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K-Theoretic Version of Fourier-Mukai Transforms Between Crepant Resolutions of Finite Quotient Singularities

by

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ATA DE DEFESA DE DOUTORADO JUNTO AO PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA DA UNIVERSIDADE FEDERAL DA PARAÍBA, REALIZADA NO DIA 25 DE AGOSTO DE 2023.

Ao vigésimo quinto dia de agosto de dois mil e vinte e três, às 12:00 horas, no Auditório do Departamento de Matemática/CCEN da Universidade Federal da Paraíba, foi aberta a sessão pública de Defesa de Tese intitulada "**Transformada de Fourier-Mukai K-teorética entre resoluções crepantes de singularidades quociente finitas**", do aluno **Ivan Junnior Serna Giraldo** que havia cumprido, anteriormente, todos os requisitos para a obtenção do grau de Doutor em Matemática, sob a orientação do Prof. Dr. Ugo Bruzzo. A Banca Examinadora, indicada pelo Colegiado do Programa de Pós-Graduação em Matemática, foi composta pelos professores: Ugo Bruzzo (Orientador), Ricardo Burity Croccia Macedo (UFPB), Valeriano Lanza (UFF), Abdelmoubine Amar Henni (UFSC) e Charles Aparecido Almeida (UFMG). O professor Ugo Bruzzo, em virtude da sua condição de orientador, presidiu os trabalhos e, depois das formalidades de apresentação, convidou o aluno a discorrer sobre o conteúdo da tese. Concluída a explanação, o candidato foi arguido pela banca examinadora que, em seguida, sem a presença do aluno, finalizando os trabalhos, reuniu-se para deliberar tendo concedido a menção: **aprovado.** E, para constar, foi lavrada a presente ata que será assinada pelos membros da Banca Examinadora.

João Pessoa, 25 de agosto de 2023.

Ugo Bruzzo ______________________________

Ricardo Burity Croccia Macedo ______________________________

Valeriano Lanza ______________________________

Abdelmoubine Amar Henni ______________________________

Charles Aparecido Almeida ______________________________

Dedication

To my mother

Acknowledgments

First of all I would like to thank my advisor Ugo Bruzzo for introducing me to this topic and for his patient guidance and continued support. I express my gratitude to the mathematics department of Universidade Federal da Paraíba where this work was done and in particular professor Fágner Dias Araruna for the help given since the beginning of my studies. I would also like to thank Professor Victor Rafael Cabanillas Zannini who always motivated me to continue my studies. To my family for their support in many ways my mother Rosa Giraldo, my father Aresio Serna, Raúl Garate and Angel Serna. And my analysts friends from Perú Mirko Sanchez, Luis Condor, Julio Lanazca and professors such as Pedro Contreras and Renato Benazic. My friends from João Pessoa Juan Nuño Ballesteros, Napoleón Caro, Lizandro Sanchez and Anselmo Raposo.

When somebody once asked whether I studied any works of the classics like Poincaré, Riemann, Siegel, Weil, etc., my answer was "For me, mathematics started with Grothendieck."

P. Scholze

Abstract

We study crepant resolutions of singularities $\mathbb{C}^3/G$, where G is a finite abelian subgroup of $\mathrm{SL}(3,\mathbb{C})$. Using derived category methods, Bridgeland, King and Reid proved that the Hilbert scheme of G-clusters $(G\text{-Hilb})(\mathbb{C}^3)$ is a crepant resolution. Following Craw-Ishii, we study the moduli spaces $\mathcal{M}_\theta$ of θ-stable G-constellations, in particular, $(G\text{-Hilb})(\mathbb{C}^3)$ is a moduli space of this type for a suitable parameters in the GIT-parameter space, while all crepant resolutions are of the form $\mathcal{M}_\theta$ for some θ.

The GIT-parameter space is divided into chambers, and for parameters in adjacent chambers, the $\mathcal{M}_\theta$ spaces are Fourier-Mukai partners. Following Craw-Ishii we study how the Fourier-Mukai transform between partners can induce a change in the tautological line bundles.

As an application, we study the case of $\mathbb{C}^3/\mathbb{Z}_4$. We outline the toric description of the singularity and its crepant resolution. Using Chern classes we determine the cohomological Fourier-Mukai transform between Fourier-Mukai partners, that are moduli spaces for adjacent chambers. In general, for the singularities $\mathbb{C}^3/G$, we also determine the cohomological Fourier-Mukai transform as a linear transformation between the cohomology rings.

Keywords: Derived categories, derived functors, Fourier-Mukai transforms, crepant resolutions, McKay correspondence, Geometric Invariant Theory, moduli spaces, K-theory, Toric Geometry.

Resumo

Estudamos a resolução de singularidades denominadas resoluções crepantes da singularidade $\mathbb{C}^3/G$, onde G é um subgrupo abeliano finito de $\mathrm{SL}(3,\mathbb{C})$. Usando métodos de categorias derivadas, Bridgeland, King e Reid provaram que o esquema de Hilbert de G-clusters (G-Hilb)($\mathbb{C}^3$) é uma resolução crepante. Seguindo Craw-Ishii, estudamos os espaços de módulos $\mathcal{M}_\theta$ de G-constelações θ-estáveis, em em particular, (G-Hilb)($\mathbb{C}^3$) é um espaço de moduli deste tipo para um parâmetro adequado no espaço de parâmetros GIT.

O espaço de parâmetros GIT é dividido em câmaras e para parâmetros em câmaras adjacentes, os espaços $\mathcal{M}_\theta$ são parceiros de Fourier-Mukai. Nós também induce uma mudança dos fibrados de linha tautológicos.

Como aplicação, estudamos o caso de $\mathbb{C}^3/\mathbb{Z}_4$. Nós delineamos a descrição tórica da singularidade e da sua resolução crepante e determinamos a transformada cohomológica de Fourier-Mukai para câmaras adjacentes. Em geral, para a singularidade $\mathbb{C}^3/G$, também determinamos a transformada cohomológica de Fourier-Mukai como uma transformação linear entre os anéis de cohomologia.

Keywords: Categorias derivadas, funtores derivados, transformadas de Fourier-Mukai, resoluçoes crepantes, Correspondencia de McKay, Teoria dos Invariantes Geométricos, espaços de moduli, Teoria K, Geometría Tórica.

Contents

Introduction

The main objects of study in algebraic geometry are the spaces of solutions of polynomial equations in several variables. The work of the so-called Italian school was groundbreaking, however at some point foundational problems started to emerge. After substantial contributions by O. Zariski and A. Weil, it was A. Grothendieck who provided a consistent framework and a vision that was needed in algebraic geometry. Using Sheaf Theory and strong new results in Homological Algebra, and using methods from Algebraic Topology, Grothendieck introduced more general spaces such as schemes, algebraic spaces, stacks, etc., generalizing the classical theory of varieties, despite these spaces and their methods being highly abstract.

In this thesis, our main focus is on the resolution of singularities, specifically the crepant resolutions of the singularities $\mathbb{C}^3/G$ when G is a finite abelian group in $\mathrm{SL}(3, \mathbb{C})$, paying particular attention to the case $G = \mathbb{Z}_4$.

Crepant resolutions are the appropriate resolutions of singularities to generalize the McKay correspondence to three dimensions. This correspondence in two dimensions was studied by Gonzalez-Sprinberg and Verdier [22] using methods from derived categories.

Derived McKay Correspondence

The McKay correspondence in two dimensions establishes a correspondence between the representation ring of the group G and the cohomology ring of the minimal resolution of singularities. We know by Bridgeland, King and Reid Theorem [8, Theorem 1.2] that the integral functor

$$\Phi_{\mathcal{U}} : D^b(Y) \xrightarrow{\sim} D^b_G(\mathbb{C}^3)$$

is an equivalence of derived categories (as triangulated categories) between the derived bounded category of $\mathrm{Coh}(Y)$, where $Y = G\text{-Hilb}(M)$, and G-

equivariant bounded derived category of $\mathrm{Coh}(\mathbb{C}^3)$. Actually according to of Ito and Nakajima [32] and Nakamura [42], Y is a crepant resolution of $\mathbb{C}^3/G$. This scheme parametrizes the G-clusters in $\mathbb{C}^3$, i.e., the G-invariant zero-dimensional subschemes of $\mathbb{C}^3$ whose space of global sections $\Gamma(\mathcal{O}_Z)$ is isomorphic to the regular representation $\mathbb{C}[G]$ as G-modules.
The functor $\Phi_{\mathcal{U}} : D^b(Y) \xrightarrow{\sim} D^b_G(\mathbb{C}^3)$ is called Fourier-Mukai transform and induces the following commutative diagram of derived categories:

$$\begin{tikzcd} D^b(Y) \arrow[r, "\Phi"] & D^b_G(\mathbb{C}^3) \\ D^b_c(Y) \arrow[u, hook] \arrow[r, "\Phi_c"'] & D^b_{c,G}(\mathbb{C}^3) \arrow[u, hook] \end{tikzcd}$$

where

$$D^b_c(Y) = \left\{ \mathcal{E}^\bullet \in D^b(Y) \;\middle|\; \mathrm{supp}(\mathcal{E}^\bullet) \subset \tau^{-1}(\pi(0)) \right\}$$

and

$$D^b_{c,G}(\mathbb{C}^3) = \left\{ \mathcal{E}^\bullet \in D^b_G(\mathbb{C}^3) \;\middle|\; \mathrm{supp}(\mathcal{E}^\bullet) \subset \{0\} \right\}.$$

These equivalences induce the following commutative diagram between K-theory Grothendieck groups:

$$\begin{tikzcd} K(Y) \arrow[r, "\phi"] & K^G(\mathbb{C}^3) \\ K_0(Y) \arrow[u, hook] \arrow[r, "\phi_0"'] & K^G_0(\mathbb{C}^3). \arrow[u, hook] \end{tikzcd}$$

For two complexes $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ in $K^G(\mathbb{C}^3)$ the G-equivariant Euler characteristic is defined by:

$$\chi^G(\mathcal{E}^\bullet, \mathcal{F}^\bullet) = \sum (-1)^i \dim \mathrm{Hom}_{D^b_G(\mathbb{C}^3)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet[i]).$$

Moreover, there exists an isomorphism between the representation ring of G and the equivariant K-theory of $\mathbb{C}^3$ given by $R(G) \cong K^G(\mathbb{C}^3), \rho \mapsto \rho \otimes \mathcal{O}_{\mathbb{C}^3}$. The G-sheaves $\{\rho \otimes \mathcal{O}_{\mathbb{C}^3}\}_{\rho \in G^*}$ make up a basis for $K^G(\mathbb{C}^3)$, and similarly one has a basis $\{\rho \otimes \mathcal{O}_0\}_{\rho \in G^*} \subset K^G_0(\mathbb{C}^3)$. With this and using the inverse of ϕ, the K-theory of a crepant resolution is generated by

$$\{\mathcal{R}_\rho\}_{\rho \in G^*} \subset K(Y).$$

This works for the n-dimensional case under the assumption that

$$\Phi_{\mathcal{U}} : D^b(Y) \xrightarrow{\sim} D^b_G(\mathbb{C}^n)$$

is an equivalence of derived categories.

Moduli Spaces of G-constellations and Crepant Resolutions of $\mathbb{C}^3/G$

More generally, Craw and Ishii in [15] introduced the notion of G-constellation, i.e., an object $\mathcal{F}$ of $\mathrm{Coh}^G(\mathbb{C}^3)$ such that $H^0(\mathcal{F})$ is isomorphic to $\mathbb{C}[G]$ as G-modules. A stability condition is imposed to these G-constellations. The notion of stability is taken from the one used for quiver representations, as introduced by King in [35]. For that purpose, they introduced the GIT parameter space:

$$\Theta = \left\{ \theta \in \mathrm{Hom}_{\mathbb{Z}}(R(G), \mathbb{Q}) \;\middle|\; \theta(R) = 0 \right\}$$

where $R(G)$ is the representation ring of G. This space has a chamber structure. The chambers are open polyhedral convex cones, are finite in number, and their union is dense in Θ. Chambers are separated by walls, which have codimension 1. Parameters lying in the same chamber produce isomorphic moduli spaces, so that one can use the notation $\mathcal{M}_C$ where C is a chamber. Moving from one chamber to an adjacent one wall-crossing phenomena occur.

Fixing a stability condition in Θ, one can define the moduli spaces $\mathcal{M}_\theta$ of θ-stable G-constellations, which come equipped with tautological line bundles $\mathcal{R}_\rho$ and a tautological bundle $\mathcal{R}$. When θ is generic, these spaces are fine moduli spaces of G-constellations, and therefore they possess a universal G-constellation. Again when θ is generic, the integral transform

$$\Phi_{\mathcal{U}} : D^b(\mathcal{M}_\theta) \xrightarrow{\sim} D^b_G(\mathbb{C}^3)$$

is an equivalence of derived categories, where the kernel of the Fourier-Mukai transform is the universal object on $Y \times \mathbb{C}^3$.

The first main theorem in [15, Theorem 1.1] is the following: If G is abelian, and $Y \to \mathbb{C}^3/G$ is a projective crepant resolution, then $Y \cong \mathcal{M}_C$ for some chamber $C \subset \Theta$. This has been recently extended to the case of general (non-abelian) G by Yamagishi [54].

The contraction morphism for a wall W is $\mathrm{Cont}_W : \mathcal{M}_\theta \longrightarrow \overline{\mathcal{M}_{\theta_0}}$ for a fixed point θ_0 that lies in W. This morphism classifies the walls into three types. For each type of wall, the exceptional locus Z of the morphism Cont_W has a closed subscheme structure. The dimension of Z is determined by the type of wall.

So, for each chamber C in the GIT-parameter space, we have a crepant resolution. We know that these chambers are separated by walls. The wall-crossing phenomenon can cause a change in the tautological line bundles, even when the adjacent chambers correspond to isomorphic varieties. This happens because we need to distinguish between varieties and moduli spaces. The latter have a universal object that depends on the chosen parameters.

For each type of wall, we have theorems that relate the tautological line bundles of the initial chamber to the adjacent chamber. Thanks to these results we can compute the cohomological Fourier-Mukai transform.

Moduli of G-Constellations as Fourier-Mukai Partners

Consider two adjacent chambers in the GIT-parameter space Θ, say C and C', with $W = \overline{C} \cap \overline{C'}$ the wall separating them. In this situation one has the diagram

$$\begin{tikzcd} D^b(\mathcal{M}_C) \arrow[r, "\Phi_1"] \arrow[dr, "\Psi_{\mathcal{P}}"'] & D^b_G(\mathbb{C}^3) \arrow[d, shift left, "\Phi_2^{-1}"] \\ & D^b(\mathcal{M}_{C'}) \arrow[u, shift left, "\Phi_2"] \end{tikzcd}$$

where $\Psi_{\mathcal{P}} = \Phi_2^{-1} \circ \Phi_1$ is a Fourier-Mukai transform. Then the equivalence

$$\Psi_{\mathcal{P}} : D^b(\mathcal{M}_C) \longrightarrow D^b(\mathcal{M}_{C'})$$

makes the varieties $\mathcal{M}_C$ and $\mathcal{M}_{C'}$ Fourier-Mukai partners. By a second result in [15, Theorem 1.2], the kernel $\mathcal{P}$ depends of the type of the wall W.

For walls of type 0, in [15, Proposition 4.3], the transform Ψ is explicitly described. There are two cases; when the divisor D parameterizes a rigid quotient $\mathcal{Q}$, this family is a locally free $\mathcal{O}_Z$-module, and the irreducible representation $\rho \in H^0(\mathcal{Q})$:

$$\Psi = \Phi_2^{-1} \circ \Phi_1 = T_{\mathcal{R}'_\rho \otimes \omega_D}$$

where $T_{\mathcal{R}'_\rho \otimes \omega_D}$ is a twist along the spherical object $\mathcal{R}'_\rho \otimes \omega_D$. For the case where the divisor D parameterizes a rigid subsheaf $\mathcal{S}$ it is analogous,

$$\Psi = \Phi_2^{-1} \circ \Phi_1 = \mathcal{O}_Y(-D) \otimes T_{\omega_D}(-)$$

and the rigid subsheaf is too a locally free $\mathcal{O}_Z$-module.

For walls of type 1 the morphism $\mathcal{M}_C \longrightarrow \mathcal{M}_{C'}$ is a classical flop along the curve $l \subset \mathcal{M}_C$; this curve is the unstable locus determined by a wall that separates the chambers. The results of Bondal and Orlov [6] can be applied in this situation. For the case when the trivial representation $\rho_0 \subset R_2$, the equivalence between Fourier-Mukai partners is

$$\Psi \cong Rq_* Lp^*$$

where the morphisms $p : \widetilde{\mathcal{M}} \longrightarrow \mathcal{M}_C$ and $q : \widetilde{\mathcal{M}} \longrightarrow \mathcal{M}_{C'}$ are the projections and $\widetilde{\mathcal{M}}$ is the blow-up of $\mathcal{M}_C$ along the curve l.

The case $\rho_0 \subset R_1$ is similar. For walls of type 3, the moduli spaces $\mathcal{M}_C$ and $\mathcal{M}_{C'}$ are isomorphic. We know that in this case, the exceptional locus is a divisor $D \subset \mathcal{M}_C$, which is contracted to a curve in Y_0 under the morphism $\mathrm{Cont}_W : \mathcal{M}_C \longrightarrow Y_0$. By forming the fiber product $\mathcal{M}_C \times_{Y_0} \mathcal{M}_C$ and considering p and q as the first and second projections onto the factors, the equivalence is written in a similar way to the previous case.

Cohomological Fourier-Mukai Transform between Crepant Resolutions of $\mathbb{C}^3/\mathbb{Z}_4$

The singularity $\mathbb{C}^3/\mathbb{Z}_4$ is interesting for several reasons. Its crepant resolution can be constructed using toric geometry, allowing us to identify the divisors that form the exceptional divisor.

The structure of the GIT-parameter space is completely determined, including the walls that separate the chambers. The parameter space has 8 chambers and has two types of walls. We obtain the equations of each wall and assign a name to them. We do this in order to apply the theorems about of the change of the tautological line bundles under wall-crossing.

First, let us consider two moduli spaces $\mathcal{M}_C$ and $\mathcal{M}_{C'}$ of θ-stable G-constellations and θ'-stable G-constellations for parameters $\theta \in C$ and $\theta' \in C'$. These varieties are Fourier-Mukai partners via

$$\Psi := \Phi_2^{-1} \circ \Phi_1 : D^b(\mathcal{M}_C) \longrightarrow D^b(\mathcal{M}_{C'})$$

where $\Phi_1 : D^b(\mathcal{M}_C) \longrightarrow D^b_G(\mathbb{C}^3)$ and $\Phi_2 : D^b(\mathcal{M}_{C'}) \longrightarrow D^b_G(\mathbb{C}^3)$ are the Fourier-Mukai transforms given by the derived McKay correspondence. The cohomological version of the Fourier-Mukai transform Ψ is given by

$$\Psi^H_{\mathcal{P}} : H^*(\mathcal{M}_C, \mathbb{Q}) \longrightarrow H^*(\mathcal{M}_{C'}, \mathbb{Q}),$$

where we take the Chern character of the kernel $\mathcal{P}$ of the Fourier-Mukai transform. The cohomology ring $H^*(\mathcal{M}_C, \mathbb{Q})$ is generated by the Chern characters of the tautological line bundles. That is, the set $\{\mathrm{Ch}(\mathcal{R}_\rho)\}_{\rho \in G^*}$ forms a basis of the rational cohomology ring $H^*(\mathcal{M}_C, \mathbb{Q})$. This ring has a decomposition

$$H^*(Y, \mathbb{Q}) = H^0(Y, \mathbb{Q}) \oplus H^2(Y, \mathbb{Q}) \oplus H^4(Y, \mathbb{Q}).$$

By the Ito-Reid theorem, we know that the exceptional prime divisors form a basis of $H^2(Y, \mathbb{Q})$ for any crepant resolution. The case of $\mathbb{C}^3/\mathbb{Z}_4$ has only one crepant resolution and we can relate the tautological line bundles of two adjacent chambers. This led us to explicitly obtain the cohomological Fourier-Mukai transform.

Finally, with this example, we notice that we could compute the matrix that defines this transformation. We obtain a matrix that defines the cohomological Fourier-Mukai transform for the general case of the singularity $\mathbb{C}^3/G$.

Organization of contents

In the first chapter, we start by reviewing fundamental facts about derived categories, which are the natural framework for derived functors. We explain the construction of abelian categories and then apply it to the case of derived categories of coherent sheaves on a scheme. With this, we can study the construction of the main functors that appear in algebraic geometry. This is a particular case of the six functors formalism, and the Grothendieck-Verdier duality used later fits into the context of this chapter.

In the second chapter, we study the derived McKay correspondence in three dimensions and describe, with all the details, the two famous theorems by Bridgeland, King, and Reid. At this point, we see how the machinery of derived categories is applied throughout. The Fourier-Mukai transforms, which are functors between derived categories, establish the equivalence of categories $D^b(Y)$ and $D^b_G(\mathbb{C}^3)$. That allows us to obtain bases for the K-theory of Y and the equivariant K-theory of $\mathbb{C}^3$. Then, in chapter 3, we observe that by taking $Y = \mathcal{M}_\theta$, the moduli space of θ-stable G-constellations, these moduli spaces have tautological bundles that generate the K-theory of $\mathcal{M}_\theta$.

In chapter 3 we study the GIT construction of the moduli spaces $\mathcal{M}_\theta$, and following Craw, we describe the description of the spaces $\mathcal{M}_\theta$ as moduli spaces of quiver representations, which provides a more concrete way of studying these spaces. This leads us to establish a relation between the universal G-constellation and the tautological bundle by working on ringed spaces rather than schemes. We conclude this chapter by explaining some details of the wall-crossing phenomenon and how it can generate changes in the tautological line bundles of the moduli spaces $\mathcal{M}_\theta$ and $\mathcal{M}'_\theta$.

In chapter 4, we review some basic facts about spherical objects that will be used to describe the Fourier-Mukai transform for Fourier-Mukai partners. We outline the proofs by Craw-Ishii, according to the type of wall. Also in this chapter, we develop a theorem that relates the Fourier-Mukai transform in its K-theoretic and cohomological versions. The Grothendieck-Riemann-Roch theorem and Chern classes serve as tools to relate the K-theoretic and cohomological versions. We will use this result in the development of the example $\mathbb{C}^3/\mathbb{Z}_4$.

Finally, in chapter 5, as an application, we study the example $\mathbb{C}^3/\mathbb{Z}_4$. We provide a description of the toric geometry of the singularity and its crepant resolution. By utilizing the theorems of Craw-Ishii that relate the tautological line bundles under wall-crossing, we obtain the explicit form of the cohomological Fourier-Mukai transform. For the general case, we also derive a simple form of the aforementioned transform.

Chapter 1

Derived Categories

Our starting point are derived categories, which were introduced by Verdier under the supervision of Grothendieck. Derived categories provide the natural setting where one can define derived functors. However, these categories have a rather complicated structure since they are not abelian. Nevertheless, they possess a triangulated category structure. In this chapter, we review the basic concepts of triangulated categories, derived categories, derived functors, derived categories of schemes, and the functors that commonly arise in algebraic geometry as they appear at the level of derived categories.

1.1 Triangulated Categories

In this section we recall the definition of a triangulated category. The main example is the derived category of a scheme. Although the derived categories are not abelian categories, they are always triangulated categories.

Definition 1.1 (Additive categories)**.** A category $\mathcal{A}$ is an additive category if for all objects A and B in $\mathcal{A}$, the set $\mathrm{Hom}(A, B)$ of maps is an abelian group satisfying the following conditions:

i. The composition of functions

$$\mathrm{Hom}(A_1, A_2) \times \mathrm{Hom}(A_2, A_3) \longrightarrow \mathrm{Hom}(A_1, A_3)$$

given by $(f, g) \longmapsto g \circ f$ is bilinear.

ii. There exists an object 0 in $\mathcal{A}$ called the zero object such that $\mathrm{Hom}(0, 0)$ is the trivial group.

iii. For two objects A_1, A_2 in $\mathcal{A}$ there is an object B in $\mathcal{A}$ and morphisms $j_i : A_i \longrightarrow B$ and $p_i : B \longrightarrow A_i$, $i = 1, 2$, which make B the direct sum and the direct product of A_1 and A_2.

Observation 1.2.

1. Functors between additive categories are always considered additive functors, where $F : \mathcal{A} \longrightarrow \mathcal{B}$ is an additive functor if the induced maps
$$\operatorname{Hom}(A, B) \longrightarrow \operatorname{Hom}(F(A), F(B))$$
are homomorphisms of abelian groups.

2. In this context, if the functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is an additive functor that is an equivalence, then it must be an additive equivalence, that is, its inverse functor $F^{-1} : \mathcal{B} \longrightarrow \mathcal{A}$ is also additive.

3. Yoneda's lemma in its most general version can in particular be adapted by considering only additive functors and additive categories.

Definition 1.3 (Abelian categories)**.** An additive category $\mathcal{A}$ is called abelian if additionally the following conditions are met:

1. Every homomorphism $f \in \operatorname{Hom}(A, B)$ has a kernel and a cokernel;

2. the map $\operatorname{Coim}(f) \longrightarrow \operatorname{im}(f)$ is an isomorphism.

These features allow one to use the notion of exact sequences.

Definition 1.4 (Triangulated categories)**.** Let $\mathcal{D}$ be an additive category; a triangulated structure over $\mathcal{D}$ is given by a functor $T : \mathcal{D} \longrightarrow \mathcal{D}$ which is an additive equivalence (the shift functor) and a set of triangles
$$A \longrightarrow B \longrightarrow C \longrightarrow T(A).$$
Which are subject to the axioms TR1-TR4 listed below, and are called distinguished triangles.

We will use the following notation to state the axioms of triangulated categories:
For all two objects $A, B \in \mathcal{D}$ and a map between them $f : A \longrightarrow B$ we denote by
$$A[1] := T(A), \text{ and } f[1] := T(f) \in \operatorname{Hom}(A[1], B[1]);$$

furthermore, a map between two triangles is given by a diagram of the form:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & A[1] \arrow[d, "f[1]"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1]. \end{tikzcd}$$

A map between two triangles is an isomorphism if f, g and h are isomorphisms.

TR1 (i) For each object $A \in \mathcal{D}$ a triangle of the form:

$$A \xrightarrow{id} A \longrightarrow 0 \longrightarrow A[1]$$

is distinguished.

(ii) If a triangle is isomorphic to a distinguished triangle, then it is also a distinguished triangle.

(iii) Every morphism $f : A \longrightarrow B$ can be completed to a distinguished triangle:

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow A[1].$$

TR2 A triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

is distinguished if and only if

$$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1]$$

is a distinguished triangle.

TR3 Given two distinguished triangles and two maps f and g such that the following diagram commutes

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] & A[1] \arrow[d, "f[1]"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd}$$

then there exists a map $h : C \longrightarrow C'$ such that

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & A[1] \arrow[d, "f[1]"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd}$$

is a morphims of distinguished triangles.

TR4 For every commutative diagram

$$\begin{tikzcd} & B \arrow[dr, "u_1"] & \\ A \arrow[ur, "u_3"] \arrow[rr, "u_2"'] & & C \end{tikzcd}$$

and for three given distinguished triangles

$$A \xrightarrow{u_3} B \xrightarrow{v_3} C' \xrightarrow{w_3} A[1]$$

$$B \xrightarrow{u_1} C \xrightarrow{v_1} A' \xrightarrow{w_1} B[1]$$

$$A \xrightarrow{u_2} C \xrightarrow{v_2} B' \xrightarrow{w_2} A[1]$$

there are two morphisms m_1 and m_3 such that the following triangle is distinguished

$$C' \xrightarrow{m_1} B' \xrightarrow{m_3} A' \xrightarrow[v_3[1]\circ w_1]{} C'[1]$$

and the following diagram is commutative

$$\begin{tikzcd}
A \arrow[r, "u_3"] \arrow[d, "1"'] & B \arrow[r, "v_3"] \arrow[d, "u_1"] & C' \arrow[r, "w_3"] \arrow[d, "m_1"] & A[1] \arrow[d, "1"] \\
A \arrow[r, "u_2"] \arrow[d, "u_3"'] & C \arrow[r, "v_2"] \arrow[d, "1"] & B' \arrow[r, "w_2"] \arrow[d, "m_3"] & A[1] \arrow[d, "u_3[1]"] \\
B \arrow[r, "u_1"] \arrow[d, "v_3"'] & C \arrow[r, "v_1"] \arrow[d, "v_2"] & A' \arrow[r, "w_1"] \arrow[d, "1"] & B[1] \arrow[d, "v_3[1]"] \\
C' \arrow[r, "m_1"] & B' \arrow[r, "m_3"] & A' \arrow[r, "v_3[1]\circ w_1"] & C'[1].
\end{tikzcd}$$

The objects of derived categories are complexes. We will first quickly review some important facts about these objects.

Definition 1.5 (Category of complexes)**.** Let $\mathcal{A}$ be an abelian category; the category of complexes $\mathrm{Kom}(\mathcal{A})$ is the category whose objects are complexes of objects in $\mathcal{A}$, and the morphisms are morphisms of complexes.

The shift of $A^\bullet \in \mathrm{Kom}(\mathcal{A})$, denoted by $A^\bullet[1]$, is the complex given by:

$$(A^\bullet[1])^i = A^{i+1}, \text{ and its differential } (d_{A^\bullet[1]})^i = -d_A^{i+1}.$$

Let $f : A^\bullet \longrightarrow B^\bullet$ be a morphism of complexes, the shift of this morphism is $f[1] : A^\bullet[1] \longrightarrow B^\bullet[1]$ defined by $f[1]^i = f^{i+1}$. So we can define the functor $T : \mathrm{Kom}(\mathcal{A}) \longrightarrow \mathrm{Kom}(\mathcal{A})$ given by $A^\bullet \longmapsto A^\bullet[1]$.

On the category of complexes we can make cohomology. Let $A^\bullet \in \mathrm{Kom}(\mathcal{A})$ the $i-$th cohomology of this complex is defined by:

$$H^i(A^\bullet) := \frac{\mathrm{Ker}(d^i)}{\mathrm{Im}(d^{i-1})} \in \mathcal{A}.$$

A complex $A^\bullet$ is called acyclic if $H^i(A^\bullet) = 0$ for all $i \in \mathbb{Z}$.

Definition 1.6. A morphism of complexes $f : A^\bullet \longrightarrow B^\bullet$ is a quasi-isomorphism if for all $i \in \mathbb{Z}$ the induced map in cohomology

$$H^i(f) : H^i(A^\bullet) \longrightarrow H^i(B^\bullet)$$

is an isomorphism.

1.2 Derived Categories

Derived categories can be described using an universal property. Hence, for an explicit construction, we need to pass through an intermediate category between the category of complexes and the derived category, which is called the homotopy category. The main triangulated categories are the homotopy category $\mathrm{K}(\mathcal{A})$ and the derived category $\mathcal{D}(\mathcal{A})$ of a given abelian category $\mathcal{A}$. In this section, we explore the triangulated structure of these categories.

Theorem 1.7 (Derived categories)**.** *Let $\mathcal{A}$ be an abelian category and $Kom(\mathcal{A})$ its associated category of complexes. There exists a category $\mathcal{D}(\mathcal{A})$, the derived category of $\mathcal{A}$, and a functor $Q : Kom(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{A})$ such that:*

i. If $f : A^\bullet \longrightarrow B^\bullet$ is an quasi-isomorphism then $Q(f)$ is an isomorphism in $\mathcal{D}(\mathcal{A})$.

ii. *Universal property: Every functor* $F : Kom(\mathcal{A}) \longrightarrow D$ *that satisfies the above condition can be uniquely factored by* Q*, i.e. then there exists a unique functor (up to isomorphism)* $G : \mathcal{D}(\mathcal{A}) \longrightarrow D$ *such that* $F \cong G \circ Q$*.*

To build the derived category from an abelian category we need to consider an intermedie category that we will define next.

Definition 1.8. Given two morphisms $f, g : A^\bullet \longrightarrow B^\bullet$ we say that they are homotopically equivalent if there exists a collection of homomorphisms $\left\{h^i : A^i \longrightarrow A^{i-1}, i \in \mathbb{Z}\right\}$ such that:

$$f^i - g^i = h^{i+1} \circ d^i_{A^\bullet} + d^{i-1}_{B^\bullet} \circ h^i.$$

In each $\mathrm{Hom}_{\mathrm{Kom}(\mathcal{A})}(A^\bullet, B^\bullet)$ we define an equivalence relation, where two morphisms $f, g : A^\bullet \longrightarrow B^\bullet$ are equivalent if and only if they are homotopically equivalent. Indeed it can be proved that this defines an equivalence relation.

Definition 1.9 (Homotopy category)**.** The homotopy category of complexes $K(\mathcal{A})$ is the category whose objects are:

$$\mathrm{Ob}(\mathrm{K}(\mathcal{A})) := \mathrm{Ob}(\mathrm{Kom}(\mathcal{A}))$$

and the set of morphisms is defined by

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(A^\bullet, B^\bullet) := \mathrm{Hom}_{\mathrm{Kom}(\mathcal{A})}(A^\bullet, B^\bullet)/ \sim_{ht}$$

where $\sim_{ht}$ denotes homotopic equivalence.

Remark 1.10. The homotopy category $\mathrm{K}(\mathcal{A})$ is not abelian but it is triangulated, and can be associated to any additive category.

Although the Theorem 1.7 is purely existential, it allows us to identify the objects of the derived category:

$$\mathrm{Ob}(\mathcal{D}(\mathcal{A})) := \mathrm{Ob}(\mathrm{Kom}(\mathcal{A})).$$

To describe the morphisms in the derived category we use commutative diagrams in homotopy category.

Diagrams of morphisms of complexes. Let $A^\bullet$, $B^\bullet$ and $C^\bullet \in \mathcal{D}(\mathcal{A})$; a roof between $A^\bullet$ and $B^\bullet$ is a diagram of the form

$$\begin{tikzcd} & C^\bullet \arrow[dl, "\phi"'] \arrow[dr, "f"] & \\ A^\bullet & & B^\bullet \end{tikzcd}$$

where ϕ is a quasi-isomorphism and f a morphism of complexes. Two such diagrams are equivalent if there is a third diagram of the same form, that is:

$$\begin{tikzcd} & & C^\bullet \arrow[dl, "\phi"'] \arrow[dr, "f"] & & \\ & C_1^\bullet \arrow[dl, "\phi_1"'] \arrow[drrr, "f_1"' near end] & & C_2^\bullet \arrow[dlll, "\phi_2" near end] \arrow[dr, "f_2"] & \\ A^\bullet & & & & B^\bullet \end{tikzcd}$$

where ϕ, ϕ_1 and ϕ_2 are quasi-isomorphisms and f, f_1 and f_2 are morphisms of complexes; also this diagram must be commutative in $\mathrm{K}(\mathcal{A})$.
Note that $\phi_1 \circ \phi$ is a quasi-isomorphism and since the commutative diagram in $\mathrm{K}(\mathcal{A})$,

$$\phi_2 \circ f = \phi_1 \circ \phi, \text{ the latter is also a quasi-isomorphism.}$$

This equivalence defines in effect an equivalence relation between diagrams of that type.
A morphism between two objects $A^\bullet$ and $B^\bullet$ in $\mathcal{D}(\mathcal{A})$ is an equivalence class $[f, \phi] : A^\bullet \longrightarrow B^\bullet$ where

$$A^\bullet \xleftarrow{\phi} C^\bullet \xrightarrow{f} B^\bullet$$

is a roof. Then:

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(A^\bullet, B^\bullet) = \{\text{equivalence classes of roofs between } A^\bullet \text{ and } B^\bullet\}.$$

The composition of two diagrams, say

$$\begin{tikzcd} & C_1^\bullet \arrow[dl, "\phi_1"'] \arrow[dr, "f"] & \\ A^\bullet & & B^\bullet \end{tikzcd} \qquad \begin{tikzcd} & C_2^\bullet \arrow[dl, "\phi_2"'] \arrow[dr, "g"] & \\ B^\bullet & & C^\bullet \end{tikzcd}$$

is given by a commutative diagram in $\mathrm{K}(\mathcal{A})$ of the following form:

$$\begin{tikzcd}
 & & C_3^\bullet \arrow[dl, "\phi"'] \arrow[dr, "h"] & & \\
 & C_1^\bullet \arrow[dl, "\phi_1"'] \arrow[dr, "f"] & & C_2^\bullet \arrow[dl, "\phi_2"'] \arrow[dr, "g"] & \\
A^\bullet & & B^\bullet & & C^\bullet
\end{tikzcd}$$

where ϕ, ϕ_1 and ϕ_2 are quasi-isomorphisms and f, g and h are morphisms of complexes. To guarantee the existence of the composition of two morphisms in the derived category, the concept of the cone of a morphism is required.

Definition 1.11 (Mapping cone)**.** The mapping cone of a morphism $f : A^\bullet \longrightarrow B^\bullet$ of complexes is a complex given by:

$$C(f)^i = A^{i+1} \oplus B^i, \text{ and } d^i_{C(f)} := \begin{pmatrix} -d_A^{i+1} & 0 \\ 0 & d_B^i \end{pmatrix}.$$

The mapping cone also allows one to define the concept of distinguished triangle in the homotopy and derived category in a natural way.

Definition 1.12 (Distinghished triangles)**.** A triangle

$$A_1^\bullet \longrightarrow A_2^\bullet \longrightarrow A_3^\bullet \longrightarrow A_1^\bullet[1]$$

in $\mathrm{K}(\mathcal{A})$ and $\mathcal{D}(\mathcal{A})$ respectively is called distinguished if its is isomorphic in $\mathrm{K}(\mathcal{A})$ and $\mathcal{D}(\mathcal{A})$ respectively to a triangle of the form:

$$A^\bullet \xrightarrow[f]{} B^\bullet \xrightarrow[\tau]{} C(f) \xrightarrow[\pi]{} A^\bullet[1].$$

Where f is a morphism of complexes and

$$\tau : B^i \longrightarrow A^{i+1} \oplus B^i, \qquad \pi : A^{i+1} \oplus B^i \longrightarrow A^{i+1},$$

are the natural injection and natural projection.

Remark 1.13. The homotopy category $\mathrm{K}(\mathcal{A})$ and derived category $\mathcal{D}(\mathcal{A})$ are triangulated but are not abelian.

1.3 Derived Functors

As not all functors between abelian categories are exact, the concept of derived functor was introduced in homological algebra. Derived functors are exact in the sense that they transform distinguished triangles into distinguished triangles.

Definition 1.14 (Injective resolutions)**.** An abelian category $\mathcal{A}$ has enough injectives if for every object $A \in \mathcal{A}$ there exists an injective object $I \in \mathcal{A}$ and a injective morphism $A \longrightarrow I$. An injective resolution of an object $A \in \mathcal{A}$ is an exact sequence:

$$0 \longrightarrow A \longrightarrow I^\bullet, \text{ where } I^i \text{ is an injective object in } \mathcal{A}.$$

Remark 1.15. If $\mathcal{A}$ has enough injectives for any $A^\bullet \in \mathrm{K}^+(\mathcal{A})$ there exists a complex $I^\bullet \in \mathrm{K}^+(\mathcal{A})$ with I^i injective objects and a quasi-isomorphism $A^\bullet \longrightarrow I^\bullet$.

Remark 1.16. Similarly one can consider of a category with has enough projectives and projective resolutions.
Let $\mathcal{I}$ be the full additive subcategory of all injective objects of $\mathcal{A}$, since it is additive we can take its homotopy category $\mathrm{K}^+(\mathcal{I})$.

Theorem 1.17. *If the abelian category $\mathcal{A}$ has enough injectives, then the natural functor $i : K^+(\mathcal{I}) \longrightarrow \mathcal{D}^+(\mathcal{A})$ is an equivalence.*

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and assume that $\mathcal{A}$ contains enough injectives. For a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ that is left exact we have the following commutative diagram:

$$\begin{tikzcd}
K^+(\mathcal{I}) \arrow[r, hook] \arrow[dr, shift left, "i"] & K^+(\mathcal{A}) \arrow[r, "K^+(F)"] \arrow[d, "\mathcal{Q}_\mathcal{A}"] & K^+(\mathcal{B}) \arrow[d, "\mathcal{Q}_\mathcal{B}"] \\
& D^+(\mathcal{A}) \arrow[ul, shift left, "i^{-1}"] & D^+(\mathcal{B}).
\end{tikzcd}$$

We define the right derived functor of F as the functor:

$$RF := \mathcal{Q}_\mathcal{B} \circ K^+(F) \circ i^{-1} : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B}).$$

This has two problems, i.e., that the definition depends on the class of injectives and on the equivalence i. The formal definition of a derived functor cannot depend on any class of objects. Moreover it might be difficult to calculate injective objects in a given category, so we could get another sufficiently large class of objects and get their localization in homotopy category, and replace the class of injective objects to define the derived functor. Therefore we need a formal definition of a derived functor that has a universal property.

Definition 1.18 (Right derived functors)**.** Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive exact left functor. A right derived functor of F is an exact functor $RF : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$ with a morphism of functors

$$\varepsilon_F : \mathcal{Q}_{\mathcal{B}} \circ K^+(F) \longrightarrow RF \circ \mathcal{Q}_{\mathcal{A}},$$

such that the following diagram is commutative

$$\begin{tikzcd} K^+(\mathcal{A}) \arrow[r, "K^+(F)"] \arrow[d, "\mathcal{Q}_{\mathcal{A}}"] & K^+(\mathcal{B}) \arrow[d, "\mathcal{Q}_{\mathcal{B}}"] \\ D^+(\mathcal{A}) \arrow[r, "R(F)"] & D^+(\mathcal{B}). \end{tikzcd}$$

Universal property. For any exact functor $G : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$ and a morphism of functors $\varepsilon_G : \mathcal{Q}_{\mathcal{B}} \circ K^+(F) \longrightarrow G \circ \mathcal{Q}_{\mathcal{A}}$, then there exists a unique morphism of functors $\eta : RF \longrightarrow G$ such that the following diagram is commutative:

$$\begin{tikzcd} & \mathcal{Q}_{\mathcal{B}} \circ K^+(\mathcal{F}) \arrow[dl, "\varepsilon_F"'] \arrow[dr, "\varepsilon"] & \\ RF \circ \mathcal{Q}_{\mathcal{A}} \arrow[rr, "\eta \circ \mathcal{Q}_{\mathcal{A}}"'] & & G \circ \mathcal{Q}_{\mathcal{A}}. \end{tikzcd}$$

Observation 1.19 (Left derived functors)**.** For right exact functors, in the above definition we substitute the lower bounded categories with the respective upper bounded categories to obtain the definition of the left derived functor, $LF : D^-(\mathcal{A}) \longrightarrow D^-(\mathcal{B})$, and morphism of functors $\varepsilon_F : LF \circ \mathcal{Q}_{\mathcal{A}} \longrightarrow \mathcal{Q}_{\mathcal{B}} \circ K^-(F)$. In this case the universal property is given by a morphism of functors $\eta : G \longrightarrow LF$.

Some functors that we will work with are defined at the level of homotopy categories and are not induced by a functor at the level of abelian category. This leads us to define a class of complexes on the homotopy categories.

Definition 1.20. Let $\mathcal{A}'$ be a thick abelian subcategory of $\mathcal{A}$ and let $F : K^*_{\mathcal{A}'}(\mathcal{A}) \longrightarrow K^*(\mathcal{B})$ be an exact functor. This functor has the property to have *has enough acyclics*, if there exists a triangulated subcategory $K^F(\mathcal{A})$ of $K^*_{\mathcal{A}'}(\mathcal{A})$ and if the following conditions are satisfied

i. There exists a functor $\mathcal{I} : K^*_{\mathcal{A}'}(\mathcal{A}) \longrightarrow K^F(\mathcal{A})$.

ii. For all $\mathcal{E}^\bullet \in K^F(\mathcal{A})$, the morphism $\mathcal{E}^\bullet \longrightarrow F(\mathcal{E}^\bullet)$ is a quasi-isomorphism, which is functorial in $\mathcal{E}$.

iii. For all complex $\mathcal{E}^\bullet \in K^F(\mathcal{A}) \cap Acyc^\bullet(\mathcal{A})$, the complex $F(\mathcal{E}^\bullet) \in Acyc^\bullet(\mathcal{B})$.

Notation. In the previous definition we denote by $Acyc^\bullet(\mathcal{A})$ the full subcategory of all complexes acyclics in the category $\mathcal{A}$.

Observation 1.21. Let $\mathcal{A}$ be an abelian category that has enough injectives and consider any left exact functor $F : \mathcal{A} \longrightarrow \mathcal{B}$. We know this functor induces naturally a functor between respectively homotopy categories $G = K^+(F) : K^+(\mathcal{A}) \longrightarrow K^+(\mathcal{B})$. This functor has enough acyclics and we take $K^G(\mathcal{A}) = K^+(\mathcal{I}_\mathcal{A})$, where $\mathcal{I}_\mathcal{A}$ is the full additive subcategory of all injective objects of $\mathcal{A}$.

Under the hypotheses of the above definition we let:

$$K^*(F) : K^*_{\mathcal{A}'}(\mathcal{A}) \longrightarrow K^*(\mathcal{B}), \text{ by } K^*(F) = F \circ \mathcal{I}.$$

This functor takes quasi-isomorphims into quasi-isomorphism (see [2, Lemma A.60]); due to this we can apply the following theorem to descend the functor to the derived category.

Theorem 1.22. *Let $F : K^*(\mathcal{A}) \longrightarrow K^*(\mathcal{B})$ be an exact functor. If this take quasi-isomorphisms into quasi-isomorphisms, then it induces an exact functor $D^*(F) : D^*(\mathcal{A}) \longrightarrow D^*(\mathcal{B})$ in the respective derived categories.*

For the case $* = +$ we obtain the right derived functor

$$RF : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B}).$$

For all $\mathcal{E}^\bullet \in D^+(\mathcal{A})$, the higher derived functors of F are

$$R^i F(\mathcal{E}^\bullet) = \mathcal{H}^i(RF(\mathcal{E}^\bullet)).$$

Alternatively one can obtain derived functors from adapted classes as in [21]. We introduce that definition to later make a comparison with the notion of a functor has enough acyclics.

Definition 1.23 (Adapted classes to a left exact functor)**.** Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a left exact functor between abelian categories. A class of objects $\mathcal{R} \subset \mathcal{A}$ is said to be *adapted to a* F if it is stable under direct sums and if the following conditions are satisfied

1. F takes a complex $\mathcal{E}^\bullet \in K^+(\mathcal{R}) \cap Acyc^\bullet(\mathcal{A})$ into a complex $F(\mathcal{E}^\bullet) \in Acyc^\bullet(\mathcal{B})$.

2. For any $A \in \mathcal{A}$ there exists $A' \in \mathcal{R}$ such that $A \hookrightarrow A'$.

Remark 1.24. In the previous definition we can also define classes of objects adapted to a right exact functor, we only have to substitute the second condition by: any $A \in \mathcal{A}$ is a quotient of some $A' \in \mathcal{R}$.

Let $\mathcal{R}$ be a class adapted to a left exact functor F. We can localize in the homotopy category as the following theorem guarantees.

Theorem 1.25 (Localization of the homotopy category)**.** *Under the hypotheses of the previous definition, if $\mathcal{S}_\mathcal{R}$ is the class of quasi-isomoprhisms in $K^+(\mathcal{R})$, then $\mathcal{S}_\mathcal{R}$ is a localizing class of morphisms in $K^+(\mathcal{R})$ and*

$$K^+(\mathcal{R})[\mathcal{S}_\mathcal{R}^{-1}] \longrightarrow D^+(\mathcal{A})$$

is an equivalence of categories.

We know that there exists a functor $\Psi : D^+(\mathcal{A}) \longrightarrow K^+(\mathcal{R})[\mathcal{S}_\mathcal{R}^{-1}]$ and a functor induced on the level of homotopy categories

$$K^+(F) : K^+(\mathcal{R})[\mathcal{S}_\mathcal{R}^{-1}] \longrightarrow D^+(\mathcal{B}).$$

We define the right derived functor by composition of these functors:

$$RF = K^+(F) \circ \Psi : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B}).$$

Observation 1.26. If the functor F has enough acyclics the subcategory $K^F(\mathcal{A})$ of $K^+(\mathcal{A})$ is already a localized subcategory. Indeed if $\mathcal{E}^\bullet \in K^F(\mathcal{A})$ there exists a quasi-isomorphism $\mathcal{E}^\bullet \longrightarrow \mathcal{I}(\mathcal{E}^\bullet)$ in $K^F(\mathcal{A})$, but this is a triangulated subcategory of $K^+_{\mathcal{A}'}(\mathcal{A})$ and by Example 3 in [29, p. 40] the last subcategory is localized in $K^+(\mathcal{A})$. This implies that $\mathcal{E}^\bullet \longrightarrow \mathcal{I}(\mathcal{E}^\bullet)$ is an isomorphism in $K^F(\mathcal{A})$.

So we have two alternative ways to construct derived functors, the first by the property of a functor to have enough acyclics and the second using adapted classes. But the universal property of derived functors leads us to the same functor in derived category.

We conclude this section enunciating Grothendieck's theorem of composition on the derived functors.

Definition 1.27 (F-acyclics complexes)**.** Under the hypotheses of Definition 1.23, a complex $\mathcal{E}^\bullet \in K^+_{\mathcal{A}'}(\mathcal{A})$ is called *F-acyclic* if the quasi-isomorphism $\mathcal{E}^\bullet \longrightarrow \mathcal{I}(\mathcal{E}^\bullet)$ induces an isomorphism $F(\mathcal{E}^\bullet) \longrightarrow RF(\mathcal{E}^\bullet)$.

Theorem 1.28 (Grothendieck)**.** *Let $\mathcal{A}'$ and $\mathcal{B}'$ thick subcategories of abelian categories $\mathcal{A}$ and $\mathcal{B}$ respectively, and*

$$F : K^+_{\mathcal{A}'}(\mathcal{A}) \longrightarrow K^+_{\mathcal{B}'}(\mathcal{B}),\ G : K^+_{\mathcal{B}'}(\mathcal{B}) \longrightarrow K^+(\mathcal{C})$$

functors at the level of homotopy categories. If F takes F-acyclics complexes into $G-$acyclics complexes then:

1. *$G \circ F$ has enough acyclics.*
2. *There exists the right derived functor of the composition*
$$R(G \circ F) : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{C}).$$
3. *There exists a natural isomorphism of functors*
$$R(G \circ F) \cong RG \circ RF.$$

1.4 Derived Categories of Coherent $\mathcal{O}_X$-modules over a Scheme X

In this section we fix a notation and we recall two theorems of equivalence between derived categories in the geometric context.

Let X be a scheme. We denote by

$$D^b(X) = D^b(\mathrm{Coh}(X))$$

the bounded derived category of abelian category of coherent sheaves. One finds the following results in [30] and [2].

Theorem 1.29. *Let X be a noetherian scheme. The natural embedding*

$$D^*(\mathrm{Qcoh}(X)) \longrightarrow D^*(\mathrm{Mod}(X)),$$

for $* = b, +$ induces an equivalence:

$$D^*(\mathrm{Qcoh}(X)) \xrightarrow{\sim} D^*_{qc}(\mathrm{Mod}(X))$$

where $D^*_{qc}(\mathrm{Mod}(X))$ denotes the derived category of complexes in $\mathrm{Mod}(X)$ with quasi-coherent cohomology.

Theorem 1.30. *Let X be a noetherian scheme. The natural embedding*

$$D^b(X) \longrightarrow D^b(\mathrm{Qcoh}(X))$$

induces an equivalence

$$D^b(X) \xrightarrow{\sim} D^b_{coh}(\mathrm{Qcoh}(X))$$

where $D^b_{coh}(\mathrm{Qcoh}(X))$ denotes the derived category of complexes in $\mathrm{Qcoh}(X)$ with coherent cohomology.

Summarizing these two results we have (see [2, Corollary A.40])

$$D^b(X) \simeq D^b_{coh}(\mathrm{Qcoh}(X)) \simeq D^b_{coh}(\mathrm{Mod}(X)).$$

1.5 Derived Functors on Schemes

We conclude this chapter by outlining the construction of the main derived functors that appear consistently in algebraic geometry.

1.5.1 Six Functors Formalism

P. Scholze in his notes [47] made an abstract study of the six functor formalism. As explained in the said reference, a particular situation is the following:
Let $\mathcal{C}$ be a category of geometric objects, say the category of schemes and an association $X \longmapsto D(X)$ from $\mathcal{C}$ to a say (triangulated categories) and six functors

$$(f^*, f_*, \otimes, \mathcal{H}om, f_!, f^!).$$

These functors have many types of compatibility, for example that the second is always to right adjoint to the previous.

In this section we will describe the construction of the functors that are used in the theory of Fourier-Mukai transforms and that are involved in some way. The main references we follow are [29], [28], [53], [21], [33], [2] and [30].

First following [30] we will describe cohomology in terms of derived categories.

Global sections.

Let X be a noetherian scheme over a field k. Since the functor of global sections

$$\Gamma : \mathrm{Mod}(X) \longrightarrow \mathrm{Vec}(k)$$

is a left exact functor and $\mathrm{Mod}(X)$ has enough injectives, there exists

$$R\Gamma : D^+(\mathrm{Mod}(X)) \longrightarrow D^+(\mathrm{Vec}(k)).$$

We can restrict the functor of global sections to the category of quasi-coherent modules then again:

$$\Gamma : \mathrm{Qcoh}(X) \longrightarrow \mathrm{Vec}(k)$$

is a left exact functor and Qcoh(X) has enough injectives, so that there exists

$$R\Gamma : D^+(\mathrm{Qcoh}(X)) \longrightarrow D^+(\mathrm{Vec}(k)).$$

By Grothendieck's theorem [28, Theorem III.2.7] for any $\mathcal{F} \in \mathrm{Qcoh}(X)$ the cohomology groups $H^i(X, \mathcal{F}) = 0$ for all $i > \dim(X)$. This implies that the right derived functor of global sections can be restricted to bounded derived categories

$$R\Gamma : D^b(\mathrm{Qcoh}(X)) \longrightarrow D^b(\mathrm{Vec}(k)).$$

In [23] Grothendieck generalizes a theorem by Serre [28, Theorem III.5.2], that is, if X is proper, then for any $\mathcal{F} \in \mathrm{Coh}(X)$, the cohomology groups $H^i(X, \mathcal{F})$ are finite-dimensional k-vector spaces. This induces

$$\Gamma : \mathrm{Coh}(X) \longrightarrow \mathrm{Vec}_{fin}(k).$$

We need to descend to bounded derived categories. Since there are embeddings:

$$D^b(X) \xrightarrow{\sim} D^b_{coh}(\mathrm{Qcoh}(X)) \hookrightarrow D^b(\mathrm{Qcoh}(X))$$

and

$$D^+(X) \xrightarrow{\sim} D^+_{coh}(\mathrm{Qcoh}(X)) \hookrightarrow D^+(\mathrm{Qcoh}(X)),$$

for the latter embedding see [29, Proposition 4.8], we can consider the restrictions of the global section functor to the derived category of X:

$$R\Gamma : D^+(\mathrm{Coh}(X)) = D^+(X) \longrightarrow D^+(\mathrm{Vec}(k)).$$

As we already discussed, the hypothesis that $\mathcal{A}$ has enough injective objects in [30, Corollary 2.68] can be replaced by that the functor F has enough acyclics, then using this corollary:

$$R\Gamma : D^+(X) \longrightarrow D^+(\mathrm{Vec}_{fin}(k)),$$

but due to the existence of an embedding between bounded derived categories mentioned above one has

$$R\Gamma : D^b(X) \longrightarrow D^b(\mathrm{Vec}(k)).$$

Finally we obtain

$$R\Gamma : D^b(X) \longrightarrow D^b(\mathrm{Vec}_{fin}(k)).$$

The higher derived functors are

$$H^i(X, \mathcal{F}^\bullet) = R^i\Gamma(\mathcal{F}^\bullet),$$

and are called hypercohomology groups. When $\mathcal{F}^\bullet$ is a complex concentrated in degree zero these are the classical cohomology groups.

1.5.2 Derived Pushforward

Let X and Y be Noetherian schemes and let $f : X \longrightarrow Y$ be a morphism of schemes. We know this morphism induces a left exact functor

$$f_* : \mathrm{Mod}(X) \longrightarrow \mathrm{Mod}(Y)$$

and by existence of enough injectives in $\mathrm{Mod}(X)$ then there exists a right derived functor:

$$Rf_* : D^+(\mathrm{Mod}(X)) \longrightarrow D^+(\mathrm{Mod}(Y)).$$

The same can be done for

$$f_* : \mathrm{Qcoh}(X) \longrightarrow \mathrm{Qcoh}(Y)$$

and the right derived functor exists

$$Rf_* : D^+(\mathrm{Qcoh}(X)) \longrightarrow D^+(\mathrm{Qcoh}(Y)).$$

By [28, III.8.6] for all $\mathcal{F} \in \mathrm{Qcoh}(X)$ the higher direct images $R^i f_* \mathcal{F} = 0$ for any $i > \dim(X)$, then we can restrict the

$$Rf_* : D^b(\mathrm{Qcoh}(X)) \longrightarrow D^b(\mathrm{Qcoh}(Y)).$$

Again Grothendieck [23, III.2.2.1] generalizes a theorem by Serre of projective morphisms to a proper morphisms. Let $f : X \longrightarrow Y$ be a proper morphism of noetherian schemes; if $\mathcal{F} \in \mathrm{Coh}(X)$ then the higher direct images $R^i f_* \mathcal{F}$ are coherent as well. We obtain a right derived functor between bounded derived categories

$$Rf_* : D^b(X) \longrightarrow D^b(Y),$$

again using the fact

$$D^b(X) \xrightarrow{\sim} D^b_{coh}(\mathrm{Qcoh}(X)).$$

If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are two morphisms of noetherian schemes, we let $\mathcal{R}_1 = \mathcal{I}$ be the class of all injective objects of $\mathrm{Qcoh}(X)$ and $\mathcal{R}_2 = \{\text{fabbly sheaves on } Y\}$. For all flabby sheaf $\mathcal{F}$ on X the sheaf $f_*\mathcal{F}$ is flabby on Y and as every injective $\mathcal{O}_X$-module is flabby sheaf on X we apply the remark of the Theorem 1.28, and we then have

$$R(g \circ f)_* \simeq Rg_* \circ Rf_*.$$

1.5.3 Derived Local Hom

Derived global hom.

Let $\mathcal{A}$ be an abelian category which has enough injectives and $\mathcal{E}^\bullet \in K^-(\mathcal{A})^{\mathrm{opp}}$. It is known that we get the functor (see [2, p. 293])

$$\mathrm{Hom}(\mathcal{E}^\bullet, -) : K^+(\mathcal{A}) \longrightarrow K^+(\mathrm{Ab})$$

which is left exact, and by Lemma A.66 in [2] if $\mathcal{F}^\bullet \in K^+(\mathcal{I}) \cap Acyc^\bullet(\mathrm{Qcoh}(X))$ the complex $\mathcal{H}\mathrm{om}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \in Acyc^\bullet(\mathrm{Qcoh}(X))$ where $\mathcal{I} \subset \mathrm{Qcoh}(X)$ is the class of all injective objects. Then take $K^F(\mathcal{A}) = K^+(\mathcal{I})$. There exists the right derived functor:

$$R_{II}\mathrm{Hom}(\mathcal{E}^\bullet, -) : D^+(\mathcal{A}) \longrightarrow D^+(\mathrm{Ab}).$$

Let us fix a complex $\mathcal{F}^\bullet \in D^+(\mathcal{A})$ and consider the functor

$$R_{II}\mathrm{Hom}(-, \mathcal{F}^\bullet) : K^-(\mathcal{A})^{\mathrm{opp}} \longrightarrow D^+(\mathrm{Ab}),$$

which takes quasi-isomorphims into quasi-isomorphims. Then by Theorem 1.22 there exists a right derived functor

$$R_I R_{II}\mathrm{Hom}(-, \mathcal{F}^\bullet) : D^-(\mathcal{A})^{\mathrm{opp}} \longrightarrow D^+(\mathrm{Ab}).$$

We obtains a bifunctor:

$$R\mathrm{Hom}(-, -) : D^-(\mathcal{A})^{\mathrm{opp}} \times D^+(\mathcal{A}) \longrightarrow D^+(\mathrm{Ab}).$$

The higher derived functors are

$$\mathrm{Ext}^i(\mathcal{E}^\bullet, \mathcal{F}^\bullet) = R^i\mathrm{Hom}(\mathcal{E}^\bullet, \mathcal{F}^\bullet).$$

For a noetherian scheme X, we take $\mathcal{A} = \mathrm{Qcoh}(X)$ and we proceed in the same way to obtain the functor

$$R\mathrm{Hom}(-, -) : D^-(\mathrm{Qcoh}(X))^{\mathrm{opp}} \times D^+(\mathrm{Qcoh}(X)) \longrightarrow D^+(\mathrm{Ab}).$$

Derived local hom.

Let X be a noetherian scheme and let us fix a complex $\mathcal{E}^\bullet \in K^-(\mathrm{Qcoh}(X))^{\mathrm{opp}}$, we know that

$$\mathcal{H}om(\mathcal{E}^\bullet, -) : K^+(\mathrm{Qcoh}(X)) \longrightarrow K^+(\mathrm{Qcoh}(X))$$

is a left exact functor. If $\mathcal{F}^\bullet \in K^+(\mathcal{I}) \cap Acyc^\bullet(\mathrm{Qcoh}(X))$ the complex $\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \in Acyc^\bullet(\mathrm{Qcoh}(X))$ (see [30, Lemma 3.25]). Due to this we can take $K^F(\mathrm{Qcoh(X)}) = K^+(\mathcal{I})$, for $F = \mathcal{H}om(\mathcal{E}^\bullet, -)$, then F-has enough acyclics. Thus the functor

$$R_{II}\mathcal{H}om(\mathcal{E}^\bullet, -) : D^+(\mathrm{Qcoh}(X)) \longrightarrow D^+(\mathrm{Qcoh}(X))$$

is the right derived functor on the second variable. Now, let us fix a complex $\mathcal{F}^\bullet \in D^+(\mathrm{Qcoh}(X))$ and define a functor:

$$R_{II}\mathcal{H}om(-, \mathcal{F}^\bullet) : K^-(\mathrm{Qcoh}(X))^{\mathrm{opp}} \longrightarrow D^+(\mathrm{Qcoh}(X)).$$

If $\mathcal{F}^\bullet$ is a complex of injective objects and $\mathcal{E}^\bullet \in Acyc^\bullet(\mathrm{Qcoh}(X))$ again by [30, Lemma 3.25] the complex $\mathcal{H}\mathrm{om}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \in Acyc^\bullet(\mathrm{Qcoh}(X))$. For an arbitrary complex $\mathcal{F}^\bullet$ there exists a quasi-isomoprhism $\mathcal{F}^\bullet \longrightarrow \mathcal{I}^\bullet$ with $\mathcal{I}^\bullet$ is a complex of injective objects and by construction of derived functor with has enough acyclics $R_{II}\mathcal{H}\mathrm{om}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) = \mathcal{H}\mathrm{om}(\mathcal{E}^\bullet, \mathcal{I}^\bullet)$ by Theorem 1.22 there exists

$$R_I R_{II}\mathcal{H}\mathrm{om}(-, \mathcal{F}^\bullet) : D^-(\mathrm{Qcoh}(X))^{\mathrm{opp}} \longrightarrow D^+(\mathrm{Qcoh}(X)),$$

the right derived functor on the first variable. We obtain a bifunctor

$$R\mathcal{H}\mathrm{om}(-,-) : D^-(\mathrm{Qcoh}(X))^{\mathrm{opp}} \times D^+(\mathrm{Qcoh}(X)) \longrightarrow D^+(\mathrm{Qcoh}(X)).$$

But we need to descend this functor to the bounded derived category. Let X be a regular scheme and $\mathcal{F}^\bullet \in D^b(X)$ then $\mathcal{F}^\bullet \cong \mathcal{L}^\bullet$ in $D^b(X)$ where $\mathcal{L}^\bullet$ is a complex of locally free sheaves (see [30, Proposition 3.26]). Then

$$R\mathcal{H}\mathrm{om}(-,-) : D^b(X)^{\mathrm{opp}} \times D^b(X) \longrightarrow D^b(X).$$

Derived Dual. Fix a complex $\mathcal{E}^\bullet \in D^-(\mathrm{Qcoh}(X))^{\mathrm{opp}}$; the derived dual of this complex is defined by $\mathcal{E}^{\bullet\vee} = R\mathcal{H}\mathrm{om}(\mathcal{E}^\bullet, \mathcal{O}_X)$. If X is a regular scheme, for any $\mathcal{E}^\bullet \in D^b(\mathrm{Qcoh}(X))^{\mathrm{opp}}$ then $\mathcal{E}^{\bullet\vee} \in D^b(\mathrm{Qcoh}(X))$.

1.5.4 Derived Tensor

Let X be noetherian scheme and let us fix a complex $\mathcal{E}^\bullet \in K^-(\mathrm{Coh}(X))$; we can define the functor (see [30, p. 79]):

$$\mathcal{E}^\bullet \otimes (-) : K^-(\mathrm{Coh}(X)) \longrightarrow K^-(\mathrm{Coh}(X));$$

this functor is exact. By Corollary A.73 in [2] for any $\mathcal{F}^\bullet \in K^-(\mathrm{Coh}(X))$ there exists a quasi-isomorphism $\mathcal{P}(\mathcal{F}^\bullet) \longrightarrow \mathcal{F}^\bullet$ where $\mathcal{P}(\mathcal{F}^\bullet)$ is a complex of flat sheaves. We denote by $K^-(\{\text{flat sheaves}\})$ the homotopy category of the complexes bounded above of flat sheaves living in $\mathrm{Coh}(X)$. By [29, Lemma II.4.1] for any $\mathcal{E}^\bullet \in K^-(\mathrm{Coh}(X))$ and $\mathcal{F}^\bullet \in K^-(\{\text{flat sheaves}\}) \cap Acyc^\bullet(\mathrm{Coh}(X))$ the tensor product $\mathcal{E}^\bullet \otimes \mathcal{F}^\bullet$ is an acyclic complex of coherent sheaves; then we can take for that

$$K^F(\mathrm{Coh}(X)) = K^-(\{\text{flat sheaves}\})$$

for $F = \mathcal{E}^\bullet \otimes (-)$. Then there exists

$$L(\mathcal{E}^\bullet \otimes (-)) : D^-(\mathrm{Coh}(X)) \longrightarrow D^-(\mathrm{Coh}(X))$$

the left derived functor in the second variable. Let us fix a complex $\mathcal{F}^\bullet \in K^-(\mathrm{Coh}(X))$ and define a functor

$$L((-) \otimes \mathcal{F}^\bullet) : K^-(\mathrm{Coh}(X)) \longrightarrow D^-(\mathrm{Coh}(X)).$$

Again by [29, Lemma II.4.1] if $\mathcal{F}^\bullet \in K^-(\mathfrak{F})$ and $\mathcal{E}^\bullet \in Acyc^\bullet(\mathrm{Coh}(X))$ then the tensor product $\mathcal{E}^\bullet \otimes \mathcal{F}^\bullet$ is an acyclic complex of coherent sheaves, and thus by Theorem 1.22 the functor $L((-)\otimes\mathcal{F}^\bullet)$ descends to derived category:

$$L((-) \otimes \mathcal{F}^\bullet) : D^-(\mathrm{Coh}(X)) \longrightarrow D^-(\mathrm{Coh}(X)).$$

This functor induces a bifunctor:

$$(-) \overset{L}{\otimes} (-) : D^-(\mathrm{Coh}(X)) \times D^-(\mathrm{Coh}(X)) \longrightarrow D^-(\mathrm{Coh}(X)).$$

We can construct the derived tensor from the class of locally free sheaves, i.e., we take $K^F(\mathrm{Coh}(X)) = K^-(\{\text{locally free sheaves}\})$ proceeding in a similar way. Let X be a regular scheme by [30, Proposition 3.26] if $\mathcal{G}^\bullet \in D^b(\mathrm{Coh}(X))$ exists a complex $\mathcal{L}^\bullet \in D^b(\mathrm{Coh}(X))$ of locally free sheaves isomorphic to a $\mathcal{G}^\bullet$. Then the derived tensor descends to a bounded derived category:

$$(-) \overset{L}{\otimes} (-) : D^b(X) \times D^b(X) \longrightarrow D^b(X).$$

1.5.5 Derived Pullback

Let X and Y two noetherian schemes and let $f : X \longrightarrow Y$ be a morphism of schemes, we know that considering f as a morphism of locally ringed spaces then the inverse image is:

$$f^{-1} : \mathfrak{Sh}_{\mathcal{O}_Y}(Y) \longrightarrow \mathfrak{Sh}_{f^{-1}\mathcal{O}_Y}(X)$$

is an exact functor, then this can descends to bounded above derived category

$$f^{-1} : D^-(\mathfrak{Sh}_{\mathcal{O}_Y}(Y)) \longrightarrow D^-(\mathfrak{Sh}_{f^{-1}\mathcal{O}_Y}(X)).$$

The usual tensor:

$$\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} (-) : \mathfrak{Sh}_{f^{-1}\mathcal{O}_Y}(X) \longrightarrow \mathfrak{Sh}_{\mathcal{O}_X}(X).$$

is a right exact functor, the derived tensor exists is also in locally ringed spaces, we can derive:

$$\mathcal{O}_X \overset{L}{\otimes}_{f^{-1}\mathcal{O}_Y} (-) : D^-(\mathfrak{Sh}_{f^{-1}\mathcal{O}_Y}(X)) \longrightarrow D^-(\mathfrak{Sh}_{\mathcal{O}_X}(X)).$$

The pullback functor is a composition of these functors above described:

$$f^* := (\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} (-)) \circ f^{-1}(-) : \mathfrak{Sh}_{\mathcal{O}_Y}(Y) \longrightarrow \mathfrak{Sh}_{\mathcal{O}_X}(X).$$

By Grothendieck's composite functor theorem for the case of left derived functors, we obtain:

$$Lf^* = (\mathcal{O}_X \overset{L}{\otimes}_{f^{-1}\mathcal{O}_Y} (-)) \circ f^{-1}(-) : D^-(\mathfrak{Sh}_{\mathcal{O}_Y}(Y)) \longrightarrow D^-(\mathfrak{Sh}_{\mathcal{O}_X}(X)),$$

finally

$$Lf^* = (\mathcal{O}_X \overset{L}{\otimes}_{f^{-1}\mathcal{O}_Y} (-)) \circ f^{-1}(-) : D^-(\mathrm{Mod}(Y)) \longrightarrow D^-(\mathrm{Mod}(X)).$$

When the morphism f of schemes is flat we do not need to derive it, because the functor induced between abelian categories is exact.

1.5.6 Exceptional Inverse Image

Let $f : X \longrightarrow Y$ be a morphism of smooth schemes over a field $\mathbb{K}$ the relative dimension of f is $\dim(f) = \dim(X) - \dim(Y)$ and the relative dualizing bundle is $\omega_f = \omega_X \otimes f^*\omega_Y^*$.
The exceptional functor is defined by

$$f^! : D^b(Y) \longrightarrow D^b(X), \qquad \mathcal{E}^\bullet \longmapsto Lf^*\mathcal{E}^\bullet \otimes \omega_f[\dim(f)].$$

Chapter 2

Derived McKay Correspondence

In this chapter, we develop in detail the two main theorems of the famous paper of Bridgeland, King and Reid. Among other things, these results guarantee the existence of crepant resolutions in 3 dimensions, and even more, they establish an equivalence between the derived category of the crepant resolution and the equivariant category of $\mathbb{C}^3$. The equivalence is established by an integral transform which is a Fourier-Mukai functor. Many properties of these transforms are based on the compatibility between the six derived functors mentioned in the previous chapter. One of these compatibilities is Grothendieck-Verdier duality, which can be found in classic texts on derived categories.

2.1 Fourier-Mukai Transforms

In this section, we see the definition of integral transforms, which are the main tool for finding equivalences between derived categories of varieties, and we explore some standard properties of these functors.

Definition 2.1 (Integral functors)**.** Let X and Y be proper algebraic varieties over a field $\mathbb{K}$, and let $\mathcal{K}^\bullet$ be an object in derived category $D^-(X \times Y)$. The functor

$$\Phi^{\mathcal{K}^\bullet} : D^-(X) \longrightarrow D^-(Y)$$

defined by

$$\Phi^{\mathcal{K}^\bullet}(\mathcal{E}^\bullet) = R\pi_{Y*}\left(\mathcal{K}^\bullet \overset{\mathbf{L}}{\otimes} \pi_X^*(\mathcal{E}^\bullet)\right)$$

is called an integral functor, where π_X and π_Y are the projections of the product $X \times Y$ onto the factors X and Y respectively, and the complex $\mathcal{K}^\bullet$ is called the kernel of the integral functor.

These functors have nice properties; for example, if $\mathcal{K}^\bullet \in D^-(X \times Y)$ and $\mathcal{L}^\bullet \in D^-(Y \times Z)$ are kernels, we can compose the respective integral functors:

$$\Phi^{\mathcal{L}^\bullet} \circ \Phi^{\mathcal{K}^\bullet} \cong \Phi^{\mathcal{L}^\bullet * \mathcal{K}^\bullet}$$

where $\mathcal{L}^\bullet * \mathcal{K}^\bullet$ is the convolution of these kernels, defined by:

$$\mathcal{L}^\bullet * \mathcal{K}^\bullet = R\pi_{XZ*}\left(\pi_{XY}^*\mathcal{K}^\bullet \overset{\mathbf{L}}{\otimes} \pi_{XZ}^*\mathcal{L}^\bullet\right) \in D^-(X \times Z)$$

Under suitable hypotheses, for example, if X and Y are smooth algebraic varieties, the functor $\Phi^{\mathcal{K}^\bullet}$ maps $D^b(X)$ to $D^b(Y)$. Another property of integral functors is that under certain conditions, they have adjoints. For example, if X is smooth and the complex $\mathcal{K}^\bullet$ has finite Tor-dimension over Y, the functor

$$\Phi^{\mathcal{K}^{\bullet\vee} \otimes \pi_X^* \omega_X[\dim(X)]} : D^b(Y) \longrightarrow D^b(X)$$

is the right adjoint of $\Phi^{\mathcal{K}^\bullet}$.

2.2 Examples of Fourier-Mukai Transforms

Some integral transforms that we will use are seen in this section as examples.

The identity $\mathrm{Id} : D^b(X) \longrightarrow D^b(X)$ is an integral transform indeed considering the embedding $\iota : X \hookrightarrow \triangle \subset X \times X$ the integral transform with kernel $\mathcal{O}_\triangle \in D^b(X \times X)$ satisfies:

$$\mathrm{Id} \cong \Phi^{\mathcal{O}_\triangle}.$$

Let L be a line bundle over X; the twist $\mathcal{E}^\bullet \longmapsto \mathcal{E}^\bullet \otimes L$ is an integral transform. To see this, let us take the complex $\iota_*(L) \in D^b(X \times X)$ as the kernel of the integral transform; then

$$L \otimes (-) \cong \Phi^{\iota_*(L)}.$$

The Serre functor $S_X : D^b(X) \longmapsto D^b(X)$ is defined by $\mathcal{E}^\bullet \longmapsto \mathcal{E}^\bullet \otimes \omega_X[n]$, where n is the dimension of X. Once again, the Serre functor it is an integral transform, indeed:

$$\begin{aligned}\Phi^{\iota_*\omega_X}\mathcal{E}^\bullet &= R\pi_{X*}\left(\iota_*\omega_X \otimes \pi_X^*\mathcal{E}^\bullet\right) = R\pi_{X*}\left(\iota_*\left(\omega_X \otimes \iota^*\pi_X^*\mathcal{E}^\bullet\right)\right)\\ &= R(\iota\circ\pi_X)_*\left(\omega_X \otimes \iota^*\pi_X^*\mathcal{E}^\bullet\right) = \omega_X \otimes \iota^*\pi_X^*\mathcal{E}^\bullet = \omega_X \otimes \mathcal{E}^\bullet\\ &= S_X(\mathcal{E}^\bullet)[-n].\end{aligned}$$

Trivially the structure sheaf $\mathcal{O}_{X\times Y}$ on $X\times Y$ defines an integral functor:

$$\Phi^{\mathcal{O}_{X\times Y}}(\mathcal{E}^\bullet) = R\pi_{Y*}\left(\mathcal{O}_{X\times Y} \overset{\mathbf{L}}{\otimes} \pi_X^*(\mathcal{E}^\bullet)\right) = R\pi_{Y*}\pi_X^*(\mathcal{E}^\bullet).$$

2.3 Derived McKay Correspondence

In this section we will see how the McKay correspondence was initially formulated in two dimensions and how it was generalized to three dimensions.

The context of the classical McKay correspondence is the following. Let $G \subset \mathrm{SL}(2,\mathbb{C})$ be an arbitrary finite group acting on $\mathbb{C}^2$; the quotient space $\mathbb{C}^2/G$ is called a Kleinian singularity. It has a unique resolution of singularities. The McKay correspondence establishes a bijection

$$\{\text{representations of } G\} \longleftrightarrow \{\ \text{Cohomology of } X\ \}$$

where X is the resolution of singularities of $\mathbb{C}^2/G$.
The first step to interpret the McKay correspondence in a categorical sense was given by Gonzalez-Sprinberg and Verdier who in [22] interpreting this correspondence as an isomorphism

$$R(G)\ \cong\ K^G(\mathbb{C}^2)$$

where $R(G)$ is the representation ring of G and $K^G(\mathbb{C}^2)$ is the equivariant K-theory of $\mathbb{C}^2$.
For the case of dimension 3, going to dimension 3 was very complicated. Nakamura in [42] introduced the scheme G-Hilb$\,\mathbb{C}^3$, which is the moduli space of G-clusters of $\mathbb{C}^3$. He proved that it is a crepant resolution of $\mathbb{C}^3/G$ when G is an abelian group, and conjectured the same for any finite group of

$\mathrm{SL}(3,\mathbb{C})$. Ito and Nakajima in [32] observed that the McKay correspondence holds in the abelian case and proved that

$$K(G\text{-Hilb}\,\mathbb{C}^3) \;\cong\; K^G(\mathbb{C}^3),$$

where this isomorphism is between the G-equivariant K-theory of $\mathbb{C}^3$ and the usual K-theory of G-Hilb $\mathbb{C}^3$.
Later Bridgeland, King and Reid in [8] proved Nakamura's conjecture using methods of derived categories and Fourier-Mukai transforms.

Before continuing we need some definitions that intervene in the situation of the problem studied by Bridgeland, King and Reid [8].

Definition 2.2 (Crepant Resolution)**.** Let X be a singular algebraic variety; a morphism $f : Y \longrightarrow X$ is called a *resolution of singularities* of X if:

i. Y is a nonsingular variety.

ii. The morphism f is proper.

iii. The morphism f is birational, in particular

$$f\big|_{Y'} : Y \setminus f^{-1}(X_{Sing}) \xrightarrow{\sim} X \setminus X_{Sing}$$

where $Y' := Y \setminus f^{-1}(X_{Sing})$ and X_{Sing} is the singular locus of X. If in addition $f^*(\omega_X) = \omega_Y$, f is called a *crepant resolution* of X.

Definition 2.3 (G-clusters)**.** Let M be a variety with a G-action, where G is a finite group. A zero-dimensional subscheme $Z \subset M$ is a G-cluster if $\Gamma(\mathcal{O}_Z) \cong \mathbb{C}[G]$ as G-representations.

The functor

$$\Lambda : \{\text{locally noetherian schemes}/\mathbb{C}\}^{opp} \longrightarrow \text{Sets}$$

is defined by

$$\Lambda(S) = \left\{ \begin{array}{c} G\text{-invariant closed subschemes } Z \subset M \times_{\mathbb{C}} S \\ \text{such that} \quad Z \longrightarrow S \quad \text{is flat and} \quad Z_s \subset M \quad \text{is} \\ \text{a} \quad G\text{-cluster} \quad \text{for each} \quad s \in S \end{array} \right\}.$$

This functor is representable by a scheme G-Hilb M called the G-Hilbert scheme, which is a closed subscheme of $\mathrm{Hilb}^{|G|} M$, the Hilbert scheme of zero-dimensional subschemes of M of length $|G|$.

2.4 Situation of the Problem

In this section, we describe the hypotheses under which the two theorems of Bridgeland, King and Reid will be developed in the following sections.

Let G be an arbitrary finite subgroup of $\mathrm{SL}(3, \mathbb{C})$ acting on a nonsingular quasiprojective complex variety M such that the canonical bundle ω_M is locally trivial in $\mathrm{Coh}^G(M)$. The quotient variety M/G is quasiprojective and usually singular. The problem is to find a crepant resolution of this singular quotient space. A natural candidate could be the irreducible component of $G\text{-Hilb}\, M$ containing free orbits. In principle we have that this irreducible component Y is quasiprojective by Grothendieck's theorem ([27, Theorem 14.139]). Since the moduli functor Λ is representable, and it is represented by $G\text{-Hilb}\, M$, there is a universal G-invariant closed subscheme $Z \hookrightarrow Y \times M$. Consider the following diagram:

$$\begin{tikzcd}
& Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\
Y \arrow[dr, "\tau"'] & & M \arrow[dl, "\pi"] \\
& X &
\end{tikzcd}$$

The hypotheses of the problem are:

i. The morphisms p, and q are the restricted projections $Z \hookrightarrow Y \times M$,

ii. the morphism π is a projection onto the quotient variety $X := M/G$

iii. τ and q are birational

iv. p and π are finite

v. p is flat.

2.5 Main Theorem of Bridgeland, King and Reid

The first main theorem of [8, Theorem 1.1] is the following one that we will describe with all the details.

Theorem 2.4. *Suppose that the scheme*

$$Y \times_X Y := \{(y_1, y_2) \in Y \times Y \mid \tau(y_1) = \tau(y_2)\}$$

has dimension $\leqslant n+1$. *Then* Y *is a crepant resolution of* $X = M/G$ *and* $\Phi : D^b(Y) \longrightarrow D^b_G(M)$ *is a equivalence of triangulated categories.*

Proof. Since the variety M is quasi projective, we need to divide the proof in two parts: first one proves the projective case, and then from there a proof for the quasi-projective case is derived.
Case 1. We suppose that M *is a projective variety.*
We start by defining the integral functor between the derived category of the candidate to be a resolution of singularities and the derived category of nonsingular variety that has a finite group action; then we will prove that it is indeed a Fourier-Mukai functor.
Let $\pi_Y : Y \times M \longrightarrow Y$ and $\pi_M : Y \times M \longrightarrow M$ be the projection morphisms; given the object $\mathcal{O}_Z \in D^b(Y \times M)$ we consider the associated integral functor

$$\Phi^{\mathcal{O}_Z}_{Y\to M} : D^b(Y) \longrightarrow D^b_G(M),$$

defined by

$$\Phi^{\mathcal{O}_Z}_{Y\to M}(-) = R\pi_{M*}(\mathcal{O}_Z \otimes \pi^*_Y(- \otimes \rho_0)).$$

The kernel $\mathcal{O}_Z$ of this integral functor is a finite Tor-dimension so the tensor does not need to be derived and the projection π_Y is flat then π^*_Y does not need to be derived either. The kernel $\mathcal{O}_Z$ has finite homological dimension, then $\mathcal{O}^\vee_Z$, the derived dual, also has finite homological dimension. By Proposition A.78 in [2] the finite homological dimension is equivalent to finite Tor-dimension. Then by Proposition 1.13 in [2], there exists $\Psi^{\mathcal{K}}_{M\to Y} \dashv \Phi^{\mathcal{O}_Z}_{Y\to M}$ where the kernel of

$$\Psi^{\mathcal{K}}_{M\to Y} : D^b_G(M) \longrightarrow D^b(Y)$$

is defined by

$$\mathcal{K} = \mathcal{O}^\vee_Z \otimes \pi^*_M \omega_M[n].$$

The composition of these integral functors

$$\Psi \circ \Phi : D^b(Y) \longrightarrow D^b(Y)$$

is

$$\Psi \circ \Phi(-) = R\pi_{2*}(\mathcal{Q} \overset{\mathbf{L}}{\otimes} \pi^*_1(-)),$$

and by Proposition 1.3 in [2] the kernel is

$$\mathcal{Q} = \mathcal{K} * \mathcal{O}_Z \in D^b(Y \times Y).$$

The morphisms $\pi_1 : Y \times Y \longrightarrow Y$ and $\pi_2 : Y \times Y \longrightarrow Y$ denote the projections onto the first and second factor respectively.

First we need to prove that the kernel of $\Psi\Phi$ is supported on the diagonal as that will help us prove isomorphisms between ext groups.
Claim: $\mathcal{Q}$ is supported on the diagonal $\triangle \subset Y \times Y$.
For a closed point $y \in Y$ we consider $i_y : \{y\} \times Y \longrightarrow Y \times Y$, a closed embedding; by the argument in Theorem 1.27 (b) in [2]

$$Li_y^* \mathcal{Q} = \Psi\Phi\mathcal{O}_y,$$

since $\mathcal{Q}$ is the kernel of $\Psi\Phi$, and an the other side $\mathcal{O}_{(y_1,y_2)} = i_{y_1*}\mathcal{O}_{y_2}$.
Then

$$\begin{aligned}
\mathrm{Hom}^i_{D^b(Y\times Y)}(\mathcal{Q}, \mathcal{O}_{(y_1,y_2)}) &= \mathrm{Hom}^i_{D^b(Y\times Y)}(\mathcal{Q}, i_{y_1*}\mathcal{O}_{y_2}) \\
&= \mathrm{Hom}^i_{D^b(Y)}(Li^*_{y_1}\mathcal{Q}, \mathcal{O}_{y_2}) \\
&= \mathrm{Hom}^i_{D^b(Y)}(\Psi\Phi\mathcal{O}_{y_1}, \mathcal{O}_{y_2}) \\
&= \mathrm{Hom}^i_{D^b_G(M)}(\Phi\mathcal{O}_{y_1}, \Phi\mathcal{O}_{y_2}) \\
&= \mathrm{Hom}^i_{D^b_G(M)}(\mathcal{O}_{Z_{y_1}}, \mathcal{O}_{Z_{y_2}}),
\end{aligned}$$

for the last identity we use the Example 5.4 (vi) in [30]. We obtain

$$\mathrm{Hom}^i_{D^b(Y\times Y)}(\mathcal{Q}, \mathcal{O}_{(y_1,y_2)}) \cong \mathrm{Hom}^i_{D^b_G(M)}(\mathcal{O}_{Z_{y_1}}, \mathcal{O}_{Z_{y_2}}).$$

For two G-clusters, say Z_1 and $Z_2 \subset M$, we have

$$G\text{-}\mathrm{Hom}_M(\mathcal{O}_{Z_1}, \mathcal{O}_{Z_2}) = \begin{cases} \mathbb{C} & \text{if } Z_1 = Z_2, \\ 0 & \text{otherwise.} \end{cases}$$

Let $(y_1, y_2) \in (Y \times Y) \setminus \Delta$; by Serre duality,

$$\begin{aligned}
G\text{-}\mathrm{Ext}^n_M(\mathcal{O}_{Z_{y_1}}, \mathcal{O}_{Z_{y_2}}) &= G\text{-}\mathrm{Ext}^{n-n}_M(\mathcal{O}_{Z_{y_2}}, \mathcal{O}_{Z_{y_1}} \otimes \omega_M)^* \\
&= G\text{-}\mathrm{Ext}^0_M(\mathcal{O}_{Z_{y_2}}, \mathcal{O}_{Z_{y_1}})^* \\
&= G\text{-}\mathrm{Hom}_M(\mathcal{O}_{Z_{y_2}}, \mathcal{O}_{Z_{y_1}})^* \\
&= 0.
\end{aligned}$$

Then for $(y_1, y_2) \in (Y \times Y) \setminus \Delta$

$$\mathrm{Hom}^i_{D^b(Y\times Y)}(\mathcal{Q}, \mathcal{O}_{(y_1,y_2)}) = 0$$

for $i = n$, unless $1 \leq p \leq n-1$. Let $\mathcal{Q}'$ be the restriction of $\mathcal{Q}$ to $(Y \times Y) \setminus \Delta$, then Proposition 5.4 in [7]

$$\mathrm{hd}(\mathcal{Q}') \leq n - 2.$$

Let $(y_1, y_2) \in (Y \times Y) \setminus (Y \times_X Y)$ be a closed point, then $\tau(y_1) \neq \tau(y_2)$,

$$\mathrm{Hom}_{D^b(Y \times Y)}(\mathcal{Q}, \mathcal{O}_{(y_1,y_2)}) = G\text{-}\mathrm{Hom}_M(\mathcal{O}_{Z_{y_2}}, \mathcal{O}_{Z_{y_1}}) = 0,$$

because the G-clusters Z_{y_1} and Z_{y_2} are disjoint. Hence $\mathrm{Supp}(\mathcal{Q}) \subset Y \times_X Y$ and therefore also $\mathrm{Supp}(\mathcal{Q}') \subset Y \times_X Y$, since $\mathcal{Q}'$ is a restriction of $\mathcal{Q}$. By Proposition 5.37. in [27]

$$\dim(Y \times Y) = 2n,$$

then by hypothesis $n - 1 \leq \mathrm{codim}(Y \times_X Y)$ and by Corollary 5.2 in [8]

$$\mathrm{codim}(\mathrm{supp}(\mathcal{Q}')) \leq \mathrm{hd}(\mathcal{Q}').$$

We have

$$n - 2 < n - 1 \leq \mathrm{codim}(Y \times_X Y) \leq \mathrm{codim}(\mathrm{supp}(\mathcal{Q}')) \leq \mathrm{hd}(\mathcal{Q}') \leq n - 2,$$

which is absurd, then $\mathcal{Q}' \cong 0$ but $\mathcal{Q}' = \mathcal{Q} \mid_{(Y \times Y) \setminus \Delta}$ and this proves the claim

$$\mathrm{supp}(\mathcal{Q}) \subset \Delta.$$

Fix a point $y \in Y$ and let $E = \Psi\Phi(\mathcal{O}_y)$; above we have proved that E is supported in y, since $E = \Psi\Phi(\mathcal{O}_y) = \mathcal{Q}_y$ and $\mathcal{Q}$ is supported on the diagonal of Y.
Now we will prove that $H^0(E) = \mathcal{O}_y$. Indeed, by adjunction $\Psi \dashv \Phi$ we know that there exists a functor morphism $\Psi\Phi \longrightarrow \mathrm{Id}_{D^b(Y)}$; then there exists a morphism of complexes $E = \Psi\Phi\mathcal{O}_y \longrightarrow \mathcal{O}_y$. With this morphism $E \xrightarrow[a]{} \mathcal{O}_y$ we can complete a distinguished triangle

$$E \xrightarrow[a]{} \mathcal{O}_y \xrightarrow[b]{} A \xrightarrow[c]{} E[1].$$

By axiom TR2 of the triangulated categories we obtain the following distinguished triangle for some $X \in D^b(Y)$

$$C \xrightarrow[\sigma]{} E \xrightarrow[a]{} \mathcal{O}_y \xrightarrow[b]{} X[1].$$

By Remark 1.1.11. in [44] the functor $\mathrm{Hom}_{D^b(Y)}(-, \mathcal{O}_y)$ is cohomological, so that

$$\mathrm{Hom}_{D^b(Y)}(\mathcal{O}_y, \mathcal{O}_y) \longrightarrow \mathrm{Hom}_{D^b(Y)}(E, \mathcal{O}_y) \longrightarrow \mathrm{Hom}_{D^b(Y)}(C, \mathcal{O}_y)$$

is an exact sequence in the abelian category of abelian groups. This induces a long exact sequence in cohomology

$$\begin{aligned} \cdots \longrightarrow \mathrm{Hom}_{D^b(Y)}(\mathcal{O}_y, \mathcal{O}_y) &\longrightarrow \mathrm{Hom}_{D^b(Y)}(E, \mathcal{O}_y) \longrightarrow \mathrm{Hom}_{D^b(Y)}(C, \mathcal{O}_y) \\ \longrightarrow \mathrm{Hom}^1_{D^b(Y)}(\mathcal{O}_y, \mathcal{O}_y) &\longrightarrow \mathrm{Hom}^1_{D^b(Y)}(E, \mathcal{O}_y) \longrightarrow \qquad \cdots \end{aligned}$$

and since $E = \Psi\Phi(\mathcal{O}_y)$

$$\begin{aligned} \cdots \longrightarrow \mathrm{Hom}_{D^b(Y)}(\mathcal{O}_y, \mathcal{O}_y) &\longrightarrow \mathrm{Hom}_{D^b(Y)}(\Psi\Phi(\mathcal{O}_y), \mathcal{O}_y) \longrightarrow \mathrm{Hom}_{D^b(Y)}(C, \mathcal{O}_y) \\ \longrightarrow \mathrm{Hom}^1_{D^b(Y)}(\mathcal{O}_y, \mathcal{O}_y) &\longrightarrow \mathrm{Hom}^1_{D^b(Y)}(\Psi\Phi(\mathcal{O}_y), \mathcal{O}_y) \longrightarrow \qquad \cdots \end{aligned}$$

by the adjunction $\Psi \dashv \Phi$ one has

$$\begin{aligned} \cdots \longrightarrow \mathrm{Hom}_{D^b(Y)}(\mathcal{O}_y, \mathcal{O}_y) &\longrightarrow \mathrm{Hom}_{D^b_G(M)}(\Phi\mathcal{O}_y, \Phi\mathcal{O}_y) \longrightarrow \mathrm{Hom}_{D^b(Y)}(C, \mathcal{O}_y) \\ \longrightarrow \mathrm{Hom}^1_{D^b(Y)}(\mathcal{O}_y, \mathcal{O}_y) &\xrightarrow[\lambda]{} \mathrm{Hom}^1_{D^b_G(M)}(\Phi\mathcal{O}_y, \Phi\mathcal{O}_y) \longrightarrow \qquad \cdots \end{aligned}$$

The morphism $\mathrm{Hom}^1_{D^b(Y)}(\mathcal{O}_y, \mathcal{O}_y) \xrightarrow[\lambda]{} \mathrm{Hom}^1_{D^b_G(M)}(\Phi\mathcal{O}_y, \Phi\mathcal{O}_y)$ is the Kodaira-Spencer map for the family $\mathcal{O}_Z$ and as for each $y \in Y, \mathcal{O}_{Z_y}$ is the structure sheaf of a G-cluster, then by the Lemma 5.3 in [3] the morphism λ is injective. Then

$$\mathrm{Hom}^i_{D^b(Y)}(C, \mathcal{O}_y) = 0, \text{ for } i = 0,$$

We claim that $H^0(C) \cong 0$.

With this we will prove that $H^0(E) \cong \mathcal{O}_y$. Indeed, in the following distinguished triangle

$$C \xrightarrow[\sigma]{} E \xrightarrow[a]{} \mathcal{O}_y \xrightarrow[b]{} X[1]$$

taking cohomology sheaves one has

$$\begin{aligned} \cdots \longrightarrow H^0(C) &\longrightarrow H^0(E) \longrightarrow H^0(\mathcal{O}_y) \\ \longrightarrow H^1(C) &\longrightarrow H^1(E) \longrightarrow H^1(\mathcal{O}_y) \longrightarrow \cdots \end{aligned}$$

but $E = \mathcal{Q}_y$ is a sheaf on Y and

$$\begin{aligned} \cdots \longrightarrow \quad 0 \quad &\longrightarrow \mathcal{Q}_y \longrightarrow \mathcal{O}_y \\ &\longrightarrow H^1(C) \longrightarrow \; 0 \; \longrightarrow \; 0 \; \longrightarrow \cdots . \end{aligned}$$

Since $\mathcal{Q}$ is a complex with support on the diagonal, $E = \mathcal{Q}_y$ is supported at the point y, then in the diagram above we have an injective morphism of sheaves supported at a point, so that

$$H^0(E) \cong \mathcal{O}_y.$$

By Corollary 5.3 [8] we conclude that Y is nonsingular at y and $E \cong \mathcal{O}_y$ but the point y is arbitrary, i.e. Y is nonsingular.

We will prove that Φ is fully faithful; indeed let $y_1, y_2 \in Y$. We already know that

$$\begin{aligned} \mathrm{Hom}^i_{D^b(Y)}(\Psi\Phi\mathcal{O}_{y_1}, \mathcal{O}_{y_2}) &= \mathrm{Hom}^i_{D^b(Y)}(\Psi\Phi\mathcal{O}_{y_1}, \mathcal{O}_{y_2}) \\ &= \mathrm{Hom}^i_{D^b_G(M)}(\Phi\mathcal{O}_{y_1}, \Phi\mathcal{O}_{y_2}) \\ &= \mathrm{Hom}^i_{D^b_G(M)}(\mathcal{O}_{Z_{y_1}}, \mathcal{O}_{Z_{y_2}}), \end{aligned}$$

and by a previous step if $E = \Psi\Phi\mathcal{O}_y$ then $E \cong \mathcal{O}_y$, so

$$\mathrm{Hom}^i_{D^b(Y)}(\Psi\Phi\mathcal{O}_{y_1}, \mathcal{O}_{y_2}) = \mathrm{Hom}^i_{D^b(Y)}(\mathcal{O}_{y_1}, \mathcal{O}_{y_2})$$

thus

$$\mathrm{Hom}^i_{D^b(Y)}(\mathcal{O}_{y_1}, \mathcal{O}_{y_2}) = \mathrm{Hom}^i_{D^b_G(M)}(\mathcal{O}_{Z_{y_1}}, \mathcal{O}_{Z_{y_2}}).$$

This isomorphism holds for any $\mathcal{O}_{y_1}, \mathcal{O}_{y_2} \in \Omega$, where Ω is the spanning class for $D^b(Y)$ [3, Example 2.2], and by Theorem 2.3 [8] Φ is fully faithful.

The property of Y being a crepant resolution relies on the fact that Φ is an equivalence; this is the next point to prove. We will apply the Theorem 2.4 [8], so let us its hypotheses. First $D^b(Y)$ is non trivial and $D^b_G(M)$ is indecomposable since G acts faithfully on M. We also need to verify that

$$\Phi S_Y(\omega) = S_M \Phi(\omega), \text{ for any } \omega \in \Omega,$$

where S_Y and S_M are the Serre functors, and Ω as we indicated before is the spanning class of $D^b(Y)$. By general hypotheses the action of the group G in

M is such that the canonical bundle ω_M is locally trivial, then for any point $x \in M$ the orbit $G.x \subset M$ has a open neighbourhood $V(G.x)$ in which:

$$\omega_M \otimes \mathcal{O}_{Z_y} \cong \mathcal{O}_{Z_y}, \text{ for any } y \in Y \text{ in } Coh^G(M).$$

Taking for $y \in Y$, $\omega = \mathcal{O}_y$

$$\Phi S_Y(\mathcal{O}_y) = \Phi(\mathcal{O}_y \otimes \omega_Y[3]) = \Phi(\mathcal{O}_y[3]) = \mathcal{O}_{Z_y}[3]$$

and

$$S_M \Phi(\mathcal{O}_y) = S_M \mathcal{O}_{Z_y} = \mathcal{O}_{Z_y} \otimes \omega_M[3] = \mathcal{O}_{Z_y}[3].$$

This verifies the identity we need, so that Φ is an equivalence.

To close the projective case, let us prove that Y is a crepant resolution. Take a point $x \in X = M/G$ and let

$$D_x(Y) = \{\mathcal{E}^\bullet \in D^b(Y) \,|\, \mathrm{supp}(\mathcal{E}^\bullet) \subset \tau^{-1}(x)\}$$

$$D_{x,G}(M) = \{\mathcal{F}^\bullet \in D^b_G(M) \,|\, \mathrm{supp}(\mathcal{F}^\bullet) \subset \pi^{-1}(x)\}.$$

These are full subcategories; we restrict Φ to them, and then

$$\Phi_x : D_x(Y) \longrightarrow D_{x,G}(M)$$

is also an equivalence. The canonical bundle ω_M is trivial as G-sheaf in a neighbourhood of $\pi^{-1}(x) \subset M$ for every $x \in X$; then $D^{b,G}(M)$ has trivial Serre functor. By the equivalence Φ_x the category $D_x(Y)$ also has trivial Serre functor, and this is true for all x. Then by Lemma 3.1 [8] Y is a crepant resolution (X has rational singularities since X is a quotient singularity).

Case 2. We suppose that M is a quasi-projective variety.
Since M is quasiprojective, we take its projective closure and get the immersion $M \hookrightarrow \overline{M}$ this induces the immersion $D^b_G(M) \hookrightarrow D^b_G(\overline{M})$. The following diagram is commutative

$$\begin{tikzcd}
D^b(Y) \arrow[r, "\Phi"] & D^b_G(M) \arrow[r] & D^b_G(\overline{M}) \\
D^b_c(Y) \arrow[u, hook] \arrow[r, "\Phi_c"'] & D^b_{c,G}(M) \arrow[u, hook] \arrow[r, hook] & D^b_{c,G}(\overline{M}) \arrow[u, hook]
\end{tikzcd}$$

where $D^b_c(Y), D^b_{c,G}(M)$ and $D^b_{c,G}(\overline{M})$ are the full subcategories of $D^b(Y)$, $D^b_G(M)$ and $D^b_G(\overline{M})$, respectively, of objects with compact support. Since the variety $\overline{M}$ is proper, then $D^b_{c,G}(\overline{M}) = D^b_G(\overline{M})$. By hypothesis M is nonsingular and by resolution of singularities $\overline{M}$ is nonsingular thus $D^b_G(\overline{M})$ has a Serre functor restricting this functor to X we get a Serre functor on $D^b_G(M)$ then $D^b_{c,G}(M)$ has Serre functor. With this we have all the functors of the previous case,so we can use the same techniques of the previous case and we conclude that:
The restricted functor Φ_c is an equivalence (i.e. a Fourier-Mukai transform), Y is a nonsingular variety and a crepant resolution.

It remains to prove that Φ is an equivalence. Again by the Proposition 1.13 in [2], Φ has a right adjoint $\Upsilon^{\omega_{Z/M}}_{M\to Y} : D^b_G(M) \longrightarrow D^b(Y)$ given by

$$\Upsilon^{\omega_{Z/M}}_{M\to Y}(-) = \left[R\pi_{Y*}(\mathcal{O}_Z \otimes \pi^*_Y\omega_Y[3] \overset{\mathbf{L}}{\otimes} \pi^*_M(-))\right]^G .$$

We prove that

$$\mathcal{O}_Z \otimes \pi^*_Y\omega_Y[3] \cong \omega_{Z/M}.$$

We start from

$$\mathcal{O}_Z^\vee = \mathrm{R}\mathcal{H}om_{Y\times M}(\mathcal{O}_Z, \mathcal{O}_{Y\times M}) = \mathrm{R}\mathcal{H}om_{Y\times M}(i_*\mathcal{O}_Z, \mathcal{O}_{Y\times M})$$

where $i : Z \hookrightarrow Y \times M$ is a closed embedding, to which we apply the Grothendieck-Verdier duality

$$\mathcal{O}_Z^\vee = i_*\mathcal{H}om_Z(\mathcal{O}_Z, i^!\mathcal{O}_{Y\times M}) = i_*i^!\mathcal{O}_{Y\times M}.$$

By Section C.1 in [2] we have $i^!\mathcal{O}_{Y\times M} = \omega_i[-3]$ but the complex $i_*i^!\mathcal{O}_{Y\times M}$; we can write $i^!\mathcal{O}_{Y\times M}$ in $Y \times M$. With this

$$\mathrm{Ker}(\Upsilon) = i^!\mathcal{O}_{Y\times M} \otimes \pi^*_Y\omega_Y[3] = \omega_i[-3] \otimes \pi^*_Y\omega_Y[3] = \omega_i \otimes \pi^*_Y\omega_Y = \omega_i$$

and then $\omega_i = \omega_{Z/M}$.

The composed functor $\Upsilon \circ \Phi : D^b(Y) \longrightarrow D^b(Y)$ defined by

$$\Upsilon \circ \Phi(-) = \mathrm{R}\pi_{2*}(\mathcal{Q} \overset{\mathbf{L}}{\otimes} \pi^*_1(-))$$

has kernel $\mathcal{Q}$ given by

$$\mathrm{Ker}(\Upsilon \circ \Phi) = \omega_{Z/M} * \mathcal{O}_Z \in D^b(Y \times Y)$$

where π_1, π_2 are the projections of $Y \times Y$ on the first and second factors. Since Φ_c is an equivalence, $\Upsilon\Phi\mathcal{O}_y = \mathcal{O}_y$ for any point $y \in Y$, but $\mathcal{Q} = \mathrm{Ker}(\Upsilon \circ \Phi)$ and then $\mathcal{Q}_y = \mathcal{O}_y$. By Corollary 5.23 [30] the composition $\Upsilon \circ \Phi$ satisfies

$$\Upsilon \circ \Phi(-) = L \otimes (-)$$

for some line bundle L in Y and $\mathcal{Q} = \triangle_* L$ where $\triangle : Y \longrightarrow Y \times Y$ is the diagonal morphism.

We prove that L is trivial; indeed by adjunction (Φ, Υ) there exists a morphism of functors $\epsilon : \mathrm{Id} \longrightarrow \Upsilon \circ \Phi$. We obtain a morphism of sheaves

$$\epsilon(\mathcal{O}_y) : \mathcal{O}_y \longrightarrow (\Upsilon \circ \Phi)(\mathcal{O}_y) = L$$

and a commutative diagram of sheaves on Y

$$\begin{tikzcd}
\mathcal{O}_Y \arrow[r, "\epsilon(\mathcal{O}_Y)"] \arrow[d, "f"'] & L \arrow[d, "L\otimes f"] \\
\mathcal{O}_y \arrow[r, "\epsilon(\mathcal{O}_y)"'] & \mathcal{O}_y
\end{tikzcd}$$

Since Φ_c is an equivalence on $D^b_c(Y)$ then ϵ is an isomorphism, and the vertical arrows are surjective so that the morphism $\epsilon(\mathcal{O}_Y)$ is an isomorphism; this proves that L is trivial. Thus $\Upsilon \circ \Phi = \mathrm{Id}_{D^b(Y)}$ and Υ is a right adjoint. We conclude that Φ is fully faithful.

We will use Lemma 2.1 of [8] to prove that $\Phi : D^b(Y) \longrightarrow D^b_G(M)$ is an equivalence; we only need to prove that,

$$\text{if } \Upsilon(E) \cong 0, \text{ then } E \cong 0 \text{ for any } E \in D^b_G(M).$$

Indeed if $\Upsilon(E) \cong 0$, for any $B \in D^b_{c,G}(M)$ there exists $A \in D^b_c(Y)$ such that $\Phi_c A = B$ since Φ_c is an equivalence, therefore

$$\mathrm{Hom}^i_{D^b_G(M)}(B, E) = \mathrm{Hom}^i_{D^b_G(M)}(\Phi A, E) = \mathrm{Hom}^i_{D^b_G(M)}(A, \Upsilon E) = 0. \quad (2.1)$$

This holds for all $i \in \mathbb{Z}$ and for all $B \in D^b_G(M)$. By absurd suppose that $E \ncong 0$ and let $W = G.x$ be a G orbit contained in $\mathrm{Supp}(E)$, then the inclusion $W \underset{i}{\hookrightarrow} E$ is a projective equivariant morphism of schemes. We consider the functors:

$$i^! : D^b_{c,G}(M) \longrightarrow D^b_{c,G}(W) = D^b_G(W),$$

and

$$i_* : D^b_G(W) \longrightarrow D^b_{c,G}(M).$$

Take $B = i_* i^! E \in D^b_{c,G}(M)$ the morphism $i_* i^! E \longrightarrow E$ is nonzero, but this contradicts (2.1), then $E \cong 0$, and we conclude that Φ is an equivalence, i.e. a Fourier-Mukai functor.

This concludes the quasi-projective case. □

2.6 Second Main Theorem of Bridgeland, King and Reid

The second main result in the mentioned paper [8] is the proof of the Nakamura conjecture.

Theorem 2.5. *Under the same general hypotheses, we suppose that M is a nonsingular quasi-projective complex variety and $dim(M) = n \leq 3$. Then G-Hilb M is irreducible and a crepant resolution of $X = M/G$, and $\Phi : D^b(Y) \longrightarrow D^b_G(M)$ is an equivalence (a Fourier-Mukai functor).*

Proof. The morphism $\tau : Y \longrightarrow X$ is birational then the exceptional locus has dimension ≤ 2. This implies that

$$\dim(Y \times_X Y) \leq 4,$$

therefore the condition of Theorem 2.4 is satisfied. Hence, Y is a crepant resolution of X and Φ is an equivalence. It remains to prove that G-Hilb M is irreducible.
First we will prove that G-Hilb M is connected. By absurd suppose that there exists a G-cluster Z of M such that $Z \notin \{Z_y \mid y \in Y\}$, that is, it is not parameterized by Y. Then $\mathcal{O}_Z \in D^b_G(M)$, and there exists $E \in D^b_c(Y)$ with $\Phi_c E = \mathcal{O}_Z$ as in particular Φ_c is an equivalence. Since Φ is an equivalence it has a quasi-inverse Ψ; using adjunction we have

$$\begin{aligned}
\mathrm{Hom}^i_{D^b(Y)}(E, \mathcal{O}_y) = \mathrm{Hom}^i_{D^b(Y)}(\Psi\Phi E, \mathcal{O}_y) &= \mathrm{Hom}^i_{D^b(M)}(\Phi E, \Phi\mathcal{O}_y) \\
= \mathrm{Hom}^i_{D^b_G(M)}(\mathcal{O}_Z, \mathcal{O}_{Z_y}) &= 0.
\end{aligned}$$

The last identity holds because $Z \neq Z_y$ so that $\mathrm{hd}(E) = 1$. This means that

$$E \cong (\cdots \to 0 \to L_1 \underset{\eta}{\to} L_2 \to 0 \to \cdots)$$

in $D^b_c(Y)$ where the only nonzero terms of the complex are L_1 and L_2 and they are locally free. The structural sheaf of Z is supported on a G-orbit $G \cdot x \subset M$, then E it is supported on a fiber $\tau^{-1}([x]) \subset Y$.
The kernel of η is a torsion sheaf then $\ker(\eta) \cong 0$. The other side $H^0(E) \cong \ker(\eta) = 0$ and $H^1(E) \cong L_2/\mathrm{Im}(\eta) = \mathrm{coker}(\eta)$, then $E \cong \mathrm{coker}(\eta)[1]$ and

$$[E] = [-\mathrm{coker}\,\eta], \text{ in } K_c(Y).$$

Let $y \in \tau^{-1}([x])$ then by definition of Hilbert-Chow morphism $\tau(y) = \mathrm{Supp}(Z_y)$ and $\tau(y) = [x]$, thus Z and Z_y are supported in same G-orbit $[x]$. By Lemma 8.1 of [8]

$$[\mathcal{O}_Z] = [\mathcal{O}_{Z_y}] \text{ in } K^G_c(M);$$

since $\Phi E = \mathcal{O}_Z, \Phi\mathcal{O}_y = \mathcal{O}_{Z_y}$ and Φ is an equivalence, this induces an isomorphism on Grothendieck groups, then

$$[E] = [\mathcal{O}_y], \text{ in } K_c(Y),$$

where $K^G_c(M), K_c(Y)$ are the Grothendieck groups of $D^b_{c,G}(M), D^b_c(Y)$ respectively. Since Y is a nonsingular quasiprojective variety there is an open inclusion $i : Y \hookrightarrow \overline{Y}$ where $\overline{Y}$ is a nonsingular projective variety, and $i_* : D^b_c(Y) \longrightarrow D^b_c(\overline{Y})$ is a full embedding. This functor induces an injective homomorphism between Grohendieck groups:

$$i_* : K_c(Y) \longrightarrow K_c(\overline{Y});$$

then

$$[E] = [\mathcal{O}_y], \text{ holds on } K_c(\overline{Y}).$$

Moreover $[E] = -[\mathrm{coker}\,\eta]$ thus $[\mathcal{O}_y] = -[\mathrm{coker}\ \eta]$. By applying Riemann-Roch theorem on $\overline{Y}$, for a sufficiently ample line bundle L on $\overline{Y}$, the Euler characteristics $\chi(\mathcal{O}_y \otimes L)$, $\chi(\mathrm{coker}\ \eta \otimes L)$ are positive, since in the last identity we multiply by the L in $K_c(\overline{Y})$

$$[\mathcal{O}_y].[L] = -[\mathrm{coker}\,\eta].[L]$$

by definition

$$[\mathcal{O}_y \otimes L] = -[\operatorname{coker} \eta \otimes L]$$

but $\chi(\mathcal{O}_y \otimes L) = -\chi(\operatorname{coker} \eta \otimes L) > 0$ this is a contradiction. As a consequence $Y = G\text{-Hilb}\, M$.

Finally $G\text{-Hilb}\, M$ is irreducible, since Y is irreducible.

□

Chapter 3

Moduli Spaces of G-constellations and Crepant Resolutions of $\mathbb{C}^3/G$

In this chapter we first review some basic facts about the construction of quotients of varieties under the action of a group. The GIT quotient is used in sections 3.2 and 3.3 for the construction of the moduli spaces of θ-stable G-constellations, which are isomorphic to moduli spaces of θ-stable McKay quiver representations. In sections 3.4 and 3.5 we prove an identity that relates the tautological bundle $\mathcal{R}$ and the universal G-constellation $\mathcal{U}$. In the last section, we study the wall-crossing phenomenon by means of the contraction morphism induced by a wall. Depending on each type of wall, we describe the results of Craw and Ishii [15] about the structure of the unstable locus Z and how the tautological line bundles change under wall-crossing.

3.1 GIT: Construction of Quotients by Group Actions

In this section we describe some important points of the construction of quotient varieties using Geometric Invariant Theory (GIT).
The principal reference for this section is [36] but also [40], [45], and [46].
Throughout this section we let
$\mathbb{K}$ be an algebraically closed field with characteristic zero, G a linear alge-

braic group over $\mathbb{K}$ and M an affine algebraic variety over $\mathbb{K}$. The algebraic group G is assumed to acts algebraically on M.

3.1.1 GIT quotient

If G is a reductive affine algebraic group the *GIT quotient* of the affine variety M by the linear action of G is defined by :

$$M /\!/ G = \text{Spec } \mathbb{K}[M]^G,$$

where $\mathbb{K}[M]^G$ is the algebra of G-invariant polynomial functions on M. The 14-th Problem of Hilbert asks when $\mathbb{K}[M]^G$ is finitely generated, and in the case G is a reductive linear group it is answered in the affirmative, so that $M /\!/ G$ is an affine algebraic variety. From now on we assume that G is reductive.
We define a map of topological spaces

$$M/G \longrightarrow M /\!/ G,$$

by

$$\mathbb{O}(x) \longmapsto \left\{ f \in \mathbb{K}[M]^G \mid f(y) = 0, \text{ for any } y \in \mathbb{O}(x) \right\}.$$

There exists a naturally morphism induced by inclusion $\mathbb{K}[M]^G \hookrightarrow \mathbb{K}[M]$

$$\Phi : M/G \longrightarrow M /\!/ G;$$

this morphism is called *affine quotient map* and is surjective (see [40, Theorem 5.9]). Two orbits, say $\mathbb{O}_1$ and $\mathbb{O}_2$, define the same point in $M /\!/ G$ if and only if:

$$\mathbb{O}_1 \cap \mathbb{O}_2 \neq \emptyset.$$

This leads us to define an equivalence relation on M, for any x and x' in M:

$$x \sim x' \text{ if and only if } \overline{\mathbb{O}(x)} \cap \overline{\mathbb{O}(x')} \neq \emptyset.$$

Theorem 3.1 (Nagata-Mumford)**.** *With the notation above: two points $x \sim x'$ if and only if cannot be separated by a equivariant polynomial function.*

We then have a topological-theoretic isomorphism

$$M /\!/ G = M/ \sim .$$

Let us remember that the quotient space $M/\sim$ is only a topological space and not an variety. The GIT quotient $M /\!\!/ G$ can be described in a more concrete way, we then set-theoretic isomorphism

$$M /\!\!/ G \xrightarrow{=} \{ \text{ closed orbits in M } \},$$

this map is defined by

$$[x] \longmapsto \text{ the unique closed orbit contained in } \overline{\mathbb{O}(x)},$$

for the latter result see [40, Corollary 5.5].

3.1.2 χ-semistable points

We denote by $\mathrm{Char}(G)$ the set of characters of the algebraic group G. Fix a character $\chi \in \mathrm{Char}(G)$, an affine variety M and define:

$$\mathbb{K}[M]^{G,\chi} = \left\{ f \in \mathbb{K}[M]^G \mid f(g.m) = \chi(g).f(m) \right\};$$

this is called the set of *semi-invariant functions*.
The ring of semi-invariant functions can be graded using the characters of group G as follows:

$$A = \bigoplus_{n \geq 0} \mathbb{K}[M]^{G,\chi^n},$$

which is a finitely generated algebra.
We define the *twisted GIT quotient*

$$M /\!\!/_\chi G = \mathrm{Proj}(A);$$

this is a quasi-projective variety and by definition there exists a projective morphism

$$\pi : M /\!\!/_\chi G \longrightarrow \mathrm{Spec}(A_0) = M /\!\!/ G.$$

By definition of twisted GIT quotient there exists a rational map

$$M \longrightarrow M /\!\!/_\chi G.$$

A point $x \in M$ is called χ-semistable if there exists $f \in A_n = \mathbb{K}[M]^{G,\chi^n}$ for some $n \geq 1$ such that $f(x) \neq 0$. We denote by $M^{ss,\chi}$ the set of all χ-semistable points in M; this set is open in M. There exists a natural surjective map of topological spaces:

$$M^{ss,\chi}/G \longrightarrow M /\!\!/_\chi G$$

defined by

$$[x] \longmapsto \{f \mid f(y) = 0, \text{ for any } y \in [x]\}.$$

Two semistable orbits $\mathbb{O}_1$ and $\mathbb{O}_2$, i.e. contained in $M^{ss,\chi}$, defines a same point in $M /\!\!/_\chi G$ if and only if

$$\overline{\mathbb{O}_1} \cap \overline{\mathbb{O}_2} \cap M^{ss,\chi} \neq \emptyset.$$

Again this leads us to define an equivalence relation, two points x and x' in $M^{ss,\chi}$:

$$x \sim_{ss} x' \text{ if and only if } \overline{\mathbb{O}(x)} \cap \overline{\mathbb{O}(x')} \cap M^{ss,\chi} \neq \emptyset.$$

We obtain a topological-theoretic isomorphism:

$$M /\!\!/_\chi G = M/\sim_{ss}.$$

The projective morphism

$$\pi : M /\!\!/_\chi G \longrightarrow M /\!\!/ G$$

can be described as follows [36, Theorem 9.19],

$$[x] \longmapsto \text{unique closed orbit in } M \text{ contained in } \overline{\mathbb{O}(x)}.$$

The twisted GIT quotient also can be described in terms of closed orbits [36, Theorem 9.20]:

$$M /\!\!/_\chi G \xrightarrow{=} \{ \text{ closed orbits in } M^{ss,\chi} \}.$$

3.1.3 χ-stable points

A point $x \in M^{ss,\chi}$ such that the stabilizer group G_x is finite and the orbit $\mathbb{O}(x)$ is closed in $M^{ss,\chi}$, is called χ-*stable* point of the affine variety M. The set of stable points of M is denoted by $M^{s,\chi}$. If a point $x \in M^{ss,\chi}$ is χ-stable we then:

$$\overline{\mathbb{O}(x)} \cap M^{ss,\chi} = \mathbb{O}(x),$$

this implies that, for two χ-stable points :

$$\overline{\mathbb{O}(x)} \cap \overline{\mathbb{O}(x')} \cap M^{ss,\chi} \neq \emptyset \text{ if and only if } \mathbb{O}(x) = \mathbb{O}(x').$$

We obtain a natural map

$$M^{s,\chi}/G \longrightarrow M /\!\!/_\chi G.$$

Some properties that $M^{s,\chi}$ satisfies:

i. $M^{s,\chi}$ is open in $M^{ss,\chi}$,

ii. if $M^{s,\chi}$ is nonempty and M is irreducible, then $M^{s,\chi}$ is dense in $M^{ss,\chi}$ and $M^{s,\chi}/G$ is dense in $M /\!\!/_\chi G$,

iii. if the variety M is nonsingular and for any $x \in M^{s,\chi}$ the stabilizer G_x is trivial, then $M^{s,\chi}/G$ is a nonsingular variety.

We summarize the morphisms between quotients that we have defined in the following diagram:

$$\begin{tikzcd}
M^{s,\chi} \arrow[r, hook] \arrow[d] & M^{ss,\chi} \arrow[r, hook] \arrow[d] & M \arrow[d] \arrow[dll, dashed] \arrow[dd, bend left, "\Phi"] \\
M^{s,\chi}/G \arrow[r] \arrow[d, "="'] & M^{ss,\chi}/G \arrow[r] \arrow[d] & M/G \arrow[d] \\
M^{s,\chi}/G \arrow[r] & M /\!\!/_\chi G \arrow[r, "\pi"'] & M /\!\!/ G.
\end{tikzcd}$$

3.2 Moduli Spaces of θ-stable G-constellations

In this section we describe the construction of the moduli spaces of θ-stable G-constellations following Craw and Ishii [15].

Let G be a finite subgroup of $\mathrm{GL}(n, \mathbb{C})$. The set of classes of irreducible representations of G is denoted by $\mathrm{Irr}(G)$ and the regular representation by $R = \bigoplus R_\rho \otimes \rho$. The representation ring is

$$\mathrm{R}(G) = \bigoplus \mathbb{Z}.\rho,$$

which can be interpreted as a Grothendieck group of classes of representations of G under isomorphism.

Definition 3.2 (G-constellations)**.** A G-constellation is an object $\mathcal{F} \in \mathrm{Coh}^G(\mathbb{C}^n)$, such that the vector space of global sections satisfies $H^0(\mathcal{F}) \cong R$ as a $\mathbb{C}[G]$-modules.

Definition 3.3. The GIT-parameter space is the set

$$\Theta = \left\{ \theta \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{R}(G), \mathbb{Q}) \;\middle|\; \theta(R) = 0 \right\};$$

an element of this set is called a stability condition.

Definition 3.4. A G-constellation $\mathcal{F}$ is called θ-stable if for every proper G-equivariant coherent sheaf, $0 \subsetneq \mathcal{F}' \subsetneq \mathcal{F}$, the inequality $\theta(\mathcal{F}') > 0$ holds. If we replace this condition by $\theta(\mathcal{F}') \geq 0$, the sheaf $\mathcal{F}$ is called θ-semistable.

Observation 3.5. The previous definition of stability was introduced by King [35] for quiver representations.

Definition 3.6 (Generic parameters)**.** A parameter $\theta \in \Theta$ is called generic if for any $\mathcal{F} \in \{\theta\text{-semistable } G\text{-constellations}\}$ one has

$$\mathcal{F} \in \{\theta\text{-stable } G\text{-constellations}\}.$$

To construct the moduli space of G-constellations as in [15, Section 2.1], let us denote $V = \mathbb{C}^n$, and define the affine scheme

$$\mathcal{N} = \Big\{ B \in \mathrm{Hom}_{\mathbb{C}[G]}(R, V \otimes R) \ \Big| \ \Lambda^2 B = 0 \Big\}.$$

Each element $B \in \mathcal{N}$ generates a G-constellation. Indeed, each $B \in \mathcal{N}$ gives a structure of $\mathbb{C}[V]$-module to the regular representation R, let us call this structure R_B, then the associated module $\widetilde{R_B} \in \mathcal{O}_{\mathrm{Spec}(\mathbb{C}[V])}$. The sheaf $\widetilde{R_B}$ is a G-equivariant coherent sheaf and its global sections are isomorphic to R, so $\widetilde{R_B}$ is a G-constellation.
As the center $\mathbb{C}^*$ acts trivially on $\mathrm{Aut}_{\mathbb{C}[G]}(R)$, one can define the quotient

$$\mathbb{P}\mathrm{Aut}_{\mathbb{C}[G]}(R) = \mathrm{Aut}_{\mathbb{C}[G]}(R)/\mathbb{C}^*;$$

this group acts on $\mathcal{N}$ by conjugation. Then $\mathrm{Aut}_{\mathbb{C}[G]}(R)$ acts on $\mathcal{N}$ by the projection homomorphism $\mathrm{Aut}_{\mathbb{C}[G]}(R) \longrightarrow \mathbb{P}\mathrm{Aut}_{\mathbb{C}[G]}(R)$. The set of characters of $\mathbb{P}\mathrm{Aut}_{\mathbb{C}[G]}(R)$ is

$$X(\mathbb{P}\mathrm{Aut}_{\mathbb{C}[G]}(R)) = \Big\{ \chi_\theta : \mathbb{P}\mathrm{Aut}_{\mathbb{C}[G]}(R) \longrightarrow \mathbb{C}^* \ , \ \chi_\theta([g_\rho]_\rho) = \prod_\rho \det(g_\rho)^{\theta(\rho)} \Big\}.$$

We can construct the GIT quotients:

$$\mathcal{M}_\theta := \mathcal{N}_\theta /_{\chi_\theta} \mathbb{P}\mathrm{Aut}_{\mathbb{C}[G]}(R), \quad \overline{\mathcal{M}_\theta} := \mathcal{N}_\theta^{ss} /_{\chi_\theta} \mathbb{P}\mathrm{Aut}_{\mathbb{C}[G]}(R)$$

the geometric quotient and categorical quotient respectively, where $\mathcal{N}_\theta, \mathcal{N}_\theta^{ss}$ are the open subsets of $\mathcal{N}$ of the χ_θ-stable and χ_θ-semistable points. By King [35, Proposition 5.3] these quotients are the fine moduli and coarse moduli spaces, respectively, of χ_θ-stable and χ_θ-semistable G-constellations. The construction of these spaces as moduli spaces of quiver representations is

more concrete, and we will describe it in the next section.
By descent theory of coherent sheaves the sheaves $R \otimes \mathcal{O}_{\mathcal{N}}$, $R_\theta \otimes \mathcal{O}_{\mathcal{N}}$ on $\mathcal{N}_\theta$ descend to locally free sheaves $\mathcal{R}$, $\mathcal{R}_\theta$ on $\mathcal{M}_\theta$, and there exists a universal G-constellation $\mathcal{U}_\theta$ on $\mathcal{M}_\theta \times \mathbb{C}^3$.

3.3 Moduli Spaces of θ-stable McKay Quiver Representations

In the previous section we reviewed the GIT construction of the moduli spaces M_θ, which are equivalent to the moduli spaces of θ-stable McKay quiver representations. In this section, we delve into the construction of these moduli spaces from the point of view of quiver representations, which is more concrete than the one presented in the previous section.

Let $G \subset \mathrm{GL}(n, \mathbb{K})$ a finite abelian subgroup of order r, such that the characteristic of the field $\mathbb{K}$ dos not divide to r. First we note that the group G can be considered as subgroup of $(\mathbb{K}^*)^n$, where $(\mathbb{K}^*)^n$ is the group of diagonal matrices with non-zero entries. We set

$$\mathbb{T}^n := (\mathbb{K}^*)^n.$$

When G is abelian its representations are one-dimensional and the set of classes of irreducible representations is also finite and

$$\mathrm{Irr}(G) = \{\rho_1, \dots, \rho_r\}$$

have the same number of elements of G.
The embedding $G \hookrightarrow \mathbb{T}^n$ into the algebraic torus induces a surjective homomorphism $\mathrm{Hom}(\mathbb{T}^n, \mathbb{K}^*) \longrightarrow \mathrm{Hom}(G, \mathbb{K}^*)$, but $\mathbb{Z}^n = \mathrm{Hom}(\mathbb{T}^n, \mathbb{K}^*)$ is the character lattice of $\mathbb{T}^n$ and $G^* = \mathrm{Hom}(G, \mathbb{K}^*)$ is the dual group as algebraic groups, and we can be identify $G^* = \mathrm{Irr}(G)$.

Definition 3.7 (McKay quiver)**.** Let $G \subset \mathrm{GL}(n, \mathbb{K})$ be a finite abelian group of order r; the *McKay quiver* of G is a quiver (I, Ω) where the set of vertices is $\mathrm{Irr}(G)$ and the set of edges is

$$\Omega = \left\{ a_i^\rho : \rho\rho_i \to \rho \mid \rho \in \mathrm{Irr}(G) \text{ and } i \in I(n) \right\}.$$

The McKay quiver of G has nr arrows.

Definition 3.8 (Representation of the McKay quiver)**.** A *representation of the McKay quiver* of dimension $d = (1, \dots, 1) \in \mathbb{Z}^r$ is an association:

$$\text{for each } \rho \in \operatorname{Irr}(G) \longmapsto R_\rho \text{ is a some one-dimensional } \mathbb{K}\text{-vector space}$$

and

$$\text{for each } a_i^\rho \in \Omega \longmapsto T_{a_i^\rho} : R_{\rho\rho_i} \to R_\rho \text{ a } \mathbb{K}\text{-linear map.}$$

To any representation of the McKay quiver one can associate a matrix $(b_i^\rho) \in \mathbb{K}^{n\times r}$ where the scalars $b_i^\rho \in \mathbb{K}$ represent the $\mathbb{K}$-linear transformations $T_{a_i^\rho}$. We write the coordinate ring of the affine space $\mathbb{A}_{\mathbb{K}}^{nr}$ as

$$\mathbb{K}[z_i^\rho : \rho \in \operatorname{Irr}(G),\ i \in I(n)].$$

The quiver representations should satisfy the following relations:

$$b_j^{\rho\rho_i} b_i^\rho = b_i^{\rho\rho_j} b_j^\rho,\ \rho \in \operatorname{Irr}(G),\ i \in I(n) \tag{3.1}$$

this comes from the relations that G-constellations must naturally satisfy [17]. We can write this relation in terms of the polynomial ring, and consider the ideal generated by these relations

$$I = \langle z_j^{\rho\rho_i} z_i^\rho - z_i^{\rho\rho_j} z_j^\rho : \rho \in \operatorname{Irr}(G),\ i \in I(n)\rangle.$$

The affine scheme defined by this ideal is denoted by Z. First note that $\mathbb{K}^*$ acts on the one-dimensional $\mathbb{K}$-vector space R_ρ by the product by a scalar, then the algebraic torus $(\mathbb{K}^*)^r$ acts diagonally on $\oplus R_\rho$, and also acts on $\operatorname{Hom}(R_{\rho\rho_i}, R_\rho)$:
for $t = (t_\rho) \in (\mathbb{K}^*)^r$ and $b_i^\rho \in \operatorname{Hom}(R_{\rho\rho_i}, R_\rho)$,

$$t.b_i^\rho := t_{\rho\rho_i}^{-1} b_i^\rho t^\rho. \tag{3.2}$$

This action can be described in matrix terms. Let

$$\{e_i^\rho \mid \rho \in G^*, i \in I(n)\} \subset \mathbb{Z}^{nr}$$

is the usual basis of $\mathbb{Z}^{nr}$; we define the block matrix:

$$A = \left[A^1, A^2, \dots, A^r\right] \in \mathbb{Z}^{nr\times nr}$$

where each block for fixed $\rho \in G^*$ has for columns the vectors $e_1^\rho, e_2^\rho, \dots, e_n^\rho \in \mathbb{Z}^{nr\times nr}$, that is

$$A^\rho = [e_1^\rho, e_2^\rho, \dots, e_n^\rho] \in \mathbb{Z}^{nr\times n}.$$

Finally we define the block matrix:

$$B = \left[B^1, B^2, \dots, B^r\right] \in \mathbb{Z}^{r \times nr}$$

where

$$B^\rho = [e_\rho - e_{\rho\rho_1}, e_\rho - e_{\rho\rho_2}, \dots, e_\rho - e_{\rho\rho_n}] \in \mathbb{Z}^{r \times n}.$$

Each block B^ρ for a fixed $\rho \in G^*$ contains the vectors $e_\rho - e_{\rho\rho_1}, e_\rho - e_{\rho\rho_2}, \dots, e_\rho - e_{\rho\rho_n} \in \mathbb{Z}^r$ as columns where

$$\{e_\rho \mid \rho \in G^*\} \subset \mathbb{Z}^r$$

is the usual basis of $\mathbb{Z}^r$.

If we put $T_B = \mathrm{Hom}(\mathbb{Z}B, \mathbb{K}^*)$, then the set of characters of T_B is:

$$X(T_B) = {T_B}^* = \mathbb{Z}B.$$

The algebraic torus $\mathbb{T}^r$ acts on $\mathbb{A}^{nr}_{\mathbb{K}}$ by 3.2 and this induces an action of T_B on $\mathbb{A}^{nr}_{\mathbb{K}}$. Now the ideal $I \subset \mathbb{K}[z_i^\rho : \rho \in \mathrm{Irr}(G),\ i \in I(n)]$ is T_B-invariant, so that T_B acts on the affine scheme

$$Z = \mathrm{Spec}\left(\mathbb{K}[z_i^\rho : \rho \in \mathrm{Irr}(G),\ i \in I(n)]/I\right).$$

The set of rational characters of T_B is

$$X(T_B) \otimes \mathbb{Q} = \mathbb{Z}B \otimes \mathbb{Q} =: \mathbb{Q}B$$

and as usual we denote this set by $X(T_B)_{\mathbb{Q}}$. In Lemma 2.4 of [16] it is proved that the semigroup $\mathbb{N}B \subset \mathbb{Z}^r$ is isomorphic to the following sublattice of $\mathbb{Z}^r$:

$$\left\{(\theta_\rho) \in \mathbb{Z}^r \mid \textstyle\sum_{\rho \in G^*} \theta_\rho = 0\right\};$$

then $\mathbb{Q}B$ is a $(n-1)$-dimensional $\mathbb{Q}$-vector space. The GIT parameter space is defined by

$$\Theta := \mathbb{Q}B = \left\{(\theta_\rho) \in \mathbb{Q}^r \mid \textstyle\sum_{\rho \in G^*} \theta_\rho = 0\right\}.$$

Since the reductive affine algebraic group T_B acts on the affine scheme Z, for a rational character $\theta \in \Theta$ as we described in the Section 3.3, we can build the twist GIT quotient of this action. We define the moduli space of θ-semistable McKay quiver representations of dimension $(1, ..., 1)$ satisfying the relations given in (3.1) it as [16]

$$\overline{\mathcal{M}_\theta} := Z /\!\!/_\theta\, T_B.$$

If the parameter θ is generic, this quotient is the fine moduli space of θ-stable McKay quiver representations and is denoted by $\mathcal{M}_\theta$.

3.4 Quasi-coherent $\mathcal{O}_X$-algebras over a Vector Bundles

In this section we are we are looking for relation between the tautological bundle $\mathcal{R}$ and the universal G-constellation $\mathcal{U}$. To achieve this, we will explore a theorem that allows us to recover a quasi-coherent algebra at the level of ringed spaces.

Theorem 3.9 (EGA). *Let S be a scheme; for any pair $(\mathcal{A}, \mathcal{B})$, where $\mathcal{A}$ is a quasi-coherent $\mathcal{O}_S$-algebra and $\mathcal{B}$ is a quasi-coherent $\mathcal{O}_{\mathcal{A}}$-algebra, there exists a pair $(X, \mathcal{E})$ where $X = \operatorname{Spec} \mathcal{A}$ and $\mathcal{E}$ is a some quasi-coherent $\mathcal{O}_X$-module such that*

$$(\mathfrak{F}(X), \mathfrak{F}(\mathcal{E})) \cong (\mathcal{A}, \mathcal{B})$$

is a bi-isomorphism.

We denote by $\widetilde{\mathcal{B}}$ a $\mathcal{O}_X$-module $\mathcal{E}$ and is called the module associated to $\mathcal{B}$.
For a $f : (X, \mathcal{O}_X) \longrightarrow (S, \mathcal{O}_S)$ morphism of ringed spaces we denote by $\mathfrak{F}(X) = f_*\mathcal{O}_X$, and $\mathfrak{F}(\mathcal{E}) = f_*\mathcal{E}$ for any $\mathcal{E} \in \mathcal{O}_X$-mod.

The proof of this theorem is based on the definition of the following morphism. Let $f = (\psi, \theta) : (X, \mathcal{O}_X) \longrightarrow (S, \mathcal{O}_S)$ be a morphism structural of locally ringed spaces; then $\mathcal{A} = \psi_*\mathcal{O}_X$.
The morphism $g = (\psi, 1_{\mathcal{A}}) : (X, \mathcal{O}_X) \longrightarrow (S, \mathcal{A})$ of locally ringed spaces is well defined, and $\mathcal{A} \longrightarrow \psi_*\mathcal{O}_X = \mathcal{A}$ is the identity; then we let

$$\mathcal{E} = g^*\mathcal{B}.$$

3.5 Relation Between the Universal G-constellation and the Tautological Bundle

In this section we prove a result that was only hinted at in [15]:

$$(\pi, 1_{\mathcal{A}})^*\mathcal{U} = \mathcal{R}$$

where $\mathcal{U}$ is the universal G-constellation, $\mathcal{R}$ is the tautological bundle and $\lambda = (\pi, 1_{\mathcal{A}})$ is the morphism at the level of ringed spaces.

Now we fix a chamber $C \subset \Theta$ in the GIT-parameter space. We denote by $\mathcal{M}_C$ the moduli space of θ-stable G-constellations for any $\theta \in C$ and $\mathcal{R}_C$ the tautological bundle on $\mathcal{M}_C$. Let us take:

$$(\mathcal{A}, \mathcal{B}) = (\mathrm{Sym}(\mathcal{O}_{\mathcal{M}_C}^{\oplus 3}), \mathcal{R}_C);$$

for convenience we denote $\mathcal{L} = \mathcal{O}_{\mathcal{M}_C}^{\oplus 3}$ the previous pair, which satisfies the conditions of the previous theorem.
Let us first observe that for a affine space over the field of complex numbers

$$\mathbb{V}(\mathcal{O}_{\mathrm{Spec}\,\mathbb{C}}^{\oplus 3}) = \mathbb{C}^3,$$

we can take the following fiber product over $\mathrm{Spec}\,\mathbb{C}$

$$\mathbb{V}(\mathcal{O}_{\mathrm{Spec}\,\mathbb{C}}^{\oplus 3}) \underset{\mathrm{Spec}\,\mathbb{C}}{\times} \mathcal{M}_C = \mathbb{C}^3 \underset{\mathrm{Spec}\,\mathbb{C}}{\times} \mathcal{M}_C.$$

The functor $\mathbb{V}(-) := \mathrm{Spec}\,\mathrm{Sym}(-)$ is compatible with base change; let $m : \mathcal{M}_C \longrightarrow \mathbb{C}$ be a morphism of projection over a point, then:

$$\mathbb{V}(m^* \mathcal{O}_{\mathrm{Spec}\,\mathbb{C}}^{\oplus 3}) = \mathbb{V}(\mathcal{O}_{\mathrm{Spec}\,\mathbb{C}}^{\oplus 3}) \times \mathcal{M}_C,$$

and we observe that

$$\begin{aligned} m^* \mathcal{O}_{\mathrm{Spec}\,\mathbb{C}}^{\oplus 3} &= m^{-1} \mathcal{O}_{\mathrm{Spec}\,\mathbb{C}}^{\oplus 3} \underset{m^{-1}\mathcal{O}_{\mathrm{Spec}\,\mathbb{C}}}{\otimes} \mathcal{O}_{\mathcal{M}_C} \\ &= \left\{ m^{-1} \mathcal{O}_{\mathrm{Spec}\,\mathbb{C}} \underset{m^{-1}\mathcal{O}_{\mathrm{Spec}\,\mathbb{C}}}{\otimes} \mathcal{O}_{\mathcal{M}_C} \right\}^{\oplus 3} \\ &= \mathcal{O}_{\mathcal{M}_C}^{\oplus 3}. \end{aligned}$$

Then

$$\mathbb{V}(\mathcal{O}_{\mathcal{M}_C}^{\oplus 3}) = \mathbb{C}^3 \times \mathcal{M}_C$$

The projection morphism is the structure morphism for the geometric vector bundle $X = \mathrm{Spec}\,\mathcal{A}$,

$$\pi : \mathrm{Spec}\,\mathcal{A} = \mathbb{C}^3 \times \mathcal{M}_C : X \longrightarrow \mathcal{M}_C;$$

we know that

$$\pi_* \mathcal{O}_{\mathrm{Spec}\,\mathcal{A}} = \mathcal{A}.$$

We define the morphism

$$\lambda = (\pi, 1_{\mathcal{A}}) : X \longrightarrow (\mathcal{M}_C, \mathcal{A})$$

of locally ringed spaces, which it is well defined because the morphism of sheaves $\mathcal{A} \longrightarrow \pi_* \mathcal{O}_{\operatorname{Spec} \mathcal{A}} = \mathcal{A}$ is the identity.
The universal G-constellation is

$$\mathcal{U}_C = \widetilde{\mathcal{R}} = \lambda^* \mathcal{R} = \lambda^{-1} \mathcal{R} \underset{\lambda^{-1}\mathcal{A}}{\otimes} \mathcal{O}_X.$$

3.6 The Wall-Crossing Phenomenon and the Contraction Morphism

Bridgeland, King, and Reid's theorem holds for moduli spaces of θ-stable G-constellations when the parameter θ is generic; in other words

$$\Phi : D^b(\mathcal{M}_\theta) \longrightarrow D^b_G(\mathbb{C}^3)$$

is a equivalence of triangulated categories and $\mathcal{M}_\theta$ is a crepant resolution of the singularity $\mathbb{C}^3/G$.

The set of generic parameters Θ^{gen} is an open and dense subset of the GIT-parameter space Θ. Furthermore, it is a finite disjoint union of open convex polyhedral cones in Θ.

Contraction morphism. Let θ and θ' be generic parameters in the chambers C and C' respectively; we put

$$\theta_t := \frac{1}{2}(1-t)\theta + \frac{1}{2}(1+t)\theta'.$$

This parameter satisfies $\theta_t \in C'$ if $0 < t \leq 1$ and $\theta_t \in C$ if $-1 \leq t < 0$, and the point θ_0 lies on the wall W that separates the chambers C and C'.

If $f : \mathcal{M}_\theta \longrightarrow \overline{\mathcal{M}_{\theta_0}}$, is the canonical morphism, the contraction morphism is defined by

$$\operatorname{Cont}_W : \mathcal{M}_\theta \longrightarrow Y_0$$

where Y_0 is the image of the morphism f.

Classification of the Walls. We classify the wall, of the space of stability parameters according to the contraction induced by this birational morphism:

I. The wall W is of type 0 if the morphism Cont_W is an isomorphism.

II. The wall W is of type 1 if Cont_W contracts a curve onto a point.

III. The wall W is of type 3 if Cont_W contracts a divisor onto a curve.

The wall W is of type 2 if Cont_W contracts a divisor to a point, but these walls do not exist [15, Proposition 3.8].

Wall Crossing. Let G be an abelian finite subgroup of $\mathrm{SL}(3,\mathbb{C})$; if we fix a parameter θ in C and a parameter θ_0 on the wall W, then there exist non-trivial subrepresentations R_1 and R_2 of the group G sucht that:

I. $\mathrm{R} \cong R_1 \oplus R_2$.

II. There exist non-trivial θ_0-semistable G-equivariant sheaves S and Q such that: $H^0(S) \cong R_1$, $H^0(Q) \cong R_2$ and $\theta_0(S) = \theta_0(Q) = 0$.

Indeed, as the parameter θ_0 belongs to the wall W, it is not generic and then there exists a G-constellation $\mathcal{F}$ which is θ_0-semistable but not θ-stable. So there exists a non-trivial G-subsheaf $\mathcal{F}'$ of $\mathcal{F}$ such that $\theta_0(\mathcal{F}') = 0$, the global section of this subsheaf is $H^0(\mathcal{F}') \cong R_1$, and there exists another nontrivial G-subsheaf F'' of $\mathcal{F}$ with $\theta_0(F'') = 0$. By additivity of the global sections $H^0(F'') \cong R_2$.

The Unstable Locus of the Wall Crossing. The set

$$Z = \left\{ \begin{array}{c} \mathcal{F} \in \mathcal{M}_C \text{ there exists a non-trivial subsheaf} \\ S \subset \mathcal{F} \text{ such that } H^0(S) \cong R_1 \end{array} \right\}$$

has a structure of closed subscheme of $\mathcal{M}_C$. To see this we define a functor:

$$h : \Big\{\text{schemes}/\mathbb{C}\Big\}^{opp} \longrightarrow \text{Sets}$$

as

$$h(T) = \left\{ \begin{array}{c} f \in \mathrm{Hom}(T, \mathcal{M}_C) \text{ such that exists a quotient} \\ \nu : (f \times Id_{\mathbb{C}^3})^* \mathcal{U}_C \twoheadrightarrow \mathcal{Q}_T \text{ flat over } T \text{ with} \\ H^0(\mathcal{Q}_t) \cong R_2 \otimes k(t), \text{ for any } \ t \in T \end{array} \right\}.$$

This functor is representable [15, Lemma 3.10] by a closed subscheme Z called the unstable locus defined by the wall W. To prove this, the Grothendieck's Quot-scheme is used.

Symmetrically, for the adjoining chamber C' the moduli space $\mathcal{M}_{C'}$ has a closed subscheme $Z' \subset \mathcal{M}_{C'}$ given by:

$$Z' = \left\{ \begin{array}{c} \mathcal{F}' \in \mathcal{M}_{C'} \text{ where exists a non-trivial subsheaf} \\ S' \subset \mathcal{F}' \text{ such that } H^0(S') \cong R_2 \end{array} \right\}.$$

Since the functor h is representable for the chamber C there exist a universal subsheaf $\mathcal{S} \subset \mathcal{R}_C\big|_Z$ and $\mathcal{Q} = (\mathcal{R}_C\big|_Z)/\mathcal{S}$, which are locally free $\mathcal{O}_Z$-modules.

3.6.1 Wall-Crossing for Walls of Type 0 and Change of Tautological Line Bundles

Let us consider two adjacent chambers C and C' separated by a wall W of type 0. The following properties are satisfied

I. the varieties associated to the moduli of these chambers are isomorphic [15, Lemma 3.3]:

$$\mathcal{M}_C \cong \mathcal{M}_{C'}.$$

II. The unstable locus determined by the wall W is a Cartier divisor D [15, Proposition 4.1], and this divisor is a compact reduced subscheme [15, Proposition 4.4, 4.5.].

III. The change of tautological bundles induced by a wall-crossing [15, Corollary 4.3] is given by:

(a) If $\rho \subset R_1$, then

$$\mathcal{R'}_\rho = \begin{cases} \mathcal{R}_\rho & \text{if } \rho \subset R_1 \\ \mathcal{R}_\rho(-D) & \text{if } \rho \subset R_2. \end{cases}$$

(b) If $\rho \subset R_2$, then

$$\mathcal{R'}_\rho = \begin{cases} \mathcal{R}_\rho(D) & \text{if } \rho \subset R_1 \\ \mathcal{R}_\rho & \text{if } \rho \subset R_2. \end{cases}$$

3.6.2 Wall-Crossing for Walls of Type 1 and Change of Tautological Line Bundles

Let us consider two adjacent chambers C and C' separated by a wall W of type 1. The following properties are satisfied

I. the variety $\mathcal{M}_C$ is a flop of the variety $\mathcal{M}_{C'}$ [15, Proposition 6.1].

II. The unstable locus $Z \subset \mathcal{M}_C$ defined by the wall W is the curve l, where this curve is contracted by the morphism Cont_W.

III. The change of tautological bundles $\mathcal{R}'_\rho$ induced by a wall-crossing [15, Corollary 6.3] is given by a proper transformation of $\mathcal{R}_\rho$ and for any irreducible representation $\rho \in G^*$:

(a) $\rho_0 \subset R_1$ if and only if $\deg(\mathcal{R}_\rho\big|_l)$ is 0 or 1.

(b) $\rho_0 \subset R_2$ if and only if $\deg(\mathcal{R}_\rho\big|_l)$ is 0 or -1.

3.6.3 Wall-Crossing for Walls of Type 3 and Change of Tautological Line Bundles

Let us consider two adjacent chambers C and C' separated by a wall W of type 3. The following properties are satisfied

I. The varieties $\mathcal{M}_C$ and $\mathcal{M}_{C'}$ can be characterized by the blow-up of $\mathcal{M}_{C'}$ along the curve l, that is the curve that was contracted by Cont_W, which was the contraction of the divisor D. Then by the [15, Proposition 6.4]:

$$\mathcal{M}_C \cong \mathcal{M}_{C'}.$$

II. The unstable locus $Z \subset \mathcal{M}_C$ defined by the wall W is the divisor D, where this divisor is contract by the morphism Cont_W.

III. The change of tautological bundles $\mathcal{R}'_\rho$ induced by a wall-crossing [15, Corollary 6.3] is given by a proper transformation of $\mathcal{R}_\rho$ and for any irreducible representation $\rho \in G^*$:

(a) $\rho_0 \subset R_1$ if and only if $\deg(\mathcal{R}_\rho\big|_l)$ is 0 or -1; in this case

$$\mathcal{R'}_\rho = \begin{cases} \mathcal{R}_\rho(-D) & \text{if } \deg(\mathcal{R}_\rho\big|_l) = -1 \\ \mathcal{R}_\rho & \text{if } \deg(\mathcal{R}_\rho\big|_l) = 0. \end{cases}$$

(b) $\rho_0 \subset R_2$ if and only if $\deg(\mathcal{R}_\rho\big|_l)$ is 0 or 1; in this case

$$\mathcal{R'}_\rho = \begin{cases} \mathcal{R}_\rho & \text{if } \deg(\mathcal{R}_\rho\big|_l) = 0 \\ \mathcal{R}_\rho(D) & \text{if } \deg(\mathcal{R}_\rho\big|_l) = 1. \end{cases}$$

Chapter 4

Moduli of G-Constellations as Fourier-Mukai Partners

In this chapter we study the Fourier-Mukai transform between derived categories of crepant resolutions of $\mathbb{C}^3/\mathbb{Z}_4$. In the first section, we establish a expression for the Fourier-Mukai transform as a composition of two functors of this kind. In the second section, we study the Fourier-Mukai transforms induced by spherical objects; this serves to determine in the following section in the following section the Fourier-Mukai transform for Fourier-Mukai partners associated with two chambers separated by a type 0 wall. In sections 4 and 5 we explore the same scenario for type 1 and type 3 walls, respectively. In the last section we prove a compatibility theorem between the K-theoretic and cohomological versions of the Fourier-Mukai transform. One needs characteristic classes and the Grothendieck-Riemann-Roch Theorem in order to establish a relation between the two versions.

4.1 Fourier-Mukai Transforms Between Derived Categories of Crepant Resolutions

In this section we consider two spaces moduli of θ-stable G-constellations, denoted $\mathcal{M}_C$ and $\mathcal{M}_{C'}$, corresponding to chambers C and C' separated by a wall W in the GIT-parameter space Θ. We know that, for walls of type 0 and 3, these moduli spaces are isomorphic as varieties. Is important to recall that tautological line bundles might change under wall-crossing. Let $\Phi_1 : D^b(\mathcal{M}_1) \longrightarrow D^b_G(\mathbb{C}^3)$ and $\Phi_2 : D^b(\mathcal{M}_2) \longrightarrow D^b_G(\mathbb{C}^3)$ be the Fourier-Mukai transforms associated to moduli spaces $\mathcal{M}_C$ and $\mathcal{M}_{C'}$ respectively; we can take the inverse of Φ_2 to obtain the Fourier-Mukai $\Phi_2^{-1} : D^b_G(\mathbb{C}^3) \longrightarrow D^b(\mathcal{M}_2)$ with kernel:

$$\mathrm{Ker}(\Phi_2^{-1}) = \mathcal{U}_2^{\vee}[3] = R\mathcal{H}om(\mathcal{U}_2[3], \mathcal{O}_{\mathcal{M}_{C'}\times\mathbb{C}^3}).$$

The moduli spaces $\mathcal{M}_C$ and $\mathcal{M}_{C'}$ are Fourier-Mukai partners; we see this equivalence in the following commutative diagram:

$$\begin{tikzcd} D^b(\mathcal{M}_1) \arrow[r, "\Phi_1"] \arrow[dr, "\Psi"'] & D^b_G(\mathbb{C}^3) \arrow[d, "\Phi_2^{-1}"] & D^b(\mathcal{M}_2); \arrow[l, "\Phi_2"'] \\ & D^b(\mathcal{M}_2) & \end{tikzcd}$$

the functor $\Psi := \Phi_2^{-1} \circ \Phi_1$ is a Fourier-Mukai between the derived categories of $\mathcal{M}_C$ and $\mathcal{M}_{C'}$. Using the following commutative diagram, we can see how to obtain the kernel:

$$\begin{tikzcd} & \mathcal{M}_1 \times \mathbb{C}^3 \times \mathcal{M}_2 \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{23}"'] \arrow[dr, "\pi_{13}"] & \\ \mathcal{M}_1 \times \mathbb{C}^3 & \mathbb{C}^3 \times \mathcal{M}_2 & \mathcal{M}_1 \times \mathcal{M}_2 \end{tikzcd}$$

if we make the convolution of the corresponding kernels

$$\mathrm{Ker}(\Psi) = \mathrm{Ker}(\Phi_2^{-1}) * \mathrm{Ker}(\Phi_1) = R\pi_{13}\left(\pi_{23}^*\mathcal{U}_2^{\vee}[3] \overset{L}{\otimes} \pi_{12}^*\mathcal{U}_1\right)$$

we obtain an object of the derived category $D^b(\mathcal{M}_1 \times \mathcal{M}_2)$.

4.2 K-Theoretic Fourier-Mukai Transform and Cohomological-Theoretic Fourier-Mukai

In this section we prove a compatibility theorem between the K-theoretic and cohomological versions of the Fourier-Mukai transform. The proof relies on the Grothendieck-Riemann-Roch theorem. We use this compatibility in the next chapter to determine the cohomological version of the Fourier-Mukai transform between Fourier-Mukai partners for the case of $\mathbb{C}^3/\mathbb{Z}_4$.

Theorem 4.1 ([30, Corollary 5.29]). *Let $e \in K(X\times Y)$. For any $a \in K(X)$ there an equality in the cohomology ring $H^*(Y, \mathbb{Q})$:*

$$\Phi^H_{v(e)}(Ch(a) \,.\, \sqrt{td(X)}) = Ch(\Phi^K_e(a)) \,.\, \sqrt{td(Y)}\,.$$

This implies that the following diagram is commutative:

$$\begin{tikzcd}
K(X) \arrow[r, "\Phi_e^K"] \arrow[d, "v"'] & K(Y) \arrow[d, "v"] \\
H^*(X,\mathbb{Q}) \arrow[r, "\Phi_{v(e)}^H"'] & H^*(Y,\mathbb{Q}).
\end{tikzcd}$$

Proof. The commutativity of the previous diagram can be deduced from the commutativity of the three squares in the following diagram:

$$\begin{tikzcd}
K(X) \arrow[r, "q^*"] \arrow[d, "v"'] & K(X\times Y) \arrow[r, ".e"] \arrow[d, "v(-)\left(p^*\sqrt{td(Y)}\right)^{-1}"] & K(X\times Y) \arrow[r, "p!"] \arrow[d, "v(-)q^*\sqrt{td(X)}"] & K(Y) \arrow[d, "v"] \\
H^*(X,\mathbb{Q}) \arrow[r, "q^*"'] & H^*(X\times Y,\mathbb{Q}) \arrow[r, ".v(e)"'] & H^*(X\times Y,\mathbb{Q}) \arrow[r, "p_*"'] & H^*(Y,\mathbb{Q})
\end{tikzcd}$$

For simplicity, for any smooth algebraic variety W we will denote

$$\lambda_W = \sqrt{td(W)}.$$

The commutativity of the first square follows from the following steps:

$$\begin{tikzcd}
K(X) \arrow[r, "q^*"] \arrow[d, "v"'] & K(X\times Y) \arrow[d, "v(-)\,.\,\left(p^*\sqrt{td(Y)}\right)^{-1}"] \\
H^*(X,\mathbb{Q}) \arrow[r, "q^*"'] & H^*(X\times Y,\mathbb{Q})
\end{tikzcd}$$

for any $a \in K(X)$ the composition of the arrows in the upper part of the diagram

$$\begin{aligned}
v(q^*a)\,.\,(p^*\lambda_Y)^{-1} &= Ch(q^*a)\,.\,\lambda_{X\times Y}\,.\,(p^*\lambda_Y)^{-1} \\
&= Ch(q^*a)\,.\,p^*\lambda_Y\,.\,q^*\lambda_X\,.\,(p^*\lambda_Y)^{-1} = Ch(q^*a)\,.\,q^*\lambda_X.
\end{aligned}$$

In the lower part

$$q^*(v(a)) = q^*(Ch(a)\,.\,\lambda_X) = q^*(Ch(a))\,.\,q^*\lambda_X$$

the compatibility between the Chern character and pullbacks

$$Ch \circ q^* = q^* \circ Ch,$$

gives us the identity.

For the second square

$$\begin{tikzcd}
K(X \times Y) \arrow[r, ".e"] \arrow[d, "v(-)\left(p^*\sqrt{td(Y)}\right)^{-1}"'] & K(X \times Y) \arrow[d, "v(-)q^*\sqrt{td(X)}"] \\
H^*(X \times Y, \mathbb{Q}) \arrow[r, ".v(e)"'] & H^*(X \times Y, \mathbb{Q})
\end{tikzcd}$$

let $d \in K(X \times Y)$ be an element of the Grothendieck group

$$\begin{aligned}
v(d)\,.\,v(e)\,.\,(p^*\lambda_Y)^{-1} &= Ch(d)\,.\,Ch(e)\,.\,(\lambda_{X\times Y})^2\,(p^*\lambda_Y)^{-1} \\
&= Ch(d)\,.\,Ch(e)\,.\,\lambda_{X\times Y}\,.\,p^*\lambda_Y\,.\,q^*\lambda_X\,.\,(p^*\lambda_Y)^{-1} \\
&= Ch(d)\,.\,Ch(e)\,.\,\lambda_{X\times Y}\,.\,q^*\lambda_X,
\end{aligned}$$

then

$$\begin{aligned}
v(d\,.\,e)\,.\,q^*\lambda_X &= Ch(d)\,.\,Ch(e)\,.\,\lambda_{X\times Y}\,.\,q^*\lambda_X \\
&= Ch(d)\,.\,Ch(e)\,.\,\lambda_{X\times Y}\,.\,q^*\lambda_X.
\end{aligned}$$

We have used the fact that the Chern character satisfies:

$$Ch(d\,.\,e) = Ch(d)\,.\,Ch(e).$$

Indeed in Lemma 42.45.3 [49] the compatibilities between the tensor product and the Chern character is discussed, and this equality was proved using finite locally free sheaves. Since every complex in the bounded derived category of a smooth variety has a resolution by locally free sheaves of finite length we can apply this result proves the commutativity of the second diagram.

Finally, for the third diagram

$$\begin{tikzcd}
K(X \times Y) \arrow[r, "p!"] \arrow[d, "v(-)q^*\sqrt{td(X)}"'] & K(Y) \arrow[d, "v"] \\
H^*(X \times Y, \mathbb{Q}) \arrow[r, "p_*"'] & H^*(Y, \mathbb{Q}),
\end{tikzcd}$$

let $d \in K(X \times Y)$ be an element of the Grothendieck group

$$v(p!d) = Ch(p!d)\,.\,\lambda_Y = \left(Ch(p!d)\,.\,\lambda_Y^2\right)\,.\,\lambda_Y^{-1}.$$

As the varieties X and Y are smooth and the morphism p is proper, we can apply the Grothendieck-Riemann-Roch [18, Theorem 5.2]

$$p_*(Ch(d)\,.\,(\lambda_{X\times Y})^2) = Ch(p!d)\,.\,\lambda_Y^2\,.$$

Then

$$\begin{aligned}\left(Ch(p!d)\,.\,\lambda_Y^2\right)\,.\,\lambda_Y^{-1} &= p_*(Ch(d)\,.\,(\lambda_{X\times Y})^2)\,.\,\lambda_Y^{-1}\\ &= p_*(Ch(d)\,.\,(\lambda_{X\times Y})^2\,.\,p^*\lambda_Y^{-1})\\ &= p_*(Ch(d)\,.\,\lambda_{X\times Y}\,.\,\lambda_{X\times Y}\,.\,p^*\lambda_Y^{-1})\\ &= p_*(Ch(d)\,.\,\lambda_{X\times Y}\,.\,p^*\lambda_Y\,.\,q^*\lambda_X\,.\,p^*\lambda_Y^{-1})\\ &= p_*(Ch(d)\,.\,\lambda_{X\times Y}\,.\,q^*\lambda_X)\\ &= p_*(v(d)\,.\,q^*\lambda_X);\end{aligned}$$

this concludes the proof. □

4.3 Fourier-Mukai Transforms Induced by Spherical Objects

In this section, we study how one can induce a Fourier-Mukai transform using certain objects in the derived category, called spherical objects. First, we introduce the concept of a spherical object and see how it can be defined in terms of distinguished triangles according to the triangulated structure of the derived category.

Definition 4.2 (Spherical Objects)**.** Let Y be a nonsingular quasi-projective variety of dimension n, and let $D^b_c(Y)$ be the full subcategory of $D^b(Y)$ whose objects have compact support. An object $\mathcal{E}^\bullet$ is called *spherical* if:

i. $\mathrm{Hom}^k_{D^b(Y)}(\mathcal{E}^\bullet, \mathcal{E}^\bullet) = \begin{cases} \mathbb{C} & \text{if } \ k = 0, n \\ 0 & \text{otherwise,} \end{cases}$

   and

ii. $\mathcal{E}^\bullet \otimes \omega_Y \cong \mathcal{E}^\bullet$.

We denote by $\mathrm{Sph}(D^b(Y))$ the set of all spherical objects.

To any $\mathcal{E}^\bullet \in \mathrm{Sph}(D^b(Y))$ we can associate an object $\mathcal{P}_{\mathcal{E}^\bullet} \in D^b(Y \times Y)$ defined by the cone of the morphism:

$$\pi_1^*\mathcal{E}^{\bullet\vee} \overset{L}{\otimes} \pi_2^*\mathcal{E}^\bullet \longrightarrow \mathcal{O}_\triangle \ ,$$

where the morphisms π_1 and π_2 are the projections onto the factors of $Y \times Y$. The embedding $X \overset{\iota}{\hookrightarrow} \triangle \subset Y \times Y$ enables us to consider $\mathcal{O}_\triangle$ as a coherent module on $Y \times Y$.
The morphism above can be constructed by the composition of the following morphisms (see [30]):

$$\pi_1^*\mathcal{E}^{\bullet\vee} \overset{L}{\otimes} \pi_2^*\mathcal{E}^\bullet \longrightarrow R\iota_*\iota^*(\pi_1^*\mathcal{E}^{\bullet\vee} \overset{L}{\otimes} \pi_2^*\mathcal{E}^\bullet) \ = R\iota_*(\iota^*\pi_1^*\mathcal{E}^{\bullet\vee} \overset{L}{\otimes} \iota^*\pi_2^*\mathcal{E}^\bullet)$$

in the last identity, the compatibility between the tensor and the inverse image in the derived category was used. We compose this morphism with the morphism $R\iota_*(\mathrm{tr}_{\mathcal{E}^\bullet})$:

$$\pi_1^*\mathcal{E}^{\bullet\vee} \overset{L}{\otimes} \pi_2^*\mathcal{E}^\bullet \longrightarrow R\iota_*(\mathcal{E}^{\bullet\vee} \overset{L}{\otimes} \mathcal{E}^\bullet) \longrightarrow R\iota_*(\mathcal{O}_X) = \mathcal{O}_\triangle.$$

We obtain

$$\pi_1^*\mathcal{E}^{\bullet\vee} \overset{L}{\otimes} \pi_2^*\mathcal{E}^\bullet \longrightarrow R\iota_*(\mathcal{O}_X) = \mathcal{O}_\triangle.$$

Definition 4.3 (Spherical Functors)**.** Let $\mathcal{E}^\bullet$ be a spherical object; the *spherical twist* along of $\mathcal{E}^\bullet$ is the functor defined by

$$T_{\mathcal{P}_{\mathcal{E}^\bullet}} = \Phi^{\mathcal{P}_{\mathcal{E}^\bullet}} : D^b(Y) \longrightarrow D^b(Y).$$

It is useful to interpret these functors in terms of the triangulated structure of $D^b(Y)$. We complete the distinguished triangle based on the cone that defines $\mathcal{P}_{\mathcal{E}^\bullet}$, that is

$$\pi_1^*\mathcal{E}^{\bullet\vee} \overset{L}{\otimes} \pi_2^*\mathcal{E}^\bullet \longrightarrow \mathcal{O}_\triangle \longrightarrow \mathcal{P}_{\mathcal{E}^\bullet} \longrightarrow *;$$

let us take $\mathcal{F}^\bullet \in D^b(Y)$ and tensor the above distinguished triangle by $\pi_1^*\mathcal{F}^\bullet$

$$\pi_1^*\mathcal{E}^{\bullet\vee} \overset{L}{\otimes} \pi_2^*\mathcal{E}^\bullet \overset{L}{\otimes} \pi_1^*\mathcal{F}^\bullet \longrightarrow \mathcal{O}_\triangle \overset{L}{\otimes} \pi_1^*\mathcal{F}^\bullet \longrightarrow \mathcal{P}_{\mathcal{E}} \overset{L}{\otimes} \pi_1^*\mathcal{F}^\bullet$$

$$\longrightarrow \pi_1^*\mathcal{E}^{\bullet\vee} \overset{L}{\otimes} \pi_2^*\mathcal{E}^\bullet \overset{L}{\otimes} \pi_1^*\mathcal{F}^\bullet[1].$$

We can apply the functor $R\pi_{2*}$, which is an exact functor between the respective triangulated categories

$$R\pi_{2*}(\pi_1^*\mathcal{E}^{\bullet\vee} \overset{L}{\otimes} \pi_2^*\mathcal{E}^\bullet \overset{L}{\otimes} \pi_1^*\mathcal{F}^\bullet) \longrightarrow R\pi_{2*}(\mathcal{O}_\triangle \overset{L}{\otimes} \pi_1^*\mathcal{F}^\bullet) \longrightarrow R\pi_{2*}(\mathcal{P}_{\mathcal{E}^\bullet} \overset{L}{\otimes} \pi_1^*\mathcal{F}^\bullet)$$
$$\longrightarrow R\pi_{2*}(\pi_1^*\mathcal{E}^{\bullet\vee} \overset{L}{\otimes} \pi_2^*\mathcal{E}^\bullet \overset{L}{\otimes} \pi_1^*\mathcal{F}^\bullet[1]).$$

Now let us use the projection formula [2, Proposition A.83.]:

$$R\pi_{2*}(\pi_1^*\mathcal{E}^{\bullet\vee} \overset{L}{\otimes} \pi_1^*\mathcal{F}^\bullet) \overset{L}{\otimes} \mathcal{E}^\bullet \longrightarrow R\pi_{2*}(\mathcal{O}_\triangle \overset{L}{\otimes} \pi_1^*\mathcal{F}^\bullet) \longrightarrow R\pi_{2*}(\mathcal{P}_{\mathcal{E}^\bullet} \overset{L}{\otimes} \pi_1^*\mathcal{F}^\bullet)$$
$$\longrightarrow R\pi_{2*}(\pi_1^*\mathcal{E}^{\bullet\vee} \overset{L}{\otimes} \pi_1^*\mathcal{F}^\bullet) \overset{L}{\otimes} \mathcal{E}^\bullet[1],$$

identifying π_1 and π_2:

$$R\pi_{1*}(\pi_1^*\mathcal{E}^{\bullet\vee} \overset{L}{\otimes} \pi_1^*\mathcal{F}^\bullet) \overset{L}{\otimes} \mathcal{E}^\bullet \longrightarrow R\pi_{1*}\mathcal{O}_\triangle \overset{L}{\otimes} R\pi_{1*}\pi_1^*\mathcal{F}^\bullet \longrightarrow R\pi_{2*}(\mathcal{P}_{\mathcal{E}^\bullet} \overset{L}{\otimes} \pi_1^*\mathcal{F}^\bullet)$$
$$\longrightarrow R\pi_{1*}(\pi_1^*\mathcal{E}^{\bullet\vee} \overset{L}{\otimes} \pi_1^*\mathcal{F}^\bullet) \overset{L}{\otimes} \mathcal{E}^\bullet[1].$$

Finally using the base change in derived category

$$R\pi_{1*}\pi_1^*\mathcal{E}^{\bullet\vee} \overset{L}{\otimes} R\pi_{1*}\pi_1^*\mathcal{F}^\bullet \overset{L}{\otimes} \mathcal{E}^\bullet \longrightarrow \mathcal{F}^\bullet \longrightarrow R\pi_{2*}(\mathcal{P}_{\mathcal{E}^\bullet} \overset{L}{\otimes} \pi_1^*\mathcal{F}^\bullet)$$
$$\longrightarrow R\pi_{1*}\pi_1^*\mathcal{E}^{\bullet\vee} \overset{L}{\otimes} R\pi_{1*}\pi_1^*\mathcal{F}^\bullet \overset{L}{\otimes} \mathcal{E}^\bullet[1],$$

we obtain

$$\mathcal{E}^{\bullet\vee} \overset{L}{\otimes} \mathcal{E}^\bullet \overset{L}{\otimes} \mathcal{F}^\bullet \longrightarrow \mathcal{F}^\bullet \longrightarrow T_{\mathcal{P}_{\mathcal{E}^\bullet}}(\mathcal{F}^\bullet) \longrightarrow \mathcal{E}^{\bullet\vee} \overset{L}{\otimes} \mathcal{E}^\bullet \overset{L}{\otimes} \mathcal{F}^\bullet[1].$$

The following isomorphism is due to [48, Lemma 3.2.]

$$R\mathrm{Hom}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \underset{\mathbb{C}}{\overset{L}{\otimes}} \mathcal{E}^\bullet \longrightarrow \mathcal{F}^\bullet \longrightarrow T_{\mathcal{P}_{\mathcal{E}^\bullet}}(\mathcal{F}^\bullet) \longrightarrow R\mathrm{Hom}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \underset{\mathbb{C}}{\overset{L}{\otimes}} \mathcal{E}^\bullet[1]; \tag{4.1}$$

this is the definition of spherical twist given in [15, Definition 7.1].

4.4 Fourier-Mukai Partners for Walls of Type 0

In this section. We consider chambers separated by walls of type 0. In this case, we can identify the Fourier-Mukai transform $\Psi = \Phi_{C'}^{-1} \circ \Phi_C$ using spherical twists.

Let C and C' be chambers in the GIT-parameter space separated by a wall W of type 0; the functor $\Psi := \Phi_{C'}^{-1} \circ \Phi_C : D^b(\mathcal{M}_C) \longrightarrow D^b(\mathcal{M}_{C'})$, as discussed in Section 4.1, is again a functor of Fourier-Mukai type, i.e. the moduli spaces associated to chambers are Fourier-Mukai partners. Craw and Ishii in [15, Section 7] proved that this functor is related to twists of spherical objects that define the wall of type 0.

Theorem 4.4 ([15, Proposition 7.3])**.** *Under the aforementioned hypotheses, there are two cases:*

I. If the divisor D parametrizes a rigid quotient $\mathcal{Q}$, then there exists an irreducible representation $\rho \in G^*$ and $\rho \in \mathcal{Q}$, and

i. If $\mathcal{R}_\rho\big|_D \ncong \mathcal{O}_D$, the autoequivalence is given by the isomorphism of functors

$$\Psi \cong T_{\mathcal{E}},$$

where $\mathcal{E} = \mathcal{R}_\rho^{-1} \otimes \omega_D$.

ii. If $\mathcal{R}_\rho\big|_D \cong \mathcal{O}_D$, then

$$\Psi \cong \mathcal{O}_Y(-D) \otimes T_{\omega_D}(-),$$

where $Y = \mathcal{M}_C$.

II. If the divisor D parametrizes a rigid subsheaf $\mathcal{S}$, then there exists an irreducible representation $\sigma \in G^*$ and $\sigma \in \mathcal{Q}$, and

i. If $\mathcal{R}_\sigma\big|_D \ncong \mathcal{O}_D$, then

$$\Psi \cong T'_{\mathcal{E}},$$

where $\mathcal{E} = \mathcal{R}_\sigma^{-1}\big|_D$.

ii. If $\mathcal{R}_\sigma\big|_D \cong \mathcal{O}_D$, then

$$\Psi \cong T'_{\omega_D}(\mathcal{O}_Y(D) \otimes -),$$

where $Y = \mathcal{M}_C$.

Proof. For the first case, let us observe that $\mathcal{E} = \mathcal{R}_\rho^{-1} \otimes \omega_D \in \mathrm{Sph}(D^b(\mathcal{M}_C))$ and the K-theory of $\mathcal{M}_C$ is generated by the set of the inverses of the tautological line bundles:

$$\left\{\mathcal{R}_\rho^{-1}, \rho \in G^*\right\}.$$

So it is only necessary to prove that

$$\Psi(\mathcal{R}_\sigma^{-1}) \cong T_{\mathcal{E}}(\mathcal{R}_\sigma^{-1}), \text{ for any } \sigma \in G^*. \tag{4.2}$$

The functor Ψ is an equivalence, then $\Psi(\mathcal{R}_\sigma^{-1}) = \mathcal{R}_\sigma^{-1}$. The wall-crossing induces a relation between the tautological line bundles of the chambers separated by a wall of type 0 (Section 3.6.1, III):

$$(\mathcal{R}'_\sigma)^{-1} = \begin{cases} \mathcal{R}_\sigma^{-1} & \text{if } \sigma \nsubseteq \mathcal{Q} \\ \mathcal{R}_\sigma^{-1}(D) & \text{if } \sigma \subseteq \mathcal{Q}. \end{cases}$$

First we compute the right-hand side of (4.2). As we saw in the previous section, this object is defined through distinguished triangles. Therefore, let us calculate the first element on the left-hand side in the distinguished triangle (4.1):

$$\begin{aligned} R\mathrm{Hom}(\mathcal{E}, \mathcal{R}_\sigma^{-1}) &= R\mathrm{Hom}(\mathcal{R}_\rho^{-1} \otimes \omega_D, \mathcal{R}_\sigma^{-1}) = R\Gamma\left(\left(\mathcal{R}_\rho^{-1} \otimes \omega_D\right)^\vee \otimes \mathcal{R}_\sigma^{-1}\right) \\ &= R\Gamma(\mathcal{R}_\rho \otimes \mathcal{R}_\sigma^{-1} \otimes \mathcal{O}_D[-1]). \end{aligned} \tag{4.3}$$

If the irreducible representation σ satisfies $\sigma \nsubseteq \mathcal{Q}$, then $(\mathcal{R}'_\sigma)^{-1} = \mathcal{R}_\sigma^{-1}$ by Proposition 5.5 in [15]

$$H^i(\mathcal{R}_\rho \otimes \mathcal{R}_\sigma^{-1}\big|_D) = 0,$$

for all $i \in \mathbb{Z}$,

$$R\mathrm{Hom}(\mathcal{E}, \mathcal{R}_\sigma^{-1}) \cong 0.$$

The distinguished triangle that defines $T_{\mathcal{E}}$ is:

$$0 \longrightarrow \mathcal{R}_\sigma^{-1} \longrightarrow T_{\mathcal{E}}(\mathcal{R}_\sigma^{-1}) \longrightarrow 0,$$

and thus we obtain the isomorphism.

If $\sigma \subseteq \mathcal{Q}$ then $(\mathcal{R}'_\sigma)^{-1}\big|_D \cong \mathcal{R}_\rho^{-1}\big|_D$, this is due to the rigidity of the subsheaf $\mathcal{Q}$, in the equation (4.3), we have:

$$\begin{aligned} R\mathrm{Hom}(\mathcal{E}, \mathcal{R}_\sigma^{-1}) &\cong R\Gamma(\mathcal{R}_\rho \otimes \mathcal{R}_\sigma^{-1} \otimes \mathcal{O}_D[-1]) \\ &\cong R\Gamma(\mathcal{R}_\rho^{-1}\big|_D \otimes \mathcal{R}_\sigma^{-1}\big|_D \otimes \mathcal{O}_D[-1]) \\ &\cong R\Gamma(\mathcal{O}_D[-1]). \end{aligned}$$

The cohomology $H^i(\mathcal{O}_D[-1]) \cong 0$, $i = 1, 2$; for that

$$R\Gamma(\mathcal{O}_D[-1]) \cong \mathbb{C}[-1].$$

The object $T_{\mathcal{E}}(\mathcal{R}_\sigma^{-1})$ is the cone of the morphism

$$\mathbb{C}[-1] \otimes \mathcal{E} \xrightarrow[\varpi]{} \mathcal{R}_\sigma^{-1},$$

This sheaf is therefore represented by a nontrivial class in the Ext group:

$$\begin{aligned} \operatorname{Ext}^1(\mathcal{R}_\sigma^{-1} \otimes \omega_D, \mathcal{R}_\sigma^{-1}) &\cong \operatorname{Ext}^1(\omega_D, \mathcal{R}_\sigma \otimes \mathcal{R}_\sigma^{-1}) \\ &= \operatorname{Ext}^1(\omega_D, \mathcal{O}_Y) \cong H^2(\omega_D)^\vee \\ &\cong H^0(\mathcal{O}_D) \cong \mathbb{C}, \end{aligned}$$

and then $T_{\mathcal{E}}(\mathcal{R}_\sigma^{-1}) \cong \mathcal{R}_\sigma^{-1}(D)$.

To establish the functoriality of the isomorphism, we observe that the crepant resolutions

$$\tau_1 : \mathcal{M}_C \longrightarrow \mathbb{C}^3/G, \qquad \tau_2 : \mathcal{M}_{C'} \longrightarrow \mathbb{C}^3/G$$

satisfy $\mathcal{M}_C \setminus Z \cong \mathcal{M}_{C'} \setminus Z'$ and $\tau_1^{-1}(0) \subseteq Z$, $\tau_2^{-1}(0) \subseteq Z'$, where Z and Z' denote the unstable loci in the respectively moduli spaces, so that

$$\mathcal{M}_C \setminus \tau_1^{-1}(0) \cong \mathcal{M}_{C'} \setminus \tau_2^{-1}(0).$$

This isomorphism induces a natural equivalence between the derived categories $D^b(\mathcal{M}_C \setminus \tau_1^{-1}(0)) \cong D^b(\mathcal{M}_{C'} \setminus \tau_2^{-1}(0))$. The restrictions of the Fourier-Mukai transforms

$$\widetilde{\Phi_C} : D^b(\mathcal{M}_C \setminus \tau_1^{-1}(0)) \longrightarrow D^b_G(\mathbb{C}^3 \setminus \{0\}),$$

$$\widetilde{\Phi_{C'}^{-1}} : D^b_G(\mathbb{C}^3 \setminus \{0\}) \longrightarrow D^b(\mathcal{M}_{C'} \setminus \tau_2^{-1}(0))$$

depend on the restricted tautological bundles $\mathcal{R}\big|_{\mathcal{M}_C \setminus \tau_1^{-1}(0)}$ and $\mathcal{R}'\big|_{\mathcal{M}_{C'} \setminus \tau_2^{-1}(0)}$, and they are isomorphic. More precisely, they depend on the restricted universal G-constellations, $\mathcal{U}\big|_{\mathcal{M}_C \setminus \tau_1^{-1}(0) \times \mathbb{C}^3 \setminus \{0\}}$ and $\mathcal{U}'\big|_{\mathcal{M}_{C'} \setminus \tau_1^{-1}(0) \times \mathbb{C}^3 \setminus \{0\}}$ respectively.
Then the restriction of the Fourier-Mukai transform Ψ

$$\widetilde{\Psi} := \widetilde{\Phi_{C'}^{-1}} \circ \widetilde{\Phi_C} : D^b(\mathcal{M}_C \setminus \tau_1^{-1}(0)) \longrightarrow D^b(\mathcal{M}_{C'} \setminus \tau_2^{-1}(0))$$

makes the upper part of the following diagram commute

$$\begin{tikzcd}
D^b(\mathcal{M}_C) \arrow[rr, "\Psi"] \arrow[d, "rest_1"'] & & D^b(\mathcal{M}_{C'}) \arrow[d, "rest_2"] \\
D^b(\mathcal{M}_C \setminus \tau_1^{-1}(0)) \arrow[rr, "\simeq", "nat"'] & & D^b(\mathcal{M}_{C'} \setminus \tau_2^{-1}(0)) \\
D^b(\mathcal{M}_C) \arrow[rr, "T_{\mathcal{E}}"'] \arrow[u, "rest_1"] & & D^b(\mathcal{M}_{C'}). \arrow[u, "rest_2"']
\end{tikzcd}$$

For the commutativity of the lower part of the diagram, let us note that the restriction of the spherical twist $T_{\mathcal{E}}$ on $D^b(\mathcal{M}_C \setminus \tau_1^{-1}(0))$ is an identity functor, indeed $\mathrm{Supp}(\mathcal{E}) \subset D \subset \tau_1^{-1}(0)$. All this proves the functoriality. $\square$

4.5 Fourier-Mukai Partners induced by Flops, Walls of Type 1

We consider, two moduli spaces of θ-stable G-constellations, $\mathcal{M}_C$ and $\mathcal{M}_{C'}$, where the chambers C and C' are separated by a wall of type 1. For chambers separated by walls of type 1, the contraction morphism by definition contracts a curve l to a point. The unstable locus $\mathcal{Z}$ of the wall crossing is the contracted curve l. In the section Section 3.6.2 I, we see that $\mathcal{M}_C$ is a flop of $\mathcal{M}_{C'}$, the curve l is a copy of $\mathbb{P}^1$, and $p : \widetilde{M} \longrightarrow \mathcal{M}_{C'}$ is the blow-up of $\mathcal{M}_C$ along the curve l; the exceptional locus is $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $q : \widetilde{M} \longrightarrow \mathcal{M}_C$ is a contraction.
So that we are under the hypotheses of Bondal and Orlov's theorem [6, Theorem 3.6] the moduli spaces $\mathcal{M}_C$ and $\mathcal{M}_{C'}$, are Fourier-Mukai partners, i.e. their have bounded derived categories that are equivalent, however the tautological bundles vary. We can visualize the flop between these moduli

spaces in the following diagram:

$$\begin{tikzcd}
 & & E \cong \mathbb{P}^1 \times \mathbb{P}^1 \arrow[d, hook] \arrow[ddll, "\pi_1"'] \arrow[ddrr, "\pi_2"] & & \\
 & & \widetilde{M} \arrow[dl, "p"'] \arrow[dr, "q"] & & \\
\mathbb{P}^1 \arrow[r, hook] & \mathcal{M}_C \arrow[rr, dashed] & & \mathcal{M}_{C'} & \mathbb{P}^1 \arrow[l, hook']
\end{tikzcd}$$

Theorem 4.5 ([15, Proposition 7.4])**.** *For a wall crossing of type* 1*, the changes in the tautological line bundles are as follows (Section 3.6.2, III): (falta)*

I. For the case of $(+1)$

$$\Psi \cong Rq_* Lp^*.$$

II. For the case of (-1)

$$\Psi \cong Rq_* \left(\mathcal{O}(E) \otimes Lp^*(-)\right).$$

These Fourier-Mukai transforms are inverses of each other according to the Bondal-Orlov theorem.

4.6 Fourier-Mukai Partners induced by Blow-ups, Walls of Type 3

Now, let us consider moduli spaces of θ-stable G-constellations $\mathcal{M}_C$ and $\mathcal{M}_{C'}$ where C and C' are chambers separated by a wall of type 3.

For walls of type 3, the contraction morphism contracts a divisor D to a curve $l \subset \overline{\mathcal{M}_{\theta_0}} =: Y_0$. In this case, the exceptional locus is isomorphic to divisor $Z \cong D$, the varieties $\mathcal{M}_C$ and $\mathcal{M}_{C'}$ are isomorphic and we denote $Y = \mathcal{M}_C = \mathcal{M}_{C'}$. This situation can be depicted in the following commutative diagram:

$$\begin{tikzcd}
Y \times_{Y_0} Y \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d] \\
Y \arrow[r] & Y_0
\end{tikzcd}$$

where $Y \longrightarrow Y_0$ is the contraction morphism.

Theorem 4.6 ([15, Proposition 7.5])**.** *For a wall crossing of type* 3*, the changes in the tautological line bundles are as follows (Section 3.6.3, III):*

I. *For the case of* (-1)

$$\Psi \cong Rq_* Lp^*.$$

II. *For the case of* $(+1)$

$$\Psi \cong Rq_* \left(\omega_{Y \times_{Y_0} Y} \otimes Lp^*(-) \right).$$

Chapter 5

Cohomological Fourier-Mukai Transform between Crepant Resolutions of $\mathbb{C}^3/\mathbb{Z}_4$

In this chapter, as an example of the developed theory, we will study the case of the singularity $\mathbb{C}^3/\mathbb{Z}_4$ following [13]. Using Toric Geometry, we can obtain the crepant resolution. Then, we describe the GIT-parameter space and provide detailed insights into the intersections between chambers and the corresponding walls. Finally, we derive the cohomological version of the Fourier-Mukai transform for Fourier-Mukai partners.

5.1 Toric Geometry of $\mathbb{C}^3/\mathbb{Z}_4$

The authors of [13] study the toric structure of the singularity $\mathbb{C}^3/\mathbb{Z}_4$. For an introduction to toric geometry techniques see [14, Chapter 3].

The action of $\mathbb{Z}_4$ on $\mathbb{C}^3$ is given by:

$$(x, y, z) \mapsto (\omega x, \omega y, \omega^2 z)$$

where $w^4 = 1$. The singularity is denoted by $Y_0 = \mathbb{C}^3/\mathbb{Z}_4$ and the singular locus is the z-axis and the origin. Let $\{e^i\}$ the usual basis of $\mathbb{R}^3$, and the corresponding dual basis is denoted by $\{\epsilon^i\}$. The lattice M is generated by

$\{u^1, u^2, u^3\}$, where

$$u^1 = \epsilon^1 - \epsilon^2, u^2 = 2\epsilon^2 - \epsilon^3, u^3 = 2\epsilon^3$$

and the dual lattice N is generated by $\{w_1, w_2, w_3\}$ where

$$w_1 = e_1, w_2 = \frac{1}{2}(e_1 + e_2), w_3 = \frac{1}{4}(e_1 + e_2 + 2e_3).$$

Inverting these last relations we obtain the rays that form the cones of the fan the singularity:

$$\Sigma(Y_0) = \{v_1, v_2, v_3\}$$

where the vector are

$$v_1 = (1, 0, 0), v_2 = (-1, 2, 0) \text{ and } v_3 = (0, -1, 2).$$

The toric divisors are given by the three rays of the fan and all divisors are Weil, this is summarized in Table 5.1.

Table 5.1: Toric Divisors in Y_0

Ray	Divisor	Fan	Variety	Type
$\mathbf{v}_1$	D_1	$(1, 0),\ (-1, 2)$	$\mathbb{C}^2/\mathbb{Z}_2$	Weil
$\mathbf{v}_2$	D_2	$(1, 0),\ (-1, 2)$	$\mathbb{C}^2/\mathbb{Z}_2$	Weil
$\mathbf{v}_3$	D_3	$(1, 0),\ (-1, 4)$	$\mathbb{C}^2/\mathbb{Z}_4$	Weil

5.2 Crepant Resolutions of $\mathbb{C}^3/\mathbb{Z}_4$

To solve the singularity $\mathbb{C}^3/\mathbb{Z}_4$ we can apply the techniques of toric geometry because the group $\mathbb{Z}_4$ is cyclic. Details may be found in [13]. An introduction to toric techniques may be found in [14, Chapter 11].

The full resolution of singularities is denoted by Y. To resolve the singularity, two rays need to be added to the fan of the singularity; these rays are

$$w_2 = (0, 1, 0), w_3 = (0, 0, 1).$$

The fan of the resolution of singularities is

$$\Sigma(Y) = \{v_1, v_2, v_3, w_2, w_3\}.$$

The two newly added rays generate the divisors that are the components of the exceptional divisor:

$$w_3 \text{ corresponds to compact divisor called } D_c$$

and

$$w_2 \text{ corresponds to non-compact divisor called } D_{nc}.$$

Some points remarkable are:

i. All cones in $\Sigma(Y)$ are smooth, so that the variety Y is smooth. This implies that all divisors are Cartier.

ii. There exists a morphism $Y \longrightarrow Y_0$, which is a resolution of singularities.

In the Table 5.2, the rays generating the divisors and the corresponding variety for these divisors are described.

Table 5.2: Toric Divisors in Y. The last column shows the components of the divisor class on the basis given by (D_{nc}, D_c).

Ray	Divisor	Fan	Variety	Components
$\mathbf{w}_3$	D_c	$(1,0),\ (-1,2),\ (0,-1),\ (0,1)$	$\mathbb{F}_2$	$(0,1)$
$\mathbf{w}_2$	D_{nc}	$(1,0),\ (-1,0),\ (0,1)$	$\mathbb{P}^1 \times \mathbb{C}$	$(1,0)$
$\mathbf{v}_1$	D_{EH}	$(1,0),\ (-1,2),\ (0,1)$	A_1	$(-\frac{1}{2}, -\frac{1}{4})$
$\mathbf{v}_3$	D_4	$(1,0),\ (-1,4),\ (0,1)$	tot $(\mathcal{O}(-4) \to \mathbb{P}^1)$	$(0, -\frac{1}{2})$
$\mathbf{v}_2$	D'_{EH}	$(1,0),\ (-1,2),\ (0,1)$	A_1	$(-\frac{1}{2}, -\frac{1}{4})$

Only two toric divisors are independent in cohomology, for instance D_c and D_{nc}. The set of divisors $Div(Y)$ is generated by $D_{EH}, D'_{EH}, D_4, D_{nc}$ and D_c and the Picard group is generated by D_{nc} and D_c.
The class of canonical divisor is $[K_Y] = 0$, this implies Y is a crepant resolution. The resolution of singularities satisfies that:

$$Y \cong \operatorname{tot}(\mathcal{O}_{\mathbb{F}_2}(-2E)),$$

where E is the exceptional divisor, i.e, Y is the total space of the canonical bundle of the second Hirzebruch surface $\mathbb{F}_2$.

5.3 Chamber Structure of GIT-Parameter Space

In this section following [13] we calculate the equations of the walls and find all the walls separating two adjacent chambers.

The GIT-parameter space is:

$$\Theta = \mathbb{R}^3;$$

this space is divided by three planes, that will form the structures of chambers and walls. The equations of these planes are

$$\begin{aligned} \pi_1 : &\quad X - Y - Z = 0 \\ \pi_2 : &\ -X + Y - Z = 0 \\ \pi_3 : &\ -X - Y + Z = 0, \end{aligned}$$

where x, y and z are the usual coordinates of $\mathbb{R}^3$.
The 8 chambers of the GIT-parameter space are convex polyhedral cones, as we know from general theory. In the Table 5.3 we describe the three half-planes that delimit the 8 chambers.

Table 5.3: The equations of the chambers

Chamber	Sign on the plane π_1	Sign on the plane π_2	Sign on the plane π_3
Chamber I	$X - Y - Z > 0$	$-X + Y - Z > 0$	$-X - Y + Z > 0$
Chamber II	$X - Y - Z > 0$	$-X + Y - Z > 0$	$-X - Y + Z < 0$
Chamber III	$X - Y - Z > 0$	$-X + Y - Z < 0$	$-X - Y + Z > 0$
Chamber IV	$X - Y - Z < 0$	$-X + Y - Z > 0$	$-X - Y + Z > 0$
Chamber V	$X - Y - Z < 0$	$-X + Y - Z < 0$	$-X - Y + Z > 0$
Chamber VI	$X - Y - Z < 0$	$-X + Y - Z > 0$	$-X - Y + Z < 0$
Chamber VII	$X - Y - Z > 0$	$-X + Y - Z < 0$	$-X - Y + Z < 0$
Chamber VIII	$X - Y - Z < 0$	$-X + Y - Z < 0$	$-X - Y + Z < 0$

To study the wall-crossing phenomenon, a name was given to each wall. Each of the planes determines four walls, and the equations of these walls are given in the Tables 5.4, 5.5 and 5.6.

Table 5.4: Equations of the walls in π_1

Name of Wall	Points	Sign in Y	Sign in Z
M_a	$(Y+Z, Y, Z)$	$Y>0$	$Z>0$
M_b	$(Y+Z, Y, Z)$	$Y>0$	$Z<0$
M_c	$(Y+Z, Y, Z)$	$Y<0$	$Z>0$
M_d	$(Y+Z, Y, Z)$	$Y<0$	$Z<0$

Table 5.5: Equations of the walls in π_2

Name of Wall	Points	Sign in X	Sign in Z
R_a	$(X, X+Z, Z)$	$X>0$	$Z<0$
R_b	$(X, X+Z, Z)$	$X>0$	$Z>0$
R_c	$(X, X+Z, Z)$	$X<0$	$Z>0$
R_d	$(X, X+Z, Z)$	$X<0$	$Z<0$

Table 5.6: Equations of the walls in π_3

Name of Wall	Points	Sign in X	Sign in Y
V_a	$(X, Y, X+Y)$	$X>0$	$Y<0$
V_b	$(X, Y, X+Y)$	$X>0$	$Y>0$
V_c	$(X, Y, X+Y)$	$X<0$	$Y>0$
V_d	$(X, Y, X+Y)$	$X<0$	$Y<0$

In Table 5.7, the first column describes the walls surrounding and forming each chamber. In the second column of this table, we describe each of the three adjacent chambers for the eight chambers of the GIT-parameter space.

Table 5.7: Walls defining chambers and adjacent chambers

Chambers	Walls of the chambers	Adjacent chambers
Chamber VII	V_a, M_a, R_a	$C_3 = V_a$, $C_8 = M_a$, $C_2 = R_a$
Chamber VIII	V_b, M_a, R_b	$C_5 = V_b$, $C_7 = M_a$, $C_6 = R_b$
Chamber VI	V_c, M_b, R_b	$C_4 = V_c$, $C_2 = M_b$, $C_8 = R_b$
Chamber II	V_d, M_b, R_a	$C_1 = V_d$, $C_6 = M_b$, $C_7 = R_a$
Chamber IV	V_c, M_d, R_c	$C_6 = V_c$, $C_1 = M_d$, $C_5 = R_c$
Chamber V	V_b, M_c, R_c	$C_8 = V_b$, $C_3 = M_c$, $C_4 = R_c$
Chamber III	V_a, M_c, R_d	$C_7 = V_a$, $C_5 = M_c$, $C_1 = R_d$
Chamber I	V_d, M_d, R_d	$C_2 = V_d$, $C_4 = M_d$, $C_3 = R_d$

5.4 Cohomological Fourier-Mukai Transform Induced by Wall-Crossing

In this section we determine an expression for the Fourier-Mukai transform at the cohomology level version for the case when the group G is Abelian and the walls W are of type 0 and 3.

Let $\mathcal{M}_C$ and $\mathcal{M}_{C'}$ be two moduli spaces of θ-stable G-constellations associated to chambers C and C' in the GIT-parameter space separated by the wall W. We know that these varieties are Fourier-Mukai partners, more precisely, there is an equivalence of categories

$$\Psi := \Phi_2^{-1} \circ \Phi_1 : D^b(\mathcal{M}_C) \longrightarrow D^b(\mathcal{M}_{C'})$$

where $\Phi_1 : D^b(\mathcal{M}_C) \longrightarrow D^b_G(\mathbb{C}^3)$ and $\Phi_2 : D^b(\mathcal{M}_{C'}) \longrightarrow D^b_G(\mathbb{C}^3)$ are the Fourier-Mukai transform given by the derived McKay correspondence. As in Chapter 4, the cohomological version for the Fourier-Mukai of Ψ is given by

$$\Psi^H_{\mathcal{P}} : H^*(\mathcal{M}_C, \mathbb{Q}) \longrightarrow H^*(\mathcal{M}_{C'}, \mathbb{Q}),$$

for any $a \in H^*(\mathcal{M}_C, \mathbb{Q})$

$$\Psi^H_{\mathcal{P}}(a) = q_*(v(\mathcal{P}) \,.\, p^*a).$$

Here $v(\mathcal{P})$ is the Mukai vector from the kernel of the $\Psi_{\mathcal{P}}$, that is:

$$v(\mathcal{P}) = Ch(\mathcal{P}) \,.\, \sqrt{\mathrm{td}(\mathcal{M}_C \times \mathcal{M}_{C'})},$$

and p and q are the projections of the product $\mathcal{M}_C \times \mathcal{M}_{C'}$ onto the factors $\mathcal{M}_C$ and $\mathcal{M}_{C'}$, respectively.

By Theorem 4.1 about the compatibility between K-theoretic and cohomological versions of Fourier-Mukai transforms,

$$\Psi^H_{\mathcal{P}}(Ch(a)\,.\,\sqrt{td(\mathcal{M}_C)}) = Ch(\Psi^K_{\mathcal{P}}(a))\,.\,\sqrt{td(\mathcal{M}_{C'})},$$

and therefore

$$\begin{aligned} Ch(\Psi^K_{\mathcal{P}}(a)) &= \Psi^H_{\mathcal{P}}(Ch(a)\,.\,\sqrt{td(\mathcal{M}_C)})\,.\,\sqrt{td(\mathcal{M}_{C'})}^{-1} \\ &= q_*\left(v(\mathcal{P})\,.\,p^*(Ch(a)\,.\,\sqrt{td(\mathcal{M}_C)}\,)\right)\,.\,\sqrt{td(\mathcal{M}_{C'})}^{-1} \\ &= q_*\left(v(\mathcal{P})\,.\,p^*(Ch(a)\,.\,\sqrt{td(\mathcal{M}_C)})\,.\,q^*\sqrt{td(\mathcal{M}_{C'})}^{-1}\right) \\ &= q_*\left(v(\mathcal{P})\,.\,p^*(Ch(a))\,.\,p^*\sqrt{td(\mathcal{M}_C)}\,.\,q^*\sqrt{td(\mathcal{M}_{C'})}^{-1}\right). \end{aligned}$$

Afetr some calculations we have

$$\begin{aligned} p^*\sqrt{td(\mathcal{M}_C)}\,.\,q^*\sqrt{td(\mathcal{M}_{C'})}^{-1} &= \left(1+\frac{1}{24}p^*c_2(\mathcal{M}_C)\right)\left(1-\frac{1}{24}q^*c_2(\mathcal{M}_{C'})\right) \\ &= 1+\frac{\lambda}{24}\left(p^*c_2(\mathcal{M}_C)-q^*c_2(\mathcal{M}_C)\right) \\ &\quad -\frac{\lambda^2}{576}\left(p^*c_2(\mathcal{M}_C)\smile q^*c_2(\mathcal{M}_C)\right). \end{aligned}$$

Taking $a = \mathcal{R}_\rho \in K(\mathcal{M}_C)$, from the general theory we know that

$$\Psi(\mathcal{R}_\rho) = \mathcal{R}'_\rho.$$

Indeed, it can be proven that $\Phi_1({\mathcal{R}_\rho}^\vee) = \rho \otimes \mathcal{O}_{\mathbb{C}^3}$, so that

$$\Psi(\mathcal{R}_\rho) = \Phi_2^{-1}\circ\Phi_1({\mathcal{R}_\rho}^\vee) = \Phi_2^{-1}(\rho\otimes\mathcal{O}_{\mathbb{C}^3}) = \mathcal{R}'_\rho.$$

The following identity holds in the cohomology ring:

$$\begin{aligned} \Psi^H_{\mathcal{P}}(Ch(\mathcal{R}_\rho)) &= Ch(\Psi^K_{\mathcal{P}}(\mathcal{R}_\rho))\,.\,\sqrt{td(\mathcal{M}_{C'})}\,.\,\sqrt{td(\mathcal{M}_C)}^{-1}, \\ &= Ch(\mathcal{R}'_\rho)\,.\,\sqrt{td(\mathcal{M}_{C'})}\,.\,\sqrt{td(\mathcal{M}_C)}^{-1}. \end{aligned}$$

This leads us to consider the following diagram between the cohomology rings of $\mathcal{M}_C$, $\mathcal{M}_{C'}$, and of the crepant resolution Y:

$$\begin{tikzcd} H^*(\mathcal{M}_C,\mathbb{Q}) \arrow[rr, "\Psi^H_{\mathcal{P}}"] \arrow[dr, "A"'] & & H^*(\mathcal{M}_{C'},\mathbb{Q}) \arrow[dl, shift right, "B"'] \\ & H^*(Y,\mathbb{Q}) \arrow[ur, shift right, "B^{-1}"'] & \end{tikzcd}$$

where A and B are linear transformations and the cohomological Fourier-Mukai is obtained as

$$\Psi^H_{\mathcal{P}} = B^{-1} . A.$$

In general, for any crepant resolution Y of $\mathbb{C}^3/G$ and G a finite abelian subgroup of $\mathrm{SL}(3, \mathbb{C})$:

$$\dim(H^*(Y, \mathbb{Q})) = \sharp(G).$$

If we put $\sharp(G) = r + 1$ by Ito-Reid's results

$$H^*(Y, \mathbb{Q}) = H^0(Y, \mathbb{Q}) \oplus H^2(Y, \mathbb{Q}) \oplus H^4(Y, \mathbb{Q})$$

We know that $h^0(Y, \mathbb{Q}) = 1$ and we suppose

$$h^2(Y, \mathbb{Q}) = r_1,\ h^4(Y, \mathbb{Q}) = r_2$$

so that

$$Ch(\mathcal{R}_0) = 1 \in H^*(Y, \mathbb{Q})$$

for $i = 1, \cdots, r$,

$$Ch(\mathcal{R}_i) = 1 + C_1(\mathcal{R}_i) + \frac{C_1(\mathcal{R}_i)^2}{2} \in H^*(Y, \mathbb{Q}).$$

We can write the components of this vector in the cohomology ring:

$$Ch(\mathcal{R}_i) = (1, a^i_1, \cdots, a^i_{r_1}, b^i_1, \cdots, b^i_{r_2}) \in H^*(Y, \mathbb{Q}).$$

So, the matrix representing the linear transformation A is:

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & a^1_1 & \cdots & a^r_1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & a^1_{r_1} & \cdots & a^r_{r_1} \\ 0 & b^1_1 & \cdots & b^r_1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & b^1_{r_2} & \cdots & b^r_{r_2} \end{bmatrix} \in \mathbb{Q}^{(r+1)^2}$$

The linear transformation B depends on the change of tautological line bundles:

$$B = \begin{bmatrix} 1 & 1 \quad \cdots \quad 1 \\ 0 & \\ \vdots & \Big[* \Big] \\ 0 & \end{bmatrix}$$

the minor matrix is represented by $[\delta_{ij}]$. We can make explicit calculations for changes on the tautological line bundles for walls of type 0 and 3.

For walls of type 0. Fixing the column $j = 1, \cdots, r$.
Case 1: if $\rho_0 \subset R_1$, $j = \rho$ and $i = 1, \ldots, r_1, 1, \ldots, r_2$

i. for $\rho \subset R_1$, then $\mathcal{R}'_\rho = \mathcal{R}_\rho$,

$$(\delta_{ij}) = (a_1^j, \ldots, a_{r_1}^j, b_1^j, \ldots, b_{r_2}^j).$$

ii. for $\rho \subset R_2$, then $\mathcal{R}'_\rho = \mathcal{R}_\rho(-D)$,

$$(\delta_{ij}) = (a_1^j - \lambda_1^j, \ldots, a_{r_1}^j - \lambda_{r_1}^j, c_1^j, \ldots, c_{r_2}^j).$$

For $\rho \subset R_1$ the matrix A does not change in the column j, for $\rho \subset R_2$ the column j of the matrix A it would only change as:

$$B = \begin{bmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ 0 & a_1^1 & \cdots & a_1^j - \lambda_1^j & \cdots & a_1^r \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_{r_1}^1 & \cdots & a_{r_1}^j - \lambda_{r_1}^j & \cdots & a_{r_1}^r \\ 0 & b_1^1 & \cdots & c_1^j & \cdots & b_1^r \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & b_{r_2}^1 & \cdots & c_{r_2}^j & \cdots & b_{r_2}^r \end{bmatrix} \in \mathbb{Q}^{(r+1)^2}$$

Case 2, if $\rho_0 \subset R_2$, $j = \rho$ and $i = 1, \ldots, r_1, 1, \ldots, r_2$.

i. For $\rho \subset R_1$, then $\mathcal{R}'_\rho = \mathcal{R}_\rho(D)$,

$$(\delta_{ij}) = (a_1^j + \lambda_1^j, \ldots, a_{r_1}^j + \lambda_{r_1}^j, c_1^j, \ldots, c_{r_2}^j).$$

ii. For $\rho \subset R_2$, then $\mathcal{R}'_\rho = \mathcal{R}_\rho$,

$$(\delta_{ij}) = (a_1^i, \ldots, a_{r_1}^i, b_1^i, \ldots, b_{r_2}^i).$$

For $\rho \subset R_2$ the matrix A does not change in the column j, for $\rho \subset R_1$ the column j of the matrix A changes in the following way

$$B = \begin{bmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ 0 & a_1^1 & \cdots & a_1^j + \lambda_1^j & \cdots & a_1^r \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_{r_1}^1 & \cdots & a_{r_1}^j + \lambda_{r_1}^j & \cdots & a_{r_1}^r \\ 0 & b_1^1 & \cdots & c_1^j & \cdots & b_1^r \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & b_{r_2}^1 & \cdots & c_{r_2}^j & \cdots & b_{r_2}^r \end{bmatrix} \in \mathbb{Q}^{(r+1)^2}.$$

Wall-crossing of type 3. Fixing the column $j = 1, \ldots, r$.
Case 1, if $\rho_0 \subset R_1$, $j = \rho$ and $i = 1, \ldots, r_1, 1, \ldots, r_2$.

i. For $deg(\mathcal{R}_\rho\big|_l) = -1$, then $\mathcal{R}'_\rho = \mathcal{R}_\rho(-D)$,

$$(\delta_{ij}) = (a_1^i - \lambda_1^i, \ldots, a_{r_1}^i - \lambda_{r_1}^i, c_1^i, \ldots, c_{r_2}^i).$$

ii. for $deg(\mathcal{R}_\rho\big|_l) = 0$, then $\mathcal{R}'_\rho = \mathcal{R}_\rho$,

$$(\delta_{ij}) = (a_1^i, \ldots, a_{r_1}^i, b_1^i, \ldots, b_{r_2}^i).$$

For the case $deg(\mathcal{R}_\rho\big|_l) = 0$ no changes occur in j-th column, for the case $deg(\mathcal{R}_\rho\big|_l) = -1$ the matrix changes only in the j-th column:

$$B = \begin{bmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ 0 & a_1^1 & \cdots & a_1^j - \lambda_1^j & \cdots & a_1^r \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_{r_1}^1 & \cdots & a_{r_1}^j - \lambda_{r_1}^j & \cdots & a_{r_1}^r \\ 0 & b_1^1 & \cdots & c_1^j & \cdots & b_1^r \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & b_{r_2}^1 & \cdots & c_{r_2}^j & \cdots & b_{r_2}^r \end{bmatrix} \in \mathbb{Q}^{(r+1)^2}.$$

Case 2, if $\rho_0 \subset R_2$, $j = \rho$ and $i = 1, \ldots, r_1, 1, \ldots, r_2$.

i. For $deg(\mathcal{R}_\rho\big|_l) = 0$, then $\mathcal{R}'_\rho = \mathcal{R}_\rho$,

$$(\delta_{ij}) = (a_1^i, \ldots, a_{r_1}^i, b_1^i, \ldots, b_{r_2}^i).$$

ii. For $deg(\mathcal{R}_\rho\big|_l) = 1$, then $\mathcal{R}'_\rho = \mathcal{R}_\rho(D)$,

$$(\delta_{ij}) = (a_1^i + \lambda_1^i, \ldots, a_{r_1}^i + \lambda_{r_1}^i, c_1^i, \ldots, c_{r_2}^i).$$

For the case $deg(\mathcal{R}_\rho\big|_l) = 0$ no changes occur in j-th column, for the case $deg(\mathcal{R}_\rho\big|_l) = 1$ the matrix changes only in the j-th column:

$$B = \begin{bmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ 0 & a_1^1 & \cdots & a_1^j + \lambda_1^j & \cdots & a_1^r \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_{r_1}^1 & \cdots & a_{r_1}^j + \lambda_{r_1}^j & \cdots & a_{r_1}^r \\ 0 & b_1^1 & \cdots & c_1^j & \cdots & b_1^r \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & b_{r_2}^1 & \cdots & c_{r_2}^j & \cdots & b_{r_2}^r \end{bmatrix} \in \mathbb{Q}^{(r+1)^2}.$$

Chapter 6

Epilogue

All the tools discussed in this thesis lead us to study the stability conditions of triangulated categories. By setting an appropriate triangulated category $\mathcal{D}$, the set of stability conditions:

$$\mathrm{Stab}(\mathcal{D})$$

has a natural topology. This topological space is a new invariant of triangulated categories.
The stability conditions are not related to the type of stability studied in this thesis, because the ones treated here refer to coherent sheaves while the stability conditions we mentioned are for a triangulated category.

Bridgeland in [4, Theorem 1.2] proved that for each connected component $\Sigma \subset \mathrm{Stab}(\mathcal{D})$ of the triangulated category $\mathcal{D}$ there exists a linear subspace $V(\Sigma) \subset \mathrm{Hom}(K(\mathcal{D}), \mathbb{C})$ and a local homeomorphism $\Sigma \longrightarrow V(\Sigma)$, then each component Σ is locally topological vector space. In the case that $\mathcal{D}$ is numerically finite [4, Corollary 1.3] Σ is a finite-dimensional complex manifold.

But in order to study the stability conditions of a triangulated category $\mathcal{D}$ of the bounded derived category of $\mathcal{O}_X$-module, where X is by example $K3$ surface, we need more general spaces to the schemes, let us remember that

$$\{\text{varieties}\} \subset \{\text{schemes}\} \subset \{\text{algebraic spaces}\} \subset \{\text{stacks}\}\,.$$

In the source [39, Theorem 2.3 (Toda, Alper, Halpern-Leistner, Heinloth)] $\sigma \in \mathrm{Stab}(\mathcal{D})$ and a phase $v \in \Lambda$, there exists a finite type Artin stack $\mathcal{M}_\sigma(v)$.

In this situation we plan to study the stability conditions for $K3$ surfaces with appropriate singularities and describe them.

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