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Master's Dissertation

LIGHTCONE FLUCTUATIONS AND AN ESTIMATION ON THE SIZE OF THE EXTRA DIMENSION IN A QUASIPERIODICALLY COMPACTIFIED KALUZA-KLEIN MODEL

BY

GIULIA ALEIXO SANTANA DO NASCIMENTO

João Pessoa, PB

February 2024

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Dissertation submitted to Universidade Federal da Paraíba in partial fulfillment of the requirements for the degree of Master of Science.

Area of concentration: Quantum Field Theory.
Advisor: Prof. Dr. Herondy Francisco Santana Mota

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Ata da Sessão Pública da Defesa de dissertação de **Mestrado** da aluna **Giulia Aleixo Santana do Nascimento**, candidata ao Título de Mestra em Física na Área de Concentração Física de Partículas Elementares e Campos.

Aos vinte e oito dias do mês de fevereiro do ano de dois mil e vinte e quatro, às 14h00, na sala virtual meet.google.com/hqe-ytzk-czs, reuniram-se os membros da Banca Examinadora constituída para avaliar a dissertação de Mestrado, na área de Física de Partículas Elementares e Campos, de **Giulia Aleixo Santana do Nascimento**. A banca foi composta pelos(as) professores(as) doutores(as): Herondy Francisco de Santana Mota (UFPB), orientador e presidente da banca examinadora, Eugênio Ramos Bezerra de Mello (UFPB) e Francisco de Assis Brito (UFCG). Dando início aos trabalhos, o Prof. Herondy Francisco de Santana Mota comunicou aos presentes a finalidade da reunião. A seguir, passou a palavra para a candidata para que a mesma fizesse, oralmente, a exposição da pesquisa de dissertação intitulada "*Flutuações do cone de Luz e uma estimativa no comprimento da dimensão extra em um modelo de Kaluza-Klein compactificado quasi-periodicamente*". Concluída a exposição, a candidata foi arguida pela Banca Examinadora, que emitiu o parecer **"aprovada"**. Assim sendo, deve a Universidade Federal da Paraíba expedir o respectivo diploma de Mestra em Física na forma da lei. E para constar, Danilo Wilson Lemos Menezes, Técnico em Assuntos Educacionais, redigiu a presente ata que vai assinada pelos membros da Banca Examinadora. João Pessoa, Paraíba, **28 de fevereiro de 2024**.

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GIULIA ALEIXO SANTANA DO NASCIMENTO

A REVIEW ON LIGHTCONE FLUCTUATIONS AND AN ESTIMATION ON THE SIZE OF THE EXTRA DIMENSION IN A QUASIPERIODICALLY COMPACTIFIED KALUZA-KLEIN MODEL

Dissertation submitted to Universidade Federal da Paraíba in partial fulfillment of the requirements for the degree of Master of Science.

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João Pessoa, February 2024.

Dedico este trabalho à minha Mãe,
o maior exemplo de amor, persistência e
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Resumo

Na presente dissertação apresentamos uma revisão do formalismo envolvido no estudo de flutuações de cone de luz, seguido por novos resultados e discussões realizados ao longo do mestrado.

Em um primeiro momento, fizemos uma revisão da Teoria Linearizada da Gravidade no regime clássico. Começamos tomando o limite de campo gravitacional fraco de modo a permitir que este varie no tempo. Como resultado, fomos capazes de expandir a métrica do nosso espaço-tempo em torno de um espaço-tempo de fundo e considerar apenas o termo linear na perturbação. Em seguida, buscamos descobrir como essa expansão modifica os elementos da Relatividade Geral, encontrar as equações de campo para a perturbação e discutir a liberdade de calibre associada à perturbação. Em seguida, passamos a trabalhar sob a hipótese de que a perturbação é quantizada e apresentamos uma revisão dos efeitos de flutuação do cone de luz. Tais flutuações levam à remoção de divergências associadas ao cone de luz clássico; em particular, mostramos explicitamente como a quantização da perturbação remove algumas dessas divergências e discutimos a possibilidade de observar o efeito de tais flutuações sobre o tempo de deslocamento de fótons.

Seguindo o caminho trilhado pela análise da perturbação quantizada, revisamos o procedimento para escrever $\langle \sigma_1^2 \rangle$ em termos da função de Hadamard para o gráviton em $(d+1)$ dimensões. No último capítulo, apresentamos novos resultados obtidos ao longo da pesquisa. Inspirados pelo trabalho dos autores na Ref. [1], buscamos ampliar a descrição de uma possível dimensão espacial extra, tornando-a compacta por meio da condição de contorno quasiperiódica. Realizamos uma discussão sobre a transição para modelos com dimensões extras e calculamos as quantidades associadas ao desvio no tempo de propagação de um fóton como consequência da topologia do espaço compactificado. Discutimos também qual deveria ser o tamanho da dimensão extra para que seja possível detectar tais desvios, utilizando como base o Near Infrared Spectrograph (NIRSpec) a bordo do telescópio James Webb. Para isto, dividimos nossa análise em duas partes: uma para o caso periódico e outra para casos nos quais o regulador de fase da condição quasiperiódica é diferente de zero. Como consequência, mostramos que o reflexo de cada caso no tempo de propagação de um fóton difere fundamentalmente um do outro.

Palavras-chave: Flutuação de Cone de Luz, Teoria Quântica de Campos, Teoria Linearizada da Gravidade, Kaluza-Klein.

Abstract

In this dissertation, we present a review of the formalism involved in the study of light cone fluctuations accompanied by new results and discussions conducted throughout this master's program.

We initially conducted a review of the Linearized Theory of Gravity in the classical regime. We began by considering the weak gravitational field limit while allowing it to vary with time we sought to determine the modification caused by this expansion on the elements of General Relativity (GR), derived the field equations for the perturbation and discussed the gauge freedom associated to the latter. Following this first contact with linearized gravity, we worked under the assumption that the perturbation is quantized and provided a review of lightcone fluctuation effects which leads to the removal of divergences associated with the classical lightcone and discussed the possibility of observing the effects of such fluctuations upon photon propagation. Continuing our analysis, we reviewed the procedure to express $\langle \sigma_1^2 \rangle$ in terms of the Hadamard function for the graviton in $(d+1)$ dimensions. Finally, we presented the new results obtained during this research. Inspired by the work in Ref. [1], we sought to expand the description of a possible extra spatial dimension by making it compact through a quasiperiodic boundary condition. We discussed the transition to models with extra dimensions and calculated quantities associated with the deviation in the photon propagation time as a consequence of the compactified space's topology. We also discussed the requirements for the size of the extra dimension in order to obtain detectable changes on a photon flight time using the Near Infrared Spectrograph (NIRSpec) aboard the James Webb Telescope as a model for detections. For this purpose, we divided our analysis into two parts: one for the periodic case and another for cases where the phase regulator of the quasiperiodic condition is nonzero. Consequently, we found that the resulting effects of each scenario upon photon propagation fundamentally differs from one another other.

Palavras-chave: Lightcone Fluctuations, Quantum Field Theory, Linerized Gravity, Kaluza-Klein models.

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Introduction

If there exists a quantized theory of gravity, divergences that commonly appear in Quantum Field Theory, such as lightcone divergences, are expected to be smoothed out. This was first conjectured by Pauli, in 1956 [3], and discussed in the following years by Deser, Isham, and others [4–6]. One approach for understanding possible implications of a full-fledged quantum theory of gravity lies in the quantization of the Linearized Theory of Gravity. The latter is obtained when one considers a perturbative approach for the weak field limit while still allowing it to vary over time. As the name suggests, in this scenario, only linear terms on the perturbation will be considered on the spacetime metric, which is now written as a background spacetime plus a symmetric perturbation $h_{\mu\nu}$. This perturbation can be considered to be quantized [7–9]. In particular, it has been shown that, in the context of a linearized quantum theory of gravity, some lightcone divergences are in fact smeared out due to nonzero metric fluctuations [9, 10]. The lightcone fluctuations effects are analogous to the Casimir effect [11–16] in the sense that the presence of boundary conditions, nontrivial topology and spacetime dimensionality can introduce perturbations to the usual quantities, causing some of the vacuum expectation values to fluctuate [1, 17–19].

On the other hand, there are more than one scenario through which metric fluctuations can arise. The subject of this dissertation are the fluctuations caused by the quantum nature of the gravitational field. By allowing the gravitational field to change over time, its own dynamical degrees of freedom will lead to spacetime fluctuations. These are called "active" fluctuations (see Refs. [9, 10, 19, 20]). However, it follows directly from Einstein's field equation that even a classical gravitational field coupled to a quantized matter field can undergo fluctuations, called "passive" fluctuations (See Ref. [10]).

Furthermore, from its symmetry properties under spatial rotations, we can see that upon quantization, a spin-2 particle is expected to arise from excited modes of the perturbation's freely propagating degrees of freedom, the graviton. The introduction of a perturbation to the background spacetime cause modifications to the two-point function of the graviton, and its effects upon photon propagation could, in principle, be observed. The smeared lightcone can be interpreted as a change in the speed of massless particles traveling through this perturbation, thereby also modifying its typical flight time [9,10,20]. If we consider a light pulse as it travels through the perturbation, it is natural to expect that the modification on the classical flight time can varies slightly from a photon to another, however, the observed spectral lines of a light pulse will exhibit a characteristic mean deviation from the original configuration and are related to the typical change in the photons flight time as $\Delta t = \Delta\lambda$. Observable effects such as this could help us understand many phenomena involving cosmic messengers and the structure of spacetime even within a particular scope such as the linearized theory of gravity. The latter can also be applied to lightcone fluctuations in order to understand horizon fluctuations, in particular, in the case of black hole horizons, some interesting developments on black hole thermodynamics can arise [20].

Several attempts have been made to unify gravity and the electromagnetic theory in the years following the formulation of General Relativity (GR). Among them, in 1921, Theodor Kaluza tried to grasp a solution describing a world beyond our usual $(3 + 1)$ dimensions introducing an extra dimension, and six years later, Oskar Klein revisited Kaluza's work under the scope of the emergent quantum mechanics. As Kaluza's work is regarded to be the first attempt to grasp a higher-dimensional world, $(d + 1)$D quantum scenarios featuring compact extra dimensions are often referred to as Kaluza-Klein models, whereas the particular case of $d = 4$ is referred to as the Original Kaluza-Klein model which we will approach in this dissertation. In recent years, such models are sought out in an attempt to describe several phenomena in many areas of physics beyond the purpose it was find intended to serve [21–25].

We know for a fact that by introducing boundary conditions to a specific direction, we tend to obtain observable physical effects on the others. The same is true for the effects of metric fluctuations resulting from modifications imposed to a particular direction. Hence, this approach could also provide us with a powerful mean to test the existence of extra dimensions. The search for the latter goes through a wide range of study subjects in physics. In particular, in Refs. [1, 18, 26], lightcone fluctuations effects have been considered assuming periodically compactified extra dimensions. Such condition imposed to the scalar part of a tensorial field in 5D is written as $\phi_{\mathbf{k}}(x, y, z, w + \ell_c) = \phi_{\mathbf{k}}(x, y, z, w)$ with ℓ_c being the circumference of the extra dimension. If we consider the idealized case of a single monochromatic plane wave solution computed at the origin, this expressions tells us that the intensity and behaviour of the tensorial field repeats itself after traversing the whole length of the extra dimension. However, although we can postulate the existence of compact extra dimensions in our search for understanding some physical phenomena, without actual verification we can not say that this perfectly repeating behaviour would actually happen. An alternative approach is to consider more complex behaviours of the perturbation as it traverses the extra dimension via a quasiperiodic boundary condition, namely $\phi_{\mathbf{k}}(x, y, z, w + \ell_c) = e^{i2\pi\alpha}\phi_{\mathbf{k}}(x, y, z, w)$ where α is the phase angle regulator varying from 0 to 1. Our calculations and discussions for this case can be found in Ref. [27], written during the course of the research conducted for this dissertation and accepted for publication in the Journal of High Energy Physics (JHEP).

As we have mentioned, attempts of detecting the existence of extra dimensions passes through many areas of study. Additionally, the transition to higher dimensional spacetimes comes with the need of reviewing our understanding of the most fundamental laws of physics [24, 28–30]. As a reflection of the interdisciplinarity of higher-dimensional models, we can find a wide range of experiments designed to detect extra dimensions. Some examples would be table-top experiments to determine the force law acting between two masses at very short distances [28, 31] and the search for heavier particles than those of the Standard Model in the Large Hadron Collider (LHC) [29, 30] which would help solve large difference between the Planck scale and the electroweak scale, known as the hierarchy problem (see. Ref [25]). Each of these, among other experiments, are designed to explore different length scale ranges. However, one common aspect of such experiments is the fact that current detection methods seem to be far from reaching the Planck scale. Therefore, the need arises to look for alternative manifestations of the existence of extra dimensions.

Inspired by this need, as our contribution for the study of lightcone fluctuations effects and as solidification of our review of the formalism involved, we will discuss the potentially observable effects arising from a quasiperiodically compactified (4+1) dimensional Kaluza-Klein model. As we discuss our results, we will look at the sensitivity range of the Near-

Infrared Spectrograph (NIRSpec) on the James Webb Space Telescope [2] in an attempt to estimate the potential size and structure of the extra spatial dimension.

This dissertation is organized as follows: In the first Chapter we will make a detailed review on how a linearized theory of gravity modifies the GR elements in a classical regimen, find the Einstein field equations for the perturbation, explore the gauge freedom associated with the perturbation, analyse the degrees of freedom for the perturbative field equation and understand their implications in the transverse and tracefree (TT) gauge. In Chapter 2 we will start working under the assumption that the perturbation is quantized and see how it leads to the smearing of some lightcone divergences and how it affects the flight time of a photon. In Chapter 3 we will obtain the mathematical expressions through which we can study the lightcone fluctuations. That is, we will obtain an expression for $\langle\sigma_1^2\rangle$ in terms of the graviton Hadamard function by expanding the quantized perturbation as a set of plane waves obeying the Klein-Gordon like equation in the TT gauge. Finally, in Chapter 4 we begin by discussing the transition to higher-dimensional models and its effects on fundamental constants related to gravity, we obtain the exact forms of $h_{\mu\nu}$, of the graviton two-point function calculated on the lightcone and of $\langle\sigma_1^2\rangle_R$ in $(4+1)$D with a quasiperiodic condition imposed on the extra dimension. In particular, we discuss the possibility of using the NIRSpec sensitivity range to verify these results and divide our analisys of $\langle\sigma_1^2\rangle_R$ for large values of $\gamma = r/\ell_c$ in two parts: one for the periodic case ($\alpha = 0$) and one for all other condition cases for α. We also plot some of these results in order to further our analysis on how to determine the potential structure of the extra dimension. Throughout this dissertation, except for when it is made clear that we are recovering the original units for our results, we will use natural units $\hbar = c = 1$, $G_d = (32\pi)^{-1}$,

Chapter 1

The Linearized Theory of Gravity

In this chapter we make a review of the initial concepts involved in the Linearized Theory of Gravity from the perturbative approach to gravitational wave solutions in the transverse and tracefree (TT) gauge. We will expose the calculations with as most details as possible, so that any student with a solid knowledge of General Relativity can also use it as a guide. Two books were adopted to make most of this review (See Refs. [7,8]), other eventual references used will be cited as needed.

1.1 Einstein's field equation for the metric perturbation

The usual path when on a first study on the principles of General Relativity is to connect the generalization of GR for curved spacetime to the Newtonian gravity by adopting the weak limit for the classical gravitational field that obeys the Einstein's field equations while considering it to be static. However, it is also possible to consider such limit while also allowing the gravitational field to change over time. By doing this we would risk violating the the compatibility between the General Relativity (GR) and the Newtonian mechanics for the low energy limit, however, it can be shown that this compatibility is in fact preserved (see Ref. [8]). In order the obtain this weak and non-static limit, one can expand the spacetime metric $g_{\mu\nu}$ as a fixed background, which for us will be the Minkowski spacetime described by $\eta_{\mu\nu}$, plus perturbative terms. Retaining terms only up to first order in the perturbation yields

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1, \tag{1.1}$$

being $h_{\mu\nu}$ the perturbative term. Throughout this chapter we will adopt the signature $(-,+,+,+)$. Eq. (1.1) gives us an expression for the spacetime metric with two covariant indexes, however, one must be careful when writing $g^{\mu\nu}$. We know that Eq. (1.1) is a linear equation of $h^{\mu\nu}$ and, since the spacetime metric must obey $g_{\mu\sigma}g^{\sigma\nu} = \delta^{\nu}_{\mu}$, we can obtain

$$\begin{aligned} g_{\mu\sigma}g^{\sigma\nu} &= (\eta_{\mu\sigma} + h_{\mu\sigma})(\eta^{\sigma\nu} \pm h^{\sigma\nu}) \\ &= \delta^{\nu}_{\mu} + h^{\nu}_{\mu} \pm h^{\nu}_{\mu} + h_{\mu\sigma}h^{\sigma\nu} = \delta^{\nu}_{\mu} \\ &\Rightarrow g^{\mu\nu} = \eta_{\mu\nu} - h_{\mu\nu}. \end{aligned} \tag{1.2}$$

Note that, as we are not interested in keeping higher order terms on $h_{\mu\nu}$, we can use $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$ instead of $g_{\mu\nu}$ and $g^{\mu\nu}$ to raise and lower indexes. Consequently, we have

$$h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\beta\nu}h_{\alpha\beta}. \tag{1.3}$$

We can understand the linearized gravity as a theory describing a symmetric tensorial field $h_{\mu\nu}$ traveling through a background field and, in this context, the spacetime is actually curved. However, since we expanded the spacetime metric around a Minkowski spacetime we will see that this will allow us to obtain and solve wave equations for $h_{\mu\nu}$ as if the actual spacetime was flat. We could have expanded $g_{\mu\nu}$ around some other background spacetime, in this case we would obtain equations describing the perturbation traveling through that fixed spacetime and we would obtain wave equations for $h_{\mu\nu}$ in that background. Additionally, $\eta_{\mu\nu}$ is invariant under Lorentz transformations, whereas

$$x^{\mu'} = \Lambda^{\mu'}{}_{\mu}x^{\mu}, \qquad h_{\mu'\nu'} = \Lambda^{\mu}{}_{\mu'}\Lambda^{\nu}{}_{\nu'}h_{\mu\nu}. \tag{1.4}$$

Our interest now lies in understanding how our theory will be modified by considering only up to first order corrections to the metric and in finding the equation of motion to which the perturbation $h_{\mu\nu}$ obeys. To that end, let us seek the new description for the GR elements that depend on the spacetime metric. Let us start from the Christoffel symbols, using Eqs. (1.1) and (1.2) on the connection definition yields

$$\begin{aligned}\Gamma^{\lambda}_{\mu\nu} &= \frac{1}{2}g^{\lambda\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \\ &= \frac{1}{2}(\eta^{\lambda\sigma} - h^{\lambda\sigma})\left[\partial_\mu(\eta_{\nu\sigma} + h_{\nu\sigma}) + \partial_\nu(\eta_{\sigma\mu} + h_{\sigma\mu}) - \partial_\sigma(\eta_{\mu\nu} + h_{\mu\nu})\right] \\ &= \frac{1}{2}\eta^{\lambda\sigma}(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}).\end{aligned} \tag{1.5}$$

We can use this result to obtain the modified Riemann tensor, namely

$$\begin{aligned}R_{\mu\nu\alpha\beta} = \eta_{\mu\lambda}R^{\lambda}{}_{\nu\alpha\beta} &= \eta_{\mu\lambda}\left(\partial_\alpha\Gamma^{\lambda}_{\beta\nu} - \partial_\beta\Gamma^{\lambda}_{\alpha\nu} + \Gamma^{\lambda}_{\alpha\sigma}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\lambda}_{\beta\sigma}\Gamma^{\sigma}_{\alpha\nu}\right) \\ &= \frac{1}{2}\left[\eta_{\mu\lambda}\eta^{\lambda\sigma}\partial_\alpha(\partial_\beta h_{\nu\sigma} + \partial_\nu h_{\sigma\beta} - \partial_\sigma h_{\beta\nu}) - \eta_{\mu\lambda}\eta^{\lambda\sigma}\partial_\beta(\partial_\alpha h_{\nu\sigma} + \partial_\nu h_{\sigma\alpha} - \partial_\sigma h_{\alpha\nu})\right] \\ &= \frac{1}{2}\delta^{\sigma}_{\mu}(\partial_\alpha\partial_\beta h_{\nu\sigma} + \partial_\alpha\partial_\nu h_{\sigma\beta} - \partial_\alpha\partial_\sigma h_{\beta\nu} - \partial_\beta\partial_\alpha h_{\nu\sigma} - \partial_\beta\partial_\nu h_{\sigma\alpha} + \partial_\beta\partial_\sigma h_{\alpha\nu}) \\ &= \frac{1}{2}(\partial_\alpha\partial_\nu h_{\mu\beta} - \partial_\alpha\partial_\mu h_{\beta\nu} - \partial_\beta\partial_\nu h_{\mu\alpha} + \partial_\beta\partial_\mu h_{\alpha\nu}).\end{aligned} \tag{1.6}$$

Note that, in Eq. (1.6) we already discarded quadratic terms on the connection, since they would only result in second order terms of the perturbation. We have also interchanged some of the partial derivatives and used the fact that the perturbation is symmetric. From Eq. (1.6) we can directly obtain the modified Ricci tensor, which reads

$$\begin{aligned}R_{\nu\beta} &= \eta^{\mu\alpha}R_{\mu\nu\alpha\beta} \\ &= \frac{1}{2}\eta^{\mu\alpha}(\partial_\alpha\partial_\nu h_{\mu\beta} - \partial_\alpha\partial_\mu h_{\beta\nu} - \partial_\beta\partial_\nu h_{\mu\alpha} + \partial_\beta\partial_\mu h_{\alpha\nu}) \\ &= \frac{1}{2}(\partial_\alpha\partial_\nu h^{\alpha}{}_{\beta} - \partial_\alpha\partial^\alpha h_{\beta\nu} - \partial_\beta\partial_\nu h^{\alpha}{}_{\alpha} + \partial_\beta\partial_\alpha h^{\alpha}{}_{\nu})\end{aligned}$$

$$\Rightarrow R_{\mu\nu} = \frac{1}{2}(\partial_\alpha\partial_\mu h^{\alpha}{}_{\nu} + \partial_\alpha\partial_\nu h^{\alpha}{}_{\mu} - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}), \tag{1.7}$$

where $h = h_\mu{}^\mu$ is the trace of the perturbation $h_{\mu\nu}$. The Ricci scalar follows directly from Eq. (1.7), that is

$$\begin{aligned}R &= \eta^{\mu\nu}R_{\mu\nu} \\ &= \frac{1}{2}\eta^{\mu\nu}\left(\partial_\alpha\partial_\mu h^{\alpha}{}_{\nu} + \partial_\alpha\partial_\nu h^{\alpha}{}_{\mu} - \partial_\mu\partial_\nu h - \Box h_{\beta\nu}\right) \\ &= \frac{1}{2}\left(\partial_\alpha\partial_\mu h^{\alpha\mu} + \partial_\alpha\partial_\nu h^{\alpha\nu} - \partial_\mu\partial^\mu h - \Box h^{\mu}{}_{\mu}\right) \\ &= \partial_\mu\partial_\nu h^{\mu\nu} - \Box h.\end{aligned} \tag{1.8}$$

Finally, the modified Einstein tensor for the linearized gravity reads

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R \\ &= \frac{1}{2}\left(\partial_\alpha\partial_\mu h^\alpha{}_\nu + \partial_\alpha\partial_\nu h^\alpha{}_\mu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}\right) - \frac{1}{2}\eta_{\mu\nu}\left(\partial_\alpha\partial_\beta h^{\alpha\beta} - \Box h\right) \\ &= \frac{1}{2}\left(\partial_\alpha\partial_\mu h^\alpha{}_\nu + \partial_\alpha\partial_\nu h^\alpha{}_\mu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta} - \eta_{\mu\nu}\Box h\right). \end{aligned} \tag{1.9}$$

Alternatively, it can be shown that the Einstein's field equation for $h_{\mu\nu}$ arises from the Fierz-Pauli action [32, 33], that is

$$S_{FP} = \frac{1}{16\pi G}\int dx \left[\partial_\mu h^{\mu\nu}\partial_\nu h - \partial_\mu h^{\lambda\sigma}\partial_\lambda h^\mu{}_\sigma + \frac{1}{2}\eta^{\mu\nu}\partial_\mu h^{\lambda\sigma}\partial_\nu h_{\lambda\sigma} - \frac{1}{2}\eta^{\mu\nu}\partial_\mu h\partial_\nu h\right], \tag{1.10}$$

which is the Einstein-Hilbert action expanded to second order in $h_{\mu\nu}$. By varying Eq. (1.10) with respect to $h_{\mu\nu}$ after adding a coupling to matter of the form $h_{\mu\nu}T^{\mu\nu}$, one obtains

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.11}$$

where $G_{\mu\nu}$ is given by Eq. (1.9) and the energy-momentum tensor $T_{\mu\nu}$ is computed to zeroth order in the perturbation [8].

The next step would be to solve the equations we found for $h_{\mu\nu}$. However, note that the expansion in Eq. (1.1) is not unique. With careful examination we can see that, just as in the electromagnetic theory one is able to find a transformation for the 4-potential $A_\mu \to A_\mu + \partial_\mu\lambda$ which keeps $F_{\mu\nu}$ invariant, in linearized gravity we also observe a gauge invariance for the perturbation $h_{\mu\nu}$. This freedom can be expressed as the existence of a vector field $\epsilon\xi^\mu$ through which the metric perturbation will be modified as

$$h^{(\epsilon)}_{\mu\nu} = h_{\mu\nu} + 2\epsilon\partial_{(\mu}\xi_{\nu)} \tag{1.12}$$

in a way such that the spacetime curvature will remain invariant. In other words, there is more than one way to write the perturbation while describing the same physical configuration. In Eq. (1.12) we introduced the notation

$$\partial_{(\mu}\xi_{\nu)} = \frac{\partial_\mu\xi_\nu + \partial_\nu\xi_\mu}{2}, \qquad \partial_{[\mu}\xi_{\nu]} = \frac{\partial_\mu\xi_\nu - \partial_\nu\xi_\mu}{2}. \tag{1.13}$$

Given Eq. (1.12), we can verify if the spacetime remains the same through the Riemann curvature tensor, that is

$$\begin{aligned}\delta R_{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma} - R^{(\epsilon)}_{\mu\nu\rho\sigma} \\ &= \frac{\epsilon}{2}\Big(\partial_\rho\partial_\nu\partial_\mu\xi_\sigma + \partial_\sigma\partial_\mu\partial_\nu\xi_\rho - \partial_\sigma\partial_\nu\partial_\mu\xi_\rho - \partial_\rho\partial_\mu\partial_\nu\xi_\sigma + \partial_\rho\partial_\nu\partial_\sigma\xi_\mu \\ &\quad + \partial_\sigma\partial_\mu\partial_\rho\xi_\nu - \partial_\sigma\partial_\nu\partial_\rho\xi_\mu - \partial_\rho\partial_\mu\partial_\sigma\xi_\nu\Big) = 0.\end{aligned} \tag{1.14}$$

In General Relativity, gravity is presented as a consequence of spacetime curvature. The same is still valid in linearized gravity, however, we are isolating a weak gravitational contribution from the background spacetime and treating it individually with gravitational effects of its own. We also state that this perturbation travels through the background spacetime causing perturbations to the metric as described by Eq. (1.1).

1.2 The decomposition of $h_{\mu\nu}$ and the Einstein's equation degrees of freedom

With Einstein's field equations for $h_{\mu\nu}$ and the gauge transformation in Eq. (1.12), we could start working on the solution for $h_{\mu\nu}$. Of course we will not do that just yet, let us first try to obtain further insight into the information $h_{\mu\nu}$ can provide us with. To that end, let us remind that in electromagnetism, given $F_{\mu\nu}$, we can assume a fixed observer and analyze the 3-vectors **E** and **B** independently. Let us see what new information comes about when we do the same for $h_{\mu\nu}$.

We know $h_{\mu\nu}$ to be a symmetric tensor. The component h_{00} is a scalar under spatial rotations. For $i \neq 0$, the components of h_{i0} are the same as those from h_{0i} and, just as the components F_{i0} of the electromagnetic tensor form **E**, they can also be used to construct some 3-vector. Finally, the spatial part h_{ij} is a symmetric tensor that can be decomposed in a diagonal tensor plus a trace-free tensor. Therefore, we can rewrite $h_{\mu\nu}$ as

$$\begin{aligned} h_{00} &= -2\Phi \\ h_{0i} &= \omega_i \\ h_{ij} &= 2s_{ij} - 2\Psi\delta_{ij}, \end{aligned} \tag{1.15}$$

where s_{ij} is the trace-free tensor given by

$$s_{ij} = \frac{1}{2}\left(h_{ij} - \frac{1}{3}\delta^{kl}h_{kl}\delta_{ij}\right), \tag{1.16}$$

and Ψ contains the information about the trace of h_{ij}, that is

$$\Psi = -\frac{1}{6}\delta^{ij}h_{ij}. \tag{1.17}$$

As we shall see, this new way of expressing the components of $h_{\mu\nu}$ will prove itself useful. Sometimes, however, it will be simpler to just keep $h_{\mu\nu}$. In terms of these new quantities, the line element of our actual spacetime is given by

$$ds^2 = -(1+2\Phi)dt^2 + \omega^i(dtdx^i + dx^i dt) + [(1-2\Psi)\delta^{ij} + 2s_{ij}]dx^i dx^j. \tag{1.18}$$

Before reviewing the elements of General Relativity in terms of these new quantities, one should pay attention to how the index of ω_i given by Eq. (1.15) is raised. Even though we are looking at ω_i as a 3-vector, it was defined from the second order tensor $h_{\mu\nu}$ and it must be consistent with this fact while also transforming as a 3-vector upon lowering and raising indexes regardless of the signature we are working with. Note that, if we were adopting the signature $(+,-,-,-)$, we should have defined ω^i as to obtain $\eta_{ji}\omega^i = -\omega_j$. In this case, it would not matter if we have defined ω^i as ${h_0}^i$ or h^{0i}. The negative signal we need will be carried by the spatial index. However, in our case we must have ω^i such that $\eta_{ji}\omega^i = \omega_j$. Therefore, we need to be careful to define $\omega^i \equiv {h_0}^i$, otherwise, we would have obtained $\eta_{00}\eta_{ii}h^{0i} = -\omega_i$. Consequently, for it to work regardless of the signature, let us define $\omega^i \equiv {h_0}^i$.

Let us now rewrite our GR elements. Combining Eqs. (1.5) and (1.15) yields

$$\begin{aligned}
\Rightarrow \Gamma^0_{00} &= \frac{1}{2}\eta_{00}\left(\partial_0 h_{00} + \partial_0 h_{00} - \partial_0 h_{00}\right) = -\frac{1}{2}\partial_0\left(-2\Phi\right)\\
&= \partial_0\Phi\\
\Rightarrow \Gamma^i_{00} &= \frac{1}{2}\eta^{i\sigma}(\partial_0 h_{0\sigma} + \partial_0 h_{\sigma 0} - \partial_\sigma h_{00}) = \frac{1}{2}\left[\partial_0\omega_i + \partial_0\omega_i - \partial_i(-2\Phi)\right]\\
&= \partial_0\omega_i + \partial_i\Phi\\
\Rightarrow \Gamma^0_{j0} &= \frac{1}{2}\eta^{0\sigma}(\partial_j h_{0\sigma} + \partial_0 h_{\sigma j} - \partial_\sigma h_{j0}) = -\frac{1}{2}\left[\partial_j(-2\Phi) + \partial_0\omega_j - \partial_0\omega_j\right]\\
&= \partial_j\Phi\\
\Rightarrow \Gamma^i_{j0} &= \frac{1}{2}\eta^{i\sigma}(\partial_j h_{0\sigma} + \partial_0 h_{\sigma j} - \partial_\sigma h_{j0}) = \frac{1}{2}\left[\partial_j\omega_i + \partial_0(2s_{ij} - 2\Phi\delta_{ij}) - \partial_i\omega_j\right]\\
&= \partial_{[j}\omega_{i]} + \frac{1}{2}\partial_0 h_{ij}\\
\Rightarrow \Gamma^0_{jk} &= \frac{1}{2}\eta^{0\sigma}(\partial_j h_{k\sigma} + \partial_k h_{\sigma j} - \partial_\sigma h_{jk}) = -\frac{1}{2}(\partial_j\omega_k + \partial_k\omega_j - \partial_0 h_{jk})\\
&= -\partial_{(j}\omega_{k)} + \frac{1}{2}\partial_0 h_{jk}\\
\Rightarrow \Gamma^i_{jk} &= \frac{1}{2}\eta^{i\sigma}(\partial_j h_{k\sigma} + \partial_k h_{\sigma j} - \partial_\sigma h_{jk}) = \frac{1}{2}(\partial_j h_{ki} + \partial_k h_{ij} - \partial_i h_{jk})\\
&= \partial_{(j}h_{k)i} - \frac{1}{2}\partial_i h_{jk}.
\end{aligned} \tag{1.19}$$

For the Riemann tensor in Eq. (1.6), one obtains

$$\begin{aligned}
\Rightarrow R_{0j0l} &= \frac{1}{2}(\partial_0\partial_j h_{0l} - \partial_0\partial_0 h_{lj} - \partial_l\partial_j h_{00} + \partial_l\partial_0 h_{0j})\\
&= \frac{1}{2}\left[\partial_0\partial_j\omega_l - \partial_0\partial_0 h_{lj} - \partial_l\partial_j(-2\Phi) + \partial_l\partial_0\omega_j\right]\\
&= \partial_0\partial_{(j}\omega_{l)} + \partial_j\partial_l\Phi - \frac{1}{2}\partial_0\partial_0 h_{jl}\\
\Rightarrow R_{0jkl} &= \frac{1}{2}(\partial_k\partial_j h_{0l} - \partial_k\partial_0 h_{lj} - \partial_l\partial_j h_{0k} + \partial_l\partial_0 h_{kj})\\
&= \frac{1}{2}(\partial_k\partial_j\omega_l - \partial_k\partial_0 h_{lj} - \partial_l\partial_j\omega_k + \partial_l\partial_0 h_{kj})\\
&= \partial_j\partial_{[k}\omega_{l]} - \partial_0\partial_{[k}h_{l]j}\\
\Rightarrow R_{ijkl} &= \frac{1}{2}(\partial_k\partial_j h_{il} - \partial_k\partial_i h_{lj} - \partial_l\partial_j h_{ik} + \partial_l\partial_i h_{kj})\\
&= \frac{1}{2}(\partial_k\partial_j h_{il} - \partial_k\partial_i h_{lj} - \partial_l\partial_j h_{ik} + \partial_l\partial_i h_{kj})\\
&= \partial_j\partial_{[k}h_{l]i} - \partial_i\partial_{[k}h_{l]j}.
\end{aligned} \tag{1.20}$$

Note that if h is the trace of $h_{\mu\nu}$, in the decomposed form one obtains $h = 2\Phi - 6\Psi$.

Hence, the decomposition of the Ricci tensor in Eq. (1.7) yields

$$\begin{aligned}
\Rightarrow R_{00} &= \frac{1}{2}(\partial_\alpha\partial_0 h^\alpha{}_0 + \partial_\alpha\partial_0 h^\alpha{}_0 - \partial_0\partial_0 h - \Box h_{00}) \\
&= \partial_0\partial_0 h^0{}_0 + \partial_i\partial_0 h^i{}_0 + \frac{1}{2}\left[-\partial_0\partial_0(2\Phi - 6\Psi) - \Box(-2\Phi)\right] \\
&= -\partial_0^2(-2\Phi) + \partial_0\partial_i\omega^i - \partial_0^2\Phi + 3\partial_0^2\Psi + \Box\Phi \\
&= \nabla^2\Phi + \partial_0\partial_i\omega^i + 3\partial_0^2\Psi \\
\Rightarrow R_{0j} &= \frac{1}{2}(\partial_\alpha\partial_0 h^\alpha{}_j + \partial_\alpha\partial_j h^\alpha{}_0 - \partial_0\partial_j h - \Box h_{0j}) \\
&= \frac{1}{2}\left[\partial_0\partial_0 h^0{}_j + \partial_i\partial_0 h^i{}_j + \partial_0\partial_j h^0{}_0 + \partial_i\partial_j h^i{}_0 - \partial_0\partial_j(2\Phi - 6\Psi) - \Box\omega_j\right] \\
&= \partial^i\partial_0(s_{ij} - \Psi\delta_{ij}) + \partial_0\partial_j\Phi + \partial_0\partial_j(-\Phi + 3\Psi) + \frac{1}{2}\left[-\partial_0^2\omega_j + \partial_j\partial_i\omega^i - \Box\omega_j\right] \\
&= \partial^i\partial_0 s_{ij} + 2\partial_0\partial_j\Psi + \frac{1}{2}\left[\partial_j\partial_i\omega^i - \nabla^2\omega_j\right] \\
\Rightarrow R_{ij} &= \frac{1}{2}(\partial_\alpha\partial_i h^\alpha{}_j + \partial_\alpha\partial_j h^\alpha{}_i - \partial_i\partial_j h - \Box h_{ij}) \\
&= \frac{1}{2}\left[\partial_0\partial_i h^0{}_j + \partial_k\partial_i h^k{}_j + \partial_0\partial_j h^0{}_i + \partial_k\partial_j h^k{}_i\right] - \partial_i\partial_j(\Phi - 3\Psi) - \Box(s_{ij} - \Psi\delta_{ij}) \\
&= \frac{1}{2}(-\partial_0\partial_i\omega_j - \partial_0\partial_j\omega_i) + \partial^k\partial_i(s_{kj} - \Psi\delta_{kj}) + \partial^k\partial_j(s_{ki} - \Psi\delta_{ki}) - \partial_i\partial_j(\Phi - 3\Psi) \\
&\quad - \Box(s_{ij} - \Psi\delta_{ij}) \\
&= -\partial_0\partial_{(i}\omega_{j)} + 2\partial^k\partial_{(i}s_{j)k} - 2\partial_i\partial_j\Psi - \partial_i\partial_j\Phi + 3\partial_i\partial_j\Psi - \Box s_{ij} + \Box\Psi\delta_{ij} \\
&= \partial_i\partial_j(\Psi - \Phi) - \partial_0\partial_{(i}\omega_{j)} + 2\partial^k\partial_{(i}s_{j)k} - \Box s_{ij} + \Box\Psi\delta_{ij},
\end{aligned} \tag{1.21}$$

whereas the Ricci scalar in Eq. (1.8) becomes

$$\begin{aligned}
R &= \partial_\mu\partial_\nu h^{\mu\nu} - \Box h \\
&= \partial_0\partial_\nu h^{0\nu} + \partial_i\partial_\nu h^{i\nu} - \Box(2\Phi - 6\Psi) \\
&= \partial_0\partial_0 h^{00} + \partial_0\partial_j h^{0j} + \partial_i\partial_0 h^{i0} + \partial_i\partial_j h^{ij} - \Box(2\Phi - 6\Psi) \\
&= \partial_0\partial_0(-2\Phi) - \partial_0\partial_j\omega^j - \partial_i\partial_0\omega^i + \partial_i\partial_j(2s^{ij} - 2\Psi\delta^{ij}) - \Box(2\Phi - 6\Psi) \\
&= -2\partial_0^2\Phi - 2\partial_0\partial_i\omega^i + 2\partial_i\partial_j s^{ij} - 2\nabla^2\Psi - 2\Box\Phi + 6\Box\Psi \\
&= 2\left[\partial_i\partial_j s^{ij} - \partial_0\partial_i\omega^i - \nabla^2(\Phi + \Psi) + 3\Box\Psi\right].
\end{aligned} \tag{1.22}$$

Finally, the decomposed Einstein tensor in Eq. (1.9) reads

$$
\begin{aligned}
\Rightarrow G_{00} &= R_{00} - \frac{1}{2}\eta_{00}R \\
&= \nabla^2\Phi + \partial_0\partial_i\omega^i + 3\partial_0^2\Psi + \left[\partial_i\partial_j s^{ij} - \partial_0\partial_i\omega^i - \nabla^2(\Phi+\Psi) + 3\Box\Psi\right] \\
&= \partial_i\partial_j s^{ij} + 2\nabla^2\Psi \\
\Rightarrow G_{0j} &= R_{0j} - \frac{1}{2}\eta_{0j}R \\
&= \partial^i\partial_0 s_{ij} + 2\partial_0\partial_j\Psi + \frac{1}{2}\left[\partial_j\partial_i\omega^i - \nabla^2\omega_j\right] \\
\Rightarrow G_{ij} &= R_{ij} - \frac{1}{2}\eta_{ij}R \\
&= \partial_i\partial_j(\Psi-\Phi) - \partial_0\partial_{(i}\omega_{j)} + 2\partial^k\partial_{(i}s_{j)k} - \Box s_{ij} + \Box\Psi\delta_{ij} \\
&\quad - \delta_{ij}\left[\partial_k\partial_l s^{kl} - \partial_0\partial_k\omega^k - \nabla^2(\Phi+\Psi) + 3\Box\Psi\right] \\
&= \partial_i\partial_j(\Psi-\Phi) + \Box\Psi\delta_{ij} + \nabla^2(\Phi+\Psi)\delta_{ij} + \nabla^2\Psi\delta_{ij} - \nabla^2\Psi\delta_{ij} - 3\Box\Psi\delta_{ij} \\
&\quad + \partial_0\partial_k\omega^k\delta_{ij} - \partial_0\partial_{(i}\omega_{j)} + 2\partial^k\partial_{(i}s_{j)k} - \Box s_{ij} - \partial_k\partial_l s^{kl}\delta_{ij} \\
&= (\nabla^2\delta_{ij} - \partial_i\partial_j)(\Phi-\Psi) + 2\partial_0^2\Psi\delta_{ij} + \delta_{ij}\partial_0\partial_k\omega^k - \partial_0\partial_{(i}\omega_{j)} \\
&\quad + 2\partial^k\partial_{(i}s_{j)k} - \Box s_{ij} - \partial_k\partial_l s^{kl}\delta_{ij}
\end{aligned}
\tag{1.23}
$$

We have started this procedure inspired by the electromagnetism approach. However, substituting the new decomposed Einstein tensor found in Eq. (1.23) in the Einstein's field equations, we find that the resulting equations reveals the existence of less degrees of freedom than we could think we would have at first sight. Substituting Eq. (1.23) in $G_{\mu\nu} = 8\pi T_{\mu\nu}$, the 00 component yields

$$\nabla^2\Psi = 4\pi G T_{00} - \frac{1}{2}\partial_i\partial_j s^{ij}, \tag{1.24}$$

whereas the $0i$ components reads

$$(\delta_{ij}\nabla^2 - \partial_j\partial_i)\omega^i = -16\pi G T_{0j} + 2\partial^i\partial_0 s_{ij} + 4\partial_0\partial_j\Psi, \tag{1.25}$$

and for the purely spatial indexes we have

$$
\begin{aligned}
(\nabla^2\delta_{ij} - \partial_i\partial_j)\Phi = (\nabla^2\delta_{ij} - \partial_i\partial_j)\Psi - 2\partial_0^2\Psi\delta_{ij} - \delta_{ij}\partial_0\partial_k\omega^k + \partial_0\partial_{(i}\omega_{j)} \\
- 2\partial^k\partial_{(i}s_{j)k} + \Box s_{ij} + \partial_k\partial_l s^{kl}\delta_{ij} + 8\pi G T_{ij}.
\end{aligned}
\tag{1.26}
$$

Note that Eq. (1.24) is an equation for Ψ with no time derivatives and by knowing T_{00}

and s_{ij} at any given time we can somehow determine Ψ. This same argument applies to Eqs. (1.25) and (1.26) as equations for ω^i and Φ if T_{0j}, T_{ij} and s_{kj} are known. The energy-momentum tensor is calculated to the zeroth order in $h_{\mu\nu}$ [8] and, therefore, the only propagating degree of freedom in Einstein's equation for linearized gravity are those contained in s_{ij}. However, this is not always the case. In alternative theories with higher order terms in the action or with extra fields, the other components of $h_{\mu\nu}$ can also represent degrees of freedom on Einstein's equations [8]. An example of the latter, in quadratic gravity, there are extra degrees of freedom that can be associated to a massive spin-0 field and to a massive spin-2 field in addition to those associated to a massless spin-2 field (see Refs. [34, 35]). Furthermore, moving forward we will also see that the tensor s_{ij} in fact leads to a spin-2 particle upon quantization within the linearized theory.

1.3 An analogy with the Lorentz force law

To gain further insight on the components of $h_{\mu\nu}$, let us consider the movement of a test particle described by the geodesic equation

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0, \tag{1.27}$$

where $\lambda = \tau/m$. However, notice that

$$p^0 = \frac{dx^0}{d\lambda} = \frac{dt}{d\lambda} = E \qquad \longrightarrow \qquad \frac{d}{d\lambda} = \frac{dt}{d\lambda}\frac{d}{dt} = E\frac{d}{dt}, \tag{1.28}$$

and therefore,

$$p^i = \gamma m v^i = E v^i = E\frac{dx^i}{dt} = \frac{dx^i}{d\lambda}. \tag{1.29}$$

With both these equations we can rewrite Eq. (1.27) as

$$\frac{dp^\mu}{dt} = -\Gamma^\mu_{\rho\sigma}\frac{p^\rho p^\sigma}{E}. \tag{1.30}$$

The expression for $\mu = 0$ in Eq. (1.30) gives us an equation for the total time derivative of the energy E. Using Eq. (1.19), we can write

$$\begin{aligned}
\frac{dE}{dt} &= -\Gamma^0_{\rho\sigma}\frac{p^\rho p^\sigma}{E} \\
&= -\Gamma^0_{0\sigma}\frac{p^0 p^\sigma}{E} - \Gamma^0_{i\sigma}\frac{p^i p^\sigma}{E} \\
&= -\Gamma^0_{00}\frac{p^0 p^0}{E} - \Gamma^0_{0j}\frac{p^0 p^j}{E} - \Gamma^0_{i0}\frac{p^i p^0}{E} - \Gamma^0_{ij}\frac{p^i p^j}{E} \\
&= -\partial_0\Phi E - 2(\partial_i\Phi)Ev^i - \left[-\partial_{(i}\omega_{j)} + \frac{1}{2}\partial_0 h_{ij}\right]Ev^iv^j \\
&= -E\left\{\partial_0\Phi + 2(\partial_i\Phi)v^i - \left[\partial_{(i}\omega_{j)} - \frac{1}{2}\partial_0 h_{ij}\right]v^iv^j\right\}.
\end{aligned} \tag{1.31}$$

Note that the energy E only contains the test particle's inertial energy. It does not contain the energy due to gravitational interactions, and therefore, does not need to be conserved. In other words, we are allowed to obtain $dE/dt \neq 0$. As for the spatial components of the geodesic equation we obtain

$$\begin{aligned}
\frac{dp^i}{dt} &= -\Gamma^i_{\rho\sigma}\frac{p^\rho p^\sigma}{E} \\
&= -\Gamma^i_{0\sigma}\frac{p^0 p^\sigma}{E} - \Gamma^i_{j\sigma}\frac{p^j p^\sigma}{E} \\
&= -\Gamma^i_{00}\frac{p^0 p^0}{E} - \Gamma^i_{0k}\frac{p^0 p^k}{E} - \Gamma^i_{j0}\frac{p^j p^0}{E} - \Gamma^i_{jk}\frac{p^j p^k}{E} \\
&= -(\partial_0\omega_i + \partial_i\Phi)E - 2(\partial_{[j}\omega_{i]} + \frac{1}{2}\partial_0 h_{ij})Ev^j - (\partial_{(j}h_{k)i} - \frac{1}{2}\partial_i h_{jk})Ev^jv^k \\
&= -E\left\{\partial_i\Phi + \partial_0\omega_i + (2\partial_{[j}\omega_{i]} + \partial_0 h_{ij})v^j + \left(\partial_{(j}h_{k)i} - \frac{1}{2}\partial_i h_{jk}\right)v^jv^k\right\}.
\end{aligned} \tag{1.32}$$

As an attempt to understand physical implications of these results, let us define two quantities inspired by the electromagnetism

$$\begin{aligned}
G^i &\equiv -\partial_i\Phi - \partial_0\omega^i \\
H^i &\equiv (\nabla\times\vec{\omega})^i = -\epsilon^{ijk}\partial_j\omega_k,
\end{aligned} \tag{1.33}$$

where we used

$$2\partial_{[j}\omega_{i]} = \partial_j\omega_i - \partial_i\omega_j = -\epsilon_{ijk}(\nabla\times\vec{\omega})^k. \tag{1.34}$$

In other words, the quantity above is the k-component of the curl of $\vec{\omega}$ if $j \leq i$, otherwise it is its negative. Therefore, we can write

$$-\epsilon_{ijk}(\nabla \times \vec{\omega})^k v^j = -[\vec{v} \times (\nabla \times \vec{\omega})]^i. \tag{1.35}$$

Substituting Eqs. (1.33) and(1.35) back in Eq. (1.32) one obtains

$$\frac{dp^i}{dt} = E\left\{G^i - (\vec{v} \times \vec{H})^i - \partial_0 h_{ij} v^j - \left(\partial_{(j} h_{k)i} + \frac{1}{2}\partial_i h_{jk}\right) v^j v^k\right\}. \tag{1.36}$$

Note that the first two terms in the r.h.s. of Eq. (1.36) takes the form of the Lorentz force law. They tell us that a test particle moving along a geodesic reacts to Φ and to ω^i in the same way a point charge reacts to the potential vector and potential scalar in the electromagnetic theory. Furthermore, we also find terms of the purely spatial h_{ij}, which we know are associated with Ψ and s_{ij} in first and second order on the the test particle's velocity 3-vector.

1.4 The transverse gauge

We are still to dive into fixing a gauge to completely specify $h_{\mu\nu}$. In the two previous sections we found that making analogies with the electromagnetic theory results in fruitful interpretations. Let us make one more. By decomposing $h_{\mu\nu}$ in Eq. (1.12) we find that the transformations

$$\begin{aligned}
\Phi &\longrightarrow \Phi + \partial_0 \xi^0 \\
\Psi &\longrightarrow \Psi - \frac{1}{3}\partial_i \xi^i \\
\omega &\longrightarrow \omega^i + \partial_0 \xi^i - \partial_i \xi^0 \\
s_{ij} &\longrightarrow s_{ij} + \partial_{(i}\xi_{j)} - \frac{1}{3}\partial_k \xi^k \delta_{ij}
\end{aligned} \tag{1.37}$$

result in the same physical problem that is, the same spacetime curvature. What we will define here as the transverse gauge is very similar to the Coulomb gauge in electromagnetism. We start by looking for a ξ^j that will result in a new $h_{\mu\nu}$ such that

$$\partial_i s^{ij} = 0. \tag{1.38}$$

To that end we make use of the transformation for s_{ij} in Eq. (1.37) and impose Eq. (1.38). After some algebraic manipulation we obtain

$$\begin{aligned} \partial_i s^{ij} + \partial_i \partial^{(i}\xi^{j)} - \frac{1}{3}\partial_i \partial^k \xi_k \delta^{ij} &= 0 \\ \partial_i s^{ij} + \frac{1}{2}\partial_i \partial^i \xi^j + \frac{1}{2}\partial_i \partial^j \xi^i - \frac{1}{3}\partial_j \partial_k \xi^k &= 0 \\ 2\partial_i s^{ij} + \nabla^2 \xi^j + \frac{1}{3}\partial_j \partial_i \xi^i &= 0, \end{aligned}$$

which gives us

$$\nabla^2 \xi^j + \frac{1}{3}\partial^j \partial_i \xi^i = -2\partial_i s^{ij} \tag{1.39}$$

as the equation for the ξ^j that will result in (1.38). However, we did not completely determine ξ^μ as the ξ^0 component is still not specified. Let us now impose

$$\partial_i \omega^i = 0 \tag{1.40}$$

and follow the same steps using the third line of Eq. (1.37) to obtain the equation for ξ^0, which reads

$$\nabla^2 \xi^0 - \partial_0 \partial_i \xi^i = \partial_i \omega^i. \tag{1.41}$$

Remember that in electromagnetism, given a transformation $A^\mu \to A^\mu + \partial^\mu f$, one obtains the same magnetic field for any scalar f. By fixing $\partial_\mu A^\mu = 0$, we still have the freedom to choose any f as long as it satisfies $\Box f = 0$. In our case, given Eqs. (1.39) and (1.41), we can see that the transverse gauge is also only an partial gauge fixing in this same sense. Both of them are second-order differential equations, therefore, we still need boundary conditions to uniquely determine ξ^μ. This will be discussed this in more details in the next section.

The equations (1.38) and (1.40) define what we call the transverse gauge. In this gauge, the decomposed Einstein's equations (1.24)-(1.26) become

$$G_{00} = 2\nabla\Psi^2 = 8\pi T_{00}, \tag{1.42}$$

$$G_{0j} = -\frac{1}{2}\nabla^2 \omega_j + 2\partial_0 \partial_j \Psi = 4\pi G T_{0j}, \tag{1.43}$$

and

$$G_{ij} = (\delta_{ij}\nabla^2 - \partial_i\partial_j)(\Phi - \Psi) + 2\partial_0^2\Psi\delta_{ij} - \partial_0\partial_{(i}\omega_{j)} - \Box s_{ij} = 8\pi G T_{ij}. \tag{1.44}$$

This partial gauge receives its name because the resulting wave vector will be orthogonal to the polarization tensor, as we will be able to verify in the next section.

1.5 The transverse traceless gauge and gravitational wave solutions for $h_{\mu\nu}$

Depending on how a certain tensor field behaves under spatial rotations, particles of different spin will be described upon quantization. The scalars Φ and Ψ would describe spin-0 particles, ω^i would describe a spin-1 particle and s_{ij} spin-2 particles. This tells us that s_{ij}, the only propagating degree of freedom in $h_{\mu\nu}$, will contain the gravitational radiation information [8]. Let us see what happens to a freely propagating degree of freedom in linearized gravity by adopting the transverse gauge. Since we are not interested in the gravitational source, only on the propagation of the perturbation, let us turn off the energy-momentum tensor, that is, we make $T_{\mu\nu} = 0$. By doing this, Eq. (1.42) becomes

$$\nabla^2\Psi = 0. \tag{1.45}$$

Additionally, note that since we are interested only in quantities that are related to spin-2 particles, we can impose boundary conditions in order obtain $\Psi = 0$, resulting in a traceless $h_{\mu\nu}$. In other words, we impose a form for ξ^μ in Eqs. (1.37) as to obtain $\Psi = 0$. As for the trace of Eq. (1.43), we now obtain

$$-\frac{1}{2}\nabla^2\omega_j + 2\partial_0\partial_j\Psi = 0 \to \nabla^2\omega_j = 0, \tag{1.46}$$

which yields $\omega_i = 0$ by using $\Psi = 0$ and making the same argument as that of Ψ. Following the same steps for Eq. (1.44) we have

$$\nabla^2\Phi = 0, \tag{1.47}$$

that again, by the same argument, yields $\Phi = 0$. Finally, for the Eq. (1.43),

$$\Box s_{ij} = 0. \tag{1.48}$$

That is, we are left with a Klein-Gordon like equation for s_{ij}. As a result, $h_{\mu\nu}$ takes the form of

$$h^{TT}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & 2s_{ij} & \\ 0 & & & \end{pmatrix}. \tag{1.49}$$

The combination of Eqs. (1.38) and (1.40) together with the boundary conditions resulting in $\Psi = \Phi = \omega_i = 0$ leads to what we call the transverse traceless (TT) gauge. In this gauge, notice that $h^{TT}_{\mu\nu}$ is traceless, transverse and purely spatial, this means that we can write the TT gauge conditions as

$$\begin{aligned} \eta^{\mu\nu} h^{TT}_{\mu\nu} &= 0, \\ \partial^{\mu} h^{TT}_{\mu\nu} &= 0, \\ h^{TT}_{0\nu} &= 0, \end{aligned} \tag{1.50}$$

while retaining only freely propagatin degrees of freedom. Additionally, Eqs. (1.48) and (1.49) tell us that the equation of motion for $h^{TT}_{\mu\nu}$ is also a Klein-Gordon like equation, that is

$$\Box h^{TT}_{\mu\nu} = 0, \tag{1.51}$$

and, therefore, can be resolved using a set of plane waves, namely

$$h^{TT}_{\mu\nu} = A_{\mu\nu} e^{ik_{\alpha}x^{\alpha}}, \tag{1.52}$$

where $A_{\mu\nu}$ is a constant, symmetric, traceless and purely spatial tensor that should result in the transverse traceless $h_{\mu\nu}$. Since $h_{\mu\nu}$ is purely spatial, we will sometimes refer to it as h_{ij}. Note that, although $h^{TT}_{\mu\nu}$ is real we can see that $e^{ik_{\alpha}x^{\alpha}}$ is a complex quantity. We shall consider only the real part in our final calculations. Finally, k^{α} is the wave vector, which again, is a constant vector.

Substituting the plane wave solution (1.52) in Eq. (1.51) we obtain a new constraint, which reads

$$\Box h^{TT}_{\mu\nu} = \partial_\beta \partial^\beta A_{\mu\nu} e^{ik_\alpha x^\alpha} = \partial_\beta (ik^\beta) A_{\mu\nu} e^{ik_\alpha x^\alpha} = -k_\beta k^\beta A_{\mu\nu} e^{ik_\alpha x^\alpha} = 0,$$

$$k_\beta k^\beta = 0. \tag{1.53}$$

Eq. (1.53) tells us that the plane wave is an acceptable solution to the equation of motion for $h^{TT}_{\mu\nu}$ only if its wave vector is null. In other words, gravitational waves propagate at the speed of light. By writing the wave vector as $k^\mu = (\omega, \mathbf{k})$, this condition becomes

$$\omega = |\mathbf{k}|. \tag{1.54}$$

Now, we know that $A_{\mu\nu}$ is such that $h^{TT}_{\mu\nu}$ is traceless and purely spatial, that is

$$\begin{aligned} A^\mu_\mu &= 0, \\ A_{0\nu} &= 0. \end{aligned} \tag{1.55}$$

To ensure that it is also transverse, we must have

$$\partial_\mu h^{\mu\nu}_{TT} = \partial_\mu A^{\mu\nu} e^{ik_\alpha x^\alpha} = (ik_\mu) A^{\mu\nu} e^{ik_\alpha x^\alpha} = 0,$$

that is,

$$k_\mu A^{\mu\nu} = 0. \tag{1.56}$$

This is exactly what we meant when we said that the transverse gauge results in a solution in which the wave vector is orthogonal to polarization tensor. We will see that the latter is given by $A_{\mu\nu}$.

In order to simplify our investigation, let us choose spatial coordinates such that the wave vector is pointing at the x^3 direction in a $(3+1)$-dimensional spacetime. By making use of Eq. (1.54), the wave vector can be written as $k^\mu = (\omega, 0, 0, \omega)$. Adding this to the constraints in Eqs. (1.55) and (1.56) we obtain

$$\begin{aligned} k_\mu A^{\mu\nu} &= k_0 A^{0\nu} + k_1 A^{1\nu} + k_2 A^{2\nu} + k_3 A^{3\nu} \\ &= \omega A^{3\nu} = 0, \end{aligned}$$

that is,

$$A_{3\nu} = 0. \tag{1.57}$$

As $A_{\mu\nu}$ must be symmetric, Eq. (1.57) tells us that its only nonzero components are A_{11}, A_{12}, A_{21} and A_{22}. Since $A_{\mu\nu}$ is also traceless, we can write it in terms of two components, that is

$$A^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & -A_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1.58}$$

As a result, we only need A_{11}, A_{12} and the frequency ω in order to completely determine $h^{TT}_{\mu\nu}$. We still can exercise our results a little more in order to obtain a more profound physical understanding of our gravitational wave solution and of its effects. It would be meaningless to study the trajectory of a single test particle since it would only tell us about the coordinates along the particle's worldline. Furthermore, it is always possible to find transverse traceless coordinates in which the particle would appear stationary in first order in $h^{TT}_{\mu\nu}$ [8]. To actually understand what is happening in our 4-dimensional spacetime we must consider the relative movement of nearby particles described by the geodesic deviation equation. Given a ensemble of particles with 4-velocity U^μ and the separation vector between them S^μ, the geodesic deviation equation reads

$$\frac{d^2 S^\mu}{d\tau^2} = R^\mu{}_{\nu\alpha\beta} U^\nu U^\alpha S^\beta. \tag{1.59}$$

Now, if we consider that the particles are moving very slowly, we can expand the 4-velocity as $U^\mu = (1, 0, 0, 0)$ plus corrections of first order in $h^{TT}_{\mu\nu}$ which we can ignore, because as we know from Eq. (1.6), the Riemann tensor is already of first order in the perturbation. As a result, the only nonzero terms will be the ones involving

$$\begin{aligned} R^\mu{}_{00\beta} &= \frac{1}{2}(\partial_0 \partial_0 h^\mu{}_\beta - \partial_0 \partial^\mu h_{\beta 0} - \partial_\beta \partial_0 h^\mu{}_0 + \partial_\beta \partial^\mu h_{00}) \\ &= \frac{1}{2} \partial_0 \partial_0 h^\mu{}_\beta. \end{aligned}$$

Furthermore, as the particles are moving slowly, we can approximate $d\tau = dt$. This way, the geodesic deviation equation becomes

$$\frac{\partial^2 S^\mu}{\partial t^2} = \frac{1}{2}\frac{\partial^2 h^{TT\mu}{}_\beta}{\partial t^2} S^\beta. \tag{1.60}$$

Since the perturbation is transverse, the only separation vector components affected by the gravitational radiation traveling in the x^3 direction will be the S^1 and the S^2 directions. We are seeking to determine the effects of the A_{11} and A_{12} components on the test particles. Let us consider these two separately. By taking $A_{12} = 0$, the geodesic deviation equations becomes

$$\frac{\partial^2 S^1}{\partial t^2} = \frac{S^1}{2}\frac{\partial^2}{\partial t^2} A_{11} e^{-i\omega(t-z)} \tag{1.61}$$

and,

$$\frac{\partial^2 S^2}{\partial t^2} = -\frac{S^2}{2}\frac{\partial^2}{\partial t^2} A_{11} e^{ik_\alpha x^\alpha}, \tag{1.62}$$

whose solutions can be written as

$$\begin{aligned} S^1 &= \left(1 + \frac{1}{2}A_{11}e^{ik_\alpha x^\alpha}\right) S^1(0), \\ S^2 &= \left(1 - \frac{1}{2}A_{11}e^{ik_\alpha x^\alpha}\right) S^2(0). \end{aligned} \tag{1.63}$$

Eq. (1.63) tells us that if the particles have a separation in the x^1 direction, they will oscillate in the x^1 direction, and if they have a separation in the x^2 direction, they will oscillate in the x^2 direction. If they are initially predisposed in a circle in the x^1x^2 plane, the circle will be stretched upwards and downwards simultaneously, come back to the initial configuration and then be stretched to both sides, come back to the initial configuration and so on... This is why the component A_{11} is usually written as h_+, and a gravitational wave in this gauge with $A_{12} = 0$ is said to contain a **plus polarization**.

Following the same steps for the case $A_{11} = 0$, we obtain

$$\begin{aligned} S^1 &= S^1(0) + \frac{1}{2}A_{12}e^{ik_\alpha x^\alpha} S^2(0), \\ S^2 &= S^2(0) - \frac{1}{2}A_{12}e^{ik_\alpha x^\alpha} S^1(0). \end{aligned} \tag{1.64}$$

from the geodesic deviation equations. This means that particles with a separation along the x^1 direction will oscillate in the x^2 direction and vice versa. If they are initially predisposed in a circle in the x^1x^2 plane, the circle will be stretched in the northeast and southwest directions simultaneously, come back to the initial configuration, it will be then stretched in the northwest and southeast directions, come back to the initial configuration again and so on... This is the reason why the component A_{12} is usually written as $h_\times$ and a gravitational wave in this gauge with $A_{11} = 0$ is said to contain a **cross polarization**.

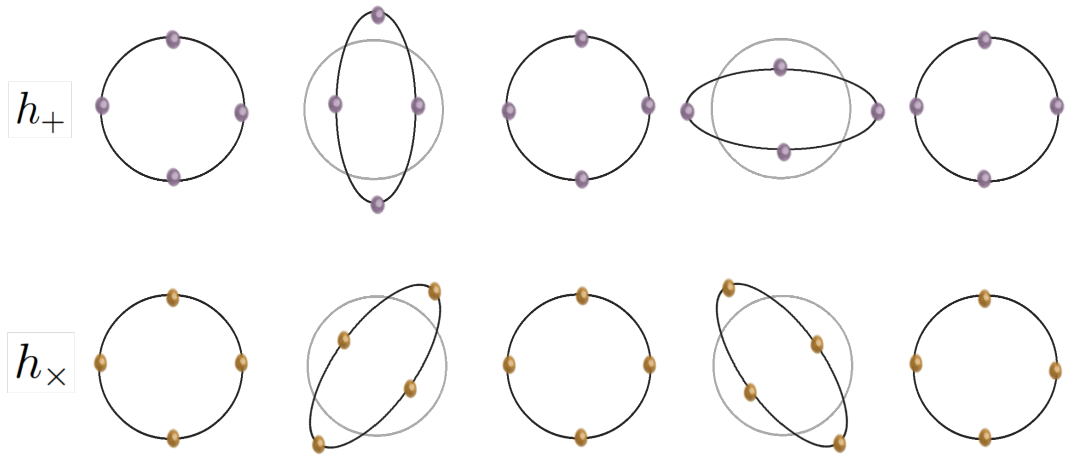

Figure 1.1: The effect of h_+ and $h_\times$ on test particles initially predisposed in a circle on the xy plane, trespassed by a gravitational wave traveling along the z-direction.

The quantities h_+ and $h_\times$ measure independent modes of polarization of the gravitational waves and its effects upon a circular disposition of test particles are depicted in Fig. 1.1. We can also use these to construct the circular polarization modes

$$\begin{aligned} h_R &= \frac{1}{\sqrt{2}}(h_+ + ih_\times), \\ h_L &= \frac{1}{\sqrt{2}}(h_+ - ih_\times), \end{aligned} \tag{1.65}$$

where the sub index R stands for "right", indicating the direction in which our hypothetical particle distribution would appear to be rotating and, as expected, L stands for "left", for the same reason. It is worth noticing that the distribution of particles would not be the one rotating, each particle would describe a small circle around its initial position.

So far we are talking about classical gravitational waves, however, note that the polarization modes depicted in Fig. 1.1 are invariant under spatial rotations of 180º on the x^1x^2 plane. If we rotate each stage of the movement in Fig. 1.1 by 90º, we would be able to tell that it was rotated. However, by rotating it 180º, an observer would not be able to tell the difference. The angle under which the polarization modes of a given field are invariant are observed to be related to the spin of the particle arising from the their quantization as $s = 360^{\text{o}}/\theta$. This is one of the indications that the particles associated with gravitational waves upon quantization should be spin-2, postulated as the gravitons.

Chapter 2

Gravitons and light cone fluctuations

From this chapter on, we will consider the perturbation to be quantized and we shall see that this assumption leads to the smearing of classical lightcone divergences. This possibility was first raised by Pauli as he commented on the work submitted by Oskar Klein to the 50th Jubilee of General Relativity [3]. This chapter will be mostly a review of Refs. [1, 9, 17]. We will show how some of the lightcone divergences are smeared out and discuss the order of magnitude on $h_{\mu\nu}$ that must be taken into account in order to observe its effects on a photon's flight time.

2.1 Lightcone divergence smearing and the change on a photon's flight time

We are still considering a linearized perturbation $h_{\mu\nu}$ propagating in a flat $(3+1)$-dimensional spacetime, so the line element is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu, \tag{2.1}$$

which is the same from the decomposed form in Eq. (1.18). Although this decomposition is still valid, $h_{\mu\nu}$ will be sufficient for our current goal. As we are solely interested in gravitational wave solutions, we will adopt the TT gauge. Let σ be one half of the geodesic separation squared between two points, then

$$\begin{aligned}\sigma &= \frac{1}{2}g_{\mu\nu}X^{\mu}X^{\nu} \\ &= \frac{1}{2}(\eta_{\mu\nu}+h_{\mu\nu})X^{\mu}X^{\nu}+\mathcal{O}(h_{\mu\nu}^2) \\ &= \frac{1}{2}(t-t')^2-\frac{1}{2}(\mathbf{x}-\mathbf{x}')^2+\frac{1}{2}h_{\mu\nu}X^{\mu}X^{\nu}+\mathcal{O}(h_{\mu\nu}^2) \\ &= \sigma_0+\sigma_1+\mathcal{O}(h_{\mu\nu}^2),\end{aligned} \tag{2.2}$$

where $\sigma_0 = (t-t')^2-(x-x')^2$, $X^{\mu} = x^{\mu}-x'^{\mu}$ and, the sub index of σ expresses the order of $h_{\mu\nu}$ on each term. For a massless scalar field in flat spacetime, the retarded Green's function reads [36]

$$G_{ret} = \theta(t-t')\frac{\delta(\sigma)}{4\pi} = \frac{\theta(t-t')}{8\pi^2}\int_{-\infty}^{\infty} ds\, e^{is(\sigma_0+\sigma_1)}. \tag{2.3}$$

Note that we used of the integral representation for the Dirac delta function in the last term on the r.h.s.

If the spacetime was free from perturbation, that is, in Minkowski spacetime, we would have $\sigma_1 = 0$ and, the presence of the δ-function would be telling us that the retarded Green's function is zero everywhere except for a singularity on the future light cone. However, in the presence of the perturbation, the singularity is now located on the perturbed light cone.

Now, considering $h_{\mu\nu}$ to be quantized [7,8], we will make a little change in our vocabulary: what we were calling perturbations traveling in a flat background we will now call gravitons in a vacuum state $|\psi\rangle$. We say that $|\psi\rangle$ is a vacuum state in the sense that, upon quantization, $h_{\mu\nu}$ becomes an operator that can be decomposed into its positive and negative frequency parts such that

$$h^{+}_{\mu\nu}|\psi\rangle = 0, \qquad \langle\psi|\, h_{\mu\nu} = 0, \tag{2.4}$$

and since σ_1 is written in first order on $h_{\mu\nu}$, it can also be decomposed in σ_1^+ and σ_1^- replicating the behaviours in Eq. (2.4). The resulting operators can be understood as representing the operators of creation and annihilation of gravitons. In the previous chapter we saw that the TT gauge results in a Klein-Gordon like equation for the perturbation, allowing us to write it in terms of a plane wave expansion (see Eqs. (1.51) and (1.52)).

Therefore, the same is valid for σ_1 and we can write

$$\sigma_1^+ = \sum_\lambda a_\lambda f_\lambda, \qquad \sigma_1^- = \sum_\lambda a_\lambda^\dagger f_\lambda^*, \tag{2.5}$$

where f and f^* are the mode functions, λ is the set of momentum vectors and polarization modes, and the coefficients a_σ and $a_\sigma^\dagger$ obey

$$\begin{aligned} &[a_\sigma, a_{\sigma'}^\dagger] = \delta_{\sigma\sigma'}, \\ &[a_\sigma, a_{\sigma'}] = [a_\sigma^\dagger, a_{\sigma'}^\dagger] = 0. \end{aligned} \tag{2.6}$$

Using the Campbell-Baker-Hausdorff (CBH) formula [37]

$$\exp\{A+B\} = \exp\left\{A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] + ...\right\}, \tag{2.7}$$

along with the fact that both σ_1^+ and σ_1^- commute with their own commutator $[\sigma_+, \sigma_-]$, we can write the exponential in Eq. (2.3) as

$$\exp\{is\sigma_1\} = \exp\left\{is\sigma_1^+ - \frac{s^2}{2}[\sigma_1^+, \sigma_1^-] + is\sigma_1^-\right\}. \tag{2.8}$$

Note that although we truncated σ up to first order in $h_{\mu\nu}$, the exponential above retains terms of all orders in $h_{\mu\nu}$. This will be crucial in order to obtain non-null contributions of the expected value of $e^{is\sigma_1}$, as we shall discuss in the next section. By expanding each exponential above we obtain

$$\begin{aligned} e^{is\sigma_1^+} |\psi\rangle &= (1 + is\sigma_1^+ + \cdots) |\psi\rangle = |\psi\rangle , \\ \langle\psi| e^{is\sigma_1^-} &= \langle\psi| (1 + is\sigma_1^- + \cdots) = \langle\psi| . \end{aligned} \tag{2.9}$$

and, using Eq. (2.6), we find that

$$[\sigma_1^+, \sigma_1^-] = \sum_{\lambda\lambda'} [a_\lambda, a_{\lambda'}^\dagger] f_\lambda f_{\lambda'}^* = \sum_\lambda f_\lambda f_\lambda^*. \tag{2.10}$$

On the other hand, note that from Eq. (2.6) we can also obtain

$$\begin{aligned}\langle\sigma_1^2\rangle &= \langle\psi|\left(\sum_\lambda a_\lambda f_\lambda + \sum_\lambda a_\lambda^\dagger f_\lambda\right)^2|\psi\rangle = \langle\psi|\sum_{\lambda\lambda'} a_\lambda a_{\lambda'}^\dagger f_\lambda f_{\lambda'}^*|\psi\rangle \\ &= \langle\psi|\sum_{\lambda\lambda'}(\delta_{\lambda\lambda'} + a_{\lambda'}^\dagger a_\lambda) f_\lambda f_{\lambda'}^*|\psi\rangle = \sum_\lambda f_\lambda f_\lambda^* \langle\psi|\psi\rangle = \sum_\lambda f_\lambda f_\lambda^*.\end{aligned} \tag{2.11}$$

Note the importance of this result. We can directly see from Eq. (2.4) that $\langle h_{\mu\nu}\rangle = \langle\sigma_1\rangle = 0$. However, Eq. (2.11) is telling us that $\langle\sigma_1^2\rangle$ and, therefore, $\langle h_{\mu\nu}^2\rangle$ may not be null. In other words, the perturbation is fluctuating and so is the spacetime metric.

Substituting Eqs. (2.9), (2.10) and (2.11) back in Eq. (2.8) we can write

$$\langle e^{is\sigma_1}\rangle = \langle e^{is\sigma_1^+} e^{-\frac{s^2}{2}[\sigma_1^+,\sigma_1^-]} e^{is\sigma_1^-}\rangle = \langle e^{-\frac{s^2}{2}[\sigma_1^+,\sigma_1^-]}\rangle = e^{-\frac{s^2}{2}\langle\sigma_1^2\rangle}. \tag{2.12}$$

Using Eq. (2.12) the expectation value in Eq. (2.3) becomes

$$\langle G_{ret}(x,x')\rangle = \frac{\theta(t-t')}{8\pi^2}\int_{-\infty}^{\infty} ds\, e^{is\sigma_0 - \frac{1}{2}s^2\langle\sigma_1^2\rangle}, \tag{2.13}$$

where we take into account that σ_0 is the classical square of the geodesic deviation and therefore does not act on the vacuum state. As we will see in Chapter 4, direct calculations of the quantity $\langle\sigma_1^2\rangle$ entail divergences, therefore, this quantity should be renormalized. As we shall also see, this is done by subtracting the respective contribution from the background spacetime.

We can now proceed to perform the integral in (2.13). Completing the square in the exponential argument, that is, substituting

$$is\sigma_0 - \frac{s^2}{2}\langle\sigma_1^2\rangle = -\left(\frac{s\sqrt{\langle\sigma_1^2\rangle}}{\sqrt{2}} + \frac{i\sigma_0}{\sqrt{2\langle\sigma_1^2\rangle}}\right)^2 - \frac{\sigma_0^2}{2\langle\sigma_1^2\rangle}$$

in Eq. (2.13) and performing the integral, one obtains

$$\langle G_{ret}(x,x')\rangle = \frac{\theta(t-t')}{8\pi^2}\sqrt{\frac{2\pi}{\langle\sigma_1^2\rangle}}\, e^{-\frac{\sigma_0^2}{2\langle\sigma_1^2\rangle}}. \tag{2.14}$$

From Eq. (2.14) we can see that the expectation value of the retarded Green's function is now finite even for $\sigma_0 = 0$, as long as $\langle\sigma_1^2\rangle \neq 0$. We can also see that the convergence of the above integral is conditioned to $\langle\sigma_1^2\rangle > 0$, otherwise its behaviour would be that of an upside-down Gaussian.

The light cone singularity has now been smeared out. We can interpret this smearing as a larger region (no longer a line) in which massless particles can travel through spacetime. This means that these particles can now have a speed that is a bit smaller or a bit greater than the usual light speed. In other words, by separating the quantized perturbation from a background Minkowski spacetime, we find that massless particles traveling through this background are now being boosted or slowed down by the metric fluctuations, that is, by the spin-2 particles arising from the quantized perturbation, the gravitons. Additionally, one could wonder whether this scenario results in some kind of causality violation. However, the graviton state defines a preferred reference frame, so this system is Lorentz invariant to begin with.

Note that the lightcone smearing is a consequence of the behaviour in Eq. (2.6) arising from the quantization of $h_{\mu\nu}$. In other words, it is mandatory for the gravitons to be in a nonclassical state, otherwise the retarded Green's function would still contain a δ-function divergence on the light cone even in the presence of the perturbation.

Now, imagine a light source emitting evenly spaced pulses. An observer at a distance r from the source will detect a variation of the order of Δt between their arrivals. The geodesic separation between the emission of the photon by the source and its arrival at the observer after traveling a distance r in a time $r + \Delta t$ is given by

$$2\sigma = (r + \Delta t)^2 - r^2 = \Delta t^2 + 2r\Delta t \quad \rightarrow \quad \sigma \approx r\Delta t, \quad \Delta t \ll r,$$

which means that the time delay or advance on the photons flight time can be taken to be on the order of

$$\Delta t \approx \frac{\sqrt{\langle\sigma_1^2\rangle}}{r}. \tag{2.15}$$

This means that spectral lines can be both shortened or broadened as they travel through the perturbation. As the lightcone fluctuates, so will the speed of the photons, therefore, Δt should be thought of as a mean deviation on the flight time of photons traveling through a perturbation instead of an actual change in the flight time of each photon. A more detailed calculation on the influence of $\langle\sigma_1^2\rangle$ in the mean deviation on a photon's flight time can be found in Ref. [1].

2.2 Higher order corrections to the metric

We found the expected value of the retarded Green function in Eq. (2.14) to contain exponential terms of $\langle \sigma_1^2 \rangle$, which is of second order on $h_{\mu\nu}$. Therefore, one could also wonder if higher order terms on the perturbation could affect these results in a non-negligible way. Let us take into account the second order correction term to the metric and see what it tells us. The geodesic separation is now terminated at σ_2, i.e.,

$$\sigma = \sigma_0 + \sigma_1 + \sigma_2. \tag{2.16}$$

We will decompose σ_1 into its positive and negative frequency parts and apply the CBH formula. As for σ_2, we will decompose it into its normal ordering part with respect to the vacuum state $|\psi\rangle$, and the part which actually contributes to the vacuum expectation value. The normal ordering, or Wick ordering part of an operator is its portion in which all creation operators are to the left and all annihilation operators to the right and, for an operator A, it is denoted by $: A :$. This can be understood by looking at the last term on the first line of Eq. (2.11) in which we discarded all terms resulting in a null expectation value. The sum of all discarded terms are exactly $: \sigma_1^2 :$. By doing this for σ_2 we obtain

$$\sigma_2 = : \sigma_2 : + \langle \sigma_2 \rangle, \tag{2.17}$$

which results in

$$e^{is:\sigma_2:} |\psi\rangle \approx (1 + is : \sigma_2 :) |\psi\rangle = |\psi\rangle .$$

This way, the equivalent of Eq. (2.13) by taking terms of second order of $h_{\mu\nu}$ will be

$$\begin{aligned} \langle G_{ret}(x, x') \rangle &= \frac{\theta(t - t')}{8\pi^2} \int_{-\infty}^{\infty} ds\, e^{is\sigma_0 - \frac{1}{2}s^2 \langle \sigma_1^2 \rangle + is\langle \sigma_2 \rangle} \\ &= \frac{\theta(t - t')}{8\pi^2} \sqrt{\frac{2\pi}{\langle \sigma_1^2 \rangle}}\, e^{-\frac{\sigma_0^2 + \langle \sigma_2 \rangle}{2\langle \sigma_1^2 \rangle}} . \end{aligned} \tag{2.18}$$

In the last step we eliminated terms of third and higher order in the perturbation that resulted from the integration. Notice that this result shows us that second order terms stemming from σ_2 will not change the form of the expected value of the retarded Green's function, they will only relocate the peak of the Gaussian.

Note that we adopted the normal ordering in Eq. (2.17) instead of using the CBH formula for σ_2 just for simplicity. We did not explicitly write σ_2 because it will not be a relevant quantity for our future calculations. However, if we did write it and used the CBH formula, we would still find that terms of second order in $h_{\mu\nu}$ would be accompanied by a factor of s in the integral, while only terms of higher order (the commutation terms of the CBH formula) would be accompanied by s^2. The latter would still be discarded after integration for being negligible when compared to the second order term in Eq. (2.14) and the result would be the same.

Additionally, although it could seem arbitrary to keep high order terms from the exponential but not from σ, the result from Eq. (2.18) already proves that higher terms will not change the form of our results.

2.3 The Hadamard function

In the previous section we chose the retarded Green function to illustrate the lightcone smearing, however, we can show that the presence of the metric fluctuations will also remove the singularity of other two-point functions that can be expressed in terms of vacuum expectation values. Among them, the Hadamard function will prove to be a valuable resource in our search for observational aspects of lightcone fluctuations. The Hadamard function for a scalar field ϕ in flat spacetime can be obtained from [36]

$$G_1(x-x') = \langle \phi(x)\phi(x') + \phi(x')\phi(x) \rangle. \tag{2.19}$$

For the massless case in flat spacetime the free field solution can be written as

$$\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3k}{(2\omega_k)^{\frac{1}{2}}} \left(a_k e^{-i\omega_k t + i\mathbf{k}\cdot\mathbf{x}} + a_k^\dagger e^{i\omega_k t - i\mathbf{k}\cdot\mathbf{x}} \right), \tag{2.20}$$

from which the Hadamard function reads [36]

$$G_1(x,x') = -\frac{1}{4\pi^2\sigma}. \tag{2.21}$$

Both the expression for the Hadamard function in Eq. (2.21) and the expression for the retarded Green function in Eq. (2.3) have the same form than those in unperturbed spacetime. This was happened because the free field solutions carry an exponential of the internal product $g_{\mu\nu}p^{\mu}x^{\nu} = (\eta_{\mu\nu} + h_{\mu\nu})p^{\mu}x^{\nu}$, resulting in both terms of $\sigma = \sigma_0 + \sigma_1$, while the measure of the integral can be taken to be $\sqrt{g} \approx 1$. Therefore, as it happened in the previous section, $\sigma = \sigma_0$ in an unperturbed scenario, and $\sigma = \sigma_0 + \sigma_1$ in the presence of metric perturbations. Since we found that higher-order terms on the perturbation will not produce relevant terms on the two-point function, we will no longer discuss them. Furthermore, the expression above also describes the asymptotic behavior of the Hadamard function for the massive case near the light cone. Let us set to find out how the lightcone divergence is removed in the presence of a quantized perturbation. We can use the identity

$$\int_0^{\infty} ds e^{\pm isx} = \pm\frac{i}{x} + \pi\delta(x), \tag{2.22}$$

to write

$$\frac{1}{\sigma_0 + \sigma_1} = -\frac{i}{2}\int_0^{\infty} ds \left[e^{is(\sigma_0+\sigma_1)} - e^{-is(\sigma_0+\sigma_1)}\right] \tag{2.23}$$

Combining the identity we found in Eq. (2.12) and Euler's formula, the expected value of the expression above can be written as

$$\left\langle \frac{1}{\sigma_0 + \sigma_1} \right\rangle = \int_0^{\infty} ds \sin(\sigma_0 s) e^{-\frac{1}{2}s^2\langle\sigma_1^2\rangle}, \tag{2.24}$$

Consequently, the expectation value of the Hadamard function becomes

$$\langle G_1(x, x')\rangle = -\frac{1}{4\pi^2}\int_0^{\infty} ds \sin(\sigma_0 s) e^{-\frac{1}{2}s^2\langle\sigma_1^2\rangle}, \tag{2.25}$$

for which we can see that once again $\langle\sigma_1^2\rangle > 0$. Let us now take a look the asymptotic behaviors of the Hadamard function. The Dawson function $F_D(x)$ reads [38]

$$F_D(x) = \frac{1}{2}\int_0^{\infty} e^{-t^2/4}\sin(xt)dt, \tag{2.26}$$

and can be expanded near the origin as

$$F_D(x) = \sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{(2n-1)!!}x^{2n+1} = x - \frac{2x^3}{3} + \frac{4x^5}{15} - \frac{8x^7}{105} + \frac{16x^9}{945}\cdots, \tag{2.27}$$

and for large arguments as

$$F_D(x) = \frac{1}{2x} + \frac{1}{4x^3} + \frac{3}{8x^5} + \frac{15}{16x^7} + \cdots . \tag{2.28}$$

Performing the change of coordinates

$$t = s\sqrt{2\langle\sigma_1^2\rangle}, \quad x = \frac{\sigma_0}{\sqrt{2\langle\sigma_1^2\rangle}}$$

in Eq. (2.25), we can write it in terms of the Dawson function, namely

$$\langle G_1(x,x')\rangle = -\frac{1}{4\pi^2}\sqrt{\frac{2}{\langle\sigma_1^2\rangle}}\, F_D\left(\frac{\sigma_0}{\sqrt{2\langle\sigma_1^2\rangle}}\right). \tag{2.29}$$

Note that by performing this change of coordinates, we are once again making the assumption that $\langle\sigma_1^2\rangle > 0$. From (2.27), the behavior near the origin, $\sigma_0 \to 0$, in first order of $h_{\mu\nu}$ for Eq. (2.29) will be given by

$$\langle G_1(x,x')\rangle = -\frac{\sigma_0}{4\pi^2\langle\sigma_1^2\rangle}, \qquad \sigma_0 \to 0, \tag{2.30}$$

from which we can see that the light cone divergence is also removed as long as $\langle\sigma_1^2\rangle \neq 0$, which we can see to be generally the case for non-coincident points from Eq. (2.11). For large distances Eq. (2.28) becomes

$$\langle G_1(x,x')\rangle = -\frac{1}{4\pi^2\sigma_0}, \quad \sigma_0 \gg \langle\sigma_1^2\rangle, \tag{2.31}$$

which is the classical form of Eq. (2.21). Alternatively, we could have performed an integration by parts on the r.h.s. of Eq. (2.25) by making the substitutions

$$\begin{aligned} u = e^{\frac{1}{2}s^2\langle\sigma_1^2\rangle} \quad &\rightarrow \quad du = s\langle\sigma_1^2\rangle e^{-\frac{s^2\langle\sigma_1^2\rangle}{2}} \\ dv = \sin(\sigma_0 s)ds \quad &\rightarrow \quad v = -\frac{1}{\sigma_0}\cos(\sigma_0 s), \end{aligned}$$

to obtain

$$\langle G_1(x,x')\rangle = -\frac{1}{4\pi^2}\left[\frac{1}{\sigma_0}\cos(\sigma_0 s)e^{-\frac{s^2\langle\sigma_1^2\rangle}{2}}\bigg|_0^\infty - \int_0^\infty \frac{s\langle\sigma_1^2\rangle}{\sigma_0}\cos(\sigma_0 s)e^{-\frac{s^2\langle\sigma_1^2\rangle}{2}}ds\right]$$
$$= -\frac{1}{4\pi^2\sigma_0}\left[1-\langle\sigma_1^2\rangle\int_0^\infty ds\, s\cos(\sigma_0 s)e^{-\frac{s^2\langle\sigma_1^2\rangle}{2}}\right].$$

By making an additional substitution $\sigma_0 s = t$ we are left with

$$\langle G_1(x,x')\rangle = -\frac{1}{4\pi^2\sigma_0}\left[1-\frac{\langle\sigma_1^2\rangle}{\sigma_0^2}\int_0^\infty dt\, t\cos t\, e^{-\frac{t^2\langle\sigma_1^2\rangle}{2\sigma_0^2}}\right], \tag{2.32}$$

which for $\sigma_0^2 \gg \langle\sigma_1^2\rangle$ also reduces to Eq. (2.31).

As we have mentioned before, in the process of performing the integral in (2.25) and in all subsequent steps we have assumed that $\langle\sigma_1^2\rangle > 0$. However, it is possible to construct an expression that will lead us to the asymptotic behaviors for negative values of $\langle\sigma_1^2\rangle$. Using the integral representation for the inverse function with a real exponential argument

$$\int_0^\infty ds\, e^{-sx} = \frac{1}{x} \tag{2.33}$$

combined with Eq. (2.12), we can write the Hadamard function for a massless scalar field with $\langle\sigma_1^2\rangle < 0$, namely

$$\langle G_1(x,x')\rangle = -\frac{1}{4\pi^2}\left\langle\frac{1}{\sigma_0+\sigma_1}\right\rangle = -\frac{1}{4\pi^2}\int_0^\infty ds e^{-s\sigma_0+\frac{1}{2}s^2\langle\sigma_1^2\rangle}. \tag{2.34}$$

Just as we did for positive values of $\langle\sigma_1^2\rangle$, we can see that the integral in Eq. (2.34) is reduced to a simple Gaussian near the origin, leaving us with

$$\langle G_1(x,x')\rangle = -\frac{1}{4\pi^2}\sqrt{\frac{\pi}{2|\langle\sigma_1^2\rangle|}}, \quad \sigma_0 \to 0. \tag{2.35}$$

Additionally, we can also perform integration by parts in Eq. (2.34) to obtain

$$\langle G_1(x,x')\rangle = -\frac{1}{4\pi^2\sigma_0}\left[1+\frac{\langle\sigma_1^2\rangle}{\sigma_0}\int_0^\infty dt\, t\, e^{-t+\frac{t^2\langle\sigma_1^2\rangle}{2\sigma_0^2}}\right], \tag{2.36}$$

which for $\sigma_0^2 \gg |\langle\sigma_1^2\rangle|$ is also reduced to Eq. (2.31). Note that Eqs. (2.35) and (2.36) could also be obtained by writing the result of the integral in Eq. (2.34) in terms of Dawson functions by extending them to the complex plane via error functions and applying the expansions on Eqs. (2.27) and (2.28), however, that would be a needlessly extensive work.

Chapter 3

Gravitons and the form of $\langle\sigma_1^2\rangle$ in the TT gauge

In this Chapter we will lay down the ground work for our calculations of $\langle\sigma_1^2\rangle$. We will find an expression for $\langle\sigma_1^2\rangle$ between the emission and detection of a photon in terms of the graviton Hadamard function in the TT gauge and by writing the perturbation as plane-wave expansion, we will also find an expression for the sum over the graviton polarization modes in $(d+1)$ dimensions. This review will be mostly based on Refs. [1,9].

3.1 $\langle\sigma_1^2\rangle$ in terms of the graviton Hadamard function

We are now interested in obtaining an expression for $\langle\sigma_1^2\rangle$. Notice that, even though we have mentioned a flat $(3+1)$-dimensional spacetime to illustrate some concepts so far, there was no actual need to particularize our results to a specific spacetime dimensionality. Since we are only interested in gravitational radiation, we will adopt the TT gauge defined by the constraints in Eq. (1.50). Consequently, $h_{\mu\nu}$ takes the form of a purely spatial tensor and for a light-like interval we have

$$0 = dt^2 - d\mathbf{x}^2 + h_{ij}dx^i dx^j \quad \rightarrow \quad dt = \sqrt{d\mathbf{x}^2 - h_{ij}dx^i dx^j}, \tag{3.1}$$

with $i, j = 1, 2, ..., d$. Eq. (3.1) allows us to write

$$\frac{dt}{|d\mathbf{x}|} = \sqrt{\frac{d\mathbf{x}^2}{|d\mathbf{x}|^2} - h_{ij}\frac{dx^i}{|d\mathbf{x}|}\frac{dx^j}{|d\mathbf{x}|}} \quad \rightarrow \quad dt = \sqrt{1 - h_{ij}n^i n^j}\, dr \approx \left(1 - \frac{1}{2}h_{ij}n^i n^j\right) dr, \tag{3.2}$$

where $dr = |d\mathbf{x}|$, and since dx^i satisfies (3.1), $n^i = dx^i/dr$ is the unit vector pointing at the spatial direction of the geodesic. From Eq. (3.2) we find that the time interval Δt a null ray will take to travel a distance $\Delta r = r_1 - r_0$ in the presence of a perturbation is given by

$$\Delta t = \Delta r - \frac{1}{2}\int_{r_0}^{r_1} h_{ij}n^i n^j \, dr. \tag{3.3}$$

If we think of this as something with the form of $\Delta t^2 - \Delta l^2 = 0$, then the r.h.s. of the expression above can be interpreted as the proper distance between the initial and final points of the null ray trajectory. We started from the light-like interval in (3.1), however, when considering a pair of arbitrary points not necessarily null separated, the square of the geodesic separation between them can be written as

$$2\sigma = (\Delta t)^2 - (\Delta l)^2 = (\Delta t)^2 - \left(\Delta r - \frac{1}{2}\int_{r_0}^{r_1} h_{ij}n^i n^j \, dr\right)^2,$$

which up to first order in $h_{\mu\nu}$ reduces to

$$2\sigma = (\Delta t)^2 - (\Delta r)^2 + \Delta r \int_{r_0}^{r_1} h_{ij}n^i n^j \, dr. \tag{3.4}$$

As $\sigma = \sigma_0 + \sigma_1$, we find that

$$\sigma_1 = \frac{1}{2}\Delta r \int_{r_0}^{r_1} h_{ij}n^i n^j \, dr. \tag{3.5}$$

Finally, we can obtain an expression for $\langle\sigma_1^2\rangle$ by averaging this result over the quantized metric perturbation h_{ij}. That is

$$\langle\sigma_1^2\rangle = \frac{1}{4}(r_1 - r_0)^2 \int_{r_0}^{r_1} dr \int_{r_0}^{r_1} dr' \, n^i n^j n^k n^l \langle h_{ij}(x) h_{kl}(x')\rangle, \tag{3.6}$$

where $\langle h_{ij}(x) h_{kl}(x')\rangle$ is the graviton two-point function, $dr = |d\mathbf{x}|$ is the spatial interval of the trajectory of a null ray from a point r_0 to a point r_1 and $n^i = dx^i/dr$ is an unitary vector pointing to the direction of the geodesic. As we mentioned before and will see explicitly in Chapter 4, direct calculations of the graviton two-point function will contain divergences over the unperturbed lightcone. However, the two-point function can be renormalized by subtracting the contribution of the Minkowski spacetime in the absence of the quantized perturbation so it is null when the quantum state of the graviton is the vacuum state and it is finite when evaluated on the lightcone.

Furthermore, it is straightforward to show that Eq. (3.6) can be also written in terms of the graviton Hadamard function as

$$\langle \sigma_1^2 \rangle = \frac{1}{8}(r_1 - r_0)^2 \int_{r_0}^{r_1} dr \int_{r_0}^{r_1} dr' \, n^i n^j n^k n^l \, G_{ijkl}(x, x'), \tag{3.7}$$

where

$$G_{ijkl}(x, x') = \langle h_{ij}(x) h_{kl}(x') + h_{ij}(x') h_{kl}(x) \rangle. \tag{3.8}$$

As we have previously mentioned, these expressions for $\langle \sigma_1^2 \rangle$ are valid for any non necessarily flat $(d+1)$-dimensional spacetime. However, from Eq. (1.51) we can see that the calculation of the two-point function will include the sum over momenta and polarization modes and, from this point on, the calculations will require certain particularisation.

3.2 Graviton two-point function in flat $(d+1)$-dimensional spacetime

Let us further develop Eq. (3.7) as we start particularize the graviton solution for our purposes. Since the quantized perturbation obeys a Klein-Gordon like equation, i.e. Eq. (1.51), it can be written in terms of a plane wave expansion, namely

$$h_{\mu\nu} = \sum_{\mathbf{k},\lambda} \left[a_{\mathbf{k},\lambda} e_{\mu\nu}(\mathbf{k},\lambda) f_{\mathbf{k}}(x) + a^\dagger_{\mathbf{k},\lambda} e_{\mu\nu}(\mathbf{k},\lambda) f^*_{\mathbf{k}}(x) \right], \tag{3.9}$$

where $a_{\mathbf{k},\lambda}$ and its Hermitian conjugate are the creation and annihilation operators associated with the behavior described in Eq. (2.4), λ labels the polarization states, $e_{\mu\nu}(\mathbf{k},\lambda)$ is the polarization tensor, $\mathbf{k}$ is the d-dimensional wave vector in Cartesian coordinates, and

$$f_{\mathbf{k}}(x) = A(2\omega)^{-\frac{1}{2}} e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}} = (2\omega)^{-\frac{1}{2}} e^{-i\omega t} \phi_{\mathbf{k}}(\mathbf{x}) \tag{3.10}$$

and its Hermitian conjugate are solutions of the Klein-Gordon equation, with A being some normalization constant. Substitution of Eq. (3.9) in Eq. (3.8) along with Eq. (2.4) yields

$$G_{ijkl}(x, x') = 2\mathrm{Re} \sum_{\mathbf{k},\lambda} e_{ij}(\mathbf{k},\lambda) e_{kl}(\mathbf{k},\lambda) f_{\mathbf{k}}(x) f^*_{\mathbf{k}}(x'). \tag{3.11}$$

As we shall further discuss in the next section, the sum over the polarization modes in $(d+1)$ is given by [17]

$$\begin{aligned}\sum_\lambda e_{ij}(\mathbf{k},\lambda)e_{kl}(\mathbf{k},\lambda) = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{d-1}\delta_{ij}\delta_{kl} \\ + \frac{2(d-2)}{d-1}\hat{k}_i\hat{k}_j\hat{k}_k\hat{k}_l + \frac{2}{d-1}(\hat{k}_i\hat{k}_j\delta_k l + \hat{k}_k\hat{k}_l\delta_{ij}) \\ - \hat{k}_i\hat{k}_l\delta_{jk} - \hat{k}_i\hat{k}_k\delta_{jl} - \hat{k}_j\hat{k}_l\delta_{ik} - \hat{k}_j\hat{k}_k\delta_{il},\end{aligned} \tag{3.12}$$

where $\hat{k}_i = k_i/\mathbf{k}$. From this definition, note that when substituted back into Eq. (3.11), the product of two quantum operators on the form of $\hat{k}_i\hat{k}_k$ can actually be written as the operator $\partial_i\partial'_j\nabla^{-2}$ acting on the spatial part of the graviton two-point function. One should pay special attention to the indexes of such quantum operators in order to know if the partial derivative acts on $\phi_\mathbf{k}(\mathbf{x})$ or on $\phi^*_\mathbf{k}(\mathbf{x}')$. Substitution of the sum in (3.12) back in Eq. (3.11) yields

$$\begin{aligned}G_{ijkl}(x,x') = 2\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{d-1}\delta_{ij}\delta_{kl}\right)D(x,x') + \frac{4(d-2)}{d-1}H_{ijkl}(x,x') \\ + \frac{4}{d-1}\left[\delta_{kl}F_{ij}(x,x') + \delta_{ij}F_{kl}(x,x')\right] \\ - 2\left[\delta_{jl}F_{ik}(x,x') + \delta_{jk}F_{il}(x,x') + \delta_{ik}F_{jl}(x,x') + \delta_{il}F_{jk}(x,x')\right],\end{aligned} \tag{3.13}$$

where $D(x,x')$ is the real part of the Wightman function for $f_\mathbf{k}(x)$. From Eq. (3.10) we can write $F_{ij}(x,x')$ and $H_{ijkl}(x,x')$ as

$$F_{ij}(x,x') = \text{Re}\,\partial_i\partial'_j\sum_\mathbf{k}\frac{e^{-i\omega\Delta t}}{2\omega^3}\phi_\mathbf{k}(\mathbf{x})\phi^*_\mathbf{k}(\mathbf{x}'), \tag{3.14}$$

and,

$$H_{ijkl}(x,x') = \text{Re}\,\partial_i\partial'_j\partial_k\partial'_l\sum_\mathbf{k}\frac{e^{-i\omega\Delta t}}{2\omega^5}\phi_\mathbf{k}(\mathbf{x})\phi^*_\mathbf{k}(\mathbf{x}'). \tag{3.15}$$

3.3 The sum over the polarization modes in $(3+1)$ and in $(d+1)$-dimensional flat spacetimes

Let us find an expression for the sum over the polarization modes in $(3+1)$D and then extend it for other spacetime dimensionalities. Let us consider the orthonormal 3-vectors

$\mathbf{e}_1(\mathbf{k})$, $\mathbf{e}_2(\mathbf{k})$ and $\mathbf{e}_3(\mathbf{k})$. As in the TT gauge the polarization tensor is perpendicular to the propagation direction, we will chose the latter to be an unitary vector pointing to the direction of the wave vector, i. e., $\mathbf{e}_3(\mathbf{k}) = \frac{\mathbf{k}}{k} = \hat{k}$. Consequently, our triad will obey

$$\begin{aligned} &\mathbf{e}_a(\mathbf{k}) \cdot \mathbf{e}_b(\mathbf{k}) = \delta_{ab} \quad \rightarrow \quad e_a^i(\mathbf{k}) e_b^i(\mathbf{k}) = \delta_{ab}, \qquad a, b = 1, 2, 3 \\ &e_a^i(\mathbf{k}) \cdot e_a^j(\mathbf{k}) = e_1^i e_1^j + e_2^i e_2^j + k^i k^j = \delta_{ij} \qquad i, j = x, y, z. \end{aligned} \tag{3.16}$$

In Chapter 1 we found a way to describe the polarization tensor for the four-dimensional case in Eq. (1.58) in terms of two independent polarization modes, the plus polarization and the cross polarization. We can express these modes as [7]

$$\begin{aligned} e^{ij}(\mathbf{k}, +) &= e_1^i \otimes e_1^j - e_2^i \otimes e_2^j \\ e^{ij}(\mathbf{k}, \times) &= e_1^i \otimes e_2^j + e_2^i \otimes e_1^j, \end{aligned} \tag{3.17}$$

where the symbol $\otimes$ indicates an inner product. Explicitly writing the summation over these polarization modes yields

$$\begin{aligned} \sum_\lambda e_{ij}(\vec{k}, \lambda) e_{kl}(\vec{k}, \lambda) &= e_{ij}(\mathbf{k}, +) e_{kl}(\mathbf{k}, +) + e_{ij}(\mathbf{k}, \times) e_{kl}(\mathbf{k}, \times) \\ &= e_1^i e_1^j e_1^k e_1^l - e_1^i e_1^j e_2^k e_2^l - e_2^i e_2^j e_1^k e_1^l + e_2^i e_2^j e_2^k e_2^l \\ &\quad + e_1^i e_2^j e_1^k e_2^l + e_1^i e_2^j e_2^k e_1^l + e_2^i e_1^j e_1^k e_2^l + e_2^i e_1^j e_2^k e_1^l \\ &= (e_1^i e_1^k + e_2^i e_2^k)(e_1^j e_1^l + e^j 2 e_2^l) + (e_1^i e_1^l + e_2^i e_2^l)(e_1^j e_1^k + e_2^j e_2^k) \\ &\quad - e_1^i e_1^j e_1^k e_1^l - e_2^i e_2^j e_2^k e_2^l - e_1^i e_1^j e_2^k e_2^l - e_2^i e_2^j e_1^k e_1^l \\ &= (e_1^i e_1^k + e_2^i e_2^k)(e_1^j e_1^l + e^j 2 e_2^l) + (e_1^i e_1^l + e_2^i e_2^l)(e_1^j e_1^k + e_2^j e_2^k) \\ &\quad - (e_1^i e_1^j + e_2^i e_2^j)(e_1^k e_1^l + e_2^k e_2^l) \\ &= (\delta_{ik} - k^i k^k)(\delta_{jl} - k^j k^l) + (\delta_{il} - k^i k^l)(\delta_{jk} - k^j k^k) \\ &\quad - (\delta_{ij} - k^i k^j)(\delta_{kl} - k^k k^l) \\ &= \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl} + k^i k^k k^j k^l + k^i k^l k^j k^k - k^i k^j k^k k^l \\ &\quad - \hat{k}^i \hat{k}^k \delta_{jl} - \hat{k}^j \hat{k}^l \delta_{ik} - \hat{k}^i \hat{k}^l \delta_{jk} - \hat{k}^j \hat{k}^k \delta_{il} + \hat{k}^i \hat{k}^j \delta_{kl} + \hat{k}^k \hat{k}^l \delta_{ij} \\ &= \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl} + k^i k^k k^j k^l \\ &\quad - \hat{k}^i \hat{k}^k \delta_{jl} - \hat{k}^j \hat{k}^l \delta_{ik} - \hat{k}^i \hat{k}^l \delta_{jk} - \hat{k}^j \hat{k}^k \delta_{il} + \hat{k}^i \hat{k}^j \delta_{kl} + \hat{k}^k \hat{k}^l \delta_{ij}, \end{aligned} \tag{3.18}$$

which is valid only in a $(3+1)$-dimensional spacetime, since it only includes the two independent polarization modes found in such scenario. Let us now see how we can expand this result for a $(d+1)$-dimensional spacetime. This sum over the polarization modes can also be denoted as a tensor

$$T_{ijkl}(\mathbf{k}) \equiv \sum_{\lambda} e_{ij}(\vec{k},\lambda) e_{kl}(\vec{k},\lambda). \tag{3.19}$$

Note from Eq. (3.16) that for any spacetime dimensionality, this sum will still be written in specific combinations of Kronecker deltas and unit wave vectors. Therefore, we can write a general expression for $T^{ijkl}(\mathbf{k})$ as the one given in Eq. (3.18) with undefined coefficients for each term. Furthermore, the polarization tensor must obey the TT gauge conditions in Eq. (1.50). For the new tensor T_{ijkl} they read

$$\begin{aligned} &k_i T^{ijkl} = k_j T^{ijkl} = k_k T^{ijkl} = k_k T^{ijkl} = 0 \quad \text{(transverse)} \\ &T^{iikl} = T^{ijkk} = 0 \quad \text{(traceless)}, \end{aligned} \tag{3.20}$$

and since $h_{\mu\nu}$ is a symmetric tensor, T_{ijkl} must also obey

$$T^{ijkl} = T^{jikl} = T^{ijlk} = T^{klij}. \tag{3.21}$$

It can be shown that

$$\begin{aligned} T_{ijkl} = &\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{d-1}\delta_{ij}\delta_{kl} + \frac{2(d-2)}{d-1}\hat{k}_i\hat{k}_j\hat{k}_k\hat{k}_l + \frac{2}{d-1}(\hat{k}_i\hat{k}_j\delta_k l + \hat{k}_k\hat{k}_l\delta_{ij}) \\ &- \hat{k}_i\hat{k}_l\delta_{jk} - \hat{k}_i\hat{k}_k\delta_{jl} - \hat{k}_j\hat{k}_l\delta_{ik} - \hat{k}_j\hat{k}_k\delta_{il}, \end{aligned} \tag{3.22}$$

being d the number of spatial dimensions, satisfies Eqs. (3.20) and (3.21). Note that in higher-dimensional spacetimes, the number of independent polarization modes also grows. We could, for example, have followed the same steps we went through in Chapter 1 in order to explicitly write the polarization tensor in $(4+1)$. We would then go through its degrees of freedom using the TT gauge and symmetry properties while searching for the equivalent of (3.17) in 5D. Finally, we would explicitly perform the sum over the polarization modes just as we did in Eq. (3.18). However, for higher-dimensional spacetimes this procedure becomes impractical.

Chapter 4

Gravitons in a five-dimensional Kaluza-Klein model with quasiperiodically compactified extra dimension

In this chapter we will make a quick discussion on the transition to higher dimensional models and the search for extra dimensions. This discussion will be mostly based on Ref. [24]. Later, we will set about computing the mean deviation on a photon flight time for a five-dimensional quasiperiodically compactified spacetime. Finally, we will discuss some observational aspects of these results and use them to estimate the size of the extra dimension having the Near-Infrared Spectrograph (NIRSpec) aboard the James Webb Space Telescope (JWST) as a model of instrument for possible detections. Except for the first section, this chapter consists of new results obtained during the course of this Master's program, which led to the work in Ref. [27] accepted for publication in the Journal of High Energy Physics (JHEP). The case $\alpha = 0$ for the quasiperiodic boundary condition in 5D was already discussed in Ref. [1], however, we revisit it under the possibility that it results in an observable shift on the flight time of photons and then move on to discuss the other condition cases for α.

4.1 Higher-dimensional models and the search for extra dimensions

It has now been over a century since we first sought a world beyond our 4 spacetime dimensions in order to address some unresolved aspects of our current models. In the years following the formulation of General Relativity many attempts have been made to unify the electromagnetic theory with gravity [39–42]. Among these, in 1921, Theodor Kaluza proposed a five-dimensional world in an attempt to incorporate the electromagnetic tensor into the metric tensor [39] and, five years later Oskar Klein revisited Kaluza's work under the scope of the emergent quantum mechanics [40]. As Kaluza's work is regarded to be the first attempt to grasp a world beyond our usual $(3+1)$ framework, scenarios involving $(d+1)$ dimensions with small unobservable extra dimensions with quantized fields are often regarded as a Kaluza-Klein model, whereas the $d=4$ case is regarded as the Kaluza-Klein original model. From now on we will now work under the assumption that we live in such five-dimensional world. Let us go through this transition step-by-step.

In a first moment, it is important to note that the transition to a higher-dimensional model comes with modifications to some physical constants. One simple way to visualize this is to look at the Poisson's equation for the gravitational potential, namely

$$\nabla^2 V_g = 4\pi G\rho. \tag{4.1}$$

The left hand side (l.h.s.) of Eq. (4.1) has the same units for any spacetime dimensionality: the Laplace operator has units of L^{-2}, being L some unit of length, whereas the gravitational potential V_g has units of energy divided by mass. As a result, for any spacetime dimensionality, the r.h.s. must be such as to preserve these same units. However, note that the matter density ρ has, by definition, units of mass divided by L^d, with d being the number of spatial dimensions. Therefore, the gravitational constant must also suffer modifications based on the spacetime dimensionality. Consequently, from now on, we will refer to it as $G^{(D)}$, instead of simply G. It is straightforward to note that c and $\hbar$ are invariant under changes on spacetime dimensionality.

The well known four-dimensional Planck length ℓ_P, on the other hand, is a quantity constructed by combinations of $G^{(4)}$, c and $\hbar$ as

$$\ell_P = \left(\frac{G^{(4)}\hbar}{c^3}\right)^{\frac{1}{2}} . \tag{4.2}$$

Therefore, for our work on a five-dimensional model, we must look for a five-dimensional Planck length $\ell_P^{(5)}$ constructed from powers of $G^{(5)}$, c and $\hbar$. From our analysis of ρ in Eq. (4.1) we can see that, for every extra spatial dimension, $G^{(D)}$ becomes one order of magnitude higher in length and, therefore, $G^{(5)}$ must have units of length times the unit of $G^{(4)}$, i.e.

$$\left[G^{(D+1)}\right] = L\ \left[G^{(D)}\right] \quad \longrightarrow \quad \left[G^{(5)}\right] = L\ \left[G^{(4)}\right] . \tag{4.3}$$

We can use Eq. (4.1) or Eq. (4.2) to infer the units of $G^{(4)}$ and rewrite the equation above as

$$\left[G^{(5)}\right] = \frac{L^3\,[c]^3}{[\hbar]^3} . \tag{4.4}$$

Therefore, the five-dimensional Planck length must be constructed as

$$\ell_P^{(5)} = \left(\frac{G^{(5)}\hbar}{c^3}\right)^{\frac{1}{3}} . \tag{4.5}$$

Furthermore, from the recurrence formula on the l.h.s. of Eq. (4.3) we can extend this analysis for any D-dimensional spacetime as

$$\ell_P^{(D)} = \left(\frac{G^{(D)}\hbar}{c^3}\right)^{\frac{1}{D-2}} . \tag{4.6}$$

Now, since $G^{(D)}$ is modified by spacetime dimensionality, Eq. (4.1) tells us that we can expect gravity to work differently if we do live in a higher-dimensional world. However, as we only observe our usual four dimensions, we can postulate extra dimensions to be compact: for distances larger than the size of the extra dimension, we would see our world as effectively four-dimensional and we would only be able to detect discrepancies from the gravitational inverse-square law at distances smaller than the size of compactification. This raises the question of how to calculate $G^{(5)}$ if all we have detected so far was gravity working in an effective four-dimensional world. Additionally, although we have postulated a compact extra dimension, note that in some models is not the size of the extra dimension itself which is small, it can be infinite in length and postulated to be curled up to a finite volume [43].

By combining Eqs. (4.2) and (4.5) we can write

$$\ell_P^{(5)} = \left(\frac{\ell_P^2 G^{(5)}}{G^{(4)}} \right)^{\frac{1}{3}}. \tag{4.7}$$

Since $G^{(5)}$ and $G^{(4)}$ do not share the same units, it does not make sense to compare them directly. However, the opposite is true for Planck lengths, they can be directly compared. We are fully aware about the need for extra-dimensions to be compact, but does it have any influence on how we compare an effective four-dimensional world with an actual five-dimensional one?

We can play with the functional dependence in Eq. (4.1) as an attempt to answer this question. Let us consider a five-dimensional world with (x^1, x^2, x^3) being three free spatial dimensions and x^4 an fourth spatial dimension compactified to a radius R and a length $\ell_c = 2\pi R$. We begin by placing an uniform distribution with total mass M in a ring around this circumference at $x^1 = x^2 = x^3 = 0$. Note that $\rho^{(5)}(x)$ is null everywhere except for $x^1 = x^2 = x^3 = 0$ and

$$M = \int \rho^{(5)}(x)\, d^4\mathbf{x} = 2\pi R\, m, \tag{4.8}$$

being m the linear mass density. Therefore, we can use the Dirac delta function definition

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1 \tag{4.9}$$

to write the mass density ρ in five dimensions as

$$\rho^{(5)}(x) = m\delta(x^1)\delta(x^2)\delta(x^3). \tag{4.10}$$

This assumption is reasonable: Eq. (4.10) satisfies Eq. (4.8) and, from Eq. (4.9), is easy to see that each delta has units of inverse of length. Therefore, Eq. (4.10) also has units of mass/L^4, consistent with a five-dimensional world.

We are interested in the comparison between the actual five-dimensional world configuration with how it is seen in an effectively four-dimensional world. If we were not able to see the ring in the extra-dimension, then we would perceive this mass distribution as a point-like mass at $x^1 = x^2 = x^3 = 0$ in a four-dimensional world and we would describe it as

$$\rho^{(4)}(x) = M\delta(x^1)\delta(x^2)\delta(x^3). \tag{4.11}$$

Comparing Eqs. (4.11) and (4.10) and using Eq. (4.8) we can write

$$\rho^{(4)}(x) = 2\pi R\, \rho^{(5)}(x) = \ell_c\, \rho^{(5)}(x). \tag{4.12}$$

Substituting this result back into Eq. (4.1) in five dimensions yields

$$\nabla^2 V_g^{(5)} = 4\pi G^{(5)}\rho^{(5)}(x) = \frac{4\pi}{\ell_c} G^{(5)}\rho^{(4)}(x), \tag{4.13}$$

whereas, in four-dimensions, it yields

$$\nabla^2 V_g^{(4)} = 4\pi G^{(4)}\rho^{(4)}(x). \tag{4.14}$$

Note that neither $V_g^{(5)}$ nor $V_g^{(4)}$ have any dependence on x^4. Therefore, the Laplacian on both equations can be written as the four-dimensional one. Since both equations describe the same physical problem and have the same functional dependence, we can compare Eqs. (4.13) and (4.14) to write

$$G^{(5)} = \ell_c\, G^{(4)}, \tag{4.15}$$

or equivalently, from Eq. (4.7),

$$\ell_P^{(5)} = \left(\ell_c \ell_P^2\right)^{\frac{1}{3}}. \tag{4.16}$$

In other words, the effective four-dimensional Planck length we observe could be a consequence of a five-dimensional world with a fundamental length scale $\ell_P^{(5)}$ in which the extra dimension is curled up to a circumference ℓ_c. So the answer for our question is yes: there is a relationship between the size of compactification of a potential extra dimension and how gravity is modified in this higher-dimensional spacetime.

The search for extra dimensions pass through many physical phenomena. For example, we can use Eq. (4.1) to find the gravitational force law between two masses and test it for smaller distances. For distances greater than the size of the extra dimension the world is seen as four-dimensional and gravity is described by the inverse-square law. However, for distances smaller than ℓ_c this law must change following the dimensional analysis of

Eq. (4.1). However, this method is extremely difficult to test since the gravitational force is too weak and other forces must be precisely canceled out. In particular, by measuring harmonic torques exerted on a detector pendulum by a rotating attractor, it has been shown that the inverse-square law holds down to separations of about 52μm [28, 31].

On the other hand, in the Large Hadron Collider (LHC), one of the most significant indicators of the existence of extra dimensions would be the production of heavier particles than those of the Standard Model (SM) [29]. However, so far the LHC probed distances down to 10^{-18}cm without any indication of the existence of anything beyond our usual 4-dimensional spacetime [30]. This distance is associated with the energy scale in which experiments are conducted in the LHC, therefore, it should be understood as a measure of how close such experiments are to the Planck scale. If we consider the fundamental length to be around this value, that is, if we were to have something like $\ell_p^{(5)} \sim 10^{-19}$cm, then Eq. (4.16) tells us that the extra dimension should be thousands of kilometers long. If that was the case, we would already have found deviations for the inverse-square law. Therefore, if we do live in a five-dimensional world, the fundamental Planck scale is still a few orders of magnitude away from our grasp via LHC.

As we shall see throughout this chapter, the change in a photon flight time caused by metric fluctuations also provides a promising test for the existence of extra dimensions. The reason for this is because it works under the possibility that the fundamental Planck scale is many orders of magnitude smaller than the four-dimensional one. However, note that the extra dimension is not completely specified just by its size. There are more than one way to mathematically obtain a small extra dimension and not all of them will necessarily result in the same physical phenomena. In this chapter we will make the extra dimension compact via a quasiperiodic boundary condition, this could help us explore not only the size of the extra dimension, but to shed light in additional ways of understanding its structure.

4.2 Calculation of $\langle\sigma_1^2\rangle_R$ for a photon propagating along the z-direction

Although the decomposition of the line element describing a flat background in Eq. (1.18) remains valid, let us describe our five-dimensional compactified spacetime once more simply as we did in Eq. (2.1) with $\mu,\nu = 0,1,2,3,4$. The extra dimension becomes compact by imposing a quasiperiodic condition to the quantized perturbation $h_{\mu\nu}$ given by Eqs. (3.9) and (3.10). It reads

$$\phi_{\mathbf{k}}(x,y,z,w+\ell_c) = e^{i2\pi\alpha}\phi_{\mathbf{k}}(x,y,z,w), \tag{4.17}$$

where ℓ_c is the compactification length and α is a parameter which regulates the phase angle and varies within the range $0 \leq \alpha < 1$. Notice that for $\alpha = 0$, Eq. (4.17) reproduces the periodic case discussed in Ref. [1], whereas $\alpha = 1/2$ corresponds to the antiperiodic case.

We can further visualize what we meant by the possible structure of the extra dimension via quasiperiodic condition if we take $x = y = z = w = 0$ in Eq. (4.17) while considering the simplest case of a plane monochromatic wave. For $\alpha = 0$, this would mean that the intensity and the behaviour of the gravitational field would repeat themselves by traversing the whole circumference of the extra dimension. However we have no indications at all that this is true. We know that the extra dimension must be small, but we have no idea about how it is structurally built to be as such, nor we know how gravity would behave in it. This is why it is important to consider the most various scenarios when looking for possible ways to describing phenomena in extra dimensions.

The imposition of (4.17) to the free solution of the five-dimensional massless Klein-Gordon equation in flat spacetime will cause the momentum coordinate parallel to the direction of compactification to suffer discretization, and it will now be given by

$$k_w = \frac{2\pi}{\ell_c}(n+\alpha), \qquad n = 0, \pm1, \pm2, \dots\ . \tag{4.18}$$

By making use of the normalization condition for the Klein-Gordon equation

$$\int f_{\mathbf{k}}(x) f^*_{\mathbf{k}'}(x)\, d^5x = \frac{1}{2\omega}\delta^{(5)}_{\mathbf{k},\mathbf{k}'}, \tag{4.19}$$

where $\delta^{(5)}_{\mathbf{k},\mathbf{k}'}$ stands for a Dirac delta for continuous momenta coordinates and for a Kronecker delta for discrete momenta coordinates, one obtains the complete set of normalized spatial solutions under the quasiperiodic boundary condition, namely

$$\phi_{\mathbf{k}}(x) = \frac{e^{i\mathbf{k}_T\cdot\mathbf{r}_T + i\frac{2\pi(n+\alpha)}{\ell_c}w}}{\sqrt{(2\pi)^3\ell_c}}, \tag{4.20}$$

where $\mathbf{k}_T = (k_x, k_y, k_z)$ and $\mathbf{r}_T = (x, y, z)$ are the momenta and their respective spatial coordinate 3-vectors relative to the directions perpendicular to the compactified dimension. In this case, the eigenvalues equation reads

$$\omega^2 = |\mathbf{k}|^2 = k_x^2 + k_y^2 + k_z^2 + \left(\frac{2\pi}{\ell_c}\right)^2 (n+\alpha)^2, \tag{4.21}$$

and the sum over all momenta coordinate possible values in $D(x, x')$ and in Eqs. (4.25), and (4.26) for this scenario should be written as

$$\sum_{\mathbf{k}} = \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \int_{-\infty}^{\infty} dk_z \sum_{n=-\infty}^{\infty}. \tag{4.22}$$

We have determined the spacetime dimensionality and now we obtained the complete and normalized set of $\phi_{\mathbf{k}}(\mathbf{x})$. There is just one ingredient missing in order to compute the graviton two-point function in Eq. (3.13) and that is to determine the trajectory of the photon. We saw from Eq. (3.7) that the only contribution of the Hadamard function for $\langle\sigma_1^2\rangle$ will be the components parallel to the lightcone. What we seek is to estimate the modification on a photon's flight time caused by the compact extra dimension. Hence, let us choose our coordinate system such that the photon is propagating along an ordinary direction perpendicular to the compactification, let us say z. With $d = 4$ and $i = j = k = l = z$, Eq. (3.13) takes the form

$$G_{zzzz}(x, x') = \frac{8}{3}\left[D(x, x') - 2F_{zz}(x, x') + H_{zzzz}(x, x')\right], \tag{4.23}$$

where

$$D(x, x') = \mathrm{Re}\sum_{\mathbf{k}} \frac{e^{-i\omega\Delta t}}{2\omega}\phi_{\mathbf{k}}(\mathbf{x})\phi^*_{\mathbf{k}}(\mathbf{x}'), \tag{4.24}$$

and, defining $\Delta z = z - z'$, we can rewrite the derivatives on Eqs. (3.14) and (3.15) to obtain

$$F_{zz}(x, x') = -\mathrm{Re}\,\partial^2_{\Delta z}\sum_{\mathbf{k}} \frac{e^{-i\omega\Delta t}}{2\omega^3}\phi_{\mathbf{k}}(\mathbf{x})\phi^*_{\mathbf{k}}(\mathbf{x}'), \tag{4.25}$$

and,

$$H_{ijkl}(x,x') = \text{Re}\,\partial^4_{\Delta z}\sum_{\mathbf{k}}\frac{e^{-i\omega\Delta t}}{2\omega^5}\phi_{\mathbf{k}}(\mathbf{x})\phi^*_{\mathbf{k}}(\mathbf{x}'). \tag{4.26}$$

Now we have all the elements needed so we can set about performing the sums over $\mathbf{k}$ for $D(x,x')$, $F_{zz}(x,x')$ and $H_{ijkl}(x,x')$. As we shall see, this calculation will be smoother if we start from $D(x,x')$ and work our way up to the functions with higher order on the derivatives. Substituting Eq. (4.20) in Eq. (4.24) and explicitly writing the sum in Eq. (4.22) yields

$$D(x,x') = \text{Re}\,\frac{1}{(2\pi)^3\ell_c}\sum_{n=-\infty}^{\infty}\int d^3\mathbf{k}_T\,\frac{e^{-i\omega\Delta t}}{2\omega}e^{i\mathbf{k}_T\cdot\Delta\mathbf{r}_T+i\frac{2\pi}{\ell_c}(n+\alpha)\Delta w}, \tag{4.27}$$

where $\Delta x^i_T = x^i - x'^i$ for $i=1,2,3$, and $\Delta w = w - w'$. We can use the integral form [19]

$$\frac{e^{-i\omega\Delta t}}{2\omega} = \frac{1}{\sqrt{\pi}}\int_0^\infty ds\; e^{-\omega^2 s^2+\frac{\Delta t^2}{4s^2}}, \tag{4.28}$$

to rewrite the terms involving ω on the r.h.s. of Eq. (4.27). By doing this and substituting ω^2 given by Eq. (4.21), we are able to perform the Gaussian integrals resulting from the components of $\mathbf{k}_T$. Our resulting expression for $D(x,x')$ is

$$D(x,x') = \;\text{Re}\,\frac{1}{8\pi^2\ell_c}\sum_{n=-\infty}^{\infty}\int_0^\infty\frac{ds}{s^3}e^{-\frac{\Delta\mathbf{x}_T^2-\Delta t^2}{4s^2}-\left(\frac{2\pi s}{\ell_c}\right)^2(n+\alpha)^2+\frac{i2\pi(n+\alpha)}{\ell_c}\Delta w}. \tag{4.29}$$

Note that the integral on the r.h.s. of Eq. (4.29) over the parameter s now has taken the form of a Bessel function, however, the resulting expression would not be very friendly towards the remaining sum over n. Hence, let us write this sum as

$$\sum_{n=-\infty}^{\infty}e^{-\left(\frac{2\pi s}{\ell_c}\right)^2(n+\alpha)^2+\frac{i2\pi(n+\alpha)}{\ell_c}\Delta w} = e^{-\left(\frac{2\pi\alpha s}{\ell_c}\right)^2+\frac{i2\pi\alpha\Delta w}{\ell_c}}\vartheta\left(i\alpha\frac{4\pi s^2}{\ell_c^2}+\frac{\Delta w}{\ell_c}, i\frac{4\pi s^2}{\ell_c^2}\right), \tag{4.30}$$

where $\vartheta(u,\mu)$ is the Jacobi Theta function, defined as

$$\vartheta(u,\mu) = \sum_{n=-\infty}^{\infty}e^{i\pi n^2\mu+i2\pi nu}, \tag{4.31}$$

which has the property [15]

$$\vartheta(u,\mu) = \vartheta\left(\frac{u}{\mu}, -\frac{1}{\mu}\right)\frac{e^{-i\pi u^2/\mu}}{(-i\mu)^{\frac{1}{2}}}, \tag{4.32}$$

which we can use to write Eq. (4.30) as

$$\sum_{n=-\infty}^{\infty} e^{-\left(\frac{2\pi s}{\ell_c}\right)^2(n+\alpha)^2+\frac{i2\pi(n+\alpha)}{\ell_c}\Delta w} = \sum_{n=-\infty}^{\infty}\frac{\ell_c}{2\pi^{\frac{1}{2}}s}\, e^{-\frac{(\Delta w+n\ell_c)^2}{4s^2}+i2\pi n\alpha}. \tag{4.33}$$

Substituting (4.33) back into Eq. (4.29), performing the integral over the parameter s and taking only the real part yields

$$D(x,x') = \frac{1}{8\pi^2}\sum_{n=-\infty}^{\infty}\frac{\cos(2\pi\alpha n)}{[\Delta\mathbf{x}_T^2+(\Delta w+n\ell_c)^2-\Delta t^2]^{\frac{3}{2}}}. \tag{4.34}$$

One can immediately see that the $n=0$ term of the sum is exactly the real part of the Wightman function of the free solution to the massless Klein-Gordon equation in (4+1)D and, therefore, must be subtracted in order to obtain the renormalized $D^R(x,x')$. Additionally, by considering a photon traveling along the z-direction, that is, $\Delta x = \Delta y = \Delta w = 0$, and noticing that the resulting summation for negative values of n provides the same terms than those for positive values of n, we can write

$$D^R(t,z,t',z') = \frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2\pi\alpha n)}{[\Delta z^2+(n\ell_c)^2-\Delta t^2]^{\frac{3}{2}}}. \tag{4.35}$$

If it were the case that the photon trajectory was not limited to the z-axis, $D^R(x,x')$ would be given by Eq. (4.34) simply by making the change $\sum\to\sum'$, where the primed summation indicates that the $n=0$ term has been subtracted.

The calculation of $F_{zz}(x,x')$ is much simpler as it takes into account most of the calculations we already did while computing $D(x,x')$. Substituting Eq. (4.20) in Eq. (4.25) explicitly writing the sum in Eq. (4.22) yields

$$F_{zz}(x,x') = -\mathrm{Re}\,\frac{\partial^2_{\Delta z}}{(2\pi)^3\ell_c}\sum_{n=-\infty}^{\infty}\int d^3\mathbf{k}_T\,\frac{e^{-i\omega\Delta t}}{2\omega^3}e^{i\mathbf{k}_T\cdot\Delta\mathbf{x}_T+i\frac{2\pi}{\ell_c}(n+\alpha)\Delta w}. \tag{4.36}$$

We can write the fraction involving terms of ω in the integrand on the r.h.s. of Eq. (4.36) in terms of the consecutive integrals

$$\frac{e^{-i\omega\Delta t}}{\omega^3} = -\int_0^{\Delta t} dt_2 \int_0^{t_2} dt_1 \frac{e^{-i\omega t_1}}{\omega} - \frac{i\Delta t}{\omega^2} + \frac{1}{\omega^3}. \tag{4.37}$$

Note that, when substituted back in Eq. (4.36), the first term on the r.h.s. of Eq. (4.37) will result in an expression similar to Eq. (4.29) and that the second term on the r.h.s. of Eq. (4.37) disappears since we are only interested in the real part of these results. Therefore, we can write

$$\begin{aligned} F_{zz}(x,x') = {} & \partial^2_{\Delta z} \int_0^{\Delta t} dt_2 \int_0^{t_2} dt_1 D(0,z,t_1,z') \\ & - \mathrm{Re}\, \frac{\partial^2_{\Delta z}}{(2\pi)^3 \ell_c} \sum_{n=-\infty}^{\infty} \int \frac{d^3\mathbf{k}_T}{2\omega^3}\, e^{i\mathbf{k}_T\cdot\Delta\mathbf{x}_T + i\frac{2\pi}{a}(n+\alpha)\Delta w}, \end{aligned} \tag{4.38}$$

where $D(0,z,t_1,z')$ is given by Eq. (4.34). Combining the integral form [19]

$$\frac{1}{2\omega^{2s}} = \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{2s-1} e^{-\omega^2\tau^2}, \tag{4.39}$$

with Eq. (4.21) and the new expression for the sum over n we found in Eq. (4.33) in the second term on the r.h.s. of Eq. (4.38), performing the integrals on both terms and considering the photon propagating along the z-direction yields

$$F_{zz}(t,z,t',z') = \frac{\partial^2_{\Delta z}}{8\pi^2} \sum_{n=-\infty}^{\infty} \cos(2\pi n\alpha) \frac{\sqrt{\Delta z^2 + (n\ell_c)^2 - \Delta t^2}}{\Delta z^2 + (n\ell_c)^2}. \tag{4.40}$$

Once again we find that the term for $n = 0$ of the sum corresponds to the Minkowski contribution, which can be easily verified by following these same steps for the free solution of the massless Klein-Gordon equation. Therefore, the renormalized F^R_{zz} for a photon traveling along the z-direction is given by

$$F^R_{zz}(t,z,t',z') = \frac{\partial^2_{\Delta z}}{4\pi^2} \sum_{n=1}^{\infty} \cos(2\pi n\alpha) \frac{\sqrt{\Delta z^2 + (n\ell_c)^2 - \Delta t^2}}{\Delta z^2 + (n\ell_c)^2}. \tag{4.41}$$

Finally, the procedure to compute $H_{zzzz}(x,x')$ will be very similar to that of $F_{zz}(x,x')$.

By substituting of Eqs. (4.20) and (4.22) in Eq. (4.26) one obtains

$$H_{zzzz}(x,x') = \mathrm{Re}\,\frac{\partial^4_{\Delta z}}{(2\pi)^3\ell_c}\sum_{n=-\infty}^{\infty}\int d^3\mathbf{k}_T\,\frac{e^{-i\omega\Delta t}}{2\omega^5}e^{i\mathbf{k}_T\cdot\Delta\mathbf{x}_T+i\frac{2\pi}{\ell_c}(n+\alpha)\Delta w}. \tag{4.42}$$

Similarly as we did for $F_{zz}(x,x')$, we can write the terms involving ω in the integrand above as the consecutive integrals

$$\frac{e^{-i\omega\Delta t}}{\omega^5} = -\int_0^{\Delta t}dt_2\int_0^{t_2}dt_1\frac{e^{-i\omega t_1}}{\omega^3} - \frac{i\Delta t}{\omega^4} + \frac{1}{\omega^5}, \tag{4.43}$$

where the first term on the r.h.s. of (4.43) will result in something resembling Eq. (4.36) and the second term will also vanish. The expressions in Eqs. (4.37) and (4.43) can be easily verified by performing the integrals on the r.h.s. of both equations. Substitution of Eq. (4.43) back in Eq. (4.42) yields

$$\begin{aligned} H_{zzzz}(x,x') &= -\partial^2_{\Delta z}\int_0^{\Delta t}dt_2\int_0^{t_2}dt_1F_{zz}(0,z,t_1,z') \\ &\quad + \mathrm{Re}\,\frac{\partial^4_{\Delta z}}{(3\pi)^2\ell_c}\sum_{n=-\infty}^{\infty}\int\frac{d^3\mathbf{k}_T}{2\omega^5}\,e^{i\mathbf{k}_T\cdot\Delta\mathbf{x}_T+i\frac{2\pi}{\ell_c}(n+\alpha)\Delta w} \\ &= I^{H1}_{zzzz} + I^{H2}_{zzzz}, \end{aligned} \tag{4.44}$$

where $F_{zz}(0,z,t_1,z')$ is given by (4.40). Performing the integrals in the first term on the r.h.s. gives us

$$\begin{aligned} I^{H1}_{zzzz} = -\frac{\partial^4_{\Delta z}}{8\pi^2}\sum_{n=-\infty}^{\infty}\cos(2\pi n\alpha)\Bigg\{&\frac{[2\Delta z^2+2(n\ell_c)^2+\Delta t^2]\sqrt{\Delta z^2+(n\ell_c)^2-\Delta t^2}}{6[\Delta z^2+(n\ell_c)^2]} \\ &-\frac{\sqrt{\Delta z^2+(n\ell_c)^2}}{3}+\frac{\Delta t}{2}\arctan\left[\frac{\Delta t}{\sqrt{\Delta z^2+(n\ell_c)^2-\Delta t^2}}\right]\Bigg\}. \end{aligned} \tag{4.45}$$

As for the second term on the r.h.s. of Eq. (4.44), by considering the photon trajectory and using Eqs. (4.39) and (4.33) one obtains

$$I^{H2}_{zzzz} = \frac{\partial^4_{\Delta z}}{8\pi^2}\frac{2\pi^{\frac{1}{2}}}{3}\sum_{n=-\infty}^{\infty}\cos(2\pi n\alpha)\int_0^{\infty}d\tau e^{-\frac{\Delta z^2+(n a)^2}{4\tau^2}}. \tag{4.46}$$

Note the integral over τ as in Eq. (4.46) does not converge. There are some ways to deal with this. One could include an exponential regulator parameter or consider the massive solution to the Klein-Gordon equation and then remove it, or similarly, take the massless limit, after performing the integral. An alternative procedure is to simply permute and execute a single derivative in Δz, from which one obtains a term of τ^{-2}, and then proceed to perform the integral. Following the latter, Eq. (4.46) becomes

$$I^{H2}_{zzzz} = \frac{\partial^3_{\Delta z}}{24\pi^2}\sum_{n=-\infty}^{\infty}\frac{\Delta z}{\sqrt{\Delta z^2+(na)^2}}, \tag{4.47}$$

which cancels out the second term on the r.h.s. of Eq. (4.45) upon substitution back in (4.44). By doing this, we obtain the $H_{zzzz}(t,z,t',z')$ for a photon propagating in the z-direction, namely

$$\begin{aligned} H^R_{zzzz}(t,z,t',z') = -\frac{\partial^4_{\Delta z}}{4\pi^2}\sum_{n=1}^{\infty}\cos(2\pi n\alpha)\Bigg\{&\frac{\Delta t}{2}\arctan\left[\frac{\Delta t}{\sqrt{\Delta z^2+(na)^2-\Delta t^2}}\right] \\ &+\frac{[2\Delta z^2+2(na)^2+\Delta t^2]\sqrt{\Delta z^2+(na)^2-\Delta t^2}}{6[\Delta z^2+(na)^2]}\Bigg\}, \end{aligned} \tag{4.48}$$

where, once again, we have subtracted the $n=0$ term corresponding to the Minkowski contribution, as it can be easily verified by retaking the same steps for the free scalar field in a five-dimensional Minkowski spacetime and we also considered the parity of the summation over n.

After such long calculations, we can finally find an expression for $G^R_{zzzz}(x,x')$. Substituting the renormalized expressions in Eqs. (4.35), (4.41) and (4.48) back into Eq. (4.23), performing the derivatives on Δz and then taking the null path $\Delta t=\Delta z$, we obtain the graviton renormalized Hadamard function calculated over the lightcone, i.e.,

$$G^R_{zzzz}(t,z,t',z')|_{\Delta t=\Delta z} = \frac{16}{3\pi^2}\sum_{n=1}^{\infty}\frac{\Delta z\,(n\ell_c)\,(5n^2\ell_c^2-3\Delta z^2)}{(\Delta z^2+n^2\ell_c^2)^5}\cos(2\pi n\alpha). \tag{4.49}$$

For a photon traveling along the z-direction, Eq. (3.7) becomes

$$\langle\sigma_1^2\rangle_R = \frac{1}{8}(b-a)^2\int_a^b dz\int_a^b dz' G^R_{zzzz}(t,z,t',z')|_{\Delta t=\Delta z}, \tag{4.50}$$

which combined with the renormalized graviton Hadarmard function in Eq. (4.49) yields

$$\langle\sigma_1^2\rangle_R = \frac{2\ell_c}{9\pi^2}\sum_{n=1}^{\infty}\frac{\gamma^8\cos(2\pi n\alpha)}{n(n^2+\gamma^2)^3}, \tag{4.51}$$

where we defined $r = b - a$, $\gamma = r/\ell_c$. The exact sum on the r.h.s. of Eq. (4.51) for a non fixed phase regulator α can be obtained. However, the resulting expression is too lengthy to be exposed here.

In order to find a more approachable expression for this sum, let us take a look at the parameters in Eq. (4.51). As we discussed in the first section of this chapter, the compactification length ℓ_c should be small enough for the Newton's gravitational law to be no longer valid. Therefore, we should have $\ell_c < 5.2 \cdot 10^{-3}$cm. On the other hand, in this work we will be looking at the deviation on the typical flight time of photons detected by instruments such as spectrographs. These instruments use a combination of mirrors and filters to divide the light in several wavelength intervals. Even if such detections are not focused on sources at cosmological distances, the distance traveled by the photon still includes the optical path inside the detector. Therefore, the distance r in this case is always much larger than the compactification length and, consequently, the cases of most interest for us will be those with $\gamma \gg 1$. The remaining parameter α, just as the size of the extra dimension ℓ_c, is a parameter to be inferred from possible detections of Δt. Hence, let us begin to explore observational implications of Eq. (4.51) for a fixed α and then look at more general cases.

4.3 The periodic case and an estimation on the size of the extra dimension

From the cosine in Eq. (4.51) and from Eq. (2.15) we can expect the periodic case, i.e. $\alpha = 0$, to result in the maximum mean deviation to a photon flight time for a fixed γ. This particular case is exactly the one discussed in Ref. [1]. However, let us revisit it here by performing the exact sum over n in Eq. (4.51) for the periodic case, that is,

$$\begin{aligned}\langle\sigma_1^2\rangle_R =&\frac{\ell_c\gamma^2}{72\pi^2}\Big\{16\gamma_e + 8\left[\psi^{(0)}(1-i\gamma)+\psi^{(0)}(1+i\gamma)\right] \\ &+ 5i\gamma\left[\psi^{(1)}(1-i\gamma)-\psi^{(1)}(1+i\gamma)\right] \\ &- \gamma^2\left[\psi^{(2)}(1-i\gamma)+\psi^{(2)}(1+i\gamma)\right]\Big\},\end{aligned} \tag{4.52}$$

where γ_e is the Euler-Mascheroni constant and $\psi^{(n)}(x)$ is a Polygamma function of order n. From Eq. (4.52) we could already compute the exact mean deviation on the photon flight time upon substitution in Eq. (2.15). However, in order to have a better understanding of what this result is indicating, let us expand (4.52) for $\gamma \gg 1$, i.e.,

$$\langle \sigma_1^2 \rangle_R = \frac{2\ell_c \gamma^2}{9\pi^2} \left[\left(\gamma_e - \frac{3}{4} + \ln\gamma \right) + \frac{1}{4\gamma^2} + \frac{1}{20\gamma^4} + \mathcal{O}(\gamma^{-6}) \right]. \tag{4.53}$$

Substituting Eq. (4.53) back into Eq. (2.15) we can approximate the mean flight time deviation as

$$\Delta t \approx \sqrt{\frac{2}{9\pi^2 \ell_c} \ln\gamma} = \ell_c^{-\frac{1}{2}} \sqrt{\frac{2\ln\gamma}{9\pi^2}}. \tag{4.54}$$

We know that the compactification length ℓ_c is expected to be smaller than 52μm [31]. However, even if ℓ_c is much smaller than this value, smaller even than the 4-dimensional Planck scale, and if r is a cosmological distance, the quantity resulting from the square root on the r.h.s. of Eq. (4.54) still would not be large enough to change the magnitude of Δt in more that a couple of orders, as it has been pointed out in Ref. [1]. As a result, we can see that $\ell_c^{-1/2}$ is actually the main quantity guiding the magnitude of the flight time mean deviation, and in an interesting way: the smaller the size of the extra dimension, the easier it would be to detect the change in the flight time of photons in a five-dimensional world.

In the previous section we saw that the compactification length can be related to the fundamental Planck length in five dimensions $\ell_P^{(5)}$, as well as to the known four-dimensional Planck length ℓ_P through Eq. (4.16). We discussed that a measurement for a value of $\ell_P^{(5)}$ depends on how gravity works in a five-dimensional world and, as a consequence, there is no guarantee it will be equal the four-dimensional Planck length ℓ_P. By recovering units of time in Eq. (4.54) and using Eqs. (4.2) and (4.16) we find that, for $\alpha = 0$ and $\gamma \gg 1$,

$$\Delta t \approx \ell_c^{-\frac{1}{2}} \frac{1}{c} \left(\frac{G\hbar}{c^3} \right)^{\frac{3}{4}} \sqrt{\frac{2\ln\gamma}{9\pi^2}} = t_P \left(\frac{\ell_P}{\ell_P^{(5)}} \right)^{\frac{3}{2}} \sqrt{\frac{2\ln\gamma}{9\pi^2}}, \tag{4.55}$$

where $t_P = \ell_P / c$ is the four-dimensional Planck time. From Eq. (4.55) we can see that, for a compactification length close to the four-dimensional Planck scale, or equivalently, from Eq. (4.16), if $\ell_P^{(5)} \sim \ell_P$, the mean flight time deviation will be of order of the four-dimensional Planck time $t_P \sim 10^{-44}$s. Such deviation would be far from observable. This is the same conclusion reached by the authors in Ref. [1].

However, we can still run some estimations if we assume that ℓ_c is such that Δt is detectable through a measurement device based on a spectrograph. When a spectrograph or radio telescope observes light emitted from a source at certain wavelength λ with a resolving power R, it will be able to discern shifts in wavelength in the order of $\Delta\lambda = \lambda/R$. Such shifts will tell us if there were changes in the expected flight time of photons emitted from a source. For this analysis, let us take the NIRSpec aboard the James Webb Space Telescope as an example.

The NIRSpec is able to detect photons in a wavelength range of 6×10^{-5}cm to 5.3×10^{-4}cm, with spectral resolutions of 100, 1000, and 2700 [2]. More specific values of the spectral resolution for each wavelength observed from the NIRSpec are depicted in Fig.4.1, reproduced without modifications from Ref. [2]. The Figure shows performances for each of the seven dispersive elements present in the grating wheel assembly (GWA) inside the NIRSpec, one prism and 6 gratings. We can see from the plot that, even though the prism alone covers all the wavelength range for which the NIRSpec was designed, it is not able to provide the optimal discernible $\Delta\lambda$. For this end one could look at the G140H data.

In order to detect the smallest deviations on the flight time as possible, we need to look for the smallest observed wavelength to resolving power ratio of the detector rather than the distance from the source. Inspired by the NIRSPec performance in Fig.4.1, supposing that a detection occurs at $\lambda = 1.4 \times 10^{-4}$cm with resolving power $R = 2700$, the perceived shift in wavelength would be of order of $\Delta\lambda \approx 5.2 \times 10^{-8}$cm. In this scenario, the smallest flight time shift the NIRSpec would be able to discern is $\Delta t \approx 1.7 \times 10^{-18}$s, where we used $\Delta t = \Delta\lambda/c$. Substituting this shift back in Eq. (4.55) we find that, in order to obtain a flight time deviation observable via NIRSpec, the extra dimension length should be of order of $\ell_c \sim 10^{-84}$cm. Equivalently from Eq. (4.16), the fundamental Planck length in five dimensions should be no greater than $\ell_P^{(5)} \sim 10^{-50}$cm, many orders of magnitude smaller than $\ell_P \simeq 10^{-33}$cm.

This is an interesting estimation in the sense that it provides a possibility to test the existence of extra dimensions at a length scale that measurements through the LHC or table-top experiments will not be able to reach in the near future. In the next section let us find a more compact expression for other values of the phase regulator α.

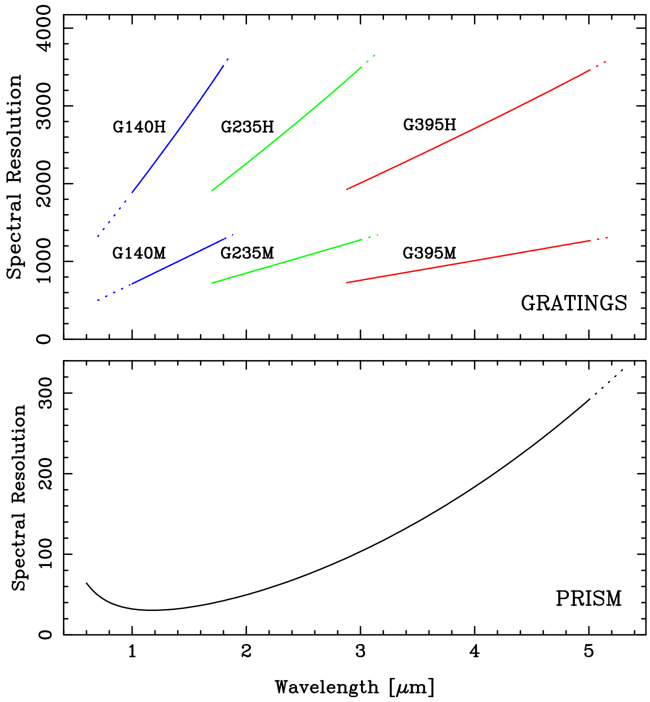

Figure 4.1: Depiction of the spectral resolution for observed wavelength for each of the seven disperser elements on the GWA from the NIRSpec, being one prism and six gratings. In order to obtain an detection throughout the NIRSpec full wavelength range, one could rely on a single exposure using the prism. The result would be a low resolution and highly contaminated image. In order to obtain high resolution data with least contamination containing the full wavelength range for which the NIRSpec was designed, each of the six gratings on the GWA work combined only with specific transmission filter from the seven present in the filter wheel assembly (FWA). Each solid line depicts the nominal spectral resolution (the resolution at the center of the nominal wavelength range for a given disperser-filter combination) for each disperser-filter combination (further details on filter-disperser pairing can be found in the original source and in the JWST documentation), and the dotted lines depicts the anticipated wavelength range. Reproduced without modifications from Jakobsen, P., Ferruit, et al., *The Near-Infrared Spectrograph (NIRSpec) on the James Webb Space Telescope* (JWST) published in A&A by EDP Sciences (Ref. [2]) under the CC BY 4.0 Legal Code.

4.4 Other values for the phase angle regulator α

Let us turn our attention for the other condition cases for α. As discussed, for our investigation based on instruments such as spectrographs, the most relevant cases are those where $\gamma \gg 1$. In other to find an expression for this limit, let us write Eq. (4.51) as

$$\langle \sigma_1^2 \rangle_R = \frac{2\ell_c \gamma^2}{9\pi^2} \sum_{n=1}^{\infty} \frac{\cos(2\pi n\alpha)}{n(\bar{n}^2 + 1)^3}, \tag{4.56}$$

where $\bar{n} = n/\gamma$. Note that the sum in Eq. (4.56) is a descending series of n and large values of γ will result in $\bar{n} < 1$ for the dominating terms of the series. Therefore, we can use the negative binomial expansion

$$(x+1)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} (-x)^k, \tag{4.57}$$

which converges only for $|x| < 1$, and

$$\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!}, & 0 \leq k < n \\ 0, & \text{else} \end{cases}, \tag{4.58}$$

to write Eq. (4.56) as

$$\langle \sigma_1^2 \rangle_R = \frac{2\ell_c \gamma^2}{9\pi^2} \sum_{k=0}^{\infty} (-1)^k \frac{(k+2)(k+1)}{2\gamma^{2k}} \sum_{n=1}^{\infty} n^{2k-1} \cos(2\pi n\alpha). \tag{4.59}$$

Note that we can write the sum in n as a combination of Polylogarithm functions, namely

$$\langle \sigma_1^2 \rangle_R = \frac{2\ell_c \gamma^2}{9\pi^2} \sum_{k=0}^{\infty} (-1)^k \frac{(k+2)(k+1)}{4\gamma^{2k}} \left[\mathrm{Li}_{1-2k}(e^{i2\pi\alpha}) + \mathrm{Li}_{1-2k}(e^{-i2\pi\alpha}) \right]. \tag{4.60}$$

We can see that the remaining sum over k indeed results in an convergent expression for large γ as long as $\alpha \neq 0$. Evaluating the Polylogarithm functions for the first two terms of the sum, we can write

$$\langle \sigma_1^2 \rangle_R = \frac{\ell_c \gamma^2}{18\pi^2} \left[-2\ln\left(1 - e^{-i2\pi\alpha}\right) - 2\ln\left(1 - e^{i2\pi\alpha}\right) + \frac{3\csc^2(\pi\alpha)}{\gamma^2} + \mathcal{O}(\gamma^{-4}) \right], \tag{4.61}$$

which is only valid for $\alpha \neq 0$. For the periodic case and large values of γ one should look at Eq. (4.53). Furthermore, we can also see from Eq. (4.61) that there are certain values of α for which the natural logarithm terms will dominate resulting in $\langle \sigma_1^2 \rangle_R < 0$. Additionally, one can see directly from Eq. (4.51) that, among all α resulting in negative $\langle \sigma_1^2 \rangle_R$, the antiperiodic case will result in the highest $|\langle \sigma_1^2 \rangle_R|$, and therefore, in the highest flight time mean deviation. The behaviors of the exact expression for $\langle \sigma_1^2 \rangle_R$ in Eq. (4.51), as well as the expansion for large γ in Eq. (4.61) after discarding terms with $k > 1$ are depicted in Fig.4.2 for $\alpha = 0.01$ in solid black and dotted red lines, respectively. It becomes clear from the plot that Eq. (4.61) is in fact a good expression for larger values

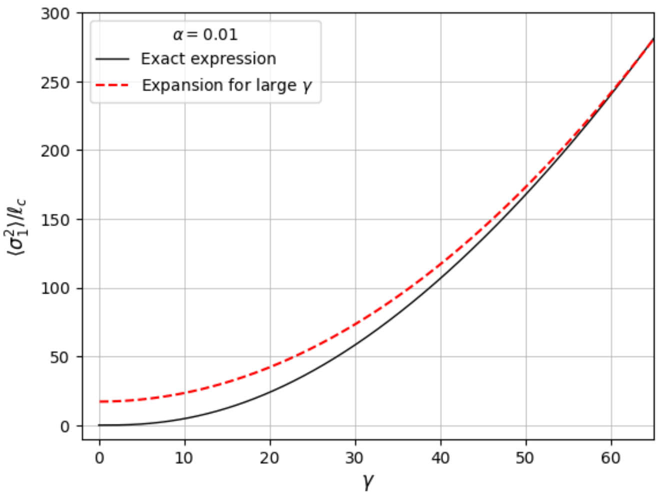

Figure 4.2: The behaviour for $\alpha = 0.01$ of the exact sum in Eq. (4.51) in terms of γ and in units of ℓ_c, is depicted in solid black line and compared with the expansion for $\gamma \gg 1$ in Eqs. (4.53) and (4.61) in dotted red line. In fact, for large γ, the curves meet for any value of α.

of γ. Our choice of α in Fig.4.2 holds no special meaning, we have numerically verified that the approximation (4.60) is good for any value of $\alpha \neq 0$. However, note that different values of α result in different paces at which our approximation will converge to the exact behaviour of $\langle \sigma_1^2 \rangle_R$. Note also that in Fig.4.2 both $\gamma = r/\ell_c$ and $\langle \sigma_1^2 \rangle/\ell_c$, are dimensionless quantities, as we can see from any of the previous equations for $\langle \sigma_1^2 \rangle$.

Furthermore, we can see that the expression for $\langle \sigma_1^2 \rangle_R$ for $\alpha \neq 0$ in Eq. (4.61) differs fundamentally from the expression for $\alpha = 0$ in Eq. (4.53). The first contains a dependence on γ with positive and negative powers, whereas the latter contains both as well as a logarithmic dependence as the most significant contribution. This will result in much larger values of $\langle \sigma_1^2 \rangle_R$ for $\alpha = 0$ than for any other value as γ increases, just as we already expected from Eq. (4.51). Considering contributions down to zeroth order of γ^{-2} in Eq. (4.61) and substituting it back in Eq. (2.15), one can also obtain an expression for the mean deviation on the flight time of photons for $\alpha \neq 0$. Recovering units of time, just as we did for the periodic case, on the resulting expression yields

$$\Delta t \approx \frac{\ell_c^{-\frac{1}{2}}}{\sqrt{18\pi^2}} \frac{1}{c} \left(\frac{G\hbar}{c^3} \right)^{\frac{3}{4}} \left| -2\ln\left(1 - e^{-i2\pi\alpha}\right) - 2\ln\left(1 - e^{i2\pi\alpha}\right) + \frac{3\csc^2(\pi\alpha)}{\gamma^2} \right|^{\frac{1}{2}}. \tag{4.62}$$

Comparing Eqs. (4.54) and (4.62) we can see that the most significant dependence on γ for $\alpha \neq 0$ comes from the last term in the square root on the r.h.s. of the expression above, in contrast to the square root of the logarithm we obtained for $\alpha = 0$. Consequently, for large values of γ, the dominating terms for $\alpha \neq 0$ will be the constant terms of natural logarithmic fixed by the quasiperiodicity phase angle regulator. Hence, similarly to the periodic case, the main quantity guiding the magnitude of Δt, for each fixed α, will be the inverse of the square root of the compactification length, i.e., $\ell_c^{-1/2}$.

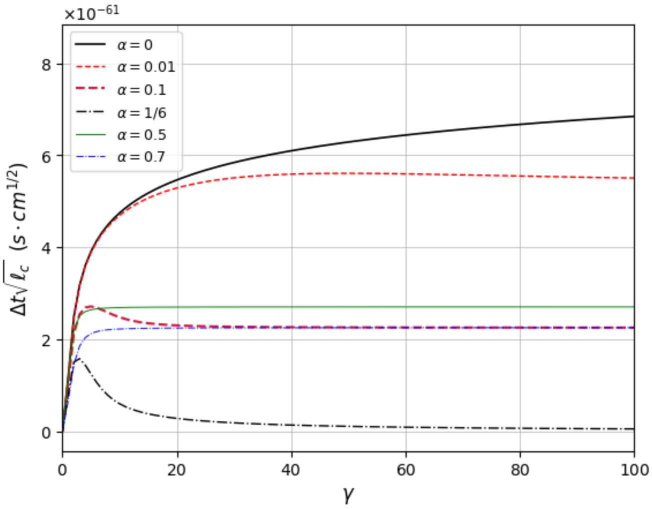

Figure 4.3: The exact behaviour of $\Delta t\sqrt{\ell_c}$, in terms of γ, for several values of α.

Although it was important to obtain a clean expression for the behavior of Δt, even if only for large values of γ, we should still look at the behavior of the exact expression obtained by combining Eqs. (4.51) and (2.15). This behavior is depicted in Fig.4.3. Note that the we chose the vertical axis to be $\Delta t\sqrt{\ell_c}$ and consequently, it has units of $s \cdot cm^{1/2}$. This choice enables us to gauge the influence of both γ and ℓ_c on the magnitude of Δt based on the observational criteria discussed in the previous section for the case $\alpha = 0$. For example, from Fig.4.3 we can see that, for γ around 10^2, the general magnitude of the result are on the order of $\Delta t\sqrt{\ell_c} \sim 10^{-61}\mathrm{s} \cdot \mathrm{cm}^{1/2}$. It follows from a straightforward dimensional analysis that, for an observable flight time shift for these values of α ($\Delta t \sim 10^{-18}$s) one must have $\ell_c \sim 10^{-84}$cm, in perfect accordance with the discussion in the previous subsection. However, note that there seems to be some values of α for which $\Delta t\sqrt{\ell_c}$ is closer to zero, potentially reducing this estimative.

Fig.4.3 also reflects the different behaviours of Eqs. (4.54) and (4.62) and, as a result, as γ increases, the periodic case will lead to a logarithmically increasing mean flight time deviation, whereas for any other $\alpha \neq 0$ case it will quickly converge to a constant Δt determined by the structure of the extra dimension, i.e., by α and ℓ_c. This is an interesting feature of our results: if the extra dimension does not exhibit a perfectly periodic behavior, then the distance that the photons travelled through the perturbation does not matter as long as $r \gg \ell_c$, that is, they will all exhibit the same characteristic Δt. The distinction between the periodic case and the others can be crucial in comprehending the potential structure of the extra dimension through measurements at a given distance r from the source. Although this distance is limited to the length of the observable universe, approximately 10^{28} cm, the functional dependence of Δt with γ could play a pivotal role in determining values for α and, consequently, for ℓ_c.

Imagine this: if we were to make a measurement of Δt at a distance r from the source and assume that we could move our detector a few meters away for another measurement while the photons still travels through the same perturbation $h_{\mu\nu}$, Fig.4.3 tells us that it could be easier to discern $\alpha = 0$ from all other condition cases because of the different behaviors of Δt between them. However, the same would not be true when trying to make the distinction between other values of α, at least not in a detection via devices such as the NIRSpec. We know that Δt converges quickly as γ increases, therefore, we would be left with two free parameters α and ℓ_c which could be chosen as to satisfy the measured Δt. On the other hand, notice that the region for smaller values of γ in Fig. 4.3 shows many different behaviors of $\Delta t\sqrt{\ell_c}$ for each α depicted and it could help us solve this ambiguity over ℓ_c and $\alpha \neq 0$. However, as we pointed out, detections for smaller values of γ falls out of the detection range for instruments such as the NIRSpec as a consequence of how they are built. Hence, this discussion does not fall within the scope of this work. However, as possible continuation of this work it would be interesting to seek the possibility of making detections of Δt within such range.

As the contribution of γ to the mean flight time deviation does not give us the means to determine the structure of the extra dimension $\alpha \neq 0$, let us look at Eq. (4.62) from another perspective. Fig.4.4 depicts the general behaviour of $\Delta t\sqrt{\ell_c}$ as a function of α, assuming three distinct values of γ. For each curve, the region in between the minima provides the values of α for which $\langle\sigma_1^2\rangle_R < 0$, whereas the values of α outside this region give $\langle\sigma_1^2\rangle_R > 0$. This can be verified from the exact expression in Eq. (4.51). As we have previously stated, it also becomes clear from the plot in Fig.4.4 that, for a fixed γ, $\alpha = 1/2$ indeed results in a maximum flight time mean deviation when compared with all other values of α for which $\langle\sigma_1^2\rangle_R < 0$. An interesting aspect also revealed by the plot is that

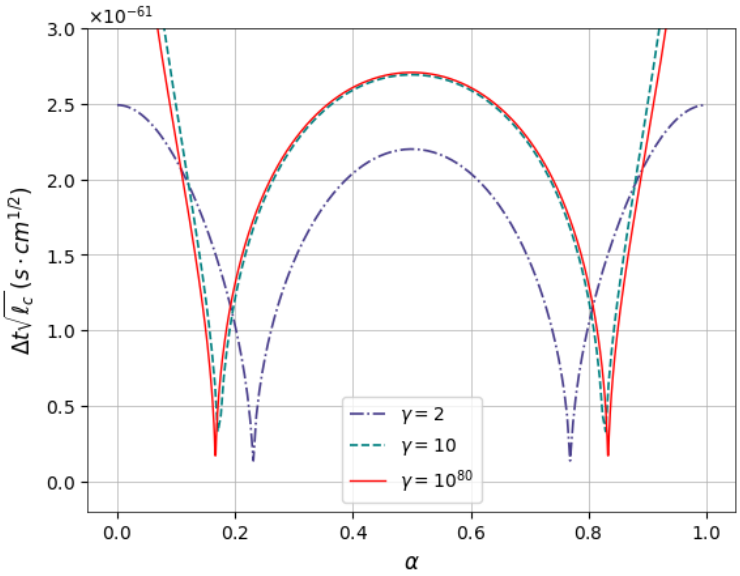

Figure 4.4: The behaviour of $\Delta t\sqrt{\ell_c}$ as a function of the quasiperiodic parameter α, for some values of γ.

there are more than one possible value for α that provides the same potential 'measured' Δt, making it difficult to completely understand the structure of the extra dimension. In addition, we can also see in Fig.4.4 the quick convergence of $\Delta t\sqrt{\ell_c}$ to constant values shown in the plot of Fig.4.3 for $\alpha \neq 0$ (note that there are no difference in the curves for sufficiently large γ).

Upon the possibility that we live in a five-dimensional world, both the size of the extra dimension, ℓ_c, and the phase angle regulator, α, are crucial parameters to be inferred from experimental data, like the ones from NIRSpec. As we have mentioned before, the extra dimension should be small. If it exists, it is expected to be at least smaller than 55μm. On the other hand, there are some factors that can modify the spectrum of a light pulse, if we assume that lightcone fluctuations are one of them and that the resulting mean flight time deviation falls right within the NIRSpec's sensitivity, then our model provides estimations on the size of the extra dimension for $\alpha = 0$, and also for most values of $\alpha \neq 0$, that are of order of $\ell_c \sim 10^{-84}$cm, resulting in an estimation of the Planck length in five dimensions that is of order of 10^{-50}cm. Such values are well beyond the scale being tested by current experiments seeking the find evidence of the existence of extra dimensions.

Chapter 5

Conclusions and discussions

In the present dissertation we discussed key concepts involved in the lightcone fluctuations effects. We started by reviewing the linearized theory of gravity from the classical perspective and then moved on to the quantum approach. We saw how the quantization of the perturbation leads to the smearing of classical lightcone divergences, resulting in a fluctuation around the usual light speed. As a result, a light pulse emitted from a source will present a characteristic deviation in its spectral lines associated with a typical change in the time flight Δt of photons. As the last piece of our review, we found expressions in $(d+1)$ for the graviton Hadamard function in Eq. (3.13), complemented by Eqs. (3.14) and (3.15), and for the expectation value of half of the geodesic distance between two points in first order on the perturbation $\langle \sigma_1^2 \rangle$ in Eq. (3.7). All these elements are used in order to compute the typical change on a photon flight time in Eq. (2.15).

In Chapter 4, we reviewed some aspects of higher-dimensional spacetimes and of the search for evidence of their existence. As consolidation of our study, we presented our new results on lightcone fluctuation effects in a quasiperiodically compactified $(4+1)$-dimensional flat spacetime and discussed some of their observational aspects using the Near Infrared Spectrograph aboard the James Webb Space Telescope. This analysis resulted in the work in Ref. [27] to be published in the Journal of High Energy Physics (JHEP). By considering a photon propagating along the z-direction, we found a renormalized expression for the graviton Hadamard function calculated on the lightcone in Eq. (4.49) by subtracting the unperturbed spacetime contribution, and we used it to compute $\langle \sigma_1^2 \rangle_R$ in Eq. (4.51), which leads to the typical deviation on the flight time of photons upon substitution in Eq. (2.15). This expression for $\langle \sigma_1^2 \rangle_R$ contains an infinite sum which is, in fact, convergent and can be performed. However, as the complete resulting expression is too extensive, we chose to look for some potential cases of interest. Our result is written in terms of the parameters α, ℓ_c and $\gamma = r/\ell_c$, being α the parameter that regulates the phase angle on the quasiperiodic boundary condition, ℓ_c the length of the extra dimension, and r the distance traversed by the photon. We found that, when looking for observations via instruments such as the NIRSpec, the optical path inside the detector is always much larger than the maximum possible size of the extra dimension, resulting in $\gamma \gg 1$.

Our focus was to find out whether and how measures of lightcone fluctuations effects could be used as a way of understanding the potential structure of an extra spatial dimension. That is, if a detection of Δt could help us estimate values for α and ℓ_c. We started this analysis by considering the periodic case $\alpha = 0$. By performing the exact sum in Eq. (4.51) and expanding it for large values of γ we obtained Eq. (4.53), from which we could approximate the mean deviation on the photons flight time as in Eqs. (4.54) and (4.55), just as found by the authors in Ref. [1] for a periodically compactified $(4+1)$ spacetime. These equations tell us that, regardless of the values that γ can assume, the main contribution for the magnitude of Δt will come from $\ell_c^{-1/2}$ and, as the authors in Ref. [1] also discussed, if the compactification length is close to the four-dimensional Planck scale, then the resulting mean flight time deviation would be on order of the four-dimensional Planck time. Nonetheless, we discussed the possibility of obtaining an observable flight time deviation within the sensitivity range of the NIRSpec and found that, in order to obtain it, the size of the extra dimension ℓ_c should be on the order of 10^{-84}cm. Furthermore, Eq. (4.54) also states that, as the size of the extra dimension becomes smaller, more significant the change on a photon's flight time becomes. These are both interesting conclusions in the sense that our results provide observational possibilities that work in a length scale that other experiments seeking to detect the existence extra dimensions, such as those discussed in Chapter 4.1, are not capable of reaching.

Finally, we discussed other condition cases for α. We used the binomial negative expansion to expand Eq. (4.51) for large values of γ with $\alpha \neq 0$ and obtained the expression in Eq. (4.61), which leads to the approximated mean deviation on the flight time in Eq. (4.62). We plotted the exact expression for $\langle \sigma_1^2 \rangle_R$ in Eq. (4.51) along with Eq. (4.61) down to the zeroth order on the perturbation with respect to γ in Figure 4.2

in order to show that Eq. (4.61) is in fact a good approximation for $\gamma \gg 1$. We also found that the expression for Δt for the periodic case in Eq. (4.54) contains a logarithm dependence on γ, whereas the expression for $\alpha \neq 0$ contains a constant term fixed by α and a inverse-square dependence on γ. This is also an interesting feature of our results: for any fixed $\alpha \neq 0$, as γ increases, the mean flight time deviation will quickly converge to a constant value whose magnitude is guided once more by $\ell_c^{-1/2}$. In other words, regardless of how long the photons traveled through the quantized perturbation, the same typical deviation on its flight time will be observed as long as $r \gg \ell_c$. On the other hand, for the case where behavior of the quantized perturbation is perfectly periodic as it travels through the extra dimension, that is for $\alpha = 0$, most significant values of Δt will be obtained as γ increases. These results were depicted in Figure 4.3. Another interesting development of the quick convergence for $\alpha \neq 0$ is that, if we were capable to detect a Δt that could be attributed to fluctuations on the metric on devices such as the NIRSpec, and if this data did not match the logarithmic growth consistent with $\alpha = 0$, we would be unable to uniquely fix values for α.

Subsequently, as γ is no longer a relevant parameter for the case $\alpha \neq 0$ in detections via isntruments such as the NIRSpec, we plotted the behavior of the exact expression for Δt obtained by substituting (4.51) in Eq. (2.15) with respect to α for some values of γ in Figure 4.4. We discussed some elements of this plot, in particular, the range of α that results in $\langle \sigma_1^2 \rangle_R < 0$ as residing between the to minima, and that leads to $\langle \sigma_1^2 \rangle_R > 0$ otherwise. From Figure 4.4 we were also able to observe the quick converge of Δt as γ increases, and that for a certain "measured" Δt, and even for a fixed ℓ_c, one would still obtain at least two possible values of α.

Although this work resulted in some interesting features and discussions, it still raises pertinent questions for future research on lightcone fluctuation effects. In particular, we can observe different behaviors for Δt in the region for smaller values of γ on Figure 4.3 that could help us solve the ambiguity over the parameters α and ℓ_c. Furthermore, it is generally the case that, when a higher number of extra dimensions is taken into account, the changes of making the phenomenon considered be detectable increases (see Ref. [24]). The calculations of lightcone fluctuation effects for higher than five dimensional quasiperiodically compactified Kaluza-Klein models are no easy task. However, for each additional extra dimension considered, Δt can be expected to be proportional to increasingly negative powers of ℓ_c, which can result in deviations on the flight time of photons which are more likely to be detected even if the length of the extra dimensions are greater than our current estimative in $(4+1)$. Our results in a flat background also paves the way to the possibility of a similar analysis for the structure of the extra dimension be reproduced in more realistic scenarios. In particular, in a cosmological background with a spatially flat $(3+1)$ Friedmann-Lemaître-Robertson-Walker metric lightcone fluctuation effects have been analyzed in Ref. [9]. As we deal with cosmological distances and relic gravitons produced in the early Universe are expected to be a potential source of metric fluctuations, we could wonder whether it is possible to generalize the results of Ref. [9] to include a compactified extra dimension, as well as a cosmological constant, and investigate the role played by the scale factor $a(t)$ in determining the structure of the extra dimension by also using the sensitivity of measure devices such as the NIRSpec. For future work we are also interested in searching for observable scenarios on lightcone fluctuations effects by including temperature corrections in the calculations, considering additional extra dimensions or working with curved backgrounds.

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