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On weighted Adams type inequalities and applications

por

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João Pessoa - PB
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sob orientação

Orientador: Prof. Dr. Manassés Xavier de Souza
Coorientador: Prof. Dr. Uberlandio Batista Severo

Tese apresentada ao Corpo Docente do Programa de Pós-Graduação em Matemática - UFPB, como requisito parcial para obtenção do título de Doutor em Matemática.

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Resumo

Este trabalho trata de algumas classes de desigualdades do tipo Adams envolvendo potenciais e pesos que podem decair a zero no infinito. A partir dessas desigualdades, estabelecemos resultados de compacidade e resultados de concentração-compacidade. Como aplicações dessas desigualdades de Adams com peso, usando métodos minimax, provamos a existência de soluções para algumas classes de problemas elípticos envolvendo o operador biharmônico em $\mathbb{R}^4$ e o operador poliharmônico em $\mathbb{R}^{2m}$, onde o termo não linear pode ter crescimento exponencial crítico no sentido de Trudinger-Moser. Além disso, em alguns casos, provamos que as soluções obtidas são limitadas em L^2, ou seja, são "bound state solutions".

Palavras-chave: Métodos variacionais; Desigualdade de Adams; Crescimento exponencial; Equações elípticas de ordem superior.

Abstract

This work deals with some classes of Adams-type inequalities involving potentials and weights that can decay to zero at infinity. From these inequalities, we establish compactness results and concentration-compactness results. As applications of these weighted Adams inequalities, using minimax methods, we prove the existence of solutions to some classes of elliptic problems involving the biharmonic operator on $\mathbb{R}^4$ and the polyharmonic operator on $\mathbb{R}^{2m}$, where the nonlinear term has critical exponential growth in the Trudinger-Moser sense. Furthermore, in some cases we prove that the solutions obtained are bounded in L^2, which are the so-called *bound state solutions*.

Keywords: Variational methods; Adams inequality; Exponential growth; Higher-order elliptic equations.

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"A educação tem raízes amargas, mas os seus frutos são doces."

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Dedicatória

À minha família, em especial a minha avó (in memoriam).

Contents

Introduction

In this thesis, we study weighted Adams-type inequalities and some of their consequences in the study of elliptic equations involving biharmonic and polyharmonic operators. These inequalities are natural generalizations for Sobolev spaces involving higher-order derivatives of the famous Trudinger-Moser inequality. Let us see a brief review of the literature on the main results involving Adams' inequalities that motivated the development of this work.

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain with $N \geq 2$. The classical Sobolev theorem states that the embedding $W_0^{1,N}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous for all $1 \leq q < \infty$, but $W_0^{1,N}(\Omega) \not\hookrightarrow L^\infty(\Omega)$. In this limiting case, the optimal embedding involves an Orlicz space, which was studied by Yudovich [74], Pohozaev [57], Peetre [56], Trudinger [66] and Moser [51]. As Ω is a bounded domain, using the Dirichlet norm $\|\nabla u\|_N := \left(\int_\Omega |\nabla u|^N\right)^{1/N}$ (equivalent to the Sobolev norm $\|u\|_{1,N} := [\int_\Omega(|\nabla u|^N + |u|^N)]^{1/N}$ in $W_0^{1,N}(\Omega)$), it was established, by Moser [51], that

$$\sup_{\{u \in W_0^{1,N}(\Omega)\,:\,\|\nabla u\|_N \leq 1\}} \int_\Omega e^{\gamma |u|^{\frac{N}{N-1}}}\,\mathrm{d}x \leq C_N |\Omega|, \tag{1}$$

for any $\gamma \leq \gamma_N := N\omega_{N-1}^{\frac{1}{N-1}}$, where $|\Omega|$ denotes the Lebesgue measure of Ω and ω_{N-1} is the measure of the unit sphere in $\mathbb{R}^N$. Moreover, γ_N is the best constant in the following sense: the integral on the left of the previous inequality is finite for any $\gamma > 0$, but if $\gamma > \gamma_N$ it can be made arbitrarily large by an appropriate choice of u and the supremum is infinity.

In the literature, (1) is known under the name *Trudinger-Moser inequality.* This result has been generalized and extended in many directions. Related results and variants can be found in several papers, see for instance, Li [41, 42] and Yang [68, 72] for generalizations to functions on compact Riemannian manifolds. When $\Omega = \mathbb{R}^N$, we cite Adachi and Tanaka [1], Adimurthi and Yang [5], Cao [14], do Ó [27, 29], de Souza and do Ó [26], Ruf [60], Li and Ruf [43], Yang and Zhu [73].

The Adams inequality is the extension of the Trudinger-Moser inequality for the higher-order derivatives and says that for $m \in \mathbb{N}$ and Ω an open bounded set of $\mathbb{R}^N$ with $m < N$, there exists a positive constant $C_{m,N}$ such that

$$\sup_{\{u \in W_0^{m,\frac{N}{m}}(\Omega)\,:\,\|\nabla^m u\|_{\frac{N}{m}} \leq 1\}} \int_\Omega e^{\gamma |u|^{\frac{N}{N-m}}} \mathrm{d}x \leq C_{m,N}|\Omega|, \tag{2}$$

for any $\gamma \leq \gamma_{m,N}$, where

$$\gamma_{m,N} := \frac{N}{\omega_{N-1}} \begin{cases} \left[\dfrac{\pi^{\frac{N}{2}} 2^m \Gamma(\frac{m}{2})}{\Gamma\left(\frac{N-m}{2}\right)}\right]^{\frac{N}{N-m}}, & \text{if } m \text{ is even,} \\ \left[\dfrac{\pi^{\frac{N}{2}} 2^m \Gamma(\frac{m+1}{2})}{\Gamma\left(\frac{N-m+1}{2}\right)}\right]^{\frac{N}{N-m}}, & \text{if } m \text{ is odd,} \end{cases}$$

and

$$\nabla^m u := \begin{cases} \Delta^{\frac{m}{2}} u, & m \text{ is even,} \\ \nabla \Delta^{\frac{m-1}{2}} u, & m \text{ is odd} \end{cases}$$

denotes the mth-order gradient of u. Furthermore, inequality (2) is sharp, that is, for any $\gamma > \gamma_{m,N}$, the integral in (2) can be made as large as possible (see Adams [2] for details). In the particular case when $N = 4$ and $m = 2$, inequality (2) becomes

$$\sup_{\{u \in H_0^2(\Omega)\,:\,\|\Delta u\|_2 \leq 1\}} \int_\Omega e^{\gamma u^2} \mathrm{d}x \begin{cases} \leq C|\Omega| & \text{if } \gamma \leq 32\pi^2, \\ = +\infty & \text{if } \gamma > 32\pi^2. \end{cases} \tag{3}$$

Similar to the classic Trudinger-Moser inequality, Adams inequality in its form (2) cannot be extended to unbounded domains. As far as the author knows, the first attempt to generalize Adams's inequality to unbounded domains was due to Ozawa in [54]. He proved the existence of two positive constants γ and C such that

$$\int_{\mathbb{R}^N} \phi_{m,N}\left(\gamma |u|^{\frac{N}{N-m}}\right) \mathrm{d}x \leq C\|u\|_{\frac{N}{m}}^{\frac{N}{m}}, \quad \text{for all } \; u \in W^{m,\frac{N}{m}}(\mathbb{R}^N) \; \text{ with } \; \|\nabla^m u\|_{\frac{N}{m}} \leq 1, \tag{4}$$

where

$$\phi_{m,N}(t) := e^t - \sum_{j=0}^{j_{\frac{N}{m}}-2} \frac{t^j}{j!}, \;\; m \in \mathbb{N}, \;\; N > m, \;\; j_{\frac{N}{m}} := \min\left\{ j \in \mathbb{N} \;:\; j \geq \frac{N}{m} \right\} \; \text{ and } \; t \geq 0.$$

In [61], Ruf and Sani established the following very useful result: let $m = 2k$ be an even integer less than N. Then, there exists a constant $C_{m,N} > 0$ such that, for any domain $\Omega \subset \mathbb{R}^N$, one has

$$\sup_{\{u\in W_0^{m,\frac{N}{m}}(\Omega)\,:\,\|(-\Delta+I)^k u\|_{\frac{N}{m}}\le 1\}} \int_\Omega \phi_{m,N}\left(\gamma_{m,N}|u|^{\frac{N}{N-m}}\right) \mathrm{d}x \le C_{m,N}, \tag{5}$$

and this inequality is also sharp. In the particular case $N = 4$, $m = 2$ and $\Omega = \mathbb{R}^4$, inequality (5) becomes

$$\sup_{\{u\in H^2(\mathbb{R}^4)\,:\,\|(-\Delta+I)u\|_2\le 1\}} \int_{\mathbb{R}^4} \left(e^{32\pi^2 u^2} - 1\right) \mathrm{d}x < \infty. \tag{6}$$

Many other improvements of the Adams inequality in bounded and unbounded domains have been proved. We can refer to Lam and Lu [38, 40], Masmoudi and Sani [47], Lu and Yang [46], Tarsi [65]. Among them, we would like to emphasize the result proved by Masmoudi and Sani [47], which complements some of the works cited previously. Precisely, in [47], it was proved that, for any $\gamma \in (0, 32\pi^2)$, there exists a constant $C = C(\gamma) > 0$ such that

$$\int_{\mathbb{R}^4} (e^{\gamma u^2} - 1)\,\mathrm{d}x \le C\|u\|_2^2, \quad \text{for all} \quad u \in H^2(\mathbb{R}^4) \quad \text{with} \quad \|\Delta u\|_2 \le 1,$$

and this inequality is not valid if $\gamma \ge 32\pi^2$.

The Trudinger-Moser inequality (1) has motivated many works on the existence of solutions to second-order nonlinear elliptic problems, we can cite for example Adimurthi et al. [3, 4]. The researcher's de Figueiredo, Miyagaki, and Ruf [23], following the ideas introduced by Brezis and Nirenberg [13], studied the existence of a solution for the equation $-\Delta u = f(x, u)$ in a bounded domain of $\mathbb{R}^2$, when the nonlinear term f has subcritical and critical exponential growth.

The Adams inequality (2) motivated the study of elliptic problems involving the polyharmonic operator in bounded domains. For example, Lakkis [37] considered the following quasilinear elliptic equation in $\Omega \subset \mathbb{R}^N$:

$$\begin{cases} (-\Delta)^m u = g(u) & \text{in } \Omega, \\ u = \nabla u = \cdots = \nabla^{m-1} u = 0, & \text{on } \partial\Omega, \end{cases}$$

with $N = 2m$ and the nonlinearity g having critical exponential growth. We can also quote the article by Lam and Lu [39], where they studied the existence of solutions for a nonlinear polyharmonic problem with nonlinearity having subcritical and critical exponential growth bounded domains in $\mathbb{R}^{2m}$. Sani in [62, 63] studied a biharmonic equation in $\mathbb{R}^4$ involving nonlinearities with subcritical and critical exponential growth.

In this context, we study Adams-type inequalities and some applications. To reach our goal, we divided this work into three chapters. In what follows, we describe each of the chapters.

In Chapter 1, motivated by the results in [30, 33], our first aim is to investigate a new Adams inequality in $\mathbb{R}^4$ involving a potential V and a weight K which can decay to zero at infinity. Moreover, as an application, we establish the existence of solutions for a class of elliptic biharmonic equations in the critical growth range.

In a more precise way, we consider functions $V(x)$ and $K(x)$ satisfying the following assumptions:

(V) $V \in C(\mathbb{R}^4)$ and there exist $\alpha, a > 0$ such that $V(x) \geq \dfrac{a}{1+|x|^\alpha}$ for all $x \in \mathbb{R}^4$;

(K) $K \in C(\mathbb{R}^4)$ and there exist $\beta, b > 0$ such that $0 < K(x) \leq \dfrac{b}{1+|x|^\beta}$ for all $x \in \mathbb{R}^4$.

In particular, we restrict our attention to the case when α and β satisfy

$$\alpha \in (0,4) \quad \text{and} \quad \beta \in [\alpha, +\infty). \tag{7}$$

Let us consider the following space:

$$E := \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^4) \,:\, |\nabla u|,\, \Delta u \in L^2(\mathbb{R}^4) \text{ and } \int_{\mathbb{R}^4} V(x) u^2 \mathrm{d}x < \infty \right\} \tag{8}$$

endowed with the inner product $\langle u, v\rangle_E := \displaystyle\int_{\mathbb{R}^4} (\Delta u \Delta v + \nabla u \nabla v + V(x) uv)\,\mathrm{d}x$ and its corresponding norm $\|u\| := \left(\displaystyle\int_{\mathbb{R}^4} (|\Delta u|^2 + |\nabla u|^2 + V(x)u^2)\,\mathrm{d}x \right)^{1/2}$.

In this context, our first result is a weighted version of Adams inequality, as follows:

Theorem 0.0.1 *Suppose that (V) and (K) hold with α and β satisfying (7). Then, for any $\gamma > 0$ and any $u \in E$, it holds*

$$\int_{\mathbb{R}^4} K(x)(e^{\gamma u^2} - 1)\,\mathrm{d}x < \infty. \tag{9}$$

Moreover, we have

$$\sup_{\substack{u\in E \\ \|u\|\leq 1}} \int_{\mathbb{R}^4} K(x)(e^{\gamma u^2} - 1)\,\mathrm{d}x = \begin{cases} < \infty, & \text{if} \quad 0 < \gamma < 32\pi^2; \\ +\infty, & \text{if} \quad \gamma > 32\pi^2. \end{cases} \tag{10}$$

As initial applications of Theorem 0.0.1, we will prove the compact embedding of the space E into $L^p_K(\mathbb{R}^4)$ for $p \geq 2, \alpha \in (0,4)$ and $\beta \in (\alpha, +\infty)$. We also will obtain a Lions-type Concentration-Compactness Principle involving exponential growth, which is a refinement of Theorem 0.0.1 (see Proposition 1.3.2). Furthermore, we will investigate the existence of weak solutions for the following class of problems:

$$\Delta^2 u - \Delta u + V(x)u = K(x)f(x,u) \quad \text{in} \quad \mathbb{R}^4, \tag{11}$$

where the potential V and the weight K satisfy the conditions (V) and (K), respectively, and the nonlinearity $f(x,s)$ has the maximal growth which allows us to study (11) by using variational methods. Precisely, motivated by (10), we say that $f(x,s)$ has *critical exponential growth* if there exists $\gamma_0 > 0$ such that

$$\lim_{|s|\to\infty} \frac{f(x,s)}{e^{\gamma s^2}} = \begin{cases} 0, & \text{for all } \gamma > \gamma_0, \\ +\infty, & \text{for all } \gamma < \gamma_0, \end{cases}$$

uniformly in $x \in \mathbb{R}^4$.

We require the following assumptions on the nonlinearity $f(x,s)$:

(H_1) $\displaystyle\lim_{s\to 0} \frac{f(x,s)}{s} = 0$, uniformly in $x \in \mathbb{R}^4$;

(H_2) the function $f : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}$ is continuous and has critical exponential growth;

(H_3) there exists $\mu > 2$ such that

$$0 < \mu F(x,s) \leq s f(x,s), \quad \text{for all} \quad (x,s) \in \mathbb{R}^4 \times \mathbb{R} \setminus \{0\};$$

(H_4) there exist constants $s_0, M_0 > 0$ such that

$$F(x,s) \leq M_0 |f(x,s)|, \quad \text{for all} \quad |s| \geq s_0 \quad \text{and} \quad x \in \mathbb{R}^4;$$

(H_5) We also consider one of the following assumptions:

(i) there exist $\theta, \theta_0 > 0$ such that $\displaystyle\liminf_{s\to\infty} \frac{F(x,s)}{s^\theta e^{\gamma_0 s^2}} \geq \theta_0$, uniformly in $x \in \mathbb{R}^4$;

(ii) $\displaystyle\liminf_{s\to\infty} \frac{sf(x,s)}{e^{\gamma_0 s^2}} \geq \theta_0'$ for some $\theta_0' > \frac{64}{\gamma_0 \tilde{K}}$, uniformly in $x \in \mathbb{R}^4$, where $\tilde{K} = \displaystyle\min_{x \in \overline{B}_1} K(x)$.

Under these conditions we can prove an existence result that says the following:

Theorem 0.0.2 *Assume* $(V), (K)$ *and* $(H_1) - (H_5)$ *hold with* $\alpha \in (0, 4)$ *and* $\beta \in (\alpha, +\infty)$*. Then, problem* (11) *has a nontrivial weak solution.*

Denoting by u_0 the weak solution obtained in the previous theorem, our next result can be stated as follows:

Theorem 0.0.3 *Assume* $(V), (K)$ *and* $(H_1) - (H_5)$ *hold with* $\alpha \in (0, 2)$ *and* $\beta \in (\alpha, +\infty)$*. Then,* $u_0 \in H^2(\mathbb{R}^4)$*, that is,* u_0 *is a bound state solution of problem* (11)*.*

In this chapter, we improve and extend some results obtained in [6, 7, 8, 9, 11, 12, 15, 18, 21, 22, 49, 63, 67, 69, 75], in the sense that we have considered nonlinearities with critical exponential growth and $V(x), K(x)$ can vanish at infinity, and they do not need to be radial. These features considered here are not treated in these previous works. Finally, as far as we know, there are few results involving biharmonic operators and vanishing potentials.

In Chapter 2, inspired by [30, 64], we study the solvability of the following biharmonic Choquard equation:

$$\Delta^2 u - \Delta u + V(x)u = \left[|x|^{-\mu} * (K(x)F(x,u))\right] K(x)f(x,u), \ x \in \mathbb{R}^4, \tag{12}$$

where the functions V and K satisfy the hypotheses (V) and (K) defined in the first chapter.

Equations like (12) arise in various branches of applied mathematics and physics, see [34, 35, 36, 55] and references therein. For instance, part of the interest is becouse that solutions of (12) are related to the existence of solitary wave solutions for Schrödinger equations of the form

$$i\frac{\partial \psi}{\partial t} = \Delta^2 \psi - \Delta \psi + W(x)\psi - \left[|x|^{-\mu} * (K(x)F(x,\psi))\right] K(x)f(x,\psi),$$

where $\psi : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{C}$ is an unknown function and $W : \mathbb{R}^4 \to \mathbb{R}$ is the potential function. For the physical interest in the influence of the biharmonic term in nonlinear Schrödinger equations, we refer the reader to [24] and references therein. For the study of problem (12) involving the biharmonic operator, it will be necessary to enlarge the range of variation of the constants α and β adopted in [64].

Precisely, we will assume the following conditions:

$$0 < \alpha < 4 \quad \text{and} \quad \frac{(8-\mu)\alpha}{8} \leq \beta < \infty, \ \text{with} \ \ \mu \in (0, 4). \tag{13}$$

Considering the space E defined in (8), we can establish the following weighted version of the Adams inequality obtained in [2]:

Theorem 0.0.4 *Suppose that* (V), (K) *and* (13) *hold. Then, for all* $\gamma > 0$ *and any* $u \in E$, *we have*

$$\int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)(e^{\gamma u^2} - 1)\,\mathrm{d}x < \infty. \tag{14}$$

Moreover, we can conclude that

$$\sup_{\{u\in E\,:\,\|u\|\le 1\}} \int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)(e^{\gamma u^2} - 1)\,\mathrm{d}x = \begin{cases} < \infty & \text{if} \quad \gamma \in (0, 32\pi^2); \\ +\infty & \text{if} \quad \gamma > 32\pi^2. \end{cases} \tag{15}$$

The proof of this result follows the same line as Theorem 0.0.1.

To control the nonlocal term $|x|^{-\mu} * (K(x)F(x,u))$ of (12), we need the well-known Hardy-Littlewood-Sobolev inequality, which we state in $\mathbb{R}^N$ and that will play an important role in Chapter 2 (see Proposition 2.1.2). We assume that the nonlinearity $f(x,s)$ satisfies the condition (H_4) and the hypotheses:

(H_1') $f \in C(\mathbb{R}^4 \times \mathbb{R})$, f has critical exponential growth, $f(x,s) = 0$ for all $(x,s) \in \mathbb{R}^4 \times (-\infty, 0]$ and $f(x,s) = o(s^{\frac{4-\mu}{4}})$ as $s \to 0^+$, uniformly in $x \in \mathbb{R}^4$;

(H_2') $0 \le H(x,t) \le H(x,s)$ for all $0 < t < s$ and for $x \in \mathbb{R}^4$, where $H(x,u) = uf(x,u) - F(x,u)$;

(H_3') $\liminf_{s\to+\infty} \dfrac{F(x,s)}{e^{\gamma_0 s^2}} =: \beta_0 > 0$, uniformly in $x \in \mathbb{R}^4$.

We emphasize that during Chapter 2 we do not use the famous Ambrosetti-Rabinowitz condition (H_3).

We are ready to state the main existence results of Chapter 2. The first result is the following:

Theorem 0.0.5 *Assume* (V) *and* (K) *hold with* $\alpha \in (0,4)$ *and* $\beta > (8-\mu)\alpha/8$. *If* f *satisfies* $(H_1') - (H_3')$ *and* (H_4), *then* (12) *admits a nontrivial weak solution in* E.

In the next result, by restricting the range of α, we will show that the solution obtained in the above theorem is $L^2(\mathbb{R}^4)$ and thus belongs to $H^2(\mathbb{R}^4)$, that is, the solution is a bound state. The statement can be expressed as follows:

Theorem 0.0.6 *Suppose that* (V) *and* (K) *hold with* $\alpha \in (0,2)$ *and* $\beta > (8-\mu)\alpha/8$. *If* f *satisfies* $(H_1') - (H_3')$ *and* (H_4), *then the solution obtained in the previous result is a bound state.*

In Chapter 3, we prove a weighted Adams inequality in $\mathbb{R}^N$, that is, much more general than the one proved in Chapter 1 (see Theorem 0.0.1). As an application we study a class of elliptic equations involving the polyharmonic operator.

We consider functions $V(x)$ and $K(x)$ satisfying the following assumptions:

(V') $V \in C(\mathbb{R}^N)$ and there exist $\alpha, a > 0$ such that $V(x) \geq \dfrac{a}{1+|x|^\alpha}$ for all $x \in \mathbb{R}^N$;

(K') $K \in C(\mathbb{R}^N)$ and there exist $\beta, b > 0$ such that $0 < K(x) \leq \dfrac{b}{1+|x|^\beta}$ for all $x \in \mathbb{R}^N$;

and we restrict our attention to the case when α and β satisfy

$$\alpha \in (0, N) \quad \text{and} \quad \beta \in [\alpha, +\infty). \tag{16}$$

Next, to present our first result, we will fix some notations. For m integer such that $m \geq 2$ and $m < N$, we consider the space

$$E := \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^N) \,:\, |\nabla^i u| \in L^{\frac{N}{m}}(\mathbb{R}^N) \ \forall\, i = 1, \dots, m \text{ and } \int_{\mathbb{R}^N} V(x)|u|^{\frac{N}{m}} \mathrm{d}x < \infty \right\}$$

and norm $\|u\| := \left(\int_{\mathbb{R}^N} (|\nabla^m u|^{N/m} + \cdots + |\nabla u|^{N/m} + V(x)|u|^{N/m}) \,\mathrm{d}x \right)^{m/N}$.

In this context, we can establish the following result.

Theorem 0.0.7 *Suppose that (V') and (K') hold with α and β satisfying (16). Then, for any $\gamma > 0$ and any $u \in E$, it holds*

$$\int_{\mathbb{R}^N} K(x)\phi_{m,N}(\gamma |u|^{\frac{N}{N-m}}) \,\mathrm{d}x < \infty. \tag{17}$$

Moreover, we have

$$\sup_{\substack{u \in E \\ \|u\| \leq 1}} \int_{\mathbb{R}^N} K(x)\phi_{m,N}(\gamma |u|^{\frac{N}{N-m}}) \,\mathrm{d}x = \begin{cases} < \infty, & \text{if } \gamma < \gamma_{m,N}; \\ +\infty, & \text{if } \gamma > \gamma_{m,N}. \end{cases} \tag{18}$$

As initial application of Theorem 0.0.7, we will prove the compact embedding of the space E into $L^p_K(\mathbb{R}^N)$ for $p \geq 2, \alpha \in (0, N)$ and $\beta \in (\alpha, +\infty)$ (see Proposition 3.3.1).

For the second part of the chapter, we assume that the integer $m \geq 2$ and the dimension N of the domain satisfy $N = 2m$. We also will obtain a Lions-type concentration-compactness principle involving exponential growth (see Proposition 3.5.3), which is a refinement of Theorem 0.0.7.

Furthermore, we will investigate the existence of weak solutions for the following class of problems:

$$\sum_{j=1}^{m}(-\Delta)^j u + V(x)u = K(x)f(x,u) \quad \text{in} \quad \mathbb{R}^{2m}, \tag{19}$$

where the potential V and the weight K satisfy the conditions (V') and (K'), respectively and α and β are such that

$$\alpha \in (0, 2m) \quad \text{and} \quad \beta \in (\alpha, +\infty). \tag{20}$$

In this case as $N = 2m$, we have the norm in space E is characterized as $\|u\|^2 = \langle u, u\rangle_E$ where

$$\langle u, v\rangle_E = \int_{\mathbb{R}^{2m}} \left(\sum_{j=1}^{m} \nabla^j u \nabla^j v + V(x)uv \right) \mathrm{d}x.$$

We assume that the nonlinearity $f(x,s)$ satisfes the assumptions:

$(\widetilde{H}_1)$ $\lim\limits_{s\to 0} \dfrac{f(x,s)}{s} = 0$, uniformly in $x \in \mathbb{R}^{2m}$;

$(\widetilde{H}_2)$ the function $f : \mathbb{R}^{2m} \times \mathbb{R} \to \mathbb{R}$ is continuous and has critical exponential growth;

$(\widetilde{H}_3)$ there exists $\mu > 2$ such that

$$0 < \mu F(x,s) \leq sf(x,s), \quad \text{for all} \quad (x,s) \in \mathbb{R}^{2m} \times \mathbb{R} \setminus \{0\};$$

$(\widetilde{H}_4)$ there exist constants $s_0, M_0 > 0$ such that

$$F(x,s) \leq M_0|f(x,s)|, \quad \text{for all} \quad |s| \geq s_0 \quad \text{and} \quad x \in \mathbb{R}^{2m};$$

$(\widetilde{H}_5)$ $\liminf\limits_{s\to\infty} \dfrac{sf(x,s)}{e^{\gamma_0 s^2}} \geq \theta_0$, for some $\theta_0 > \frac{2m\gamma_{m,2m}}{\omega_{2m}\widetilde{K}\gamma_0}$ uniformly with respect to $x \in \mathbb{R}^{2m}$, where $\widetilde{K} = \min\limits_{x\in\overline{B}_1} K(x)$.

Hence, we prove the following existence result:

Theorem 0.0.8 *Assume $(V'), (K')$, $(\widetilde{H}_1) - (\widetilde{H}_5)$ and (20) hold. Then, problem (19) has a nontrivial weak solution.*

We do not resort to the introduction and for the sake of the independence of the chapters, we will present again, in each chapter, the main results and the related assumptions.

Notation and terminology

- $\tilde{C}, C, C_i$, $i = 1, 2, ...$, denote positive (possibly different) constants;
- $C(\varepsilon)$ denotes positive constant which depends on the parameter ε;
- $B_R(x)$ denotes the open ball centered at $x \in \mathbb{R}^N$ and radius R and $B_R = B_R(0)$;
- for a subset $\Omega \subset \mathbb{R}^N$, we denote by $\partial\Omega$, $\overline{\Omega}$, $|\Omega|$ and Ω^c, the boundary, the closure, the Lebesgue measure and the complement of Ω in $\mathbb{R}^N$, respectively;
- χ_Ω denotes the characteristic function of a set $\Omega \subset \mathbb{R}^N$, that is, $\chi_\Omega(x) = 1$ if $x \in \Omega$ and $\chi_\Omega(x) = 0$ if $x \in \Omega^c$;
- $o_n(1)$ denotes a sequence that converges to zero;
- for $1 \leq p \leq \infty$, the standard norm in $L^p(\mathbb{R}^N)$ is denoted by $\|\cdot\|_p$;
- $u_n \rightharpoonup u$ and $u_n \to u$ denote weak and strong convergence, respectively, in a normed space;
- $C_0^\infty(\Omega)$ denotes the space of infinitely differentiable real functions whose support is compact in $\Omega \subset \mathbb{R}^N$.

Chapter 1

On a weighted Adams type inequality and an application to a biharmonic equation

This chapter deals with an improvement of a class of Adams-type inequalities involving a potential V and a weight K which can decay to zero at infinity as $(1+|x|^{\alpha})^{-1}$, $\alpha \in (0,4)$ and $(1+|x|^{\beta})^{-1}$, $\beta \in [\alpha, +\infty)$, respectively.

As an application of this result and by using minimax methods, we establish the existence of solutions for the following class of problems:

$$\Delta^2 u - \Delta u + V(x)u = K(x)f(x,u) \quad \text{in} \quad \mathbb{R}^4,$$

where $\Delta^2 u = \Delta(\Delta u)$ is the operator biharmonic and the nonlinear term $f(x,u)$ can have critical exponential growth. Furthermore, when $\alpha \in (0,2)$ we prove that the solutions belong to the Sobolev space $H^2(\mathbb{R}^4)$ (bound state solutions).

1.1 Introduction and main results

The main purpose of this chapter is two-fold: motivated by the results in [30, 33], our first aim is to investigate a new Adams inequality in $\mathbb{R}^4$ involving a potentials V and a weights K which can decay to zero at infinity. Moreover, as an application we establish the existence of solutions for a class of elliptic biharmonic equations in the critical growth range.

In a more precise way, we consider some weight functions $V(x)$ and $K(x)$ satisfying the following assumptions:

(V) $V \in C(\mathbb{R}^4)$ and there exist $\alpha, a > 0$ such that $V(x) \geq \dfrac{a}{1+|x|^\alpha}$ for all $x \in \mathbb{R}^4$;

(K) $K \in C(\mathbb{R}^4)$ and there exist $\beta, b > 0$ such that $0 < K(x) \leq \dfrac{b}{1+|x|^\beta}$ for all $x \in \mathbb{R}^4$.

We use the notation $\|\cdot\|_{L^p_K}$ for the norm of the weighted Lebesgue space

$$L^p_K(\mathbb{R}^4) = \left\{ u : \mathbb{R}^4 \to \mathbb{R} : u \text{ is measurable and } \int_{\mathbb{R}^4} K(x)|u|^p \mathrm{d}x < \infty \right\},$$

that is, $\|u\|_{L^p_K(\mathbb{R}^4)} = \left(\int_{\mathbb{R}^4} K(x)|u|^p \mathrm{d}x \right)^{1/p}$.

In particular, we restrict our attention to the case when α and β satisfy

$$\alpha \in (0,4) \quad \text{and} \quad \beta \in [\alpha, +\infty). \tag{1.1}$$

Next, in order to present our first result, we will fix some notations. Consider the space

$$E := \left\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^4) \ : \ |\nabla u|,\ \Delta u \in L^2(\mathbb{R}^4) \text{ and } \int_{\mathbb{R}^4} V(x) u^2 \mathrm{d}x < \infty \right\}$$

endowed with the inner product

$$\langle u, v \rangle_E := \int_{\mathbb{R}^4} \left(\Delta u \Delta v + \nabla u \nabla v + V(x) uv \right) \mathrm{d}x$$

and its corresponding norm

$$\|u\| := \left(\int_{\mathbb{R}^4} \left(|\Delta u|^2 + |\nabla u|^2 + V(x) u^2 \right) \mathrm{d}x \right)^{1/2}.$$

By using condition (V), it follows that the space E equipped with the inner product $\langle \cdot, \cdot \rangle_E$ is a Hilbert space and we have that $C_0^\infty(\mathbb{R}^4)$ is dense in E (see Appendix A).

We cite again the following results that were presented in the Introduction as they will be useful for the proof of our first theorem. The first inequality concerns the particular case of (2) when $N = 4$ and $m = 2$

$$\sup_{\{u \in H_0^2(\Omega) : \|\Delta u\|_2 \leq 1\}} \int_\Omega e^{\gamma u^2} \mathrm{d}x \begin{cases} \leq C|\Omega| & \text{if } \gamma \leq 32\pi^2, \\ = +\infty & \text{if } \gamma > 32\pi^2. \end{cases} \tag{1.2}$$

Masmoudi and Sani [47], proved that, for any $\gamma \in (0, 32\pi^2)$, there exists a constant $C = C(\gamma) > 0$ such that

$$\int_{\mathbb{R}^4} (e^{\gamma u^2} - 1)\,\mathrm{d}x \leq C\|u\|_2^2, \quad \text{for all} \quad u \in H^2(\mathbb{R}^4) \quad \text{with} \quad \|\Delta u\|_2 \leq 1, \tag{1.3}$$

and this inequality is not valid if $\gamma \geq 32\pi^2$.

In this context, we can establish our first result.

Theorem 1.1.1 *Suppose that (V) and (K) hold with α and β satisfying (1.1). Then, for any $\gamma > 0$ and any $u \in E$, it holds*

$$\int_{\mathbb{R}^4} K(x)(e^{\gamma u^2} - 1)\,\mathrm{d}x < \infty. \tag{1.4}$$

Moreover, we have

$$\sup_{\substack{u \in E \\ \|u\| \leq 1}} \int_{\mathbb{R}^4} K(x)(e^{\gamma u^2} - 1)\,\mathrm{d}x = \begin{cases} < \infty, & \text{if} \quad 0 < \gamma < 32\pi^2; \\ +\infty, & \text{if} \quad \gamma > 32\pi^2. \end{cases} \tag{1.5}$$

Remark 1.1.2 *We highlight that inequality (1.5) in Theorem 1.1.1 treats only the subcritical case. The critical case $\gamma = 32\pi^2$ is still an open question.*

As initial applications of Theorem 1.1.1, we will prove the compact embedding of the space E into $L^p_K(\mathbb{R}^4)$ for $p \geq 2, \alpha \in (0, 4)$ and $\beta \in (\alpha, +\infty)$ (see Proposition 1.3.1). We also will obtain a Lions-type concentration-compactness principle involving exponential growth (see Proposition 1.3.2), which is a refinement of Theorem 1.1.1.

The existence of solutions for elliptic equations involving the biharmonic operator has been the object of study in recent years by several researchers, mainly motivated by the wide variety of applications. For example, we can cite the modeling of thin elastic plates, clamped plates and in the study of the Paneitz-Branson equation and the Willmore equation (see [34]).

Due to the applicability of problems involving the biharmonic operator and motivated by the Theorem 1.1.1, we will investigate the existence of weak solutions for the following class of problems

$$\Delta^2 u - \Delta u + V(x)u = K(x)f(x, u) \quad \text{in} \quad \mathbb{R}^4, \tag{1.6}$$

where the potential V and the weight K satisfy the conditions (V) and (K), respectively, and the nonlinearity $f(x, s)$ has the maximal growth which allows us to study

(1.6) by using a variational method. Precisely, motivated by (1.5), we say that $f(x,s)$ has *critical exponential growth* if there exists $\gamma_0 > 0$ such that

$$\lim_{|s|\to\infty} \frac{f(x,s)}{e^{\gamma s^2}} = \begin{cases} 0, & \text{for all } \gamma > \gamma_0, \\ +\infty, & \text{for all } \gamma < \gamma_0, \end{cases}$$

uniformly in $x \in \mathbb{R}^4$. In this context, we say that $u \in E$ is a weak solution for (1.6) if

$$\int_{\mathbb{R}^4} (\Delta u \Delta v + \nabla u \nabla v + V(x)uv)\,\mathrm{d}x = \int_{\mathbb{R}^4} K(x)f(x,u)v\,\mathrm{d}x, \quad \text{for all } v \in E. \tag{1.7}$$

We will assume sufficient conditions on f so that weak solutions of (1.6) become critical points of the functional $I : E \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^4} K(x)F(x,u)\,\mathrm{d}x,$$

where $F(x,s) := \int_0^s f(x,t)\mathrm{d}t$.

We require the following assumptions on the nonlinearity $f(x,s)$:

(f_1) $\displaystyle\lim_{s\to 0} \frac{f(x,s)}{s} = 0$, uniformly in $x \in \mathbb{R}^4$;

(f_2) the function $f : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}$ is continuous and has critical exponential growth;

(f_3) there exists $\mu > 2$ such that

$$0 < \mu F(x,s) \le sf(x,s), \quad \text{for all} \quad (x,s) \in \mathbb{R}^4 \times \mathbb{R} \setminus \{0\};$$

(f_4) there exist constants $s_0, M_0 > 0$ such that

$$F(x,s) \le M_0|f(x,s)|, \quad \text{for all} \quad |s| \ge s_0 \quad \text{and} \quad x \in \mathbb{R}^4;$$

(f_5) We also consider one of the following assumptions:

(i) there exist $\theta, \theta_0 > 0$ such that $\displaystyle\liminf_{s\to\infty} \frac{F(x,s)}{s^\theta e^{\gamma_0 s^2}} \ge \theta_0$, uniformly in $x \in \mathbb{R}^4$;

(ii) $\displaystyle\liminf_{s\to\infty} \frac{sf(x,s)}{e^{\gamma_0 s^2}} \ge \theta_0'$ for some $\theta_0' > \frac{64}{\gamma_0 \tilde{K}}$, uniformly in $x \in \mathbb{R}^4$, where $\tilde{K} = \displaystyle\min_{x \in \overline{B}_1} K(x)$.

Our existence result is stated below.

Theorem 1.1.3 *Assume $(V), (K)$ and $(f_1) - (f_5)$ hold with $\alpha \in (0,4)$ and $\beta \in (\alpha, +\infty)$. Then, problem (1.6) has a nontrivial weak solution.*

Denoting by u_0 the weak solution obtained in Theorem 1.1.3, our next result can be stated as follows:

Theorem 1.1.4 *Assume $(V),(K)$ and $(f_1)-(f_5)$ hold with $\alpha \in (0,2)$ and $\beta \in (\alpha,+\infty)$. Then $u_0 \in H^2(\mathbb{R}^4)$, that is, u_0 is a bound state solution of problem (1.6).*

Regarding this issue, we would like to mention interesting works that we found in the literature that have treated the existence of solutions through variational methods. We begin by citing some works that consider the biharmonic operator and nonlinearities with polynomial growth at infinity, e.g., [6, 7, 8, 9, 12, 15, 18, 21, 22, 67, 75]. On the other hand, elliptic problems involving nonlinearities with exponential growth were initially studied in [3, 23]. There were introduced the notions of critical exponential growth to problems that can be treated by using variational methods. This notion was motivated by the Trudinger-Moser inequality (1).

Sani [63], Aouaoui and Albuquerque [11], Miyagaki et al. [49] and Yang [69] have applied these ideas to treat some fourth-order problems. In these works, following the ideas from [3, 23] and motivated by Adams' inequality (2), Sani [63] studied a class of problems involving the biharmonic operator and a class of spherically symmetric potentials (or even coercive) and bounded from below by a positive constant. Aouaoui and Albuquerque [11] considered potentials and weights that are radial and can have a singularity at the origin and can vanish at infinity.

In [69], Yang considered a class of nonhomogeneous problems similar to (1.6) and studied the case where the potential is bounded from below by a positive constant and satisfies the integrability condition $1/V \in L^1(\mathbb{R}^4)$. For other related results, we would like to mention the works [19, 33, 48, 50, 52, 58, 59, 77].

In this chapter, we improve and extend some results obtained in [6, 7, 8, 9, 11, 12, 15, 18, 21, 22, 49, 63, 67, 69, 75], in the sense that we have considered nonlinearities with critical exponential growth and potentials that can vanish at infinity which are not radially symmetric. These features considered here are not treated in these previous works. Finally, as far as we know, there are few results involving biharmonic operators and vanishing potentials.

The outline of the chapter is as follows: in Section 1.2 we prove the weighted Adams' inequality. In Section 1.3 we prove that the embedding $E \hookrightarrow L^p_K(\mathbb{R}^4)$ is compact for all $p \in [2, +\infty)$ (see Proposition 1.3.1). Moreover, we prove a version of the Concentration-Compactness Principle due to P. -L. Lions [45] to the space E (see Proposition 1.3.2). Section 1.4 contains the variational framework related to problem (1.6) and we also check the geometric properties of the functional I. Section 1.5 deals with the Palais-Smale compactness condition. In Section 1.6 we estimate the minimax level. In Section 1.7 we complete the proof of Theorem 1.1.3 and in Section 1.8 we prove some auxiliary results and Theorem 1.1.4.

1.2 Proof of Theorem 1.1.1

This section is devoted to the proof of Theorem 1.1.1.

Proof . We begin proving the first part of (1.5). The proof will be divided into two steps.

Step 1: Let $u \in E$ be such that $\|u\| \leq 1$. First, we want to estimate the weighted Trudinger-Moser functional

$$TM(u, \gamma, R) = \int_{B_R} K(x)(e^{\gamma u^2} - 1)\, \mathrm{d}x$$

for some $R > 0$, independently of u, that will be chosen during the proof. From condition (K), we have

$$\int_{B_R} K(x)(e^{\gamma u^2} - 1)\, \mathrm{d}x \leq b \int_{B_R} (e^{\gamma u^2} - 1)\, \mathrm{d}x. \tag{1.8}$$

Consider a cutoff function $\varphi \in C_0^\infty(B_{2R})$ such that

$$0 \leq \varphi \leq 1 \text{ in } B_{2R}, \quad \varphi \equiv 1 \text{ in } B_R, \quad |\nabla \varphi| \leq \frac{C}{R} \text{ in } B_{2R} \text{ and } |\Delta \varphi| \leq \frac{C}{R^2} \text{ in } B_{2R},$$

for some constant $C > 0$. Then, we have

$$\begin{aligned} |\Delta(\varphi u)|^2 &= |\Delta \varphi|^2 u^2 + 4(\varphi \Delta u)\nabla \varphi \nabla u + 2(\varphi \Delta u) u \Delta \varphi + 4(\nabla \varphi \cdot \nabla u)^2 \\ &+ 4\nabla \varphi \nabla u (u \Delta \varphi) + |\Delta u|^2 \varphi^2, \end{aligned}$$

and by Young's inequality $a_1b_1 \le \varepsilon a_1^2 + \varepsilon^{-1}b_1^2$, with $\varepsilon \in (0,1)$ and $a_1, b_1 \ge 0$, we obtain

$$\begin{aligned}\int_{B_{2R}} |\Delta(\varphi u)|^2 \, \mathrm{d}x \le \frac{C^2}{R^4}\int_{B_{2R}} u^2 \mathrm{d}x + 4\varepsilon \int_{B_{2R}} |\Delta u|^2 \mathrm{d}x + \frac{4C^2}{\varepsilon R^2}\int_{B_{2R}} |\nabla u|^2 \mathrm{d}x \\ + 2\varepsilon \int_{B_{2R}} |\Delta u|^2 \mathrm{d}x + \frac{2C^2}{\varepsilon R^4}\int_{B_{2R}} u^2 \mathrm{d}x + \frac{4C^2}{R^2}\int_{B_{2R}} |\nabla u|^2 \mathrm{d}x \\ + \frac{4\varepsilon C^2}{R^2}\int_{B_{2R}} |\nabla u|^2 \mathrm{d}x + \frac{4C^2}{\varepsilon R^4}\int_{B_{2R}} u^2 \mathrm{d}x + \int_{B_{2R}} |\Delta u|^2 \mathrm{d}x \\ = (1+6\varepsilon)\int_{B_{2R}} |\Delta u|^2 \, \mathrm{d}x + \left(\frac{4C^2}{\varepsilon R^2} + \frac{4C^2}{R^2} + \frac{4\varepsilon C^2}{R^2}\right)\int_{B_{2R}} |\nabla u|^2 \, \mathrm{d}x \\ + \left(\frac{C^2}{R^4} + \frac{6C^2}{R^4\varepsilon}\right)\int_{B_{2R}} u^2 \, \mathrm{d}x.\end{aligned}$$

Thus, by using (V), it follows that

$$\begin{aligned}\int_{B_{2R}} |\Delta(\varphi u)|^2 \, \mathrm{d}x \le (1+6\varepsilon)\int_{B_{2R}} |\Delta u|^2 \, \mathrm{d}x + \left(\frac{4C^2}{\varepsilon R^2} + \frac{4C^2}{R^2} + \frac{4\varepsilon C^2}{R^2}\right)\int_{B_{2R}} |\nabla u|^2 \, \mathrm{d}x \\ + \left(1 + \frac{6}{\varepsilon}\right)\frac{C^2}{a}\frac{1+(2R)^\alpha}{R^4}\int_{B_{2R}} V(x)u^2 \, \mathrm{d}x.\end{aligned}$$

Fixed $\varepsilon \in (0,1)$ such that $\gamma(1+6\varepsilon) \le 32\pi^2$ and since $\alpha \in (0,4)$, we can choose $\bar{R} = \bar{R}(\varepsilon, a, \alpha) > 0$ sufficiently large satisfying

$$\frac{4C^2}{\varepsilon R^2} + \frac{4C^2}{R^2} + \frac{4\varepsilon C^2}{R^2} \le 1 + 6\varepsilon \quad \text{and} \quad \left(1 + \frac{6}{\varepsilon}\right)\frac{C^2}{a}\frac{1+(2R)^\alpha}{R^4} \le 1 + 6\varepsilon,$$

for all $R \ge \bar{R}$. Thus,

$$\int_{B_{2R}} |\Delta(\varphi u)|^2 \, \mathrm{d}x \le (1+6\varepsilon)\|u\|^2 \le 1 + 6\varepsilon.$$

Therefore, defining

$$v := \frac{\varphi u}{\sqrt{1+6\varepsilon}}$$

we have that $\|\Delta v\|_2^2 = \frac{1}{1+6\varepsilon}\int_{B_{2R}} |\Delta(\varphi u)|^2 \, \mathrm{d}x \le 1$, and applying (1.2), we get

$$\int_{B_R} (e^{\gamma u^2} - 1) \, \mathrm{d}x = \int_{B_R} (e^{\gamma(\varphi u)^2} - 1) \, \mathrm{d}x \le \int_{B_{2R}} e^{\gamma(1+6\varepsilon)v^2} \mathrm{d}x \le CR^2.$$

The previous inequality combined with (1.8) implies that

$$\int_{B_R} K(x)(e^{\gamma u^2} - 1)\mathrm{d}x \le b\int_{B_R} (e^{\gamma u^2} - 1) \, \mathrm{d}x \le CR^2, \text{ for all } u \in E \text{ with } \|u\| \le 1. \quad (1.9)$$

Step 2: Now, we estimate the weighted Trudinger-Moser functional in the exterior of a large ball.

For any $n \geq n_0$ fixed, where n_0 will be chosen during the proof, we consider B_n^c the exterior de B_n and the covering of B_n^c formed by all annuli A_n^σ with $\sigma > n$ given by

$$A_n^\sigma := \{x \in B_n^c : |x| < \sigma\} = \{x \in \mathbb{R}^4 : n < |x| < \sigma\}.$$

By the Besicovitch covering Lemma [25], for any $\sigma > n_0$, there exist a sequence of points $(x_k) \in A_{\tilde{n}}^\sigma$ and an universal constant $\theta > 0$ such that

$$A_{\tilde{n}}^\sigma \subseteq \bigcup_k U_k^{1/2}, \quad \text{where} \quad U_k^{1/2} := B\left(x_k, \frac{1}{2}\frac{|x_k|}{3}\right)$$

and

$$\sum_k \chi_{U_k}(x) \leq \theta \quad \text{for any} \quad x \in \mathbb{R}^4, \quad \text{where} \quad U_k := B\left(x_k, \frac{|x_k|}{3}\right)$$

where χ_{U_k} is its characteristic function. Let $u \in E$ be such that $\|u\| \leq 1$. We start with the estimate of the weighted exponential integral of u in A_{3n}^σ with $n \geq n_0$ and $\sigma > 3n$. Note that

$$A_{3n}^\sigma \subset A_{\tilde{n}}^\sigma \subseteq \bigcup_k U_k^{1/2}$$

and defining the set of indices $K_{n,\sigma} := \{k \in \mathbb{N} : U_k^{1/2} \cap B_{3n}^c \neq \emptyset\ \}$, we have

$$A_{3n}^\sigma \subseteq \bigcup_{k \in K_{n,\sigma}} U_k^{1/2}.$$

Therefore,

$$\int_{A_{3n}^\sigma} K(x)\left(e^{\gamma u^2} - 1\right) \mathrm{d}x \leq \sum_{k \in K_{n,\sigma}} \int_{U_k{}^{1/2}} K(x)\left(e^{\gamma u^2} - 1\right) \mathrm{d}x. \tag{1.10}$$

Since $\frac{2}{3}|x_k| \leq |y| \leq \frac{4}{3}|x_k|$ for all $y \in U_k$, from (V) and (K), we obtain

$$V(y) \geq \frac{a}{1 + |y|^\alpha} \geq \frac{a}{1 + \left(\frac{4}{3}\right)^\alpha |x_k|^\alpha}, \quad \text{for all} \quad y \in U_k \tag{1.11}$$

and

$$K(y) \leq \frac{b}{1 + |y|^\beta} \leq \frac{b}{1 + \left(\frac{2}{3}\right)^\beta |x_k|^\beta}, \quad \text{for all} \quad y \in U_k. \tag{1.12}$$

Besides, if $U_k \cap B_{3n}^c \neq \emptyset$ then $U_k \subset B_n^c$, which implies that

$$\bigcup_{k \in K_{n,\sigma}} U_k^{1/2} \subseteq \bigcup_{k \in K_{n,\sigma}} U_k \subseteq B_n^c \subseteq B_{\tilde{n}}^c. \tag{1.13}$$

Next, let us fix $k \in K_{n,\sigma}$. From (1.12), we obtain

$$\int_{U_k{}^{1/2}} K(x)\left(e^{\gamma u^2} - 1\right) \mathrm{d}x \leq \frac{b}{1 + \left(\frac{2}{3}\right)^\beta |x_k|^\beta} \int_{U_k{}^{1/2}} (e^{\gamma u^2} - 1)\, \mathrm{d}x. \tag{1.14}$$

Again, consider a cutoff function $\varphi_k \in C_0^\infty(U_k)$ such that

$$0 \le \varphi_k \le 1 \ \text{ in } U_k, \quad \varphi_k \equiv 1 \ \text{ in } \ U_k^{1/2}, \quad |\nabla \varphi_k| \le \frac{C}{|x_k|} \ \text{ in } U_k \ \text{ and } \ |\Delta \varphi_k| \le \frac{C}{|x_k|^2} \ \text{ in } U_k,$$

for some constant $C > 0$. Proceeding as before, it follows that

$$\int_{U_k} |\Delta(\varphi_k u)|^2 \, \mathrm{d}x \le (1+6\varepsilon) \int_{U_k} |\Delta u|^2 \, \mathrm{d}x + \left(\frac{4C^2}{\varepsilon |x_k|^2} + \frac{4C^2}{|x_k|^2} + \frac{4\varepsilon C^2}{|x_k|^2} \right) \int_{U_k} |\nabla u|^2 \, \mathrm{d}x$$
$$+ \left(\frac{C^2}{|x_k|^4} + \frac{6C^2}{|x_k|^4 \varepsilon} \right) \int_{U_k} u^2 \, \mathrm{d}x,$$

and by (1.11)

$$\int_{U_k} |\Delta(\varphi_k u)|^2 \, \mathrm{d}x \le (1+6\varepsilon) \int_{U_k} |\Delta u|^2 \, \mathrm{d}x + \left(\frac{4C^2}{\varepsilon |x_k|^2} + \frac{4C^2}{|x_k|^2} + \frac{4\varepsilon C^2}{|x_k|^2} \right) \int_{U_k} |\nabla u|^2 \, \mathrm{d}x$$
$$+ \left(1 + \frac{6}{\varepsilon} \right) \frac{C^2}{a} \frac{1 + \left(\frac{4}{3}\right)^\alpha |x_k|^\alpha}{|x_k|^4} \int_{U_k} V(x) u^2 \, \mathrm{d}x.$$

Since $k \in K_{n,\sigma}$, in view of (1.13), we obtain that $x_k \in B_{n_0}^c$. Once $\alpha \in (0,4)$, we can choose $n_0 = n_0(\varepsilon, a, \alpha) > 0$ sufficiently large such that

$$\left(\frac{4C^2}{\varepsilon |x_k|^2} + \frac{4C^2}{|x_k|^2} + \frac{4\varepsilon C^2}{|x_k|^2} \right) \le 1 + 6\varepsilon \quad \text{and} \quad \left(1 + \frac{6}{\varepsilon} \right) \frac{C^2}{a} \frac{1 + \left(\frac{4}{3}\right)^\alpha |x_k|^\alpha}{|x_k|^4} \le 1 + 6\varepsilon,$$

for all $k \in K_{n,\sigma}$ and $n \ge n_0$. Thus,

$$\int_{U_k} |\Delta(\varphi_k u)|^2 \, \mathrm{d}x \le (1+6\varepsilon)\|u\|^2 \le 1 + 6\varepsilon.$$

Therefore, defining $v_k = \varphi_k u / \sqrt{1+6\varepsilon}$ one has

$$\|\Delta v_k\|_2^2 = \frac{1}{1+6\varepsilon} \int_{U_k} |\Delta(\varphi_k u)|^2 \, \mathrm{d}x \le 1.$$

Now, applying (1.3), we get

$$\int_{U_k^{1/2}} (e^{\gamma u^2} - 1) \, \mathrm{d}x = \int_{U_k^{1/2}} (e^{\gamma (\varphi_k u)^2} - 1) \, \mathrm{d}x \le \int_{\mathbb{R}^4} (e^{\gamma(1+6\varepsilon) v_k^2} - 1) \, \mathrm{d}x \le C \int_{\mathbb{R}^4} |v_k|^2 \, \mathrm{d}x.$$

By the previous inequality and (1.11), we have

$$\int_{U_k^{1/2}} (e^{\gamma u^2} - 1) \, \mathrm{d}x \le \frac{C}{1+6\varepsilon} \int_{U_k} u^2 \, \mathrm{d}x \le \frac{C}{1+6\varepsilon} \frac{1 + \left(\frac{4}{3}\right)^\alpha |x_k|^\alpha}{a} \int_{U_k} V(x) u^2 \, \mathrm{d}x. \quad (1.15)$$

Combining the estimates (1.10), (1.14), (1.15) and by using (1.13), we get

$$\int_{A_{3n}^\sigma} K(x) \left(e^{\gamma u^2} - 1 \right) \mathrm{d}x \le \frac{C}{1+6\varepsilon} \frac{b}{a} \sum_{k \in K_{n,\sigma}} \frac{1 + \left(\frac{4}{3}\right)^\alpha |x_k|^\alpha}{1 + \left(\frac{2}{3}\right)^\beta |x_k|^\beta} \int_{U_k} V(x) u^2 \, \mathrm{d}x$$
$$\le \frac{C}{1+6\varepsilon} \frac{b}{a} \sum_{k \in K_{n,\sigma}} \frac{1 + \left(\frac{4}{3}\right)^\alpha |x_k|^\alpha}{1 + \left(\frac{2}{3}\right)^\beta |x_k|^\beta} \int_{B_n^c} V(x) u^2 \chi_{U_k} \, \mathrm{d}x.$$

By (1.13) again, we obtain

$$\frac{1+\left(\frac{4}{3}\right)^{\alpha}|x_k|^{\alpha}}{1+\left(\frac{2}{3}\right)^{\beta}|x_k|^{\beta}} \le \mathcal{B}_n := \sup_{x\in B_n^c} \frac{1+\left(\frac{4}{3}\right)^{\alpha}|x|^{\alpha}}{1+\left(\frac{2}{3}\right)^{\beta}|x|^{\beta}}, \quad \text{for all} \quad k \in K_{n,\sigma}.$$

Therefore,

$$\int_{A_{3n}^{\sigma}} K(x)\left(e^{\gamma u^2}-1\right) \mathrm{d}x \le \frac{C}{1+6\varepsilon}\frac{b}{a}\mathcal{B}_n \sum_{k\in K_{n,\sigma}} \int_{B_n^c} V(x)u^2 \chi_{U_k}\ \mathrm{d}x.$$

Applying the Besicovitch covering lemma, we reach

$$\int_{A_{3n}^{\sigma}} K(x)\left(e^{\gamma u^2}-1\right)\ \mathrm{d}x \le \frac{C}{1+6\varepsilon}\frac{b}{a}\mathcal{B}_n\theta \int_{B_n^c} V(x)u^2\ \mathrm{d}x.$$

Letting $\sigma \to \infty$, we deduce the existence of $n_0 = n_0(\varepsilon, a, \alpha) > 1$ sufficiently large so that

$$\int_{B_{3n}^c} K(x)\left(e^{\gamma u^2}-1\right) \mathrm{d}x \le Cb\mathcal{B}_n\theta \int_{B_n^c} V(x)u^2\ \mathrm{d}x \le Cb\mathcal{B}_n\theta\|u\|^2 \le Cb\mathcal{B}_n\theta, \tag{1.16}$$

for any $n \ge n_0$. Note that

$$\lim_{n\to\infty} \mathcal{B}_n = \lim_{n\to\infty} \frac{1+\left(\frac{4}{3}\right)^{\alpha} n^{\alpha}}{1+\left(\frac{2}{3}\right)^{\beta} n^{\beta}} = \begin{cases} 0, & \text{if } \beta > \alpha \\ 2^{\alpha}, & \text{if } \beta = \alpha. \end{cases} \tag{1.17}$$

Thus, from (1.9) and (1.16), we conclude the proof of the first part of (1.5).

For the second part of (1.5), we consider the Moser sequence $\tilde{\omega}_n$ defined by

$$\tilde{\omega}_n(x) = \begin{cases} \sqrt{\frac{\log n}{8\pi^2}} - \frac{n^2}{\sqrt{32\pi^2\log n}}|x|^2 + \frac{1}{\sqrt{32\pi^2\log n}}, & \text{for } |x| \le \frac{1}{n}, \\ \frac{1}{\sqrt{8\pi^2\log n}} \log\frac{1}{|x|}, & \text{for } \frac{1}{n} < |x| \le 1, \\ \zeta_n(x), & \text{for } |x| > 1. \end{cases}$$

Here ζ_n is a compactly supported smooth function in $\overline{B}_2(0)$ satisfying

$$\zeta_n|_{\partial B_1(0)} = \zeta_n|_{\partial B_2(0)} = 0, \quad \frac{\partial \zeta_n}{\partial \nu}|_{\partial B_1(0)} = \frac{1}{\sqrt{8\pi^2\log n}}, \quad \frac{\partial \zeta_n}{\partial \nu}|_{\partial B_2(0)} = 0$$

and $\zeta_n, |\nabla\zeta_n|, \Delta\zeta_n$ are $O(1/\sqrt{\log n})$. For any $n \in \mathbb{N}$, we have that $\tilde{\omega}_n \in E$ and straightforward calculations show that

$$\|\tilde{\omega}_n\|_2^2 = O(1/\log n), \quad \|\nabla\tilde{\omega}_n\|_2^2 = O(1/\log n) \quad \text{and} \quad \|\Delta\tilde{\omega}_n\|_2^2 = 1 + O(1/\log n).$$

Consequently, by (V), it follows that $\|\tilde{\omega}_n\|^2 = 1 + \delta_n$, where $\delta_n \to 0$ and $\delta_n = O(1/\log n)$, as $n \to \infty$. Setting

$$\omega_n := \frac{\tilde{\omega}_n}{\|\tilde{\omega}_n\|}, \tag{1.18}$$

we have $\omega_n \in E$ and $\|\omega_n\| = 1$. Observe that $\omega_n \geq \frac{1}{\|\tilde{\omega}_n\|}\sqrt{\log n/(8\pi^2)}$ for all $x \in \mathbb{R}^4$ with $|x| \leq \frac{1}{n}$ and defining $\tilde{K} := \min\limits_{x \in \overline{B}_1} K(x)$, for all $\gamma > 32\pi^2$, we have

$$\begin{aligned}
\int_{\mathbb{R}^4} K(x)\left(e^{\gamma\omega_n^2} - 1\right) \mathrm{d}x &\geq \tilde{K} \int_{B_{1/n}} \left(e^{\gamma\omega_n^2} - 1\right) \mathrm{d}x \\
&\geq \tilde{K} \int_{B_{1/n}} \left(e^{\frac{\gamma}{\|\tilde{\omega}_n\|^2}\frac{\log n}{8\pi^2}} - 1\right) \mathrm{d}x \\
&= \frac{\pi^2}{2n^4}\tilde{K}\left(e^{\frac{\gamma}{\|\tilde{\omega}_n\|^2}\frac{\log n}{8\pi^2}} - 1\right) \to \infty \quad \text{as} \quad n \to \infty,
\end{aligned}$$

that is,

$$\lim_{n \to \infty} \int_{\mathbb{R}^4} K(x)\left(e^{\gamma\omega_n^2} - 1\right) \mathrm{d}x = \infty. \tag{1.19}$$

Taking into account that

$$\sup_{\substack{u \in E \\ \|u\| \leq 1}} \int_{\mathbb{R}^4} K(x)\left(e^{\gamma u^2} - 1\right) \mathrm{d}x \geq \int_{\mathbb{R}^4} K(x)\left(e^{\gamma\omega_n^2} - 1\right) \mathrm{d}x,$$

then our sharpness result can be derived from (1.19).

To finish the proof of the theorem, it remains to show that (1.4) holds. For every $\gamma > 0$ and $u \in E$, by density of $C_0^\infty(\mathbb{R}^4)$ in E (see Appendix A), there exists $u_0 \in C_0^\infty(\mathbb{R}^4)$ such that

$$\|u - u_0\| \leq \frac{1}{\sqrt{\gamma}}.$$

Since $u^2 \leq 2(u - u_0)^2 + 2u_0^2$, choosing $R > 0$ such that $\operatorname{supp}(u_0) \subseteq B_R$, we have

$$\begin{aligned}
\int_{\mathbb{R}^4} K(x)\left(e^{\gamma u^2} - 1\right) \mathrm{d}x &\leq \int_{\mathbb{R}^4} K(x)\left(e^{2\gamma(u-u_0)^2}e^{2\gamma u_0^2} - 1\right) \mathrm{d}x \\
&\leq \frac{1}{2}\int_{\mathbb{R}^4} K(x)\left(e^{4\gamma\|u-u_0\|^2\frac{|u-u_0|^2}{\|u-u_0\|}} - 1\right) \mathrm{d}x \\
&\quad + \frac{1}{2}\int_{B_R} K(x)\left(e^{4\gamma u_0^2} - 1\right) \mathrm{d}x.
\end{aligned}$$

Thus, by using (1.5) and since $4\gamma\|u - u_0\|^2 \leq 4 < 32\pi^2$, we reach that (1.4) holds. Therefore, the result is proved. ■

1.3 Some consequences of Theorem 1.1.1

A first important consequence that we have obtained is the following compactness result:

Proposition 1.3.1 *If (V) and (K) hold with $\alpha \in (0,4)$ and $\beta \in [\alpha, +\infty)$, then for all $p \in [2, +\infty)$ the embedding*

$$E \hookrightarrow L^p_K(\mathbb{R}^4) \tag{1.20}$$

is continuous. Moreover, if $\beta > \alpha$ then the above embedding is compact.

Proof . We will proceed in two steps, on a ball of radius $R > 0$ and on its complement. Let $u \in E$ and observe that by condition (K) we have

$$\left(\int_{B_R} K(x)|u|^p \mathrm{d}x\right)^{\frac{1}{p}} \leq \left(\int_{B_R} \frac{b}{1+|x|^\beta}|u|^p \mathrm{d}x\right)^{\frac{1}{p}} \leq b^{\frac{1}{p}}\|u\|_{L^p(B_R)}. \tag{1.21}$$

By the embedding $H^2(B_R) \hookrightarrow L^p(B_R)$ for all $p \in [1, +\infty)$, we get

$$\begin{aligned} \|u\|_{L^p(B_R)} \leq C_1\|u\|_{H^2(B_R)} &= C_1\left[\int_{B_R}(|\Delta u|^2 + |\nabla u|^2 + u^2)\mathrm{d}x\right]^{1/2} \\ &\leq C_1\left[\int_{B_R}\left(|\Delta u|^2 + |\nabla u|^2 + \left(\frac{1+R^\alpha}{a}\right)V(x)u^2\right)\mathrm{d}x\right]^{1/2} \\ &\leq C_R\left[\int_{B_R}(|\Delta u|^2 + |\nabla u|^2 + V(x)u^2)\mathrm{d}x\right]^{1/2}, \end{aligned} \tag{1.22}$$

because $V(x) \geq a/(1+|x|^\alpha) \geq a/(1+R^\alpha)$. Thus, for each $R > 0$, it follows, from (1.21) and (1.22), that

$$\int_{B_R} K(x)|u|^p \mathrm{d}x \leq bC_R^p\left[\int_{B_R}(|\Delta u|^2 + |\nabla u|^2 + V(x)u^2)\mathrm{d}x\right]^{p/2} \leq bC_R^p\|u\|^p. \tag{1.23}$$

For each $p \in [2, \infty)$, there is $C_p > 0$ such that

$$|s|^p \leq C_p(e^{s^2} - 1), \quad \text{for all} \quad s \in \mathbb{R}$$

and proceeding as in the proof of Theorem 1.1.1, we obtain

$$\begin{aligned} \int_{A_{3n}^\sigma} K(x)|u|^p \,\mathrm{d}x &\leq C_p\int_{A_{3n}^\sigma} K(x)(e^{u^2} - 1)\,\mathrm{d}x \\ &\leq C\frac{b}{a}\sum_{k\in K_{n,\sigma}} \frac{1+\left(\frac{4}{3}\right)^\alpha |x_k|^\alpha}{1+\left(\frac{2}{3}\right)^\beta |x_k|^\beta}\int_{B_n^c} V(x)u^2\chi_{U_k}\,\mathrm{d}x. \end{aligned}$$

for all $n \geq n_0$. Letting $\sigma \to +\infty$ we reach

$$\int_{B_{3n}^c} K(x)|u|^p \, \mathrm{d}x \leq C\frac{b}{a}\mathcal{B}_n\theta \int_{B_n^c} V(x)u^2 \, \mathrm{d}x \leq C\frac{b}{a}\mathcal{B}_n\theta\|u\|^2 \tag{1.24}$$

Taking $R = 3n_0$, from (1.24) we conclude that

$$\int_{B_R^c} K(x)|u|^p \, \mathrm{d}x \leq C\frac{b}{a}\mathcal{B}_{n_0}\theta\|u\|^2. \tag{1.25}$$

Now, if $(u_m) \subset E$ is such that $u_m \to 0$ in E, then by (1.23) and (1.25) we obtain

$$\int_{\mathbb{R}^4} K(x)|u_m|^p \, \mathrm{d}x = \int_{B_R} K(x)|u_m|^p \, \mathrm{d}x + \int_{B_R^c} K(x)|u_m|^p \, \mathrm{d}x \to 0 \quad \text{as} \quad m \to \infty$$

and the continuity of the embedding is proved for all $p \in [2, \infty)$.

Next, suppose that $\beta > \alpha$ and $(u_m) \subset E$ is such that $u_m \rightharpoonup 0$ in E. Since (u_m) is bounded in E and

$$\lim_{n\to\infty} \mathcal{B}_n = \lim_{n\to\infty} \frac{1 + \left(\frac{4}{3}\right)^\alpha n^\alpha}{1 + \left(\frac{2}{3}\right)^\beta n^\beta} = 0, \text{ if } \beta > \alpha,$$

in view of (1.24), for all $\varepsilon > 0$, there exists $n_1 \geq n_0$ such that

$$\int_{B_{3n_1}^c} K(x)|u_m|^p \, \mathrm{d}x \leq \frac{\varepsilon}{2}, \quad \text{for all} \quad m \in \mathbb{N}.$$

Taking $R = 3n_1$ and since (u_m) is also bounded in $H^2(B_R)$, by the compact embedding $H^2(B_R) \hookrightarrow L^p(B_R)$ for all $p \in [2, \infty)$, it follows from (1.21) that $\int_{B_R} K(x)|u_m|^p \mathrm{d}x \to 0$ as $m \to \infty$ and therefore there exists $m_0 \in \mathbb{N}$ such that

$$\int_{B_R} K(x)|u_m|^p \, \mathrm{d}x \leq \frac{\varepsilon}{2}, \quad \text{for all} \quad m \geq m_0.$$

Hence, for all $m \geq m_0$ one has

$$\int_{\mathbb{R}^4} K(x)|u_m|^p \, \mathrm{d}x = \int_{B_R} K(x)|u_m|^p \, \mathrm{d}x + \int_{B_R^c} K(x)|u_m|^p \, \mathrm{d}x \leq \varepsilon$$

which shows that $u_m \to 0$ in $L_K^p(\mathbb{R}^4)$. Therefore, the compact embedding is proved for $\beta > \alpha$. ■

The next result is a Lions-type Concentration-Compactness Principle (see [45]) and the proof follows the same lines as in Lemma 2.6 of [32]. This result will be crucial to show that the functional I satisfies the Palais-Smale condition.

Proposition 1.3.2 *Suppose that $(V), (K)$ hold with $\alpha \in (0,4)$ and $\beta \in [\alpha, +\infty)$. If $(u_n) \subset E$ satisfies $\|u_n\| = 1$, for all $n \in \mathbb{N}$, and $u_n \rightharpoonup u$ in E with $\|u\| < 1$, then for all $p \in \left(0, \frac{32\pi^2}{1-\|u\|^2}\right)$ we have*

$$\sup_n \int_{\mathbb{R}^4} K(x)(e^{pu_n^2} - 1)\,\mathrm{d}x < \infty.$$

Proof . Since $u_n \rightharpoonup u$ in E and $\|u_n\| = 1$, we have

$$\|u_n - u\|^2 = \|u_n\|^2 - 2\langle u_n, u\rangle_E + \|u\|^2 \to 1 - \|u\|^2 < \frac{32\pi^2}{p}.$$

Thus, for $n \in \mathbb{N}$ enough large, we get $p\|u_n - u\|^2 < \gamma < 32\pi^2$ for some $\gamma > 0$. Choosing $q > 1$ close to 1 and $\varepsilon > 0$ small satisfying

$$pq(1+\varepsilon^2)\|u_n - u\|^2 < \gamma,$$

by invoking Theorem 1.1.1, we obtain

$$\begin{aligned} \int_{\mathbb{R}^4} K(x)(e^{pq(1+\varepsilon^2)(u_n-u)^2} - 1)\mathrm{d}x &= \int_{\mathbb{R}^4} K(x)(e^{pq(1+\varepsilon^2)\|u_n-u\|^2\left(\frac{u_n-u}{\|u_n-u\|}\right)^2} - 1)\mathrm{d}x \\ &\leq \int_{\mathbb{R}^4} K(x)(e^{\gamma\left(\frac{|u_n-u|}{\|u_n-u\|}\right)^2} - 1)\mathrm{d}x \leq C. \end{aligned} \tag{1.26}$$

Now, notice that $pu_n^2 \leq p(1+\varepsilon^2)(u_n-u)^2 + p\left(1+\frac{1}{\varepsilon^2}\right)u^2$, and by Young's inequality one has $ab - 1 \leq \frac{a^q-1}{q} + \frac{b^r-1}{r}$ for $\frac{1}{q} + \frac{1}{r} = 1$. Thus,

$$\begin{aligned} e^{pu_n^2} - 1 &\leq \left(e^{p(1+\varepsilon^2)(u_n-u)^2} e^{p\left(1+\frac{1}{\varepsilon^2}\right)u^2}\right) - 1 \\ &\leq \frac{1}{q}\left(e^{pq(1+\varepsilon^2)(u_n-u)^2} - 1\right) + \frac{1}{r}\left(e^{pr\left(1+\frac{1}{\varepsilon^2}\right)u^2} - 1\right). \end{aligned} \tag{1.27}$$

Therefore, (1.26) and (1.27) imply that

$$\begin{aligned} \int_{\mathbb{R}^4} K(x)\left(e^{pu_n^2} - 1\right)\mathrm{d}x &\leq \frac{1}{q}\int_{\mathbb{R}^4} K(x)\left(e^{pq(1+\epsilon^2)(u_n-u)^2} - 1\right)\mathrm{d}x \\ &+ \frac{1}{r}\int_{\mathbb{R}^4} K(x)\left(e^{pr\left(1+\frac{1}{\epsilon^2}\right)u^2} - 1\right)\mathrm{d}x \leq C, \end{aligned}$$

for n sufficiently large, which concludes the proof. ■

The next sections are dedicated to the study of the problem (1.6) that is an application of Theorem 1.1.1.

1.4 The variational framework

The purpose of this section is to prove some geometric properties of the Euler-Lagrange functional associated to problem (1.6). We begin by considering the functional $I : E \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^4} K(x)F(x,u)\,\mathrm{d}x.$$

Notice that from (f_1) and (f_2), for each $\gamma > \gamma_0$, $\varepsilon > 0$ and $q \geq 2$, there exists $C(\gamma, q, \varepsilon) > 0$ such that

$$|f(x,s)| \leq \varepsilon|s| + C(\gamma,q,\varepsilon)|s|^{q-1}(e^{\gamma s^2} - 1) \tag{1.28}$$

and by (f_3)

$$|F(x,s)| \leq \frac{\varepsilon}{2}s^2 + C(\gamma,q,\varepsilon)|s|^q(e^{\gamma s^2} - 1), \text{ for all } (x,s) \in \mathbb{R}^4 \times \mathbb{R}. \tag{1.29}$$

Thus, given $u \in E$, by Hölder's inequality with $p > 1$ and $1/p + 1/p' = 1$, we can find $C > 0$ such that

$$\begin{aligned}\int_{\mathbb{R}^4} K(x)F(x,u)\,\mathrm{d}x \leq\ & \frac{\varepsilon}{2}\int_{\mathbb{R}^4} K(x)u^2\,\mathrm{d}x + C\left(\int_{\mathbb{R}^4} K(x)|u|^{pq}\,\mathrm{d}x\right)^{\frac{1}{p}} \\ & \times \left(\int_{\mathbb{R}^4} K(x)(e^{p'\gamma u^2} - 1)\,\mathrm{d}x\right)^{\frac{1}{p'}}.\end{aligned}$$

In view of (1.30) combined with the continuous embedding $E \hookrightarrow L^{pq}_K(\mathbb{R}^4)$ and (1.4), we reach $K(x)F(x,u) \in L^1(\mathbb{R})$, for all $u \in E$. Consequently, I is well-defined and by standard arguments $I \in C^1(E, \mathbb{R})$ with

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^4}(\Delta u\Delta v + \nabla u\nabla v + V(x)uv)\,\mathrm{d}x - \int_{\mathbb{R}^4} K(x)f(x,u)v\,\mathrm{d}x, \quad \text{for} \quad u, v \in E. \tag{1.30}$$

Hence, a critical point of I is a weak solution of problem (1.6) and reciprocally.

The geometric conditions of the Mountain Pass Theorem for the functional I are established by the next lemma.

Lemma 1.4.1 *Suppose that $(V), (K)$, $(f_1) - (f_3)$ and (1.1) hold. Then,*

(i) there exist $\tau, \rho > 0$ such that $I(u) \geq \tau$ for all $\|u\| = \rho$.

(ii) there exists $e \in E$ with $\|e\| > \rho$ such that $I(e) < 0$.

Proof . (i) Here we consider $\gamma > \gamma_0$, $\varepsilon > 0$ and $q > 2$. In view of (1.30), the continuous embedding $E \hookrightarrow L^2_K(\mathbb{R}^4)$ and (1.5), we can find $C_2 = C_2(\gamma, q, \varepsilon) > 0$ such that

$$\int_{\mathbb{R}^4} K(x)F(x,u)\,\mathrm{d}x \leq \varepsilon\, C_1\|u\|^2 + C_2\|u\|^q, \tag{1.31}$$

for all $u \in E$ with $\|u\| = \rho$, where $\rho > 0$ satisfies $\gamma p'\rho^2 < 32\pi^2$. By using (1.31), we get

$$I(u) \geq \frac{1}{2}\|u\|^2 - C_1\varepsilon\|u\|^2 - C_2\|u\|^q = \left(\frac{1}{2} - C_1\varepsilon\right)\rho^2 - C_2\rho^q.$$

Thus, if $u \in E$ with $\|u\| = \rho$, choosing $\varepsilon > 0$ sufficiently small such that $\frac{1}{2} - C_1\varepsilon > 0$ we get

$$I(u) \geq \tilde{C}_1\rho^2 - C_2\rho^q.$$

Since $q > 2$ we may choose $\rho > 0$ small enough such that $\tau := \tilde{C}_1\rho^2 - C_2\rho^q > 0$. Thus, there exists $\tau > 0$ satisfying $I(u) \geq \tau$ whenever $\|u\| = \rho$.

(ii) Let $u \in C_0^\infty(B_R) \setminus \{0\}$ be such that $u \geq 0$. By (f_3), there exist positive constants C_1 and C_2 such that

$$F(x,s) \geq C_1 s^\mu - C_2, \quad \text{for all} \quad (x,s) \in \overline{B}_R \times [0,\infty).$$

Then, for $t > 0$, we get

$$I(tu) \leq \frac{t^2}{2}\|u\|^2 - C_1 t^\mu \int_{B_R} K(x)u^\mu\,\mathrm{d}x + C_2\int_{B_R} K(x)\,\mathrm{d}x.$$

Since $\mu > 2$, we have $I(tu) \to -\infty$ as $t \to \infty$. Setting $e = tu$ with t large enough, the proof is finished. ■

1.5 The Palais-Smale compactness condition

In this section, we show that the functional I satisfies the Palais-Smale condition for certain energy levels. We recall that the functional I satisfies the Palais-Smale condition at the level c, denoted by $(PS)_c$ condition, if any sequence $(u_n) \subset E$ verifying

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0 \;\text{ as }\; n \to \infty, \tag{1.32}$$

has a strongly convergent subsequence in E. We begin by proving some auxiliary results.

Lemma 1.5.1 *Suppose that* $(V), (K)$, $(f_1) - (f_3)$ *and* (1.1) *hold. Then, any* $(PS)_c$-*sequence* (u_n) *for* I *is bounded in* $(E, \|\cdot\|)$ *and satisfies*

$$\sup_n \int_{\mathbb{R}^4} K(x)f(x,u_n)u_n \, dx < \infty. \tag{1.33}$$

Proof . Since (u_n) is $(PS)_c$-sequence for I, we have

$$I(u_n) = \frac{1}{2}\|u_n\|^2 - \int_{\mathbb{R}^4} K(x)F(x,u_n) \, dx \to c \tag{1.34}$$

and

$$|\langle I'(u_n), v\rangle| = \left|\langle u_n, v\rangle_E - \int_{\mathbb{R}^4} K(x)f(x,u_n)v \, dx\right| \le \varepsilon_n\|v\|, \tag{1.35}$$

for all $v \in E$, where $\varepsilon_n \to 0$ as $n \to \infty$. Note that (1.34) guarantees $(I(u_n)) \subset \mathbb{R}$ is bounded and hence, there exists a constant $C > 0$ such that

$$\frac{1}{2}\|u_n\|^2 \le C + \int_{\mathbb{R}^4} K(x)F(x,u_n) \, dx, \tag{1.36}$$

for all $n \in \mathbb{N}$. By the condition (f_3), we have

$$\int_{\mathbb{R}^4} K(x)F(x,u_n) \, dx \le \frac{1}{\mu}\int_{\mathbb{R}^4} K(x)f(x,u_n)u_n \, dx. \tag{1.37}$$

By choosing $v = u_n$ in (1.35), we obtain

$$\int_{\mathbb{R}^4} K(x)f(x,u_n)u_n \, dx \le \|u_n\|^2 + \varepsilon_n\|u_n\|. \tag{1.38}$$

From (1.36), (1.37) and (1.38), we get

$$\left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2 \le C + \frac{\varepsilon_n}{\mu}\|u_n\|$$

and once $\mu > 2$, it follows that (u_n) is bounded in E. This together with (1.38) implies (1.33). ■

The previous lemma guarantees that, up to a subsequence, there exists $u \in E$ such that $u_n \rightharpoonup u$ in E. Moreover, in view of (1.33), we can apply [23, Lemma 2.1] to conclude that

$$K(x)f(x,u_n) \to K(x)f(x,u) \quad \text{in} \quad L^1_{\text{loc}}(\mathbb{R}^4). \tag{1.39}$$

Now let us see the following convergence result that can be found in [33, Lemma 5.4]. We have added the proof here for the reader's convenience.

Lemma 1.5.2 *Suppose that $(V),(K)$ and $(f_1)-(f_4)$ are satisfied with $\alpha \in (0,4)$ and $\beta \in (\alpha,+\infty)$. Let $(u_n) \subset E$ be a Palais-Smale sequence of I at the level c with $u_n \rightharpoonup u$ in E. Then,*

$$\int_{\mathbb{R}^4} K(x)F(x,u_n)\,\mathrm{d}x \to \int_{\mathbb{R}^4} K(x)F(x,u)\,\mathrm{d}x \quad \textit{as} \quad n\to\infty.$$

Proof . Note that by (f_3) and (f_4), we have

$$0 \le \lim_{|s|\to\infty} \frac{F(x,s)}{sf(x,s)} \le \lim_{|s|\to+\infty} \frac{M_0}{|s|} = 0$$

and for any $\varepsilon > 0$ there exists $s_0' = s_0'(\varepsilon) > 0$ such that

$$F(x,s) \le \varepsilon s f(x,s) \quad \text{for all} \quad |s| \ge s_0'. \tag{1.40}$$

Using (1.33), for some $C > 0$ we obtain

$$\int_{\mathbb{R}^4} K(x)f(x,u)u\,\mathrm{d}x \le C \quad \text{and} \quad \int_{\mathbb{R}^4} K(x)f(x,u_n)u_n\,\mathrm{d}x \le C \quad \text{for all} \quad n\in\mathbb{N}.$$

From (1.40) and by the previous inequalities, fixed $\varepsilon > 0$, we reach

$$\int_{\{|u|\ge s_0'\}} K(x)F(x,u)\,\mathrm{d}x \le \varepsilon \int_{\{|u|\ge s_0'\}} K(x)f(x,u)u\,\mathrm{d}x$$

and

$$\int_{\{|u_n|\ge s_0'\}} K(x)F(x,u_n)\,\mathrm{d}x \le \varepsilon \int_{\{|u_n|\ge s_0'\}} K(x)f(x,u_n)u_n\,\mathrm{d}x.$$

Defining $\ell_n(x) := K(x)\chi_{\{|u_n|<s_0'\}}F(x,u_n)$ and $\ell(x) := K(x)\chi_{\{|u|<s_0'\}}F(x,u)$, we have that $\{\ell_n\}$ is a sequence of measurable functions and $\ell_n(x) \to \ell(x)$ for a.e $x \in \mathbb{R}^4$, because $u_n \to u$ a.e. in $\mathbb{R}^4$. Using (1.29) with $\gamma > \gamma_0$, $\varepsilon > 0$ and $q = 2$, for any $|s| \le s_0'$ we obtain

$$F(x,s) \le \frac{\varepsilon}{2}s^2 + C(\gamma,\varepsilon)s^2(e^{\gamma s^2}-1) \le C(\gamma,\varepsilon,s_0')s^2.$$

So writing

$$g_n(x) := C(\gamma,\varepsilon,s_0')K(x)u_n^2 \quad \text{and} \quad g(x) := C(\gamma,\varepsilon,s_0')K(x)u^2,$$

we have $0 \le \ell_n(x) \le g_n(x)$ and $g_n(x) \to g(x)$ a.e. in $\mathbb{R}^4$, and by virtue of the compact embedding $E \hookrightarrow L^2_K(\mathbb{R}^4)$, we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^4} g_n(x)\,\mathrm{d}x = \int_{\mathbb{R}^4} g(x)\,\mathrm{d}x.$$

Hence, applying the Generalized Lebesgue Dominated Convergence Theorem, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^4}\ell_n(x)\,\mathrm{d}x=\int_{\mathbb{R}^4}\ell(x)\,\mathrm{d}x.$$

In conclusion, for any fixed $\varepsilon>0$, denoting by

$$A_n:=\left|\int_{\mathbb{R}^4}K(x)F(x,u_n)\,\mathrm{d}x-\int_{\mathbb{R}^4}K(x)F(x,u)\,\mathrm{d}x\right|,$$

we obtain

$$\begin{aligned}A_n&\leq\int_{\{|u_n|\geq s_0'\}}K(x)F(x,u_n)\,\mathrm{d}x+\int_{\{|u|\geq s_0'\}}K(x)F(x,u)\,\mathrm{d}x\\&\quad+\left|\int_{\{|u_n|<s_0'\}}K(x)F(x,u_n)\,\mathrm{d}x-\int_{\{|u|<s_0'\}}K(x)F(x,u)\,\mathrm{d}x\right|\\&\leq 2C\varepsilon+\left|\int_{\mathbb{R}^4}\ell_n(x)\,\mathrm{d}x-\int_{\mathbb{R}^4}\ell(x)\,\mathrm{d}x\right|\end{aligned}$$

and passing to the limit as $n\to\infty$, we get $0\leq\lim_{n\to\infty}A_n\leq 2C\varepsilon$. Since $\varepsilon>0$ is arbitrary, we conclude the proof of the lemma. ∎

Next, we shall prove the main compactness result of this chapter.

Proposition 1.5.3 *Under the hypotheses* $(V),(K)$ *and* $(f_1)-(f_4)$ *with* $\alpha\in(0,4)$ *and* $\beta\in(\alpha,+\infty)$, *the functional* I *satisfies* $(PS)_c$ *condition for any* $0\leq c<16\pi^2/\gamma_0$.

Proof . Let $(u_n)\subset E$ be an arbitrary Palais-Smale sequence of I at the level c. By Lemma 1.5.1, up to a subsequence, $u_n\rightharpoonup u$ weakly in E. We shall show that, up to a subsequence, $u_n\to u$ strongly in E. For this, we have two cases to consider:

<u>*Case 1:*</u> $u=0$. In this case, by Lemma 1.5.2, we have

$$\int_{\mathbb{R}^4}K(x)F(x,u_n)\,\mathrm{d}x\to 0\quad\text{as}\quad n\to\infty.$$

Since

$$I(u_n)=\frac{1}{2}\|u_n\|^2-\int_{\mathbb{R}^4}K(x)F(x,u_n)\,\mathrm{d}x=c+o_n(1),$$

we get

$$\lim_{n\to\infty}\|u_n\|^2=2c.$$

Hence, we can deduce that for n large there exist $r>1$ sufficiently close to 1, $\gamma>\gamma_0$ close to γ_0 and $\tilde{r}>r$ sufficiently close to r such that $\tilde{r}\gamma\|u_n\|^2<32\pi^2$. Thus, by (1.5)

$$\int_{\mathbb{R}^4}K(x)(e^{\gamma u_n^2}-1)^r\,\mathrm{d}x\leq C\int_{\mathbb{R}^4}K(x)(e^{\tilde{r}\gamma\|u_n\|^2\left(\frac{u_n}{\|u_n\|}\right)^2}-1)\,\mathrm{d}x\leq C.\tag{1.41}$$

We claim that

$$\int_{\mathbb{R}^4} K(x)f(x,u_n)u_n \,\mathrm{d}x \to 0 \quad \text{as} \quad n \to \infty.$$

Indeed, since $f(x,s)$ satisfies (f_1) and (f_2), for $\gamma > \gamma_0$ and $\varepsilon > 0$, there exists $C(\gamma,\varepsilon) > 0$ such that

$$|f(x,s)| \leq \varepsilon|s| + C(\gamma,\varepsilon)(e^{\alpha s^2} - 1), \quad \text{for all} \quad (x,s) \in \mathbb{R}^4 \times \mathbb{R}.$$

Choosing $r > 1$ close to 1 such that $r' \geq 2$, where $1/r + 1/r' = 1$, by Hölder's inequality we obtain

$$\left| \int_{\mathbb{R}^4} K(x)f(x,u_n)u_n \,\mathrm{d}x \right| \leq C \int_{\mathbb{R}^4} K(x)u_n^2 \,\mathrm{d}x + C \left(\int_{\mathbb{R}^4} K(x)(e^{\gamma u_n^2} - 1)^r \,\mathrm{d}x \right)^{\frac{1}{r}} \left(\int_{\mathbb{R}^4} K(x)|u_n|^{r'} \,\mathrm{d}x \right)^{\frac{1}{r'}} \to 0,$$

as $n \to \infty$, where we have used (3.46) and the compact embedding $E \hookrightarrow L^p_K(\mathbb{R}^4)$, for all $[2,\infty)$. Therefore, once $\langle I'(u_n), u_n \rangle = o_n(1)$, we conclude that, up to a subsequence, $u_n \to 0$ strongly in E.

<u>*Case 2:*</u> $u \neq 0$. In this case, since (u_n) is a Palais-Smale sequence of I at the level c, we may define

$$v_n = \frac{u_n}{\|u_n\|} \quad \text{and} \quad v = \frac{u}{\lim\limits_{n\to\infty} \|u_n\|}.$$

Thus, $v_n \rightharpoonup v$ in E, $\|v_n\| = 1$ and $\|v\| \leq 1$. If $\|v\| = 1$, we conclude the proof. If $\|v\| < 1$, we claim that there exist $r > 1$ sufficiently close to 1, $\gamma > \gamma_0$ close to γ_0 and $\sigma > 0$ such that

$$r\gamma\|u_n\|^2 \leq \sigma < \frac{32\pi^2}{1 - \|v\|^2} \tag{1.42}$$

for $n \in \mathbb{N}$ large. Indeed, since $I(u_n) = c + o_n(1)$, we have

$$\frac{1}{2} \lim_{n\to\infty} \|u_n\|^2 = c + \int_{\mathbb{R}^4} K(x)F(x,u) \,\mathrm{d}x. \tag{1.43}$$

Setting

$$A(u) = \left(c + \int_{\mathbb{R}^4} K(x)F(x,u) \,\mathrm{d}x \right) (1 - \|v\|^2),$$

from (1.43) and by the definition of v, we obtain $A(u) = c - I(u)$, which together with (1.43) imply

$$\frac{1}{2} \lim_{n\to\infty} \|u_n\|^2 = \frac{A(u)}{1 - \|v\|^2} = \frac{c - I(u)}{1 - \|v\|^2}. \tag{1.44}$$

Hence, from (3.49) and the fact $c - I(u) < c$, we conclude

$$\frac{1}{2}\lim_{n\to\infty}\|u_n\|^2 = \frac{c - I(u)}{1-\|v\|^2} < \frac{c}{1-\|v\|^2} < \frac{16\pi^2}{\gamma_0(1-\|v\|^2)} \tag{1.45}$$

because $0 \le c < 16\pi^2/\gamma_0$. Therefore, by using (1.45) we reach that (1.42) holds. Thus, by Proposition 1.3.2, we get

$$\int_{\mathbb{R}^4} K(x)(e^{\gamma u_n^2} - 1)^r \,\mathrm{d}x \le C.$$

By using Hölder's inequality, the compact embedding $E \hookrightarrow L^p_K(\mathbb{R}^4)$, for all $[2,\infty)$, and arguing similar to Case 1, it follows that

$$\int_{\mathbb{R}^4} K(x)f(x,u_n)(u_n - u)\,\mathrm{d}x \to 0 \quad \text{as} \quad n\to\infty.$$

This convergence and the fact that $\langle I'(u_n), u_n - u\rangle = o_n(1)$ show that $\|u_n\|^2 = \langle u_n, u\rangle_E + o_n(1)$. Since $u_n \rightharpoonup u$ in E, we conclude $u_n \to u$ strongly in E what concludes completes the proof. ■

1.6 The minimax level

In this section, we prove an estimate for the minimax level associated to the functional I.

Lemma 1.6.1 *Suppose that $(V), (K)$, $(f_1) - (f_3)$, (f_5) and (1.1) hold. Then, there exists $n \in \mathbb{N}$ such that*

$$\max_{t\ge 0} I(t\omega_n) < \frac{16\pi^2}{\gamma_0}$$

where ω_n is defined in (1.18).

Proof . Let us first study the case where $(f_5)(i)$ is valid. Assume by contradiction that

$$\max_{t\ge 0} I(t\omega_n) \ge \frac{16\pi^2}{\gamma_0}, \quad \text{for all} \quad n\in\mathbb{N}.$$

For each $n \in \mathbb{N}$, let $t_n > 0$ be such that

$$\frac{t_n^2}{2} - \int_{\mathbb{R}^4} K(x)F(x,t_n\omega_n)\,\mathrm{d}x = \max_{t\ge 0} I(t\omega_n) \ge \frac{16\pi^2}{\gamma_0}.$$

By (f_3), we obtain

$$\frac{16\pi^2}{\gamma_0} \le \frac{t_n^2}{2} - \int_{\mathbb{R}^4} K(x)F(x,t_n\omega_n)\,\mathrm{d}x \le \frac{t_n^2}{2}$$

and therefore

$$t_n^2 \geq \frac{32\pi^2}{\gamma_0}, \quad \text{for all} \quad n \in \mathbb{N}. \tag{1.46}$$

At $t = t_n$, we have

$$0 = \frac{d}{dt}\left[\frac{t^2}{2} - \int_{\mathbb{R}^4} K(x)F(x, t\omega_n)\,\mathrm{d}x\right]\Bigg|_{t=t_n} = t_n - \int_{\mathbb{R}^4} K(x)f(x, t_n\omega_n)\omega_n\,\mathrm{d}x,$$

which implies that

$$t_n^2 = \int_{\mathbb{R}^4} K(x)f(x, t_n\omega_n)t_n\omega_n\,\mathrm{d}x \quad \text{for all} \quad n \in \mathbb{N}. \tag{1.47}$$

Now, we will prove that (t_n) is bounded sequence. In fact, in view of assumptions (f_3) and (f_5), given $\varepsilon \in (0, \theta_0)$, there exists $R = R_\varepsilon > 0$ such that

$$f(x, s)s \geq \mu F(x, s) \geq \mu(\theta_0 - \varepsilon)s^\theta e^{\gamma_0 s^2}, \text{ for all } s \geq R \text{ and } x \in \mathbb{R}^4. \tag{1.48}$$

Since

$$t_n\omega_n(x) \geq \frac{t_n}{\|\tilde{\omega}_n\|}\sqrt{\frac{\log n}{8\pi^2}} \quad \text{in} \quad B_{1/n} \tag{1.49}$$

and

$$\frac{t_n}{\|\tilde{\omega}_n\|}\sqrt{\frac{\log n}{8\pi^2}} \to +\infty \quad \text{as } n \to \infty,$$

by (1.49) we have $t_n\omega_n \to \infty$ as $n \to \infty$ in $B_{1/n}$. Taking $n \in \mathbb{N}$ sufficiently large so that $t_n\omega_n(x) \geq R$, for all $x \in B_{1/n}$, it follows, from (1.47), (1.48) and (1.49), that

$$\begin{aligned}
t_n^2 &\geq \int_{B_{1/n}} K(x)f(x, t_n\omega_n)t_n\omega_n\,\mathrm{d}x \\
&\geq \mu(\theta_0 - \varepsilon)\int_{B_{1/n}} K(x)(t_n\omega_n)^\theta e^{\gamma_0 t_n^2\omega_n^2}\,\mathrm{d}x \\
&\geq \mu(\theta_0 - \varepsilon)\tilde{K}\int_{B_{1/n}} \left(\frac{t_n}{\|\tilde{\omega}_n\|}\right)^\theta \left(\frac{\log n}{8\pi^2}\right)^{\frac{\theta}{2}} e^{\gamma_0 \frac{t_n^2}{\|\tilde{\omega}_n\|^2}\frac{\log n}{8\pi^2}}\,\mathrm{d}x \\
&= \frac{\pi^2}{2n^4}\mu(\theta_0 - \varepsilon)\tilde{K}\left(\frac{t_n}{\|\tilde{\omega}_n\|}\right)^\theta \left(\frac{\log n}{8\pi^2}\right)^{\frac{\theta}{2}} e^{\gamma_0 \frac{t_n^2}{\|\tilde{\omega}_n\|^2}\frac{\log n}{8\pi^2}},
\end{aligned}$$

where $\tilde{K} := \min_{x \in \overline{B}_1} K(x)$. Thus, we may write

$$t_n^2 \geq \frac{\pi^2}{2}\mu(\theta_0 - \varepsilon)\tilde{K}\left(\frac{t_n}{\|\tilde{\omega}_n\|}\right)^\theta \frac{1}{(8\pi^2)^{\frac{\theta}{2}}} e^{\left(\frac{t_n^2}{\|\tilde{\omega}_n\|^2}\frac{\gamma_0}{8\pi^2} - 4\right)\log n + \frac{\theta}{2}\log(\log(n))}, \tag{1.50}$$

which implies that (t_n) is bounded. Therefore, there exists $C > 0$ such that $\frac{32\pi^2}{\gamma_0} \leq t_n^2 \leq C$. Thus, from (1.50) it follows that

$$\left(\frac{t_n^2}{\|\tilde{\omega}_n\|^2}\frac{\gamma_0}{8\pi^2} - 4\right)\log n + \frac{\theta}{2}\log(\log n) \leq \tilde{C}$$

for some constant $\tilde{C} > 0$, that is,

$$\left[\frac{t_n^2\gamma_0 - 32\pi^2}{(1+\delta_n)8\pi^2} - \frac{4\delta_n}{(1+\delta_n)}\right]\log n + \frac{\theta}{2}\log(\log n) \leq \tilde{C}.$$

By (1.46), $(t_n^2\gamma_0 - 32\pi^2)\log n/[(1+\delta_n)8\pi^2] > 0$ and consequently

$$-\frac{4\delta_n}{(1+\delta_n)}\log n + \frac{\theta}{2}\log(\log n) \leq \tilde{C}.$$

However, $\theta > 0$, $\delta_n \to 0$, $\delta_n \log n \to C_2$ and $\log(\log n) \to \infty$ as $n \to \infty$, which leads us to a contradiction. This completes the proof with the condition $(f_5)(i)$.

Now we will prove the lemma under the condition $(f_5)(ii)$ (the proof this case is inspired by [63, Lemma 8]). We start by remembering that

$$\theta_0' > \frac{64}{\gamma_0 \tilde{K}} \tag{1.51}$$

where $\tilde{K} = \min_{x\in\overline{B}_1} K(x)$. From $(f_5)(ii)$, for any $\varepsilon > 0$ there exists $R = R(\varepsilon) > 0$ such that

$$sf(x,s) \geq (\theta_0' - \varepsilon)e^{\gamma_0 s^2}, \text{ for all } s \geq R \text{ and } x \in \mathbb{R}^4. \tag{1.52}$$

Proceeding as in the previous case, suposse by contradiction that $\max_{t\geq 0} I(t\omega_n) \geq 16\pi^2/\gamma_0$, for all $n \in \mathbb{N}$ and let $t_n > 0$ such that

$$\max_{t\geq 0} I(t\omega_n) = \frac{t_n^2}{2} - \int_{\mathbb{R}^4} K(x)F(x,t_n\omega_n)\mathrm{d}x.$$

We claim that (t_n) is a bounded sequence. Indeed, taking $n \in \mathbb{N}$ sufficiently large so that $t_n\omega_n(x) \geq R$, for all $x \in B_{1/n}$, it follows, from (1.47), (1.49) and (1.52), that

$$\begin{aligned} t_n^2 &\geq \int_{B_{1/n}} K(x)f(x,t_n\omega_n)t_n\omega_n \,\mathrm{d}x \geq (\theta_0' - \varepsilon)\int_{B_{1/n}} K(x)e^{\gamma_0 t_n^2\omega_n^2}\,\mathrm{d}x \\ &\geq (\theta_0' - \varepsilon)\tilde{K}\int_{B_{1/n}} e^{\gamma_0 \frac{t_n^2}{\|\tilde{\omega}_n\|^2}\frac{\log n}{8\pi^2}}\,\mathrm{d}x = \frac{\pi^2}{2}(\theta_0' - \varepsilon)\tilde{K}e^{\log n\left(\gamma_0\frac{t_n^2}{\|\tilde{\omega}_n\|^2}\frac{1}{8\pi^2} - 4\right)}. \end{aligned} \tag{1.53}$$

Consequently,

$$1 \geq \frac{\pi^2}{2}(\theta_0' - \varepsilon)\tilde{K}e^{\left(\gamma_0\frac{t_n^2}{\|\tilde{\omega}_n\|^2}\frac{\log n}{8\pi^2} - 4\log n - \log t_n^2\right)}$$

and it turns out that (t_n) is a bounded sequence. In view of (1.46), from (1.53) we can conclude that

$$\lim_{n\to\infty} t_n^2 = \frac{32\pi^2}{\gamma_0}. \tag{1.54}$$

Consider the sets defined by

$$A_n := \{x \in B_1 : t_n\omega_n \geq R\} \quad \text{and} \quad C_n := B_1 \setminus A_n$$

where $R > 0$ is given in (1.52). It follows from (1.47) and (1.52) that

$$\begin{aligned} t_n^2 &\geq \int_{B_1} K(x)f(x, t_n\omega_n)t_n\omega_n \,\mathrm{d}x \\ &\geq (\theta_0' - \varepsilon)\tilde{K}\int_{B_1} e^{\gamma_0 t_n^2\omega_n^2}\,\mathrm{d}x + \tilde{K}\int_{C_n} f(x, t_n\omega_n)t_n\omega_n\,\mathrm{d}x \\ &- (\theta_0' - \varepsilon)\tilde{K}\int_{C_n} e^{\gamma_0 t_n^2\omega_n^2}\,\mathrm{d}x. \end{aligned} \tag{1.55}$$

By definition of C_n and since $\omega_n \to 0$ almost everywhere in B_1, we reach

$$\chi_{C_n} \to 1 \quad \text{a.e. in} \quad B_1.$$

Using the Lebesgue dominated convergence Theorem, we get

$$\int_{C_n} f(x, t_n\omega_n)t_n\omega_n\,\mathrm{d}x \to 0 \quad \text{and} \quad \int_{C_n} e^{\gamma_0 t_n^2\omega_n^2}\,\mathrm{d}x \to \frac{\pi^2}{2}. \tag{1.56}$$

Observe that by (1.46) and the definition of ω_n we have

$$\int_{B_1} e^{\gamma_0 t_n^2\omega_n^2}\,\mathrm{d}x \geq \int_{B_1\setminus B_{1/n}} e^{32\pi^2\omega_n^2}\,\mathrm{d}x = 2\pi^2\int_{1/n}^{1} e^{\frac{4}{\|\tilde{\omega}_n\|^2}\frac{1}{\log n}(\log\frac{1}{s})^2} s^3\,\mathrm{d}s.$$

Making the change of variable

$$t = \frac{1}{\|\tilde{\omega}_n\|\log n}\log\frac{1}{s}$$

we can we estimate

$$\int_{B_1} e^{\gamma_0 t_n^2\omega_n^2}\,\mathrm{d}x \geq 2\pi^2\|\tilde{\omega}_n\|\log n\int_0^{1/\|\tilde{\omega}_n\|} e^{\log n(4t^2 - 4\|\tilde{\omega}_n\|t)}\,\mathrm{d}t.$$

Consider $g : [0, 1/\|\tilde{\omega}_n\|] \to \mathbb{R}$ the function defined by $g(t) = \log n(4t^2 - 4\|\tilde{\omega}_n\|t)$. Then we have that

$$g'(0) = -4\|\tilde{\omega}_n\|\log n \quad \text{and} \quad g'(1/\|\tilde{\omega}_n\|) = \frac{8}{\|\tilde{\omega}_n\|}\log n - 4\log n\|\tilde{\omega}_n\|.$$

Let $\varepsilon > 0$ be sufficiently small, thus

$$g(t) = -4t\|\tilde{\omega}_n\|\log n + o(t), \quad t \in [0, \varepsilon]$$

and

$$g(t) = 4\log n\left(\frac{2}{\|\tilde{\omega}_n\|} - \|\tilde{\omega}_n\|\right)\left(t - \frac{1}{\|\tilde{\omega}_n\|}\right) + o(t), \quad t \in [1/\|\tilde{\omega}_n\| - \varepsilon, 1/\|\tilde{\omega}_n\|].$$

Hence, choosing $\varepsilon = \frac{1}{2\|\tilde{\omega}_n\|}$ we have that

$$\begin{aligned}\int_{B_1} e^{\gamma_0 t_n^2 \omega_n^2}\,\mathrm{d}x &\geq 2\pi^2\|\tilde{\omega}_n\|\log n \int_0^{1/2\|\tilde{\omega}_n\|} e^{-4t\|\tilde{\omega}_n\|\log n}\,\mathrm{d}t \\ &\quad + 2\pi^2\|\tilde{\omega}_n\|\log n \int_{1/2\|\tilde{\omega}_n\|}^{1/\|\tilde{\omega}_n\|} e^{4\log n\left(\frac{2}{\|\tilde{\omega}_n\|} - \|\tilde{\omega}_n\|\right)\left(t - \frac{1}{\|\tilde{\omega}_n\|}\right)}\,\mathrm{d}t \\ &= \frac{\pi^2}{2}(1 - e^{-2\log n}) + \frac{\pi^2\|\tilde{\omega}_n\|^2}{(4 - 2\|\tilde{\omega}_n\|^2)}\left(1 - e^{-2\log n\frac{(2 - \|\tilde{\omega}_n\|^2)}{\|\tilde{\omega}_n\|^2}}\right).\end{aligned}$$

Since $\|\tilde{\omega}_n\|^2 \to 1$ as $n \to \infty$, we get that

$$\lim_{n\to\infty} \int_{B_1} e^{\gamma_0 t_n^2 \omega_n^2}\,\mathrm{d}x \geq \pi^2. \tag{1.57}$$

Therefore, combining (1.54)-(1.57) and calculating the limit we obtain that

$$\frac{32\pi^2}{\gamma_0} = \lim_{n\to\infty} t_n^2 \geq (\theta_0' - \varepsilon)\tilde{K}\pi^2 - (\theta_0' - \varepsilon)\tilde{K}\frac{\pi^2}{2} = (\theta_0' - \varepsilon)\tilde{K}\frac{\pi^2}{2}.$$

By the arbitrariness of $\varepsilon > 0$, we can let $\varepsilon \to 0$, thus

$$\frac{32\pi^2}{\gamma_0} \geq \theta_0'\tilde{K}\frac{\pi^2}{2}$$

contradicting (1.51) and this concludes the proof. ■

1.7 Proof of Theorem 1.1.3

Initially, it follows from Lemma 1.4.1 that the functional I satisfies the geometric conditions of the Mountain Pass Theorem. As a consequence, the minimax level

$$c := \inf_{g\in\Gamma} \max_{t\in[0,1]} I(g(t))$$

is positive, where $\Gamma := \{g \in C([0,1], E) : g(0) = 0, g(1) = e\}$. We also have by Proposition 1.5.3 that the functional I satisfies the $(PS)_c$ condition. Thus, by invoking the Mountain Pass Theorem, I has a critical point $u_0 \in E$ at the minimax level c.

1.8 Bound state solution

This section is dedicated to the proof of Theorem 1.1.4. In particular, we will prove that any weak solution of problem (1.6) is a bound state solution, that is, belongs to $H^2(\mathbb{R}^4)$. In order to reach this goal, we will adapt some arguments presented by Ambrosetti, Felli and Malchiodi in [10]. The lemma below is a direct consequence of inequalities (1.16), (1.17) and will be useful in the later lemma.

Lemma 1.8.1 *Let* $(V),(K)$ *with* $\alpha \in (0,4)$, $\beta \in (\alpha, +\infty)$ *and* $\gamma \in (0, 32\pi^2)$. *Then, for each* $v \in E \setminus \{0\}$ *with* $\|v\| \leq 1$ *and any* $\varepsilon > 0$, *there exists a constant* $\bar{n} = \bar{n}(\gamma, a, \alpha) > 1$ *independent of* v *such that*

$$\int_{B_{3n}^c} K(x)(e^{\gamma v^2} - 1)\mathrm{d}x \leq \varepsilon \int_{B_n^c} (|\Delta v|^2 + |\nabla v|^2 + V(x)v^2)\mathrm{d}x, \quad \textit{for all} \quad n \geq \bar{n}.$$

The next result is inspired by arguments in [10, Lemma 11].

Lemma 1.8.2 *Suppose that* (V) *and* (K) *hold with* $\alpha \in (0,2)$ *and* $\beta \in (\alpha, +\infty)$. *Let* $\gamma > 0$ *and* $u \in E\setminus\{0\}$. *Then, for any* $\varepsilon > 0$, *there exists a constant* $\bar{R} = \bar{R}(u, \gamma, a, \alpha) > 0$ *such that*

$$\int_{B_R^c} K(x)(e^{\gamma u^2} - 1)\mathrm{d}x \leq \varepsilon \int_{B_R^c} (|\Delta u|^2 + |\nabla u|^2 + V(x)u^2)\mathrm{d}x, \quad \textit{for all} \quad R \geq \bar{R}. \tag{1.58}$$

Proof . Let $R > 1$ and $\overline{\psi}_R : \mathbb{R}_+ \to [0,1]$ be a smooth nondecreasing function such that

$$\overline{\psi}_R(r) = \begin{cases} 0, & \text{if } \ r \leq R - R^{\alpha/2}, \\ 1, & \text{if } \ r \geq R \end{cases}$$

and satisfying

$$|\overline{\psi}'_R| \leq \frac{2}{R^{\alpha/2}} \quad \text{and} \quad |\overline{\psi}''_R| \leq \frac{2}{R^{\alpha}}.$$

By using polar coordinates, for $(r,\theta) \in [0, +\infty) \times \mathbb{S}^3$ we define

$$\overline{u}_R(r,\theta) = \begin{cases} 0, & \text{if } r \leq R - R^{\alpha/2}, \\ \overline{\psi}_R(r)u(2R - r, \theta), & \text{if } R - R^{\alpha/2} \leq r \leq R, \\ u(r,\theta), & \text{if } r \geq R. \end{cases}$$

In the annulus $A_R = \{x \in \mathbb{R}^4 : R - R^{\alpha/2} \leq |x| \leq R\}$, we have

$$\nabla \overline{u}_R = \overline{\psi}'_R(r)u(2R - r,\theta)\mathbf{e}_r - \overline{\psi}_R(r)u_r(2R - r,\theta)\mathbf{e}_r + \frac{1}{r}\overline{\psi}_R(r)u_\theta(2R - r,\theta)\mathbf{e}_\theta,$$

where $\mathbf{e}_r = x/|x|$ and $\mathbf{e}_\theta$ is a unit vector tangent to the unit sphere. Moreover,

$$\begin{aligned}\Delta \overline{u}_R &= \frac{1}{r}\overline{\psi}'_R(r)u(2R-r,\theta) - \frac{1}{r}\overline{\psi}_R(r)u_r(2R-r,\theta) + \overline{\psi}''_R(r)u(2R-r,\theta)\\ &\quad - 2\overline{\psi}'_R(r)u_r(2R-r,\theta) + \overline{\psi}_R(r)u_{rr}(2R-r,\theta) + \frac{1}{r^2}\overline{\psi}_R(r)u_{\theta\theta}(2R-r,\theta).\\ &= \overline{\psi}_R(r)u_{rr}(2R-r,\theta) + \frac{1}{r^2}\overline{\psi}_R(r)u_{\theta\theta}(2R-r,\theta)\\ &\quad - \left(2\overline{\psi}'_R(r) + \frac{1}{r}\overline{\psi}_R(r)\right)u_r(2R-r,\theta) + \left(\overline{\psi}''_R(r) + \frac{1}{r}\overline{\psi}'_R(r)\right)u(2R-r,\theta).\end{aligned}$$

Thus, in A_R we obtain

$$|\nabla \overline{u}_R|^2 \leq C_1|\nabla u(2R-r,\theta)|^2 + \frac{C_2}{R^\alpha}u^2(2R-r,\theta)$$

and

$$|\Delta \overline{u}_R|^2 \leq C_3|\Delta u(2R-r,\theta)|^2 + \frac{C_4}{R^\alpha}|\nabla u(2R-r,\theta)|^2 + \frac{C_5}{R^\alpha}u^2(2R-r,\theta).$$

So, by integrating in A_R and making the change of variable $(r,\theta) \mapsto (2R-r,\theta)$ we have

$$\begin{aligned}\int_{A_R} |\Delta \overline{u}_R|^2 &\leq C_6 \int_{R\leq|x|\leq R+R^{\alpha/2}} \left(|\Delta u|^2 + |\nabla u|^2 + R^{-\alpha}u^2\right)\mathrm{d}x\\ &\leq C_7 \int_{R\leq|x|\leq R+R^{\alpha/2}} \left(|\Delta u|^2 + |\nabla u|^2 + V(x)u^2\right)\mathrm{d}x.\end{aligned}$$

Since $\overline{u}_R = u(r,\theta)$ for $|x| \geq R$, it follows that

$$\int_{A_R} |\Delta \overline{u}_R|^2 \leq C_8 \int_{B_R^c} \left(|\Delta u|^2 + |\nabla u|^2 + V(x)u^2\right)\mathrm{d}x. \tag{1.59}$$

Analogously, we obtain the estimates

$$\int_{A_R} |\nabla \overline{u}_R|^2 \leq C_9 \int_{R\leq|x|\leq R+R^{\alpha/2}} \left(|\nabla u|^2 + V(x)u^2\right)\mathrm{d}x \leq C_{10}\int_{B_R^c}\left(|\nabla u|^2 + V(x)u^2\right)\mathrm{d}x. \tag{1.60}$$

and

$$\int_{A_R} V(x)\overline{u}_R^2 \leq C_{11}\int_{R\leq|x|\leq R+R^{\alpha/2}} V(x)\overline{u}_R^2 \leq C_{11}\int_{B_R^c} V(x)u^2\mathrm{d}x. \tag{1.61}$$

Thereby, by (1.59), (1.60) and (1.61), we deduce that

$$\int_{A_R} |\Delta \overline{u}_R|^2 + |\nabla \overline{u}_R|^2 + V(x)\overline{u}_R^2\,\mathrm{d}x \leq C\int_{B_R^c} |\Delta u|^2 + |\nabla u|^2 + V(x)u^2\mathrm{d}x.$$

Recalling that $\overline{u}_R \equiv 0$ when $|x| \le R - R^{\alpha/2}$ and $\overline{u}_R \equiv u$ for $|x| \ge R$, we obtain

$$\begin{aligned} \|\overline{u}_R\|^2 &= \int_{B^c_{R-R^{\alpha/2}}} |\Delta \overline{u}_R|^2 + |\nabla \overline{u}_R|^2 + V(x)\overline{u}_R^2 \, \mathrm{d}x \\ &\le (1+C) \int_{B^c_R} |\Delta u|^2 + |\nabla u|^2 + V(x)u^2 \mathrm{d}x. \end{aligned} \tag{1.62}$$

Since $u \in E$, there exists $\bar{R} = \bar{R}(u,\gamma) > 1$ such that

$$\int_{B^c_{\bar{R}}} \left(|\Delta \overline{u}_R|^2 + |\nabla \overline{u}_R|^2 + V(x)\overline{u}_R^2\right) \mathrm{d}x = \int_{B^c_{\bar{R}}} \left(|\Delta u|^2 + |\nabla u|^2 + V(x)u^2\right) \mathrm{d}x < \frac{32\pi^2}{(1+C)\gamma},$$

which combined with (1.62) shows that $\gamma\|\overline{u}_R\|^2 < 32\pi^2$ for all $R \ge \bar{R}$. Choosing $\bar{R} = \bar{R}(u,\gamma,a,\alpha) > 0$ sufficiently large such that $\bar{R} - \bar{R}^{\alpha/2} \ge 3\bar{n}$, by Lemma 1.8.1 and for $v = \overline{u}_R/\|\overline{u}_R\|$, we have

$$\begin{aligned} \int_{B^c_{\bar{R}}} K(x)(e^{\gamma u^2} - 1)\mathrm{d}x &= \int_{B^c_{\bar{R}}} K(x)(e^{\gamma\|\overline{u}_R\|^2\left(\frac{\overline{u}_R}{\|\overline{u}_R\|}\right)^2} - 1)\mathrm{d}x \\ &\le \int_{B^c_{R-R^{\alpha/2}}} K(x)(e^{\gamma\|\overline{u}_R\|^2\left(\frac{\overline{u}_R}{\|\overline{u}_R\|}\right)^2} - 1)\mathrm{d}x \\ &\le \varepsilon \int_{B^c_{\bar{R}}} \left(|\Delta u|^2 + |\nabla u|^2 + V(x)u^2\right) \mathrm{d}x, \end{aligned}$$

for all $R \ge \bar{R}$. Therefore, (1.58) holds and the proof is done. ■

From now on, $u_0 \in E$ will denote a nontrivial weak solution of (1.6).

Lemma 1.8.3 *Suppose that* (V) *and* (K) *hold with* $\alpha \in (0,2)$ *and* $\beta \in (\alpha, +\infty)$. *Then, there exists* $\bar{R} > 0$ *such that for any* $n \in \mathbb{N}$ *satisfying* $R_n := n^{\frac{2}{2-\alpha}} \ge \bar{R}$ *we have*

$$\int_{B^c_{R_{n+1}}} (|\Delta u_0|^2 + |\nabla u_0|^2 + V(x)u_0^2)\mathrm{d}x \le \frac{3}{4} \int_{B^c_{R_n}} (|\Delta u_0|^2 + |\nabla u_0|^2 + V(x)u_0^2)\mathrm{d}x.$$

Proof . Arguing as in [10, Lemma 17], let $\chi_n : \mathbb{R}^4 \to [0,1]$ be a smooth function such that

$$\chi_n \equiv 0 \text{ in } B_{R_n}, \quad \chi_n \equiv 1 \text{ in } B_{R_{n+1}}, \quad |\nabla \chi_n| \le \frac{C}{R_{n+1}} \text{ in } A_n \quad \text{and} \quad |\Delta \chi_n| \le \frac{C}{R^2_{n+1}} \text{ in } A_n$$

where $A_n := \{x \in \mathbb{R}^4 : R_n \le |x| \le R_{n+1}\}$. Note that by construction $\chi_n u_0 \in E$ and therefore

$$\int_{B^c_{R_{n+1}}} \left(|\Delta u_0|^2 + |\nabla u_0|^2 + V(x)u_0^2\right) \mathrm{d}x \le \int_{B^c_{R_n}} \left(|\Delta u_0|^2 + |\nabla u_0|^2 + V(x)u_0^2\right) \chi_n \mathrm{d}x. \tag{1.63}$$

By using $u_0\chi_n$ as a test function in (1.30), since $\chi_n \equiv 0$ in B_{R_n} and $\chi_n \le 1$, we obtain

$$
\begin{aligned}
&\int_{B_{R_n}^c} \chi_n \left(|\Delta u_0|^2 + |\nabla u_0|^2 + V(x)u_0^2\right) \mathrm{d}x \\
&= \int_{B_{R_n}^c} [K(x)f(x,u_0)\chi_n u_0 - (\Delta u_0 \Delta\chi_n u_0 + 2\Delta u_0 \nabla\chi_n \nabla u_0 + \nabla u_0 \nabla\chi_n u_0)]\mathrm{d}x \\
&\le \int_{B_{R_n}^c} K(x)f(x,u_0)u_0 \ \mathrm{d}x - \int_{B_{R_n}^c} (\Delta u_0 \Delta\chi_n u_0 + 2\Delta u_0 \nabla\chi_n \nabla u_0 + \nabla u_0 \nabla\chi_n u_0)\,\mathrm{d}x.
\end{aligned}
$$

Using Young's inequality with $\varepsilon \in (0,1)$, we can estimate the second integral of (1.64) by

$$
\begin{aligned}
&\int_{B_{R_n}^c} (\Delta u_0 \Delta\chi_n u_0 + 2\Delta u_0 \nabla\chi_n \nabla u_0 + \nabla u_0 \nabla\chi_n u_0)\,\mathrm{d}x \\
&\le \int_{B_{R_n}^c} \left(\varepsilon|\Delta u_0|^2 + \frac{C^2}{\varepsilon R_{n+1}^4}u_0^2 + 2\varepsilon|\Delta u_0|^2 + \frac{2C^2}{\varepsilon R_{n+1}^2}|\nabla u_0|^2 + \varepsilon|\nabla u_0|^2 + \frac{C^2}{\varepsilon R_{n+1}^2}u_0^2\right) \mathrm{d}x \\
&\le 3\varepsilon \int_{B_{R_n}^c} |\Delta u_0|^2 \ \mathrm{d}x + \left(\varepsilon + \frac{2C^2}{\varepsilon R_{n+1}^2}\right) \int_{B_{R_n}^c} |\nabla u_0|^2 \ \mathrm{d}x \\
&\quad + \left(\frac{2C^2}{a}\frac{1+R_{n+1}^\alpha}{\varepsilon R_{n+1}^2}\right) \int_{B_{R_n}^c} V(x)u_0^2 \ \mathrm{d}x.
\end{aligned}
$$

Fixed $\varepsilon = 1/6$, we can choose R_{n+1} sufficiently large so that

$$
\varepsilon + \frac{2C^2}{\varepsilon R_{n+1}^2} \le \frac{1}{2} \quad \text{and} \quad \frac{2C^2}{a}\frac{1+R_{n+1}^\alpha}{\varepsilon R_{n+1}^2} \le \frac{1}{2}.
$$

Therefore,

$$
\begin{aligned}
\int_{B_{R_n}^c} &(\Delta u_0 \Delta\chi_n u_0 + 2\Delta u_0 \nabla\chi_n \nabla u_0 + \nabla u_0 \nabla\chi_n u_0)\,\mathrm{d}x \\
&\le \frac{1}{2}\int_{B_{R_n}^c} (|\Delta u_0|^2 + |\nabla u_0|^2 + V(x)u_0^2)\mathrm{d}x. \qquad (1.64)
\end{aligned}
$$

Now, let us estimate the first integral of (1.64). By (1.29), we have

$$
|f(x,s)s| \le C(\gamma)|s|(e^{\gamma s^2}-1) \ \text{ for all } \ |s| \ge \tilde{R} \ \text{ and } \ x \in \mathbb{R}^4,
$$

for some $\tilde{R} > 1$. By the previous inequality and Hölder's inequality, we find

$$
\begin{aligned}
\int_{B_{R_n}^c} K(x)f(x,u_0)u_0 \ \mathrm{d}x &\le C(\gamma)\int_{B_{R_n}^c} K(x)|u_0|(e^{\gamma u_0^2}-1) \ \mathrm{d}x \\
&\le C(\gamma)\left(\int_{B_{R_n}^c} K(x)u_0^2 \ \mathrm{d}x\right)^{1/2}\left(\int_{B_{R_n}^c} K(x)(e^{2\gamma u_0^2}-1)\mathrm{d}x\right)^{1/2}.
\end{aligned}
$$

According to (1.58), we have

$$\int_{B_{R_n}^c} K(x)(e^{2\gamma u_0^2}-1)\mathrm{d}x \leq \varepsilon \int_{B_{R_n}^c} (|\Delta u_0|^2+|\nabla u_0|^2+V(x)u_0^2)\mathrm{d}x.$$

Moreover, if $\tilde{R}$ is sufficiently large then for all $n\in\mathbb{N}$ such that $R_n \geq \tilde{R}$, we obtain

$$\sup_{x\in B_{R_n}^c} \frac{K(x)}{V(x)} \leq \sup_{x\in B_{\tilde{R}}^c} \frac{K(x)}{V(x)} \leq \sup_{x\in B_{\tilde{R}}^c} \frac{b(1+|x|^\alpha)}{a(1+|x|^\beta)} \leq \frac{b(1+\tilde{R}^\alpha)}{a(1+\tilde{R}^\beta)} := \mathcal{B}(\tilde{R})$$

and hence

$$\int_{B_{R_n}^c} K(x)u_0^2\,\mathrm{d}x \leq \mathcal{B}(\tilde{R})\int_{B_{R_n}^c} V(x)u_0^2\,\mathrm{d}x.$$

Therefore,

$$\int_{B_{R_n}^c} K(x)f(x,u_0)u_0\,\mathrm{d}x \leq \tilde{C}(\gamma)\mathcal{B}^{1/2}(\tilde{R})\int_{B_{R_n}^c} (|\Delta u_0|^2+|\nabla u_0|^2+V(x)u_0^2)\,\mathrm{d}x. \quad (1.65)$$

Since $\beta>\alpha$, one sees that $\lim_{\tilde{R}\to+\infty}\mathcal{B}(\tilde{R})=0$, which implies that $\tilde{C}(\gamma)\mathcal{B}(\tilde{R})\leq 1/4$ for $\tilde{R}>0$ sufficiently large. Thus, combining (1.63)-(1.65) we finished the proof. ■

Let us see the last result before proof of Theorem 1.1.4.

Lemma 1.8.4 *Suppose that (V) and (K) hold with $\alpha\in(0,2)$ and $\beta\in(\alpha,+\infty)$. Then, there exist $\tilde{R}>0$ and $C>0$ such that, for any $\varrho>2\tilde{R}$, there holds*

$$\int_{B_\varrho^c}(|\Delta u_0|^2+|\nabla u_0|+V(x)u_0^2)\,\mathrm{d}x \leq Ce^{\left(\log\frac{3}{4}\right)\varrho^{(2-\alpha)/2}}.$$

Proof . Let $\tilde{R}$ and (R_n) be as in Lemma 1.8.3. Considering $\varrho>2\tilde{R}$, there exist $n_1,n_2\in\mathbb{N}$, $n_1>n_2$ such that

$$R_{n_1}\leq\tilde{R}\leq R_{n_1+1} \quad\text{and}\quad R_{n_2-1}\leq\varrho\leq R_{n_2}$$

and then

$$n_2-n_1=R_{n_2}^{(2-\alpha)/2}-R_{n_1}^{(2-\alpha)/2}\geq\varrho^{(2-\alpha)/2}-\tilde{R}^{(2-\alpha)/2}>\tilde{R}(2^{(2-\alpha)/2}-1)>2$$

for $\tilde{R}>0$ sufficiently large. Hence, $n_2-n_1\geq 3$ and in particular $R_{n_2-2}\geq R_{n_1+1}\geq\tilde{R}$.

By Lemma 1.8.3, we have

$$\begin{aligned}
\int_{B_\varrho^c}(|\Delta u_0|^2+|\nabla u_0|+V(x)u_0^2)\ \mathrm{d}x &\leq \int_{B^c_{R_{n_2-1}}}(|\Delta u_0|^2+|\nabla u_0|+V(x)u_0^2)\ \mathrm{d}x\\
&\leq \frac{3}{4}\int_{B^c_{R_{n_2-2}}}(|\Delta u_0|^2+|\nabla u_0|+V(x)u_0^2)\ \mathrm{d}x\\
&\leq \left(\frac{3}{4}\right)^{n_2-n_1-2}\int_{B^c_{R_{n_1+1}}}(|\Delta u_0|^2+|\nabla u_0|+V(x)u_0^2)\ \mathrm{d}x\\
&\leq \left(\frac{3}{4}\right)^{n_2-n_1-2}\int_{B^c_{\tilde R}}(|\Delta u_0|^2+|\nabla u_0|+V(x)u_0^2)\ \mathrm{d}x\\
&\leq \frac{16}{9}e^{\left(\log\frac{3}{4}\right)(\varrho^{(2-\alpha)/2}-\tilde R^{(2-\alpha)/2})}\int_{B^c_{\tilde R}}(|\Delta u_0|^2+|\nabla u_0|+V(x)u_0^2)\ \mathrm{d}x\\
&= C(\tilde R)e^{\left(\log\frac{3}{4}\right)\varrho^{(2-\alpha)/2}}\int_{B^c_{\tilde R}}(|\Delta u_0|^2+|\nabla u_0|+V(x)u_0^2)\ \mathrm{d}x.
\end{aligned}$$

which proves the lemma. ■

We highlight that the following proof is inspired by [64, Theorem 1.11], which in turn simplifies the proof given in [10, Theorem 16] by not using Borel's finite cover Lemma.

Proof of Theorem 1.1.4: Since $u_0 \in E$, for any $x \in B_2$ we have

$$\int_{B_2} u_0^2\ \mathrm{d}x \leq \frac{(1+2^\alpha)}{a}\int_{B_2} V(x)u_0^2\ \mathrm{d}x < \infty.$$

In order to conclude that $u_0 \in L^2(\mathbb{R}^4)$, it is enough to prove that $\int_{B_2^c} u_0^2\ \mathrm{d}x < \infty$.

Let $\Sigma_j := \{x \in \mathbb{R}^4 : 2^j \leq |x| < 2^{j+1}\}$ for $j \in \mathbb{N}\cup\{0\}$. Since

$$2^{(j+2)\alpha}V(x) \geq (1+|x|^\alpha)V(x) \geq a$$

on Σ_j, we get

$$\begin{aligned}
\int_{\Sigma_j} u_0^2\ \mathrm{d}x &\leq \frac{2^{(j+2)\alpha}}{a}\int_{\Sigma_j} V(x)u_0^2\ \mathrm{d}x \leq \frac{2^{(j+2)\alpha}}{a}\int_{\Sigma_j}(|\Delta u_0|^2+|\nabla u_0|^2+V(x)u_0^2)\ \mathrm{d}x\\
&\leq \frac{2^{(j+2)\alpha}}{a}\int_{B^c_{2^j}}(|\Delta u_0|^2+|\nabla u_0|^2+V(x)u_0^2)\ \mathrm{d}x
\end{aligned}$$

and it follows from Lemma 1.8.4, taking $\varrho := 2^j \geq 2\tilde R$, that

$$\int_{\Sigma_j} u_0^2\ \mathrm{d}x \leq \frac{2^{(j+2)\alpha}}{a}Ce^{\left(\log\frac{3}{4}\right)2^{(2-\alpha)j/2}}. \tag{1.66}$$

Therefore, there exists an integer $j_0 > 0$ such that (1.66) holds for all $j \geq j_0 + 1$. Hence,

$$\begin{aligned}\int_{B_2^c} u_0^2 \,\mathrm{d}x &= \sum_{j=1}^{\infty} \int_{\Sigma_j} u_0^2 \,\mathrm{d}x = \sum_{j=1}^{j_0} \int_{\Sigma_j} u_0^2 \,\mathrm{d}x + \sum_{j=j_0+1}^{\infty} \int_{\Sigma_j} u_0^2 \,\mathrm{d}x \\ &\leq \sum_{j=1}^{j_0} \int_{\Sigma_j} u_0^2 \,\mathrm{d}x + \frac{C}{a} \sum_{j=j_0+1}^{\infty} 2^{(j+2)\alpha} e^{\left(\log \frac{3}{4}\right) 2^{(2-\alpha)j/2}} < \infty\end{aligned}$$

since $\alpha \in (0, 2)$ and $\log \frac{3}{4} < 0$. This completes the proof.

Chapter 2

On a biharmonic Choquard equation involving critical exponential growth

In this chapter, we will study the solvability of the following biharmonic Choquard equation:

$$\Delta^2 u - \Delta u + V(x)u = \left[|x|^{-\mu} * (K(x)F(x,u))\right] K(x)f(x,u),\ x \in \mathbb{R}^4,$$

where the functions V and K may decay to zero at infinity like $(1+|x|^{\alpha})^{-1}$, $\alpha \in (0,4)$, and $(1+|x|^{\beta})^{-1}$, $\beta > (8-\mu)\alpha/8$, respectively and $\mu \in (0,4)$. The nonlinear term f is a continuous function that behaves like $e^{\gamma_0 s^2}$ at infinity, for some $\gamma_0 > 0$ and F is the primitive of f. By establishing a weighted version of the Adams inequality involving V and K, we investigate the existence of nontrivial solutions for the problem above. Furthermore, we establish that the nontrivial solution is a bound state solution when $\alpha \in (0,2)$.

2.1 Introduction and main results

This chapter concerns the existence of solutions to the following fourth-order elliptic equation:

$$\Delta^2 u - \Delta u + V(x)u = \left[|x|^{-\mu} * (K(x)F(x,u))\right] K(x)f(x,u), \quad x \in \mathbb{R}^4, \tag{2.1}$$

where V and K are positive continuous functions, which can vanish at infinity, f is a nonnegative continuous function with critical exponential growth at infinity (see

Definition (2.8)), F is the primitive of f, $*$ denotes the convolution operator and $\mu \in (0,4)$.

Equations like (2.1) arise in various branches of applied mathematics and physics, see [34, 35, 36, 55] and references therein. For instance, part of the interest is becouse solutions of (2.1) are related to the existence of solitary wave solutions for Schrödinger equations of the form

$$i\frac{\partial \psi}{\partial t} = \Delta^2 \psi - \Delta \psi + W(x)\psi - \left[|x|^{-\mu} * (K(x)F(x,\psi))\right] K(x) f(x,\psi),$$

where $\psi : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{C}$ is an unknown function and $W : \mathbb{R}^4 \to \mathbb{R}$ is the potential function. For the physical interest in the influence of the biharmonic term in nonlinear Schrödinger equations we can cite [24] and references therein.

Motivated by these physical aspects, Equation (2.1) has attracted a lot of attention from many researchers and some existence and multiplicity results have been obtained. In [69], Yang considered a class of nonhomogeneous problems and studied the case where the potential is bounded from below by a positive constant and satisfies the integrability condition $1/V \in L^1(\mathbb{R}^4)$. Generally, the conditions imposed on the potential V are to overcome the loss of compactness of the Sobolev embedding $H^2(\mathbb{R}^4) \hookrightarrow L^s(\mathbb{R}^4)$ for $s \geq 2$.

Miyagaki et al. [49] studied the existence of ground state solution for fourth-order elliptic equations of the form

$$\Delta^2 u - \Delta u + u = Q(x)(f_1(u) + f_2(u)) \quad \text{in} \quad \mathbb{R}^4,$$

where f_1 is a continuous nonnegative function with polynomial growth at infinity, f_2 is a continuous nonnegative function with exponential growth and Q is a positive bounded continuous function that can vanish at infinity in the sense that if $\{A_n\}$ is a sequence of Borel sets of $\mathbb{R}$ with $\sup_{n\in\mathbb{N}} |A_n| \leq R$, for some $R > 0$, then

$$\lim_{r\to\infty} \int_{A_n \cap B_r^c(0)} Q(x)\,\mathrm{d}x = 0, \quad \text{uniformly in} \quad n \in \mathbb{N}. \tag{2.2}$$

Recently, Chen and Wang in [20] studied the existence of a normalized ground state solution for the following biharmonic Choquard-type problem:

$$\begin{cases} \Delta^2 u - \beta \Delta u = \lambda u + (I_\mu * F(u)) f(u) \quad \text{in} \quad \mathbb{R}^4, \\ \displaystyle\int_{\mathbb{R}^4} |u|^2 \mathrm{d}x = c^2, \end{cases}$$

where $\beta \geq 0$ is small, $c > 0$, $\lambda \in \mathbb{R}$, $I_\mu(x) = 1/|x|^\mu$ with $\mu \in (0,4)$ and f has critical exponential growth, among other standard conditions. For others related results with (2.1), we would like to mention the works [19, 33, 77].

The present work has been motivated by some aforementioned works and by a paper due to Shen, Radulescu and Yang [64] that studied the existence of solutions for a class of Schrödinger equations of the type

$$-\Delta u + V(x)u = \left[|x|^{-\mu} * (K(x)G(u))\right] K(x)g(u), \quad x \in \mathbb{R}^2,$$

where the potential V and the weight K may decay to zero at infinity like $(1+|x|^\alpha)^{-1}$ and $(1+|x|^\beta)^{-1}$, respectively, for $\alpha \in (0,2)$, $\beta > (4-\mu)\alpha/4$, $\mu \in (0,2)$ and G is the primitive of g, which fulfills a critical exponential growth in the Trudinger-Moser sense.

Therefore, inspired by [30, 64], throughout this work we will assume the following hypotheses on the functions V and K:

(V) $V \in C(\mathbb{R}^4)$ and there exist $\alpha, a > 0$ such that $V(x) \geq \dfrac{a}{1+|x|^\alpha}$ for all $x \in \mathbb{R}^4$;

(K) $K \in C(\mathbb{R}^4)$ and there exist $\beta, b > 0$ such that $0 < K(x) \leq \dfrac{b}{1+|x|^\beta}$ for all $x \in \mathbb{R}^4$.

We observe that the above conditions allow that V and K to vanish at infinity. For the study of problem (2.1) involving the biharmonic operator it will be necessary to enlarge the range of variation of the constants α and β adopted in [64]. Precisely, we will assume the following conditions:

$$0 < \alpha < 4 \quad \text{and} \quad \frac{(8-\mu)\alpha}{8} \leq \beta < \infty, \ \text{with} \ \ \mu \in (0,4). \tag{2.3}$$

Next, we will introduce some notations and definitions that will be used throughout of the chapter. Consider the space defined as follows

$$E := \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^4) \ : \ |\nabla u|, \, \Delta u \in L^2(\mathbb{R}^4) \ \text{ and } \ \int_{\mathbb{R}^4} V(x)u^2 \mathrm{d}x < \infty \right\}$$

endowed with the inner product

$$\langle u, v\rangle_E := \int_{\mathbb{R}^4} (\Delta u \Delta v + \nabla u \nabla v + V(x)uv) \, \mathrm{d}x$$

and its corresponding norm

$$\|u\| := \left[\int_{\mathbb{R}^4} (|\Delta u|^2 + |\nabla u|^2 + V(x)u^2) \, \mathrm{d}x \right]^{1/2}.$$

We use the notation $\|\cdot\|_{L^p(\mathbb{R}^4,K)}$ for the norm of the weighted Lebesgue space

$$L^p(\mathbb{R}^4,K)=\left\{u:\mathbb{R}^4\to\mathbb{R}: u \text{ is measurable and } \int_{\mathbb{R}^4}K(x)|u|^p\mathrm{d}x<\infty\right\},$$

that is, $\|u\|_{L^p(\mathbb{R}^4,K)}=\left(\int_{\mathbb{R}^4}K(x)|u|^p\mathrm{d}x\right)^{1/p}$.

In this context, we can establish the following weighted version of the Adams inequality:

Theorem 2.1.1 *Suppose that* (V), (K) *and* (2.3) *hold. Then, for all* $\gamma>0$ *and any* $u\in E$ *we have*

$$\int_{\mathbb{R}^4}K^{\frac{8}{8-\mu}}(x)(e^{\gamma u^2}-1)\,\mathrm{d}x<\infty. \tag{2.4}$$

Moreover, if we consider the supremum

$$\sup_{\{u\in E:\,\|u\|\le 1\}}\int_{\mathbb{R}^4}K^{\frac{8}{8-\mu}}(x)(e^{\gamma u^2}-1)\,\mathrm{d}x=\begin{cases}<\infty & \text{if }\ \gamma\in(0,32\pi^2);\\ +\infty & \text{if }\ \gamma>32\pi^2.\end{cases} \tag{2.5}$$

To control the nonlocal term $|x|^{-\mu}*(K(x)F(x,u))$, we need the well-known Hardy-Littlewood-Sobolev inequality, which we state in $\mathbb{R}^N$ and that will play an important role in this chapter.

Proposition 2.1.2 *(Hardy-Littlewood-Sobolev inequality [44, Theorem 4.3]). Suppose that* $s,r>1$ *and* $0<\mu<N$ *with* $\frac{1}{s}+\frac{\mu}{N}+\frac{1}{r}=2$, $g\in L^s(\mathbb{R}^N)$ *and* $h\in L^r(\mathbb{R}^N)$. *Then, there exists a sharp constant* $C=C(s,N,\mu,r)>0$, *independent of* g *and* h, *such that*

$$\int_{\mathbb{R}^N}[|x|^{-\mu}*g(x)]h(x)\mathrm{d}x\le C\|g\|_s\|h\|_r. \tag{2.6}$$

As an application of Theorem 2.1.1 and Proposition 2.1.2, we will investigate the existence of a weak solution for problem (2.1). We say that $u\in E$ is a weak solution for (2.1), if for all $v\in E$ it holds the equality

$$\int_{\mathbb{R}^4}(\Delta u\Delta v+\nabla u\nabla v+V(x)uv)\,\mathrm{d}x=\int_{\mathbb{R}^4}[|x|^{-\mu}*(K(x)F(x,u))]K(x)f(x,u)v\,\mathrm{d}x. \tag{2.7}$$

We are interested in the case that the nonlinearity $f(x,s)$ has the maximal growth which allows us to study (2.1) by using a variational framework considering the space E. More specifically, we assume sufficient conditions so that weak solutions of (2.1) become critical points of the functional $I:E\to\mathbb{R}$ defined by

$$I(u)=\frac{1}{2}\|u\|^2-\frac{1}{2}\int_{\mathbb{R}^4}[|x|^{-\mu}*(K(x)F(x,u))]K(x)F(x,u)\,\mathrm{d}x,$$

where $F(x,s) := \displaystyle\int_0^s f(x,t)\mathrm{d}t$ (see details in Section 2.5). Precisely, we say that $f(x,s)$ has *critical exponential growth* (at infinity) if there exists $\gamma_0 > 0$ such that

$$\lim_{|s|\to+\infty} \frac{f(x,s)}{e^{\gamma s^2}} = \begin{cases} 0, & \text{for all } \gamma > \gamma_0, \\ +\infty, & \text{for all } \gamma < \gamma_0, \end{cases} \tag{2.8}$$

uniformly in $x \in \mathbb{R}^4$.

Now, we can to establish our main assumptions on the nonlinearity $f(x,s)$:

(f_1) $f \in C(\mathbb{R}^4 \times \mathbb{R})$, f has critical exponential growth, $f(x,s) = 0$ for all $(x,s) \in \mathbb{R}^4 \times (-\infty, 0]$ and $f(x,s) = o(s^{\frac{4-\mu}{4}})$ as $s \to 0^+$, uniformly in $x \in \mathbb{R}^4$;

(f_2) $0 \le H(x,t) \le H(x,s)$ for all $0 < t < s$ and for $x \in \mathbb{R}^4$, where $H(x,u) = uf(x,u) - F(x,u)$;

(f_3) $F(x,s) \ge 0$ for all $(x,s) \in \mathbb{R}^4 \times [0,+\infty)$ and there exist constants $s_0, M_0 > 0$ such that

$$0 < sF(x,s) \le M_0 f(x,s), \text{ for all } s \ge s_0 \text{ and } x \in \mathbb{R}^4;$$

(f_4) $\displaystyle\liminf_{s\to+\infty} \frac{F(x,s)}{e^{\gamma_0 s^2}} := \beta_0 > 0$, uniformly in $x \in \mathbb{R}^4$.

We are ready to state the main existence results of the present chapter. The first theorem is the following:

Theorem 2.1.3 *Assume (V) and (K) hold with $\alpha \in (0,4)$ and $\beta > (8-\mu)\alpha/8$. If f satisfies $(f_1) - (f_4)$, then (2.1) admits a nontrivial weak solution in E.*

In the next result, by restricting the range of α, we will show that the solution obtained in Theorem 2.1.3 is in $L^2(\mathbb{R}^4)$ and thus belongs to $H^2(\mathbb{R}^4)$, that is, the solution is a bound state.

Theorem 2.1.4 *Suppose that (V) and (K) hold with $\alpha \in (0,2)$ and $\beta > (8-\mu)\alpha/8$. If f satisfies $(f_1) - (f_4)$, then the solution obtained in Theorem 2.1.3 is a bound state solution of (2.1),*

Compared to the works cited above, the main novelties of our work are the following:

(i) Theorem 2.1.1 is a weighted version of the well known Adams inequality, see Theorem 2.2.1 below. We highlight that inequality (2.5) in Theorem 2.1.1 treats only the subcritical case ($\gamma < 32\pi^2$). The critical case $\gamma = 32\pi^2$ is still an open question. The technique exploited here uses suitable cutoff functions, which restrict some estimates in the derivatives and hold only when $\gamma < 32\pi^2$;

(ii) naturally Theorem 2.1.1 can be used to extend results in the literature for potentials that may decay at infinity. For example, concerning the work [64], we seek to improve the hypotheses considered in the study of the existence of solutions, namely, we use (f_2) instead of the well-known Ambrosetti-Rabinowitz condition. This generated an additional difficulty to prove the boundedness of Cerami sequences (see Lemma 2.7.2). Moreover, by using condition (f_4), we present a simplified proof of the minimax level estimate (see Lemma 2.6.1);

(iii) we prove a compactness condition for the Euler-Lagrange functional associated with Equation (2.1) (see Lemma 2.7.3), which is generally very delicate to obtain for problems involving critical exponential growth and potentials vanishing at infinity. Moreover, we are not requiring that the functions V and K be radial;

(iv) as far as the authors know, Theorem 2.1.3 has not been obtained yet even for the case where the potential $V(x)$ is bounded from below by a positive constant or coercive. In this sense, our results complement the papers [63, 49, 69]. Furthermore, Theorem 2.1.1 extends some results in [30, 64] to the biharmonic operator;

(v) since $\alpha \in (0, 4)$ and $\beta > (8 - \mu)\alpha/8$, assumption (K) addresses situations where the function $K^{\frac{8-\mu}{8}}$ is not integrable, that is, it does not satisfy a condition like (2.2), which has been used in many works such as [12, 22, 49, 75]. Moreover, when $\alpha \in (0, 2)$, we managed to prove that the solution obtained is a bound state.

The outline of the chapter is as follows: in Section 2.2 we present some results that will be used throughout the chapter. Section 2.3 is devoted to the proof of Theorem 2.1.1 (a weighted Adams inequality). In Section 2.4 we prove two consequences of Theorem 2.1.1. First, we prove a version of the Concentration-Compactness Principle due to P.-L. Lions [45] to the space E (see Proposition 2.4.1) and then we prove that the embedding $E \hookrightarrow L^{\frac{8p}{8-\mu}}(\mathbb{R}^4, K^{\frac{8}{8-\mu}})$ is compact for all $p \geq (8 - \mu)/4$ (see Proposition

2.4.2). Section 2.5 contains the variational framework related to problem (2.1) and we also check the geometric properties of the functional I. In Section 2.6, we estimate the minimax level associated with I. Section 2.7 deals with the Cerami compactness condition. In Section 2.8, we complete the proof of Theorem 2.1.3 and in Section 2.9 we prove some auxiliary results and Theorem 2.1.4.

2.2 Preliminary results

We initially bring some results from the literature that will help us. First, we recall the well known Adams inequality [2, Theorem 3] for bounded domains in $\mathbb{R}^4$.

Theorem 2.2.1 *Let Ω be a bounded domain of $\mathbb{R}^4$. Then, there exists a constant $C > 0$ such that*

$$\sup_{\{u\in H_0^2(\Omega)\,:\,\|\Delta u\|_2\le 1\}} \int_\Omega e^{\gamma u^2}\,dx \begin{cases} \le C|\Omega| & \text{if } \gamma \le 32\pi^2, \\ = +\infty & \text{if } \gamma > 32\pi^2. \end{cases}$$

The second result is due to [47, Theorem 2.2], who have proved in dimension four the analogous of the Adachi-Tanaka inequality [1] for the space $H^2(\mathbb{R}^4)$.

Theorem 2.2.2 *For any $\alpha \in (0, 32\pi^2)$, there exists a constant $C = C(\alpha) > 0$ such that*

$$\int_{\mathbb{R}^4} (e^{\alpha u^2} - 1)\mathrm{d}x \le C\|u\|_2^2, \text{ for all } \ u \in H^2(\mathbb{R}^4) \text{ with } \|\Delta u\|_2 \le 1,$$

and this inequality is false if $\alpha \ge 32\pi^2$.

The next result is similar to [71, Lemma 4.1].

Lemma 2.2.3 *There exists a constant $C > 0$ such that for all $y \in \mathbb{R}^4$, $R > 0$ and any $u \in H_0^2(B_R(y))$ satisfying $\|\Delta u\|_{L^2(B_R(y))} \le 1$, there holds*

$$\int_{B_R(y)} (e^{32\pi^2 u^2} - 1)\mathrm{d}x \le CR^4 \int_{B_R(y)} |\Delta u|^2 \mathrm{d}x.$$

Proof . Let $\tilde{u} := \frac{u}{\|\Delta u\|_{L^2(B_R(y))}}$ with $u \neq 0$ and $\|\Delta u\|_{L^2(B_R(y))} \le 1$. Then

$$e^{32\pi^2 u^2} - 1 = \sum_{k=1}^{\infty} \frac{(32\pi^2)^k \|\Delta u\|_{L^2(B_R(y))}^{2k} |\tilde{u}|^{2k}}{k!} \le \|\Delta u\|_{L^2(B_R(y))}^2 (e^{32\pi^2 \tilde{u}^2} - 1). \tag{2.9}$$

Note that $\tilde{u} \in H_0^2(B_R(y))$ and $\|\Delta \tilde{u}\|_2 = 1$. Thus, it follows from Theorem 2.2.1 that

$$\int_{B_R(y)} (e^{32\pi^2 \tilde{u}^2} - 1)\mathrm{d}x \le CR^4.$$

Therefore, integrating (2.9) in $B_R(y)$ and using the previous inequality, we obtain the desired result. ■

2.3 Proof of Theorem 2.1.1

In this subsection, we prove our weighted version of the Adams inequality.

Proof . We will start by proving the first part of (2.5). The proof will be divided into two steps.

Step 1: Set $u \in E$ be such that $\|u\| \leq 1$. First, we want to estimate the functional

$$\mathcal{A}(u, \mu, \gamma, R) = \int_{B_R} K^{\frac{8}{8-\mu}}(x)(e^{\gamma u^2} - 1)\ \mathrm{d}x$$

for some $R > 0$, independent of u, that will be chosen during the proof. From condition (K) we get

$$\int_{B_R} K^{\frac{8}{8-\mu}}(x)(e^{\gamma u^2} - 1)\ \mathrm{d}x \leq b^{\frac{8}{8-\mu}} \int_{B_R} (e^{\gamma u^2} - 1)\ \mathrm{d}x. \tag{2.10}$$

Consider a cutoff function $\varphi \in C_0^\infty(B_{2R})$ satisfying the conditions

$$0 \leq \varphi \leq 1 \ \text{ in } B_{2R}, \ \ \varphi \equiv 1 \ \text{ in } \ B_R, \ |\nabla\varphi| \leq \frac{C}{R} \ \text{ in } \ B_{2R} \ \text{ and } \ |\Delta\varphi| \leq \frac{C}{R^2} \ \text{ in } \ B_{2R}, \tag{2.11}$$

for some constant $C > 0$. Now, notice that

$$\begin{aligned} |\Delta(\varphi u)|^2 &= |\Delta\varphi|^2 u^2 + 4(\varphi\Delta u)\nabla\varphi\nabla u + 2(\varphi\Delta u)u\Delta\varphi \\ &\quad + 4(\nabla\varphi\cdot\nabla u)^2 + 4\nabla\varphi\nabla u(u\Delta\varphi) + |\Delta u|^2\varphi^2, \end{aligned}$$

and by Young's inequality $a_1 b_1 \leq \varepsilon a_1^2 + \varepsilon^{-1} b_1^2$, with $\varepsilon \in (0,1)$ and $a_1, b_1 \geq 0$, we estimate

$$\begin{aligned} \int_{B_{2R}} |\Delta(\varphi u)|^2\ \mathrm{d}x \leq{}& \frac{C^2}{R^4}\int_{B_{2R}} u^2 \mathrm{d}x + 4\varepsilon \int_{B_{2R}} |\Delta u|^2 \mathrm{d}x + \frac{4C^2}{\varepsilon R^2}\int_{B_{2R}} |\nabla u|^2 \mathrm{d}x \\ &+ 2\varepsilon \int_{B_{2R}} |\Delta u|^2 \mathrm{d}x + \frac{2C^2}{\varepsilon R^4}\int_{B_{2R}} u^2 \mathrm{d}x + \frac{4C^2}{R^2}\int_{B_{2R}} |\nabla u|^2 \mathrm{d}x \\ &+ \frac{4\varepsilon C^2}{R^2}\int_{B_{2R}} |\nabla u|^2 \mathrm{d}x + \frac{4C^2}{\varepsilon R^4}\int_{B_{2R}} u^2 \mathrm{d}x + \int_{B_{2R}} |\Delta u|^2 \mathrm{d}x \\ ={}& (1+6\varepsilon)\int_{B_{2R}} |\Delta u|^2\ \mathrm{d}x + \left(\frac{4C^2}{\varepsilon R^2} + \frac{4C^2}{R^2} + \frac{4\varepsilon C^2}{R^2}\right)\int_{B_{2R}} |\nabla u|^2\ \mathrm{d}x \\ &+ \left(\frac{C^2}{R^4} + \frac{6C^2}{\varepsilon R^4}\right)\int_{B_{2R}} u^2\ \mathrm{d}x. \end{aligned}$$

Thus, by (V) we have $V(x)(1+(2R)^\alpha) \geq V(x)(1+|x|^\alpha) \geq a > 0$ for $x \in B_{2R}$ and therefore

$$\begin{aligned} \int_{B_{2R}} |\Delta(\varphi u)|^2\ \mathrm{d}x \leq{}& (1+6\varepsilon)\int_{B_{2R}} |\Delta u|^2\ \mathrm{d}x + \left(\frac{4C^2}{\varepsilon R^2} + \frac{4C^2}{R^2} + \frac{4\varepsilon C^2}{R^2}\right)\int_{B_{2R}} |\nabla u|^2\ \mathrm{d}x \\ &+ \left(1 + \frac{6}{\varepsilon}\right)\frac{C^2}{a}\frac{1+(2R)^\alpha}{R^4}\int_{B_{2R}} V(x)u^2\ \mathrm{d}x. \end{aligned}$$

Fixed $\varepsilon \in (0,1)$ such that $\gamma(1+6\varepsilon) \le 32\pi^2$ and since $\alpha \in (0,4)$, we can choose $\bar{R} = \bar{R}(\varepsilon, a, \alpha) > 0$ sufficiently large such that

$$\left(1+\varepsilon+\frac{1}{\varepsilon}\right)\frac{4C^2}{R^2} \le 1+6\varepsilon \quad \text{and} \quad \left(1+\frac{6}{\varepsilon}\right)\frac{C^2}{a}\frac{1+(2R)^\alpha}{R^4} \le 1+6\varepsilon,$$

for all $R \ge \bar{R}$. Consequently,

$$\int_{B_{2R}} |\Delta(\varphi u)|^2 \,\mathrm{d}x \le (1+6\varepsilon)\|u\|^2 \le 1+6\varepsilon.$$

Therefore, defining $v := \varphi u/\sqrt{1+6\varepsilon}$, we obtain $\|\Delta v\|_2^2 = \dfrac{1}{1+6\varepsilon}\displaystyle\int_{B_{2R}} |\Delta(\varphi u)|^2 \,\mathrm{d}x \le 1$ and by invoking Theorem 2.2.1, we reach

$$\int_{B_R} (e^{\gamma u^2} - 1)\,\mathrm{d}x = \int_{B_R} (e^{\gamma(\varphi u)^2} - 1)\,\mathrm{d}x \le \int_{B_{2R}} e^{\gamma(1+6\varepsilon)v^2}\mathrm{d}x \le CR^4.$$

This combined with (2.10) implies, for all $u \in C_0^\infty(\mathbb{R}^4)$ with $\|u\| \le 1$, that

$$\int_{B_R} K^{\frac{8}{8-\mu}}(x)(e^{\gamma u^2} - 1)\mathrm{d}x \le b^{\frac{8}{8-\mu}} \int_{B_R} (e^{\gamma u^2} - 1)\,\mathrm{d}x \le CR^4. \tag{2.12}$$

<u>Step 2:</u> Let us to estimate $\mathcal{A}(u,\mu,\gamma,R) = \displaystyle\int_{B_R^c} K^{\frac{8}{8-\mu}}(x)(e^{\gamma u^2} - 1)\,\mathrm{d}x$ for some R large.

For any fixed $n \ge \tilde{n}$, with $\tilde{n} \in \mathbb{N}$ to be chosen during the proof, we consider B_n^c the exterior de B_n and the covering of B_n^c formed by all annuli A_n^σ with $\sigma > n$ defined by

$$A_n^\sigma := \{x \in B_n^c : |x| < \sigma\} = \{x \in \mathbb{R}^4 : n < |x| < \sigma\}.$$

Besicovitch covering Lemma [25] ensures that for any $\sigma > \tilde{n}$, there exist a sequence of points $(x_k) \in A_{\tilde{n}}^\sigma$ and $\theta > 0$ universal constant such that

$$A_{\tilde{n}}^\sigma \subseteq \bigcup_k U_k^{1/2}, \quad \text{where} \quad U_k^{1/2} := B\left(x_k, \frac{1}{2}\frac{|x_k|}{3}\right),$$

$$\sum_k \chi_{U_k}(x) \le \theta \quad \text{for any} \quad x \in \mathbb{R}^4, \quad \text{where} \quad U_k := B\left(x_k, \frac{|x_k|}{3}\right)$$

and χ_{U_k} is its characteristic function. Let $u \in C_0^\infty(\mathbb{R}^4)$ be such that $\|u\| \le 1$. We start with the estimate of the weighted exponential integral of u in A_{3n}^σ with $n \ge \tilde{n}$ and $\sigma > 3n$. Observe that

$$A_{3n}^\sigma \subset A_{\tilde{n}}^\sigma \subseteq \bigcup_k U_k^{1/2}$$

and defining the set of indices $K_{n,\sigma} := \{k \in \mathbb{N} : U_k^{1/2} \cap B_{3n}^c \neq \emptyset \}$, we have

$$A_{3n}^\sigma \subseteq \bigcup_{k \in K_{n,\sigma}} U_k^{1/2}$$

and hence we obtain the following estimate:

$$\int_{A_{3n}^\sigma} K^{\frac{8}{8-\mu}}(x)\left(e^{\gamma u^2}-1\right) \mathrm{d}x \leq \sum_{k\in K_{n,\sigma}} \int_{{U_k}^{1/2}} K^{\frac{8}{8-\mu}}(x)\left(e^{\gamma u^2}-1\right) \mathrm{d}x. \tag{2.13}$$

Since $\frac{2}{3}|x_k| \leq |y| \leq \frac{4}{3}|x_k|$ for all $y \in U_k$, from (V) and (K), we deduce

$$V(y) \geq \frac{a}{1+|y|^\alpha} \geq \frac{a}{1+\left(\frac{4}{3}\right)^\alpha |x_k|^\alpha}, \quad \text{for all } \ y \in U_k \tag{2.14}$$

and

$$K(y) \leq \frac{b}{1+|y|^\beta} \leq \frac{b}{1+\left(\frac{2}{3}\right)^\beta |x_k|^\beta}, \quad \text{for all } \ y \in U_k. \tag{2.15}$$

Furthermore, if $U_k \cap B_{3n}^c \neq \emptyset$ then $U_k \subset B_n^c$, implying that

$$\bigcup_{k\in K_{n,\sigma}} U_k^{1/2} \subseteq \bigcup_{k\in K_{n,\sigma}} U_k \subseteq B_n^c \subseteq B_{\tilde{n}}^c. \tag{2.16}$$

Let us fix $k \in K_{n,\sigma}$. From (2.15), we obtain

$$\int_{{U_k}^{1/2}} K^{\frac{8}{8-\mu}}(x)\left(e^{\gamma u^2}-1\right) \mathrm{d}x \leq \frac{b^{\frac{8}{8-\mu}}}{\left[1+\left(\frac{2}{3}\right)^\beta |x_k|^\beta\right]^{\frac{8}{8-\mu}}} \int_{{U_k}^{1/2}} (e^{\gamma u^2}-1)\,\mathrm{d}x. \tag{2.17}$$

Consider now a cutoff function $\varphi_k \in C_0^\infty(U_k)$ such that

$$0 \leq \varphi_k \leq 1 \ \text{ in } U_k, \quad \varphi_k \equiv 1 \ \text{ in } \ U_k^{1/2}, \quad |\nabla \varphi_k| \leq \frac{C}{|x_k|} \ \text{ in } \ U_k \ \text{ and } \ |\Delta \varphi_k| \leq \frac{C}{|x_k|^2} \ \text{ in } \ U_k,$$

for some constant $C > 0$. Proceeding as before, we have

$$\int_{U_k} |\Delta(\varphi_k u)|^2 \,\mathrm{d}x \leq (1+6\varepsilon)\int_{U_k} |\Delta u|^2 \,\mathrm{d}x + \left(\frac{4C^2}{\varepsilon|x_k|^2}+\frac{4C^2}{|x_k|^2}+\frac{4\varepsilon C^2}{|x_k|^2}\right)\int_{U_k} |\nabla u|^2 \,\mathrm{d}x$$
$$+\left(\frac{C^2}{|x_k|^4}+\frac{6C^2}{|x_k|^4\varepsilon}\right)\int_{U_k} u^2 \,\mathrm{d}x,$$

and in view (2.14) it follows that

$$\int_{U_k} |\Delta(\varphi_k u)|^2 \,\mathrm{d}x \leq (1+6\varepsilon)\int_{U_k} |\Delta u|^2 \,\mathrm{d}x + \left(\frac{4C^2}{\varepsilon|x_k|^2}+\frac{4C^2}{|x_k|^2}+\frac{4\varepsilon C^2}{|x_k|^2}\right)\int_{U_k} |\nabla u|^2 \,\mathrm{d}x$$
$$+\left(1+\frac{6}{\varepsilon}\right)\frac{C^2}{a}\frac{1+\left(\frac{4}{3}\right)^\alpha |x_k|^\alpha}{|x_k|^4}\int_{U_k} V(x)u^2 \,\mathrm{d}x.$$

Since $k \in K_{n,\sigma}$, in view of (2.16), we obtain $x_k \in B^c_{\tilde{n}}$. Once $\alpha \in (0,4)$ we can choose $\tilde{n} = \tilde{n}(\varepsilon, a, \alpha) > 0$ sufficiently large satisfying

$$\left(1+\varepsilon+\frac{1}{\varepsilon}\right)\frac{4C^2}{|x_k|^2} \le 1+6\varepsilon \quad \text{and} \quad \left(1+\frac{6}{\varepsilon}\right)\frac{C^2}{a}\frac{1+\left(\frac{4}{3}\right)^\alpha |x_k|^\alpha}{|x_k|^4} \le 1+6\varepsilon,$$

for all $k \in K_{n,\sigma}$ and $n \ge \tilde{n}$. Hence,

$$\int_{U_k} |\Delta(\varphi_k u)|^2 \,\mathrm{d}x \le (1+6\varepsilon)\|u\|^2 \le 1+6\varepsilon.$$

Therefore, setting $v_k = \varphi_k u/\sqrt{1+6\varepsilon}$ we have

$$\|\Delta v_k\|_2^2 = \frac{1}{1+6\varepsilon}\int_{U_k} |\Delta(\varphi_k u)|^2 \,\mathrm{d}x \le 1.$$

Fixed $\varepsilon \in (0,1)$ such that $\gamma(1+6\varepsilon) < 32\pi^2$, by applying Theorem 2.2.2 we obtain

$$\int_{U_k^{1/2}} (e^{\gamma u^2}-1)\,\mathrm{d}x = \int_{U_k^{1/2}} (e^{\gamma(\varphi_k u)^2}-1)\,\mathrm{d}x \le \int_{\mathbb{R}^4} (e^{\gamma(1+6\varepsilon)v_k^2}-1)\,\mathrm{d}x \le C\int_{\mathbb{R}^4} |v_k|^2 \,\mathrm{d}x.$$

By the previous inequality, the definition of v_k and (2.14) we get

$$\int_{U_k^{1/2}} (e^{\gamma u^2}-1)\,\mathrm{d}x \le \frac{C}{1+6\varepsilon}\int_{U_k} u^2 \,\mathrm{d}x \le \frac{C}{1+6\varepsilon}\frac{1+\left(\frac{4}{3}\right)^\alpha |x_k|^\alpha}{a}\int_{U_k} V(x)u^2\mathrm{d}x. \quad (2.18)$$

By estimates (2.13), (2.17), (2.18) and in view of (2.16), we have

$$\begin{aligned}\int_{A^\sigma_{3n}} K^{\frac{8}{8-\mu}}(x)\left(e^{\gamma u^2}-1\right)\mathrm{d}x &\le \frac{C}{1+6\varepsilon}\frac{b^{\frac{8}{8-\mu}}}{a}\sum_{k\in K_{n,\sigma}} \frac{1+\left(\frac{4}{3}\right)^\alpha |x_k|^\alpha}{\left(1+\left(\frac{2}{3}\right)^\beta |x_k|^\beta\right)^{\frac{8}{8-\mu}}}\int_{U_k} V(x)u^2 \,\mathrm{d}x.\\ &\le \frac{C}{1+6\varepsilon}\frac{b^{\frac{8}{8-\mu}}}{a}\sum_{k\in K_{n,\sigma}} \frac{1+\left(\frac{4}{3}\right)^\alpha |x_k|^\alpha}{\left(1+\left(\frac{2}{3}\right)^\beta |x_k|^\beta\right)^{\frac{8}{8-\mu}}}\int_{B^c_n} V(x)u^2\chi_{U_k} \,\mathrm{d}x.\end{aligned}$$

By using again (2.16), we deduce

$$\frac{1+\left(\frac{4}{3}\right)^\alpha |x_k|^\alpha}{\left(1+\left(\frac{2}{3}\right)^\beta |x_k|^\beta\right)^{\frac{8}{8-\mu}}} \le \mathcal{B}_n := \sup_{x\in B^c_n} \frac{1+\left(\frac{4}{3}\right)^\alpha |x|^\alpha}{\left(1+\left(\frac{2}{3}\right)^\beta |x|^\beta\right)^{\frac{8}{8-\mu}}}, \quad \text{for all} \quad k \in K_{n,\sigma}$$

and therefore

$$\int_{A^\sigma_{3n}} K^{\frac{8}{8-\mu}}(x)\left(e^{\gamma u^2}-1\right)\mathrm{d}x \le \frac{C}{1+6\varepsilon}\frac{b^{\frac{8}{8-\mu}}}{a}\mathcal{B}_n \sum_{k\in K_{n,\sigma}}\int_{B^c_n} V(x)u^2\chi_{U_k} \,\mathrm{d}x.$$

Applying the Besicovitch covering Lemma, we have

$$\int_{A^\sigma_{3n}} K^{\frac{8}{8-\mu}}(x)\left(e^{\gamma u^2}-1\right)\,\mathrm{d}x \le \frac{C}{1+6\varepsilon}\frac{b^{\frac{8}{8-\mu}}}{a}\mathcal{B}_n\theta\int_{B^c_n} V(x)u^2 \,\mathrm{d}x.$$

Taking $\sigma \to \infty$, there exists $\tilde{n} = \tilde{n}(\varepsilon, a, \alpha) >> 1$ such that

$$\begin{aligned}\int_{B_{3n}^c} K^{\frac{8}{8-\mu}}(x)\left(e^{\gamma u^2}-1\right)\mathrm{d}x &\leq Cb^{\frac{8}{8-\mu}}\mathcal{B}_n\theta\int_{B_n^c} V(x)u^2\,\mathrm{d}x\\ &\leq Cb^{\frac{8}{8-\mu}}\mathcal{B}_n\theta\|u\|^2 \leq Cb\mathcal{B}_n\theta,\end{aligned} \tag{2.19}$$

for any $n \geq \tilde{n}$. Notice that

$$\lim_{n\to\infty}\mathcal{B}_n = \lim_{n\to\infty}\frac{1+\left(\frac{4}{3}\right)^{\alpha}n^{\alpha}}{\left(1+\left(\frac{2}{3}\right)^{\beta}n^{\beta}\right)^{\frac{8}{8-\mu}}} = \begin{cases}0, & \text{if } \beta > \frac{(8-\mu)\alpha}{8}\\ 2^{\alpha}, & \text{if } \beta = \frac{(8-\mu)\alpha}{8}.\end{cases} \tag{2.20}$$

Therefore, from estimates (2.12) and (2.19), we prove the first part of (2.5).

Next, we will now prove the second part of (2.5). Consider the Moser's sequence $\tilde{\omega}_n$ defined by

$$\tilde{\omega}_n(x) = \begin{cases}\sqrt{\frac{\log n}{8\pi^2}} - \frac{n^2}{\sqrt{32\pi^2\log n}}|x|^2 + \frac{1}{\sqrt{32\pi^2\log n}}, & \text{for } |x| \leq \frac{1}{n},\\ \frac{1}{\sqrt{8\pi^2\log n}}\log\frac{1}{|x|}, & \text{for } \frac{1}{n} < |x| \leq 1,\\ \zeta_n(x), & \text{for } |x| > 1,\end{cases}$$

where ζ_n is a smooth function compactly supported in $\overline{B}_2$ and satisfying $\zeta_n|_{\partial B_1} = \zeta_n|_{\partial B_2} = 0$,

$$\frac{\partial\zeta_n}{\partial\nu}|_{\partial B_1} = \frac{1}{\sqrt{8\pi^2\log n}},\ \frac{\partial\zeta_n}{\partial\nu}|_{\partial B_2} = 0 \quad\text{and}\quad \zeta_n,\ |\nabla\zeta_n|,\ \Delta\zeta_n \ \text{ are all }\ O(1/\sqrt{\log n}).$$

Observe that $\tilde{\omega}_n \in E$ for any $n \in \mathbb{N}$, and straightforward calculations show that

$$\|\tilde{\omega}_n\|_2^2 = O(1/\log n),\quad \|\nabla\tilde{\omega}_n\|_2^2 = O(1/\log n)\quad\text{and}\quad \|\Delta\tilde{\omega}_n\|_2^2 = 1 + O(1/\log n).$$

Besides, by (V) it follows that $\|\tilde{\omega}_n\|^2 = 1 + \delta_n$, where $\delta_n \to 0$ and $\delta_n = O(1/\log n)$, as $n \to \infty$. Setting

$$\omega_n := \frac{\tilde{\omega}_n}{\|\tilde{\omega}_n\|}, \tag{2.21}$$

we have $\omega_n \in E$ and $\|\omega_n\| = 1$. Notice further that $\omega_n(x) \geq \frac{1}{\|\tilde{\omega}_n\|}\sqrt{\frac{\log n}{8\pi^2}}$ for all $x \in \mathbb{R}^4$ with $|x| \leq 1/n$. Defining $\tilde{K} := \min_{x\in\overline{B}_1} K(x)$, we obtain, for all $\gamma > 32\pi^2$, that

$$\begin{aligned}\int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)(e^{\gamma\omega_n^2}-1)\mathrm{d}x \geq \tilde{K}^{\frac{8}{8-\mu}}\int_{B_{1/n}}(e^{\gamma\omega_n^2}-1)\,\mathrm{d}x \geq\ \tilde{K}^{\frac{8}{8-\mu}}\int_{B_{1/n}}(e^{\frac{\gamma}{\|\tilde{\omega}_n\|^2}\frac{\log n}{8\pi^2}}-1)\,\mathrm{d}x\\ = \frac{\pi^2}{2n^4}\tilde{K}^{\frac{8}{8-\mu}}\left(e^{\frac{\gamma}{\|\tilde{\omega}_n\|^2}\frac{\log n}{8\pi^2}}-1\right) \to +\infty,\end{aligned}$$

as $n \to \infty$ and therefore

$$\lim_{n\to\infty} \int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)\left(e^{\gamma\omega_n^2}-1\right) \mathrm{d}x = +\infty. \tag{2.22}$$

Since

$$\sup_{\substack{u\in E\\ \|u\|\le 1}} \int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)\left(e^{\gamma u^2}-1\right) \mathrm{d}x \ge \int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)\left(e^{\gamma\omega_n^2}-1\right) \mathrm{d}x,$$

the result can be obtained from (2.22).

To finalize the proof, it remains to show that (2.4) holds. In this way, for all $\gamma > 0$ and $u \in E$, using the density of $C_0^\infty(\mathbb{R}^4)$ in E, there exists $u_0 \in C_0^\infty(\mathbb{R}^4)$ such that $\|u-u_0\| \le 1/\sqrt{\gamma}$. Since $u^2 \le 2(u-u_0)^2+2u_0^2$, choosing $R > 0$ such that $\mathrm{supp}(u_0) \subset B_R$, we get

$$\begin{aligned}\int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)\left(e^{\gamma u^2}-1\right) \mathrm{d}x &\le \int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)\left(e^{2\gamma(u-u_0)^2}e^{2\gamma u_0^2}-1\right) \mathrm{d}x\\ &\le \frac{1}{2}\int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)\left(e^{4\gamma\|u-u_0\|^2\frac{|u-u_0|^2}{\|u-u_0\|}}-1\right) \mathrm{d}x + \frac{1}{2}\int_{B_R} K^{\frac{8}{8-\mu}}(x)\left(e^{4\gamma u_0^2}-1\right) \mathrm{d}x.\end{aligned}$$

Hence, using (2.5) and once $4\gamma\|u-u_0\|^2 \le 4 < 32\pi^2$, if follows that (2.4) holds. Thus, the theorem is proved. ■

2.4 Some applications of Theorem 2.1.1

The next result is a Lions-type concentration-compactness principle (see [45]) and the proof follows the same lines as in Lemma 2.6 of [32]. This result will be crucial to show that the functional I satisfies the Cerami compactness condition.

Proposition 2.4.1 *Suppose that* $(V), (K)$ *and* (2.3) *hold. If* $(u_n) \subset E$ *satisfies* $\|u_n\| = 1$, *for all* $n \in \mathbb{N}$, *and* $u_n \rightharpoonup u$ *in* E *with* $u \in E$ *and* $\|u\| < 1$, *then for all* $p \in \left(0, \frac{32\pi^2}{1-\|u\|^2}\right)$ *we have*

$$\sup_n \int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)(e^{pu_n^2}-1)\, \mathrm{d}x < \infty. \tag{2.23}$$

Proof . Since $u_n \rightharpoonup u$ in E and $\|u_n\| = 1$, we have

$$\|u_n - u\|^2 = \|u_n\|^2 - 2\langle u_n, u\rangle_E + \|u\|^2 \to 1 - \|u\|^2 < \frac{32\pi^2}{p}.$$

For $n \in \mathbb{N}$ enough large, we get $p\|u_n - u\|^2 < \gamma < 32\pi^2$ for some $\gamma > 0$. Choosing $q > 1$ close to 1 and $\varepsilon > 0$ small such that

$$pq(1+\varepsilon^2)\|u_n - u\|^2 < \gamma,$$

by Theorem 2.1.1, we obtain

$$\begin{aligned}\int_{\mathbb{R}^4} & K^{\frac{8}{8-\mu}}(x)\left(e^{pq(1+\varepsilon^2)(u_n-u)^2}-1\right)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)(e^{pq(1+\varepsilon^2)\|u_n-u\|^2\left(\frac{u_n-u}{\|u_n-u\|}\right)^2}-1)\mathrm{d}x \\ &\leq \int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)(e^{\gamma\left(\frac{|u_n-u|}{\|u_n-u\|}\right)^2}-1)\mathrm{d}x \\ &\leq C.\end{aligned} \tag{2.24}$$

By using Young's inequality, it follows that $ab-1 \leq \frac{a^q-1}{q}+\frac{b^r-1}{r}$ for $1/q+1/r=1$. From this and since $pu_n^2 \leq p(1+\varepsilon^2)(u_n-u)^2+p\left(1+\frac{1}{\varepsilon^2}\right)u^2$, we have

$$\begin{aligned}e^{pu_n^2}-1 &\leq \left(e^{p(1+\varepsilon^2)(u_n-u)^2}e^{p\left(1+\frac{1}{\varepsilon^2}\right)u^2}\right)-1 \\ &\leq \frac{1}{q}\left(e^{pq(1+\varepsilon^2)(u_n-u)^2}-1\right)+\frac{1}{r}\left(e^{pr\left(1+\frac{1}{\varepsilon^2}\right)u^2}-1\right).\end{aligned} \tag{2.25}$$

Therefore, inequalities (2.24) and (2.25) imply that

$$\begin{aligned}\int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)\left(e^{pu_n^2}-1\right)\mathrm{d}x &\leq \frac{1}{q}\int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)\left(e^{pq(1+\epsilon^2)(u_n-u)^2}-1\right)\mathrm{d}x \\ &+\frac{1}{r}\int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)\left(e^{pr\left(1+\frac{1}{\epsilon^2}\right)u^2}-1\right)\mathrm{d}x \leq C,\end{aligned}$$

for n sufficiently large, completing the proof. ∎

Finally, we establish the following compactness result:

Proposition 2.4.2 *Suppose that (V), (K) and (2.3) hold. Then, for all $p \geq \frac{8-\mu}{4}$, the embedding*

$$E \hookrightarrow L^{\frac{8p}{8-\mu}}(\mathbb{R}^4, K^{\frac{8}{8-\mu}}) \tag{2.26}$$

is continuous. Moreover, if the condition (2.3) is replaced by

$$0<\alpha<4 \quad \text{and} \quad \beta>\frac{(8-\mu)\alpha}{8}, \quad \text{where} \quad 0<\mu<4, \tag{2.27}$$

then the above embeddings are compact for every $p \geq \frac{8-\mu}{4}$.

Proof . Let $u \in E$. By condition (K), it follows that

$$\|u\|_{L^{\frac{8p}{8-\mu}}(B_R,K^{\frac{8}{8-\mu}})} = \left(\int_{B_R} K^{\frac{8}{8-\mu}}(x)|u|^{\frac{8p}{8-\mu}}\mathrm{d}x\right)^{\frac{8-\mu}{8p}} \leq b^{\frac{1}{p}}\|u\|_{L^{\frac{8p}{8-\mu}}(B_R)}. \tag{2.28}$$

Now, from the embedding $H^2(B_R) \hookrightarrow L^q(B_R)$ for all $q \in [1,\infty)$, we have $H^2(B_R) \hookrightarrow L^{\frac{8p}{8-\mu}}(B_R)$. Thus,

$$\begin{aligned}
\|u\|_{L^{\frac{8p}{8-\mu}}(B_R)} &\le C_1\|u\|_{H^2(B_R)} = C_1\left(\int_{B_R} |\Delta u|^2 + |\nabla u|^2 + u^2 \mathrm{d}x\right)^{\frac{1}{2}} \\
&\le C_1\left(\int_{B_R} |\Delta u|^2 + |\nabla u|^2 + \left(\frac{1+R^\alpha}{a}\right) V(x)u^2 \mathrm{d}x\right)^{\frac{1}{2}} \\
&\le C_2\left(\int_{B_R} |\Delta u|^2 + |\nabla u|^2 + V(x)u^2 \mathrm{d}x\right)^{\frac{1}{2}} := C_2\|u\|_{E(B_R)},
\end{aligned} \tag{2.29}$$

where we have used that $V(x) \ge \frac{a}{1+|x|^\alpha} \ge \frac{a}{1+R^\alpha}$ for $x \in B_R$. It follows, from (2.28) and (2.29), that

$$\|u\|_{L^{\frac{8p}{8-\mu}}(B_R, K^{\frac{8}{8-\mu}})} \le C\|u\|_{E(B_R)}.$$

Therefore,

$$E_{|B_R} \hookrightarrow H^2(B_R) \hookrightarrow\hookrightarrow L^{\frac{8p}{8-\mu}}(B_R) \hookrightarrow L^{\frac{8p}{8-\mu}}(B_R, K^{\frac{8}{8-\mu}})$$

showing the compactness of the embedding in a ball B_R.

Now let us see the compact embedding in B_R^c for $R > 0$ sufficiently large. Let (u_m) be a sequence in E such that $u_m \rightharpoonup u$ in E. We will show that $u_m \to u$ in $L^{\frac{8p}{8-\mu}}(B_R^c, K^{\frac{8}{8-\mu}})$ after passing to a subsequence if necessary. Without loss of generality, we may assume that $u \equiv 0$.

Using the fact that for any $q \in [2,\infty)$, there is $C_q > 0$, such that $|u_m|^q \le C_q(e^{u_m^2} - 1)$ and that $\frac{8p}{8-\mu} \ge 2$, proceeding as in the proof of Theorem 2.1.1, we obtain

$$\begin{aligned}
\int_{A_{3n}^\sigma} K^{\frac{8}{8-\mu}}(x)|u_m|^{\frac{8p}{8-\mu}} \mathrm{d}x &\le C_{p,\mu}\int_{A_{3n}^\sigma} K^{\frac{8}{8-\mu}}(x)(e^{u_m^2} - 1) \mathrm{d}x \\
&\le C_{p,\mu}\frac{b^{\frac{8}{8-\mu}}}{a}\sum_{k\in K_{n,\sigma}} \frac{1+\left(\frac{4}{3}\right)^\alpha |x_k|^\alpha}{\left(1+\left(\frac{2}{3}\right)^\beta |x_k|^\beta\right)^{\frac{8}{8-\mu}}}\int_{B_n^c} V(x)u_m^2\chi_{U_k} \mathrm{d}x.
\end{aligned}$$

From Besicovitch covering Lemma and the definition of $\mathcal{B}_n$, letting $\sigma \to \infty$ we have

$$\int_{B_{3n}^c} K^{\frac{8}{8-\mu}}(x)|u_m|^{\frac{8p}{8-\mu}} \mathrm{d}x \le C_{p,\mu}\frac{b^{\frac{8}{8-\mu}}}{a}\mathcal{B}_n\theta\int_{B_n^c} V(x)u_m^2 \mathrm{d}x \le C_{p,\mu}\frac{b^{\frac{8}{8-\mu}}}{a}\mathcal{B}_n\theta\|u_m\|^2, \tag{2.30}$$

and provided that $u_m \rightharpoonup 0$ in E, we get $\|u_m\|^2 \le C$, for some $C > 0$ and for all $m \in \mathbb{N}$. Thereby,

$$\int_{B_{3n}^c} K^{\frac{8}{8-\mu}}(x)|u_m|^{\frac{8p}{8-\mu}} \mathrm{d}x \le C_{p,\mu}\frac{b^{\frac{8}{8-\mu}}}{a}\mathcal{B}_n\theta.$$

Since $\beta > \frac{(8-\mu)\alpha}{8}$, by (2.20), for all $\varepsilon > 0$ there exists $m_0 > 0$ such that

$$\int_{B_{3n}^c} K^{\frac{8}{8-\mu}}(x)|u_m|^{\frac{8p}{8-\mu}}\mathrm{d}x \leq \varepsilon,$$

for all $m \geq m_0$. Consequently, we conclude that $u_m \to 0$ in $L^{\frac{8p}{8-\mu}}(B_R^c, K^{\frac{8}{8-\mu}})$, ending the proof. ■

2.5 The variational framework

The purpose of this section is to prove some geometric properties of the Euler-Lagrange functional $I : E \to \mathbb{R}$ associated to problem (2.1) given by

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{R}^4}[|x|^{-\mu} * (K(x)F(x,u))]K(x)F(x,u)\,\mathrm{d}x.$$

First, notice that by (f_1), for each $\gamma > \gamma_0$, $\varepsilon > 0$ and $q \geq 1$, there exists $C = C(\gamma, q, \varepsilon) > 0$ such that

$$|f(x,s)| \leq \varepsilon|s|^{\frac{4-\mu}{4}} + C|s|^{q-1}(e^{\gamma s^2} - 1) \tag{2.31}$$

and

$$|F(x,s)| \leq \varepsilon|s|^{\frac{8-\mu}{4}} + C|s|^{q}(e^{\gamma s^2} - 1), \tag{2.32}$$

for all $(x,s) \in \mathbb{R}^4 \times \mathbb{R}$. We will verify that I is well defined on the space E. Indeed, given $u \in E$, from (2.6), (2.32) and the continuous embedding $E \hookrightarrow L^2(\mathbb{R}^4, K^{\frac{8}{8-\mu}})$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^4}[|x|^{-\mu}*(K(x)F(x,u))]K(x)F(x,u)\ \mathrm{d}x \\
&\qquad\leq\ C\left(\int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)F^{\frac{8}{8-\mu}}(x,u)\ \mathrm{d}x\right)^{\frac{8-\mu}{4}} \\
&\qquad\leq\ C\left(\int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)|u|^{\frac{8-\mu}{4}\cdot\frac{8}{8-\mu}}\ \mathrm{d}x\right)^{\frac{8-\mu}{4}} \\
&\qquad\quad + C\left(\int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)|u|^{\frac{8q}{8-\mu}}(e^{\frac{8\gamma}{8-\mu}u^2} - 1)\ \mathrm{d}x\right)^{\frac{8-\mu}{4}} \\
&\qquad\leq\ C_1\|u\|^{\frac{8-\mu}{2}} + C_2\left(\int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)|u|^{\frac{8q}{8-\mu}}(e^{\frac{8\gamma}{8-\mu}u^2} - 1)\ \mathrm{d}x\right)^{\frac{8-\mu}{4}}.
\end{aligned} \tag{2.33}$$

By using Hölder's inequality with $p>1$ and $1/p+1/p'=1$, the continuous embedding $E \hookrightarrow L^{\frac{8pq}{8-\mu}}(\mathbb{R}^4, K^{\frac{8}{8-\mu}})$ and (2.4), we get

$$\begin{aligned}
&\left(\int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)|u|^{\frac{8q}{8-\mu}}(e^{\frac{8\gamma}{8-\mu}u^2}-1)\,\mathrm{d}x\right)^{\frac{8-\mu}{4}}\\
&\le \left(\int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)|u|^{\frac{8pq}{8-\mu}}\mathrm{d}x\right)^{\frac{8-\mu}{4p}} \times \left(\int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)(e^{\frac{8\gamma p'}{8-\mu}u^2}-1)\,\mathrm{d}x\right)^{\frac{8-\mu}{4p'}} \qquad (2.34)\\
&\le \|u\|^{2q}\left(\int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)(e^{\frac{8\gamma p'}{8-\mu}u^2}-1)\,\mathrm{d}x\right)^{\frac{8-\mu}{4p'}} < \infty.
\end{aligned}$$

Thus, from (2.33) and (2.34), we reach

$$\begin{aligned}
&\int_{\mathbb{R}^4}[|x|^{-\mu}*(K(x)F(x,u))]K(x)F(x,u)\mathrm{d}x\\
&\qquad \le C\|u\|^{\frac{8-\mu}{2}} + C\|u\|^{2q}\left(\int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)(e^{\frac{8\gamma p'}{8-\mu}u^2}-1)\,\mathrm{d}x\right)^{\frac{8-\mu}{4p'}} < \infty.
\end{aligned} \qquad (2.35)$$

Consequently, I is well-defined and by standard arguments we can see that $I \in C^1(E,\mathbb{R})$ with

$$\langle I'(u),v\rangle = \int_{\mathbb{R}^4}(\Delta u\Delta v+\nabla u\nabla v+V(x)uv)\,\mathrm{d}x - \int_{\mathbb{R}^4}[|x|^{-\mu}*(K(x)F(x,u))]K(x)f(x,u)v\mathrm{d}x,$$

for all $u,v\in E$. Hence, a critical point of I is a weak solution of problem (2.1).

The geometric conditions of the Mountain Pass Theorem for the functional I are established by the next lemma.

Lemma 2.5.1 *Suppose that* $(V),(K)$, (f_1), (f_4) *and* (2.3) *hold. Then*

(i) there exist $\tau,\rho>0$ *such that* $I(u)\ge\tau$ *for all* $\|u\|=\rho$*;*

(ii) there exists $e\in E$*, with* $\|e\|>\rho$*, such that* $I(e)<0$*.*

Proof . *(i)* In view of (2.35) and (2.5), it follows that

$$\int_{\mathbb{R}^4}[|x|^{-\mu}*(K(x)F(x,u))]K(x)F(x,u)\,\mathrm{d}x \le C_1\|u\|^{\frac{8-\mu}{2}} + C_2\|u\|^{2q}$$

for all $u\in E$ with $\|u\|=\rho$, where $p'>1, q>1$ and $\rho>0$ satisfies $\frac{8}{8-\mu}\gamma p'\rho^2 < 32\pi^2$. Hence,

$$I(u) \ge \frac{1}{2}\|u\|^2 - C_1\|u\|^{\frac{8-\mu}{2}} - C_2\|u\|^{2q} = \|u\|^2\left(\frac{1}{2} - C_1\|u\|^{\frac{4-\mu}{2}} - C_2\|u\|^{2q-2}\right).$$

Therefore, choosing $\rho>0$ sufficiently small such that $1/2 - C_1\rho^{\frac{4-\mu}{2}} - C_2\rho^{2(q-1)} := \sigma > 0$, we get $I(u)\ge\rho^2\sigma =: \tau$ whenever $\|u\|=\rho$ and item *(i)* is proved.

(ii) Consider a nonnegative function $u \in C_0^\infty(B_1) \setminus \{0\}$. Denoting by $K_1 = \min_{x \in \overline{B}_1} K(x)$, we have

$$\int_{\mathbb{R}^4} [|x|^{-\mu} * (K(x)F(x,tu))]K(x)F(x,tu)\ \mathrm{d}x$$
$$= \int_{B_1} \left(\int_{B_1} K(y) \frac{F(y,tu)}{|x-y|^\mu} \mathrm{d}y \right) K(x)F(x,tu)\mathrm{d}x$$
$$\geq \frac{K_1^2}{2^\mu} \left(\int_{B_1} F(x,tu)\mathrm{d}x \right)^2.$$

Hence, by (f_4) it follows that

$$I(tu) \leq \frac{t^2}{2}\|u\|^2 - \frac{K_1^2}{2^{\mu+1}} \left(\int_{B_1} F(x,tu)\mathrm{d}x \right)^2 \to -\infty$$

as $t \to +\infty$. Setting $e = tu$ with t large enough, the proof is complete. ■

From Lemma 2.5.1, the functional I satisfies the geometric conditions of the Mountain Pass Theorem. As a consequence, the minimax level

$$c_M := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) \tag{2.36}$$

is positive, where $\Gamma := \{\gamma \in C([0,1], E) : \gamma(0) = 0, I(\gamma(1)) < 0\}$.

2.6 The minimax level

In this section, we prove an estimate for the minimax level c_M associated to the functional I.

Lemma 2.6.1 *Suppose that* $(V), (K)$, $(f_1), (f_3), (f_4)$ *and* (2.3) *hold. Then*

$$c_M < \frac{2\pi^2(8-\mu)}{\gamma_0}.$$

Proof . It is enough to prove that there exists $n \in \mathbb{N}$ such that

$$\max_{t \geq 0} I(t\omega_n) < \frac{2\pi^2(8-\mu)}{\gamma_0},$$

where ω_n is defined in (2.21). Arguing by contradiction, we assume that

$$\max_{t \geq 0} I(t\omega_n) \geq \frac{2\pi^2(8-\mu)}{\gamma_0},$$

for all $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$, let $t_n > 0$ such that

$$\frac{t_n^2}{2} - \frac{1}{2} \int_{\mathbb{R}^4} |x|^{-\mu} * (K(x)F(x,t_n\omega_n))K(x)F(x,t_n\omega_n)\ \mathrm{d}x = \max_{t \geq 0} I(t\omega_n) \geq \frac{2\pi^2(8-\mu)}{\gamma_0}.$$

Since $F(x, t_n\omega_n) \geq 0$ for all $x \in \mathbb{R}^4$, we obtain

$$t_n^2 \geq \frac{4\pi^2(8-\mu)}{\gamma_0}, \quad \text{for all} \quad n \in \mathbb{N}. \tag{2.37}$$

At $t = t_n$, we have

$$\frac{d}{dt}[I(t\omega_n)]|_{t=t_n} = t_n - \int_{\mathbb{R}^4} |x|^{-\mu} * (K(x)F(x, t_n\omega_n))K(x)f(x, t_n\omega_n)\omega_n \, \mathrm{d}x = 0,$$

which implies

$$t_n^2 = \int_{\mathbb{R}^4} |x|^{-\mu} * (K(x)F(x, t_n\omega_n))K(x)f(x, t_n\omega_n)t_n\omega_n \, \mathrm{d}x, \quad \text{for all} \quad n \in \mathbb{N}. \tag{2.38}$$

Now we will prove that (t_n) is bounded sequence. Indeed, in view of (f_3) and (f_4), for all $\varepsilon \in (0, \beta_0)$, there exists $R = R_\varepsilon > 0$ such that

$$F(x, s)f(x, s)s \geq M_0^{-1}(\beta_0 - \varepsilon)^2 s^2 e^{2\gamma_0 s^2}, \quad \text{for all } s \geq R. \tag{2.39}$$

Since

$$t_n\omega_n(x) \geq \frac{t_n}{\|\tilde{\omega}_n\|}\sqrt{\frac{\log n}{8\pi^2}} \quad \text{for } x \in B_{1/n} \tag{2.40}$$

and

$$\frac{t_n}{\|\tilde{\omega}_n\|}\sqrt{\frac{\log n}{8\pi^2}} \to +\infty \quad \text{as } n \to \infty,$$

we conclude for $x \in B_{1/n}$ that $t_n\omega_n(x) \to \infty$ as $n \to \infty$. Taking $n \in \mathbb{N}$ sufficiently large such that $t_n\omega_n(x) \geq R$ for all $x \in B_{1/n}$, and using (2.38)- (2.40), we reach

$$\begin{aligned}
t_n^2 &\geq \int_{B_{1/n}} |x|^{-\mu} * (K(x)F(x, t_n\omega_n))K(x)f(x, t_n\omega_n)t_n\omega_n \, \mathrm{d}x \\
&\geq K_1^2 \int_{B_{1/n}} \left(\int_{B_{1/n}} \frac{F(y, t_n\omega_n)}{|x-y|^\mu} \mathrm{d}y \right) f(x, t_n\omega_n)t_n\omega_n \mathrm{d}x \\
&\geq M_0^{-1}(\beta_0 - \varepsilon)^2 (t_n\omega_n)^2 e^{2\gamma_0(t_n\omega_n)^2} K_1^2 \int_{B_{1/n}} \int_{B_{1/n}} \frac{1}{|x-y|^\mu} \, \mathrm{d}y\mathrm{d}x \\
&\geq M_0^{-1}(\beta_0 - \varepsilon)^2 \left(\frac{t_n}{\|\tilde{\omega}_n\|} \right)^2 \left(\frac{\log n}{8\pi^2} \right) e^{\gamma_0 \frac{t_n^2}{\|\tilde{\omega}_n\|^2} \frac{\log n}{4\pi^2}} K_1^2 \int_{B_{1/n}} \int_{B_{1/n}} \frac{1}{|x-y|^\mu} \, \mathrm{d}y\mathrm{d}x \\
&\geq \frac{\pi^4}{2^{\mu+2} n^{8-\mu} M_0} (\beta_0 - \varepsilon)^2 K_1^2 \frac{t_n^2}{\|\tilde{\omega}_n\|^2} \left(\frac{\log n}{8\pi^2} \right) e^{\gamma_0 \frac{t_n^2}{\|\tilde{\omega}_n\|^2} \frac{\log n}{4\pi^2}},
\end{aligned}$$

where $K_1 = \min_{x \in \overline{B_1}} K(x)$. Thus, we may write

$$1 \geq \frac{\pi^4}{2^{\mu+2} M_0} (\beta_0 - \varepsilon)^2 K_1^2 \frac{1}{(1+\delta_n)8\pi^2} e^{\left(\frac{t_n^2}{1+\delta_n} \frac{\gamma_0}{4\pi^2} - (8-\mu) \right) \log n + \log(\log n)}. \tag{2.41}$$

By (2.37) and once $\log(\log n) > 0$ we get

$$0 \geq \log\left(\frac{\pi^4}{2^{\mu+2}M_0}(\beta_0-\varepsilon)^2K_1^2\frac{1}{(1+\delta_n)8\pi^2}\right) + \left(\frac{t_n^2}{1+\delta_n}\frac{\gamma_0}{4\pi^2}-(8-\mu)\right)\log n. \quad (2.42)$$

In view of (2.42), we see that (t_n) is bounded, because if $t_n \to +\infty$ then letting $n \to \infty$ in (2.42) and since $\delta_n \to 0$, we obtain a contradiction. Thus, up to a subsequence, by (2.37), there exists a positive constant t_0 such that

$$\lim_{n\to\infty} t_n^2 = t_0^2 \geq \frac{4\pi^2(8-\mu)}{\gamma_0}.$$

We claim that $t_0^2 = 4\pi^2(8-\mu)/\gamma_0$. To prove this, it suffices suppose that $t_0^2 > 4\pi^2(8-\mu)/\gamma_0$ and letting $n \to \infty$ in (2.42), again we reach a contradiction.

Finally, passing to the limit as $n \to \infty$ in (2.41), there holds

$$1 \geq \frac{\pi^4}{2^{\mu+2}M_0}(\beta_0-\varepsilon)^2K_1^2\frac{1}{8\pi^2}\lim_{n\to\infty} e^{\log(\log n)} = +\infty$$

which is an absurd. This completes the proof of the lemma. ■

2.7 On the Cerami compactness condition

In this section, we show that the functional I satisfies the Cerami condition for certain energy levels. We recall that the functional I satisfies the Cerami condition at the level c, denoted by $(Ce)_c$ condition, if any sequence $(u_n) \subset E$ verifying

$$I(u_n) \to c \quad \text{and} \quad (1+\|u_n\|)\|I'(u_n)\|_* \to 0 \quad \text{as} \quad n \to \infty, \quad (2.43)$$

has a strongly convergent subsequence in E.

We begin by proving some auxiliary results. First we will present a convergence result, whose proof follows the same lines of [64, Lemma 3.4], and we omit it.

Lemma 2.7.1 *Suppose that $(V), (K)$, $(f_1)-(f_4)$ and (2.27) hold. If $(u_n) \subset E$ is such that $u_n \rightharpoonup u$ in E as $n \to \infty$ and there is a constant $C > 0$ satisfying*

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^4} [|x|^{-\mu} * (K(x)F(x,u_n))]K(x)f(x,u_n)u_n \mathrm{d}x \leq C, \quad (2.44)$$

then, up to a subsequence, there holds

$$\int_{\mathbb{R}^4} [|x|^{-\mu} * (K(x)F(x,u_n))]K(x)F(x,u_n)\mathrm{d}x \to \int_{\mathbb{R}^4} [|x|^{-\mu} * (K(x)F(x,u))]K(x)F(x,u)\mathrm{d}x. \quad (2.45)$$

Moreover, for all $\psi \in C_0^\infty(\mathbb{R}^4)$*, up to a subsequence, we have*

$$\int_{\mathbb{R}^4} [|x|^{-\mu} * (K(x)F(x,u_n))]K(x)f(x,u_n)\psi \mathrm{d}x \\ \to \int_{\mathbb{R}^4} [|x|^{-\mu} * (K(x)F(x,u))]K(x)f(x,u)\psi \mathrm{d}x. \tag{2.46}$$

Now we can prove the following result:

Lemma 2.7.2 *Suppose that* $(V), (K)$*,* $(f_1) - (f_4)$ *and* (2.3) *hold. Then, any* $(Ce)_{c_M}$*-sequence* (u_n) *for* I *is bounded in* E*.*

Proof . Consider $(u_n) \subset E$ such that

$$I(u_n) = \frac{1}{2}\|u_n\|^2 - \frac{1}{2}\int_{\mathbb{R}^4} [|x|^{-\mu} * (K(x)F(x,u_n))]K(x)F(x,u_n)\ \mathrm{d}x \to c_M \tag{2.47}$$

and for all $v \in E$

$$(1 + \|u_n\|) \left| \langle u_n, v\rangle_E - \int_{\mathbb{R}^4} [|x|^{-\mu} * (K(x)F(x,u_n))]K(x)f(x,u_n)v\ \mathrm{d}x \right| \le o_n(1)\|v\|. \tag{2.48}$$

We argue by a contradiction by assuming that, up to a subsequence, $\|u_n\| \to +\infty$. Let $v_n := \sigma u_n/\|u_n\|$ where

$$\sigma = \sqrt{c_M + \frac{2\pi^2(8-\mu)}{\gamma_0}}.$$

By Lemma 2.6.1, it follows that

$$2c_M < \|v_n\|^2 = \sigma^2 = c_M + \frac{2\pi^2(8-\mu)}{\gamma_0} < \frac{4\pi^2(8-\mu)}{\gamma_0}. \tag{2.49}$$

Now, choose $\gamma > \gamma_0$ sufficiently close to γ_0 and $r > 1$ close to 1, where $1/r + 1/r' = 1$ such that

$$\gamma_1 := \frac{8\gamma r\|v_n\|^2}{8-\mu} < 32\pi^2. \tag{2.50}$$

By (f_2), together with (2.6), (2.31), (2.26), (2.50) and (2.5), it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^4}[|x|^{-\mu}*(K(x)F(x,v_n))]K(x)f(x,v_n)v_n\,\mathrm{d}x \\
&\le C\left(\int_{\mathbb{R}^4}K^{\frac{8}{8-\mu}}(x)(f(x,v_n)v_n)^{\frac{8}{8-\mu}}\mathrm{d}x\right)^{\frac{8-\mu}{4}} \\
&\le C\left(\int_{\mathbb{R}^4}K^{\frac{8}{8-\mu}}(x)|v_n|^2\,\mathrm{d}x\right)^{\frac{8-\mu}{4}} \\
&\quad + C\left(\int_{\mathbb{R}^4}K^{\frac{8}{8-\mu}}(x)|v_n|^{\frac{8q}{8-\mu}}(e^{\frac{8\gamma}{8-\mu}{v_n}^2}-1)\,\mathrm{d}x\right)^{\frac{8-\mu}{4}} \\
&\le C\|v_n\|^{\frac{8-\mu}{2}}+C\left(\int_{\mathbb{R}^4}K^{\frac{8}{8-\mu}}(x)|v_n|^{\frac{8qr'}{8-\mu}}\mathrm{d}x\right)^{\frac{8-\mu}{4r'}} \\
&\quad\times\left(\int_{\mathbb{R}^4}K^{\frac{8}{8-\mu}}(x)\left(e^{\frac{8\gamma r}{8-\mu}{v_n}^2}-1\right)\,\mathrm{d}x\right)^{\frac{8-\mu}{4r}} \\
&\le C\|v_n\|^{\frac{8-\mu}{2}}+C\|v_n\|^{2q}\left(\int_{\mathbb{R}^4}K^{\frac{8}{8-\mu}}(x)\left(e^{\frac{8\gamma r\|v_n\|^2}{8-\mu}\left(\frac{v_n}{\|v_n\|}\right)^2}-1\right)\mathrm{d}x\right)^{\frac{8-\mu}{4r}} \\
&\le C\sigma^{\frac{8-\mu}{2}}+C\sigma^{2q}\left(\sup_{\substack{u\in E\\ \|u\|\le 1}}\int_{\mathbb{R}^4}K^{\frac{8}{8-\mu}}(x)(e^{\gamma_1 u^2}-1)\,\mathrm{d}x\right)^{\frac{8-\mu}{4r}}\le C.
\end{aligned} \tag{2.51}$$

Since $\|v_n\|=\sigma$, up to a subsequence, there exists $v\in E$ such that $v_n\rightharpoonup v$ in E. We will analyze two possibilities:

Case 1: $v\equiv 0$;

In view of (2.51) and (2.45), we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^4}[|x|^{-\mu}*(K(x)F(x,v_n))]K(x)F(x,v_n)\mathrm{d}x=0$$

and this implies that

$$\lim_{n\to\infty}I(v_n)=\frac{\sigma^2}{2}-\frac{1}{2}\lim_{n\to\infty}\int_{\mathbb{R}^4}[|x|^{-\mu}*(K(x)F(x,v_n))]K(x)F(x,v_n)\mathrm{d}x=\frac{\sigma^2}{2}. \tag{2.52}$$

Note that $\sigma/\|u_n\|\in(0,1)$ for $n\in\mathbb{N}$ sufficiently large, once $\|u_n\|\to+\infty$ as $n\to\infty$. Let $t_n\in(0,1]$ be such that $\max\limits_{t\in(0,1]}I(tu_n)$ is achieved. Thus, $\langle I'(t_nu_n),t_nu_n\rangle=0$ and by

(f_2) we get

$$\begin{aligned}
I(v_n) &= I\left(\frac{\sigma u_n}{\|u_n\|}\right) \\
&\leq \max_{t\in(0,1]} I(tu_n) = I(t_nu_n) = I(t_nu_n) - \frac{1}{2}\langle I'(t_nu_n), t_nu_n\rangle \\
&= \frac{1}{2}\int_{\mathbb{R}^4}[|x|^{-\mu} * (K(x)F(x,t_nu_n))]K(x)(f(x,t_nu_n)t_nu_n - F(x,t_nu_n))\mathrm{d}x \qquad (2.53) \\
&\leq \frac{1}{2}\int_{\mathbb{R}^4}[|x|^{-\mu} * (K(x)F(x,u_n))]K(x)(f(x,u_n)u_n - F(x,u_n))\mathrm{d}x \\
&= I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle.
\end{aligned}$$

Thereby, from (2.47), (2.48) and (2.52), letting $n \to \infty$ in (2.53), we obtain $\sigma^2 \leq 2c_M$, which contradicts (2.49).

<u>Case 2:</u> $v \neq 0$;

In this case, there exists $R > 0$ such that $\Upsilon \cap B_R$ has positive Lebesgue measure, where $\Upsilon := \{x \in \mathbb{R}^4 : v(x) \neq 0\}$. Notice that

$$\begin{aligned}
&\frac{1}{\|u_n\|^2}\int_{\mathbb{R}^4}[|x|^{-\mu} * (K(x)F(x,u_n))]K(x)F(x,u_n)\mathrm{d}x \\
&\qquad = \int_{\mathbb{R}^4}\left(\int_{\mathbb{R}^4}\frac{K(y)F(y,u_n)}{|x-y|^{\mu}\|u_n\|}\mathrm{d}y\right)\frac{K(x)F(x,u_n)}{\|u_n\|}\mathrm{d}x \\
&\qquad \geq \frac{K_R^2}{(2R)^{\mu}}\left(\int_{\Upsilon\cap B_R}\frac{F(x,u_n)}{|u_n|}|v_n|\mathrm{d}x\right)^2
\end{aligned}$$

where $K_R := \min_{x\in\overline{B}_R} K(x)$. Since $\|u_n\| \to +\infty$, by definition of v we have $|u_n| \to \infty$ in $\Upsilon \cap B_R$. Moreover, by assumption (f_4) we have $F(x,u_n)/|u_n| \to +\infty$ as $|u_n| \to +\infty$. By applying Fatou's Lemma we reach

$$\frac{1}{\|u_n\|^2}\int_{\mathbb{R}^4}[|x|^{-\mu} * (K(x)F(x,u_n))]K(x)F(x,u_n)\mathrm{d}x \to +\infty \quad \text{as} \quad n \to \infty.$$

Therefore, since $I(u_n) \to c_M$ as $n \to \infty$, we obtain

$$\begin{aligned}
0 &= \liminf_{n\to\infty}\frac{I(u_n)}{\|u_n\|^2} \\
&\leq \frac{1}{2} - \frac{1}{2}\limsup_{n\to\infty}\frac{1}{\|u_n\|^2}\int_{\mathbb{R}^4}[|x|^{-\mu} * (K(x)F(x,u_n))]K(x)F(x,u_n)\mathrm{d}x \\
&= -\infty
\end{aligned}$$

which is a contradiction. Thus, we complete the proof of the lemma. ■

We are ready to prove the main compactness result of this section.

Proposition 2.7.3 *Under hypotheses* $(V), (K)$, $(f_1)-(f_4)$ *and* (2.27) *the functional* I *satisfies* $(Ce)_{c_M}$ *condition.*

Proof . Let $(u_n) \subset E$ be a Cerami sequence of I at the level c_M. By Lemma 2.7.2, up to a subsequence, there exists $u \in E$ such that $u_n \rightharpoonup u$ weakly in E. Taking $v = u_n$ in (2.48) we have (2.44) holds. Then, by Lemma 2.7.1, 2.45 and (2.46) must occur. It follows from (2.46) that $I'(u) = 0$ and this combined with (f_2) shows that

$$\begin{aligned} I(u) &= I(u) - \frac{1}{2}\langle I'(u), u\rangle \\ &= \frac{1}{2}\int_{\mathbb{R}^4} [|x|^{-\mu} * (K(x)F(x,u))]K(x)\left(f(x,u)u - F(x,u)\right) \mathrm{d}x \geq 0. \end{aligned} \tag{2.54}$$

Since $c_M > 0$, we have two cases to consider.

<u>*Case 1:*</u> $u = 0$;

In this case, taking $u \equiv 0$ in (2.45), we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^4} [|x|^{-\mu} * (K(x)F(x,u_n))]K(x)F(x,u_n)\mathrm{d}x = 0. \tag{2.55}$$

This together with (2.47) and Lemma 2.6.1 implies that

$$\limsup_{n\to\infty} \|u_n\|^2 = 2c_M < \frac{4\pi^2(8-\mu)}{\gamma_0}. \tag{2.56}$$

We claim that

$$\lim_{n\to\infty} \int_{\mathbb{R}^4} [|x|^{-\mu} * (K(x))F(x,u_n)]K(x)f(x,u_n)u_n\mathrm{d}x = 0. \tag{2.57}$$

Indeed, by using (f_2), (2.6), (2.31), (2.56) and Proposition 2.4.2, we get

$$\begin{aligned} &\int_{\mathbb{R}^4} [|x|^{-\mu} * (K(x)F(x,u_n))]K(x)f(x,u_n)u_n \, \mathrm{d}x \\ &\leq C\left(\int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)(f(x,u_n)u_n)^{\frac{8}{8-\mu}}\mathrm{d}x\right)^{\frac{8-\mu}{4}} \\ &\leq C\left(\int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)|u_n|^2 \, \mathrm{d}x\right)^{\frac{8-\mu}{4}} + C\left(\int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)|u_n|^{\frac{8q}{8-\mu}}(e^{\frac{8\gamma}{8-\mu}{u_n}^2} - 1) \, \mathrm{d}x\right)^{\frac{8-\mu}{4}} \\ &\leq C\|u_n\|^{\frac{8-\mu}{2}}_{L^2(\mathbb{R}^4, K^{\frac{8}{8-\mu}})} + C\|u_n\|^{2q}_{L^{\frac{8qr'}{8-\mu}}(\mathbb{R}^4, K^{\frac{8}{8-\mu}})}\left(\int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)(e^{\frac{8\gamma r}{8-\mu}{u_n}^2} - 1) \, \mathrm{d}x\right)^{\frac{8-\mu}{4r}} \to 0, \end{aligned}$$

as $n \to \infty$, with $\gamma > \gamma_0$ sufficiently close to γ_0 and $r > 1$ close to 1, where $1/r + 1/r' = 1$. From this and once $\langle I'(u_n), u_n\rangle \to 0$ we obtain $\|u_n\| \to 0$, which contradicts (2.56).

<u>*Case 2:*</u> $u \neq 0$;

In this case, since (u_n) is $(Ce)_{c_M}$ sequence for I, we may define

$$v_n := \frac{u_n}{\|u_n\|} \quad \text{and} \quad v := \frac{u}{\lim\limits_{n\to\infty} \|u_n\|}.$$

Thus, $v_n \rightharpoonup v$ weakly in E, $\|v_n\| = 1$ and $\|v\| \leq 1$. If $\|v\| = 1$ then we conclude the proof. Suppose that $\|v\| < 1$. From Lemma 2.6.1, (2.54), (2.48) and (2.45) we obtain

$$\begin{aligned}\frac{4\pi^2(8-\mu)}{\gamma_0} > 2c_M \geq 2(c_M - I(u))& \\ &= \limsup_{n\to\infty}(\|u_n\|^2 - \|u\|^2) \\ &= \limsup_{n\to\infty} \|u_n\|^2 \left(1 - \left\|\frac{u}{\|u_n\|}\right\|^2\right).\end{aligned}$$

By the definition of v and Fatou's Lemma, we get

$$\limsup_{n\to\infty} \|u_n\|^2 < \frac{4\pi^2(8-\mu)}{\gamma_0(1-\|v\|^2)}.$$

Choosing $\gamma > \gamma_0$ sufficiently close to γ_0 and $r > 1$ sufficiently close to 1 such that $1/r + 1/r' = 1$, we can deduce that

$$\frac{8r\gamma\|u_n\|^2}{8-\mu} \leq \eta < \frac{32\pi^2}{1-\|v\|^2} \tag{2.58}$$

for $n \in \mathbb{N}$ large and for some $\eta > 0$. By using that $|u_n|^2 = \|u_n\|^2|v_n|^2$, (2.58) and (2.23) we reach

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)\left(e^{\frac{8r\gamma}{8-\mu}|u_n|^2} - 1\right) \mathrm{d}x \leq \sup_{n\in\mathbb{N}} \int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)\left(e^{\eta|v_n|^2} - 1\right) \mathrm{d}x < \infty. \tag{2.59}$$

Now, we claim that

$$\int_{\mathbb{R}^4} [|x|^{-\mu} * (K(x)F(x,u_n))]K(x)f(x,u_n)(u_n - u)\,\mathrm{d}x \to 0 \quad \text{as} \quad n \to \infty. \tag{2.60}$$

Indeed, from (2.6) one has

$$\begin{aligned}&\left|\int_{\mathbb{R}^4} [|x|^{-\mu} * (K(x)F(x,u_n))]K(x)f(x,u_n)(u_n - u)\,\mathrm{d}x\right| \\ &\quad \leq C\|K(x)F(x,u_n)\|_{\frac{8}{8-\mu}}\|K(x)f(x,u_n)(u_n-u)\|_{\frac{8}{8-\mu}} =: CI_{1,n}I_{2,n}.\end{aligned} \tag{2.61}$$

By using (2.32), (2.59), Hölder's inequality and Proposition 2.4.2, we obtain

$$\begin{aligned}I_{1,n}^{\frac{8}{8-\mu}} &\leq C\int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)|u_n|^{\frac{8}{8-\mu}\cdot\frac{8-\mu}{4}}\mathrm{d}x + C\int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)|u_n|^{\frac{8q}{8-\mu}}(e^{\frac{8\gamma}{8-\mu}|u_n|^2} - 1)\mathrm{d}x \\ &\leq C\|u_n\|^2 + C\left(\int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)|u_n|^{\frac{8qr'}{8-\mu}}\mathrm{d}x\right)^{\frac{1}{r'}}\left(\int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)(e^{\frac{8r\gamma}{8-\mu}|u_n|^2} - 1)\mathrm{d}x\right)^{\frac{1}{r}} \\ &\leq C\|u_n\|^2 + C\|u_n\|^{\frac{8q}{8-\mu}}\left(\int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)(e^{\frac{8r\gamma}{8-\mu}|u_n|^2} - 1)\mathrm{d}x\right)^{\frac{1}{r}} \leq \tilde{C}.\end{aligned}$$

Similarly, choosing $\varepsilon > 0$ and $q = r' \geq \frac{8-\mu}{4}$, we have $r = \frac{q}{q-1}$ and by using (2.31), Hölder's inequality and Proposition 2.4.2, we get

$$\begin{aligned}
I_{2,n}^{\frac{8}{8-\mu}} &\leq \varepsilon \int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)|u_n|^{\frac{4-\mu}{4}\cdot\frac{8}{8-\mu}}|u_n - u|^{\frac{8}{8-\mu}}\mathrm{d}x \\
&\quad + C(\gamma, q, \varepsilon)\int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)|u_n - u|^{\frac{8}{8-\mu}}|u_n|^{\frac{8(q-1)}{8-\mu}}(e^{\frac{8\gamma}{8-\mu}|u_n|^2} - 1)\mathrm{d}x \\
&\leq \varepsilon\left(\int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)|u_n|^2\mathrm{d}x\right)^{\frac{4-\mu}{8-\mu}}\left(\int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)|u_n - u|^2\mathrm{d}x\right)^{\frac{4}{8-\mu}} \\
&\quad + C(\gamma, q, \varepsilon)\left(\int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)|u_n - u|^{\frac{8q}{8-\mu}}|u_n|^{\frac{8q(q-1)}{8-\mu}}\mathrm{d}x\right)^{\frac{1}{q}} \\
&\quad \times\left(\int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)(e^{\frac{8r\gamma}{8-\mu}|u_n|^2} - 1)\mathrm{d}x\right)^{\frac{q-1}{q}} \\
&\leq \varepsilon\|u_n\|^{\frac{2(4-\mu)}{(8-\mu)}}\left(\int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)|u_n - u|^2\mathrm{d}x\right)^{\frac{4}{8-\mu}} \\
&\quad + C\|u_n\|^{\frac{4(q-1)}{(8-\mu)}}\left(\int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)|u_n - u|^{\frac{16q}{8-\mu}}\mathrm{d}x\right)^{\frac{1}{2q}}\left(\int_{\mathbb{R}^4} K^{\frac{8}{8-\mu}}(x)(e^{\frac{8r\gamma}{8-\mu}|u_n|^2} - 1)\mathrm{d}x\right)^{\frac{q-1}{q}}.
\end{aligned}$$

Hence, combining the previous estimate with (2.59) and Proposition 2.4.2, up to a subsequence, we have $I_{2,n} \to 0$ as $n \to \infty$. From this and according to (2.61) and (2.7), it follows that (2.60) holds.

Finally, note that the convexity of the functional $J(v) := \frac{1}{2}\|v\|^2$ guarantees that

$$\begin{aligned}
\frac{1}{2}\|u\|^2 = J(u) &\geq J(u_n) + \langle J'(u_n), u - u_n\rangle \\
&= \frac{1}{2}\|u_n\|^2 + \int_{\mathbb{R}^4} \Delta u_n \Delta(u - u_n) + \nabla u_n \nabla(u - u_n) + V(x)u_n(u - u_n)\mathrm{d}x \\
&= \frac{1}{2}\|u_n\|^2 + \langle I'(u_n), u - u_n\rangle \\
&\quad - \int_{\mathbb{R}^4} [|x|^{-\mu} * (K(x)F(x, u_n))]K(x)f(x, u_n)(u_n - u)\mathrm{d}x.
\end{aligned}$$

By virtue of $\langle I'(u_n), u - u_n\rangle \to 0$ as $n \to \infty$ and (2.60), we conclude $\limsup_{n\to\infty} \|u_n\|^2 \leq \|u\|^2$ and consequenlty $u_n \to u$ strongly in E. ■

2.8 Proof of Theorem 2.1.3

From Lemma 2.5.1, the functional I satisfies the geometric conditions of the Mountain Pass Theorem and in view of Lemma 2.6.1 and Proposition 2.7.3 we have that I satisfies the $(Ce)_{c_M}$ condition. Thus, by the Mountain Pass Theorem (with

Cerami condition, see [16, 17]) the functional I has a critical point u_0 nontrivial at the minimax level c_M.

2.9 Bound state solution

In this last section, we prove Theorem 2.1.4. We will show that any weak solution of problem (2.1) is a bound state solution, that is, belongs to $H^2(\mathbb{R}^4)$. Therefore, it remains to verify that $u_0 \in L^2(\mathbb{R}^4)$. To do this, we will adapt some arguments presented by Ambrosetti, Felli and Malchiodi in [10]. The first result is the following:

Lemma 2.9.1 *Let $(V), (K)$ with $\alpha \in (0,4)$, $\beta > (8-\mu)\alpha/8$ and $\gamma \in (0, 32\pi^2)$, then for each $v \in E \setminus \{0\}$ with $\|v\| \leq 1$ and any $\varepsilon > 0$, there exists $\bar{n} = \bar{n}(\gamma, a, \alpha) > 1$ independent of v such that for every $n \geq \bar{n}$,*

$$\int_{B_{3n}^c} K^{\frac{8}{8-\mu}}(x)(e^{\gamma v^2} - 1)\mathrm{d}x \leq \varepsilon.$$

The proof follows directly from inequalities (2.19) and (2.20). The next lemma is inspired by the arguments of [10, Lemma 11].

Lemma 2.9.2 *Suppose that (V) and (K) hold with $\alpha \in (0,2)$ and $\beta > (8-\mu)\alpha/8$. Let $\gamma > 0$ and $u \in E \backslash \{0\}$. Then, for any $\varepsilon > 0$ there exists $\bar{R} = \bar{R}(u, \gamma, a, \alpha)$ such that*

$$\int_{B_R^c} K^{\frac{8}{8-\mu}}(x)(e^{\gamma u^2} - 1)\mathrm{d}x \leq \varepsilon, \quad \textit{for all} \;\; R \geq \bar{R}. \tag{2.62}$$

Proof . Let $R > 1$ and $\overline{\psi}_R : \mathbb{R}^+ \to [0,1]$ be a smooth nondecreasing function given by

$$\overline{\psi}_R(r) = \begin{cases} 0, & \text{if } r \leq R - R^{\alpha/2}, \\ 1, & \text{if } r \geq R \end{cases}$$

and satisfying

$$|\overline{\psi}'_R| \leq \frac{2}{R^{\alpha/2}} \quad \text{and} \quad |\overline{\psi}''_R| \leq \frac{2}{R^{\alpha}}.$$

In polar coordinates $(r, \theta) \in [0, +\infty) \times \mathbb{S}^3$, we define

$$\overline{u}_R(r, \theta) := \begin{cases} 0, & \text{if } \; r \leq R - R^{\alpha/2}, \\ \overline{\psi}_R(r) u(2R - r, \theta), & \text{if } \; R - R^{\alpha/2} \leq r \leq R, \\ u(r, \theta), & \text{if } \; r \geq R. \end{cases}$$

In the annulus $A_R = \{x \in \mathbb{R}^4 : R - R^{\alpha/2} \leq |x| \leq R\}$, we have

$$\nabla \overline{u}_R = \overline{\psi}'_R(r)u(2R-r,\theta)\mathbf{e}_r - \overline{\psi}_R(r)u_r(2R-r,\theta)\mathbf{e}_r + \frac{1}{r}\overline{\psi}_R(r)u_\theta(2R-r,\theta)\mathbf{e}_\theta,$$

where $\mathbf{e}_r = x/|x|$ and $\mathbf{e}_\theta$ is a unit vector tangent to the unit sphere. Furthermore,

$$\begin{aligned}\Delta \overline{u}_R &= \frac{1}{r}\overline{\psi}'_R(r)u(2R-r,\theta) - \frac{1}{r}\overline{\psi}_R(r)u_r(2R-r,\theta) + \overline{\psi}''_R(r)u(2R-r,\theta)\\ &\quad - 2\overline{\psi}'_R(r)u_r(2R-r,\theta) + \overline{\psi}_R(r)u_{rr}(2R-r,\theta) + \frac{1}{r^2}\overline{\psi}_R(r)u_{\theta\theta}(2R-r,\theta)\\ &= \overline{\psi}_R(r)u_{rr}(2R-r,\theta) + \frac{1}{r^2}\overline{\psi}_R(r)u_{\theta\theta}(2R-r,\theta)\\ &\quad - \left(2\overline{\psi}'_R(r) + \frac{1}{r}\overline{\psi}_R(r)\right)u_r(2R-r,\theta) + \left(\overline{\psi}''_R(r) + \frac{1}{r}\overline{\psi}'_R(r)\right)u(2R-r,\theta).\end{aligned}$$

Hence, in the annulus A_R we obtain

$$|\nabla \overline{u}_R|^2 \leq C_1|\nabla u(2R-r,\theta)|^2 + \frac{C_2}{R^\alpha}u^2(2R-r,\theta)$$

and

$$|\Delta \overline{u}_R|^2 \leq C_3|\Delta u(2R-r,\theta)|^2 + \frac{C_4}{R^\alpha}|\nabla u(2R-r,\theta)|^2 + \frac{C_5}{R^\alpha}u^2(2R-r,\theta).$$

So, by integrating in A_R and making the change of variable $(r,\theta) \mapsto (2R-r,\theta)$, we reach

$$\begin{aligned}\int_{A_R} |\Delta \overline{u}_R|^2 &\leq C_6 \int_{R\leq|x|\leq R+R^{\alpha/2}} \left(|\Delta u|^2 + |\nabla u|^2 + R^{-\alpha}u^2\right) \mathrm{d}x\\ &\leq C_7 \int_{R\leq|x|\leq R+R^{\alpha/2}} \left(|\Delta u|^2 + |\nabla u|^2 + V(x)u^2\right) \mathrm{d}x.\end{aligned}$$

Since $\overline{u}_R = u(r,\theta)$ for $|x| \geq R$, it follows that

$$\int_{A_R} |\Delta \overline{u}_R|^2 \leq C_8 \int_{B_R^c} \left(|\Delta u|^2 + |\nabla u|^2 + V(x)u^2\right) \mathrm{d}x. \tag{2.63}$$

Similarly, we obtain

$$\int_{A_R} |\nabla \overline{u}_R|^2 \leq C_9 \int_{R\leq|x|\leq R+R^{\alpha/2}} \left(|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x \leq C_{10} \int_{B_R^c} \left(|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x \tag{2.64}$$

and

$$\int_{A_R} V(x)\overline{u}_R^2 \leq C_{11} \int_{R\leq|x|\leq R+R^{\alpha/2}} V(x)\overline{u}_R^2 \leq C_{11} \int_{B_R^c} V(x)u^2 \mathrm{d}x. \tag{2.65}$$

Thereby, by (2.63), (2.64) and (2.65), we deduce that

$$\int_{A_R} |\Delta \overline{u}_R|^2 + |\nabla \overline{u}_R|^2 + V(x)\overline{u}_R^2 \,\mathrm{d}x \leq C \int_{B_R^c} |\Delta u|^2 + |\nabla u|^2 + V(x)u^2 \mathrm{d}x.$$

Recalling that $\overline{u}_R \equiv 0$ when $|x| \le R - R^{\alpha/2}$ and $\overline{u}_R \equiv u$ for $|x| \ge R$, we see that

$$\begin{aligned} \|\overline{u}_R\|^2 &= \int_{B^c_{R-R^{\alpha/2}}} |\Delta \overline{u}_R|^2 + |\nabla \overline{u}_R|^2 + V(x)\overline{u}_R^2 \,\mathrm{d}x \\ &\le (1+C)\int_{B^c_R} |\Delta u|^2 + |\nabla u|^2 + V(x)u^2 \mathrm{d}x. \end{aligned} \tag{2.66}$$

Since $u \in E$, there exists $\bar{R} = \bar{R}(u,\gamma) > 1$ such that

$$\int_{B^c_{\bar{R}}} |\Delta \overline{u}_R|^2 + |\nabla \overline{u}_R|^2 + V(x)\overline{u}_R^2 \mathrm{d}x = \int_{B^c_{\bar{R}}} |\Delta u|^2 + |\nabla u|^2 + V(x)u^2 \mathrm{d}x < \frac{32\pi^2}{(1+C)\gamma}.$$

This combined with (2.66) shows that $\gamma\|\overline{u}_R\|^2 < 32\pi^2$ for all $R \ge \bar{R}$. Choosing $\bar{R} = \bar{R}(u,\gamma,a,\alpha) > 0$ sufficiently large such that $\bar{R} - \bar{R}^{\alpha/2} \ge 3\bar{n}$, by Lemma 2.9.1 for $v = \overline{u}_R/\|\overline{u}_R\|$ we have

$$\begin{aligned} \int_{B^c_R} K^{\frac{8}{8-\mu}}(x)(e^{\gamma u^2} - 1)\mathrm{d}x &= \int_{B^c_R} K^{\frac{8}{8-\mu}}(x)(e^{\gamma\|\overline{u}_R\|^2\left(\frac{\overline{u}_R}{\|\overline{u}_R\|}\right)^2} - 1)\mathrm{d}x \\ &\le \int_{B^c_{R-R^{\alpha/2}}} K^{\frac{8}{8-\mu}}(x)(e^{\gamma\|\overline{u}_R\|^2\left(\frac{\overline{u}_R}{\|\overline{u}_R\|}\right)^2} - 1)\mathrm{d}x \le \varepsilon, \forall\ R \ge \bar{R}, \end{aligned}$$

which concludes the proof of (2.62). ■

From now on, we will denote by $u_0 \in E$ a nontrivial weak solution of (2.1).

Lemma 2.9.3 *Suppose that (V) and (K) hold with $\alpha \in (0,2)$ and $\beta > (8-\mu)\alpha/8$. There exists $\bar{R} > 0$ such that for any $n \in \mathbb{N}^+$ satisfying $R_n := n^{\frac{2}{2-\alpha}} \ge \bar{R}$ we have*

$$\int_{B^c_{R_{n+1}}} (|\Delta u_0|^2 + |\nabla u_0|^2 + V(x)u_0^2)\mathrm{d}x \le \frac{3}{4}\int_{B^c_{R_n}} (|\Delta u_0|^2 + |\nabla u_0|^2 + V(x)u_0^2)\mathrm{d}x.$$

Proof . Arguing as in [[10], Lemma 17], set $\chi_n : \mathbb{R}^4 \to [0,1]$ be a smooth function such that

$$\chi_n(x) \equiv 0 \text{ in } B_{R_n},\ \ \chi_n(x) \equiv 1 \text{ in } B_{R_{n+1}},\ \ |\nabla \chi_n| \le \frac{C}{R_{n+1}} \text{ in } A_n \text{ and } |\Delta \chi_n| \le \frac{C}{R^2_{n+1}} \text{ in } A_n$$

where $A_n = \{x \in \mathbb{R}^4 : R_n \le |x| \le R_{n+1}\}$. Note that by construction $\chi_n u_0 \in E$ and

$$\int_{B^c_{R_{n+1}}} |\Delta u_0|^2 + |\nabla u_0|^2 + V(x)u_0^2 \,\mathrm{d}x \le \int_{B^c_{R_n}} \left(|\Delta u_0|^2 + |\nabla u_0|^2 + V(x)u_0^2\right)\chi_n \,\mathrm{d}x. \tag{2.67}$$

If we use $\chi_n u_0$ as test function in (2.7), since $\chi_n(x) \equiv 0$ in B_{R_n} and that $\chi_n(x) \le 1$, we obtain

$$
\begin{aligned}
\int_{B_{R_n}^c} & \chi_n \left(|\Delta u_0|^2 + |\nabla u_0|^2 + V(x)u_0^2\right) \mathrm{d}x \\
&= \int_{B_{R_n}^c} [|x|^{-\mu} * (K(x)F(x,u_0))]K(x)f(x,u_0)\chi_n u_0 \mathrm{d}x \\
&\quad - \int_{B_{R_n}^c} (\Delta u_0 \Delta\chi_n u_0 + 2\Delta u_0 \nabla\chi_n \nabla u_0 + \nabla u_0 \nabla\chi_n u_0)\, \mathrm{d}x \\
&\leq \int_{B_{R_n}^c} [|x|^{-\mu} * (K(x)F(x,u_0))]K(x)f(x,u_0)u_0 \mathrm{d}x \\
&\quad - \int_{B_{R_n}^c} (\Delta u_0 \Delta\chi_n u_0 + 2\Delta u_0 \nabla\chi_n \nabla u_0 + \nabla u_0 \nabla\chi_n u_0)\, \mathrm{d}x.
\end{aligned} \tag{2.68}
$$

Using Young's inequality with $\varepsilon \in (0,1)$, we can estimate the second part of (2.68) by

$$
\begin{aligned}
&\int_{B_{R_n}^c} (\Delta u_0 \Delta\chi_n u_0 + 2\Delta u_0 \nabla\chi_n \nabla u_0 + \nabla u_0 \nabla\chi_n u_0)\, \mathrm{d}x \\
&\leq \int_{B_{R_n}^c} \left(\varepsilon|\Delta u_0|^2 + \frac{C^2}{\varepsilon R_{n+1}^4} u_0^2 + 2\varepsilon|\Delta u_0|^2 + \frac{2C^2}{\varepsilon R_{n+1}^2}|\nabla u_0|^2 + \varepsilon|\nabla u_0|^2 + \frac{C^2}{\varepsilon R_{n+1}^2} u_0^2\right) \mathrm{d}x \\
&\leq 3\varepsilon \int_{B_{R_n}^c} |\Delta u_0|^2 \mathrm{d}x + \left(\varepsilon + \frac{2C^2}{\varepsilon R_{n+1}^2}\right) \int_{B_{R_n}^c} |\nabla u_0|^2 \mathrm{d}x + \frac{2C^2}{a}\frac{1+R_{n+1}^{\alpha}}{\varepsilon R_{n+1}^2} \int_{B_{R_n}^c} V(x)u_0^2\, \mathrm{d}x.
\end{aligned}
$$

Fixed $\varepsilon = 1/6$, we can choose R_{n+1} sufficiently large such that

$$
\varepsilon + \frac{2C^2}{\varepsilon R_{n+1}^2} \leq \frac{1}{2} \quad \text{and} \quad \frac{2C^2}{a}\frac{1+R_{n+1}^{\alpha}}{\varepsilon R_{n+1}^2} \leq \frac{1}{2}.
$$

Therefore,

$$
\begin{aligned}
\int_{B_{R_n}^c} & (\Delta u_0 \Delta\chi_n u_0 + 2\Delta u_0 \nabla\chi_n \nabla u_0 + \nabla u_0 \nabla\chi_n u_0)\, \mathrm{d}x \\
&\leq \frac{1}{2}\int_{B_{R_n}^c} (|\Delta u_0|^2 + |\nabla u_0|^2 + V(x)u_0^2)\mathrm{d}x.
\end{aligned} \tag{2.69}
$$

Now let us estimate the first integral of (2.68). By (2.32), we have

$$
|f(x,s)s| \leq C(\alpha)|s|(e^{\gamma s^2} - 1) \text{ for all } |s| \geq \tilde{R},
$$

for some $\tilde{R} > 1$. From (2.6), (f_2), the previous inequality, Hölder's inequality and by applying (2.62) for $\varepsilon \in (0, \tilde{C}^{4/(\mu-4)})$, we find

$$\begin{aligned}
&\int_{B^c_{R_n}} [|x|^{-\mu} * (K(x)F(x,u_0))]K(x)f(x,u_0)u_0 \, \mathrm{d}x \\
&\qquad \le C\left(\int_{B^c_{R_n}} K^{\frac{8}{8-\mu}}(x)|f(x,u_0)u_0|^{\frac{8}{8-\mu}} \, \mathrm{d}x\right)^{\frac{8-\mu}{4}} \\
&\qquad \le \tilde{C}\left(\int_{B^c_{R_n}} K^{\frac{8}{8-\mu}}(x)|u_0|^{\frac{8}{8-\mu}}(e^{\frac{8\gamma}{8-\mu}u_0^2}-1) \, \mathrm{d}x\right)^{\frac{8-\mu}{4}} \\
&\qquad \le \tilde{C}\int_{B^c_{R_n}} K^{\frac{8}{8-\mu}}(x)u_0^2 \, \mathrm{d}x \left(\int_{B^c_{R_n}} K^{\frac{8}{8-\mu}}(x)(e^{\frac{8\gamma}{4-\mu}u_0^2}-1)\mathrm{d}x\right)^{\frac{4-\mu}{4}} \\
&\qquad \le \int_{B^c_{R_n}} K^{\frac{8}{8-\mu}}(x)u_0^2 \, \mathrm{d}x.
\end{aligned}$$

Moreover, if $\tilde{R}$ is sufficiently large then for all $n \in \mathbb{N}$ such that $R_n \ge \tilde{R}$, we have

$$\sup_{x\in B^c_{R_n}} \frac{K^{\frac{8}{8-\mu}}(x)}{V(x)} \le \sup_{x\in B^c_{\tilde{R}}} \frac{K^{\frac{8}{8-\mu}}(x)}{V(x)} \le \sup_{x\in B^c_{\tilde{R}}} \frac{b^{\frac{8}{8-\mu}}(1+|x|^\alpha)}{a(1+|x|^\beta)^{\frac{8}{8-\mu}}} \le \frac{b^{\frac{8}{8-\mu}}(1+\tilde{R}^\alpha)}{a(1+\tilde{R}^\beta)^{\frac{8}{8-\mu}}} =: \mathcal{B}(\tilde{R}).$$

Thus,

$$\begin{aligned}
\int_{B^c_{R_n}} [|x|^{-\mu} * (K(x)F(x,u_0))]K(x)f(x,u_0)u_0\mathrm{d}x &\le \mathcal{B}(\tilde{R})\int_{B^c_{R_n}} V(x)u_0^2 \, \mathrm{d}x \\
&\le \mathcal{B}(\tilde{R})\int_{B^c_{R_n}} (|\Delta u_0|^2 + |\nabla u_0|^2 + V(x)u_0^2) \, \mathrm{d}x.
\end{aligned} \tag{2.70}$$

Since $\beta > (8-\mu)\alpha/8$, one sees that $\lim_{\tilde{R}\to+\infty} \mathcal{B}(\tilde{R}) = 0$, which implies that $\mathcal{B}(\tilde{R}) \le 1/4$ for $\tilde{R} > 0$ sufficiently large. Therefore, combining the equations (2.67)-(2.70) we finished the proof. ■

Lemma 2.9.4 *Suppose that (V) and (K) hold with $\alpha \in (0,2)$ and $\beta > (8-\mu)\alpha/8$. There exists $\tilde{R} > 0$ and constant $C > 0$ such that for any $\varrho > 2\tilde{R}$, there holds*

$$\int_{B^c_\varrho} (|\Delta u_0|^2 + |\nabla u_0| + V(x)u_0^2) \, \mathrm{d}x \le Ce^{\left(\log\frac{3}{4}\right)\varrho^{(2-\alpha)/2}}.$$

Proof . Let $\tilde{R}$ and $(R_n)_n$ be as in Lemma 2.9.3. Considering $\varrho > 2\tilde{R}$, there exist $\tilde{n} > \bar{n}$ positive integers such that

$$R_{\tilde{n}} \le \tilde{R} \le R_{\tilde{n}+1} \quad \text{and} \quad R_{\bar{n}-1} \le \varrho \le R_{\bar{n}}$$

and therefore

$$\bar{n} - \tilde{n} = R_{\bar{n}}^{(2-\alpha)/2} - R_{\tilde{n}}^{(2-\alpha)/2} \ge \varrho^{(2-\alpha)/2} - \tilde{R}^{(2-\alpha)/2} > \tilde{R}(2^{(2-\alpha)/2}-1) > 2$$

for $\tilde{R} > 0$ sufficiently large. Hence, $\bar{n} - \tilde{n} \geq 3$ and in particular $R_{\bar{n}-2} \geq R_{\tilde{n}+1} \geq \tilde{R}$. It follows from Lemma 2.9.3 that

$$\begin{aligned}
&\int_{B_\varrho^c} (|\Delta u_0|^2 + |\nabla u_0| + V(x)u_0^2) \,\mathrm{d}x \\
&\qquad \leq \int_{B^c_{R_{\bar{n}-1}}} (|\Delta u_0|^2 + |\nabla u_0| + V(x)u_0^2) \,\mathrm{d}x \\
&\qquad \leq \frac{3}{4} \int_{B^c_{R_{\bar{n}-2}}} (|\Delta u_0|^2 + |\nabla u_0| + V(x)u_0^2) \,\mathrm{d}x \\
&\qquad \leq \left(\frac{3}{4}\right)^{\bar{n}-\tilde{n}-2} \int_{B^c_{R_{\tilde{n}+1}}} (|\Delta u_0|^2 + |\nabla u_0| + V(x)u_0^2) \,\mathrm{d}x \\
&\qquad \leq \left(\frac{3}{4}\right)^{\bar{n}-\tilde{n}-2} \int_{B^c_{\tilde{R}}} (|\Delta u_0|^2 + |\nabla u_0| + V(x)u_0^2) \,\mathrm{d}x \\
&\qquad \leq \frac{16}{9} e^{\left(\log \frac{3}{4}\right)(\varrho^{(2-\alpha)/2} - \tilde{R}^{(2-\alpha)/2})} \int_{B^c_{\tilde{R}}} (|\Delta u_0|^2 + |\nabla u_0| + V(x)u_0^2) \,\mathrm{d}x \\
&\qquad = C(\tilde{R}) e^{\left(\log \frac{3}{4}\right)\varrho^{(2-\alpha)/2}} \int_{B^c_{\tilde{R}}} (|\Delta u_0|^2 + |\nabla u_0| + V(x)u_0^2) \,\mathrm{d}x
\end{aligned}$$

and proof of the lemma is done. ■

We highlight that the following proof was inspired by [64, Theorem 1.11], which in turn simplifies the proof given in [[10], Theorem 16], because we do not use Borel's finite cover lemma.

Proof of Theorem 2.1.4: Note that as $u_0 \in E$, for any $x \in B_2$, we have

$$\int_{B_2} u_0^2 \,\mathrm{d}x \leq \frac{(1+2^\alpha)}{a} \int_{B_2} V(x)u_0^2 \,\mathrm{d}x < \infty.$$

To conclude that $u_0 \in L^2(\mathbb{R}^4)$, it is enough to prove that $\int_{B_2^c} u_0^2 \,\mathrm{d}x < \infty$.

Let $\Sigma_j := \{x \in \mathbb{R}^4 : 2^j \leq |x| < 2^{j+1}\}$ for all $j \in \mathbb{N} \cup \{0\}$. Since $2^{(j+2)\alpha}V(x) \geq (1+|x|^\alpha)V(x) \geq a$ for $x \in \Sigma_j$, we get

$$\begin{aligned}
\int_{\Sigma_j} u_0^2 \,\mathrm{d}x \;&\leq\; \frac{2^{(j+2)\alpha}}{a} \int_{\Sigma_j} V(x)u_0^2 \,\mathrm{d}x \leq \frac{2^{(j+2)\alpha}}{a} \int_{\Sigma_j} (|\Delta u_0|^2 + |\nabla u_0|^2 + V(x)u_0^2) \,\mathrm{d}x \\
&\leq\; \frac{2^{(j+2)\alpha}}{a} \int_{B^c_{2^j}} (|\Delta u_0|^2 + |\nabla u_0|^2 + V(x)u_0^2) \,\mathrm{d}x.
\end{aligned}$$

It follows from Lemma 2.9.3, taking $\varrho := 2^j \geq 2\tilde{R}$, that

$$\int_{\Sigma_j} u_0^2 \,\mathrm{d}x \leq \frac{2^{(j+2)\alpha}}{a} C e^{\left(\log \frac{3}{4}\right) 2^{(2-\alpha)j/2}}. \tag{2.71}$$

Hence, there exists an integer $j_0 > 0$ such that (2.71) holds for all $j \geq j_0+1$. Therefore,

$$\begin{aligned}\int_{B_2^c} u_0^2 \, \mathrm{d}x = \sum_{j=1}^{\infty} \int_{\Sigma_j} u_0^2 \, \mathrm{d}x &= \sum_{j=1}^{j_0} \int_{\Sigma_j} u_0^2 \, \mathrm{d}x + \sum_{j=j_0+1}^{\infty} \int_{\Sigma_j} u_0^2 \, \mathrm{d}x \\ &\leq \sum_{j=1}^{j_0} \int_{\Sigma_j} u_0^2 \, \mathrm{d}x + \frac{C}{a} \sum_{j=j_0+1}^{\infty} 2^{(j+2)\alpha} e^{\left(\log \frac{3}{4}\right) 2^{(2-\alpha)j/2}} < \infty,\end{aligned}$$

because $\alpha \in (0,2)$ and $\log \frac{3}{4} < 0$. This completes the proof of Theorem 2.1.4.

Chapter 3

On a weighted Adams type inequality and an application to a polyharmonic equation

In this chapter, we have two objectives. The first deals with the improvement of a class of Adams-type inequalities involving a potential V and a weight K, which can decay to zero at infinity as $(1+|x|^{\alpha})^{-1}$, $\alpha \in (0,N)$, and $(1+|x|^{\beta})^{-1}$, $\beta \in [\alpha,+\infty)$ for all $x \in \mathbb{R}^N$, respectively. The second objective is, using minimax methods and the Adams inequality obtained in the first moment, to establish the existence of solutions for the following class of problems:

$$\sum_{j=1}^{m}(-\Delta)^j u + V(x)u = K(x)f(x,u) \quad \text{in} \quad \mathbb{R}^{2m},$$

where $(-\Delta)^j$ denote the polyharmonic operator, m is a positive integer and the nonlinear term $f(x,u)$ can have critical exponential growth.

3.1 Introduction and main results

In a more precise way, throughout this chapter, we consider some weight functions $V(x)$ and $K(x)$ satisfying the following assumptions:

(V) $V \in C(\mathbb{R}^N)$ and there exist $\alpha, a > 0$ such that $V(x) \geq \dfrac{a}{1+|x|^{\alpha}}$ for all $x \in \mathbb{R}^N$;

(K) $K \in C(\mathbb{R}^N)$ and there exist $\beta, b > 0$ such that $0 < K(x) \leq \dfrac{b}{1+|x|^\beta}$ for all $x \in \mathbb{R}^N$.

In particular, we restrict our attention to the case when α and β satisfy

$$\alpha \in (0, N) \quad \text{and} \quad \beta \in [\alpha, +\infty). \tag{3.1}$$

Next, in order to present our first result, we will fix some notations. Consider the space

$$E := \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^N) \,:\, |\nabla^i u| \in L^{\frac{N}{m}}(\mathbb{R}^N) \ \forall\, i = 1, \ldots, m \text{ and } \int_{\mathbb{R}^N} V(x)|u|^{\frac{N}{m}} \mathrm{d}x < \infty \right\}$$

and norm

$$\|u\| := \left(\int_{\mathbb{R}^N} (|\nabla^m u|^{N/m} + \cdots + |\nabla u|^{N/m} + V(x)|u|^{N/m})\, \mathrm{d}x \right)^{m/N}.$$

We use the notation $\|\cdot\|_{L^p_K(\mathbb{R}^N)}$ for the norm of the weighted Lebesgue space

$$L^p_K(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \to \mathbb{R} : u \text{ is measurable and } \int_{\mathbb{R}^N} K(x)|u|^p \mathrm{d}x < \infty \right\},$$

that is, $\|u\|_{L^p_K(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} K(x)|u|^p \mathrm{d}x \right)^{1/p}$.

Before presenting our first theorem, let us recall here what inequality (2) says, for $m \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^N$ a bounded domain with $m < N$, there exists a positive constant $C_{m,N}$ such that

$$\sup_{\{u \in W_0^{m,\frac{N}{m}}(\Omega)\,:\, \|\nabla^m u\|_{\frac{N}{m}} \leq 1\}} \int_\Omega e^{\gamma |u|^{\frac{N}{N-m}}} \mathrm{d}x \leq C_{m,N}|\Omega|, \tag{3.2}$$

for any $\gamma \leq \gamma_{m,N}$.

In this context, we can establish our first result, as follows.

Theorem 3.1.1 *Suppose that (V) and (K) hold with α and β satisfying (3.1). Then, for any $\gamma > 0$ and any $u \in E$, it holds*

$$\int_{\mathbb{R}^N} K(x)\phi_{m,N}(\gamma |u|^{\frac{N}{N-m}})\, \mathrm{d}x < \infty. \tag{3.3}$$

Moreover, we have

$$\sup_{\substack{u \in E \\ \|u\| \leq 1}} \int_{\mathbb{R}^N} K(x)\phi_{m,N}(\gamma |u|^{\frac{N}{N-m}})\, \mathrm{d}x = \begin{cases} < \infty, & \text{if } \ \gamma < \gamma_{m,N}; \\ +\infty, & \text{if } \ \gamma > \gamma_{m,N}. \end{cases} \tag{3.4}$$

Remark 3.1.2 *We highlight that inequality* (3.4) *in Theorem* 3.1.1 *treats only the subcritical case. The critical case* $\gamma = \gamma_{m,N}$ *is still an open question.*

Remark 3.1.3 *Note that when* $m = 2$ *and* $N = 4$, $\gamma_{2,4} = 32\pi^2$ *and Theorem* 3.1.1 *coincides with Theorem* 1.1.1.

As initial applications of Theorem 3.1.1, we will prove the compact embedding of the space E into $L^p_K(\mathbb{R}^N)$ for $p \geq 2, \alpha \in (0, N)$ and $\beta \in (\alpha, +\infty)$ (see Proposition 3.3.1). From now on, we assume that the integer $m \geq 2$ and the dimension N of the domain satisfy $N = 2m$. We also will obtain a Lions-type concentration-compactness principle involving exponential growth (see Proposition 3.5.3), which is a refinement of Theorem 3.1.1. Furthermore, we will investigate the existence of weak solution for the following class of problems

$$\sum_{j=1}^{m}(-\Delta)^j u + V(x)u = K(x)f(x,u) \quad \text{in} \quad \mathbb{R}^{2m}, \tag{3.5}$$

where the potential V and the weight K satisfy the conditions (V) and (K), respectively and α, β are such that

$$\alpha \in (0, 2m) \quad \text{and} \quad \beta \in (\alpha, +\infty). \tag{3.6}$$

The nonlinearity $f(x, s)$ has the maximal growth which allows us to study (3.5) by using a variational method. Precisely, motivated by (3.4), we say that $f(x, s)$ has *critical exponential growth* if there exists $\gamma_0 > 0$ such that

$$\lim_{|s|\to\infty} \frac{f(x,s)}{e^{\gamma s^2}} = \begin{cases} 0, & \text{for all } \gamma > \gamma_0, \\ +\infty, & \text{for all } \gamma < \gamma_0, \end{cases}$$

uniformly in $x \in \mathbb{R}^{2m}$. In this context, we say that $u \in E$ is a weak solution for (3.5) if

$$\int_{\mathbb{R}^{2m}} \left(\sum_{j=1}^{m} \nabla^j u \nabla^j v + V(x)uv \right) \mathrm{d}x = \int_{\mathbb{R}^{2m}} K(x)f(x,u)v \,\mathrm{d}x, \quad \text{for all } v \in E. \tag{3.7}$$

We will assume sufficient conditions on f so that weak solutions of (3.5) become critical points of the functional $I : E \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^{2m}} K(x)F(x,u)\,\mathrm{d}x,$$

where $F(x,s) := \int_0^s f(x,t)\mathrm{d}t$. In this case as $N = 2m$, we have the norm in space E can be characterized as $\|u\|^2 = \langle u, u\rangle_E$ where

$$\langle u, v\rangle_E = \int_{\mathbb{R}^{2m}} \left(\sum_{j=1}^{m} \nabla^j u \nabla^j v + V(x)uv \right) \mathrm{d}x.$$

We require the following assumptions on the nonlinearity $f(x,s)$:

(f_1) $\lim_{s\to 0} \frac{f(x,s)}{s} = 0$, uniformly in $x \in \mathbb{R}^{2m}$;

(f_2) the function $f : \mathbb{R}^{2m} \times \mathbb{R} \to \mathbb{R}$ is continuous and has critical exponential growth;

(f_3) there exists $\mu > 2$ such that

$$0 < \mu F(x,s) \le sf(x,s), \quad \text{for all} \quad (x,s) \in \mathbb{R}^{2m} \times \mathbb{R} \setminus \{0\};$$

(f_4) there exist constants $s_0, M_0 > 0$ such that

$$0 < F(x,s) \le M_0|f(x,s)|, \quad \text{for all} \quad |s| \ge s_0 \quad \text{and} \quad x \in \mathbb{R}^{2m};$$

(f_5) $\liminf_{s\to\infty} \frac{sf(x,s)}{e^{\gamma_0 s^2}} \ge \theta_0$, for some $\theta_0 > \frac{2m\gamma_{m,2m}}{\omega_{2m}\widetilde{K}\gamma_0}$ uniformly with respect to $x \in \mathbb{R}^{2m}$, where $\widetilde{K} = \min_{x\in\overline{B}_1} K(x)$.

Our existence result is stated as follows:

Theorem 3.1.4 *Assume $(V), (K)$, $(f_1) - (f_5)$ and (3.6) hold. Then, problem (3.5) has a nontrivial weak solution.*

In [76], Zhao and Chang establish a singular Adams-type inequality on the whole $\mathbb{R}^{2m}$ for $m \ge 2$. In addition, they deduce that for further conditions on f, the problem

$$(-\Delta)^m u + \sum_{\gamma=0}^{m-1} (-1)^\gamma \nabla^\gamma \cdot (a_\gamma(x)\nabla^\gamma u) = \frac{f(x,u)}{|x|^\beta} + \varepsilon h(x) \quad \text{in} \quad \mathbb{R}^{2m}$$

has at least two distinct weak solutions where $m \ge 2$ is an even integer and $a_\gamma(x)$ are continuous functions satisfying: there exist positive constants $a_\gamma, \gamma = 0, 1, 2 \ldots, m-1$, such that $a_\gamma(x) \ge a_\gamma$ for all $x \in \mathbb{R}^{2m}$ and $\frac{1}{a_0(x)} \in L^1(\mathbb{R}^{2m})$.

Do Ó and Macedo in [31] studied a Adams type inequality for the Sobolev space $W^{m,2}(\mathbb{R}^{2m})$ and establish od the existence of a nontrivial radial solution to the following class of polyharmonic equations:

$$(-\Delta)^m u(x) + u(x) = f(|x|, u), \quad \text{in} \quad \mathbb{R}^{2m},$$

where the nonlinearity is superlinear and has critical exponential growth at infinity.

The outline of the chapter is as follows: in Section 3.2 we present some preliminary results. In Section 3.3 we prove the weighted Adams' inequality (Theorem 3.1.1) and that the embedding $E \hookrightarrow L^p_K(\mathbb{R}^N)$ is compact for all $p \in [N/m, +\infty)$ (Proposition 3.3.1). Section 3.4 contains the variational framework related to problem (3.5) and we also check the geometric properties of the functional I. Section 3.5 we prove a version of the Concentration-Compactness Principle due to P. -L. Lions [45] to the space E (see Proposition 3.5.3) and deals with the Palais-Smale compactness condition. In Section 3.6 we estimate the minimax level. Finally, in Section 3.7 we complete the proof of Theorem 3.1.4.

3.2 Some preliminary results

The following lemmas are adaptations of Lemma 2.1 and Lemma 2.2 respectively obtained from Yang in [70]. Adaptation is simply replacing $N \geq 2$ with $j_{\frac{N}{m}} \geq 2$ in the results.

Lemma 3.2.1 *Let $s \geq 0$ and $p \geq 1$ be real numbers. Then, there holds*

$$[\phi_{m,N}(s)]^p \leq \phi_{m,N}(ps)$$

Lemma 3.2.2 *For all $j_{\frac{N}{m}} \geq 2, s \geq 0, t \geq 0, \mu > 1$ and $\nu > 1$ with $\frac{1}{\mu} + \frac{1}{\nu} = 1$ there holds*

$$\phi_{m,N}(s+t) \leq \frac{1}{\mu}\phi_{m,N}(\mu s) + \frac{1}{\nu}\phi_{m,N}(\nu s).$$

3.3 Proof of Theorem 3.1.1 and Compactness Result

In the first subsection, we prove Theorem 3.1.1.

3.3.1 Proof of Theorem 3.1.1

Proof . We begin proving the first part of (3.4). The proof will be divided into two steps.

Step 1: Let $u \in E$ be such that $\|u\| \leq 1$. First, we want to estimate the weighted Adams functional

$$AD(u, \gamma, R) = \int_{B_R} K(x)\phi_{m,N}(\gamma |u|^{\frac{N}{N-m}})\,\mathrm{d}x$$

for some $R > 0$, independently of u, that will be chosen during the proof. From condition (K), we have

$$\int_{B_R} K(x)\phi_{m,N}(\gamma|u|^{\frac{N}{N-m}})\ \mathrm{d}x \le b\int_{B_R} \phi_{m,N}(\gamma|u|^{\frac{N}{N-m}})\ \mathrm{d}x. \tag{3.8}$$

Consider a cutoff function $\varphi \in C_0^\infty(B_{2R})$ such that

$$0 \le \varphi \le 1 \ \text{ in } B_{2R}, \quad \varphi \equiv 1 \ \text{ in } \ B_R, \quad \text{and} \ \ |\nabla^i \varphi| \le \frac{C}{R^i} \ \text{ in } \ B_{2R},$$

for all $i = 1, \ldots, m$ and for some constant $C > 0$. From Leibniz's formula we have

$$|\nabla^m(\varphi u)|^{\frac{N}{m}} = \left|\sum_{j=0}^{m} \binom{m}{j} \nabla^j\varphi\nabla^{m-j}u\right|^{\frac{N}{m}}.$$

Using the elementary inequality

$$(a+b)^q \le (1+\varepsilon)^q a^q + (1+\varepsilon^{-1})^q b^q,$$

with $\varepsilon > 0, a, b \ge 0$ and $q \ge 1$, we obtain that

$$\begin{aligned}
\int_{B_{2R}} |\nabla^m(\varphi u)|^{\frac{N}{m}}\ \mathrm{d}x &= \int_{B_{2R}} \left|\varphi\nabla^m u + \sum_{j=1}^{m}\binom{m}{j}\nabla^j\varphi\nabla^{m-j}u\right|^{\frac{N}{m}}\ \mathrm{d}x \\
&\le (1+\varepsilon)^{\frac{N}{m}}\int_{B_{2R}} \varphi^{\frac{N}{m}}|\nabla^m u|^{\frac{N}{m}}\mathrm{d}x + (1+\varepsilon^{-1})^{\frac{N}{m}}\int_{B_{2R}}\left|\sum_{j=1}^{m}\binom{m}{j}\nabla^j\varphi\nabla^{m-j}u\right|^{\frac{N}{m}}\ \mathrm{d}x \\
&\le (1+\varepsilon)^{\frac{N}{m}}\int_{B_{2R}} |\nabla^m u|^{\frac{N}{m}}\mathrm{d}x \\
&\quad + (1+\varepsilon^{-1})^{\frac{N}{m}}\int_{B_{2R}}\left|\sum_{j=1}^{m-1}\binom{m}{j}\nabla^j\varphi\nabla^{m-j}u + m(\nabla^m\varphi)u\right|^{\frac{N}{m}}\ \mathrm{d}x \\
&\le (1+\varepsilon)^{\frac{N}{m}}\int_{B_{2R}} |\nabla^m u|^{\frac{N}{m}}\mathrm{d}x + (1+\varepsilon^{-1})^{\frac{2N}{m}}\int_{B_{2R}}\left|\sum_{j=1}^{m-1}\binom{m}{j}\nabla^j\varphi\nabla^{m-j}u\right|^{\frac{N}{m}}\ \mathrm{d}x \\
&\quad + (1+\varepsilon)^{\frac{N}{m}}(1+\varepsilon^{-1})^{\frac{N}{m}} m^{\frac{N}{m}}\int_{B_{2R}} |\nabla^m\varphi|^{\frac{N}{m}}|u|^{\frac{N}{m}}\mathrm{d}x.
\end{aligned}$$

Repeating the same process m times for the second term of the previous inequality, we estimate

$$\begin{aligned}
\int_{B_{2R}} |\nabla^m(\varphi u)|^{\frac{N}{m}}\ \mathrm{d}x \le (1+\varepsilon)^{\frac{N}{m}}\int_{B_{2R}}|\nabla^m u|^{\frac{N}{m}}\mathrm{d}x + (1+\varepsilon^{-1})^N m^{\frac{N}{m}}\frac{C^{\frac{N}{m}}}{R^{\frac{N}{m}}}\int_{B_{2R}}|\nabla^{m-1}u|^{\frac{N}{m}}\mathrm{d}x \\
+ \cdots + (1+\varepsilon)^{\frac{N}{m}}(1+\varepsilon^{-1})^{\frac{2N}{m}} m^{\frac{N}{m}}\frac{C^{\frac{N}{m}}}{R^{\frac{(m-1)N}{m}}}\int_{B_{2R}}|\nabla u|^{\frac{N}{m}}\mathrm{d}x \\
+ (1+\varepsilon)^{\frac{N}{m}}(1+\varepsilon^{-1})^{\frac{N}{m}} m^{\frac{N}{m}}\frac{C^{\frac{N}{m}}}{R^N}\int_{B_{2R}}|u|^{\frac{N}{m}}\mathrm{d}x.
\end{aligned}$$

Now, using the condition (V), it follows that

$$\int_{B_{2R}} |\nabla^m(\varphi u)|^{\frac{N}{m}}\,\mathrm{d}x \le (1+\varepsilon)^{\frac{N}{m}} \int_{B_{2R}} |\nabla^m u|^{\frac{N}{m}}\mathrm{d}x + (1+\varepsilon^{-1})^N m^{\frac{N}{m}} \frac{C^{\frac{N}{m}}}{R^{\frac{N}{m}}} \int_{B_{2R}} |\nabla^{m-1}u|^{\frac{N}{m}}\mathrm{d}x$$
$$+ \cdots + (1+\varepsilon)^{\frac{N}{m}}(1+\varepsilon^{-1})^{\frac{2N}{m}} m^{\frac{N}{m}} \frac{C^{\frac{N}{m}}}{R^{\frac{(m-1)N}{m}}} \int_{B_{2R}} |\nabla u|^{\frac{N}{m}}\mathrm{d}x$$
$$+ (1+\varepsilon)^{\frac{N}{m}}(1+\varepsilon^{-1})^{\frac{N}{m}} \frac{m^{\frac{N}{m}}C^{\frac{N}{m}}}{a}\frac{1+(2R)^\alpha}{R^N} \int_{B_{2R}} V(x)|u|^{\frac{N}{m}}\,\mathrm{d}x.$$

Fixed $\varepsilon \in (0,1)$ such that $\gamma(1+\varepsilon)^{\frac{N}{N-m}} \le \gamma_{m,N}$ and since $\alpha \in (0,N)$, we can choose $\bar{R} = \bar{R}(\varepsilon, a, \alpha) > 0$ sufficiently large satisfying

$$(1+\varepsilon^{-1})^N m^{\frac{N}{m}} \frac{C^{\frac{N}{m}}}{R^{\frac{N}{m}}} \le (1+\varepsilon)^{\frac{N}{m}},\ \ldots\ , (1+\varepsilon)^{\frac{N}{m}}(1+\varepsilon^{-1})^{\frac{2N}{m}} m^{\frac{N}{m}} \frac{C^{\frac{N}{m}}}{R^{\frac{(m-1)N}{m}}} \le (1+\varepsilon)^{\frac{N}{m}}$$

and

$$(1+\varepsilon)^{\frac{N}{m}}(1+\varepsilon^{-1})^{\frac{N}{m}} \frac{m^{\frac{N}{m}}C^{\frac{N}{m}}}{a}\frac{1+(2R)^\alpha}{R^N} \le (1+\varepsilon)^{\frac{N}{m}},$$

for all $R \ge \bar{R}$. Thus,

$$\int_{B_{2R}} |\nabla^m(\varphi u)|^{\frac{N}{m}}\,\mathrm{d}x \le (1+\varepsilon)^{\frac{N}{m}} \|u\|^{\frac{N}{m}} \le (1+\varepsilon)^{\frac{N}{m}}.$$

Therefore, defining

$$v := \frac{\varphi u}{1+\varepsilon}$$

we have that $v \in E$ and

$$\|\nabla^m v\|_{\frac{N}{m}}^{\frac{N}{m}} = \frac{1}{(1+\varepsilon)^{\frac{N}{m}}} \int_{B_{2R}} |\nabla^m(\varphi u)|^{\frac{N}{m}}\,\mathrm{d}x \le 1.$$

Applying Adams inequality (3.2), we get

$$\int_{B_R} \phi_{m,N}(\gamma|u|^{\frac{N}{N-m}})\,\mathrm{d}x = \int_{B_R} \phi_{m,N}(\gamma|\varphi u|^{\frac{N}{N-m}})\,\mathrm{d}x \le \int_{B_{2R}} e^{\gamma(1+\varepsilon)^{\frac{N}{N-m}}|v|^{\frac{N}{N-m}}}\mathrm{d}x \le C\,|B_{2R}|$$

where $|B_{2R}|$ denotes the Lebesgue measure of the ball of radius $2R$ in $\mathbb{R}^N$. The previous inequality combined with (3.8) implies that

$$\int_{B_R} K(x)\phi_{m,N}(\gamma|u|^{\frac{N}{N-m}})\mathrm{d}x \le b\int_{B_R} \phi_{m,N}(\gamma|u|^{\frac{N}{N-m}})\,\mathrm{d}x \le C|B_{2R}|, \tag{3.9}$$

for all $u \in E$ with $\|u\| \le 1$.

Step 2: Now, we estimate the weighted Adams functional in the exterior of a large ball.

For any $n \geq n_0$ fixed, where n_0 will be chosen during the proof, we consider B_n^c the exterior de B_n and the covering of B_n^c formed by all annuli A_n^σ with $\sigma > n$ given by

$$A_n^\sigma := \{x \in B_n^c : |x| < \sigma\} = \{x \in \mathbb{R}^N : n < |x| < \sigma\}.$$

By the Besicovitch covering Lemma [25], for any $\sigma > n_0$, there exist a sequence of points $(x_k) \in A_{\tilde{n}}^\sigma$ and a universal constant $\theta > 0$ such that

$$A_{\tilde{n}}^\sigma \subseteq \bigcup_k U_k^{1/2}, \quad \text{where} \quad U_k^{1/2} := B\left(x_k, \frac{1}{2}\frac{|x_k|}{3}\right)$$

and

$$\sum_k \chi_{U_k}(x) \leq \theta \quad \text{for any} \quad x \in \mathbb{R}^N, \quad \text{where} \quad U_k := B\left(x_k, \frac{|x_k|}{3}\right)$$

and χ_{U_k} is its characteristic function. Let $u \in E$ be such that $\|u\| \leq 1$. We start with the estimate of the weighted exponential integral of u in A_{3n}^σ with $n \geq n_0$ and $\sigma > 3n$. Note that

$$A_{3n}^\sigma \subset A_{\tilde{n}}^\sigma \subseteq \bigcup_k U_k^{1/2}$$

and defining the set of indices $K_{n,\sigma} := \{k \in \mathbb{N} : U_k^{1/2} \cap B_{3n}^c \neq \emptyset \}$, we have

$$A_{3n}^\sigma \subseteq \bigcup_{k \in K_{n,\sigma}} U_k^{1/2}.$$

Therefore,

$$\int_{A_{3n}^\sigma} K(x)\phi_{m,N}(\gamma|u|^{\frac{N}{N-m}})\,\mathrm{d}x \leq \sum_{k \in K_{n,\sigma}} \int_{{U_k}^{1/2}} K(x)\phi_{m,N}(\gamma|u|^{\frac{N}{N-m}})\,\mathrm{d}x. \tag{3.10}$$

Since $\frac{2}{3}|x_k| \leq |y| \leq \frac{4}{3}|x_k|$ for all $y \in U_k$, from (V) and (K), we obtain

$$V(y) \geq \frac{a}{1+|y|^\alpha} \geq \frac{a}{1+\left(\frac{4}{3}\right)^\alpha |x_k|^\alpha}, \quad \text{for all} \quad y \in U_k \tag{3.11}$$

and

$$K(y) \leq \frac{b}{1+|y|^\beta} \leq \frac{b}{1+\left(\frac{2}{3}\right)^\beta |x_k|^\beta}, \quad \text{for all} \quad y \in U_k. \tag{3.12}$$

Besides, if $U_k \cap B_{3n}^c \neq \emptyset$ then $U_k \subset B_n^c$, which implies that

$$\bigcup_{k \in K_{n,\sigma}} U_k^{1/2} \subseteq \bigcup_{k \in K_{n,\sigma}} U_k \subseteq B_n^c \subseteq B_{\tilde{n}}^c. \tag{3.13}$$

Next, let us fix $k \in K_{n,\sigma}$. From (3.12), we obtain

$$\int_{{U_k}^{1/2}} K(x)\phi_{m,N}(\gamma|u|^{\frac{N}{N-m}})\,\mathrm{d}x \leq \frac{b}{1+\left(\frac{2}{3}\right)^\beta |x_k|^\beta} \int_{{U_k}^{1/2}} \phi_{m,N}(\gamma|u|^{\frac{N}{N-m}})\,\mathrm{d}x. \tag{3.14}$$

Again, consider a cutoff function $\varphi_k \in C_0^\infty(U_k)$ such that

$$0 \leq \varphi_k \leq 1 \text{ in } U_k, \quad \varphi_k \equiv 1 \text{ in } U_k^{1/2}, \quad |\nabla^i \varphi_k| \leq \frac{C}{|x_k|^i} \text{ in } U_k,$$

for all $i = 1, \ldots, m$ and for some constant $C > 0$. Proceeding as before, it follows that

$$\begin{aligned}\int_{U_k} |\nabla^m(\varphi_k u)|^{\frac{N}{m}} \,\mathrm{d}x \leq (1+\varepsilon)^{\frac{N}{m}} \int_{U_k} |\nabla^m u|^{\frac{N}{m}} \,\mathrm{d}x + (1+\varepsilon^{-1})^N m^{\frac{N}{m}} \frac{C^{\frac{N}{m}}}{|x_k|^{\frac{N}{m}}} \int_{U_k} |\nabla^{m-1} u|^{\frac{N}{m}} \mathrm{d}x \\ + \cdots + (1+\varepsilon)^{\frac{N}{m}} (1+\varepsilon^{-1})^{\frac{2N}{m}} m^{\frac{N}{m}} \frac{C^{\frac{N}{m}}}{|x_k|^{\frac{(m-1)N}{m}}} \int_{U_k} |\nabla u|^{\frac{N}{m}} \mathrm{d}x \\ + (1+\varepsilon)^{\frac{N}{m}} (1+\varepsilon^{-1})^{\frac{N}{m}} \frac{m^{\frac{N}{m}} C^{\frac{N}{m}}}{|x_k|^N} \int_{U_k} |u|^{\frac{N}{m}} \,\mathrm{d}x\end{aligned}$$

and by (3.11)

$$\begin{aligned}\int_{U_k} |\nabla^m(\varphi_k u)|^{\frac{N}{m}} \,\mathrm{d}x \leq (1+\varepsilon)^{\frac{N}{m}} \int_{U_k} |\nabla^m u|^{\frac{N}{m}} \,\mathrm{d}x + (1+\varepsilon^{-1})^N m^{\frac{N}{m}} \frac{C^{\frac{N}{m}}}{|x_k|^{\frac{N}{m}}} \int_{U_k} |\nabla^{m-1} u|^{\frac{N}{m}} \mathrm{d}x \\ + \cdots + (1+\varepsilon)^{\frac{N}{m}} (1+\varepsilon^{-1})^{\frac{2N}{m}} m^{\frac{N}{m}} \frac{C^{\frac{N}{m}}}{|x_k|^{\frac{(m-1)N}{m}}} \int_{U_k} |\nabla u|^{\frac{N}{m}} \mathrm{d}x \\ + (1+\varepsilon)^{\frac{N}{m}} (1+\varepsilon^{-1})^{\frac{N}{m}} m^{\frac{N}{m}} \frac{C^{\frac{N}{m}}}{a} \frac{1 + \left(\frac{4}{3}\right)^\alpha |x_k|^\alpha}{|x_k|^N} \int_{U_k} V(x) |u|^{\frac{N}{m}} \,\mathrm{d}x.\end{aligned}$$

Since $k \in K_{n,\sigma}$, in view of (3.13), we obtain that $x_k \in B_{n_0}^c$. Once $\alpha \in (0, N)$, we can choose $n_0 = n_0(\varepsilon, a, \alpha) > 0$ sufficiently large such that

$$(1+\varepsilon^{-1})^N m^{\frac{N}{m}} \frac{C^{\frac{N}{m}}}{|x_k|^{\frac{N}{m}}} \leq (1+\varepsilon)^{\frac{N}{m}}, \quad \ldots \quad , (1+\varepsilon)^{\frac{N}{m}} (1+\varepsilon^{-1})^{\frac{2N}{m}} m^{\frac{N}{m}} \frac{C^{\frac{N}{m}}}{|x_k|^{\frac{(m-1)N}{m}}} \leq (1+\varepsilon)^{\frac{N}{m}}$$

and

$$(1+\varepsilon)^{\frac{N}{m}} (1+\varepsilon^{-1})^{\frac{N}{m}} m^{\frac{N}{m}} \frac{C^{\frac{N}{m}}}{a} \frac{1 + \left(\frac{4}{3}\right)^\alpha |x_k|^\alpha}{|x_k|^N} \leq (1+\varepsilon)^{\frac{N}{m}},$$

for all $k \in K_{n,\sigma}$ and $n \geq n_0$. Thus,

$$\int_{U_k} |\nabla^m(\varphi_k u)|^{\frac{N}{m}} \,\mathrm{d}x \leq (1+\varepsilon)^{\frac{N}{m}} \|u\|^{\frac{N}{m}} \leq (1+\varepsilon)^{\frac{N}{m}}.$$

Defining $v_k = \frac{\varphi_k u}{1+\varepsilon}$ we have $v_k \in W_0^{m,\frac{N}{m}}(U_k) \subset W^{m,\frac{N}{m}}(\mathbb{R}^N)$ and one has

$$\|\nabla^m v_k\|_{\frac{N}{m}}^{\frac{N}{m}} = \frac{1}{(1+\varepsilon)^{\frac{N}{m}}} \int_{U_k} |\nabla^m(\varphi_k u)|^{\frac{N}{m}} \,\mathrm{d}x \leq 1.$$

Now, fixed $\varepsilon \in (0,1)$ such that $\gamma(1+\varepsilon)^{\frac{N}{N-m}} < \gamma_{m,N}$ we can apply (4) and we find

$$\begin{aligned}\int_{U_k^{1/2}} \phi_{m,N}(\gamma |u|^{\frac{N}{N-m}}) \,\mathrm{d}x &= \int_{U_k^{1/2}} \phi_{m,N}(\gamma |\varphi_k u|^{\frac{N}{N-m}}) \,\mathrm{d}x \\ &\leq \int_{\mathbb{R}^N} \phi_{m,N}(\gamma (1+\varepsilon)^{\frac{N}{N-m}} |v_k|^{\frac{N}{N-m}}) \,\mathrm{d}x \leq C \int_{\mathbb{R}^N} |v_k|^{\frac{N}{m}} \,\mathrm{d}x.\end{aligned}$$

The previous inequality combined with the definition of v_k and (3.11) leads us to

$$\begin{aligned}\int_{U_k^{1/2}} \phi_{m,N}(\gamma|u|^{\frac{N}{N-m}})\,\mathrm{d}x &\leq \frac{C}{(1+\varepsilon)^{\frac{N}{m}}}\int_{U_k}|u|^{\frac{N}{m}}\,\mathrm{d}x\\ &\leq \frac{C}{(1+\varepsilon)^{\frac{N}{m}}}\frac{1+\left(\frac{4}{3}\right)^{\alpha}|x_k|^{\alpha}}{a}\int_{U_k}V(x)|u|^{\frac{N}{m}}\,\mathrm{d}x. \quad (3.15)\end{aligned}$$

Combining the estimates (3.10), (3.14), (3.15) and by using (3.13), we get

$$\begin{aligned}\int_{A_{3n}^{\sigma}} K(x)\phi_{m,N}(\gamma|u|^{\frac{N}{N-m}})\,\mathrm{d}x &\leq \frac{C}{(1+\varepsilon)^{\frac{N}{m}}}\frac{b}{a}\sum_{k\in K_{n,\sigma}}\frac{1+\left(\frac{4}{3}\right)^{\alpha}|x_k|^{\alpha}}{1+\left(\frac{2}{3}\right)^{\beta}|x_k|^{\beta}}\int_{U_k}V(x)|u|^{\frac{N}{m}}\,\mathrm{d}x\\ &\leq \frac{C}{(1+\varepsilon)^{\frac{N}{m}}}\frac{b}{a}\sum_{k\in K_{n,\sigma}}\frac{1+\left(\frac{4}{3}\right)^{\alpha}|x_k|^{\alpha}}{1+\left(\frac{2}{3}\right)^{\beta}|x_k|^{\beta}}\int_{B_n^c}V(x)|u|^{\frac{N}{m}}\chi_{U_k}\,\mathrm{d}x.\end{aligned}$$

By (3.13) again, we obtain

$$\frac{1+\left(\frac{4}{3}\right)^{\alpha}|x_k|^{\alpha}}{1+\left(\frac{2}{3}\right)^{\beta}|x_k|^{\beta}} \leq \mathcal{B}_n := \sup_{x\in B_n^c}\frac{1+\left(\frac{4}{3}\right)^{\alpha}|x|^{\alpha}}{1+\left(\frac{2}{3}\right)^{\beta}|x|^{\beta}}, \quad \text{for all} \quad k\in K_{n,\sigma}.$$

Therefore,

$$\int_{A_{3n}^{\sigma}} K(x)\phi_{m,N}(\gamma|u|^{\frac{N}{N-m}})\mathrm{d}x \leq \frac{C}{(1+\varepsilon)^{\frac{N}{m}}}\frac{b}{a}\mathcal{B}_n\sum_{k\in K_{n,\sigma}}\int_{B_n^c}V(x)|u|^{\frac{N}{m}}\chi_{U_k}\,\mathrm{d}x.$$

By invoking the Besicovitch Covering Lemma, we reach

$$\int_{A_{3n}^{\sigma}} K(x)\phi_{m,N}(\gamma|u|^{\frac{N}{N-m}})\,\mathrm{d}x \leq \frac{C}{(1+\varepsilon)^{\frac{N}{m}}}\frac{b}{a}\mathcal{B}_n\theta\int_{B_n^c}V(x)|u|^{\frac{N}{m}}\,\mathrm{d}x.$$

Letting $\sigma\to\infty$, we deduce the existence of $n_0 = n_0(\varepsilon, a, \alpha) > 1$ sufficiently large so that

$$\begin{aligned}\int_{B_{3n}^c} K(x)\phi_{m,n}(\gamma|u|^{\frac{N}{N-m}})\mathrm{d}x &\leq C\frac{b}{a}\mathcal{B}_n\theta\int_{B_n^c}V(x)|u|^{\frac{N}{m}}\,\mathrm{d}x\\ &\leq C\frac{b}{a}\mathcal{B}_n\theta\|u\|^{\frac{N}{m}} \leq C\frac{b}{a}\mathcal{B}_n\theta, \quad (3.16)\end{aligned}$$

for any $n\geq n_0$. Note that

$$\lim_{n\to\infty}\mathcal{B}_n = \lim_{n\to\infty}\frac{1+\left(\frac{4}{3}\right)^{\alpha}n^{\alpha}}{1+\left(\frac{2}{3}\right)^{\beta}n^{\beta}} = \begin{cases}0, & \text{if } \beta>\alpha\\ 2^{\alpha}, & \text{if } \beta=\alpha.\end{cases} \quad (3.17)$$

Thus, from (3.9) and (3.16), we conclude the proof of the first part of (3.4).

For the second part of (3.4), inspired by [53], we consider the Moser sequence $\widetilde{M}_n$ defined by

$$\widetilde{M}_n(x) = \begin{cases} \left(\frac{\log n}{\gamma_{m,N}}\right)^{1-\frac{m}{N}} + \frac{N\gamma_{m,N}^{\frac{m}{N}-1}}{2(\log n)^{\frac{m}{N}}} \sum_{j=1}^{m-1} \frac{(1-n^{\frac{2}{N}}|x|^2)^j}{j}, & \text{for } |x| \leq \frac{1}{\sqrt[N]{n}}, \\ -N\gamma_{m,N}^{\frac{m}{N}-1}(\log n)^{-\frac{m}{N}} \log |x|, & \text{for } \frac{1}{\sqrt[N]{n}} < |x| \leq 1, \\ \zeta_n(x), & \text{for } |x| \geq 1. \end{cases}$$

Here ζ_n is a compactly supported smooth function in $\overline{B}_2(0)$ satisfying for $j = 1, 2, \ldots, m-1$ and such that

$$\zeta_n|_{\partial B_1(0)} = \zeta_n|_{\partial B_2(0)} = 0, \quad \frac{\partial^j \zeta_n}{\partial \nu^j}|_{\partial B_1(0)} = (-1)^j (j-1)! \gamma_{m,N}^{\frac{m}{N}-1} (\log n)^{-\frac{m}{N}}, \quad \frac{\partial^j \zeta_n}{\partial \nu^j}|_{\partial B_2(0)} = 0$$

and $\zeta_n, |\nabla^j \zeta_n|$ and $|\nabla^m \zeta_n|$ are all $O((\log n)^{-\frac{m}{N}})$.

For any $n \in \mathbb{N}$, we have that $\tilde{M}_n \in E$ and straightforward calculations show that for $j = 1, 2, \ldots, m-1$,

$$\|\widetilde{M}_n\|_{\frac{N}{m}}^{\frac{N}{m}} = O(1/\log n), \quad \|\nabla^j \widetilde{M}_n\|_{\frac{N}{m}}^{\frac{N}{m}} = O(1/\log n) \quad \text{and} \quad \|\nabla^m \widetilde{M}_n\|_{\frac{N}{m}}^{\frac{N}{m}} = 1+O(1/\log n).$$

Consequently, by condition (V) it follows that $\|\widetilde{M}_n\|^{\frac{N}{m}} = 1 + \delta_n$, where $\delta_n \to 0$ and $\delta_n = O(1/\log n)$, as $n \to \infty$. Defining

$$M_n := \frac{\widetilde{M}_n}{\|\widetilde{M}_n\|}, \tag{3.18}$$

we obtain that $M_n \in E$ and $\|M_n\| = 1$. Notice that

$$M_n(x) \geq \frac{1}{\|\widetilde{M}_n\|} \left(\frac{\log n}{\gamma_{m,N}}\right)^{1-\frac{m}{N}} \quad \text{for} \quad x \in B_{\frac{1}{\sqrt[N]{n}}} \tag{3.19}$$

and defining $\widetilde{K} := \min_{x \in \overline{B}_1} K(x)$, for all $\gamma > \gamma_{m,N}$, we have

$$\begin{aligned} \int_{\mathbb{R}^N} K(x)\phi_{m,N}(\gamma|M_n|^{\frac{N}{N-m}})\mathrm{d}x &\geq \tilde{K} \int_{B_{\frac{1}{\sqrt[N]{n}}}} \phi_{m,N}(\gamma|M_n|^{\frac{N}{N-m}})\ \mathrm{d}x \\ &\geq \widetilde{K} \int_{B_{\frac{1}{\sqrt[N]{n}}}} \phi_{m,N}\left(\frac{\gamma}{\|\widetilde{M}_n\|^{\frac{N}{N-m}}} \frac{\log n}{\gamma_{m,N}}\right) \mathrm{d}x \\ &= \frac{\omega_N}{N}\left(\frac{1}{\sqrt[N]{n}}\right)^N \widetilde{K}\phi_{m,N}\left(\frac{\gamma}{\|\widetilde{M}_n\|^{\frac{N}{N-m}}} \frac{\log n}{\gamma_{m,N}}\right) \\ &\geq \frac{\omega_N}{N}\widetilde{K} e^{\log n\left(\frac{\gamma}{\|\widetilde{M}_n\|^{\frac{N}{N-m}}\gamma_{m,N}}-1\right)} \to \infty \quad \text{as} \quad n \to \infty. \end{aligned}$$

Thereby,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)\phi_{m,N}(\gamma|M_n|^{\frac{N}{N-m}})\mathrm{d}x = \infty. \tag{3.20}$$

Taking into account that

$$\sup_{\substack{u\in E\\ \|u\|\le 1}}\int_{\mathbb{R}^N} K(x)\phi_{m,N}(\gamma|u|^{\frac{N}{N-m}})\mathrm{d}x \ge \int_{\mathbb{R}^N} K(x)\phi_{m,N}(\gamma|M_n|^{\frac{N}{N-m}})\mathrm{d}x,$$

then our sharpness result can be derived from (3.20).

To finish the proof of the theorem, it remains to show that (3.3) holds. For every $\gamma > 0$ and $u \in E$, by density of $C_0^\infty(\mathbb{R}^N)$ in E, there exists $u_0 \in C_0^\infty(\mathbb{R}^N)$ such that

$$\gamma\|u-u_0\|^{\frac{N}{N-m}} < \frac{\gamma_{m,N}}{2}. \tag{3.21}$$

Using the inequality $|u|^p \le (1+\varepsilon)|u-u_0|^p + c(\varepsilon,p)|u_0|^p$, for all $\varepsilon > 0, p > 1$ and $c(\varepsilon,p)$ is a constant depending only on ε and p, and by Lemma 3.2.2 we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} K(x)\phi_{m,N}(\gamma|u|^{\frac{N}{N-m}})\mathrm{d}x\\
&\quad\le \int_{\mathbb{R}^N} K(x)\phi_{m,N}\left((1+\varepsilon)\gamma|u-u_0|^{\frac{N}{N-m}} + c\left(\varepsilon, N/(N-m)\right)|u_0|^{\frac{N}{N-m}}\right)\mathrm{d}x\\
&\quad\le \frac{1}{\mu}\int_{\mathbb{R}^N} K(x)\phi_{m,N}\left(\mu(1+\varepsilon)\gamma|u-u_0|^{\frac{N}{N-m}}\right)\mathrm{d}x\\
&\qquad+\frac{1}{\nu}\int_{\mathbb{R}^N} K(x)\phi_{m,N}\left(\nu c\left(\varepsilon, N/(N-m)\right)|u_0|^{\frac{N}{N-m}}\right)\mathrm{d}x,
\end{aligned} \tag{3.22}$$

where $\mu,\nu > 1$ and $\frac{1}{\mu}+\frac{1}{\nu} = 1$. Choosing $\varepsilon > 0$ sufficiently small and μ close to 1 such that

$$\mu(1+\varepsilon)\frac{\gamma_{m,N}}{2} \le \gamma_{m,N}$$

and from (3.21) we get $\mu(1+\varepsilon)\gamma\|u-u_0\|^{\frac{N}{N-m}} < \mu(1+\varepsilon)\frac{\gamma_{m,N}}{2} \le \gamma_{m,N}$. Thus, in view (3.4) we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} K(x)\phi_{m,N}\left(\mu(1+\varepsilon)\gamma|u-u_0|^{\frac{N}{N-m}}\right)\mathrm{d}x\\
&\quad= \int_{\mathbb{R}^N} K(x)\phi_{m,N}\left(\left(\mu(1+\varepsilon)\gamma\|u-u_0\|^{\frac{N}{N-m}}\right)\frac{|u-u_0|^{\frac{N}{N-m}}}{\|u-u_0\|^{\frac{N}{N-m}}}\right)\mathrm{d}x \le C.
\end{aligned} \tag{3.23}$$

Moreover, since $u_0 \in C_0^\infty(\mathbb{R}^N)$, we have that u_0 has compact support. Thereby,

$$\int_{\mathbb{R}^N} K(x)\phi_{m,N}\left(\nu c\left(\varepsilon, \frac{N}{N-m}\right)|u_0|^{\frac{N}{N-m}}\right)\mathrm{d}x < \infty. \tag{3.24}$$

Combining (3.22), (3.23) and (3.24) we reach that (3.3) holds. Therefore, the theorem is proved. ∎

3.3.2 The compactness result

The first important consequence we get from Theorem 3.1.1 is the following compactness result:

Proposition 3.3.1 *If (V) and (K) hold with (3.1), then for all $p \in [N/m, +\infty)$ the embedding*

$$E \hookrightarrow L^p_K(\mathbb{R}^N) \tag{3.25}$$

is continuous. Moreover, if $\beta > \alpha$ then the above embedding is compact.

Proof . We will proceed in two steps, on a ball of radius $R > 0$ and on its complement. Let $u \in E$ and observe that by condition (K) we have

$$\left(\int_{B_R} K(x)|u|^p \mathrm{d}x\right)^{\frac{1}{p}} \leq \left(\int_{B_R} \frac{b}{1+|x|^\beta}|u|^p \mathrm{d}x\right)^{\frac{1}{p}} \leq b^{\frac{1}{p}}\|u\|_{L^p(B_R)}. \tag{3.26}$$

By the embedding $W^{m,\frac{N}{m}}(B_R) \hookrightarrow L^p(B_R)$ for all $p \in [1, +\infty)$, we get

$$\begin{aligned}
\|u\|_{L^p(B_R)} &\leq C_1\|u\|_{W^{m,\frac{N}{m}}(B_R)} = C_1\left[\int_{B_R} (|\nabla^m u|^{\frac{N}{m}} + \ldots + |\nabla u|^{\frac{N}{m}} + |u|^{\frac{N}{m}})\mathrm{d}x\right]^{m/N} \\
&\leq C_1\left[\int_{B_R} \left(|\nabla^m u|^{\frac{N}{m}} + \ldots + |\nabla u|^{\frac{N}{m}} + \left(\frac{1+R^\alpha}{a}\right) V(x)|u|^{\frac{N}{m}}\right) \mathrm{d}x\right]^{m/N} \\
&\leq C_R\left[\int_{B_R} \left(|\nabla^m u|^{\frac{N}{m}} + \ldots + |\nabla u|^{\frac{N}{m}} + V(x)|u|^{\frac{N}{m}}\right) \mathrm{d}x\right]^{m/N}, \qquad (3.27)
\end{aligned}$$

because $V(x) \geq a/(1+|x|^\alpha) \geq a/(1+R^\alpha)$. Thus, for each $R > 0$, it follows, from (3.26) and (3.27), that

$$\begin{aligned}
\int_{B_R} K(x)|u|^p \mathrm{d}x &\leq bC_R^p\left[\int_{B_R} \left(|\nabla^m u|^{\frac{N}{m}} + \ldots + |\nabla u|^{\frac{N}{m}} + V(x)|u|^{\frac{N}{m}}\right) \mathrm{d}x\right]^{\frac{pm}{N}} \\
&\leq bC_R^p\|u\|^p. \qquad (3.28)
\end{aligned}$$

Proceeding as in the proof of Theorem 3.1.1 for the function $|u|^p$ instead of $\phi_{m,N}(\gamma|u|^{\frac{N}{N-m}})$, where $p \in [N/m, \infty)$ we obtain

$$\int_{A^\sigma_{3n}} K(x)|u|^p \;\mathrm{d}x \leq \frac{C}{(1+\varepsilon)^{\frac{N}{m}}}\frac{b}{a}\mathcal{B}_n\theta\int_{B^c_n} V(x)|u|^{\frac{N}{m}} \;\mathrm{d}x.$$

for all $n \geq n_0$. Letting $\sigma \to +\infty$ we reach

$$\int_{B^c_{3n}} K(x)|u|^p \;\mathrm{d}x \leq C\frac{b}{a}\mathcal{B}_n\theta\int_{B^c_n} V(x)|u|^{\frac{N}{m}} \;\mathrm{d}x \leq C\frac{b}{a}\mathcal{B}_n\theta\|u\|^{\frac{N}{m}}. \tag{3.29}$$

Taking $R = 3n_0$, from (3.29) we have that

$$\int_{B_R^c} K(x)|u|^p \, \mathrm{d}x \leq C\frac{b}{a}\mathcal{B}_{n_0}\theta\|u\|^{\frac{N}{m}}. \tag{3.30}$$

Now, if $(u_m) \subset E$ is such that $u_m \to 0$ in E, then from estimates (3.28) and (3.30) we conclude

$$\int_{\mathbb{R}^N} K(x)|u_m|^p \, \mathrm{d}x = \int_{B_R} K(x)|u_m|^p \, \mathrm{d}x + \int_{B_R^c} K(x)|u_m|^p \, \mathrm{d}x \to 0 \quad \text{as} \quad m \to \infty$$

and the continuity of the embedding is proved for all $p \in [N/m, \infty)$.

Now, suppose that $\beta > \alpha$ and $(u_m) \subset E$ is such that $u_m \rightharpoonup 0$ in E. Since (u_m) is a bounded sequence in E and from (3.17) for $\beta > \alpha$

$$\lim_{n\to\infty} \mathcal{B}_n = \lim_{n\to\infty} \frac{1 + \left(\frac{4}{3}\right)^{\alpha} n^{\alpha}}{1 + \left(\frac{2}{3}\right)^{\beta} n^{\beta}} = 0.$$

Thus, in view of (3.29), for all $\varepsilon > 0$ there exists $n_1 \geq n_0$ such that

$$\int_{B_{3n_1}^c} K(x)|u_m|^p \, \mathrm{d}x \leq \frac{\varepsilon}{2}, \quad \text{for all} \quad m \in \mathbb{N}.$$

Choosing $R = 3n_1$ and since (u_m) is also bounded in $W^{m,\frac{N}{m}}(B_R)$, by the compact embedding $W^{m,\frac{N}{m}}(B_R) \hookrightarrow L^p(B_R)$ for all $p \in [N/m, \infty)$, it follows from (3.26) that $\int_{B_R} K(x)|u_m|^p \mathrm{d}x \to 0$ as $m \to \infty$ and therefore there exists $m_0 \in \mathbb{N}$ such that

$$\int_{B_R} K(x)|u_m|^p \, \mathrm{d}x \leq \frac{\varepsilon}{2}, \quad \text{for all} \quad m \geq m_0.$$

Hence, for all $m \geq m_0$ one has

$$\int_{\mathbb{R}^N} K(x)|u_m|^p \, \mathrm{d}x = \int_{B_R} K(x)|u_m|^p \, \mathrm{d}x + \int_{B_R^c} K(x)|u_m|^p \, \mathrm{d}x \leq \varepsilon$$

which guarantees that $u_m \to 0$ in $L^p_K(\mathbb{R}^N)$ and the compact embedding is proved as $\beta > \alpha$. ■

3.4 The variational framework

The purpose of this section is to prove some geometric properties of the Euler-Lagrange functional associated to problem (3.5). We emphasize that in the context of problem (3.5), the space E is defined by

$$E := \left\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^{2m}) \, : |\nabla^i u| \in L^2(\mathbb{R}^{2m}) \ \forall \ i = 1, \ldots, m \text{ and } \int_{\mathbb{R}^{2m}} V(x)u^2 \mathrm{d}x < \infty \right\}$$

and norm

$$\|u\| := \left(\int_{\mathbb{R}^{2m}} (|\nabla^m u|^2 + \cdots + |\nabla u|^2 + V(x)u^2)\,\mathrm{d}x\right)^{\frac{1}{2}}.$$

We begin by considering the functional $I : E \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^{2m}} K(x)F(x,u)\,\mathrm{d}x.$$

Notice that from (f_1), (f_2) and (f_3), for each $\gamma > \gamma_0$, $\varepsilon > 0$ and $q \geq 2$, there exists $C(\gamma, q, \varepsilon) > 0$ such that

$$|f(x,s)| \leq \varepsilon|s| + C(\gamma,q,\varepsilon)|s|^{q-1}(e^{\gamma s^2} - 1), \quad \text{for all } \ (x,s) \in \mathbb{R}^{2m} \times \mathbb{R} \tag{3.31}$$

and

$$|F(x,s)| \leq \frac{\varepsilon}{2}|s|^2 + C(\gamma,q,\varepsilon)|s|^{q}(e^{\gamma s^2} - 1), \quad \text{for all } \ (x,s) \in \mathbb{R}^{2m} \times \mathbb{R}. \tag{3.32}$$

Thus, given $u \in E$, using Hölder's inequality with $p, p' > 1$ satisfying $1/p + 1/p' = 1$ and Lemma 3.2.1, we can find $C > 0$ such that

$$\begin{aligned} \int_{\mathbb{R}^{2m}} K(x)F(x,u)\,\mathrm{d}x \leq & \ \frac{\varepsilon}{2}\int_{\mathbb{R}^{2m}} K(x)u^2\,\mathrm{d}x + C\left(\int_{\mathbb{R}^{2m}} K(x)|u|^{pq}\,\mathrm{d}x\right)^{\frac{1}{p}} \\ & \times \left(\int_{\mathbb{R}^{2m}} K(x)(e^{p'\gamma u^2} - 1)\,\mathrm{d}x\right)^{\frac{1}{p'}}. \end{aligned} \tag{3.33}$$

Since $pq \geq 2$, combining (3.33) with the continuous embedding given by Proposition 3.3.1 and (3.3), we have that $K(x)F(x,u) \in L^1(\mathbb{R})$, for all $u \in E$. Consequently, I is well-defined and by standard arguments $I \in C^1(E,\mathbb{R})$ with

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^{2m}} (\nabla^m u \nabla^m v + \cdots + \nabla u \nabla v + V(x)uv)\,\mathrm{d}x - \int_{\mathbb{R}^{2m}} K(x)f(x,u)v\,\mathrm{d}x,$$

for all $u, v \in E$. Hence, a critical point of I is a weak solution of problem (3.5) and reciprocally.

The geometric conditions of the Mountain Pass Theorem for the functional I is established by the next lemma.

Lemma 3.4.1 *Suppose that* $(V), (K)$ *with* $\alpha \in (0, 2m)$ *and* $\beta \in [\alpha, +\infty)$ *and* (f_1)–(f_3) *hold. Then,*

(i) there exist $\tau, \rho > 0$ *such that* $I(u) \geq \tau$ *for all* $\|u\| = \rho$;

(ii) there exists $e \in E$ *with* $\|e\| > \rho$ *such that* $I(e) < 0$.

Proof . (i) Here we consider $\gamma > \gamma_0$, $\varepsilon > 0$ and $q > 2$. In view of (3.33), the continuous embedding $E \hookrightarrow L^2_K(\mathbb{R}^{2m})$ and (3.4), we can find $C_2 = C_2(\gamma, q, \varepsilon) > 0$ such that

$$\int_{\mathbb{R}^{2m}} K(x)F(x,u)\, \mathrm{d}x \leq \varepsilon\, C_1\|u\|^2 + C_2\|u\|^q, \tag{3.34}$$

for all $u \in E$ with $\|u\| = \rho$, where $\rho > 0$ satisfies $p'\gamma\rho^2 < \gamma_{m,2m}$. The inequality (3.34) implies that

$$I(u) \geq \frac{1}{2}\|u\|^2 - C_1\varepsilon\|u\|^2 - C_2\|u\|^q = \left(\frac{1}{2} - C_1\varepsilon\right)\rho^2 - C_2\rho^q.$$

Thus, if $u \in E$ with $\|u\| = \rho$, choosing $\varepsilon > 0$ sufficiently small such that $\frac{1}{2} - C_1\varepsilon > 0$ we get

$$I(u) \geq \tilde{C}_1\rho^2 - C_2\rho^q.$$

Since $q > 2$ we may choose $\rho > 0$ small enough such that $\tau := \tilde{C}_1\rho^2 - C_2\rho^q > 0$. Thus, there exists $\tau > 0$ satisfying $I(u) \geq \tau$ whenever $\|u\| = \rho$.

(ii) Let $u \in C_0^\infty(B_R) \setminus \{0\}$ be such that $u \geq 0$. By (f_3), there exist positive constants C_1 and C_2 such that

$$F(x,s) \geq C_1 s^\mu - C_2, \quad \text{for all} \quad (x,s) \in \overline{B}_R \times [0,\infty).$$

Then, for $t > 0$, we get

$$I(tu) \leq \frac{t^2}{2}\|u\|^2 - C_1 t^\mu \int_{B_R} K(x)u^\mu\, \mathrm{d}x + C_2 \int_{B_R} K(x)\, \mathrm{d}x.$$

Since $\mu > 2$, we have $I(tu) \to -\infty$ as $t \to \infty$. By setting $e = tu$ to t large enough, we immediately achieve the desired result. ■

3.5 The Palais-Smale compactness condition

In this section, we show that the functional I satisfies the Palais-Smale condition for certain energy levels. We recall that the functional I satisfies the Palais-Smale condition at the level c, denoted by $(PS)_c$ condition, if any sequence $(u_n) \subset E$ verifying

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0 \;\; \text{as} \;\; n \to \infty, \tag{3.35}$$

has a strongly convergent subsequence in E. We begin by proving some auxiliary results.

Lemma 3.5.1 *Suppose that* $(V), (K)$ *with* $\alpha \in (0, 2m)$ *and* $\beta \in [\alpha, +\infty)$ *and* $(f_1)-(f_3)$ *hold. Then, any* $(PS)_c$*-sequence* (u_n) *for* I *is bounded in* $(E, \|\cdot\|)$ *and satisfies*

$$\sup_n \int_{\mathbb{R}^{2m}} K(x)f(x,u_n)u_n \,\mathrm{d}x < \infty. \tag{3.36}$$

Proof . Since (u_n) is $(PS)_c$-sequence for I, we have

$$I(u_n) = \frac{1}{2}\|u_n\|^2 - \int_{\mathbb{R}^{2m}} K(x)F(x,u_n)\ \mathrm{d}x \to c \tag{3.37}$$

and

$$\langle I'(u_n), v\rangle = \int_{\mathbb{R}^{2m}} \left(\sum_{j=1}^{m} \nabla^j u_n \nabla^j v + V(x)u_n v \right) \mathrm{d}x - \int_{\mathbb{R}^{2m}} K(x)f(x,u_n)v \,\mathrm{d}x \leq \varepsilon_n \|v\|, \tag{3.38}$$

for all $v \in E$, where $\varepsilon_n \to 0$ as $n \to \infty$. Note that (3.37) guarantees $(I(u_n)) \subset \mathbb{R}$ is bounded and hence, there exists a constant $C > 0$ such that

$$\frac{1}{2}\|u_n\|^2 \leq C + \int_{\mathbb{R}^{2m}} K(x)F(x,u_n)\ \mathrm{d}x, \tag{3.39}$$

for all $n \in \mathbb{N}$ and from the condition (f_3), we have

$$\int_{\mathbb{R}^{2m}} K(x)F(x,u_n)\ \mathrm{d}x \leq \frac{1}{\mu}\int_{\mathbb{R}^{2m}} K(x)f(x,u_n)u_n\ \mathrm{d}x. \tag{3.40}$$

By choosing $v = u_n$ in (3.38), we obtain

$$\int_{\mathbb{R}^{2m}} K(x)f(x,u_n)u_n\ \mathrm{d}x \leq \|u_n\|^2 + \varepsilon_n\|u_n\|. \tag{3.41}$$

It follows from (3.39), (3.40) and (3.41) that

$$\left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2 \leq C + \frac{\varepsilon_n}{\mu}\|u_n\|$$

and since $\mu > 2$ we obtain that (u_n) is bounded in E. This together with (3.41) implies (3.36). ∎

The previous lemma guarantees that, up to a subsequence, there exists $u \in E$ such that $u_n \rightharpoonup u$ in E. Moreover, in view of (3.36), we can apply [23, Lemma 2.1] to conclude that

$$K(x)f(x,u_n) \to K(x)f(x,u) \quad \text{in} \quad L^1_{\mathrm{loc}}(\mathbb{R}^{2m}).$$

Now let us see the following convergence result that can be found in [33, Lemma 5.4]. We have added the proof here for the reader's convenience.

Lemma 3.5.2 *Suppose that* $(V), (K)$ *with* (3.6) *and* $(f_1) - (f_4)$ *are satisfied. Let* $(u_n) \subset E$ *be a Palais-Smale sequence of* I *at the level* c. *Then there exist a subsequence of* (u_n), *still denoted by* (u_n), *and* $u \in E$ *such that* $u_n \rightharpoonup u$ *in* E, $u_n \to u$ *in* $L^p_K(\mathbb{R}^{2m})$ *for all* $p \geq 2$ *and*

$$\int_{\mathbb{R}^{2m}} K(x)F(x, u_n)\ \mathrm{d}x \to \int_{\mathbb{R}^{2m}} K(x)F(x, u)\ \mathrm{d}x.$$

Furthermore, u *is a weak solution of problem* (3.5).

Proof . Note that by (f_3) and (f_4), we have

$$0 \leq \lim_{|s|\to\infty} \frac{F(x,s)}{sf(x,s)} \leq \lim_{|s|\to+\infty} \frac{M_0}{|s|} = 0$$

and for any $\varepsilon > 0$ there exists $s'_0 = s'_0(\varepsilon) > 0$ such that

$$F(x,s) \leq \varepsilon s f(x,s) \quad \text{for all} \quad |s| \geq s'_0. \tag{3.42}$$

Using (3.36), for some constant $C > 0$ we obtain

$$\int_{\mathbb{R}^{2m}} K(x)f(x,u)u\ \mathrm{d}x \leq C \quad \text{and} \quad \int_{\mathbb{R}^{2m}} K(x)f(x,u_n)u_n\ \mathrm{d}x \leq C \quad \text{for all} \quad n \in \mathbb{N}.$$

From (3.42) and by the previous inequalities, fixed $\varepsilon > 0$, we have

$$\int_{\{|u|\geq s'_0\}} K(x)F(x,u)\,\mathrm{d}x \leq \varepsilon \int_{\{|u|\geq s'_0\}} K(x)f(x,u)u\,\mathrm{d}x$$

and

$$\int_{\{|u_n|\geq s'_0\}} K(x)F(x,u_n)\,\mathrm{d}x \leq \varepsilon \int_{\{|u_n|\geq s'_0\}} K(x)f(x,u_n)u_n\,\mathrm{d}x.$$

Defining $\ell_n(x) := K(x)\chi_{\{|u_n|<s'_0\}}F(x,u_n)$ and $\ell(x) := K(x)\chi_{\{|u|<s'_0\}}F(x,u)$, we have that $\{\ell_n\}$ is a sequence of measurable functions and $\ell_n(x) \to \ell(x)$ for a.e $x \in \mathbb{R}^{2m}$, because $u_n \to u$ a.e. in $\mathbb{R}^{2m}$. Using (3.32) with $\gamma > \gamma_0$, $\varepsilon > 0$ and $q = 2$, for any $|s| \leq s'_0$ we obtain

$$|F(x,s)| \leq \frac{\varepsilon}{2}s^2 + C(\gamma,\varepsilon)s^2(e^{\gamma s^2} - 1) \leq C(\gamma,\varepsilon,s'_0)s^2.$$

So writing

$$g_n(x) := C(\gamma,\varepsilon,s'_0)K(x)|u_n|^2 \quad \text{and} \quad g(x) := C(\gamma,\varepsilon,s'_0)K(x)|u|^2,$$

we have $0 \leq \ell_n(x) \leq g_n(x)$ and $g_n(x) \to g(x)$ a.e. in $\mathbb{R}^{2m}$, and by virtue of the compact embedding $E \hookrightarrow L^2_K(\mathbb{R}^{2m})$, we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^{2m}} g_n(x)\ \mathrm{d}x = \int_{\mathbb{R}^{2m}} g(x)\ \mathrm{d}x.$$

Hence, applying the Generalized Lebesgue Dominated Convergence Theorem, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^{2m}} \ell_n(x)\ \mathrm{d}x = \int_{\mathbb{R}^{2m}} \ell(x)\ \mathrm{d}x.$$

In conclusion, for any fixed $\varepsilon > 0$, denoting by

$$A_n := \left|\int_{\mathbb{R}^{2m}} K(x)F(x,u_n)\ \mathrm{d}x - \int_{\mathbb{R}^{2m}} K(x)F(x,u)\ \mathrm{d}x\right|,$$

we obtain

$$\begin{aligned} A_n &\le \int_{\{|u_n|\ge s_0'\}} K(x)F(x,u_n)\ \mathrm{d}x + \int_{\{|u|\ge s_0'\}} K(x)F(x,u)\ \mathrm{d}x \\ &\quad + \left|\int_{\{|u_n|< s_0'\}} K(x)F(x,u_n)\ \mathrm{d}x - \int_{\{|u|< s_0'\}} K(x)F(x,u)\ \mathrm{d}x\right| \\ &\le 2C\varepsilon + \left|\int_{\mathbb{R}^{2m}} \ell_n(x)\ \mathrm{d}x - \int_{\mathbb{R}^{2m}} \ell(x)\ \mathrm{d}x\right| \end{aligned}$$

and passing to the limit as $n\to\infty$, we get $0 \le \lim_{n\to\infty} A_n \le 2C\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have that $\int_{\mathbb{R}^{2m}} K(x)F(x,u_n)\ \mathrm{d}x \to \int_{\mathbb{R}^{2m}} K(x)F(x,u)\ \mathrm{d}x$.

Moreover, since $u_n \rightharpoonup u$ in E and $K(x)f(x,u_n) \to K(x)f(x,u)$ in $L^1(\mathbb{R}^{2m})$, we get from (3.38) that for all $\varphi \in C_0^\infty(\mathbb{R}^{2m})$,

$$\int_{\mathbb{R}^{2m}} \left(\sum_{j=1}^{m} \nabla^j u \nabla^j \varphi + V(x)u\varphi\right) \mathrm{d}x - \int_{\mathbb{R}^{2m}} K(x)f(x,u)\varphi\ \mathrm{d}x = 0.$$

Since $C_0^\infty(\mathbb{R}^N)$ is dense in E, the above equation implies that u is a weak solution of (3.5). This completes the proof of the lemma. ■

The next result is a Lions-type Concentration-Compactness Principle (see [45]) and the proof follows the same lines as in Lemma 2.6 of [32]. This result will be crucial to study of the compactness of Palais-Smale sequences.

Proposition 3.5.3 *Suppose that $(V), (K)$ hold with $\alpha \in (0,2m)$ and $\beta \in [\alpha,+\infty)$. If $(u_n) \subset E$ satisfies $\|u_n\| = 1$, for all $n \in \mathbb{N}$, and $u_n \rightharpoonup u$ in E with $\|u\| < 1$, then for all $p \in \left(0, \frac{\gamma_{m,2m}}{1-\|u\|^2}\right)$ we have*

$$\sup_n \int_{\mathbb{R}^{2m}} K(x)(e^{p|u_n|^2} - 1)\,\mathrm{d}x < \infty. \tag{3.43}$$

Proof . Since $u_n \rightharpoonup u$ in E and $\|u_n\| = 1$, we obtain using the Hilbert's structure of $L^2(\mathbb{R}^{2m})$ that

$$\|u_n - u\|^2 = \|u_n\|^2 - 2\langle u_n, u\rangle + \|u\|^2 \le 1 - \|u\|^2 + o_n(1) < \frac{\gamma_{m,2m}}{p}.$$

Thus, for $n \in \mathbb{N}$ enough large, we get $p\|u_n - u\|^2 < \gamma < \gamma_{m,2m}$ for some $\gamma > 0$. Choose $q > 1$ close to 1 and $\varepsilon > 0$ small satisfying

$$pq(1+\varepsilon^2)\|u_n - u\|^2 < \gamma.$$

By inequality (3.4) of Theorem 3.1.1, we obtain

$$\begin{aligned}\int_{\mathbb{R}^{2m}} K(x)\left(e^{pq(1+\varepsilon^2)|u_n-u|^2} - 1\right) \mathrm{d}x &= \int_{\mathbb{R}^{2m}} K(x)\left(e^{pq(1+\varepsilon^2)\|u_n-u\|^2\left(\frac{u_n-u}{\|u_n-u\|}\right)^2} - 1\right) \mathrm{d}x \\ &\leq \int_{\mathbb{R}^{2m}} K(x)\left(e^{\gamma\left(\frac{|u_n-u|}{\|u_n-u\|}\right)^2} - 1\right) \mathrm{d}x \leq C. \qquad (3.44)\end{aligned}$$

On the other hand, observe that by elementary inequality $p|u_n|^2 \leq p(1+\varepsilon^2)|u_n-u|^2 + p(1+1/\varepsilon^2)|u|^2$ and Lemma 3.2.2 for $1/\mu + 1/\nu = 1$

$$\begin{aligned}\int_{\mathbb{R}^{2m}} K(x)\left(e^{p|u_n|^2} - 1\right) \mathrm{d}x &\leq \int_{\mathbb{R}^{2m}} K(x)\left[\left(e^{p(1+\varepsilon^2)|u_n-u|^2 + p\left(1+\frac{1}{\varepsilon^2}\right)|u|^2}\right) - 1\right] \mathrm{d}x \\ &\leq \frac{1}{\mu}\int_{\mathbb{R}^{2m}} K(x)\left(e^{p\mu(1+\varepsilon^2)|u_n-u|^2} - 1\right) \mathrm{d}x \qquad (3.45) \\ &\quad + \frac{1}{\nu}\int_{\mathbb{R}^{2m}} K(x)\left(e^{p\nu\left(1+\frac{1}{\varepsilon^2}\right)|u|^2} - 1\right) \mathrm{d}x.\end{aligned}$$

Therefore, for n sufficiently large, we can conclude by the inequalities (3.44), (3.45) and using (3.3) that (3.43) holds. ■

Next, we shall prove the main compactness result of this chapter.

Proposition 3.5.4 *Under the hypotheses (V), (K) with (3.6) and $(f_1)-(f_4)$ hold, the functional I satisfies $(PS)_c$ condition for any $0 \leq c < \frac{\gamma_{m,2m}}{2\gamma_0}$.*

Proof . Let $(u_n) \subset E$ be an arbitrary Palais-Smale sequence of I at the level c. By Lemma 3.5.1, (u_n) is bounded sequence in E so, up to a subsequence, $u_n \rightharpoonup u$ weakly in E. We shall show that, up to a subsequence, $u_n \to u$ strongly in E. For this, we have three cases to consider:

<u>*Case 1:*</u> $c > 0$ and $u = 0$. In this case, by Lemma 3.5.2, we have

$$\int_{\mathbb{R}^{2m}} K(x)F(x,u_n)\,\mathrm{d}x \to 0 \quad \text{as} \quad n \to \infty.$$

Since

$$I(u_n) = \frac{1}{2}\|u_n\|^2 - \int_{\mathbb{R}^{2m}} K(x)F(x,u_n)\,\mathrm{d}x = c + o_n(1),$$

we have that

$$\lim_{n\to\infty} \|u_n\|^2 = 2c.$$

Hence, we can deduce that for n large there exist $r > 1$ sufficiently close to 1, $\gamma > \gamma_0$ close to γ_0 and $\tilde{r} > r$ sufficiently close to r such that $\tilde{r}\gamma\|u_n\|^2 < \gamma_{m,2m}$. Thus, by (3.4)

$$\int_{\mathbb{R}^{2m}} K(x)(e^{\gamma u_n^2} - 1)^r \, \mathrm{d}x \le C \int_{\mathbb{R}^{2m}} K(x) \left(e^{\tilde{r}\gamma\|u_n\|^2 (\frac{u_n}{\|u_n\|})^2} - 1 \right) \mathrm{d}x \le C. \tag{3.46}$$

We claim that

$$\int_{\mathbb{R}^{2m}} K(x) f(x, u_n) u_n \, \mathrm{d}x \to 0 \quad \text{as} \quad n \to \infty.$$

Indeed, for $\gamma > \gamma_0$, $\varepsilon > 0$ and $q = 2$, we have from (3.31) that there exists $C(\gamma, \varepsilon) > 0$ such that

$$f(x, s) \le \varepsilon|s| + C(\gamma, \varepsilon)(e^{\gamma s^2} - 1), \quad \text{for all} \quad (x, s) \in \mathbb{R}^{2m} \times \mathbb{R}.$$

Choosing $r > 1$ close to 1 such that $r' \ge 2$, where $1/r + 1/r' = 1$, it follows by Hölder's inequality that

$$\left| \int_{\mathbb{R}^{2m}} K(x) f(x, u_n) u_n \, \mathrm{d}x \right| \le C \int_{\mathbb{R}^{2m}} K(x) u_n^2 \, \mathrm{d}x$$
$$+ C \left(\int_{\mathbb{R}^{2m}} K(x)(e^{\gamma u_n^2} - 1)^r \, \mathrm{d}x \right)^{\frac{1}{r}} \left(\int_{\mathbb{R}^{2m}} K(x)|u_n|^{r'} \, \mathrm{d}x \right)^{\frac{1}{r'}} \to 0,$$

as $n \to \infty$, where we have used (3.46) and the compact embedding $E \hookrightarrow L^p_K(\mathbb{R}^{2m})$, for $p \in [2, \infty)$. Therefore, once $\langle I'(u_n), u_n \rangle = o_n(1)$, we conclude that, up to a subsequence, $u_n \to 0$ strongly in E.

Case 2: $c > 0$ and $u \ne 0$. In this case, since (u_n) is a Palais-Smale sequence of I at the level c, we may define

$$v_n = \frac{u_n}{\|u_n\|} \quad \text{and} \quad v = \frac{u}{\lim\limits_{n\to\infty} \|u_n\|}.$$

Thus, $v_n \rightharpoonup v$ in E, $\|v_n\| = 1$ and $\|v\| \le 1$. Case $\|v\| = 1$, we conclude the proof. If $\|v\| < 1$, we claim that there exist $r > 1$ sufficiently close to 1, $\gamma > \gamma_0$ close to γ_0 and $\sigma > 0$ satisfies

$$r\gamma\|u_n\|^2 \le \sigma < \frac{\gamma_{m,2m}}{1 - \|v\|^2} \tag{3.47}$$

for $n \in \mathbb{N}$ large. Really, using that $I(u_n) = c + o_n(1)$ and Lemma 3.5.2, we have

$$\frac{1}{2} \lim_{n\to\infty} \|u_n\|^2 = c + \int_{\mathbb{R}^{2m}} K(x) F(x, u) \, \mathrm{d}x. \tag{3.48}$$

Consider

$$A(u) := \left(c + \int_{\mathbb{R}^{2m}} K(x)F(x,u)\,\mathrm{d}x\right)(1 - \|v\|^2),$$

from (3.48) and by the definition of v, we obtain $A(u) = c - I(u)$, which together with (3.48) imply that

$$\frac{1}{2}\lim_{n\to\infty}\|u_n\|^2 = \frac{A(u)}{1-\|v\|^2} = \frac{c - I(u)}{1-\|v\|^2}. \tag{3.49}$$

Hence, from (3.49) and the fact $c - I(u) < c$, we conclude

$$\frac{1}{2}\lim_{n\to\infty}\|u_n\|^2 = \frac{c - I(u)}{1-\|v\|^2} < \frac{c}{1-\|v\|^2} < \frac{\gamma_{m,2m}}{2\gamma_0(1-\|v\|^2)} \tag{3.50}$$

because $0 \le c < \frac{\gamma_{m,2m}}{2\gamma_0}$. Therefore, by using (3.50) we conclude that (3.47) holds. Thus, by Proposition 3.5.3, we get

$$\int_{\mathbb{R}^{2m}} K(x)(e^{\gamma u_n^2} - 1)^r\,\mathrm{d}x \le C.$$

By using Hölder's inequality, the compact embedding $E \hookrightarrow L^p_K(\mathbb{R}^{2m})$ for $p \in [2,\infty)$ and arguing similar to Case 1, it follows that

$$\int_{\mathbb{R}^{2m}} K(x)f(x,u_n)(u_n - u)\,\mathrm{d}x \to 0 \quad \text{as} \quad n \to \infty.$$

This convergence and the fact that $\langle I'(u_n), u_n - u\rangle = o_n(1)$ show that

$$\|u_n\|^2 = \langle u_n, u\rangle_E + o_n(1).$$

Since $u_n \rightharpoonup u$ in E, we conclude $u_n \to u$ strongly in E.

Case 3: $c = 0$. Observe that

$$0 \le I(u) \le \liminf_{n\to\infty} I(u_n) = 0,$$

consequently, $I(u) = 0$. And from Lemma 3.5.2, $K(x)F(x,u_n) \to K(x)F(x,u)$ in $L^1(\mathbb{R}^{2m})$, thereby $\|u_n\| \to \|u\|$, in other words, $u_n \to u$ in E which completes the proof. ■

3.6 The minimax level

In this section, we provide an estimate for the minimax level associated to the functional I.

Lemma 3.6.1 *Suppose that* $(V),(K)$ *with* $\alpha \in (0,2m)$ *and* $\beta \in [\alpha,+\infty)$, $(f_1)-(f_3)$ *and* (f_5) *hold. Then, there exists* $n \in \mathbb{N}$ *such that*

$$\max_{t\geq 0} I(tM_n) < \frac{\gamma_{m,2m}}{2\gamma_0}$$

where M_n *is defined in* (3.18).

Proof . Assume by contradiction that

$$\max_{t\geq 0} I(tM_n) \geq \frac{\gamma_{m,2m}}{2\gamma_0}, \quad \text{for all} \quad n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, let $t_n > 0$ such that

$$\frac{t_n^2}{2} - \int_{\mathbb{R}^{2m}} K(x)F(x,t_nM_n)\ \mathrm{d}x = \max_{t\geq 0} I(tM_n) \geq \frac{\gamma_{m,2m}}{2\gamma_0}.$$

By hypothesis (f_3), we obtain

$$\frac{\gamma_{m,2m}}{2\gamma_0} \leq \frac{t_n^2}{2} - \int_{\mathbb{R}^{2m}} K(x)F(x,t_nM_n)\ \mathrm{d}x \leq \frac{t_n^2}{2}$$

and therefore

$$t_n^2 \geq \frac{\gamma_{m,2m}}{\gamma_0}, \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.51}$$

At $t = t_n$, we have

$$0 = \frac{d}{dt}\left[\frac{t_n^2}{2} - \int_{\mathbb{R}^{2m}} K(x)F(x,tM_n)\ \mathrm{d}x\right]\Bigg|_{t=t_n} = t_n - \int_{\mathbb{R}^{2m}} K(x)f(x,t_nM_n)M_n\ \mathrm{d}x,$$

which implies that

$$t_n^2 = \int_{\mathbb{R}^{2m}} K(x)f(x,t_nM_n)t_nM_n\ \mathrm{d}x \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.52}$$

From condition (f_5), for any $\varepsilon > 0$ there exists $R = R(\varepsilon) > 0$ such that

$$sf(x,s) \geq (\theta_0 - \varepsilon)e^{\gamma_0 s^2}, \text{ for all } s \geq R \text{ and } x \in \mathbb{R}^{2m}. \tag{3.53}$$

We claim that (t_n) is a bounded sequence. Indeed, taking $n \in \mathbb{N}$ sufficiently large so that $t_nM_n(x) \geq R$, for all $x \in B_{\frac{1}{2m\sqrt[]{n}}}$, it follows, from (3.19) (3.52) and (3.53), that

$$\begin{aligned} t_n^2 &\geq \int_{B_{\frac{1}{2m\sqrt{n}}}} K(x)f(x,t_nM_n)t_nM_n\ \mathrm{d}x \geq (\theta_0-\varepsilon)\int_{B_{\frac{1}{2m\sqrt{n}}}} K(x)e^{\gamma_0 t_n^2 M_n^2}\ \mathrm{d}x \\ &\geq (\theta_0-\varepsilon)\widetilde{K}_n \int_{B_{\frac{1}{2m\sqrt{n}}}} e^{\gamma_0 \frac{t_n^2}{\|\widetilde{M}_n\|^2}\frac{\log n}{\gamma_{m,2m}}}\ \mathrm{d}x \\ &= \frac{\omega_{2m}}{2m}(\theta_0-\varepsilon)\widetilde{K}_n e^{\log n\left(\frac{\gamma_0 t_n^2}{\gamma_{m,2m}\|\widetilde{M}_n\|^2}-1\right)}. \end{aligned} \tag{3.54}$$

where we are denoting by $\widetilde{K}_n = \min_{x \in \overline{B}_{\frac{1}{2m\sqrt{n}}}} K(x)$. Thereby,

$$1 \geq \frac{\omega_{2m}}{2m}(\theta_0 - \varepsilon)\widetilde{K}_n e^{\left(\frac{\gamma_0 t_n^2 \log n}{\gamma_{m,2m}\|\widetilde{M}_n\|^2} - \log n - \log t_n^2\right)}$$

for $n \geq 1$ sufficiently large and hence (t_n) is a bounded sequence. Even more, we have

$$\lim_{n\to\infty} t_n^2 = \frac{\gamma_{m,2m}}{\gamma_0}. \tag{3.55}$$

Suppose by contradiction that this does not happen. Since that (3.51) holds, we must have

$$\lim_{n\to\infty} t_n^2 > \frac{\gamma_{m,2m}}{\gamma_0}.$$

However, letting $n \to \infty$ in (3.54), since $\|\widetilde{M}_n\| \to 1$, we get a contradition with the boundedness of the sequence (t_n). So, (3.55) holds.

Now, we consider the sets defined by

$$A_n := \{x \in B_1 : t_n M_n \geq R\} \quad \text{and} \quad C_n := B_1 \setminus A_n$$

where $R > 0$ is given in (3.53). Using (3.52) and (3.53) we get that

$$\begin{aligned} t_n^2 &\geq \int_{B_1} K(x) f(x, t_n M_n) t_n M_n \,\mathrm{d}x \\ &\geq (\theta_0 - \varepsilon)\widetilde{K}_n \int_{B_1} e^{\gamma_0 t_n^2 M_n^2} \,\mathrm{d}x + \widetilde{K}_n \int_{C_n} f(x, t_n M_n) t_n M_n \,\mathrm{d}x \\ &\quad - (\theta_0 - \varepsilon)\widetilde{K}_n \int_{C_n} e^{\gamma_0 t_n^2 M_n^2} \,\mathrm{d}x. \end{aligned} \tag{3.56}$$

By definition of C_n and since $\omega_n \to 0$ almost everywhere in B_1, we reach $\chi_{C_n} \to 1$ almost everywhere in B_1. Using the Lebesgue Dominated Convergence Theorem, we get

$$\int_{C_n} f(x, t_n M_n) t_n M_n \,\mathrm{d}x \to 0 \quad \text{and} \quad \int_{C_n} e^{\gamma_0 t_n^2 M_n^2} \,\mathrm{d}x \to \frac{\omega_{2m}}{2m} \quad \text{as } n \to \infty. \tag{3.57}$$

Observe that by (3.51) and the definition of M_n we have

$$\begin{aligned} \int_{B_1} e^{\gamma_0 t_n^2 M_n^2} \,\mathrm{d}x \;&\geq\; \int_{B_1 \setminus B_{\frac{1}{2m\sqrt{n}}}} e^{\gamma_{m,2m} M_n^2} \,\mathrm{d}x \\ &=\; \omega_{2m} \int_{1/\sqrt[2m]{n}}^{1} e^{\frac{4m^2}{\|\widetilde{M}_n\|^2} \frac{1}{\log n} (\log \frac{1}{s})^2} s^{2m-1} \,\mathrm{d}s. \end{aligned}$$

Making the change of variable

$$t = \frac{1}{\|\tilde{\omega}_n\| \log n} \log \frac{1}{s},$$

we can estimate

$$\int_{B_1} e^{\gamma_0 t_n^2 M_n^2}\, \mathrm{d}x \geq \omega_{2m} \|\widetilde{M}_n\| \log n \int_0^{\frac{1}{2m\|\widetilde{M}_n\|}} e^{\log n (4m^2t^2 - 2m\|\widetilde{M}_n\| t)}\, \mathrm{d}t.$$

Consider the function $g : \left[0, \frac{1}{2m\|\tilde{\omega}_n\|}\right] \to \mathbb{R}$ defined by $g(t) = (4m^2t^2 - 2m\|\tilde{\omega}_n\| t)\log n$. Then we have that

$$g'(0) = -2m\|\tilde{\omega}_n\| \log n \quad \text{and} \quad g'\left(\frac{1}{N2m\|\tilde{\omega}_n\|}\right) = \frac{4m}{\|\tilde{\omega}_n\|} \log n - 2m\|\tilde{\omega}_n\| \log n.$$

Let $\varepsilon > 0$ sufficiently small, thus

$$g(t) = -2mt\|\tilde{\omega}_n\| \log n + o(t), \quad t \in [0, \varepsilon]$$

and

$$g(t) = 2m \log n \left(\frac{2}{\|\tilde{\omega}_n\|} - \|\tilde{\omega}_n\|\right)\left(t - \frac{1}{2m\|\tilde{\omega}_n\|}\right) + o(t), \quad t \in \left[\frac{1}{2m\|\tilde{\omega}_n\|} - \varepsilon, \frac{1}{2m\|\tilde{\omega}_n\|}\right].$$

Hence, choosing $\varepsilon = \frac{1}{4m\|\tilde{\omega}_n\|}$ we have that

$$\begin{aligned}
\int_{B_1} e^{\gamma_0 t_n^2 M_n^2}\, \mathrm{d}x \quad &\geq \quad \omega_{2m} \|\widetilde{M}_n\| \log n \int_0^{\frac{1}{4m\|\widetilde{M}_n\|}} e^{-2mt\|\widetilde{M}_n\| \log n}\, \mathrm{d}t \\
&+ \quad \sigma_{2m} \|\widetilde{M}_n\| \log n \int_{\frac{1}{4m\|\widetilde{M}_n\|}}^{\frac{1}{2m\|\widetilde{M}_n\|}} e^{2m \log n \left(\frac{2}{\|\widetilde{M}_n\|} - \|\widetilde{M}_n\|\right)\left(t - \frac{1}{2m\|\widetilde{M}_n\|}\right)}\, \mathrm{d}t \\
&= \quad \frac{\omega_{2m}}{2m}(1 - e^{-\frac{1}{2}\log n}) + \frac{\omega_{2m}}{2m} \frac{\|\widetilde{M}_n\|^2}{\left(2 - \|\widetilde{M}_n\|^2\right)} \left(1 - e^{-\log n \left(\frac{1}{\|\widetilde{M}_n\|^2} - \frac{1}{2}\right)}\right).
\end{aligned}$$

Since $\|\widetilde{M}_n\|^2 \to 1$ we obtain that

$$\lim_{n\to\infty} \int_{B_1} e^{\gamma_0 t_n^2 M_n^2}\, \mathrm{d}x \geq \lim_{n\to\infty} \frac{\omega_{2m}}{2m} + \frac{\omega_{2m}}{2m} = \frac{\omega_{2m}}{m}. \tag{3.58}$$

Therefore, combining (3.55)-(3.58) and calculating the limit, we conclude

$$\frac{\gamma_{m,2m}}{\gamma_0} = \lim_{n\to\infty} t_n^2 \geq (\theta_0 - \varepsilon)\widetilde{K}\frac{\omega_{2m}}{m} - (\theta_0 - \varepsilon)\widetilde{K}\frac{\omega_{2m}}{2m} = \frac{(\theta_0 - \varepsilon)\widetilde{K}\omega_{2m}}{2m}.$$

By the arbitrariness of $\varepsilon > 0$, we deduce that

$$\theta_0 \leq \frac{2m\gamma_{m,2m}}{\omega_{2m}\widetilde{K}\gamma_0}.$$

This contradicts the hypothesis (f_5) and ends the proof of the lemma. ■

3.7 Proof of Theorem 3.1.4

In view of Lemma 3.4.1, the functional I satisfies the geometric conditions of the Mountain Pass Theorem. As a consequence, the minimax level

$$c := \inf_{g \in \Gamma} \max_{t \in [0,1]} I(g(t))$$

is positive, where $\Gamma := \{g \in C([0,1], E) : g(0) = 0, g(1) = e\}$. We also have by Proposition 3.5.4 that the functional I satisfies the $(PS)_c$ condition. Then, using the Mountain Pass Theorem, I has a critical point $u_0 \in E$ at the minimax level c. Hence, u_0 is a nontrivial weak solution of problem (3.5).

Appendix A

Auxiliary results

Lemma A.0.1 *E is a Banach space with the norm $\|\cdot\|$.*

Proof . Let $(u_n) \subset E$ be a Cauchy sequence, then (Δu_n), (∇u_n) and $(V^{\frac{1}{2}}(x)u_n)$ are Cauchy sequences in $L^2(\mathbb{R}^4)$. Since $L^2(\mathbb{R}^4)$ is complete, there are w_1, w_2 and w_3 in $L^2(\mathbb{R}^4)$ such that

$$\Delta u_n \to w_1, \quad \nabla u_n \to w_2 \quad \text{and} \quad V^{\frac{1}{2}}(x)u_n \to w_3 \text{ in } L^2(\mathbb{R}^4) \text{ respectively.}$$

Note that (u_n) is Cauchy in L^1_{loc}. Indeed, for all $R > 0$ we have from Hölder inequality, the condition (V) and the fact that $(V^{\frac{1}{2}}(x)u_n)$ Cauchy in $L^2(\mathbb{R}^4)$ that

$$\begin{aligned}
\int_{B_R} |u_n - u_m| \mathrm{d}x &\leq |B_R|^{\frac{1}{2}} \left(\int_{B_R} |u_n - u_m|^2 \mathrm{d}x \right)^{1/2} \\
&\leq C \left[\int_{B_R} |u_n - u_m|^2 \left(V(x) \frac{1 + |x|^\alpha}{a} \right) \mathrm{d}x \right]^{1/2} \\
&\leq C \left(\int_{B_R} V(x) |u_n - u_m|^2 \mathrm{d}x \right)^{1/2} \\
&\leq \epsilon^{1/2},
\end{aligned}$$

for all $m, n \in \mathbb{N}$ sufficient large; so there is $u \in L^1_{loc}$ such that $u_n \to u$ in L^1_{loc}. Observe that $w_3 = V^{\frac{1}{2}} u$ almost everywhere in $\mathbb{R}^4$ and for all $\varphi \in C_0^\infty(\mathbb{R}^4)$ we have

$$\int_{\mathbb{R}^4} w_1 \varphi \, \mathrm{d}x = \lim_{n \to \infty} \int_{\mathbb{R}^4} \Delta u_n \varphi \, \mathrm{d}x = \lim_{n \to \infty} \int_{\mathbb{R}^4} u_n \Delta \varphi \, \mathrm{d}x = \int_{\mathbb{R}^4} u \Delta \varphi \, \mathrm{d}x$$

and

$$\int_{\mathbb{R}^4} w_2 \varphi \, \mathrm{d}x = \lim_{n \to \infty} \int_{\mathbb{R}^4} \nabla u_n \varphi \, \mathrm{d}x = - \lim_{n \to \infty} \int_{\mathbb{R}^4} u_n \nabla \varphi \, \mathrm{d}x = - \int_{\mathbb{R}^4} u \nabla \varphi \, \mathrm{d}x$$

implying that $w_1 = \Delta u$ and $w_2 = \nabla u$ almost everywhere in $\mathbb{R}^4$. Since $w_1, w_2 \in L^2(\mathbb{R}^4)$ we obtain that $|\Delta u|, |\nabla u| \in L^2(\mathbb{R}^4)$.

Now by Lemma de Fatou, we have that

$$\int_{\mathbb{R}^4} V(x)u^2 \mathrm{d}x \leq \liminf_{n\to\infty} \int_{\mathbb{R}^4} V(x)u_n^2 \,\mathrm{d}x = \int_{\mathbb{R}^4} w_3^2 \,\mathrm{d}x < +\infty$$

because $w_3 \in L^2(\mathbb{R}^4)$. Therefore, $u \in E$ and

$$\|u_n - u\|_E^2 = \lim_{n\to\infty} \int_{\mathbb{R}^4} |\Delta(u_n - u)|^2 + |\nabla(u_n - u)|^2 + (V(x)^{\frac{1}{2}}(u_n - u))^2 \mathrm{d}x = 0,$$

since $\Delta u_n \to w_1 = \Delta u, \nabla u_n \to w_2 = \nabla u$ and $V(x)^{\frac{1}{2}} u_n \to w_3 = V(x)^{\frac{1}{2}} u$ in $L^2(\mathbb{R}^4)$. ■

Lemma A.0.2 *$C_0^\infty(\mathbb{R}^4)$ is dense in $(E, \|\cdot\|)$.*

Proof . We show first that $C_0^\infty(\mathbb{R}^4)$ is dense in

$$E_0 := \{u \in E : u \text{ has a compact support}\}.$$

Indeed, let $u \in E_0$ and $K = \operatorname{supp}(u)$, since $C_0^\infty(K)$ is dense in $H_0^2(K)$, then for all $\varepsilon > 0$, there is $\varphi_\varepsilon \in C_0^\infty(K)$ such that

$$\|u - \varphi_\varepsilon\|_{H_0^2(K)} < \frac{\varepsilon}{\sqrt{2\max\{1, \|V\|_{L^\infty(K)}\}}}.$$

Notice that

$$\begin{aligned}
\|u - \varphi_\varepsilon\|_E^2 &\leq \int_{\mathbb{R}^4} |\Delta u - \Delta\varphi_\varepsilon|^2 + |\nabla u - \nabla\varphi_\varepsilon|^2 + V(x)|u - \varphi_\varepsilon|^2 \mathrm{d}x \\
&= \int_K |\Delta u - \Delta\varphi_\varepsilon|^2 + |\nabla u - \nabla\varphi_\varepsilon|^2 + V(x)|u - \varphi_\varepsilon|^2 \mathrm{d}x \\
&\leq \max\{1, \|V\|_{L^\infty(K)}\}\|u - \varphi_\varepsilon\|_{H_0^2(K)}^2 \\
&< \max\{1, \|V\|_{L^\infty(K)}\}\frac{\varepsilon^2}{2\max\{1, \|V\|_{L^\infty(K)}\}} = \frac{\varepsilon^2}{2} < \varepsilon^2.
\end{aligned}$$

Hence, given $u \in E_0$ there exists $\varphi \in C_0^\infty(K) \subset C_0^\infty(\mathbb{R}^4)$ such that $\|u - \varphi\|_E < \varepsilon$. Therefore, $C_0^\infty(\mathbb{R}^4)$ is dense in $(E_0, \|\cdot\|_E)$.

Now, we prove that E_0 is dense in E. In fact, for every $R > 1$, consider a function $\varphi_R \in C_0^\infty(\mathbb{R}^4, [0,1])$ satisfying $\varphi_R(x) \equiv 1$ for $|x| \leq R$, $\varphi_R(x) \equiv 0$ for $|x| \geq 2R$, $|\nabla\varphi_R| \leq \frac{C}{R}$ for every $x \in \mathbb{R}^4$ and $|\Delta\varphi_R| \leq \frac{C}{R^2}$ for all $x \in \mathbb{R}^4$ for some constant $C > 0$. Given the function $u \in E$, we have that $\varphi_R u \in E_0$ for each fixed $R > 1$. We will prove that

$$\|\varphi_R u - u\|^2 = \int_{\mathbb{R}^4} |\Delta(\varphi_R u - u)|^2 + |\nabla(\varphi_R u - u)|^2 + V(x)|\varphi_R u - u|^2 \mathrm{d}x \to 0 \text{ as } R \to \infty. \tag{A.1}$$

Observe that $V^{1/2}(x)(\varphi_R - 1)u \to 0$ a.e in $\mathbb{R}^4$ as $R \to \infty$ and $V^{1/2}(x)|\varphi_R u - u| \leq 2V^{1/2}(x)u \in L^2(\mathbb{R}^4)$, then by Lebesgue Dominated Convergence Theorem, one has

$$\int_{\mathbb{R}^4} V(x)|\varphi_R u - u|^2 \mathrm{d}x \to 0 \quad \text{as} \quad R \to \infty. \tag{A.2}$$

Since $\nabla(\varphi_R u) = \nabla(\varphi_R)u + \varphi_R \nabla u$, we have that

$$\int_{\mathbb{R}^4} |\nabla(\varphi_R u - u)|^2 \mathrm{d}x \leq 2\left(\int_{\mathbb{R}^4} |\nabla(\varphi_R)u|^2 \mathrm{d}x + \int_{\mathbb{R}^4} |(\varphi_R - 1)\nabla u|^2 \mathrm{d}x\right). \tag{A.3}$$

From the fact that $|\nabla u| \in L^2(\mathbb{R}^4)$, since $u \in E$, we can derive by Lebesgue Dominated Convergence Theorem that

$$\int_{\mathbb{R}^4} |(\varphi_R - 1)\nabla u|^2 \mathrm{d}x \to 0 \quad \text{as} \quad R \to \infty. \tag{A.4}$$

Let's see that it is also valid $\int_{\mathbb{R}^4} |\nabla(\varphi_R)u|^2 \mathrm{d}x \to 0$ as $R \to \infty$. Indeed, define

$$\lambda := \inf\left\{\int_{B_2 \setminus B_1} |\nabla v|^2 \mathrm{d}x : v \in H_0^1(B_2 \setminus B_1) \text{ and } \int_{B_2 \setminus B_1} |v|^2 \mathrm{d}x = 1\right\},$$

that is, the first eigenvalue with the Dirichlet condition in the annulus. Using $x = Ry$ and $u_R(y) = u(Ry)$, we obtain that

$$\begin{aligned}
\int_{\mathbb{R}^4} |\nabla(\varphi_R)u|^2 \mathrm{d}x &= \int_{B_{2R} \setminus B_R} |\nabla(\varphi_R)u|^2 \mathrm{d}x \leq \frac{C}{R^2} \int_{B_{2R} \setminus B_R} |u|^2 \mathrm{d}x \\
&= CR^2 \int_{B_2 \setminus B_1} |u_R(y)|^2 \mathrm{d}y \leq \frac{CR^2}{\lambda} \int_{B_2 \setminus B_1} |\nabla u_R(y)|^2 \mathrm{d}y \\
&= \frac{CR^2}{\lambda} \int_{B_2 \setminus B_1} |R \nabla u(Ry)|^2 \mathrm{d}y = \frac{C}{\lambda} \int_{B_{2R} \setminus B_R} |\nabla u|^2 \mathrm{d}x.
\end{aligned}$$

Therefore, since $|\nabla u|^2$ is integrable, we concluded that

$$\int_{\mathbb{R}^4} |\nabla(\varphi_R)u|^2 \mathrm{d}x \leq \frac{C}{\lambda} \int_{B_{2R} \setminus B_R} |\nabla u|^2 \mathrm{d}x \leq \frac{C}{\lambda} \int_{B_R^c} |\nabla u|^2 \mathrm{d}x \to 0 \text{ as } R \to +\infty. \tag{A.5}$$

Finally, let's see that

$$\int_{\mathbb{R}^4} |\Delta(\varphi_R u) - u|^2 \mathrm{d}x \to 0 \quad \text{as} \quad R \to \infty. \tag{A.6}$$

Using that $\Delta(\varphi_R u) = \Delta(\varphi_R)u + 2\nabla\varphi_R \nabla u + \varphi_R \Delta u$ we obtain

$$\begin{aligned}
\int_{\mathbb{R}^4} |\Delta(\varphi_R u - u)|^2 \mathrm{d}x &\leq 2\left(\int_{\mathbb{R}^4} (\Delta(\varphi_R)u + 2\nabla\varphi_R \nabla u)^2 \mathrm{d}x + \int_{\mathbb{R}^4} |(\varphi_R - 1)\Delta u|^2 \mathrm{d}x\right) \\
&\leq 4\int_{\mathbb{R}^4} |\Delta(\varphi_R)u|^2 \mathrm{d}x + 16\int_{\mathbb{R}^4} |\nabla\varphi_R \nabla u|^2 \mathrm{d}x + 2\int_{\mathbb{R}^4} |(\varphi_R - 1)\Delta u|^2 \mathrm{d}x
\end{aligned} \tag{A.7}$$

and the third integral of (A.7) converges to zero by Lebesgue Dominated Convergence Theorem as $R \to \infty$. The first integral of (A.7) can be estimated as follows

$$\int_{\mathbb{R}^4} |\Delta(\varphi_R)u|^2 \mathrm{d}x \leq \frac{C}{a} \frac{1+(2R)^\alpha}{R^4} \int_{\mathbb{R}^4} V(x)u^2 \mathrm{d}x$$

and by the definition of E and that $\alpha \in (0,4)$ we obtain that $\displaystyle\int_{\mathbb{R}^4} |\Delta(\varphi_R)u|^2 \mathrm{d}x \to 0$ as $R \to \infty$. From the fact that $|\nabla u| \in L^2(\mathbb{R}^4)$, we concluded that

$$\int_{\mathbb{R}^4} |\nabla \varphi_R \nabla u|^2 \mathrm{d}x \leq \frac{C}{R^2} \int_{\mathbb{R}^4} |\nabla u|^2 \mathrm{d}x \to 0 \quad \text{as} \quad R \to \infty.$$

So, (A.6) is valid. Combining (A.2)-(A.6) the convergence (A.1) is proven. Thus E_0 is dense in E, and combined with $C_0^\infty(\mathbb{R}^4)$ being dense in E_0 we concluded the proof that $C_0^\infty(\mathbb{R}^4)$ is dense in E. ■

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