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Studying Riemannian Immersions into Semi-Riemannian Spaces via Parabolicity, Liouville type results and other maximum principles

by

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Abstract

This thesis studies the geometry of complete Riemannian submanifolds immersed in certain semi-Riemannian spaces via parabolicity criteria related to modified Cheng-Yau's operators and to a linearized differential operator which can be regarded as a natural extension of the standard Laplacian, via generalization of a Liouville-type result and versions of maximum principle. In this regard, via parabolicity criteria and from appropriate Simons type formulas concerning spacelike submanifolds immersed with parallel normalized mean curvature vector in Einstein Manifolds we prove new characterization results. In the case of submanifolds of semi-Riemannian warped products, under standard convergence conditions and appropriated constraints on the higher order mean curvatures, we also obtain uniqueness and nonexistence results via parabolicity and p-integrability criteria, for $p \geq 1$, generalization of a Liouville-type result, a version of maximum principle at infinity for vector fields and a maximum principle related to polynomial volume growth. Applications are also presented to cases in which the ambient space is either an Einstein manifold, the Steady State models, Schwarzschild and Reeissner-Nordström spaces, and a particular investigation of entire graphs constructed over the fiber of the ambient space.

Keywords: Riemannian Submanifolds, Semi-Riemannian Spaces, Parabolicity Criteria, Maximum Principles.

Resumo

Esta tese estuda a geometria de subvariedades Riemannianas completas imersas em certos espaços semi-Riemannianos via critérios de parabolicidade relacionados ao operador de Cheng-Yau modificado e a um operador diferencial linearizado que pode ser considerado como uma extensão natural do Laplaciano padrão, via generalização de um resultado tipo-Liouville e versões do princípio máximo. Neste sentido, através de critérios de parabolicidade e de fórmulas apropriadas do tipo Simons relativas a subvariedades imersas com vetor de curvatura média normalizado paralelo em variedades Einstein, provamos novos resultados de caracterização. No caso de subvariedades de produtos *warped* semi-Riemannianos, sob condições de convergência e restrições apropriadas nas curvaturas médias de ordem superior, também obtemos resultados de unicidade e inexistência via critérios de parabolicidade e de p-integrabilidade, para $p \geq 1$, generalização de um resultado do tipo-Liouville, uma versão do princípio máximo no infinito para campos vetoriais e um princípio máximo relacionado ao crescimento de volume polinomial. Também são apresentadas aplicações aos casos em que o espaço ambiente é uma variedade de Einstein, os modelos de Steady-Statede espaços Schwarzschild e Reeissner-Nordström, e uma investigação particular de gráficos inteiros construídos sobre a fibra do espaço ambiente.

Palavras-chave: Subvariedades Riemannianas, Espaços Semi-Riemannianos, Critérios de Parabolicidade, Princípios do Máximo.

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"*When peace like a river attendeth my way, when sorrows like sea-billows roll, whatever my lot, Christ has taught me to know; It is well, it is well with my soul.*"

Horatio Spafford

Dedicatory

To my persecuted family.

Contents

Introduction

This Thesis is divided into two parts where we propose to study n-dimensional complete Riemannian submanifolds immersed in semi-Riemannian spaces from the following themes: characterizations of complete linear Weingarten Riemannian submanifolds immersed with parallel normalized mean curvature vector in $(n+p)$-dimensional Einstein manifolds and rigidity and nonexistence of Riemannian immersions in semi-Riemannian warped products.

Part I: Parabolicity of complete linear Weingarten submanifolds in semi-Riemannian manifolds

Initially, we established new characterization results related to n-dimensional complete linear Weingarten Riemannian submanifolds M^n with parallel normalized mean curvature vector immersed in $(n+p)$-dimensional Riemannian space form $\mathbb{Q}_c^{n+p}$ of constant sectional curvature $c \in \{-1, 0, 1\}$, in the $(n+p)$-dimensional de Sitter space $\mathbb{S}_p^{n+p}$ and in an arbitrary Einstein manifold $\mathcal{E}_p^{n+p}$ of index $p \geq 1$ via parabolicity criteria with respect to a modified Cheng-Yau's operator L defined on M^n. This criteria are consequence of a general result concerning divergent type operators due to Pigola, Rigoli and Setti [111, Theorem 2.6] (see also [21, Lemma 6.9]).

Following Chen [55], we say that a manifold M^n to have parallel normalized mean curvature vector if the mean curvature vector H of M is nonzero and the corresponding normalized mean curvature vector field $\frac{h}{H}$ is parallel as a section of the normal bundle. We also recall that a submanifold is said to be linear Weingarten when its mean curvature function H and its normalized scalar curvature R satisfy a linear relation of the type

$$R = aH + b,$$

for some constants $a, b \in \mathbb{R}$. Manifolds satisfying this relation were introduced by Weingartein [124, 125] in the study of surfaces in Euclidean space $\mathbb{R}^3$. Linear Weingarten manifolds can be regarded as a natural generalization of manifolds with constant scalar curvature.

In the case of constant scalar curvature, Cheng [58] showed that the totally umbilical sphere $\mathbb{S}^n(r)$, totally geodesic Euclidean space $\mathbb{R}^n$ and generalized cylinder $\mathbb{R} \times \mathbb{S}^{n-1}(r)$ are the only n-dimensional $(n > 2)$ complete submanifolds with constant scalar curvature and parallel normalized mean curvature vector in the Euclidean space $\mathbb{R}^{n+p}$, which satisfy a suitable constrain on the norm of the second fundamental form. Later on, Guo and Li [79] generalized previous results of [90] showing that the only n-dimensional compact (without boundary) submanifolds in the unit sphere $\mathbb{S}^{n+p}$ with constant scalar curvature, parallel normalized mean curvature vector and whose second fundamental form satisfies some appropriate boundedness are the totally umbilical sphere $\mathbb{S}^n(r)$ and the Clifford torus $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r)$. Afterwards, de Lima, Araújo, dos Santos and Velásquez [39] obtained an Omori-type maximum principle for the Cheng-Yau's operator and applied it to establish an extension of the results of [58, 79] for n-dimensional complete submanifolds immersed with parallel normalized mean curvature vector in $\mathbb{Q}_c^{n+p}$, with positive constant normalized scalar curvature.

Next, these same authors [73] used the Hopf's strong maximum principle and a maximum principle at infinity due to Caminha [48] to obtain versions of the results of [39, 58, 79] for the context of n-dimensional complete linear Weingarten submanifolds immersed with parallel normalized mean curvature vector in $\mathbb{Q}_c^{n+p}$. In [36], Araújo and de Lima studied compact linear Weingarten surfaces immersed with flat normal bundle and parallel normalized mean curvature vector in $\mathbb{Q}_c^{2+p}$. In this setting, they got a version of the classical Liebmann's rigidity theorem showing that such a surface with nonnegative Gaussian curvature must be isometric to a totally umbilical round sphere. In [37], they obtained another version of this Liebmann's result when the ambient space is the hyperbolic space (see also [38] for other characterizations concerning linear Weingarten submanifolds in the hyperbolic space).

Motivated by the works described above, in Chapter 2 we will study the L-parabolicity of a complete linear Weingarten submanifold M^n immersed with parallel normalized mean curvature vector in $\mathbb{Q}_c^{n+p}$ to obtain L-parabolicity criterion(see Proposition A) and we apply it to prove that M^n must be either totally umbilical or isometric to a hyperbolic cylinder $\mathbb{H}^1\left(-\sqrt{1+r^2}\right)\times\mathbb{S}^{n-1}(r)$, when $c=-1$, a circular cylinder $\mathbb{R}\times\mathbb{S}^{n-1}(r)$, when $c=0$, and a Clifford torus, when $c=1$ for a certain radius r (see Theorems 2.4.1 and 2.4.2).

When the ambient is de Sitter space $\mathbb{S}_1^{n+1}$, Goddard [80] conjectured that every complete spacelike hypersurface with constant mean curvature H constant must be totally umbilical. Ramanathan in [112] proved that complete spacelike hypersurfaces with constant mean curvature $0\leq H\leq 1$ in $\mathbb{S}_1^3(1)$ are totally umbilical, but for $H>1$ this not occurs, proving that Goddard's conjecture is false, as can be seen from an example due to Dajczer-Nomizu in [65]. In [3], Akutagawa showed that Goddard's conjecture is true provided that $n=2$ and $H^2\leq 1$ or $n\geq 3$ and $H^2<\frac{4(n-1)}{n^2}$. In the case M^n compact. Montiel [100] also proved that this conjecture is true and exhibited examples of complete spacelike hypersurfaces in $\mathbb{S}_1^{n+1}(1)$ with constant H satisfying $H^2\geq\frac{4(n-1)}{n^2}$ and being not umbilical, the so-called hyperbolic cylinders, which are isometric to a Riemannian product of the type $\mathbb{S}^{n-1}(c_1)\times\mathbb{H}^1(c_2)$, where $c_1>0$ and $c_2<0$ satisfy $\frac{1}{c_1}+\frac{1}{c_2}=1$. Moreover, the characterization theses hyperbolic cylinders as the only complete non-compact spacelike hypersurfaces in $\mathbb{S}_1^{n+1}(1)$ with constant mean curvature $H^2=\frac{4(n-1)}{n^2}$ and having more than one topological connected component in the complement of balls with radii increasing to infinity, was also obtained by Montiel in [101].

Liu [92] showed that, when $b=0$ the totally umbilical round spheres are the only n-dimensional compact (without boundary) linear Weingarten spacelike submanifolds of $\mathbb{S}_p^{n+p}$ with nonnegative sectional curvature and flat normal bundle. Later on, Yang and Hou [127] applied the Omori-Yau's generalized maximum principle to show that a linear Weingarten spacelike submanifold in $\mathbb{S}_p^{n+p}$, with $a>0$, $b<1$, having parallel normalized mean curvature vector and such that the squared norm of its second fundamental form satisfies a suitable boundedness, must be either totally umbilical or isometric to a certain hyperbolic cylinder. Liu and Zhang [94] used the classical strong maximum principle of Hopf to obtain other classifications for complete linear Weingarten spacelike submanifolds in $\mathbb{S}_p^{n+p}$ having parallel normalized mean curvature.

More recently, Araújo, de Lima, dos Santos and Velásquez [41] obtained other characterization results related to complete linear Weingarten spacelike submanifolds with parallel normalized mean curvature vector in $\mathbb{S}_p^{n+p}$ under suitable constraints on the values of the mean curvature and of the norm of the traceless part of the second fundamental form, now through an extension of Hopf's maximum principle for complete Riemannian manifolds.

Proceeding, in Chapter 3, where the ambient space is the de Sitter space $\mathbb{S}_p^{n+p}$, we also establish a suitable parabolicity criterion related to a Cheng-Yau's modified operator L (see Proposition C) and we use it to revisit the results of [41, 94, 127] obtaining new characterizations concerning n-dimensional complete linear Weingarten spacelike submanifolds immersed with parallel normalized mean curvature vector in $\mathbb{S}_p^{n+p}$ (see Theorems 3.3.1, 3.3.2, 3.3.3, 3.3.4).

It should be noted that the study of de Sitter space $\mathbb{S}_1^{n+1}$ is of interest from both a geometric and a physical point of view, since it is a vacuum solution of the Einstein field equations with positive cosmological constant, that is,

$$\mathrm{Ric} - \frac{1}{2}Rg + \Lambda g = 0,$$

where g is the metric, Ric is the Ricci tensor and Λ is the cosmological constant

$$\Lambda = \left(\frac{n-1}{2(n+1)}R\right) = \frac{n(n-1)}{2}.$$

In relativity theory, $\mathbb{S}_1^{n+1}$ is called de Sitter spacetime. (see, for instance, [82, Section 5.2], [49, Section 4.1] and [118, Section 2] for more details concerning the geometry of $\mathbb{S}_1^4$).

Finally, in Chapter 4 we also deal with complete linear Weingarten spacelike submanifolds M^n immersed with parallel normalized mean curvature vector in an Einstein manifold $\mathcal{E}_p^{n+p}$ of index p, now supposing that such submanifolds have a flat normal bundle, that is, its the normal curvature tensor vanishes identically.

In [68], H. de Lima and J. de Lima obtained characterization results for linear Weingarten spacelike hypersurfaces immersed in a locally symmetric Einstein manifold $\mathcal{E}_1^{n+1}$ of index 1 considering restrictions on the square length of the second fundamental form and some appropriate curvature constraints of the ambient space which were inspired by the works of Nishikawa [106] and Choi et al. [61, 62]. Later, Araújo, de Lima, dos Santos and Velásquez [40] extended these results for the context of an n-dimensional spacelike submanifold M^n immersed with parallel normalized mean curvature vector in a locally symmetric semi-Riemannian manifold L_p^{n+p} of index p. For this, they assumed the existence of real constants c_1, c_2 and c_3 such that the sectional curvature $\overline{K}$ and curvature tensor $\overline{R}$ of L_p^{n+p} satisfy the following conditions:

$$\overline{K}(u,\eta) = \frac{c_1}{n}, \tag{1}$$

for any $u \in TM$ and $\eta \in TM^\perp$;

$$\overline{K}(u,v) \geq c_2, \tag{2}$$

for any $u, v \in TM$;

$$\overline{K}(\eta,\xi) = \frac{c_3}{p}, \tag{3}$$

for $\eta, \xi \in TM^\perp$; and

$$\langle \overline{R}(\xi,u)\eta, u\rangle = 0, \tag{4}$$

for $u \in TM$ and $\xi, \eta \in TM^\perp$ with $\langle \xi, \eta\rangle = 0$. We note that, when $p = 1$, conditions (3) and (4) are automatically satisfied. Afterwards, also assuming this set of constraints, Araújo, Barboza, de Lima and Velásquez [35] applied the techniques developed by Hou and Yang in [127] and by Liu and Zhang in [95] to get sufficient conditions guaranteeing that such a spacelike submanifold M^n is either totally umbilical or isometric to an isoparametric hypersurface of a totally geodesic submanifold $L_1^{n+1} \hookrightarrow L_p^{n+p}$, with two distinct principal curvatures, one of which is simple.

In [93], Liu and Xie used the Omori-Yau's generalized maximum principle to obtain classification results concerning complete spacelike hypersurface M^n with constant mean curvature in $\mathcal{E}_1^{n+1}$ without asking that the ambient space is locally symmetric but also assuming the curvature constraints (1) and (2).

As in Chapters 2 and 3, in the Chapter 4 we establish a L-parabolicity criterion (see Proposition D) and we apply it to obtain sufficient conditions which guarantee that complete linear Weingarten spacelike submanifolds immersed with parallel normalized mean curvature vector and flat normal bundle in an Einstein manifold $\mathcal{E}_p^{n+p}$ of index p obeying curvature constraints (1), (2), (3) and (4), must be an isoparametric hypersurface of $\mathcal{E}_1^{n+1}$ with two distinct principal curvatures one of which is simple (see Theorem 4.3.1 and Corollary 4.3.1).

It is worth to point out that the works [40, 35] contain similar results under the assumption that the ambient space is locally symmetric. But, according to Example 1.1 of [93], the semi-Riemannian product space $\mathbb{R}^p_p \times M^n$, where M^n is a Ricci flat Riemannian manifold, is an Einstein manifold of index p. Moreover, supposing that the sectional curvature K_M of M^n is such that $K_M(u, v) \geq c_2$ for any $u, v \in TM$ and some constant c_2 and considering the spacelike submanifold given by the inclusion $\iota : M^n \hookrightarrow \mathbb{R}^p_p \times M^n$, we can verify that the curvature constraints (1), (2), (3) and (4) are satisfied. However, if M^n is not locally symmetric, then $\mathbb{R}^p_p \times M^n$ is not a locally symmetric manifold.

Part II: Rigidity and nonexistence of complete hypersurfaces in a semi-Riemannian warped product

Let $\overline{M}^{n+1}$ be a semi-Riemannian warped product of the type $\overline{M}^{n+1} = \epsilon I \times_f M^n$, where M^n is an n-dimensional connected oriented Riemannian manifold, $I \subseteq \mathbb{R}$ is an open interval, $f : I \to \mathbb{R}$ is a positive smooth function and $\epsilon = \pm 1$, being $\epsilon = 1$ when $\overline{M}^{n+1}$ is a Riemannian space and $\epsilon = -1$ when $\overline{M}^{n+1}$ is a Lorentzian space. In the Lorentzian case, $\overline{M}^{n+1}$ is called a generalized Robertson-Walker (GRW) spacetime.

This thematic has been treated by several authors along the last years, which have used a considerable amount of analytical tools in their investigations. For instance, Alías and Dajczer [12] studied complete surfaces immersed in a warped product $\mathbb{R} \times_\rho M^2$ such that the fiber M^2 is a complete surface with nonnegative Gaussian curvature. Under certain restrictions on the constant mean curvature, they showed nonexistence results of surfaces contained between two leaves M_{t_1}, M_{t_2} with $t_1 < t_2$ of the foliation $M_t = \{t\} \times M^2$, as well as results in which these immersions are leaves of the trivial totally umbilical foliation. These same authors in [13] generalized the results of Montiel [102] for compact hypersurfaces of constant mean curvature immersed into $\mathbb{R} \times_\rho M^n$, treating the case of complete hypersurfaces via Omori-Yau maximum principle.

Romero, Rubio and Salamanca [114] presented results for complete noncompact maximal hypersurfaces in spatially parabolic GRW spacetimes, that is, the fiber is a complete noncompact Riemannian manifold such that the only superharmonic functions on it which are bounded from below are the constants. In [115], under curvature assumptions on the Riemannian fiber of the ambient space and some conditions on the warping function, these authors also studied complete maximal hypersurfaces in spatially open GRW spacetimes via different maximum principles. Assuming that the ambient spacetime satisfies the Null Convergence Condition (NCC), which means that the Ricci curvature is nonnegative along null directions, Pelegrín and Rigoli [109] also obtained uniqueness and nonexistence results for complete spacelike hypersurfaces of constant mean curvature immersed in a GRW spacetime.

Alías, Impera and Rigoli [17, 18] investigated compact and complete noncompact hypersurfaces with constant higher order mean curvatures H_r, $2 \leq r \leq n$, immersed into semi-Riemannian warped products $\overline{M}^{n+1}$ via a generalized version of the Omori-Yau maximum principle for a divergence-type operator $\mathcal{L}_r$ associated to each globally defined Newton transformation T_r, $0 \leq r \leq n$, which can be regarded as a natural extension of the standard Laplacian operator.

Motivated by the work of Alías and Dajczer [13], García-Martínez, Impera and Rigoli [77] proved height estimates for compact hypersurfaces of constant positive higher order mean curvature in Riemannian warped product spaces with boundary contained in a slice and applied such estimates in the study of properly immersed complete hypersurfaces in pseudo-hyperbolic spaces $\mathbb{R} \times_{e^t} M^n$ or $\mathbb{R} \times_{\cosh t} M^n$ contained in certain half-spaces. Next, under appropriated constraints on the higher order mean curvatures, Aquino, Araújo and de Lima [31] established new sufficient conditions to guarantee the rigidity of hypersurfaces immersed in $\epsilon I \times M^n$ via generalized version of the Omori-Yau maximum principle.

Furthermore, many works have approached problems in the context of entire graphs in semi-Riemannian warped products. For instance, Caminha and de Lima [47] obtained necessary conditions for the existence of complete vertical graphs with constant mean curvature in the hyperbolic and steady state spaces $\epsilon\mathbb{R}\times_{e^t}\mathbb{R}^n$. For this, they deduced suitable formulas for the Laplacian of the height function and of a support-like function naturally attached to the graph. Later on, Alías, Colares and de Lima [11] considered restrictions on H_r to obtain uniqueness results for entire graphs in a warped product satisfying a standard curvature condition. As an application, they obtained rigidity results for minimal and radial graphs over the Euclidean sphere. See also the works [32, 63, 67] for similar rigidity results.

In the Chapter 6, under appropriate differential inequalities involving higher order mean curvatures and assuming that the ambient space obeys suitable curvature constraints, we establish new rigidity and nonexistence results concerning complete spacelike hypersurfaces in a GRW spacetime and, afterwards, we treat the case of complete two-sided hypersurfaces in a Riemannian warped product. Applications to the cases that the ambient space is either an Einstein manifold, a steady state type spacetime or a pseudo-hyperbolic space are given, and a particular investigation of entire graphs construct over the fiber of the ambient space is also made. Our approach is based on a parabolicity criterion related to a linearized differential operator which is a divergence-type operator and can be regarded as a natural extension of the standard Laplacian. This criterion also is obtained as a application of [111, Theorem 2.6].

In Chapter 7, our strategy is to study Riemannian immersions in semi-Riemannian warped products $\overline{M}^{n+1} = \epsilon I \times_\rho M^n$ by applying suitable maximum principles for the operator $\mathcal{L}_r$. Firstly, we will consider criteria of integrability from extension of a result due to Yau in [128, Proposition 2.1] and of generalization of a Liouville-type result due to Pigola, Rigoli and Setti [111, Theorem 1.1] to obtain uniqueness results via integrability on the norm of the gradient of a arbitrary primitive of the warping function ρ. Furthermore, motivated by recent works of Alías, Caminha and Nascimento in [7, 8], we also obtain rigidity and nonexistence results from versions of a maximum principle for vector fields [7, Theorem 2.2] and a maximum principle related to polynomial volume growth [8, Theorem 2.1]. In both works, these authors obtained Bernstein-type results for connected, oriented, complete noncompact Riemannian hypersurfaces with constant mean curvature immersed into a semi-Riemannian warped product. In particular, we obtain nonexistence results of complete spacelike hypersurfaces with polynomial volume growth in GRW spacetimes from appropriate constraints on the hyperbolic angle between the unit timelike vectors N of the hypersurface and ∂_t, that is, the standard unit vector field tangent to I.

We also present applications considering that ambient space is either an Einstein manifold, a steady state-type spacetime or a pseudo-hyperbolic space. Besides, we obtain results in the Schwarzschild and Reissner-Nordström spaces. We recall that, given a mass parameter $\mathfrak{m} > 0$, the *Schwarzschild space* is defined as being the product

$\overline{M}^{n+1} = (r_0(\mathfrak{m}), +\infty) \times \mathbb{S}^n$ furnished with the metric $\bar{g} = V_{\mathfrak{m}}(r)^{-1}dr^2 + r^2 g_{\mathbb{S}^n}$, where $g_{\mathbb{S}^n}$ is the standard metric of $\mathbb{S}^n$,

$$V_{\mathfrak{m}}(r) = 1 - 2\mathfrak{m}r^{1-n}$$

stands for its potential function and

$$r_0(\mathfrak{m}) = (2\mathfrak{m})^{1/(n-1)}$$

is the unique positive root of $V_{\mathfrak{m}}(r) = 0$. Its importance lies in the fact that the manifold $\mathbb{R} \times \overline{M}^{n+1}$ equipped with the Lorentzian static metric $-V_{\mathfrak{m}}(r)dt^2 + \bar{g}$ is a solution of the Einstein field equation in vacuum with zero cosmological constant (see, for instance, [56, Section 4.7] and [107, Chapter 13] for more details concerning Schwarzschild geometry).

As it was observed in [64, Example 1.3], $\overline{M}^{n+1}$ can be reduced in the form $(0, +\infty) \times_\rho \mathbb{S}^n$ with metric (5.1.1) via the following change of variables:

$$t = \int_{r_0(\mathfrak{m})}^{r} \frac{d\sigma}{\sqrt{V_{\mathfrak{m}}(\sigma)}}, \qquad \rho(t) = r(t), \quad t \in (0, +\infty). \tag{5}$$

is the largest positive zero of $V_{\mathfrak{m},\mathfrak{q}}(r)$.

Moreover, given an electric charge $\mathfrak{q} \in \mathbb{R}$ with $|\mathfrak{q}| \leq \mathfrak{m}$, the *Reissner-Nordström space* is defined as being the product $\overline{M}^{n+1} = (r_0(\mathfrak{m}, \mathfrak{q}), +\infty) \times \mathbb{S}^n$ endowed with the metric $\bar{g} = V_{\mathfrak{m},\mathfrak{q}}(r)^{-1}dr^2 + r^2 g_{\mathbb{S}^n}$, where $g_{\mathbb{S}^n}$ is the standard metric of $\mathbb{S}^n$,

$$V_{\mathfrak{m},\mathfrak{q}}(r) = 1 - 2\mathfrak{m}r^{1-n} + \mathfrak{q}^2 r^{2-2n}$$

stands for its potential function and

$$r_0(\mathfrak{m}, \mathfrak{q}) = \left(\frac{\mathfrak{q}^2}{\mathfrak{m} - \sqrt{\mathfrak{m}^2 - \mathfrak{q}^2}} \right)^{1/(n-1)}$$

is the largest positive zero of $V_{\mathfrak{m},\mathfrak{q}}(r)$. The importance of this model lies in the fact that the manifold $\mathbb{R} \times \overline{M}^{n+1}$ equipped with the Lorentzian static metric $-V_{\mathfrak{m},\mathfrak{q}}(r)dt^2 + \bar{g}$ is a charged black-hole solution of the Einstein field equation in vacuum with zero cosmological constant (see, for instance, [56, Remark 4.5] and [123, Section 12.3]). As the Schwarzschild space, $\overline{M}^{n+1}$ can be reduced in the form $(0, +\infty) \times_\rho \mathbb{S}^n$ with metric (5.1.1) via the same change of variables as in (5), just considering $V_{\mathfrak{m},\mathfrak{q}}$ instead of $V_{\mathfrak{m}}$.

At this point, it is worth to point out that Brendle [44] proved an analogue of Alexandrov's theorem for a class of warped product manifolds and obtained rigidity results for compact, embedded hypersurfaces of nonzero constant mean curvature in these spaces. Inspired by this work, Neto [105] also obtained rigidity results replacing the assumption of embeddedness by stability.

Finally, we also establish nonparametric versions for our results in the context of entire graphs constructed over the fiber of the ambient space (see Section 7.2).

This Thesis work was based on the following articles: previously published [26, 27, 28, 25], recently have been accepted for publication [29] and the submitted [30].

Part I

Parabolicity of complete linear Weingarten submanifolds in semi-Riemannian manifolds

Chapter 1

Preliminaries I

In this chapter, we deal with some necessary background and definitions for developing the study the geometry of submanifolds complete linear Weingarten immersed in semi-Riemannian space forms

1.1 Semi-Riemannian space forms

Let $\mathbb{Q}_c^{n+p}$, $n \geq 2$ e $p \geq 1$, be a complete Riemannian manifold, connected $(n+p)$-dimensional of constant sectional curvature $c \in \{-1, 0, 1\}$. We recall that the $\mathbb{Q}_c^{n+p}$ is called *Riemannian space form* and, according to the sign of c, we can determine that universal covering manifold of $\mathbb{Q}_c^{n+p}$ (with the covering metric) is isometric to Hyperbolic space $\mathbb{H}^{n+p}(c = -1)$, Euclidean sphere $\mathbb{S}^{n+p}(c = 1)$ and Euclidean space $\mathbb{R}^{n+p}(c = 0)$(see Theorem 4.1 of [74]).

We also consider the $(n + p + 1)$-dimensional real vector space $\mathbb{R}^{n+p+1}$ endowed with an inner product of index p given by

$$\langle x, y\rangle = -\sum_{i=1}^{p} x_i y_i + \sum_{j=p+1}^{n+p+1} x_j y_j,$$

denoted by $\mathbb{R}_p^{n+p+1}$, where $x = (x_1, x_2, \ldots, x_{n+p+1})$ is the natural coordinate. The semi-Riemannian manifold $\mathbb{R}_p^{n+p+1}$ is called an $n + p + 1$-*dimensional semi-Euclidean space* and it has identically zero sectional curvature c. We define the $(n + p)$-dimensional *de Sitter space* $\mathbb{S}_p^{n+p}$ as the following hyperquadric of $\mathbb{R}_p^{n+p+1}$

$$\mathbb{S}_p^{n+p}(c) := \left\{(x_1, x_2, \ldots, x_{n+p+1}) \in \mathbb{R}_p^{n+p+1} : \langle x, x\rangle = -\sum_{i=1}^{p} x_i^2 + \sum_{i=p+1}^{n+p+1} {x_i}^2 = \frac{1}{c}\right\} \quad (c > 0),$$

The induced metric $\langle\,,\rangle$ makes $\mathbb{S}_p^{n+p}(c)$ a complete semi-Riemannian manifold, connected $(n+p)$-dimensional of index p with constant sectional curvature c equal to 1, and in this case, this space is abbreviated by $\mathbb{S}_p^{n+p}$.

Similarly, the $n+p$-dimensional *anti-de Sitter space* of index p is defined by

$$\mathbb{H}_p^{n+p}(c) := \left\{(x_1, x_2, \ldots, x_{n+p+1}) \in \mathbb{R}_p^{n+p+1} : \langle x, x\rangle = -\sum_{i=1}^{p} x_i^2 + \sum_{i=p+1}^{n+p+1} {x_i}^2 = -\frac{1}{c}\right\} \quad (c < 0),$$

and when we consider $c = -1$, this space is abbreviated by $\mathbb{H}_p^{n+p}$.

These three spaces $\mathbb{R}_p^{n+p+1}$, $\mathbb{S}_p^{n+p}(c)$ and $\mathbb{H}_p^{n+p}(c)$ are complete and of constant curvature c and are called *semi-Riemannian space forms*.

1.2 Parabolicity and modified Cheng-Yau's operator

The main purpose of this first part is to study the geometry of complete Riemnanian submanifolds M^n immersed in a $(n+p)-$dimensional semi-Riemannian manifold $\overline{M}^{n+p}$, having parallel normalized mean curvature vector field h, that is, the mean curvature function H is positive and the corresponding normalized mean curvature vector field $\frac{h}{H}$ is parallel as a section of the normal bundle.

We recall that a Riemannian manifold M^n is said to be parabolic (with respect to the Laplacian operator) if the constant functions are the only subharmonic functions on M^n which are bounded from above; that is, for a smooth function $f : M^n \to \mathbb{R}$

$$\Delta f \geq 0 \quad \text{and} \quad f \leq f^* < +\infty \quad \text{implies} \quad f = \text{constant}.$$

It is well known that $\mathbb{R}$ is parabolic, since there is no non-constant negative harmonic function defined in Euclidean space. When a complete, connected, noncompact Riemannian surface of nonnegative Gaussian curvature, a classical result of Huber in [84] guarantee the parabolicity of such surface. For examples, consider $\mathbb{R}^2$ and $\mathbb{S}^1\times\mathbb{R}$ (see also [60] and [78]). Other examples are obtained from the parabolicity criterion due to Alías and Caminha in [6] when consider a product manifold $M_1 \times M_2$, where M_1 is a connected, compact, oriented Riemannian manifold and M_2 is a parabolic Riemannian manifold.

Extending this previous concept for a class of second-order differential of operators on M^n which is substantially more general than the Laplacian operator, we consider the following operator defined by

$$\mathcal{L}(f) := \mathrm{tr}(\mathcal{P} \circ \nabla^2 f)$$

for every $f \in C^\infty(M)$, where $\mathcal{P} : TM \to TM$ is a symmetric operator on M^n and $\nabla^2 f$ stands for the self-adjoint linear operator metrically equivalent to the Hessian of f. In this setting, a Riemannian manifold M^n is said to be $\mathcal{L}$-parabolic or parabolic with respect to the operator $\mathcal{L}$ if the constant functions are the only smooth functions $f : M^n \to \mathbb{R}$ which are bounded from above and satisfying $\mathcal{L}f \geq 0$. In other words, for a smooth function $f : M^n \to \mathbb{R}$ such that $f^* = \sup_M f$,

$$\mathcal{L}f \geq 0 \quad \text{and} \quad f \leq f^* < +\infty \quad \text{implies} \quad f = \text{constant}.$$

It is well-known that the Laplacian operator Δ is an elliptic operator. The differential operator $\mathcal{L}$ is elliptic (respectively semi-elliptic) if and only if the operator $\mathcal{P}$ is positive definite (respectively positive semi-definite).

In views of these, let us consider the self-adjoint differential operator $\square$ introduced by Cheng and Yau in [59], acting on a smooth function $f : M^n \to \mathbb{R}$ given by

$$\square f := \sum_{i,j} (nH\delta_{ij} - h_{ij}^{n+1}) f_{ij} = nH\Delta f - \sum_{i,j} h_{ij}^{n+1} f_{ij}, \tag{1.2.1}$$

where f_{ij} denote the components of the Hessian of f, $h^{n+1} = (h_{ij}^{n+1})$ denotes the second fundamental form of M^n in the direction of normal vector field $e_{n+1} = \frac{h}{H}$. From (1.2.1), we also have

$$\square f = \mathrm{tr}(P_1 \circ \nabla^2 f), \tag{1.2.2}$$

where $P_1 : TM \to TM$ is the operator given by

$$P_1 = nHI - h^{n+1}, \tag{1.2.3}$$

I being the identity in the algebra of smooth vector fields on M^n.

The following concept will play a fundamental role in the first part of this work.

Definition 1.2.1 *Let M^n be a submanifold in a semi-Riemannian. We say that M^n is a linear Weingarten submanifold if its mean curvature function H and normalized scalar curvature R satisfy $R = aH + b$ for some $a, b \in \mathbb{R}$.*

As mentioned in the introduction, manifolds satisfying this relation were introduced by Weingartein [124, 125] in the study of isometric deformations between surfaces in Euclidean space $\mathbb{R}^3$ preserving the mean curvature, and can be regarded as a natural generalization of manifolds with constant scalar curvature. In such a sense, from of linear relation the mean curvature H and Gaussian curvature K of a surface in Euclidean space $\mathbb{R}^3$ satisfying $aH + bK = c$, where $a, b, c \in \mathbb{R}$, López in [96] presented examples of linear Weingarten surfaces such that the constants a, b, c satisfy $a^2 + 4bc < 0$, the so called rotational linear Weingarten surfaces of hyperbolic type and obtained a family of complete hyperbolic linear Weingarten surfaces in $\mathbb{R}^3$ that consists of surfaces with self-intersections whose generating curves are periodic. Recently, Silva in [66] also showed examples of surfaces in Euclidean, Lorentzian and Hyperbolic 3-space such that $a^2 + 4bc = 0$, the so called tubular surfaces which are the surfaces obtained by the moviment of a circle of constant radius $r > 0$ along a central curve and proved that every Polynomial Weingarten tubular surface is linear.

In order to study of linear Weingarten submanifolds M^n in a semi-Riemannian manifold, we will consider the following modified Cheng-Yau's operator

$$L := \Box - \varepsilon \frac{n-1}{2} a\Delta, \tag{1.2.4}$$

where the constant $\varepsilon = \pm 1$ is chosen appropriately, such that, $\varepsilon = 1$ when the ambient space is Riemannian (see, for instance, [52, 51, 53]), and $\varepsilon = -1$ when ambient space has index $p \geq 1$.

Note that, equivalently, for all smooth function $f : M^n \to \mathbb{R}$, the definition (1.2.4) can be rewritten as follows:

$$Lf = \operatorname{tr}(P \circ \nabla^2 f),$$

where $P : TM \to TM$ be the self-adjoint operator given by

$$P = \left(nH - \varepsilon \frac{n-1}{2} a \right) I - h^{n+1}. \tag{1.2.5}$$

In the next Chapters we will obtain our parabolicity criteria with respect to modified Cheng-Yau's operator L and in Part II this work, a parabolicity criterion with respect to a general class of second order differential operators on M defined in (6.1.1). Our approach is based on a parabolicity criterion related to a linearized differential operator which is a divergence-type operator and obtained as a application of Theorem 2.6 in [111] (see also Lemma 6.9 in [21]).

Theorem 1.2.1 (cf. [111]) *Let (M, g) be a complete Riemannian manifold and let o be a fixed reference point on M. We denote by $r(x)$ the Riemannian distance from x to o, and by B_r the geodesic ball of radius r centered at o. Let h be the symmetric tensor field on M satisfying the following bounds*

$$h_-(r) \leq h(X, X) \leq h_+(r), \forall X \in T_xM, |X| = 1, x \in \partial B_r$$

for some positive continuous functions $h_\pm$ defined on $[0, +\infty)$. If

$$(h_+(r)\mathrm{vol}(\partial B_r))^{-1} \notin L^1(+\infty), \tag{1.2.6}$$

then M is parabolic with respect to the differential operator defined by

$$\mathcal{L}_h u = \mathrm{div}\left(h(\nabla u, \cdot)^\sharp\right), \tag{1.2.7}$$

where $\sharp$ denotes the musical isomorphism.

Chapter 2

L-parabolic complete linear Weingarten submanifolds in a Riemannian space form

Our main goal in this chapter is to present the results obtained in article [29]. We start with the geometry of n-dimensional complete linear Weingarten submanifolds immersed with parallel normalized mean curvature vector in an $(n+p)$-dimensional Riemannian space form $\mathbb{Q}_c^{n+p}$ of constant sectional curvature $c \in \{-1, 0, 1\}$ and via a parabolicity criterion related to modified Cheng-Yau's operator L defined in (1.2.4) we apply it to prove that such a submanifold must be either totally umbilical or isometric to a hyperbolic cylinder, when $c = -1$, a circular cylinder, when $c = 0$, and a Clifford torus, when $c = 1$.

2.1 Submanifolds immersed in a Riemannian space form

Let M^n be an n-dimensional connected submanifold immersed in a Riemannian space form $\mathbb{Q}_c^{n+p}$, with constant sectional curvature $c \in \{-1, 0, 1\}$. We choose a local field of orthonormal frame $\{e_1, \ldots, e_{n+p}\}$ in $\mathbb{Q}_c^{n+p}$ adapted to M^n, that is, at each point of M^n, $e_1, \ldots, e_n$ are tangent to M^n and $e_{n+1}, \ldots, e_{n+p}$ are normal to M^n. Let $\omega_1, \ldots, \omega_{n+p}$ be the corresponding dual coframe and let $\{\omega_{AB}\}$ denote the connection 1-forms on $\mathbb{Q}_c^{n+p}$.

Moreover, throughout of first part this work, the following convention will be used on the range of indices:

$$1 \leq A, B, C, \ldots \leq n+p, \quad 1 \leq i, j, k, \ldots \leq n \quad \text{and} \quad n+1 \leq \alpha, \beta, \gamma, \ldots \leq n+p.$$

Restricting the 1-forms to submanifold M^n, we note that, since e_α is normal to M for all $\alpha = n+1, ..., n+p$,

$$\omega_\alpha = 0.$$

Since

$$0 = d\omega_\alpha = \sum_i \omega_{\alpha i} \wedge \omega_i, \tag{2.1.1}$$

from Cartan's lemma we can write

$$\omega_{\alpha i} = \sum_j h^\alpha_{ij} \omega_j,$$

where (2.1.1) becomes $\sum_{i,j} h^\alpha_{ij} \omega_i \wedge \omega_j = 0$ which implies that

$$h^\alpha_{ij} = h^\alpha_{ji}.$$

Note that

$$-\left(\overline{\nabla}_{e_i} e_\alpha\right)^\top = -\sum_j \omega_{\alpha j}(e_i) e_j = \sum_j h^\alpha_{ij} e_j.$$

This gives the second fundamental form of M^n and its square length,

$$B = \sum_{\alpha,i,j} h^\alpha_{ij} \omega_i \otimes \omega_j \otimes e_\alpha \quad \text{and} \quad S = |B|^2 = \sum_{\alpha,i,j} (h^\alpha_{ij})^2. \tag{2.1.2}$$

Furthermore, the mean curvature vector h and the mean curvature function H of M^n are defined, respectively, by

$$h := \frac{1}{n} \sum_\alpha \left(\sum_i h^\alpha_{ii} \right) e_\alpha \qquad \text{and} \qquad H := |h| = \frac{1}{n} \sqrt{\sum_\alpha \left(\sum_i h^\alpha_{ii} \right)^2}.$$

The components R_{ijkl} of the curvature tensor R and the components $R^\perp_{\alpha\beta kl}$ of the normal curvature tensor $R^\perp$ of M^n are given, respectively, by

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l$$

and

$$d\omega_{\alpha\beta} = \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\alpha} - \frac{1}{2} \sum_{k,l} R^\perp_{\alpha\beta kl} \omega_k \wedge \omega_l.$$

Therefore, the Gauss equation of M^n is

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_{\alpha}(h^{\alpha}_{ik}h^{\alpha}_{jl} - h^{\alpha}_{il}h^{\alpha}_{jk}). \tag{2.1.3}$$

The components of the Ricci tensor R_{ij} and the normalized scalar curvature R of M^n are given, respectively, by

$$R_{ij} = c(n-1)\delta_{ij} + \sum_{\alpha}\left(\sum_{k} h^{\alpha}_{kk}\right)h^{\alpha}_{ij} - \sum_{\alpha,k} h^{\alpha}_{ik}h^{\alpha}_{kj} \quad \text{and} \quad R = \frac{1}{(n-1)}\sum_{i} R_{ii} \tag{2.1.4}$$

The Ricci and Codazzi equations of M^n are given, respectively, by

$$R^{\perp}_{\alpha\beta ij} = \sum_{k}(h^{\alpha}_{ik}h^{\beta}_{kj} - h^{\alpha}_{jk}h^{\beta}_{ki}) \tag{2.1.5}$$

and

$$h^{\alpha}_{ijk} = h^{\alpha}_{ikj} = h^{\alpha}_{jik}, \tag{2.1.6}$$

where the components h^{α}_{ijk} of the covariant derivative ∇B satisfy

$$\sum_{k} h^{\alpha}_{ijk}\omega_k = dh^{\alpha}_{ij} + \sum_{k} h^{\alpha}_{ki}\omega_{kj} + \sum_{k} h^{\alpha}_{kj}\omega_{ki} + \sum_{\beta} h^{\beta}_{ij}\omega_{\beta\alpha}. \tag{2.1.7}$$

From (2.1.4), we get the following relation

$$S = n^2H^2 + n(n-1)(c-R). \tag{2.1.8}$$

By exterior differentiation of (2.1.7), we reach at the following Ricci formula

$$h^{\alpha}_{ijkl} - h^{\alpha}_{ijlk} = \sum_{m} h^{\alpha}_{mj}R_{mikl} + \sum_{m} h^{\alpha}_{im}R_{mjkl} + \sum_{\beta} h^{\beta}_{ij}R^{\perp}_{\beta\alpha kl}. \tag{2.1.9}$$

The Laplacian Δh^{α}_{ij} of the components h^{α}_{ij} of the second fundamental form is defined by

$$\Delta h^{\alpha}_{ij} := \sum_{k} h^{\alpha}_{ijkk}. \tag{2.1.10}$$

Consequently, combining (2.1.2) with Codazzi equation (2.1.6) we obtain

$$\frac{1}{2}\Delta S = \sum_{\alpha,i,j} h^{\alpha}_{ij}\left(\sum_{k} h^{\alpha}_{ijkk}\right) + \sum_{\alpha,i,j,k}(h^{\alpha}_{ijk})^2 = |\nabla B|^2 + \sum_{\alpha,i,j,k} h^{\alpha}_{ij}h^{\alpha}_{kijk}. \tag{2.1.11}$$

Here, we deal with submanifold M^n immersed with parallel normalized mean curvature vector in $\mathbb{Q}^{n+p}_c$, which means that the mean curvature function H is positive and h is parallel as a section of the normal bundle. In particular, we can choose a orthonormal frame $\{e_1, \ldots, e_{n+p}\}$ of $T\mathbb{Q}_c^{n+p}$ such that $e_{n+1} = \frac{h}{H}$. Then we get

$$H^{n+1} := \frac{1}{n}\mathrm{tr}(h^{n+1}) = H \quad \text{and} \quad H^\alpha := \frac{1}{n}\mathrm{tr}(h^\alpha) = 0, \ \text{ for } \alpha \geq n+2, \tag{2.1.12}$$

where h^α denotes the matrix (h_{ij}^α). Thus, using the equations equations (2.1.3)-(2.1.6), together with (2.1.9)-(2.1.11), we can deduce the following Simon-type formula (see Proposition 3.1 of [73])

$$\begin{aligned}\frac{1}{2}\Delta S = |\nabla B|^2 + \sum_{i,j,\alpha} nH_{ij}^\alpha h_{ij}^\alpha + cn(S - nH^2) + n \sum_{\beta,i,j,k} Hh_{ij}^{n+1} h_{jk}^\beta h_{ki}^\beta \\ - \sum_{i,j,k,l}\left(\sum_\alpha h_{ij}^\alpha h_{kl}^\alpha\right)^2 - \sum_{i,j,\alpha,\beta}(R_{\alpha\beta ij}^\perp)^2.\end{aligned} \tag{2.1.13}$$

2.2 A parabolicity criterion for linear Weingarten submanifolds

Now, we consider the appropriate Cheng–Yau's modified operator defined in (1.2.4), the following result provides sufficient conditions which guarantee the L-parabolicity of a linear Weingarten submanifold in a Riemannian space form $\mathbb{Q}_c^{n+p}$. This criterion is obtained as an application of Theorem 2.6 in [111] (cf.Theorem 1.2.1).

Proposition A *Let M^n be a complete linear Weingarten submanifold immersed with parallel normalized mean curvature vector in $\mathbb{Q}_c^{n+p}$ with $c \in \{-1, 0, 1\}$, such that $R = aH + b$ with $b \geq c$. If H is bounded on M^n and, for some reference point $o \in M^n$ and $\delta > 0$,*

$$\int_\delta^{+\infty} \frac{dt}{\mathrm{vol}(\partial B_t)} = +\infty, \tag{2.2.1}$$

where B_t is the geodesic ball of radius t in M^n centered at o, then M^n is L-parabolic.

Proof. Let us consider on M^n the symmetric $(0,2)$-tensor field ξ given by

$$\xi(X, Y) := \langle PX, Y\rangle,$$

for all $X, Y \in TM$ or, equivalently, $\xi(\nabla u, \cdot)^\sharp = P(\nabla u)$ for all smooth function $u : M^n \to \mathbb{R}$, where $\sharp : T^*M \to TM$ denotes the musical isomorphism and P is the operator defined in (1.2.5).

Lemma 5 of [15] guarantees that Cheng-Yau's operator $\Box$ defined in (1.2.1) can be seen as a divergence type operator. Thus, from definition (1.2.4), we get

$$L(u) = \mathrm{div}(P(\nabla u)). \tag{2.2.2}$$

This implies that

$$L(u) = \mathrm{div}\left(\xi(\nabla u, \cdot)^{\sharp}\right).$$

Now, note that since $R = aH + b$ and $b \geq c$, Lemma 3.3 of [33] gives that the operator L is semi-elliptic and, consequently, P is positive semi-definite. We may choose a local frame of orthonormal vector fields $\{e_1, ..., e_n\}$ on M^n such that $h_{ij}^{n+1} = \lambda_i^{n+1}\delta_{ij}$. Thus, we obtain

$$\sum_{i,j} \left(h_{ij}^{n+1}\right)^2 \leq \sum_{\alpha,i,j} \left(h_{ij}^{\alpha}\right)^2 = S.$$

It follows from (2.1.8) that

$$n^2H^2 \geq \left(\lambda_i^{n+1}\right)^2 + n(n-1)aH,$$

for all $i = 1, \dots, n$. Moreover, since

$$\left(\lambda_i^{n+1}\right)^2 \leq n^2H^2 - n(n-1)aH \leq \left(nH - \frac{n-1}{2}a\right)^2$$

which together with the assumption that the normalized mean curvature vector is parallel, we get

$$-nH + \frac{n-1}{2}a \leq \lambda_i^{n+1} \leq nH - \frac{n-1}{2}a, \quad i = 1, \dots, n.$$

Hence, for all $i \in \{1, \dots, n\}$, we obtain

$$0 \leq \sigma_i \leq 2nH - (n-1)a,$$

where $\sigma_i := nH - \frac{n-1}{2}a - \lambda_i^{n+1}$ are the eigenvalues of the operator P (see Lemma 3.4 of [14]). Consequently, we can define a positive continuous function ξ_+ on $[0, +\infty)$ by

$$\xi_+(t) := 2n \sup_{\partial B_t} H - (n-1)a.$$

Then it follows from the assumption that H is bounded on M^n that

$$\xi_+(t) \leq 2n \sup_{M} H - (n-1)a < +\infty.$$

From this last inequality we reach the following estimate:

$$\int_{\delta}^{+\infty} \frac{dt}{\xi_{+}(t)\mathrm{vol}(\partial \mathrm{B}_t)} \geq \left(2n \sup_M H - (n-1)a\right)^{-1} \int_{\delta}^{+\infty} \frac{dt}{\mathrm{vol}(\partial B_t)}.$$

This together with hypothesis (2.2.1)implies that

$$\int_{\delta}^{+\infty} \frac{dt}{\xi_{+}(t)\mathrm{vol}(\partial \mathrm{B}_t)} = +\infty.$$

Therefore, applying Theorem 2.6 of [111], we conclude that M^n is L-parabolic. ■

Remark 2.2.1 *It is worth to comment that we can reason as in the proof of Proposition A to infer that an isometric immersion satisfying (2.2.1) is $\mathcal{L}$-parabolic for $\mathcal{L} = \mathrm{div}(\mathcal{P}(\nabla\cdot))$, where $\mathcal{P}$ is a positive semi-definite tensor such that $\sup \mathcal{P} < +\infty$ and $\mathrm{div}\mathcal{P} \equiv 0$.*

Motivated by the work of Cheng and Yau in [59], we will also consider the following symmetric tensor

$$\Phi = \sum_{\alpha,i,j} \Phi^{\alpha}_{ij}\omega_i \otimes \omega_j e_{\alpha}, \tag{2.2.3}$$

where $\Phi^{\alpha}_{ij} = h^{\alpha}_{ij} - H^{\alpha}\delta_{ij}$ and H^{α} is defined by (2.1.12). Consequently, we have that

$$\Phi^{n+1}_{ij} = h^{n+1}_{ij} - H\delta_{ij} \quad \text{and} \quad \Phi^{\alpha}_{ij} = h^{\alpha}_{ij}, \quad n+2 \leq \alpha \leq n+p. \tag{2.2.4}$$

Let $|\Phi|^2 = \sum_{\alpha,i,j}(\Phi^{\alpha}_{ij})^2$ be the square of the length of Φ. Then, by an easy computation we show that $\mathrm{tr}\,(\Phi) = 0$. From of relation (2.1.8) we obtain

$$|\Phi|^2 = S - nH^2 = nH^2(n-1) + n(n-1)(c - R). \tag{2.2.5}$$

2.3 Auxiliary Lemmas

In this section, we will recall some important well-known lemmas which will be used to prove our characterization results in the next section. The first one is an algebraic Lemma obtained in [116].

Lemma 2.3.1 (cf. [116]) *Let $A, B : \mathbb{R}^n \to \mathbb{R}^n$ be symmetric linear maps such that $AB - BA = 0$ and $\mathrm{tr}(A) = \mathrm{tr}(B) = 0$. Then*

$$\left|\mathrm{tr}(A^2B)\right| \leq \frac{n-2}{\sqrt{n(n-1)}} N(A)\sqrt{N(B)},$$

where $N(A) = \mathrm{tr}(AA^t)$, for any matrix $A = (a_{ij})$. Moreover, the equality holds if, and only if, $(n-1)$ of the eigenvalues x_i of A and corresponding eigenvalues y_i of B satisfy

$$|x_i| = \sqrt{\frac{N(A)}{n(n-1)}},\ x_i y_i \geq 0 \quad \text{and} \quad y_i = \sqrt{\frac{N(B)}{n(n-1)}} \left(\text{resp.} -\sqrt{\frac{N(B)}{n(n-1)}}\right).$$

The second one is also algebraic lemma, whose proof can be found in [89].

Lemma 2.3.2 (cf. [89]) *Let $B^1, B^2, ..., B^p$ be p symmetric $(n \times n)$-matrices $(p \geq 2)$. If $S_{\alpha\beta} = \operatorname{tr}\left((B^\alpha)^t B^\beta\right)$, $S_\alpha = S_{\alpha\alpha} = N(B^\alpha)$ and $S = \sum_\alpha S_\alpha$,then*

$$\sum_{\alpha,\beta} N\left(B^\alpha B^\beta - B^\beta B^\alpha\right) + \sum_{\alpha,\beta} S_{\alpha\beta}^2 \leq \frac{3}{2} S^2, \tag{2.3.1}$$

where $N(B) = \operatorname{tr}(BB^t)$, for any matrix $B = (b_{ij})$.

Consider the classic algebraic lemma, the inequality case is obtained in Lemma 2.1 [108], and the equality is due to Lemma 1 of [24].

Lemma 2.3.3 (cf. [108] and [24]) *Let $\mu_i, i = 1, 2, ..., n$ be real numbers satisfying*

$$\sum_{i=1}^n \mu_i = 0 \quad \text{and} \quad \sum_{i=1}^n \mu_i^2 = k^2,$$

where $k = const.$ nonnegative.Then we have

$$-\frac{n-2}{\sqrt{n(n-1)}} k^3 \leq \sum_{i=1}^n \mu_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}} k^3.$$

and equality holds in the right-hand (left-hand) side if and only if $(n-1)$ of the μ_i's are nonpositive and equal ($(n-1)$ of the μ_i's are nonnegative and equal).

The next key lemma due to [73] is very applicable.

Lemma 2.3.4 (cf. [73]) *Let M^n be a linear Weingarten submanifold immersed in a Riemannian space form $\mathbb{Q}_c^{n+p}$, with $R = aH + b$ for some $a, b \in \mathbb{R}$. Suppose that*

$$(n-1)a^2 + 4n(b-c) \geq 0. \tag{2.3.2}$$

Then

$$|\nabla B|^2 \geq n^2 |\nabla H|^2. \tag{2.3.3}$$

Moreover, the equality holds in (2.3.3) on M^n if, and only if, M^n is an isoparametric submanifold of $\mathbb{Q}_c^{n+p}$.

Remark 2.3.1 *It is worth mentioning also Lemma 3.1 obtaneid in [36] for the case of surfaces. In this case, the authors concluded that the when the equality occurs in (2.3.3) on M^2, the mean curvature function H on M^2 is constant.*

2.4 Characterization Results

Now we are ready to present and prove the main results of this chapter. In what follows, we will apply Proposition A to get new characterization results concerning complete linear Weingarten submanifolds immersed with parallel normalized mean curvature vector in a Riemannian space form $\mathbb{Q}_c^{n+p}$ with constant sectional curvature $c \in \{-1, 0, 1\}$. These results are mostly inspired in those ones obtained in [16, 34, 36, 73, 79].

Initially, we will consider the case of linear Weingarten surfaces immersed with codimension p in $\mathbb{Q}_c^{2+p}$.

Theorem 2.4.1 *Let M^2 be a complete linear Weingarten surface immersed with parallel normalized mean vector and flat normal bundle in $\mathbb{Q}_c^{2+p}$, such that its Gaussian curvature K and mean curvature H satisfy $K = aH + b$ with $b \geq c$. Suppose in addition that there exits a point $q \in M^2$ such that $K(q) > 0$ and that H is bounded on M^2. If hypothesis (2.2.1) is satisfied and K is nonnegative on M^2, then M^2 is totally umbilical.*

Proof. Since we assume that M^2 has flat normal bundle, that is, $R^{\perp} \equiv 0$, for each fixed α, taking a local orthonormal frame $\{e_1, e_2\}$ on M^2 such that $h_{ij}^{\alpha} = \lambda_i^{\alpha}\delta_{ij}$, we obtain from (2.1.9), (2.1.12) and (2.1.13) that

$$\begin{aligned}\frac{1}{2}\Delta S &= |\nabla B|^2 + \sum_{i,j} 2H_{ij}^3 h_{ij}^3 + \sum_{i,j,k,m,\alpha} h_{ij}^{\alpha} h_{mi}^{\alpha} R_{mkjk} + \sum_{i,j,k,m,\alpha} h_{ij}^{\alpha} h_{km}^{\alpha} R_{mijk} \\ &= |\nabla B|^2 + \sum_i \lambda_i^3 (2H)_{ii} + \frac{1}{2}\sum_{i,j,\alpha} R_{ijij}(\lambda_i^{\alpha} - \lambda_j^{\alpha})^2. \end{aligned} \qquad (2.4.1)$$

On the other hand, it follows from of modified Cheng-Yau's operator defined in (1.2.4) acting on the mean curvature function H that

$$\begin{aligned} L(2H) &= \square(2H) - a\Delta H \\ &= 2H\Delta(2H) - \sum_i \lambda_i^3 (2H)_{ii} - a\Delta H \\ &= \frac{1}{2}\Delta(2H)^2 - \sum_i (2H)_i^2 - \sum_i \lambda_i^3 (2H)_{ii} - a\Delta H \\ &= \Delta R + \frac{1}{2}\Delta S - 4|\nabla H|^2 - \sum_i \lambda_i^3 (2H)_{ii} - a\Delta H. \end{aligned} \qquad (2.4.2)$$

Since $R = aH + b$, combining (2.4.1) and (2.4.2) we obtain

$$L(2H) = |\nabla B|^2 - 4|\nabla H|^2 + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i^{\alpha} - \lambda_j^{\alpha})^2. \qquad (2.4.3)$$

Moreover, Gauss equation (2.1.3) gives

$$R_{ijij} = c + \sum_{\beta} \lambda_i^{\beta} \lambda_j^{\beta}. \tag{2.4.4}$$

This together with (2.1.8) and (2.2.5) implies that

$$\begin{aligned}
\frac{1}{2}\sum_{i,j,\alpha} R_{ijij}(\lambda_i^{\alpha}-\lambda_j^{\alpha})^2 &= \sum_{\alpha} R_{1212}(\lambda_1^{\alpha}-\lambda_2^{\alpha})^2 \\
&= 2|\Phi|^2\left(c+\sum_{\beta}\left(\frac{|h^{\beta}|^2}{2}-|\Phi^{\beta}|^2\right)\right) \\
&= 2|\Phi|^2\left(c+\frac{S}{2}-|\Phi|^2\right) \qquad (2.4.5)\\
&= |\Phi|^2(-|\Phi|^2+2H^2+2c) \qquad (2.4.6)\\
&= 2K|\Phi|^2.
\end{aligned}$$

On the other hand, by our assumption $b \geq c$, Lemma 3.1 of [36](cf. Remark 2.3.1) guarantees that

$$|\nabla B|^2 = \sum_{\alpha,i,j,k} (h_{ijk}^{\alpha})^2 \geq 4|\nabla H|^2. \tag{2.4.7}$$

Thus, since the Gaussian curvature K of M^2 is nonnegative, from (2.4.3), (2.4.5) and (2.4.7) we obtain

$$L(2H) \geq 2K|\Phi|^2 \geq 0. \tag{2.4.8}$$

Moreover, Proposition A assures that M^2 is L-parabolic. This together with the assumption H is bounded, the above inequalitywe get H is constant on M^2 which implies, in particular, that $L(2H) = 0$ on M^2. Therefore, since $K = aH + b$ and taking into account the existence of a point $q \in M^2$ such that $K(q) > 0$, from (2.4.8) we get that $|\Phi| \equiv 0$ and, hence, M^2 is totally umbilical. ∎

We recall that, for $n \geq 2$, umbilical submanifolds of a space form $\mathbb{Q}^{n+p}(c)$ have constant curvature $K = c + ||H||^2$, and, besides the totally geodesic submanifolds, if $c \geq 0$ they are the spheres; or, if $c < 0$ we have the geodesic spheres ($K > 0$) or we have the horospheres ($K = 0$) or we have the hyperspheres equidistant ($K < 0$).

Lobos in [97], based on a theorem due to Ferus-Strübing (see [75, 76, 119]), defined and classified a basic class of parallel immersions into space forms, the umbilical manifolds and their extrinsic products.

Finally, our approach allows us to establish the following characterization result concerning n-dimensional complete linear Weingarten submanifolds, now considering the case $n \geq 3$.

Theorem 2.4.2 *Let M^n be a complete linear Weingarten submanifold immersed with parallel normalized mean curvature vector in $\mathbb{Q}_c^{n+p}$ ($n \geq 4$ and $p \geq 1$ or $n \geq 3$ and $p = 1$), such that $R = aH + b$ with $b \geq c$. Suppose in addition that, when $c \in \{-1, 0\}$, $R > 0$ and, when $c = 1$, $R \geq 1$. If hypothesis (2.2.1) is satisfied, H is bounded on M^n and*

$$\sup_M |\Phi|^2 \leq \frac{n(n-1)R^2}{(n-2)(nR-(n-2)c)}, \tag{2.4.9}$$

then

i. either $|\Phi| \equiv 0$ and M^n is totally umbilical,

ii. or $|\Phi|^2 \equiv \dfrac{n(n-1)R^2}{(n-2)(nR-(n-2)c)}$ and M^n is isometric to a

(a) hyperbolic cylinder $\mathbb{H}^1\left(-\sqrt{1+r^2}\right) \times \mathbb{S}^{n-1}(r) \hookrightarrow \mathbb{H}^{n+1} \hookrightarrow \mathbb{H}^{n+p}$, when $c = -1$;

(b) circular cylinder $\mathbb{R} \times \mathbb{S}^{n-1}(r) \hookrightarrow \mathbb{R}^{n+1} \hookrightarrow \mathbb{R}^{n+p}$, when $c = 0$;

(c) Clifford torus $\mathbb{S}^1\left(\sqrt{1-r^2}\right) \times \mathbb{S}^{n-1}(r) \hookrightarrow \mathbb{S}^{n+1} \hookrightarrow \mathbb{S}^{n+p}$, when $c = 1$;

where $r = \sqrt{\dfrac{n-2}{nR}}$.

Proof. In the first place, we will consider the case $n \geq 4$ and $p \geq 1$. Because of relations (2.1.8) and (2.2.5) together with Simon-type formula (2.1.13), we have that the modified Cheng-Yau's operator defined in (1.2.4) acting on the mean curvature function H is expressed as follows:

$$\begin{aligned} L(nH) &= \sum_{i,j}(nH\delta_{ij} - h_{ij}^{n+1})(nH)_{ij} - \frac{n-1}{2}a\Delta(nH) \\ &= \frac{1}{2}n^2\Delta H^2 - n^2|\nabla H|^2 - \frac{n-1}{2}a\Delta(nH) - n\sum_{i,j} h_{ij}^{n+1}H_{ij} \\ &= \left(|\nabla B|^2 - n^2|\nabla H|^2\right) + cn|\Phi|^2 + n\sum_{\beta,i,j,k} Hh_{ij}^{n+1}h_{jk}^{\beta}h_{ki}^{\beta} \\ &\quad -\sum_{i,j,k,l}\left(\sum_\alpha h_{ij}^\alpha h_{kl}^\alpha\right)^2 - \sum_{\alpha,\beta,i,j}(R_{\alpha\beta ij}^{\perp})^2. \end{aligned} \tag{2.4.10}$$

By our assumption $b \geq c$, Lemma 4.1 of [73](cf. Lemma 2.3.4) assure that

$$L(nH) \geq \ cn|\Phi|^2 + n\sum_{\beta,i,j,k} Hh_{ij}^{n+1}h_{jk}^{\beta}h_{ki}^{\beta}$$

$$-\sum_{i,j,k,l}\left(\sum_{\alpha}h_{ij}^{\alpha}h_{kl}^{\alpha}\right)^{2}-\sum_{\alpha,\beta,i,j}(R_{\alpha\beta ij}^{\perp})^{2}. \tag{2.4.11}$$

To estimate the above inequality, we consider

$$I:=\sum_{\beta,i,j,k}Hh_{ij}^{n+1}h_{jk}^{\beta}h_{ki}^{\beta} \tag{2.4.12}$$

and

$$II:=\sum_{i,j,k,l}\left(\sum_{\alpha}h_{ij}^{\alpha}h_{kl}^{\alpha}\right)^{2}+\sum_{\alpha,\beta,i,j}(R_{\alpha\beta ij}^{\perp})^{2}. \tag{2.4.13}$$

Hence, (2.4.11) is rewritten as

$$L(nH)\geq cn|\Phi|^{2}+nI-II. \tag{2.4.14}$$

Firstly, we estimate (2.4.12):

Combining (2.1.12) with (2.2.4), we get

$$\begin{aligned}\sum_{i,j,k,\beta}Hh_{ij}^{n+1}h_{jk}^{\beta}h_{ki}^{\beta}=&H\mathrm{tr}(\Phi^{n+1})^{3}+3H^{2}|\Phi^{n+1}|^{2}+nH^{4}\\&+\sum_{\beta=n+2}^{n+p}\sum_{i,j,k}H\Phi_{ij}^{n+1}\Phi_{jk}^{\beta}\Phi_{ki}^{\beta}+\sum_{\beta=n+2}^{n+p}H^{2}|\Phi^{\beta}|^{2}.\end{aligned} \tag{2.4.15}$$

Taking into account that the matrices Φ^{β} are symmetric and traceless and Φ^{n+1} commutes with all the matrices Φ^{β}, we can use Lemma 2.6 of [116](cf. Lemma 2.3.1) for Φ^{β} and Φ^{n+1} in order to obtain

$$\left|\mathrm{tr}((\Phi^{\beta})^{2}\Phi^{n+1})\right|\leq\frac{n-2}{\sqrt{n(n-1)}}|\Phi^{\beta}|^{2}|\Phi^{n+1}|. \tag{2.4.16}$$

This together with (2.4.12) and the right-hand side of (2.4.15) implies

$$\begin{aligned}I\ \geq\ & -\frac{n-2}{\sqrt{n(n-1)}}H|\Phi^{n+1}|^{3}+2H^{2}|\Phi^{n+1}|^{2}+H^{2}|\Phi|^{2}+nH^{4}\\&-\frac{n-2}{\sqrt{n(n-1)}}\sum_{\beta=n+2}^{n+p}H|\Phi^{n+1}||\Phi^{\beta}|^{2}\\=\ & 2H^{2}|\Phi^{n+1}|^{2}+H^{2}|\Phi|^{2}+nH^{4}-\frac{n-2}{\sqrt{n(n-1)}}H|\Phi^{n+1}||\Phi|^{2}.\end{aligned} \tag{2.4.17}$$

Hence, from (2.4.15) and (2.4.17) we have

$$I=\sum_{\beta,i,j,k}Hh_{ij}^{n+1}h_{jk}^{\beta}h_{ki}^{\beta}\ \geq\ 2H^{2}|\Phi^{n+1}|^{2}+H^{2}|\Phi|^{2}+nH^{4} \tag{2.4.18}$$

$$-\frac{n-2}{\sqrt{n(n-1)}}H|\Phi^{n+1}||\Phi|^2.$$

Secondly, we estimate (2.4.13):

Note that

$$\begin{aligned} II &= \sum_{i,j,k,l}\left(\sum_{\alpha} h_{ij}^{\alpha}h_{kl}^{\alpha}\right)^2 + \sum_{\alpha,\beta,i,j}(R^{\perp}_{\alpha\beta ij})^2 \\ &= \sum_{i,j,k,l}\left(\sum_{\alpha,\beta} h_{ij}^{\alpha}h_{kl}^{\alpha}h_{ij}^{\beta}h_{kl}^{\beta}\right) + \sum_{\alpha\neq n+1,\beta\neq n+1,i,j}(R^{\perp}_{\alpha\beta ij})^2 \qquad (2.4.19) \\ &= \sum_{\alpha,\beta}\left[\mathrm{tr}(B^{\alpha}B^{\beta})\right]^2 + \sum_{\alpha,\beta,i,j,m,l} h_{ij}^{\alpha}h_{im}^{\alpha}h_{ml}^{\beta}h_{lj}^{\beta} \\ &\quad - \sum_{\alpha,\beta,i,j,k,m} h_{ij}^{\alpha}h_{km}^{\alpha}h_{jm}^{\beta}h_{ik}^{\beta} - \sum_{\alpha,\beta,i,j,k} h_{ji}^{\alpha}h_{ik}^{\beta}R^{\perp}_{\beta\alpha ji} \qquad (2.4.20) \end{aligned}$$

Hence, from Ricci equation (2.1.5) we get

$$\begin{aligned} II &= \left[\mathrm{tr}(B^{n+1}B^{n+1})\right]^2 + 2\sum_{\beta\neq n+1}\left[\mathrm{tr}(B^{n+1}B^{\beta})\right]^2 \qquad (2.4.21) \\ &\quad + \sum_{\alpha\neq n+1,\beta\neq n+1}\left[\mathrm{tr}(B^{\alpha}B^{\beta})\right]^2 + \sum_{\alpha\neq n+1,\beta\neq n+1}|B^{\alpha}B^{\beta}-B^{\beta}B^{\alpha}|^2. \end{aligned}$$

It follows from Theorem 1 of [89](cf. Lemma 2.3.2) and (2.2.4) that

$$\begin{aligned} \sum_{\alpha\neq n+1,\beta\neq n+1}[\mathrm{tr}(B^{\alpha}B^{\beta})]^2 + \sum_{\alpha\neq n+1,\beta\neq n+1}|B^{\alpha}B^{\beta}-B^{\beta}B^{\alpha}|^2 &\leq \frac{3}{2}\left(\sum_{\beta\neq n+1}\mathrm{tr}(B^{\beta}B^{\beta})\right)^2 \\ &\leq \frac{3}{2}\left(\sum_{\beta\neq n+1}|\Phi^{\beta}|^2\right)^2. \qquad (2.4.22) \end{aligned}$$

Hence, combining (2.4.21) with (2.4.22) we obtain

$$\begin{aligned} II &\leq [\mathrm{tr}(B^{n+1}B^{n+1})]^2 + 2\sum_{\beta\neq n+1}[\mathrm{tr}(B^{n+1}B^{\beta})]^2 + \frac{3}{2}\left(\sum_{\beta\neq n+1}|\Phi^{\beta}|^2\right)^2 \\ &= |\Phi^{n+1}|^4 + 2nH^2|\Phi^{n+1}|^2 + n^2H^4 + 2\sum_{\beta\neq n+1}[\mathrm{tr}(\Phi^{n+1}\Phi^{\beta})]^2 + \frac{3}{2}(|\Phi|^2-|\Phi^{n+1}|^2)^2 \\ &\leq \frac{5}{2}|\Phi^{n+1}|^4 + 2nH^2|\Phi^{n+1}|^2 + n^2H^4 + 2|\Phi^{n+1}|^2(|\Phi|^2-|\Phi^{n+1}|^2) + \frac{3}{2}|\Phi|^4 \\ &\quad -3|\Phi|^2|\Phi^{n+1}|^2 \\ &= \frac{1}{2}|\Phi^{n+1}|^4 + 2nH^2|\Phi^{n+1}|^2 + n^2H^4 - |\Phi|^2|\Phi^{n+1}|^2 + \frac{3}{2}|\Phi|^4. \qquad (2.4.23) \end{aligned}$$

Therefore, from (2.4.11)-(2.4.14), (2.4.18) and (2.4.23) we get

$$L(nH) \geq cn|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi^{n+1}||\Phi|^2 + nH^2|\Phi|^2 - \frac{1}{2}|\Phi^{n+1}|^4$$

$$
\begin{aligned}
&+|\Phi|^2|\Phi^{n+1}|^2 - \frac{3}{2}|\Phi|^4 \\
=\,& |\Phi|^2\left(-|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| + n(H^2+c)\right) \qquad (2.4.24)\\
&+(|\Phi|-|\Phi^{n+1}|)\Big(\frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi|^2 \\
&-\frac{1}{2}(|\Phi|-|\Phi^{n+1}|)(|\Phi|+|\Phi^{n+1}|)^2\Big).
\end{aligned}
$$

Note that the relations (2.1.8) and (2.2.5) implies that

$$
H^2 = \frac{1}{n(n-1)}|\Phi|^2 + (R-c). \qquad (2.4.25)
$$

Substituting this into (2.4.24), we obtain

$$
\begin{aligned}
L(nH) \geq\,& (|\Phi|-|\Phi^{n+1}|)\Big(\frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi|^2 \qquad (2.4.26)\\
&-\frac{1}{2}(|\Phi|-|\Phi^{n+1}|)(|\Phi|+|\Phi^{n+1}|)^2\Big) + \frac{1}{n-1}|\Phi|^2 Q_R\left(|\Phi|\right),
\end{aligned}
$$

where $Q_R(x)$ is the function introduced by Alías, García-Martínez and Rigoli in [16] which is given by

$$
Q_R(x) = -(n-2)x^2 - (n-2)x\sqrt{x^2+n(n-1)(R-c)} + n(n-1)R. \qquad (2.4.27)
$$

On the other hand, we quote the following algebraic inequality (3.5) of [79]

$$
(|\Phi|-|\Phi^{n+1}|)(|\Phi|+|\Phi^{n+1}|)^2 \leq \frac{32}{27}|\Phi|^3. \qquad (2.4.28)
$$

Again using the relation (2.1.8) we also have $S \leq n^2H^2$ and, consequently, (2.2.5) gives us

$$
H \geq \frac{1}{\sqrt{n(n-1)}}|\Phi|. \qquad (2.4.29)
$$

Thus, (2.4.28) together with (2.4.29) implies that

$$
\frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi|^2 - \frac{1}{2}(|\Phi|-|\Phi^{n+1}|)(|\Phi|+|\Phi^{n+1}|)^2 \geq \left(\frac{n-2}{n-1} - \frac{16}{27}\right)|\Phi|^3. \qquad (2.4.30)
$$

Since $n \geq 4$ we have that

$$
\frac{n-2}{n-1} - \frac{16}{27} > 0. \qquad (2.4.31)
$$

Then, from (2.4.26), (2.4.30) and (2.4.31) we obtain

$$
L(nH) \geq (|\Phi|-|\Phi^{n+1}|)\left(\frac{n-2}{n-1} - \frac{16}{27}\right)|\Phi|^3 + \frac{1}{n-1}|\Phi|^2 Q_R\left(|\Phi|\right)
$$

$$\geq \frac{1}{n-1}|\Phi|^2 Q_R\left(|\Phi|\right). \tag{2.4.32}$$

In the next place, let us consider the case $n \geq 3$ and $p = 1$. In this case, for simplicity, we will just denote $h_{ij} := h_{ij}^{n+1}$. We choose a (local) orthonormal frame $\{e_1, \dots, e_n\}$ on M^n such that $h_{ij} = \lambda_i \delta_{ij}$ and $\Phi_{ij} = \mu_i \delta_{ij}$. Thus,

$$L(nH) \;=\; |\nabla B|^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2. \tag{2.4.33}$$

From Gauss equation (2.1.3) we have $R_{ijij} = c + \lambda_i\lambda_j$. Thus, combining this with the relation (2.2.5) we obtain

$$L(nH) = |\nabla B|^2 - n^2|\nabla H|^2 + nc|\Phi|^2 - S^2 + nH\sum_i \lambda_i^3. \tag{2.4.34}$$

On the other hand, it is straightforward to check that

$$\sum_i \mu_i = 0, \quad \sum_i \mu_i^2 = |\Phi|^2 \quad \text{and} \quad \sum_i \mu_i^3 = \sum_i \lambda_i^3 - 3H|\Phi|^2 - nH^3. \tag{2.4.35}$$

Thus, from (2.4.34) and (2.4.35) we get

$$L(nH) = |\nabla B|^2 - n^2|\nabla H|^2 + nH\sum_i \mu_i^3 + |\Phi|^2(-|\Phi|^2 + nH^2 + nc). \tag{2.4.36}$$

Since $b \geq c$, we can apply Lemma 4.1 of [73](cf. Lemma 2.3.4) and Lemma 2.1 of [108](cf. Lemma 2.3.3)to the real numbers $\mu_1, \dots, \mu_n$ to obtain

$$L(nH) \geq |\Phi|^2\left(-|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| + nH^2 + nc\right). \tag{2.4.37}$$

Consequently, from (2.4.25) and (2.4.37) we also reach at

$$L(nH) \geq \frac{1}{n-1}|\Phi|^2 Q_R\left(|\Phi|\right), \tag{2.4.38}$$

where $Q_R(x)$ is the function given by (2.4.27).

It follows from our constraints on R that $Q_R(0) = n(n-1)R > 0$ and the function $Q_R(x)$ is strictly decreasing for $x \geq 0$, with $Q_R(x^*) = 0$ at

$$x^* = R\sqrt{\frac{n(n-1)}{(n-2)(nR-(n-2)c)}} > 0.$$

Thus, since hypothesis (2.4.9) is satisfied, we obtain that $Q_R\left(|\Phi|\right) \geq 0$. Hence, from (2.4.32) and (2.4.38) we have

$$L(nH) \geq \frac{1}{n-1}|\Phi|^2 Q_R\left(|\Phi|\right) \geq 0.$$

Moreover, Proposition A assures that M^n is L-parabolic. Consequently, from the boundedness of H, we get that it is constant on M^n. This implies, in particular, that $L(nH) = 0$ on M^n. Thus, returning to (2.4.10) and (2.4.33) we have

$$|\nabla B|^2 = n^2 |\nabla H|^2.$$

Therefore, for $n \geq 4$ and $p \geq 1$, Lemma 4.1 of [73](cf. Lemma 2.3.4) guarantees that M^n is an isoparametric submanifold of $\mathbb{Q}_c^{n+p}$. In the case $n \geq 3$ and $p = 1$, Lemma 1 of [24](cf. Lemma 2.3.3) assures that M^n is an isoparametric hypersurface with two distinct principal curvatures one of which is simple.

Now, let us suppose that M^n is not totally umbilical. When $n \geq 4$, taking into account the inequality (2.4.31), from estimate (2.4.32) we conclude that $|\Phi| = |\Phi^{n+1}|$ and, consequently, $\Phi^\alpha = 0$, for all $\alpha \geq n+2$. Thus, since M^n has parallel normalized mean curvature vector we can apply Theorem 1 of [129] to conclude that M^n is isometrically immersed in a $(n+1)$-dimensional totally geodesic submanifold $\mathbb{Q}_c^{n+1}$ of $\mathbb{Q}_c^{n+p}$.Therefore, in both cases, by the classical results on isoparametric hypersurfaces of real space forms [50, 87, 117], we conclude that either $|\Phi|^2 \equiv 0$ and M^n is totally umbilical or

$$|\Phi|^2 \equiv \frac{n(n-1)R^2}{(n-2)(nR-(n-2)c)}$$

and that M^n must be isometric to a following standard product embeddings:

(a) hyperbolic cylinder $\mathbb{H}^1\left(-\sqrt{1+r^2}\right) \times \mathbb{S}^{n-1}(r) \hookrightarrow \mathbb{H}^{n+1} \hookrightarrow \mathbb{H}^{n+p}$, when $c = -1$.

(b) circular cylinder $\mathbb{R} \times \mathbb{S}^{n-1}(r) \hookrightarrow \mathbb{R}^{n+1} \hookrightarrow \mathbb{R}^{n+p}$, when $c = 0$;

(c) Clifford torus $\mathbb{S}^1\left(\sqrt{1-r^2}\right) \times \mathbb{S}^{n-1}(r) \hookrightarrow \mathbb{S}^{n+1} \hookrightarrow \mathbb{S}^{n+p}$, when $c = 1$;

To conclude this proof, we now consider the value of constant sectional curvature $c \in \{-1, 0, 1\}$ of $\mathbb{Q}_c^{n+p}$. In this setting, if $c = -1$, then for a given $r > 0$, in the Hyperbolic space $\mathbb{H}^{n+p}$, we have that

$$\mathbb{H}^1\left(-\sqrt{1+r^2}\right) \times \mathbb{S}^{n-1}(r) \hookrightarrow \mathbb{H}^{n+1} \hookrightarrow \mathbb{H}^{n+p}$$

has constant principal curvatures given by

$$k_1 = \frac{r}{\sqrt{1+r^2}} \quad \text{and} \quad k_2 = \cdots = k_n = \frac{\sqrt{1+r^2}}{r}.$$

Thus,

$$H = \frac{n(1+r^2)-1}{nr\sqrt{1+r^2}} \quad \text{and} \quad |\Phi|^2 = \frac{n-1}{nr^2(1+r^2)}.$$

When $c = 0$, for a given radius $r > 0$, in the the Euclidean space $\mathbb{R}^{n+p}$ we have that

$$\mathbb{R} \times \mathbb{S}^{n-1}(r) \hookrightarrow \mathbb{R}^{n+1} \hookrightarrow \mathbb{R}^{n+p}$$

has constant principal curvatures given by

$$k_1 = 0, k_2 = \cdots = k_n = \frac{1}{r}$$

which implies that

$$H = \frac{n-1}{nr} \quad \text{and} \quad |\Phi|^2 = \frac{n-1}{nr^2}.$$

In the case $c = 1$, for a given radius $0 < r < 1$, in the Euclidean sphere $\mathbb{S}^{n+p}$ we have that

$$\mathbb{S}^1\left(\sqrt{1-r^2}\right) \times \mathbb{S}^{n-1}(r) \hookrightarrow \mathbb{S}^{n+1} \hookrightarrow \mathbb{S}^{n+p}$$

has constant principal curvatures given by

$$k_1 = \frac{r}{\sqrt{1-r^2}}, k_2 = \cdots = k_n = -\frac{\sqrt{1-r^2}}{r}$$

Thus, in this case,

$$H = \frac{nr^2-(n-1)}{nr\sqrt{1-r^2}} \quad \text{and} \quad |\Phi|^2 = \frac{n-1}{nr^2(1-r^2)}.$$

Finally, note that for all $c \in \{-1, 0, 1\}$, the relation (2.2.5) implies that and constant scalar curvature of $\mathbb{Q}_c^{n+p}$ is given by $R = \dfrac{n-2}{nr^2} > 0$. This finishes the proof of theorem. ■

Chapter 3

L-parabolic complete linear Weingarten spacelike submanifolds in the de Sitter space

In this chapter, we present the results concerning the article [28]. By following the same strategy of previous chapter, we can prove our characterization results for complete linear Weingarten spacelike submanifold immersed with parallel normalized mean curvature vector in de Sitter space $\mathbb{S}_p^{n+p}$ of index p. In this setting, imposing appropriate restrictions on the values of the mean curvature function H, we establish a parabolicity criterion related to a suitable Cheng-Yau's modified operator L defined in (1.2.4) and we use it to obtain sufficient conditions which guarantee that such a spacelike submanifold must be either totally umbilical or isometric to certain hyperbolic cylinders of $\mathbb{S}_p^{n+p}$.

3.1 Spacelike Submanifolds immersed in the de Sitter space

An n-dimensional submanifold M^n of $\mathbb{S}_p^{n+p}$ is said to be *spacelike* if the induced metric on M^n from that of the ambient space $\mathbb{S}_p^{n+p}$ is positive definite. We choose a local field of semi-Riemannian orthonormal frames $\{e_1, \ldots, e_{n+p}\}$ in $\mathbb{S}_p^{n+p}$ adapted on M^n. We will use the same indices convention as in the previous chapter:

$$1 \leq A, B, C, \ldots \leq n+p, \quad 1 \leq i, j, k, \ldots \leq n \quad \text{and} \quad n+1 \leq \alpha, \beta, \gamma, \ldots \leq n+p,$$

and taking the corresponding dual coframes $\omega_1, \ldots, \omega_{n+p}$, the semi-Riemannian metric of $\mathbb{S}_p^{n+p}$ is given by

$$ds^2 = \sum_i \omega_i^2 - \sum_\alpha \omega_\alpha^2 = \sum_A \epsilon_A \, \omega_A^2,$$

where $\epsilon_i = 1$ and $\epsilon_\alpha = -1, 1 \le i \le n$ and $n+1 \le \alpha \le n+p$.

Denote by $\{\omega_{AB}\}$ the connection forms of $\mathbb{S}_p^{n+p}$, we have that the structure equations of $\mathbb{S}_p^{n+p}$ are given by

$$d\omega_A = \sum_B \epsilon_B \, \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{3.1.1}$$

$$d\omega_{AB} = \sum_C \epsilon_C \, \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} \epsilon_C \epsilon_D K_{ABCD} \, \omega_C \wedge \omega_D, \tag{3.1.2}$$

where $K_{ABCD} = \epsilon_A \epsilon_B (\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC})$.

Restricting those forms to M^n, that is, applying in $\mathbb{S}_p^{n+p}$, we note that $\omega_\alpha = 0$ on M^n and the Riemannian metric of M^n is written as $ds^2 = \sum_i \omega_i^2$. Thus, from (3.1.1) we obtain

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0 \tag{3.1.3}$$

and

$$\sum_i \omega_{\alpha i} \wedge \omega_i = d\omega_\alpha = 0. \tag{3.1.4}$$

Hence, from Cartan's Lemma we obtain

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j \quad \text{and} \quad h_{ij}^\alpha = h_{ji}^\alpha. \tag{3.1.5}$$

This gives the second fundamental form of M^n and its square length,

$$B = \sum_{\alpha,i,j} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha, \quad \text{and} \quad S = |B|^2 = \sum_{\alpha,i,j} (h_{ij}^\alpha)^2, \tag{3.1.6}$$

respectively. Furthermore, the mean curvature vector h and the mean curvature function H of M^n are defined, respectively, by in

$$h = \frac{1}{n} \sum_\alpha \left(\sum_i h_{ii}^\alpha \right) e_\alpha \quad \text{and} \quad H = |h| = \frac{1}{n} \sqrt{\sum_\alpha \left(\sum_i h_{ii}^\alpha \right)^2}. \tag{3.1.7}$$

From (3.1.2) and (3.1.3), the structure equations of M^n are given by

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0 \quad \text{and}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \tag{3.1.8}$$

where R_{ijkl} are the components of the curvature tensor of M^n. From previous structure equations, we obtain the Gauss equation

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \sum_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}). \tag{3.1.9}$$

The components of the Ricci curvature R_{ij} and the normalized scalar curvature R of M^n are given by

$$R_{ij} = (n-1)\delta_{ij} - \sum_\alpha \left(\sum_k h^\alpha_{kk}\right) h^\alpha_{ij} + \sum_{\alpha,k} h^\alpha_{ik}h^\alpha_{kj} \tag{3.1.10}$$

and

$$R = \frac{1}{n(n-1)}\sum_i R_{ii}, \tag{3.1.11}$$

respectively. Combining (3.1.10) with (3.1.11) we compute the following relation

$$S = n^2H^2 + n(n-1)(R-1). \tag{3.1.12}$$

We also have the structure equations of the normal bundle of M^n

$$d\omega_\alpha = -\sum_\beta \omega_{\alpha\beta} \wedge \omega_\beta, \quad \omega_{\alpha\beta} + \omega_{\beta\alpha} = 0$$

and

$$d\omega_{\alpha\beta} = -\sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} - \frac{1}{2}\sum_{k,l} R_{\alpha\beta kl}\omega_k \wedge \omega_l, \tag{3.1.13}$$

where $R_{\alpha\beta jk}$ satisfy Ricci equation

$$R_{\alpha\beta ij} = \sum_l \left(h^\alpha_{il}h^\beta_{lj} - h^\alpha_{jl}h^\beta_{li}\right). \tag{3.1.14}$$

From (3.1.5) we obtain Codazzi equation

$$h^\alpha_{ijk} = h^\alpha_{ikj} = h^\alpha_{kij}, \tag{3.1.15}$$

where h^α_{ijk} are the components of the covariant derivative ∇B, which satisfy

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} + \sum_k h^\alpha_{ik}\omega_{kj} + \sum_k h^\alpha_{jk}\omega_{ki} - \sum_\beta h^\beta_{ij}\omega_{\beta\alpha}. \tag{3.1.16}$$

Differentiating (3.1.16) exteriorly we obtain the following Ricci formula for the second fundamental form

$$h^{\alpha}_{ijkl} - h^{\alpha}_{ijlk} = \sum_{m} h^{\alpha}_{mj} R_{mikl} + \sum_{m} h^{\alpha}_{im} R_{mjkl} + \sum_{k,\beta} h^{\beta}_{ik} R_{\alpha\beta jk}. \tag{3.1.17}$$

From these formulas, we can compute the Laplacian Δh^{α}_{ij} of the components h^{α}_{ij} of second fundamental form as follows:

$$\Delta h^{\alpha}_{ij} = \sum_{k} h^{\alpha}_{kkij} + \sum_{k,l} h^{\alpha}_{kl} R_{lijk} + \sum_{k,l} h^{\alpha}_{li} R_{lkjk} + \sum_{k,\beta} h^{\beta}_{ik} R_{\alpha\beta jk}. \tag{3.1.18}$$

As we considered in the previous chapter, here we also take $H > 0$ and from of orthonormal frame $\{e_1, \dots, e_{n+p}\}$ of $T\mathbb{S}^{n+p}_{p}$ such that $e_{n+1} = \frac{h}{H}$. Thus, we get

$$H^{n+1} := \frac{1}{n}\mathrm{tr}(h^{n+1}) = H \quad \text{and} \quad H^{\alpha} := \frac{1}{n}\mathrm{tr}(h^{\alpha}) = 0,\ \alpha \geq n+2, \tag{3.1.19}$$

where $h^{\alpha} = (h^{\alpha}_{ij})$ denotes the second fundamental form of M^n in direction e_{α} for every $n+1 \leq \alpha \leq n+p$. Thus, from (3.1.9), (3.1.14), (3.1.18) and (3.1.19) we have

$$\begin{aligned}
\Delta h^{n+1}_{ij} &= nH_{ij} + nh^{n+1}_{ij} - nH\delta_{ij} \\
&\quad + \sum_{\beta,m,k} h^{n+1}_{mk} h^{\beta}_{mk} h^{\beta}_{ij} - 2\sum_{\beta,m,k} h^{n+1}_{mk} h^{\beta}_{mj} h^{\beta}_{ik} \\
&\quad + \sum_{\beta,m,k} h^{n+1}_{mi} h^{\beta}_{mk} h^{\beta}_{jk} - nH\sum_{m} h^{n+1}_{mi} h^{n+1}_{mj} \\
&\quad + \sum_{\beta,m,k} h^{n+1}_{mj} h^{\beta}_{mk} h^{\beta}_{ik}
\end{aligned}$$

and for all $n+2 \leq \alpha \leq n+p$,

$$\begin{aligned}
\Delta h^{\alpha}_{ij} &= nH^{\alpha}_{ij} + nh^{\alpha}_{ij} + \sum_{\beta,m,k} h^{\alpha}_{mk} h^{\beta}_{km} h^{\beta}_{ij} \\
&\quad -2\sum_{\beta,m,k} h^{\alpha}_{mk} h^{\beta}_{jm} h^{\beta}_{ik} + \sum_{\beta,m,k} h^{\alpha}_{mi} h^{\beta}_{mk} h^{\beta}_{kj} \\
&\quad -nH\sum_{m} h^{\alpha}_{mi} h^{n+1}_{jm} + \sum_{\beta,m,k} h^{\alpha}_{mj} h^{\beta}_{mk} h^{\beta}_{ik}.
\end{aligned}$$

Hence, it follows from (2.1.11) the following Simons-type formula

$$\begin{aligned}
\frac{1}{2}\Delta S &= \sum_{\alpha,i,j,k} (h^{\alpha}_{ijk})^2 + n\sum_{\alpha,i,j} h^{\alpha}_{ij} H^{\alpha}_{ij} + n(S - nH^2) + \sum_{\alpha,\beta} \left(\mathrm{tr}(h^{\alpha}h^{\beta})\right)^2 \\
&\quad - nH\sum_{\alpha} \mathrm{tr}\left(h^{n+1}(h^{\alpha})^2\right) + \sum_{\alpha,\beta} N\left(h^{\alpha}h^{\beta} - h^{\beta}h^{\alpha}\right),
\end{aligned} \tag{3.1.20}$$

where $N(A) = \mathrm{tr}(AA^t)$, for all matrix $A = (a_{ij})$.

3.2 A parabolicity criterion for linear Weingarten spacelike submanifolds

Our next result provides sufficient conditions which guarantee the parabolicity of a linear Weingarten spacelike submanifold in de Sitter space $\mathbb{S}_p^{n+p}$ with respect to modified Cheng-Yau's operator L defined in (1.2.4) for $\varepsilon = -1$. With respect to the ellipcity of this operator, the following results are stated in [127] and [19].

Lemma 3.2.1 (cf. [19]) *Let M^n be a spacelike submanifold in the de Sitter space $\mathbb{S}_p^{n+p}$ with $H > 0$. Let μ_- and μ_+ be, respectively, the minimum and the maximum of the eigenvalues of the operator P_1 at every point $p \in M^n$. If $R < 1$(resp., $R \leq 1$ on M^n), then the operator $\square$ is elliptic (resp., semi-elliptic), with $\mu_- > 0$ (resp., $\mu_- \geq 0$). and $\mu_+ < 2nH$ (resp., $\mu_+ \leq 2nH$).*

Proposition B (cf. [127]) *Let M^n be an n-dimensional spacelike linear Weingarten submanifold in de Sitter space $\mathbb{S}_p^{n+p}$ with $R = aH + b$. If $a \neq 0$, $b < 1$, then L is elliptic.*

As a application of Theorem 2.6 in [111] (cf.Theorem 1.2.1), we obtain the following L-parabolicity criterion whose its proof is similar to proof of Proposition A, where we used the results above.

Proposition C *Let M^n be a complete linear Weingarten spacelike submanifold immersed with parallel normalized mean curvature vector in $\mathbb{S}_p^{n+p}$ such that $R = aH + b$ for some $a, b \in \mathbb{R}$ with $b \leq 1$. If H is bounded and, for some reference point $o \in M^n$ and some $\delta > 0$,*

$$\int_\delta^{+\infty} \frac{dt}{\operatorname{vol}(\partial B_t)} = +\infty, \tag{3.2.1}$$

where B_t is the geodesic ball of radius t in M^n centered at the reference point o, then M^n is L-parabolic.

Let Φ be the traceless symmetric tensor defined in (2.2.3). Then, considering a complete spacelike submanifold M^n immersed in $\mathbb{S}_p^{n+p}$, it can be easily checked the following relation

$$|\Phi|^2 = S - nH^2 = n(n-1)H^2 + n(n-1)(R-1). \tag{3.2.2}$$

3.3 Characterization Results

In this section, we revisit the results of [41, 94, 127] and we characterize complete linear Weingarten submanifolds immersed with parallel normalized mean curvature vector in $\mathbb{S}_p^{n+p}$ applying Proposition C. For this, we consider also the result due to [127].

Lemma 3.3.1 (cf. [127]) *Let M^n be a spacelike linear Weingarten submanifold in de Sitter space $\mathbb{S}_p^{n+p}$ with $R = aH + b$ for some $a, b \in \mathbb{R}$. If $b \leq 1$, then*

$$|\nabla B|^2 \geq n^2 |\nabla H|^2. \tag{3.3.1}$$

Moreover, suppose that the equality in (2.3.3) holds, then M^n is constant on M^n.

In this setting, we prepare the following theorem.

Theorem 3.3.1 *Let M^n be a complete linear Weingarten spacelike submanifold immersed with parallel normalized mean curvature vector in $\mathbb{S}_p^{n+p}$, such that $R = aH + b$ for some $a, b \in \mathbb{R}$ with $b \leq 1$. Suppose that M^n has nonnegative sectional curvature and that H is bounded on M^n. If hypothesis (3.2.1) is satisfied, then M^n is either totally umbilical or isometric to a product $M_1 \times M_2 \times \ldots \times M_k$, where the factors M_i are totally umbilical spacelike submanifolds of $\mathbb{S}_p^{n+p}$ which are mutually perpendicular along their intersections.*

Proof. Since $N(B) := \operatorname{tr}(BB^t)$, for any matrix $B = (b_{ij})$, from Ricci equation (3.1.14) we can verify that

$$\sum_{\alpha,\beta,i,j,k} h_{ij}^\alpha h_{ki}^\beta R_{\alpha\beta jk} = \frac{1}{2} \sum_{\alpha,\beta} N(h^\alpha h^\beta - h^\beta h^\alpha). \tag{3.3.2}$$

Thus, by the assumption on the normalized mean curvature vector and combining (3.1.18), (3.1.20) with (3.3.2), the Cheng-Yau's modified operator L defined in (1.2.4) acting on the mean curvature function H is

$$\begin{aligned} L(nH) &= \sum_{\alpha,i,j,k} (h_{ijk}^\alpha)^2 - n^2|\nabla H|^2 + \sum_{i,j,k,m} \left(h_{ij}^\alpha h_{km}^\alpha R_{mijk} + h_{ij}^\alpha h_{mi}^\alpha R_{mkjk}\right) \\ &\quad + \frac{1}{2} \sum_{\alpha,\beta} N\big(h^\alpha h^\beta - h^\beta h^\alpha\big)\,. \end{aligned} \tag{3.3.3}$$

Now, we estimate the above equation. For this, note that, since we are also supposing that $b \leq 1$, from Proposition 2.2 of [127](cf. Lemma 3.3.1) we have

$$|\nabla B|^2 = \sum_{\alpha,i,j,k} (h_{ijk}^\alpha)^2 \geq n^2 |\nabla H|^2. \tag{3.3.4}$$

On the other hand, since the matrix $h^\alpha := (h_{ij}^\alpha)$ can be diagonalized, for each fixed α we consider a local orthonormal frame $\{e_i\}$ such that $h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}$, where λ_i^α

denote the eigenvalue of h^α. Thus, considering the third term of (3.3.3) we obtain the following estimate

$$\sum_{i,j,k,m} h^\alpha_{ij} h^\alpha_{km} R_{mijk} + \sum_{i,j,k,m} h^\alpha_{ij} h^\alpha_{mi} R_{mkjk} \geq \frac{1}{2} \sum_{i,j} (\lambda^\alpha_i - \lambda^\alpha_j)^2 R_{ijij}. \tag{3.3.5}$$

Note that for all matrix $B = (b_{ij})$ we have $(B^t)_{ij} = \sum_k b_{ik} b^t_{kj}$ and

$$N(B) = \mathrm{tr}(BB^t) = \sum_{i,k} b_{ik} b^t_{ki} = \sum_{i,k} \left(b^t_{ki}\right)^2 \geq 0.$$

Consequently, the last term of (3.3.3) is

$$N\big(h^\alpha h^\beta - h^\beta h^\alpha\big) \geq 0. \tag{3.3.6}$$

By assumption that the sectional curvature of M^n is nonnegative, substituting (3.3.4), (3.3.5), (3.3.6) into (3.3.3), we obtain

$$L(nH) \geq \frac{1}{2} \sum_{i,j} (\lambda^\alpha_i - \lambda^\alpha_j)^2 R_{ijij} \geq 0. \tag{3.3.7}$$

Moreover, Proposition C assures that M^n is L-parabolic. Thus, since H is bounded, H is constant on M^n. It follows from (3.3.6) that

$$h^\alpha h^\beta = h^\beta h^\alpha \quad \text{for all} \quad \alpha, \beta > n+1,$$

which implies that the normal bundle of M^n is flat. Hence, all the matrices h^α can be diagonalized simultaneously. Note also that, $h^\alpha_{ijk} = 0$, for all i, j, k, α and thus the second fundamental form B is parallel. In particular, it implies that λ^α_i is constant for all i, α. Therefore,

$$\sum_{i,j} (\lambda^\alpha_i - \lambda^\alpha_j)^2 R_{ijij} = 0.$$

Thus by the assumption of nonnegative sectional curvature, we obtain that M^n has parallel mean curvature vector and constant scalar curvature R. Therefore, we can apply Theorem 1.11 of [54](see also Lemmas 5.1, 5.3 and Theorem 1.3 of [85]) to conclude the proof. ■

Now, imposing a suitable constraint on the mean curvature function, we prove our second characterization result for complete linear Weingarten spacelike submanifold of $\mathbb{S}^{n+p}_p$.

Theorem 3.3.2 *Let M^n, $n \geq 3$, be a complete linear Weingarten spacelike submanifold immersed with parallel normalized mean curvature vector in $\mathbb{S}_p^{n+p}$, such that $R = aH + b$ for some $a, b \in \mathbb{R}$ with $b \leq 1$. Suppose that*

$$H^2 \leq \frac{4(n-1)}{Q(p)}, \tag{3.3.8}$$

where

$$Q(p) = p(n-2)^2 + 4(n-1).$$

If hypothesis (3.2.1) is satisfied, then M^n is either totally umbilical or isometric to the hyperbolic cylinder $\mathbb{S}^{n-1}(c_1) \times \mathbb{H}^1(c_2)$, where $c_1 = \dfrac{n-2}{n-1}$ and $c_2 = -(n-2)$.

Proof. From the assumption that M^n has parallel normalized mean curvature vector e_{n+1}, that is, $\nabla^{\perp} e_{n+1} = 0$, where $\nabla^{\perp}$ denote the normal connection of M^n in $\mathbb{S}_p^{n+p}$. Thus, $\omega_{\alpha n+1} = 0$, for all $\alpha > n+1$. This together with the structure equation (3.1.13) implies that

$$R_{n+1\alpha ij} = 0, \quad \text{for all} \quad 1 \leq i, j \leq n \quad \text{and} \quad \alpha > n+1.$$

It follows from Ricci equation (3.1.14) that $h^{\alpha} h^{n+1} = h^{n+1} h^{\alpha}$ for all α, that is, h^{n+1} commutes with all the matrices h^{α}. Thus, using the traceless symmetric tensor defined in (2.2.3) we obtain that

$$\Phi^{n+1} = h^{n+1} - H^{n+1} I \quad \text{and} \quad \Phi^{\alpha} = h^{\alpha} \quad \text{for all} \quad \alpha > n+1, \tag{3.3.9}$$

which implies that Φ^{n+1} commutes with all the matrices Φ^{α}. Since that the matrices Φ^{α} are symmetric and traceless, we can use Lemma 2.6 of [116](cf. Lemma 2.3.1) for Φ^{α} and Φ^{n+1} in order to obtain

$$\left|\mathrm{tr}((\Phi^{\alpha})^2 \Phi^{n+1})\right| \leq \frac{n-2}{\sqrt{n(n-1)}} N(\Phi^{\alpha}) \sqrt{N(\Phi^{n+1})}. \tag{3.3.10}$$

On the other hand, using Cauchy-Schwarz inequality we get that

$$p \sum_{\alpha,\beta} [\mathrm{tr}(\Phi^{\alpha} \Phi^{\beta})]^2 \geq p \sum_{\alpha} [\mathrm{tr}(\Phi^{\alpha})^2]^2 = p \sum_{\alpha} [N(\Phi^{\alpha})]^2 \geq \left(\sum_{\alpha} N(\Phi^{\alpha}) \right)^2 = |\Phi|^4. \tag{3.3.11}$$

Furthermore, combining (3.2.2) with (3.1.20) it follows that

$$\square(nH) = \sum_{\alpha,i,j,k} (h_{ijk}^{\alpha})^2 - n^2 |\nabla H|^2 - nH \sum_{\alpha} \mathrm{tr}\left(h^{n+1} (h^{\alpha})^2\right)$$

$$+\sum_{\alpha,\beta} N\left(h^\alpha h^\beta - h^\beta h^\alpha\right) + \sum_{\alpha,\beta}\left(\mathrm{tr}(h^\alpha h^\beta)\right)^2$$
$$+n(S - nH^2) - \frac{n-1}{2} a\Delta(nH). \tag{3.3.12}$$

Consequently, since

$$N(\Phi^{n+1}) = \mathrm{tr}(\Phi^{n+1})^2 \leq |\Phi|^2 \quad \text{and} \quad \sum_\alpha N(\Phi^\alpha) = |\Phi|^2,$$

substituting (3.3.10), (3.3.11) and (3.3.12) into Cheng-Yau's modified operator L (1.2.4) acting on the mean curvature function H, we obtain

$$L(nH) \geq |\Phi|^2 P_H(|\Phi|),$$

where

$$P_H(x) = \frac{x^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}} Hx - n\left(H^2 - 1\right). \tag{3.3.13}$$

Note that, when $H^2 < \dfrac{4(n-1)}{Q(p)}$, by a direct computation, it is not difficult to verify that $P_H(|\Phi|) > 0$.

In the case $H^2 = \dfrac{4(n-1)}{Q(p)}$, we can write $H = \dfrac{2\sqrt{n-1}}{\sqrt{Q(p)}}$ and the polynomial P_H just has a real root, namely

$$C(n,p) = \frac{p(n-2)\sqrt{n}}{\sqrt{Q(p)}}.$$

Hence, in this case,

$$P_H(|\Phi|) = \left(\frac{|\Phi|}{\sqrt{p}} - \frac{(n-2)\sqrt{np}}{\sqrt{Q(p)}}\right)^2 \geq 0.$$

This implies that, in both cases, we have

$$L(nH) \geq |\Phi|^2 P_H(|\Phi|) \geq 0.$$

Moreover, Proposition C assures that M^n is L-parabolic. Thus, from the boundedness of H, we get that it is constant on M^n implying, in particular, that $L(nH) = 0$ on M^n.

Consequently, if $|\Phi|^2 = 0$, then M^n is totally umbilical. Otherwise, if $P_H(|\Phi|) = 0$, the inequalities (3.3.10) and (3.3.11) are, in facts, equalities. In particular, $N(\Phi^{n+1}) = \mathrm{tr}(\Phi^{n+1})^2 = |\Phi|^2$. It follows from (3.2.2) that

$$\mathrm{tr}(\Phi^{n+1})^2 = |\Phi|^2 = S - nH^2. \tag{3.3.14}$$

Note that, the assumption implies that M^n has parallel normalized mean curvature vector together with (3.1.19) implies that

$$\mathrm{tr}(\Phi^{n+1})^2 = S - \sum_{\alpha>n+1}\sum_{i,j}(h_{ij}^{\alpha})^2 - nH^2. \tag{3.3.15}$$

Hence, combining (3.3.14) and (3.3.15) we conclude that $\sum_{\alpha>n+1}\sum_{i,j}(h_{ij}^{\alpha})^2 = 0$.

On the other hand, using that inequality (3.3.11) is, in fact, an equality, we get

$$p|\Phi|^4 = pN(\Phi^{n+1})^2 = p\sum_{\alpha\geq n+1}[N(\Phi^{\alpha})]^2 = |\Phi|^4.$$

Thus, since in this case $|\Phi| > 0$, we conclude that $p = 1$.

Moreover, since the mean curvature function H is constant on M^n and by assumption $b \leq 1$, Proposition 2.2 of [127](cf. Lemma 3.3.1) guarantee that

$$\sum_{i,j,k}(h_{ijk}^{n+1})^2 = n^2|\nabla H|^2 = 0,$$

that is, $h_{ijk}^{n+1} = 0$ for all i, j, k. Therefore, M^n is an isoparametric spacelike hypersurface of $\mathbb{S}_1^{n+1}$. We also note that the inequality in (3.3.8) cannot be strict, otherwise, $P_H(|\Phi|) > 0$. This implies that

$$H = \frac{2\sqrt{n-1}}{n} \quad \text{and} \quad |\Phi| = \frac{n-2}{\sqrt{n}}. \tag{3.3.16}$$

Since the equality occurs in (3.3.10), we have that also happens the equality in Lemma 2.6 of [116](cf. Lemma 2.3.1). Thus, M^n must be either totally umbilical or an isoparametric spacelike hypersurface of $\mathbb{S}_1^{n+1}$ with two distinct principal curvatures one of which is simple. In this last case, we can apply Theorem 5.1 of [1] to conclude that M^n is isometric to a hyperbolic cylinder $\mathbb{S}^{n-k}(c_1) \times \mathbb{H}^k(c_2)$, where $k \in \{1, n-1\}$, $c_1 > 0$, $c_2 < 0$ and $\frac{1}{c_1} + \frac{1}{c_2} = 1$.

With a straightforward computation it is not difficult to verify that, for a suitable choice of the normal vector field, in $\mathbb{S}_1^{n+1}$ we have that

$$\mathbb{S}^{n-k}(c_1) \times \mathbb{H}^k(c_2) \hookrightarrow \mathbb{S}^{n+1}$$

has principal curvatures given by

$$\lambda_1 = \cdots = \lambda_{n-k} = \sqrt{1-c_1} \quad \lambda_{n-k+1} = \cdots = \lambda_n = \sqrt{1-c_2}. \tag{3.3.17}$$

Hence,

$$nH = (n-k)\sqrt{1-c_1} + k\sqrt{1-c_2} \tag{3.3.18}$$

and

$$S = (n-k))(1-c_1) + k(1-c_2). \tag{3.3.19}$$

Substituting (3.3.18) and (3.3.19) into relation (3.2.2), we obtain that

$$|\Phi|^2 = \frac{n}{4k(n-k)}((n-2k)H \pm \sqrt{n^2H^2 - 4k(n-k)})^2. \tag{3.3.20}$$

This together with (3.3.16) implies that M^n must be isometric to $\mathbb{S}^{n-1}(c_1) \times \mathbb{H}^1(c_2)$, with $c_1 = \dfrac{n-2}{n-1}$ and $c_2 = -(n-2)$. ■

In the case $n = 2$, as consequence of Theorem 3.3.2 and Cheng's Theorem [57] we obtain

Corollary 3.3.1 *Let M^2 be a complete linear Weingarten spacelike surface immersed with parallel normalized mean curvature vector in $\mathbb{S}^{2+p}_p$, such that $R = aH + b$ for some $a, b \in \mathbb{R}$ with $b \leq 1$. If $H^2 \leq 1$ and hypothesis (3.2.1) is satisfied, then M^2 is totally umbilical.*

We recall that Montiel in [100] (see also Section 3 of [103]) classified all totally umbilical spacelike hypersurfaces of de Sitter space $\mathbb{S}^{n+1}_1$. Such hypersurfaces are obtained by intersecting the de Sitter space $\mathbb{S}^{n+1}_1$ with affine hyperplanes $\{p \in \mathbb{R}^{n+2}_1; \langle p, a\rangle = \tau\}$ of the ambient space $\mathbb{R}^{n+2}_1$, where $a \in \mathbb{R}^{n+2}_1 - \{0\}$ and $\tau^2 > \langle a, a\rangle = c$ with $c \in \{1, 0, -1\}$.

$$M_\tau = \{x \in \mathbb{S}^{n+1}_1 : \langle p, a\rangle = \tau\}.$$

Then, for $p \in M_\tau$, the unit (timelike) normal fields on M_τ is given by

$$N_\tau(p) = \frac{1}{\sqrt{\tau^2 - c}}(a - \tau p).$$

Hence, the second fundamental form of M_τ is

$$A_\tau X = \frac{\tau}{\tau^2 - c} X,$$

for all vector field X tangent to M_τ. So, M^n_τ has constant mean curvature $H^2 = \dfrac{\tau^2}{\tau^2 - c}$. In fact, it can be verified that:

i. If $c = 1$, that is, a is a spacelike vector, then M^n_τ is isometric to an n-dimensional hyperbolic space of constant sectional curvature $\dfrac{-1}{\tau^2 - 1}$ and H^2 ranges all possible values in $(1, \infty)$;

ii. If $c = 0$, that is, a is a null vector, then M^n_τ is isometric to an n-dimensional Euclidean space $\mathbb{R}^n$ and $H^2 = 1$;

iii. If $c = -1$, that is, a is a timelike vector, then M^n_τ is isometric to an n-dimensional sphere with constant sectional curvature $\dfrac{1}{\tau^2+1}$ and H^2 takes all possible values in $[0, 1)$;

Now, when considering the opposite inequality to given in (3.3.8) and under appropriate restriction on the norm of traceless symmetric tensor defined in (2.2.3), we present our third characterization result.

Theorem 3.3.3 *Let M^n, $n \geq 3$, be a complete linear Weingarten spacelike submanifold immersed with parallel normalized mean curvature vector in $\mathbb{S}^{n+p}_p$, such that $R = aH + b$ for some $a, b \in \mathbb{R}$ with $b \leq 1$. Suppose that*

$$\frac{4(n-1)}{Q(p)} \leq H^2 < 1 \quad \textit{and} \quad |\Phi| \leq C_-(n, p, H), \tag{3.3.21}$$

where is the real root of P_H given by

$$C_-(n,p,H) = \frac{\sqrt{n}}{2\sqrt{n-1}}\left(p(n-2)H - \sqrt{pQ(p)H^2 - 4p(n-1)}\right).$$

If hypothesis (3.2.1) is satisfied, then either M^n is totally umbilical or isometric to the hyperbolic cylinder $\mathbb{S}^{n-1}(c_1) \times \mathbb{H}^1(c_2)$, where $0 < c_1 < \dfrac{n(n-2)}{(n-1)^2}$ and $-n(n-2) < c_2 < 0$ with $\dfrac{1}{c_1} + \dfrac{1}{c_2} = 1$.

Proof. From hypothesis (3.3.21), by a direct computation, it is not difficult to verify that the polynomial P_H defined in (3.3.13) has two distinct real roots, which are given by

$$C_\pm(n,p,H) = \frac{\sqrt{n}}{2\sqrt{n-1}}\left(p(n-2)H \pm \sqrt{pQ(p)H^2 - 4p(n-1)}\right).$$

We observe that $C_-(n, p, H)$ is positive if, and only if,

$$\frac{4(n-1)}{Q(p)} \leq H^2 < 1.$$

Hence, we have that $P_H(|\Phi|) \geq 0$ for $|\Phi| \leq C_-(n, p, H)$, and $P_H(|\Phi|) = 0$ if, and only if, $|\Phi| = C_-(n, p, H)$. Consequently, proceeding in a similar way of the last part of proof of Theorem 3.3.2, we obtain that M^n must be an isoparametric spacelike hypersurface of $\mathbb{S}^{n+1}_1$ and, from Theorem 5.1 of [1] to conclude that M^n is isometric to a hyperbolic cylinder $\mathbb{S}^{n-1}(c_1) \times \mathbb{H}^1(c_2)$ with $c_1 > 0$, $c_2 < 0$ and $\frac{1}{c_1} + \frac{1}{c_2} = 1$. Moreover, combining the assumption (3.3.21) with equation (3.3.20), we have $0 < c_1 < \frac{n(n-2)}{(n-1)^2}$ and $-n(n-2) < c_2 < 0$. ∎

We closed our chapter by presenting a new version of Theorem 1.1 of [94] and Theorem 1.4 of [127].

Theorem 3.3.4 *Let M^n be a complete spacelike linear Weingarten submanifold immersed with parallel normalized mean curvature vector in $\mathbb{S}_p^{n+p}$, such that $R = aH + b$ for some $a, b \in \mathbb{R}$ with $b \leq 1$. If $S \leq 2\sqrt{n-1}$ and hypothesis (3.2.1) is satisfied, then either*

i. M^n is totally umbilical, or

ii. $S = 2\sqrt{n-1}$. Moreover, when $b < 1$, M^n is isometric to a hyperbolic cylinder $\mathbb{S}^{n-1}(c_1) \times \mathbb{H}^1(c_2)$ with $\frac{1}{c_1} + \frac{1}{c_2} = 1$.

Proof. Since $R = aH + b$, Cheng-Yau's modified operator L (1.2.4) acting on the mean curvature function H can be estimated as follows

$$L(nH) \geq \sum_{\alpha,i,j,k} (h_{ijk}^{\alpha})^2 - n^2|\nabla H|^2 + n(S - nH^2)\left(n - \frac{n}{2\sqrt{n-1}}S\right) \quad (3.3.22)$$

This together with Proposition 2.2 of [127](cf. Lemma 3.3.1) and relation (3.2.2) implies that

$$L(nH) \geq |\Phi|^2 \left(n - \frac{n}{2\sqrt{(n-1)}}S\right). \quad (3.3.23)$$

From assumption $S \leq 2\sqrt{n-1}$ and (3.3.23) we obtain that $L(nH) \geq 0$. This hypothesis with relation (3.1.12) also gives us that H is bounded on M^n.

Moreover, Proposition C guarantee that M^n is L-parabolic. Thus, we obtain that H is constant on M^n. Consequently, returning to (3.3.23) we obtain

$$|\Phi|^2 \left(n - \frac{n}{2\sqrt{(n-1)}}S\right) = 0 \text{ on } M^n.$$

Consequently, if $|\Phi|^2 = 0$, then M^n is totally umbilical. Otherwise, $S = 2\sqrt{n-1}$. In the case when $b < 1$, proceeding in a similar way of the last part of proof of Theorem 3.3.2 we conclude the proof of theorem.

∎

Chapter 4

L-parabolic complete linear Weingarten submanifolds immersed in an Einstein manifold

This chapter aims to present the results related to the articles [26, 27], where the results of the second reference correspond to a natural improvement of the previous ones obtained in the first one. We start with the geometry of a spacelike submanifold immersed in a semi-Riemannian manifold and, by virtue of the work of [40], we obtain a Simons-type formula and a characterization result via parabolicity criterion for complete linear Weingarten spacelike submanifolds immersed with parallel normalized mean curvature vector and flat normal bundle in an Einstein manifold $\mathcal{E}_p^{n+p}$ of index p satisfying the curvature conditions (1), (2), (3) and (4).

4.1 Spacelike submanifolds immersed in a semi-Riemannian manifold

Let L_p^{n+p} be an $(n+p)$-dimensional semi-Riemannian manifold of index p and let M^n be a spacelike submanifold immersed in L_p^{n+p}. In this context, we choose a local field of semi-Riemannian orthonormal frames $\{e_1, \ldots, e_{n+p}\}$ in L_p^{n+p} adapted to M^n. Using the same indices convention as in the previous chapters:

$$1 \leq A, B, C, \ldots \leq n+p, \quad 1 \leq i, j, k, \ldots \leq n \quad \text{and} \quad n+1 \leq \alpha, \beta, \gamma, \ldots \leq n+p.$$

Let $\omega_1, \ldots, \omega_{n+p}$ be its corresponding dual coframes so that the semi-Riemannian metric of L_p^{n+p} is given by $ds^2 = \sum_A \epsilon_A \omega_A^2$.Then the structure equations of L_p^{n+p} are given by

$$d\omega_A = -\sum_B \epsilon_B \, \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{4.1.1}$$

$$d\omega_{AB} = -\sum_C \epsilon_C \, \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} \epsilon_C \epsilon_D \overline{R}_{ABCD} \, \omega_C \wedge \omega_D, \tag{4.1.2}$$

where $\overline{R}_{ABCD}$ denote the components of the curvature tensor of L_p^{n+p}. In this setting, denoting by $\overline{R}_{CD}$ and $\overline{R}$ the the components of the Ricci tensor and the scalar curvature of L_p^{n+p}, respectively, we also have $\overline{R}_{CD} = \displaystyle\sum_B \epsilon_B \overline{R}_{CBDB}$, and $\overline{R} = \sum_A \epsilon_A \overline{R}_{AA}$. Moreover, the components $\overline{R}_{ABCD;E}$ of the covariant derivative of the curvature tensor of L_p^{n+p} are defined by

$$\sum_E \epsilon_E \overline{R}_{ABCD;E} \omega_E = d\overline{R}_{ABCD} - \sum_E \epsilon_E \left(\overline{R}_{EBCD} \omega_{EA} + \overline{R}_{AECD} \omega_{EB} \right.$$
$$\left. + \overline{R}_{ABED} \omega_{EC} + \overline{R}_{ABCE} \omega_{ED} \right).$$

Restricting all the tensors to M^n, i.e., applying in L_p^{n+p}, we have $\omega_\alpha = 0$ on M^n. Hence, from (4.1.1) and we obtain

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0 \quad \text{and} \tag{4.1.3}$$

$$\sum_i \omega_{\alpha i} \wedge \omega_i = d\omega_\alpha = 0. \tag{4.1.4}$$

Thus, from Cartan's Lemma we obtain

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j \quad \text{and} \quad h_{ij}^\alpha = h_{ji}^\alpha, \tag{4.1.5}$$

As in the previous chapters, let

$$B = \sum_{\alpha,i,j} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha, \quad \text{and} \quad S = |B|^2 = \sum_{\alpha,i,j} (h_{ij}^\alpha)^2, \tag{4.1.6}$$

be the second fundamental form of M^n and its square length, respectively. Furthermore, the mean curvature vector h and the mean curvature function H of M^n are defined, respectively, by in

$$h = \frac{1}{n} \sum_\alpha \left(\sum_i h_{ii}^\alpha \right) e_\alpha \quad \text{and} \quad H = |h| = \frac{1}{n} \sqrt{\sum_\alpha \left(\sum_i h_{ii}^\alpha \right)^2}. \tag{4.1.7}$$

From (4.1.1) and (4.1.2), we deduce that the connection forms $\{\omega_{ij}\}$ of M^n are characterized by the following structure equations

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0 \quad \text{and}$$
$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \tag{4.1.8}$$

where R_{ijkl} are the components of the curvature tensor of M^n. From previous structure equations, we obtain Gauss equation (see Theorem 4.5 of [107])

$$R_{ijkl} = \overline{R}_{ijkl} - \sum_\beta (h^\beta_{ik}h^\beta_{jl} - h^\beta_{il}h^\beta_{jk}). \tag{4.1.9}$$

Hence, combining (4.1.6), (4.1.7) with (4.1.9) we also get the following relation

$$S = n^2H^2 + n(n-1)R - \sum_{i,j} \overline{R}_{ijij}, \tag{4.1.10}$$

where R stands for the normalized scalar curvature of M^n. Moreover, the first covariant derivatives h^α_{ijk} of h_{ij} satisfy

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} - \sum_k h^\alpha_{ik}\omega_{kj} - \sum_k h^\alpha_{jk}\omega_{ki} - \sum_\beta h^\beta_{ij}\omega_{\beta\alpha}. \tag{4.1.11}$$

Then, by exterior differentiation of (4.1.5) we get Codazzi equation (see Theorem 4.33 of [107])

$$h^\alpha_{ijk} - h^\alpha_{ikj} = \overline{R}_{\alpha ijk}. \tag{4.1.12}$$

The second covariant derivatives h^α_{ijkl} of h^α_{ij} are given by

$$\sum_l h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk} - \sum_l h^\alpha_{ljk}\omega_{li} - \sum_l h^\alpha_{ilk}\omega_{lj} - \sum_l h^\alpha_{ijl}\omega_{lk} - \sum_\beta h^\beta_{ijk}\omega_{\beta\alpha}.$$

Taking the exterior derivative in (4.1.11), we obtain the following Ricci formula

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = -\sum_m h^\alpha_{im}R_{mjkl} - \sum_m h^\alpha_{mj}R_{mikl}. \tag{4.1.13}$$

Restricting the covariant derivative $\overline{R}_{ABCD;E}$ of $\overline{R}_{ABCD}$ to M^n, we get

$$\overline{R}_{\alpha ijk;l} = \overline{R}_{\alpha ijk;l} + \sum_\beta \overline{R}_{\alpha\beta jk}h^\beta_{il} + \sum_\beta \overline{R}_{\alpha i\beta k}h^\beta_{jl} \tag{4.1.14}$$

$$+\sum_{\beta}\overline{R}_{\alpha ij\beta}h^{\beta}_{kl}+\sum_{m,k}\overline{R}_{mijk}h^{\alpha}_{lm},$$

where $\overline{R}_{\alpha ijkl}$ denotes the covariant derivative of $\overline{R}_{\alpha ijk}$ as a tensor on M^n. Note that, when we suppose that M^n has flat normal bundle, that is, $R^{\perp}=0$ (equivalently, $R_{\alpha\beta jk}=0$), $\overline{R}_{\alpha\beta jk}$ satisfy Ricci equation

$$\overline{R}_{\alpha\beta ij}=\sum_{k}(h^{\alpha}_{ik}h^{\beta}_{kj}-h^{\alpha}_{kj}h^{\beta}_{ik}). \tag{4.1.15}$$

In next section, let us consider the particular case when L^{n+p}_p is the Einstein manifold. From now on, we will denote this ambient space by $\mathcal{E}^{n+p}_p$ of index p. We recall that a semi-Riemannian manifold is called an Einstein manifold when its Ricci curvature can be written as a multiple of its metric.

4.2 A Simons-type formula

Taking into account the previous digression, when deal with a spacelike submanifold M^n immersed with parallel normalized mean curvature vector in an Einstein manifold $\mathcal{E}^{n+p}_p$, we can choose a orthonormal frame $\{e_1,\dots,e_{n+p}\}$ of $T\mathcal{E}^{n+p}_p$ such that $e_{n+1}=\frac{h}{H}$. It follows that

$$H^{n+1}:=\frac{1}{n}\mathrm{tr}(h^{n+1})=H\quad\text{and}\quad H^{\alpha}:=\frac{1}{n}\mathrm{tr}(h^{\alpha})=0,\ \alpha\geq n+2, \tag{4.2.1}$$

where h^{α} denotes the matrix (h^{α}_{ij}).

Thus, by similar method to the proofs of Lemma 2 in [40] and Lemma 3.1 in [93], we can show the following Simons-type formula for a spacelike submanifold immersed into an Einstein manifold $\mathcal{E}^{n+p}_p$.

Lemma 4.2.1 *Let M^n be a spacelike submanifold immersed with parallel normalized mean curvature vector and flat normal bundle in an Einstein manifold $\mathcal{E}^{n+p}_p$ of index p. Suppose that there exists an orthogonal basis for TM that diagonalizes simultaneously all operators B_{η} with $\eta\in TM^{\perp}$, where $\langle B_{\eta}u,v\rangle:=\langle B(u,v),\eta\rangle$ for any u, $v\in TM$. Then*

$$\begin{aligned}\frac{1}{2}\Delta S &= |\nabla B|^2+2\left(\sum_{i,j,k,m,\alpha}h^{\alpha}_{ij}h^{\alpha}_{km}\overline{R}_{mijk}+\sum_{i,j,k,m,\alpha}h^{\alpha}_{ij}h^{\alpha}_{jm}\overline{R}_{mkik}\right)\\ &\quad+\sum_{i,j,k,\alpha,\beta}h^{\alpha}_{ij}h^{\beta}_{jk}\overline{R}_{\alpha i\beta k}-\sum_{i,j,k,\alpha,\beta}h^{\alpha}_{ij}h^{\beta}_{jk}\overline{R}_{\alpha k\beta i}\end{aligned} \tag{4.2.2}$$

$$+\sum_{i,j,k,\alpha,\beta} h_{ij}^{\alpha} h_{ij}^{\beta} \overline{R}_{\alpha k\beta k} - \sum_{i,j,k,\alpha,\beta} h_{ij}^{\alpha} h_{kk}^{\beta} \overline{R}_{\alpha i\beta j} + n\sum_{i,j} h_{ij}^{n+1} H_{ij}$$
$$-nH \sum_{i,j,m,\alpha} h_{ij}^{\alpha} h_{mi}^{\alpha} h_{mj}^{n+1} + \sum_{\alpha,\beta} [\mathrm{tr}(h^{\alpha}h^{\beta})]^2 + \frac{3}{2}\sum_{\alpha,\beta} N(h^{\alpha}h^{\beta} - h^{\beta}h^{\alpha}),$$

where $N(A) = \mathrm{tr}(AA^t)$*, for any matrix* $A = (a_{ij})$*.*

Proof. Since

$$\left[(h_{ij}^{\alpha})^2\right]_k = 2h_{ij}^{\alpha} h_{ijk}^{\alpha},$$

where k denotes the index of the derivative in the tangent direction e_k, together with the linearity of the Laplacian operator, a straightforward calculation shows that

$$\frac{1}{2}\Delta S = \sum_{\alpha,i,j,k} h_{ij}^{\alpha} \left(\sum_k h_{ijkk}^{\alpha}\right) + \sum_{\alpha,i,j,k} (h_{ijk}^{\alpha})^2. \tag{4.2.3}$$

Combining the equation (4.1.6) with Codazzi equation (4.1.12) we have

$$\frac{1}{2}\Delta S = \sum_{i,j,k,\alpha} h_{ij}^{\alpha} \overline{R}_{\alpha ijkk} + \sum_{\alpha,i,j,k} h_{ij}^{\alpha} h_{kijk}^{\alpha} + |\nabla B|^2. \tag{4.2.4}$$

Note that, since $(\mathcal{E}_p^{n+p}, \overline{g})$ is an Einstein manifold, the components of its Ricci tensor satisfy $\overline{R}_{AB} = \lambda \overline{g}_{AB}$, for some constant $\lambda \in \mathbb{R}$. Moreover, by the assumption that there exists an orthogonal basis for TM that diagonalizes simultaneously all B_η with $\eta \in TM^{\perp}$, we can consider $\{e_1, \ldots, e_n\}$ a local orthonormal frame on M^n such that $h_{ij}^{\alpha} = \lambda_i^{\alpha}\delta_{ij}$ for all $\alpha \in \{n+1, \ldots, n+p\}$. Thus, if we proceed as in [93], then from the differential Bianchi identity and from Ricci's Lemma $\overline{g}_{AB}$ behave as constants in covariant differentiations, that is, $\overline{g}_{AB;C} \equiv 0$ we get

$$\begin{aligned}
\sum_{i,k,\alpha} \lambda_i^{\alpha} \overline{R}_{\alpha iik;k} &= -\sum_{i,k,\alpha} \lambda_i^{\alpha} \left(\overline{R}_{ikik;\alpha} + \overline{R}_{k\alpha ik;i}\right) \\
&= -\sum_{i,\alpha} \lambda_i^{\alpha} \left(\overline{R}_{ii;\alpha} - \overline{R}_{\alpha i;i}\right) \\
&= -\sum_{i,\alpha} \lambda_i^{\alpha} \left(\lambda \overline{g}_{ii;\alpha} - \lambda \overline{g}_{\alpha i;i}\right) = 0
\end{aligned} \tag{4.2.5}$$

and

$$\sum_{i,k,\alpha} \lambda_i^{\alpha} \overline{R}_{\alpha kik;i} = \sum_{i,\alpha} \lambda_i^{\alpha} \overline{R}_{\alpha i;i} = \sum_{i,\alpha} \lambda_i^{\alpha} \lambda \overline{g}_{\alpha i;i} = 0, \tag{4.2.6}$$

where $\overline{R}_{ijkl;m}$ are the covariant derivatives of $\overline{R}_{ijkl}$ on $\mathcal{E}_p^{n+p}$. It follows from (4.2.5) and (4.2.6) that

$$\sum_{i,j,k,\alpha} \left(\overline{R}_{\alpha ijk;k} + \overline{R}_{\alpha iki;j}\right) h_{ij}^{\alpha} = 0. \tag{4.2.7}$$

Therefore, using the equations (4.1.9), (4.1.13) – (4.2.1) together with (4.2.4) and (4.2.7), we can reason as in the proof of Lemma 2 in [40] to obtain formula (4.2.2). ■

Remark 4.2.1 Several authors obtained interesting results on the local symmetry of Einstein Manifolds. In this direction, Tod [121] showed that four-dimensional Einstein manifolds which are also D'Atri spaces are necessarily locally symmetric. Brendle [43] proved that a compact Einstein manifold of dimension $n \geq 4$ having nonnegative isotropic curvature must be locally symmetric, extending a previous result of Micallef and Wang for $n = 4$ (see Theorem 4.4 of [99]). See also [126] for another sufficient conditions for an Einstein manifold to be locally symmetric. Recently, Peterson and Wink [110] showed that Einstein manifolds with $\lfloor \frac{n-1}{2} \rfloor$-nonnegative curvature operators are locally symmetric.

At this point we also observe that, denoting by $\overline{R}_{CD}$ the components of the Ricci tensor of $\mathcal{E}_p^{n+p}$, the scalar curvature $\overline{R}$ of $\mathcal{E}_p^{n+p}$ is given by

$$\overline{R} \;=\; \sum_{A}^{n+p} \epsilon_A \overline{R}_{AA} = \sum_{i,j} \overline{R}_{ijij} - 2\sum_{i,\alpha} \overline{R}_{i\alpha i\alpha} + \sum_{\alpha,\beta} \overline{R}_{\alpha\beta\alpha\beta}.$$

Thus, if $\mathcal{E}_p^{n+p}$ satisfies curvature conditions (1) and (3) then we obtain

$$\overline{R} = n(n-1)\overline{\mathcal{R}} - 2pc_1 + (p-1)c_3, \tag{4.2.8}$$

where

$$\overline{\mathcal{R}} := \frac{1}{n(n-1)} \sum_{i,j} \overline{R}_{ijij}.$$

In views of these, since the scalar curvature of an Einstein manifold is constant, from (4.2.8) we conclude that $\overline{\mathcal{R}}$ is a constant naturally attached to an Einstein manifold $\mathcal{E}_p^{n+p}$ satisfying (1) and (3).

4.3 A parabolicity criterion for spacelike submanifolds immersed in an Einstein manifold

The next result provides sufficient conditions which guarantee the L-parabolicity of a linear Weingarten spacelike submanifold immersed in $\mathcal{E}_p^{n+p}$, whose its proof is similar to proof of Proposition A.

In this setting, denoting by $\overline{\mathrm{Ric}}$ the Ricci tensor of Einstein manifold $\mathcal{E}_p^{n+p}$, $\overline{\mathrm{Ric}} = \lambda\langle\, ,\rangle$ for some constant $\lambda \in \mathbb{R}$ and together with Lemma 3.1 of [4](cf. Lemma 5.2.1) we get

$$\begin{aligned}\langle \mathrm{div} P_1, \nabla f\rangle &= \sum_i \langle \overline{\mathrm{R}}(e_{n+1}, e_i)e_i, \nabla f\rangle = \overline{\mathrm{Ric}}(e_{n+1}, \nabla f)\\ &= \lambda\langle e_{n+1}, \nabla f\rangle = 0, \end{aligned} \tag{4.3.1}$$

where P_1 is defined in (1.2.3) and $\overline{\mathrm{R}}$ denotes the curvature tensor of $\mathcal{E}_p^{n+p}$. Choosing a local orthonormal frame $\{e_1, \ldots, e_n\}$ on M^n, we have

$$\begin{aligned}\mathrm{div}(P_1(\nabla f)) &= \sum_i \langle (\nabla_{e_i} P_1)(\nabla f), e_i\rangle + \langle P_1(\nabla_{e_i}\nabla f), e_i\rangle\\ &= \langle \mathrm{div} P_1, \nabla f\rangle + \square f. \end{aligned} \tag{4.3.2}$$

Thus, combining (4.3.1) with (4.3.2) we obtain $\square f = \mathrm{div}(P_1(\nabla f))$. Consequently, we get

$$L(f) = \mathrm{div}(P(\nabla f)). \tag{4.3.3}$$

Therefore, we can apply Theorem 2.6 of [111](cf. Theorem 1.2.1), to obtain our L-parabolicity criterion as follows:

Proposition D *Let M^n be a complete linear Weingarten spacelike submanifold immersed with parallel normalized mean curvature vector in an Einstein manifold $\mathcal{E}_p^{n+p}$ of index p, such that $R = aH + b$ for some constants $a, b \in \mathbb{R}$ with $b \leq \overline{\mathcal{R}}$. If H is bounded on M^n and, for some reference point $o \in M^n$ and some $\delta > 0$,*

$$\int_\delta^{+\infty} \frac{dt}{\mathrm{vol}(\partial B_t)} = +\infty, \tag{4.3.4}$$

where B_t denotes the geodesic ball of radius t in M^n centered at o, then M^n is L-parabolic.

When considering again the traceless symmetric tensor Φ defined in (2.2.3)from relation (4.1.10), for a complete spacelike submanifold M^n immersed in $\mathcal{E}_p^{n+p}$ we obtain the following relation

$$|\Phi|^2 = S - nH^2 = nH^2(n-1) + n(n-1)(R - \overline{\mathcal{R}}). \tag{4.3.5}$$

The following lemma is an immediate consequence of the equations (4.2.5)- (4.2.7) obtained in the proof of Theorem 4.2.1 and Lemma 1 of [40].

Lemma 4.3.1 *Let M^n be a linear Weingarten spacelike submanifold immersed in an Einstein manifold $\mathcal{E}_p^{n+p}$ satisfying conditions (1) and (3), such that $R = aH + b$ for some $a, b \in R$. Suppose that there exists an orthogonal basis for TM that diagonalizes simultaneously all operators B_η with $\eta \in TM^\perp$ and $b \leq \overline{\mathcal{R}}$. Then,*

$$|\nabla B|^2 \geq n^2 |\nabla H|^2. \tag{4.3.6}$$

Moreover, if the equality holds in (4.3.6) on M^n, then H is constant on M^n.

4.3.1 Main characterization Result

From Proposition D, we obtain the following characterization result for complete linear Weingarten spacelike submanifolds.

Theorem 4.3.1 *Let M^n be a complete linear Weingarten spacelike submanifold immersed with parallel normalized mean curvature vector and flat normal bundle in an Einstein manifold $\mathcal{E}_p^{n+p}$ of index p satisfying conditions (1), (2), (3) and (4), such that $R = aH + b$ for some constants $a, b \in \mathbb{R}$ with $b \leq \overline{\mathcal{R}}$. Suppose that there exists an orthogonal basis for TM that diagonalizes simultaneously all operators B_η with $\eta \in TM^\perp$, where $\langle B_\eta u, v\rangle := \langle B(u, v), \eta\rangle$ for any u, $v \in TM$. When $c := \frac{c_1}{n} + 2c_2 > 0$, assume in addition that $H^2 \geq \frac{4(n-1)c}{Q(p)}$, where $Q(p) := p(n-2)^2 + 4(n-1)$. If H is bounded on M^n, $|\Phi| \geq C(n, p, H)$, where*

$$C(n, p, H) := \frac{\sqrt{n}}{2\sqrt{n-1}} \left(p(n-2)H + \sqrt{pQ(p)H^2 - 4p(n-1)c} \right),$$

and hypothesis (4.3.4) is satisfied, then $p = 1$ and M^n is an isoparametric hypersurface of $\mathcal{E}_1^{n+1}$ with two distinct principal curvatures one of which is simple.

Proof. Since we assume that M has parallel normalized mean curvature vector, from Ricci equation (4.1.15) it follows that $h^\alpha h^{n+1} = h^{n+1} h^\alpha$ for all α, that is, h^{n+1} commutes with all the matrices h^α. Thus, from traceless symmetric tensor Φ and (2.2.4)we have that Φ^{n+1} commutes with all the matrices Φ^α. Since the matrices Φ^α are symmetric and traceless, applying Lemma 2.6 of [116](cf. Lemma 2.3.1) with $A = \Phi^\alpha$ and $B = \Phi^{n+1}$ in order to obtain

$$\left|\mathrm{tr}((\Phi^\alpha)^2 \Phi^{n+1})\right| \leq \frac{n-2}{\sqrt{n(n-1)}} N(\Phi^\alpha) \sqrt{N(\Phi^{n+1})}. \tag{4.3.7}$$

Let us recall that, by Cauchy-Schwarz inequality we also have that

$$p \sum_{\alpha,\beta} [\mathrm{tr}(\Phi^\alpha \Phi^\beta)]^2 \;\; \geq \;\; p \sum_\alpha [\mathrm{tr}(\Phi^\alpha)^2]^2 = p \sum_\alpha [N(\Phi^\alpha)]^2$$

$$\geq \left(\sum_{\alpha} N(\Phi^{\alpha})\right)^2 = |\Phi|^4. \tag{4.3.8}$$

On the other hand, taking into account our set of constraints on $M^n \hookrightarrow \mathcal{E}_p^{n+p}$ together with Lemma 4.2.1, we can reason as in the proof of Proposition 1 in [40] to obtain

$$L(nH) \geq |\Phi|^2 P_{H,p,c}(|\Phi|), \tag{4.3.9}$$

where

$$P_{H,p,c}(x) = \frac{x^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}} Hx - n(H^2 - c).$$

Now we analyze the sign of the constant c. First notice that in the case $c > 0$, if $H^2 \geq \frac{4(n-1)c}{Q(p)}$, then the polynomial $P_{H,p,c}$ has (at last) a positive real root given by $C(n,p,H)$. Thus, since $|\Phi| \geq C(n,p,H)$, we get $P_{H,p,c}(|\Phi|) \geq 0$, with $P_{H,p,c}(|\Phi|) = 0$ if, and only if, $|\Phi| = C(n,p,H)$. On the other hand, if $c \leq 0$ we have that $P_{H,p,c}(|\Phi|) \geq 0$ without any restriction on the values of the mean curvature function H. Consequently, in both cases, from (4.3.9) we get that $L(nH) \geq 0$.

Let us observe that in virtue of Proposition A the submanifold M^n is L-parabolic. It follows from the boundedness of H, that it is constant on M^n implying, in particular, that $L(nH) = 0$ on M^n. Since $|\Phi| > 0$, we obtain that $P_{H,p,c}(|\Phi|) = 0$. Thus, inequalities (4.3.7) and (4.3.8) are, in fact, equalities. This implies, in particular, that $N(\Phi^{n+1}) = \operatorname{tr}(\Phi^{n+1})^2 = |\Phi|^2$. Consequently, from (2.2.5) we obtain

$$\operatorname{tr}(\Phi^{n+1})^2 = |\Phi|^2 = S - nH^2. \tag{4.3.10}$$

Considering again that M^n has parallel normalized mean curvature vector, from (4.2.1) we also have

$$\operatorname{tr}(\Phi^{n+1})^2 = S - \sum_{\alpha>n+1} \sum_{i,j} (h_{ij}^{\alpha})^2 - nH^2. \tag{4.3.11}$$

By comparing the last two above expressions we conclude that $\sum_{\alpha>n+1} \sum_{i,j} (h_{ij}^{\alpha})^2 = 0$.

Now, returning to (4.3.8) we get

$$p|\Phi|^4 = pN(\Phi^{n+1})^2 = p \sum_{\alpha \geq n+1} [N(\Phi^{\alpha})]^2 = |\Phi|^4.$$

It immediately follows that $p = 1$. Moreover, by the assumption $b \leq \overline{\mathcal{R}}$, Lemma 1 of [40](cf. Lemma 4.3.1) together with fact that H is constant on M^n implies that

$$\sum_{i,j,k} (h_{ijk}^{n+1})^2 = n^2 |\nabla H|^2 = 0,$$

that is, $h_{ijk}^{n+1} = 0$ for all i, j, k. Therefore, we have that M^n must be an isoparametric spacelike hypersurface of $\mathcal{E}_1^{n+1}$. ∎

We close this Chapter quoting the following consequence of Theorem 4.3.1.

Corollary 4.3.1 *Let M^n be a complete linear Weingarten spacelike hypersurface immersed in an Einstein manifold $\mathcal{E}_1^{n+1}$ of index 1 satisfying conditions (1) and (2), such that $R = aH + b$ for some constants $a, b \in \mathbb{R}$ with $b \le \overline{\mathcal{R}}$. When $c := \dfrac{c_1}{n} + 2c_2 > 0$, assume in addition that $H^2 \ge \frac{4(n-1)c}{(n-2)^2+4(n-1)}$. If H is bounded on M^n,*

$$|\Phi| \ge \frac{\sqrt{n}}{2\sqrt{n-1}}\left((n-2)H + \sqrt{n^2H^2 - 4(n-1)c}\right)$$

and hypothesis (4.3.4) is satisfied, then M^n is an isoparametric hypersurface of $\mathcal{E}_1^{n+1}$ with two distinct principal curvatures one of which is simple.

Remark 4.3.1 *Considering the particular case where $p = 1$, we also obtain in [26] a characterization result similar to Corollary 4.3.1, now assuming that H has a strict sign and $S \ge 2\sqrt{n-1}c$ when $c := \dfrac{c_1}{n} + 2c_2 > 0$.*

Part II

Rigidity and nonexistence of complete hypersurfaces in a semi-Riemannian warped product

Chapter 5

Preliminaries II

In a similar way to Chapter 1, here, we recall some basic facts related to Riemannian immersions in semi-Riemannian warped products and we quote the auxiliaries lemmas which will be used to prove our main results.

5.1 Semi-Riemannian warped product

We start given a description of our ambient space. Let $\overline{M}^{n+1}$ be a connected semi-Riemannian manifold with metric $\overline{g} = \langle\ ,\ \rangle$ of index $\nu \leq 1$, and semi-Riemannian connection $\overline{\nabla}$. For a vector field $X \in \mathfrak{X}(\overline{M})$, let $\epsilon_X = \langle X, X\rangle$. We will say that X is a *unit* vector field if $\epsilon_X = \pm 1$, and *timelike* if $\epsilon_X = -1$.

Now, let $(M^n, \langle, \rangle_M)$ be a connected, n-dimensional oriented Riemannian manifold and let $I \subseteq \mathbb{R}$ denote an open interval. The product manifold $\overline{M}^{n+1} = I \times M^n$ endowed with the semi-Riemannian metric

$$\langle \cdot, \cdot \rangle_{\overline{M}} = \epsilon \pi_I^*(dt^2) + \rho^2(\pi_I)\pi_M^*(\langle \cdot, \cdot \rangle_M), \tag{5.1.1}$$

of index $\nu \leq 1$, where $\epsilon := \epsilon_X = \langle X, X\rangle_{\overline{M}}$ for any smooth vector field $X \in \mathfrak{X}(\overline{M})$, $\rho \in C^\infty(I)$ is a positive smooth function, π_I and π_M denote the canonical projections onto the factors I and M^n, respectively. Such a space is a particular case of a semi-Riemannian *warped product* with fiber $(M^n, \langle, \rangle_M)$, base $(I, \epsilon dt^2)$ and warping function ρ, and, from now on, we will just write $\overline{M}^{n+1} = \epsilon I \times_\rho M^n$ to denote it.

In the Lorentzian setting $\nu = 1$ or, equivalently, when $\epsilon = -1$, according to the terminology established in [22], $\overline{M}^{n+1}$ is the so-called generalized Robertson-Walker (GRW) spacetime.

Remark 5.1.1 *Related to Riemannian setting $\nu = 0$, we can point out an interesting fact about Riemannian space form $\mathbb{Q}_c^{n+1}$ endowed with a non-trivial closed and conformal vector field X. From [102, Proposition 2] we have:*

1. *The Euclidean space $\mathbb{R}^n \backslash \{0\}$ is naturally isometric to product $\mathbb{R}^+ \times \mathbb{S}^{n-1}$ endowed with metric $dr^2 + r^2 d\sigma_n^2$, where $d\sigma_n^2$ is the constant curvature one metric on the sphere $\mathbb{S}^n$.*

2. *Let $a \in \mathbb{S}^{n+1}$ be an point arbitrary. The hyperfsurfaces $\mathbb{S}^{n+1} \backslash \{a, -a\}$ are isometric to $(0, \pi) \times \mathbb{S}^n$ endowed with metric $d\theta^2 + \sin\theta^2 d\sigma_n^2$.*

3. *The Hyperbolic space $\mathbb{H}^{n+1}$ is isometric to product manifold $\mathbb{R}^+ \times_{\sinh r} \mathbb{S}^n, \mathbb{R} \times_{e^t} \mathbb{R}^n$ and $\mathbb{R} \times_{\cosh t} \mathbb{H}^n$.*

In the Lorentzian setting, we can quote the ambient space of Chapter 3. From of works of [82, 49, 118] we have that the de Sitter space $\mathbb{S}_1^{n+1}$ is isometric to product $\mathbb{R} \times \mathbb{S}^n$ with metric $-dt^2 + \cosh^2(t) d\sigma_n^2$.

5.1.1 Riemannian immersions

Let us consider $\psi : \Sigma^n \to \overline{M}^{n+1}$ *Riemannian immersions*, that is, immersions from a connected, n-dimensional orientable differentiable manifold Σ^n into a semi-Riemannian warped product $\overline{M}^{n+1} = \epsilon I \times_\rho M^n$, such that the induced metric on Σ^n from the metric (5.1.1) is positive definite. When $\nu = 1$, we will refer to $(\Sigma^n, \langle\, , \rangle)$ as a *spacelike hypersurface* of $\overline{M}^{n+1}$. In this setting, ∇ will stand for the Levi-Civita connection of Σ^n, while $\overline{\nabla}$ will represent the Levi-Civita connection of $\overline{M}^{n+1}$.

We orient Σ^n by the choice of a unit normal vector field N on it. Let ∂_t be the standard unit vector field tangent to I. So, we have that $\epsilon = \epsilon_{\partial_t} = \epsilon_N$. Let $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ be the Weingarten operator of Σ^n with respect to N, which is defined by

$$AX = -\overline{\nabla}_X N.$$

At each $p \in \Sigma^n$, A restricts to a self-adjoint linear map $A_p : T_p\Sigma \to T_p\Sigma$. For $0 \leq r \leq n$, let $S_r(p)$ denote the r-th elementary symmetric function on the eigenvalues of A_p; this way one gets n smooth functions $S_r : \Sigma^n \to \mathbb{R}$, such that the characteristic polynomial of A can be written in terms of the $S_k's$ as

$$\det(tI - A) = \sum_{k=0}^{n} (-1)^k S_k t^{n-k},$$

where $S_0 = 1$ by construction. If $p \in \Sigma^n$ and $\{e_k\}$ is a basis of $T_p\Sigma$ formed by eigenvectors of A_p, with corresponding eigenvalues $\{\lambda_k\}$, one immediately sees that $S_r = \sigma_r(\lambda_1, \dots, \lambda_n)$, where $\sigma_r \in \mathbb{R}[X_1, \dots, X_n]$ is the r-th elementary symmetric polynomial on the indeterminates $X_1, \dots, X_n$.

We define the r-th mean curvature H_r of the hypersurface by

$$\binom{n}{r} H_r = \epsilon^r S_r = \sigma_r(\epsilon\lambda_1, \dots, \epsilon\lambda_n), \tag{5.1.2}$$

for every $0 \leq r \leq n$. In particular, $H_0 = 1$ and

$$H_1 = \epsilon_{_N} \frac{1}{n} \sum_{i=1}^{n} \lambda_i = \epsilon_{_N} \frac{1}{n} \text{trace}(A) = H$$

is the usual mean curvature of Σ^n.

For $t_0 \in I$, we orient the *slice* $\Sigma^n_{t_0} = \{t_0\} \times M^n$ by choosing ∂_t. Note that Σ_{t_0} has constant r-th mean curvature

$$H_r = (-\epsilon)^r \left((\log\rho)'\right)^r (t_0) = (-\epsilon)^r \left(\frac{\rho'(t_0)}{\rho}\right)^r \tag{5.1.3}$$

with respect to ∂_t (see, for instance, Example 5.6 of [4] and Section 2 of [18]).

We recall that a slab of a warped product $\epsilon I \times_\rho M^n$ is a region of the type

$$[t_1, t_2] \times M^n = \{(t, q) \in I \times_\rho M^n : t_1 \leq t \leq t_2\}. \tag{5.1.4}$$

Now, for $0 \leq r \leq n$, one defines the r-th Newton transformation T_r on Σ^n by

$$T_0 = I \text{ and } \; T_r = \epsilon^r S_r I - \epsilon A T_{r-1} \quad (1 \leq r \leq n), \tag{5.1.5}$$

where I is the identity operator. With a trivial induction, from (5.1.5) we verify that

$$T_r = \epsilon^r (S_r I - S_{r-1} A + S_{r-2} A^2 - \cdots + (-1)^r A^r), \tag{5.1.6}$$

so that Cayley-Hamilton theorem gives $T_n = 0$. Moreover, since for every r, T_r is a polynomial function in A, it is also self-adjoint and commutes with A. Therefore, all bases of $T_p\Sigma$ diagonalizing A at $p \in \Sigma^n$ also diagonalize all of the T_r at p. So, let $\{e_1, \dots, e_n\}$ be an orthonormal frame on $T_p\Sigma$ which diagonalizes A_p, $A_p(e_i) = \lambda_i(p)e_i$, then from (5.1.6) we have that

$$(T_r)_p e_i = \epsilon^r \sum_{i_1 < \ldots < i_r, i_j \neq i} \lambda_{i_1}(p) \ldots \lambda_{i_r}(p) e_i. \tag{5.1.7}$$

For each Newton transformation T_r, $0 \leq r \leq n$, we associate a second order linear differential operator $L_r : C^\infty(\Sigma) \to C^\infty(\Sigma)$ given by

$$L_r(f) = \mathrm{tr}(T_r \circ \nabla^2 f), \tag{5.1.8}$$

where $\nabla^2 f : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ denotes the self-adjoint linear operator equivalent to the Hessian operator of f, defined by

$$\langle \nabla^2 f(X), Y\rangle = \langle \nabla_X \nabla f, Y\rangle,$$

for all vector fields $X, Y \in \mathfrak{X}(\Sigma)$.

The divergence of T_r on Σ^n is defined by

$$\mathrm{div} T_r := \mathrm{tr}(\nabla T_r) = \sum_{i=1}^{n} (\nabla_{e_i} T_r)(e_i) \quad \text{and} \quad \mathrm{div} T_0 = \mathrm{div} I = 0, \tag{5.1.9}$$

where $\{e_1, \cdots, e_n\}$ be a local orthonormal frame on Σ^n. Thus, for any $f \in C^\infty(\Sigma)$, we have that

$$\begin{aligned} \mathrm{div}(T_r(\nabla f)) &= \sum_{i=1}^{n} \langle (\nabla_{e_i} T_r)(\nabla f), e_i\rangle + \sum_{i=1}^{n} \langle T_r(\nabla_{e_i} \nabla f), e_i\rangle \\ &= \langle \mathrm{div} T_r, \nabla f\rangle + L_r(f). \end{aligned} \tag{5.1.10}$$

We close this subsection recalling a terminology introduced in [5]. We say that a Riemannian immersion $\psi : \Sigma^n \to \epsilon I \times_\rho M^n$ is *bounded away from the future infinity* of $\epsilon I \times_\rho M^n$ if there exists $\bar{t} \in I$ such that

$$\psi(\Sigma) \subset \{(t, x) \in \epsilon I \times_\rho M^n ; t \leq \bar{t}\},$$

and we say that it is *bounded away from the past infinity* of $\epsilon I \times_\rho M^n$ if there exists $\underline{t} \in I$ such that

$$\psi(\Sigma) \subset \{(t, x) \in \epsilon I \times_\rho M^n ; t \geq \underline{t}\}.$$

5.2 Some auxiliary lemmas

In order to establish our main results, we will quote some auxiliary lemmas. The lemma below gives us the divergences of Newton transformations $\mathrm{div} T_r$, defined in (5.1.9), by Alías, Brasil and Colares in [4, Lemma 3.1] for a spacelike hypersurface in a conformally stationary spacetime. Lima in [88, Theorem 2.4] obtained a similar result in the Riemannian setting.

Lemma 5.2.1 (cf. [4]) *The divergences of the Newton transformations T_r are given by the following inductive formula:*

$$\begin{cases} \operatorname{div} T_0 = 0, \\ \\ \operatorname{div} T_r = A(\operatorname{div} T_{r-1}) + \sum_{i=1}^{n} \left(\overline{\mathrm{R}}(N, T_{r-1}E_i)E_i\right)^{\top}, \end{cases}$$

where $\overline{\mathrm{R}}$ stands for the curvature tensor of the ambient spacetime. Equivalently, for every tangent field $X \in \mathfrak{X}(M)$ it follows that

$$\langle \operatorname{div} T_r, X\rangle = \sum_{j=1}^{r}\sum_{i=1}^{n}\langle \overline{\mathrm{R}}(N, T_{r-j}E_i)E_i, A^{j-i}X\rangle.$$

We note that the gradient of π_I on $\overline{M}^{n+1}$ is given by

$$\overline{\nabla}\pi_I = \epsilon\langle\overline{\nabla}\pi_I, \partial_t\rangle\partial_t = \epsilon\partial_t. \tag{5.2.1}$$

Now, let $h := (\pi_I)|_{\Sigma}$ and $\Theta := \langle N, \partial_t\rangle$ two particular functions naturally attached to Σ^n, namely, the (vertical) *height function* and the *angle function*, respectively. In this context, the following computation is obtained from (5.2.1) and definition of h.

$$\nabla h = (\overline{\nabla}\pi_I)^{\top} = \epsilon\partial_t^{\top}, \tag{5.2.2}$$

where $\partial_t^{\top} = \epsilon\partial_t - \Theta N$ is the tangential component of ∂_t on Σ^n. In particular, from (5.2.2) we get

$$|\nabla h|^2 = \epsilon\left(1 - \langle N, \partial_t\rangle^2\right) = \epsilon\left(1 - \Theta^2\right), \tag{5.2.3}$$

where $|\ \ |$ denotes the norm of a vector field on Σ^n.

In our setting, the divergence of the Newton transformation T_r in $\overline{M}^{n+1}$ is obtained from lemma above and equation (3.12) of the proof of [11, Theorem 2] (see also [88, Theorem 2.4]) .

Lemma 5.2.2 *Let $\psi : \Sigma^n \to \epsilon I \times_\rho M^n$ be a Riemannian immersion. Then*

$$\langle \operatorname{div} T_1, \nabla h\rangle = -\epsilon(\mathrm{Ric}_M(N^*, N^*) + \epsilon(n-1)(\log\rho)''(h)|\nabla h|^2)\Theta, \tag{5.2.4}$$

where Ric_M denotes the Ricci curvature of the fiber M^n and $N^* = N - \epsilon\Theta\partial_t$ is the projection of the N onto M^n. Moreover, when M^n has constant sectional curvature κ,

$$\langle \operatorname{div} T_r, \nabla h\rangle = -\epsilon(n-r)\left(\frac{\kappa}{\rho^2(h)} + \epsilon(\log\rho)''(h)\right)\langle T_{r-1}\nabla h, \nabla h\rangle\Theta. \tag{5.2.5}$$

For an appropriate choice of the orientation N of Σ^n, the relation (5.1.10) guarantees that the operator L_r is elliptic if and only if the Newton transformation T_r is positive definite. Note that, for $r = 0$, $L_0 = \Delta$ is always elliptic, where Δ denotes the Laplace-Beltrami operator. In this context, the following two lemmas establish sufficient conditions to the ellipticity of the operator L_1 and L_r when $r \geq 2$ (see, for instance, [9, Lemmas 3.2 and 3.3]).

Lemma 5.2.3 *Let $\psi : \Sigma^n \to \epsilon I \times_\rho M^n$ be a Riemannian immersion in a semi-Riemannian warped product $\overline{M}^{n+1} = \epsilon I \times_\rho M^n$. If $H_2 > 0$ on Σ^n, then L_1 is elliptic or, equivalently, T_1 is positive definite (for an appropriate choice of the Gauss map N).*

Lemma 5.2.4 *Let $\psi : \Sigma^n \to \epsilon I \times_\rho M^n$ be a Riemannian immersion in a semi-Riemannian warped product $\overline{M}^{n+1} = \epsilon I \times_\rho M^n$. If there exists an elliptic point of Σ^n, with respect to an appropriate choice of the Gauss map N, and $H_{r+1} > 0$ on Σ^n, for $2 \leq r \leq n-1$, then for all $1 \leq j \leq r$ the operator L_j is elliptic or, equivalently, T_j is positive definite (for an appropriate choice of the Gauss map N, if j is odd).*

We recall that a point is said to be *elliptic* in a Riemannian immersion when all principal curvatures have the same sign. The next lemma gives a sufficient condition to guarantee the existence of an elliptic point in a Riemannian immersion. For its proof, see [4, Lemma 5.4] and [11, Lemma 4].

Lemma 5.2.5 *Let $\psi : \Sigma^n \to \epsilon I \times_\rho M^n$ be a Riemannian immersion in a semi-Riemannian warped product $\overline{M}^{n+1} = \epsilon I \times_\rho M^n$. If $-\epsilon\rho(h)$ attains a local minimum at some $p \in \Sigma^n$, such that $\rho'(h(p)) \neq 0$, then p is an elliptic point for Σ^n.*

It follows from [9, Lemma 4.1] and [18, Proposition 6] the next formula:

Lemma 5.2.6 *Let $\psi : \Sigma^n \to \epsilon I \times_\rho M^n$ be a Riemannian immersion and let $g : I \to \mathbb{R}$ be any primitive of the warping function ρ. Then, for every $r = 0, \cdots, n-1$,*

$$L_r(g(h)) = \epsilon b_r \left(\rho'(h) H_r + H_{r+1}\rho(h)\Theta\right),$$

where

$$b_r = (n-r)\binom{n}{r} = (r+1)\binom{n}{r+1}. \tag{5.2.6}$$

Along this second part, we will always denote by $u \in C^\infty(\Sigma)$ an arbitrary primitive g of the warping function ρ restricted to the Riemannian immersion $\psi : \Sigma^n \to \overline{M}^{n+1}$, that is,

$$u := g(h).$$

When the ambient space is a GRW spacetime $\overline{M}^{n+1} = -I \times_\rho M^n$, since ∂_t is a unitary timelike vector field globally defined on $\overline{M}^{n+1}$, there exists a unique timelike unitary normal vector field N globally defined on a spacelike hypersurface $\Sigma^n \hookrightarrow \overline{M}^{n+1}$ which is in the same time-orientation as ∂_t. We then say that N is *future-pointing* and, from the Cauchy-Schwarz inequality for timelike vectors, we have that the angle function

$$\Theta = \langle N, \partial_t \rangle \leq -1. \tag{5.2.7}$$

In the Riemannian setting, we recall that a hypersurface is said to be *two-sided* if its normal bundle is trivial, that is, there is on it a globally defined unit normal vector field N.

5.3 Entire graphs

Let $\Omega \subseteq (M^n, \langle\,,\rangle_M)$ be a connected domain and let $w \in C^\infty(\Omega)$ be a smooth function such that $w(\Omega) \subseteq I$, then $\Sigma(w)$ will denote the (vertical) graph over Ω determined by w, that is,

$$\Sigma(w) = \{(w(x), x) : x \in \Omega\} \subset \epsilon I \times_\rho M^n.$$

The metric induced on Ω from the metric of the ambient space via $\Sigma(w)$ is

$$\langle\,,\rangle = \epsilon dw^2 + \rho^2(w)\langle\,,\rangle_M. \tag{5.3.1}$$

We observe that for a graph $\Sigma(w)$, its height function h is nothing but the function w seen as a function on $\Sigma(w)$. Therefore, in what follows, Dw stands for the gradient of w, as a function on M^n, while $\nabla w = \nabla h$ stands for the gradient of the height function, as a function on $\Sigma(w)$.

The graph is said to be *entire* when $\Omega = M^n$. In the case $\epsilon = 1$, when the function $\rho(w)$ is bounded on M^n, the entire graph $\Sigma(w)$ is complete. In particular, this occurs when $\Sigma(w)$ lies between two slices of $I \times_\rho M^n$. While in the case $\epsilon = -1$, a graph $\Sigma(w)$ is a spacelike hypersurface if and only if $|Dw|^2_{M^n} < \rho^2(w)$, where $|Dw|_{M^n}$ stands for the norm of Dw with respect to the metric $\langle\,,\rangle_M$ in Ω.

From [22, Lemma 3.1], in the case that M^n is simply connected, every complete spacelike hypersurface Σ^n in $-I \times_\rho M^n$ such that the warping function ρ is bounded on Σ^n is an entire spacelike graph. In particular, this happens for complete spacelike hypersurfaces contained in a timelike bounded region. However, in contrast to the case of graphs in a Riemannian warped product, an entire spacelike graph in a GRW spacetime is not necessarily complete, in the sense that its induced Riemannian metric (5.3.1) is not necessarily complete on M^n. For instance, Albujer constructed explicit examples of noncomplete entire maximal spacelike graphs (that is, whose mean curvature is identically zero) in the Lorentzian product space $-\mathbb{R} \times \mathbb{H}^2$ (see [2, Section 3]).

Given an entire graph $\Sigma(w) \subset \epsilon I \times_\rho M^n$, its orientation N which corresponds to the choices made in Sections 6.2, 6.3 and 7.1 is described by

$$N = \frac{\rho(w)}{W(w)} \left(\epsilon \partial_t - \frac{1}{\rho^2(w)} Dw \right), \tag{5.3.2}$$

where $W(w) := \sqrt{\rho^2(w) + \epsilon |Dw|^2_{M^n}}$. Moreover, from (5.3.2) we obtain the corresponding Weingarten operator

$$AX = \frac{1}{\rho(w)W(w)} D_X Dw - \epsilon \frac{\rho'(w)}{W(w)} X + \epsilon \left(\frac{-\langle D_X Dw, Dw \rangle_M}{\rho(w) W^3(w)} - \epsilon \frac{\rho'(w) \langle Dw, X \rangle_M}{W^3(w)} \right) Dw, \tag{5.3.3}$$

for any vector field X tangent to Ω, where D is the Levi-Civita connection of $(M^n, \langle \, , \rangle_M)$.

On the other hand, we have that

$$N = N^* + \epsilon \Theta \partial_t, \tag{5.3.4}$$

where N^* denotes the projection of N onto the tangent bundle of the fiber M^n. Consequently, combining (5.2.2) with (5.3.4) we get

$$(N^*)^\top = \epsilon \Theta \nabla h \tag{5.3.5}$$

and

$$|\nabla h|^2 = \rho^2(h) \langle N^*, N^* \rangle_M. \tag{5.3.6}$$

Thus, from (5.3.2), (5.3.5) and (5.3.6) we obtain that

$$|\nabla h| = \frac{|Dw|_{M^n}}{W(w)}. \tag{5.3.7}$$

Chapter 6

Rigidity and nonexistence of Riemannian immersions in semi-Riemannian warped products via parabolicity

Our main goal in this chapter is to present the results obtained in article [30]. We study complete Riemannian immersions in semi-Riemannian warped products obeying suitable curvature constraints. Under appropriate differential inequalities involving higher order mean curvatures, we establish rigidity and nonexistence results concerning these immersions. Applications to the cases that the ambient space is either an Einstein manifold, a steady state type spacetime or a pseudo-hyperbolic space are given, and a particular investigation of entire graphs construct over the fiber of the ambient space is also made. Our approach is based on a parabolicity criterion related to a linearized differential operator which is a divergence-type operator and can be regarded as a natural extension of the standard Laplacian.

6.1 A parabolicity criterion for Riemannian immersions

In the Part I this thesis, we study the parabolicity of Riemannian submanifolds with respect to modified Cheng-Yau's operator L defined in (1.2.4). Here, considering the setting of the previous chapter, from (5.1.10) we define the operator $\mathcal{L}_r : C^\infty(\Sigma) \to$

$C^\infty(\Sigma)$ by

$$\mathcal{L}_r(\xi) := \mathrm{div}(T_r(\nabla\xi)). \tag{6.1.1}$$

In particular, from [4, Corollary 3.2], when the ambient spacetime $\overline{M}^{n+1}$ has constant sectional curvature, the Newton transformations T_r are divergence-free, that is, $\mathrm{div}Tr = 0$. Consequently, $\mathcal{L}_r(\xi) = L_r(\xi)$ for all $1 \le r \le n-1$. For instance, in Chapter 3, since the Sitter spacetime $\mathbb{S}_1^{n+1}$ has constant sectional curvature, the Cheng-Yau's operator $\square$ defined in (1.2.1) is the divergence-free operator (6.1.1) in the particular case $r = 1$.

With respect to operator $\mathcal{L}_r$, the following concept is due to [17, Definition $5 \cdot 4$] and [18, Definition 30].

Definition 6.1.1 *A Riemannian immersion $\psi : \Sigma^n \to \epsilon I \times_\rho M^n$ is said $\mathcal{L}_r$-parabolic if the only bounded from above smooth solutions of the differential inequality $\mathcal{L}_r\xi \ge 0$ are the constant ones.*

As in the previous parabolicity criteria (Propositions A, C and D), in the next result we will consider the boundedness of the r-th mean curvature H_r of ψ, $1 \le r \le n$. Moreover, under appropriate conditions on the Newton transformation T_r defined in (5.1.5), we present the following result provides sufficient conditions which guarantee the $\mathcal{L}_r$-parabolicity of Riemannian immersions in a semi-Riemannian warped product, whose its proof is similar to proof of Proposition A.

Proposition E *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete Riemannian immersion in $\overline{M}^{n+1} = \epsilon I \times_\rho M^n$. Suppose that the Newton transformation T_r is positive semi-definite and $\sup_\Sigma H_r < +\infty$, for some $0 \le r \le n$. If, for some reference point $o \in \Sigma^n$,*

$$\int_0^{+\infty} \frac{dt}{\mathrm{vol}(\partial B_t)} = +\infty, \tag{6.1.2}$$

where B_t is the geodesic ball of radius t in Σ^n centered at the origin o, then Σ^n is $\mathcal{L}_r$-parabolic.

6.2 Rigidity and nonexistence of spacelike hypersurfaces

In this section, taking into account the previous digression, we can state and prove our first rigidity result.

Theorem 6.2.1 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface immersed into a GRW spacetime $\overline{M}^{n+1} = -I \times_\rho M^n$. Suppose that Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$ and $H > 0$ on Σ^n. If hypothesis (6.1.2) is satisfied and*

$$H \geq \frac{\rho'}{\rho}(h), \tag{6.2.1}$$

then Σ^n is a slice of $\overline{M}^{n+1}$.

Proof. Since $u := g(h)$, by the definition of operator (6.1.1) and Lemma 5.2.6 we have

$$\mathcal{L}_0(u) = -n(\rho'(h) + \rho(h)H\Theta). \tag{6.2.2}$$

It follows from assumptions $H > 0$ and $\Theta \leq -1$ on Σ^n that

$$\mathcal{L}_0(u) \geq n\rho(h)\left(H - \frac{\rho'}{\rho}(h)\right). \tag{6.2.3}$$

Thus, combining the inequalities (6.2.1) and (6.2.3) we obtain that $\mathcal{L}_0(u) \geq 0$.

Moreover, Proposition E guarantees that Σ^n is $\mathcal{L}_0$-parabolic. But, since Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$, we have that the primitive u is bounded from above. Consequently, u is constant on Σ^n. Therefore, we conclude that the height function h is constant and, hence, Σ^n must be a slice of $\overline{M}^{n+1}$. ∎

Next, we will consider a natural extension of the $(n+1)$-dimensional steady state spacetime $-\mathbb{R} \times_{e^t} \mathbb{R}^n$, the so-called *steady state-type spacetime* $\overline{M}^{n+1} = -\mathbb{R} \times_{e^t} M^n$, where M^n is a connected n-dimensional Riemannian manifold (see [5, Section 4]). It is worth to note that when a steady state-type spacetime admits a complete spacelike hypersurface which is bounded away from the future infinity, [5, Lemma 7] guarantees that its Riemannian fiber M^n is necessarily complete. In this setting, Theorem 6.2.1 reads as follows.

Corollary 6.2.1 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface immersed into a steady state-type spacetime $\overline{M}^{n+1} = -\mathbb{R} \times_{e^t} M^n$. Suppose that Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$. If $H \geq 1$ and hypothesis (6.1.2) is satisfied, then Σ^n is a slice of $\overline{M}^{n+1}$.*

For $r = 1$, we will suppose that the GRW spacetime obeys a suitable curvature constraint.

Theorem 6.2.2 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface immersed into a GRW spacetime $\overline{M}^{n+1} = -I \times_\rho M^n$ which obeys the following curvature constraint*

$$\mathrm{Ric}_M \leq (n-1)\inf_I(\rho\rho'' - (\rho')^2)\langle\,,\rangle_M, \tag{6.2.4}$$

where Ric_M stands for the Ricci tensor of M^n. Suppose that Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$, $H > 0$ with $\sup_\Sigma H < +\infty$, and $H_2 > 0$. If hypothesis (6.1.2) is satisfied and

$$\frac{H_2}{H} \geq \frac{\rho'}{\rho}(h), \tag{6.2.5}$$

then Σ^n is a slice of $\overline{M}^{n+1}$.

Proof. From operator (6.1.1) together with Lemma 5.2.6 and equation (5.2.4) of Lemma 5.2.2, we obtain that

$$\begin{aligned} \mathcal{L}_1(u) &= -\rho(h)((n-1)(\log\rho)''(h)|\nabla h|^2 - \mathrm{Ric}_M(N^*, N^*))\Theta \\ &\quad -c_1(\rho'(h)H + \rho(h)H_2\Theta), \end{aligned} \tag{6.2.6}$$

where $N^* = N + \Theta\partial_t$.

On the other hand, from (5.2.3) we have that

$$\langle N^*, N^*\rangle_M = \frac{1}{\rho^2(h)}|\nabla h|^2.$$

Thus, using curvature constraint (6.2.4) and $\Theta \leq -1$, from (6.2.6) we get that

$$\mathcal{L}_1(u) \geq c_1\rho(h)H\left(\frac{H_2}{H} - \frac{\rho'}{\rho}(h)\right).$$

It follows from (6.2.5) we have that $\mathcal{L}_1(u) \geq 0$. Moreover, by assumptions $\sup_\Sigma H < +\infty$ and (6.1.2), Proposition E assures that Σ^n is $\mathcal{L}_1$-parabolic.

Consequently, since Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$, we obtain that u is constant on Σ^n. Therefore, we conclude that the height function h is constant, which means that Σ^n is a slice of $\overline{M}^{n+1}$. ■

When the ambient spacetime is an Einstein manifold, Theorem 6.2.2 reads as it follows:

Corollary 6.2.2 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface immersed into a Einstein GRW spacetime $\overline{M}^{n+1} = -I \times_\rho M^n$. Suppose that Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$. If $H > 0$ with $\sup_\Sigma H < +\infty$, $H_2 > 0$ and hypotheses (6.1.2) and (6.2.5) are satisfied, then Σ^n is a slice of $\overline{M}^{n+1}$.*

Proof. From [42, Corollary 9.107] (see also [23, Section 2]) we have that $\overline{M}^{n+1}$ is an Einstein manifold with Ricci tensor $\overline{\mathrm{Ric}} = \overline{c}\,\overline{g}, \overline{c} \in \mathbb{R}$, if and only if the fiber $(M^n, g_{_M})$ has constant Ricci curvature $\mathrm{Ric}_M = c\langle , \rangle_M$ and the warping function ρ satisfies the differential equations

$$\frac{\rho''}{\rho} = \frac{\overline{c}}{n} \quad \text{and} \quad \frac{\overline{c}(n-1)}{n} = \frac{c + (n-1)(\rho')^2}{\rho^2}. \tag{6.2.7}$$

and thus, we obtain $(n-1)(\log \rho)'' = \dfrac{c}{\rho^2}$. Therefore, in this case, we have that

$$\mathrm{Ric}_M = (n-1)\inf_I(\rho\rho'' - (\rho')^2)\langle\, , \rangle_M.$$

Consequently, the result follows by applying Theorem 6.2.2. ∎

Considering once more a steady state-type spacetime, from Theorem 6.2.2 we get the following consequence.

Corollary 6.2.3 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface immersed into a steady state-type spacetime $\overline{M}^{n+1} = -\mathbb{R} \times_{e^t} M^n$ whose fiber M^n has nonpositive Ricci curvature. Suppose that Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$. If $\sup_\Sigma H < +\infty$, $H_2 \geq H > 0$ and hypothesis (6.1.2) is satisfied, then Σ^n is a slice of $\overline{M}^{n+1}$.*

When $2 \leq r \leq n-1$, we will assume that the Riemannian fiber of the GRW spacetime has constant sectional curvature. In this case, $\overline{M}^{n+1} = -I \times_\rho M^n$ is classically called a Robertson-Walker (RW) spacetime. In this setting, we will use Lemma 5.2.5 to guarantee the ellipticity of the operator $\mathcal{L}_r$.

Theorem 6.2.3 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface immersed into a RW spacetime $\overline{M}^{n+1} = -I \times_\rho M^n$ whose fiber M^n has constant sectional curvature κ satisfying the following curvature constraint*

$$\kappa \leq \inf_I(\rho\rho'' - (\rho')^2). \tag{6.2.8}$$

Suppose that Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$, $H_r > 0$ with $\sup_\Sigma H_r < +\infty$, and $H_{r+1} > 0$ for some $2 \leq r \leq n-1$. Assume in addition that $\rho(h)$ attains a local minimum at some point $p \in \Sigma^n$ such that $\rho'(h(p)) \neq 0$. If hypothesis (6.1.2) is satisfied and

$$\frac{H_{r+1}}{H_r} \geq \frac{\rho'}{\rho}(h), \tag{6.2.9}$$

then Σ^n is a slice of $\overline{M}^{n+1}$.

Proof. Since M^n has constant sectional curvature κ, from Lemma 5.2.6 together with equation (5.2.5) of Lemma 5.2.2 and (6.1.1) we obtain

$$\begin{aligned}\mathcal{L}_r(u) &= \rho(h)(n-r)\left(\frac{\kappa}{\rho^2(h)} - (\log\rho)''(h)\right)\langle T_{r-1}\nabla h, \nabla h\rangle\Theta \\ &\quad -c_r(\rho'(h)H_r + \rho(h)H_{r+1}\Theta).\end{aligned} \tag{6.2.10}$$

On the other hand, since $\rho(h)$ attains a local minimum at some point $p \in \Sigma^n$ such that $\rho'(h(p)) \neq 0$, Lemma 5.2.5 guarantees that p is an elliptic point of Σ^n. Using the assumption $H_{r+1} > 0$, Lemma 5.2.4 we get that the operator L_j is elliptic or, equivalently, T_j is positive definite for all $1 \leq j \leq r$.

Thus, combining the curvature constraint (6.2.8) and the assumption $\Theta \leq -1$, from (6.2.10) we obtain the following estimate

$$\begin{aligned}\mathcal{L}_r(u) &\geq \frac{1}{\rho(h)}(n-r)\left((\rho\rho'' - (\rho')^2)(h) - \kappa\right)\langle T_{r-1}\nabla h, \nabla h\rangle + c_r(\rho(h)H_{r+1} - \rho'(h)H_r) \\ &\geq c_r\rho(h)H_r\left(\frac{H_{r+1}}{H_r} - \frac{\rho'}{\rho}(h)\right).\end{aligned} \tag{6.2.11}$$

Hence, from inequalities (6.2.9) and (6.2.11) we get that $\mathcal{L}_r(u) \geq 0$ on Σ^n. Moreover, from hypotheses $\sup_\Sigma H_r < +\infty$ and (6.1.2), we have that Σ^n is $\mathcal{L}_r$-parabolic. Therefore, since Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$, we conclude that Σ^n must be a slice of $\overline{M}^{n+1}$. ∎

Let us recall that a GRW spacetime $-I \times_\rho M^n$ is said to be *static* when the warping function ρ is constant. In this case, we can suppose, without loss of generality, that $\rho \equiv 1$. In this case, we obtain the following nonexistence result:

Corollary 6.2.4 *Let $\overline{M}^{n+1} = -I \times M^n$ be a static RW spacetime whose fiber M^n has nonpositive constant sectional curvature κ. There is no complete spacelike hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$ bounded away from the future infinity of $\overline{M}^{n+1}$ such that, for some $2 \leq r \leq n-1$, $H_r > 0$ with $\sup_\Sigma H_r < +\infty$, $H_{r+1} > 0$, having an elliptic point and satisfying hypothesis* (6.1.2).

Proof. By contradiction, let us suppose the existence of such a complete spacelike hypersurface Σ^n. Since $p \in \Sigma^n$ is a elliptic point and $H_{r+1} > 0$, Lemma 5.2.4 guarantees that T_j is positive definite for all $1 \leq j \leq r$. Hence, since $\kappa \leq 0$ and $\Theta \leq -1$, from inequality (6.2.10) we have

$$\mathcal{L}_r(u) \geq (n-r)\langle T_{r-1}\nabla h, \nabla h\rangle - c_r H_{r+1}\Theta \geq c_r H_{r+1} > 0.$$

It follows from boundeness of H_r that Σ^n is $\mathcal{L}_r$-parabolic. Therefore, since Σ^n also is bounded away from the future infinity of ambient space, Σ^n must be a slice of $\overline{M}^{n+1}$. This implies that $H_r = 0$ and so, we reach at a contradiction.

■

Proceeding, we will consider also the case when the spacelike hypersurface is bounded away from the past infinity of a GRW spacetime whose Riemannian fiber has constant sectional curvature obeying a curvature constraint which corresponds to the so-called *strong null convergence condition* (SNCC) which was originally conceived by Alías and Colares in [9]. For a throughout discussion concerning the SNCC see also, for instance, see [104].

Theorem 6.2.4 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface immersed into a RW spacetime $\overline{M}^{n+1} = -I \times_\rho M^n$ whose fiber M^n has constant sectional curvature κ satisfying the SNCC*

$$\kappa \geq \sup_I (\rho\rho'' - (\rho')^2). \tag{6.2.12}$$

Suppose that Σ^n is bounded away from the past infinity of $\overline{M}^{n+1}$, $H_{r-1} > 0$ and $H_r > 0$ with $\sup_\Sigma H_r < +\infty$, for some $2 \leq r \leq n-1$. Assume in addition that the sectional curvature of Σ^n, K_Σ, is such that

$$K_\Sigma \leq \frac{\rho''}{\rho}(h). \tag{6.2.13}$$

If hypothesis (6.1.2) is satisfied and

$$\frac{H_{r+1}}{H_r} \leq -\frac{1}{\Theta}\frac{\rho'}{\rho}(h), \tag{6.2.14}$$

then Σ^n is a slice of $\overline{M}^{n+1}$.

Proof. We consider the self-adjoint operator defined by $\mathcal{T}_{r-1} : \mathfrak{X}(\Sigma^n) \to \mathfrak{X}(\Sigma^n)$ by $\mathcal{T}_{r-1} := H_{r-1}T_{r-1}$. Choose a local orthonormal frame $\{e_1, \cdots, e_n\}$ such that $Ae_i(p) = \lambda_i e_i(p)$. It follows from (5.1.7) that

$$T_{r-1}e_i = (-1)^{r-1} \sum_{i_1<\cdots<i_{r-1}, i_j \neq i} \lambda_{i_1} \cdots \lambda_{i_{r-1}} e_i.$$

This implies that, for all $i \in 1, \cdots, n$, we get

$$\langle \mathcal{T}_{r-1}e_i, e_i \rangle = \binom{n}{r-1}^{-1} \sum_{i_1<\cdots<i_{r-1}, i_j \neq i, j_1<\cdots<j_{r-1}} (\lambda_{i_1}\lambda_{j_1}) \cdots (\lambda_{i_{r-1}}\lambda_{j_{r-1}}).$$

Denote by K_Σ and $\overline{K}$ the sectional curvatures of Σ^n and $\overline{M}^{n+1}$, respectively. Thus, from Gauss equation we obtain

$$K_\Sigma(e_i, e_j) = K(e_i, e_j) - \lambda_i\lambda_j, \tag{6.2.15}$$

From [107, Proposition 7.42] (see also [9, Equation (6.6)]), given arbitrary vector fields U, V, W in $\overline{M}^{n+1}$ we can compute the following relation

$$\begin{aligned}\overline{R}(U,V)W &= R_M(U^*,V^*)W^* + ((\log\rho)'(h))^2(\langle U,W\rangle V - \langle V,W\rangle U)\\ &\quad -(\log\rho)''(h)\langle W,\partial_t\rangle(\langle U,\partial_t\rangle V - \langle V,\partial_t\rangle U)\\ &\quad -(\log\rho)''(h)(\langle U,W\rangle\langle V,\partial_t\rangle - \langle U,\partial_t\rangle\langle V,W\rangle)\partial_t, \end{aligned} \tag{6.2.16}$$

where $U^* = (\pi_{M^n})_*U = U + \langle U,\partial_t\rangle\partial_t$. Hence, for an orthonormal basis $\{X,Y\}$ of an arbitrary 2-plane tangent to Σ^n, the above relation (6.2.16) we get

$$\begin{aligned}\overline{K}(X,Y) &= \frac{1}{\rho^2(h)}K_M(X^*,Y^*)|X^*\wedge Y^*|^2\\ &\quad +((\log\rho)'(h))^2(\langle X,X\rangle\langle Y,Y\rangle - \langle Y,X\rangle\langle X,Y\rangle)\\ &\quad -(\log\rho)''(h)\langle X,\partial_t\rangle(\langle X,\partial_t\rangle\langle Y,Y\rangle - \langle Y,\partial_t\rangle\langle X,Y\rangle)\\ &\quad -(\log\rho)''(h)(\langle X,X\rangle\langle Y,\partial_t\rangle - \langle X,\partial_t\rangle\langle Y,X\rangle)\langle\partial_t,Y\rangle\\ &= \frac{1}{\rho^2(h)}K_M(X^*,Y^*)|X^*\wedge Y^*|^2 + ((\log\rho)'(h))^2 \qquad (6.2.17)\\ &\quad -(\log\rho)''(h)(\langle X,\partial_t\rangle^2 + \langle Y,\partial_t\rangle^2). \qquad (6.2.18)\end{aligned}$$

Note that

$$\begin{aligned}|X^*\wedge Y^*|^2 &= |X^*|^2|Y^*|^2 - \langle X^*,Y^*\rangle^2 = \langle X^*,X^*\rangle\langle Y^*,Y^*\rangle - \langle X^*,Y^*\rangle^2\\ &= (1+\langle X,\partial_t\rangle^2)(1+\langle Y,\partial_t\rangle^2) - \langle X,\partial_t\rangle^2\langle Y,\partial_t\rangle^2\\ &= 1 + \langle X,\partial_t\rangle^2 + \langle Y,\partial_t\rangle^2.\end{aligned} \tag{6.2.19}$$

On the other hand, from (5.2.2)

$$\langle X,\partial_t\rangle^2 = \langle X, -\nabla h - \Theta N\rangle^2 = \langle X.\nabla h\rangle^2. \tag{6.2.20}$$

Combining (6.2.19) with (6.2.20), we have that $|X^*\wedge Y^*|^2 = 1 + \langle X,\nabla h\rangle^2 + \langle Y,\nabla h\rangle^2$. Thus,

$$\overline{K}(X,Y) = \frac{1}{\rho^2(h)}K_M(X^*,Y^*)(1+\langle X,\nabla h\rangle^2 + \langle Y,\nabla h\rangle^2) + ((\log\rho)'(h))^2$$

$$
\begin{aligned}
& -(\log\rho)''(h)(\langle X,\nabla h\rangle^2+\langle Y,\nabla h\rangle^2) \\
= & \ \frac{1}{\rho^2(h)}K_M(X^*,Y^*)+((\log\rho)'(h))^2 \\
& +\left(\frac{1}{\rho^2(h)}K_M(X^*,Y^*)-(\log\rho)''(h)\right)(\langle X,\nabla h\rangle^2+\langle Y,\nabla h\rangle^2) \\
= & \ \frac{1}{\rho^2(h)}K_M(X^*,Y^*)+\left(\frac{\rho'}{\rho}(h)\right)^2 \\
& +\frac{1}{\rho^2(h)}(K_M(X^*,Y^*)-\rho\rho''+(\rho')^2)(\langle X,\nabla h\rangle^2+\langle Y,\nabla h\rangle^2). \quad (6.2.21)
\end{aligned}
$$

This together with the convergence condition null (6.2.12) we deduce the following inequality

$$K(X,Y)\geq\frac{\rho''}{\rho}(h). \tag{6.2.22}$$

It follows from (6.2.15) and (6.2.22) that

$$\lambda_i\lambda_j=\overline{K}(e_i,e_j)-K_\Sigma(e_i,e_j)\geq\frac{\rho''}{\rho}(h)-K_\Sigma(e_i,e_j). \tag{6.2.23}$$

Thus, from (6.2.13) and (6.2.23) we have $\lambda_i\lambda_j\geq 0$, for all $i,j\in\{1,2,\cdots,n\}$ with $i\neq j$. Hence,

$$\langle\mathcal{T}_{r-1}e_i,e_i\rangle=\sum(\lambda_{j_1}\lambda_{i_1})\cdots(\lambda_{j_{r-1}}\lambda_{i_{r-1}})\geq 0. \tag{6.2.24}$$

Therefore, conclude that the operator $\mathcal{T}_{r-1}$ is positive semi-definite. Consequently, since H_{r-1} and H_r are positive, $\Theta\leq -1$ and the convergence condition (6.2.12) is satisfied, from (6.2.10) we obtain

$$
\begin{aligned}
\mathcal{L}_r(u) &= \rho(h)(n-r)\left(\frac{\kappa}{f^2(h)}-(\log\rho)''(h)\right)\frac{1}{H_{r-1}}\langle\mathcal{T}_{r-1}\nabla h,\nabla h\rangle\Theta \\
&\quad -c_r(\rho'(h)H_r+\rho(h)H_{r+1}\Theta) \\
&\leq -c_r\rho(h)H_r\Theta\left(\frac{H_{r+1}}{H_r}+\frac{1}{\Theta}\frac{\rho'}{\rho}(h)\right). \quad (6.2.25)
\end{aligned}
$$

Hence, considering inequality (6.2.14) into (6.2.25) we get that $\mathcal{L}_r(u)\leq 0$ on Σ^n. Moreover, hypotheses $\sup_\Sigma H_r<+\infty$ and (6.1.2) assure that Σ^n is $\mathcal{L}_r$-parabolic. Since Σ^n is bounded away from the past infinity of $\overline{M}^{n+1}$, we conclude that Σ^n must be a slice of $\overline{M}^{n+1}$. ■

Remark 6.2.1 Concerning Theorem 6.2.4, we observe that when Σ^n has a elliptic point, hypothesis (6.2.13) can be dropped. Furthermore, we point out that inequality (6.2.14) was already used in [32, Theorem 4] to obtain an extension of [10, Theorem 3.7] and in [113, Theorem 4.1].

6.3 Rigidity and nonexistence of two-sided hypersurfaces

Similarly to the case of spacelike hypersurfaces in GRW spacetimes, in this Section we will establish rigidity and nonexistence results concerning complete two-sided hypersurfaces immersed in a Riemannian warped product. We recall that a hypersurface is said to be *two-sided* if its normal bundle is trivial, that is, there is on it a globally defined unit normal vector field N.

Theorem 6.3.1 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface immersed into a Riemannian warped product $\overline{M}^{n+1} = I \times_\rho M^n$. Suppose that Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$ and that $-1 \le \Theta \le 0$. If hypothesis (6.1.2) is satisfied and*

$$0 < H \le \frac{\rho'}{\rho}(h), \tag{6.3.1}$$

then Σ^n is a slice of $\overline{M}^{n+1}$.

Proof. Taking into account that $H > 0$ and $-1 \le \Theta \le 0$, from Lemma 5.2.6 together with (6.3.1) we obtain that

$$\mathcal{L}_0(u) = n(\rho'(h) + \rho(h)H\Theta)\rangle \ge n\rho(h)\left(\frac{\rho'}{\rho}(h) - H\right). \tag{6.3.2}$$

Hence, combining the inequalities (6.3.1) and (6.3.2), we obtain that $\mathcal{L}_0(u) \ge 0$. Moreover, by hypothesis (6.1.2) we have that Σ^n is $\mathcal{L}_0$-parabolic. So, since Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$, we obtain that u is constant on Σ^n and, therefore, Σ^n is a slice of $\overline{M}^{n+1}$. ■

When the warping function ρ is either exponential or hyperbolic cosine, following the terminology introduced by [120], the corresponding warped product $\mathbb{R} \times_{e^t} M^n$ or $\mathbb{R} \times_{\cosh t} M^n$ has been referred to as a *pseudo-hyperbolic space*. Tashiro's terminology is due to the fact that with suitable choices of the fiber M^n we obtain warped products which are isometric to the hyperbolic space. For more details about these spaces see, for instance, [12, 13, 77, 102]. In this context, we get the following applications of Theorem 6.3.1.

Corollary 6.3.1 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface immersed into a pseudo-hyperbolic space $\overline{M}^{n+1} = \mathbb{R} \times_{e^t} M^n$. Suppose that Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$ and that $-1 \le \Theta \le 0$. If hypothesis (6.1.2) is satisfied and $0 < H \le 1$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

Corollary 6.3.2 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface immersed into a pseudo-hyperbolic space $\overline{M}^{n+1} = \mathbb{R} \times_{\cosh t} M^n$. Suppose that Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$ and that $-1 \leq \Theta \leq 0$. If hypothesis (6.1.2) is satisfied and $0 < H \leq \tanh(h)$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

In our next result, we will suppose that the ambient space obeys a suitable curvature constraint which is the opposite of that assumed in the results of [102].

Theorem 6.3.2 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface immersed into a Riemannian warped product $\overline{M}^{n+1} = I \times_\rho M^n$, which obeys the following curvature constraint*

$$\mathrm{Ric}_M \geq (n-1) \sup_I ((\rho')^2 - \rho\rho'')\langle\, , \rangle_M, \tag{6.3.3}$$

where Ric_M stands for the Ricci tensor of M^n. Suppose that Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$ and that $-1 \leq \Theta \leq 0$. If hypothesis (6.1.2) is satisfied, $H > 0$ with $\sup_\Sigma H < +\infty$, $H_2 > 0$ and

$$\frac{H_2}{H} \leq \frac{\rho'}{\rho}(h), \tag{6.3.4}$$

then Σ^n is a slice of $\overline{M}^{n+1}$.

Proof. From Lemma 5.2.6 and equation (5.2.4) of Lemma 5.2.2 we obtain that

$$\begin{aligned}\mathcal{L}_1(u) &= \rho(h)(-\mathrm{Ric}_M(N^*, N^*) - (n-1)(\log\rho)''(h)|\nabla h|^2)\Theta \\ &\quad + c_1(\rho'(h)H + \rho(h)H_2\Theta),\end{aligned} \tag{6.3.5}$$

where $N^* = N - \Theta\partial_t$.

From (6.2) and curvature constraint (6.3.3) we obtain

$$(n-1)\left(\frac{(\rho')^2 - \rho\rho''}{\rho^2}\right)(h)|\nabla h|^2 - \mathrm{Ric}_M(N^*, N^*) \leq 0. \tag{6.3.6}$$

Thus, since we are assuming $-1 \leq \Theta \leq 0$, substituting (6.3.6) into (6.3.5) we get

$$\mathcal{L}_1(u) \geq c_1\rho(h)H\left(\frac{\rho'}{\rho}(h) - \frac{H_2}{H}\right).$$

Hence, using the assumption (6.3.4) we reach at $\mathcal{L}_1(u) \geq 0$. Moreover, since Lemma 5.2.3 gives that P_1 is positive definite, we can apply Proposition E to guarantee that Σ^n is $\mathcal{L}_1$-parabolic. So, since Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$, we get that the function u is constant. Therefore, we conclude that Σ^n must be a slice of $\overline{M}^{n+1}$. ∎

We can reason as in the proof of Corollary 6.2.2, obtaining the following consequence of Theorem 6.3.2:

Corollary 6.3.3 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface immersed into an Einstein warped product $\overline{M}^{n+1} = I \times_\rho M^n$. Suppose that Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$ and that $-1 \leq \Theta \leq 0$. If $H > 0$ with $\sup_\Sigma H < +\infty$, $H_2 > 0$ and hypotheses (6.1.2) and (6.3.4) are satisfied, then Σ^n is a slice of $\overline{M}^{n+1}$.*

When the ambient is a pseudo-hyperbolic space, Theorem 6.3.2 leads us to the following applications:

Corollary 6.3.4 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface immersed into a pseudo-hyperbolic space $\overline{M}^{n+1} = \mathbb{R} \times_{e^t} M^n$ whose fiber M^n has nonpositive Ricci curvature. Suppose that Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$ and that $-1 \leq \Theta \leq 0$. If hypothesis (6.1.2) is satisfied, $H > 0$ with $\sup_\Sigma H < +\infty$, $H_2 > 0$ and $\dfrac{H_2}{H} \leq 1$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

Corollary 6.3.5 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface immersed into a pseudo-hyperbolic space $\overline{M}^{n+1} = \mathbb{R} \times_{\cosh t} M^n$ whose Ricci tensor of the fiber M^n is such that $\mathrm{Ric}_M \leq -(n-1)\langle\, ,\rangle_M$. Suppose that Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$ and that $-1 \leq \Theta \leq 0$. If hypothesis (6.1.2) is satisfied, $H > 0$ with $\sup_\Sigma H < +\infty$, $H_2 > 0$ and $\dfrac{H_2}{H} \leq \tanh(h)$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

In our next results, we deal with higher order mean curvatures.

Theorem 6.3.3 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface immersed into a Riemannian warped product $\overline{M}^{n+1} = I \times_\rho M^n$ whose fiber M^n has constant sectional curvature κ and it obeys the curvature constraint*

$$\kappa \geq \sup_I((\rho')^2 - \rho\rho''). \tag{6.3.7}$$

Suppose that Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$, $-1 \leq \Theta \leq 0$ and that the sectional curvature of Σ^n, K_Σ, is such that

$$K_\Sigma \geq \frac{1}{\rho^2(h)}(\kappa - (\rho'(h))^2). \tag{6.3.8}$$

If hypothesis (6.1.2) is satisfied, $H_{r-1} > 0$, $H_r > 0$ with $\sup_\Sigma H_r < +\infty$, and

$$\frac{H_{r+1}}{H_r} \leq \frac{\rho'}{\rho}(h), \tag{6.3.9}$$

for some $2 \leq r \leq n-1$, then Σ^n is a slice of $\overline{M}^{n+1}$.

Proof. Since the fiber M^n has constant sectional curvature κ, from (6.1.1) toghether with Lemma 5.2.6 and equation (5.2.5) of Lemma 5.2.2 we obtain

$$\begin{aligned}\mathcal{L}_r(u) &= -(n-r)\rho(h)\left(\frac{\kappa}{f^2(h)} + (\log\rho)''(h)\right)\langle T_{r-1}\nabla h, \nabla h\rangle\Theta \\ &\quad + c_r(\rho'(h)H_r + \rho(h)H_{r+1}\Theta). \end{aligned} \tag{6.3.10}$$

As in the proof of Theorem 6.2.4 when we consider the self-adjoint operator defined by $\mathcal{T}_{r-1} : \mathfrak{X}(\Sigma^n) \to \mathfrak{X}(\Sigma^n)$ by $\mathcal{T}_{r-1} := H_{r-1}T_{r-1}$. Choose a local orthonormal frame $\{e_1, \cdots, e_n\}$ such that $Ae_i(p) = \lambda_i e_i(p)$. It follows from (5.1.7) that

$$T_{r-1}e_i = (-1)^{r-1} \sum_{i_1<\cdots<i_{r-1}, i_j\neq i} \lambda_{i_1}\cdots\lambda_{i_{r-1}} e_i.$$

This implies that, for any $i \in \{1, \cdots, n\}$

$$\langle \mathcal{T}_{r-1}e_i, e_i\rangle = \binom{n}{r-1}^{-1} \sum_{i_1<\cdots<i_{r-1}, i_j\neq i, j_1<\cdots<j_{r-1}} (\lambda_{i_1}\lambda_{j_1})\cdots(\lambda_{i_{r-1}}\lambda_{j_{r-1}}).$$

Denote by K_Σ and $\overline{K}$ the sectional curvatures of Σ^n and $\overline{M}^{n+1}$, respectively. Thus, from Gauss equation we obtain

$$K_\Sigma(e_i, e_j) = K(e_i, e_j) + \lambda_i\lambda_j, \tag{6.3.11}$$

From [107, Proposition 7.42], given arbitrary vector fields U, V, W in $\overline{M}^{n+1}$ we can compute the following relation

$$\begin{aligned}\overline{R}(U,V)W &= R_M(U^*, V^*)W^* - ((\log\rho)'(h))^2(\langle U, W\rangle V - \langle V, W\rangle U) \\ &\quad -(\log\rho)''(h)\langle W, \partial_t\rangle(\langle U, \partial_t\rangle V - \langle V, \partial_t\rangle U) \\ &\quad -(\log\rho)''(h)(\langle U, W\rangle\langle V, \partial_t\rangle - \langle U, \partial_t\rangle\langle V, W\rangle)\partial_t, \end{aligned} \tag{6.3.12}$$

where $U^* = (\pi_{M^n})_* U = U + \langle U, \partial_t\rangle\partial_t$. Hence, for an orthonormal basis $\{X, Y\}$ of an arbitrary 2-plane tangent to Σ^n, the above relation (6.3.12) we get

$$\begin{aligned}\overline{K}(X,Y) &= \frac{1}{\rho^2(h)}K_M(X^*, Y^*)|X^*\wedge Y^*|^2 \\ &\quad -((\log\rho)'(h))^2 - (\log\rho)''(h)(\langle X, \nabla h\rangle^2 + \langle Y, \nabla h\rangle^2).\end{aligned}$$

Note that

$$|X^*\wedge Y^*|^2 = 1 - \langle X, \nabla h\rangle^2 + \langle Y, \nabla h\rangle^2.$$

Thus,

$$\begin{aligned}
\overline{K}(X,Y) &= \frac{1}{\rho^2(h)}K_M(X^*,Y^*)(1-\langle X,\nabla h\rangle^2+\langle Y,\nabla h\rangle^2)\\
&\quad -((\log\rho)'(h))^2-(\log\rho)''(h)(\langle X,\nabla h\rangle^2+\langle Y,\nabla h\rangle^2)\\
&= \frac{1}{\rho^2(h)}K_M(X^*,Y^*)-((\log\rho)'(h))^2\\
&\quad -\left(\frac{1}{\rho^2(h)}K_M(X^*,Y^*)+(\log\rho)''(h)\right)(\langle X,\nabla h\rangle^2+\langle Y,\nabla h\rangle^2)\\
&= \frac{1}{\rho^2(h)}K_M(X^*,Y^*)-\left(\frac{\rho'}{\rho}(h)\right)^2\\
&\quad -\left(\frac{1}{\rho^2(h)}K_M(X^*,Y^*)+\frac{\rho\rho''-(\rho')^2}{\rho^2}(h)\right)(\langle X,\nabla h\rangle^2+\langle Y,\nabla h\rangle^2).
\end{aligned}$$

This together with the convergence condition (6.3.7) and assumption (6.3.8) implies the

$$\lambda_i\lambda_j\geq 0,\quad \text{for all}\quad i,j\in\{1,...,n\},\quad i\neq j.$$

Thus, we conclude that $\langle\mathcal{T}_{r-1}e_i,e_i\rangle\geq 0$ and $\mathcal{T}_{r-1}$, is positive semi-definite. Since H_{r-1} and H_r are positive and $-1\leq\Theta\leq 0$, from (6.3.7) and (6.3.10) we get

$$\begin{aligned}
\mathcal{L}_r(u) &= -(n-r)\rho(h)\left(\frac{\kappa}{f^2(h)}+(\log\rho)''(h)\right)\frac{1}{H_{r-1}}\langle\mathcal{T}_{r-1}\nabla h,\nabla h\rangle\Theta\\
&\quad +c_r(\rho'(h)H_r+\rho(h)H_{r+1}\Theta)\\
&\geq c_r\rho(h)H_r\left(\frac{\rho'}{\rho}(h)-\frac{H_{r+1}}{H_r}\right).
\end{aligned}\tag{6.3.13}$$

Hence, considering (6.3.9) into (6.3.13) we conclude that $\mathcal{L}_r(u)\geq 0$ on Σ^n. Consequently, since we are assuming that Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$, we can apply Proposition E to obtain that h is constant on Σ^n. Therefore, Σ^n must be a slice of $\overline{M}^{n+1}$. ■

From Theorem 6.3.3 we get the following nonexistence result:

Corollary 6.3.6 *Let $\overline{M}^{n+1}=I\times M^n$ be a Riemannian warped product whose fiber M^n has constant nonnegative sectional curvature κ. There is no complete two-sided hypersurface $\psi:\Sigma^n\to\overline{M}^{n+1}$ bounded away from the future infinity of $\overline{M}^{n+1}$, with $-1\leq\Theta\leq 0$, satisfying hypothesis (6.1.2) and such that $K_\Sigma\geq\kappa$, $H_{r-1},H_r>0$ and $H_{r+1}>0$ with $\sup_\Sigma H_r<+\infty$, for some $2\leq r\leq n-1$.*

Related to the higher order mean curvatures, we also establish the following result:

Theorem 6.3.4 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface immersed into a Riemannian warped product $\overline{M}^{n+1} = I \times_\rho M^n$ whose fiber M^n has constant sectional curvature κ satisfying*

$$\kappa \leq \inf_I ((\rho')^2 - \rho\rho''). \tag{6.3.14}$$

Suppose that Σ^n is bounded away from the past infinity of $\overline{M}^{n+1}$ and that $-1 \leq \Theta < 0$. Assume in addition that $\rho(h)$ attains a local maximum at some point $p \in \Sigma^n$ such that $\rho'(h(p)) \neq 0$. If hypothesis (6.1.2) is satisfied, $H_r > 0$ with $\sup_\Sigma H_r < +\infty$, $H_{r+1} > 0$ and

$$\frac{H_{r+1}}{H_r} \geq -\frac{1}{\Theta}\frac{\rho'}{\rho}(h), \tag{6.3.15}$$

for some $2 \leq r \leq n-1$, then Σ^n is a slice of $\overline{M}^{n+1}$.

Proof. Since we are assuming that $\rho(h)$ attains a local maximum at some point $p \in \Sigma^n$ such that $\rho'(h(p)) \neq 0$, Lemma 5.2.5 guarantees that p is an elliptic point of Σ^n. Using the assumption $H_{r+1} > 0$, from Lemma 5.2.4 the operator L_j is elliptic or, equivalently, T_j is positive definite for all $1 \leq j \leq r$. Thus, the curvature constraint (6.3.14) together with $-1 \leq \Theta < 0$ and (6.3.10) implies

$$\mathcal{L}_r(u) \leq c_r \rho(h) H_r \Theta \left(\frac{H_{r+1}}{H_r} + \frac{1}{\Theta}\frac{\rho'}{\rho}(h) \right).$$

Hence, from hypothesis (6.3.15) we have that $\mathcal{L}_r(u) \leq 0$ on Σ^n. Therefore, since Σ^n is bounded away from the past infinity of $\overline{M}^{n+1}$, we can apply once more Proposition E to conclude that Σ^n must be a slice of $\overline{M}^{n+1}$. ∎

6.4 Applications to entire graphs

When $\overline{M}^{n+1} = -I \times_\rho M^n$ is a GRW spacetime, we can restate Theorem 6.2.3 in the context of entire graphs as follows:

Corollary 6.4.1 *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a RW spacetime whose fiber M^n has constant sectional curvature κ satisfying curvature constraint (6.2.8) and let $\Sigma(w)$ be an entire graph determined by a bounded function $w \in C^\infty(M)$ such that, for some $2 \leq r \leq n-1$, $H_r > 0$ with $\sup_M H_r < +\infty$ and $H_{r+1} > 0$. Suppose that $\rho(w)$ attains a local minimum at some point $x \in M^n$ such that $\rho'(w(x)) \neq 0$ and that $|Dw|^2_M \leq \alpha\rho^2(w)$, for some constant $0 < \alpha < 1$. If $\Sigma(w)$ satisfies hypothesis (6.1.2) and*

$$\frac{H_{r+1}}{H_r} \geq \frac{\rho'}{\rho}(w), \tag{6.4.1}$$

then $w \equiv t_0$ for some $t_0 \in I$.

Proof. As in the beginning of the proof of Corollary 5.1 in [10], the assumption that $|Dw|_M^2 \leq \alpha\rho^2(w)$, for some constant $0 < \alpha < 1$, guarantees that $\Sigma(w)$ is a complete spacelike hypersurface. Therefore, since we are also assuming that hypotheses (6.1.2) and (6.4.1) are satisfied, we can apply Theorem 6.2.3 to conclude the result. ∎

Taking into account (5.3.2), it is not difficult to see that we can also reformulate Theorem 6.2.4 in the context of entire graphs as follows:

Corollary 6.4.2 *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a RW spacetime whose fiber M^n has constant sectional curvature κ satisfying the SNCC (6.2.12) and let $\Sigma(w)$ be an entire graph determined by a bounded function $w \in C^\infty(M)$ such that, for some $2 \leq r \leq n-1$, $H_{r-1} > 0$ and $H_r > 0$ with $\sup_\Sigma H_r < +\infty$. Suppose that the sectional curvature of $\Sigma(w)$ satisfies (6.2.13) and that $|Dw|_M^2 \leq \alpha\rho^2(w)$, for some constant $0 < \alpha < 1$. If hypothesis (6.1.2) is satisfied and*

$$\frac{H_{r+1}}{H_r} \leq \frac{\rho'}{\rho^2}(w)W(w),$$

then $w \equiv t_0$ for some $t_0 \in I$.

When the ambient space $\overline{M}^{n+1} = I \times_\rho M^n$ is a Riemannian warped product, all results in Subsection 6.3 can be also rewritten for the context of entire graphs. In particular, we quote the following versions of Theorems 6.3.3 and 6.3.4:

Corollary 6.4.3 *Let $\overline{M}^{n+1} = I \times_\rho M^n$ be a Riemannian warped product whose fiber M^n has constant sectional curvature κ obeying the curvature constraint (6.3.7) and let be $\Sigma(w)$ be an entire graph determined by a bounded function $w \in C^\infty(M)$ such that, for some $2 \leq r \leq n-1$, $H_{r-1} > 0$, $H_r > 0$ with $\sup_\Sigma H_r < +\infty$. Suppose that the sectional curvature of $\Sigma(w)$ satisfies (6.3.8) and that $|Dw|_M < +\infty$. If hypothesis (6.1.2) is satisfied and*

$$\frac{H_{r+1}}{H_r} \leq \frac{\rho'}{\rho}(w),$$

then $w \equiv t_0$ for some $t_0 \in I$.

Corollary 6.4.4 *Let $\overline{M}^{n+1} = I \times_\rho M^n$ be a Riemannian warped product whose fiber M^n has constant sectional curvature κ obeying the curvature constraint (6.3.14) and let be $\Sigma(w)$ be an entire graph determined by a bounded function $w \in C^\infty(M)$ such that, for some $2 \leq r \leq n-1$, $H_r > 0$ with $\sup_\Sigma H_r < +\infty$ and $H_{r+1} > 0$. Suppose that $\rho(w)$ attains a local maximum at some point $x \in M^n$ such that $\rho'(w(x)) \neq 0$ and that $|Dw|_M < +\infty$. If hypothesis (6.1.2) is satisfied and*

$$\frac{H_{r+1}}{H_r} \geq \frac{\rho'}{\rho^2}(w)W(w),$$

then $w \equiv t_0$ for some $t_0 \in I$.

Chapter 7

Rigidity and nonexistence of complete hypersurfaces via Liouville type results and other maximum principles

To close this Thesis, we present the results concerning the article [25]. As in the previous chapter, here we investigate complete hypersurfaces with some positive higher order mean curvature in a semi-Riemannian warped product space. Now, under standard curvature conditions on the ambient space and appropriate constraints on the higher order mean curvatures, we establish rigidity and nonexistence results via Liouville type results and suitable maximum principles related to the divergence of smooth vector fields on a complete noncompact Riemannian manifold. Applications to standard warped product models, like the Schwarzschild, Reissner-Nordström and pseudo-hyperbolic spaces, as well as steady state type spacetimes, are given and a particular study of entire graphs is also presented.

7.1 Rigidity and nonexistence results

This section is devoted to present rigidity and nonexistence results concerning complete Riemannian immersions in a semi-Riemannian warped product. Our approach is based on a criteria of integrability due to Yau in [128], a Liouville-type result due to Pigola, Rigoli and Setti in [111], a version of maximum principle at infinity for vector fields due to Alías, Caminha and Nascimento in [7] and a maximum principle related to polynomial volume growth also obtained by these same authors in [8].

7.1.1 Via integrability

We recall the f-*divergence operator* on a Riemannian manifold Σ^n endowed with a (smooth) weight function $f : \Sigma^n \to \mathbb{R}$ is defined by

$$\mathrm{div}_f(X) = e^f \mathrm{div}(e^{-f} X),$$

where X is a tangent vector field on Σ^n and div stands for the standard divergence operator of Σ^n. From this, for all smooth function $u : \Sigma^n \to \mathbb{R}$, we define the *drift Laplacian* of u by

$$\Delta_f u = \mathrm{div}_f(\nabla u) = \Delta u - \langle \nabla u, \nabla f \rangle.$$

Let us also consider the set of Lebesgue integrable functions on Σ^n

$$\mathfrak{L}^p_f(\Sigma) := \left\{ w : \Sigma^n \to \mathbb{R} : \int_\Sigma |w|^p e^{-f} d\Sigma < +\infty \right\},$$

with respect to the modified volume element $e^{-f} d\Sigma$, where $d\Sigma$ is the volume element induced by the metric of Σ^n. In this setting, we get the following consequence of [128, Proposition 2.1].

Lemma 7.1.1 *Let u be a smooth function on a complete Riemannian manifold Σ^n endowed with weight function f, such that $\Delta_f u$ does not change sign on Σ^n. If $|\nabla u| \in \mathfrak{L}^1_f(\Sigma)$, then $\Delta_f u$ vanishes identically on Σ^n.*

Remark 7.1.1 We observe that Lemma 7.1.1 can be regarded as a consequence of the version of Stokes' Theorem due to Karp in [86]. Indeed, using [86, Theorem], condition $|\nabla u| \in \mathfrak{L}^1_f(\Sigma)$ can be weakened to the following hypothesis

$$\liminf_{r \to +\infty} \frac{1}{r} \int_{B(2r) \setminus B(r)} |\nabla u| e^{-f} d\Sigma = 0,$$

where $B(r)$ stands for the geodesic ball of radius r center at some fixed origin $o \in \Sigma^n$. Moreover, [86, Corollary 1 and Remark] give some geometric conditions which guarantee this hypothesis.

In our first rigidity result, we will suppose that the Riemannian fiber M^n of the GRW spacetime obeys a suitable curvature constraint and that the Newton transformation T_1 satisfies $T_1(\nabla u) = \varphi(\nabla u)$, for some $0 < \varphi \in C^\infty(\Sigma)$ and $u := g(h) \in C^\infty(\Sigma)$ an arbitrary primitive g of the warping function ρ. We observe that the totally umbilical hypersurfaces satisfy this last condition. Recently, this condition has been used extensively in the study of a class of hypersurfaces of warped products whose gradient of their height functions are principal directions. See, for instance, the works [71, 69, 70, 98, 122]. In this setting, considering again the curvature constraint (6.2.4), we can present the following

Theorem 7.1.1 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface immersed into a GRW spacetime $\overline{M}^{n+1} = -I \times_\rho M^n$, which obeys the following curvature constraint*

$$\mathrm{Ric}_M \leq (n-1) \inf_I (\rho\rho'' - (\rho')^2)\langle , \rangle_M, \tag{7.1.1}$$

where Ric_M stands for the Ricci tensor of M^n. Suppose that H_2 is positive and $T_1(\nabla u) = \varphi(\nabla u)$, for some $0 < \varphi \in C^\infty(\Sigma)$. If $|\nabla u| \in \mathfrak{L}^1_{-\ln\varphi}(\Sigma)$ and

$$\frac{H_2}{H_1} \geq \frac{\rho'}{\rho}(h), \tag{7.1.2}$$

then Σ^n is a slice of $\overline{M}^{n+1}$.

Proof. Since the function $\varphi \in C^\infty(\Sigma)$ is positive, from definition (6.1.1) for $r = 1$, we obtain

$$\begin{aligned} \frac{1}{\varphi}\mathcal{L}_1(u) &= \frac{1}{\varphi}\mathrm{div}(T_1(\nabla u)) = e^{-\ln\varphi}\mathrm{div}\,(\varphi\nabla u) \\ &= e^{-\ln\varphi}\mathrm{div}\left(e^{\ln\varphi}\nabla u\right) \\ &= \Delta_{-\ln\varphi} u. \end{aligned} \tag{7.1.3}$$

On the other hand, from Lemma 5.2.6 and equation (5.2.4) of Lemma 5.2.2 we obtain that

$$\begin{aligned} \mathcal{L}_1(u) &= \rho(h)\left(\mathrm{Ric}_M(N^*, N^*) - (n-1)(\log\rho)''(h)|\nabla h|^2\right)\langle N, \partial_t\rangle \\ &\quad -b_1(\rho'(h)H_1 + \rho(h)H_2\langle N, \partial_t\rangle), \end{aligned} \tag{7.1.4}$$

where $N^* = N + \langle N, \partial_t\rangle\partial_t$. Since $H_2 > 0$, Lemma 5.2.3 guarantees that, with respect to N, $H_1 > 0$ and T_1 is positive definite. So, once N is future-pointing and $|\nabla h|^2 = \rho^2(h)\langle N^*, N^*\rangle_M$, using curvature constraint (7.1.1) into (7.1.4), we obtain

$$\mathcal{L}_1(u) \geq \rho(h) b_1 H_1 \left(\frac{H_2}{H_1} - \frac{\rho'}{\rho}(h)\right). \tag{7.1.5}$$

Hence, taking into account hypothesis (7.1.2), from (7.1.3) and (7.1.5) we have that u is $(-\ln\varphi)$-subharmonic function. Moreover, since $|\nabla u| \in \mathfrak{L}^1_{-\ln\varphi}(\Sigma)$, Lemma 7.1.1

gives that $\Delta_{-\ln\varphi} u$ vanishes identically on Σ^n, that is, u is constant on Σ^n. Therefore, we conclude that the height function h is constant and, hence, Σ^n is a slice of $\overline{M}^{n+1}$. ■

As application of Theorem 7.1.1, we can consider the case when ambient is an Einstein manifold to obtain the following rigidity result, whose its proof is similar to proof of Corollary 6.2.2.:

Corollary 7.1.1 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface immersed into a Einstein GRW spacetime $\overline{M}^{n+1} = -I \times_\rho M^n$. Suppose that T_1 is positive definite with $T_1(\nabla u) = \varphi(\nabla u)$, for some $0 < \varphi \in C^\infty(\Sigma)$. If $|\nabla u| \in \mathfrak{L}^1_{-\ln\varphi}(\Sigma)$ and hypothesis (7.1.2) is satisfied, then Σ^n is a slice of $\overline{M}^{n+1}$.*

Next, as in Corollary 6.2.1, we will consider the *steady state-type spacetime* $\overline{M}^{n+1} = -\mathbb{R} \times_{e^t} M^n$. In this case, we also obtain the following application from Theorem 7.1.1:

Corollary 7.1.2 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface immersed into a steady state-type spacetime $\overline{M}^{n+1} = -\mathbb{R} \times_{e^t} M^n$ whose fiber M^n has nonpositive Ricci curvature. Suppose that T_1 is positive definite with $T_1(\nabla u) = \varphi(\nabla u)$, for some $0 < \varphi \in C^\infty(\Sigma)$. If $|\nabla u| \in \mathfrak{L}^1_{-\ln\varphi}(\Sigma)$ and $H_2 \geq H_1$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

In order to extend this reasoning to the higher order mean curvatures, we will assume that the Riemannian fiber M^n has constant sectional curvature, that is, $\overline{M}^{n+1} = -I \times_\rho M^n$ is Robertson-Walker (RW) spacetime.

Theorem 7.1.2 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface immersed into a RW spacetime $\overline{M}^{n+1} = -I \times_\rho M^n$ whose constant sectional curvature κ of fiber M^n obeys the curvature constraint*

$$\kappa \leq \inf_I (\rho\rho'' - (\rho')^2). \tag{7.1.6}$$

Suppose that, for some $2 \leq r \leq n-1$, H_{r+1} is positive and $\rho(h)$ attains a local minimum at a point $q \in \Sigma^n$ such that $\rho'(h(q)) \neq 0$. If T_r satisfies $T_r(\nabla u) = \varphi(\nabla u)$, for some $0 < \varphi \in C^\infty(\Sigma)$, $|\nabla u| \in \mathfrak{L}^1_{-\ln\varphi}(\Sigma)$ and

$$\frac{H_{r+1}}{H_r} \geq \frac{\rho'}{\rho}(h), \tag{7.1.7}$$

then Σ^n is a slice of $\overline{M}^{n+1}$.

Proof. Since the fiber M^n has constant sectional curvature κ, from (6.1.1) jointly with Lemma 5.2.6 and equation (5.2.5) of Lemma 5.2.2 we obtain

$$\begin{aligned}\mathcal{L}_r(u) &= (n-r)\rho(h)\left(\frac{\kappa}{\rho^2(h)} - (\log\rho)''(h)\right)\langle T_{r-1}\nabla h, \nabla h\rangle\langle N, \partial_t\rangle \\ &\quad -b_r(\rho'(h)H_r + \rho(h)H_{r+1}\langle N, \partial_t\rangle).\end{aligned} \tag{7.1.8}$$

On the other hand, since $H_{r+1} > 0$ and $\rho(h)$ attains a local maximum at some point $q \in \Sigma^n$ such that $\rho'(h(q)) \neq 0$, from Lemma 5.2.4 and Lemma 5.2.5, the Newton transformations T_j is positive definite and H_j is positive for all $1 \leq j \leq r$. Thus, taking into account that N is future-pointing, from (7.1.6) and (7.1.8) we obtain

$$\mathcal{L}_r(u) \geq b_r\rho(h)H_r\left(\frac{H_{r+1}}{H_r} - \frac{\rho'}{\rho}(h)\right). \tag{7.1.9}$$

But, we can reason as in (7.1.3) to deduce that

$$\frac{1}{\varphi}\mathcal{L}_r(u) = \Delta_{-\ln\varphi}u. \tag{7.1.10}$$

Hence, considering (7.1.7) into (7.1.9) we have that u is $(-\ln\varphi)$-subharmonic function. Thus, since $|\nabla u| \in \mathfrak{L}^1_{-\ln\varphi}(\Sigma)$, from Lemma 7.1.1 we get that u is constant on Σ^n. Therefore, we conclude that Σ^n is a slice of $\overline{M}^{n+1}$. ∎

Remark 7.1.2 Concerning Theorem 7.1.2, we observe that if we substitute the assumptions (7.1.6) and (7.1.7) by $\kappa \leq \sup_I(\rho\rho'' - (\rho')^2)$ and $\frac{H_{r+1}}{H_r} \leq -\frac{1}{\langle N, \partial_t\rangle}\frac{\rho'}{\rho}(h)$, respectively, we will also obtain the rigidity result.

When the ambient space is a Riemannian warped product and the complete hypersurface is two-sided, we obtain the following result:

Theorem 7.1.3 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface immersed into a Riemannian warped product $\overline{M}^{n+1} = I \times_\rho M^n$, which obeys the following curvature constraint*

$$\mathrm{Ric}_M \geq (n-1)\sup_I((\rho')^2 - \rho\rho'')\langle,\rangle_M, \tag{7.1.11}$$

where Ric_M stands for the Ricci tensor of M^n. Suppose that H_2 is positive, $-1 \leq \langle N, \partial_t\rangle \leq 0$ and $T_1(\nabla u) = \varphi(\nabla u)$, for some $0 < \varphi \in C^\infty(\Sigma)$. If $|\nabla u| \in \mathfrak{L}^1_{-\ln\varphi}(\Sigma)$ and

$$\frac{H_2}{H_1} \leq \frac{\rho'}{\rho}(h), \tag{7.1.12}$$

then Σ^n is a slice of $\overline{M}^{n+1}$.

Proof. From Lemma 5.2.6 and equation (5.2.4) of Lemma 5.2.2 we obtain that

$$\mathcal{L}_1(u) = \rho(h)(-\mathrm{Ric}_M(N^*, N^*) - (n-1)(\log\rho)''(h)|\nabla h|^2)\langle N, \partial_t\rangle + b_1(\rho'(h)H_1 + \rho(h)H_2\langle N, \partial_t\rangle), \quad (7.1.13)$$

where $N^* = N - \langle N, \partial_t\rangle\partial_t$.

We also note that, since $H_2 > 0$, Lemma 5.2.3 guarantee that, with respect to N, $H_1 > 0$ and T_1 positive definite. So, once $|\nabla h|^2 = \rho^2(h)\langle N^*, N^*\rangle_M$ and we are assuming $-1 \leq \langle N, \partial_t\rangle \leq 0$, using curvature constraint (7.1.11) into (7.1.13) we obtain

$$\mathcal{L}_1(u) \geq \rho(h)b_1H_1\left(\frac{\rho'}{\rho}(h) - \frac{H_2}{H_1}\right). \quad (7.1.14)$$

Hence, from (7.1.3), (7.1.12) and (7.1.14) we get that u is $(-\ln\varphi)$-subharmonic function. Thus, since $|\nabla u| \in \mathfrak{L}^1_{-\ln\varphi}(\Sigma)$, from Lemma 7.1.1 we have that $\Delta_{-\ln\varphi}u$ vanishes identically on Σ^n, that is, u is constant on Σ^n. Therefore, Σ^n is a slice of $\overline{M}^{n+1}$. ■

In particular, when $\overline{M}^{n+1}$ is an Einstein manifold, Theorem 7.1.3 reads as follows:

Corollary 7.1.3 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface immersed into an Einstein warped product $\overline{M}^{n+1} = I \times_\rho M^n$. Suppose that H_1 and H_2 satisfy (7.1.12), H_2 is positive, $-1 \leq \langle N, \partial_t\rangle \leq 0$ and $T_1(\nabla u) = \varphi(\nabla u)$, for some $0 < \varphi \in C^\infty(\Sigma)$. If $|\nabla u| \in \mathfrak{L}^1_{-\ln\varphi}(\Sigma)$ and hypothesis (7.1.12) is satisfied, then Σ^n is a slice of $\overline{M}^{n+1}$.*

Proof. From [42, Corollary 9.107] (see also [45, Section 6]) we have that $\overline{M}^{n+1}$ is an Einstein manifold with Ricci tensor $\overline{\mathrm{Ric}} = \overline{c}\,\overline{g}$, for some constant $\overline{c} \in \mathbb{R}$, if and only if the fiber $(M^n, g_{_M})$ has constant Ricci curvature $\mathrm{Ric}_M = c$ and the warping function ρ satisfies the differential equations

$$-\frac{\rho''}{\rho} = \frac{\overline{c}}{n} \quad \text{and} \quad \frac{\overline{c}(n-1)}{n} = \frac{c - (n-1)(\rho')^2}{\rho^2}. \quad (7.1.15)$$

Hence, from (7.1.15) we obtain $-(n-1)(\log\rho)'' = \dfrac{c}{\rho^2}$. Therefore, in this case, we have that

$$\mathrm{Ric}_M = (n-1)\sup_I((\rho')^2 - \rho\rho'')\langle\,,\rangle_M$$

and, consequently, the result follows by applying Theorem 7.1.14. ■

When $\overline{M}^{n+1}$ is a *pseudo-hyperbolic space*, we get the following applications of Theorem 7.1.3:

Corollary 7.1.4 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface immersed into a pseudo-hyperbolic space $\overline{M}^{n+1} = \mathbb{R} \times_{e^t} M^n$ whose fiber M^n has nonnegative Ricci curvature. Suppose that $H_1 \geq H_2 > 0$, $-1 \leq \langle N, \partial_t \rangle \leq 0$ and $T_1(\nabla u) = \varphi(\nabla u)$, for some $0 < \varphi \in C^\infty(\Sigma)$. If $|\nabla u| \in \mathfrak{L}^1_{-\ln\varphi}(\Sigma)$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

Corollary 7.1.5 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface immersed into a pseudo-hyperbolic space $\overline{M}^{n+1} = \mathbb{R} \times_{\cosh t} M^n$ whose Ricci tensor of the fiber M^n is such that $\mathrm{Ric}_M \geq -(n-1)\langle , \rangle_M$. Suppose that H_2 is positive, $-1 \leq \langle N, \partial_t \rangle \leq 0$ and $T_1(\nabla u) = \varphi(\nabla u)$, for some $0 < \varphi \in C^\infty(\Sigma)$. If $|\nabla u| \in \mathfrak{L}^1_{-\ln\varphi}(\Sigma)$ and $\dfrac{H_2}{H} \leq \tanh(h)$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

When the ambient is either the *Schwarzschild space* or *Reissner-Nordström space*, Theorem 7.1.3 reads as follows.

Corollary 7.1.6 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface immersed into either a Schwarzschild space or a Reissner-Nordström space $\overline{M}^{n+1} = (0, +\infty) \times_{r(t)} \mathbb{S}^n$, where $r(t)$ is defined in (5). Suppose that H_2 is positive, $-1 \leq \langle N, \partial_t \rangle \leq 0$ and $T_1(\nabla u) = \varphi(\nabla u)$, for some $0 < \varphi \in C^\infty(\Sigma)$. If $|\nabla u| \in \mathfrak{L}^1_{-\ln\varphi}(\Sigma)$ and*

$$\frac{H_2}{H_1} \leq \frac{r'}{r}(t), \tag{7.1.16}$$

then Σ^n is a slice of $\overline{M}^{n+1}$.

In our next result, we will deal with higher order mean curvatures.

Theorem 7.1.4 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface immersed into a Riemannian warped product $\overline{M}^{n+1} = I \times_\rho M^n$ whose fiber M^n has constant sectional curvature κ obeying the curvature constraint*

$$\kappa \geq \sup_I((\rho')^2 - \rho\rho''). \tag{7.1.17}$$

Suppose that, for some $2 \leq r \leq n-1$, H_{r+1} is positive, $-1 \leq \langle N, \partial_t \rangle \leq 0$ and $\rho(h)$ attains a local maximum at a point $q \in \Sigma^n$ such that $\rho'(h(q)) \neq 0$. If T_r satisfies $T_r(\nabla u) = \varphi(\nabla u)$, for some $0 < \varphi \in C^\infty(\Sigma)$, $|\nabla u| \in \mathfrak{L}^1_{-\ln\varphi}(\Sigma)$ and

$$\frac{H_{r+1}}{H_r} \leq \frac{\rho'}{\rho}(h), \tag{7.1.18}$$

then Σ^n is a slice of $\overline{M}^{n+1}$.

Proof. Since the fiber M^n has constant sectional curvature κ, from (6.1.1) jointly with Lemma 5.2.6 and equation (5.2.5) of Lemma 5.2.2 we obtain

$$\begin{aligned}\mathcal{L}_r(u) &= -(n-r)\rho(h)\left(\frac{\kappa}{\rho^2(h)} + (\log\rho)''(h)\right)\langle T_{r-1}\nabla h, \nabla h\rangle\langle N, \partial_t\rangle \\ &\quad + b_r(\rho'(h)H_r + \rho(h)H_{r+1}\langle N, \partial_t\rangle).\end{aligned} \tag{7.1.19}$$

On the other hand, since $H_{r+1} > 0$ and $\rho(h)$ attains a local maximum at some point $q \in \Sigma^n$ such that $\rho'(h(q)) \neq 0$, from Lemma 5.2.4 and Lemma 5.2.5 the Newton transformation T_j is positive definite and H_j is positive for all $1 \leq j \leq r$. Thus, taking into account that $-1 \leq \langle N, \partial_t\rangle \leq 0$, from (7.1.17) and (7.1.19) we obtain

$$\mathcal{L}_r(u) \geq \rho(h)b_rH_r\left(\frac{\rho'}{\rho}(h) - \frac{H_{r+1}}{H_r}\right). \tag{7.1.20}$$

Hence, considering (7.1.18) into (7.1.20) and using (7.1.10), we get that u is $(-\ln\varphi)$-subharmonic function. Thus, since $|\nabla u| \in \mathfrak{L}^1_{-\ln\varphi}(\Sigma)$, from Lemma 7.1.1 we have that u is constant on Σ^n. Therefore, we conclude that Σ^n is a slice of $\overline{M}^{n+1}$. ■

Remark 7.1.3 Concerning Theorem 7.1.4, we observe that if we substitute the assumptions (7.1.17) and (7.1.18) by $\kappa \leq \inf_I((\rho')^2 - \rho\rho'')$ and $\dfrac{H_{r+1}}{H_r} \geq -\dfrac{1}{\langle N, \partial_t\rangle}\dfrac{\rho'}{\rho}(h)$, respectively, we will also obtain the rigidity result.

7.1.2 Via p–integrability, for $p > 1$.

We start quoting a consequence of [111, Theorem 1.1].

Lemma 7.1.2 *Let Σ be complete Riemannian manifold and let $u \in C^\infty(\Sigma)$ be nonnegative. If u is $(-\ln\varphi)$-subharmonic function, where $0 < \varphi \in C^\infty(\Sigma)$, and $u \in \mathfrak{L}^p_{-\ln\varphi}(\Sigma)$, for some $p > 1$, then u is constant.*

Remark 7.1.4 According to the ideas discussed in [32, Remark 3], $\int_\Sigma |u|^p\varphi d\Sigma$ can be considered as a sort of total (p,φ)-energy associated to u. In this setting, the hypothesis $u \in \mathfrak{L}^p_{-\ln\varphi}(\Sigma)$ in Lemma 7.1.2 is equivalent to Σ^n having finite (p,φ)-energy associated to u.

Now, we are in position to present our next rigidity result.

Theorem 7.1.5 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface immersed into a GRW spacetime $\overline{M}^{n+1} = -I \times_\rho M^n$ which obeys the curvature constraint (7.1.1). Suppose that H_1 and H_2 satisfy (7.1.2), H_2 is positive and $T_1(\nabla u) = \varphi(\nabla u)$, for some $0 < \varphi \in C^\infty(\Sigma)$. Assume in addition that u is nonnegative and bounded from above. If $u \in \mathfrak{L}^p_{-\ln\varphi}(\Sigma)$, for some $p > 1$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

Proof. Since $H_2 > 0$, Lemma 5.2.3 guarantees that $H_1 > 0$ and T_1 is positive definite. Following similar steps of the proof of Theorem 7.1.1, we obtain that $\mathcal{L}_1(u) = \varphi\Delta_{-\ln\varphi}u$. Thus, from (7.1.1) and (7.1.2) we get that u is $(-\ln\varphi)$-subharmonic function. Thus, since $u \in \mathfrak{L}^p_{-\ln\varphi}(\Sigma)$ and $0 \leq u \leq \sup_\Sigma u < +\infty$, from Lemma 7.1.2 we have that u is constant on Σ^n. Therefore, we conclude that the height function h is constant, which means that Σ^n is a slice of $\overline{M}^{n+1}$. ∎

As application of Theorem 7.1.5, we obtain the following result when the ambient spacetime is an Einstein manifold:

Corollary 7.1.7 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface immersed into an Einstein GRW spacetime $\overline{M}^{n+1} = -I \times_\rho M^n$. Suppose that H_1 and H_2 satisfy (7.1.2), H_2 is positive and $T_1(\nabla u) = \varphi(\nabla u)$, for some $0 < \varphi \in C^\infty(\Sigma)$. Assume in addition that u is nonnegative and bounded from above. If $u \in \mathfrak{L}^p_{-\ln\varphi}(\Sigma)$, for some $p > 1$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

When the ambient space is an steady state-type spacetime, we also obtain the following application for Theorem 7.1.5:

Corollary 7.1.8 *Let $\Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface immersed into a steady state-type spacetime $\overline{M}^{n+1} = -\mathbb{R} \times_{e^t} M^n$ whose fiber M^n has nonpositive Ricci curvature. Suppose that H_1 and H_2 satisfy (7.1.2), H_2 is positive and $T_1(\nabla u) = \varphi(\nabla u)$, for some $0 < \varphi \in C^\infty(\Sigma)$. Assume in addition that u is nonnegative and bounded from above. If $u \in \mathfrak{L}^p_{-\ln\varphi}(\Sigma)$, for some $p > 1$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

In the case $2 \leq r \leq n-1$, we will consider the ambient space is a RW spacetime.

Theorem 7.1.6 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a spacelike hypersurface immersed into a RW spacetime $\overline{M}^{n+1} = -I \times_\rho M^n$ whose constant sectional curvature κ of the fiber M^n satisfies the curvature constraint (7.1.6). Suppose that, for some $2 \leq r \leq n-1$, H_r and H_{r+1} satisfy (7.1.7), H_{r+1} is positive, $\rho(h)$ attains a local minimum at some point $q \in \Sigma^n$ such that $\rho'(h(q)) \neq 0$, and that T_r satisfies $T_r(\nabla u) = \varphi(\nabla u)$, for some $0 < \varphi \in C^\infty(\Sigma)$. Assume in addition that u is nonnegative and bounded from above. If $u \in \mathfrak{L}^p_{-\ln\varphi}(\Sigma)$, for some $p > 1$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

Proof. Since the warping function $\rho(h)$ attains a local minimum at some point $q \in \Sigma^n$ such that $\rho'(h(q)) \neq 0$ and $H_{r+1} > 0$, Lemma 5.2.4 and Lemma 5.2.5 guarantee that T_{r-1} and T_r are positive definite.

On the other hand, since the fiber M^n has constant sectional curvature κ, from Lemma 5.2.6 jointly equation (5.2.5) of Lemma 5.2.2 and (6.1.1) we obtain

$$\begin{aligned}\mathcal{L}_r(u) &= \rho(h)(n-r)\left(\frac{\kappa}{\rho^2(h)} - (\log\rho)''(h)\right)\langle T_{r-1}\nabla h, \nabla h\rangle\langle N, \partial_t\rangle \\ &\quad -b_r(\rho'(h)H_r + \rho(h)H_{r+1}\langle N, \partial_t\rangle).\end{aligned} \tag{7.1.21}$$

Thus, taking into account curvature constraint (7.1.6) and that N is future-pointing, from (7.1.21) we obtain

$$\mathcal{L}_r(u) \geq b_r H_r \rho(h)\left(\frac{H_{r+1}}{H_r} - \frac{\rho'}{\rho}(h)\right). \tag{7.1.22}$$

Consequently, from (7.1.7) and (7.1.22) we get that $\mathcal{L}_r(u) = \varphi\Delta_{-\ln\varphi}u \geq 0$ on Σ^n, that is, u is $(-\ln\varphi)$-subharmonic function. So, since $u \in \mathfrak{L}^p_{-\ln\varphi}(\Sigma)$ and $0 \leq u \leq \sup_\Sigma u < +\infty$, Lemma 7.1.2 assures that u is constant on Σ^n. Therefore, we conclude that the height function h is constant, which means that Σ^n is a slice of $\overline{M}^{n+1}$. ■

As a consequence of Theorem 7.1.6 we obtain the following nonexistence result when the GRW spacetime $-I \times_\rho M^n$ is *static*:

Corollary 7.1.9 *Let $\overline{M}^{n+1} = -I \times M^n$ be a static RW spacetime whose constant sectional curvature κ of the fiber M^n is nonpositive. There is no complete spacelike hypersurface $\Sigma^n \hookrightarrow \overline{M}^{n+1}$ such that $H_r > 0$, T_r is positive definite with $T_r(\nabla h) = \varphi(\nabla h)$, for all $0 \leq j \leq r$ and some $0 < \varphi \in C^\infty(\Sigma)$, the height function h is nonnegative, bounded from above with $h \in \mathfrak{L}^p_{-\ln\varphi}(\Sigma)$, for some $p > 1$, and hypothesis (7.1.7) is satisfied.*

Proof. Suppose, by contradiction, that there is such a hypersurface Σ^n satisfying these assumptions. So, from Theorem 7.1.6 Σ^n must be a slice of $\overline{M}^{n+1}$. Thus, since that $\rho(h) \equiv 1$, we obtain that $H_r = (\rho'(h))^r = 0$ which contradicts the assumption $H_r > 0$. ■

Remark 7.1.5 Concerning Theorem 7.1.6, we observe that if u is just nonnegative and we substitute the hypothesis (7.1.6) by *null convergence condition* (NCC) (see [104]), and replace the assumptions (7.1.7) by $\dfrac{H_{r+1}}{H_r} \leq -\dfrac{1}{\langle N, \partial_t\rangle}\dfrac{\rho'}{\rho}(h)$, respectively, we will also obtain the rigidity result.

For complete two-sided hypersurfaces immersed in a Riemannian warped product, we obtain the following:

Theorem 7.1.7 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface immersed into a Riemannian warped product $\overline{M}^{n+1} = I \times_\rho M^n$, which obeys the curvature constraint (7.1.11). Suppose that H_1 and H_2 satisfy (7.1.12), H_2 is positive, $-1 \leq \langle N, \partial_t \rangle \leq 0$ and $T_1(\nabla u) = \varphi(\nabla u)$, for some $0 < \varphi \in C^\infty(\Sigma)$. Assume in addition that u is nonnegative and bounded from above. If $u \in \mathfrak{L}^p_{-\ln\varphi}(\Sigma)$, for some $p > 1$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

Proof. Following similar steps of the proof of Theorem 7.1.3, we obtain that u is $(-\ln\varphi)$-subharmonic function. Thus, since $u \in \mathfrak{L}^p_{-\ln\varphi}(\Sigma)$ and $0 \leq u \leq \sup_\Sigma u < +\infty$, Lemma 7.1.2 guarantees that u is constant on Σ^n. Therefore, we conclude that Σ^n is a slice of $\overline{M}^{n+1}$. ∎

It follows the applications of Theorem 7.1.7 when $\overline{M}^{n+1}$ is either an Einstein space or a pseudo-hyperbolic space.

Corollary 7.1.10 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface immersed into an Einstein warped product $\overline{M}^{n+1} = I \times_\rho M^n$. Suppose that H_1 and H_2 satisfy (7.1.12), H_2 is positive, $-1 \leq \langle N, \partial_t \rangle \leq 0$ and $T_1(\nabla u) = \varphi(\nabla u)$, for some $0 < \varphi \in C^\infty(\Sigma)$. Assume in addition that u is nonnegative and bounded from above. If $u \in \mathfrak{L}^p_{-\ln\varphi}(\Sigma)$, for some $p > 1$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

Corollary 7.1.11 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface immersed into a pseudo-hyperbolic space $\overline{M}^{n+1} = \mathbb{R} \times_{e^t} M^n$ whose fiber M^n has nonnegative Ricci curvature. Suppose that $H_1 \geq H_2 > 0$, $-1 \leq \langle N, \partial_t \rangle \leq 0$ and $T_1(\nabla u) = \varphi(\nabla u)$, for some $0 < \varphi \in C^\infty(\Sigma)$. Assume in addition that u is nonnegative and bounded from above. If $u \in \mathfrak{L}^p_{-\ln\varphi}(\Sigma)$, for some $p > 1$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

Corollary 7.1.12 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface immersed into a pseudo-hyperbolic space $\overline{M}^{n+1} = \mathbb{R} \times_{\cosh t} M^n$ whose Ricci tensor of the fiber M^n is such that $\mathrm{Ric}_M \geq -(n-1)\langle , \rangle_M$. Suppose that $H_2 > 0$, $-1 \leq \langle N, \partial_t \rangle \leq 0$ and $T_1(\nabla u) = \varphi(\nabla u)$, for some $0 < \varphi \in C^\infty(\Sigma)$. Assume in addition that u is nonnegative and bounded from above. If $u \in \mathcal{L}^p_{-\ln\varphi}(\Sigma)$, for some $p > 1$, and $\dfrac{H_2}{H_1} \leq \tanh(h)$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

When the ambient is the Schwarzschild space or Reissner-Nordström space, we get the following applications of Theorem 7.1.7:

Corollary 7.1.13 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface immersed into either a Schwarzschild space or a Reissner-Nordström space $\overline{M}^{n+1} = (0, +\infty) \times_{r(t)} \mathbb{S}^n$, where $r(t)$ is defined in (5). Suppose that H_1 and H_2 satisfy (7.1.16), H_2 is positive,*

$-1 \leq \langle N, \partial_t \rangle \leq 0$ and $T_1(\nabla u) = \varphi(\nabla u)$, for some $0 < \varphi \in C^\infty(\Sigma)$. Assume in addition that u is nonnegative and bounded from above. If $u \in \mathcal{L}^p_{-\ln\varphi}(\Sigma)$, for some $p > 1$, then Σ^n is a slice of $\overline{M}^{n+1}$.

Now, we will consider the context of higher order mean curvatures.

Theorem 7.1.8 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface immersed into a Riemannian warped product $\overline{M}^{n+1} = I \times_\rho M^n$ whose fiber M^n has constant sectional curvature κ obeying the curvature constraint (7.1.17). Suppose that, for some $2 \leq r \leq n-1$, H_r and H_{r+1} satisfy (7.1.18), H_{r+1} is positive, $-1 \leq \langle N, \partial_t \rangle \leq 0$, $\rho(h)$ attains a local maximum at some point $q \in \Sigma^n$ such that $\rho'(h(q)) \neq 0$ and T_r satisfies $T_r(\nabla u) = \varphi(\nabla u)$, for some $0 < \varphi \in C^\infty(\Sigma)$. Assume in addition that u is nonnegative and bounded from above. If $u \in \mathcal{L}^p_{-\ln\varphi}(\Sigma)$, for some $p > 1$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

Proof. Following similar steps of the proof of Theorem 7.1.4, we obtain that u is $(-\ln\varphi)$-subharmonic function. Hence, since $u \in \mathfrak{L}^p_{-\ln\varphi}(\Sigma)$ and $0 \leq u \leq \sup_\Sigma u < +\infty$, Lemma 7.1.2 gives that u is constant on Σ^n. Therefore, we conclude that Σ^n is a slice of $\overline{M}^{n+1}$. ■

Remark 7.1.6 Concerning Theorem 7.1.8, we observe that if u is just nonnegative and we substitute hypothesis (7.1.17) by $\kappa \leq \inf_I((\rho')^2 - \rho\rho'')$ and replace assumption (7.1.18) by $\dfrac{H_{r+1}}{H_r} \geq -\dfrac{1}{\langle N, \partial_t \rangle}\dfrac{\rho'}{\rho}(h)$, we will also obtain the rigidity result.

7.1.3 Via a version of maximum principle at infinity

Let Σ^n be a (connected) complete noncompact Riemannian manifold, and let $d(\cdot, o) : \Sigma \to [0, +\infty)$ stand for the Riemannian distance of Σ^n, measured from a fixed point $o \in \Sigma^n$. According to [7], we say that $f \in C^0(\Sigma)$ converges to zero at infinity when it satisfies

$$\lim_{d(x,o)\to+\infty} f(x) = 0.$$

In this setting, from [7, Theorem 2.2], we have the following lemma:

Lemma 7.1.3 *Let $(\Sigma^n, \langle , \rangle)$ be a complete noncompact Riemmanian manifold and let $X \in \mathfrak{X}(\Sigma)$ be a smooth vector field on Σ^n with* div$X \geq 0$. *If there exists a nonnegative, non-identically vanishing function $f \in C^\infty(\Sigma)$ such that f converges to zero at infinity and $\langle \nabla f, X \rangle \geq 0$, then $\langle \nabla f, X \rangle \equiv 0$ on Σ^n.*

Returning to the context of a Riemannian immersion $\psi : \Sigma^n \to \overline{M}^{n+1}$ in a warped product $\overline{M}^{n+1} = \epsilon I \times_\rho M^n$, we say that Σ^n is *asymptotic to a slice* $\Sigma_{t_*} := \{t_*\} \times M^n$ *at infinity* when the function $f := h - t_*$ converges to zero at infinity.

Keeping in mind this previous digression, we present our next rigidity result.

Theorem 7.1.9 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete noncompact spacelike hypersurface immersed into a GRW spacetime $\overline{M}^{n+1} = -I \times_\rho M^n$. Suppose that Σ^n lies above a slice $\Sigma_{t_*} := \{t_*\} \times M^n$ and is asymptotic to it at infinity. If H_1 is positive and satisfies*

$$H_1 \geq \frac{\rho'}{\rho}(h), \tag{7.1.23}$$

then Σ^n is the slice Σ_{t_*}.

Proof. Since $H_1 > 0$ and N is future-pointing, from (6.1.1) and Lemma 5.2.6 we have

$$\Delta u = -n(\rho'(h) + H_1\rho(h)\langle N, \partial_t\rangle) \geq n\rho(h)\left(H_1 - \frac{\rho'}{\rho}(h)\right). \tag{7.1.24}$$

Thus, from inequalities (7.1.23) and (7.1.24) we obtain that u is a subharmonic function. Moreover, since Σ^n lies above the slice Σ_{t_*} and is asymptotic to it at infinity, the function $f := h - t_*$ is nonnegative converging to zero at infinity. Hence, considering the smooth vector field $X = \nabla u$, from Lemma 7.1.3 we have $\langle \nabla f, \nabla u\rangle = \rho\langle \nabla h, \nabla h\rangle \equiv 0$. Therefore, we conclude that the height function h is constant and, since $h - t_*$ converges to zero at infinity, Σ^n must be the slice Σ_{t_*}. ∎

When $\overline{M}^{n+1}$ is a steady state-type spacetime, Theorem 7.1.9 reads as follows.

Corollary 7.1.14 *Let $\Sigma^n \hookrightarrow \overline{M}^{n+1}$ be a complete noncompact spacelike hypersurface immersed into a steady state-type spacetime $\overline{M}^{n+1} = -\mathbb{R} \times_{e^t} M^n$ such that Σ^n lies above a slice $\Sigma_{t_*} := \{t_*\} \times M^n$ and is asymptotic to it at infinity. If $H_1 \geq 1$ on Σ^n, then Σ^n is the slice Σ_{t_*}.*

For $r = 1$, we will suppose that the GRW spacetime obeys curvature constraint (7.1.1).

Theorem 7.1.10 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete noncompact spacelike hypersurface immersed into a GRW spacetime $\overline{M}^{n+1} = -I \times_\rho M^n$ which obeys the curvature constraint (7.1.1). Suppose that Σ^n lies above a slice $\Sigma_{t_*} := \{t_*\} \times M^n$ and is asymptotic to it at infinity. If H_1 and H_2 satisfy (7.1.2) with H_2 positive, then Σ^n is the slice Σ_{t_*}.*

Proof. Considering the smooth vector field $X := T_1(\nabla u) \in \mathfrak{X}(\Sigma)$, from (6.1.1), jointly with Lemma 5.2.6 and equation (5.2.4) of Lemma 5.2.2, we obtain that

$$\begin{aligned} \operatorname{div} X = \mathcal{L}_1(u) &= \rho(h)((n-1)(\operatorname{Ric}_M(N^*, N^*) - (\log f)''(h)|\nabla h|^2)\langle N, \partial_t\rangle \\ &\quad - b_1(\rho'(h)H + \rho(h)H_2\langle N, \partial_t\rangle), \end{aligned} \tag{7.1.25}$$

where $N^* = N + \langle N, \partial_t\rangle\partial_t$. But, from (5.2.3) we have that $|\nabla h|^2 = \rho^2(h)\langle N^*, N^*\rangle_M$. So, using curvature constraint (7.1.1) and taking into account that $H_2 > 0$ and N is future-pointing, from (7.1.25) we get

$$\mathcal{L}_1(u) \geq b_1 H_1 \rho(h)\left(\frac{H_2}{H} - \frac{\rho'}{\rho}(h)\right). \tag{7.1.26}$$

By Cauchy-Schwarz inequality we have $H_1^2 \geq H_2 > 0$, and H does not vanish on Σ^n. Thus, we may assume that $H_1 > 0$ and from (7.1.2) and (7.1.26) we obtain that u is $\mathcal{L}_1$-subharmonic function, that is, $\operatorname{div} X = \operatorname{div}(T_1(\nabla u)$ on Σ^n.

On the other hand, since Σ^n lies above the slice Σ_{t_*} and is asymptotic to it at infinity, the function $f := h - t_*$ is nonnegative converging to zero at the infinity. Moreover, using again that $H_2 > 0$, Lemma 5.2.3 guarantees that T_1 is positive definite. Then,

$$\langle \nabla f, X\rangle = \langle \nabla h, T_1(\nabla u)\rangle = g'(h)\langle \nabla h, T_1(\nabla h)\rangle = \rho(h)\langle \nabla h, T_1(\nabla h)\rangle \geq 0.$$

Hence, from Lemma 7.1.3 we have $\langle \nabla f, X\rangle \equiv 0$. Consequently, the height function h is constant and, therefore, Σ^n must be the slice Σ_{t_*}. ∎

When the ambient spacetime is an Einstein manifold, Theorem 7.1.10 gives the following consequence.

Corollary 7.1.15 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete noncompact spacelike hypersurface immersed into a Einstein GRW spacetime $\overline{M}^{n+1} = -I \times_\rho M^n$ such that Σ^n lies above a slice $\Sigma_{t_*} := \{t_*\} \times M^n$ and is asymptotic to it at infinity. If H_1 and H_2 satisfy (7.1.2) with H_2 positive, then Σ^n is the slice Σ_{t_*}.*

Considering once more a steady state-type spacetime, from Theorem 7.1.10 we obtain the next corollary.

Corollary 7.1.16 *Let $\Sigma^n \hookrightarrow \overline{M}^{n+1}$ be a complete noncompact spacelike hypersurface immersed into a steady state-type spacetime $\overline{M}^{n+1} = -\mathbb{R} \times_{e^t} M^n$ whose fiber M^n has nonpositive Ricci curvature. Suppose that Σ^n lies above a slice $\Sigma_{t_*} := \{t_*\} \times M^n$ and is asymptotic to it at infinity. If H_1 and H_2 satisfy (7.1.2) with H_2 positive, then Σ^n is the slice Σ_{t_*}.*

For $2 \le r \le n-1$, we will consider that $\overline{M}^{n+1}$ is a Robertson-Walker (RW) spacetime.

Theorem 7.1.11 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete noncompact spacelike hypersurface immersed into a RW spacetime $\overline{M}^{n+1} = -I \times_\rho M^n$ whose Riemannian fiber M^n has constant sectional curvature κ satisfying the curvature constraint (7.1.6). Suppose that Σ^n lies above a slice $\Sigma_{t_*} := \{t_*\} \times M^n$ and is asymptotic to it at infinity, and that $\rho(h)$ attains a local minimum at some point $q \in \Sigma^n$ such that $\rho'(h(q)) \neq 0$. If, for some $2 \le r \le n-1$, H_r and H_{r+1} satisfy (7.1.7) with H_{r+1} positive, then Σ^n is the slice Σ_{t_*}.*

Proof. Considering the smooth vector field $X := T_r(\nabla u) \in \mathfrak{X}(\Sigma)$, since M^n has constant sectional curvature κ, from Lemma 5.2.6 jointly (6.1.1) and equation (5.2.5) of Lemma 5.2.2 we obtain

$$\begin{aligned} \mathrm{div} X = \mathcal{L}_r(u) &= \rho(h)(n-r)\left(\frac{\kappa}{\rho^2(h)} - (\log\rho)''(h)\right)\langle T_{r-1}\nabla h, \nabla h\rangle\langle N, \partial_t\rangle \\ &\quad -b_r(\rho'(h)H_r + \rho(h)H_{r+1}\langle N, \partial_t\rangle). \end{aligned} \tag{7.1.27}$$

Since $\rho(h)$ attains a local minimum at some point $q \in \Sigma$ such that $\rho'(h(q)) \neq 0$ and $H_{r+1} > 0$, Lemma 5.2.4 guarantees that T_{r-1} and T_r are positive definite. Thus, since N is future-pointing, from (7.1.27) we obtain

$$\mathcal{L}_r(u) \ \ge \ b_r H_r \rho(h)\left(\frac{H_{r+1}}{H_r} - \frac{\rho'}{\rho}(h)\right). \tag{7.1.28}$$

Since $H_r > 0$, from inequalities (7.1.7) and (7.1.28) we get that u is $\mathcal{L}_r$-subharmonic function and, consequently, $\mathrm{div} X \ge 0$.

On the other hand, since Σ^n lies above the slice Σ_{t_*} and is asymptotic to it at infinity, the function $f := h - t_*$ is nonnegative converging to zero at the infinity. Thus,

$$\langle \nabla f, X\rangle = \langle \nabla h, T_r(\nabla u)\rangle = g'(h)\langle \nabla h, T_r(\nabla h)\rangle = \rho(h)\langle \nabla h, T_r(\nabla h)\rangle \ge 0.$$

Hence, from Lemma 7.1.3 we have $\langle \nabla f, X\rangle \equiv 0$. Consequently, $\nabla h \equiv 0$ and, therefore, Σ^n must be the slice Σ_{t_*}. ∎

When $\overline{M}^{n+1} = -I \times M^n$ is a static RW spacetime, we obtain the following nonexistence result.

Corollary 7.1.17 *Let $\overline{M}^{n+1} = -I \times M^n$ be a static RW spacetime whose Riemannian fiber M^n has nonpositive constant sectional curvature. There is no complete noncompact spacelike hypersurface $\Sigma^n \hookrightarrow \overline{M}^{n+1}$ lying above a slice $\Sigma_{t_*} := \{t_*\} \times M^n$ and being asymptotic to it at infinity, and such that H_r, H_{r+1} are positive, for some $2 \le r \le n-1$, and T_j is positive definite for $0 \le j \le r$.*

Now, we will consider also the case when the RW spacetime has Riemannian fiber with constant sectional curvature obeying a curvature constraint which corresponds to the so-called *strong null convergence condition* (SNCC).

Theorem 7.1.12 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete noncompact spacelike hypersurface immersed into a RW spacetime $\overline{M}^{n+1} = -I \times_\rho M^n$ whose constant sectional curvature κ of the Riemannian fiber M^n satisfies the SNCC*

$$\kappa \geq \sup_I (\rho\rho'' - (\rho')^2). \tag{7.1.29}$$

Suppose that Σ^n lies below a slice $\Sigma_{t_*} := \{t_*\} \times M^n$ and is asymptotic to it at infinity. If T_j is positive definite for all $0 \leq j \leq r$, for some $2 \leq r \leq n-1$, H_r is positive and

$$\frac{H_{r+1}}{H_r} \leq -\frac{1}{\langle N, \partial_t\rangle}\frac{\rho'}{\rho}(h), \tag{7.1.30}$$

then Σ^n is the slice of Σ_{t_*}.

Proof. Since the sectional curvature of the Riemannian fiber M^n is constant, H_r is positive, T_{r-1} is positive definite and N is future-pointing, taking into account that (7.1.29) is satisfied, from (7.1.27) we obtain

$$\mathcal{L}_r(u) \;\leq\; -b_r H_r \rho(h)\langle N, \partial_t\rangle \left(\frac{H_{r+1}}{H_r} + \frac{1}{\langle N, \partial_t\rangle}\frac{\rho'}{\rho}(h)\right). \tag{7.1.31}$$

Thus, considering inequality (7.1.30) into (7.1.31) we get that u is $\mathcal{L}_r$-superharmonic function.

On the other hand, since Σ^n lies below the slice Σ_{t_*} and is asymptotic to it at infinity, the function $f := t_* - h$ is nonnegative converging to zero at the infinity. Moreover, since T_r is positive definite we obtain

$$\langle \nabla f, T_r(\nabla u)\rangle = -\langle \nabla h, T_r(\nabla u)\rangle = -g'(h)\langle \nabla h, T_r(\nabla h)\rangle = -\rho(h)\langle \nabla h, T_r(\nabla h)\rangle \leq 0.$$

Therefore, choosing the smooth vector field $X := -T_r(\nabla u)$, from Lemma 7.1.3 we have $\langle \nabla f, X\rangle \equiv 0$. Hence, $\nabla h \equiv 0$ and, consequently, Σ^n must be the slice Σ_{t_*}. ■

When the ambient space is a Riemannian warped product, we obtain the following result.

Theorem 7.1.13 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete noncompact two-sided hypersurface immersed into a Riemannian warped product $\overline{M}^{n+1} = I \times_\rho M^n$. Suppose* *that Σ^n lies above a slice $\Sigma_{t_*} := \{t_*\} \times M^n$ and is asymptotic to it at infinity. If $-1 \leq \langle N, \partial_t \rangle \leq 0$ and H_1 is positive satisfying*

$$H_1 \leq \frac{\rho'}{\rho}(h), \tag{7.1.32}$$

then Σ^n is the slice Σ_{t_*}.

Proof. Since $H_1 > 0$ and $-1 \leq \langle N, \partial_t \rangle \leq 0$, from Lemma 5.2.6 jointly with (7.1.32) we obtain that

$$\Delta u = n(\rho'(h) + \rho(h)H_1\langle N, \partial_t \rangle) \geq n\rho(h) \left(\frac{\rho'}{\rho}(h) - H_1 \right). \tag{7.1.33}$$

Thus, using inequality (7.1.32) in (7.1.33), we obtain that u is subharmonic function. Moreover, since Σ^n lies above the slice Σ_{t_*} and is asymptotic to it at infinity, the function $f := h - t_*$ is nonnegative converging to zero at infinity. Hence, Lemma 7.1.3 gives $\langle \nabla f, \nabla u \rangle = \rho(h)\langle \nabla h, \nabla h \rangle \equiv 0$. Consequently, $|\nabla h| \equiv 0$ and, therefore, Σ^n must be the slice Σ_{t_*}. ∎

When the ambient is a pseudo-hyperbolic space we have the following applications of Theorem 7.1.13.

Corollary 7.1.18 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete noncompact two-sided hypersurface immersed into a pseudo-hyperbolic space $\overline{M}^{n+1} = \mathbb{R} \times_{e^t} M^n$. Suppose that Σ^n lies above a slice $\Sigma_{t_*} := \{t_*\} \times M^n$ and is asymptotic to it at infinity. If $-1 \leq \langle N, \partial_t \rangle \leq 0$ and $0 < H \leq 1$, then Σ^n is the slice Σ_{t_*}.*

Corollary 7.1.19 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete noncompact two-sided hypersurface immersed into a pseudo-hyperbolic space $\overline{M}^{n+1} = \mathbb{R} \times_{\cosh t} M^n$. Suppose that Σ^n lies above a slice $\Sigma_{t_*} := \{t_*\} \times M^n$ and is asymptotic to it at infinity. If $-1 \leq \langle N, \partial_t \rangle \leq 0$ and $0 < H \leq \tanh(h)$, then Σ^n is the slice Σ_{t_*}.*

In our next result, we will suppose that the ambient space obeys a suitable curvature constraint.

Theorem 7.1.14 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete noncompact two-sided hypersurface immersed into a Riemannian warped product $\overline{M}^{n+1} = I \times_\rho M^n$ which obeys the curvature constraint (7.1.11). Suppose that Σ^n lies above a slice $\Sigma_{t_*} := \{t_*\} \times M^n$ and is asymptotic to it at infinity. If H_1 and H_2 satisfy (7.1.12) with H_2 positive and $-1 \leq \langle N, \partial_t \rangle \leq 0$, then Σ^n is the slice Σ_{t_*}.*

Proof. From (6.1.1), Lemma 5.2.6 and equation (5.2.4) of Lemma 5.2.2 we obtain that

$$\begin{aligned}\mathcal{L}_1(u) &= \rho(h)(-\mathrm{Ric}_M(N^*, N^*) - (n-1)(\log \rho)''(h)|\nabla h|^2)\langle N, \partial_t\rangle \\ &\quad + b_1(\rho'(h)H + \rho(h)H_2\langle N, \partial_t\rangle), \end{aligned} \tag{7.1.34}$$

where $N^* = N - \langle N, \partial_t\rangle\partial_t$. Taking into account that $|\nabla h|^2 = \rho^2(h)\langle N^*, N^*\rangle_M$, $-1 \leq \langle N, \partial_t\rangle \leq 0$, $H_2 > 0$ and using curvature constraint (7.1.11), we obtain

$$\mathcal{L}_1(u) \geq b_1 H_1 \rho(h) \left(\frac{\rho'}{\rho}(h) - \frac{H_2}{H_1} \right).$$

So, since $H_1 > 0$, from hypothesis (7.1.12) we conclude that u is a $\mathcal{L}_1$-subharmonic function.

On the other hand, since Σ^n lies above the slice Σ_{t_*} and is asymptotic to it at infinity, the function $f := h - t_*$ is nonnegative converging to zero at the infinity. Moreover, since $H_2 > 0$, Lemma 5.2.3 gives that T_1 is positive definite. Thus, choosing the smooth vector field $X := T_1(\nabla u) \in \mathfrak{X}(\Sigma)$, we get

$$\langle \nabla f, X\rangle = \langle \nabla h, T_1(\nabla u)\rangle = g'(h)\langle \nabla h, T_1(\nabla h)\rangle = \rho(h)\langle \nabla h, T_1(\nabla h)\rangle \geq 0.$$

Hence, from Lemma 7.1.3 we have $\langle \nabla f, X\rangle \equiv 0$. Consequently, $|\nabla h| \equiv 0$ and, therefore, Σ^n must be the slice Σ_{t_*}. ■

As application of Theorem 7.1.14, we can consider the case when ambient is an Einstein manifold to obtain the following rigidity result:

Corollary 7.1.20 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete noncompact two-sided hypersurface immersed into an Einstein warped product $\overline{M}^{n+1} = I \times_\rho M^n$. Suppose that Σ^n lies above a slice $\Sigma_{t_*} := \{t_*\} \times M^n$ and is asymptotic to it at infinity. If H_1 and H_2 satisfy (7.1.12) with H_2 positive and $-1 \leq \langle N, \partial_t\rangle \leq 0$, then Σ^n is the slice Σ_{t_*}.*

When the ambient is a pseudo-hyperbolic space, Theorem 7.1.14 leads us to the following applications:

Corollary 7.1.21 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete noncompact two-sided hypersurface immersed into a pseudo-hyperbolic space $\overline{M}^{n+1} = \mathbb{R} \times_{e^t} M^n$ whose fiber M^n has nonnegative Ricci curvature. Suppose that Σ^n lies above a slice $\Sigma_{t_*} := \{t_*\} \times M^n$ and is asymptotic to it at infinity. If $H_1 \geq H_2$ with H_2 positive and $-1 \leq \langle N, \partial_t\rangle \leq 0$, then Σ^n is the slice Σ_{t_*}.*

Corollary 7.1.22 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete noncompact two-sided hypersurface immersed into a pseudo-hyperbolic space $\overline{M}^{n+1} = \mathbb{R} \times_{\cosh t} M^n$ whose Ricci tensor of the fiber M^n is such that $\mathrm{Ric}_M \geq -(n-1)\langle\, ,\rangle_M$. Suppose that Σ^n lies above a slice $\Sigma_{t_*} := \{t_*\} \times M^n$ and is asymptotic to it at infinity. If H_1 and H_2 satisfy $\dfrac{H_2}{H_1} \leq \tanh(h)$ with H_2 positive and $-1 \leq \langle N, \partial_t\rangle \leq 0$, then Σ^n is the slice Σ_{t_*}.*

In our next results, we deal with higher order mean curvatures.

Theorem 7.1.15 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete noncompact two-sided hypersurface immersed into a Riemannian warped product $\overline{M}^{n+1} = I \times_\rho M^n$ whose fiber M^n has constant sectional curvature κ obeying curvature constraint (7.1.17). Suppose that Σ^n lies above a slice $\Sigma_{t_*} := \{t_*\} \times M^n$ and is asymptotic to it at infinity. Assume in addition that Σ^n has an elliptic point. If, for some $2 \leq r \leq n-1$, H_r and H_{r+1} satisfy (7.1.18) with H_{r+1} positive and $-1 \leq \langle N, \partial_t\rangle \leq 0$, then Σ^n is the slice Σ_{t_*}.*

Proof. Since the fiber M^n has constant sectional curvature κ, from (6.1.1) jointly with Lemma 5.2.6 and equation (5.2.5) of Lemma 5.2.2 we obtain

$$\begin{aligned}\mathcal{L}_r(u) \;&=\; -(n-r)\rho(h)\left(\frac{\kappa}{\rho^2(h)} + (\log\rho)''(h)\right)\langle T_{r-1}\nabla h, \nabla h\rangle\langle N, \partial_t\rangle \\ &\quad + b_r(\rho'(h)H_r + \rho(h)H_{r+1}\langle N, \partial_t\rangle). \qquad (7.1.35)\end{aligned}$$

By the assumption of the existence of an elliptic point in Σ^n and since $H_{r+1} > 0$, Lemma 5.2.4 guarantees that T_{r-1} and T_r are positive definite and $H_r > 0$. Thus, since $-1 \leq \langle N, \partial_t\rangle \leq 0$, from (7.1.17) and (7.1.35) we get

$$\mathcal{L}_r(u) \;\geq\; b_r\rho(h)H_r\left(\frac{\rho'}{\rho}(h) - \frac{H_{r+1}}{H_r}\right). \qquad (7.1.36)$$

Hence, considering (7.1.18) into (7.1.36) we conclude that u is a $\mathcal{L}_r$-subharmonic function. On the other hand, since Σ^n lies above the slice Σ_{t_*} and is asymptotic to it at infinity, the function $f := h - t_*$ is nonnegative converging to zero at the infinity. Moreover, since T_r is positive definite, choosing the smooth vector field $X := T_r(\nabla u)$ we get

$$\langle\nabla f, X\rangle = \langle\nabla h, T_r(\nabla u)\rangle = g'(h)\langle\nabla h, T_r(\nabla h)\rangle = \rho(h)\langle\nabla h, T_r(\nabla h)\rangle \geq 0.$$

Thus, from Lemma 7.1.3 we have $\langle\nabla f, X\rangle \equiv 0$. Consequently, $|\nabla h| \equiv 0$ and, therefore, Σ^n must be the slice Σ_{t_*}. ■

Related to the higher order mean curvatures, we also establish the following result:

Theorem 7.1.16 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete noncompact two-sided hypersurface immersed into a Riemannian warped product $\overline{M}^{n+1} = I \times_\rho M^n$ whose fiber M^n has constant sectional curvature κ satisfying*

$$\kappa \leq \inf_I((\rho')^2 - \rho\rho''). \tag{7.1.37}$$

Suppose that Σ^n lies below a slice $\Sigma_{t_*} := \{t_*\} \times M^n$ and is asymptotic to it at infinity. If $-1 \leq \langle N, \partial_t\rangle < 0$ and, for some $2 \leq r \leq n-1$, H_r is positive satisfying

$$\frac{H_{r+1}}{H_r} \geq -\frac{1}{\langle N, \partial_t\rangle}\frac{\rho'}{\rho}(h), \tag{7.1.38}$$

and T_j is positive definite for all $0 \leq j \leq r$, then Σ^n is the slice Σ_{t_*}.

Proof. Since we are assuming that T_j is positive definite for $1 \leq j \leq r$, taking into account curvature constraint (7.1.37) and that $-1 \leq \langle N, \partial_t\rangle < 0$, from (7.1.35) we obtain

$$\mathcal{L}_r(u) \leq b_r\rho(h)H_r\langle N, \partial_t\rangle\left(\frac{H_{r+1}}{H_r} + \frac{1}{\langle N, \partial_t\rangle}\frac{\rho'}{\rho}(h)\right).$$

Hence, since $H_r > 0$, from hypothesis (7.1.38) we conclude that u is a $\mathcal{L}_r$-superharmonic function on Σ^n.

On the other hand, since Σ^n lies below the slice Σ_{t_*} and is asymptotic to it at infinity, the function $f := t_* - h$ is nonnegative converging to zero at the infinity. Moreover, choosing the smooth vector field $X := -T_r(\nabla u)$, we get

$$\langle \nabla f, X\rangle = \langle \nabla h, T_r(\nabla u)\rangle = g'(h)\langle \nabla h, T_r(\nabla h)\rangle = \rho(h)\langle \nabla h, T_r(\nabla h)\rangle \geq 0.$$

Hence, from Lemma 7.1.3 we have $\langle \nabla f, X\rangle \equiv 0$. Consequently, $|\nabla h| \equiv 0$ and, therefore, Σ^n must be the slice Σ_{t_*}. ■

7.1.4 Via polynomial volume growth

Let Σ^n be a Riemannian manifold, and let us denote by $B(p, r)$ the geodesic ball centered at $p \in \Sigma^n$ and with radius r. Given a polynomial function $\sigma : (0, +\infty) \to (0, +\infty)$, we say that Σ^n has *polynomial volume growth like* σ if there exists $p \in \Sigma^n$ such that

$$\text{vol}(B(p, r)) = \mathcal{O}(\sigma(r)),$$

as $r \to +\infty$, where vol denotes the canonical Riemannian volume of Σ^n. According to [8], if $p, q \in \Sigma^n$ are at a distance d from each other, we can verify that

$$\frac{\text{vol}(B(p, r))}{\sigma(r)} \geq \frac{\text{vol}(B(q, r-d))}{\sigma(r-d)} \cdot \frac{\sigma(r-d)}{\sigma(r)}.$$

Consequently, the choice of p in the notion of volume growth is immaterial, and we will just say that Σ^n has *polynomial volume growth*. In this setting, from [8, Theorem 2.1] we have the following

Lemma 7.1.4 *Let Σ^n be a connected, oriented, complete noncompact Riemannian manifold and let $X \in \mathfrak{X}(\Sigma)$ be a vector field on Σ^n, with $|X| \leq c < +\infty$, for some positive constant $c \in \mathbb{R}$. Assume in addition that $f \in C^\infty(\Sigma^n)$ is such that $\langle \nabla f, X\rangle \geq 0$ and $\operatorname{div} X \geq af$ on Σ^n, for some positive constant $a \in \mathbb{R}$. If Σ^n has polynomial volume growth, then $f \leq 0$ on Σ^n.*

In what follows, we will apply Lemma 7.1.4 to the vector field $X = T_r(\nabla\hat{\Theta})$ for $0 \leq r \leq n-1$, where $\hat{\Theta} = \rho(h)\langle N, \partial_t\rangle \in C^\infty(\Sigma)$.

Theorem 7.1.17 *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime obeying the null convergence condition*

$$\operatorname{Ric}_M \geq (n-1)\sup_I(\rho\rho'' - (\rho')^2)\langle,\rangle_M, \tag{7.1.39}$$

where Ric_M stands the Ricci tensor of fiber M^n. There is not complete spacelike hypersurface Σ^n immersed into $\overline{M}^{n+1}$ with polynomial volume growth such that H_1, H_2 are constant with $H_2 > 0$, $\rho'(h)$ nonpositive and $|\nabla\hat{\Theta}|$ bounded on Σ^n.

Proof. Suppose, by contradiction, the existence of such a spacelike hypersurface. Thus, from (6.1.1) and [9, Corollary 8.2] we obtain

$$\begin{aligned}
\Delta\hat{\Theta} &= n\rho(h)\langle\nabla h, \nabla H_1\rangle + \rho'(h)nH_1 \\
&\quad +\hat{\Theta}\left(\operatorname{Ric}_M(N^*, N^*) - (n-1)(\log\rho)''(h)||\nabla h||^2\right) \\
&\quad +\hat{\Theta}n\left(nH_1^2 - (n-1)H_2\right).
\end{aligned}$$

Since $H_2 > 0$, Lemma 5.2.3 guarantees that, with respect to the future pointing Gauss map N, $H_1 > 0$ on Σ^n. Moreover, since H_1 and H_2 are constant,

$$n^2H^2 - n(n-1)H_2 = |A|^2 \tag{7.1.40}$$

is also constant. Consequently, since $\rho'(h)$ is nonpositive, from (7.1.39) we obtain

$$\Delta\hat{\Theta} \leq \hat{\Theta}n^2(H_1^2 - H_2),$$

with $H_1^2 \geq H_2 > 0$ by the Newton inequalities. Hence,

$$\Delta(-\hat{\Theta}) \geq -\hat{\Theta}a,$$

where $a = n^2(H_1^2 - H_2) \in \mathbb{R}$.

When Σ^n is compact, Divergence Theorem assures the nonexistence of it. In the case that Σ^n is noncompact, since $|\nabla\hat{\Theta}|$ is bounded on it, jointly with (7.1.40) we conclude that there exists a constant $C > 0$ such that $|\nabla\hat{\Theta}| \leq C$ on Σ^n. We also have that $\langle\nabla\hat{\Theta}, \nabla\hat{\Theta}\rangle \geq 0$ and, taking into account that we are also supposing that Σ^n has polynomial volume growth, Lemma 7.1.4 implies $-\hat{\Theta} \leq 0$ on Σ^n. Therefore, since $\rho > 0$, from (5.2.7) we arrive at a contradiction. ∎

In the particular case that the ambient space is a static GRW spacetime, Corollary 7.1.17 reads as follows.

Corollary 7.1.23 *Let $\overline{M}^{n+1} = -I \times M^n$ be a GRW spacetime whose Riemannian fiber M^n has nonnegative Ricci curvature. There is not complete spacelike hypersurface Σ^n immersed into $\overline{M}^{n+1}$ with polynomial volume growth, such that H_1 and H_2 are constant with $H_2 > 0$, and $|\nabla\hat{\Theta}|$ is bounded on Σ^n.*

Remark 7.1.7 *Related to the previous nonexistence results, we point out that if Σ^n is contained in a slab and $|\nabla h|$ is bounded on it, then $\rho(h)$ is bounded and, since $\nabla\hat{\Theta} = \rho(h)A\nabla h$ and taking into account the constancy of H_1 and H_2, we guarantee the boundedness of $|\nabla\hat{\Theta}|$.*

When $r = 1$, we will suppose again that the constant sectional curvature κ of the Riemannian fiber M^n of the RW spacetime $-I \times_\rho M^n$ obeys the *strong null convergence condition* (SNCC)

$$\kappa \geq \sup_I(\rho\rho'' - (\rho')^2). \tag{7.1.41}$$

Theorem 7.1.18 *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a RW spacetime whose Riemannian fiber M^n has constant sectional curvature κ satisfying the strong null convergence condition (7.1.41). There is not complete spacelike hypersurface Σ^n immersed into $\overline{M}^{n+1}$ with polynomial volume growth, such that H_2 and H_3 are positive with H_2 constant, $\rho'(h)$ is nonpositive and $|T_1(\nabla\hat{\Theta})|$ is bounded on Σ^n.*

Proof. Suppose, by contradiction, the existence of such a spacelike hypersurface. So, we consider the smooth vector field $X = T_1\nabla\hat{\Theta}$. Since M^n has constant sectional curvature κ, from (6.1.1) jointly with equation (5.2.5) of Lemma 5.2.2 and [9, Corollary 8.4], we obtain

$$\mathcal{L}_1\hat{\Theta} \;=\; (n-1)\left(\frac{\kappa}{\rho^2(h)} - (\log\rho)''(h)\right)\langle\nabla h, \nabla\hat{\Theta}\rangle\langle N, \partial_t\rangle$$

$$
\begin{aligned}
&+\binom{n}{2}\rho(h)\langle\nabla h, \nabla H_2\rangle + \rho'(h)b_1H_2 \\
&+\hat{\Theta}\left(\frac{\kappa}{\rho^2(h)} - (\log\rho)''(h)\right)\left(||\nabla h||^2 b_1 H_1 - \langle T_1\nabla h, \nabla h\rangle\right) \\
&+\hat{\Theta}\binom{n}{2}\left(nH_1H_2 - (n-2)H_3\right).
\end{aligned}
$$

But, since $\nabla\hat{\Theta} = \rho A\nabla h$, from (5.1.5) we get

$$
\begin{aligned}
\mathcal{L}_1\hat{\Theta} &= \hat{\Theta}(n-2)\left(\frac{\kappa}{\rho^2(h)} - (\log\rho)''(h)\right)\langle T_1\nabla h, \nabla h\rangle + \rho'(h)b_1H_2 \\
&\quad +\hat{\Theta}\binom{n}{2}\left(nH_1H_2 - nH_3 + 2H_3\right).
\end{aligned}
$$

Hence, since $H_2 > 0$, Lemma 5.2.3 guarantees that, with respect to the future pointing Gauss map N, $H_1 > 0$ and T_1 positive definite. Thus, from (7.1.41) jointly with our assumption over $\rho'(h)$, we obtain

$$
\mathcal{L}_1\hat{\Theta} \leq \hat{\Theta}\binom{n}{2}\left(n(H_1H_2 - H_3) + 2H_3\right).
$$

Since $H_2^2 \geq H_1H_3$ (see, for instance, [81, Theorems 51 and 144]), from Newton inequalities we have

$$
H_1H_2 - H_3 \geq \frac{H_2}{H_1}(H_1^2 - H_2) \geq 0.
$$

Consequently,

$$
\mathcal{L}_1(-\hat{\Theta}) \geq -\hat{\Theta}\binom{n}{2}\left(n(H_1H_2 - H_3) + 2H_3\right).
$$

Hence, since $H_3 > 0$, there exists a positive constant $a \in \mathbb{R}$ such that

$$
\mathcal{L}_1(-\hat{\Theta}) \geq -\hat{\Theta}a.
$$

Note that, if Σ^n is compact, Divergence Theorem gives the nonexistence of Σ^n. In the case that Σ^n is complete noncompact, using again that T_1 is positive definite, we have that $\langle\nabla\hat{\Theta}, T_1\nabla\hat{\Theta}\rangle \geq 0$. Then, since we are also supposing that $|T_1\nabla\hat{\Theta}|$ bounded on Σ^n and that Σ^n has polynomial volume growth, Lemma 7.1.4 assures that $-\hat{\Theta} \leq 0$ on Σ^n. Therefore, we reach a contradiction. ■

Similarly to Corollary 7.1.23 we have the following nonexistence result when the ambient space is a static GRW spacetime.

Corollary 7.1.24 *Let $\overline{M}^{n+1} = -I \times M^n$ be a RW spacetime whose Riemannian fiber M^n has nonnegative sectional curvature κ. There is not complete spacelike hypersurface Σ^n immersed into $\overline{M}^{n+1}$ with polynomial volume growth, such that H_2 and H_3 are positive with H_2 constant, and $|T_1(\nabla\Theta)|$ is bounded on Σ^n.*

Next, we consider the case the r-th Newton Transformation T_r is positive semi-definite, for some $2 \leq r \leq n-1$.

Theorem 7.1.19 *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a RW spacetime whose Riemannian fiber M^n has constant sectional curvature κ satisfying the strong null convergence condition (7.1.41). There is not complete spacelike hypersurface Σ^n immersed into $\overline{M}^{n+1}$ with polynomial volume growth, such that H_{r+1} is constant, for some $2 \leq r \leq n-1$, H_s is positive, for $r \leq s \leq r+2$, $\rho'(h)$ is nonpositive, the r-th Newton transformation T_r is positive semi-definite and $|T_r(\nabla\hat{\Theta})|$ is bounded on Σ^n.*

Proof. Suppose, by contradiction, the existence of such a spacelike hypersurface. We consider the smooth vector field $X = T_r\nabla\hat{\Theta}$. Since M^n has constant sectional curvature κ, from (6.1.1) jointly with equation (5.2.5) of Lemma 5.2.2 and [9, Corollary 8.4] we obtain

$$\begin{aligned}\mathcal{L}_r\hat{\Theta} &= (n-r)\left(\frac{\kappa}{\rho^2(h)} - (\log\rho)''(h)\right)\langle T_{r-1}\nabla h, \nabla\hat{\Theta}\rangle\langle N, \partial_t\rangle \\ &\quad + \binom{n}{r+1}\rho(h)\langle\nabla h, \nabla H_{r+1}\rangle + \rho'(h)b_r H_{r+1} \\ &\quad + \hat{\Theta}\left(\frac{\kappa}{\rho^2(h)} - (\log\rho)''(h)\right)\left(||\nabla h||^2 b_r H_r - \langle T_r\nabla h, \nabla h\rangle\right) \\ &\quad + \hat{\Theta}\binom{n}{r+1}\left(nH_1H_{r+1} - (n-r-1)H_{r+2}\right). \end{aligned} \tag{7.1.42}$$

Thus, since $\nabla\hat{\Theta} = \rho A\nabla h$, from (5.1.5) and (7.1.42) we get

$$\begin{aligned}\mathcal{L}_r\hat{\Theta} &= \hat{\Theta}(n-r-1)\left(\frac{\kappa}{\rho^2(h)} - (\log\rho)''(h)\right)\langle T_r\nabla h, \nabla h\rangle + \rho'(h)b_r H_{r+1} \\ &\quad + \hat{\Theta}\binom{n}{r+1}\left(nH_1H_{r+1} - (n-r-1)H_{r+2}\right). \end{aligned} \tag{7.1.43}$$

Hence, since T_r positive semi-definite, from (7.1.41), (7.1.43) and our assumptions on H_r and $\rho'(h)$, we deduce

$$\mathcal{L}_r(\hat{\Theta}) \;\leq\; \hat{\Theta}\binom{n}{r+1}\left(nH_1H_{r+1} - (n-r-1)H_{r+2}\right).$$

Consequently,

$$\mathcal{L}_r(-\hat{\Theta}) \;\geq\; -\hat{\Theta}\binom{n}{r+1}\left(nH_1H_{r+1} - (n-r-1)H_{r+2}\right).$$

So, since $H_{r+2} > 0$, from Newton inequality $H_1H_{r+1} - H_{r+2} \geq 0$ there exists a positive constant $a \in \mathbb{R}$ such that

$$\mathcal{L}_r(-\hat{\Theta}) \;\geq\; -\hat{\Theta}a$$

If Σ is compact, then Divergence Theorem guarantees the nonexistence of such a spacelike hypersurface. When Σ^n is complete noncompact, using again that T_r is positive semi-definite, we have that $\langle\nabla\hat{\Theta}, T_r\nabla\hat{\Theta}\rangle \geq 0$. Hence, since we are also supposing that Σ^n has polynomial volume growth, Lemma 7.1.4 assures that $-\hat{\Theta} \leq 0$ on Σ^n. Therefore, we arrive at a contradiction. ■

It is not difficult to verify that, from [42, Corollary 9.107] and equation (7.1.43), Theorem 7.1.19 also holds for GRW spacetimes with constant sectional curvature. More precisely,

Corollary 7.1.25 *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime with constant sectional curvature. There is not complete spacelike hypersurface Σ^n immersed into $\overline{M}^{n+1}$ with polynomial volume growth, such that H_{r+1} is constant, for some $2 \leq r \leq n-1$, H_s is positive, for $r \leq s \leq r+2$, $\rho'(h)$ is nonpositive, the r-th Newton transformation T_r is positive semi-definite and $|T_r(\nabla\hat{\Theta})|$ is bounded on Σ^n.*

When the ambient space is a static RW spacetime, we have the following result.

Corollary 7.1.26 *Let $\overline{M}^{n+1} = -I \times M^n$ be a static RW spacetime whose Riemannian fiber M^n has nonnegative sectional curvature. There is not complete spacelike hypersurface Σ^n immersed into $\overline{M}^{n+1}$ with polynomial volume growth such that H_{r+1} constant, for some $2 \leq r \leq n-1$, H_s is positive, for $r \leq s \leq r+2$, the r-th Newton transformation T_r is positive semi-definite and $|T_r(\nabla\Theta)|$ is bounded on Σ^n.*

Next, we will consider the case that the ambient space is a Riemannian warped product.

Theorem 7.1.20 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete noncompact two-sided hypersurface immersed into a Riemannian warped product $\overline{M}^{n+1} = I \times_\rho M^n$ which obeys curvature constraint (7.1.11). Assume that the Weingarten operator A is bounded and Σ^n has polynomial volume growth. If H_1 is constant, $\rho \leq 1$, $\rho'(h)H_1 \geq 0$ and $\hat{\Theta} = \langle N, K\rangle$ satisfies*

$$\hat{\Theta} \geq \frac{1 - \rho' n H_1}{|A|^2 + 1}, \tag{7.1.44}$$

then Σ^n is totally geodesic.

Proof. We define the function $\xi = 1 - \hat{\Theta}$. Since $\rho \leq 1$, we have $\xi \geq 0$. We consider the vector field $X = AK^\top \in \mathfrak{X}(\Sigma)$, where $K^\top$ stands for the orthogonal projection of K onto $\mathfrak{X}(\Sigma)$. From [107, Corollary 7.43], we obtain

$$\begin{aligned} \operatorname{div}(X) &= \langle N, K\rangle \left(\operatorname{Ric}_M(N^*, N^*) + (n-1)(\log\rho)''(1 - \langle N, \partial_t\rangle^2 + |A|^2\right) \\ &\quad - K^\top(nH_1) + \rho' n H_1. \end{aligned} \tag{7.1.45}$$

Using that $K \in \mathfrak{X}(\overline{M})$ is a non-vanishing conformal closed field with conformal factor $\rho' \in C^\infty(\overline{M})$, is not difficult verify that the gradient of $\hat{\Theta}$ is given by $\nabla\hat{\Theta} = -AK^\top$. Thus, since A is self-adjoint we get

$$\langle \nabla\xi, X\rangle = -\langle \nabla\hat{\Theta}, X\rangle = \langle AK^\top, AK^\top\rangle = |AK^\top|^2 \geq 0.$$

Moreover, since H_1 is constant, from (7.1.45) we obtain

$$\operatorname{div}(X) = \langle N, K\rangle \left(\operatorname{Ric}_M(N^*, N^*) + (n-1)(\log\rho)''(1 - \langle N, \partial_t\rangle^2 + |A|^2\right) + \rho' n H_1.$$

Thus, from (5.2.3) and constraint (7.1.44) we obtain

$$\operatorname{div}(X) \geq \rho' n H_1 + \langle N, K\rangle |A|^2, \tag{7.1.46}$$

and it thus follows from assumption (7.1.44) that $\operatorname{div}(X) \geq \xi$ on Σ^n.

Since we are assuming that A and ρ are bounded, we have that $X = AK^\top$ is also bounded on Σ^n. Moreover, since Σ^n has polynomial volume growth, Lemma 7.1.4 guarantees that $\xi \leq 0$ on Σ^n. Hence, $\xi \equiv 0$, that is, $\hat{\Theta} \equiv 1$. Therefore, if $\rho' H_1 \geq 0$ then from of assumption (7.1.44) and inequality (7.1.46) we conclude that Σ^n must be totally geodesic in $\overline{M}^{n+1}$. ∎

For $2 \leq r \leq n$, assuming that the fiber of the ambient space has constant curvature, we obtain the following nonexistence result.

Theorem 7.1.21 *Let $\overline{M}^{n+1} = I \times_\rho M^n$ be a Riemannian warped product whose fiber M^n has constant sectional curvature κ satisfying*

$$\kappa \geq \sup_I \left((\rho')^2 - \rho\rho''\right). \tag{7.1.47}$$

There is not a complete noncompact hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$ with polynomial volume growth immersed into $\overline{M}^{n+1} = I \times_\rho M^n$, such that the Weingarten operator A with respect to unit normal vector field N is bounded and T_{r-1} is positive semi-definite, for some $2 \leq r \leq n-1$, H_r is constant, H_j is positive, for all $1 \leq j \leq r+1$, and the warping function $\rho \leq 1$ and the support function $\hat{\Theta} = \langle N, K\rangle$ satisfy

$$\hat{\Theta} \geq \frac{1 - \rho' r S_r}{\mathrm{tr}(A^2 T_{r-1}) + 1}. \tag{7.1.48}$$

Proof. Suppose, by contradiction, the existence of such a hypersurface immersed into $\overline{M}^{n+1}$. Thus, we define the function $\xi := 1 - \hat{\Theta}$. Since $\rho \leq 1$, we have $\xi \geq 0$. Consider the vector field $X = AT_{r-1}K^\top \in \mathfrak{X}(\Sigma)$, where $K^\top$ stands for the orthogonal projection of K onto the tangent bundle of Σ^n. Using that $K \in \mathfrak{X}(\overline{M})$ is a non-vanishing conformal closed field with conformal factor $\rho' \in C^\infty(\overline{M})$, we show that the gradient of $\hat{\Theta}$ is given by $\nabla\hat{\Theta} = -\rho A \nabla h$. Since H_r is constant, from [18, Corollary 26 and Lemma 28] we get

$$\begin{aligned}\mathcal{L}_{r-1}(-\hat{\Theta}) &= \hat{\Theta}(n-r)\left(\frac{\kappa}{\rho^2(h)} + (\log\rho)''(h)\right)\langle T_{r-1}\nabla h, \nabla h\rangle + \rho'(h) b_{r-1} H_r \\ &\quad + \hat{\Theta}\mathrm{tr}(A^2 T_{r-1}).\end{aligned} \tag{7.1.49}$$

Consequently, Since T_{r-1} is positive semi-definite, from (7.1.49)

$$\mathcal{L}_{r-1}(-\hat{\Theta}) \geq \rho'(h) r S_r + \hat{\Theta}\mathrm{tr}(A^2 T_{r-1}). \tag{7.1.50}$$

Thus, from (7.1.48) and (7.1.50) we obtain

$$\mathcal{L}_{r-1}(-\hat{\Theta}) \geq 1 - \hat{\Theta} = \xi.$$

Hence, $\mathrm{div}(X) \geq \xi$ on Σ^n. Since we are assuming that $|A|$ and ρ are bounded on Σ^n, we have that X is also is bounded on Σ^n. Moreover, since A is self-adjoint we get

$$\langle \nabla\xi, X\rangle = -\langle\nabla\hat{\Theta}, X\rangle = \langle AK^\top, AT_{r-1}K^\top\rangle = \langle K^\top, A^2 T_{r-1} K^\top\rangle \geq 0.$$

Then, since Σ^n has polynomial volume growth, [8, Theorem 2.1] guarantees that $\xi \leq 0$ on Σ^n. Therefore, $\xi \equiv 0$, that is, $\hat{\Theta} = 1$. Hence, from assumption (7.1.48) and inequality (7.1.50) we obtain

$$\rho'(h) r S_r = -\mathrm{tr}(A^2 T_{r-1}),$$

that is, $\rho'(h) \leq -\dfrac{H_{r+1}}{H_r}$. Thus, from assumptions under H_{r+1} and ρ' on Σ^n, we arrive at a contradiction. ■

In particular, when the ambient space has constant sectional curvature, we can state the following nonexistence result.

Corollary 7.1.27 *Let $\overline{M}^{n+1} = I \times_\rho M^n$ a Riemannian warped product of constant sectional curvature. There is not a complete noncompact hypersurface with polynomial volume growth immersed into $\overline{M}^{n+1} = I \times_\rho M^n$, such that the Weingarten operator A with respect to unit normal vector field N is bounded, T_{r-1} is positive semi-definite, for some $2 \leq r \leq n-1$, H_r is constant, H_j is positive, for all $1 \leq j \leq r+1$, and the warping function $\rho \leq 1$ and the support function $\hat{\Theta} = \langle N, K \rangle$ satisfy (7.1.48).*

7.2 Applications to entire graphs

In what follows, for simplification of notation, we will just identify

$$T_r\left(\left(\pi_M|_{\Sigma(w)}\right)_*^{-1}(Dw)\right) = \left(\varphi \circ \pi_M|_{\Sigma(w)}\right)(Dw)$$

by

$$T_r(Dw) = \varphi Dw.$$

In this setting, we can establish nonparametric versions for the results of Section 7.1. For instance, from Theorem 7.1.2 we obtain the following:

Corollary 7.2.1 *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a RW spacetime whose Riemannian fiber M^n has constant sectional curvature κ obeying the curvature constraint (7.1.6) and let $\Sigma(w)$ be an entire graph determined by a bounded function $w \in C^\infty(M)$. Suppose that, for some $2 \leq r \leq n-1$, H_{r+1} is positive and $\rho(w)$ attains a local minimum at a point $q \in \Sigma(w)$ such that $\rho'(w(q)) \neq 0$. If T_r satisfies $T_r(Dw) = \varphi Dw$, for some $0 < \varphi \in C^\infty(\Sigma(w))$, $|Dw|_M \in \mathfrak{L}^1_{-\ln\varphi}(M)$, $|Dw|_M^2 \leq \alpha\rho^2(w)$, for some constant $0 < \alpha < 1$, and*

$$\frac{H_{r+1}}{H_r} \geq \frac{\rho'}{\rho}(w), \tag{7.2.1}$$

then $w \equiv t_0$ for some $t_0 \in I$.

Proof. As in the beginning of the proof of Corollary 5.1 in [10], the assumption that $|Dw|_M^2 \leq \alpha\rho^2(w)$, for some constant $0 < \alpha < 1$, guarantees that $\Sigma(w)$ is a complete spacelike hypersurface. Moreover, it follows from (5.3.1) that $d\Sigma(w) = \sqrt{|G|}dM$, where dM and $d\Sigma(w)$ stand for the Riemannian volume elements of $(M^n, \langle\, ,\rangle_M)$ and $(\Sigma(w), \langle\, ,\rangle)$, respectively, and $G = \det(g_{ij})$ with

$$g_{ij} = \langle e_i, e_j\rangle = -e_i(w)e_j(w) + \rho^2(w)\delta_{ij}. \tag{7.2.2}$$

Here, $\{e_1, \cdots, e_n\}$ denotes a local orthonormal frame with respect to the metric $\langle\, ,\rangle_M$. It is not difficult to verify that

$$|G| = \rho^{2n-2}(w)(\rho^2(w) - |Dw|^2_{M^n}). \tag{7.2.3}$$

Indeed, in the points where Dw does not vanish, it is enough to take $e_1 = \dfrac{Dw}{|Dw|_{M^n}}$ and (7.2.3) can be easily deduced using (7.2.2). Consequently, from (7.2.3) we get

$$d\Sigma(w) = \rho^{n-1}(w)\sqrt{\rho^2(w) - |Dw|^2_{M^n}}\, dM. \tag{7.2.4}$$

It follows from (5.3.7) that

$$|Dw| d\Sigma(w) = \rho^{n-1}(w)|Dw|_{M^n} dM.$$

Therefore, since $|Dw|_M \in \mathfrak{L}^1_{-\ln\varphi}(M)$, from condition (7.2.1) we can apply Theorem 7.1.17 to conclude the result. ■

When the ambient space $\overline{M}^{n+1} = I \times_\rho M^n$ is a Riemannian warped product, all results in Subsection 7.1.1 can be also rewritten for the context of entire graphs. In particular, we quote the following nonparametric versions of Theorem 7.1.4.

Corollary 7.2.2 *Let $\overline{M}^{n+1} = I \times_\rho M^n$ be a Riemannian warped product whose fiber M^n has constant sectional curvature κ obeying the curvature constrain (7.1.17) and let $\Sigma(w)$ be an entire graph determined by a bounded function $w \in C^\infty(M)$. Suppose that, for some $2 \le r \le n-1$, H_{r+1} is positive, $-1 \le \langle N, \partial_t\rangle \le 0$ and $\rho(w)$ attains a local maximum at a point $q \in \Sigma(w)$ such that $\rho'(w(q)) \neq 0$. If T_r satisfies $T_r(Dw) = \varphi(Dw)$, for some $0 < \varphi \in C^\infty(\Sigma(w))$, $|Dw| \in \mathfrak{L}^1_{-\ln\varphi}(M)$, $|Dw|^2_M \le \alpha\rho^2(w)$, for some constant $0 < \alpha < 1$, and*

$$\frac{H_{r+1}}{H_r} \le \frac{\rho'}{\rho}(w),$$

then $w \equiv t_0$ for some $t_0 \in I$.

Taking into account equation (7.2.4), if $w \in C^\infty(M)$ is bounded and M^n has polynomial volume growth, we conclude that $\Sigma(w)$ also has polynomial volume growth. Thus, we can also restate Theorem 7.1.17 in the context of entire graphs.

Corollary 7.2.3 *Let* $\overline{M}^{n+1} = -I \times_\rho M^n$ *be a GRW spacetime obeying the NCC* (7.1.39) *and with Riemannian fiber* M^n *having polynomial volume growth. There is not entire graph* $\Sigma(w)$ *determined by a bounded function* $w \in C^\infty(M)$ *such that* H_1, H_2 *are constants with* H_2 *positive,* $\rho'(w)$ *is nonpositive,* $|Dw|^2_M \leq \alpha\rho^2(w)$*, for some constant* $0 < \alpha < 1$*, and* $|\nabla\hat{\Theta}|$ *is bounded on* $\Sigma(w)$*.*

From (5.3.2), it is not difficult to see that we can also reformulate Theorem 7.1.19 in the context of entire graphs as follows:

Corollary 7.2.4 *Let* $\overline{M}^{n+1} = -I \times_\rho M^n$ *be a RW spacetime whose Riemannian fiber* M^n *has polynomial volume growth and constant sectional curvature* κ *satisfying the SNCC* (7.1.41)*. There is not entire graph* $\Sigma(w)$ *determined by a bounded function* $w \in C^\infty(M)$ *such that* H_{r+1} *constant, for some* $2 \leq r \leq n-1$*,* H_s *is positive for all* $r \leq s \leq r+2$*,* $\rho'(w)$ *is nonpositive,* $|Dw|^2_M \leq \alpha\rho^2(w)$*, for some constant* $0 < \alpha < 1$*, and* T_r *is positive semi-definite with* $|T_r(\nabla\hat{\Theta})|$ *bounded on* $\Sigma(w)$*.*

When the ambient space $\overline{M}^{n+1} = I \times_\rho M^n$ is a Riemannian warped product, using (5.3.3) and denoting by D^2w the Hessian of $w \in C^\infty(M)$ with respect to the metric $\langle\, ,\rangle_M$, we obtain nonparametric versions of Theorems 7.1.20 and 7.1.21.

Corollary 7.2.5 *Let* $\overline{M}^{n+1} = I \times_\rho M^n$ *be a Riemannian warped product whose fiber* M^n *has polynomial volume growth and which obeys the curvature constraint* (7.1.11)*. Let* $\Sigma(w)$ *be entire graph determined by a bounded function* $w \in C^\infty(M)$ *such that* $|Dw|^2_M \leq \alpha\rho^2(w)$*, for some constant* $0 < \alpha < 1$*, and* D^2w *is bounded. If* H_1 *is constant,* $\rho(w) \leq 1$*,* $\rho'(w)H_1 \geq 0$ *and* $\hat{\Theta} = \langle N, K\rangle$ *satisfies* (7.1.44)*, then* $\Sigma(w)$ *is totally geodesic.*

Corollary 7.2.6 *Let* $\overline{M}^{n+1} = I \times_\rho M^n$ *be a Riemannian warped product whose fiber* M^n *has polynomial volume growth and constant sectional curvature* κ *satisfying* (7.1.47)*. There is not an entire graph* $\Sigma(w)$ *determined by a bounded function* $w \in C^\infty(M)$ *such that* T_{r-1} *is positive semi-definite, for some* $2 \leq r \leq n-1$*,* $\rho(w) \leq 1$*,* $|Dw|^2_M \leq \alpha\rho^2(w)$*, for some constant* $0 < \alpha < 1$*,* D^2w *is bounded and* $\hat{\Theta} = \langle N, K\rangle$ *satisfies* (7.1.48)*.*

Bibliography

[1] N. Abe, N. Koike and S. Yamaguchi, *Congruence theorems for proper semi-Riemannian hypersurfaces in a real space form*, Yokohama Math. J. **35** 123–136, (1987). 42, 44

[2] A. L. Albujer, *New examples of entire maximal graphs in* $\mathbb{H}^2 \times \mathbb{R}_1$, Diff. Geom. Appl. **26** (2008), 456–462. 64

[3] K. Akutagawa, *On spacelike hypersurfaces with constant mean curvature in the de Sitter space*, Math. Z. **196** (1987), 13–19. 3

[4] L. J. Alías, A. Brasil, Jr. and A. G. Colares, *Integral formulae for spacelike hypersurfaces in conformally stationary spacetimes and applications*, Proc. Edinburgh Math. Soc. **46** (2003), 465–488. 51, 59, 60, 61, 62, 66

[5] A. L. Albujer and L. J. Alías, *Spacelike hypersurfaces with constant mean curvature in the steady state space*, Proc. Amer. Math. Soc. **137** (2009), 711–721. 60, 67

[6] L. J. Alías and A. Caminha, *On the scarcity of non-totally geodesic complete spacelike hypersurfaces of constant mean curvature in a Lie group with bi-invariant Lorentzian metric.* Dif. Geo. and Appl., **51** (2017) 49–64. 13

[7] L. J. Alías, A. Caminha and F.Y. do Nascimento *A maximum principle at infinity with applications to geometric vector fields*,J. Math. Anal. Appl. **474** (2019) 242–247. 8, 81, 92

[8] L. J. Alías, A. Caminha and F.Y. do Nascimento *A maximum principle related to volume growth and applications*, Fond. Ann. Mat. Pura Appl. **474** (2020) 1637–1650. 8, 81, 100, 101, 107

[9] L. J. Alías and A. G. Colares, *Uniqueness of spacelike hypersurfaces with constant higher order mean curvature in generalized Robertson-Walker spacetimes*, Math. Proc. Camb. Philos. Soc. **143** (2007), 703–729. 62, 71, 72, 101, 102, 104

[10] L. J. Alías, A. G. Colares and H. F. de Lima, *On the rigidity of complete spacelike hypersurfaces immersed in a generalized Robertson-Walker spacetime*, Bull. Braz. Math. Soc. **44** (2013), 195–217. 73, 80, 108

[11] L. J. Alías, A. G. Colares and H. F. de Lima, *Uniqueness of entire graphs in warped products*, J. Math. Anal. Appl. **430** (2015), 60–75. 7, 61, 62

[12] L. J. Alías and M. Dajczer, *Uniqueness of constant mean curvature surfaces properly immersed in a slab*, Comment. Math. Helv. **81** (2006), 653–663. 6, 74

[13] L. J. Alías and M. Dajczer, *Constant mean curvature hypersurfaces in warped product spaces*, Proc. Edinb. Math. Soc. **50** (2007), 511–526. 6, 7, 74

[14] L. J. Alías, H. F. de Lima, J. Meléndez and F. R. dos Santos, *Rigidity of linear Weingarten hypersurfaces in locally symmetric manifolds*, Math. Nachr.**289** (2016),1309–1324. 21

[15] L. J. Alías, H. F. de Lima and F.R. dos Santos, *Characterizations of spacelike submanifolds with constant scalar curvature in the de Sitter Space*, Mediterranean J. Math. **15**:12, (2018). 21

[16] L. J. Alías, S. C. García-Martínez and M. Rigoli, *A maximum principle for hypersurfaces with constant scalar curvature and applications*, Ann. Glob. Anal. Geom. **41** (2012), 307–320. 24, 29

[17] L. J. Alías, D. Impera and M. Rigoli, *Spacelike hypersurfaces of constant higher order mean curvature in generalized Robertson-Walker spacetimes*, Math. Proc. Camb. Phil. Soc. **152**, (2012) 365–383 7, 66

[18] L. J. Alías, D. Impera and M. Rigoli, *Hypersurfaces of constant higher order mean curvature in warped products*, Trans. Amer. Math. Soc. **365** (2013), 591–621. 7, 59, 62, 66, 107

[19] L. J. Alías, H. F. de Lima and F.R. dos Santos, *Characterizations of spacelike submanifolds with constant scalar curvature in the de Sitter Space*, Mediterranean J. Math. **15** (2018). 37

[20] L. J. Alías, H. F. de Lima and F.R. dos Santos, *New characterizations of linear Weingarten spacelike hypersurfaces in the de Sitter Space*, Pacific J. Math. **292** (2018), 1–19.

[21] L. J. Alías, P. Mastrolia and M. Rigoli, *Maximum principles and geometric applications*, Springer Monographs in Mathematics xvii+570pp, Springer, Cham, 2016. 1, 16

[22] L. J. Alías, A. Romero and M. Sánchez, *Uniqueness of complete spacelike hypersurfaces with constant mean curvature in generalized Robertson-Walker spacetimes*, Gen. Rel. Grav. **27** (1995), 71–84. 57, 63

[23] L. J. Alías, A. Romero and M. Sánchez, *Spacelike hypersurfaces of constant mean curvature and Calabi-Bernstein type problems*, Tohoku Math. J. **49** (1997), 337–345. 69

[24] H. Alencar and M. do Carmo, *Hypersurfaces with constant mean curvature in spheres*, Proc. Am. Math. Soc. **120** (1994), 1223–1229. 23, 31

[25] R. Antonia, G. M. Bisci, H. F. de Lima and M.S. Santos, *Rigidity and nonexistence of complete hypersurfaces via Liouville type results and other maximum principles, with applications to entire graphs*, to appear in Asymp. Anal. Preprint (2023): 1-36. 10, 81

[26] R. Antonia and H. F. de Lima, *L-parabolic linear Weingarten spacelike hypersurfaces in a locally symmetric Einstein spacetime*, Rend. Circ. Mat. Palermo, II. Ser (2021). 10, 46, 55

[27] R. Antonia and H. F. de Lima, *L-parabolic linear Weingarten spacelike submanifolds immersed in an Einstein manifold*, Port. Math. **79** (2022), 211–224 10, 46

[28] R. Antonia and H. F. de Lima, *On the parabolicity of linear Weingarten spacelike submanifolds in the de Sitter space*, Quaestiones Mathematicae, **45**(2022), 1589–1602. 10, 33

[29] R. Antonia, H. F. de Lima and M.S. Santos, *Characterizing linear Weingarten submanifolds in a Riemannian space form via L-parabolicity*, to appear in Publ. Math. Debrecen. 10, 17

[30] R. Antonia, H. F. de Lima and M.S. Santos, *Rigidity and nonexistence of Riemannian immersions in semi-Riemannian warped products via parabolicity*, Preprint. 10, 65

[31] C. P. Aquino, J. G. Araújo and H. F. de Lima, *Rigidity of complete hypersurfaces in warped product spaces via higher order mean curvatures*, Beitr. Alg. Geom. **57** (2016), 391–405. 7

[32] C. P. Aquino, J. G. Araújo, M. Batista and H. F. de Lima, *Uniqueness of spacelike hypersurfaces in a GRW spacetime via higher order mean curvatures*, Bull. Braz. Math. Soc. New Series **48** (2017), 45–61. 8, 73, 88

[33] C. P. Aquino, H. F. de Lima and M.A.L. Velásquez, *A new characterization of complete linear Weingarten hypersurfaces in real space forms*, Pacific J. Math. **261** (2013), 33–43. 21

[34] C. P. Aquino, H. F. de Lima and M.A.L. Velásquez, *Generalized maximum principles and the characterization of linear Weingarten hypersurfaces in space forms*, Michigan Math. J. **63** (2014), 27–40. 24

[35] J. G. Araújo, W.F.C. Barboza, H. F. de Lima and M.A.L. Velásquez, *On the linear Weingarten spacelike submanifolds immersed in a locally symmetric semi-Riemannian space*, Beitr. Algebra Geom. **61** (2019), 267–282. 5, 6

[36] J. G. Araújo and H. F. de Lima, *Revisiting Liebmann's theorem in higher codimension*, Bull. Polish Acad. Sc. Math. **67** (2019), 179–185. 2, 23, 24, 25

[37] J. G. Araújo and H. F. de Lima, *LW-surfaces with higher codimension and Liebmann's Theorem in the hyperbolic space*, Boll. Un. Mat. Ital. **12** (2019), 613–621. 2

[38] J. G. Araújo and H. F. de Lima, *Linear Weingarten submanifolds in the hyperbolic space*, Beitr. Algebra Geom. **60**, (2019), 339–349. 2

[39] J. G. Araújo, H. F. de Lima, F.R. dos Santos and M.A.L. Velásquez, *Submanifolds with constant scalar curvature in a space form*, J. Math. Anal. Appl. **447** (2017), 488–498. 2

[40] J. G. Araújo, H. F. de Lima, F.R. dos Santos and M.A.L. Velásquez, *Characterizations of complete linear Weingarten spacelike submanifolds in a locally symmetric semi-Riemannian manifold*, Extracta Math. **32** (2017), 55–81. 4, 6, 46, 49, 51, 52, 54

[41] J. G.S. Araújo, H. F. de Lima, F.R. dos Santos and M.A.L. Velásquez, *Complete linear Weingarten spacelike submanifolds with higher codimension in the de Sitter space*, Int. J. Geom. Meth. Mod. Phys. **16**:1950050, (2019). 3, 4, 37

[42] A. L. Besse, *Einstein Manifolds*, Springer-Verlag, Berlin, 1987. 69, 86, 105

[43] S. Brendle, *Einstein manifolds with nonnegative isotropic curvature are locally symmetric*, Duke Math. J. **151** (2009), 1–21. 51

[44] S. Brendle, *Constant mean curvature surfaces in warped product manifolds,* Publ. Math. Inst. Hautes Etudes Sci. **117** (2013), 247–269. 9

[45] M. Caballero, A. Romero and R. M. Rubio, *Constant mean curvature spacelike hypersurfaces in Lorentzian manifolds with a timelike gradient conformal vector field*, Class. Quan. Grav. **28** (2011), 145009–145022. 86

[46] F. Camargo, A. Caminha, H. F. de Lima and U.L. Parente, *Generalized maximum principles and the rigidity of complete spacelike hypersurfaces*, Math. Proc. Camb. Philos. Soc. **153** (2012), 541–556.

[47] A. Caminha and H. F. de Lima, *Complete vertical graph with constant mean curvature in semi-Riemannian warped products*, Bull. Belg. Math. Soc. Simon Stevin **16** (2009), 91–105. 7

[48] A. Caminha, *The geometry of closed conformal vector fields on Riemmanian spaces*, Bull. Brazilian Math. Soc. **42** (2011), 277–300. 2

[49] J. Carot and J. da Costa, *On the geometry of warped spacetimes.*Class. Quantum Grav. **10** (1993) 4, 58

[50] É. Cartan, *Familles de surfaces isoparamétriques dans les espaces à courbure constante*, Ann. Mat. Pura Appl. **17** (1938), 177–191. 31

[51] X. L. Chao and P. J. Wang, *Linear Weingarten hypersurfaces in Riemannian space forms*, Bull. Korean Math. Soc. **51** (2014), 567–577. 15

[52] X. L. Chao, *Linear Weingarten hypersurfaces in a unit sphere*, Bull. Iranian Math. Soc. **41** (2015), 353–362. 15

[53] X. L. Chao and P. J. Wang, *Linear Weingarten submanifolds in unit sphere*, Arch. Math. **106** (2016), 581–590. 15

[54] R. M. B. Chaves and L. A. M. Sousa Jr., *On complete spacelike submanifolds in the De Sitter space with parallel mean curvature vector*, Rev. Union Mat. Argentina **47** 85–98, (2006). 39

[55] B. Y. Chen, *Surfaces with parallel normalized mean curvature vector*, Monatsh. Math. **90** (1980), 185–194. 1

[56] B. Y. Chen, *Differential Geometry of Warped Product Manifolds and Submanifolds*, World Scientific, London, (2017). 9

[57] Q. M. Cheng, *Complete space-like submanifolds with parallel mean curvature vector*, Math. Z. **206** 333–339, (1991). 43

[58] Q. M. Cheng, *Submanifolds with constant scalar curvature*, Proc. Roy. Soc. Edinburgh Sect. A **132** (2002), 1163–1183. 2

[59] S. Y. Cheng and S. T. Yau, *Hypersurfaces with constant scalar curvature*, Math. Ann. **225** 195–204, (1977). 14, 22

[60] S. Y. Cheng and S. T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. **28**, 333–354, (1975). 13

[61] S. M. Choi, S. M. Lyu and Y. J. Suh, *Complete space-like hypersurfaces in a Lorentz manifold*, Math. J. Toyama Univ. **22** (1999), 53–76. 4

[62] Y. S. Choi, Y. J. Suh and H. Y. Yang, *On spacelike hypersurfaces with constant mean curvature in a Lorentz manifold*, Houston J. Math. **28** (2002), 47–70. 4

[63] A. G. Colares and H. F. de Lima, *Some rigidity theorems in semi-Riemannian warped products*, Kodai Math. J. **35** (2012), 268–282. 8

[64] G. Colombo, L. Mari and M. Rigoli, *Remarks on mean curvature flow solitons in warped products*, Disc. Cont. Dynam. Syst. **13** (2020), 1957–1991. 9

[65] M. Dajczer and K. Nomizu, *On the flat surfaces in* $\mathbb{S}^3_1$ *and* $\mathbb{H}^3_1$, Manifolds and Lie Groups Birkauser, Boston, 1981. 3

[66] F. G. da Silva *Polynomial Weingarten surfaces of tubular type.*(2022) 15

[67] H. F. de Lima and J.R. de Lima, *Complete hypersurfaces immersed in a semi-Riemannian warped product*, Diff. Geom. Appl. **30** (2012), 136–143. 8

[68] H. F. de Lima and J.R. de Lima, *Characterizations of linear Weingarten spacelike hypersurfaces in Einstein spacetimes*, Glasgow Math. J. **55** (2013), 567–579. 4

[69] R. F. de Lima, F. Manfio and J. P. dos Santos, *Hypersurfaces of constant higher order mean curvature in* $M \times \mathbb{R}$, Ann. Mat. Pur. Appl. **201** (2022), 2979–3028 83

[70] R. F. de Lima, F. Manfio and J. P. dos Santos, *Einstein hypersurfaces of warped product spaces*, Results Math. **77** (2022), 228. 83

[71] R. F. de Lima and P. Roitman, *Helicoids and catenoids in* $\mathbb{M} \times \mathbb{R}$, Ann. Mat. Pur. Appl. **200** (2021), 2385–2421. 83

[72] H. F. de Lima, M. S. Santos and M. A. L. Velàsquez, *Mean Curvature Flow Solitons in Warped Products: Nonexistence, rigidity ans stability*, preprint (2022).

[73] H. F. de Lima, F.R. dos Santos, J. G. Araújo and M.A.L. Velásquez, *Linear Weingarten submanifolds immersed in a space form*, Kodai Math. J. **40** (2017), 214–228. 2, 20, 23, 24, 26, 30, 31

[74] M. P. do Carmo, *Geometria riemanniana.* IMPA, (2008). 12

[75] D. Ferus.*Immersions with parallel second fundamental form.* Math. Z. **140**, (1974), 87–92. [F4]

[76] D. Ferus.*Symmetric Submanifolds of Euclidean Space.* Math. Ann. **247**, (1980), 81–93. 25

25

[77] S. C. García-Martínez, D. Impera and M. Rigoli, *A sharp height estimate for compact hypersurfaces with constant k-mean curvature in warped product spaces*, Proc. Edinb. Math. Soc. **58** (2015), 403–419. 7, 74

[78] A. Grigor'yan, *Heat Kernel and Analysis on Manifolds*, AMS/IP Studies in Adv. Math., **47**, AMS, Providence, (2009). 13

[79] X. Guo and H. Li, *Submanifolds with cosntant scalar curvature in a unit sphere*, Tohoku Math. J. **65** (2013), 331-339. 2, 24, 29

[80] A. J. Goddard, *Some remarks on the existence of spacelike hypersurfaces of constant mean curvature*, Math. Proc. Cambridge Phil. Soc. **82** (1977), 489–495. 3

[81] G. Hardy, J. E. Littekwood and G. Pólya, *Inequalities.* Camb. Math. Lib., **2** (1989). 103

[82] S. Hawking and G. Ellis, *The Large Scale Structure of Space-Time* (Cambridge Monographs on Mathematical Physics). Cambridge: Camb. Uni. Press. (1973). 4, 58

[83] Z.H. Hou, and D. Yang, *Linear Weingarten spacelike hypersurfaces in de Sitter space*, Bull. Belg. Math. Soc. Simon Stevin **17** (2010), 769–780.

[84] A. Huber, *On subharmonic functions and differential geometry in the large*, Comment. Math. Helv., **32** (1957) 13–72,. 13

[85] T. Ishihara, *Maximal spacelike submanifolds of a pseudo-Riemannian space of constant curvature.* Mich. Math. Jour., **35**(1988), 345–352. 39

[86] L. Karp, *On Stokes' theorem for noncompact manifolds*, Proc. Amer. Math. Soc. **82** (1981), 487–490. 82

[87] T. Levi-Civita, *Famiglia di superfici isoparametriche nell'ordinario spazio Euclideo*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. **26** (1937), 355–362. 31

[88] B. P. Lima, *O Princípio de Omori-Yau para os operadores L_r e aplicações.* (2000). 60, 61

[89] A. M. Li and J. M. Li, *An intrinsic rigidity theorem for minimal submanifolds in a sphere*, Arch. Math. **58** (1992), 582–594. 23, 28

[90] H. Li, *Hypersurfaces with constant scalar curvature in space forms*, Math. Ann. **305** (1996), 665–672. 2

[91] H. Li, Y. J. Suh, and G. Wei, *Linear Weingartein Hypersurfaces in a unit Sphere* Bull. Korean Math. Soc. **46** (2009), 321–329.

[92] X. Liu, *Spacelike submanifolds in de Sitter spaces*, J. Phys. A: Math. Gen. **34** 5463–5468, (2001). 3

[93] J. Liu and X. Xie, *Complete spacelike hypersurfaces with CMC in Lorentz Einstein manifolds*, Bull. Korean Math. Soc. **58** (2021), 1053–1068. 5, 6, 49, 50

[94] J. Liu and J. Zhang, *Complete spacelike submanifolds in de Sitter spaces with $R = aH + b$*, Bull. Australian Math. Soc. **87** 386–399, (2013). 3, 4, 37, 45

[95] J. Liu and J. Zhang, *Complete spacelike submanifolds in de Sitter spaces with $R = aH + b$*, Bull. Austral. Math. Soc. **87** (2013), 386–399. 5

[96] R. López, *Rotational Linear Weingarten surfaces of hyperbolic type*, Isr. Jour. of Math., **167**, (2008), 283–301 . 15

[97] G. A. Lobos. *Classificação das imersões paralelas em formas espaciais.* Dissertação de mestrado, Universidade Estadual de Campinas - UNICAMP, Campinas/SP - Brasil, (1994). 25

[98] F. Manfio and R. Tojeiro, *Hypersurfaces with constant sectional curvature of* $\mathbb{S}^n \times \mathbb{R}$ *and* $\mathbb{H}^n \times \mathbb{R}$, Illinois J. Math. **55** (2011), 397–415. 83

[99] M. J. Micallef and M. Y. Wang, *Metrics with nonnegative isotropic curvature*, Duke Math. J. **72** (1993), 649–672. 51

[100] S. Montiel, *An integral inequality for compact spacelike hypersurfaces in the de Sitter space and applications to the case of constant mean curvature*, Indiana Univ. Math. J. **37** (1988), 909–917. 3, 43

[101] S. Montiel, *A characterization of hyperbolic cylinders in the de Sitter space*, Tôhoku Math. J. **48** (1996), 23–31. 3

[102] S. Montiel, *Unicity of constant mean curvature hypersurfaces in some Riemannian manifolds*, Ind. Univ. Math. J. **48** (1999), 711–748. 6, 58, 74, 75

[103] S. Montiel, *Complete non-compact spacelike hypersurfaces of constant mean curvature in de Sitter spaces.* J. Math. Soc. of Jap. **55** (2003), 915–938. 43

[104] S. Montiel, *Uniqueness of spacelike hypersurfaces of constant mean curvature in foliated spacetimes*, Math. Ann. **314** (1999), 529–553. 71, 90

[105] G. S. Neto. *Stability of constant mean curvature surfaces in three-dimensional warped product manifolds*, Ann. Gl. Anal.Geo. **56** (2019), 57–86. 10

[106] S. Nishikawa, *On maximal spacelike hypersurfaces in a Lorentzian manifold*, Nagoya Math. J. **95** (1984), 117–124. 4

[107] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity,* New York: Academic Press (1983). 9, 48, 72, 77, 106

[108] M. Okumura, *Hypersurfaces and a pinching problem on the second fundamental tensor*, American J. Math. **96** (1974), 207–213. 23, 30

[109] J. A. S. Pelegrín, and M. Rigoli, *Constant Mean Curvature Spacelike Hypersurfaces in Spatially Open GRW Spacetimes*, J. Geom. Anal. **29** (2018), 3293–3307. 7

[110] P. Petersen and M. Wink,*New curvature conditions for the Bochner Technique*, Invent. math. **224** (2021), 33–54. 51

[111] S. Pigola, M. Rigoli and A. G. Setti, *A Liouville-type result for quasi-linear elliptic equations on complete Riemannian manifolds*, J. Funct. Anal. **219** (2005), 400–432. 1, 8, 16, 20, 22, 37, 52, 81, 88

[112] J. Ramanathan, *Complete spacelike hypersurfaces of constant mean curvature in de Sitter space*, Indiana Univ. Math. J. **36** (1987), 349–359. 3

[113] A. Romero and R. M. Rubio, *On the mean curvature of spacelike surfaces in certain three-dimensional Robertson-Walker spacetimes and Calabi-Bernstein type problems*, Ann. Glob. Anal. Geom. **37** (2010), 21–31. 73

[114] A. Romero, R. M. Rubio, and J. J. Salamanca, *Uniqueness of complete maximal hypersurfaces in spatially parabolic generalized Robertson–Walker spacetimes*, Class. Quant. Grav. **30**, p. 115007, (2013). 6

[115] A. Romero, R. M. Rubio, and J. J. Salamanca, *Complete maximal hypersurfaces in certain spatially open generalized Robertson–Walker spacetimes*, Rev. La Real Acad. C. Exactas, Físicas y Naturales. Serie A. Mat., **109**, 451–460. 7

[116] W. Santos, *Submanifolds with parallel mean curvature vector in spheres*, Tohoku Math. J. **46** (1994), 403–415. 22, 27, 40, 42, 53

[117] B. Segre, *Famiglie di ipersuperficie isoparametriche negli spazi euclidei ad un qualunque numero di dimensioni*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. **27** (1938), 203–207. 31

[118] M. Spradlin, A.Strominger and A.Volovich, *Les Houches lectures on de Sitter space*. arXiv:hep-th/0110007 (2001) ,423–453. 4, 58

[119] W. Strübing, *Symmetric submanifolds of riemannian manifolds.* Math. Ann. **245**, (1979), 37–44. 25

[120] Y. Tashiro, *Complete Riemannian manifolds and some vector fields*, Trans. Am. Math. Soc. **117** (1965), 251–275. 74

[121] K.P. Tod, *Four-dimensional D'Atri Einstein spaces are locally symmetric*, Diff. Geom. Appl. **11** (1999), 55–67. 51

[122] R. Tojeiro. *On a class of hypersurfaces in* $\mathbb{S}^n \times \mathbb{R}$ *and* $\mathbb{H}^n \times \mathbb{R}$ Bull. Braz. Math. Soc. **41** (2010), 199–209. 83

[123] R. M. Wald, *General Relativity*, Univ. Chic. Press, Chicago and London, 1984. 9

[124] J . Weingarten, *Ueber eine Klasse auf einander abwickelbarer Flächen*, Jour. für die Reine und Ang. Math., **59** (1861), 382–393. 2, 15

[125] J. Weingarten, *Ueber die Flächen, derer Normalen eine gegebene Fäche berühren*, Jour. für die Reine und Ang. Math., **62** (1863), 61–63. 2, 15

[126] H. Xu and J.R. Gu, *Rigidity of Einstein manifolds with positive scalar curvature*, Math. Ann. **358** (2014), 169–193. 51

[127] D. Yang and Z. Hou, *Linear Weingarten spacelike submanifolds in de Sitter space*, J. Geom. **103**, (2012), 177–190. 3, 4, 5, 37, 38, 42, 45

[128] S. T. Yau, *Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry.* Ind. Univ. Math. J. **25** (1976), 659–670. 8, 81, 82

[129] S.T. Yau, *Submanifolds with constant mean curvature I*, American J. Math. **96** (1974), 346–366. 31