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Free Boundary Problems with Gradient Activation and Oscillatory Singularities

Aelson Oliveira Sobral

João Pessoa – Paraíba

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Thesis submitted to the post-graduate Program of the Mathematical Department of the Federal University of Paraíba in partial fulfillment of the requirements for the Ph.D. degree in Mathematics.

Advisor: Damião Júnio Araujo
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ATA DE DEFESA DE DOUTORADO JUNTO AO PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA DA UNIVERSIDADE FEDERAL DA PARAÍBA, REALIZADA NO DIA 12 DE MARÇO DE 2024.

Ao décimo segundo dia de março de dois mil e vinte e quatro, às 08:00 horas, no Auditório do Departamento de Matemática/CCEN da Universidade Federal da Paraíba, foi aberta a sessão pública de Defesa de Tese intitulada "**Problemas de Fronteira Livre com Ativação Gradiente e Singularidades Oscilatórias**", do aluno **Aelson Oliveira Sobral** que havia cumprido, anteriormente, todos os requisitos para a obtenção do grau de Doutor em Matemática, sob a orientação do Prof. Dr. Damião Júnio Gonçalves Araújo. A Banca Examinadora, indicada pelo Colegiado do Programa de Pós-Graduação em Matemática, foi composta pelos professores: Damião Júnio Gonçalves Araújo (Orientador), Eduardo Vasconcelos Oliveira Teixeira (Coorientador/UCF), Felipe Wallison Chaves Silva (UFPB), Disson Soares dos Prazeres (UFS), José Miguel Urbano (KAUST/Arábia Saudita) e Serena Dipierro (UWA/Austrália). O professor Damião Júnio Gonçalves Araújo, em virtude da sua condição de orientador, presidiu os trabalhos e, depois das formalidades de apresentação, convidou o aluno a discorrer sobre o conteúdo da tese. Concluída a explanação, o candidato foi arguido pela banca examinadora que, em seguida, sem a presença do aluno, finalizando os trabalhos, reuniu-se para deliberar tendo concedido a menção: **aprovado.** E, para constar, foi lavrada a presente ata que será assinada pelos membros da Banca Examinadora.

João Pessoa, 12 de março de 2024.

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To my wife, Ires.

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Abstract

Esta tese oferece uma análise aprofundada de duas categorias distintas de problemas de fronteira livre, fundamentais para a compreensão de sistemas complexos regidos por equações diferenciais.

No primeiro segmento do estudo, mergulhamos no âmbito das equações elípticas altamente degeneradas. Esta parte foca em um modelo caracterizado por um processo de difusão não linear, que se torna a força motriz em áreas onde o gradiente excede um limiar específico. Esta investigação não apenas ilumina o comportamento dessas soluções, mas também explora os pontos de convergência com outras tendências emergentes de pesquisa, enriquecendo assim o discurso neste campo.

A segunda parte da tese transita para a exploração de modelos variacionais de fronteira livre, particularmente aqueles marcados por singularidades oscilatórias. Este segmento é fundamental para abordar problemas em que a natureza oscilatória resulta em um espectro de geometrias de fronteira livre. Por meio de uma pesquisa meticulosa, conduzimos uma análise extensiva e estabelecemos uma nova fórmula de monotonicidade. Esta fórmula é instrumental para considerar os aspectos variáveis desses modelos. De forma significativa, demonstramos que, quando o poder singular varia de acordo com um padrão W^{1,n^+}, então a fronteira livre se manifesta localmente como uma superfície $C^{1,\delta}$, exceto por um conjunto negligenciável, caracterizado por uma co-dimensão de Hausdorff de pelo menos 2.

Esta tese tem como objetivo contribuir substancialmente para o campo da análise matemática e equações diferenciais, oferecendo perspectivas e metodologias novas no estudo de problemas de fronteira livre.

Palavras-chave: Problemas de fronteira livre sem restrição, Estimativas de regularidade, Problemas de fronteira livre singulares.

Abstract

This thesis provides an in-depth analysis of two distinct categories of free boundary problems, which are fundamental in understanding complex systems governed by differential equations.

In the first segment of the study, we delve into the realm of highly degenerate elliptic equations. This part focuses on a model characterized by a nonlinear diffusion process, which becomes the driving force in areas where the gradient exceeds a specific threshold. This investigation not only sheds light on the behavior of these solutions but also explores the convergence points with other emerging research trends, thereby enriching the discourse in this field.

The second part of the thesis transitions into an exploration of free boundary variational models, particularly those marked by oscillatory singularities. This segment is pivotal in addressing problems where the oscillatory nature results in a spectrum of free boundary geometries. Through meticulous research, we conduct an extensive analysis and establish a novel monotonicity formula. This formula is instrumental in considering the variable aspects of these models. Significantly, we demonstrate that when the singular power varies in a W^{1,n^+} fashion, then the free boundary is locally a $C^{1,\delta}$ surface, up to a negligible set of Hausdorff co-dimension at least 2.

This thesis aims to contribute substantially to the field of mathematical analysis and differential equations, offering novel perspectives and methodologies in the study of free boundary problems.

Keywords: Unconstrained free boundary problems, Regularity estimates, Singular free boundary problems

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Introduction

Free Boundary Problems (FBPs) stand out as a captivating subject of study of mathematical analysis. At the heart of these problems lies an unknown function, u, which solves an Partial Differential Equation (PDE) within an unknown domain, Ω. The boundary of this domain, denoted as $\partial\Omega$, is referred to as the *free boundary*. What makes FBPs particularly fascinating and challenging is the dynamic nature of the region in which the diffusive process occurs. This region's dependence on the solution itself injects an element of unpredictability and complexity into mathematical analysis, setting FBPs apart as a uniquely intriguing aspect of mathematical theory.

The emerging interest in FBPs can be attributed to their extensive applicability in various physical phenomena, bridging abstract mathematical concepts with tangible real-world situations. These problems underpin a myriad of phenomena, ranging from the melting of ice to the complexities of superconductivity, from the dynamics of flame propagation to the growth patterns of species, and from the principles of fluid mechanics to the challenges of shape optimization and material science. The presence of FBPs is even observed in practical scenarios, such as when attaching a membrane to a wire above an obstacle, where the membrane adjusts itself to minimize energy, exemplifying a FBP well known as the obstacle problem. This interplay between the mathematical modeling and their manifestation in both nature and daily life highlights the importance of their study. It is through the meticulous mathematical analysis of FBPs that we gain a precise understanding of these diverse and fascinating phenomena.

It turns out that most of these captivating problems can be encapsulated within a unifying mathematical framework, elegantly capturing the essence of these problems:

$$\begin{cases} F(x,u,Du,D^2u) = 0 & \text{in} \quad B_1 \cap \Omega \\ G(x,u,Du,D^2u) = 0 & \text{in} \quad B_1 \setminus \Omega, \end{cases} \tag{0.1}$$

where the free boundary is $\partial\Omega \cap B_1$. In this concise representation, it reveals the unknown pair (u,Ω), where u belongs to a specific functional space, rendering the equations meaningful, and F and G are, usually, functions with some elliptic structure. For a more in-depth discussion on this elegant and unified approach to FBPs, the [44] survey offers a wealth of knowledge, further bridging the gap between abstract theory and the palpable reality of the phenomena FBPs represent.

Diving deeper into the mathematical analysis of FBPs, the study bifurcates into two well-defined yet interconnected branches. When solutions to (0.1) do not enjoy a further (a priori) structure on the free boundary, they are often referred to as *unconstrained*. The first part of this thesis focuses on degenerate unconstrained FBPs, where we delve into the regularity estimates for viscosity solutions. The second part pivots to examining singular FBPs, coming from the minimization of an Alt-Philips-type functional, where the emphasis relies on the regularity and geometric characteristics of the free boundary $\partial\Omega$. This dual approach, examining both the solutions and the free boundaries, presents a comprehensive understanding of the complexities inherent in FBPs, each aspect bringing its unique challenges.

In Chapter 1, we set some notations and important aspects of the problems to be treated in the thesis. In particular, we discuss the different notions of solutions to be adopted in the sequel and scaling properties for the problems.

In Chapter 2, we investigate diffusion models triggered by a gradient threshold, results obtained in collaboration with Damião Araújo, and Eduardo Teixeira, [8]. These are self-regulatory systems in which a diffusive agent is prompted whenever the density difference becomes much larger than the displacement. Mathematically, this leads to the analysis of a class of *highly* degenerate elliptic partial differential equations of the form

$$\mathcal{H}(Du, D^2u) = f, \tag{0.2}$$

where the operator $\mathcal{H}$ collapses in a subset $C \subset \mathbb{R}^n$, and is elliptic for $\mathbb{R}^n \setminus C$.

Problems of that nature appear, for instance, in the theory of superconductivity, when examining vortices in the mean-field model, e.g. [12, 27, 41] and [25]. Variational interpretations are related to minimization issues in random surfaces and tilings, see [28, 50] and [33] for such a connection, as well as to problems in congested traffic dynamics, see [16] as well as [14, 29, 61]. Fully nonlinear equations of this type also appear as limiting free boundary problems, obtained when the degeneracy parameter of the equation tends to infinity, which can be understood as a free boundary version of the infinity Laplacian operator.

Note the region where a PDE governs the system depends upon the solution itself, more precisely on its gradient. That is, the correct way to interpret (0.2) is as an (unconstrained) free boundary problem, viz.

$$\mathcal{H}(Du, D^2u) = f, \quad \text{in } \left\{x \in \Omega \,\middle|\, Du(x) \in \mathbb{R}^n \setminus C\right\}. \tag{0.3}$$

We will further discuss this point of view in subsection 2.4.1.

To simplify the presentation, we focus on the case $C = B_\kappa$, for $\kappa \geq 0$, leading to the free boundary problem

$$(|Du| - \kappa)_+^q F(D^2u) = f \quad \text{in } \{|Du| > \kappa\}. \tag{0.4}$$

The operator F is uniformly elliptic and the parameter, $q \geq 0$, prescribes the degeneracy degree of the model along the free boundary $\partial\{|Du| > \kappa\}$. It is worth noting that the problem is still (very) degenerate even if $q = 0$, due to the diffusion collapse in the (a priori unknown) region $\{|Du| \leq \kappa\}$.

It is also important to highlight that no information upon the sets $\{|Du| \leq \kappa\}$ and $\{|Du| > \kappa\}$ can be a priori inferred. In particular, the free boundary, $\partial\{|Du| > \kappa\}$ can be very irregular, and thus out of the scope of known elliptic boundary regularity estimates.

The case $\kappa = q = 0$ falls into the theory launched by [24], where the authors investigated fully nonlinear elliptic equations of the form

$$F(D^2u) = g(x,u)\chi_{\{|Du|\neq 0\}}. \tag{0.5}$$

Solutions of (0.5) are understood in a very weak viscosity sense, where one disregards smooth test functions that touch with zero gradient. In [24], the authors manage to show that solutions of (0.5) satisfy (ordinary) viscosity inequalities, and thus the classical fully nonlinear regularity theory applies. In the case $F = \Delta$ and $g(x,u) = cu$, the authors obtain the sharp $C^{1,1}$-regularity of solutions to (0.5); see also [22] for related advances on similar problems.

In parallel to the approach adopted in [24], in this thesis we introduce the concept of κ-grad viscosity solutions of (0.4), see Definition 1.2. The idea is to interpret (0.4) by disregarding test functions touching u at point x_0 with not sufficient large slope. That is, the corresponding viscosity inequalities are enforced only at points x_0 for which one can touch by a smooth test function φ verifying $|D\varphi(x_0)| > \kappa$.

When $\kappa > 0$, the optimal (local) regularity one can hope for a solution of (0.4) is Lipschitz continuity. This is because any function whose gradient norm is less than κ automatically satisfies the equation. Also, one can easily construct 1D-examples of solutions of (0.4) that are merely Lipschitz continuous. On the other hand, κ-grad viscosity solutions of (0.4) are entitled to the regularity theory developed in [48]. In particular, solutions are locally of class $C^{0,\alpha}$, for some $0 < \alpha \ll 1$, depending on dimension, ellipticity constants, and κ.

The first main result of this chapter is the sharp Lipschitz regularity estimate for κ-grad viscosity solutions of (0.4), see Theorem 2.2. The proof relies on carefully crafting special jets, as in [30], whose gradient at touching points is sufficiently large. We perform a meticulous analysis, identifying all possible dependencies along the process. In particular, we prove that the Lipschitz norm of solutions of (0.4) does not depend upon the degree of degeneracy, q. We mention that this remark is new (and sharp) even in the case that the PDE holds everywhere, say for the family of PDEs:

$$|Du|^q F(D^2u) = f, \quad \text{in } B_1 \subset \mathbb{R}^n, \tag{0.6}$$

with $c < f < c^{-1}$. Indeed, a result proven in [5], see also [47] and [4], assures that viscosity solutions of (0.6) are locally of class $C^{1,\frac{1}{1+q}}$ (at least for $q \gg 1$) and that such a regularity is optimal. Hence, insofar as uniform-in-q estimates are concerned, gradient bounds are the best one can hope for solutions u_q of (0.6).

While Lipschitz estimates are indeed optimal in regards to the local regularity of solutions to (0.4), one could inquire about C^1 regularity within the PDE region, viz. $\Omega_u := \{|Du| > \kappa\}$, up the free boundary,

$$\Gamma_u := \partial\{|Du| > \kappa\}.$$

This problem is particularly challenging, as it seems hard to say anything about the structure of Γ_u, unless further information is given; see [25] for the case $q = \kappa = 0$ and $F = \Delta$.

It is worth noting that, the continuity of $(|Du| - \kappa)_+$ implies Ω_u must be an open set, and that the PDE $(|Du| - \kappa)_+^q F(D^2u) = f$ holds in the traditional viscosity sense within Ω_u.

The considerations above give rise to a slightly stronger, though necessary, notion of solutions to (0.4), see Definition 1.4. Under such a regime, the second main result we prove in this chapter yields a universal modulus of continuity of the gradient of u in Ω_u, up to the free boundary, Γ_u, see Theorem 2.3. The proof combines several ingredients and it will be delivered in Section 2.3. The idea relies on an interplay between interior C^{1,α_d} regularity estimates at points that are d-away (concerning the gradient level-set distance) from the free-boundary, Γ_u, and how $0 < \alpha_d \ll 1$ deteriorates as $d \to 0$. This is attained by introducing a sort of DeGiorgi's improvement of oscillation technique at the gradient level. This is particularly useful to gauge regularity for points sufficiently close to the free boundary, concerning the gradient level-sets. For points far from the free boundary (again concerning the gradient level-set distance), the equation is elliptic, and thus, up to rescaling, u is close to a F-harmonic function; uniform $C^{1,\alpha}$ regularity estimates are then obtained *a*la Caffarelli, [18]; see also [65] for a didatical account of this method.

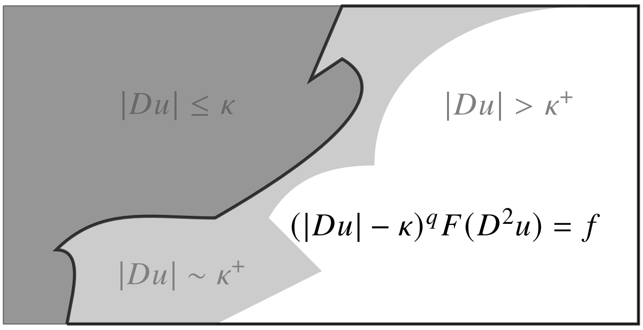

Figure 1 – This figure is a representation of the geometry of the problem. The white region, $\{|Du| > \kappa\}$, displays the part of the domain in which a diffusion PDE drives the system. The system is dormant in the dark grey zone, $\{|Du| \le \kappa\}$. The analysis in the intermediary light grey sector, $\{\kappa < |Du| < \kappa + \mu\}$, for some $0 < \mu \ll 1$, is critical for the proof of Theorem 2.3. It is worth highlighting, however, that the topology of such a region can be much more complicated and their corresponding boundaries highly irregular. This is why Theorem 2.3 is a non-trivial (somewhat striking) result.

This chapter is organized as follows. In Section 2.2.1, we prove the uniform Lipschitz estimate, Theorem 2.2. In Section 2.2.2, we establish compactness for the scaled PDE. In Section 2.3, we split the analysis between the region close and far away from the free boundary to attain the universal C^1 regularity theorem. In the last Section 2.4, we discuss further applications of the methods introduced in this chapter.

In Chapter 3 of the thesis, we develop a variational framework for the analysis of free boundary problems that include a continuum of singularities, [9], results obtained in collaboration with Damião Araújo, José Miguel Urbano, and Eduardo Teixeira. The mathematical setup leads to the minimization of an energy-functional of type

$$\mathscr{E}(v, O) = \int_O F(Dv, v, x)\, dx, \tag{0.7}$$

whose Lagrangian, $F(\vec{p}, v, x)$, is non-differentiable concerning the v argument, and the degree of singularity varies with respect to the spatial variable x. The singularity oscillation exerts an intricate influence on the free boundary's trace and shape in a notably unpredictable manner. This dynamic not only alters the geometric behavior of the solution but also significantly impacts the regularity of the free boundary. As a consequence, the associated Euler-Lagrange equation gives rise to a rich new class of singular elliptic partial differential equations, which, in their own right, present an array of intriguing and independent mathematical challenges and interests.

Singular elliptic PDEs, particularly those involving free boundaries, find applications in a variety of fields, including thin film flows, image segmentation, shape optimization, and biological invasion models in ecology, to cite just a few. Mathematically, such models lead to the analysis of an elliptic PDE of the form

$$\Delta u = \mathfrak{s}(x, u)\chi_{\{u>0\}}, \tag{0.8}$$

within a domain $\Omega \subset \mathbb{R}^n$. The defining characteristic of the PDE above lies in the singular term $\mathfrak{s}\colon \Omega \times (0, \infty) \to \mathbb{R}$, which becomes arbitrarily large near the zero level set of the solution, *i.e.*,

$$\lim_{v\to 0} \mathfrak{s}(x, v) = \infty. \tag{0.9}$$

Fine regularity properties of solutions to (0.8), along with geometric measure estimates and eventually the differentiability of their free boundaries, $\partial\{u > 0\}$, are inherently intertwined with *quantitative* information concerning the blow-up rate outlined in (0.9). Heuristically, solutions of PDEs with a faster singular blow-up rate will exhibit reduced regularity along their free boundaries. Existing methods for treating these singular PDE models, in various forms, rely to some extent on the *uniformity* of the blow-up rate prescribed in (0.9).

We investigate a broader class of variational free boundary problems, extending our focus to encompass oscillatory blow-up rates. That is, we are interested in PDE models involving singular terms with fluctuating asymptotic behavior,

$$\Delta u \sim u^{-p(x)}, \tag{0.10}$$

for some function $p\colon \Omega \to [0, 1)$. As anticipated, the analysis will be variational, *i.e.*, we will investigate local minimizers of a given non-differentiable functional, as described in (0.7), which exhibit a spectrum of oscillatory exponents of non-differentiability.

The investigation of the static case, *i.e.*, of PDE models in the form of $\Delta u \sim u^{-p_0}$, where $0 < p_0 < 1$, has a rich historical lineage, tracing its roots to the classical Alt-Phillips problem, as documented in [3, 58, 59]. This elegant problem has served as a source of inspiration, sparking significant advancements in the domain of free boundary problems, as exemplified by works like [7, 35, 39, 40, 62, 66–68], to cite just a few. Remarkably, the Alt-Phillips model serves as a bridge connecting the classical obstacle problem, which pertains to the case $p_0 = 0$, and the cavitation problem, achieved as the limit when $p_0 \nearrow 1$. Each intermediary model exhibits its unique geometry. That is, solutions present a precise geometric behavior at a free boundary point, viz. $u \sim \mathrm{dist}^{\beta}(x, \partial\{u > 0\})$, for a critical, well-defined and uniform exponent $\beta(p_0)$.

Mathematically, the oscillation of the singular exponent brings several new challenges, as the model prescribes multiple free boundary geometries. The main difficulty in analyzing free boundary problems with oscillatory singularities relies on quantifying how the local free boundary geometry fluctuations affect the regularity of the solution u as well as the behavior of its associated free boundary $\partial\{u > 0\}$. In essence, the main quest in this chapter is to understand how changes in the free boundary geometry directly influence its local behavior.

From the applied viewpoint, the model studied in this chapter accounts for the heterogeneity of external factors influencing the reaction rates within the porous catalyst region where the gas density $u(x)$ is distributed. To be more specific, when examining the theory of diffusion and reaction within catalysts modeled in an isotropic, homogeneous medium, the task at hand involves the minimization of an energy-functional, which takes the form

$$\mathcal{J}(v, \mathcal{O}) = \int_{\mathcal{O}} \frac{1}{2}|Dv|^2 dx + \int_{\mathcal{O}} f(x, v) dx. \tag{0.11}$$

Minimizers of $\mathcal{J}$ describe the density distribution of the gas in a stationary situation. The term $\int_{\mathcal{O}} f(x, v) dx$ corresponds to the rupture law along the free boundary. It models the complexities of the catalytic reaction, dictated by the abrupt shifts and discontinuities in the reaction rates as they intersect the catalyst's surface. Mathematically, such factors prompt the non-differentiability of the term $f(x, v)$ concerning the $v-$argument.

The singularity of $\partial_v f(x, v)$ along $v = 0$ carries critical information about the model's behavior. It is a no-static feature of the model, dynamically shifting in response to several external factors, including temperature, pressure, and the roughness of the catalyst's surface. Such considerations require mathematical models allowing for non-differentiable terms whose singularity may vary concerning the spatial variable x.

In this chapter, our focus is directed toward fine regularity properties of local minimizers of the energy-functional

$$J_{\delta(x)}^{\gamma(x)}(v) := \int \frac{1}{2}|Dv|^2 + \delta(x)(v^+)^{\gamma(x)} dx, \tag{0.12}$$

where the functions $\gamma(x)$ and $\delta(x)$ possess specific properties that will be elaborated upon in due course. In connection with the theory of singular elliptic PDEs, minimizers of (0.12) are distributional solutions of

$$\begin{cases} \Delta u &= \delta(x)\gamma(x)u^{\gamma(x)-1} \quad \text{in} \quad \{u > 0\} \\ Du &= 0 \quad\quad\quad\quad\quad\quad\; \text{on} \quad \partial\{u > 0\}, \end{cases}$$

with the free boundary condition being observed by local regularity estimates, to be shown in this chapter.

This chapter is organized as follows. We establish the existence of minimizers as well as local $C^{1,\alpha_\star}$-regularity, for some $0 < \alpha_\star < 1$, independent of the modulus of continuity of $\gamma(x)$ in sections 3.1 and 3.2. Non-degeneracy estimates are obtained in Section 3.3. In Section 3.4, we obtain gradient estimates near the free boundary, quantifying the magnitude of $Du(y)$ in terms of the pointwise value $u(y)$. We highlight that the results established in Sections 1.3 and 3.4 are all independent of the continuity of $\gamma(x)$. However, when $\gamma(x)$ varies randomly, regularity estimates of u and its non-degeneracy properties along the free boundary have different homogeneities, and thus no further regularity properties of the free boundary are expected to hold. We tackle this issue in Section 3.5, where under a very weak condition on the modulus of continuity of $\gamma(x)$, we establish sharp *pointwise* growth estimates of u. The estimates from Section 3.5 imply that near a free boundary point $x_0 \in \partial\{u > 0\}$, the minimizer u behaves *precisely* as $\sim d^{\frac{2}{2-\gamma(x_0)}}$, with universal estimates. Section 3.6 is devoted to Hausdorff's estimates of the free boundary. In Section 3.7, we obtain a Weiss-type monotonicity formula which yields blow-up classification, and in Section 3.8, we discuss the regularity of the free boundary $\partial\{u > 0\}$.

1

Preliminaries

In this chapter, we lay the foundational framework essential for navigating the subsequent sections of the thesis. We first establish some standard notation. Then, we delve into the intricate structural aspects of Fully Nonlinear models, and minimization problems.

1.1 Notation

Problems are modeled in the n-dimensional Euclidean space, $\mathbb{R}^n$. The open ball of radius r centered at the point x_0 is denoted by $B_r(x_0)$. We shall omit the center of the ball for $x_0 = 0$. When writing integrals, the symbol

$$\int_U f(x)dx,$$

means the integral of f concerning the Lebesgue measure, in the measurable set U. We omit the symbol dx when there is no misunderstanding. When calculating surface integrals in a $(n-1)$-dimensional set U', we will write

$$\int_{U'} f(x)d\mathcal{H}^{n-1},$$

and omit the symbol whenever there is no misunderstanding.

We will write $|U|$ to denote the Lebesgue measure of a set U. Given a number $s > 0$, the symbol $\mathcal{H}^s(U)$ denotes the s-dimensional Hausdorff measure. Properties concerning these measures will be used without further comments.

The space of all $n \times n$ symmetric matrices is denoted by $\mathrm{Sym}(n)$. For a matrix $M \in \mathrm{Sym}(n)$, we denote by $\mathrm{Spec}(M)$ the set of all eigenvalues of M.

For an open set U, a natural number k and an exponent $\alpha \in (0, 1]$, we define $C^{k,\alpha}(U)$, to be the set of functions that are continuously differentiable up to order k, and the $k-$th derivatives are Hölder continuous of order α. The usual norm associated with this set of functions is

$$\|v\|_{C^{k,\alpha}(U)} := \sum_{i=0}^{k} \|D^i v\|_{L^\infty(U)} + [D^k v]_{C^{0,\alpha}(U)}.$$

Here, $\|w\|_{L^\infty(U)}$ denotes the supremum norm, and

$$[w]_{C^{0,\alpha}(U)} := \sup_{x,y\in U} \frac{|w(x) - w(y)|}{|x-y|^\alpha}.$$

When $\alpha = 1$, we shall also write $\mathrm{Lip}(U)$ instead of $C^{0,1}$. We recall that when $k = 0$, then $D^k v = v$.

Likewise, we define the Sobolev spaces $W^{k,p}(U)$, for $p > 1$ to be the set of weakly differentiable functions u, whose weak derivatives lie in $L^p(u)$. Classical immersion theorems, equivalences, and properties shall also be used without further comments.

Given $a \in \mathbb{R}$, we denote

$$a^+ := \max\{a, 0\} \quad \text{and} \quad a^- := \max\{-a, 0\}.$$

To simplify notation, we will also write a_+ and a_- to denote the same objects.

Given a finite set $F \subset \mathbb{N}$, we write $\#F$ to denote its cardinality, that is, the number of elements in F.

1.2 Fully nonlinear models

The main structure our problem requires is the long-standing notion of uniform ellipticity. To define so, we need the notion of *Pucci Extremal Operators*. Given constants $0 < \lambda \le \Lambda$, let

$$\mathcal{A}_{\lambda,\Lambda} := \left\{ A \in \mathrm{Sym}(n) \,\middle|\, \mathrm{Spec}(A) \subseteq [\lambda, \Lambda] \right\}.$$

The so-called *Pucci Extremal Operators* $\mathcal{M}^+_{\lambda,\Lambda}$ and $\mathcal{M}^-_{\lambda,\Lambda}$, acting on $\mathrm{Sym}(n)$, are defined as

$$\mathcal{M}^+_{\lambda,\Lambda}(M) := \sup_{A\in\mathcal{A}_{\lambda,\Lambda}} \mathrm{Trace}(AM) \quad \text{and} \quad \mathcal{M}^-_{\lambda,\Lambda}(M) := \inf_{A\in\mathcal{A}_{\lambda,\Lambda}} \mathrm{Trace}(AM),$$

we will simply write $\mathcal{M}^+$ and $\mathcal{M}^-$ whenever the ellipticity constants, λ and Λ, are understood.

There is an equivalent way of defining such operators. Given a matrix $M \in \mathrm{Sym}(n)$, we denote by $e_i(M)$ the $i-$th eigenvalue of the symmetric matrix M and $e_i(M)^+$, $e_i(M)^-$ its positive and negative part, respectively. Then,

$$\begin{aligned} \mathcal{M}^+(M) &= \Lambda \sum e_i(M)^+ - \lambda \sum e_i(M)^-, \\ \mathcal{M}^-(M) &= \lambda \sum e_i(M)^+ - \Lambda \sum e_i(M)^-. \end{aligned}$$

Definition 1.1. *Given constants* $0 < \lambda \le \Lambda$, *we say that* $F\colon Sym(n) \to \mathbb{R}$ *is* (λ, Λ)*-elliptic if*

$$\mathcal{M}^-(M - N) \le F(M) - F(N) \le \mathcal{M}^+(M - N),$$

for every $M, N \in Sym(n)$.

This sort of monotonicity assumption is what allows the regularity theory developed here to hold.

The notion of *viscosity solutions* has been presented to the mathematical community a while ago in a paper by Crandall, Ishii and Lions, [30]. Such a notion was a breakthrough in the analysis of partial differential equations and ever since, a great amount of work has been done. Important extensions of this notion to equations with L^p ingredients have also been done, pushing the theory beyond the boundaries of continuous viscosity solutions, see [20]. In particular, in [24], the authors consider the usual notion of viscosity solution type, but they disregard touching function whose slope is zero. Nevertheless, they still managed to prove their solutions satisfied ordinary viscosity inequalities and were thus entitled to the classical regularity theory, see [19]. Inspired by them, we propose the following definition:

Definition 1.2. *(κ-grad viscosity solutions) Let $G : \mathbb{R}^n \times Sym(n) \longrightarrow \mathbb{R}$ be a continuous function. Given a nonnegative κ, we say that u is a κ-grad viscosity subsolution to*

$$G(Du, D^2u) = f \tag{1.1}$$

if for every x_0 and φ such that $(u - \varphi)$ attain a local maximum at x_0 with $|D\varphi(x_0)| > \kappa$ there holds

$$G(D\varphi(x_0), D^2\varphi(x_0)) \geq f(x_0).$$

We say u is a κ-grad viscosity supersolution for (1.1), if for every x_0 and φ such that $(u - \varphi)$ attain a local minimum at x_0 with $|D\varphi(x_0)| > \kappa$ there holds

$$G(D\varphi(x_0), D^2\varphi(x_0)) \leq f(x_0).$$

We say u is a κ-grad viscosity solution for (1.1) if u is both a κ-grad subsolution and supersolution.

Similarly, we will say that a continuous $v\colon B_1 \longrightarrow \mathbb{R}$ satisfies $|Dv|(x_0) > \kappa$ (in the viscosity sense) if there exists a C^2 function φ touching v from above (or below) at x_0 satisfying $|D\varphi(x_0)| > \kappa$.

Definition 1.3. *Given a continuous function $v\colon \overline{B}_1 \longrightarrow \mathbb{R}$ we define*

$$\Omega_v = \left\{x \in B_1 \;\middle|\; |Dv| > \kappa\right\}.$$

The interior boundary of this set will be denoted by Γ_v, i.e.

$$\Gamma_v := \partial\Omega_v \cap B_1$$

For the PDE model we will investigate in this paper, Γ_u will represent the free boundary of the problem, whereas Ω_u is the region in which the system is driven by a (fully nonlinear, degenerate) elliptic equation.

We note that the notion of κ-grad viscosity solutions is indeed very weak. It enlarges the set where we search for solutions by disregarding test functions whose slope at a touching point is less than or equal to κ. In particular, this definition gives very little information about the set Ω_u, where the PDE is placed. If one seeks for further regularity of solutions to (0.4) within Ω_u, a bit more structure is naturally required. This is the contents of the next definition:

Definition 1.4. *We say u is an effective viscosity solution of* (0.4)*, if the set Ω_u defined as $\{x \in B_1 \mid |Du| > \kappa\}$ is open and u satisfies*

$$(|Du| - \kappa)_+^q F(D^2u) = f \quad in \quad \Omega_u,$$

in the classical viscosity sense.

As a byproduct of the results to be proven in Chapter 2, κ-grad viscosity solutions of (0.4) can be easily obtained through a limiting process. More precisely, let u_j be a bounded family of viscosity solutions to

$$\left((|Du_j| - \kappa)_+^q + 1/j\right) F(D^2u_j) = f, \quad \text{in } \Omega. \tag{1.2}$$

The regularity estimates established in this paper are uniform with respect to the approximation parameter j. Hence, up to a subsequence, one can pass the limit as $j \to \infty$ in (1.2). It is standard to verify that the limit function will enjoy the same regularity estimate of u_j, i.e. Lipschitz continuous, and it solves (0.4) in the κ-grad viscosity sense. The C^1 regularity of u_j, up to the free boundary, viz. the corresponding Theorem 2.3, is too uniform concerning the parameter j.

Next, we comment on the scaling properties of the model, which shall be used throughout the entire evolution of the paper.

Remark 1.1. *Let u be a κ-grad (resp. effective) viscosity solution of* (0.4) *in B_1. Assume $\kappa > 0$ and define the constants:*

$$A = \frac{1}{\max(1, \|u\|_\infty)} \quad and \quad B = \frac{\tau \cdot \max(1, \|u\|_\infty)}{\kappa},$$

for an arbitrary $\tau > 0$. In the sequel, define

$$w(x) := Au(Bx).$$

One verifies that w is a τ-grad (resp. effective) viscosity solution of the re-scaled model:

$$(|Dw| - \tau)_+^q \overline{F}(D^2w) = \overline{f},$$

in the ball $B_{1/B}$, where

$$\overline{F}(M) = (AB^2)F((AB^2)^{-1}M)$$

and

$$\overline{f}(x) = A^{q+1}B^{q+2}f(Bx).$$

Indeed, if $\varphi \in C^2$ touches w from above(or below) at a point x with $|D\varphi(x)| > \tau$, then the function $\overline{\varphi}(x) = A^{-1}\varphi(B^{-1}x)$ touches u from above(or below) at Bx with $|D\overline{\varphi}(Bx)| > \kappa$.

Given the previous remark, all results in this paper will be proven, with no loss of generality, for a normalized solution, $-1 \leq u \leq 1$.

1.3 Minimization problems

In this section, we bring some important preliminaries to deal with certain minimization problems. We start by describing precisely the mathematical setup of our problem. We assume $\Omega \subset \mathbb{R}^n$ is a bounded domain and $\delta, \gamma : \Omega \to \mathbb{R}_0^+$ are bounded mensurable functions.

For each subset $\mathcal{O} \subset \Omega$, we denote

$$\gamma_\star(\mathcal{O}) := \operatorname*{ess\,inf}_{y \in \mathcal{O}} \gamma(y) \quad \text{and} \quad \gamma^\star(\mathcal{O}) := \operatorname*{ess\,sup}_{y \in \mathcal{O}} \gamma(y). \tag{1.3}$$

In the case of balls, we adopt the simplified notation

$$\gamma_\star(x, r) := \gamma_\star(B_r(x)) \qquad \text{and} \qquad \gamma^\star(x, r) := \gamma^\star(B_r(x)).$$

Throughout the whole paper, we shall assume

$$0 < \gamma_\star(\Omega) \leq \gamma^\star(\Omega) \leq 1. \tag{1.4}$$

For a non-negative boundary datum $0 \leq \varphi \in H^1(\Omega) \cap L^\infty(\Omega)$, we consider the problem of minimizing the functional

$$\mathcal{J}_\gamma^\delta(v, \Omega) := \int_\Omega \frac{1}{2} |Dv|^2 + \delta(x)(v^+)^{\gamma(x)} dx \tag{1.5}$$

among competing functions

$$v \in \mathcal{A} := \left\{ v \in H^1(\Omega) \ : \ v - \varphi \in H_0^1(\Omega) \right\}.$$

We say $u \in \mathcal{A}$ is a minimizer of (1.5) if

$$\mathcal{J}_\gamma^\delta(u, \Omega) \leq \mathcal{J}_\gamma^\delta(v, \Omega), \quad \forall v \in \mathcal{A}.$$

Note that minimizers as above are, in particular, local minimizers in the sense that, for any open subset $\Omega' \subset \Omega$,

$$\mathcal{J}_\gamma^\delta(u, \Omega') \leq \mathcal{J}_\gamma^\delta(v, \Omega'), \quad \forall v \in H^1(\Omega') \ : \ v - u \in H_0^1(\Omega').$$

Scaling

Some of the arguments used recurrently in this paper rely on a scaling feature of the functional (1.5) that we detail in the sequel for future reference. Let $x_0 \in \Omega$ and consider two parameters $A, B \in (0, 1]$. If $u \in H^1(\Omega)$ is a minimizer of $\mathcal{J}_\gamma^\delta(v, B_A(x_0))$, then

$$w(x) := \frac{u(x_0 + Ax)}{B}, \quad x \in B_1 \tag{1.6}$$

is a minimizer of the functional

$$\mathcal{J}_{\tilde{\gamma}}^{\tilde{\delta}}(v, B_1) := \int_{B_1} \frac{1}{2} |Dv|^2 + \tilde{\delta}(x) v^{\tilde{\gamma}(x)} dx,$$

with

$$\tilde{\delta}(x) := B^{\gamma(x_0+Ax)} \left(\frac{A}{B}\right)^2 \delta(x_0 + Ax) \quad \text{and} \quad \tilde{\gamma}(x) := \gamma(x_0 + Ax).$$

Indeed, by changing variables,

$$\int_{B_A(x_0)} \frac{1}{2} |Du(x)|^2 + \delta(x) u(x)^{\gamma(x)} dx$$

$$\begin{aligned}
&= A^n \int_{B_1} \frac{1}{2} |Du(x_0 + Ax)|^2 \, dx + A^n \int_{B_1} \delta(x_0 + Ax) u(x_0 + Ay)^{\gamma(x_0+Ax)} dy \\
&= A^n \int_{B_1} \frac{1}{2} \left| \left(\frac{B}{A}\right) Dw(x) \right|^2 + \delta(x_0 + Ax) \left[Bw(x)\right]^{\gamma(x_0+Ax)} \, dx \\
&= A^{n-2} B^2 \int_{B_1} \frac{1}{2} |Dw(x)|^2 + B^{\gamma(x_0+Ax)-2} A^2 \delta(x_0 + Ax) \left[w(x)\right]^{\gamma(x_0+Ax)} \, dx \\
&= A^{n-2} B^2 \int_{B_1} \frac{1}{2} |Dw(x)|^2 + \tilde{\delta}(x) \left[w(x)\right]^{\tilde{\gamma}(x)} \, dx.
\end{aligned}$$

Observe that since $0 < B \leq 1$, $\tilde{\delta}$ satisfies

$$\|\tilde{\delta}\|_{L^\infty(B_1)} \leq B^{\gamma_\star(x_0,A)-2} A^2 \|\delta\|_{L^\infty(B_A(x_0))}.$$

In particular, choosing $A = r$ and $B = r^\beta$, with $0 < r \leq 1$ and

$$\beta = \frac{2}{2 - \gamma_\star(x_0, A)},$$

we obtain $\|\tilde{\delta}\|_{L^\infty(B_1)} \leq \|\delta\|_{L^\infty(B_r(x_0))}$.

2

Regularity in diffusion models with gradient activation

In this chapter, we embark on an in-depth examination of viscosity solutions in the κ-grad sense, to

$$(|Du| - \kappa)_+^q F(D^2u) = f \in L^\infty.$$

Our analysis is particularly focused on seeking regularity issues for such solutions. The challenge lies in devising analytical tools capable of discerning between the states of degeneracy and non-degeneracy; it is worth pointing out that the degeneracy set $\{|Du| \leq \kappa\}$ has no structure whatsoever.

The program initiates in Section 2.1 with the proof of the ABP estimate, venturing beyond conventional approaches due to the absence of additional PDE information within the degeneracy set. Our discourse offers a detailed exposition on the utilization of inf-sup convolutions, enhancing the understanding of this critical analytical tool.

Progressing to Section 2.2, we delve into the optimal Lipschitz regularity for solutions of both the original and scaled PDEs. This investigation is crucial, as the precise dependence of Lipschitz estimates underpins the regularity achievements discussed in Section 2.3. Here, by refining the solution concept, we introduce a methodology to effectively distinguish between degeneracy and non-degeneracy phases, facilitating gradient continuity up to the free boundary.

Concluding the chapter in Section 2.4, we elucidate the connections between our model and current research trends, highlighting the broader implications and potential applications of our findings. This section not only bridges our work with ongoing research efforts but also sets the stage for future explorations into the complex interplay of viscosity solutions and nonlinear PDE analysis.

2.1 Aleksandrov-Bakelman-Pucci estimate

In this section, we discuss the validity of the Aleksandrov-Bakelman-Pucci estimate for solutions of 0.4. The usual estimate should not hold, as no further PDE information can be gathered within the set where $\{|Du| < \kappa\}$. The key idea, however, is that such information can also be understood as PDE information.

It is worth mentioning that the ABP estimate presented here appeared in [46] for a more general class of equations. The issue, however, is that it requires the equation to hold everywhere in an open set, let us say, Ω. The gap we fill with our proof of the ABP estimate is that the region we only have PDE information is $\{|Du| > \kappa\}$. A similar issue was pointed out in [48] for the L^{ϵ}-estimate.

The main result of this chapter is the following:

Theorem 2.1. *Let u be a κ-grad viscosity solution of* (0.4) *in an open domain Ω. Then*

$$\|u\|_{L^\infty(\Omega)} \leq \sup_{\partial\Omega} |u| + C\left(\|f\|_{L^n}^{\frac{1}{q+1}} + \kappa\right),$$

for a constant $C = C(n, q, diam(\Omega), \lambda, \Lambda)$.

We split the proof of this result into two parts: first, we assume solutions are C^2 smooth and proceed through an approximation argument. In general, the first step is where lies the key ideas of the proof.

To ease the presentation, let us define

$$\begin{aligned}
\Gamma^+_{r,s}(u) &= \{y \in \Omega : \exists \xi \in B_r \backslash B_s \quad \text{such that} \quad u(x) \leq \ell_{u(y),\xi}(x) \quad \text{in} \quad \Omega\},\\
\Gamma^-_{r,s}(u) &= \{y \in \Omega : \exists \xi \in B_r \backslash B_s \quad \text{such that} \quad u(x) \geq \ell_{u(y),\xi}(x) \quad \text{in} \quad \Omega\},
\end{aligned}$$

where $\ell_{a,b}(x) := a + b \cdot x$. These are the contact sets of the solution with its concave(convex) envelope whose slope lies in the annulus $B_r \backslash B_s$.

To bring a comprehensive proof of the ABP estimate, and taking into advantage that the proof is modular, we will split the analysis into a few lemmas and put it all together at the end.

2.1.1 Geometric argument for classical functions

First, we begin with the geometric part of the argument, which does not require any PDE information.

Lemma 2.1. *Let $u \in C(\overline{\Omega}) \cap C^2(\Omega)$. Then,*

$$\frac{\sup_\Omega u^- - \sup_{\partial\Omega} u^-}{diam(\Omega)} \leq \left(\frac{2(q+1)|\partial B_1|}{\lambda^n n^{n-1}}\right)^{\frac{1}{(q+1)n}} \left\|(|Du| - \kappa)^q_+ \mathcal{M}^-(D^2u)\right\|^{\frac{1}{q+1}}_{L^n\left(\Gamma^+_{r,2\kappa}(-u)\right)} + 5\kappa,$$

where

$$r := \frac{\sup_\Omega u^- - \sup_{\partial\Omega} u^-}{diam(\Omega)}.$$

Proof. First, we may assume $r \geq 2\kappa$, otherwise the Lemma would be automatically true. Since $r \geq 2\kappa$, it holds that $-u(\overline{x}) = \sup_\Omega u^- > 0$. Define the mapping $F : \mathbb{R}^n \to \mathbb{R}$ as

$$F(\xi) = (|\xi| - \kappa)_+^q \, \xi.$$

By definition of the quantity r, it follows that

$$B_r \backslash B_{2\kappa} \subseteq Du\left(\Gamma^+_{r,2\kappa}(-u)\right). \tag{2.1}$$

Indeed, let $\xi \in B_r \backslash B_{2\kappa}$ and consider the affine function $\ell_{h,\xi}$, where

$$h = \sup_{y \in \Omega} \{-u(y) - \xi \cdot y\}.$$

The constant h is precisely the value we need to lower the hyperplane so that $-u \leq \ell_{h,\xi}$ in Ω and touches $-u$ from above at some $z \in \overline{\Omega}$. Observe that $z \in \Omega$, otherwise if $z \in \partial\Omega$, then

$$\begin{aligned} \sup_\Omega u^- = -u(\overline{x}) &\leq \ell_{h,\xi}(\overline{x}) = \ell_{h,\xi}(z) + \xi \cdot (\overline{x} - z) \\ &= -u(z) + \xi \cdot (\overline{x} - z) < \sup_{\partial\Omega} u^- + r \operatorname{diam}(\Omega) = \sup_\Omega u^-, \end{aligned}$$

which is a contradiction, and so, it follows that $z \in \Omega$. As a consequence,

$$\xi = -Du(z) \quad \text{for some } z \in \Omega,$$

and so (2.1) follows. Since $u \in C^2(\Omega)$, it also holds $D^2u \geq 0$. Now, from (2.1), it follows that

$$|F(B_r \backslash B_{2\kappa})| \leq \left|F\left(Du\left(\Gamma^+_{r,2\kappa}(-u)\right)\right)\right|. \tag{2.2}$$

Let us estimate the LHS of (2.2) from below. Observe that

$$\begin{aligned} |F(B_r \backslash B_{2\kappa})| &= \int_{F(B_r \backslash B_{2\kappa})} dx \\ &= \int_{B_r \backslash B_{2\kappa}} |\det(DF(\xi))| d\xi \\ &= \int_{B_r \backslash B_{2\kappa}} \left|\det\left(q(|\xi| - \kappa)_+ \frac{\xi}{|\xi|} \otimes \xi + (|\xi| - \kappa)_+ I\right)\right| d\xi \\ &= \int_{B_r \backslash B_{2\kappa}} \left(q(|\xi| - \kappa)_+^{nq-1}|\xi| + (|\xi| - \kappa)_+^{qn}\right) d\xi \\ &= |\partial B_1| \int_{2\kappa}^r \left(q(t - \kappa)_+^{nq-1} t + (t - \kappa)_+^{qn}\right) t^{n-1} dt \\ &= \frac{|\partial B_1|}{n} \left((r - \kappa)_+^{qn} r^n - \kappa^{qn}(2\kappa)^n\right). \end{aligned}$$

Now, let us estimate the RHS of (2.2) from above. This will be achieved through a change of variables.

$$\begin{aligned} \left|F\left(Du\left(\Gamma^+_{r,2\kappa}(-u)\right)\right)\right| &= \int_{F(Du(\Gamma^+_{r,2\kappa}(-u))} dx = \int_{\Gamma^+_{r,2\kappa}(-u)} |\det\left(D(F\circ Du\right)|dy \\ &= \int_{\Gamma^+_{r,2\kappa}(-u)} |\det DF(Du(y))|\,\det D^2u(y)dy \\ &\leq \int_{\Gamma^+_{r,2\kappa}(-u)} |\det DF(Du(y))|\left(\frac{\Delta u}{n}\right)^n dy \\ &\leq \int_{\Gamma^+_{r,2\kappa}(-u)} |\det DF(Du(y))|\left(\frac{\mathcal{M}^-(D^2u)}{\lambda n}\right)^n dy. \end{aligned}$$

Observe now that in the set $\Gamma^+_{r,2\kappa}(-u)$, we have

$$2\kappa \leq |Du| \leq r,$$

and so, we have

$$|Du| \leq 2(|Du|-\kappa)_+.$$

Now, since

$$DF(\xi) = q\,(|\xi|-\kappa)_+^{q-1}\frac{\xi}{|\xi|}\otimes\xi + (|\xi|-\kappa)_+^q\,I,$$

we obtain by direct computations that

$$\det\left(DF(\xi)\right) = \left(q\frac{|\xi|}{(|\xi|-\kappa)_+}+1\right)(|\xi|-\kappa)_+^{qn} \leq 2(q+1)\,(|\xi|-\kappa)_+^{qn}.$$

Thus,

$$\left|F\left(Du(\Gamma^+_{r,2\kappa}(-u))\right)\right| \leq \frac{2(q+1)}{\lambda^n n^n}\int_{\Gamma^+_{r,2\kappa}(-u)} \left((|Du|-\kappa)_+^q\,\mathcal{M}^-(D^2u)\right)^n dy.$$

Putting all together, the Lemma is proven. □

The previous lemma is interesting in itself, but it only gives us quantitative information whenever u solves an elliptic PDE.

2.1.2 Estimate for viscosity solutions

First, we will prove the estimate to the case where u is semiconvex.

Lemma 2.2. *Let $u \in C(\overline{\Omega})$ be a semiconcave κ-grad viscosity supersolution of* (0.4). *Then,*

$$\frac{\sup_\Omega u^- - \sup_{\partial\Omega} u^-}{diam(\Omega)} \leq \left(\frac{2(q+1)|\partial B_1|}{\lambda^n n^{n-1}}\right)^{\frac{1}{(q+1)n}} \|f^+\|^{\frac{1}{q+1}}_{L^n\left(\Gamma^+_{r,2\kappa}(-u)\right)} + 5\kappa.$$

Proof. First, recall that if u is semiconcave, then it follows that

$$\begin{aligned} (|Du|-\kappa)_+^q F(D^2u) \leq f \quad &\text{a.e in} \quad \Omega\cap\{|Du|>\kappa\}, \\ D^2u \leq C_0 I_n \quad &\text{a.e in} \quad \Omega, \end{aligned}$$

for some constant $C_0 > 0$. Now, let u_η be a standard mollification of u, see (appendix). It then follows that $u_\eta \to u$ is locally uniform in Ω. Furthermore, $Du_\eta \to Du$ and $D^2u_\eta \to D^2u$ a.e in Ω, with $D^2u_\eta \leq C_0 I_n$ in Ω. Let

$$r_\eta := \frac{\sup_\Omega u_\eta^- - \sup_{\partial\Omega} u_\eta^-}{\operatorname{diam}(\Omega)}.$$

As $u_\eta \in C(\overline{\Omega}) \cap C^2(\Omega)$, we can apply Lemma 2.1 in order to obtain

$$\frac{\sup_\Omega u_\eta^- - \sup_{\partial\Omega} u_\eta^-}{\operatorname{diam}(\Omega)} \leq 5\kappa + \left(\frac{2(q+1)|\partial B_1|}{\lambda^n n^{n-1}}\right)^{\frac{1}{(q+1)n}} \left\|(|Du_\eta| - \kappa)_+^q \mathcal{M}^-(D^2u_\eta)\right\|_{L^n\left(\Gamma^+_{r_\eta,2\kappa}(-u_\eta)\right)}^{\frac{1}{q+1}}.$$

Recall, from the proof of Lemma 2.1, that within $\Gamma^+_{r_\eta,2\kappa}(-u_\eta)$, it also holds $D^2u_\eta \geq 0$, and so $|D^2u_\eta| \leq C_0$. Thus, it follows that

$$\left|(|Du_\eta| - \kappa)_+^q \mathcal{M}^-(D^2u_\eta)\right| \leq C_0' \quad \text{in} \quad \Gamma^+_{r_\eta,2\kappa}(-u_\eta),$$

and by dominated convergence theorem, we can pass to the limit in order to obtain

$$\frac{\sup_\Omega u^- - \sup_{\partial\Omega} u^-}{\operatorname{diam}(\Omega)} \leq 5\kappa + \left(\frac{2(q+1)|\partial B_1|}{\lambda^n n^{n-1}}\right)^{\frac{1}{(q+1)n}} \left\|(|Du| - \kappa)_+^q \mathcal{M}^-(D^2u)\right\|_{L^n\left(\Gamma^+_{r,2\kappa}(-u)\right)}^{\frac{1}{q+1}},$$

where

$$r := \frac{\sup_\Omega u^- - \sup_{\partial\Omega} u^-}{\operatorname{diam}(\Omega)}.$$

As the equation holds at almost every point for u, we have

$$\frac{\sup_\Omega u^- - \sup_{\partial\Omega} u^-}{\operatorname{diam}(\Omega)} \leq \left(\frac{2(q+1)|\partial B_1|}{\lambda^n n^{n-1}}\right)^{\frac{1}{(q+1)n}} \left\|f^+\right\|_{L^n\left(\Gamma^+_{r,2\kappa}(-u)\right)}^{\frac{1}{q+1}} + 5\kappa,$$

and the Lemma is proved. □

Now we perform a regularization argument to bypass the *a priori* semiconcavity assumed in Lemma 2.2.

Proposition 2.1. *Let u be a κ-grad viscosity supersolution of* (0.4). *Then, the conclusion of Lemma* 2.2 *holds.*

Proof. Given $\epsilon > 0$, we consider u_ϵ to be the inf-convolution of u defined as

$$u_\epsilon(x) := \inf_{y\in\Omega}\left\{u(y) + \frac{|x-y|^2}{2\epsilon}\right\}.$$

It is classical that u_ϵ is semiconcave and by Lemma 2.3 below, u_ϵ solves

$$(|Du_\epsilon| - \kappa)_+^q F(D^2u_\epsilon) = f_\epsilon \quad \text{in} \quad \Omega_\epsilon,$$

where

$$f_\epsilon(x) := \sup_{|y-x|<2\sqrt{\epsilon\|u\|_\infty}} f(y),$$
$$\Omega_\epsilon := \left\{x \in \Omega \,\middle|\, \operatorname{dist}(x,\partial\Omega) > 2\sqrt{\epsilon\|u\|_\infty}\right\}.$$

Now, as u_ϵ falls in the hypothesis of Lemma 2.2, it holds that

$$\frac{\sup_{\Omega_\epsilon} u_\epsilon^- - \sup_{\partial\Omega_\epsilon} u_\epsilon^-}{\operatorname{diam}(\Omega_\epsilon)} \le \left(\frac{2(q+1)|\partial B_1|}{\lambda^n n^{n-1}}\right)^{\frac{1}{(q+1)n}} \|f_\epsilon^+\|_{L^n\left(\Gamma^+_{r_\epsilon,2\kappa}(-u_\epsilon)\right)}^{\frac{1}{q+1}} + 5\kappa,$$

where

$$r_\epsilon := \frac{\sup_{\Omega_\epsilon} u_\epsilon^- - \sup_{\partial\Omega_\epsilon} u_\epsilon^-}{\operatorname{diam}(\Omega_\epsilon)}.$$

Since $u_\epsilon \to u$, $f_\epsilon \to f$ locally uniform and $\Omega_\epsilon \to \Omega$ we can pass to the limit to have the Proposition proved. □

Observe that everything could have been done to the subsolution case. Nevertheless, small modifications should be taken care and we just comment on. The precise estimate we want to prove is the following.

Proposition 2.2. *Let u be a κ-grad viscosity subsolution of* (0.4). *Then,*

$$\frac{\sup_\Omega u^+ - \sup_{\partial\Omega} u^+}{diam(\Omega)} \le \left(\frac{2(q+1)|\partial B_1|}{\lambda^n n^{n-1}}\right)^{\frac{1}{(q+1)n}} \|f^-\|_{L^n\left(\Gamma^+_{r,2\kappa}(u)\right)}^{\frac{1}{q+1}} + 5\kappa.$$

Proof. It is enough to assume $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ and the quantity

$$r := \frac{\sup_\Omega u^+ - \sup_{\partial\Omega} u^-}{\operatorname{diam}(\Omega)} \ge 2\kappa.$$

It can be proved that

$$B_r \backslash B_{2\kappa} \subseteq Du\left(\Gamma^+_{r,2\kappa}(u)\right).$$

The only difference is that we take

$$h = \sup_{y\in\Omega}\{u(y) - \xi\cdot y\},$$

so that $\ell_{h,\xi}$ touches u from above at some $z \in \Omega$ whenever $\xi \in B_r\backslash B_{2\kappa}$. The proof is finished when we estimate

$$\begin{aligned}
\left|F\left(Du\left(\Gamma^+_{r,2\kappa}(u)\right)\right)\right| &= \int_{F(Du(\Gamma^+_{r,2\kappa}(u))} dx = \int_{\Gamma^+_{r,2\kappa}(u)} |\det\left(D(F\circ Du)\right|dy \\
&= \int_{\Gamma^+_{r,2\kappa}(u)} |\det DF(Du(y))| - \det D^2u(y)dy \\
&\le \int_{\Gamma^+_{r,2\kappa}(u)} |\det DF(Du(y))| \left(\frac{-\Delta u}{n}\right)^n dy \\
&\le \int_{\Gamma^+_{r,2\kappa}(u)} |\det DF(Du(y))| \left(\frac{-\mathcal{M}^+(D^2u)}{\lambda n}\right)^n dy.
\end{aligned}$$

The rest of the proof follows seamlessly with f^- instead of f^+. □

2.1.3 Inf-sup convolutions

It will be necessary, to proceed with the regularization argument, to make use of the so-called inf-sup convolutions. Given a continuous function $u \in C(\overline{\Omega})$ and a parameter $\epsilon > 0$ we define the sup-convolution of u as

$$u^\epsilon(x) := \sup_{y\in\Omega} \left\{ u(y) - \frac{|x-y|^2}{2\epsilon} \right\}.$$

Similarly, we define the inf-convolution as

$$u_\epsilon(x) := \inf_{y\in\Omega} \left\{ u(y) + \frac{|x-y|^2}{2\epsilon} \right\}.$$

It is classical, see [19], that such functions are approximations of the function u from above and from below, respectively. Furthermore, they enjoy the property of being semiconvex and semiconcave, respectively. When elliptic PDE information from u is available, one can carry such information to these approximations as in the following lemma.

Lemma 2.3. *Let $u \in C(\overline{\Omega})$ be a κ-grad viscosity solution of* (0.4). *Then, u_ϵ is a κ-grad viscosity supersolution of*

$$(|Du_\epsilon| - \kappa)_+^q F(D^2 u_\epsilon) = f_\epsilon \quad in \quad \Omega_\epsilon,$$

where

$$\begin{aligned} f_\epsilon(x) &:= \sup_{|y-x|<2\sqrt{\epsilon\|u\|_\infty}} f(y), \\ \Omega_\epsilon &:= \left\{ x \in \Omega \,\middle|\, dist(x,\partial\Omega) > 2\sqrt{\epsilon\|u\|_\infty} \right\}. \end{aligned}$$

Similarly, u^ϵ is a κ-grad viscosity subsolution of

$$(|Du^\epsilon| - \kappa)_+^q F(D^2 u^\epsilon) = f^\epsilon \quad in \quad \Omega_\epsilon,$$

where

$$f^\epsilon(x) := \inf_{|y-x|<2\sqrt{\epsilon\|u\|_\infty}} f(y).$$

Proof. We prove only the supersolution case. First, observe that

$$\begin{aligned} u_\epsilon(x) &= \inf_{y\in\Omega} \left\{ u(y) + \frac{|x-y|^2}{2\epsilon} \right\} = u(x^*) + \frac{|x-x^*|^2}{2\epsilon} \\ &\geq -\|u\|_\infty + \frac{|x-x^*|^2}{2\epsilon}, \end{aligned}$$

and so, since $u_\epsilon \leq u$,

$$|x-x^*|^2 \leq 4\epsilon\|u\|_\infty < \text{dist}(x,\partial\Omega). \tag{2.3}$$

As a consequence, the infimum is always attained in Ω_ϵ.

Now, let $\varphi \in C^2(B_r(x_0))$ touching u_ϵ from below at x_0 with $|D\varphi(x_0)| > \kappa$. Denoting by x_0^* the point where the infimum is attained, then

$$\begin{aligned} \varphi(x_0) &= u_\epsilon(x_0) = u(x_0^*) + \frac{|x_0 - x_0^*|^2}{2\epsilon} \\ \varphi(x) &\leq u_\epsilon(x) \leq u(y) + \frac{|x-y|^2}{2\epsilon}, \end{aligned}$$

for every $x \in B_r(x_0)$ and $y \in \Omega$. Hence, these conditions together, imply that the function

$$\Psi(x) := \varphi(x + x_0 - x_0^*) - \frac{|x_0 - x_0^*|^2}{2\epsilon}$$

touches u from below at x_0^*. Furthermore, it holds that

$$D\Psi(x_0^*) = D\varphi(x_0) \quad \text{and} \quad D^2\Psi(x_0^*) = D^2\varphi(x_0),$$

and so, by (2.3),

$$(|D\varphi(x_0)| - \kappa)_+^q F(D^2\varphi(x_0)) \leq f(x_0^*) \leq f_\epsilon(x_0),$$

as desired. □

2.2 Optimal regularity and compactness for scaled PDE's

In this section, we will prove (optimal) Lipschitz regularity for viscosity solutions of (0.4). This is done through the Ishii-Lions method.

2.2.1 Uniform Lipschitz estimates

This section discusses the proof of sharp Lipschitz regularity of κ-grad viscosity solution of (0.4). The main result is the following:

Theorem 2.2. *Let u be a κ-grad viscosity solution of* (0.4) *in B_1. Then u is Lipschitz continuous in $B_{1/2}$, with universal bounds. More precisely, there exists a constant C depending only on n, λ, Λ, κ, $\|f\|_\infty$ and $\|u\|_\infty$, but not on q, such that*

$$\sup_{x,y\in B_{1/2}} \frac{|u(x)-u(y)|}{|x-y|} \leq C.$$

As commented, Theorem 2.2 is optimal, even in the case $q = 0$. It is also important to highlight that the Lipschitz bound does not depend on the degeneracy parameter, q. This is interesting (and new) even in the case when the PDE holds everywhere in the domain. We will further discuss this in Section 2.4.

The first key Lemma in the proof of Theorem 2.2 fosters useful bounds for barriers, to be crafted, at maximum points of the double-variable function $w(x, y) := u(x) - u(y)$.

Lemma 2.4. *Let u be a κ-grad viscosity solution of* (0.4) *and consider double-variable functions:*

$$w(x,y) = u(x) - u(y) \quad \text{and} \quad \varphi(x,y) := L\phi(|x-y|) + K(|x|^2 + |y|^2),$$

for positive parameters L, K and $\phi \in C^2(\mathbb{R}^+)$ a nonnegative function. Let $(\overline{x}, \overline{y})$ be an interior maximum point for $w - \varphi$ such that $\overline{x} \neq \overline{y}$. Then,

$$-4\phi''(|\overline{x}-\overline{y}|)L \leq$$
$$4n\frac{\Lambda}{\lambda}K + \frac{1}{\lambda}\|f\|_\infty \left[(|D_x\varphi(\overline{x},\overline{y})| - \kappa)_+^{-q} + \left(\left|D_y\varphi(\overline{x},\overline{y})\right| - \kappa\right)_+^{-q} \right].$$

Proof. Consider

$$D_x\varphi(\overline{x},\overline{y}),\ D_y\varphi(\overline{x},\overline{y}) \in \mathbb{R}^n \setminus \overline{B}_\kappa.$$

From Jensen-Ishii's Lemma [30, Theorem 3.2], there exist $X, Y \in \mathcal{S}(n)$, such that

$$(|D_x\varphi(\overline{x},\overline{y})| - \kappa)_+^q F(X) \geq f(\overline{x}) \quad \text{and} \quad (|D_y\varphi(\overline{x},\overline{y})| - \kappa)_+^q F(Y) \leq f(\overline{y}). \tag{2.4}$$

In addition,

$$\begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leq \begin{bmatrix} Z & -Z \\ -Z & Z \end{bmatrix} + (2K + \iota) I_{2n\times 2n}, \tag{2.5}$$

where $Z = LD_x^2\phi(|\cdot|)(\overline{x}-\overline{y})$. Estimate (2.5) applied to vectors (ξ,ξ), provides $spec(X-Y) \subset (-\infty, 4K+2\iota]$. On the other hand, now choosing $(\hat{\eta}, -\hat{\eta})$, for $\hat{\eta} = (\overline{x}-\overline{y})/|\overline{x}-\overline{y}|$, gives

$$
\begin{aligned}
(X-Y)\hat{\eta}\cdot\hat{\eta} &\leq 4Z\hat{\eta}\cdot\hat{\eta} + (4K+2\iota) \\
&= 4L\phi''(|\overline{x}-\overline{y}|) + 4K + 2\iota.
\end{aligned}
$$

This implies that at least one eigenvalue of $(X-Y)$ should be less than

$$4L\phi''(|\overline{x}-\overline{y}|) + 4K + 2\iota.$$

Therefore,

$$
\begin{aligned}
\mathcal{M}^+(X-Y) &\leq \Lambda(n-1)(4K+2\iota) + \lambda(4L\phi''(|\overline{x}-\overline{y}|) + 4K + 2\iota) \\
&= n\Lambda(4K+2\iota) + 4\lambda L\phi''(|\overline{x}-\overline{y}|).
\end{aligned}
$$

From (1.1) and (2.4), we conclude

$$-\|f\|_\infty \left[(|D_x\varphi(\overline{x},\overline{y})| - \kappa)_+^{-q} + \left(\left|D_y\varphi(\overline{x},\overline{y})\right| - \kappa\right)_+^{-q} \right] \leq \mathcal{M}^+(X-Y),$$

and the Lemma is proven. □

We are ready to deliver a proof of Theorem 2.2; extra care is required to keep track of all constants' dependence.

Proof of Theorem 2.2. The idea is to show the existence of universal positive parameters L and K, such that

$$u(x) - u(y) \leq L|x-y| + K\left(|x|^2 + |y|^2\right), \tag{2.6}$$

for each $(x,y) \in B_{1/2} \times B_{1/2}$.

Let us denote

$$\phi(t) = \frac{3t - 2t^{3/2}}{3} \tag{2.7}$$

for $t \in [0,1]$. We further define

$$M := \sup_{x,y\in\overline{B}_{1/2}} \left(u(x) - u(y) - L\phi(|x-y|) - K\left(|x|^2 - |y|^2\right)\right).$$

Note that showing $M \leq 0$ yields (2.6). The strategy is then to assume that $M > 0$ and verify that this implies a constraint to the size of L and K.

Let $(\overline{x},\overline{y})$ be the point in which M is attained. Since $\phi(0) = 0$, we easily see that $\overline{x} \neq \overline{y}$. Additionally,

$$L\phi(|\overline{x}-\overline{y}|) + K\left(|\overline{x}|^2 + |\overline{y}|^2\right) < u(\overline{x}) - u(\overline{y}) \leq 2.$$

This implies that, choosing K universally large, there holds $|\overline{x}-\overline{y}| \leq 1/4$. Also,

$$\frac{1}{2} \leq \phi'(|\overline{x}-\overline{y}|) \leq 1$$

and thus, for $L \geq 4K$, we have

$$\frac{L}{4} \leq \frac{L}{2} - K \leq \min\{|D_x\varphi(\overline{x}, \overline{y})|, |D_y\varphi(\overline{x}, \overline{y})|\}, \tag{2.8}$$

where, hereafter,

$$\varphi(x, y) := L\phi(|x - y|) + K(|x|^2 + |y|^2).$$

From Lemma 2.4 and the fact that $\phi''(|\overline{x} - \overline{y}|) < -1$, we derive

$$L \leq n\frac{\Lambda}{\lambda}K + \frac{1}{4\lambda}\|f\|_\infty \left((|D_x\varphi(\overline{x}, \overline{y})| - \kappa)_+^{-q} + \left(\left|D_y\varphi(\overline{x}, \overline{y})\right| - \kappa\right)_+^{-q}\right). \tag{2.9}$$

Taking in account the last two estimates, we obtain

$$L - n\frac{\Lambda}{\lambda}K \leq \frac{1}{\lambda}\|f\|_\infty \left(\frac{L}{4} - \kappa\right)_+^{-q}$$

For $L > 4(1 + \kappa)$, we conclude

$$L < Kn\frac{\Lambda}{\lambda} + \frac{1}{\lambda}\|f\|_\infty. \tag{2.10}$$

Thus, if one selects

$$L > \max\left\{4(1 + \kappa), 4K, Kn\frac{\Lambda}{\lambda} + \frac{1}{\lambda}\|f\|_\infty\right\},$$

we conclude M cannot be a positive quantity and the proof of Theorem 2.2 is complete. □

2.2.2 Compactness for scaled PDEs

In this section, we establish equicontinuity estimates for normalized solutions of

$$(|\xi + \vartheta Du| - \kappa)_+^q F(D^2u) = f. \tag{2.11}$$

The main goal is to obtain estimates that are independent of $\xi \in \mathbb{R}^n$ and of $\vartheta > 0$.

We note that the aforementioned equation is understood in the κ-grad viscosity sense for the auxiliary function $v(x) = \xi \cdot x + \vartheta u(x)$ with respect to the PDE

$$(|Dv| - \kappa)_+^q F_\vartheta(D^2v) = f_\vartheta, \tag{2.12}$$

where $F_\vartheta(M) = \vartheta F(\vartheta^{-1}M)$ and $f_\vartheta = \vartheta f$. That is, saying u verifies (2.11) means that v is a κ-grad viscosity solution of (2.12). With that understood, we pass to discuss the first technical lemma needed to obtain uniform compactness for such PDEs.

Lemma 2.5. *Assume u is normalized and satisfies (2.11) with $\|f\|_\infty \leq 1$. Given $\mu \in (0, 1)$, there exists a constant C depending only on n, λ and Λ, such that if*

$$|\xi| \geq \kappa + 2\mu \quad \textit{and} \quad \vartheta \leq \frac{\mu^{q+1}}{2C}, \tag{2.13}$$

then

$$\sup_{x,y \in B_{1/2}} \frac{|u(x) - u(y)|}{|x - y|} \leq C\mu^{-q}.$$

Proof. The proof follows the lines of reasoning employed in Section 2.2.1. We will only comment on the necessary amendments.

Consider ϕ as defined in (2.7) and

$$M := \sup_{x,y\in\overline{B}_{1/2}} \left(u(x) - u(y) - L\phi(|x-y|) - K\left(|x|^2 - |y|^2\right)\right).$$

Let $(\overline{x}, \overline{y})$ be the pair where M is attained and assume $M > 0$. First, we localize the points where M is attained by choosing K large enough.

The auxiliary function $v(x) = \xi \cdot x + \vartheta u(x)$ solves (2.12) in the κ-grad viscosity sense, thus we can apply Lemma 2.4 with $\varphi(x,y) := L\phi(|x-y|) + K(|x|^2 + |y|^2)$ as to reach

$$L \leq n\frac{\Lambda}{\lambda}K + \frac{1}{4\lambda}\left((|\vartheta D_x\varphi(\overline{x},\overline{y}) + \xi| - \kappa)_+^{-q} + \left(\left|\vartheta D_y\varphi(\overline{x},\overline{y}) - \xi\right| - \kappa\right)_+^{-q}\right). \tag{2.14}$$

From (2.13) and the estimate

$$\max\{|D_x\varphi(\overline{x},\overline{y})|, |D_y\varphi(\overline{x},\overline{y})|\} \leq 2L,$$

there holds

$$\min\{|\xi + \vartheta D_x\varphi(\overline{x},\overline{y})|, |\xi - \vartheta D_y\varphi(\overline{x},\overline{y})|\} \geq \kappa + \mu.$$

Therefore, from estimate (2.14), we can further estimate

$$L < n\frac{\Lambda}{\lambda}K + \frac{1}{2\lambda}\mu^{-q} \leq \overline{C}\mu^{-q},$$

for $\overline{C} = C(n, \lambda, \Lambda)$. The conclusion is that if $L \geq \overline{C}\mu^{-q}$, then $M \leq 0$, which is equivalent to the thesis of the Lemma. □

2.3 Regularity estimates up to the free boundary

In this Section, we establish gradient continuity for effective viscosity solutions of (0.4), viz Definition 1.4. Some of the technical lemmas to be presented here, though, are still valid for the weaker notion of solutions, according to Definition 1.2. We will state such results in their more general form for future reference.

We further comment that in this section we will deal with the solutions of (0.4) for a universally small $\kappa > 0$, to be chosen later in the proof. According to Remark 1.1, this is not restrictive. The main result of this section reads as follows:

Theorem 2.3. *Let u be an effective viscosity solution of* (0.4) *in B_1. Then, there exists a modulus of continuity σ, depending on κ, q, n, λ, Λ, $\|f\|_{Lip}$ and $\|u\|_\infty$, such that*

$$(|Du| - \kappa)_+ \in C^{0,\sigma}(B_{1/2}).$$

We comment that the main new information given by Theorem 2.3 is that u is uniformly in C^1_{loc} in Ω_u, up to the free boundary Γ_u; a non-trivial result, as no information can be retrieved from the local structure of Γ_u. Throughout this section, we shall obtain a slightly stronger result, from which Theorem 2.3 follows as a consequence. We state it here for future reference.

Proposition 2.3. *Let u be an effective viscosity solution of* (0.4). *Then, given $0 < \mu < 1$, there exist constants $\alpha_\mu \in (0,1)$ and $C_\mu > 0$, depending only upon n, q, λ, Λ, $\|f\|_{Lip}$, $\|u\|_\infty$ and μ, such that*

$$\|(|Du| - (\kappa+\mu))_+\|_{C^{0,\alpha_\mu}(B_{1/2})} \leq C_\mu.$$

Critical to Proposition 2.3 is the fact that, while the Hölder exponent α_μ may degenerate as $\mu \longrightarrow 0$, the estimate is local, i.e. holds within $B_{1/2}$, and not only in the region where the PDE drives the system.

The proof of Theorem 2.3 will be divided into two main steps: given $0 < \mu < 1$, we slice Ω_u as follows

$$\Omega_u = \left\{x \in B_1 \,\middle|\, \kappa < |Du| < \kappa+\mu\right\} \cup \left\{x \in B_1 \,\middle|\, |Du| > \kappa+\mu\right\}.$$

At points μ-close (in the sense of level set of $|Du|$) to the free boundary Γ_u, we employ a *De Giorgi* based argument to get improvement of oscillation for functions of Du, which corresponds to subsection 2.3.1. At points μ-far away from the free boundary, the equation is uniformly elliptic, so one can proceed with an approximation argument.

2.3.1 Improvement of oscillation near the free boundary

Hereafter in this section we assume the source term f to be a Lipschitz continuous function. Note that if u is an effective viscosity solution of (0.4), then it is locally of class $C^{1,\alpha}$ in $\{|Du| > \kappa\}$.

Lemma 2.6. *Let u be an effective viscosity solution of* (0.4) *with* $f \in \mathrm{Lip}(\bar{B}_1)$. *For a unit vector $e \in \partial B_1$, consider w to be defined as*

$$w = (\partial_e u - (\kappa + \mu))_+.$$

Then, w satisfies

$$\mathcal{M}^+(D^2 w) + q\,\mu^{-q-1}\,\|f\|_\infty |Dw| \geq -\mu^{-q}\,\|f\|_{Lip},$$

in the viscosity sense in B_1.

Proof. To ease notation, let $\mathcal{G}$ be defined by $\mathcal{G}(\xi) = (|\xi| - \kappa)_+^q$. Notice that for $|\xi| > \kappa$

$$|D\mathcal{G}(\xi)| \leq q(|\xi| - \kappa)_+^{q-1}. \tag{2.15}$$

Differentiating the equation with respect to $e \in \partial B_1$ inside the open set $\left\{x \in B_1 \,\middle|\, w > 0\right\}$, we obtain

$$D\mathcal{G}(Du) \cdot D(\partial_e u) F(D^2 u) + \mathcal{G}(Du) F_{ij}(D^2 u) \partial_{ij}(\partial_e u) = \partial_e f.$$

Taking into account that $F(D^2 u) = f[\mathcal{G}(Du)]^{-1}$ and dividing the above equation by $\mathcal{G}(Du)$ we get

$$D\mathcal{G}(Du) \cdot D(\partial_e u)[\mathcal{G}(Du)]^{-2} f + F_{ij}(D^2 u)\partial_{ij}(\partial_e u) = \partial_e f\,[\mathcal{G}(Du)]^{-1}. \tag{2.16}$$

Now, from (2.15) and the fact that

$$\left\{x \in B_1 \,\middle|\, w(x) > 0\right\} \subset \left\{x \in B_1 \,\middle|\, |Du(x)| > \kappa + \mu\right\},$$

we obtain

$$\begin{aligned} D\mathcal{G}(Du) \cdot D(\partial_e u)[\mathcal{G}(Du)]^{-2} f &\leq \|f\|_\infty\,[\mathcal{G}(Du)]^{-2}\,|D\mathcal{G}(Du)|\,|D(\partial_e u)| \\ &\leq q\,\mu^{-q-1}\,\|f\|_\infty\,|D(\partial_e u)| \\ &= q\,\mu^{-q-1}\,\|f\|_\infty\,|Dw|. \end{aligned}$$

Moreover, by definition of $\mathcal{G}$, we have

$$\partial_e f\,[\mathcal{G}(Du)]^{-1} \geq -\|Df\|_\infty\,\mu^{-q}.$$

Hence, ellipticity of F yields

$$q\,\mu^{-q-1}\,\|f\|_\infty\,|Dw| + \mathcal{M}^+(D^2 w) \geq -\mu^{-q}\,\|Df\|_\infty, \tag{2.17}$$

as desired. □

Next, we obtain an oscillation improvement of the gradient, away from (but arbitrarily near) the free boundary Γ_u. In order to ease presentation throughout this section, we adopt the following notation for a vector $e \in \partial B_1$:

$$w_e := (\partial_e u - (\kappa + \mu))_+ \quad \text{and} \quad w_M := (|Du| - (\kappa + \mu))_+$$

Lemma 2.7. *Assume u is an effective viscosity solution of (0.4), with $f \in Lip(\bar{B}_1)$. Assume that for some $\eta > 0$, there holds*

$$\sup_{e \in \partial B_1} \left| \left\{ x \in B_{1/8} \;\middle|\; w_e \geq (1-\eta)\|w_M\|_{L^\infty(B_{1/4})} \right\} \right| \leq (1-\eta) \left| B_{1/8} \right|. \tag{2.18}$$

Then, there exist parameters $\overline{c}$, depending only on n, λ, Λ, q, μ, $\|f\|_\infty$, and θ, depending on n, λ, and Λ, such that

$$\|w_M\|_{L^\infty(B_{1/4})} \leq \max\left\{ (1 - \overline{c}\,\eta^{1+\frac{1}{\theta}})\|w_M\|_{L^\infty(B_{1/4})}\,,\; \left(\overline{c}\,\eta^{\frac{1}{\theta}+1}\right)^{-1} \|f\|_{Lip} \right\}.$$

Proof. Let us call

$$\mathcal{A} := \left\{ x \in B_{1/8} \;\middle|\; w_e \geq (1-\eta)d \right\}.$$

Easily one notes that

$$\overline{w} := \|w_M\|_\infty - w_e \geq 0,$$

where $\|w_M\|_\infty := \|w_M\|_{L^\infty(B_{1/4})}$. Combining Lemma 2.6 and the weak Harnack inequality, see for instance [51, Theorem 4.5], we obtain

$$\|\overline{w}\|_{L^\theta(B_{1/8})} \leq C\left(\inf_{B_{1/8}} \overline{w} + \|f\|_{Lip} \right),$$

for some $\theta = \theta(n, \lambda, \Lambda)$, and $C = C(\mu, \|f\|_\infty, n, \lambda, \Lambda, q)$. From the last inequality and (2.18), we obtain

$$\overline{w} + \|f\|_{Lip} \geq C^{-1} \left(\int_{B_{1/8}} \overline{w}^{\,\theta} dx \right)^{1/\theta} \geq C^{-1} \left(\int_{\mathcal{A}^c} (\|w_M\|_\infty - w_e)^\theta \, dx \right)^{1/\theta},$$

and thus,

$$\overline{w} + \|f\|_{Lip} \;\geq\; C^{-1} |\mathcal{A}^c|^{\frac{1}{\theta}} \eta \|w_M\|_\infty \;\geq\; c_1 \eta^{\frac{1}{\theta}+1} \eta \|w_M\|_\infty,$$

for some $c_1 = c_1(\mu, \|f\|_\infty, n, \lambda, \Lambda, q)$. This implies that

$$\|w_M\|_\infty - w_e \geq c_1 \eta^{\frac{1}{\theta}+1} \|w_M\|_\infty - \|f\|_{Lip},$$

which translates into

$$\|w_M\|_\infty - w_e \geq c_1 \eta^{\frac{1}{\theta}+1} \|w_M\|_\infty - \|f\|_{Lip} \quad \text{in } B_{1/8}. \tag{2.19}$$

Next, we split the analysis into two cases. First, we assume

$$c_1 \eta^{\frac{1}{\theta}+1} \|w_M\|_\infty \geq 2\|f\|_{Lip}.$$

By (2.19), we have

$$w_e \leq \left(1 - \frac{c_1 \eta^{\frac{1}{\theta}+1}}{2} \right) \|w_M\|_\infty \quad \text{in } \; B_{1/8},$$

and hence,

$$\|w_M\|_{L^\infty(B_{1/16})} \leq \left(1 - \overline{c}\eta^{\frac{1}{\theta}+1}\right) \|w_M\|_\infty.$$

Next, we assume that

$$c_1\eta^{\frac{1}{\theta}+1}\|w_M\|_\infty < 2\|f\|_{Lip}.$$

From this,

$$\|w_M\|_{L^\infty(B_{1/16})} \leq \|w_M\|_\infty \leq \left(\overline{c}\eta^{\frac{1}{\theta}+1}\right)^{-1} \|f\|_{\mathrm{Lip}}.$$

The proof is complete. □

Iterating the previous Lemma in dyadic balls we obtain the following:

Proposition 2.4. *Assume u is an effective viscosity solution of (0.4) and let μ, η be positive constants. For some integer $k > 0$, we assume that the following holds*

$$\sup_{e\in\partial B_1} \left|\{x \in B_{2^{-(2i+1)}} \mid w_e \geq (1-\eta)\|w_M\|_{L^\infty\left(B_{2^{-2i}}\right)}\}\right| \leq (1-\eta)\left|B_{2^{-(2i+1)}}\right|$$

for all $i = 1, \cdots, k$. Then, there exists constants $\overline{C} > 0$ and $\alpha \in (0,1)$, depending only on μ, η, $\|f\|_{\mathrm{Lip}}$, n, λ, Λ, q such that

$$\|w_M\|_{L^\infty\left(B_{2^{-2i}}\right)} \leq \max\left(\|Du\|_{L^\infty(B_{1/4})}, \overline{C}\right) 2^{-2(i-1)\alpha},$$

for all $i = 1, \ldots, k+1$.

Proof. We argue by induction. Case $i = 1$ is obvious. We assume that Proposition 2.4 holds for $i = k$. Let $r = 2^{-2(k-1)}$ and consider

$$u_r(x) := u(rx)/r.$$

Easily one notes that w solves (0.4), for $F_r(M) = rF(r^{-1}M)$ and $f_r(x) = rf(rx)$. In addition,

$$\begin{aligned}&\left|\left\{x \in B_{2^{-2k-1}} \mid w_e \geq (1-\eta)\|w_M\|_{L^\infty\left(B_{2^{-2k}}\right)}\right\}\right| \\ &= 2^{-2(k-1)n}\left|\left\{x \in B_{1/8} \mid (\partial_e u_r(x) - (\kappa+\mu))_+ \geq (1-\eta)d\right\}\right|,\end{aligned}$$

where

$$d = \|(|Du_r| - (\kappa+\mu))_+\|_{L^\infty(B_{1/4})} = \|w_M\|_{L^\infty\left(B_{2^{-2k}}\right)}.$$

Hence, u_r is under assumptions of Lemma 2.7. Therefore,

$$\begin{aligned}&\|(|Du_r| - (\kappa+\mu))_+\|_{L^\infty\left(B_{1/16}\right)} \leq \\ &\max\left\{(1-\overline{c}\,\eta^{\frac{1}{\theta}+1})\|(|Du_r| - (\kappa+\mu))_+\|_{L^\infty(B_{1/4})}, \left(\overline{c}\,\eta^{\frac{1}{\theta}+1}\right)^{-1} r\|f\|_{\mathrm{Lip}}\right\}.\end{aligned}$$

From this,

$$\begin{aligned}&\|w_M\|_{L^\infty\left(B_{2^{-2(k+1)}}\right)} \leq \\ &\max\left\{(1-\overline{c}\,\eta^{\frac{1}{\theta}+1})\|w_M\|_{L^\infty\left(B_{2^{-2k}}\right)}, \left(\overline{c}\,\eta^{\frac{1}{\theta}+1}\right)^{-1} 2^{-2(k-1)}\|f\|_{\mathrm{Lip}}\right\}.\end{aligned} \tag{2.20}$$

In what follows, we choose

$$\alpha := \frac{-\ln\left(1-\overline{c}\eta^{\frac{1}{\theta}+1}\right)}{2\ln 2}.$$

Hence $1-\eta^{\frac{1}{\theta}+1}\overline{c} = 2^{-2\alpha}$. Utilizing the result for $i=k$, we obtain

$$\|w_M\|_{L^\infty\left(B_{2^{-2(k+1)}}\right)} \leq \max\left(\|Du\|_{L^\infty(B_{1/4})}, \overline{C}\right) 2^{-2k\alpha},$$

which completes the proof. □

2.3.2 Regularity estimates far from the free boundary

In what follows, for a given $\mu \in (0,1)$, we denote

$$\vartheta_\mu := \frac{\mu^{1+q}}{2C},$$

where $C>0$ is the universal constant given by Lemma 2.5.

Lemma 2.8. *Let u be a solution of (2.11), under the conditions*

$$\vartheta \in (0, \vartheta_\mu), \quad \textit{and} \quad |\xi| \geq \kappa + 2\mu.$$

Given $\varepsilon > 0$, there exists $\varsigma > 0$ depending on ε, μ and q such that, if

$$\max\left(\|u\|_\infty, \varsigma^{-1}\|f\|_\infty\right) \leq 1,$$

then, there exists a κ-grad viscosity solution to

$$\mathcal{F}(D^2h) = 0 \quad in \ \{|Dh| > \kappa\} \cap B_{1/2}, \tag{2.21}$$

with $\mathcal{F}$ satisfying (1.1), such that

$$\|u-h\|_{L^\infty(B_{1/2})} \leq \varepsilon.$$

Proof. Let us assume, seeking a contradiction, that the thesis of Lemma fails. That is, for some $\varepsilon_0 > 0$, there exists a sequence

$$(u_k, \vartheta_k, \xi_k, \varsigma_k, f_k, F_k)_{k\in\mathbb{N}},$$

where u_k is a normalized solution of (2.11), according to Definition 1.2, with the corresponding parameters given above and

$$\varsigma_k = \mathrm{o}(1),$$

as $k \longrightarrow \infty$; however,

$$|u_k - h| > \varepsilon_0 \quad \text{in } B_{1/2}, \tag{2.22}$$

for all h satisfying (2.21). From Lemma 2.5, we have

$$\|Du_k\|_{L^\infty(B_{1/2})} \leq C\mu^{-q}.$$

From this, and the fact that $\vartheta_k \leq \vartheta_\mu$ and $|\xi_k| \geq \kappa + 2\mu$, one has

$$|F_k(D^2 u_k)| \leq \|f_k\|_\infty (|\xi_k + \vartheta_k D u_k| - \kappa)_+^{-q} \leq \varsigma_k \mu^{-q}.$$

Now both F_k and u_k are uniformly bounded and equicontinuous, hence, up to a subsequence, $F_k \to F_\infty$ and $u_k \to u_\infty$ locally uniformly. By stability u_∞ solves

$$F_\infty(D^2 u_\infty) = 0 \quad \text{in} \;\; B_{1/2},$$

in the κ-grad viscosity sense. This leads to a contradiction on (2.22) for $k \gg 1$ large enough. □

The previous Lemma gives proximity to functions that are κ-grad viscosity solutions, and thus only entitled to local Lipschitz regularity. Next, we show that those functions are actually close to $C^{1,\alpha}$ functions.

Lemma 2.9. *Given $\varepsilon > 0$, there exists small positive parameters κ and ς, depending on n, λ, Λ and ϵ such that if*

$$\max\left(\|u\|_\infty, \varsigma^{-1}\|f\|_\infty\right) \leq 1,$$

and u is a κ-grad viscosity solution to

$$(|Du| - \kappa)_+^q F(D^2 u) = f,$$

then, there exists $h \in C^{1,\alpha}$ with universal bounds satisfying

$$\sup_{B_{1/2}} |u - h| < \epsilon$$

Proof. Assume, seeking a contradiction, that the Lemma thesis does not hold true. This means there are sequences $u_k, F_k, f_k, \varsigma_k, \kappa_k$ with κ_k and ς_k converging to zero, such that u_k is a κ_k-grad viscosity solution to

$$(|Du_k| - \kappa_k)_+^q F_k(D^2 u_k) = f_k,$$

but

$$\sup_{B_{1/2}} |u_k - h| > \epsilon_0,$$

for some $\epsilon_0 > 0$ and every h in the set of $C^{1,\alpha}$ functions (with universal bounds to be set a posteriori).

Since $\varsigma_k \to 0$, we have $f_k \to 0$. As $\|u_k\|_\infty \leq 1$, Theorem 2.2 yields equicontinuity, and thus, up to a subsequence, we can assume $u_k \to u_\infty$. Passing a further subsequence, if necessary, $F_k \to F_\infty$, and, by stability, u_∞ is a 0-grad viscosity solution to

$$|Du_\infty|^q F_\infty(D^2 u_\infty) = 0.$$

Notice that since the equation is homogeneous, u_∞ solves

$$|Du_\infty|^q F_\infty(D^2 u_\infty) = 0,$$

and by [47, Lemma 6], there holds

$$F_\infty(D^2u_\infty) = 0$$

in the classical viscosity sense. The contradiction follows as in the proof of Lemma 2.8. □

Next, we use iteration arguments to obtain the following result.

Proposition 2.5. *Let u be a κ-grad viscosity solution of (0.4). There exists constants $\rho_0, \gamma \in (0,1)$ depending on n, λ, Λ, and small positive constants ς_0, τ_0 depending only on μ, n, λ, Λ and q, such that, if*

$$\|f\|_\infty \leq \varsigma_0, \quad \text{and} \quad |u(x) - (\xi \cdot x + b)| \leq \tau_0 \quad \text{in } B_1,$$

for some $\xi \in \mathbb{R}^n$, such that

$$\kappa + 3\mu \leq |\xi|,$$

then, for each positive integer k, there exists an affine function

$$\ell_k = \xi_k \cdot x + b_k,$$

such that

$$|\xi_k - \xi_{k-1}| \leq C\tau_0\rho_0^{(k-1)\gamma}, \quad |b_k - b_{k-1}| \leq C\tau_0\rho_0^{(k-1)(1+\gamma)}$$

and

$$|u - \ell_k| \leq \tau_0\rho_0^{(k-1)(1+\gamma)} \quad \text{in } B_{\rho_0^{k-1}},$$

for some $C \geq 1$ depending on n, λ, Λ.

Proof. We argue inductively. Case $k = 1$ follows from the assumptions, taking $\xi_0 = \xi_1 = \xi$ and $b_0 = b_1 = b$. Assume that the thesis of the Proposition holds for $k = j$. Define the following function

$$u_j(y) := \frac{(u - \ell_j)(\rho_0^{j-1}y)}{\tau_0\rho_0^{(j-1)(1+\gamma)}} \quad \text{in } B_1.$$

Note that u_j solves

$$(|\xi_j + \tau_0\rho_0^{(j-1)\gamma}Du_j| - \kappa)_+^q F_j(D^2u_j) = f_j \quad \text{in } B_1,$$

where

$$F_j(M) = \tau_0^{-1}\rho_0^{(j-1)(1-\gamma)}F(\tau_0\rho_0^{(j-1)(\gamma-1)}M) \text{ and } f_j(x) = \rho_0^{(j-1)(1-\gamma)}\tau_0^{-1}f(\rho_0^{j-1}x).$$

From the induction thesis, $k = j$, we have $\|u_j\|_\infty \leq 1$. In the sequel, we make the following choice

$$\tau_0 \leq \min\left\{\frac{1}{4C}\mu, \vartheta_\mu\right\}. \tag{2.23}$$

In addition, take ς_0 sufficiently small, such that

$$\|f_j\|_\infty \leq \tau_0^{-1}\varsigma_0 = \varsigma,$$

where ς is given by Lemma 2.8, for $\varepsilon = \rho_0^{1+\gamma}/2$. Additionally, from (2.23)

$$\sum_{i=1}^{j} |\xi_i - \xi_{i-1}| \le C\tau_0 \sum_{i=1}^{j} \rho_0^{(i-1)\gamma} \le C\tau_0 \sum_{i=1}^{\infty} \frac{1}{2^i} \le \frac{1}{4}\mu,$$

provided $\rho_0^{\gamma} \le 1/2$. This implies that

$$|\xi_j| \ge |\xi| - \sum_{i=1}^{j} |\xi_i - \xi_{i-1}| \ge \kappa + 2\mu.$$

In view of these estimates, we can apply Lemma 2.8 for u_j in combination with Lemma 2.9, as to obtain the existence of a (λ, Λ)-harmonic function h, such that

$$\|u_j - h\|_{L^\infty(B_{1/2})} \le \frac{\rho_0^{1+\gamma}}{2}.$$

Since h is universally bounded, we apply classical regularity estimates, to obtain

$$|h(x) - Dh(0)\cdot x - h(0)| \le C'|x|^{1+\alpha'} \quad \text{for } x \in B_{1/4},$$

for constants C' and α' depending upon n, λ and Λ. Therefore, selecting

$$\gamma = \alpha'/2 \quad \text{and} \quad \rho_0 \le \min\left\{ \left(\frac{1}{2}\right)^{\frac{1}{\gamma}}, \left(\frac{2}{C'}\right)^{\frac{1}{\gamma}} \right\},$$

we obtain

$$|h(x) - Dh(0)\cdot x - h(0)| \le \rho_0^{1+\gamma}/2, \quad \text{for } x \in B_{\rho_0}.$$

By the triangle inequality,

$$|u_j(x) - Dh(0)\cdot x - h(0)| \le \rho_0^{1+\gamma}, \quad \text{for } x \in B_{\rho_0}.$$

Finally, we define

$$\ell_{j+1}(x) := \ell_j(x) - \tau_0 \rho_0^{(j-1)(1+\gamma)} \ell(\rho_0^{-(j-1)} x)$$

where $\ell(x) = Dh(0)\cdot x + h(0)$. Hence,

$$|u - \ell_{j+1}| \le \tau_0 \rho_0^{j(1+\gamma)} \quad \text{in } B_{\rho_0^j},$$

which completes the proof. □

Corollary 2.1. *Under the assumptions of Proposition 2.5, there exists a constant C depending only on n, λ and Λ, such that*

$$|Du(x) - Du(0)| \le \tau_0 C|x|^{\gamma},$$

for each $x \in B_{1/2}$.

Proof. Recall that

$$\rho_0^k|\xi_{k+1}-\xi_k|+|b_{k+1}-b_k|\le 2C\tau_0\rho_0^{k(1+\gamma)}, \tag{2.24}$$

implies that sequences ξ_k and b_k converge. Labeling,

$$\lim_{k\to\infty}\xi_k=\xi_\infty, \quad \text{and} \quad \lim_{k\to\infty}b_k=b_\infty,$$

from (2.24), we obtain

$$|\xi_\infty-\xi_k|\le\frac{C\tau_0}{1-\rho_0}\rho_0^{k\gamma} \quad \text{and} \quad |b_\infty-b_k|\le\frac{C\tau_0}{1-\rho_0}\rho_0^{k(1+\gamma)}.$$

Next, given $r<1$, consider integer $k>0$ such that $\rho_0^{k+1}\le r\le\rho_0^k$. Hence, denoting

$$\ell_\infty(x):=\xi_\infty\cdot x+b_\infty,$$

we apply Proposition 2.5, obtaining so

$$|u(x)-\ell_\infty(x)|\le|u(x)-\ell_k(x)|+|\ell_k(x)-\ell_\infty(x)|\le\tau_0\frac{1}{\rho_0^{1+\gamma}}\left(1+\frac{2C}{1-\rho_0}\right)r^{1+\gamma},$$

for each $x\in B_{\rho_0^k}$. This implies that

$$\sup_{x\in B_r}|u-\ell_\infty|(x)\le\tau_0\overline{C}r^{1+\gamma}.$$

and some constant $\overline{C}=\overline{C}(n,\lambda,\Lambda)$. Therefore,

$$|u(x)-\ell_\infty(x)|\le\tau_0\overline{C}|x|^{1+\gamma},$$

for $|x|<1$. Notice that if we make $x=0$ we get $b_\infty=u(0)$. Furthermore, for $s<1$ we get

$$\left|\frac{u(se_i)-u(0)}{s}-\xi_\infty\cdot\vec{e}_i\right|\le\tau_0\overline{C}s^\gamma.$$

where e_i is a n-dimensional canonical vector. Passing to the limit when $s\to 0$ we obtain that $\xi_\infty\cdot e_i=\partial_{e_i}u(0)$ for every $i=1,\cdots,n$, and so $\xi_\infty=Du(0)$. Therefore,

$$|u(x)-u(0)-Du(0)\cdot x|\le\tau_0\overline{C}|x|^{1+\gamma},$$

for $|x|<1$. In particular,

$$|Du(x)-Du(0)|\le\tau_0\overline{C}|x|^\gamma,$$

for $x\in B_{1/2}$. □

Proof of Proposition 2.3

First, for $r \leq 1/2$ and $x_0 \in B_{1/4}$, we define

$$u_r(x) := \frac{1}{r}u(x_0 + rx),$$

for $x \in B_1$. Note that we have

$$\|Du_r\|_{L^\infty(B_{1/2})} \leq \|Du\|_{L^\infty(B_{1/2})} \leq C, \tag{2.25}$$

where the last estimate is due to Theorem 2.2, for some C depending on dimension, ellipticity and $\|f\|_\infty$. Additionally, we observe that u_r solves

$$(|Du_r| - \kappa)_+^q F_r(D^2u_r) = f_r \quad \text{in } B_1$$

for $F_r(M) = rF(r^{-1}M)$ and $f_r(x) = rf(x_0 + rx)$. Next, consider

$$r := \frac{1}{1 + \|f\|_\infty}\varsigma_0$$

for ς_0 as in Proposition 2.5. In the sequel, let

$$w_e := (Du_r \cdot e - (\kappa + \mu))_+ \quad \text{and} \quad w_M := (|Du_r| - (\kappa + \mu))_+.$$

Let $\eta \in (0, 1)$ to be chosen later. Define $i_\star \in \mathbb{N}$ to be the smallest parameter i such that

$$\sup_{e \in \partial B_1} \left|\left\{x \in B_{2^{-(2i+1)}} \,\middle|\, w_e \geq (1 - \eta)\|w_M\|_{L^\infty\left(B_{2^{-2i}}\right)}\right\}\right| \geq (1 - \eta)|B_{2^{-(2i+1)}}|.$$

If $i_\star = \infty$, Proposition 2.3 follows directly from Proposition 2.4. If, on the other hand, $i_\star < +\infty$, for constants $\overline{C} > 0$ and $\alpha \in (0, 1)$, there holds

$$\|w_M\|_{L^\infty\left(B_{2^{-2i}}\right)} \leq \overline{C}2^{-2(i-1)\alpha}, \tag{2.26}$$

for all $i = 1, 2, \cdots, i_\star$. Thus, we can estimate

$$\|w_M\|_{L^\infty\left(B_{2^{-2i}}\right)} \leq \|w_M\|_{L^\infty\left(B_{2^{-2i_\star}}\right)} \leq \overline{C}2^{-2(i_\star-1)\alpha} \leq 4\overline{C}2^{-i\alpha}$$

for $i = i_\star + 1, \cdots, 2i_\star$. From the definition of $i_\star$, there exists at least one direction $e \in \partial B_1$ for which

$$\left|\left\{x \in B_{2^{-2i_\star-1}} \,\middle|\, w_e \geq (1 - \eta)\|w_M\|_{L^\infty\left(B_{2^{-2i_\star}}\right)}\right\}\right| \geq (1 - \eta)|B_{2^{-2i_\star-1}}|.$$

Therefore, for

$$\overline{v}(x) := 2^{2i_\star+1}(u_r(2^{-2i_\star-1}x) - u_r(0)) \quad \text{for } x \in B_1,$$

we have that

$$|\{x \in B_1 \,|\, (\partial_e\overline{v}(x) - (\kappa + \mu))_+ \geq (1 - \eta)d_\star\}| \geq (1 - \eta)|B_1|,$$

where $d_\star := \|(\partial_e \overline{v}(x) - (\kappa+\mu))_+\|_{L^\infty(B_1)}$. Additionally, by (2.25) there holds

$$|D\overline{v}| \le \kappa + \mu + d_\star \le C \quad \text{and} \quad \overline{v}(0) = 0.$$

Considering $\epsilon = \tau_0/C$ and applying [29, Lemma 4.1], we choose η (depending only on the choice of ϵ) to find (ξ, b), such that

$$|\xi| = \kappa + \mu + d_\star \quad \text{and} \quad |\overline{v}(x) - b - \xi \cdot x| \le \epsilon(\kappa + \mu + d_\star) \le \tau_0.$$

We can now apply Proposition 2.5 to obtain

$$|D\overline{v}(x) - D\overline{v}(0)| \le C_1 |x|^\gamma$$

for $x \in B_{1/2}$. Recall that for $x \in B_{2^{-2i_\star - 2}}$ we have

$$|w_M(x) - w_M(0)| \le |D\overline{v}\left(2^{2i_\star+1}x\right) - D\overline{v}(0)|,$$

and thus

$$|w_M(x) - w_M(0)| \le C_1 2^{-i\gamma},$$

for each $x \in B_{2^{-2i}}$ and $i \ge 2i_\star + 1$.

We are ready to conclude the proof. Setting

$$C' = 8\max\{\overline{C}, C_1\} \quad \text{and} \quad \overline{\alpha} = \frac{1}{2}\min\{\alpha, \gamma\},$$

we conclude

$$\|(|Du(x)| - (\kappa+\mu))_+ - (|Du(x_0)| - (\kappa+\mu))_+\|_{L^\infty\left(B_{r2^{-2i}}(x_0)\right)} \le C' 2^{-2i\overline{\alpha}},$$

for every $i \in \mathbb{N}$. Given $x \in B_r(x_0)$, we take integer $j > 0$, such

$$r2^{-2(j+1)} \le |x - x_0| \le r2^{-2j}.$$

This implies that

$$2^{-2j\overline{\alpha}} \le \left(\frac{4|x - x_0|}{r}\right)^{\overline{\alpha}}.$$

We then obtain

$$|(|Du(x)| - (\kappa+\mu))_+ - (|Du(x_0)| - (\kappa+\mu))_+| \le C''|x - x_0|^{\overline{\alpha}},$$

for $x \in B_r(x_0)$, and constant $C'' > 0$ depends upon μ, q, n, λ, Λ and $\|f\|_\infty$. For $x \in B_{1/2} \setminus B_r(x_0)$, we estimate

$$\begin{aligned} |(|Du(x)| - (\kappa+\mu))_+ - (|Du(x_0)| - (\kappa+\mu))_+| &\le 2\|Dv\|_{L^\infty(B_1)} \\ &\le C\,|x - x_0|^{\overline{\alpha}} \end{aligned}$$

where C is another constant that depends only on μ, q, n, λ, Λ and $\|f\|_\infty$. Since x_0 was taken arbitrarily, the proof of Proposition 2.3 is finally complete.

Concluding the proof of Theorem 2.3

Recall that u is an effective viscosity solution of

$$(|Du| - \kappa)_+^q F(D^2 u) = f.$$

By Theorem 2.2, we have $\|Du\|_\infty \leq C$, for a positive constant

$$C = C(n, \lambda, \Lambda, \kappa, \|u\|_\infty, \|f\|_\infty).$$

By Proposition 2.3, given any $\mu > 0$, there exist constants $C_\mu > 0$ and $\alpha_\mu \in (0, 1)$ depending upon μ and universal data, such that:

$$\|(|Du| - (\kappa + \mu))_+\|_{C^{0,\alpha_\mu}(B_{1/4})} \leq C_\mu.$$

To ease notation define

$$v_\mu(x) = (|Du(x)| - (\kappa + \mu))_+ \quad \text{and} \quad v(x) = (|Du| - \kappa)_+$$

By triangle inequality we can estimate:

$$\begin{aligned} |v(x) - v(y)| &\leq |v_\mu(x) - v(x)| + |v_\mu(y) - v(y)| + |v_\mu(x) - v_\mu(y)| \\ &\leq 2\mu + C_\mu |x - y|^{\alpha(\mu)}, \end{aligned}$$

for every $\mu \in (0, 1)$. Since such an estimate holds for all $\mu > 0$, we obtain

$$|v(x) - v(y)| \leq \sigma(|x - y|),$$

where

$$\sigma(t) := \min_{\mu \in (0,1)} \{2\mu + C_\mu t^{\alpha(\mu)}\}.$$

It is easy to see that σ, as defined above, is indeed a modulus of continuity and that Du is σ-continuous within the region $\{|Du| \geq \kappa\}$.

2.4 Connection to other trends of research

In this section, we will bring a comprehensive discussion on how our very general problem intersects with other trends of research.

2.4.1 Unconstrained free boundary problems

Initially we revisit the theory of unconstrained free boundary problems, as in the work of Figalli and Shahgholian, [43].

Let Ω be an open set of $\mathbb{R}^n$ and $w \in W^{2,p}(B_1)$ be a viscosity solution of

$$\begin{cases} F(D^2w) = 1 & \text{in } B_1 \cap \Omega \\ |D^2w| \le K & \text{in } B_1 \backslash \Omega, \end{cases}$$

where F is convex and uniformly elliptic. The main result proven in [43] is a sharp $C^{1,1}$ regularity of solutions. It is worth comparing such an improved estimate with the results of [64], where $C^{1,\text{log-Lip}}$ regularity is proven for $F(D^2u) = f \in L^\infty$; see also [21] for related results.

Theorem 2.2 can also be viewed as an unconstrained free boundary problem; the first-order counterpart of [43]. More precisely, solutions of

$$\begin{cases} F(D^2w) = 1 & \text{in } B_1 \cap \Omega \\ |Dw| \le K & \text{in } B_1 \backslash \Omega, \end{cases}$$

are K-grad viscosity solutions in the sense investigated in this paper. In particular, in the case of linear equations, say $F = \Delta$, Theorem 2.2 applied to w_e implies the sharp $C^{1,1}$-regularity of unconstrained free boundary problems at the hessian level. Furthermore, Theorem 2.3, applied to w_e, yields to the existence of a modulus of continuity σ such that $D^2w \in C^{0,\sigma}(\{|D^2w| \ge K\} \cap B_{1/2})$.

2.4.2 Flame propagation with an obstacle

Singularly perturbed PDEs of the flame propagation type have received warm attention since the pioneering work [11], see for instance [6, 15, 23, 26, 52–54, 56, 63] and references therein. For free boundary problems driven by operators in non-divergence form, introducing a heavy penalization term, $\beta_\epsilon(u)$, allows for an existence theory, as long as one can obtain strong enough estimates that are uniform concerning the regularizing parameter ϵ, see for instance [7, 31, 49, 60].

Typically, β_ϵ is an approximation of the Dirac delta function, δ_0, in L^1. One can think of

$$\beta_\epsilon(s) := \frac{1}{\epsilon}\beta\left(\frac{s}{\epsilon}\right),$$

where β is a fixed, smooth function with bounded support. The main goal is to obtain uniform-in-ϵ regularity estimates for u and its free boundary.

Here we are interested in a new type of flame propagation model, which carries activation fronts. Mathematically this gives raise to a free boundary problem of the singularly perturbed type for which the jump discontinuity happens along the coincidence set $\Lambda_\epsilon := \{u_\epsilon = \varphi\}$, for a given obstacle function φ.

The starting point of this program is to prove that solutions are uniformly-in-ϵ Lipschitz continuous, provided the obstacle, φ, is Lipschitz. This is the result we discuss here as the final application of the methods introduced in this paper.

Hereafter u_ϵ denotes a viscosity solution of the PDE

$$F(D^2u_\epsilon) = \beta_\epsilon(u_\epsilon - \varphi), \tag{2.27}$$

with $u_\epsilon \geq \varphi$ and $\varphi \in C^{0,1}$.

Existence of minimal solutions

We dedicate this section to discuss the existence of minimal solutions to

$$F(D^2u) = \beta_\epsilon(u - \varphi),$$

with $\varphi \in C^{0,1}$. From [60], it is clear that the diffusion operator F could be more general. We will keep it as it is to ease the presentation.

We also point out that the Classical Perron method cannot be applied directly due to the lack of monotonicity in the variable u.

Theorem 2.4. *Let $g \in L^\infty(\mathbb{R}) \cap C^{0,1}(\mathbb{R})$. Assume F is a (λ, Λ)-elliptic operator and that $F(D^2u) = g(u - \varphi)$ admits a Lipschitz viscosity subsolution u_* and a Lipschitz viscosity supersolution u^* such that $u_* = u^* = \Psi \in C^{1,\gamma}(\partial\Omega)$. Define the set of functions,*

$$\mathcal{F} := \left\{ w \in C(\overline{\Omega}) \;\middle|\; u_* \leq w \leq u^* \quad \text{and } w \text{ is a supersolution to} \quad F(D^2u) = g(u - \varphi) \right\}.$$

Then,

$$v(x) := \inf_{w \in \mathcal{F}} w(x)$$

is a continuous viscosity solution to $F(D^2u) = g(u - \varphi)$ and $u = \Psi$ continuously on $\partial\Omega$.

Proof. The proof is a small adaptation of [60, Theorem 2.1]. By Lipschitz continuity of g, let $\theta > 0$ be such that $|Dg| < \theta/2$. Define $h(z) = \theta z - g(z)$. Given a Lipschitz function f, define

$$G_f[u] := F(D^2u) - \theta(u - \varphi) + f.$$

Now, G_f is a uniformly elliptic operator and strictly monotone in the variable u. By Classical Perron's method, one obtains the existence of solutions to

$$\begin{cases} G_f[u] = 0 & \text{in } \Omega \\ u = \Psi & \text{in } \partial\Omega. \end{cases}$$

By classical regularity theory, $u \in C^{1,\gamma}(\overline{\Omega})$ with a estimate depending on the parameters. We will now iterate this argument. Set $u_0 = u_*$ and let u_{k+1} be the solution to

$$\begin{cases} G_{f_k}[u] = 0 & \text{in } \Omega \\ u = \Psi & \text{in } \partial\Omega, \end{cases}$$

where $f_k(x) = h(u_k(x) - \varphi(x))$. Observe that

$$\begin{aligned}
G_{f_0}[u] &:= F(D^2u) - \theta(u-\varphi) + h(u_0 - \varphi) = F(D^2u) - h(u-\varphi) + h(u_0-\varphi) - g(u-\varphi),\\
G_{f_k}[u] &:= F(D^2u) - \theta(u-\varphi) + h(u_k - \varphi) = G_{f_{k-1}}[u] + h(u_k-\varphi) - h(u_{k-1}-\varphi),\\
G^*[u] &:= F(D^2u) - \theta(u-\varphi) + h(u^* - \varphi) = F(D^2u) - h(u-\varphi) + h(u^*-\varphi) - g(u-\varphi),\\
G^*[u] &:= F(D^2u) - \theta(u-\varphi) + h(u^* - \varphi) = G_{f_k}[u] + h(u^*-\varphi) - h(u_k-\varphi).
\end{aligned}$$

The proof is done once we show the following claim.

Claim: The sequence $\{u_k\}_{k\in\mathbb{N}}$ is increasing in k and satisfies $u_* \le u_k \le u^*$ in Ω for every $k \in \mathbb{N}$. Indeed, assume the claim is true. Recall that u_k is an uniformly bounded sequence that satisfies

$$|F(D^2u_k)| = |\theta(u_k - \varphi) + f_{k-1}| \le C(\theta, \varphi, \|u_*\|_\infty, \|u^*\|_\infty),$$

and thus are bounded in $C^{0,1}$ with a universal estimate that does not depend on k. By Arzelá-Ascoli theorem, it converges through a subsequence, and so

$$G_{f_k}[u] \to G[u] := F(D^2u) - \theta(u-\varphi) + h(u-\varphi) \quad \text{as} \quad k \to \infty.$$

By the claim, we can define the pointwise limit $v(x) = \lim_{k\to\infty} u_k(x)$, and we get that v solves

$$F(D^2u) = g(u-\varphi),$$

in the viscosity sense. Comparison principle allow us to show that v is actually the least supersolution, that is

$$v(x) = \inf_{w\in\mathcal{F}} w(x).$$

Now let us get back to the claim. We will show, by induction, that $u_k \le u_{k+1}$ for every $k \in \mathbb{N}$. By definition of G_{f_0} and our assumptions, it holds

$$G_{f_0}[u_1] = 0 \le G_{f_0}[u_0]$$

in the viscosity sense. Comparison principle thus implies that $u_0 \le u_1$ and the case $k = 0$ is done. Assuming it holds up to k, that is $u_{k-1} \le u_k$. Since $h' \ge 0$, we have $h(u_{k-1} - \varphi) \le h(u_k - \varphi)$, and so

$$\begin{aligned}
G_{f_k}[u_k] &= G_{f_{k-1}}[u_k] + h(u_k-\varphi) - h(u_{k-1}-\varphi)\\
&\ge G_{f_{k-1}}[u_k] = 0.
\end{aligned}$$

Hence, $G_{f_k}[u_{k+1}] = 0 \le G_{f_k}[u_k]$ and by the comparison principle once more, $u_k \le u_{k+1}$. The same could have been done to show $u_k \le u^*$ for every $k \in \mathbb{N}$ and the Theorem is proved. □

The purpose of the previous theorem is to ensure that the problem

$$\begin{cases} F(D^2u) = \beta_\epsilon(u-\varphi) & \text{in } \Omega \\ u = \Psi & \text{in } \partial\Omega, \end{cases}$$

has a solution for every $\epsilon > 0$. When $\varphi = 0$, an application of the ABP estimate, see [60, Lemma 2.3], leads to $u \geq 0$, so it would be natural to expect that our solutions satisfy $u \geq \varphi$. It turns out that this is not a straightforward task as we need to force the obstacle φ to satisfy differential inequalities. Nevertheless, if we can assure the existence of functions $u_*, u^* \geq \varphi$ such that

$$\begin{cases} F(D^2u_*) = c_\epsilon & \text{in } \Omega \\ u_* = \Psi & \text{in } \partial\Omega, \end{cases} \qquad \begin{cases} F(D^2u^*) = 0 & \text{in } \Omega \\ u^* = \Psi & \text{in } \partial\Omega, \end{cases}$$

then it follows, by classical regularity estimates, $u^*, u_* \in C^{0,1}$, and thus an application of Theorem 2.4 leads to the desired.

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and $\Psi \in C^{1,\gamma}(\partial\Omega)$. Then, for each $\epsilon > 0$, the problem*

$$\begin{cases} F(D^2u) = \beta_\epsilon(u-\varphi) & \textit{in } \Omega \\ u = \Psi & \textit{in } \partial\Omega, \end{cases}$$

admits a minimal solution $u_\epsilon \in C(\overline{\Omega})$.

Proof. Let $c_\varphi = \|\varphi\|_\infty$ and $c_\epsilon = \sup \beta_\epsilon$. By classical Perron's method, consider u_1 and u_2 to be the solutions to

$$\begin{cases} F(D^2u_1) = c_\epsilon & \text{in } \Omega \\ u_1 = \Psi - c_\varphi & \text{in } \partial\Omega, \end{cases} \qquad \begin{cases} F(D^2u_2) = 0 & \text{in } \Omega \\ u_2 = \Psi - c_\varphi & \text{in } \partial\Omega. \end{cases}$$

Assuming $\Psi - c_\varphi \geq 0$, applying the classical ABP estimate we get that both u_1 and u_2 are nonnegative. Now consider $u^* = u_1 + c_\varphi$ and $u_* = u_2 + c_\varphi$. They solve

$$\begin{cases} F(D^2u^*) = c_\epsilon & \text{in } \Omega \\ u^* = \Psi & \text{in } \partial\Omega, \end{cases} \qquad \begin{cases} F(D^2u_*) = 0 & \text{in } \Omega \\ u_* = \Psi & \text{in } \partial\Omega, \end{cases}$$

and, by the comparison principle, $u^* \geq u_*$. Furthermore,

$$u_* = u_2 + c_\varphi \geq c_\varphi \geq \varphi.$$

By classical regularity estimates, both u_* and u^* are Lipschitz, and by Theorem 2.4 we can find a minimal solution u_ϵ to our problem between u_* and u^*. □

We remark that the minimality of the solutions is not used in the proof of Lipschitz estimates. This will be important when trying to prove nondegeneracy estimates.

Lipschitz estimates

The most important step, to analyze the limiting problem as $\epsilon \to 0$, is to obtain Lipschitz estimates independent of the parameter ϵ. It is clear that such family u_ϵ is in $C^{1,\alpha}$, but the estimate is not uniform in the parameter ϵ. As pointed out in [60], uniform-in-ϵ Lipschitz estimates are sharp.

Theorem 2.6. *Given $\Omega' \Subset \Omega$, there exists a constant C' such that any bounded family $\{u_\epsilon\}_{\epsilon>0}$ of solutions of* (2.27) *satisfies*

$$\|Du_\epsilon\|_{L^\infty(\Omega')} \leq C'\left(n, \lambda, \Lambda, \beta, [\varphi]_{C^{0,1}}, \Omega'\right).$$

Proof. The key feature of the model is its distinct behavior within the regions

$$\Omega_1 := \Omega' \cap \{u_\epsilon - \varphi \leq \epsilon\} \quad \text{and} \quad \Omega_2 := \Omega' \cap \{u_\epsilon - \varphi > \epsilon\}.$$

By means of a standard covering argument, we can restrict the analysis to the case $\Omega = B_1$ and $\Omega' = B_{1/2}$.

Case I: Let $x_0 \in \Omega_1$ be fixed. We will prove the existence of a constant $C_1' > 0$ that does not depend on $\epsilon > 0$ such that

$$|Du_\epsilon(x_0)| \leq C_1'\left(n, \lambda, \Lambda, \beta, [\varphi]_{C^{0,1}}\right).$$

For that, define the auxiliary function, $v\colon B_2 \to \mathbb{R}$, as:

$$v(z) := \epsilon^{-1}[u_\epsilon(x_0 + \epsilon z) - u_\epsilon(x_0)].$$

Direct calculations show that v solves

$$F_\epsilon(D^2 v) = \beta(v - \tilde{\varphi}),$$

in B_2, where $\tilde{\varphi}(z) := \epsilon^{-1}(\varphi(x_0 + \epsilon z) - u_\epsilon(x_0))$ and $F_\epsilon(M) := \epsilon F(\epsilon^{-1} M)$. Note that the equation for v is uniformly elliptic and therefore Lipschitz estimates are available. In particular, we can estimate

$$|Du_\epsilon(x_0)| = |Dv(0)| \leq C\|v\|_{L^\infty(B_{3/2})},$$

for a constant $C > 0$ depending only on n, λ, Λ and $\|\beta\|_\infty$.

Now we turn to get uniform (in the parameter ϵ) estimates for $\|v\|_{L^\infty(B_{3/2})}$. Recall that since $u_\epsilon \geq \varphi$ and $u_\epsilon(x_0) - \varphi(x_0) \leq \epsilon$, we get for $z \in B_2$,

$$
\begin{aligned}
v(z) &= \epsilon^{-1}[u_\epsilon(x_0+\epsilon z) - u_\epsilon(x_0)] \\
&\geq \epsilon^{-1}[\varphi(x_0+\epsilon z) - \varphi(x_0) - \epsilon] \\
&\geq -\epsilon^{-1}|\varphi(x_0+\epsilon z) - \varphi(x_0)| - 1 \\
&\geq -[\varphi]_{C^{0,1}}|z| - 1 \\
&\geq -2\left([\varphi]_{C^{0,1}} + 1\right) = -\kappa
\end{aligned}
$$

Harnack inequality applied to the non-negative function $w := v + \kappa \geq 0$ yields

$$
\begin{aligned}
\sup_{B_{3/2}} w &\leq C\left(w(0) + \|\beta\|_{L^\infty(\mathbb{R})}\right) \\
&\leq C\left(\kappa + \|\beta\|_{L^\infty(\mathbb{R})}\right).
\end{aligned}
$$

Combining all such estimates we finally end up with

$$
\|Du_\epsilon\|_{L^\infty(\Omega_1)} \leq K.
$$

for K depending on n, λ, Λ, $\|\beta\|_\infty$ and $[\varphi]_{C^{0,1}}$.

Case II: The estimate for $x_0 \in \Omega_2$.

We simply note that, because of the estimate obtained in Case I, u_ϵ satisfies

$$
F(D^2u_\epsilon) = 0 \quad \text{in } \{|Du_\epsilon| > K\}.
$$

Theorem 2.2 then gives the desired local Lipschitz estimate, independently of the parameter $\epsilon > 0$. □

It is worth pointing out that after we get the estimate as in **Case I**, then the same barrier construction as in [60] would work. This proof, however, is way shorter than theirs. The idea behind it is, somehow, reminiscent of the doubling variables technique presented in this thesis. Once it is known that the estimates hold at some region, it can be used to push the points of maxima away from this region.

2.4.3 PDE models with infinite degree of degeneracy

Next, we would like to discuss connections with limiting free boundary problems, obtaining when the degree of degeneracy tends to infinity. More precisely, let us look at the non-variational q-Laplacian equation:

$$
|Du|^q F(D^2u) = f \quad \text{in} \quad B_1 \tag{2.28}
$$

This model has received warm attention in the last two decades, see for instance [5,13,32,37,38,47] and references therein.

The type of results we are interested in the section are uniform-in-q regularity results. The first result of this type in this thesis is described in Section 2.1. More precisely, if $\kappa = 0$, then Theorem 2.1 says that solutions of (2.28) satisfies

$$\|u\|_{L^\infty(B_1)} \leq \sup_{\partial B_1} |u| + C\|f\|_{L^n}^{\frac{1}{q+1}},$$

for a constant

$$C = o\left((1+q)^{\frac{1}{(1+q)n}}\right).$$

If, say, u has a fixed bounded boundary data g (to successfully assure existence) and taking into account that

$$\lim_{q\to\infty} (1+q)^{\frac{1}{(1+q)n}} = 1,$$

then we obtain that solutions to (2.28) are uniform bounded in the parameter $q \in [0, +\infty]$.

It is important to mention that this estimate, with precisely the same dependence as ours, was obtained in [10].

Uniform-in-q Lipschitz estimates

The prime goal we want to achieve is to analyze the limiting problem as $q \to \infty$, and to do so, we need compactness. First and foremost, solutions are bounded, depending only on the boundary data and RHS, so we may assume they are normalized.

An important Corollary of the analysis carried out in Section 2.2.1 is the following sharp regularity estimate:

Corollary 2.2. *Let $q \geq 0$, $f \in L^\infty(B_1)$, and u_q be a normalized viscosity solution of*

$$|Du_q|^q F(D^2u_q) = f \text{ in } B_1.$$

Then, there exists a constant C, depending only on dimension, ellipticity, and $\|f\|_{L^\infty(B_1)}$, but independent of q, such that

$$\|Du_q\|_{L^\infty(B_{1/2})} \leq C.$$

An equivalent way, by applying directly our results, is to observe that

$$F(D^2u_q) = \overline{f} \quad \text{for} \quad \overline{f} = f|Du_q|^{-q}$$

in the viscosity sense, and $|\overline{f}|_\infty \leq |f|_\infty$ within $\{|Du_q| > 1\}$ (or, in the 1-grad sense), and thus entitled to Theorem 2.2. Therefore, solutions are bounded in $C^{0,1}(\overline{B}_{1/2})$. It is worth pointing out that to get to the limiting problem, C^0 compactness is enough, This can be achieved by the same argument with the regularity results from [48].

Uniform-in-q Gradient regularity

It is well known that solutions to the nonvariational q-Laplacian are in C^{1,α_q}, for some $\alpha_q \leq \frac{1}{1+q}$. The counterpart of every regularity estimate for this problem, however, is that the estimate degenerates as $q \to \infty$. Here, we provide a regularity result that allows us to pass to the limit when $q \to \infty$.

Theorem 2.7. *Let $q \geq 0$, $f \in \mathrm{Lip}(\overline{B}_1)$, and u_q be a normalized viscosity solution of*

$$|Du_q|^q F(D^2 u_q) = f \text{ in } B_1.$$

Then, given $0 < \mu < 1$, there exists constants $0 < \alpha_\mu < 1$ and $C_\mu > 0$ depending on data, μ but independent of q such that

$$\|(|Du_q| - (1+\mu))\|_{C^{0,\alpha_\mu}(B_{1/2})} \leq C_\mu.$$

The rationale to prove this result was already presented in Section 2.3.2. As a courtesy to the reader, explain the main differences and the adapted proofs whenever it is necessary. A careful analysis is needed to ensure everything is uniform in the degeneracy parameter q.

The first, and important, step is the (uniform-in-q) Lipschitz estimates, as Lipschitz rescalings will be used in the proof. As before, a compactness estimate for scaled PDEs will be necessary and we shall use the following version of Lemma 2.5.

Lemma 2.10. *Assume u is normalized and satisfies*

$$|\xi + \vartheta Du|^q F(D^2 u) = f \quad \text{with} \quad \|f\|_\infty \leq 1.$$

Given $\mu \in (0,1)$, there exists a constant C depending only on n, λ and Λ, such that if

$$|\xi| \geq 1 + 2\mu \quad \text{and} \quad \vartheta \leq \frac{\mu}{2C}, \tag{2.29}$$

then

$$\sup_{x,y \in B_{1/2}} \frac{|u(x) - u(y)|}{|x-y|} \leq C.$$

Proof. Consider ϕ as defined in (2.7) and

$$M := \sup_{x,y \in \overline{B}_{1/2}} \left(u(x) - u(y) - L\phi(|x-y|) - K\left(|x|^2 - |y|^2\right)\right).$$

Let $(\overline{x}, \overline{y})$ be the pair where M is attained and assume $M > 0$. First, we localize the points where M is attained by choosing K large enough.

We can apply Lemma 2.4 (with $\kappa = 0$) with $\varphi(x,y) := L\phi(|x-y|) + K(|x|^2 + |y|^2)$ as to reach

$$L \leq n\frac{\Lambda}{\lambda}K + \frac{1}{4\lambda}\left(|\vartheta D_x\varphi(\overline{x},\overline{y}) + \xi|^{-q} + \left|\vartheta D_y\varphi(\overline{x},\overline{y}) - \xi\right|^{-q}\right). \tag{2.30}$$

From (2.29) and the estimate

$$\max\{|D_x\varphi(\overline{x},\overline{y})|, |D_y\varphi(\overline{x},\overline{y})|\} \leq 2L,$$

there holds

$$\min\{|\xi + \vartheta D_x\varphi(\overline{x},\overline{y})|, |\xi - \vartheta D_y\varphi(\overline{x},\overline{y})|\} \geq 1+\mu.$$

Therefore, from estimate (2.30), we can further estimate

$$L < n\frac{\Lambda}{\lambda}K + \frac{1}{2\lambda}(1+\mu)^{-q} \leq \overline{C},$$

for $\overline{C} = C(n,\lambda,\Lambda)$. The conclusion is that if $L \geq \overline{C}$, then $M \leq 0$, which is equivalent to the thesis of the Lemma. □

It is worthwhile to mention, although not important to the subsequent analysis, that the previous Lipschitz estimates do not depend upon the parameter $\mu \in (0,1)$. The one obtained in Lemma 2.5, however, not only depended on μ, but was blowing up as $q \to \infty$.

Following the program developed before, we need to prove some sort of regularity estimates near and far from the free boundary. It is important to observe that, at this point, there is no free boundary as in (0.4). A preliminary analysis unveils that the free boundary, in the limiting problem, will be as if $\kappa = 1$ in (0.4). That explains, in some sense, why we are proving Theorem 2.7 that way.

The first step of the dichotomy to be obtained further is to prove a version of Proposition 2.4, which reflects the estimates near the free boundary. We recall that, given a vector $e \in \partial B_1$, we define

$$w_e := (\partial_e u - (1+\mu))_+ \quad \text{and} \quad w_M := (|Du| - (1+\mu))_+.$$

Proposition 2.6. *Assume u solves*

$$|Du|^q F(D^2u) = f \quad in \quad B_1,$$

and let μ, η be positive constants. For some integer $k > 0$, we assume that the following holds

$$\sup_{e\in\partial B_1}\left|\{x \in B_{2^{-(2i+1)}} \mid w_e \geq (1-\eta)\|w_M\|_{L^\infty(B_{2^{-2i}})}\}\right| \leq (1-\eta)\left|B_{2^{-(2i+1)}}\right|$$

for all $i = 1,\cdots,k$. Then, there exists constants $\overline{C} > 0$ and $\alpha \in (0,1)$, depending only on μ, η, $\|f\|_{\text{Lip}}$, n, λ, Λ, such that

$$\|w_M\|_{L^\infty(B_{2^{-2i}})} \leq \max\left(\|Du\|_{L^\infty(B_{1/4})}, \overline{C}\right) 2^{-2(i-1)\alpha},$$

for all $i = 1,\ldots,k+1$.

The proof of this proposition is exactly as the proof of Proposition 2.4 with $\kappa = 1$. The only difference is the equation we apply weak-harnack to, which comes from a version of Lemma 2.6, which is when we make $\kappa = 0$ and change μ by $1 + \mu$. We remark that it will be independent of the parameter q because

$$\frac{q}{(1+\mu)^{q+1}} \to 0 \quad \text{as} \quad q \to \infty.$$

Now we proceed to obtain regularity estimates far from the free boundary. In what follows, for a given $\mu \in (0, 1)$, we denote

$$\vartheta_\mu := \frac{\mu}{2C},$$

where $C > 0$ is the universal constant given by Lemma 2.10.

Lemma 2.11. *Let u be a solution of*

$$|\xi + \vartheta Du|^q F(D^2 u) = f, \tag{2.31}$$

under the conditions

$$\vartheta \in (0, \vartheta_\mu), \quad \text{and} \quad |\xi| \geq 1 + 2\mu.$$

Given $\varepsilon > 0$, there exists $\varsigma > 0$ depending on ε, μ such that, if

$$\max\left(\|u\|_\infty, \varsigma^{-1}\|f\|_\infty\right) \leq 1,$$

then, there exists a viscosity solution to

$$\mathcal{F}(D^2 h) = 0 \quad \text{in} \ B_{1/2}, \tag{2.32}$$

with $\mathcal{F}$ satisfying (1.1), such that

$$\|u - h\|_{L^\infty(B_{1/2})} \leq \varepsilon.$$

Proof. The proof is essentially the same as the proof of Lemma 2.8, except that the parameter q needs to vary. Let us assume, seeking a contradiction, that the thesis of Lemma fails. That is, for some $\varepsilon_0 > 0$, there exists a sequence

$$(\vartheta_k, \varsigma_k, q_k, \xi_k, u_k, f_k, F_k)_{k\in\mathbb{N}},$$

where u_k is a normalized solution of (2.31), with the corresponding parameters given above and

$$\varsigma_k = \mathrm{o}(1),$$

as $k \longrightarrow \infty$; however,

$$|u_k - h| > \varepsilon_0 \quad \text{in} \ B_{1/2}, \tag{2.33}$$

for all h satisfying (2.32). From Lemma 2.10, we have

$$\|Du_k\|_{L^\infty(B_{1/2})} \leq C.$$

From this, and the fact that $\vartheta_k \le \vartheta_\mu$ and $|\xi_k| \ge 1 + 2\mu$, one has

$$|F_k(D^2u_k)| \le \|f_k\|_\infty\,|\xi_k + \vartheta_k Du_k|^{-q_k} \le \varsigma_k(1+\mu)^{-q_k}.$$

Now both F_k and u_k are uniformly bounded and equicontinuous, hence, up to a subsequence, $F_k \to F_\infty$ and $u_k \to u_\infty$ locally uniformly. By stability u_∞ solves

$$F_\infty(D^2u_\infty) = 0 \quad \text{in } \; B_{1/2},$$

in the viscosity sense. This leads to a contradiction on (2.33) for $k \gg 1$ large enough. □

As before, by employing iteration arguments, we have the following result.

Proposition 2.7. *Let u be a viscosity solution of*

$$|Du|^q F(D^2u) = f \quad in \quad B_1.$$

There exists constants $\rho_0, \gamma \in (0,1)$ depending on n, λ, Λ, and small positive constants ς_0, τ_0 depending only on μ, n, λ, Λ, such that, if

$$\|f\|_\infty \le \varsigma_0, \quad and \quad |u(x) - (\xi\cdot x + b)| \le \tau_0 \quad in \;\; B_1,$$

for some $\xi \in \mathbb{R}^n$, such that

$$1 + 3\mu \le |\xi|,$$

then, for each positive integer k, there exists an affine function

$$\ell_k = \xi_k \cdot x + b_k,$$

such that

$$|\xi_k - \xi_{k-1}| \le C\tau_0\rho_0^{(k-1)\gamma}, \quad |b_k - b_{k-1}| \le C\tau_0\rho_0^{(k-1)(1+\gamma)}$$

and

$$|u - \ell_k| \le \tau_0\rho_0^{(k-1)(1+\gamma)} \quad in \;\; B_{\rho_0^{k-1}},$$

for some $C \ge 1$ depending on n, λ, Λ.

The proof is equal to the one we did before, with the obvious modifications. As consequence, Corollary 2.1 also holds. Notice that they are independent of the parameter q due to the approximation lemma. Moreover, as a consequence of the results independent of the parameter q, we can prove Theorem 2.7 by the very same proof of Proposition 2.3 as if $\kappa = 0$ and exchanging μ by $1 + \mu$.

Limiting equation

Now, let us briefly explain how we draw the limiting problem and how the regularity estimates independent of q are useful. Consider $\{u_q\}_{q>0}$ to be a family of normalized viscosity solutions to (2.28). First, let us assume $f \geq c > 0$, for some constant c. By Corollary 2.2, up to a subsequence, it holds that $u_q \to u_\infty$, for some Lipschitz function u_∞. We will show that u_∞ solves,

$$\min\Big(|Du_\infty| - 1,\ F(D^2u_\infty)\Big) = 0$$

in the viscosity sense. Indeed, let $\varphi \in C^2$ touching u_∞ from above at x_0. By uniform convergence, a vertical translation of φ touches u_q from above at x_q, and so

$$|D\varphi(x_q)|^q F(D^2\varphi(x_q)) \geq f(x_q) \geq 0.$$

Observe that

$$F(D^2\varphi(x_q)) \geq 0$$

for every q, and so, by continuity,

$$F(D^2\varphi(x_0)) \geq 0.$$

Moreover, we must have $|D\varphi(x_0)| \geq 1$. Indeed, if $|D\varphi(x_0)| < 1$, then by continuity we would have $|D\varphi(x_q)| < 1$ for large q. Since,

$$|D\varphi(x_q)|^q F(D^2\varphi(x_q)) \geq f(x_q) \geq c > 0,$$

passing to the limit as $q \to \infty$, we obtain

$$0 \geq c,$$

a contradiction. This proves that u_∞ is a subsolution. Let us now prove the supersolution side. Let $\varphi \in C^2$ touching u_∞ from below at x_0. We need to show that

$$\min\Big(|Du_\infty(x_0)| - 1, F(D^2\varphi(x_0))\Big) \leq 0.$$

Since it is a minimum, we may assume $|D\varphi(x_0)| > 1$, otherwise the inequality would be automatically true. By continuity, $|D\varphi(x_q)| > 1$ for large q and so,

$$|D\varphi(x_q)|^q F(D^2\varphi(x_q)) \leq f(x_q) \leq \|f\|_\infty,$$

and so, passing to the limit as $q \to \infty$, it holds that

$$F(D^2\varphi(x_0)) \leq 0.$$

When the RHS has no sign assumption, the only information we can obtain for the limiting problem is that u_∞ satisfies:

$$F(D^2u_\infty) = 0, \qquad \text{in } \{|Du_\infty| > 1\},$$

that is, u_∞ is a 1-grad F-harmonic function. Due to our Lipschitz regularity results, $u_\infty \in C^{0,1}$.

As a further consequence, by considering another subsequence if necessary, from Theorem 2.7, that the limiting solution u_∞ has a continuous gradient up to the free boundary. Moreover, the set $\{|Du_\infty| > 1\}$ in an open set and u_∞ is an effective viscosity solution.

3

On free boundary problems shaped by oscillatory singularities

In this chapter, we delve into the geometric characteristics of minimizers associated with the functional (1.5). Our exploration embarks on a thorough investigation into the existence of minimizers under the broadest possible conditions, alongside an analysis of non-degeneracy and gradient estimates within this environment. We dedicate the first four sections of this chapter to meticulously unfold these implications and the program will develop as further structure on the oscillatory singularities are in force.

Commencing with an environment where the oscillatory singularities satisfies a weak Dini continuity assumption, we achieve precise gradient regularity and establish positive density estimates for the free boundary in Section 3.5. These findings lay the groundwork for advancing our understanding of the functional's behavior under finely tuned conditions.

Transitioning to Section 3.6, we tackle the ambitious goal of deriving Hausdorff measure estimates, a task demanding exceptional regularity in the underlying parameters due to the necessity of differentiating the equation twice. This section underscores the intricate balance required in handling an array of parameters, highlighting the complexity and depth of our analytical approach.

Our journey reaches a pivotal moment in Section 3.7, where, under the assumption that the oscillatory parameters exhibit a Sobolev regularity, we introduce a novel Weiss-type monotonicity formula. This innovative analytical tool not only enables the classification of blow-ups but also sets the stage for addressing the classification of minimal cones in lower dimensions. Leveraging Federer's reduction argument, we culminate our analysis in Section 3.8 by establishing the regularity of the free boundary.

This chapter represents a comprehensive endeavor to dissect and understand the intricate geometric properties of minimizers, navigating through the complexities of non-degeneracy, gradient estimates, and the free boundary's geometric and regularity properties. Through meticulous analysis and innovative methodologies, we contribute significantly to the ongoing dialogue in the field, paving the way for further research and exploration.

3.1 Existence of minimizers

We start by proving the existence of non-negative minimizers of functional (1.5) and deriving global L^∞-bounds.

Proposition 3.1. *Under the conditions above, namely* (1.4)*, there exists a minimizer $u \in \mathcal{A}$ of the energy-functional* (1.5)*. Furthermore, u is non-negative in Ω and $\|u\|_{L^\infty(\Omega)} \leq \|\varphi\|_{L^\infty(\Omega)}$.*

Proof. Let

$$m = \inf_{v \in \mathcal{A}} \mathcal{J}_\gamma^\delta(v, \Omega)$$

and choose a minimizing sequence $u_k \in \mathcal{A}$ such that, as $k \to \infty$,

$$\mathcal{J}_\gamma^\delta(u_k, \Omega) \longrightarrow m.$$

Then, for $k \gg 1$, we have

$$\begin{aligned} \|Du_k\|^2_{L^2(\Omega)} &= 2\mathcal{J}_\gamma^\delta(u_k, \Omega) - 2\int_\Omega \delta(x)(u_k^+)^{\gamma(x)} dx \\ &\leq 2(m+1) + 2\|\delta\|_{L^\infty(\Omega)} \left(|\Omega| + \|u_k\|_{L^1(\Omega)}\right) \\ &\leq 2(m+1) + 2\|\delta\|_{L^\infty(\Omega)} \left(|\Omega| + \sqrt{|\Omega|}\,\|u_k\|_{L^2(\Omega)}\right). \end{aligned}$$

From Poincaré inequality, we also have

$$\begin{aligned} \|u_k\|_{L^2(\Omega)} &\leq \|u_k - \varphi\|_{L^2(\Omega)} + \|\varphi\|_{L^2(\Omega)} \\ &\leq C\|Du_k - D\varphi\|_{L^2(\Omega)} + \|\varphi\|_{L^2(\Omega)} \\ &\leq C\|Du_k\|_{L^2(\Omega)} + C\|D\varphi\|_{L^2(\Omega)} + \|\varphi\|_{L^2(\Omega)}, \end{aligned}$$

and so

$$\|u_k\|_{L^2(\Omega)} \leq C^2(4\epsilon)^{-1} + \epsilon\|Du_k\|^2_{L^2(\Omega)} + C\|D\varphi\|_{L^2(\Omega)} + \|\varphi\|_{L^2(\Omega)}, \tag{3.1}$$

with $\epsilon > 0$ to be chosen. We thus obtain

$$\|Du_k\|^2_{L^2(\Omega)} \leq C_1 + 2\epsilon\|\delta\|_{L^\infty(\Omega)} \sqrt{|\Omega|}\,\|Du_k\|^2_{L^2(\Omega)},$$

with

$$C_1 = C_1\left(m, \|\delta\|_{L^\infty(\Omega)}, |\Omega|, C, \epsilon, \|\varphi\|_{H^1(\Omega)}\right).$$

Choosing

$$\epsilon = \frac{1}{4\|\delta\|_{L^\infty(\Omega)}\sqrt{|\Omega|}},$$

we conclude

$$\|Du_k\|^2_{L^2(\Omega)} \leq 2C_1$$

and thus, using again (3.1), that $\{u_k\}_k$ is bounded in $H^1(\Omega)$. Consequently, for a subsequence (relabeled for convenience) and a function $u \in H^1(\Omega)$, we have

$$u_k \longrightarrow u,$$

weakly in $H^1(\Omega)$, strongly in $L^2(\Omega)$ and pointwise for a.e. $x \in \Omega$. Using Mazur's theorem, it is standard to conclude that $u \in \mathcal{A}$.

The weak lower semi-continuity of the norm gives

$$\int_\Omega \frac{1}{2}|Du|^2\,dx \leq \liminf_{k\to\infty} \int_\Omega \frac{1}{2}|Du_k|^2\,dx$$

and the pointwise convergence and Lebesgue's dominated convergence give

$$\int_\Omega \delta(x)(u_k^+)^{\gamma(x)}dx \longrightarrow \int_\Omega \delta(x)(u^+)^{\gamma(x)}dx.$$

We conclude that

$$\mathcal{J}_\gamma^\delta(u,\Omega) \leq \liminf_{k\to\infty} \mathcal{J}_\gamma^\delta(u_k,\Omega) = m$$

and so u is a minimizer.

We now turn to the bounds on the minimizer. That u is non-negative for a non-negative boundary datum is trivial since $(u^+)^+ = u^+$, and testing the functional against $u^+ \in \mathcal{A}$ immediately gives the result. For the upper bound, test the functional with $v = \min\left\{u, \|\varphi\|_{L^\infty(\Omega)}\right\} \in \mathcal{A}$ to get, by the minimality of u,

$$\begin{aligned} 0 \leq \int_\Omega |D(u-v)|^2\,dx &= \int_{\Omega\cap\{u>\|\varphi\|_{L^\infty(\Omega)}\}} |Du|^2\,dx \\ &= \int_\Omega |Du|^2 - |Dv|^2\,dx \\ &\leq 2\int_\Omega \delta(x)\left[(v^+)^{\gamma(x)} - (u^+)^{\gamma(x)}\right]dx \\ &\leq 0. \end{aligned}$$

We conclude that $v = u$ in Ω and thus $\|u\|_{L^\infty(\Omega)} \leq \|\varphi\|_{L^\infty(\Omega)}$. □

Remark 3.1. *If the boundary datum φ changes sign, the existence theorem above still applies, but the minimizer is no longer non-negative. Uniqueness may, in general, fail, even in the case of $\gamma \equiv \gamma_0 < 1$.*

3.2 Local $C^{1,\alpha}$–regularity estimates

Our first main regularity result yields local $C^{1,\alpha}$–regularity estimates for minimizers of the energy-functional (1.5), under no further assumption on $\gamma(x)$ other than (1.4).

Theorem 3.1. *Let u be a minimizer of the energy-functional (1.5) under assumption (1.4). For each subdomain $\Omega' \Subset \Omega$, there exists a constant $C > 0$, depending only on n, $\|\delta\|_\infty$, $\gamma_\star(\Omega')$, $\operatorname{dist}(\Omega', \partial\Omega)$ and $\|u\|_\infty$, such that*

$$\|u\|_{C^{1,\alpha}(\Omega')} \le C,$$

for $\alpha = \dfrac{\gamma_\star(\Omega')}{2 - \gamma_\star(\Omega')}$.

For the proof of Theorem 3.1, we will argue along the lines of [45, 55], but several adjustments are needed, and we will mainly comment on those. We start by noting that, without loss of generality, one can assume that the minimizer satisfies the bound

$$\|u\|_{L^\infty(\Omega)} \le 1. \tag{3.2}$$

Indeed, u minimizes (1.5) if, and only if, the auxiliary function

$$\overline{u}(x) := \frac{u(x)}{M},$$

minimizes the functional

$$v \mapsto \int_\Omega \frac{1}{2}|Dv|^2 + \overline{\delta}(x)(v^+)^{\gamma(x)}\, dx,$$

where

$$\overline{\delta}(x) := M^{\gamma(x)-2}\delta(x).$$

Taking $M = \max\{1, \|u\|_{L^\infty(\Omega)}\}$, places the new function $\overline{u}$ under condition (3.2); any regularity estimate proven for $\overline{u}$ automatically translates to u.

Next, we gather some useful estimates, which can be found in [55, Lemma 2.4 and Lemma 4.1, respectively]. We adjust the statements of the lemmata to fit the setup treated here. Given a ball $B_R(x_0) \Subset \Omega$, we denote the harmonic replacement (or lifting) of u in $B_R(x_0)$ by h, *i.e.*, h is the solution of the boundary value problem

$$\Delta h = 0 \;\text{ in }\; B_R(x_0) \qquad \text{and} \qquad h - u \in H^1_0(B_R(x_0)).$$

By the maximum principle, we have $h \ge 0$ and

$$\|h\|_{L^\infty(B_R(x_0))} \le \|u\|_{L^\infty(B_R(x_0))}. \tag{3.3}$$

Lemma 3.1. *Let $\psi \in H^1(B_R)$ and h be the harmonic replacement of ψ in B_R. There exists c, depending only on n, such that*

$$c\int_{B_R} |D\psi - Dh|^2\, dx \le \int_{B_R} |D\psi|^2 - |Dh|^2\, dx. \tag{3.4}$$

Lemma 3.2. *Let $\psi \in H^1(B_R)$ and h be the harmonic replacement of ψ in B_R. Given $\beta \in (0,1)$, there exists C, depending only on n and β, such that*

$$\int_{B_r} |D\psi - (D\psi)_r|^2 \, dx \le C\left(\frac{r}{R}\right)^{n+2\beta} \int_{B_R} |D\psi - (D\psi)_R|^2 \, dx + C\int_{B_R} |D\psi - Dh|^2 \, dx,$$

for each $0 < r \le R$.

We are ready to prove the local regularity result.

Proof of Theorem 3.1. We prove the result for the case of balls $B_R(x_0) \Subset \Omega$. Without loss of generality, assume $x_0 = 0$ and denote $B_R := B_R(0)$. Since u is a local minimizer, by testing (1.5) against its harmonic replacement, we obtain the inequality

$$\int_{B_R} |Du|^2 - |Dh|^2 \, dx \le 2\int_{B_R} \delta(x)\left(h(x)^{\gamma(x)} - u(x)^{\gamma(x)}\right) dx. \tag{3.5}$$

Next, with the aid of [55, Lemma 2.5], one obtains

$$h(x)^{\gamma(x)} - u(x)^{\gamma(x)} \le |u(x) - h(x)|^{\gamma(x)},$$

and, using (1.4), together with (3.2) and (3.3), we get

$$|u(x) - h(x)|^{\gamma(x)} \le |u(x) - h(x)|^{\gamma_\star(0,R)}, \quad \text{a.e. in } B_R. \tag{3.6}$$

This readily leads to

$$\int_{B_R} \delta(x)\left(h(x)^{\gamma(x)} - u(x)^{\gamma(x)}\right) dx \le \|\delta\|_{L^\infty(\Omega)} \int_{B_R} |u(x) - h(x)|^{\gamma_\star(0,R)} \, dx.$$

In addition, by combining Hölder and Sobolev inequalities, we obtain

$$\begin{aligned}\int_{B_R} |u-h|^{\gamma_\star(0,R)} \, dx &\le C|B_R|^{1-\frac{\gamma_\star(0,R)}{2^*}} \left(\int_{B_R} |u-h|^{2^*} \, dx\right)^{\frac{\gamma_\star(0,R)}{2^*}} \\ &\le C|B_R|^{1-\frac{\gamma_\star(0,R)}{2^*}} \left(\int_{B_R} |Du-Dh|^{2} \, dx\right)^{\frac{\gamma_\star(0,R)}{2}}\end{aligned} \tag{3.7}$$

for $2^* = \dfrac{2n}{n-2}$.

Therefore, using Lemma 3.1, together with (3.5), (3.6) and (3.7), we get

$$\int_{B_R} |Du - Dh|^2 \, dx \le C|B_R|^{\frac{2(2^*-\gamma_\star(0,R))}{2^*(2-\gamma_\star(0,R))}} = CR^{n+2\frac{\gamma_\star(0,R)}{2-\gamma_\star(0,R)}}. \tag{3.8}$$

Finally, by taking

$$\beta = \frac{\gamma_\star(0,R)}{2-\gamma_\star(0,R)} \in (0,1),$$

in Lemma 3.2, we conclude

$$\int_{B_r} |Du - (Du)_r|^2\, dx$$

$$\leq C\left(\frac{r}{R}\right)^{n+2\frac{\gamma_\star(0,R)}{2-\gamma_\star(0,R)}} \int_{B_R} |Du-(Du)_R|^2\, dx + CR^{n+2\frac{\gamma_\star(0,R)}{2-\gamma_\star(0,R)}},$$

for each $0 < r \leq R$. Campanato's embedding theorem completes the proof. □

Hereafter, in this paper, we assume $\Omega = B_1 \subset \mathbb{R}^n$ and, according to what was argued around (3.2), fix a normalized, non-negative minimizer, $0 \leq u \leq 1$, of the energy-functional (1.5).

Remark 3.2. *It is worth noting that the proof of Theorem 3.1 does not rely on the non-negativity property of u. Therefore, the same conclusion applies to the two-phase problem, and the proof remains unchanged.*

3.3 Non-degeneracy

We now turn our attention to local non-degeneracy estimates. We will assume $\delta(x)$ is bounded below away from zero, namely that it satisfies the condition

$$\operatorname*{ess\,inf}_{x\in B_1} \delta(x) =: \delta_0 > 0. \tag{3.9}$$

Theorem 3.2. *Assume (3.9) is in force. For any $y \in \overline{\{u>0\}}$ and $0 < r \ll 1$, we have*

$$\sup_{\partial B_r(y)} u \geq c\, r^{\frac{2}{2-\gamma^\star(y,r)}}, \tag{3.10}$$

where $c > 0$ depends only on n, δ_0 and $\gamma_\star(0,1)$.

Proof. With $y \in \{u > 0\}$ and $0 < r \ll 1$ fixed, define the auxiliary function φ by

$$\varphi(x) := u(x)^{2-\gamma^\star(y,r)} - c|x-y|^2,$$

for $c > 0$ to be chosen later. Note that in $\{u > 0\} \cap B_r(y)$, we have

$$\begin{aligned}
\Delta\varphi &= (2-\gamma^\star(y,r))\Big((1-\gamma^\star(y,r))u^{-\gamma^\star(y,r)}|Du|^2 + u^{1-\gamma^\star(y,r)}\Delta u\Big) - 2nc \\
&= (2-\gamma^\star(y,r))\Big((1-\gamma^\star(y,r))u^{-\gamma^\star(y,r)}|Du|^2 + \delta(x)\gamma(x)u^{\gamma(x)-\gamma^\star(y,r)}\Big) \\
&\quad -2nc \\
&\geq (2-\gamma^\star(y,r))\delta(x)\gamma(x)u^{\gamma(x)-\gamma^\star(y,r)} - 2nc.
\end{aligned}$$

Hence, choosing $c > 0$ small enough such that

$$0 < c \leq \min\left\{1, \frac{\delta_0 \gamma_\star(0,1)}{2n}\right\},$$

we obtain $\Delta\varphi \geq 0$ in $\{u > 0\} \cap B_r(y)$. In addition, since $\varphi(y) > 0$, by the Maximum Principle,

$$\partial\left(\{u > 0\} \cap B_r(y)\right) \cap \{\varphi > 0\} \neq \emptyset.$$

Consequently, since $\frac{1}{2-\gamma^\star(y,r)} \leq 1$

$$\sup_{\partial B_r(y)} u > c^{\frac{1}{2-\gamma^\star(y,r)}}\, r^{\frac{2}{2-\gamma^\star(y,r)}} \geq c\, r^{\frac{2}{2-\gamma^\star(y,r)}},$$

and the proof is complete for $y \in \{u > 0\}$; the general case follows by continuity. □

3.4 Gradient estimates near the free boundary

In this section, we study gradient oscillation estimates for minimizers of (1.5) in regions relatively close to the free boundary. We first show that pointwise flatness implies an L^∞−estimate.

Lemma 3.3. *Let u be a local minimizer of the energy-functional* (1.5) *in B_1. Assume that*

$$\gamma_\star(0,1) > 0.$$

There exists a constant $C > 1$, depending only on $\gamma_\star(0,1)$ and universal parameters, such that, if

$$u(x) \leq \frac{1}{C} r^{\frac{2}{2-\gamma_\star(x,r)}}, \tag{3.11}$$

for $x \in B_{1/2}$ and $r \leq 1/4$, then

$$\sup_{B_r(x)} u \leq C r^{\frac{2}{2-\gamma_\star(x,r)}}.$$

Proof. We suppose the thesis of the lemma fails. Then, for each integer $k > 0$, there exist a minimizer u_k of (1.5) in B_1, $x_k \in B_{1/2}$ and $0 < r_k < 1/4$, such that

$$u_k(x_k) \leq \frac{1}{k} r_k^{\frac{2}{2-\gamma_k}},$$

but

$$k\, r_k^{\frac{2}{2-\gamma_k}} < \sup_{B_{r_k}(x_k)} u_k =: s_k \leq 1,$$

where $\gamma_k := \gamma_\star(x_k, r_k)$. Note that from the last two estimates,

$$u_k(x_k) \leq \frac{1}{k} r_k^{\frac{2}{2-\gamma_k}} < \frac{1}{k^2} s_k,$$

and

$$\frac{r_k^{\frac{2}{2-\gamma_k}}}{s_k} < \frac{1}{k}. \tag{3.12}$$

In the sequel, define

$$\varphi_k(x) := \frac{u_k(x_k + r_k x)}{s_k} \quad \text{in } B_1.$$

Hence,

$$\sup_{B_1} \varphi_k = 1, \quad \text{and} \quad \varphi_k(0) < \frac{1}{k^2}. \tag{3.13}$$

In addition, note that φ_k minimizers

$$v \longmapsto \int_{B_1} \frac{1}{2}|Dv|^2 + \delta_k(x) v^{\gamma_k(x)} dx,$$

for

$$\delta_k(x) := \delta(x_k + r_k x)\frac{r_k^2}{s_k^{2-\gamma(x_k+r_kx)}} \quad \text{and} \quad \gamma_k(x) := \gamma(x_k + r_k x).$$

From (3.12), we obtain

$$s_k^{\gamma(x_k+r_kx)-2} r_k^2 \le s_k^{\gamma(x_k+r_kx)-2}\left(\frac{s_k}{k}\right)^{2-\gamma_k} = s_k^{\gamma(x_k+r_kx)-\gamma_k}\left(\frac{1}{k}\right)^{2-\gamma_k} \le \frac{1}{k},$$

for each $x \in B_1$. The last estimate is guaranteed since, for each k,

$$\gamma_k = \inf_{y \in B_{r_k}(x_k)} \gamma(y) = \inf_{x \in B_1} \gamma(x_k + r_k x) \le \gamma(x_k + r_k x).$$

Hence,

$$\|\delta_k\|_{L^\infty(B_1)} \le \|\delta\|_{L^\infty(B_1)} k^{-1}.$$

Next, we apply Theorem 3.1 for the lower bound

$$\inf_{y \in B_1} \gamma_k(y) = \inf_{y \in B_1} \gamma(x_k + r_k y) = \inf_{x \in B_{r_k}(x_k)} \gamma(x) = \gamma_\star(x_k, r_k) \ge \gamma_\star(0,1) =: \theta,$$

and observe that the sequence $\{\varphi_k\}_k$ is $C^{1,\frac{\theta}{2-\theta}}$–equicontinuous. Therefore, up to a subsequence, φ_k converges uniformly to φ_∞ in $B_{1/2}$, as $k \to \infty$. Taking into account the estimates above, we conclude that φ_∞ minimizers the functional

$$v \longmapsto \int_{B_1} \frac{1}{2}|Dv|^2 \, dx.$$

In particular, φ_∞ is harmonic in B_1, and $\varphi_\infty(0) = 0$. Therefore, by the strong maximum principle, one has $\varphi_\infty \equiv 0$ in B_1. But this contradicts

$$\sup_{B_1} \varphi_\infty = 1,$$

and the proof of the lemma is complete. □

Next, we prove a pointwise gradient estimate.

Lemma 3.4. *Let u be a local minimizer of energy-functional (1.5) in B_1. Assume γ is lower semi-continuous in Ω and that*

$$\gamma_\star(0,1) > 0.$$

There exists a small universal parameter $\tau > 0$ and a constant $\overline{C}$, depending only on $\gamma_\star(0,1)$ and universal parameters, such that if

$$0 \le u \le \tau \quad \text{in} \;\; B_1, \tag{3.14}$$

then

$$|Du(x)|^2 \le \overline{C}\,[u(x)]^{\gamma_\star(0,1)}, \tag{3.15}$$

for each $x \in B_{1/2}$.

Proof. The case $x \in \partial\{u > 0\} \cap B_{1/2}$ follows from Theorem 3.1. In fact, since solutions are locally $C^{1,\beta}$, for some $\beta > 0$, the fact that u attains at each $x \in \partial\{u > 0\}$ its minimum value implies that $|Du(x)| = 0$.

We now consider $x \in \{u > 0\} \cap B_{1/2}$ and choose

$$\tau := \frac{1}{C}\left(\frac{1}{4}\right)^{\frac{2}{2-\gamma^\star(0,1)}},$$

for C as in Lemma 3.3. Note that

$$\lim_{s\to 0^+} s^{\frac{2}{2-\gamma_\star(x,s)}} = 0,$$

for each $x \in B_{1/2}$. From this and the fact that $\gamma_\star(x,\cdot)$ is continuous, we select $r > 0$ such that

$$r^{\frac{2}{2-\gamma_\star(x,r)}} = Cu(x) \le \left(\frac{1}{4}\right)^{\frac{2}{2-\gamma^\star(0,1)}},$$

the inequality following from (3.14). This implies, in particular, that

$$r \le \left(\frac{1}{4}\right)^{\frac{2-\gamma_\star(x,r)}{2-\gamma^\star(0,1)}} \le \frac{1}{4},$$

since the exponent in the above expression is greater than 1. We can now apply Lemma 3.3 since condition (3.11) holds trivially, obtaining

$$\sup_{B_r(x)} u \le C\, r^{\frac{2}{2-\gamma_\star(x,r)}}.$$

Define

$$v(y) := u(x+ry)\, r^{-\frac{2}{2-\gamma_\star(x,r)}} \quad \text{in} \;\; B_1,$$

and observe that it satisfies the uniform bound

$$\sup_{B_1} v \le C.$$

Additionally, by the scaling properties of section 2, v is a minimizer of a scaled functional as (1.5) in B_1, and so, by Theorem 3.1,

$$|Dv(0)| \leq L,$$

for some L, depending only on $\gamma_\star(0,1)$ and universal parameters. This translates into

$$\begin{aligned} |Du(x)| &\leq Lr^{\frac{\gamma_\star(x,r)}{2-\gamma_\star(x,r)}} \\ &= L\,(Cu(x))^{\frac{\gamma_\star(x,r)}{2-\gamma_\star(x,r)}\cdot\frac{2-\gamma_\star(x,r)}{2}} \\ &\leq L\sqrt{C}\,[u(x)]^{\frac{\gamma_\star(x,r)}{2}}, \end{aligned}$$

recalling that $C > 1$. Since $\gamma_\star(x,r) \geq \gamma_\star(0,1)$ and $0 \leq u \leq 1$, the proof follows with $\overline{C} = L^2C$, which depends only on $\gamma_\star(0,1)$ and universal parameters. □

Remark 3.3. *We have proved Lemma 3.4 under the assumption that (3.14) holds. Observe, however, that the conclusion is trivial otherwise. Indeed, if $u(x) > \tau$, then by Lipschitz regularity we have*

$$|Du(x)|^2 \leq L^2 = L^2\left(\frac{\tau}{\tau}\right)^{\gamma_\star(0,1)} \leq \frac{L^2}{\tau^{\gamma_\star(0,1)}}[u(x)]^{\gamma_\star(0,1)}.$$

Remark 3.4. *It is worthwhile mentioning that the lower semi-continuity assumption on $\gamma(x)$ in Lemma 3.4 can be removed. To do so, one has to prove a weaker version of Lemma 3.3, with $2/(2-\gamma_*(0,1))$ replacing $2/(2-\gamma_*(x,r))$. The reasoning follows seamlessly.*

3.5 Weak Dini-continuous exponents and sharp estimates

The local regularity result in Theorem 3.1 yields a $(1+\alpha)$−growth control for a minimizer u near its free boundary. More precisely, if z_0 is a free boundary point then $u(z_0) = Du(z_0) = 0$. Consequently, with $r = |y - z_0|$, we have, by continuity,

$$\begin{aligned} u(y) &\leq \sup_{x\in B_r(z_0)} |u(x) - u(z_0) - Du(z_0)\cdot(x - z_0)| \\ &\leq Cr^{1+\alpha} \\ &= C|y - z_0|^{\frac{2}{2-\gamma_\star(z_0,r)}}. \end{aligned}$$

However, such an estimate is suboptimal and a key challenge is to understand how the oscillation of $\gamma(x)$ impacts the prospective (point-by-point) $C^{1,\alpha}$ regularity of minimizers along the free boundary.

In this section, we assume γ is continuous at a free boundary point z_0, with a modulus of continuity ω satisfying

$$\omega(1) + \lim_{t\to 0}\omega(t)\ln\left(\frac{1}{t}\right) \leq \tilde{C}, \tag{3.16}$$

for a constant $\tilde{C} > 0$. Such a condition often appears in models involving variable exponent PDEs as a critical (minimal) assumption for the theory; see, for instance, [1] for functionals with $p(x)$-growth and [17] for the non-variational theory.

Note that assumption (3.16) is weaker than the classical notion of Dini continuity. In fact, if (3.16) is violated then, for a constant $M > 0$ and $0 < t_0 \ll 1$, we have

$$\omega(t) \ln\left(\frac{1}{t}\right) \geq M, \quad \forall t \in (0, t_0)$$

and then

$$\int_0^1 \frac{\omega(t)}{t} dt \geq \int_0^{t_0} \frac{M}{t \ln\left(\frac{1}{t}\right)} dt = M \int_{-\ln t_0}^{+\infty} \frac{dy}{y} = +\infty,$$

so γ is not Dini continuous.

We are ready to state a sharp pointwise regularity estimate for local minimizers of (1.5) under (3.16). We define the subsets

$$\Omega(u) := \left\{x \in B_1 \,\middle|\, u(x) > 0\right\} \quad \text{and} \quad F(u) := \partial\Omega(u),$$

corresponding to the non-coincidence set and the free boundary of the problem, respectively.

Theorem 3.3. *Let u be a local minimizer of (1.5) in B_1 and $z_0 \in F(u) \cap B_{1/2}$. Assume γ satisfies (3.16) at z_0. Then, there exist universal constants $r_0 > 0$ and $C' > 1$ such that*

$$u(y) \leq C' \left|y - z_0\right|^{\frac{2}{2-\gamma(z_0)}}, \tag{3.17}$$

for all $y \in B_{r_0}(z_0)$.

Proof. Since (3.16) is in force, let $r_0 \ll 1$ be such that, for $r < r_0$,

$$\omega(r) \ln\left(\frac{1}{r}\right) \leq 2\left[\tilde{C} - \omega(1)\right] =: C^*. \tag{3.18}$$

Fix $y \in B_{r_0}(z_0)$ and let

$$r := |y - z_0| < r_0.$$

Apply Theorem 3.1 to u over $B_r(z_0)$, to get

$$\sup_{x \in B_r(z_0)} u(x) \leq C\, r^{\frac{2}{2-\gamma_\star(z_0,r)}}.$$

In particular, by continuity, it follows that

$$u(y) \leq C\, r^{\frac{2}{2-\gamma_*(z_0,r)}}. \tag{3.19}$$

In view of (3.16), we can estimate

$$\gamma(z_0) - \gamma_\star(z_0, r) \leq \omega(r),$$

and, since the function $g\colon [0,1] \to [0,1]$ given by

$$g(t) := \frac{2}{2-t}$$

satisfies $\frac{1}{2} \leq g'(t) \leq 2$, for all $t \in [0,1]$, we have

$$\begin{aligned} g\left(\gamma(z_0)\right) - g\left(\gamma_\star(z_0,r)\right) &\leq 2\left(\gamma(z_0) - \gamma_\star(z_0,r)\right) \\ &\leq 2\,\omega(r). \end{aligned}$$

Combining (3.19) with this inequality, and taking (3.18) into account, we reach

$$\begin{aligned} u(y) &\leq C\,r^{-[g(\gamma(z_0))-g(\gamma_\star(z_0,r))]}\,r^{\frac{2}{2-\gamma(z_0)}} \\ &\leq C\,r^{-2\omega(r)}\,r^{\frac{2}{2-\gamma(z_0)}} \\ &\leq C\,e^{2C^*}\,r^{\frac{2}{2-\gamma(z_0)}} \\ &= C'\,|y-z_0|^{\frac{2}{2-\gamma(z_0)}}, \end{aligned}$$

as desired. □

We also obtain a sharp strong non-degeneracy result.

Theorem 3.4. *Let u be a local minimizer of (1.5) in B_1 and $z_0 \in F(u) \cap B_{1/2}$. Assume (3.9) and that (3.16) is in force at z_0. Then, there exist universal constants $r_0 > 0$ and $c^* > 0$ such that*

$$\sup_{\partial B_r(z_0)} u \geq c^*\,r^{\frac{2}{2-\gamma(z_0)}},$$

for every $0 < r < r_0$.

Proof. As before, let $r_0 \ll 1$ be such that (3.18) holds and fix $r < r_0$. From Theorem 3.2, we know

$$\sup_{\partial B_r(z_0)} u \geq c\,r^{\frac{2}{2-\gamma^\star(z_0,r)}},$$

with $c > 0$ depending only on n, δ_0 and $\gamma_\star(0,1)$.

Now, observe that

$$\frac{2}{2-\gamma^\star(z_0,r)} = \frac{2}{2-\gamma(z_0)} + \frac{2}{2-\gamma^\star(z_0,r)} - \frac{2}{2-\gamma(z_0)}$$

and

$$\begin{aligned} \frac{2}{2-\gamma^\star(z_0,r)} - \frac{2}{2-\gamma(z_0)} &= \frac{2(\gamma^\star(z_0,r)-\gamma(z_0))}{(2-\gamma^\star(z_0,r))\,(2-\gamma(z_0))} \\ &\leq 2(\gamma^\star(z_0,r)-\gamma(z_0)) \\ &\leq 2\omega(r). \end{aligned}$$

Thus,

$$\begin{aligned} r^{\frac{2}{2-\gamma^{\star}(z_0,r)}} &\geq r^{2\omega(r)} r^{\frac{2}{2-\gamma(z_0)}} \\ &= e^{2\omega(r)\ln r}\, r^{\frac{2}{2-\gamma(z_0)}} \\ &\geq e^{-2C^*}\, r^{\frac{2}{2-\gamma(z_0)}}, \end{aligned}$$

due to (3.18), and the result follows with $c^* := c\, e^{-2C^*}$. □

With sharp regularity and non-degeneracy estimates at hand, we can now prove the positive density of the non-coincidence set.

Theorem 3.5. *Let u be a local minimizer of (1.5) in B_1 and $z_0 \in F(u) \cap B_{1/2}$. Assume (3.9) and that (3.16) is in force at z_0. There exists a constant $\mu_0 > 0$, depending on n, δ_0, $\gamma_\star(0,1)$ and the constant from (3.16), such that*

$$\frac{|B_r(z_0) \cap \Omega(u)|}{|B_r(z_0)|} \geq \mu_0,$$

for every $0 < r < r_0$. In particular, $F(u)$ is porous and there exists an $\epsilon > 0$ such that $\mathcal{H}^{n-\epsilon}(F(u) \cap B_{1/2}) = 0$.

Proof. Fix $r < r_0$, with r_0 as in Theorem 3.3. It follows from the non-degeneracy (Theorem 3.4) that there exists $y \in \partial B_r(z_0)$ such that

$$u(y) \geq c^* r^{\frac{2}{2-\gamma(z_0)}}.$$

Now, let $z \in F(u)$ be such that

$$|z-y| = \operatorname{dist}(y, F(u)) =: d.$$

Then, we have

$$c^* r^{\frac{2}{2-\gamma(z_0)}} \leq u(y) \leq \sup_{B_d(z)} u \leq C d^{\frac{2}{2-\gamma(z)}}.$$

Furthermore, observe that

$$|z-z_0| \leq |z-y| + |y-z_0| \leq d + r,$$

and so, since $d \leq r$, we have $|z - z_0| \leq 2r$. Therefore, one can proceed as in Theorem 3.3 to obtain

$$c^* r^{\frac{2}{2-\gamma(z_0)}} \leq u(y) \leq C d^{\frac{2}{2-\gamma(z_0)}}.$$

This implies that

$$r \leq \left(\frac{C}{c^*}\right)^{\frac{2-\gamma(z_0)}{2}} d \leq \max\left\{1, \frac{C}{c^*}\right\} d.$$

So for $\kappa = \min\{1, c^*/C\}$, we have

$$B_{\kappa r}(y) \subset B_d(y) \subset \Omega(u).$$

Since also $B_{\kappa r}(y) \subset B_{2r}(z_0)$, we conclude

$$|B_{2r}(z_0) \cap \Omega(u)| \geq \left(\frac{\kappa}{2}\right)^n \alpha(n)(2r)^n,$$

where $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$, and the result follows with $\mu_0 = \left(\frac{\kappa}{2}\right)^n$. □

Next we establish an optimized version of Lemma 3.4, assuming that $\gamma(x)$ satisfies condition (3.16). First, observe that if $x \in \Omega(u) \cap B_{1/2}$ is such that

$$u(x) \leq \frac{1}{C} r^{\frac{2}{2-\gamma(x)}},$$

for $r \leq 1/4$, then (3.11) also holds at x. Therefore, Lemma 3.3 applies and we also have

$$\sup_{B_r(x)} u \leq C r^{\frac{2}{2-\gamma_*(x,r)}}.$$

Condition (3.16) comes into play, and proceeding as in the proof of Theorem 3.4, for a larger constant C_1, we have

$$\sup_{B_r(x)} u \leq C_1 r^{\frac{2}{2-\gamma(x)}}, \tag{3.20}$$

for r universally small. This remark leads to the following result.

Lemma 3.5. *Let u be a local minimizer of the energy-functional (1.5) in B_1. Assume (3.9) and (3.16) are in force. There exists a constant C, depending on $\gamma_\star(0,1)$ and universal parameters, such that*

$$|Du(x)|^2 \leq C\,[u(x)]^{\gamma(x)},$$

for each $x \in B_{1/2}$.

Proof. The proof is essentially the same as the proof of Lemma 3.4, except for the steps we highlight below. By Remark 3.3, it is enough to prove the result at points such that $0 \leq u(x) \leq \tau$. First, we choose r so that

$$r^{\frac{2}{2-\gamma(x)}} = Cu(x),$$

which can be taken small enough depending on τ. As a consequence, (3.20) implies that the function, defined in B_1 by

$$v(y) := u(x+ry)r^{-\frac{2}{2-\gamma(x)}},$$

is uniformly bounded. What remains to be shown is that the parameters in the functional that v minimizes are also controlled. Due to the scaling properties from section 1.3, we have

$$\|\tilde{\delta}\|_{L^\infty(B_1)} \leq r^{\frac{2}{2-\gamma(x)}\gamma_*(x,r)-2} r^2 \|\delta\|_{L^\infty(B_1)} \leq r^{\gamma_*(x,r)-\gamma(x)} \|\delta\|_{L^\infty(B_1)}.$$

Condition (3.16) comes into play once more so that the power

$$r^{\gamma_*(x,r)-\gamma(x)}$$

can be uniformly bounded. Consequently, Lipschitz estimates are also available for v, and the lemma follows. □

Example 3.1. *We conclude this section with an insightful observation leading to a class of intriguing free boundary problems. Initially, it is worth noting that the proof of the existence of a minimizer can be readily adapted for more general energy functionals of the form*

$$J(v) = \int \frac{1}{2}|Dv|^2 + \delta(x)(v^+)^{\gamma(x,v(x))}\, dx, \tag{3.21}$$

provided $\gamma \colon \Omega \times \mathbb{R} \to \mathbb{R}$ *is a Carathéodory function. We further emphasize that our local* $C^{1,\alpha}$ *regularity result, Theorem 3.1, also applies to this class of functionals.*

To illustrate the applicability of these results, let us consider the following toy model, where the oscillatory singularity $\gamma(v)$ *is given only globally measurable and bounded, such that* $\gamma(v) \geq 1/6$, *and*

$$\gamma(x, v) = \frac{1}{2} - \frac{3}{(\ln(v))^2} \quad \textit{for} \quad 0 < v \ll 1, \tag{3.22}$$

see figure 2. One easily checks that γ *is Dini continuous along the surface*

$$\{\gamma(x,u) = 0\} \subset F(u),$$

for any minimizer u *of the corresponding functional* J *in* (3.21). *Since*

$$\gamma_\star(0,1) = \frac{1}{6},$$

the local regularity estimate obtained in Theorem 3.1, gives that minimizers are locally of class $C^{12/11}$. *In contrast, observe that*

$$\gamma \equiv \frac{1}{2} \quad \textit{at} \quad F(u),$$

and so, Theorem 3.3 asserts that local minimizers are precisely of class $C^{4/3}$ *at free boundary points. A wide range of meaningful examples can be constructed out of functions obtained in [4, Section 2].*

Applying similar reasoning, we can provide examples of energy functionals for which minimizers are locally of class $C^{1,\epsilon}$, *for* $0 < \epsilon \ll 1$, *whereas along the free boundary, they are* $C^{1,1-\epsilon}$*–regular. We anticipate revisiting the analysis of such models in future investigations.*

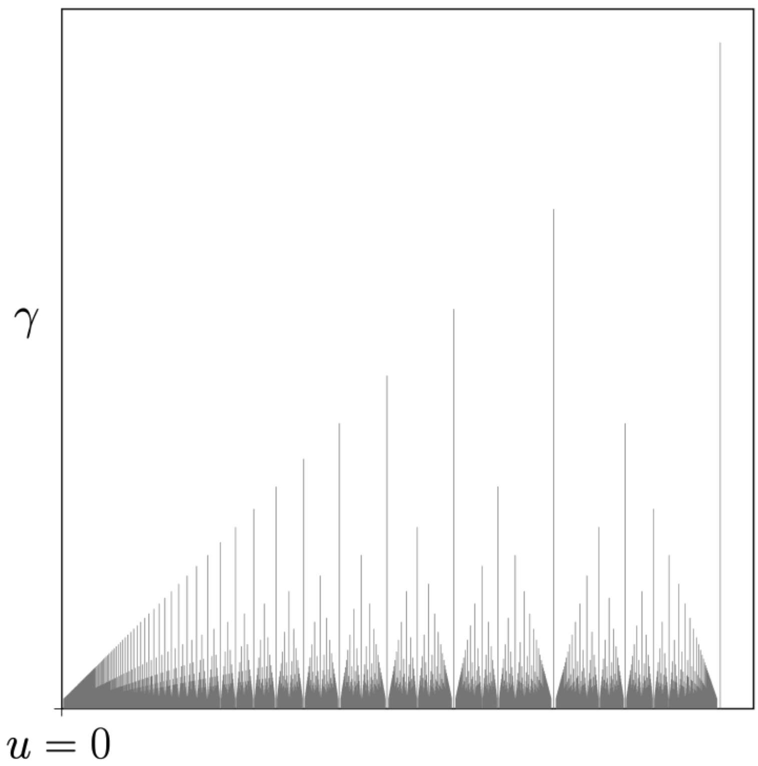

Figure 2 – The graph above illustrates a power singularity $\gamma(x, u)$, characterized by pronounced measurement imprecision arising from inherent randomness in the microstructure composition of the material. Despite this inherent uncertainty, our regularity results, applicable both locally and at free boundary points, offer universal and accurate estimates. Remarkably, these estimates remain independent of the substantial oscillations observed in the function $\gamma(x, u)$.

3.6 Hausdorff measure estimates

In this section, we prove Hausdorff measure estimates for the free boundary under the stronger regularity assumptions on the data

$$\delta(x) \in W^{2,\infty}(B_1) \quad \text{and} \quad \gamma(x) \in W^{2,\infty}(B_1). \tag{3.23}$$

Differentiability of the free boundary will be obtained in Section 3.8, assuming only $\delta, \gamma \in W^{1,q}(B_1)$, for some $q > n$.

Furthermore, we shall also assume

$$\gamma^*(0, 1) := \gamma^*(B_1(0)) < 1. \tag{3.24}$$

We will need a few preliminary results, as in [3]. We begin with a slightly different pointwise gradient estimate with respect to Lemma 3.5.

Lemma 3.6. *Let u be a local minimizer of the energy-functional (1.5) in B_1. Assume (3.9), (3.16), (3.24) and (3.23) are in force and let $x_0 \in F(u) \cap B_{1/2}$. There exists a constant c_1, depending only on n, δ_0, $\gamma_\star(0,1)$, $\|D\delta\|_\infty$, $\|D^2\delta\|_\infty$, $\|D\gamma\|_\infty$ and $\|D^2\gamma\|_\infty$, such that*

$$|Du(x)|^2 \leq 2\delta(x)\,[u(x)]^{\gamma(x)} + c_1 u(x),$$

for each $x \in B_{1/8}(x_0)$.

Proof. Consider $\zeta\colon [0,3\tau] \to \mathbb{R}$, defined by

$$\zeta(t) = \begin{cases} 0 & \text{if } \ t \in [0,\tau] \\ & \\ K_1\,(t-\tau)^3 & \text{if } \ t \in [\tau, 3\tau], \end{cases}$$

and define, for $\tau = 1/8$ and $K > 0$ a large constant to be chosen later,

$$w(y) := |Du(y)|^2 - 2\delta(y)[u(y)]^{\gamma(y)} - Ku(y) - \zeta(|y-x_0|)[u(y)]^{\gamma(y)},$$

for $y \in \Omega(u) \cap B_{3\tau}(x_0)$. By Lemma 3.5, we can suitably choose $K_1 > 0$ so that $w \leq 0$ on $\partial B_{3\tau}(x_0)$, and so $w \leq 0$ on $\partial[\Omega(u) \cap B_{3\tau}(x_0)]$. We will show that $w \leq 0$ in $\Omega(u) \cap B_{3\tau}(x_0)$. To do so, we assume, to the contrary, that w attains a positive maximum at $p \in \Omega(u) \cap B_{3\tau}(x_0)$. Since w is smooth within $\Omega(u)$ and p is a point of maximum for w, we have $\Delta w(p) \leq 0$. To reach a contradiction, we will show that $\Delta w(p) > 0$, for τ small and K large.

We will omit the point p whenever possible to ease the notation. We also rotate the coordinate system so that e_1 is in the direction of $Du(p)$. We then have

$$\begin{aligned} 0 &= \partial_1 w(p) \\ &= 2Du \cdot D\partial_1 u - 2\partial_1\delta u^\gamma - 2\delta\left(\gamma u^{\gamma-1}\partial_1 u + \partial_1\gamma u^\gamma \ln(u)\right) - K\partial_1 u \\ &\quad -\partial_1\zeta u^\gamma - \zeta\left(\gamma u^{\gamma-1}\partial_1 u + \partial_1\gamma\, u^\gamma \ln(u)\right) \\ &= \partial_1 u\left[2\partial_{11}u - \frac{u^\gamma}{\partial_1 u}(2\partial_1\delta + \partial_1\zeta) - u^{\gamma-1}\gamma(2\delta+\zeta) - K\right] \\ &\quad +\partial_1 u\left[-\frac{u^\gamma}{\partial_1 u}\partial_1\gamma\ln(u)(2\delta+\zeta)\right]. \end{aligned}$$

Since $\partial_1 u(p) > 0$, we obtain

$$2\partial_{11}u = \frac{u^\gamma}{\partial_1 u}(2\partial_1\delta + \partial_1\zeta) + u^{\gamma-1}\gamma(2\delta+\zeta) + K + \frac{u^\gamma}{\partial_1 u}\partial_1\gamma\ln(u)(2\delta+\zeta).$$

Moreover, since $w(p) > 0$, it also holds that $\partial_1 u(p) > \sqrt{2\delta(p)}u(p)^{\frac{\gamma(p)}{2}}$, and so

$$\frac{u^\gamma}{\partial_1 u} \leq \frac{u^{\frac{\gamma}{2}}}{\sqrt{2\delta}} \leq \frac{1}{\sqrt{2\delta_0}}u^{\frac{\gamma}{2}}.$$

This implies that

$$2\partial_{11}u \geq 2\delta\gamma u^{\gamma-1} + K + \zeta\gamma u^{\gamma-1} - C_1 u^{\frac{\gamma}{2}} - C_2 u^{\frac{\gamma}{2}}|\ln(u)|,$$

for constants $C_1 = C_1(\delta_0, \|D\delta\|_\infty, K_1)$ and $C_2 = C_2(\delta_0, \|D\gamma\|_\infty, K_1)$. For a small $\eta^* > 0$ so that $\gamma/2 - \eta^* > 0$ and a larger constant C_3, we then have

$$2\partial_{11}u \geq 2\delta\gamma u^{\gamma-1} + K + \zeta\gamma u^{\gamma-1} - C_3 u^{\frac{\gamma}{2}-\eta^*}.$$

Writing $K = \eta K + (1-\eta)K$, for $\eta \in (0,1)$, we obtain, for large K,

$$2\partial_{11}u \geq 2\delta\gamma u^{\gamma-1} + \eta K + \zeta\gamma u^{\gamma-1},$$

and as a consequence, squaring and dropping positive terms,

$$2\left(\partial_{11}u\right)^2 \geq 2\left(\delta\gamma u^{\gamma-1}\right)^2 + 2\delta\gamma\eta K u^{\gamma-1} + 2\delta\zeta\left(\gamma u^{\gamma-1}\right)^2. \tag{3.25}$$

Now, we calculate Δw at the point p. By direct computations, we obtain

$$\begin{aligned} \Delta w &= 2\sum_{k,j}(\partial_{k,j}u)^2 + 2Du\cdot D(\Delta u) - 2u^\gamma\Delta\delta - 4D\delta\cdot D(u^\gamma) \\ &\quad -2\delta\,\Delta(u^\gamma) - K\Delta u - u^\gamma\Delta\zeta - 2D\zeta\cdot D(u^\gamma) - \zeta\Delta(u^\gamma). \end{aligned}$$

Moreover,

$$\begin{aligned} D(u^\gamma) &= u^\gamma\ln(u)D\gamma + \gamma u^{\gamma-1}Du, \\ \Delta(u^\gamma) &= u^\gamma\ln(u)\Delta\gamma + u^\gamma(\ln(u))^2|D\gamma|^2 + 2\gamma u^{\gamma-1}\ln(u)D\gamma\cdot Du \\ &\quad +2u^{\gamma-1}D\gamma\cdot Du + \gamma(\gamma-1)u^{\gamma-2}|Du|^2 + \gamma u^{\gamma-1}\Delta u. \end{aligned}$$

Observe that, by Lemma 3.5 and since $\gamma < 1$,

$$|D(u^\gamma)| \leq C_4\gamma u^{2\gamma-1}.$$

We also have

$$\Delta(u^\gamma) \leq C_5 u^{\gamma-1} + \delta\gamma^2 u^{2\gamma-2}\left[\frac{(\gamma-1)}{\gamma\delta}\frac{|Du|^2}{u^\gamma} + 1\right],$$

for a constant $C_5 = C_5(\|D\gamma\|_\infty, \|D^2\gamma\|_\infty, \gamma_\star(0,1))$. One can now further estimate Δw from below to obtain

$$\begin{aligned} \Delta w &\geq 2(\partial_{11}u)^2 - C_6u^{\gamma-1} + 2\delta\gamma(\gamma-1)u^{\gamma-2}|Du|^2 \\ &\quad -2\delta^2\gamma^2u^{2\gamma-2}\left[\frac{(\gamma-1)}{\gamma\delta}\frac{|Du|^2}{u^\gamma} + 1\right] - K\delta\gamma u^{\gamma-1} \\ &\quad -\delta\zeta\gamma^2u^{2\gamma-2}\left[\frac{(\gamma-1)}{\gamma\delta}\frac{|Du|^2}{u^\gamma} + 1\right] \\ &= 2(\partial_{11}u)^2 - 2\delta^2\gamma^2u^{2\gamma-2} - C_6u^{\gamma-1} \\ &\quad -K\delta\gamma u^{\gamma-1} - \delta\zeta\gamma^2u^{2\gamma-2}\left[\frac{(\gamma-1)}{\gamma\delta}\frac{|Du|^2}{u^\gamma} + 1\right]. \end{aligned}$$

By (3.25), it follows that

$$\begin{aligned}
\Delta w &\geq 2\delta\gamma\eta K u^{\gamma-1} + 2\delta\zeta\left(\gamma u^{\gamma-1}\right)^2 - C_6 u^{\gamma-1} \\
&\quad -K\delta\gamma u^{\gamma-1} - \delta\zeta\gamma^2 u^{2\gamma-2}\left[\frac{(\gamma-1)}{\gamma\delta}\frac{|Du|^2}{u^\gamma} + 1\right] \\
&= u^{\gamma-1}\left[2\delta\gamma\eta K - C_6 - K\delta\gamma\right] + 2\delta\zeta\left(\gamma u^{\gamma-1}\right)^2 \\
&\quad -\delta\zeta\gamma^2 u^{2\gamma-2}\left[\frac{(\gamma-1)}{\gamma\delta}\frac{|Du|^2}{u^\gamma} + 1\right].
\end{aligned}$$

Since $\gamma < 1$, we conclude

$$\Delta w \geq u^{\gamma-1}\left[2\delta\gamma\eta K - C_6 - K\delta\gamma\right].$$

Now we fix any $1/2 < \eta < 1$ and choose K so large that the above expression is positive. This leads to a contradiction, as discussed before. Since ζ vanishes on $B_\tau(x_0)$, the result is proved. □

The second preliminary result concerns the integrability of a negative power of the minimizer.

Lemma 3.7. *Let u be a local minimizer of the energy-functional* (1.5) *in* B_1. *Assume* (3.9), (3.16), (3.23) *and* (3.24) *are in force. If* $0 \in F(u)$, *then*

$$u(x)^{-\frac{\gamma(x)}{2}} \in L^1(\Omega(u) \cap B_{1/2}).$$

Proof. Observe that it is enough to show that

$$u(x)^{-\frac{\gamma(x)}{2}} \in L^1(\Omega(u) \cap B_\tau(z)), \tag{3.26}$$

for some small $\tau > 0$ and every $z \in F(u)$. Indeed, once this is proved, we can cover $F(u) \cap B_{1/2}$ with finitely many balls with radius $\tau > 0$, say $\{B_\tau(z_i)\}$. Then,

$$\int_{\Omega(u)\cap(\cup B_\tau(z_i))} u^{-\frac{\gamma(x)}{2}} \leq \sum_i \int_{\Omega(u)\cap B_\tau(z_i)} u^{-\frac{\gamma(x)}{2}} \leq C.$$

Also, by continuity of u, we have

$$u \geq c \quad \text{in} \quad \left(\Omega(u) \cap B_{1/2}\right) \setminus \cup_i B_\tau(z_i),$$

from which the statement in the lemma follows.

To prove (3.26), we follow closely the argument in [59, Lemma 2.5]. Set

$$w = u^{2-\frac{3}{2}\gamma(x)}.$$

First, take $\rho \in C^\infty(\mathbb{R}^+)$, satisfying $\rho' \geq 0$, $\rho \equiv 0$ in $[0, 1/2]$ and $\rho(t) = t$ in $[1, \infty)$. For $\delta > 0$, let $\rho_\delta(t) = \delta\rho(\delta^{-1}t)$. If $\delta < \epsilon$, then

$$\frac{1}{\epsilon}\int_{\{0\leq u<\epsilon\}\cap B_\tau(z_i)} Dw \cdot Du\, \rho'_\delta(u) = \frac{1}{\epsilon}\int_{B_\tau(z_i)} Dw \cdot D\rho_\delta(\min(u,\epsilon)) =: A. \tag{3.27}$$

Integrating by parts, we obtain

$$A = -\frac{1}{\epsilon}\int_{\{0<u\}\cap B_\tau(z_i)} \rho_\delta(\min(u,\epsilon))\Delta w + \int_{\partial B_\tau(z_i)} \frac{\rho_\delta(\min(u,\epsilon))}{\epsilon}\partial_\nu w.$$

Now we choose $\delta = \epsilon/2$, observing that $\rho_\delta(u) = 0$ in the set $\{0 < u \le \epsilon/4\}$. Therefore,

$$\begin{aligned} A &= -\frac{1}{2}\int_{\{\epsilon/4<u\le\epsilon\}\cap B_\tau(z_i)} \rho\left(\frac{2}{\epsilon}u\right)\Delta w - \int_{\{\epsilon<u\}\cap B_\tau(z_i)} \Delta w \\ &\quad + \int_{\partial B_\tau(z_i)} \frac{\rho_\delta(\min(u,\epsilon))}{\epsilon}\partial_\nu w. \end{aligned}$$

By Lemma 3.5, we have

$$\begin{aligned} |Dw(x)| &\le 2|D\gamma(x)|u(x)^{2-\frac{3}{2}\gamma(x)}\ln(u(x)) \\ &\quad + \left(2-\frac{3}{2}\gamma(x)\right)u(x)^{1-\frac{3}{2}\gamma(x)}|Du(x)| \\ &\le C\left(|D\gamma(x)|+1\right), \end{aligned}$$

for some universal constant $C > 0$, and so

$$A \le C\tau^{n-1} - \frac{1}{2}\int_{\{\epsilon/4<u\le\epsilon\}\cap B_\tau(z_i)} \rho\left(\frac{2}{\epsilon}u\right)\Delta w - \int_{\{\epsilon<u\}\cap B_\tau(z_i)} \Delta w. \tag{3.28}$$

By direct computations, it follows that

$$\begin{aligned} \Delta w(x) &= a(x) + \left(2-\frac{3}{2}\gamma(x)\right)\left(\left(1-\frac{3}{2}\gamma(x)\right)u(x)^{-\frac{3}{2}\gamma(x)}|Du(x)|^2\right. \\ &\quad \left. + u(x)^{1-\frac{3}{2}\gamma(x)}\Delta u(x)\right), \end{aligned}$$

where

$$\begin{aligned} \frac{2}{3}a(x) &= -w(x)\ln(u(x))\Delta\gamma(x) - \ln(u(x))D\gamma(x)\cdot Du(x) \\ &\quad -2u(x)^{1-\frac{3}{2}\gamma(x)}D\gamma(x)\cdot Du(x) \\ &\quad -\left(2-\frac{3}{2}\gamma(x)\right)\ln(u(x))u(x)^{1-\gamma(x)}D\gamma(x)\cdot Du(x). \end{aligned}$$

By Lemma 3.6, there exists a universal constant $c > 0$ such that

$$\begin{aligned} \left(\left(1-\frac{3}{2}\gamma(x)\right)u(x)^{-\frac{3}{2}\gamma(x)}|Du(x)|^2 + u(x)^{1-\frac{3}{2}\gamma(x)}\Delta u(x)\right) &= \\ u(x)^{\frac{-\gamma(x)}{2}}\left(\left(1-\frac{3}{2}\gamma(x)\right)\frac{|Du(x)|^2}{u(x)^{\gamma(x)}} + \delta(x)\gamma(x)\right) &\ge \\ u(x)^{\frac{-\gamma(x)}{2}}\delta(x)\ \left(2(1-\gamma(x)) + c(1-g(x))u(x)\right) &\ge \\ \delta_0 u(x)^{\frac{-\gamma(x)}{2}}\left(2(1-\gamma^*(0,1)) - c_1\tau^{\frac{2}{2-\gamma(z_i)}}\right), \end{aligned}$$

where, for the last inequality, we used Theorem 3.3. Since $\gamma^*(0,1) < 1$, we can choose $\tau > 0$ small enough, such that

$$2(1-\gamma^*(0,1)) - c_1\tau^{\frac{2}{2-\gamma(z_i)}} \ge (1-\gamma^*(0,1)),$$

and so

$$\Delta w(x) \geq a(x) + c_2 u(x)^{-\frac{\gamma(x)}{2}}.$$

Furthermore, notice that

$$|a(x)| \leq C(|D\gamma(x)| + |D^2\gamma(x)| + 1),$$

for some positive universal constant $C > 0$. Therefore, by (3.28), we have

$$\begin{aligned} A &\leq -\frac{1}{2}\int_{\{\epsilon/4<u\leq\epsilon\}\cap B_\tau(z_i)} \rho\left(\frac{2}{\epsilon}u\right)\left(a(x) + c_2 u(x)^{-\frac{\gamma(x)}{2}}\right) \\ &\quad - \int_{\{\epsilon<u\}\cap B_\tau(z_i)} \left(a(x) + c_2 u(x)^{-\frac{\gamma(x)}{2}}\right) \\ &\leq C\left(\|D\gamma\|_{L^1}, \|D^2\gamma\|_{L^1}\right) - \overline{c}\int_{\{\epsilon/4<u\}\cap B_\tau(z_i)} u(x)^{-\frac{\gamma(x)}{2}}. \end{aligned}$$

Now, we estimate the left-hand side of (3.27). By Lemma 3.5 and since $\gamma^*(0,1) < 1$, we obtain

$$\begin{aligned} Dw \cdot Du &\geq -2u(x)^{2-\frac{3}{2}\gamma(x)}|\ln(u(x))||D\gamma(x)|\,|Du(x)| \\ &\geq -Cu(x)^{2-\gamma(x)}|\ln(u(x))||D\gamma(x)| \\ &\geq -Cu(x)^{2-\gamma^*(0,1)}|\ln(u(x))||D\gamma(x)| \\ &\geq -C_1 u(x)\,|D\gamma(x)|, \end{aligned}$$

for some universal constant C_1. Thus, from (3.27), we have

$$\begin{aligned} &-C_1\frac{1}{\epsilon}\int_{\{0\leq u<\epsilon\}\cap B_\tau(z_i)} u(x)|D\gamma(x)|\,\rho'_\delta(u) \\ &\leq C\left(\|D\gamma\|_{L^1}, \|D^2\gamma\|_{L^1}\right) - \overline{c}\int_{\{\epsilon/4<u\}\cap B_\tau(z_i)} u(x)^{-\frac{\gamma(x)}{2}}. \end{aligned}$$

Since $\rho'_\delta \leq 1$, we obtain

$$\int_{\{\epsilon/4<u\}\cap B_\tau(z_i)} u(x)^{-\frac{\gamma(x)}{2}} \leq C\left(\overline{c}, C_1, \|D\gamma\|_{L^1}, \|D^2\gamma\|_{L^1}\right).$$

We get the result by passing to the limit as $\epsilon \to 0$. □

We are now ready to state and prove the main result of this section.

Theorem 3.6. *Let u be a local minimizer of the energy-functional (1.5) in B_1. Assume (3.9), (3.16), (3.23) and (3.24) are in force. Then, there exists a universal constant $C > 0$, depending only on n, δ_0, $\gamma_\star(0,1)$, $\|D\delta\|_\infty$, $\|D^2\delta\|_\infty$, $\|D\gamma\|_\infty$ and $\|D^2\gamma\|_\infty$, such that*

$$\mathcal{H}^{n-1}(F(u) \cap B_{1/2}) < C.$$

Proof. Assume $0 \in F(u)$. It is enough to prove that for some small r,

$$\mathcal{H}^{n-1}(F(u) \cap B_r) < \infty.$$

Given a small parameter $\epsilon > 0$, we cover $F(u) \cap B_r$ with finitely many balls $\{B_\epsilon(x_i)\}_{i \in F_\epsilon}$ with finite overlap, that is,

$$\sum_{i \in F_\epsilon} \mathcal{X}_{B_\epsilon(x_i)} \leq c,$$

for a constant $c > 0$ that depends only on the dimension n. It then follows that

$$\mathcal{H}^{n-1}(F(u) \cap B_r) \leq \overline{c} \liminf_{\epsilon \to 0} \epsilon^{n-1} \#(F_\epsilon).$$

Since $x_i \in F(u)$, by Theorem 3.3, we have

$$\Omega(u) \cap B_\epsilon(x_i) \subset \left\{0 < u \leq \overline{M} \epsilon^{\beta_i}\right\} \cap B_\epsilon(x_i),$$

where $\beta_i = 2/(2 - \gamma(x_i))$. By assumption (3.16), it follows that

$$\Omega(u) \cap B_\epsilon(x_i) \subset \left\{0 < u \leq \overline{M}_1 \epsilon^{\beta^*(x_i, \epsilon)}\right\} \cap B_\epsilon(x_i),$$

for a universal constant $\overline{M}_1 > \overline{M}$. Let us assume, to simplify, that $\overline{M}_1 = 1$. Now, observe that

$$\bigcup_{i \in F_\epsilon} \left(B_\epsilon(x_i) \cap \left\{0 < u(x) \leq \epsilon^{\beta^*(x_i, \epsilon)}\right\}\right) \subseteq B_{2r} \cap \left\{0 < u(x)^{\frac{1}{\beta(x)}} < \epsilon\right\}.$$

Since the covering $\{B_\epsilon(x_i)\}_{i \in F_\epsilon}$ has finite overlap, it then follows that

$$\sum_{i \in F_\epsilon} |\Omega(u) \cap B_\epsilon(x_i)| \leq c \left|B_{2r} \cap \left\{0 < u(x)^{\frac{1}{\beta(x)}} < \epsilon\right\}\right|.$$

By Theorem 3.5, this implies that

$$|\Omega(u) \cap B_\epsilon(x_i)| \geq \mu_0 \epsilon^n,$$

and so

$$\epsilon^{n-1} \#(F_\epsilon) \leq \frac{c}{\mu_0} \frac{\left|B_{2r} \cap \left\{0 < u(x)^{\frac{1}{\beta(x)}} < \epsilon\right\}\right|}{\epsilon},$$

which readily leads to

$$\mathcal{H}^{n-1}(F(u) \cap B_r) \leq \frac{\overline{c}\, c}{\mu_0} \liminf_{\epsilon \to 0} \frac{|\left|B_{2r} \cap \left\{0 < u(x)^{\frac{1}{\beta(x)}} < \epsilon\right\}\right|}{\epsilon}.$$

We will show below that the right-hand side of the inequality above can be bounded above uniformly in ϵ. To do so, let

$$v(x) := u(x)^{\frac{1}{\beta(x)}}.$$

Observe that

$$\int_{B_{2r} \cap \{0 < v \leq \epsilon\}} |Dv|^2 = \int_{B_{2r}} D(\min(v, \epsilon)) \cdot Dv =: I.$$

Integrating by parts, we get

$$I = -\int_{B_{2r}} \min(v,\epsilon)\Delta v + \int_{\partial B_{2r}} \min(v,\epsilon)\partial_\nu v,$$

and so,

$$\int_{B_{2r}\cap\{0<v\le\epsilon\}} |Dv|^2 + v\Delta v = -\epsilon \int_{B_{2r}\cap\{v>\epsilon\}} \Delta v + \int_{\partial B_{2r}} \min(v,\epsilon)\partial_\nu v. \tag{3.29}$$

By direct computations, we readily obtain

$$Dv(x) = g(x)D\left(\frac{1}{\beta(x)}\right) + \frac{1}{\beta(x)}u(x)^{\frac{1}{\beta(x)}-1}Du(x)$$

and

$$\Delta v(x) = A(x) + B(x) + \frac{\delta(x)\,\gamma(x)}{\beta(x)}u(x)^{-\frac{1}{\beta(x)}},$$

where $g(x) = v(x)\ln(u(x))$, with

$$\begin{aligned} A(x) \;&=\; g(x)\Delta\left(\frac{1}{\beta(x)}\right) + D\left(\frac{1}{\beta(x)}\right)\cdot Dg(x) \\ &\quad +u(x)^{\frac{1}{\beta(x)}-1} D\left(\frac{1}{\beta(x)}\right)\cdot Du(x), \end{aligned}$$

and

$$B(x) = \frac{1}{\beta(x)}D\left(u^{\frac{1}{\beta(x)}-1}\right)\cdot Du(x).$$

Let us first bound (3.29) from below. To do so, we estimate

$$\begin{aligned} |Dv|^2 + v\Delta v \;&=\; g(x)^2\left|D\left(\frac{1}{\beta(x)}\right)\right|^2 + \frac{1}{\beta(x)^2}u(x)^{2\left(\frac{1}{\beta(x)}-1\right)}|Du|^2 \\ &\quad +2\frac{1}{\beta(x)}g(x)D\left(\frac{1}{\beta(x)}\right)\cdot Du(x) \\ &\quad +(A(x)+B(x))u(x)^{\frac{1}{\beta(x)}} + \frac{\delta(x)\,\gamma(x)}{\beta(x)} \\ &\ge\; B(x)u(x)^{\frac{1}{\beta(x)}} + \frac{1}{\beta(x)^2}u(x)^{2\left(\frac{1}{\beta(x)}-1\right)}|Du|^2 \\ &\quad +2\frac{1}{\beta(x)}g(x)D\left(\frac{1}{\beta(x)}\right)\cdot Du(x) \\ &\quad +A(x)u(x)^{\frac{1}{\beta(x)}} + \frac{\delta_0\,\gamma_\star(0,1)}{2}. \end{aligned}$$

By Lemma 3.5, we have

$$\begin{aligned} &B(x)u(x)^{\frac{1}{\beta(x)}} + \frac{1}{\beta(x)^2}u(x)^{2\left(\frac{1}{\beta(x)}-1\right)}|Du|^2 \\ &\quad\ge\; \frac{1}{\beta(x)}u^{\frac{2}{\beta(x)}-1}\ln(u(x))D\left(\frac{1}{\beta(x)}\right)\cdot Du(x) \\ &\quad\ge\; -u(x)^{\frac{1}{\beta(x)}}\ln(u(x))|D\gamma(x)| \\ &\quad\ge\; -u^{\frac{1}{2\beta(x)}}|D\gamma(x)|, \end{aligned}$$

which implies

$$\begin{aligned} |Dv|^2 + v\Delta v \quad \geq \quad & -Cu^{\frac{1}{2\beta(x)}}|D\gamma(x)| \\ & +2\frac{1}{\beta(x)}g(x)D\left(\frac{1}{\beta(x)}\right)\cdot Du(x) \\ & +A(x)u(x)^{\frac{1}{\beta(x)}} + \frac{\delta_0\,\gamma_\star(0,1)}{2}, \end{aligned}$$

for some universal constant C. Using Lemma 3.5 once more, we can show that

$$\left|2\frac{1}{\beta(x)}g(x)D\left(\frac{1}{\beta(x)}\right)\cdot Du(x)\right| \leq C_1 u(x)^{\frac{1}{\beta(x)}}|D\gamma(x)|$$

and

$$|A(x)| \leq C_1(|D\gamma(x)| + |D^2\gamma(x)| + |D\gamma(x)||\ln(u(x))|),$$

for some universal constant C_1, and so

$$\begin{aligned} |Dv|^2 + v\Delta v \quad \geq \quad & -C_2u^{\frac{1}{2\beta(x)}}|D\gamma(x)| - C_1u^{\frac{1}{\beta(x)}}\left(|D\gamma(x)| + |D^2\gamma(x)|\right) \\ & +\frac{\delta_0\,\gamma_\star(0,1)}{2}, \end{aligned}$$

for a universal constant C_2. We can now estimate the left-hand side of (3.29) as

$$\begin{aligned} \int_{B_{2r}\cap\{0<v\leq\epsilon\}} |Dv|^2 + v\Delta v \quad \geq \quad & -C_2\|D\gamma\|_\infty\epsilon^{1/2}|B_{2r}\cap\{0<v\leq\epsilon\}| \\ & -C_1\epsilon(\|D\gamma\|_{L^1(B_{2r})} + \|D^2\gamma\|_{L^1(B_{2r})}) \\ & +\frac{\delta_0\,\gamma_\star(0,1)}{2}|B_{2r}\cap\{0<v\leq\epsilon\}| \\ \geq \quad & \frac{\delta_0\,\gamma_\star(0,1)}{4}|B_{2r}\cap\{0<v\leq\epsilon\}| - C_3\epsilon, \end{aligned}$$

for ϵ small enough and depending only on universal constants. By Lemma 3.5, there exists a constant $C_4 > 0$ such that $|Dv| \leq C_4$, and so (3.29) implies

$$\frac{\delta_0\,\gamma_\star(0,1)}{4}|B_{2r}\cap\{0<v\leq\epsilon\}| - C_2\epsilon \leq -\epsilon\int_{B_{2r}\cap\{v>\epsilon\}}\Delta v + C_4\epsilon,$$

and so

$$\frac{\delta_0\,\gamma_\star(0,1)}{4}\frac{|B_{2r}\cap\{0<v\leq\epsilon\}|}{\epsilon} \leq C_2 + C_4 - \int_{B_{2r}\cap\{v>\epsilon\}}\Delta v.$$

The proof will then be complete as long as this remaining integral is uniformly bounded in $\epsilon > 0$.

Recalling the expression for Δv, we have

$$
\begin{aligned}
-\Delta v &\leq |A(x)| + B(x) + \frac{\delta(x)\,\gamma(x)}{\beta(x)} u(x)^{-\frac{1}{\beta(x)}} \\
&\leq C_1(|D\gamma(x)| + |D^2\gamma(x)| + |D\gamma(x)||\ln(u(x))|) \\
&\quad + u(x)^{-\frac{\gamma(x)}{2}} \left(\frac{1}{2} \ln(u(x)) D\gamma(x) \cdot Du(x) + \frac{\gamma(x)}{2} \frac{1}{u(x)} |Du(x)|^2 \right) \\
&\quad - \frac{\delta(x)\,\gamma(x)}{\beta(x)} u(x)^{-\frac{1}{\beta(x)}} \\
&\leq C_5(|D\gamma(x)| + |D^2\gamma(x)|) + C_6|D\gamma(x)||\ln(u(x))| + C_7 u(x)^{-\frac{\gamma(x)}{2}} \\
&\leq C_5(|D\gamma(x)| + |D^2\gamma(x)|) + C_8|D\gamma|_\infty u(x)^{-\frac{\gamma(x)}{2}},
\end{aligned}
$$

where we used Lemma 3.6 and the fact that $|\ln(u(x))|$ can be bounded above by $u(x)^{-\frac{\gamma(x)}{2}}$. This implies that

$$
\int_{B_{2r}\cap\{v>\epsilon\}} \Delta v \leq C_5(\|D\gamma\|_{L^1} + \|D^2\gamma\|_{L^1}) + C_8|D\gamma|_\infty + \int_{B_{2r}\cap\{v>\epsilon\}} u(x)^{-\frac{\gamma(x)}{2}},
$$

from which the conclusion of the theorem follows because of Lemma 3.7. □

3.7 Monotonicity formula and classification of blow-ups

In this section, we obtain a monotonicity formula valid for local minimizers of the energy-functional (1.5). Given $z_0 \in B_1$, let

$$
\gamma := \gamma(z_0) \qquad \text{and} \qquad \beta := \frac{2}{2-\gamma}.
$$

Now, for a Lipschitz function v and $z_0 \in F(v)$, define

$$
\begin{aligned}
W_{v,z_0}(r) &:= r^{-(n+2(\beta-1))} \int_{B_r(z_0)} \frac{1}{2}|Dv|^2 + \delta(x) v^{\gamma(x)} \chi_{\{v>0\}} \\
&\quad - \beta r^{-((n-1)+2\beta)} \int_{\partial B_r(z_0)} v^2 \\
&\quad - \int_0^r \beta t^{-(n+\beta\gamma+1)} \int_{B_t(z_0)} (\gamma(x)-\gamma)\delta(x) v^{\gamma(x)} \chi_{\{v>0\}} \\
&\quad - \int_0^r t^{-(n+\beta\gamma+1)} \int_{B_t(z_0)} (D\gamma(x)\cdot(x-z_0))\,\delta(x) v^{\gamma(x)} \ln(v) \chi_{\{v>0\}} \\
&\quad - \int_0^r t^{-(n+\beta\gamma+1)} \int_{B_t(z_0)} (D\delta(x)\cdot(x-z_0)) v^{\gamma(x)} \chi_{\{v>0\}}. \qquad (3.30)
\end{aligned}
$$

For our formula to hold, we will further need to assume that, for some $0 < r_0 < 1$,

$$
t \to t^{-n} \int_{B_t(z_0)} |D\delta(x)| dx \in L^1(0, r_0) \qquad (3.31)
$$

and

$$t \to t^{-n} \ln t \int_{B_t(z_0)} |D\gamma(x)| dx \in L^1(0, r_0). \tag{3.32}$$

We remark that sufficient conditions for these to hold are $|D\delta| \in L^q(B_1)$ and $|D\gamma| \in L^q(B_1)$, for $q > n$. Indeed, we readily have

$$t^{-n} \ln t \int_{B_t(z_0)} |D\gamma(x)| dx \le C(n,q) \|D\gamma\|_{L^q\left(B_{r_0}(z_0)\right)} t^{-\frac{n}{q}} \ln t,$$

and

$$\int_0^{r_0} t^{-\frac{n}{q}} \ln t \, dt < \infty \quad \Longleftarrow \quad q > n.$$

Remark 3.5. *If we assume $\gamma \in W^{1,q}$, for $q > n$, then γ is Hölder continuous and therefore condition (3.16) is automatically satisfied. We also point out that these integrability conditions are important to assure that $W_{u,z_0}(r) < \infty$, for every $0 < r$ and $z_0 \in F(u)$ such that $B_r(z_0) \Subset B_1$, for u a local minimizer of (1.5).*

We are now ready to state and prove the monotonicity formula for local minimizers of our oscillatory exponent functional.

Theorem 3.7. *Let u be a local minimizer of (1.5) and assume (3.31) and (3.32) are in force. If $z_0 \in F(u)$, then*

$$\frac{d}{dr} W_{u,z_0}(r) \ge 0.$$

Proof. Without loss of generality, we consider $z_0 = 0$. Let

$$\begin{aligned} \overline{W}_u(r) &= r^{-(n+2(\beta-1))} \int_{B_r} \frac{1}{2}|Du|^2 + \delta(x)\, u^{\gamma(x)} \chi_{\{u>0\}} \\ &\quad -\beta r^{-((n-1)+2\beta)} \int_{\partial B_r} u^2, \end{aligned}$$

and define

$$u_r(x) := \frac{u(rx)}{r^\beta} \quad \text{and} \quad \gamma_r(x) := \gamma(rx).$$

By scaling,

$$\overline{W}_u(r) = \int_{B_1} \frac{1}{2}|Du_r|^2 + \delta(rx)\, r^{\beta(\gamma_r(x)-\gamma)} u_r^{\gamma_r(x)} \chi_{\{u_r>0\}} - \beta \int_{\partial B_1} u_r^2,$$

where we used that, by definition of the parameter β, we have

$$2(\beta - 1) = \beta\gamma.$$

Differentiating $\overline{W}_u$ with respect to r leads to

$$\begin{aligned} \frac{d}{dr}\overline{W}_u(r) &= \int_{B_1} Du_r \cdot D\left(\frac{d}{dr}u_r\right) + \frac{d}{dr}\left(\delta(rx)\, r^{\beta(\gamma_r(x)-\gamma)} u_r^{\gamma_r(x)}\right) \chi_{\{u_r>0\}} \\ &\quad -\beta \int_{\partial B_1} 2u_r \frac{d}{dr}u_r. \end{aligned}$$

Integrating by parts, we obtain

$$\frac{d}{dr}\overline{W}_u(r) = (A) + (B) + (C) + (D) + (E),$$

for

$$\begin{aligned}
(A) &:= \int_{B_1} -\Delta u_r \cdot \frac{d}{dr} u_r, \\
(B) &:= 2\int_{\partial B_1} (\partial_\nu u_r - \beta u_r)\frac{d}{dr} u_r, \\
(C) &:= \int_{B_1} \frac{d}{dr}\left(r^{\beta(\gamma_r(x)-\gamma)}\right)\delta(rx)\, u_r^{\gamma_r(x)}\chi_{\{u_r>0\}}, \\
(D) &:= \int_{B_1} r^{\beta(\gamma_r(x)-\gamma)}\delta(rx)\,\frac{d}{dr}\left(u_r^{\gamma_r(x)}\right)\chi_{\{u_r>0\}}, \\
(E) &:= \int_{B_1} (D\delta(rx)\cdot x)\; r^{\beta(\gamma_r(x)-\gamma)} u_r^{\gamma_r}\chi_{\{u_r>0\}}.
\end{aligned}$$

In order to simplify the notation, we write $\gamma_r = \gamma_r(x)$ and notice that

$$\begin{aligned}
(D) &= \int_{B_1} r^{\beta(\gamma_r-\gamma)}\delta(rx)\left(\gamma_r u_r^{\gamma_r-1}\frac{d}{dr}u_r + u_r^{\gamma_r}\ln(u_r)\frac{d}{dr}\gamma_r\right)\chi_{\{u_r>0\}} \\
&= (D.1) + (D.2).
\end{aligned}$$

Since u is a minimizer to (1.5), it follows that $(D.1) + (A) = 0$, and so

$$\frac{d}{dr}\overline{W}_u(r) = (B) + (C) + (D.2) + (E).$$

By direct computations, it follows that

$$\frac{d}{dr}u_r(x) = r^{-\beta}\left(Du(rx)\cdot x - \beta r^{\beta-1}u_r(x)\right).$$

Since ν is the normal vector at ∂B_1, we obtain

$$\partial_\nu u_r(x) = r^{1-\beta}\partial_\nu u(rx) = r^{1-\beta}Du(rx)\cdot x,$$

which implies that

$$\frac{d}{dr}u_r = \frac{1}{r}\left(\partial_\nu u_r - \beta u_r\right).$$

Hence,

$$(B) = \frac{2}{r}\int_{\partial B_1} |\partial_\nu u_r - \beta u_r|^2.$$

Moreover,

$$\begin{aligned}
(C) &= \int_{B_1} \beta(\gamma_r-\gamma)r^{\beta(\gamma_r-\gamma)-1}\delta(rx)\,u_r^{\gamma_r}\chi_{\{u_r>0\}} \\
&\quad + \int_{B_1} r^{\beta(\gamma_r-\gamma)}\beta\ln(r)\delta(rx)\,u_r^{\gamma_r}\left(\frac{d}{dr}\gamma_r\right)\chi_{\{u_r>0\}},
\end{aligned}$$

and

$$(D.2) = \int_{B_1} r^{\beta(\gamma_r-\gamma)}\delta(rx)\, u_r^{\gamma_r}\,(\ln(u(rx)) - \beta\ln(r))\left(\frac{d}{dr}\gamma_r\right)\chi_{\{u_r>0\}}.$$

Therefore,

$$\begin{aligned}(C)+(D.2) &= r^{-(n+\beta\gamma+1)}\int_{B_r}\beta(\gamma(x)-\gamma)\delta(x)u^{\gamma(x)}\chi_{\{u>0\}}\\ &\quad + r^{-(n+\beta\gamma+1)}\int_{B_r}\delta(x)\ln(u)u^{\gamma(x)}(D\gamma(x)\cdot x)\chi_{\{u>0\}}.\end{aligned}$$

This implies that

$$\begin{aligned}\frac{d}{dr}\overline{W}_u(r) &= \frac{2}{r}\int_{\partial B_1}|\partial_\nu u_r - \beta u_r|^2\\ &\quad + r^{-(n+\beta\gamma+1)}\int_{B_r}\beta(\gamma(x)-\gamma)\delta(x)u^{\gamma(x)}\chi_{\{u>0\}}\\ &\quad + r^{-(n+\beta\gamma+1)}\int_{B_r}\delta(x)\ln(u)u^{\gamma(x)}(D\gamma(x)\cdot x)\chi_{\{u>0\}}\\ &\quad + r^{-(n+\beta\gamma+1)}\int_{B_r}(D\delta(x)\cdot x)u^{\gamma(x)}\chi_{\{u>0\}}.\end{aligned}$$

Now, recalling the definition of $W_{u,0}(r)$, we have

$$\begin{aligned}\frac{d}{dr}W_{u,0}(r) &= \frac{d}{dr}\overline{W}_u(r) - r^{-(n+\beta\gamma+1)}\int_{B_r}\beta(\gamma(x)-\gamma)\delta(x)u^{\gamma(x)}\chi_{\{u>0\}}\\ &\quad - r^{-(n+\beta\gamma+1)}\int_{B_r}\delta(x)\ln(u)u^{\gamma(x)}(D\gamma(x)\cdot x)\chi_{\{u>0\}}\\ &\quad - r^{-(n+\beta\gamma+1)}\int_{B_r}(D\delta(x)\cdot x)u^{\gamma(x)}\chi_{\{u>0\}},\end{aligned}$$

which implies, by our previous computations, that

$$\frac{d}{dr}W_{u,0}(r) = \frac{2}{r}\int_{\partial B_1}|\partial_\nu u_r - \beta u_r|^2 \geq 0.$$

□

As a consequence of the monotonicity formula, we obtain the homogeneity of blow-ups.

Definition 3.1 (Blow-up). *Given a point $z_0 \in F(u)$, we say that u_0 is a blow-up of u at z_0 if the family $\{u_r\}_{r>0}$, defined by*

$$u_r(x) := \frac{u(z_0+rx)}{r^{\beta(z_0)}}, \quad \textit{with} \quad \beta(z_0) := \frac{2}{2-\gamma(z_0)},$$

converges, through a subsequence, to u_0, when $r \to 0$.

We say u_0 is $\beta(z_0)$-homogeneous if

$$u_0(\lambda x) = \lambda^{\beta(z_0)}u_0(x), \quad \forall\lambda > 0,\ \forall x \in \mathbb{R}^n.$$

Unlike in the constant case $\gamma(x) \equiv \gamma_0$, the homogeneity property of blow-ups will vary depending on the free boundary point we are considering. This is the object of the following result.

Corollary 3.1. *Let u be a local minimizer of (1.5) and assume (3.31) and (3.32) are in force. If u_0 is a blow-up of u at a point $z_0 \in F(u) \cap B_{1/2}$, then u_0 is $\beta(z_0)$-homogeneous.*

Proof. Without loss of generality, we assume $z_0 = 0$. Recall

$$\beta := \frac{2}{2-\gamma} \quad \text{where} \quad \gamma := \gamma(0).$$

In order to ease the notation, for each $j \in \mathbb{N}$, we will write γ_j instead of $\gamma(\lambda_j x)$, and define

$$\begin{aligned}
W_v^j(r) \quad := \quad & r^{-(n+2(\beta-1))} \int_{B_r} \frac{1}{2}|Dv|^2 + \lambda_j^{\beta(\gamma_j-\gamma)} v^{\gamma_j} \delta(\lambda_j x) \chi_{\{v>0\}} \\
- \quad & \beta r^{-((n-1)+2\beta)} \int_{\partial B_r} v^2 \\
- \quad & \int_0^r \beta t^{-(n+\beta\gamma+1)} \int_{B_t} (\gamma_j - \gamma) \lambda_j^{\beta(\gamma_j-\gamma)} \delta(\lambda_j x) v^{\gamma_j} \chi_{\{v>0\}} \\
- \quad & \int_0^r t^{-(n+\beta\gamma+1)} \int_{B_t} \left(D\gamma(\lambda_j x)\cdot x\right) \lambda_j^{\beta(\gamma_j-\gamma)+1} \delta(\lambda_j x) v^{\gamma_j} \ln(\lambda_j^{\beta} v) \chi_{\{v>0\}} \\
- \quad & \int_0^r t^{-(n+\beta\gamma+1)} \int_{B_t} \left(D\delta(\lambda_j x)\cdot x\right) \lambda_j^{\beta(\gamma_j-\gamma)+1} v^{\gamma_j} \chi_{\{v>0\}}
\end{aligned}$$

and

$$W_v^{\infty}(r) := r^{-(n+2(\beta-1))} \int_{B_r} \frac{1}{2}|Dv|^2 + \delta(0) v^{\gamma(0)} \chi_{\{v>0\}} - \beta r^{-((n-1)+2\beta)} \int_{\partial B_r} v^2.$$

We now show that

$$W_{u_0}^{\infty}(r) = \lim_{j\to\infty} W_{u_j}^j(r) \quad \text{as long as} \quad \lim_{j\to\infty} \lambda_j^{\beta(\gamma_j-\gamma)} \to 1.$$

Indeed,

$$\begin{aligned}
W_{u_j}^j(r) \quad = \quad & r^{-(n+2(\beta-1))} \int_{B_r} \frac{1}{2}|Du_j|^2 + \lambda_j^{\beta(\gamma_j-\gamma)} \delta(\lambda_j x) u_j^{\gamma_j} \chi_{\{u_j>0\}} \\
& -\beta r^{-((n-1)+2\beta)} \int_{\partial B_r} u_j^2 \\
& -\int_0^r \beta t^{-(n+\beta\gamma+1)} \int_{B_t} (\gamma_j - \gamma) \lambda_j^{\beta(\gamma_j-\gamma)} \delta(\lambda_j x) u_j^{\gamma_j} \chi_{\{u_j>0\}} \\
& -\int_0^r t^{-(n+\beta\gamma+1)} \int_{B_t} \left(D\gamma(\lambda_j x)\cdot x\right) \lambda_j^{\beta(\gamma_j-\gamma)+1} \delta(\lambda_j x) u_j^{\gamma_j} \ln(u(\lambda_j x)) \chi_{\{u_j>0\}} \\
& -\int_0^r t^{-(n+\beta\gamma+1)} \int_{B_t} \left(D\delta(\lambda_j x)\cdot x\right) \lambda_j^{\beta(\gamma_j-\gamma)+1} u_j^{\gamma_j} \chi_{\{u_j>0\}}
\end{aligned}$$

and scaling back to u, we obtain

$$\begin{aligned} W^j_{u_j}(r) &= (\lambda_j r)^{-(n+2(\beta-1))}\int_{B_{\lambda_j r}} \frac{1}{2}|Du|^2+\delta(x)u^{\gamma(x)}\chi_{\{u>0\}} \\ &\quad -\beta(\lambda_j r)^{-((n-1)+2\beta)}\int_{\partial B_{\lambda_j r}} u^2 \\ &\quad -\int_0^r \beta t^{-(n+\beta\gamma+1)}\lambda_j^{-(n+\beta\gamma)}\int_{B_{\lambda_j t}}(\gamma(x)-\gamma)\delta(x)u^{\gamma(x)}\chi_{\{u>0\}} \\ &\quad -\int_0^r t^{-(n+\beta\gamma+1)}\lambda_j^{-(n+\beta\gamma)}\int_{B_{\lambda_j t}}(D\gamma(x)\cdot x)\,\delta(x)u^{\gamma(x)}\ln\left(u(x)\right)\chi_{\{u>0\}} \\ &\quad -\int_0^r t^{-(n+\beta\gamma+1)}\lambda_j^{-(n+\beta\gamma)}\int_{B_{\lambda_j t}}(D\delta(x)\cdot x)\,u^{\gamma(x)}\chi_{\{u>0\}}. \end{aligned}$$

Changing variables in the last three integrals, we reach

$$\begin{aligned} W^j_{u_j}(r) &= (\lambda_j r)^{-(n+2(\beta-1))}\int_{B_{\lambda_j r}} \frac{1}{2}|Du|^2+\delta(x)u^{\gamma(x)}\chi_{\{u>0\}} \\ &\quad -\beta(\lambda_j r)^{-((n-1)+2\beta)}\int_{\partial B_{\lambda_j r}} u^2 \\ &\quad -\int_0^{\lambda_j r} \beta t^{-(n+\beta\gamma+1)}\int_{B_t}(\gamma(x)-\gamma)\delta(x)u^{\gamma(x)}\chi_{\{u>0\}} \\ &\quad -\int_0^{\lambda_j r} t^{-(n+\beta\gamma+1)}\int_{B_t}(D\gamma(x)\cdot x)\,\delta(x)u^{\gamma(x)}\ln(u)\chi_{\{u>0\}} \\ &\quad -\int_0^{\lambda_j r} t^{-(n+\beta\gamma+1)}\int_{B_t}(D\delta(x)\cdot x)\,u^{\gamma(x)}\chi_{\{u>0\}}, \end{aligned}$$

and so

$$W^j_{u_j}(r) = W_u(\lambda_j r).$$

Therefore

$$W^\infty_{u_0}(r) = \lim_{j\to\infty} W^j_{u_j}(r) = \lim_{j\to\infty} W_u(\lambda_j r) = W_u(0^+),$$

where the last inequality is guaranteed by the monotonicity of the functional at the minimizer u. We conclude that $W^\infty_{u_0}$ is constant. We note that u_0 is a minimizer to the functional

$$\int_{B_R}\frac{1}{2}|Dv|^2+\delta(0)v^{\gamma(0)}\chi_{\{v>0\}}, \tag{3.33}$$

for every $R > 0$, and thus entitled to the regularity results from [3]. In particular, it follows, from [3, Lemma 7.1], that u_0 is $\beta(0)$-homogeneous. □

Remark 3.6. *To assure the existence of blow-ups, one needs to guarantee that the family* $(u_r)_{r>0}$, *defined as*

$$u_r(x) = \frac{u(z_0+rx)}{r^{\beta(z_0)}} \quad \textit{for} \quad \beta(z_0) = \frac{2}{2-\gamma(z_0)},$$

is locally bounded in $C^{1,\beta(z_0)-1}$*. Indeed, by Theorem 3.3, there exists a constant* $C' > 1$ *such that*

$$\|u_r\|_{L^\infty(B_1)} \le C'.$$

Moreover, by applying Theorem 3.1 to u *over* $B_r(z_0)$*, we obtain*

$$\operatorname{osc}_{B_r(z_0)}|Du| := \left(\sup_{B_r(z_0)} |Du|\right) - \left(\inf_{B_r(z_0)} |Du|\right) \le Cr^{\frac{\gamma_*(z_0,2r)}{2-\gamma_*(z_0,2r)}}.$$

Proceeding as at the end of the proof of Theorem 3.3, we use condition (3.16) to obtain

$$Cr^{\frac{\gamma_*(z_0,2r)}{2-\gamma_*(z_0,2r)}} \le \overline{C}r^{\frac{\gamma(z_0)}{2-\gamma(z_0)}},$$

which implies

$$\operatorname{osc}_{B_r(z_0)} \le \overline{C}r^{\frac{\gamma(z_0)}{2-\gamma(z_0)}}.$$

As a consequence, the family $\{u_r\}_{r>0}$ *is locally bounded in* $C^{1,\beta(z_0)-1}$.

Given the above, blow-up limits of minimizers of the variable singularity functional (1.5) are global minimizers of an energy-functional with constant singularity, namely $\gamma(z_0)$. Corollary 3.1 further yields that blow-ups are $\beta(z_0)$-homogeneous.

The pivotal insight here is that the blow-up limits of minimizers of the variable singularity functional are entitled to the same theoretical framework applicable to the constant coefficient case. In particular, in dimension $n = 2$, blow-up profiles are thoroughly classified due to [3, Theorem 8.2]. More precisely, if u_0 is the blow-up of u at $z_0 \in F(u)$, for u a local minimizer of (1.5) and $0 < \gamma(z_0) < 1$, then u_0 verifies

$$\frac{\beta(z_0)}{\sqrt{2}} u_0(x)^{\frac{1}{\beta(z_0)}} = \delta(z_0)((x - x_0) \cdot \nu)_+ \quad \text{for} \quad x \in \mathbb{R}^n,$$

for some $\nu \in \partial B_1$.

Classifying minimal cones in lower dimensions is crucial, chiefly because of Federer's dimension reduction argument that we will utilize in our upcoming session.

3.8 Free boundary regularity

In this final section, we investigate the regularity of the free boundary. For models with constant exponent γ, differentiability of the free boundary was obtained in [3], following the developments of [2]. Although it may seem plausible, the task of amending the arguments from [2, 3] to the case of oscillatory exponents – the object of study of this paper – proved quite intricate. More recently, similar free boundary regularity estimates have been obtained via a linearization argument in [35] (see also [33]). Here, we will adopt the latter strategy, *i.e.*, and proceed through an approximation technique, where the tangent models are the ones with constant γ.

More precisely, given a point $z_0 \in F(u) \cap B_{1/2}$, let us define

$$c_0(z_0) = \left[\frac{(\alpha(z_0) - 1)\alpha(z_0)}{\gamma(z_0)\delta(z_0)}\right]^{\frac{1}{\gamma(z_0)-2}}$$

and

$$w = c_0^{-\frac{1}{\alpha}} u^{1/\alpha},$$

for $\alpha := \alpha(z_0) = 2/(2 - \gamma(z_0))$. We note that since the equation holds within the set where u is positive, we have

$$\delta(x)\,\gamma(x)u^{\gamma(x)-1} = c_0\alpha w^{\alpha-2}\left[w\Delta w + (\alpha - 1)|Dw|^2\right],$$

and so

$$w\Delta w = \delta(x)\frac{\gamma(x)}{\alpha}c_0^{\gamma(x)-2}w^{\alpha(\gamma(x)-1)+2-\alpha} - (\alpha - 1)|Dw|^2.$$

Since

$$\alpha(\gamma(x) - 1) + 2 - \alpha = \alpha(\gamma(x) - \gamma(z_0)),$$

we can rewrite the equation as

$$\Delta w = \frac{h(x, w, Dw)}{w}, \tag{3.34}$$

where $h\colon B_1 \times \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ is defined as

$$h(x, s, \xi) = \delta(x)\frac{\gamma(x)}{\alpha}c_0^{\gamma(x)-2}s^{\alpha(\gamma(x)-\gamma(z_0))} - (\alpha - 1)|\xi|^2.$$

The crucial insight here is that given appropriate continuity conditions on $\gamma(x)$, we can achieve a uniform approximation of the classical Alt-Philips problem. To put it differently, the oscillatory exponent model will be uniformly close to the classical Alt-Philips functional. Since minimizers of the latter have smooth free boundaries, one should be able to infer the free boundary regularity of the former via compactness methods. To put this strategy into practice, though, we must first introduce and discuss some necessary tools.

We first remark that defining w_r as

$$w_r(x) = \frac{w(z_0 + rx)}{r}, \tag{3.35}$$

direct calculations yield

$$\Delta w_r = \frac{h_r(x, w_r, Dw_r)}{w_r},$$

where

$$\begin{aligned} h_r(y, s, \xi) \quad &= \quad \delta(z_0 + rx)\frac{\gamma(z_0 + rx)}{\alpha}c_0^{\gamma(z_0+rx)-2}(rs)^{\alpha(\gamma(z_0+rx)-\gamma(z_0))} \\ &\quad -(\alpha - 1)|\xi|^2. \end{aligned}$$

We can now pass to the limit as $r \to 0$, and in view of the choice of c_0, we reach

$$h_r(y, s, \xi) \to \overline{h}(z_0, \xi),$$

where $\overline{h}(z_0, \xi)$ is given by

$$\overline{h}(x_0, \xi) = (\alpha(z_0) - 1)(1 - |\xi|^2).$$

The second key remark is that if the exponent function $\gamma(x)$ is assumed to be Hölder continuous, say, of order $\mu \in (0, 1)$, then for a fixed $s > 0$, the above convergence does not depend on the free boundary point, $z_0 \in F(u) \cap B_{1/2}$. Indeed, we can estimate

$$\begin{aligned} |\alpha(z_0)(\gamma(z_0 + rx) - \gamma(z_0)) \ln(rs)| &\leq Cr^{\mu}|(\ln(r) + \ln(s)| \\ &\leq C([\gamma]_{C^{0,\mu}}, |\ln(s)|)r^{\frac{\mu}{2}}, \end{aligned}$$

which implies that

$$\lim_{r \to 0} (rs)^{\alpha(z_0)(\gamma(z_0+rx)-\gamma(z_0))} = 1,$$

uniformly in $z_0 \in F(u) \cap B_{1/2}$. Arguing similarly, one also obtains that

$$\lim_{r \to 0} \delta(z_0 + rx) \frac{\gamma(z_0 + rx)}{\alpha} c_0^{\gamma(z_0+rx)-2} = \alpha(z_0) - 1,$$

uniformly in $z_0 \in F(u) \cap B_{1/2}$. Here, we only need the uniform continuity of the ingredients involved.

The insights above are critical to ensure the linearized problem is uniformly close to the one with constant exponent as treated in [35]. To be more precise, we borrow the following improvement of flatness result, [35, Proposition 6.1], available for the constant exponent case.

Lemma 3.8. *Let $\overline{w}$ be a viscosity solution to*

$$\Delta \overline{w} = \frac{h(z_0, D\overline{w})}{\overline{w}} \quad in \quad \{\overline{w} > 0\}, \tag{3.36}$$

with $0 \in F^{vis}(\overline{w})$ and $z_0 \in B_{1/2}$. There exist $\epsilon_0, \eta > 0$ such that if $\epsilon \leq \epsilon_0$ and

$$(x_n - \epsilon)_+ \leq \overline{w} \leq (x_n + \epsilon)_+ \quad in \quad B_1,$$

then

$$\left(x \cdot \nu - \frac{\epsilon}{2}\eta\right)_+ \leq \overline{w} \leq \left(x \cdot \nu + \frac{\epsilon}{2}\eta\right)_+ \quad in \quad B_\eta,$$

with $|\nu| = 1$ and $|\nu - e_n| \leq C\epsilon$, for $C > 0$ universal.

It's important to note that in [35], and thus in Lemma 3.8, being a free boundary point conveys additional information. This is encoded in the free boundary condition held in the viscosity sense, as defined in [35, Definition 1.1]. We display the precise definition below for the readers' convenience.

Definition 3.2. *We say that $x_0 \in F^{vis}(\overline{w})$ in the viscosity sense if $x_0 \in F(\overline{w})$, and if $\psi \in C^2$ is such that ψ^+ touches $\overline{w}$ from below (resp., from above) at x_0, with $|D\psi(x_0)| \neq 0$, then*

$$|D\psi(x_0)| \leq 1 \quad (resp., |D\psi(x_0)| \geq 1).$$

Next, we will argue that, as the solutions we address in this paper arise from a variational problem, we can still employ the flatness improvement technique outlined in Lemma 3.8. The rationale behind this is explained in the sequel.

Let u be a minimizer to the functional (1.5) and $z_0 \in F(u)$. The *distorted* solution w, as defined before, solves (3.34). By optimal regularity, Theorem 3.3, Lipschitz rescalings of w defined as in (3.35) converge to a viscosity solution to (3.36), say $\overline{w}$. The rescalings are related to a sequence of the form

$$u_r(x) = \frac{u(z_0 + rx)}{r^\alpha}, \quad \text{for} \quad \alpha = \frac{2}{2 - \gamma(z_0)},$$

which is a minimizer to a scaled functional that converges to the one with constant $\gamma(x) \equiv \gamma(z_0)$. Thus, we get

$$\overline{w} = c_0^{-\frac{1}{\alpha}} \overline{u}^{\frac{1}{\alpha}},$$

for a minimizer $\overline{u}$ of the functional with constant exponent.

What is left to show is that $\overline{w}$ satisfies the free boundary condition as in Definition 3.2. However, as pointed out in [33], see also [36], this is a consequence of a one-dimensional analysis. For a free boundary point $x_0 \in F(\overline{u})$, there holds

$$\overline{u}(x_0 + t\nu) \approx c_0 t^\alpha,$$

where $t \geq 0$ small and ν is the unit normal pointing towards $\{\overline{u} > 0\}$.

With this well understood, we proceed with the discussion of another delicate issue in the program, namely the necessity to control the dependence of the constant C, appearing in Lemma 3.8, as the free boundary point z_0 varies. The results in [35] guarantee that this dependence will be contingent on the dimension and the C^1–norm of $\overline{h}(z_0, \xi)$ within a neighborhood of ∂B_1. Importantly, this norm remains uniformly bounded due to our assumptions regarding the range of the function $\gamma(x)$.

The discussions presented above bring us to the next crucial tool required in the proof of the free boundary regularity.

Lemma 3.9. *Let w be a solution to (3.34), $0 \in F(w)$ and $r, \epsilon > 0$ be two positive small parameters such that*

$$(x_n - \epsilon r)_+ \leq w \leq (x_n + \epsilon r)_+ \quad \text{in} \quad B_r.$$

Then, there exists $\eta > 0$ small enough such that

$$(x \cdot \nu - \eta \epsilon r)_+ \leq w \leq (x \cdot \nu + \eta \epsilon r)_+ \quad \text{in} \quad B_{\eta r}.$$

Proof. By considering $w_r(x) = r^{-1} w(rx)$, the flatness assumption reads as

$$(x_n - \epsilon)_+ \leq w_r \leq (x_n + \epsilon)_+ \quad \text{in} \quad B_1.$$

We will prove that there exist $\epsilon_0, \eta > 0$ such that

$$(x \cdot \nu - \eta\epsilon)_+ \leq w_r \leq (x \cdot \nu + \eta\epsilon)_+ \quad \text{in} \quad B_\eta,$$

for $r > 0$ small enough. By Theorem 3.3, it follows that w_r is bounded and Lipschitz continuous. Thus, $w_r \longrightarrow \overline{w}$, for some sequence $r \to 0$. By Lemma 3.8, there exist $\epsilon_0, \eta > 0$ such that

$$\left(x \cdot \nu - \frac{\epsilon}{2}\eta\right)_+ \leq \overline{w} \leq \left(x \cdot \nu + \frac{\epsilon}{2}\eta\right)_+ \quad \text{in} \quad B_\eta.$$

Observe that since we can restrict to the set where $\overline{w}$ is positive, for r small enough, we obtain

$$(x \cdot \nu - \epsilon\eta)_+ \leq w_r \leq (x \cdot \nu + \epsilon\eta)_+ \quad \text{in} \quad B_\eta,$$

as desired. □

Notice that, by taking $w_\eta = \eta^{-1}w(\eta x)$, the conclusion of Lemma 3.9 says that w_η satisfies

$$(x \cdot \nu - \epsilon r)_+ \leq w_\eta \leq (x \cdot \nu + \epsilon r)_+ \quad \text{in} \quad B_r.$$

By further composing with an orthogonal linear transformation, Lemma 3.9 leads to the existence of $\nu' \in \partial B_1$ such that $|\nu' - \nu| \leq C\epsilon/2$ and

$$(x \cdot \nu' - \eta\epsilon r)_+ \leq w_\eta \leq (x \cdot \nu' + \eta\epsilon r)_+ \quad \text{in} \quad B_{\eta r}.$$

Therefore,

$$(x \cdot \nu' - \eta^2\epsilon r)_+ \leq w \leq (x \cdot \nu' + \eta^2\epsilon r)_+ \quad \text{in} \quad B_{\eta^2 r}.$$

By induction, one gets a sequence $(\nu_k)_{k\in\mathbb{N}} \subset \partial B_1$ such that

$$|\nu_k - \nu_{k-1}| \leq C2^{-k}\epsilon$$

and

$$(x \cdot \nu_k - \eta^k\epsilon r)_+ \leq w \leq (x \cdot \nu_k + \eta^k\epsilon r)_+ \quad \text{in} \quad B_{\eta^k r}.$$

As a consequence, $F(w)$ is $C^{1,\delta}$ at 0.

We conclude by commenting on Federer's classical dimension reduction argument, [42], and how one can adapt it to the free boundary problem investigated in this paper.

We start by arguing, as explored above, that when $\gamma(x)$ is a continuous function, blow-ups converge to minimizers of the functional with constant exponent $\gamma(z_0)$. Now, at least in dimension $n = 2$, it is possible to classify them using ODE techniques, see [3]. Hence, a successful implementation of Federer's reduction argument will imply that the singular part of the free boundary, $\mathrm{Sing}(F(u))$, satisfies

$$\mathcal{H}^{n-2+s}(\mathrm{Sing}(F(u))) = 0 \quad \text{for every} \quad s > 0.$$

This, in particular, will allow us to conclude the portion of the free boundary to which Lemma 3.9 can be applied has total measure.

Here are the ingredients needed. Let $z_0 \in F(u)$ and define

$$u_r(x) := \frac{u(z_0 + rx)}{r^{\beta(z_0)}}, \quad \text{with } \beta(z_0) = \frac{2}{2-\gamma(z_0)}.$$

Such a family converges, up to a subsequence, to some function u_0 that is a minimizer to the Alt-Philips functional with constant exponent $\gamma(z_0)$. The first step is to establish the convergence of the singular sets of the family $\{u_r\}$ as $r \to 0$. This is a consequence of the sharp non-degeneracy, Theorem 3.2, and that the set of regular points is locally an open set because of our Lemma 3.9. Next, as a consequence of optimal regularity estimates and monotonicity formula, Corollary 3.1, blow-up limits of the family $\{u_r\}_r$ are homogeneous of degree $\beta(z_0)$. The final step of Federer's routine is to prove a dimension reduction result to the singular set of a global $\beta(z_0)$-homogeneous minimizer of the Alt-Philips functional with constant parameters. To do so, one must prove a sort of translation invariance of global minimizers. This part follows using similar arguments found in [36], and thus we omit it here.

The comprehensive discussion above leads to the regularity of the free boundary, which can be briefly summarized in the following theorem. We say a function belongs to W^{1,n^+} if it belongs to $W^{1,q}$, for some $q > n$.

Theorem 3.8. *Let u be a local minimizer of* (1.5) *and assume*

$$\gamma(x) \in W^{1,n^+}.$$

Then, the free boundary $F(u)$ is locally a $C^{1,\delta}$ surface, up to a negligible singular set of Hausdorff dimension less or equal to $n-2$.

Proof. With all the ingredients from the preceding discussion available, the proof is standard, and we only highlight the main steps.

We start by decomposing the free boundary as the disjoint union of its regular points and its singular points, that is,

$$F(u) = \text{Reg}(u) \cup \text{Sing}(u).$$

The set $\text{Reg}(u)$ stands for the points where blow-ups can be classified. More precisely, $z_0 \in \text{Reg}(u)$, if for a sequence of radii r_n converging to zero and a unitary vector ν, there holds

$$u_{r_n}(x) := \frac{u(z_0 + r_n x)}{r_n^{\frac{2}{2-\gamma(z_0)}}} \longrightarrow c_0 (x \cdot \nu)_+^{\frac{2}{2-\gamma(z_0)}}.$$

The set $\text{Sing}(u)$ is simply the complement of $\text{Reg}(u)$. That is

$$\text{Sing}(u) := F(u) \backslash \text{Reg}(u).$$

The dimension reduction argument mentioned earlier assures that

$$\mathcal{H}^{n-2+s}(\mathrm{Sing}(u)) = 0,$$

for all $s > 0$. Thus, one can estimate the Hausdorff dimension of the singular set as

$$\dim_{\mathcal{H}}(\mathrm{Sing}(u)) := \inf\{d : \mathcal{H}^d(\mathrm{Sing}(u)) = 0\} \leq n - 2 + s,$$

for every $s > 0$, and so

$$\dim_{\mathcal{H}}(\mathrm{Sing}(u)) \leq n - 2.$$

In particular, we conclude that $\mathrm{Sing}(u)$ is a negligible set with respect to the Hausdorff measure $\mathcal{H}^{n-1}$, *i.e.*,

$$\mathcal{H}^{n-1}(F(u) \setminus \mathrm{Reg}(u)) = 0.$$

Now, we show that $\mathrm{Reg}(u)$ is locally $C^{1,\delta}$, for some $\delta > 0$ universal. Consider $z_0 \in \mathrm{Reg}(u)$ and let u_0 be a blow-up limit of u at z_0. In other words, for a sequence $r = o(1)$, and up to a change of coordinates, there holds

$$u_r(x) := \frac{u(z_0 + rx)}{r^{\frac{2}{2-\gamma(z_0)}}} \longrightarrow c_0(x_n)_+^{\frac{2}{2-\gamma(z_0)}},$$

in the $C^{1,\frac{\gamma(z_0)}{2-\gamma(z_0)}}_{loc}(\mathbb{R}^n)$ topology. By such a convergence, one deduces that

$$c_0(x_n - \epsilon)_+^{\frac{2}{2-\gamma(z_0)}} \leq u_r(x) \leq c_0(x_n + \epsilon)_+^{\frac{2}{2-\gamma(z_0)}} \quad \text{in} \quad B_1.$$

As a consequence, we obtain

$$(x_n - \epsilon r)_+^{\frac{2}{2-\gamma(z_0)}} \leq c_0^{-1} u(z_0 + x) \leq (x_n + \epsilon r)_+^{\frac{2}{2-\gamma(z_0)}} \quad \text{in} \quad B_1.$$

Next, we define

$$w(x) := c_0^{-\frac{1}{\alpha(z_0)}} u(z_0 + x)^{\frac{1}{\alpha(z_0)}}, \quad \text{for } \alpha(z_0) = 2/(2 - \gamma(z_0)),$$

which is a function satisfying the assumptions of Lemma 3.9. Hence, scaling back to u the thesis of Lemma 3.9 and repeating the process inductively, keeping in mind the remarks previously noted, we conclude that $F(u)$ is $C^{1,\delta}$ at z_0.

By Hölder continuity of $\gamma(x)$ and the computations made at the beginning of the section, the proximity condition in Lemma 3.9 is uniform in $z_0 \in F(u) \cap B_{1/2}$. By the boundedness assumption on $\gamma(x)$, the constant C in Lemma 3.8 is universally bounded, and therefore $F(u)$ is locally $C^{1,\delta}$, with universal estimates. □

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