4}{2-s}}}dr\\
&= O\left(\epsilon^2\int_0^{\delta}r^{1-\theta}dr\right)\\
&= O(\epsilon^2).
\end{aligned}$$

Thus, for each case above and (2.20), when $n \geq 4$ we get that

$$\int_M\frac{\tilde{a}}{\rho^{\theta}}u_{\epsilon}^2dv_g=\begin{cases}O(\epsilon^{4-\theta}), & \text{if } n\geq 6,\\ O(\epsilon^{4-\theta}), & \text{if } n=5 \text{ and } \theta>1,\\ O(\epsilon^3\ln(\epsilon^{-1})), & \text{if } n=5 \text{ and } \theta=1,\\ O(\epsilon^3), & \text{if } n=5 \text{ and } \theta<1,\\ O(\epsilon^2), & \text{if } n=4.\end{cases} \tag{2.23}$$

The estimate of $\int_M \frac{\tilde{c}}{\rho^{\gamma}}u_{\epsilon}^2dv_g$ is completely analogous. So, we have

$$\int_M\frac{\tilde{c}}{\rho^{\gamma}}u_{\epsilon}^2dv_g=\begin{cases}O(\epsilon^{4-\theta}), & \text{if } n\geq 6,\\ O(\epsilon^{4-\theta}), & \text{if } n=5 \text{ and } \gamma>1,\\ O(\epsilon^3\ln(\epsilon^{-1})), & \text{if } n=5 \text{ and } \gamma=1,\\ O(\epsilon^3), & \text{if } n=5 \text{ and } \gamma<1,\\ O(\epsilon^2), & \text{if } n=4.\end{cases} \tag{2.24}$$

Now, let us estimate $\int_M \frac{f(x)}{d_g(x,x_0)^s} u_\epsilon^{2^\star(s)} dv_g$. First, it is easy to see that

$$\int_{M\setminus B_\delta(x_0)} \frac{f(x)}{d_g(x,x_0)^s} u_\epsilon^{2^\star(s)} dv_g = O(\epsilon^{n-s}). \tag{2.25}$$

In coordinate normal we can write the following expansions

$$f(x)\eta^{2^\star(s)}(r) = f(x_0) + \frac{1}{2}\partial_{ij} f(x_0) x^i x^j + O(r^3), \tag{2.26}$$

With that,

$$\begin{aligned}
\int_{B_\delta(x_0)} \frac{f(x)}{d_g(x,x_0)^s} u_\epsilon^{2^\star(s)} dv_g &= f(x_0) \int_{B_\delta(0)} \frac{\epsilon^{n-s}}{|y|^s(\epsilon^{2-s} + |y|^{2-s})^{\frac{2(n-s)}{2-s}}} \sqrt{\det(g)} dy \\
&\quad + \frac{1}{2}\partial_{ij} f(x_0) \int_{B_\delta(0)} \frac{\epsilon^{n-s} y^i y^j}{|y|^s(\epsilon^{2-s} + |y|^{2-s})^{\frac{2(n-s)}{2-s}}} \sqrt{\det(g)} dy + O(\epsilon^3) \\
&= f(x_0)\omega_{n-1} \int_0^\delta \frac{\epsilon^{n-s} r^{n-1}}{r^s(\epsilon^{2-s} + r^{2-s})^{\frac{2(n-s)}{2-s}}} \left(1 - \frac{R_g(x_0)}{6n} r^2 + O(1) r^4\right) dr \\
&\quad + \frac{\Delta f(x_0)}{2n}\omega_{n-1} \int_0^\delta \frac{\epsilon^{n-s} r^{n-1}}{r^s(\epsilon^{2-s} + r^{2-s})^{\frac{2(n-s)}{2-s}}} \left(1 + O(1) r^2\right) dr + O(\epsilon^3),
\end{aligned}$$

Taking the variable change $t = \frac{r}{\epsilon}$, we have

$$\begin{aligned}
f(x_0)\omega_{n-1} &\int_0^{\frac{\delta}{\epsilon}} \frac{t^{n-1}}{t^s(1+t^{2-s})^{\frac{2(n-s)}{2-s}}} \left(1 - \frac{R_g(x_0)}{6n}(\epsilon t)^2 + O(1)(\epsilon t)^4\right) dt \\
&= f(x_0) \int_{\mathbb{R}^n} \frac{|\Phi|^{2^\star(s)}}{|y|^s} dy - \frac{R_g(x_0)}{6n} f(x_0)\epsilon^2 \int_{\mathbb{R}^n} \frac{|\Phi|^{2^\star(s)}}{|y|^{s-2}} dy + O(\epsilon^{n-s})
\end{aligned} \tag{2.27}$$

and also,

$$\frac{\Delta f(x_0)}{2n}\omega_{n-1}\epsilon^2 \int_0^{\frac{\delta}{\epsilon}} \frac{t^{n-1}(1+O(1)(\epsilon t)^2)}{t^s(1+t^{2-s})^{\frac{2(n-s)}{2-s}}} dt = \frac{1}{2}\Delta f(x_0)\epsilon^2 \int_{\mathbb{R}^n} \frac{|\Phi|^{2^\star(s)}}{|y|^{s-2}} dy + O(\epsilon^{n+2-s}). \tag{2.28}$$

So, by (2.27) and (2.28) we reach

$$\int_M \frac{f(x) u_\epsilon^{2^\star(s)}}{d_g(x,x_0)^s} dv_g = f(x_0) \|\Phi\|^{2^\star(s)}_{L^{2^\star(s)s^*}_{d_g,s}(\mathbb{R}^n)} + \epsilon^2 \frac{f(x_0)}{2n} \left(\frac{\Delta f(x_0)}{f(x_0)} - \frac{R_g(x_0)}{3}\right) \int_{\mathbb{R}^n} \frac{|\Phi|^{2^\star(s)}}{|y|^{s-2}} dy + o(\epsilon^2). \tag{2.29}$$

Next, let us calculate the quotient

$$\frac{\int_M |\nabla_g u_\epsilon|^2 dv_g + \int_M \left(\frac{\alpha}{2^\star(s)} \frac{\tilde{a}}{\rho^\theta} u_\epsilon^2 + \frac{\beta}{2^\star(s)} \frac{\tilde{c}}{\rho^\gamma} u_\epsilon^2\right) dv_g + \int_M h u_\epsilon^2 dv_g}{\left(\int_M \frac{f(x)}{d_g(x,x_0)^s} u_\epsilon^{2^\star(s)} dv_g\right)^{\frac{2}{2^\star(s)}}},$$

for $n \geq 5$ we write the constants $k_1(n,s) = \int_{\mathbb{R}^n} |y|^2 |\nabla \Phi|^2 dy$ and $k_2(n,s) = \int_{\mathbb{R}^n} \frac{|\Phi|^{2^\star(s)}}{|y|^{s-2}} dy$, so by (2.18), (2.19), (2.23), (2.24) and (2.29), we get that

$$\frac{\int_M |\nabla_g u_\epsilon|^2 dv_g + \int_M \left(\frac{\alpha}{2^\star(s)} \frac{\tilde{a}}{\rho^\theta} u_\epsilon^2 + \frac{\beta}{2^\star(s)} \frac{\tilde{c}}{\rho^\gamma} u_\epsilon^2 \right) dv_g + \int_M h u_\epsilon^2 dv_g}{\left(\int_M \frac{f(x)}{d_g(x,x_0)^s} u_\epsilon^{2^\star(s)} dv_g \right)^{\frac{2}{2^\star(s)}}}$$
$$= \frac{\|\nabla \Phi\|_{L^2}^2 - \frac{k_1(n,s)}{6n} R_g(x_0)\epsilon^2 + h(x_0)\|\Phi^2\|_{L^2}^2 \epsilon^2 + o(\epsilon^2)}{f(x_0)^{\frac{2}{2^\star(s)}} \left(\|\Phi\|_{L^{2^\star(s)}_{d_g,s}}^{2^\star(s)} + \epsilon^2 \left(\frac{\Delta f(x_0)}{2n f(x_0)} - \frac{R_g(x_0)}{6n} \right) k_2(n,s) + o(\epsilon^2) \right)^{\frac{2}{2^\star(s)}}}$$
$$= \frac{K(n,s)^{-1}}{f(x_0)^{\frac{2}{2^\star(s)}}}$$
$$+ \frac{\left(\left(\frac{R_g(x_0)}{6n} - \frac{\Delta_g f(x_0)}{2n f(x_0)} \right) \frac{2k_2(n,s)\|\nabla \Phi\|_{L^2}^2}{2^\star(s)\|\Phi\|_{L^{2^\star(s)}_{d_g,s}}^{2^\star(s)}} - \frac{k_1(n,s)}{6n} R_g(x_0) + h(x_0)\|\Phi^2\|_{L^2}^2 \right) \epsilon^2 + o(\epsilon^2)}{f(x_0)^{\frac{2}{2^\star(s)}} \|\Phi\|_{L^{2^\star(s)}_{d_g,s}}^2 \left(1 + \frac{2}{2^\star(s)} \left(\frac{\Delta f(x_0)}{2n f(x_0)} - \frac{R_g(x_0)}{6n} \right) \frac{C_2(n,s)}{\|\Phi\|_{L^{2^\star(s)}_{d_g,s}}^{2^\star(s)}} \epsilon^2 + o(\epsilon^2) \right)}$$

so, we reach that

$$\frac{\int_M |\nabla_g u_\epsilon|^2 dv_g + \int_M \left(\frac{\alpha}{2^\star(s)} \frac{\tilde{a}}{\rho^\theta} u_\epsilon^2 + \frac{\beta}{2^\star(s)} \frac{\tilde{c}}{\rho^\gamma} u_\epsilon^2 \right) dv_g + \int_M h u_\epsilon^2 dv_g}{\left(\int_M \frac{f(x)}{d_g(x,x_0)^s} u_\epsilon^{2^\star(s)} dv_g \right)^{\frac{2}{2^\star(s)}}}$$
$$= \frac{K(n,s)^{-1}}{f(x_0)^{\frac{2}{2^\star(s)}}} + \frac{K(n,s)^{-1}}{f(x_0)^{\frac{2}{2^\star(s)}}} \left(\frac{R_g(x_0)}{6n} - \frac{\Delta_g f(x_0)}{2n f(x_0)} \right) \frac{2k_2(n,s)}{2^\star(s)\|\Phi\|_{L^{2^\star(s)}_{d_g,s}}^{2^\star(s)}} \epsilon^2$$
$$- \frac{K(n,s)^{-1}}{f(x_0)^{\frac{2}{2^\star(s)}}} \frac{k_1(n,s)}{6n\|\nabla \Phi\|_{L^2}^2} R_g(x_0)\epsilon^2 + h(x_0) \frac{\|\Phi^2\|_{L^2(\mathbb{R}^n)}^2}{\|\nabla \Phi\|_{L^2}^2} \epsilon^2 + o(\epsilon^2).$$

Now, using that (see [35, Section 3])

$$\frac{k_1(n,s)}{\|\Phi\|_{L^2}^2} = \frac{\int_{\mathbb{R}^n} |y|^2 |\nabla \Phi|^2 dy}{\|\Phi\|_{L^2}^2} = \frac{n(n-2)(n+2-s)}{2(2n-2-s)},$$
$$\frac{k_2(n,s)}{\|\Phi\|_{L^2}^2} = \frac{\int_{\mathbb{R}^n} |y|^{2-s} |\Phi|^{2^\star(s)} dy}{\|\Phi\|_{L^2}^2} = \frac{n(n-4)}{2(n-2)(2n-2-s)},$$
$$\frac{\|\nabla \Phi\|_{L^2}^2}{\|\Phi\|_{L^{2^\star(s)}_{d_g,s}}^{2^\star(s)}} = (n-2)(n-s),$$

then for $n \geq 5$ we have

$$\frac{\int_M |\nabla_g u_\epsilon|^2 dv_g + \int_M \left(\frac{\alpha}{2^\star(s)}\frac{\tilde{a}}{\rho^\theta}u_\epsilon^2 + \frac{\beta}{2^\star(s)}\frac{\tilde{c}}{\rho^\gamma}u_\epsilon^2\right) dv_g + \int_M h u_\epsilon^2 dv_g}{\left(\int_M \frac{f(x)}{d_g(x,x_0)^s} u_\epsilon^{2^\star(s)} dv_g\right)^{\frac{2}{2^\star(s)}}}$$
$$= \frac{K(n,s)^{-1}}{f(x_0)^{\frac{2}{2^\star(s)}}}\left[1 - \frac{(n-2)(n-4)}{4(2n-2-s)}\frac{\Delta_g f(x_0)}{f(x_0)}\right]$$
$$- \frac{K(n,s)^{-1}}{f(x_0)^{\frac{2}{2^\star(s)}}}\frac{\|\Phi\|_{L^2}^2}{\|\nabla\Phi\|_{L^2}^2}\left[\frac{(n-2)(6-s)}{12(2n-2-s)}R_g(x_0) - h(x_0)\right]\epsilon^2 + o(\epsilon^2).$$

Similarly for $n = 4$ we get

$$\frac{\int_M |\nabla_g u_\epsilon|^2 dv_g + \int_M \left(\frac{\alpha}{2^\star(s)}\frac{\tilde{a}}{\rho^\theta}u_\epsilon^2 + \frac{\beta}{2^\star(s)}\frac{\tilde{c}}{\rho^\gamma}u_\epsilon^2\right) dv_g + \int_M h u_\epsilon^2 dv_g}{\left(\int_M \frac{f(x)}{d_g(x,x_0)^s} u_\epsilon^{2^\star(s)} dv_g\right)^{\frac{2}{2^\star(s)}}}$$
$$= \frac{K(4,s)^{-1}}{f(x_0)^{\frac{2}{2^\star(s)}}} + \frac{\omega_3(6h(x_0) - R_g(x_0))}{6f(x_0)^{\frac{2}{2^\star(s)}}\|\Phi\|^2_{L^{2^\star(s)}_{d_g,s}}}\epsilon^2\ln(\epsilon^{-1}) + o(\epsilon^2\ln(\epsilon^{-1})).$$

As $h(x_0) < \frac{n(n-2)(6-s)}{12(2n-2-s)}R_g(x_0) + \frac{(n-2)(n-4)}{4(2n-2-s)}\frac{\Delta_g f(x_0)}{f(x_0)}$ for $n \geq 4$, then for ϵ small enough, we reach that

$$\frac{\int_M |\nabla_g u_\epsilon|^2 dv_g + \int_M \left(\frac{\alpha}{2^\star(s)}\frac{\tilde{a}}{\rho^\theta}u_\epsilon^2 + \frac{\beta}{2^\star(s)}\frac{\tilde{c}}{\rho^\gamma}u_\epsilon^2\right) dv_g + \int_M h u_\epsilon^2 dv_g}{\left(\int_M \frac{f(x)}{d_g(x,x_0)^s} u_\epsilon^{2^\star(s)} dv_g\right)^{\frac{2}{2^\star(s)}}} < \frac{K(n,s)^{-1}}{f(x_0)^{\frac{2}{2^\star(s)}}}.$$

Therefore, for ϵ small enough we have that

$$\frac{E_h(\xi u_\epsilon, \zeta u_\epsilon)}{\left(\int_M \frac{f(x)}{d_g(x,x_0)^s}|\xi u_\epsilon|^\alpha|\zeta u_\epsilon|^\beta dv_g\right)^{\frac{2}{2^\star(s)}}}$$
$$= \frac{(\xi^2+\zeta^2)}{(\xi^\alpha\zeta^\beta)^{\frac{2}{2^\star(s)}}}\frac{\int_M |\nabla_g u_\epsilon|^2 dv_g + \int_M \left(\frac{\alpha}{2^\star(s)}\frac{\tilde{a}}{\rho^\theta}u_\epsilon^2 + \frac{\beta}{2^\star(s)}\frac{\tilde{c}}{\rho^\gamma}u_\epsilon^2\right) dv_g + \int_M h u_\epsilon^2 dv_g}{\left(\int_M \frac{f(x)}{d_g(x,x_0)^s} u_\epsilon^{2^\star(s)} dv_g\right)^{\frac{2}{2^\star(s)}}}$$
$$< \frac{\mathcal{K}^s_{(\alpha,\beta)}}{f(x_0)^{\frac{2}{2^\star(s)}}},$$

so if $\tau := \left(\int_M \frac{f(x)}{d_g(x,x_0)^s}|\xi u_\epsilon|^\alpha|\zeta u_\epsilon|^\beta dv_g\right)^{1/2^\star(s)}$ then $(\xi\tau^{-1}u_\epsilon, \zeta\tau^{-1}u_\epsilon) \in \Lambda^{(\alpha,\beta)}_{s,f}$ and

$$K^{(\alpha,\beta)}_{f,s} \leq E_h(\xi\tau^{-1}u_\epsilon, \zeta\tau^{-1}u_\epsilon) = \frac{E_h(\xi u_\epsilon, \zeta u_\epsilon)}{\left(\int_M \frac{f(x)}{d_g(x,x_0)^s}|\xi u_\epsilon|^\alpha|\zeta u_\epsilon|^\beta dv_g\right)^{\frac{2}{2^\star(s)}}} < \frac{\mathcal{K}^s_{(\alpha,\beta)}}{f(x_0)^{\frac{2}{2^\star(s)}}}.$$

From now on we will estimate the terms of the functional in the case which $n = 3$. Since in this case $\tilde{a} \equiv \tilde{c} \equiv 0$ and a, b and c satisfy the hypothesis $(\mathcal{H}_1)$ and (2.4) then

$-\Delta_g + h$ is a coercive operator. Indeed, given any $u \in H^1(M)$, by $(\mathcal{H}_1)$ for the pair $(\sqrt{\alpha}u, \sqrt{\beta}u)$ we have

$$\int_M |\nabla_g u|^2 dv_g + \int_M \left[\frac{\alpha}{2^\star(s)}a(x) + \frac{2\sqrt{\alpha\beta}}{2^\star(s)}b(x) + \frac{\beta}{2^\star(s)}c(x)\right] u^2 dv_g \geq C_0\|u\|^2_{H^1}.$$

As $h = \frac{\alpha}{2^\star(s)}a + \frac{2\sqrt{\alpha\beta}}{2^\star(s)}b + \frac{\beta}{2^\star(s)}c$, it follows the claim. With that, there exist $G_{x_0} : M\backslash\{x_0\} \to \mathbb{R}$ the Green function this operator. So,

$$-\Delta_g G_{x_0} + hG_{x_0} = \delta_{x_0},$$

where δ_{x_0} is the Dirac mass at x_0. It is well known that for x close to x_0 we can write

$$G_{x_0}(x) = \frac{1}{\omega_2 d_g(x, x_0)} + m(x_0) + o(1).$$

Next, we will use Druet's idea [16]. By using the cut-off function η, we can write G_{x_0} as follows:

$$\omega_2 G_{x_0}(x) = \frac{\eta(d_g(x, x_0))}{d_g(x, x_0)} + w_h(x), \tag{2.30}$$

where $w_h \in C^\infty_{loc}(M\backslash\{x_0\})$. In $M\backslash B_\delta(x_0)$, we have

$$-\Delta_g w_h + hw_h = \Delta_g\left(\frac{\eta}{d_g(x, x_0)}\right) - h\frac{\eta}{d_g(x, x_0)}. \tag{2.31}$$

And, in $B_\delta(x_0)$, we write in normal coordinates

$$-\Delta_g w_h + hw_h = -\frac{\partial_r(\ln(\det(g)))}{2d_g(x, x_0)^2} - h\frac{1}{d_g(x, x_0)}. \tag{2.32}$$

In particular, we have that the right side of the above equation is in $L^p(M)$ for all $1 < p < 3$, so by standard elliptic theory, $w_h \in C^{0,\theta}(M) \cap H^{2,p}(M) \cap C^{2,\gamma}(M\backslash\{x_0\})$ and moreover $w_h(x_0) = \omega_2 m(x_0)$ (for more details see Druet [16, Section 2]). Since we assume that or $h \leq \frac{1}{8}R_g$ and $h(x_0) < \frac{1}{8}R_g(x_0)$ or $h \equiv \frac{1}{8}R_g$ and (M, g) is not conformally equivalent to the standard sphere $\mathbb{S}^n$, then in any case we get that $w_h(x_0) > 0$ (for more details see [43], the second case follows from the positive mass theorem).

In this case, we consider the test function

$$v_\epsilon = u_\epsilon + \epsilon^{\frac{1}{2}} w_h. \tag{2.33}$$

Now, let's estimate $\int_M (|\nabla_g v_\epsilon|^2 + hv_\epsilon^2)dv_g$ and $\int_M \frac{f(x)}{d_g(x,x_0)^s} v_\epsilon^{2^\star(s)} dv_g$. First, from [35, Section 4] we have

$$\int_M (|\nabla_g v_\epsilon|^2 + hv_\epsilon^2)dv_g = \int_{\mathbb{R}^n} |\nabla \Phi|^2 dy + \epsilon\omega_2 w_h(x_0) + o(\epsilon). \tag{2.34}$$

So let us focus on estimating the other integral. As $s \in (0,2)$ then $6-2s > 2$ and therefore there is a positive constant $C(s)$ such that for all $y, z \in \mathbb{R}$ we have

$$\left||y+z|^{6-2s} - |y|^{6-2s} - (6-2s)yz|y|^{4-2s}\right| \leq C(s)(|y|^{4-2s}z^2 + |z|^{6-2s}).$$

With that, we can write

$$\begin{aligned}
\int_M \frac{f(x)}{d_g(x,x_0)^s} v_\epsilon^{2^\star(s)} dv_g &= \int_M \frac{f(x)}{d_g(x,x_0)^s}(u_\epsilon + \epsilon^{\frac{1}{2}} w_h)^{6-2s} dv_g \\
&= \int_{B_\delta(x_0)} \frac{f(x)}{d_g(x,x_0)^s}(u_\epsilon + \epsilon^{\frac{1}{2}} w_h)^{6-2s} dv_g + O(\epsilon^{3-s}) \\
&= \int_{B_\delta(x_0)} \frac{f(x)[u_\epsilon^{6-2s} + (6-2s)u_\epsilon^{5-2s}\epsilon^{\frac{1}{2}} w_h]}{d_g(x,x_0)^s} dv_g \\
&\quad + \tilde{C}(s) \int_{B_\delta(x_0)} \frac{f(x)[u_\epsilon^{4-2s}\epsilon w_h^2 + \epsilon^{3-s}|w_h|^{6-2s}]}{d_g(x,x_0)^s} dv_g + O(\epsilon^{3-s}).
\end{aligned}$$

Let's calculate:

- $\displaystyle\int_{B_\delta(x_0)} \frac{f(x)u_\epsilon^{6-2s}}{d_g(x,x_0)^s} dv_g = f(x_0)\int_{\mathbb{R}^3} \frac{|\Phi|^{6-2s}}{|y|^s} dy + o(\epsilon),$
- $\displaystyle\int_{B_\delta(x_0)} \frac{f(x)(6-2s)u_\epsilon^{5-2s}\epsilon^{\frac{1}{2}} w_h}{d_g(x,x_0)^s} dv_g = \epsilon(6-2s)f(x_0)w_h(x_0)\int_{\mathbb{R}^3} \frac{|\Phi|^{5-2s}}{|y|^s} dy + o(\epsilon),$
- $\displaystyle\int_{B_\delta(x_0)} \frac{f(x)[u_\epsilon^{4-2s}\epsilon w_h^2 + \epsilon^{3-s}|w_h|^{6-2s}]}{d_g(x,x_0)^s} dv_g = O(1)\int_0^\delta \frac{u_\epsilon^{4-2s}}{d_g(x,x_0)^s} dv_g + o(\epsilon) = o(\epsilon).$

Since $-\Delta\Phi = (3-s)\frac{\Phi^{5-2s}}{|y|^s}$ in $\mathbb{R}^3$ then

$$\int_{\mathbb{R}^3} \frac{|\Phi|^{5-2s}}{|y|^s} dy = (3-s)^{-1}\int_{\mathbb{R}^3} -\Delta\Phi dy = (3-s)^{-1}\lim_{r\to\infty}\int_{\partial B_r(0)} \partial_\nu \Phi d\sigma = (3-s)^{-1}\omega_2.$$

From the calculations above we reach that

$$\int_M \frac{f(x)}{d_g(x,x_0)^s} v_\epsilon^{2^\star(s)} dv_g = f(x_0)\|\Phi\|_{L^{2^\star(s)}_{d_g,s}}^{2^\star(s)} + 2\omega_2 f(x_0)w_h(x_0)\epsilon + o(\epsilon). \tag{2.35}$$

Hence, by (2.34) and (2.35), we get

$$\frac{\int_M(|\nabla_g v_\epsilon|^2 + h v_\epsilon^2)dv_g}{\left(\int_M \frac{f(x)}{d_g(x,x_0)^s} v_\epsilon^{2^\star(s)}\right)^{\frac{2}{2^\star(s)}} dv_g} = \frac{K(3,s)^{-1}}{f(x_0)^{\frac{2}{2^\star(s)}}}\left[1 - \frac{2\omega_2 w_h(x_0)}{\|\Phi\|_{L^{2^\star(s)}_{d_g,s}}^{2^\star(s)}}\epsilon + o(\epsilon)\right].$$

Since $w_h(x_0) > 0$. So for ϵ small enough (as in the case $n \geq 4$) it follows that

$$K_{f,s}^{(\alpha,\beta)} \leq E_h(\xi\tau_0^{-1}v_\epsilon, \zeta\tau_0^{-1}v_\epsilon) = \frac{E_h(\xi v_\epsilon, \zeta v_\epsilon)}{\left(\int_M \frac{f(x)}{d_g(x,x_0)^s}|\xi v_\epsilon|^\alpha|\zeta v_\epsilon|^\beta dv_g\right)^{\frac{2}{2^\star(s)}}} < \frac{\mathcal{K}^s_{(\alpha,\beta)}}{f(x_0)^{\frac{2}{2^\star(s)}}}$$

where $\tau_0 = \left(\int_M \frac{f(x)}{d_g(x,x_0)^s} v_\epsilon^{2^\star(s)}\right)^{\frac{1}{2^\star(s)}}$, that is, $(\xi\tau_0^{-1}v_\epsilon, \zeta\tau_0^{-1}v_\epsilon) \in \Lambda_{s,f}^{(\alpha,\beta)}$. Which completes the proof of the lemma. ■

We now have the tools for the proof of Theorem 2.1.1.

Proof of Theorem 2.1.1. Since by assumed assumptions we have that $f(x_0)^{\frac{2}{2^\star(s)}} K_{s,f}^{(\alpha,\beta)} < \mathcal{K}^s_{(\alpha,\beta)}$, then from Proposition 2.3.1 there exists $(u_0, v_0) \in \Lambda_{s,f}^{\alpha,\beta}$ such that $E_h(u_0, v_0) = K_{s,f}^{(\alpha,\beta)}$. Denote by $G(u,v) = \int_M \frac{f(x)}{d_g(x,x_0)^s}|u|^\alpha|v|^\beta dv_g - 1$, where $(u,v) \in H$. Then, there is a Lagrange multiplier λ that satisfies

$$E_h'(u_0, v_0)\cdot(\varphi,\psi) - \lambda G'(u_0,v_0)\cdot(\varphi,\psi) = 0, \text{ for all } (\varphi,\psi) \in H. \tag{2.36}$$

Taking $\varphi = u_0$ and $\psi = v_0$ above, then $2E_h(u_0,v_0) = 2^\star(s)\lambda$, hence $\lambda = \frac{2}{2^\star(s)}K_{s,f}^{(\alpha,\beta)} > 0$. Therefore, by (2.36), we have that (u_0, v_0) is a weak solution of the system

$$\begin{cases} -\Delta_g u + \frac{\tilde{a}(x)}{\rho(x)^\theta}u + au + bv &= K_{s,f}^{(\alpha,\beta)} \frac{2\alpha}{2^\star(s)} f(x)u|u|^{\alpha-2}|v|^\beta \text{ in } M, \\ -\Delta_g v + \frac{\tilde{c}(x)}{\rho(x)^\gamma}v + bu + cv &= K_{s,f}^{(\alpha,\beta)} \frac{2\beta}{2^\star(s)} f(x)v|v|^{\beta-2}|u|^\alpha \text{ in } M. \end{cases} \tag{2.37}$$

It is easy to see that the pair $((2K_{s,f}^{(\alpha,\beta)})^{1/(2^\star(s)-2)}u_0, (2K_{s,f}^{(\alpha,\beta)})^{1/(2^\star(s)-2)}v_0)$ is a pair of weak solutions of system (2.1). This completes the proof of the theorem. ■

To prove that from the hypotheses assumed in Corollary 2.1.2 we obtain a positive solution of the system (2.1), let us consider the constrained set

$$\Lambda_{s,f,+}^{\alpha,\beta} := \left\{(u,v) \in H \ : \ \int_M \frac{f(x)}{d_g(x,x_0)^s}(u^+)^\alpha(v^+)^\beta dv_g = 1\right\}$$

and

$$K_{f,h,+}^{(\alpha,\beta)} := \inf_{(u,v)\in\Lambda_{s,f,+}^{\alpha,\beta}} E_h(u,v).$$

Note that, if $b \le 0$ in M, it is easy to see that $E_h(|u|,|v|) \le E_h(u,v)$, and that if $(u,v) \in \Lambda_{s,f}^{\alpha,\beta}$ then $(|u|,|v|) \in \Lambda_{s,f,+}^{\alpha,\beta}$, therefore, we deduce that $K_{f,h,+}^{(\alpha,\beta)} \le K_{s,f}^{(\alpha,\beta)}$. Then, by Lemma 2.4.2, we get that $f(x_0)^{\frac{2}{2^\star(s)}}K_{f,h,+}^{(\alpha,\beta)} < \mathcal{K}^s_{(\alpha,\beta)}$. Moreover, we claim that $K_{s,h,+}^{(\alpha,\beta)} > 0$, indeed,

$$\begin{aligned} E_h(u,v) \ge C_0||(u,v)||_H^2 &\ge \frac{C}{f(x_0)^{\frac{2}{2^\star(s)}}}\left(\int_M \frac{f(x)}{d_g(x,x_0)^s}(u^+)^\alpha(v^+)^\beta dv_g\right)^{\frac{2}{2^\star(s)}} \\ &\ge \frac{C}{f(x_0)^{\frac{2}{2^\star(s)}}} > 0, \end{aligned}$$

for all $(u,v) \in \Lambda^{\alpha,\beta}_{s,f,+}$.

Proof of Corollary 2.1.2. Let $\{(u_m, v_m)\} \subset \Lambda^{\alpha,\beta}_{s,f,+}$ be a minimizing sequence for $K^{(\alpha,\beta)}_{f,h,+}$. Argues as in Proposition 2.3.1, we obtain a pair $(u,v) \in \Lambda^{\alpha,\beta}_{s,f,+}$ such that $E_h(u,v) = K^{(\alpha,\beta)}_{f,h,+}$, with $u \neq 0$ and $v \neq 0$, where $u_m \rightharpoonup u$ and $v_m \rightharpoonup v$ in $H^1(M)$. Now, we claim that $u \geq 0$ and $v \geq 0$ in M. Indeed, if we consider $G_+(u,v) = \int_M \frac{f(x)}{d_g(x,x_0)^s}(u^+)^\alpha (v^+)^\beta dv_g - 1$, there is a Lagrange multiplier λ such that

$$E_h'(u,v)\cdot(\varphi,\psi) - \lambda G_+'(u,v)\cdot(\varphi,\psi) = 0, \text{ for all } (\varphi,\psi) \in H. \tag{2.38}$$

Taking $\varphi = u^-$ and $\psi = v^-$ as test functions above, we have

$$2E_h(u^-, v^-) + 2\int_M b(u^+v^- + u^-v^+)dv_g = 0.$$

Since $b \leq 0$, it follows that $E_h(u^-, v^-) \leq 0$, hence $u^- = v^- = 0$. Thus, we conclude that $u \geq 0$ and $v \geq 0$. Considering $\varphi = u$ and $\psi = v$ as test functions in (2.38), we get $2E_h(u,v) = 2^\star(s)\lambda > 0$, and consequently $\lambda = \frac{2}{2^\star(s)} K^{(\alpha,\beta)}_{f,A,+} > 0$. Therefore, $((2K^{(\alpha,\beta)}_{f,h,+})^{1/(2^\star(s)-2)}u, (2K^{(\alpha,\beta)}_{f,h,+})^{1/(2^\star(s)-2)}v)$ is a pair of non-negative weak solutions of system (2.1). ■

2.5 Proof of Theorem 2.1.4

In the present section, we prove our second theorem. Here we assume only that $\tilde{a}$ and $\tilde{c}$ are functions Hölder continuous. Consequently, the estimation of singular term of functional E_h will be different, where we use the same test function of previous section in the case $n \geq 4$. Therefore, under the assumptions of Theorem 2.1.4 we will prove in the following auxiliary lemma that $f(x_0)^{\frac{2}{2^\star(s)}} K^{(\alpha,\beta)}_{s,f} < \mathcal{K}^s_{(\alpha,\beta)}$.

Lemma 2.5.1 *If we assume the same assumptions of Theorem 2.1.4. Then we have*

$$f(x_0)^{\frac{2}{2^\star(s)}} K^{(\alpha,\beta)}_{s,f} < \mathcal{K}^s_{(\alpha,\beta)}.$$

Proof. Using the same test function (2.17), we have

$$K^{(\alpha,\beta)}_{s,f} \leq E_h(\xi\tau^{-1}u_\epsilon, \zeta\tau^{-1}u_\epsilon) = \frac{E_h(\xi u_\epsilon, \zeta u_\epsilon)}{\left(\int_M \frac{f(x)}{d_g(x,x_0)^s}|\xi u_\epsilon|^\alpha |\zeta u_\epsilon|^\beta dv_g\right)^{\frac{2}{2^\star(s)}}}$$

$$= \kappa(\alpha,\beta)\frac{\int_M |\nabla_g u_\epsilon|^2 dv_g + \int_M \left(\frac{\alpha}{2^\star(s)}\frac{\tilde{a}}{\rho^\theta}u_\epsilon^2 + \frac{\beta}{2^\star(s)}\frac{\tilde{c}}{\rho^\gamma}u_\epsilon^2\right)dv_g + \int_M h u_\epsilon^2 dv_g}{\left(\int_M \frac{f(x)}{d_g(x,x_0)^s}u_\epsilon^{2^\star(s)}dv_g\right)^{\frac{2}{2^\star(s)}}},$$

where $\kappa(\alpha,\beta) := \left[\left(\frac{\alpha}{\beta}\right)^{\beta/2^\star(s)} + \left(\frac{\beta}{\alpha}\right)^{\alpha/2^\star(s)}\right]$, $\left(\frac{\xi}{\zeta}\right)^2 = \frac{\alpha}{\beta}$ and $\tau := \left(\int_M \frac{f(x)u_\epsilon^{2^\star(s)}}{d_g(x,x_0)^s} dv_g\right)^{1/2^\star(s)}$.

From [35, Section 3] we have that

$$\int_M |\nabla_g u_\epsilon|^2 dv_g = \begin{cases} \int_{\mathbb{R}^n} |\nabla\Phi|^2 dy - \frac{\int_{\mathbb{R}^n} |y|^2 |\nabla\Phi|^2 dy}{6n} R_g(x_0)\epsilon^2 + o(\epsilon^2), & \text{if } n \geq 5, \\ \int_{\mathbb{R}^4} |\nabla\Phi|^2 dy - \frac{\omega_3}{6} R_g(x_0)\epsilon^2 \ln(\epsilon^{-1}) + O(\epsilon^2), & \text{if } n = 4, \\ \int_{\mathbb{R}^3} |\nabla\Phi|^2 dy + O(\epsilon), & \text{if } n = 3, \end{cases} \tag{2.39}$$

and

$$\int_M h(x) u_\epsilon^2 dv_g = \begin{cases} h(x_0) \int_{\mathbb{R}^n} \Phi^2 dy \epsilon^2 + o(\epsilon^2) \text{ if } n \geq 5, \\ h(x_0)\omega_3 \epsilon^2 \ln(\epsilon^{-1}) + O(\epsilon^2) \text{ if } n = 4. \\ O(\epsilon), \quad \text{if} \quad n = 3. \end{cases} \tag{2.40}$$

As already calculated in Lemma 2.4.2 we have

$$\int_M \frac{f(x)u_\epsilon^{2^\star(s)}}{d_g(x,x_0)^s} dv_g = \begin{cases} f(x_0) \|\Phi\|^{2^\star(s)}_{L^{2^\star(s)}_{d_g,s}(\mathbb{R}^n)} + O(\epsilon^2), & \text{if } n \geq 4, \\ f(x_0) \|\Phi\|^{2^\star(s)}_{L^{2^\star(s)}_{d_g,s}(\mathbb{R}^3)} + O(\epsilon), & \text{if} \quad n = 3. \end{cases} \tag{2.41}$$

Now let us estimate $\int_M \frac{\tilde{a}}{\rho^\theta} u_\epsilon^2 dv_g$. So, for $n \geq 4$ we have that

$$\begin{aligned} \int_M \frac{\tilde{a}}{\rho^\theta} u_\epsilon^2 dv_g &= \int_{B_\delta(x_0)} \frac{\tilde{a}}{\rho^\theta} u_\epsilon^2 dv_g + \int_{M \setminus B_\delta(x_0)} \frac{\tilde{a}}{\rho^\theta} u_\epsilon^2 dv_g \\ &= \int_{B_\delta(x_0)} \frac{\tilde{a}}{\rho^\theta} u_\epsilon^2 dv_g + O(\epsilon^{n-2}). \end{aligned}$$

As in normal coordinate we can write $\sqrt{\det(g)}(y) = (1 + O(|y|^2))$ and by hypothesis $\rho(\exp_{x_0}(y)) = |y| + O(|y|^{1+\mu})$ in $B_\delta(0)$ (δ is small enough), since I^p_q denotes the $\int_0^\infty \frac{t^p}{(1+t)^q} dt$, then we get

$$\begin{aligned} \int_{B_\delta(x_0)} \frac{\tilde{a}}{\rho^\theta} u_\epsilon^2 dv_g &= \int_{B_\delta(0)} \frac{(\tilde{a} \circ \exp_{x_0})(y)}{(|y| + O(|y|^{1+\mu}))^\theta} ((u_\epsilon \circ \exp_{x_0})(y))^2 \sqrt{\det(g)} dy \\ &= \int_{B_\delta(0)} \frac{(\tilde{a} \circ \exp_{x_0})(y)}{|y|^\theta (1 + O(|y|^\mu))} ((u_\epsilon \circ \exp_{x_0})(y))^2 \sqrt{\det(g)} dy \\ &= \tilde{a}(x_0) \int_{B_\delta(0)} \frac{((u_\epsilon \circ \exp_{x_0})(y))^2}{|y|^\theta (1 + O(|y|^\mu))} dy + \int_{B_\delta(0)} \frac{\tilde{b}(y)}{|y|^\theta (1 + O(|y|^\mu))} ((u_\epsilon \circ \exp_{x_0})(y))^2 dy \\ &= \tilde{a}(x_0) \int_{B_\delta(0)} \frac{((u_\epsilon \circ \exp_{x_0})(y))^2}{|y|^\theta} dy + \int_{B_\delta(0)} \frac{\left[\tilde{b}(y) + O(|y|^\mu)\right]}{|y|^\theta (1 + O(|y|^\mu))} ((u_\epsilon \circ \exp_{x_0})(y))^2 dy \\ &= \tilde{a}(x_0)\epsilon^{n-2} \int_{B_\delta(0)} \frac{1}{|y|^\theta} \frac{1}{(\epsilon^{2-s} + |y|^{2-s})^{\frac{2(n-2)}{2-s}}} dy + o(\epsilon^{2-\theta}) \\ &= \frac{\tilde{a}(x_0)\omega_{n-1}}{2-s} I^{\frac{n-2+s-\theta}{2-s}}_{\frac{2(n-2)}{2-s}} \epsilon^{2-\theta} + o(\epsilon^{2-\theta}), \end{aligned}$$

where $\tilde{b}(y) = [(\tilde{a} \circ \exp_{x_0})(y) - \tilde{a}(x_0)](1 + O(|y|^2))$. So for $n \geq 4$ we reach

$$\int_M \frac{\tilde{a}}{\rho^\theta} u_\epsilon^2 dv_g = \frac{\tilde{a}(x_0)\omega_{n-1}}{2-s} I_{\frac{2(n-2)}{2-s}}^{\frac{n-2+s-\theta}{2-s}} \epsilon^{2-\theta} + o(\epsilon^{2-\theta}). \tag{2.42}$$

Similarly, also we have

$$\int_M \frac{\tilde{c}}{\rho^\gamma} u_\epsilon^2 dv_g = \frac{\tilde{c}(x_0)\omega_{n-1}}{2-s} I_{\frac{2(n-2)}{2-s}}^{\frac{n-2+s-\gamma}{2-s}} \epsilon^{2-\gamma} + o(\epsilon^{2-\gamma}). \tag{2.43}$$

In case $n = 3$. We get that

$$\int_M \frac{\tilde{a}}{\rho^\theta} u_\epsilon^2 dv_g = \begin{cases} \dfrac{\tilde{a}(x_0)\omega_2}{2-s} I_{\frac{2}{2-s}}^{\frac{1-\theta+s}{2-s}} \epsilon^{2-\theta} + o(\epsilon^{2-\theta}), & \text{if } \theta > 1 \\ \tilde{a}(x_0)\omega_2 \epsilon \ln(\epsilon^{-1}) + O(\epsilon), & \text{if } \theta = 1 \\ O(\epsilon), & \text{if } \theta < 1. \end{cases} \tag{2.44}$$

Similarly,

$$\int_M \frac{\tilde{c}}{\rho^\gamma} u_\epsilon^2 dv_g = \begin{cases} \dfrac{\tilde{c}(x_0)\omega_2}{2-s} I_{\frac{2}{2-s}}^{\frac{1-\gamma+s}{2-s}} \epsilon^{2-\gamma} + o(\epsilon^{2-\gamma}), & \text{if } \gamma > 1 \\ \tilde{c}(x_0)\omega_2 \epsilon \ln(\epsilon^{-1}) + O(\epsilon), & \text{if } \gamma = 1 \\ O(\epsilon), & \text{if } \gamma < 1. \end{cases} \tag{2.45}$$

Hence, we can estimate the quotient

$$\frac{\int_M |\nabla_g u_\epsilon|^2 dv_g + \int_M \left(\frac{\alpha}{2^\star(s)} \frac{\tilde{a}}{\rho^\theta} u_\epsilon^2 + \frac{\beta}{2^\star(s)} \frac{\tilde{c}}{\rho^\gamma} u_\epsilon^2 \right) dv_g + \int_M h u_\epsilon^2 dv_g}{\left(\int_M \frac{f(x)}{d_g(x,x_0)^s} u_\epsilon^{2^\star(s)} dv_g \right)^{\frac{2}{2^\star(s)}}}.$$

In fact, according to the above estimates, for $n \geq 4$ we reach

$$\begin{aligned} &\frac{\int_M |\nabla_g u_\epsilon|^2 dv_g + \int_M \left(\frac{\alpha}{2^\star(s)} \frac{\tilde{a}}{\rho^\theta} u_\epsilon^2 + \frac{\beta}{2^\star(s)} \frac{\tilde{c}}{\rho^\gamma} u_\epsilon^2 \right) dv_g + \int_M h u_\epsilon^2 dv_g}{\left(\int_M \frac{f(x)}{d_g(x,x_0)^s} u_\epsilon^{2^\star(s)} dv_g \right)^{\frac{2}{2^\star(s)}}} \\ &= \frac{||\nabla \Phi||_2^2 + \frac{\alpha \tilde{a}(x_0)\omega_{n-1}}{2^\star(s)(2-s)} I_{\frac{2(n-2)}{2-s}}^{\frac{n-2+s-\theta}{2-s}} \epsilon^{2-\theta} + o(\epsilon^{2-\theta}) + \frac{\beta \tilde{c}(x_0)\omega_{n-1}}{2^\star(s)(2-s)} I_{\frac{2(n-2)}{2-s}}^{\frac{n-2+s-\gamma}{2-s}} \epsilon^{2-\gamma} + o(\epsilon^{2-\gamma})}{f(x_0)^{\frac{2}{2^\star(s)}} \|\Phi\|^2_{L^{2^\star(s)}_{d_g,s}(\mathbb{R}^n)} (1 + O(\epsilon^2))} \\ &= \frac{K(n,s)^{-1}}{f(x_0)^{\frac{2}{2^\star(s)}}} + \alpha \tilde{a}(x_0) C_1(n,s,f,\theta)\epsilon^{2-\theta} + \beta \tilde{c}(x_0) C_2(n,s,f,\gamma)\epsilon^{2-\gamma} + o(\epsilon^{2-\theta}) + o(\epsilon^{2-\gamma}), \end{aligned}$$

where $C_1(n,s,f,\theta) = \dfrac{\omega_{n-1} f(x_0)^{-\frac{2}{2^\star(s)}}}{2^\star(s)(2-s)\|\Phi\|^2_{L^{2^\star(s)}_{d_g,s}}} I_{\frac{2(n-2)}{2-s}}^{\frac{n-2+s-\theta}{2-s}}$ and $C_2(n,s,f,\gamma) =$

$\frac{\omega_{n-1} f(x_0)^{-\frac{2}{2^\star(s)}}}{2^\star(s)(2-s)\|\Phi\|^2_{L^{2^\star(s)}_{d_g,s}}} I^{\frac{n-2+s-\gamma}{2-s}}_{\frac{2(n-2)}{2-s}}$ are positive constants. So, for $n \geq 4$ we get that

$$\frac{\int_M |\nabla_g u_\epsilon|^2 dv_g + \int_M \left(\frac{\alpha}{2^\star(s)} \frac{\tilde{a}}{\rho^\theta} u_\epsilon^2 + \frac{\beta}{2^\star(s)} \frac{\tilde{c}}{\rho^\gamma} u_\epsilon^2 \right) dv_g + \int_M h u_\epsilon^2 dv_g}{\left(\int_M \frac{f(x)}{d_g(x,x_0)^s} u_\epsilon^{2^\star(s)} dv_g \right)^{\frac{2}{2^\star(s)}}}$$

$$= \begin{cases} \frac{K(n,s)^{-1}}{f(x_0)^{\frac{2}{2^\star(s)}}} + \tilde{a}(x_0)\alpha C_1(n,s,f,\theta)\epsilon^{2-\theta} + o(\epsilon^{2-\theta}), & \text{if } \theta > \gamma, \\ \frac{K(n,s)^{-1}}{f(x_0)^{\frac{2}{2^\star(s)}}} + [\alpha\tilde{a}(x_0) + \beta\tilde{c}(x_0)] C_1(n,s,f,\theta)\epsilon^{2-\theta} + o(\epsilon^{2-\theta}), & \text{if } \theta = \gamma, \\ \frac{K(n,s)^{-1}}{f(x_0)^{\frac{2}{2^\star(s)}}} + \tilde{c}(x_0)\beta C_2(n,s,f,\gamma)\epsilon^{2-\gamma} + o(\epsilon^{2-\gamma}), & \text{if } \theta < \gamma. \end{cases} \quad (2.46)$$

Since for $n \geq 4$, $\tilde{a}(x_0) < 0$ if $\theta > \gamma$, or $\alpha\tilde{a}(x_0) + \beta\tilde{c}(x_0) < 0$ if $\theta = \gamma$, or $\tilde{c}(x_0) < 0$ if $\gamma > \theta$. Then for ϵ small enough we reach that $f(x_0)^{\frac{2}{2^\star(s)}} K^{(\alpha,\beta)}_{s,f} < \mathcal{K}^s_{(\alpha,\beta)}$, for $n \geq 4$.

Now, when $n = 3$ we have

$$\frac{\int_M |\nabla_g u_\epsilon|^2 dv_g + \int_M \left(\frac{\alpha}{2^\star(s)} \frac{\tilde{a}}{\rho^\theta} u_\epsilon^2 + \frac{\beta}{2^\star(s)} \frac{\tilde{c}}{\rho^\gamma} u_\epsilon^2 \right) dv_g + \int_M h u_\epsilon^2 dv_g}{\left(\int_M \frac{f(x)}{d_g(x,x_0)^s} u_\epsilon^{2^\star(s)} dv_g \right)^{\frac{2}{2^\star(s)}}}$$

$$= \begin{cases} \frac{K(3,s)^{-1}}{f(x_0)^{\frac{2}{2^\star(s)}}} + \tilde{a}(x_0)\alpha C_1(s,f,\theta)\epsilon^{2-\theta} + o(\epsilon^{2-\theta}), & \text{if } \theta > 1 \text{ and } \theta > \gamma, \\ \frac{K(3,s)^{-1}}{f(x_0)^{\frac{2}{2^\star(s)}}} + \tilde{a}(x_0)\frac{\alpha\omega_2}{2^\star(s)}\epsilon\ln(\epsilon^{-1}) + o(\epsilon\ln(\epsilon^{-1})), & \text{if } \theta = 1 > \gamma, \\ \frac{K(3,s)^{-1}}{f(x_0)^{\frac{2}{2^\star(s)}}} + \tilde{c}(x_0)\beta C_2(s,f,\gamma)\epsilon^{2-\gamma} + o(\epsilon^{2-\gamma}), & \text{if } \gamma > 1 \text{ and } \gamma > \theta, \\ \frac{K(3,s)^{-1}}{f(x_0)^{\frac{2}{2^\star(s)}}} + \tilde{c}(x_0)\frac{\beta\omega_2}{2^\star(s)}\epsilon\ln(\epsilon^{-1}) + o(\epsilon\ln(\epsilon^{-1})), & \text{if } \gamma = 1 > \theta, \\ \frac{K(3,s)^{-1}}{f(x_0)^{\frac{2}{2^\star(s)}}} + [\alpha\tilde{a}(x_0) + \beta\tilde{c}(x_0)] C_1(s,f,\theta)\epsilon^{2-\theta} + o(\epsilon^{2-\theta}), & \text{if } \theta = \gamma > 1, \\ \frac{K(3,s)^{-1}}{f(x_0)^{\frac{2}{2^\star(s)}}} + \left[\frac{\alpha\tilde{a}(x_0)}{2^\star(s)} + \frac{\beta\tilde{c}(x_0)}{2^\star(s)} \right] \omega_2\epsilon\ln(\epsilon^{-1}) + o(\epsilon\ln(\epsilon^{-1})), & \text{if } \theta = \gamma = 1, \\ \frac{K(3,s)^{-1}}{f(x_0)^{\frac{2}{2^\star(s)}}} + O(\epsilon), & \text{if } \theta < 1 \text{ and } \gamma < 1, \end{cases}$$

where $C_1(s,f,\theta) = \frac{\omega_2 f(x_0)^{-\frac{2}{2^\star(s)}}}{2^\star(s)(2-s)\|\Phi\|^2_{L^{2^\star(s)}_{d_g,s}}} I^{\frac{1+s-\theta}{2-s}}_{\frac{2}{2-s}}$ and $C_2(s,f,\gamma) = \frac{\omega_2 f(x_0)^{-\frac{2}{2^\star(s)}}}{2^\star(s)(2-s)\|\Phi\|^2_{L^{2^\star(s)}_{d_g,s}}} I^{\frac{1+s-\gamma}{2-s}}_{\frac{2}{2-s}}$ are positive constants. As for $n = 3$, $\tilde{a}(x_0) < 0$ if $\theta > \gamma$ and $\theta \geq 1$, or $\alpha\tilde{a}(x_0) + \beta\tilde{c}(x_0) < 0$ if $\theta = \gamma \geq 1$, or $\tilde{c}(x_0) < 0$ if $\gamma > \theta$ and $\gamma \geq 1$. This implies that when $n = 3$ and ϵ is small enough we get $f(x_0)^{\frac{2}{2^\star(s)}} K^{(\alpha,\beta)}_{s,f} < \mathcal{K}^s_{(\alpha,\beta)}$. ■

Proof of Theorem 2.1.4. Arguing in the same way as it was done in the proof of Theorem 2.1.1, we establish the existence of a pair of nontrivial solutions for system (2.1). ■

Chapter 3

On a Class of Quasilinear Elliptic Systems involving Critical Growth

In this chapter, inspired by ideas from Chen and Liu [11], Demengel and Hebey [12], and Druet [15], we establish the existence of solutions for a class of generalized coupled elliptic systems involving the p-Laplace-Beltrami operator. Here, we focus on the case where $p \neq 2$, as we have already studied the case of $p = 2$ in Chapters 1 and 2. The main tool used here is the Mountain Pass Theorem. We encounter an additional difficulty due to the lack of the Hilbert space structure of $H^{1,p}(M)$. This is because if $u_m \rightharpoonup u$ in $H^{1,p}(M)$, it does not necessarily hold that $|\nabla_g u_m|^{p-2}\nabla_g u_m \rightharpoonup |\nabla_g u|^{p-2}\nabla_g u$ in $H^{1,p}(M)$.

3.1 Introduction

Let (M, g) be a smooth closed Riemannian manifold of dimension $n \geq 2$. We are concerned with the existence of solutions of the following Hardy-Sobolev system:

$$\begin{cases} -\Delta_{p,g}u + a(x)|u|^{p-2}u + b(x)[(p-1)|u|^{p-2} + |v|^{p-2}]v = \dfrac{\alpha}{p^\star(s)}\dfrac{f(x)u|u|^{\alpha-2}|v|^{\beta}}{d_g(x,x_0)^s}, \\ -\Delta_{p,g}v + b(x)[(p-1)|v|^{p-2} + |u|^{p-2}]u + c(x)|v|^{p-2}v = \dfrac{\beta}{p^\star(s)}\dfrac{f(x)v|v|^{\beta-2}|u|^{\alpha}}{d_g(x,x_0)^s}, \end{cases} \tag{3.1}$$

where $\Delta_{p,g}$ is the p-Laplace-Beltrami operator, $p \in (1, n)$ with $p \neq 2$, $a, b, c \in C^{0,\varrho}(M)$ for some $\varrho \in (0, 1)$ with $b \equiv 0$ when $1 < p < 2$, $x_0 \in M$, $s \in [0, p)$, $f \in C^\infty(M)$ with

$f(x_0) = \max_M f > 0$ and $\alpha > 1$, $\beta > 1$ are real constants satisfying $\alpha + \beta = p^\star(s)$, where $p^\star(s) = \frac{p(n-s)}{(n-p)}$ is the critical Hardy-Sobolev exponent.

Next, we will present some works that, along with the previous chapters, motivated us to study the above problem. First is the paper by Druet [15], in which the author considers a generalized elliptic Yamabe-type equation

$$-\Delta_{p,g} u + a(x)|u|^{p-2}u = f(x)|u|^{p^*-2}u \ \text{ in } \ M,$$

the author proved some existence results for this equation on compact manifolds. Another paper that also motivated us, was the work of Chen and Liu in [11], who investigated the Hardy-Sobolev equation

$$-\Delta_{p,g} u + a(x)u = f(x)\frac{|u|^{p^\star(s)-2}u}{d_g(x,x_0)^s} \ \text{ in } \ M$$

They proved a Hardy-Sobolev inequality on compact Riemannian manifolds and an existence result for this equation. Two sufficient conditions (when $p \neq 2$) which both of the above-mentioned works present (see Druet [15] for $s = 0$), the following:

$$(i)\ 2 < p < \sqrt{n} \text{ and } 0 < R_g(x_0) + \frac{3(n+2-3p)}{(3p-s)}\frac{\Delta f(x_0)}{f(x_0)};$$
$$(ii)\ 1 < p < \min\left\{2, \frac{n+2}{3}\right\} \text{ and } a(x_0) < 0.$$

Here in this chapter, in particular, we were able to improve the estimates made by them in their results, and thus expand the interval for $p \neq 2$. More precisely, we improved the conditions for:

$$(i)'\ 2 < p \leq \frac{n+2}{3};$$
$$(ii)'\ 1 < p < 2 \text{ for all } n \geq 4, \text{ and } 1 < p \leq \sqrt{n} \text{ when } n = 3, 2;$$

because $\sqrt{n} < \frac{n+2}{3}$, for all $n \geq 5$, moreover, when $n = 4, 3$ or $n = 2$ we have that, $\sqrt{4} = 2 = \frac{6}{3} = \frac{4+2}{3}$, $\frac{5}{3} = \frac{3+2}{3} < \sqrt{3}$ and $\frac{4}{3} = \frac{2+2}{3} < \sqrt{2}$, respectively.

Now, before presenting our main results, we need to introduce some notations and definitions. Throughout this work, we will denote by $H^{1,p}(M)$ the Sobolev space of all functions in $L^p(M)$ with one derivative (in the weak sense) in $L^p(M)$. We equip $H^{1,p}(M)$ with the standard $\|\cdot\|_{H^{1,p}}-$norm, that is, $\|u\|^p_{H^{1,p}} = \|\nabla u\|^p_p + \|u\|^p_p$, where

$\|\cdot\|_q$ denotes the norm of the Lebesgue space $L^q(M)$, whenever $q \geq 1$. The norm of $L^q(M) \times L^q(M)$ will be defined by $\|(u,v)\|_q = \left(\|u\|_q^q + \|v\|_q^q\right)^{1/q}$.

Here, we will work in the space $H^p = H^{1,p}(M) \times H^{1,p}(M)$ endowed with the norm

$$\|(u,v)\| = \left(\|u\|_{H^{1,p}}^p + \|v\|_{H^{1,p}}^p\right)^{1/p}.$$

As is know $p^\star(s) = \frac{p(n-s)}{(n-p)}$ is the critical exponent of the continuous embedding of $H^{1,p}(M)$ in the weighted Lebesgue space $L^q_{d_g,s}(M) := L^q(M, d_g(\cdot, x_0)^{-s})$ for $1 \leq q \leq p^\star(s)$, which is a compact embedding when $1 \leq q < p^\star(s)$, see [11, Lemma 3.1]. We equip $L^q_{d_g,s}(M)$ with the norm

$$\|u\|_{q,s} = \left(\int_M \frac{|u|^q}{d_g(x,x_0)^s} dv_g\right)^{\frac{1}{q}}.$$

In this context, we say that a pair of functions $(u,v) \in H^p$ is a weak solution of (3.1), if for every $(\varphi,\psi) \in H^p$, it holds

$$\begin{aligned}\int_M \big(\langle|\nabla_g u|^{p-2}\nabla_g u, \nabla_g\varphi\rangle + \langle|\nabla_g v|^{p-2}\nabla_g v, \nabla\psi\rangle + a(x)|u|^{p-2}u\varphi + c(x)|v|^{p-2}v\psi \\ + b(x)[\,(p-1)|u|^{p-2} + |v|^{p-2}]v\varphi + (p-1)|v|^{p-2} + |u|^{p-2}u\psi\,]\big)\, dv_g \quad (3.2)\\ = \int_M \frac{\alpha}{p^\star(s)}\frac{f(x)}{d_g(x,x_0)^s}|u|^{\alpha-2}|v|^\beta u\varphi dv_g + \int_M \frac{\beta}{p^\star(s)}\frac{f(x)}{d_g(x,x_0)^s}|v|^{\beta-2}|u|^\alpha v\psi dv_g,\end{aligned}$$

when $1 < p < 2$, we assume $b \equiv 0$.

Recalling the relation (1.3) established by Alves et al. [2], which can be easily obtained when $p \neq 2$ and $s \geq 0$, it is as follows:

$$\mathcal{K}^{p,s}_{(\alpha,\beta)} = \kappa(\alpha,\beta)K(n,p,s), \quad (3.3)$$

where $\kappa(\alpha,\beta) := \left[\left(\frac{\alpha}{\beta}\right)^{\beta/p^\star(s)} + \left(\frac{\beta}{\alpha}\right)^{\alpha/p^\star(s)}\right]$ and $K(n,p,s)$ is the best Hardy-Sobolev constant defined by

$$K(n,p,s) = \inf_{u\in H^{1,p}(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n}|\nabla u|^p dx}{\left(\int_{\mathbb{R}^n}\frac{|u|^{p^\star(s)}}{|x|^s}dx\right)^{\frac{p}{p^\star(s)}}},$$

and $\mathcal{K}^{p,s}_{(\alpha,\beta)}$ is defined by

$$\mathcal{K}^{p,s}_{(\alpha,\beta)} = \inf_{(u,v)\in[H^{1,p}(\mathbb{R}^n)]^2\setminus\{0\}} \frac{\int_{\mathbb{R}^n}(|\nabla u|^p + |\nabla v|^p)\,dx}{\left(\int_{\mathbb{R}^n}\frac{|u|^\alpha|v|^\beta}{|x|^s}dx\right)^{\frac{p}{p^\star(s)}}}. \quad (3.4)$$

Throughout this chapter we assume some very general hypotheses about the functions a, b, c and f that will allow us to obtain some existence results for system (3.1) through variational methods. We assume that the functions a, b and c are Hölder continuous and f is a smooth function on M. In addition, these functions satisfy the following coercivity condition, that is, there exists $C_0 > 0$ such that

$$\int_M \left[|\nabla_g u|^p + |\nabla_g v|^p + a(x)|u|^p + b(x)uv(|u|^{p-2} + |v|^{p-2}) + c(x)|v|^p\right] dv_g \geq C_0\|(u, v)\|^p, \tag{3.5}$$

for all $(u, v) \in H^p$, where $b \equiv 0$ when $1 < p < 2$.

Our first result in this chapter can be stated as follows:

Teorema 3.1.1 *Let (M, g) be a closed Riemannian manifold of dimension $n \geq 5$, $p \in (2, n)$ and $s \in [0, p)$. Let a, b and c be functions Hölder continuous in M satisfying (3.5) and f smooth function on M such that $f(x_0) = \max_M f > 0$. If, in addition, $2 < p \leq \frac{n+2}{3}$ and*

$$0 < R_g(x_0) + \frac{3(n + 2 - 3p)}{(3p - s)} \frac{\Delta f(x_0)}{f(x_0)}. \tag{3.6}$$

Then, system (3.1) has a pair of nontrivial solution.

A consequence of Theorem 3.1.1 is the following result.

Corollary 3.1.2 *Suppose the same assumptions of Theorem 3.1.1. If, in addition, we assume that the function $b \leq 0$ in M. Then, system (3.1) has a pair of non-negative nontrivial solution.*

For our next theorem, we consider $b \equiv 0$ and we will write $h := \frac{\alpha}{p^\star(s)} a + \frac{\beta}{p^\star(s)} c$. Thus, we can state the following result.

Teorema 3.1.3 *Let (M, g) be a closed Riemannian manifold of dimension $n \geq 2$, $p \in (1, 2)$ and $s \in [0, p)$. Let a, b and c be functions Hölder continuous in M satisfying (3.5) and f a smooth function such that $f(x_0) = \max_M f > 0$. If, in addition, we assume that $h(x_0) < 0$. Then, system (3.1) has a pair of non-negative nontrivial solution, when $n \geq 4$, and the same conclusion holds when $n = 2, 3$, for $p \leq \sqrt{n}$.*

The results will be proved using the Mountain Pass Theorem. A tricky part is to estimate the minimax level to overcome the lack of compactness caused by the critical growth of the nonlinearities. We achieved this goal by following some ideas developed in [15, 11]. Here, we encounter additional difficulties due to the strong coupling of the system and the involvement of the p-Laplace-Beltrami operator.

The chapter is organized as follows. In Sect. 3.2 we prove a Hardy-Sobolev type inequality, important to prove the main results. In Sect. 3.3 we prove a compactness theorem for (PS) sequence. In Sect. 3.4 we estimate the minimax level. In Sect. 3.5 we prove the main theorems.

3.2 Preliminary results

In this section, we prove some auxiliary results Specifically, we establish a Hardy-Sobolev type inequality where the best constant is $(\mathcal{K}^{p,s}_{(\alpha,\beta)})^{-1}$, although it is generally not the optimal constant (as we will discuss shortly). We also present a Brézis-Lieb type lemma. For this purpose, we mention the work by Chen and Liu [11], where the authors established that the best constant for the Hardy-Sobolev inequality is $K(n,p,s)$. However, this constant is generally not optimal. More precisely, for any given $\varepsilon > 0$, there exists a positive constant $B\varepsilon$ such that the inequality

$$\left(\int_M \frac{|u|^{p^\star(s)}}{d_g(x,x_0)^s} dv_g\right)^{\frac{p}{p^\star(s)}} \leq [K(n,p,s)+\varepsilon]\|\nabla_g u\|_p^p + B_\varepsilon \|u\|_p^p, \tag{3.7}$$

holds for all $u \in H^{1,p}(M)$. But in general the inequality above does not hold for $\varepsilon = 0$ when $p \neq 2$ (for more details, see [11, Theorems 1,6 and 1.7]).

Initially, we establish an inequality that will be used in the proof of the main results.

Teorema 3.2.1 *Let $\mathcal{K}^{p,s}_{(\alpha,\beta)}$ be the constant defined in (3.4) when $\alpha+\beta = p^\star(s)$. Then, given any $\varepsilon > 0$ there is a positive constant B_ε such that*

$$\left(\int_M \frac{|u|^\alpha |v|^\beta}{d_g(x,x_0)^s} dv_g\right)^{\frac{p}{p^\star(s)}} \leq [(\mathcal{K}^{p,s}_{(\alpha,\beta)})^{-1} + \varepsilon]\|(|\nabla_g u|, |\nabla_g v|)\|_p^p + B_\varepsilon \|(u,v)\|_p^p, \tag{3.8}$$

for all $(u,v) \in H^p$.

Proof. Given $u, v \in H^{1,p}(M)$, since $\frac{\alpha}{p^\star(s)} + \frac{\beta}{p^\star(s)} = 1$, by Hölder's inequality,

$$\int_M \frac{|u|^\alpha |v|^\beta}{d_g(x,x_0)^s} dv_g \leq \left(\int_M \frac{|u|^{p^\star(s)}}{d_g(x,x_0)^s} dv_g\right)^{\alpha/p^\star(s)} \left(\int_M \frac{|v|^{p^\star(s)}}{d_g(x,x_0)^s} dv_g\right)^{\beta/p^\star(s)},$$

that is,

$$\left(\int_M \frac{|u|^\alpha |v|^\beta}{d_g(x,x_0)^s} dv_g\right)^{\frac{p}{p^\star(s)}} \leq \left(\|u\|^p_{p^\star(s),s}\right)^{\alpha/p^\star(s)} \left(\|v\|^p_{p^\star(s),s}\right)^{\beta/p^\star(s)}.$$

On the other hand, by Young's inequality,

$$\left(\|u\|^p_{p^\star(s),s}\right)^{\alpha/p^\star(s)}\left(\|v\|^p_{p^\star(s),s}\right)^{\beta/p^\star(s)} \le \frac{\alpha}{p^\star(s)}\epsilon\|u\|^p_{p^\star(s),s} + \frac{\beta}{p^\star(s)}\epsilon^{-\alpha/\beta}\|v\|^p_{p^\star(s),s}.$$

Choosing $\epsilon = \kappa(\alpha,\beta)^{-1}\frac{p^\star(s)}{\alpha}$, by a simple calculation, we get

$$\frac{\alpha}{p^\star(s)}\epsilon = \frac{\beta}{p^\star(s)}\epsilon^{-\alpha/\beta} = \kappa(\alpha,\beta)^{-1},$$

and consequently,

$$\left(\int_M \frac{|u|^\alpha|v|^\beta}{d_g(x,x_0)^s}dv_g\right)^{\frac{p}{p^\star(s)}} \le \kappa(\alpha,\beta)^{-1}\left(\|u\|^p_{p^\star(s),s} + \|v\|^p_{p^\star(s),s}\right). \tag{3.9}$$

Using Hardy-Sobolev inequality (3.7) for arbitrary $\varepsilon > 0$ in (3.9), we can find $\tilde{B}_\varepsilon > 0$ such that

$$\begin{aligned}\left(\int_M \frac{|u|^\alpha|v|^\beta}{d_g(x,x_0)^s}dv_g\right)^{\frac{p}{p^\star(s)}} \le&\kappa(\alpha,\beta)^{-1}[K(n,p,s)+\varepsilon\kappa(\alpha,\beta)]\|(|\nabla_g u|,|\nabla_g v|)\|^p_p \\ &+\kappa(\alpha,\beta)^{-1}\tilde{B}_\varepsilon\|(u,v)\|^2_2.\end{aligned}$$

Therefore, we get that

$$\left(\int_M \frac{|u|^\alpha|v|^\beta}{d_g(x,x_0)^s}dv_g\right)^{\frac{p}{p^\star(s)}} \le [(\mathcal{K}^{p,s}_{(\alpha,\beta)})^{-1}+\varepsilon]\|(|\nabla_g u|,|\nabla_g v|)\|^p_p + B_\varepsilon\|(u,v)\|^p_p$$

for all $(u,v)\in H^p$, where $B_\varepsilon = \kappa(\alpha,\beta)^{-1}\tilde{B}_\varepsilon$. ■

An immediate consequence of this result is the following inequality.

Corollary 3.2.2 *Let $C_\varepsilon = \max\{(\mathcal{K}^{p,s}_{(\alpha,\beta)})^{-1}+\varepsilon, B_\varepsilon\}$, then taking $\varepsilon = 1$, we find $C > 0$ such that*

$$\left(\int_M \frac{|u|^\alpha|v|^\beta}{d_g(x,x_0)^s}dv_g\right)^{\frac{p}{p^\star(s)}} \le C\|(u,v)\|^p,$$

for all $(u,v)\in H^p$.

One question that arises is: is the constant $(\mathcal{K}^{p,s}_{(\alpha,\beta)})^{-1} = \kappa(\alpha,\beta)^{-1}K(n,p,s)$ optimal? The answer is yes for the case $p=2$, while for the case $p\ne 2$, the answer is generally no, as will be proved in the following proposition.

Proposition 3.2.3 *Let (M,g) be a smooth closed Riemannian manifold of dimension $n>4$, $x_0\in M$, $2<p\le\frac{n+2}{3}$ and $s\in(0,p)$. If we assume that $R_g(x_0)>0$. Then the following optimal inequality*

$$\left(\int_M \frac{|u|^\alpha|v|^\beta}{d_g(x,x_0)^s}dv_g\right)^{\frac{p}{p^\star(s)}} \le (\mathcal{K}^{p,s}_{(\alpha,\beta)})^{-1}\|(|\nabla_g u|,|\nabla_g v|)\|^p_p + B\|(u,v)\|^p_p, \tag{3.10}$$

is not valid for all $(u,v)\in H^p$.

Proof. Suppose by contradiction that (3.10) holds. Let $\varphi \in H^{1,p}(M)\backslash\{0\}$ and write $u = \xi\varphi$ and $v = \zeta\varphi$, where $\xi, \zeta \in \mathbb{R}_+$ will be chosen later. Thus, by (3.10) it follows that

$$(\xi^\alpha\zeta^\beta)^{\frac{p}{p^\star(s)}} \left(\int_M \frac{|\varphi|^{p^\star(s)}}{d_g(x,x_0)^s} dv_g\right)^{\frac{p}{p^\star(s)}} \leq (\xi^p + \zeta^p)\left\{(\mathcal{K}^{p,s}_{(\alpha,\beta)})^{-1}\|\nabla_g\varphi\|_p^p + B\|\varphi\|_p^p\right\}. \quad (3.11)$$

Note that choosing ξ, ζ such that $\left(\frac{\xi}{\zeta}\right)^p = \frac{\alpha}{\beta}$ then $\frac{\xi^p+\zeta^p}{(\xi^\alpha\zeta^\beta)^{\frac{p}{p^\star(s)}}} = \kappa(\alpha,\beta)$ and so by (3.11), we get that

$$\left(\int_M \frac{|\varphi|^{p^\star(s)}}{d_g(x,x_0)^s} dv_g\right)^{\frac{p}{p^\star(s)}} \leq K(n,p,s)\|\nabla_g\varphi\|_p^p + \kappa(\alpha,\beta)B\|\varphi\|_p^p. \quad (3.12)$$

Thus, since φ is arbitrary we have a contradiction, because (3.12) is not valid. ∎

Remark 3.2.4 *In the case $s = 0$, based on the argument developed in the previous prove, the optimal type inequality (3.10) is valid when the optimal inequality*

$$\left(\int_M |\varphi|^{p^\star} dv_g\right)^{p/p^\star} \leq K(n,p)\|\nabla_g\varphi\|_p^p + \tilde{B}\|\varphi\|_p^p, \quad \forall\varphi \in H^{1,p}(M)$$

is valid and it is not valid when above inequality is not valid. Details about the above inequality can be read at [28, Chapter 4].

Another result that will be important later on is the following Brézis-Lieb type lemma.

Lemma 3.2.5 *Let $u_m \rightharpoonup u$ and $v_m \rightharpoonup v$ in $H^{1,p}(M)$ and let $\ell \in L^\infty(M)$. Then we have*

$$(i)\ \int_M \frac{\ell(x)}{d_g(x,x_0)^s}\left[|u_m|^\alpha|v_m|^\beta - |u|^\alpha|v|^\beta - |u_m-u|^\alpha|v_m-v|^\beta\right] dv_g = o_m(1);$$

$$(ii)\ \int_M \ell(x)u_mv_m(|u_m|^{p-2} + |v_m|^{p-2})dv_g - \int_M \ell(x)uv(|u|^{p-2}+|v|^{p-2})dv_g$$
$$= \int_M \ell(x)(u_m-u)(v_m-v)(|u_m-u|^{p-2} + |v_m-v|^{p-2})dv_g + o_m(1), \text{ when } p \in (2,n),$$

where $o_m(1) \to 0$ as $m \to \infty$.

Proof. In the proof of (i) we follow the idea of [10]. We start by writing the following

$$\int_M \frac{\ell(x)|u_m|^\alpha|v_m|^\beta}{d_g(x,x_0)^s}dv_g - \int_M \frac{\ell(x)|u_m-u|^\alpha|v_m-v|^\beta}{d_g(x,x_0)^s}dv_g$$
$$= \int_M \frac{\ell(x)}{d_g(x,x_0)^s}\left[|u_m|^\alpha\left(|v_m|^\beta - |v_m-v|^\beta\right) + |v_m-v|^\beta\left(|u_m|^\alpha - |u_m-u|^\alpha\right)\right] dv_g$$
$$= -\int_M \left[\frac{\ell(x)|u_m|^\alpha}{d_g(x,x_0)^s}\int_0^1 \frac{d}{dt}|v_m - tv|^\beta dt\right] dv_g$$
$$- \int_M \left[\frac{\ell(x)|v_m-v|^\beta}{d_g(x,x_0)^s}\int_0^1 \frac{d}{dt}|u_m-tu|^\alpha dt\right] dv_g,$$

so, we get that

$$\int_M \frac{\ell(x)|u_m|^\alpha |v_m|^\beta}{d_g(x,x_0)^s}dv_g - \int_M \frac{\ell(x)|u_m-u|^\alpha |v_m-v|^\beta}{d_g(x,x_0)^s}dv_g$$
$$= \beta \int_M \left[|u_m|^\alpha \int_0^1 |v_m - tv|^{\beta-2}(v_m - tv)dt\right] \frac{\ell(x)v}{d_g(x,x_0)^s}dv_g$$
$$+ \alpha \int_M \left[|v_m - v|^\beta \int_0^1 |u_m - tu|^{\alpha-2}(u_m - tu)dt\right] \frac{\ell(x)u}{d_g(x,x_0)^s}dv_g,$$

Next, to complete the proof of (i), we show the following limits

- $\lim_{m\to\infty} \beta \int_M \left[|u_m|^\alpha \int_0^1 |v_m - tv|^{\beta-2}(v_m - tv)dt\right] \frac{\ell(x)v}{d_g(x,x_0)^s}dv_g = \int_M \frac{\ell(x)|u|^\alpha|v|^\beta}{d_g(x,x_0)^s}dv_g;$
- $\lim_{m\to\infty} \alpha \int_M \left[|v_m - v|^\beta \int_0^1 |u_m - tu|^{\alpha-2}(u_m - tu)dt\right] \frac{\ell(x)u}{d_g(x,x_0)^s}dv_g = 0.$

For this purpose, since $\ell(x)v, \ell(x)u \in L^{p^\star(s)}_{d_g,s}(M)$, then it is enough to show that $w_m \rightharpoonup w$ and $\tilde{w}_m \rightharpoonup 0$ in $L^{\frac{p^\star(s)}{p^\star(s)-1}}_{d_g,s}(M)$, where the functions are given by

$$w_m(x) := |u_m(x)|^\alpha \int_0^1 |v_m(x) - tv(x)|^{\beta-2}(v_m(x) - tv(x))dt,$$
$$\tilde{w}_m(x) := |v_m(x) - v(x)|^\beta \int_0^1 |u_m(x) - tu(x)|^{\alpha-2}(u_m(x) - tu(x))dt,$$
$$w(x) := |u(x)|^\alpha \int_0^1 (1-t)^{\beta-1}|v(x)|^{\beta-2}v(x)dt = \beta^{-1}|u(x)|^\alpha|v(x)|^{\beta-2}v(x).$$

Let us to prove only that $\tilde{w}_m \rightharpoonup 0$, the other weak converge is to similar. Suppose otherwise, then there exist $\phi \in L^{p^\star(s)}_{d_g,s}(M)$ and $A > 0$ such that (up to a subsequence)

$$\left|\int_M \frac{\tilde{w}_m \phi}{d_g(x,x_0)^s}dv_g\right| > A \ \ \forall m, \tag{3.13}$$

since $u_m \rightharpoonup u$ and $v_m \rightharpoonup v$ in $H^{1,p}(M)$, then there exist subsequences such that $u_{m_k}(x) \to u(x)$ and $v_{m_k}(x) \to v(x)$ a.e in M, it is easy to see that this gives us $\tilde{w}_{m_k}(x) \to 0$ a.e in M. Now, note that

$$\int_M \frac{|\tilde{w}_{m_k}|^{\frac{p^\star(s)}{p^\star(s)-1}}}{d_g(x,x_0)^s}dv_g = \int_M \left(|v_{m_k} - v|^\beta \left|\int_0^1 |u_{m_k} - tu|^{\alpha-2}(u_{m_k} - tu)dt\right|\right)^{\frac{p^\star(s)}{p^\star(s)-1}} \frac{dv_g}{d_g(x,x_0)^s}$$
$$\leq \int_M \left(|v_{m_k} - v|^\beta \int_0^1 |u_{m_k} - tu|^{\alpha-1}dt\right)^{\frac{p^\star(s)}{p^\star(s)-1}} \frac{dv_g}{d_g(x,x_0)^s}$$
$$= \int_M \left(|v_{m_k} - v|^\beta \int_0^1 |(1-t)u_{m_k} + t(u_{m_k} - u)|^{\alpha-1}dt\right)^{\frac{p^\star(s)}{p^\star(s)-1}} \frac{dv_g}{d_g(x,x_0)^s}$$
$$\leq \int_M \left[|v_{m_k} - v|^\beta \left(\int_0^1 |(1-t)u_{m_k} + t(u_{m_k} - u)|^\alpha dt\right)^{\frac{\alpha-1}{\alpha}}\right]^{\frac{p^\star(s)}{p^\star(s)-1}} \frac{dv_g}{d_g(x,x_0)^s},$$

where using Holder's inequality for $\frac{\alpha-1}{\alpha}+\frac{1}{\alpha}=1$ and $\alpha>1$. So, by triangle inequality we get that

$$\begin{aligned}
&\int_M \frac{|\tilde{w}_{m_k}|^{\frac{p^\star(s)}{p^\star(s)-1}}}{d_g(x,x_0)^s}dv_g \\
&\quad\le \int_M \left\{|v_{m_k}-v|^\beta\left[\|(1-t)u_{m_k}\|_{L^\alpha(0,1)}+\|t(u_{m_k}-u)\|_{L^\alpha(0,1)}\right]^{\alpha-1}\right\}^{\frac{p^\star(s)}{p^\star(s)-1}}\frac{dv_g}{d_g(x,x_0)^s} \\
&\quad= \left(\frac{1}{\alpha+1}\right)^{\frac{\alpha-1}{\alpha}}\int_M\left[|v_{m_k}-v|^\beta\left(|u_{m_k}|+|u_{m_k}-u|\right)^{\alpha-1}\right]^{\frac{p^\star(s)}{p^\star(s)-1}}\frac{dv_g}{d_g(x,x_0)^s},
\end{aligned}$$

again by Holder's inequality for $\frac{\alpha-1}{p^\star(s)-1}+\frac{\beta}{p^\star(s)}=1$, we reach that

$$\left(\int_M \frac{|\tilde{w}_{m_k}|^{\frac{p^\star(s)}{p^\star(s)-1}}}{d_g(x,x_0)^s}dv_g\right)^{\frac{p^\star(s)-1}{p^\star(s)}} \le C||v_{m_k}-v||^\beta_{p^\star(s),s}|||u_{m_k}|+|u_{m_k}-u|||^{\alpha-1}_{p^\star(s),s},$$

hence, $\{\tilde{w}_{m_k}\}$ is bounded in $L^{\frac{p^\star(s)}{p^\star(s)-1}}_{d_g,s}(M)$, because the right side above is bounded. From [36, Lemma 4.8] follows that $\tilde{w}_{m_k}\rightharpoonup 0$ in $L^{\frac{p^\star(s)}{p^\star(s)-1}}_{d_g,s}(M)$, which contradicts (3.13). Therefore, $\tilde{w}_m\rightharpoonup 0$ in $L^{\frac{p^\star(s)}{p^\star(s)-1}}_{d_g,s}(M)$. The prove of $w_m\rightharpoonup w$ is to similar.

For (ii), note that by Hölder's inequality

$$\begin{aligned}
&\left|\int_M \ell(x)(u_mv_m|u_m|^{p-2}-uv|u|^{p-2})dv_g\right| \\
&\qquad= \left|\int_M \ell(x)[v_m(u_m|u_m|^{p-2}-u|u|^{p-2})-u|u|^{p-2}(v_m-v)]dv_g\right| \\
&\qquad\le ||\ell||_\infty\int_M\left(|v_m|\left|u_m|u_m|^{p-2}-u|u|^{p-2}\right|+|u|^{p-1}|v_m-v|\right)dv_g \\
&\qquad\le ||\ell||_\infty\left[||v_m||_p\left\|u_m|u_m|^{p-2}-u|u|^{p-2}\right\|_{\frac{p}{p-1}}+||u||_p^{p-1}||v_m-v||_p\right],
\end{aligned}$$

since $u_m|u_m|^{p-2}\to u|u|^{p-2}$ in $L^{\frac{p}{p-1}}(M)$ and $v_m\to v$ in $L^p(M)$, thus the right-hand side of the above inequality converges to zero, hence

$$\int_M \ell(x)(u_mv_m|u_m|^{p-2}-uv|u|^{p-2})dv_g=o_m(1),$$

similarly we get

$$\begin{aligned}
&\int_M \ell(x)(u_mv_m|u_m|^{p-2}-uv|u|^{p-2})dv_g=o_m(1), \\
&\int_M \ell(x)(u_m-u)(v_m-v)(|u_m-u|^{p-2}+|v_m-v|^{p-2})dv_g=o_m(1).
\end{aligned}$$

This completes the proof of the lemma. ■

3.3 A Compactness Theorem

In this section, let us to prove a compactness result for $(PS)_\tau$ sequence at level τ, where the minimax level τ satisfies appropriate estimate with respect the best constant $\mathcal{K}^{p,s}_{(\alpha,\beta)}$ defined in (3.4).

For this, we first consider the functional $E_h : H^p \to \mathbb{R}$ given by

$$\begin{aligned} E_h(u,v) =& \frac{1}{p}\int_M \left(|\nabla_g u|^p + |\nabla_g v|^p + a|u|^p + buv(|u|^{p-2} + |v|^{p-2}) + c|v|^p\right) dv_g \\ & - \frac{1}{p^\star(s)}\int_M \frac{f(x)|u|^\alpha |v|^\beta}{d_g(x,x_0)^s} dv_g, \end{aligned} \tag{3.14}$$

while in the case $1 < p < 2$ we consider the same functional with $b \equiv 0$.

It is easy to see that E_h is well defined and is of class $C^1(H^p, \mathbb{R})$ with

$$\begin{aligned} & E_h'(u,v)(\varphi,\psi) = \\ & \int_M \{\langle |\nabla_g u|^{p-2}\nabla_g u, \nabla_g \varphi\rangle + \langle |\nabla_g v|^{p-2}\nabla_g v, \nabla\psi\rangle + a(x)|u|^{p-2}u\varphi + c(x)|v|^{p-2}v\psi \\ & + \frac{b(x)}{p}\Big[((p-1)|u|^{p-2} + |v|^{p-2})v\varphi + ((p-1)|v|^{p-2} + |u|^{p-2})u\psi\,]\} \,dv_g \\ & - \int_M \frac{\alpha}{p^\star(s)}\frac{f(x)}{d_g(x,x_0)^s}|u|^{\alpha-2}|v|^\beta u\varphi dv_g + \int_M \frac{\beta}{p^\star(s)}\frac{f(x)}{d_g(x,x_0)^s}|v|^{\beta-2}|u|^\alpha v\psi dv_g. \end{aligned}$$

In this context, we say that a pair of functions $(u,v) \in H^p$ is a weak solution of (3.1) if only if is a critical point of E_h.

The following lemma is an immediate consequence of the coercivity hypothesis (3.5).

Lemma 3.3.1 *The functional E_h satisfies the geometry of Mountain Pass Theorem, that is, there exists $\rho > 0$ and $R > 0$ such that*

$$E_h(u,v) \geq \rho \quad \textit{whenever} \quad ||(u,v)|| = R,$$

and there exist $(\tilde{u},\tilde{v}) \in H^p$ satisfying $||(\tilde{u},\tilde{v})|| > R$ and $E_h(\tilde{u},\tilde{v}) < 0$.

Remark 3.3.2 *Let $(\tilde{u},\tilde{v})$ be a pair satisfying $||(\tilde{u},\tilde{v})|| > R$ and $E_h(\tilde{u},\tilde{v}) < 0$ given in the previous lemma, then we consider the set*

$$\Gamma := \{\gamma \in C([0,1], H^p) : \gamma(0) = 0 \textit{ and } \gamma(1) = (\tilde{u},\tilde{v})\},$$

and so we can define the minimax level

$$\tau = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} E_h(\gamma(t)) \geq \rho > 0. \tag{3.15}$$

Lemma 3.3.3 *If* $\{(u_m, v_m)\}$ *is a sequence in* H^p *such that*

$$E_h(u_m, v_m) \to \tau \quad \text{and} \quad E'_h(u_m, v_m) \to 0, \tag{3.16}$$

as $m \to \infty$. *Then* $\{(u_m, v_m)\}$ *is bounded in* H^p.

Proof. It is enough to note that

$$\begin{aligned} &E_h(u_m, v_m) - \frac{1}{p^\star(s)} E'_h(u_m, u_m)(u_m, v_m) \\ &= \tfrac{p-s}{p(n-s)} \int_M \left(|\nabla_g u_m|^p + |\nabla_g v_m|^p + a|u_m|^p + b u_m v_m (|u_m|^{p-2} + |v_m|^{p-2}) + c|v_m|^p\right) dv_g, \end{aligned}$$

thus, by the coercivity hypothesis and (3.16) follows that

$$C_1 + o_m(||(u_m, v_m)||) \geq C_0 ||(u_m, v_m)||^p,$$

this implies that $\{(u_m, v_m)\}$ is bounded in H^p. ∎

Remark 3.3.4 *Let* $\ell \in L^\infty(M)$, *remember that if* $\{u_m\}$ *is a bounded sequence in* $H^{1,p}(M)$ *such that* u_m *and* $\nabla_g u_m$ *converge almost everywhere in* M *to* u *and* $\nabla_g u$ *respectively. Then by Brézis-Lieb [7] one has that*

$$\begin{aligned} &(i) \int_M \ell(x)|u_m|^p dv_g - \int_M \ell(x)|u|^p dv_g = \int_M \ell(x)|u_m - u|^p dv_g + o_m(1); \\ &(ii) \int_M |\nabla_g u_m|^p dv_g - \int_M |\nabla_g u|^p dv_g = \int_M |\nabla_g (u_m - u)|^p dv_g + o_m(1). \end{aligned}$$

Now we can to prove that a $(PS)_\tau$ sequence admits a convergent subsequence when the minimax level τ satisfies an appropriate estimate. More precisely:

Proposition 3.3.5 *If* $\{(u_m, v_m)\}$ *is a sequence in* H^p *such that* $E_h(u_m, v_m) \to \tau$ *and* $E'_h(u_m, v_m) \to 0$ *as* $m \to \infty$, *with*

$$\tau < \frac{p-s}{p(n-s)} \frac{(\mathcal{K}^{p,s}_{(\alpha,\beta)})^{\frac{(p^\star(s)-p)}{p^\star(s)}}}{f(x_0)^{\frac{p}{p^\star(s)-p}}}. \tag{3.17}$$

Then $\{(u_m, v_m)\}$ *has a strongly convergent subsequence. Moreover, if* (u, v) *is the limit of this subsequence, then* (u, v) *is a critical point of* E_h *and* $E_h(u, v) = \tau$.

Proof. Firstly, by hypothesis (3.17) we can choose $\varepsilon_0 > 0$ small enough such that

$$\tau < \frac{p-s}{p(n-s)} \frac{[(\mathcal{K}^{p,s}_{(\alpha,\beta)})^{-1} + \varepsilon_0]^{-\frac{(p^\star(s)-p)}{p^\star(s)}}}{f(x_0)^{\frac{p}{p^\star(s)-p}}}, \tag{3.18}$$

just choose $\varepsilon_0 < \left[\left(\frac{(p-s)}{p(n-s)}\tau^{-1}f(x_0)^{-\frac{p}{p^\star(s)-p}}\right)^{\frac{p^\star(s)}{p^\star(s)-p}} - (\mathcal{K}^{p,s}_{(\alpha,\beta)})^{-1}\right]$.

Now, Lemma 3.3.3 gives us that $\{(u_m, v_m)\}$ is bounded in H^p, so up to a subsequence for some $(u,v) \in H^p$ we have

$$\begin{aligned}
&(u_m, v_m) \rightharpoonup (u,v) \text{ in } H^p;\\
&(u_m, v_m) \to (u,v) \text{ in } L^p(M) \times L^p(M);\\
&(u_m, v_m) \to (u,v) \text{ in } L^q_{d_g,s}(M) \times L^q_{d_g,s}(M), \forall q \in [p, p^\star(s));\\
&(u_m(x), v_m(x)) \rightharpoonup (u(x), v(x)) \text{ a.e in } M.
\end{aligned} \tag{3.19}$$

We claim that, the pair (u,v) is a critical point of functional E_h. Indeed, the sequences $\{|\nabla_g u_m|^{p-2}\nabla_g u_m\}$ and $\{|\nabla_g v_m|^{p-2}\nabla_g v_m\}$ are bounded in $L^{\frac{p}{p-1}}(M)$, thus, we can assume that

$$|\nabla_g u_m|^{p-2}\nabla_g u_m \rightharpoonup \Sigma_1 \text{ and } |\nabla_g v_m|^{p-2}\nabla_g v_m \rightharpoonup \Sigma_2 \text{ in } L^{\frac{p}{p-1}}(M).$$

Now, since $\{f|u_m|^{\alpha-2}u_m|v_m|^\beta\}$ and $\{|v_m|^{\beta-2}v_m|u_m|^\alpha\}$ are bounded in $L_{d_g,s}^{\frac{p^\star(s)}{p^\star(s)-1}}(M)$ and converges almost everywhere in M to $f|u|^{\alpha-2}u|v|^\beta$ and $f|v|^{\beta-2}v|u|^\alpha$ respectively, then

$$f|u_m|^{\alpha-2}u_m|v_m|^\beta \rightharpoonup f|u|^{\alpha-2}u|v|^\beta \text{ and } |v_m|^{\beta-2}v_m|u_m|^\alpha \rightharpoonup f|v|^{\beta-2}v|u|^\alpha \text{ in } L_{d_g,s}^{\frac{p^\star(s)}{p^\star(s)-1}}(M).$$

As $E'_h(u_m, v_m)(\varphi,\psi) \to 0$ for any $(\varphi,\psi) \in H^p$ and since

$$\begin{aligned}
&E'_h(u_m, v_m)(\varphi,\psi)\\
&= \int_M \Big\{\langle|\nabla_g u_m|^{p-2}\nabla_g u_m, \nabla_g\varphi\rangle + \langle|\nabla_g v_m|^{p-2}\nabla_g v_m, \nabla_g\psi\rangle + a(x)|u_m|^{p-2}u_m\varphi\\
&\qquad + c(x)|v_m|^{p-2}v_m\psi + \tfrac{b(x)}{p}((p-1)|u_m|^{p-2} + |v_m|^{p-2})v_m\varphi\\
&\qquad + \tfrac{b(x)}{p}((p-1)|v_m|^{p-2} + |u_m|^{p-2})u_m\psi\Big\}\, dv_g\\
&\qquad - \int_M \tfrac{\alpha}{p^\star(s)}\tfrac{f(x)}{d_g(x,x_0)^s}|u_m|^{\alpha-2}|v_m|^\beta u_m\varphi dv_g + \int_M \tfrac{\beta}{p^\star(s)}\tfrac{f(x)}{d_g(x,x_0)^s}|v_m|^{\beta-2}|u_m|^\alpha v_m\psi dv_g,
\end{aligned}$$

letting $m \to \infty$, by the weak convergences above and (3.18), we get that

$$\begin{aligned}
&0 = \int_M \big\{\langle\Sigma_1, \nabla_g\varphi\rangle + \langle\Sigma_2, \nabla_g\psi\rangle + a(x)|u|^{p-2}u\varphi + c(x)|v|^{p-2}v\psi\\
&+ \frac{b(x)}{p}\Big[((p-1)|u|^{p-2} + |v|^{p-2})v\varphi + ((p-1)|v|^{p-2} + |u|^{p-2})u\psi\,]\big\}\, dv_g\\
&- \int_M \frac{\alpha}{p^\star(s)}\frac{f(x)}{d_g(x,x_0)^s}|u|^{\alpha-2}|v|^\beta u\varphi dv_g + \int_M \frac{\beta}{p^\star(s)}\frac{f(x)}{d_g(x,x_0)^s}|v|^{\beta-2}|u|^\alpha v\psi dv_g.
\end{aligned} \tag{3.20}$$

Now, we want to prove that $\Sigma_1 = |\nabla_g u|^{p-2}\nabla_g u$ and $\Sigma_2 = |\nabla_g v|^{p-2}\nabla_g v$. To achieve this purpose, we will adapt the ideas of [12] and [11].

Given $\delta > 0$ arbitrary, according to Egoroff's Theorem there exists $U_\delta \subset M$ such that $vol(M\backslash U_\delta) < \delta$, and (u_m, v_m) converge uniformly to (u, v) in U_δ. So for any $\epsilon > 0$ there exists m_0 large enough such that

$$|u_m(x) - u(x)| < \epsilon \ \text{ and } \ |v_m(x) - v(x)| < \epsilon,\ \forall x \in U_\delta.$$

We define the function $T_\epsilon : \mathbb{R} \to \mathbb{R}$ by

$$T_\epsilon(r) = \begin{cases} r, \text{ if } |r| \leq \epsilon \\ \epsilon sign(r), \text{ if } |r| > \epsilon. \end{cases}$$

From Lindqvist's inequality [40, Section 4], there exists a positive constant $C(p)$ depending only of p such that, for all $X, Y \in \mathbb{R}^n$, one has that

$$(|X|^{p-2}X - |Y|^{p-2}Y)\cdot(X - Y) \geq \begin{cases} C(p)|X - Y|^p, \text{ if } p \geq 2, \\ C(p)\frac{|X-Y|^2}{(|X|+|Y|)^{2-p}}, \text{ if } 1 < p < 2, \end{cases}$$

here $X \cdot Y$ denotes the standard scalar product in $\mathbb{R}^n$. Thus, we get that

$$\begin{aligned} \langle |\nabla_g u_m|^{p-2}\nabla_g u_m - |\nabla_g u|^{p-2}\nabla_g u, \nabla_g(T_\epsilon \circ (u_m - u))\rangle \geq 0, \\ \langle |\nabla_g v_m|^{p-2}\nabla_g v_m - |\nabla_g v|^{p-2}\nabla_g v, \nabla_g(T_\epsilon \circ (v_m - v))\rangle \geq 0, \end{aligned}$$

a.e in M, since $\nabla_g(T_\epsilon \circ w) = T'_\epsilon(w)\nabla_g w$ and $T'_\epsilon \geq 0$. Now, for $m \geq m_0$ we have $T'_\epsilon(u_m - u) = T'_\epsilon(v_m - v) = 1$ for all $x \in U_p$ and so, we reach that

$$\begin{aligned} &\int_M \langle |\nabla_g u_m|^{p-2}\nabla_g u_m - |\nabla_g u|^{p-2}\nabla_g u, \nabla_g(T_\epsilon \circ (u_m - u))\rangle dv_g \\ &\quad \geq \int_{U_\delta} \langle |\nabla_g u_m|^{p-2}\nabla_g u_m - |\nabla_g u|^{p-2}\nabla_g u, \nabla_g(u_m - u)\rangle dv_g \geq 0, \\ &\int_M \langle |\nabla_g v_m|^{p-2}\nabla_g v_m - |\nabla_g v|^{p-2}\nabla_g v, \nabla_g(T_\epsilon \circ (v_m - v))\rangle dv_g \\ &\quad \geq \int_{U_\delta} \langle |\nabla_g v_m|^{p-2}\nabla_g v_m - |\nabla_g v|^{p-2}\nabla_g v, \nabla_g(v_m - v)\rangle dv_g \geq 0. \end{aligned} \tag{3.21}$$

On the other hand, as $T_\epsilon \circ (u_m - u)$ and $T_\epsilon \circ (v_m - v)$ goes to zero weakly in $H^{1,p}(M)$ then

$$\begin{aligned} \int_M \langle |\nabla_g u|^{p-2}\nabla_g u, \nabla_g(T_\epsilon \circ (u_m - u))\rangle dv_g \to 0, \\ \int_M \langle |\nabla_g v|^{p-2}\nabla_g v, \nabla_g(T_\epsilon \circ (v_m - v))\rangle dv_g \to 0. \end{aligned} \tag{3.22}$$

As $T_\epsilon \circ (u_m - u), T_\epsilon \circ (v_m - v) \in H^{1,p}(M)$, then we have

$$\left| \int_M \left[\langle |\nabla_g u_m|^{p-2} \nabla_g u_m, \nabla_g (T_\epsilon \circ (u_m - u)) \rangle + \langle |\nabla_g v_m|^{p-2} \nabla_g v_m, \nabla_g (T_\epsilon \circ (v_m - v)) \rangle \right] dv_g \right|$$
$$\leq |E'_h(u_m, v_m)(T_\epsilon \circ (u_m - u), T_\epsilon \circ (v_m - v))|$$
$$+ \Bigg| - \int_M \left[a(x)|u_m|^{p-2} u_m (T_\epsilon \circ (u_m - u)) + c(x)|v_m|^{p-2} v_m (T_\epsilon \circ (v_m - v)) \right] dv_g$$
$$- \int_M \frac{b(x)}{p} ((p-1)|u_m|^{p-2} + |v_m|^{p-2}) v_m (T_\epsilon \circ (u_m - u)) dv_g$$
$$- \int_M \frac{b(x)}{p} ((p-1)|v_m|^{p-2} + |u_m|^{p-2}) u_m (T_\epsilon \circ (v_m - v)) dv_g$$
$$+ \int_M \frac{\alpha}{p^\star(s)} \frac{f(x)}{d_g(x, x_0)^s} |u_m|^{\alpha-2} |v_m|^\beta u_m (T_\epsilon \circ (u_m - u)) dv_g$$
$$+ \int_M \frac{\beta}{p^\star(s)} \frac{f(x)}{d_g(x, x_0)^s} |v_m|^{\beta-2} |u_m|^\alpha v_m (T_\epsilon \circ (v_m - v)) dv_g \Bigg|,$$

since $|T_\epsilon \circ (u_m - u)|, |T_\epsilon \circ (v_m - v)| \leq \epsilon$ and the fact that the sequences are bounded, we reach that

$$\left| \int_M \left[\langle |\nabla_g u_m|^{p-2} \nabla_g u_m, \nabla_g (T_\epsilon \circ (u_m - u)) \rangle + \langle |\nabla_g v_m|^{p-2} \nabla_g v_m, \nabla_g (T_\epsilon \circ (v_m - v)) \rangle \right] dv_g \right|$$
$$\leq C\epsilon + o_m(||(T_\epsilon \circ (u_m - u), T_\epsilon \circ (v_m - v))||),$$

now, by (3.21), (3.22) and inequality above, we get that

$$\limsup_{m \to \infty} \left\{ \int_{U_\delta} \langle |\nabla_g u_m|^{p-2} \nabla_g u_m - |\nabla_g u|^{p-2} \nabla_g u, \nabla_g (u_m - u) \rangle dv_g \right.$$
$$\left. + \int_{U_\delta} \langle |\nabla_g v_m|^{p-2} \nabla_g v_m - |\nabla_g v|^{p-2} \nabla_g v, \nabla_g (v_m - v) \rangle dv_g \right\} \leq C\epsilon.$$

Since $\epsilon > 0$ is arbitrary and each term above is nonnegative it follows that

$$\langle |\nabla_g u_m|^{p-2} \nabla_g u_m - |\nabla_g u|^{p-2} \nabla_g u, \nabla_g (u_m - u) \rangle \to 0 \text{ in } L^1(U_\delta),$$
$$\langle |\nabla_g v_m|^{p-2} \nabla_g v_m - |\nabla_g v|^{p-2} \nabla_g v, \nabla_g (v_m - v) \rangle \to 0 \text{ in } L^1(U_\delta),$$

hence, up to a subsequence, these sequences converge almost everywhere in U_δ. The Lindqvist's inequality gives us that $\nabla_g u_m \to \nabla_g u$ ($\nabla_g v_m \to \nabla_g v$) a.e in U_δ. This happens for any $\delta > 0$, then $\nabla_g u_m \to \nabla_g u$ ($\nabla_g v_m \to \nabla_g v$) a.e in M. Clearly, this in turn implies that

$$|\nabla_g u_m|^{p-2} \nabla_g u_m \to |\nabla_g u|^{p-2} \nabla_g u \text{ and } |\nabla_g v_m|^{p-2} \nabla_g v_m \to |\nabla_g v| \nabla_g v \text{ a.e in } M.$$

Since $\{|\nabla_g u_m|^{p-2}\nabla_g u_m\}$ and $\{|\nabla_g v_m|^{p-2}\nabla_g v_m\}$ are bounded in $L^{\frac{p}{p-1}}(M)$, from Lemma 4.8 in [36] it follows that

$$|\nabla_g u_m|^{p-2}\nabla_g u_m \rightharpoonup |\nabla_g u|^{p-2}\nabla_g u \text{ in } L^{\frac{p}{p-1}}(M),$$
$$|\nabla_g v_m|^{p-2}\nabla_g v_m \rightharpoonup |\nabla_g v|^{p-2}\nabla_g v \text{ in } L^{\frac{p}{p-1}}(M),$$

thus, $\Sigma_1 = |\nabla_g u|^{p-2}\nabla_g u$ and $\Sigma_2 = |\nabla_g v|^{p-2}\nabla_g v$. This prove that (u, v) is a critical point of E_h.

As consequence we get that

$$E_h(u, v) = \tfrac{p-s}{p(n-s)} \int_M \left(|\nabla_g u|^p + |\nabla_g v|^p + a|u|^p + buv(|u|^{p-2} + |v|^{p-2}) + c|v|^p\right) dv_g \geq 0.$$

Now, writing $\tilde{u}_m = u_m - u$ and $\tilde{v}_m = v_m - v$, we want to prove that $\|(\tilde{u}_m, \tilde{v}_m)\|$ converge to zero as $m \to \infty$. By Lemma 3.2.5 and Brézis-Lieb lemma (since $\nabla_g u_m \to \nabla_g u$ and $\nabla_g v_m \to \nabla_g v$ a.e in M) we have

$$\begin{aligned}
&E'_h(\tilde{u}_m, \tilde{v}_m)(\tilde{u}_m, \tilde{v}_m)\\
&\quad= \int_M \left(|\nabla_g \tilde{u}_m|^p + |\nabla_g \tilde{v}_m|^p + a|\tilde{u}_m|^p + b\tilde{u}_m\tilde{v}_m(|\tilde{u}_m|^{p-2} + |\tilde{v}_m|^{p-2}) + c|\tilde{v}_m|^p\right) dv_g\\
&\qquad- \int_M \frac{f(x)|\tilde{u}_m|^\alpha|\tilde{v}_m|^\beta}{d_g(x, x_0)^s} dv_g\\
&\quad= \int_M (|\nabla_g u_m|^p + |\nabla_g v_m|^p) dv_g - \int_M (|\nabla_g u|^p + |\nabla_g v|^p) dv_g\\
&\qquad+ \int_M (a(x)|u_m|^p + c(x)|v_m|^p) dv_g - \int_M (a(x)|u|^p + c(x)|v|^p) dv_g\\
&\qquad+ \int_M b(x)u_m v_m(|u_m|^{p-2} + |v_m|^{p-2}) dv_g - \int_M b(x)uv(|u|^{p-2} + |v|^{p-2}) dv_g\\
&\qquad- \int_M \frac{f(x)|u_m|^\alpha|v_m|^\beta}{d_g(x, x_0)^s} dv_g + \int_M \frac{f(x)|u|^\alpha|v|^\beta}{d_g(x, x_0)^s} dv_g + o_m(1)\\
&\quad= E'_h(u_m, v_m)(u_m, v_m) - E'_h(u, v)(u, v) + o_m(1)\\
&\quad= o_m(1).
\end{aligned}$$

Consequently, up to a subsequence, the above expression gives us

$$\lim_{m\to\infty} \int_M (|\nabla_g \tilde{u}_m|^p + |\nabla_g \tilde{v}_m|^p) dv_g = \lim_{m\to\infty} \int_M \frac{f(x)|\tilde{u}_m|^\alpha|\tilde{v}_m|^\beta}{d_g(x, x_0)^s} dv_g =: \tau_0. \qquad (3.23)$$

On the other hand, with the same argument used above, we get that

$$E_h(\tilde{u}_m, \tilde{v}_m) = E_h(u_m, v_m) - E_h(u, v) + o_m(1).$$

Since $E_h(u_m,v_m)\to\tau$ as $m\to\infty$, then
$$\lim_{m\to\infty}\left\{\frac{1}{p}\int_M(|\nabla_g\tilde{u}_m|^p+|\nabla_g\tilde{v}_m|^p)dv_g-\frac{1}{p^\star(s)}\int_M\frac{f(x)|\tilde{u}_m|^\alpha|\tilde{v}_m|^\beta}{d_g(x,x_0)^s}dv_g\right\}=\tau-E_h(u,v),$$
hence, $\left(\frac{1}{p}-\frac{1}{p^\star(s)}\right)\tau_0=\tau-E_h(u,v)\leq\tau$, because $E_h(u,v)\geq 0$.

We claim that $\tau_0=0$. Otherwise, if $\tau_0>0$ then by Lemma 3.2.1 for $\varepsilon_0>0$ (such that (3.18) holds) there exists $B_{\varepsilon_0}>0$ such that
$$\left(\int_M\frac{f(x)|\tilde{u}_m|^\alpha|\tilde{v}_m|^\beta}{d_g(x,x_0)^s}dv_g\right)^{\frac{p}{p^\star(s)}}\leq f(x_0)^{\frac{p}{p^\star(s)}}\left(\int_M\frac{|\tilde{u}_m|^\alpha|\tilde{v}_m|^\beta}{d_g(x,x_0)^s}dv_g\right)^{\frac{p}{p^\star(s)}}$$
$$\leq f(x_0)^{\frac{p}{p^\star(s)}}[(\mathcal{K}^{p,s}_{(\alpha,\beta)})^{-1}+\varepsilon_0]\|(|\nabla_g\tilde{u}_m|,|\nabla_g\tilde{v}_m|)\|_p^p+B_{\varepsilon_0}\|(\tilde{u}_m,\tilde{v}_m)\|_p^p,$$
thus, since $\lim_{m\to\infty}\|(\tilde{u}_m,\tilde{v}_m)\|_p^p=0$ and by (3.23), passing to the limit in the inequality above we have
$$f(x_0)^{\frac{p}{p^\star(s)}}[(\mathcal{K}^{p,s}_{(\alpha,\beta)})^{-1}+\varepsilon_0]^{-1}\leq\tau_0^{1-\frac{p}{p^\star(s)}},$$
so,
$$f(x_0)^{-\frac{p}{p^\star(s)-p}}[(\mathcal{K}^{p,s}_{(\alpha,\beta)})^{-1}+\varepsilon_0]^{-\frac{(p^\star(s)-p)}{p^\star(s)}}\leq\tau_0\leq\left(\frac{pp^\star(s)}{p^\star(s)-p}\right)\tau=\frac{p(n-s)}{p-s}\tau.$$
However, this contradicts (3.18). Thus, we get that $\tau_0=0$, consequently one has $E(u,v)=\tau$ and
$$\lim_{m\to\infty}\int_M(|\nabla_g\tilde{u}_m|^p+|\nabla_g\tilde{v}_m|^p)dv_g=0.$$
This finish the proof. ■

3.4 The Minimax Level

First, we state an auxiliary lemma where we will show that under the assumptions of Theorems 3.1.1 and 3.1.3, there is a pair $(\tilde{u},\tilde{v})\in H^p$ such that $||(\tilde{u},\tilde{v})||>R$, $E_h(\tilde{u},\tilde{v})<0$ and
$$\tau=\inf_{\gamma\in\Gamma}\sup_{t\in[0,1]}E_h(\gamma(t))<\frac{(p-s)}{p(n-s)}\frac{(\mathcal{K}^{p,s}_{(\alpha,\beta)})^{\frac{(p^\star(s)-p)}{p^\star(s)}}}{f(x_0)^{\frac{p}{p^\star(s)-p}}}.$$
To achieve this, let $\delta\in(0,i_g/2)$ small enough such that $f>0$ in $B_{2\delta}(x_0)$ (geodesic ball centered in x_0 and with radius 2δ). We consider the following cut-off function $\eta\in C_0^\infty([-2\delta,2\delta])$, with $\eta=1$ in $[-\delta,\delta]$, $0\leq\eta\leq 1$ in $\mathbb{R}$, and define the function
$$u_\epsilon(x)=\eta(d_g(x,x_0))\left(\frac{\epsilon^{\frac{p-s}{p(p-1)}}}{\epsilon^{\frac{p-s}{p-1}}+d_g(x,x_0)^{\frac{p-s}{p-1}}}\right)^{\frac{n-p}{p-s}}. \tag{3.24}$$

Remark 3.4.1 *As it is known the function* $\Phi_p(y) = (1+|y|^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}}$, *with* $y \in \mathbb{R}^n$ *is an extremal for*

$$K(n,p,s)^{-1} = \inf_{u\in H^{1,p}(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n} |\nabla u|^p dx}{\left(\int_{\mathbb{R}^n} \frac{|u|^{p^\star(s)}}{|x|^s} dx\right)^{\frac{p}{p^\star(s)}}}.$$

For this result we refer to [23].

Remark 3.4.2 *Note that, as* $s \in [0,p)$, *if* $p \leq \frac{n+2}{3}$ *then* $p < \frac{n+2-s}{2}$, *for all* $n \geq 3$. *While, if* $p \leq \sqrt{n}$ *then* $p < \frac{n+2}{3}$, *for all* $n \geq 5$, *moreover,* $\sqrt{4} = 2 = \frac{6}{3} = \frac{4+2}{3}$, *and* $\sqrt{3} > \frac{5}{3} = \frac{3+2}{3}$ *and* $\sqrt{2} > \frac{4}{3}$ *(which correspond to the cases* $n=4$, $n=3$ *and* $n=2$ *respectively).*

Lemma 3.4.3 *If the functions* a, b *and* c *satisfy the assumptions of Theorems 3.1.1 and 3.1.3. Then*

$$\tau < \frac{(p-s)}{p(n-s)} \frac{\left(\mathcal{K}^{p,s}_{(\alpha,\beta)}\right)^{\frac{(p^\star(s)-p)}{p^\star(s)}}}{f(x_0)^{\frac{p}{p^\star(s)-p}}},$$

for some pair $(\tilde{u},\tilde{v}) \in H^p$ *with* $||(\tilde{u},\tilde{v})|| > R$ *and* $E_h(\tilde{u},\tilde{v}) < 0$, *where* τ *is defined in* (3.15).

Proof. Here we are inspired by ideas from [11, 15]. Since $a, b, c \in C^{0,\varrho}(M)$, then $h \in C^{0,\varrho}(M)$. Using the test function u_ϵ, let us estimate $\int_M hu_\epsilon^p dv_g$. As $\delta > 0$ is small enough, we can write $h(x)\eta^p = h(x_0) + d(x,x_0)^\varrho O(1)$, for all $x \in B_\delta(x_0)$, thus

$$\int_M hu_\epsilon^p dv_g = \int_{B_\delta(x_0)} hu_\epsilon^p dv_g + \int_{M\setminus B_\delta(x_0)} hu_\epsilon^p dv_g.$$

It is easy to see that $\int_{M\setminus B_\delta(x_0)} hu_\epsilon^p dv_g = O(\epsilon^{\frac{n-p}{p-1}})$. The first integral will be estimated in three cases: $p^2 < n, p^2 = n$ and $p^2 > n$.

(*i*) $p^2 < n$

$$\begin{aligned}\int_{B_\delta(x_0)} hu_\epsilon^p dv_g &= \int_{B_\delta(0)} (h(x_0) + O(1)r^\varrho)(1+O(1)r^2)\left(\frac{\epsilon^{\frac{p-s}{p(p-1)}}}{\epsilon^{\frac{p-s}{p-1}} + r^{\frac{p-s}{p-1}}}\right)^{\frac{(n-p)p}{p-s}} dy \\ &= h(x_0)\omega_{n-1}\int_0^{\frac{\delta}{\epsilon}} \frac{r^{n-1}}{(1+r^{\frac{p-s}{p-1}})^{\frac{(n-p)p}{p-s}}} dr\epsilon^p + O(1)\int_0^{\frac{\delta}{\epsilon}} \frac{r^{n-1+\varrho}}{(1+r^{\frac{p-s}{p-1}})^{\frac{(n-p)p}{p-s}}} dr\epsilon^{p+\varrho} \\ &= h(x_0)\|\Phi_p\|_p^p\epsilon^p - h(x_0)\omega_{n-1}\int_{\frac{\delta}{\epsilon}}^\infty \frac{r^{n-1}}{(1+r^{\frac{p-s}{p-1}})^{\frac{(n-p)p}{p-s}}} dr\epsilon^p + O(\epsilon^{p+\varrho}) \\ &= h(x_0)\|\Phi_p\|_p^p\epsilon^p + o(\epsilon^p).\end{aligned}$$

(ii) $p^2 = n$

$$\int_{B_\delta(x_0)} hu_\epsilon^p dv_g = h(x_0)\omega_{n-1}\int_0^{\frac{\delta}{\epsilon}} \frac{r^{n-1}}{(1+r^{\frac{p-s}{p-1}})^{\frac{(n-p)p}{p-s}}} dr\epsilon^p + O(\epsilon^{p+\varrho}).$$

Thus, we assume that ϵ is small enough, we can write

$$\begin{aligned}
\int_0^{\frac{\delta}{\epsilon}} \frac{r^{n-1}}{(1+r^{\frac{p-s}{p-1}})^{\frac{(n-p)p}{p-s}}} dr &= \int_0^{\delta} \frac{r^{n-1}}{(1+r^{\frac{p-s}{p-1}})^{\frac{(n-p)p}{p-s}}} dr + \int_\delta^{\frac{\delta}{\epsilon}} \frac{r^{n-1}}{(1+r^{\frac{p-s}{p-1}})^{\frac{(n-p)p}{p-s}}} dr \\
&= O(1) + \int_\delta^{\frac{\delta}{\epsilon}} r^{-1} dr + \int_\delta^{\frac{\delta}{\epsilon}} r^{n-1}\left[\frac{1}{(1+r^{\frac{p-s}{p-1}})^{\frac{(n-p)p}{p-s}}} - \frac{1}{r^n}\right] dr \\
&= O(1) + \ln(\epsilon^{-1}) + \int_\delta^{\frac{\delta}{\epsilon}} r^{n-1}\left[\frac{1-(1+r^{\frac{s-p}{p-1}})^{\frac{(n-p)p}{p-s}}}{(1+r^{\frac{p-s}{p-1}})^{\frac{(n-p)p}{p-s}}}\right] dr \\
&= O(1) + \ln(\epsilon^{-1}) + \int_\delta^{\frac{\delta}{\epsilon}} r^{n-1}\left[\frac{O(r^{-\frac{p-s}{p-1}})}{(1+r^{\frac{p-s}{p-1}})^{\frac{(n-p)p}{p-s}}}\right] dr \\
&= O(1) + \ln(\epsilon^{-1}) + O(\epsilon^{\frac{p-s}{p-1}}),
\end{aligned}$$

so, we get that

$$\int_{B_\delta(x_0)} hu_\epsilon^p dv_g = h(x_0)\omega_{n-1}\epsilon^p \ln(\epsilon^{-1}) + o(\epsilon^p \ln(\epsilon^{-1})).$$

(iii) $p^2 > n$

In this case it is easy to see that

$$\int_{B_\delta(x_0)} hu_\epsilon^p dv_g = O(\epsilon^{\frac{n-p}{p-1}}).$$

Therefore, we have

$$\int_M hu_\epsilon^p dv_g = \begin{cases} h(x_0)\|\Phi_p\|_p^p \epsilon^p + o(\epsilon^p) \;\text{ if } p < \sqrt{n}, \\ h(x_0)\omega_{n-1}\epsilon^p \ln(\epsilon^{-1}) + o(\epsilon^p \ln(\epsilon^{-1})) \;\text{ if } p = \sqrt{n} \\ O(\epsilon^{\frac{n-p}{p-1}}) \;\text{ if } p > \sqrt{n}. \end{cases} \tag{3.25}$$

Now, let us estimate $\int_M |\nabla_g u_\epsilon^p| dv_g$. For that, using the normal coordinate we can write the following expansion,

$$\bullet \int_{\mathbb{S}^{n-1}} \sqrt{\det(g)} d\sigma = \omega_{n-1}\left(1 - \frac{R_g(x_0)}{6n} r^2 + O(1) r^3\right),$$

moreover rewrite the gradient norm

$$\bullet |\nabla_g v_\epsilon|^p = \epsilon^{\frac{n-p}{p-1}} \left(\frac{n-p}{p-1}\right)^p (\epsilon^{\frac{n-s}{p-1}} + r^{\frac{p-s}{p-1}})^{-\frac{(n-s)p}{p-s}} r^{\frac{(1-s)p}{p-1}},$$

where $v_\epsilon(x) = \left(\frac{\epsilon^{\frac{p-s}{p(p-1)}}}{\epsilon^{\frac{p-s}{p-1}} + d_g(x,x_0)^{\frac{p-s}{p-1}}} \right)^{\frac{n-p}{p-s}}$. Thus,

$$\int_M |\nabla_g u_\epsilon|^p dv_g = \int_{B_\delta(x_0)} |\nabla_g v_\epsilon|^p dv_g + \int_{M\setminus B_\delta(x_0)} |\nabla_g u_\epsilon|^p dv_g.$$

It is easy to see that $\int_{M\setminus B_\delta(x_0)} |\nabla_g u_\epsilon|^p dv_g = O(\epsilon^{\frac{n-p}{p-1}})$. Now, let us calculate the integral $\int_{B_\delta(x_0)} |\nabla_g u_\epsilon|^p dv_g$, by the expressions above, we get that

$$\begin{aligned}
&\int_{B_\delta(x_0)} |\nabla_g u_\epsilon|^p dv_g \\
&= \left(\frac{n-p}{p-1}\right)^p \omega_{n-1} \int_0^{\frac{\delta}{\epsilon}} \frac{r^{\frac{(1-s)p}{p-1}+n-1}}{(1+r^{\frac{p-s}{p-1}})^{\frac{(n-s)p}{p-s}}} \left(1 - \frac{R_g(x_0)}{6n}(\epsilon r)^2 + O((\epsilon r)^3)\right) dr \\
&= \|\nabla \Phi_p\|_p^p - \frac{R_g(x_0)}{6n} \left(\frac{n-p}{p-1}\right)^p \omega_{n-1} \int_0^{\frac{\delta}{\epsilon}} \frac{r^{\frac{(1-s)p}{p-1}+n+1}}{(1+r^{\frac{p-s}{p-1}})^{\frac{(n-s)p}{p-s}}} dr \epsilon^2 + O(\epsilon^{\frac{n-p}{p-1}}).
\end{aligned}$$

Next, let us estimate the second term in the cases $p < \frac{n+2}{3}, p = \frac{n+2}{3}$ and $p > \frac{n+2}{3}$.

$(i)'$ $p < \frac{n+2}{3}$

$$\int_0^{\frac{\delta}{\epsilon}} \frac{r^{\frac{(1-s)p}{p-1}+n+1}}{(1+r^{\frac{p-s}{p-1}})^{\frac{(n-s)p}{p-s}}} dr = \int_0^{\infty} \frac{r^{\frac{(1-s)p}{p-1}+n+1}}{(1+r^{\frac{p-s}{p-1}})^{\frac{(n-s)p}{p-s}}} dr + O(\epsilon^{\frac{n+2-3p}{p-1}}).$$

$(ii)'$ $p = \frac{n+2}{3}$

$$\begin{aligned}
\int_0^{\frac{\delta}{\epsilon}} \frac{r^{\frac{(1-s)p}{p-1}+n+1}}{(1+r^{\frac{p-s}{p-1}})^{\frac{(n-s)p}{p-s}}} dr &= \int_0^{\delta} \frac{r^{\frac{(1-s)p}{p-1}+n+1}}{(1+r^{\frac{p-s}{p-1}})^{\frac{(n-s)p}{p-s}}} dr + \int_\delta^{\frac{\delta}{\epsilon}} \frac{r^{\frac{(1-s)p}{p-1}+n+1}}{(1+r^{\frac{p-s}{p-1}})^{\frac{(n-s)p}{p-s}}} dr \\
&= O(1) + \ln(\epsilon^{-1}) + \int_\delta^{\frac{\delta}{\epsilon}} \left[\frac{r^{\frac{(1-s)p}{p-1}+n+1}}{(1+r^{\frac{p-s}{p-1}})^{\frac{(n-s)p}{p-s}}} - \frac{1}{r} \right] dr \\
&= O(1) + \ln(\epsilon^{-1}) + \int_\delta^{\frac{\delta}{\epsilon}} r^{\frac{(1-s)p}{p-1}+n+1} \left[\frac{O(r^{-\frac{p-s}{p-1}})}{(1+r^{\frac{p-s}{p-1}})^{\frac{(n-s)p}{p-s}}} \right] dr \\
&= O(1) + \ln(\epsilon^{-1}) + O(\epsilon^{\frac{p-s}{p-1}}).
\end{aligned}$$

$(iii)'$ $p > \frac{n+2}{3}$

$$\int_0^{\frac{\delta}{\epsilon}} \frac{r^{\frac{(1-s)p}{p-1}+n+1}}{(1+r^{\frac{p-s}{p-1}})^{\frac{(n-s)p}{p-s}}} dr = O\left(\int_0^{\frac{\delta}{\epsilon}} r^{\frac{(1-s)p}{p-1}+n+1-\frac{(n-s)p}{p-1}} dr \right) = O(\epsilon^{\frac{n+2-3p}{p-1}}).$$

So we get that

$$\int_M |\nabla_g u_\epsilon|^p dv_g = \begin{cases} \|\nabla \Phi_p\|_p^p - \frac{R_g(x_0)}{6n} C_1(n,p,s)\epsilon^2 + o(\epsilon^2), \text{ if } p < \frac{n+2}{3}, \\ \|\nabla \Phi_p\|_p^p - \frac{R_g(x_0)}{6n} \omega_{n-1} \left(\frac{n-p}{p-1}\right)^p \epsilon^2 \ln(\epsilon^{-1}) + O(\epsilon^2), \text{ if } p = \frac{n+2}{3}, \\ \|\nabla \Phi_p\|_p^p + O(\epsilon^{\frac{n-p}{p-1}}), \text{ if } p > \frac{n+2}{3}, \end{cases} \tag{3.26}$$

where $C_1(n,p,s) = \int_{\mathbb{R}^n} |\nabla \Phi_p|^p |y|^2 dy$.

Finally, let us estimate $\int_M \frac{f(x)u_\epsilon^{p^\star(s)}}{d_g(x,x_0)^s} dv_g$. For that it is easy to see that

$$\int_{M\setminus B_\delta(x_0)} \frac{f(x)u_\epsilon^{p^\star(s)}}{d_g(x,x_0)^s} dv_g = O(\epsilon^{\frac{n-s}{p-1}}).$$

In normal coordinate we can write the following expansion

$$f(x)\eta^{p^\star(s)}(r) = f(x_0) + \frac{1}{2}\partial_{ij} f(x_0) x^i x^j + O(r^3). \tag{3.27}$$

With that, we reach

$$\begin{aligned}
\int_{B_\delta(x_0)} \frac{f(x)u_\epsilon^{p^\star(s)}}{d_g(x,x_0)^s} dv_g =& f(x_0)\int_{B_\delta(0)} \frac{\epsilon^{\frac{n-s}{p-1}}}{|y|^s(\epsilon^{\frac{p-s}{p-1}} + |y|^{\frac{p-s}{p-1}})^{\frac{2(n-s)}{2-s}}} \sqrt{\det(g)} dy \\
&+ \frac{1}{2}\partial_{ij} f(x_0) \int_{B_\delta(0)} \frac{\epsilon^{\frac{n-s}{p-1}} y^i y^j}{|y|^s(\epsilon^{\frac{p-s}{p-1}} + |y|^{\frac{p-s}{p-1}})^{\frac{p(n-s)}{p-s}}} \sqrt{\det(g)} dy + O(\epsilon^3) \\
=& f(x_0)\omega_{n-1} \int_0^\delta \frac{\epsilon^{\frac{n-s}{p-1}} r^{n-1}}{r^s(\epsilon^{\frac{p-s}{p-1}} + r^{\frac{p-s}{p-1}})^{\frac{p(n-s)}{p-s}}} \left(1 - \frac{R_g(x_0)}{6n} r^2 + O(r^3)\right) dr \\
&+ \frac{\Delta f(x_0)}{2n}\omega_{n-1} \int_0^\delta \frac{\epsilon^{\frac{n-s}{p-1}} r^{n+1}}{r^s(\epsilon^{\frac{p-s}{p-1}} + r^{\frac{p-s}{p-1}})^{\frac{p(n-s)}{p-s}}} \left(1 + O(r^2)\right) dr + O(\epsilon^3).
\end{aligned}$$

Taking the variable change $t = \frac{r}{\epsilon}$, we have

$$f(x_0)\omega_{n-1} \int_0^{\frac{\delta}{\epsilon}} \frac{t^{n-1}}{t^s(1+t^{\frac{p-s}{p-1}})^{\frac{p(n-s)}{p-s}}} dt = f(x_0)\int_{\mathbb{R}^n} \frac{|\Phi_p|^{p^\star(s)}}{|y|^s} dy + O(\epsilon^{\frac{n-s}{p-1}}). \tag{3.28}$$

Now, note that:

$(i)''$ $p < \frac{n+2-s}{2}$

$$\omega_{n-1}\int_0^{\frac{\delta}{\epsilon}} \frac{t^{n+1}}{t^s(1+t^{\frac{p-s}{p-1}})^{\frac{p(n-s)}{p-s}}} dt = \int_{\mathbb{R}^n} \frac{|\Phi|^{p^\star(s)}}{|y|^{s-2}} dy + O(\epsilon^{\frac{n+2-s-2p}{p-1}}).$$

$(ii)''$ $p = \frac{n+2-s}{2}$

$$\int_0^{\frac{\delta}{\epsilon}} \frac{t^{n+1}}{t^s(1+t^{\frac{p-s}{p-1}})^{\frac{p(n-s)}{p-s}}} dt = O(1) + \ln(\epsilon^{-1}) + O(\epsilon^{\frac{p-s}{p-1}}).$$

$(iii)''$ $p > \frac{n+2-s}{2}$

$$\int_0^{\frac{\delta}{\epsilon}} \frac{t^{n+1}}{t^s(1+t^{\frac{p-s}{p-1}})^{\frac{p(n-s)}{p-s}}} dt = O(\epsilon^{\frac{n+2-s-2p}{p-1}}).$$

Thus, we reach that

$$\int_M \frac{f(x)u_\epsilon^{p^\star(s)}}{d_g(x,x_0)^s}dv_g$$
$$= \begin{cases} f(x_0)\|\Phi\|^{p^\star(s)}_{L^{p^\star(s)}_{d_g,s}} + \frac{C_2(n,p,s)}{2nf(x_0)}\left(\frac{\Delta f(x_0)}{f(x_0)} - \frac{R_g(x_0)}{3}\right)\epsilon^2 + O(\epsilon^{\frac{n-s}{p-1}}), \text{ if } p < \frac{n+2-s}{2}, \\ f(x_0)\|\Phi\|^{p^\star(s)}_{L^{p^\star(s)}_{d_g,s}} + \left(\frac{\Delta f(x_0)}{2n} - \frac{f(x_0)R_g(x_0)}{6n}\right)\epsilon^2\ln(\epsilon^{-1}) + O(\epsilon^2), \text{ if } p = \frac{n+2-s}{2}, \\ f(x_0)\|\Phi\|^{p^\star(s)}_{L^{p^\star(s)}_{d_g,s}} + O(\epsilon^{\frac{n-s}{p-1}}), \text{ if } p > \frac{n+2-s}{2}, \end{cases} \quad (3.29)$$

where $C_2(n,p,s) := \int_{\mathbb{R}^n} \frac{|\Phi|^{p^\star(s)}}{|y|^{s-2}}dy$.

Next, let us calculate the quotient

$$\frac{\int_M |\nabla_g u_\epsilon|^p dv_g + \int_M hu_\epsilon^p dv_g}{\left(\int_M \frac{f(x)u_\epsilon^{p^\star(s)}}{d_g(x,x_0)^s}dv_g\right)^{\frac{p}{p^\star(s)}}}.$$

Starting when $n \geq 4$ and $p \leq \frac{n+2}{3}$.

First, for $p < \sqrt{n}$, by (3.25), (3.26) and (3.29) we can compute the following

$$\frac{\int_M |\nabla_g u_\epsilon|^p dv_g + \int_M hu_\epsilon^p dv_g}{\left(\int_M \frac{f(x)u_\epsilon^{p^\star(s)}}{d_g(x,x_0)^s}dv_g\right)^{\frac{p}{p^\star(s)}}}$$
$$= \frac{\|\nabla\Phi\|_p^p - \frac{C_1(n,p,s)}{6n}R_g(x_0)\epsilon^2 + h(x_0)\|\Phi_p\|_p^p\epsilon^p + o(\epsilon^{\min\{p,2\}})}{f(x_0)^{\frac{p}{p^\star(s)}}\left(\|\Phi_p\|^{p^\star(s)}_{L^{p^\star(s)}_{d_g,s}} + \left(\frac{\Delta f(x_0)}{2nf(x_0)} - \frac{R_g(x_0)}{6n}\right)C_2(n,p,s)\epsilon^2 + o(\epsilon^2)\right)^{\frac{p}{p^\star(s)}}}$$
$$= \begin{cases} \frac{K(n,p,s)^{-1}}{f(x_0)^{\frac{p}{p^\star(s)}}}\left\{1 + \frac{h(x_0)\|\Phi_p\|_p^p}{\|\nabla\Phi\|_p^p}\epsilon^p + o(\epsilon^p)\right\}, \text{ if } 1 < p < 2 \\ \frac{K(n,p,s)^{-1}}{f(x_0)^{\frac{p}{p^\star(s)}}}\left\{1 - \kappa_p\left[\left(\frac{p^\star(s)C_1(n,p,s)\|\Phi_p\|^{p^\star(s)}_{p^\star(s)}}{pC_2(n,p,s)\|\nabla\Phi_p\|_p^p} - 1\right)\frac{R_g(x_0)}{3} + \frac{\Delta f(x_0)}{f(x_0)}\right]\epsilon^2 + o(\epsilon^2)\right\}, \text{ if } p > 2, \end{cases}$$

where $\kappa_p = \kappa_p(n,p,s) := \frac{pC_2(n,p,s)}{2np^\star(s)\|\Phi_p\|^{p^\star(s)}_{p^\star(s)}} > 0$ when $p > 2$.

From [11, section 5] we have that $\frac{C_1(n,p,s)\|\Phi_p\|^{p^\star(s)}_{p^\star(s)}}{C_2(n,p,s)\|\nabla\Phi_p\|_p^p} = \frac{(n-p)(n+2-s)}{(n-s)(n+2-3p)}$ for $0 < s < p$ (the case $s = 0$ was calculated by Druet in [15, subsection 5.2]), therefore, for $n \geq 4$ and $p < \sqrt{n}$ we reach that

$$\frac{\int_M |\nabla_g u_\epsilon|^p dv_g + \int_M hu_\epsilon^p dv_g}{\left(\int_M \frac{f(x)u_\epsilon^{p^\star(s)}}{d_g(x,x_0)^s}dv_g\right)^{\frac{p}{p^\star(s)}}} =$$
$$\begin{cases} \frac{K(n,p,s)^{-1}}{f(x_0)^{\frac{p}{p^\star(s)}}}\left\{1 + \frac{h(x_0)\|\Phi_p\|_p^p}{\|\nabla\Phi\|_p^p}\epsilon^p + o(\epsilon^p)\right\}, \text{ if } 1 < p < 2 \\ \frac{K(n,p,s)^{-1}}{f(x_0)^{\frac{p}{p^\star(s)}}}\left\{1 - \kappa_p\left[\frac{(3p-s)}{(n+2-3p)}\frac{R_g(x_0)}{3} + \frac{\Delta f(x_0)}{f(x_0)}\right]\epsilon^2 + o(\epsilon^2)\right\}, \text{ if } p > 2. \end{cases} \quad (3.30)$$

In the case $n \geq 5$ and $\sqrt{n} \leq p \leq \frac{n+2}{3}$, in a similar way we get that

$$\frac{\int_M |\nabla_g u_\epsilon|^2 dv_g + \int_M h u_\epsilon^2 dv_g}{\left(\int_M \frac{f(x)}{d_g(x,x_0)^s} u_\epsilon^{p^\star(s)} dv_g\right)^{\frac{p}{p^\star(s)}}}$$

$$= \begin{cases} \frac{K(n,p,s)^{-1}}{f(x_0)^{\frac{p}{p^\star(s)}}} \left\{1 - \kappa_p \left[\frac{(3p-s)}{(n+2-3p)} \frac{R_g(x_0)}{3} + \frac{\Delta f(x_0)}{f(x_0)}\right] \epsilon^2 + o(\epsilon^2)\right\}, \text{ if } 2 < p < \frac{n+2}{3}, \\ \frac{K(n,p,s)^{-1}}{f(x_0)^{\frac{p}{p^\star(s)}}} \left\{1 - \frac{R_g(x_0)\omega_{n-1} p^p}{6n\|\nabla \Phi\|_p^p} \epsilon^2 \ln(\epsilon^{-1}) + o(\epsilon^2 \ln(\epsilon^{-1}))\right\}, \text{ if } 2 < p = \frac{n+2}{3}. \end{cases}$$

Now, when $n = 3$ (resp. $n = 2$).

$(i)''$ $p < \sqrt{3}$ (resp. $\sqrt{2}$)

$$\frac{\int_M |\nabla_g u_\epsilon|^2 dv_g + \int_M h u_\epsilon^2 dv_g}{\left(\int_M \frac{f(x)}{d_g(x,x_0)^s} u_\epsilon^{p^\star(s)} dv_g\right)^{\frac{p}{p^\star(s)}}} = \frac{K(n,p,s)^{-1}}{f(x_0)^{\frac{p}{p^\star(s)}}} \left\{1 + \frac{h(x_0)\|\Phi_p\|_p^p}{\|\nabla \Phi\|_p^p} \epsilon^p + o(\epsilon^p)\right\}.$$

$(ii)''$ $p = \sqrt{3}$ (resp. $\sqrt{2}$)

$$\frac{\int_M |\nabla_g u_\epsilon|^2 dv_g + \int_M h u_\epsilon^2 dv_g}{\left(\int_M \frac{f(x)}{d_g(x,x_0)^s} u_\epsilon^{p^\star(s)} dv_g\right)^{\frac{p}{p^\star(s)}}} = \frac{K(n,p,s)^{-1}}{f(x_0)^{\frac{p}{p^\star(s)}}} \left\{1 + \frac{h(x_0)\omega_2}{\|\nabla \Phi\|_p^p} \epsilon^p \ln(\epsilon^{-1}) + o(\epsilon^p \ln(\epsilon^{-1}))\right\}.$$

Therefore, by the hypothesis of Theorems 3.1.1 and 3.1.3 in their respective cases, and $\epsilon > 0$ small enough in the above estimates we reach that

$$\frac{\int_M |\nabla_g u_\epsilon|^p dv_g + \int_M h u_\epsilon^p dv_g}{\left(\int_M \frac{f(x) u_\epsilon^{p^\star(s)}}{d_g(x,x_0)^s} dv_g\right)^{\frac{p}{p^\star(s)}}} < \frac{K(n,p,s)^{-1}}{f(x_0)^{\frac{p}{p^\star(s)}}}. \tag{3.31}$$

Now, to complete the proof, let ξ, ζ be positive real number with $\left(\frac{\xi}{\zeta}\right)^p = \frac{\alpha}{\beta}$ and we define for any $t \geq 0$ the following functional:

$$\Phi(t) = E_h(t\xi u_\epsilon, t\zeta u_\epsilon) = \frac{t^p}{p} X_{u_\epsilon} - \frac{t^{p^\star(s)}}{p^\star(s)} Y_{u_\epsilon},$$

where $X_{u_\epsilon} = (\xi^p + \zeta^p) \int_M |\nabla_g u_\epsilon|^p dv_g + \int_M (\xi^p a(x) + \xi\zeta(\xi^{p-2} + \zeta^{p-2}) b(x) + \zeta^p c(x)) |u_\epsilon|^p dv_g$ and $Y_\epsilon = \xi^\alpha \zeta^\beta \int_M \frac{f(x)|u_\epsilon|^{p^\star(s)}}{d_g(x,x_0)^s} dv_g$. We want to find $t_0 > 0$ such that $\Phi'(t_0) = 0$, this occurs when $t^{p-1} X_{u_\epsilon} - t^{p^\star(s)-1} Y_{u_\epsilon} = 0$. Hence,

$$t_0 = \left(\frac{X_{u_\epsilon}}{Y_{u_\epsilon}}\right)^{\frac{1}{p^\star(s)-p}},$$

this implies that t_0 is a critical point of Φ, more precisely, is a maximum point of Φ

since $\Phi(t) \to -\infty$ as $t \to \infty$. Thus, we get that

$$\begin{aligned} \Phi(t_0) &= \left(\frac{1}{p} - \frac{1}{p^\star(s)}\right)\left(\frac{X_{u_\epsilon}}{Y_{u_\epsilon}^{\frac{p}{p^\star(s)}}}\right)^{\frac{p^\star(s)}{p^\star(s)-p}} \\ &= \frac{p-s}{p(n-s)}\left(\frac{(\xi^p+\zeta^p)}{(\xi^\alpha\zeta^\beta)^{\frac{p}{p^\star(s)}}}\frac{\int_M |\nabla_g u_\epsilon|^p dv_g + \int_M h u_\epsilon^p dv_g}{\left(\int_M \frac{f(x)u_\epsilon^{p^\star(s)}}{d_g(x,x_0)^s}dv_g\right)^{\frac{p}{p^\star(s)}}}\right)^{\frac{p^\star(s)}{p^\star(s)-p}}, \end{aligned} \tag{3.32}$$

and by estimate (3.31), we reach

$$\Phi(t_0) < \frac{p-s}{p(n-s)}\frac{(\mathcal{K}^{p,s}_{(\alpha,\beta)})^{\frac{p^\star(s)}{p^\star(s)-p}}}{f(x_0)^{\frac{p}{p^\star(s)-p}}}.$$

Now, choose $t_1 > t_0$ large such that $\Phi(t_1) < 0$ and $||(\tilde{u},\tilde{v})|| > R$ with $\tilde{u} = t_1\xi u_\epsilon$ and $\tilde{v} = t_1\zeta u_\epsilon$. We have

$$\begin{aligned} 0 < \tau = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} E_h(\gamma(t)) &\le \sup_{t\in[0,1]} E_h(t\tilde{u}, t\tilde{v}) \\ &= \sup_{t\in[0,1]} \Phi(tt_1) < \Phi(t_0) < \frac{p-s}{p(n-s)}\frac{(\mathcal{K}^{p,s}_{(\alpha,\beta)})^{\frac{p^\star(s)}{p^\star(s)-p}}}{f(x_0)^{\frac{p}{p^\star(s)-p}}}. \end{aligned}$$

This completes the proof of the lemma. ■

3.5 Proof of Theorems 3.1.1 and 3.1.3

Proof of Theorems 3.1.1. Lemma 3.4.3 gives us that

$$0 < \tau < \frac{(p-s)}{p(n-s)}\frac{(\mathcal{K}^{p,s}_{(\alpha,\beta)})^{\frac{(p^\star(s)-p)}{p^\star(s)}}}{f(x_0)^{\frac{p}{p^\star(s)-p}}},$$

and from Proposition 3.3.5 it follows that E_h satisfies the Palais-Smale condition at the level τ. Thus, by the Mountain Pass Theorem [49, Theorem 2.10] we reach that τ is a critical value of E_h, that is, there exists (u,v) in H such that

$$E_h'(u,v) \equiv 0 \quad \text{and} \quad E_h(u,v) = \tau,$$

this implies that (u,v) is pair of nontrivial weak solutions of system (3.1). Which completes the proof of Theorem 3.1.1. ■

To prove that from the hypotheses assumed in Corollary 3.1.2 we obtain a pair of non-negative solution of system (3.1), let us consider the functional

$$\begin{aligned} E_{h,+}(u,v) =& \frac{1}{p}\int_M \left(|\nabla_g u|^p + |\nabla_g v|^p + a|u|^p + buv(|u|^{p-2} + |v|^{p-2}) + c|v|^p\right) dv_g \\ &- \frac{1}{p^\star(s)}\int_M \frac{f(x)(u^+)^\alpha (v^+)^\beta}{d_g(x,x_0)^s} dv_g. \end{aligned} \tag{3.33}$$

Proof of Corollary 3.1.2. It is easy to see that the functional $E_{h,+}$ satisfies the same properties of E_h. Using the same test function we have that the Lemma 3.4.3 is true for $E_{h,+}$, from Theorem 3.1.1 one has that there exists $(u,v) \in H^p$ a nontrivial critical point of $E_{h,+}$. Now, let us to prove that u and v are nonnegative. Since $E'_{h,+}(u,v)(u^-,v^-) = 0$, i.e,

$$\begin{aligned} 0 &= E'_{h,+}(u,v)(u^-,v^-) \\ &= \int_M \left\{|\nabla_g u^-|^p + |\nabla_g v^-|^p + a|u^-|^p + c|v^-|^p\right\} \\ &+ \int_M \frac{b(x)}{p}[((p-1)|u|^{p-2} + |v|^{p-2})vu^- + ((p-1)|v|^{p-2} + |u|^{p-2})uv^-]dv_g \\ &- \int_M \frac{\alpha}{p^\star(s)}\frac{f(x)}{d_g(x,x_0)^s}(u^+)^{\alpha-1}(v^+)^\beta u^- dv_g + \int_M \frac{\beta}{p^\star(s)}\frac{f(x)}{d_g(x,x_0)^s}(v^+)^{\beta-1}(u^+)^\alpha v^- dv_g. \\ &= \int_M \left[|\nabla_g u^-|^p + |\nabla_g v^-|^p + a(x)|u^-|^p + b(x)u^-v^-(|u^-|^{p-2} + |v^-|^{p-2}) + c(x)|v^-|^p\right] dv_g \\ &+ \int_M \frac{b(x)}{p}\left[((p-1)|u^-|^{p-2} + |v^+|^{p-2})v^+u^- + ((p-1)|v^-|^{p-2} + |u^+|^{p-2})u^+v^-\right] dv_g \end{aligned}$$

and as $b \leq 0$, it follows that

$$\int_M \left[|\nabla_g u^-|^p + |\nabla_g v^-|^p + a|u^-|^p + b(x)u^-v^-(|u^-|^{p-2} + |v^-|^{p-2}) + c|v^-|^p\right] dv_g \leq 0,$$

by coercivity hypothesis one has that $u^- = 0$ and $v^- = 0$. Therefore, (u,v) is a pair of non-negative weak solutions of system (3.1). This ends the proof. ■

Proof of Theorem 3.1.3. In the case $1 < p < 2$, where we assume $b \equiv 0$, the proof follows a similar approach to that of Corollary 3.1.2. Thus, we establish the existence of a pair (u,v) in H^p of non-negative nontrivial weak solutions of system (3.1). This concludes the proof. ■

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