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Mean Curvature Flow Solitons in a GRW Spacetime and CMC Free Boundary Hypersurfaces in Rotational Domains

by

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João Pessoa - PB
July/2024

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Thesis submitted to the Post-graduate Program of the Departament of Mathematics of the Federal University of Paraíba in partial fulfillment of the necessary requirements for the degree of Ph.D in Mathematics.

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Abstract

In this work, we study two themes. First, we study a n-dimensional spacelike mean curvature flow solitons related to the closed conformal timelike vector field $\mathcal{K} = f(t)\partial_t$ ($t \in I \subset \mathbb{R}$) which is globally defined on an generalized Robertson-Walker (GRW) spacetime $-I \times_f M^{n+p}$ with warping function $f \in C^\infty(I)$ and Riemannian fiber M^{n+p}, these are particular cases of trapped submanifolds, and we obtain rigidity and non-existence results for this submanifold class via applications of suitable generalized maximum principles and under certain constraints on f and on the curvatures of M^{n+p}. Then, we work with the existence and uniqueness of free boundary constant mean curvature hypersurfaces in rotational domains, these are domains whose boundary is generated by a rotation of a graph. We classify the CMC free boundary hypersurfaces as topological disks or annulus, under some conditions in the generatrix function and a gap condition on the umbilicity tensor.

Keywords: Mean curvature flow solitons; Generalized Robertson-Walker spacetime; CMC free boundary hypersurfaces; Rotational domains.

Resumo

Nesse trabalho, estudamos dois temas. Primeiro, estudamos solitons do fluxo da curvatura média de dimensão n relacionados a um campo de vetores tipo-tempo, conforme e fechado $\mathcal{K} = f(t)\partial_t$ ($t \in I \subset \mathbb{R}$) o qual é globalmente definido em um espaço-tempo Robertson-Walker generalizado (GRW) $-I \times_f M^{n+p}$ com função warping $f \in C^\infty(I)$ e fibra Riemanniana M^{n+p}, estes são casos particulares de subvariedades trapped, e obtemos resuldados de rigidez e não-existência para esta classe de subvariedades via aplicações de princípios do máximo generalizados adequados e certas restrições em f e nas curvaturas de M^{n+p}. Depois, trabalhamos com a existência e unicidade de hipersuperfícies de fronteira livre com curvatura média constante em domínios rotacionais, que são domínios cuja fronteira é gerada pela rotação de um gráfico. Classificamos hipersuperfícies de fronteira livre CMC como um disco topológico ou um anel, sob algumas condições na função geratiz e a condição de gap no tensor de umbilicidade.

Palavras-chave: Solitons do fluxo da curvatura média; Espaço-tempo Robertson-Walker generalizado; Hipersuperfície de fronteira livre CMC; Domínio rotacional

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"O que sabemos é uma gota; o que ignoramos é um oceano. Mas o que seria o oceano se não infinitas gotas?"

Isaac Newton

Dedicatória

À memória de Vovó Tudinha.

Contents

Introduction

Throughout this work, we dealt with two different contexts. First, we study nonexistence and rigidity of an $n-$dimesional spacelike mean curvature flow solitons in a GRW spacetime, and then we work with the existence and uniqueness of freeboundary CMC hypersurfaces in rotational domains.

It is known that the *spacelike mean curvature flow* related to a spacelike immersion $x : \Sigma^n \looparrowright \overline{M}^{n+k}$ in a $(n+k)$-dimensional Lorentzian manifold $\overline{M}^{n+k}$ is a one-parameter family of smooth spacelike immersions $X_t = X(t, \cdot) : \Sigma^n \looparrowright \overline{M}^{n+k}$ with corresponding images $\Sigma^n_t = X_t(\Sigma^n)$ satisfying the following evolution equation

$$\begin{cases} \dfrac{\partial X}{\partial t} = \vec{H} \\ \\ X(0, q) = x(q) \end{cases} \tag{1}$$

on some time interval, where $\vec{H}$ stands for the (non-normalized) mean curvature vector of the spacelike submanifold Σ^n_t in $\overline{M}^{n+k}$.

The relevance of this concept is due, for instance, to the fact that *spacelike mean curvature flow solitons*, which correspond to the singularities of (1), can be regarded as a natural way of foliating spacetimes by almost null like hypersurfaces and particular examples may give insight into the structure of certain spacetimes at null infinity and have possible applications in General Relativity (for more details, we recommend the references [38, 37, 39, 50]).

More recently, Lambert and Lotay [54] proved longtime existence and convergence results for spacelike solutions to mean curvature flow in the n-dimensional pseudo-Euclidean space $\mathbb{R}^n_m$ of index m, which are entire or defined on bounded domains and satisfying Neumann or Dirichlet boundary conditions. In [46], Guilfoyle and Klingenberg proved the longtime existence of mean curvature flow of a smooth n-dimensional spacelike submanifold of an $(n+m)$-dimensional manifold whose metric satisfies the so-called timelike curvature condition.

Meanwhile, Alías, de Lira and Rigoli [9] introduced the general definition of self-similar mean curvature flow in a Riemannian manifold $\overline{M}^{n+1}$ endowed with a conformal vector field $\mathcal{K}$ and establishing the corresponding notion of mean curvature flow soliton. In particular, when $\overline{M}^{n+1}$ is a Riemannian warped product of the type $I \times_f M^n$ and $\mathcal{K} = f(t)\partial_t$, they applied weak maximum principles to guarantee that a complete n-dimensional mean curvature flow soliton is a slice of $\overline{M}^{n+1}$. In [32], Colombo, Mari, and Rigoli also studied some properties of mean curvature flow solitons in general Riemannian manifolds and warped products, focusing on splitting and rigidity results under various geometric conditions, ranging from the stability of the soliton to the fact that the image of its Gauss map is contained in suitable regions of the sphere. Moreover, they also investigated the case of entire mean curvature flow graphs. In [36], de Lima, Santos, and Velásquez investigated several aspects of the geometry of mean curvature flow solitons Σ^m immersed into a Riemannian warped product $I \times_f M^n$. They applied suitable maximum principles to guarantee that such a mean curvature flow soliton is a slice of the ambient space and obtained nonexistence results concerning these geometric objects. In particular, when $m = n$ they also studied entire graphs constructed over the fiber M^n which are mean curvature flow solitons. In [60], Mari, Oliveira, Savas-Halilaj and et al. studied examples of conformal solitons for the mean curvature flow in hyperbolic space $\mathbb{H}^{n+1}$.

When the ambient space is a Lorentzian product space of the type $-I \times M^n$, Batista and de Lima [19] constructed new examples of rotationally symmetric spacelike translating solitons embedded in such an ambient space when the Rimannian fiber M has non-positive sectional curvature. For more examples of translating solitons in a product space see ([67], [68] and [55]). When the ambient space is a warped product space see [14].

More recently, de Lima, Gomes, Santos, et al. [35] investigated several geometric aspects of complete spacelike mean curvature flow solitons of codimension 1 in a GRW spacetime $-I \times_f M^n$. In addition to obtaining several uniqueness and nonexistence results, they also studied the stability of spacelike mean curvature flow solitons concerning an appropriate stability operator.

Going a step further, here, in Part I of this work, we extend the ideas and techniques developed in [9, 19, 32, 35] to study n-dimensional spacelike mean curvature flow solitons related to the closed conformal timelike vector field $\mathcal{K} = f(t)\partial_t$ ($t \in I \subset \mathbb{R}$) which is globally defined on a generalized Robertson-Walker (GRW) spacetime $-I \times_f M^{n+p}$ with warping function $f \in C^\infty(I)$ and Riemannian fiber M^{n+p}. Theorems 3.0.1 and 4.3.1, for example, are extensions to higher codimension of results in [35]. In this setting, we apply suitable maximum principles to obtain nonexistence and rigidity concerning these spacelike mean curvature flow solitons, under certain constraints on f and on the curvatures of M^{n+p}. Furthermore, in codimension 1 (that is, when $p = 0$), we also obtain new Calabi-Bernstein type results concerning the spacelike mean curvature flow soliton equation in a GRW spacetime.

Also, it is worth pointing out another motivation for the study of spacelike mean curvature flow solitons in a GRW spacetime: They correspond to particular cases of *trapped submanifolds*, which means that the mean curvature vector is timelike (see Proposition A). The concept of trapped submanifold was first introduced by Penrose [69] to study singularities of spacetime, giving rise to some of the famous singularity theorems (see [48, 49, 72]). In General Relativity, a trapped surface is a two-dimensional embedded spatial surface such that the product of the traces of their two future-directed null second fundamental forms is everywhere positive and its existence indicates the presence of a black hole (see [20, 52, 73]).

The results presented in Part I of this work were obtained by the author in collaboration with Freitas, de Lima, and Santos in [43].

In Part II of this work, we shift our focus to an n-dimensional constant mean curvature (CMC) hypersurface Σ with a smooth boundary that is compact, oriented, and immersed in a Riemannian manifold M^{n+1} also possessing a smooth boundary ∂M. Here, $\partial\Sigma \subset \partial M$, and the boundary of Σ meets the boundary of M orthogonally. In this situation, we say that Σ is a *free boundary CMC hypersurface in M*. Such hypersurfaces are stationary for the area functional for variations preserving the enclosed volume (see, for example, [71, Section 1]). When $H = 0$, we say that Σ is a free boundary minimal hypersurface. In the particular case where the domain M is the unitary ball $\mathbb{B}^3$ in the Euclidean space, the simplest examples of CMC free boundary surfaces are the equatorial disk, the critical catenoid (minimal surfaces) and the spherical caps.

We remember that after its initial motivation by Courant [33] and preliminary developments (for example, [65], [75] and [71]), this topic has received plenty of attention, mainly after the 2010 decade and the undeniable contributions of Fraser and Schoen ([42], [41]). These underlying works reveal, in particular, several similarities between free boundary minimal surfaces in an Euclidean unit ball and closed minimal surfaces in the sphere. In this sense, the classical results and strategies to obtain rigidity results in the last one could indicate interest directions to get related developments in the free boundary case.

In this direction, the so-called gap results in the second fundamental form give an important characterization of CMC surfaces in the sphere. A series of contributions from Simons [74], Lawson [56], and Chern, do Carmo and Kobayashi [30] get the following gap result for the second fundamental form A of the immersion:

Theorem A (Chern-do Carmo-Kobayashi [30], Lawson [56], Simons [74]) *Let Σ be a closed minimal hypersurface in the unit sphere $\mathbb{S}^{n+1}$. Assume that the second fundamental form A on Σ satisfies*

$$|A|^2 \leq n.$$

Then

1. *either $|A|^2 = 0$ and Σ is an equator;*

2. *or $|A|^2 = n$ and Σ is a Clifford minimal hypersurface.*

In the study of CMC hypersurfaces in the sphere, Alencar and do Carmo [4] also obtained a gap result, but now considering the umbilicity tensor $\phi = A - Hg$.

Theorem B (Alencar-do Carmo [4]) *Let Σ be a closed, CMC hypersurface in the unit sphere $\mathbb{S}^{n+1}$. If*

$$\|\phi\|^2 \leq C_H,$$

1. *either $\|\phi\|^2 \equiv 0$ and Σ^n is totally umbilical in $\mathbb{S}^{n+1}$,*

2. *$\|\phi\|^2 \equiv C_H$ and Σ^n is an $H(r)$-torus in $\mathbb{S}^{n+1}$.*

Here, C_H is related to a root of a polynomial whose coefficients depend on the mean curvature H and the dimension n[3]*.*

[3] For details, see the Introduction of [4].

We could describe some contributions by starting from these two characterizations and studying similar phenomena for free boundary CMC hypersurfaces in the ball. In [11], Ambrozio and Nunes proved that if Σ is a compact free boundary minimal surface in $\mathbb{B}^3$ and for all points x in Σ,

$$|A|^2(x)\langle x, N(x)\rangle^2 \leq 2, \tag{2}$$

then Σ is a flat equatorial disk or a critical catenoid. In higher dimensions, some similar gap results to (2) can be obtained for 2-dimensional surfaces in the ball (see [17]) and, with a topological rigidity, in submanifolds of any codimension in higher dimensional balls (see [18, Theorem 3.7]). Also, some gaps result just in the second fundamental form, as that in Theorem A, was obtained in [25], [16] and [15].

The question arises: Does an analogous result hold to these in the context of free boundary CMC non-minimal surfaces? Barbosa, Cavalvante, and Pereira answered this question in [13]. More specifically, in this work they proved that if Σ is a compact free boundary CMC surface in $\mathbb{B}^3$ and for all points x in Σ,

$$|\phi|^2\langle \vec{x}, N\rangle^2 \leq \frac{1}{2}(2 + H\langle \vec{x}, N\rangle^2),$$

where H is the non-normalized mean curvature, then Σ is a totally umbilical disc or a part of a Delaunay surface.

In [21], Bettiol, Piccione, and Santoro have studied the existence of CMC disks and Delaunay annuli that are free boundary in a ball. Furthermore, in addition to the studies involving free boundary surfaces in the unit ball, investigations of this kind have also been conducted in other domains. For example, when the ambient space is a wedge (López [57]), a slab (Ainouz and Souam [1]), a cone (Choe [31]) or a cylinder (Lopez and Pyo [58]).

Regarding rigidity conclusions starting from a gap condition, Andrade, Barbosa, and Pereira [12] established some results for balls conforming to the Euclidean ball. More recently, when the ambient space is a strictly convex domain in a 3-dimensional Riemannian manifold with sectional curvature bounded above, and Σ is a CMC free boundary surface in this region, Min and Seo [63] establish a pinching condition on the length of the umbilicity tensor on Σ. This criterion ensures that the surface is topologically equivalent to a disk or an annulus. In the particular case where the domain is a geodesic ball of a 3-dimensional space form, they concluded that Σ is a spherical cap or a Delaunay surface. In [15], Barbosa, Freitas, Melo, et al., investigated the existence of compact free boundary minimal hypersurfaces immersed in domains whose boundary is a regular level set, in particular giving some gap results for free boundary minimal hypersurfaces immersed in an Euclidean ball and a rotational ellipsoid.

Part II of this work explores some gap results for CMC free boundary surfaces in rotation domains described below. By considering a curve $\alpha(t) = (f(t), t)$, where f is a positive real-valued smooth function, we generate a hypersurface $\partial\Omega$ starting from the revolution of this curve in an appropriate axis. In this sense, we can describe a domain Ω such that $\partial\Omega \subset F^{-1}(1)$ is a revolution hypersurface and $F : \mathbb{R}^n \times I \to \mathbb{R}$ is a smooth function given by

$$F(x, y) = \frac{1}{2}\left(|x|^2 - f^2(y)\right) + 1.$$

Furthermore, we consider a hypersurface Σ, which is a free boundary CMC surface in Ω and we use the following condition for the function f of the profile curve

$$(f')^2 + ff'' + 1 \leq 0.$$

Then, we get gap results for CMC free boundary surfaces in 3-dimensional rotational domains and minimal free boundary hypersurfaces in $(n + 1)$-dimensional rotational domains.

Some considerations about the inequality condition in f are necessary at this point. The inequality condition in f has an intriguing interpretation in terms of the principal curvatures of the profile curve, and the equality case is characterized by the Euclidean ball (see Remarks 6.0.2 and 6.0.3). In particular, this condition implies that the boundary of the domain is convex (see demonstration of Lemma 7.1.1). About the existence of free boundary minimal disks in convex regions of $\mathbb{R}^3$, we refer Struwe [75] when he shows that there is at least one such disk. More recently, Haslhofer and Ketover [47] show these regions admit at least two free boundary minimal disks. On the other hand, by studying the existence of free boundary minimal annuli inside convex subsets of 3-dimensional Riemannian manifolds with nonnegative Ricci curvature we refer [61] by Máximo, Nunes, and Smith.

Furthermore, we construct new examples of CMC surfaces that are free boundary on the rotational ellipsoid, exploring the technique from [13]. This permits seeing examples of catenoids, nodoids, and onduloids in this domain.

The results presented in Part II were obtained by the author with Freitas and Santos in [44].

Part I

Nonexistence and rigidity of spacelike mean curvature flow solitons

Chapter 1

Preliminaries I

1.1 Generalized Robertson-Walker spacetimes

In this part of the work, our ambient spacetime will be taken as an $(n+p+1)$-dimensional generalized Robertson-Walker (GRW) spacetime given by $-I \times_f M^{n+p}$, that is, the time-oriented warped product manifold $I \times_f M^{n+p}$, where $(M^{n+p}, \langle\, ,\rangle_M)$ is a connected, $(n+p)$-dimensional, oriented Riemannian manifold, with $n \geq 2$ and $p \geq 0$, $I \subset \mathbb{R}$ is an open interval and $f : I \to \mathbb{R}$ is a positive smooth function, endowed with the following Lorentzian warped metric

$$\langle\, ,\rangle := -dt^2 + f(t)^2 \langle\, ,\rangle_M, \tag{1.1.1}$$

where dt^2 stands for the standard metric of $I \subset \mathbb{R}$ (cf. [10]). When M^{n+p} has constant sectional curvature, $-I \times_f M^{n+p}$ has been known in the mathematical literature as a Robertson-Walker (RW) spacetime, having a strong allusion to its study in the case $n = 3$, where it is an exact solution of Einstein's field equations (see, for instance, [66, Chapter 12]).

In this setting, we will consider the closed conformal timelike vector field

$$\mathcal{K}(t, y) \;=\; f(t)\partial_t|_{(t,y)}, \qquad (t, y) \in -I \times_f M^{n+p}, \tag{1.1.2}$$

which is globally defined on $-I \times_f M^{n+p}$, where $\partial_t = \frac{\partial}{\partial_t}$ stands for the coordinate timelike vector field tangent to I. From the relationship between the Levi-Civita connections of $-I \times_f M^{n+p}$ and those of I and M^{n+p} (see [66, Proposition 7.35]), it follows that

$$\overline{\nabla}_V \mathcal{K} = f'(\pi_I)V, \tag{1.1.3}$$

for all $V \in \mathfrak{X}(-I \times_f M^{n+p})$, where $\overline{\nabla}$ is the Levi-Civita connection of $-I \times_f M^{n+p}$ and π_I is the projection of $-I \times_f M^{n+p}$ onto its factor I. In fact, for every vector field $V \in \mathfrak{X}(-I \times_f M^{n+p})$ we can write $V = -a\partial_t + V^*$, where $a = \langle V, \partial_t\rangle \in C^\infty(M^{n+p})$ and $V^* \in \mathfrak{X}(M^{n+p})$. Then,

$$\begin{aligned}
\overline{\nabla}_V \mathcal{K} &= \overline{\nabla}_{(-a\partial_t + V^*)} f(t)\partial_t \\
&= -a\overline{\nabla}_{\partial_t} f(t)\partial_t + \overline{\nabla}_{V^*} f(t)\partial_t \\
&= -af'(t)\partial_t + f(t)\left(\frac{\partial_t(f(t))}{f(t)}\right)V^* \\
&= f'(t)(-a\partial_t + V^*) = f'(\pi_I)V
\end{aligned}$$

It follows from (1.1.2) and (1.1.3) that

$$\begin{aligned}
\overline{\nabla}_V \partial_t &= \overline{\nabla}_V \left(\frac{1}{f(t)}\mathcal{K}\right) \\
&= \frac{1}{f(t)} f'(t)V - a\partial_t\left(\frac{1}{f(t)}\right)\mathcal{K} + V^*\left(\frac{1}{f(t)}\right)\mathcal{K} \\
&= \frac{1}{f(t)} f'(t)V + a\left(\frac{\partial_t(f(t))}{f(t)^2}\right)\mathcal{K} \\
&= -\frac{1}{f(t)^2}\langle V, \overline{\nabla} f\rangle \mathcal{K} + \frac{1}{f(t)} f'(t)V.
\end{aligned} \tag{1.1.4}$$

On the other hand, a simple computation shows that the gradient of the projection $\pi_I(t, y) = t$ is given by

$$\overline{\nabla}\pi_I = -\langle \overline{\nabla}\pi_I, \partial_t\rangle\partial_t = -\partial_t. \tag{1.1.5}$$

Then,

$$\overline{\nabla} f = \overline{\nabla}(f \circ \pi_I) = -f'(t)\partial_t.$$

Thus, using (1.1.4) and (1.1.5) we obtain

$$\overline{\nabla}_V \partial_t = \frac{f'(\pi_I)}{f(\pi_I)}\{V + \langle V, \partial_t\rangle\partial_t\}. \tag{1.1.6}$$

1.2 Spacelike mean curvature flow soliton

In what follows, we deal with a connected spacelike submanifold $x : \Sigma^n \looparrowright -I \times_f M^{n+p}$ of codimension $(p+1)$, which means that the induced metric on Σ^n

from the metric of $-I \times_f M^{n+p}$ is positive definite and there exists an orthonormal basis $\{N_1, \ldots, N_{p+1}\}$ of the normal bundle $\mathfrak{X}^\perp(\Sigma)$ constituted by a future (resp. past) timelike normal unit vector field N_{p+1}, that is, $\langle N_{p+1}, \partial_t\rangle \leq -1$ (resp. $\langle N_{p+1}, \partial_t\rangle \geq -1$) on Σ^n, and spacelike normal unit vector fields $N_1, \ldots, N_p$.

In this section and throughout Part I of this work, we will denote by A and $\vec{H} = -\mathrm{tr}(A)$ the second fundamental form and the non-normalized mean curvature vector of the spacelike submanifold $x : \Sigma^n \looparrowright -I \times_f M^{n+p}$, respectively. Now, we are in a position to define our object of study which is inspired in [9, Definition (1.1)] and [32, Definition (1.1)].

Definition 1.2.1 *A spacelike submanifold $x : \Sigma^n \looparrowright -I \times_f M^{n+p}$ is said to be a spacelike mean curvature flow soliton with respect to $\mathcal{K} = f(t)\partial_t$ and with soliton constant $c \in \mathbb{R}$ if its mean curvature vector $\vec{H}$ satisfies*

$$\vec{H} = c\mathcal{K}^\perp, \tag{1.2.1}$$

where $\mathcal{K}^\perp$ stands from the orthogonal projection of $\mathcal{K}$ along Σ^n. When $f \equiv 1$, such a spacelike mean curvature flow soliton is called a spacelike translation soliton with respect to ∂_t.

1.2.1 Some previous computations and basic results

Given a spacelike mean curvature flow soliton $x : \Sigma^n \looparrowright -I \times_f M^{n+p}$, its *height function* h is the restriction of the projection $\pi_I(t, y) = t$ to Σ^n, that is, $h : \Sigma^n \to I$ is given by

$$h = \pi_I|_{\Sigma^n} = \pi_I \circ x. \tag{1.2.2}$$

From (1.1.5) and (1.2.2), we get that the gradient of h on Σ^n is

$$\nabla h = (\overline{\nabla}\pi_I)^\top = -\partial_t^\top = -\partial_t + \partial_t^\perp, \tag{1.2.3}$$

where $\partial_t = \partial_t^\top + \partial_t^\perp$. Here $\partial_t^\top \in \mathfrak{X}(\Sigma^n)$ and $\partial_t^\perp \in \mathfrak{X}^\perp(\Sigma^n)$ denote, respectively, the tangential and normal components of ∂_t with respect to Σ^n. Since $\partial_t^\perp \in \mathfrak{X}^\perp(\Sigma^n)$, we can write

$$\partial_t^\perp = \sum_{i=1}^{p+1} \varepsilon_i \Theta_i N_i, \tag{1.2.4}$$

where $\varepsilon_i = \mathrm{sgn}(\langle N_i, N_i\rangle)$ and $\Theta_i = \langle N_i, \partial_t\rangle$. Thus, from (1.2.3) and (1.2.4)

$$\nabla h = -\partial_t + \sum_{i=1}^{p+1} \varepsilon_i \Theta_i N_i. \tag{1.2.5}$$

Consequently, from (1.2.5) we obtain the following relation

$$|\nabla h|^2 = -1 - \sum_{i=1}^{p+1} \varepsilon_i \Theta_i^2, \tag{1.2.6}$$

where $|\cdot|$ stands from the norm of a tangent vector field on Σ^n, by considering its induced metric. Moreover, from (1.1.6) and (1.2.5) we deduce that, for any $X \in \mathfrak{X}(\Sigma^n)$, the Hessian of h in the metric $\langle\,,\rangle$ is given by

$$\begin{aligned}
\nabla^2 h(X,X) &= \langle \nabla_X \nabla h, X\rangle \\
&= -\langle \nabla_X \partial_t, X\rangle + \left\langle \nabla_X \left(\sum_{i=1}^{p+1} \varepsilon_i \Theta_i N_i\right), X\right\rangle \\
&= -\frac{f'(h)}{f(h)}\{|X|^2 + \langle X, \nabla h\rangle^2\} - \sum_{i=1}^{p+1} \varepsilon_i \langle A_{N_i} X, X\rangle \Theta_i,
\end{aligned} \tag{1.2.7}$$

where A_{N_i} denotes the Weingarten operator with respect to N_i.

In the sequence of this manuscript, we will also consider the function

$$u = g(h) \in C^\infty(\Sigma^n), \tag{1.2.8}$$

where $g : I \to \mathbb{R}$ is an arbitrary primitive of the warping function f. Since $g' = f > 0$, $u = g(h)$ can be thought as a reparametrization of the height function. In particular, from (1.2.3) we have that the gradient of u on Σ^n is given by

$$\nabla u = f(h)\nabla h = -f(h)\partial_t^\top = -\mathcal{K}^\top, \tag{1.2.9}$$

where $\mathcal{K}^\top$ denotes the tangential component of the closed conformal vector field $\mathcal{K}$ defined in (1.1.2).

In order to calculate the Laplacian of the function u, we consider $X \in \mathfrak{X}(\Sigma)$ and, using (1.2.9), we evaluate the Hessian of u

$$\begin{aligned}
\nabla^2 u(X,X) = \langle \overline{\nabla}_X \nabla u, X\rangle = \langle \overline{\nabla}_X (f(h)\nabla h), X\rangle &= f(h)\langle \overline{\nabla}_X \nabla h, X\rangle + X(f(h))\langle \nabla h, X\rangle \\
&= f(h)\nabla^2 h(X,X) + \langle \nabla(f(h)), X\rangle\langle \nabla h, X\rangle \\
&= f(h)\nabla^2 h(X,X) + f'(h)\langle \nabla h, X\rangle^2.
\end{aligned} \tag{1.2.10}$$

Then, replacing (1.2.7) into (1.2.10) we get

$$\begin{aligned}\nabla^2 u(X,X) &= f(h)\left(-\frac{f'(h)}{f(h)}\{|X|^2+\langle X,\nabla h\rangle^2\}-\sum_{i=1}^{p+1}\varepsilon_i\langle A_{N_i}X,X\rangle\Theta_i\right)\\ &\quad + f'(h)\langle\nabla h,X\rangle^2\\ &= -f'(h)|X|^2 - f(h)\sum_{i=1}^{p+1}\varepsilon_i\langle A_{N_i}X,X\rangle\Theta_i. \qquad (1.2.11)\end{aligned}$$

Therefore, considering a local orthonormal frame $\{E_1,\ldots,E_n\}$ in Σ^n, from (1.2.11), we conclude that the standard Laplacian of u is given by

$$\begin{aligned}\Delta u &= \sum_{j=1}^{n}\nabla^2 u(E_j,E_j)\\ &= -nf'(h) - f(h)\sum_{j=1}^{n}\sum_{i=1}^{p+1}\varepsilon_i\langle A(E_j,E_j),N_i\rangle\langle N_i,\partial_t\rangle\\ &= -nf'(h) + f(h)\sum_{i=1}^{p+1}\varepsilon_i\langle \vec{H},N_i\rangle\langle N_i,\partial_t^\perp\rangle\\ &= -nf'(h) + cf^2(h)\sum_{i=1}^{p+1}\varepsilon_i\langle \partial_t,N_i\rangle\langle N_i,\partial_t^\perp\rangle\\ &= -nf'(h) + cf^2(h)\langle\partial_t^\perp,\partial_t^\perp\rangle\\ &= -nf'(h) + cf^2(h)\langle\partial_t+\nabla h,\partial_t+\nabla h\rangle\\ &= -nf'(h) + cf^2(h)\{-1+\langle\nabla h,\partial_t\rangle\}. \qquad (1.2.12)\end{aligned}$$

Now, we consider the following *drift Laplacian* on Σ^n

$$\Delta_{-cu}(\varphi) = \Delta\varphi + \langle\nabla(cu),\nabla\varphi\rangle, \qquad (1.2.13)$$

where $\varphi\in C^\infty(\Sigma^n)$. Then, since (1.2.9), (1.2.12) and (1.2.13), we have

$$\begin{aligned}\Delta_{-cu}(u) &= \Delta u + \langle\nabla(cu),\nabla u\rangle\\ &= -nf'(h) - cf^2(h) + cf^2(h)\langle\nabla h,\partial_t\rangle - cf^2(h)\langle\nabla h,\partial_t\rangle = -\zeta_c(h), \qquad (1.2.14)\end{aligned}$$

where, adopting the terminology introduced in [9] and [32], $\zeta_c(h) = nf'(h)+cf^2(h)$ is called the *soliton function* associated to the mean curvature flow soliton $x:\Sigma^n \looparrowright -I\times_f M^{n+p}$.

Remark 1.2.1 It is easy to check that if the slice $M_{t_*}=\{t_*\}\times M^{n+p}$ is a spacelike mean curvature flow soliton with respect to $\mathcal{K}=f(t)\partial_t$ then the soliton constant c is given by

$$c = -n\frac{f'(t_*)}{f(t_*)^2}. \tag{1.2.15}$$

Consequently, t_* is implicitly given by the condition $\zeta_c(t_*) = 0$. In fact, in the slice M_{t_*} the height function h is constant with $h = t_*$, therefore u is also constant, then $\Delta_{-cu}u = 0$. From (1.2.14), we get $\zeta_c(h) = nf'(t_*) + cf^2(t_*) = 0$. Consequently, we obtain (1.2.15).

Example 1 Let us consider the $(n+1)$-dimensional de Sitter space $\mathbb{S}_1^{n+1}$ given by

$$\mathbb{S}_1^{n+1} = \{x \in \mathbb{R}_1^{n+2} : \langle x, x\rangle = 1\},$$

where $\mathbb{R}_1^{n+2}$ is the $(n+2)$-dimensional Minkowski space. From [64, Example 4.2], $\mathbb{S}_1^{n+1}$ is isometric to the GRW spacetime $-\mathbb{R} \times_{\cosh t} \mathbb{S}^n$, where $\mathbb{S}^n$ denotes the n-dimensional unit Euclidean sphere endowed with its standard metric. From (1.2.15) we see that the slices $\{\sinh^{-1}(\frac{-n\pm\sqrt{n^2-4c^2}}{2c})\} \times \mathbb{S}^n$ are spacelike mean curvature flow soliton with respect to $\mathcal{K} = \cosh t\,\partial_t$ and with soliton constant $0 < |c| \le \frac{n}{2}$.

Example 2 Let us consider the $(n+1)$-dimensional anti-de Sitter space $\mathbb{H}_1^{n+1}$ given by

$$\mathbb{H}_1^{n+1} = \{x \in \mathbb{R}_2^{n+2} : \langle x, x\rangle = -1\}.$$

Motivated by [64, Example 4.3], we will consider the open subset of $\mathbb{H}_1^{n+1}$ which is isometric to the GRW spacetime $-(-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos t} \mathbb{H}^n$. From (1.2.15) we see that the slices $\{\sin^{-1}(\frac{-n\pm\sqrt{n^2+4c^2}}{2c})\} \times \mathbb{H}^n$ are spacelike mean curvature flow solitons with respect to $\mathcal{K} = \cos t\,\partial_t$ and with soliton constant $c \neq 0$.

Example 3 The 4-dimensional Einstein-de Sitter spacetime $-\mathbb{R}^+ \times_{t^{\frac{2}{3}}} \mathbb{R}^3$, where $\mathbb{R}^3$ stands for the 3-dimensional Euclidean space endowed with its canonical metric, is a classical exact solution to the Einstein field equation without cosmological constant. Here, let us consider the $(n+1)$-dimensional Einstein-de Sitter spacetime $-\mathbb{R}^+ \times_{t^{\frac{2}{3}}} \mathbb{R}^n$. From (1.2.15) we conclude that the slices $\{(-\frac{2n}{3c})^{\frac{3}{5}}\} \times \mathbb{R}^n$ are spacelike mean curvature flow solitons with respect to $\mathcal{K} = t^{\frac{2}{3}}\partial_t$ and with soliton constant $c < 0$.

According to [69], we recall that a spacelike submanifold of a spacetime is called *trapped* if its mean curvature vector is timelike.

We have an important fact about spacelike mean curvature flow solitons in a GRW spacetime: They correspond to particular cases of trapped submanifolds. See the next Proposition.

Proposition A *Let $x : \Sigma^n \looparrowright -I \times_f M^{n+p}$ be a spacelike mean curvature flow soliton with respect to $\mathcal{K} = f(t)\partial_t$ and with soliton constant $c \neq 0$. Then, Σ^n is a trapped submanifold of $-I \times_f M^{n+p}$.*

Proof. Indeed, from (1.2.1) and (1.2.3) we have that

$$\langle \vec{H}, \vec{H} \rangle = c^2 f^2 \langle \partial_t^\perp, \partial_t^\perp \rangle = c^2 f^2 \langle \partial_t - \partial_t^\top, \partial_t - \partial_t^\top \rangle = -c^2 f^2 (1 + |\nabla h|^2) < 0. \qquad (1.2.16)$$

Therefore, from (1.2.16) we conclude that $\vec{H}$ is timelike, which means that Σ^n is a trapped submanifold of $-I \times_f M^{n+p}$. ■

We close this subsection establishing the following key lemma.

Lemma 1.2.1 *Let $x : \Sigma^n \looparrowright -I \times_f M^{n+p}$ be a spacelike mean curvature flow soliton with respect to $\mathcal{K} = f(t)\partial_t$ and with soliton constant $c \neq 0$. Then*

$$\langle A_{\vec{H}} X, Y \rangle + c\nabla^2 u(X, Y) = -cf'(h)\langle X, Y \rangle, \qquad (1.2.17)$$

for all $X, Y \in \mathfrak{X}(\Sigma)$, where $A_{\vec{H}}$ denotes the Weingarten operator with respect to $\vec{H}$. Furthermore,

$$\nabla |\vec{H}|^2 = -2cA_{\vec{H}}(\nabla u),$$

where $|\vec{H}|^2 := -\langle \vec{H}, \vec{H} \rangle$.

Proof. Considering the function $u = g(h)$ as in (1.2.8), from (1.2.11) we get

$$c\nabla^2 u(X, Y) = -cf'(h)\langle X, Y \rangle - cf(h) \sum_{i=1}^{p+1} \varepsilon_i \langle A_{N_i} X, Y \rangle \Theta_i, \qquad (1.2.18)$$

for all $X, Y \in \mathfrak{X}(\Sigma)$.

Observe that we can write

$$\vec{H} = \sum_{i=1}^{p+1} \varepsilon_i H_{N_i} N_i,$$

where $\{N_1, \ldots, N_{p+1}\}$ is a local orthonormal frame on $\mathfrak{X}^\perp(\Sigma)$ and $H_{N_i} = \langle \vec{H}, N_i \rangle$. Thus, since $\vec{H} = cf(h)\partial_t^\perp$, using (1.2.18), we conclude that

$$-cf'(h)\langle X, Y \rangle = \sum_{i=1}^{p+1} \varepsilon_i \langle A_{N_i} X, Y \rangle \langle N_i, cf(h)\partial_t \rangle + c\nabla^2 u(X, Y)$$

$$
\begin{aligned}
&= \sum_{i=1}^{p+1} \varepsilon_i \langle A_{N_i} X, Y\rangle H_{N_i} + c\nabla^2 u(X,Y) \\
&= \sum_{i=1}^{p+1} \varepsilon_i \langle A(X,Y), N_i\rangle H_{N_i} + c\nabla^2 u(X,Y) \\
&= \sum_{i=1}^{p+1} \langle A(X,Y), \varepsilon_i H_{N_i} N_i\rangle + c\nabla^2 u(X,Y) \\
&= \langle A(X,Y), \vec{H}\rangle + c\nabla^2 u(X,Y) \\
&= \langle A_{\vec{H}} X, Y\rangle + c\nabla^2 u(X,Y)
\end{aligned}
$$

and, hence, we obtain (1.2.17).

On the other hand, we note that

$$
\begin{aligned}
c\bar{\nabla}_X \mathcal{K} &= c\bar{\nabla}_X (f(t)\partial_t) \\
&= c\bar{\nabla}_X (f(t)\partial_t^{\top}) + c\bar{\nabla}_X (f(t)\partial_t^{\perp}) \\
&= c\nabla_X (f(t)\partial_t^{\top}) + cA(X, f(t)\partial_t^{\top}) + c\bar{\nabla}_X \left(\frac{\vec{H}}{c}\right) \\
&= c\nabla_X (f(t)\partial_t)^{\top} + cA(X, f(t)\partial_t^{\top}) + \bar{\nabla}_X \vec{H}.
\end{aligned}
$$

In particular, from (1.1.3) we have

$$
0 = (cf'X)^{\perp} = (c\bar{\nabla}_X \mathcal{K})^{\perp} = (c\nabla_X (f(t)\partial_t)^{\top} + cA(X, f(t)\partial_t^{\top}) + \bar{\nabla}_X \vec{H})^{\perp}. \tag{1.2.19}
$$

Thus, from (1.2.19) we get

$$
-cA(X, \nabla u) = cA(X, f(t)\partial_t^{\top}) = -(\bar{\nabla}_X \vec{H})^{\perp}.
$$

Finally, we obtain

$$
-c\langle A_{\vec{H}} \nabla u, X\rangle = -c\langle A(X, \nabla u), \vec{H}\rangle = -\langle (\bar{\nabla}_X \vec{H})^{\perp}, \vec{H}\rangle = -\frac{1}{2} X(\langle \vec{H}, \vec{H}\rangle) = \frac{1}{2}\langle \nabla |\vec{H}|^2, X\rangle.
$$

Therefore,

$$
\nabla |\vec{H}|^2 = -2cA_{\vec{H}}(\nabla u).
$$

■

1.2.2 Spacelike submanifolds contained in slices

In this subsection, we establish a necessary condition for a spacelike mean curvature flow soliton to be contained in a slice of a GRW spacetime.

Proposition B *Let* $x : \Sigma^n \looparrowright -I \times_f M^{n+p}$ $(p > 0)$ *be a spacelike mean curvature flow soliton with respect to* $\mathcal{K} = f(t)\partial_t$ *and with soliton constant* c. *If there exists* $t \in I$ *such that* $x(\Sigma^n)$ *is contained in a slice* $\{t\} \times M^{n+p}$, *then* $\phi := \pi_M \circ x : \Sigma^n \looparrowright M^{n+p}$ *is a minimal submanifold, where* π_M *denotes the projection of* $-I \times_f M^{n+p}$ *onto its Riemannian fiber* M^{n+p}.

Proof. Let us denote by $\langle , \rangle_\Sigma$ the Riemannian metric induced on Σ^n via ϕ. For a fixed $t \in I$, we can consider $\phi_t : \Sigma^n \looparrowright -I \times_f M^{n+p}$ given by

$$\phi_t(q) = (t, \phi(q)),\ \forall q \in \Sigma^n.$$

So, ϕ_t is a spacelike immersion of Σ^n into $-I \times_f M^{n+p}$, with codimension $p+1$, which is contained in the slice $M_t = \{t\} \times M$, and such that the metric induced on Σ^n via ϕ_t from the Lorentzian metric (1.1.1) is given by

$$\langle , \rangle_t = \phi_t^*(\langle , \rangle) = f^2(t)\langle , \rangle_\Sigma. \tag{1.2.20}$$

Now, our objective is to compute the second fundamental form A_t of the immersion ϕ_t. Let us consider $\{N_i\}_{i=1}^p$ a local orthonormal frame of the normal bundle of ϕ. It follows that the vector fields

$$\eta_i(\phi_t(q)) = \frac{1}{f(t)} N_i(\phi(q)),\ 1 \le i \le p, \tag{1.2.21}$$

and

$$\eta_{p+1}(\phi_t(q)) = \partial_t|_{\phi_t(q)}$$

define a local orthonormal frame of normal vector fields along the immersion ϕ_t. Observe that the Weingarten operator A_{η_i} of ϕ_t with respect to η_i is given by

$$A_{\eta_i} X = \frac{1}{f(t)} A_{N_i} X,\ X \in \mathfrak{X}(\Sigma), \tag{1.2.22}$$

where A_{N_i} stands for the Weingarten operator of $\phi : \Sigma^n \looparrowright M^{n+p}$ with respect to the normal direction N_i. Furthermore, it follows from (1.1.3) that

$$A_{\eta_{p+1}} X = -(\bar{\nabla}_X \partial_t)^\top = -\frac{f'(t)}{f(t)} X,\ X \in \mathfrak{X}(\Sigma). \tag{1.2.23}$$

Note that, given $X, Y \in \mathfrak{X}(\Sigma)$, from (1.2.20) and (1.2.21) we get

$$\langle A_{N_i} X, Y\rangle \eta_i = f(t)\langle A_{N_i} X, Y\rangle_\Sigma N_i, \forall\ 1 \le i \le p. \tag{1.2.24}$$

Thus, from (1.2.22), (1.2.23) and (1.2.24), we have that the second fundamental form A_t of the immersion ϕ_t is given by

$$\begin{aligned} A_t(X,Y) &= \sum_{i=1}^{p+1} \varepsilon_i \langle A_{\eta_i} X, Y\rangle \eta_i \\ &= \frac{1}{f(t)} \sum_{i=1}^{p} \varepsilon_i \langle A_{N_i} X, Y\rangle \eta_i + \frac{f'(t)}{f(t)} \langle X, Y\rangle \partial_t \\ &= \sum_{i=1}^{p} \varepsilon_i \langle A_{N_i} X, Y\rangle_\Sigma N_i + \frac{f'(t)}{f(t)} \langle X, Y\rangle \partial_t \\ &= A_\phi(X,Y) + \frac{f'(t)}{f(t)} \langle X, Y\rangle \partial_t, \end{aligned} \tag{1.2.25}$$

where A_ϕ denotes the second fundamental form of the immersion ϕ. Taking traces in both sides of (1.2.25) with respect to the metric $\langle , \rangle_t$, we see that the mean curvature vector $\vec{H}_t$ of ϕ_t is

$$\vec{H}_t = \frac{1}{f(t)^2} \vec{H}_\phi - n \frac{f'(t)}{f(t)} \partial_t, \tag{1.2.26}$$

where $\vec{H}_\phi = -\mathrm{tr}_{\langle,\rangle_\Sigma}(A_\phi)$ is the mean curvature vector of $\phi(\Sigma)$.

Since ∂_t is normal to $\phi_t(\Sigma)$, we have that $\mathcal{K} = \mathcal{K}^\perp = f(t)\partial_t$ along ϕ_t. So, from (1.2.26) we conclude that $\phi_t : \Sigma^n \looparrowright -I \times_f M^{n+p}$ is a mean complete curvature flow soliton with respect to $\mathcal{K} = f(t)\partial_t$ if and only if $\vec{H}_\phi \equiv 0$, that is, $\phi := \pi_M \circ x : \Sigma^n \looparrowright M^{n+p}$ is a minimal submanifold. ■

1.3 Omori-Yau's maximum principle

In this section, for the sake of completeness, we will present a Omori-Yau maximum principle due to Chen and Qiu in [28, Theorem 1].

First, let us consider the operator $\Delta_V := \Delta - \langle V, \nabla\rangle$ for a vector field V on a Riemannian manifold (M, g), and denote by $\mathrm{Ric}_V := \mathrm{Ric} - \frac{1}{2} L_V g$ the Bakry–Emery Ricci tensor. In this context, Chen and Qiu established the following Omori-Yau maximum principle.

Theorem 1.3.1 *Let (M^m, g) be a complete Riemannian manifold, V a C^1 vector field on M. If $\mathrm{Ric}_V \geq -F(r)g$, where r is the distance function on M from a fixed point*

$x_0 \in M$, $F : \mathbb{R} \to \mathbb{R}$ is a positive continuous function satisfying

$$\varphi(t) := \int_{\rho_0+1}^{t} \frac{dr}{\int_{\rho_0}^{r} F(s)ds + 1} \to +\infty \quad (t \to +\infty)$$

for some positive constant ρ_0. Let $f \in C^2(M)$ with $\lim\limits_{x\to\infty} \frac{f(x)}{\varphi(r(x))} = 0$, then there exist points $\{x_j\} \subset M$, such that

$$\lim_{j\to\infty} f(x_j) = \sup f, \quad \lim_{j\to\infty} |\nabla f|(x_j) = 0, \quad \lim_{j\to\infty} \Delta_V f(x_j) \le 0. \tag{1.3.1}$$

Proof. For any $u \in C^3(M)$, we know that the Bochner formula holds, namely

$$\frac{1}{2}\Delta|\nabla u|^2 = |\operatorname{Hess}(u)|^2 + \operatorname{Ric}(\nabla u, \nabla u) + \langle \nabla \Delta u, \nabla u\rangle.$$

This implies that

$$\frac{1}{2}\Delta_V|\nabla u|^2 = |\operatorname{Hess}(u)|^2 + \operatorname{Ric}(\nabla u, \nabla u) + \langle \nabla \Delta u, \nabla u\rangle + \frac{1}{2}\left\langle V, \nabla|\nabla u|^2\right\rangle.$$

Choose $\{e_1, \ldots, e_m\}$ a local orthonormal frame field on M at the considered point. Direct computation gives

$$\begin{aligned}
\left\langle V, \nabla|\nabla u|^2\right\rangle &= 2V^i u_{ji} u_j,\\
\nabla\langle V, \nabla u\rangle &= \left(V_i^j u_j + V^k u_{ki}\right) e_i\\
\frac{1}{2}\left(L_V g\right)(\nabla u, \nabla u) &= V^j u_i u_{ij} - \left(V^i u_{ji} - u_i V_i^j\right) u_j = u_i V_i^j u_j.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\frac{1}{2}\Delta_V|\nabla u|^2 =& |\operatorname{Hess}(u)|^2 + \operatorname{Ric}(\nabla u, \nabla u) + \langle \nabla \Delta_V u, \nabla u\rangle\\
&- \langle \nabla\langle V, \nabla u\rangle, \nabla u\rangle + \frac{1}{2}\left\langle V, \nabla|\nabla u|^2\right\rangle\\
=& |\operatorname{Hess}(u)|^2 + \operatorname{Ric}(\nabla u, \nabla u) + \langle \nabla \Delta_V u, \nabla u\rangle\\
&- \left\langle \left(V_i^j u_j + V^k u_{ki}\right) e_i, u_l \epsilon_l\right\rangle + V^i u_{ji} u_j\\
=& |\operatorname{Hess}(u)|^2 + \operatorname{Ric}_V(\nabla u, \nabla u) + \langle \nabla \Delta_V u, \nabla u\rangle.
\end{aligned} \tag{1.3.2}$$

If x is not on the cut locus of x_0, and for $r \ge r_0$ (r_0 is a positive constant), let $\sigma : [0, r] \to M$ be a minimal unit speed geodesic with $\sigma(0) = x_0, \sigma(r) = x$. Set $\varphi_V(s) = (\Delta_V r) \circ \sigma(s), s \in (0, r]$. Applying the Bochner formula (1.3.2), we have

$$|\operatorname{Hess}(r)|^2 + \operatorname{Ric}_V(\nabla r, \nabla r) + \langle \nabla \Delta_V r, \nabla r\rangle = \frac{1}{2}\Delta_V|\nabla r|^2 = 0.$$

Therefore,

$$\langle \nabla \Delta_V r, \nabla r\rangle \leq -\operatorname{Ric}_V(\nabla r, \nabla r)$$

Computing both sides of the above inequality along $\sigma(s)$ gives

$$\varphi_V'(s) \leq -\operatorname{Ric}_V(\sigma', \sigma') \leq F(r(\sigma(s))) = F(s) \text{ on } (0, r]. \tag{1.3.3}$$

Choosing $\rho_0 > 0$ small enough so that the geodesic ball $B_{\rho_0}(x_0)$ lies in the neighborhood of the normal coordinate at x_0. Let $C_1 = \max_{\partial B_{\rho_0}(x_0)} \Delta_V r$. Then integrating (1.3.3) from ρ_0 to $r(x)$, we have

$$\Delta_V r(x) \leq \int_{\rho_0}^{r} F(s)ds + C_1.$$

For simplicity, we denote $W(t) := \int_{\rho_0}^{t} F(s)ds + 1$, then $\varphi(t) = \int_{\rho_0+1}^{t} \frac{dr}{W(r)}$, and we have

$$\Delta_V r(x) \leq CW(r)$$

for some constant $C > 0$.

Let $\{\varepsilon_j\}$ be a sequence of positive real numbers such that $\varepsilon_j \to 0$ as $j \to \infty$. Define

$$f_j(x) = f(x) - \varepsilon_j \varphi(r(x)), \quad \forall j.$$

By the condition on f, $f_j \to -\infty$ as $x \to \infty$, so f_j attains its maximum at some point $x_j \in M$. Consequently, we have

$$\nabla f_j(x_j) = 0 \text{ and } \Delta_V f_j(x_j) \leq 0.$$

To prove the last two conclusions in (1.3.1), if $\{r(x_j)\}$ is bounded, then there is a subsequence of $\{x_j\}$ converging to some point $x \in M$, at which f attains its maximum, in this case, the conclusions follow easily. Now we assume that $\{r(x_j)\} \to +\infty$ as $j \to +\infty$. Without loss of generality, we can suppose that x_j is not on the cut locus of x_0. Otherwise, we can use Calabi's trick to remedy it (cf. [22, 29]).

Since

$$\varphi'(t) = \frac{1}{W(t)} > 0, \quad \varphi''(t) \leq 0, \quad \text{for} \quad t \in [\rho_0, +\infty).$$

We obtain

$$|\nabla f|(x_j) = \varepsilon_j |\nabla \varphi(r)|(x_j) = \varepsilon_j \varphi'(r)(x_j)|\nabla r|(x_j) \to 0 \text{ as } j \to \infty,$$

and

$$\begin{aligned}\Delta_V f(x_j) &\le \varepsilon_j \left\{\varphi'(r(x_j))\,\Delta_V r(x_j) + \varphi''(r(x_j))\,|\nabla r(x_j)|^2\right\}\\ &\le \varepsilon_j \varphi'(r(x_j))\,\Delta_V r(x_j)\\ &\le \varepsilon_j \frac{1}{W(r(x_j))}\cdot CW(r(x_j)) \to 0 \text{ as } j\to\infty.\end{aligned}$$

It remains to prove $\lim\limits_{j\to+\infty} f(x_j) = \sup f$. If there exists a subsequence $\{xj_k\} \neq \{x_j\}$, such that $\lim\limits_{k\to+\infty} f(x_{j_k}) = \sup f$, then by still denoting $\{x_{j_k}\}$ as x_j, the proof is completed. Otherwise, we claim that $\lim\limits_{j\to+\infty} f(x_j) = \sup f$ (if $\sup f = \infty$, then we claim that $\lim\limits_{j\to+\infty} \sup f(x_j) = \infty$). Indeed, if this were not true, there would exist $\hat{x} \in M$ and $\delta > 0$, such that

$$f(\hat{x}) > f(x_j) + \delta \tag{1.3.4}$$

for each $j \ge j_0$ sufficiently large.

Since

$$f(x_j) - \varepsilon_j\varphi(r(x_j)) = f_j(x_j) \ge f_j(\hat{x}) = f(\hat{x}) - \varepsilon_j\varphi(r(\hat{x})), \tag{1.3.5}$$

we then have

$$f(x_j) \ge f(\hat{x}) + \varepsilon_j(\varphi(r(x_j)) - \varphi(r(\hat{x}))).$$

If $r(x_j) \to +\infty$ as $j \to +\infty$, then for j large enough, we have $\varphi(r(x_j)) > \varphi(r(\hat{x}))$, that is $f(x_j) > f(\hat{x})$, which contradicts (1.3.4).

If $\{x_j\}$ lies in a compact set, then for some subsequence of j, x_j converges to a point $\bar{x}$, so that $f(\hat{x}) \ge f(\bar{x}) + \delta$. On the other hand, we can deduce from (1.3.5) that

$$f(\bar{x}) \ge f(\hat{x}).$$

This is also a contradiction. This proves (1.3.1). ∎

Chapter 2

Nonexistence and rigidity of spacelike mean curvature flow solitons

2.1 Nonexistence and rigidity results via integrability

In what follows, let Σ^n be an oriented submanifold. So, we denote the space of Lebesgue integrable functions on Σ^n by

$$\mathcal{L}^1(\Sigma^n) = \left\{\varphi \in C^\infty(\Sigma^n) : \int_{\Sigma^n} |\varphi| d\Sigma \ll +\infty\right\},$$

where $d\Sigma$ stands for the volume element induced by the metric of Σ^n. Furthermore, we denote by $\mathcal{L}^1_{-cu}(\Sigma^n)$ the set of Lebesgue integrable functions on Σ^n with respect to the modified volume element

$$d\mu = e^{cu} d\Sigma. \tag{2.1.1}$$

It follows from (1.2.13) that

$$\Delta_{-cu}(\varphi) = e^{-cu}\mathrm{div}_\Sigma(e^{cu}\nabla\varphi), \tag{2.1.2}$$

for all $\varphi \in C^\infty(\Sigma^n)$, where div_Σ is the standard divergence relative to the metric of Σ^n. In this context, a smooth function φ on Σ^n is said to be $(-cu)$-*subharmonic* (respectively, $(-cu)$-*superharmonic*) if $\Delta_{-cu}(\varphi) \geq 0$ (respectively, $\Delta_{-cu}(\varphi) \leq 0$) on Σ^n.

Taking into account these previous considerations, it is not difficult to see that the following extension of a result due to Yau in [78, Page 660] holds.

Lemma 2.1.1 *Let Σ^n be an n-dimensional complete oriented Riemannian manifold. If $\varphi \in C^\infty(\Sigma^n)$ is a $(-cu)$-subharmonic function (or a $(-cu)$-superharmonic function) on Σ^n and $|\nabla\varphi| \in \mathcal{L}^1_{-cu}(\Sigma^n)$, then $\Delta_{-cu}(\varphi) = 0$ on Σ^n.*

We recall that a *slab* of $-I \times_f M^{n+p}$ is a timelike bounded region of the type

$$[t_*; t^*] \times M^{n+p} = \{(t; p) \in -I \times_f M^{n+p} : t_* \leq t \leq t^* \text{ and } p \in M^{n+p}\}.$$

So, we are in position to establish the following result.

Theorem 2.1.1 *Let $x : \Sigma^n \looparrowright -I \times_f M^{n+p}$ be a complete spacelike mean curvature flow soliton with respect to $\mathcal{K} = f(t)\partial_t$ and with soliton constant $c \in \mathbb{R}$, lying in a slab of $-I \times_f M^{n+p}$ and whose height function h satisfies $|\nabla h| \in \mathcal{L}^1(\Sigma^n)$. If the soliton function $\zeta_c(h) = nf'(h) + cf^2(h)$ does not change the sign, then there exists $t \in I$ such that $x(\Sigma^n)$ is contained in the slice $\{t\} \times M^{n+p}$.*

Proof. Since $x : \Sigma^n \looparrowright -I \times_f M^{n+p}$ lies in a slab of $-I \times_f M^{n+p}$, we have that h is bounded on Σ^n and, therefore, $u = g(h)$ is bounded on Σ^n and, consequently, the same happens with the function e^{cu}. Then, from $|\nabla h| \in \mathcal{L}^1(\Sigma^n)$ and (2.1.1), we get $|\nabla h| \in \mathcal{L}^1_{-cu}(\Sigma^n)$. Therefore, $|\nabla u| \in \mathcal{L}^1_{-cu}(\Sigma^n)$. Furthermore, from (1.2.14), $\Delta_{-cu}(u) = -\zeta_c(h)$ and, since ζ_c does not change the sign on Σ^n, we get that $\Delta_{-cu}(u)$ does not change the sign on Σ^n. Thus, from Lemma 2.1.1, $\Delta_{-cu}(u) = 0$ em Σ^n.

Now, we observe that

$$\Delta_{-cu}(u^2) = 2u\Delta_{-cu}(u) + 2|\nabla u|^2 = 2|\nabla u|^2 \geq 0 \quad \text{and} \quad |\nabla(u^2)| = 2|u||\nabla u| \in \mathcal{L}^1_{-cu}(\Sigma^n).$$

Hence, we can apply Lemma 2.1.1 once more to get that $\Delta_{-cu}(u^2) = 0$, implying that $|\nabla u| = 0$ on Σ^n. Therefore, $u = g(h)$ is constant on Σ^n and, since $g' = f > 0$, that is, the function g is increasing, we conclude that h is also constant, which means that $x(\Sigma^n)$ is contained in a slice $\{t\} \times M^{n+p}$, for some $t \in I$. ■

From Theorem 2.1.1 we obtain the following nonexistence result, when the ambient spacetime is a Lorentzian product space.

Corollary 2.1.1 *There does not exist complete spacelike translation soliton $x : \Sigma^n \looparrowright -I \times M^{n+p}$ with respect to ∂_t and with soliton constant $c \neq 0$, lying in a slab of $-I \times M^{n+p}$ and such that $|\nabla h| \in \mathcal{L}^1(\Sigma^n)$.*

Proof. Supposing that there exists such a spacelike translation soliton $x : \Sigma^n \looparrowright -I \times M^{n+p}$, from Theorem 2.1.1 we get that there exists $t \in I$ such that $x(\Sigma^n)$ is contained in a slice $\{t\} \times M^{n+p}$. Since $f = 1$, from (1.2.15) we have that $c = 0$, leading us to a contradiction. ■

Our next result also deals with the nonexistence of spacelike mean curvature flow solitons.

Theorem 2.1.2 *There does not exist a complete spacelike mean curvature flow soliton $x : \Sigma^n \looparrowright -I \times_f M^{n+p}$ with respect to $\mathcal{K} = f(t)\partial_t$ with soliton constant $c \in \mathbb{R}$, lying in a slab of $-I \times_f M^{n+p}$, whose height function h satisfies $|\nabla h| \in \mathcal{L}^1(\Sigma^n)$ and such that*

(a) $cf'(h) \geq 0$ on Σ^n and

(b) either $c \neq 0$ or the set where $f'(h) \neq 0$ is a dense subset on Σ^n.

Proof. Let us suppose for the sake of contradiction the existence of such a complete mean curvature flow soliton $x : \Sigma^n \looparrowright -I \times_f M^{n+p}$. Since $x : \Sigma^n \looparrowright -I \times_f M^{n+p}$ lies in a slab of $-I \times_f M^n$ and $|\nabla h| \in \mathcal{L}^1(\Sigma^n)$, we conclude in the same way as in the proof of Theorem 2.1.1 that $|\nabla u| \in \mathcal{L}^1_{-cu}(\Sigma^n)$.

Furthermore, from item (a) we have that either $c \geq 0$ and $f'(h) \geq 0$ or $c \leq 0$ and $f'(h) \leq 0$. In both cases, from (1.2.14) we get that

$$\Delta_{-cu}(u) = -(nf'(h) + cf^2(h)) = -\zeta_c(h)$$

does not change the sign on Σ^n. Consequently, Lemma 2.1.1 guarantees that $\Delta_{-cu}(u) = 0$ em Σ^n.

On the other hand, item (a) jointly with one of the options in item (b) give

$$-\zeta_c(h) = -(nf'(h) + cf^2(h)) \neq 0$$

in a dense subset of Σ^n. Therefore, from (1.2.14) we get $\Delta_{-cu}(u) \neq 0$ on Σ^n. So, we reach a contradiction. ■

Remark 2.1.1 We observe that these previous results remain valid if we assume that $c \neq 0$ and $\sqrt{|A_{\vec{H}}|} \in \mathcal{L}^1(\Sigma)$ instead of supposing that $|\nabla h| \in \mathcal{L}^1(\Sigma^n)$. Indeed, from (1.2.6), (1.2.9) and (1.2.16) we have that

$$c^2|\nabla u|^2 = |\vec{H}|^2 - c^2 f^2.$$

Since $|\vec{H}|^2 \leq \sqrt{n}|A_{\vec{H}}|$, we have that

$$c^2|\nabla u|^2 \leq \sqrt{n}|A_{\vec{H}}|.$$

Hence, we conclude that $|\nabla u| \in \mathcal{L}^1(\Sigma^n)$.

2.2 Nonexistence and rigidity results via asymptotic convergence

Let Σ^n be a connected, complete noncompact Riemannian manifold and let $d(\cdot, o) : \Sigma \to [0, +\infty)$ stand for the Riemannian distance of Σ^n, measured from a fixed point $o \in \Sigma^n$. According to [5], we say that $f \in C^0(\Sigma)$ *converges to zero at infinity* if it satisfies

$$\lim_{d(x,o)\to+\infty} f(x) = 0.$$

Considering this setting, we have the following key lemma which corresponds to [5, Theorem 2.2].

Lemma 2.2.1 *Let Σ^n be a connected, oriented, complete noncompact Riemannian manifold and let $X \in \mathfrak{X}(\Sigma)$ be a vector field on Σ^n. Assume that there exists a non-negative, non-identically vanishing function $f \in C^\infty(\Sigma)$ converging to zero at infinity and such that $\langle \nabla f, X\rangle \geq 0$. If $\mathrm{div}_\Sigma X \geq 0$ on Σ^n, then:*

(a) $\langle \nabla f, X\rangle \equiv 0$ *on* Σ^n*;*

(b) $\mathrm{div}_\Sigma X \equiv 0$ *on* $\Sigma^n \setminus f^{-1}(0)$*;*

(c) $\mathrm{div}_\Sigma X \equiv 0$ *on* Σ^n *if* $f^{-1}(0)$ *has zero Lebesgue measure.*

As an application of Lemma 2.2.1 we obtain the following result.

Theorem 2.2.1 *Let $x : \Sigma^n \looparrowright -I \times_f M^{n+p}$ be a complete noncompact spacelike mean curvature flow soliton with respect to $\mathcal{K} = f(t)\partial_t$ and with soliton constant $c \in \mathbb{R}$. Suppose that there exists $t_0 \in I$ such that $x(\Sigma^n)$ is above (below) the slice $\{t_0\} \times M^{n+p}$ and converges asymptotically to it at infinity. If the soliton function $\zeta_c(h)$ is nonpositive, then $x(\Sigma^n)$ is contained in the slice $\{t_0\} \times M^{n+p}$.*

Proof. Let us suppose for the sake of contradiction that $x(\Sigma^n)$ is not contained in the slice $\{t_0\} \times M^{n+p}$. Defining on Σ^n the function $\rho = h - t_0$, if $x(\Sigma^n)$ is above the slice $\{t_0\} \times M^{n+p}$ and asymptotic to it at infinity, we have that $\rho \geq 0$ and ρ is a non-identically vanishing function converging to zero at the infinity.

Let us consider the vector field $X = e^{cu}\nabla u$, and observe that

$$\langle X, \nabla\rho\rangle = \langle e^{cu}\nabla u, \nabla h\rangle = e^{cu} f(h)|\nabla h|^2 \geq 0.$$

Furthermore, from (1.2.14) and (2.1.2) we have that

$$e^{-cu}\mathrm{div}_\Sigma X = e^{-cu}\mathrm{div}_\Sigma(e^{cu}\nabla u) = \Delta_{-cu}(u) = -\zeta_c(h).$$

Therefore, $\mathrm{div}_\Sigma X \geq 0$, because we are supposing that $\zeta_c(h) \leq 0$. Thus, from Lemma 2.2.1 we get $\langle X, \nabla\rho\rangle \equiv 0$. Consequently, h is constant in Σ^n and, since $x(\Sigma^n)$ is asymptotic to slice $\{t_0\} \times M^{n+p}$ at infinity, we concluded that $h = t_0$. Therefore, $\rho \equiv 0$, and we arrived at a contradiction. If $x(\Sigma^n)$ is below the slice $\{t_0\} \times M^{n+p}$ and asymptotic to it at infinity, we can define the function $\rho = t_0 - h$ and reason as in the previous case. ■

When the ambient spacetime is a Lorentzian product space, from Theorem 2.2.1 we obtain the following consequence.

Corollary 2.2.1 *There does not exist a complete noncompact spacelike mean curvature flow soliton* $x : \Sigma^n \looparrowright -I \times M^{n+p}$ *with respect to* $\mathcal{K} = f(t)\partial_t$, *with soliton constant* $c \neq 0$ *and such that* $x(\Sigma^n)$ *is above (below) the slice* $\{t_0\} \times M^{n+p}$ *and converges asymptotically to it at infinity.*

From now on, in this subsection, we denote by N the future-pointing Gauss map of Σ^n and will always assume such a timelike orientation for Σ^n. Then, $\langle N, N\rangle = -1$. Furthermore, from the inverse Cauchy-Schwarz inequality (see [66, Proposition 5.30]), we have that $\langle N, \partial_t\rangle \leq -1$, with the equality holding at a point $p \in \Sigma^n$ if, and only if, $N = \partial_t$ at p. Therefore, on Σ^n the hyperbolic angle Θ verifies

$$\Theta = \langle N, \partial_t\rangle \leq -1. \tag{2.2.1}$$

Naturally attached to a spacelike mean curvature flow soliton $x : \Sigma^n \looparrowright -I \times M^n$ with respect to $\mathcal{K} = f(t)\partial_t$ and with soliton constant $c \in \mathbb{R}$, we can consider the support function

$$\begin{array}{rcl} \varphi_{\mathcal{K}} : \Sigma^n & \rightarrow & \mathbb{R} \\ q & \mapsto & \varphi_{\mathcal{K}}(q) = \langle \mathcal{K}(q), N(q) \rangle_{x(q)}. \end{array} \tag{2.2.2}$$

From (2.2.1), we have that

$$\varphi_{\mathcal{K}} = f(h)\langle \partial_t, N \rangle = f(h)\Theta \leq -f(h) < 0.$$

Furthermore, we have the following relationship between the gradient of the height function h and angle Θ

$$\nabla h = -\partial_t^\top = -\partial_t - \Theta N.$$

Then, we get

$$|\nabla h|^2 = \Theta^2 - 1.$$

From [23, Proposition 2.1] and (1.2.6) we have

$$\Delta(\varphi_{\mathcal{K}}) = \{\overline{\mathrm{Ric}}(N, N) + |A|^2\}\varphi_{\mathcal{K}} - \{nN(f') - Hf'\} + \langle \mathcal{K}, \nabla H \rangle, \tag{2.2.3}$$

where ∇H is the gradient of $H = \langle \vec{H}, N \rangle$ in the metric of Σ^n, $\overline{\mathrm{Ric}}$ is the Ricci tensor of $-I \times_f M^n$ and $|A|$ is the Hilbert-Schmidt norm of A.

Besides, we get that

$$N(f') = \langle \nabla f'(h), N \rangle = f''(h)\langle \nabla h, N \rangle = -f''\Theta = -\frac{f''}{f}\varphi_{\mathcal{K}}. \tag{2.2.4}$$

On the other hand, since $N = N^* - \Theta\partial_t$, where $N^* = (\pi_M)_*(N)$ is the orthogonal projection of N onto M^n, it follows from [66, Corollary 7.43] that

$$\begin{aligned} \overline{\mathrm{Ric}}(N, N) &= \overline{\mathrm{Ric}}(N^*, N^*) + \Theta^2\overline{\mathrm{Ric}}(\partial_t, \partial_t) \\ &= \mathrm{Ric}_M(N^*, N^*) + \langle N^*, N^* \rangle \left\{ \frac{f''}{f} + (n-1)\frac{(f')^2}{f^2} \right\} - \frac{nf''}{f}\Theta^2 \\ &= \mathrm{Ric}_M(N^*, N^*) - \left\{ \frac{f''}{f} + (n-1)\frac{(f')^2}{f^2} \right\} \\ &- (n-1)\left(\frac{f'}{f}\right)'\Theta^2, \end{aligned} \tag{2.2.5}$$

where Ric_M denotes the Ricci tensor of M^n. We note that it was used the relation $\langle N^*, N^* \rangle = \Theta^2 - 1$ in the last equality above.

Thus, inserting (2.2.4) and (2.2.5) into (2.2.3), we get that

$$\begin{aligned}
\Delta(\varphi_{\mathcal{K}}) &= \left\{\mathrm{Ric}_M(N^*,N^*) + |A|^2 - \left\{\frac{f''}{f} + (n-1)\frac{(f')^2}{f^2}\right\} - (n-1)\left(\frac{f'}{f}\right)'\Theta^2\right\}\varphi_{\mathcal{K}} \\
&+ \left\{n\frac{f''}{f}\varphi_{\mathcal{K}} + Hf'\right\} + \langle K, \nabla H\rangle \\
&= \left\{\mathrm{Ric}_M(N^*,N^*) + |A|^2 + (n-1)\frac{(f''f - (f')^2)}{f^2} - (n-1)\left(\frac{f'}{f}\right)'\Theta^2\right\}\varphi_{\mathcal{K}} \\
&+ Hf' + \langle K, \nabla H\rangle \\
&= \left\{\mathrm{Ric}_M(N^*,N^*) + (n-1)(\ln f)''(1-\Theta^2) + |A|^2\right\}\varphi_{\mathcal{K}} + Hf' + \langle K, \nabla H\rangle \\
&= \{\mathrm{Ric}_M(N^*,N^*) - (n-1)(\ln f)''|\nabla h|^2 + |A|^2\}\varphi_{\mathcal{K}} + Hf' + \langle \mathcal{K}, \nabla H\rangle \quad (2.2.6)
\end{aligned}$$

From equations (1.2.1) and (2.2.2), we have that $H = c\,\varphi_{\mathcal{K}}$, and from (1.2.9) we get $\nabla u = -\mathcal{K}^\top$, where u is the reparametrization of the height function h given in (1.2.8). Consequently, we can rewrite (2.2.6) in the following way

$$\Delta(\varphi_{\mathcal{K}}) = \{cf'(h) + \mathrm{Ric}_M(N^*,N^*) - (n-1)(\ln f)''(h)|\nabla h|^2 + |A|^2\}\varphi_{\mathcal{K}} + \langle\nabla(cu), \nabla(\varphi_{\mathcal{K}})\rangle. \quad (2.2.7)$$

Then, from (2.2.7) and (1.2.13) we conclude that the drift Laplacian Δ_{-cu} acting on $\varphi_{\mathcal{K}} = \frac{H}{c}$ is given by

$$\Delta_{-cu}\left(\frac{H}{c}\right) = \{\tilde{\zeta}_c + \mathrm{Ric}_M(N^*,N^*) - (n-1)(\ln f)''(h)|\nabla h|^2\}\frac{H}{c}, \quad (2.2.8)$$

where $\tilde{\zeta}_c \in C^\infty(\Sigma^n)$ is the function defined by

$$\tilde{\zeta}_c(q) = cf'(h(q)) + |A(q)|^2,$$

for every $q \in \Sigma^n$, which will be called the *second soliton function* associated to the spacelike mean curvature flow soliton $x : \Sigma^n \looparrowright -I \times_f M^n$. Such nomenclature for $\tilde{\zeta}_c$ is motivated by [9, Equation (6.11)].

In our next result, we will assume that the GRW spacetime $-I \times_f M^n$ satisfies the *null convergence condition* (NCC)

$$\mathrm{Ric}_M \geq (n-1)(ff'' - f'^2)\langle\,,\rangle_M, \quad (2.2.9)$$

where Ric_M denotes the Ricci tensor of the Riemannian fiber M^n.

Theorem 2.2.2 *Let $-I \times_f M^n$ be a GRW spacetime obeying the NCC (2.2.9), with equality holding only in isolated points of I. Let $x : \Sigma^n \looparrowright -I \times_f M^n$ be a complete* *noncompact spacelike mean curvature flow soliton with respect to $K = f(t)\partial_t$ and with soliton constant $c \in \mathbb{R}$. Suppose that there exists $t_0 \in I$ such that $x(\Sigma^n)$ is above the slice $\{t_0\} \times M^n$ and converges asymptotically to it at infinity. If the second soliton function $\tilde{\zeta}_c = |A|^2 + cf'(h)$ is nonnegative and $c\langle A(\nabla u), \nabla u\rangle \leq 0$, then $x(\Sigma^n)$ is the slice $\{t_0\} \times M^n$.*

Proof. Let us suppose for the sake of contradiction that $x(\Sigma^n) \neq \{t_0\} \times M^n$. Defining the function $\rho = h - t_0$, since $x(\Sigma^n)$ is above the slice $\{t_0\} \times M^n$ and asymptotic to it at infinity, we have that $\rho \geq 0$ and ρ is a non-identically vanishing function converging to zero at the infinity.

Now, let us consider the field vector $X = e^{cu}\nabla(-\frac{H}{c})$, and observe that, by hypothesis,

$$\langle X, \nabla\rho\rangle = \left\langle e^{cu}\nabla\left(-\frac{H}{c}\right), \nabla\rho\right\rangle = -\langle e^{cu} c A(\nabla u), \nabla h\rangle = -\frac{e^{cu}}{f(h)} c\langle A(\nabla u), \nabla u\rangle \geq 0.$$

Furthermore, from (2.2.9) we obtain

$$\begin{aligned}
\mathrm{Ric}_M(N^*, N^*) - (n-1)(\ln f)''(h)|\nabla h|^2 &\geq \\
\geq (n-1)(f(h)f''(h) - f'(h)^2)|N^*|^2_M - (n-1)(\ln f)''(h)|\nabla h|^2 \\
= (n-1)(f(h)f''(h) - f'(h)^2)|N + \Theta\partial_t|^2_M - (n-1)\left(\frac{f'}{f}\right)'(h)|\nabla h|^2 \\
= (n-1)\left\{(f(h)f''(h) - f'(h)^2)\frac{|\nabla h|^2}{f(h)^2} - \left(\frac{f(h)f''(h) - f'(h)^2}{f(h)^2}\right)|\nabla h|^2\right\} &= 0.
\end{aligned}$$

Here, we use that

$$|N + \Theta\partial_t|^2_M = \langle N, N\rangle + 2\Theta^2 + \Theta^2\langle\partial_t, \partial_t\rangle = \Theta^2 - 1 = |\nabla h|^2.$$

Then, also using the fact that the second soliton function $\tilde{\zeta}_c$ is nonnegative and $-\frac{H}{c} > 0$, from (2.2.8) we have that

$$\Delta_{-cu}\left(-\frac{H}{c}\right) = \{\tilde{\zeta}_c + \mathrm{Ric}_M(N^*, N^*) - (n-1)(\ln f)''(h)|\nabla h|^2\}\left(-\frac{H}{c}\right) \geq 0. \quad (2.2.10)$$

On the other hand, since

$$\Delta_{-cu}\left(-\frac{H}{c}\right) = e^{-cu}\mathrm{div}_\Sigma\left(e^{cu}\nabla\left(-\frac{H}{c}\right)\right),$$

we get that $\mathrm{div}_\Sigma X = \mathrm{div}_\Sigma(e^{cu}\nabla(-\frac{H}{c})) \geq 0$. Then, by Lemma 2.2.1, $\mathrm{div}_\Sigma X = 0$ in $\Sigma^n \setminus \rho^{-1}(0)$. Therefore, $\Delta_{-cu}(-\frac{H}{c}) = 0$ in $\Sigma^n \setminus \rho^{-1}(0)$. Thus, from (2.2.10) we have that

$$\tilde{\zeta}_c = 0 \qquad \text{and} \qquad \mathrm{Ric}_M(N^*, N^*) - (n-1)(\ln f)''(h)|\nabla h|^2 = 0,$$

in $\Sigma^n \setminus \rho^{-1}(0)$. But, taking into account that the equality in (2.2.9) occurs only in isolated points of I, we obtain that $|\nabla h| = 0$ on Σ^n and, consequently, h is constant on $\Sigma^n \setminus \rho^{-1}(0)$. Since $x(\Sigma^n)$ is asymptotic to slice $\{t_0\} \times M^n$ at infinity, we conclude that $h = t_0$, which implies $\rho \equiv 0$. Hence, we arrived at a contradiction. ■

Chapter 3

Higher order mean curvatures of spacelike mean curvature flow solitons

Motivated by [34] (see also [24, 45, 53, 70]), we devote this chapter to study the r-mean curvature of a spacelike mean curvature flow soliton in a GRW spacetime.

Let $x : \Sigma^n \looparrowright -I \times_f M^{n+p}$ be a spacelike submanifold and let $\{E_1, \ldots, E_n\}$ be a local orthonormal frame on Σ^n. For $r \in \{1, \ldots, n\}$, denoting $\alpha_{ij} := A(E_i, E_j)$, the r-mean curvature of Σ^n is defined by

$$H_r = \begin{pmatrix} n \\ r \end{pmatrix}^{-1} \frac{1}{r!} \sum_{\substack{i_1,\ldots,i_r \\ j_1,\ldots,j_r}} \delta^{i_1,\ldots,i_r}_{j_1,\ldots,j_r} \langle \alpha_{i_1 j_1}, \alpha_{i_2 j_2} \rangle \ldots \langle \alpha_{i_{r-1} j_{r-1}}, \alpha_{i_r j_r} \rangle,$$

for r even, and

$$\vec{H}_r = \begin{pmatrix} n \\ r \end{pmatrix}^{-1} \frac{1}{r!} \sum_{\substack{i_1,\ldots,i_r \\ j_1,\ldots,j_r}} \delta^{i_1,\ldots,i_r}_{j_1,\ldots,j_r} \langle \alpha_{i_1 j_1}, \alpha_{i_2 j_2} \rangle \ldots \langle \alpha_{i_{r-2} j_{r-2}}, \alpha_{i_{r-1} j_{r-1}} \rangle \alpha_{i_r j_r},$$

for r odd, where

$$\delta^{i_1,\ldots,i_r}_{j_1,\ldots,j_r} = \begin{cases} 0, \text{ if } i_k = i_l \text{ or } j_k = j_l, \text{ for some } k \neq l, \text{ or if } \{i_1, \ldots, i_r\} \neq \{j_1, \ldots, j_r\} \text{ as sets;} \\ \text{sign of the permutation } (i_1, \ldots, i_r) \to (j_1, \ldots, j_r), \text{ otherwise.} \end{cases}$$

We also define $H_0 = 1$.

Now, let $\{E^1, \ldots, E^n\}$ the dual coframe of $\{E_1, \ldots, E_n\}$. The r-th Newton transformation T_r, $r \in \{1, \ldots, n\}$, is the $(0,2)$-tensor given by

$$T_r = \sum_{i,j} (T_r)_{ij} E^i \otimes E^j,$$

where

$$(T_r)_{ij} = \frac{1}{r!} \sum_{\substack{i_1, \ldots, i_r \\ j_1, \ldots, j_r}} \delta^{i,i_1,\ldots,i_r}_{j,j_1,\ldots,j_r} \langle \alpha_{i_1 j_1}, \alpha_{i_2 j_2} \rangle \ldots \langle \alpha_{i_{r-1} j_{r-1}}, \alpha_{i_r j_r} \rangle,$$

for r even, and

$$(T_r)_{ij} = \frac{1}{r!} \sum_{\substack{i_1, \ldots, i_r \\ j_1, \ldots, j_r}} \delta^{i,i_1,\ldots,i_r}_{j,j_1,\ldots,j_r} \langle \alpha_{i_1 j_1}, \alpha_{i_2 j_2} \rangle \ldots \langle \alpha_{i_{r-2} j_{r-2}}, \alpha_{i_{r-1} j_{r-1}} \rangle \alpha_{i_r j_r},$$

for r odd. By convention, $T_0 = \langle \cdot, \cdot \rangle$.

Associated to each globally defined r-th Newton tensor $T_r : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ with $0 \leq r \leq n$ even, we consider the second-order differential operator $L_r : C^\infty(\Sigma) \to C^\infty(\Sigma)$ given by

$$L_r u := \langle \nabla^2 u, T_r \rangle = \mathrm{div}(T_r(\nabla u)) - (\mathrm{div} T_r)(\nabla u).$$

We observe that L_r is elliptic if, and only if, T_r is positive definite. In particular, L_0 is just the Laplace-Beltrami operator Δ.

The proof of the next auxiliary lemma can be, for instance, found in [24, Lemma 3.3].

Lemma 3.0.1 *If $0 \leq r \leq n$ is even, we get that*

(i) $\mathrm{tr}(T_r) = k(r) H_r$;

(ii) $\sum_{ij} T_r(E_i, E_j) A(E_i, E_j) = k(r) \vec{H}_{r+1}$, *where* $k(r) := (n - r) \begin{pmatrix} n \\ r \end{pmatrix}$.

Now, we can state and proof our next result.

Theorem 3.0.1 *Let $-I\times_f M^{n+p}$ be a GRW spacetime and let $x:\Sigma^n \looparrowright -I\times_f M^{n+p}$ be a compact spacelike mean curvature flow soliton with respect to $K = f(t)\partial_t$ and with soliton constant $c \neq 0$. Suppose that $r \in \{1,\ldots,n\}$ is even and the r-th Newton transformation T_r is positive definite. If $c > 0$, then*

$$\min_\Sigma \left\langle \frac{\vec{H}_{r+1}}{H_r}, \vec{H} \right\rangle \leq -cnf'(h_*) \text{ and } \max_\Sigma \left\langle \frac{\vec{H}_{r+1}}{H_r}, \vec{H} \right\rangle \geq -cnf'(h^*),$$

where h_* and h^* are, respectively, the minimum and maximum of the height function h on Σ^n. Moreover, if $c < 0$, then

$$\min_\Sigma \left\langle \frac{\vec{H}_{r+1}}{H_r}, \vec{H} \right\rangle \leq -cnf'(h^*) \text{ and } \max_\Sigma \left\langle \frac{\vec{H}_{r+1}}{H_r}, \vec{H} \right\rangle \geq -cnf'(h_*).$$

Proof. Remember that Lemma 1.2.1 gives

$$\langle A_{\vec{H}}X, Y\rangle + c\nabla^2 u(X,Y) = -cf'(h)\langle X, Y\rangle.$$

Then, using this and considering a local orthonormal frame $\{E_1,\ldots,E_n\}$ on Σ^n, we get

$$\sum_{ij}\langle (T_r)_{ij}\alpha_{ij}, \vec{H}\rangle + cL_r u = -cf'(h)\sum_{ij}(T_r)_{ij}\langle E_i, E_j\rangle.$$

Therefore, from Lemma 3.0.1 we have that

$$k(r)\langle \vec{H}_{r+1}, \vec{H}\rangle + cL_r u = -ncf'(h)k(r)H_r.$$

Thus,

$$cL_r u = -ncf'(h)k(r)H_r - k(r)\langle \vec{H}_{r+1}, \vec{H}\rangle. \tag{3.0.1}$$

Let us consider $c > 0$ and let p_0 be a point of minimum of the height function h. Since $g' = f > 0$, that is, g is strictly increasing, we have that $h(p_0) = h_*$ is also a minimum of the function $u = g(h)$. Since T_r is positive definite, this implies that L_r is elliptic and, hence, $L_r u(p_0) \geq 0$. Then, from (3.0.1) we obtain

$$\min_\Sigma \left\langle \frac{\vec{H}_{r+1}}{H_r}, \vec{H} \right\rangle \leq \left\langle \frac{\vec{H}_{r+1}}{H_r}, \vec{H} \right\rangle (p_0) = -ncf'(h_*) - \frac{\mathrm{cL_r}u(p_0)}{k(r)H_r} \leq -ncf'(h_*).$$

Analogously, taking a point of maximum of h, we are able to conclude that

$$\max_\Sigma \left\langle \frac{\vec{H}_{r+1}}{H_r}, \vec{H} \right\rangle \geq -ncf'(h^*).$$

The proof of the case $c < 0$ follows the same steps of the case $c > 0$. ■

As a first consequence of Theorem 3.0.1 we obtain the following result.

Corollary 3.0.1 *Let $-I \times_f M^{n+p}$ be a GRW spacetime and let $x : \Sigma^n \looparrowright -I \times_f M^{n+p}$ be a compact spacelike mean curvature flow soliton with respect to $\mathcal{K} = f(t)\partial_t$ and with soliton constant $c \neq 0$. Assume that $f'' \leq 0$ on $[h_*, h^*]$, where h_* and h^* are, respectively, the minimum and maximum of the height function h on Σ^n. If $\left\langle \frac{\vec{H}_{r+1}}{H_r}, \vec{H} \right\rangle$ is constant, then $x(\Sigma^n)$ is contained in a slice $\{t\} \times M^{n+p}$.*

Proof. Since $f'' \leq 0$ on $[h_*, h^*]$, we have that $f'(h^*) \leq f'(h_*)$. From Theorem 3.0.1,

$$\min_{\Sigma} \left\langle \frac{\vec{H}_{r+1}}{H_r}, \vec{H} \right\rangle \leq -cnf'(h_*) \leq -cnf'(h^*) \leq \max_{\Sigma} \left\langle \frac{\vec{H}_{r+1}}{H_r}, \vec{H} \right\rangle, \text{ when } c > 0,$$

and

$$\min_{\Sigma} \left\langle \frac{\vec{H}_{r+1}}{H_r}, \vec{H} \right\rangle \leq -cnf'(h^*) \leq -cnf'(h_*) \leq \max_{\Sigma} \left\langle \frac{\vec{H}_{r+1}}{H_r}, \vec{H} \right\rangle, \text{ when } c < 0.$$

Using that $\left\langle \frac{\vec{H}_{r+1}}{H_r}, \vec{H} \right\rangle$ is constant, we get

$$f'(h_*) = f'(h^*) = -\frac{1}{cn} \left\langle \frac{\vec{H}_{r+1}}{H_r}, \vec{H} \right\rangle = \text{constant}.$$

Since f' is not increasing on $[h_*, h^*]$, we have that

$$f'(h) = -\frac{1}{cn} \left\langle \frac{\vec{H}_{r+1}}{H_r}, \vec{H} \right\rangle \text{ on } [h_*, h^*].$$

Thus,

$$nf'(h)H_r + \frac{1}{c}\langle \vec{H}_{r+1}, \vec{H} \rangle = 0 \text{ on } \Sigma^n.$$

Therefore, from (3.0.1)

$$L_r(u) = -k(r)\left(nf'(h)H_r + \frac{1}{c}\langle \vec{H}_{r+1}, \vec{H} \rangle \right) = 0$$

on the compact manifold Σ^n, which implies that u is constant and, therefore, h is also constant on Σ^n; that is, $x(\Sigma^n)$ is contained in a slice $\{t\} \times M^{n+p}$. ■

Since the parallelism of $\vec{H}$ implies that $|\vec{H}|^2$ is constant, considering $r = 0$ in Corollary 3.0.1 we get:

Corollary 3.0.2 *Let $-I \times_f M^{n+p}$ be a GRW spacetime and let $x : \Sigma^n \looparrowright -I \times_f M^{n+p}$ be a compact spacelike mean curvature flow soliton with respect to $\mathcal{K} = f(t)\partial_t$ and with soliton constant $c \neq 0$. Assume that $f'' \leq 0$ on $[h_*, h^*]$, where h_* and h^* are, respectively, the minimum and maximum of the height function h on Σ^n. If the mean curvature vector $\vec{H}$ is parallel, then $x(\Sigma^n)$ is contained in a slice $\{t\} \times M^{n+p}$.*

When $p = 0$, Theorem 3.0.1 reads as the following rigidity result.

Corollary 3.0.3 *Let* $-I \times_f M^n$ *be a GRW spacetime and let* $x : \Sigma^n \looparrowright -I \times_f M^n$ *be a compact spacelike mean curvature flow soliton with respect to* $\mathcal{K} = f(t)\partial_t$ *and with soliton constant* $c \neq 0$*. Assume that* $f'' \leq 0$ *on* $[h_*, h^*]$*, where* h_* *and* h^* *are, respectively, the minimum and maximum of the height function* h *on* Σ^n*. If* H_2 *is constant, then* $x(\Sigma^n)$ *is contained in a slice* $\{t\} \times M^n$*.*

Chapter 4

Further rigidity and nonexistence results

We will establish further rigidity and nonexistence results concerning spacelike mean curvature flow solitons in a GRW spacetime under appropriate constraints on the length of the mean curvature vector.

This chapter is motivated by the work of Chen and Qiu in [28], Batista and de Lima in [19], and de Lima, Gomes, Santos, et al in [35].

4.1 The drift Laplacian of the length of the mean curvature vector

Here we are considering the same drift Laplacian defined in (1.2.13). We start deducing the following suitable formula.

Proposition C *Let $x : \Sigma^n \looparrowright -I \times_f M^{n+p}$ be a spacelike mean curvature flow soliton with respect to $\mathcal{K} = f(t)\partial_t$ and with soliton constant $c \neq 0$. Let $\vec{H}$ be its mean curvature vector and suppose that the Riemannian fiber M^{n+p} has constant sectional curvature k. Then,*

$$\frac{1}{2}\Delta_{-cu}|\vec{H}|^2 = cf'|\vec{H}|^2 + \langle A_{\vec{H}}, A_{\vec{H}}\rangle - \langle(\bar{\nabla}_{(\cdot)}\vec{H})^\perp, (\bar{\nabla}_{(\cdot)}\vec{H})^\perp\rangle$$
$$+ (n-1)\left(\frac{k}{f^2} - \frac{1}{f^2}(f''f - f'^2)\right)|\vec{H}|^2\left(-1 + \frac{|\vec{H}|^2}{c^2 f^2}\right),$$

where $|\vec{H}|^2 := -\langle \vec{H}, \vec{H} \rangle$.

Proof. From Lemma 1.2.1 we have that $\nabla|\vec{H}|^2 = -2cA_{\vec{H}}(\nabla u) = 2cA_{\vec{H}}(\mathcal{K}^\top)$. Then, taking a local geodesic frame $\{E_1, \ldots, E_n\}$ in Σ^n, we have

$$\begin{aligned}
\frac{1}{2}\langle \nabla_{E_i}\nabla|\vec{H}|^2, E_j\rangle &= \frac{1}{2}E_i\langle \nabla|\vec{H}|^2, E_j\rangle - \frac{1}{2}\langle \nabla|\vec{H}|^2, \nabla_{E_i}E_j\rangle \\
&= cE_i\langle A_{\vec{H}}(\mathcal{K}^\top), E_j\rangle \\
&= cE_i(A_{\vec{H}}(\mathcal{K}^\top, E_j)) \\
&= c[(\nabla_{E_i}A_{\vec{H}})(\mathcal{K}^\top, E_j) + A_{\vec{H}}(\nabla_{E_i}(\mathcal{K}^\top), E_j) + A_{\vec{H}}(\mathcal{K}^\top, \nabla_{E_i}E_j)] \\
&= c(\nabla_{E_i}A_{\vec{H}})(\mathcal{K}^\top, E_j) + cA_{\vec{H}}(\nabla_{E_i}(\mathcal{K}^\top), E_j).
\end{aligned}$$

We note that (1.1.3) and (1.2.1) imply

$$\begin{aligned}
\langle \bar{\nabla}_{E_i}\vec{H}, A(\mathcal{K}^\top, E_j)\rangle &= c\langle \bar{\nabla}_{E_i}(\mathcal{K} - \mathcal{K}^\top), A(\mathcal{K}^\top, E_j)\rangle \\
&= c\langle \bar{\nabla}_{E_i}\mathcal{K}, A(\mathcal{K}^\top, E_j)\rangle - c\langle \bar{\nabla}_{E_i}\mathcal{K}^\top, A(\mathcal{K}^\top, E_j)\rangle \\
&= cf'\langle E_i, A(\mathcal{K}^\top, E_j)\rangle - c\langle \bar{\nabla}_{E_i}\mathcal{K}^\top, A(\mathcal{K}^\top, E_j)\rangle \\
&= -c\langle (\bar{\nabla}_{E_i}\mathcal{K}^\top)^\perp, A(\mathcal{K}^\top, E_j)\rangle \\
&= -c\langle A(\mathcal{K}^\top, E_i), A(\mathcal{K}^\top, E_j)\rangle.
\end{aligned}$$

Here we are also denoting by E_i the extension of the field E_i to $-I \times_f M^{n+p}$.

We also observe that

$$\begin{aligned}
(\nabla_{E_i}A_{\vec{H}})(\mathcal{K}^\top, E_j) &= E_i\langle A(\mathcal{K}^\top, E_j), \vec{H}\rangle - A_{\vec{H}}(\nabla_{E_i}\mathcal{K}^\top, E_j) - A_{\vec{H}}(\mathcal{K}^\top, \nabla_{E_i}E_j) \\
&= \langle \bar{\nabla}_{E_i}A(\mathcal{K}^\top, E_j), \vec{H}\rangle + \langle A(\mathcal{K}^\top, E_j), \bar{\nabla}_{E_i}\vec{H}\rangle - \langle A(\nabla_{E_i}\mathcal{K}^\top, E_j), \vec{H}\rangle \\
&\quad - \langle A(\mathcal{K}^\top, \nabla_{E_i}E_j), \vec{H}\rangle \\
&= \langle (\bar{\nabla}_{E_i}A)(\mathcal{K}^\top, E_j), \vec{H}\rangle + \langle A(\mathcal{K}^\top, E_j), \bar{\nabla}_{E_i}\vec{H}\rangle.
\end{aligned}$$

Thus, using Codazzi's equation we get

$$\begin{aligned}
(\nabla_{E_i}A_{\vec{H}})(\mathcal{K}^\top, E_j) - (\nabla_{\mathcal{K}^\top}A_{\vec{H}})(E_i, E_j) &= \langle (\bar{\nabla}_{E_i}A)(\mathcal{K}^\top, E_j) - (\bar{\nabla}_{\mathcal{K}^\top}A)(E_i, E_j), \vec{H}\rangle \\
&\quad + \langle A(\mathcal{K}^\top, E_j), \bar{\nabla}_{E_i}\vec{H}\rangle - \langle A(E_i, E_j), \bar{\nabla}_{\mathcal{K}^\top}\vec{H}\rangle \\
&= \langle \bar{R}(\mathcal{K}^\top, E_i)E_j, \vec{H}\rangle + \langle A(\mathcal{K}^\top, E_j), \bar{\nabla}_{E_i}\vec{H}\rangle \\
&\quad - \langle A(E_i, E_j), \bar{\nabla}_{\mathcal{K}^\top}\vec{H}\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{2}\langle \nabla_{E_i}\nabla|\vec{H}|^2, E_j\rangle &= c(\nabla_{E_i}A_{\vec{H}})(\mathcal{K}^\top, E_j) + cA_{\vec{H}}(\nabla_{E_i}\mathcal{K}^\top, E_j)\\
&= c(\nabla_{\mathcal{K}^\top}A_{\vec{H}})(E_i, E_j) + c\langle \bar{R}(\mathcal{K}^\top, E_i)E_j, \vec{H}\rangle + c\langle A(\mathcal{K}^\top, E_j), \bar{\nabla}_{E_i}\vec{H}\rangle\\
&- c\langle A(E_i, E_j), \bar{\nabla}_{\mathcal{K}^\top}\vec{H}\rangle + cA_{\vec{H}}(\nabla_{E_i}\mathcal{K}^\top, E_j)\\
&= c(\nabla_{\mathcal{K}^\top}A_{\vec{H}})(E_i, E_j) + c\langle \bar{R}(E_i, \mathcal{K}^\top)\vec{H}, E_j\rangle\\
&- c^2\langle A(\mathcal{K}^\top, E_i), A(\mathcal{K}^\top, E_j)\rangle - c\langle A(E_i, E_j), \bar{\nabla}_{\mathcal{K}^\top}\vec{H}\rangle\\
&+ cA_{\vec{H}}(\nabla_{E_i}\mathcal{K}^\top, E_j).\end{aligned}$$

Hence, taking traces through the choice of a local geodesic frame $\{E_1, \ldots, E_n\}$ of Σ^n, we obtain

$$\begin{aligned}\frac{1}{2}\Delta|\vec{H}|^2 &= \frac{1}{2}\sum_i \langle \nabla_{E_i}\nabla|\vec{H}|^2, E_i\rangle\\
&= c\sum_i (\nabla_{\mathcal{K}^\top}A_{\vec{H}})(E_i, E_i) + c\sum_{i,j} g^{ij}\langle \bar{R}(E_i, \mathcal{K}^\top)\vec{H}, E_j\rangle\\
&- c^2\sum_{i,j}\langle A(\mathcal{K}^\top, E_i), A(\mathcal{K}^\top, E_j)\rangle - c\sum_i \langle A(E_i, E_i), \bar{\nabla}_{\mathcal{K}^\top}\vec{H}\rangle\\
&+ c\sum_{i,j} g^{ij}A_H(\nabla_{E_i}\mathcal{K}^\top, E_j). \qquad (4.1.1)\end{aligned}$$

But, we have

$$\begin{aligned}\sum_i (\nabla_{E_j}A_{\vec{H}})(E_i, E_i)(p) &= \sum_i (E_j(A_{\vec{H}}(E_i, E_i)))(p) = \sum_i (E_j\langle A(E_i, E_i), \vec{H}\rangle)(p)\\
&= E_j(|\vec{H}|^2)(p) = \langle E_j, \nabla|\vec{H}|^2\rangle(p).\end{aligned}$$

Then, we get

$$c\sum_i (\nabla_{\mathcal{K}^\top}A_{\vec{H}})(E_i, E_i) = c\langle \mathcal{K}^\top, \nabla|\vec{H}|^2\rangle. \qquad (4.1.2)$$

Using once more (1.2.1), we obtain

$$\begin{aligned}c\sum_{i,j} g^{ij}\langle \bar{R}(E_i, \mathcal{K}^\top)\vec{H}, E_j\rangle &= c\sum_{i,j} g^{ij}\langle \bar{R}(E_i, \mathcal{K} - \vec{H}/c)\vec{H}, E_j\rangle \qquad (4.1.3)\\
&= c\sum_{i,j} g^{ij}\langle \bar{R}(E_i, \mathcal{K})\vec{H}, E_j\rangle - \sum_{i,j} g^{ij}\langle \bar{R}(E_i, \vec{H})\vec{H}, E_j\rangle.\end{aligned}$$

Furthermore, we have that

$$-c\sum_i \langle A(E_i, E_i), \bar{\nabla}_{\mathcal{K}^\top}\vec{H}\rangle = c\langle \vec{H}, \bar{\nabla}_{\mathcal{K}^\top}\vec{H}\rangle$$

$$
\begin{aligned}
&= \frac{1}{2}c(\mathcal{K}^\top)\langle \vec{H}, \vec{H}\rangle \\
&= -\frac{1}{2}c(\mathcal{K}^\top)(|\vec{H}|^2) \\
&= -\frac{1}{2}c\langle \mathcal{K}^\top, \nabla|\vec{H}|^2\rangle,
\end{aligned} \tag{4.1.4}
$$

and

$$
\begin{aligned}
c\sum_{i,j} g^{ij} A_{\vec{H}}(\nabla_{E_i}\mathcal{K}^\top, E_j) &= c\sum_{i,j} g^{ij}\langle A(\nabla_{E_i}\mathcal{K}^\top, E_j), \vec{H}\rangle \\
&= c\sum_{i,j} g^{ij}\langle A_{\vec{H}}(\nabla_{E_i}\mathcal{K}^\top), E_j\rangle \\
&= -c\sum_{i,j} g^{ij}\langle \bar{\nabla}_{\nabla_{E_i}\mathcal{K}^\top}\vec{H}, E_j\rangle \\
&= -c\sum_{i,j,l} g^{ij}\langle \bar{\nabla}_{\langle \nabla_{E_i}\mathcal{K}^\top, E_l\rangle E_l}\vec{H}, E_j\rangle \\
&= -c\sum_{i,j,l} g^{ij}\langle \nabla_{E_i}\mathcal{K}^\top, E_l\rangle\langle \bar{\nabla}_{E_l}\vec{H}, E_j\rangle \\
&= c\sum_{i,j,l} g^{ij}(A_{\vec{H}})^l_j\langle \nabla_{E_i}\mathcal{K}^\top, E_l\rangle.
\end{aligned} \tag{4.1.5}
$$

Thus, using (4.1.2), (4.1.3), (4.1.4) and (4.1.5) in (4.1.1), we get

$$
\begin{aligned}
\frac{1}{2}\Delta|\vec{H}|^2 &= c\langle \mathcal{K}^\top, \nabla|\vec{H}|^2\rangle + c\sum_{i,j} g^{ij}\langle \bar{R}(E_i, \mathcal{K})\vec{H}, E_j\rangle - \sum_{i,j} g^{ij}\langle \bar{R}(E_i, \vec{H})\vec{H}, E_j\rangle \\
&- c^2\langle A(\mathcal{K}^\top, \cdot), A(\mathcal{K}^\top, \cdot)\rangle - \frac{1}{2}c\langle \mathcal{K}^\top, \nabla|\vec{H}|^2\rangle + c\sum_{i,j,l} g^{ij}(A_{\vec{H}})^l_j\langle \nabla_{E_i}\mathcal{K}^\top, E_l\rangle \\
&= \frac{c}{2}\langle \mathcal{K}^\top, \nabla|\vec{H}|^2\rangle + c\sum_{i,j} g^{ij}\langle \bar{R}(E_i, \mathcal{K})\vec{H}, E_j\rangle + \sum_{i,j} g^{ij}\langle \bar{R}(\vec{H}, E_i)\vec{H}, E_j\rangle \\
&- c^2\langle A(\mathcal{K}^\top, \cdot), A(\mathcal{K}^\top, \cdot)\rangle + c\sum_{i,j,l} g^{ij}(A_{\vec{H}})^l_j\langle \bar{\nabla}_{E_i}\mathcal{K}, E_l\rangle \\
&- \sum_{i,j,l} g^{ij}(A_{\vec{H}})^l_j\langle \bar{\nabla}_{E_i}\vec{H}, E_l\rangle \\
&= \frac{c}{2}\langle \mathcal{K}^\top, \nabla|\vec{H}|^2\rangle + c\sum_{i,j} g^{ij}\langle \bar{R}(E_i, \mathcal{K})\vec{H}, E_j\rangle + \mathrm{tr}\bar{R}(\vec{H}, \cdot)\vec{H} \\
&- c^2\langle A(\mathcal{K}^\top, \cdot), A(\mathcal{K}^\top, \cdot)\rangle + cf'\sum_{i,j,l} g^{ij}(A_{\vec{H}})^l_j g_{il} + \sum_{i,j,l} g^{ij}(A_{\vec{H}})^l_j\langle \bar{\nabla}_{E_i}E_l, \vec{H}\rangle.
\end{aligned} \tag{4.1.6}
$$

Now, we observe that

$$
cf'\sum_{i,j,l} g^{ij}(A_{\vec{H}})^l_j g_{il} = cf'\sum_{l}\langle A(E_l, E_l), \vec{H}\rangle = cf'|\vec{H}|^2. \tag{4.1.7}
$$

We also have that

$$\begin{aligned}\sum_{i,j,l} g^{ij}(A_{\vec{H}})_j^l\langle\bar{\nabla}_{E_i}E_l,\vec{H}\rangle &= \sum_{i,j,l} g^{ij}(A_{\vec{H}})_j^l\langle(\bar{\nabla}_{E_i}E_l)^\perp,\vec{H}\rangle\\ &= \sum_{i,j,l} g^{ij}(A_{\vec{H}})_j^l\langle A(E_i,E_l),\vec{H}\rangle\\ &= \sum_{i,j,l} g^{ij}(A_{\vec{H}})_j^l(A_{\vec{H}})_i^l = \langle A_{\vec{H}},A_{\vec{H}}\rangle.\end{aligned} \tag{4.1.8}$$

Moreover, from (1.1.3) we get

$$0 = c(f'U)^\perp = c(\bar{\nabla}_U\mathcal{K})^\perp = c(\bar{\nabla}_U\mathcal{K}^\top)^\perp + c(\bar{\nabla}_U\mathcal{K}^\perp)^\perp = cA(\mathcal{K}^\top,U) + (\bar{\nabla}_U\vec{H})^\perp.$$

Consequently,

$$c^2\langle A(\mathcal{K}^\top,\cdot),A(\mathcal{K}^\top,\cdot)\rangle = \langle(\bar{\nabla}_{(\cdot)}\vec{H})^\perp,(\bar{\nabla}_{(\cdot)}\vec{H})^\perp\rangle. \tag{4.1.9}$$

Going back to (4.1.6) using (4.1.7), (4.1.8) and (4.1.9), we reach at

$$\begin{aligned}\frac{1}{2}\Delta|\vec{H}|^2 = \frac{c}{2}\langle\mathcal{K}^\top,\nabla|\vec{H}|^2\rangle + c\sum_{i,j} g^{ij}\langle\bar{R}(E_i,\mathcal{K})\vec{H},E_j\rangle + \mathrm{tr}\bar{R}(\vec{H},\cdot)\vec{H}\\ - \langle(\bar{\nabla}_{(\cdot)}\vec{H})^\perp,(\bar{\nabla}_{(\cdot)}\vec{H})^\perp\rangle + cf'|\vec{H}|^2 + \langle A_{\vec{H}},A_{\vec{H}}\rangle.\end{aligned}$$

Then, we conclude that

$$\begin{aligned}\frac{1}{2}\Delta_{-cu}|\vec{H}|^2 &= \frac{1}{2}\Delta|\vec{H}|^2 - \frac{c}{2}\langle\nabla|\vec{H}|^2,\mathcal{K}^\top\rangle\\ &= cf'|\vec{H}|^2 - \langle(\bar{\nabla}_{(\cdot)}\vec{H})^\perp,(\bar{\nabla}_{(\cdot)}\vec{H})^\perp\rangle + \langle A_{\vec{H}},A_{\vec{H}}\rangle\\ &+ c\sum_{i,j} g^{ij}\langle\bar{R}(E_i,\mathcal{K})\vec{H},E_j\rangle + \mathrm{tr}\bar{R}(\vec{H},\cdot)\vec{H}.\end{aligned}$$

On the other hand, using again (1.1.3), we obtain

$$\begin{aligned}\bar{R}(E_j,\vec{H})\mathcal{K} &= -\bar{\nabla}_{E_j}\bar{\nabla}_{\vec{H}}\mathcal{K} + \bar{\nabla}_{\vec{H}}\bar{\nabla}_{E_j}\mathcal{K} + \bar{\nabla}_{[E_j,\vec{H}]}\mathcal{K}\\ &= -\bar{\nabla}_{E_j}f'\vec{H} + \bar{\nabla}_{\vec{H}}f'E_j + f'[E_j,\vec{H}]\\ &= -f'\bar{\nabla}_{E_j}\vec{H} - E_j(f')\vec{H} + f'\bar{\nabla}_{\vec{H}}E_j + \vec{H}(f')E_j\\ &+ f'\bar{\nabla}_{E_j}\vec{H} - f'\bar{\nabla}_{\vec{H}}E_j\\ &= -\langle E_j,\bar{\nabla}f'\rangle\vec{H} + \langle\vec{H},\bar{\nabla}f'\rangle E_j.\end{aligned}$$

Consequently,

$$c\sum_{i,j} g^{ij}\langle\bar{R}(E_i,\mathcal{K})\vec{H},E_j\rangle = c\sum_{i,j} g^{ij}\langle\bar{R}(E_j,\vec{H})\mathcal{K},E_i\rangle$$

$$= c\sum_{i,j} g^{ij}\langle -\langle E_j, \bar\nabla f'\rangle \vec H + \langle \vec H, \bar\nabla f'\rangle E_j, E_i\rangle$$

$$= c\sum_{i,j} g^{ij}\langle \vec H, \bar\nabla f'\rangle g_{ij}$$

$$= nc\langle \vec H, \bar\nabla f'\rangle.$$

Hence, we deduce that

$$\frac{1}{2}\Delta_{-cu}|\vec H|^2 = cf'|\vec H|^2 + \langle A_{\vec H}, A_{\vec H}\rangle - \langle (\bar\nabla_{(\cdot)}\vec H)^\perp, (\bar\nabla_{(\cdot)}\vec H)^\perp\rangle + nc\langle \bar\nabla f', \vec H\rangle$$
$$+ \operatorname{tr}\bar R(\vec H, \cdot)\vec H. \tag{4.1.10}$$

We observe that

$$\bar\nabla f' = -f''(t)\partial_t = -\frac{f''(t)}{f(t)}\mathcal{K}.$$

Thus, from (1.2.1) we get

$$\langle \vec H, \partial_t\rangle = \left\langle \vec H, \frac{\mathcal{K}^\perp}{f}\right\rangle = \frac{1}{cf}\langle \vec H, \vec H\rangle = -\frac{1}{cf}|\vec H|^2. \tag{4.1.11}$$

So,

$$c\langle \bar\nabla f', \vec H\rangle = -c\frac{f''}{f}\langle \mathcal{K}, \vec H\rangle = -c\frac{f''}{f}\langle \mathcal{K}^\perp, \vec H\rangle = \frac{f''}{f}|\vec H|^2. \tag{4.1.12}$$

Furthermore, since $\partial_t = \partial_t^\top + \partial_t^\perp$, from (1.2.1) we also have equation

$$\partial_t^\perp = \frac{1}{f}\mathcal{K}^\perp = \frac{1}{cf}\vec H.$$

Then,

$$|\partial_t^\top|^2 = -1 + |\partial_t^\perp|^2 = -1 + \frac{|\vec H|^2}{c^2 f^2} \geq 0. \tag{4.1.13}$$

Now, using the properties of the curvature tensor, we have

$$\bar R(\vec H, E_i)\vec H = \bar R(\vec H^*, E_i^*)\vec H^* - \langle \vec H, \partial_t\rangle \bar R(\vec H^*, E_i^*)\partial_t - \langle E_i, \partial_t\rangle \bar R(\vec H^*, \partial_t)\vec H^*$$
$$+ \langle E_i, \partial_t\rangle\langle \vec H, \partial_t\rangle \bar R(\vec H^*, \partial_t)\partial_t - \langle \vec H, \partial_t\rangle \bar R(\partial_t, E_i^*)\vec H^* + \langle \vec H, \partial_t\rangle^2 \bar R(\partial_t, E_i^*)\partial_t$$
$$+ \langle \vec H, \partial_t\rangle\langle E_i, \partial_t\rangle \bar R(\partial_t, \partial_t)\vec H^* - \langle \vec H, \partial_t\rangle^2\langle E_i, \partial_t\rangle \bar R(\partial_t, \partial_t)\partial_t, \tag{4.1.14}$$

where $\vec H^* = (\pi_M)_*(\vec H)$ and $E_i^* = (\pi_M)_*(E_i)$. But, [66, Proposition 7.42] gives

(i) $\bar R(\vec H^*, E_i^*)\partial_t = 0$;

(ii) $\bar{R}(\vec{H}^*, \partial_t)\vec{H}^* = -\bar{R}(\partial_t, \vec{H}^*)\vec{H}^* = \langle \vec{H}^*, \vec{H}^* \rangle \frac{f''}{f}\partial_t = (-|\vec{H}|^2 + \langle \vec{H}, \partial_t \rangle^2)\frac{f''}{f}\partial_t$;

(iii) $\bar{R}(\vec{H}^*, \partial_t)\partial_t = \frac{f''}{f}\vec{H}^* = \frac{f''}{f}(\vec{H} + \langle \vec{H}, \partial_t \rangle \partial_t)$;

(iv) $\bar{R}(\partial_t, E_i^*)\vec{H}^* = -\langle E_i^*, \vec{H}^* \rangle \dfrac{f''}{f}\partial_t$

$$= -(\langle \vec{H}, E_i \rangle + \langle \vec{H}, \partial_t \rangle \langle E_i, \partial_t \rangle)\frac{f''}{f}\partial_t = -\langle \vec{H}, \partial_t \rangle \langle E_i, \partial_t \rangle \frac{f''}{f}\partial_t;$$

(v) $\bar{R}(\partial_t, E_i^*)\partial_t = -\bar{R}(E_i^*, \partial_t)\partial_t = -\frac{f''}{f}E_i^* = -\frac{f''}{f}(E_i + \langle E_i, \partial_t \rangle \partial_t)$.

Hence, we can use these items in (4.1.14) to get

$$\begin{aligned}
\bar{R}(\vec{H}, E_i)\vec{H} = \bar{R}(\vec{H}^*, E_i^*)\vec{H}^* &- \langle E_i, \partial_t \rangle(-|\vec{H}|^2 + \langle \vec{H}, \partial_t \rangle^2)\frac{f''}{f}\partial_t \\
&+ \langle E_i, \partial_t \rangle \langle \vec{H}, \partial_t \rangle \frac{f''}{f}(\vec{H} + \langle \vec{H}, \partial_t \rangle \partial_t) + \langle \vec{H}, \partial_t \rangle^2 \langle E_i, \partial_t \rangle \frac{f''}{f}\partial_t \\
&- \frac{f''}{f}\langle \vec{H}, \partial_t \rangle^2 (E_i + \langle \partial_i, \partial_t \rangle \partial_t).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\langle \bar{R}(\vec{H}, E_i)\vec{H}, E_i \rangle &= \langle \bar{R}(\vec{H}^*, E_i^*)\vec{H}^*, E_i^* \rangle - \langle E_i, \partial_t \rangle \langle \bar{R}(\vec{H}^*, E_i^*)\vec{H}^*, \partial_t \rangle + \langle E_i, \partial_t \rangle^2 |\vec{H}|^2 \frac{f''}{f} \\
&- \langle E_i, \partial_t \rangle^2 \langle \vec{H}, \partial_t \rangle^2 \frac{f''}{f} + \langle E_i, \partial_t \rangle \langle \vec{H}, \partial_t \rangle \langle \vec{H}, E_i \rangle \frac{f''}{f} \\
&+ \langle E_i, \partial_t \rangle^2 \langle \vec{H}, \partial_t \rangle^2 \frac{f''}{f} + \langle \vec{H}, \partial_t \rangle^2 \langle E_i, \partial_t \rangle^2 \frac{f''}{f} - \langle \vec{H}, \partial_t \rangle^2 |E_i|^2 \frac{f''}{f} \\
&- \langle E_i, \partial_t \rangle^2 \langle \vec{H}, \partial_t \rangle^2 \frac{f''}{f} \\
&= \langle \bar{R}(\vec{H}^*, E_i^*)\vec{H}^*, E_i^* \rangle + \langle E_i, \partial_t \rangle^2 |\vec{H}|^2 \frac{f''}{f} - \langle \vec{H}, \partial_t \rangle^2 |E_i|^2 \frac{f''}{f}. \quad (4.1.15)
\end{aligned}$$

Moreover, we observe that

$$\begin{aligned}
\bar{R}(\vec{H}^*, E_i^*)\vec{H}^* &= R_M(\vec{H}^*, E_i^*)\vec{H}^* - \frac{\langle \nabla f, \nabla f \rangle}{f^2}\{\langle \vec{H}^*, \vec{H}^* \rangle E_i^* - \langle E_i^*, \vec{H}^* \rangle \vec{H}^*\} \\
&= R_M(\vec{H}^*, E_i^*)\vec{H}^* - \left(\frac{f'}{f}\right)^2 |\vec{H}|^2 E_i - \left(\frac{f'}{f}\right)^2 |\vec{H}|^2 \langle E_i, \partial_t \rangle \partial_t \\
&+ \left(\frac{f'}{f}\right)^2 \langle \vec{H}, \partial_t \rangle^2 E_i - \left(\frac{f'}{f}\right)^2 \langle \vec{H}, \partial_t \rangle \langle E_i, \partial_t \rangle \vec{H}.
\end{aligned}$$

Consequently,

$$\langle \bar{R}(\vec{H}^*, E_i^*)\vec{H}^*, E_i^* \rangle = \langle R_M(\vec{H}^*, E_i^*)\vec{H}^*, E_i^* \rangle - \left(\frac{f'}{f}\right)^2 |\vec{H}|^2 |E_i|^2$$

$$+\left(\frac{f'}{f}\right)^2\langle\vec{H},\partial_t\rangle^2|E_i|^2-\left(\frac{f'}{f}\right)^2|\vec{H}|^2\langle E_i,\partial_t\rangle^2.$$

Returning to (4.1.15), we obtain

$$\begin{aligned}\langle\bar{R}(\vec{H},E_i)\vec{H},E_i\rangle &= \langle R_M(\vec{H}^*,E_i^*)\vec{H}^*,E_i^*\rangle-\left(\frac{f'}{f}\right)^2|\vec{H}|^2|E_i|^2+\left(\frac{f'}{f}\right)^2\langle\vec{H},\partial_t\rangle^2|E_i|^2\\ &-\left(\frac{f'}{f}\right)^2|\vec{H}|^2\langle E_i,\partial_t^\top\rangle^2+\langle E_i,\partial_t^\top\rangle^2|\vec{H}|^2\frac{f''}{f}-\langle\vec{H},\partial_t\rangle^2|E_i|^2\frac{f''}{f}.\end{aligned}$$

Thus,

$$\begin{aligned}\sum_i\langle\bar{R}(\vec{H},E_i)\vec{H},E_i\rangle &= \sum_i\langle R_M(\vec{H}^*,E_i^*)\vec{H}^*,E_i^*\rangle-n\left(\frac{f'}{f}\right)^2|\vec{H}|^2+n\left(\frac{f'}{f}\right)^2\langle\vec{H},\partial_t\rangle^2\\ &-\left(\frac{f'}{f}\right)^2|\vec{H}|^2|\partial_t^\top|^2+|\partial_t^\top|^2|\vec{H}|^2\frac{f''}{f}-n\langle\vec{H},\partial_t\rangle^2\frac{f''}{f}\\ &=\sum_i\langle R_M(\vec{H}^*,E_i^*)\vec{H}^*,E_i^*\rangle-n\left(\frac{f'}{f}\right)^2|\vec{H}|^2\\ &-\frac{1}{f^2}(f''f-f'^2)(n\langle\vec{H},\partial_t\rangle^2-|\vec{H}|^2|\partial_t^\top|^2).\end{aligned}$$

We also note that

$$\begin{aligned}\langle R_M(\vec{H}^*,E_i^*)\vec{H}^*,E_i^*\rangle &= \frac{1}{f^2}(\langle\vec{H}^*,\vec{H}^*\rangle_M\langle E_i^*,E_i^*\rangle_M-\langle\vec{H}^*,E_i^*\rangle_M^2)k\\ &=\frac{1}{f^2}(-|\vec{H}|^2|E_i|^2-|\vec{H}|^2\langle E_i,\partial_t\rangle^2+\langle\vec{H},\partial_t\rangle^2|E_i|^2)k.\end{aligned}$$

Thus,

$$\sum_i\langle R_M(\vec{H}^*,E_i^*)\vec{H}^*,E_i^*\rangle=\frac{1}{f^2}(-n|\vec{H}|^2-|\vec{H}|^2|\partial_t^\top|^2+n\langle\vec{H},\partial_t\rangle^2)k.$$

Hence,

$$\begin{aligned}\mathrm{tr}\bar{R}(\vec{H},\cdot)\vec{H} &= \sum_i\langle\bar{R}(\vec{H},E_i)\vec{H},E_i\rangle\\ &=\frac{1}{f^2}(-n|\vec{H}|^2-|\vec{H}|^2|\partial_t^\top|^2+n\langle\vec{H},\partial_t\rangle^2)k-n\left(\frac{f'}{f}\right)^2|\vec{H}|^2\\ &-\frac{1}{f^2}(f''f-f'^2)(n\langle\vec{H},\partial_t\rangle^2-|\vec{H}|^2|\partial_t^\top|^2)\\ &=-n\left(\frac{k}{f^2}+\left(\frac{f'}{f}\right)^2\right)|\vec{H}|^2 \qquad (4.1.16)\\ &+\left(\frac{k}{f^2}-\frac{1}{f^2}(f''f-f'^2)\right)(n\langle\vec{H},\partial_t\rangle^2-|\vec{H}|^2|\partial_t^\top|^2).\end{aligned}$$

Using (4.1.11), (4.1.12), (4.1.13) and (4.1.16), the two last terms in (4.1.10) can be easily written in the following form

$$
\begin{aligned}
nc\langle \bar{\nabla} f', \vec{H}\rangle + \operatorname{tr}\bar{R}(\vec{H},\cdot)\vec{H} &= n\frac{f''}{f}|\vec{H}|^2 - n\left(\frac{k}{f^2} + \left(\frac{f'}{f}\right)^2\right)|\vec{H}|^2 \\
&\quad + \left(\frac{k}{f^2} - \frac{1}{f^2}(f''f - f'^2)\right)\left(n\langle \vec{H}, \partial_t\rangle^2 - |\vec{H}|^2|\partial_t^\top|^2\right) \\
&= -n\left(\frac{k}{f^2} - \frac{1}{f^2}(f''f - f'^2)\right)|\vec{H}|^2 \\
&\quad + \left(\frac{k}{f^2} - \frac{1}{f^2}(f''f - f'^2)\right)\left(n\frac{1}{c^2f^2}|\vec{H}|^4 + \left(1 - \frac{|\vec{H}|^2}{c^2f^2}\right)|\vec{H}|^2\right) \\
&= -(n-1)\left(\frac{k}{f^2} - \frac{1}{f^2}(f''f - f'^2)\right)|\vec{H}|^2 \\
&\quad + \left(\frac{k}{f^2} - \frac{1}{f^2}(f''f - f'^2)\right)\left((n-1)\frac{|\vec{H}|^4}{c^2f^2}\right) \\
&= (n-1)\left(\frac{k}{f^2} - \frac{1}{f^2}(f''f - f'^2)\right)|\vec{H}|^2\left(-1 + \frac{|\vec{H}|^2}{c^2f^2}\right).
\end{aligned}
$$

Therefore, we conclude that

$$
\begin{aligned}
\frac{1}{2}\Delta_{-cu}|\vec{H}|^2 &= cf'|\vec{H}|^2 + \langle A_{\vec{H}}, A_{\vec{H}}\rangle - \langle (\bar{\nabla}_{(\cdot)}\vec{H})^\perp, (\bar{\nabla}_{(\cdot)}\vec{H})^\perp\rangle \\
&\quad + (n-1)\left(\frac{k}{f^2} - \frac{1}{f^2}(f''f - f'^2)\right)|\vec{H}|^2\left(-1 + \frac{|\vec{H}|^2}{c^2f^2}\right).
\end{aligned}
$$

■

When the ambient spacetime is a Lorentzian product space, Proposition C reads as follows.

Corollary 4.1.1 *Let $x : \Sigma^n \looparrowright -I \times M^{n+p}$ be a spacelike translation soliton with respect to ∂_t and with soliton constant $c \neq 0$. Let $\vec{H}$ be its mean curvature vector and suppose that the Riemannian fiber M^{n+p} has constant sectional curvature k. Then,*

$$
\frac{1}{2}\Delta_{-cu}|\vec{H}|^2 = \langle A_{\vec{H}}, A_{\vec{H}}\rangle - \langle (\bar{\nabla}_{(\cdot)}\vec{H})^\perp, (\bar{\nabla}_{(\cdot)}\vec{H})^\perp\rangle + (n-1)k|\vec{H}|^2\left(-1 + \frac{|\vec{H}|^2}{c^2}\right).
$$

When a GRW spacetime $-I \times_f M^{n+p}$ has constant sectional curvature $\bar{k}$, it follows from [66, Corollary 7.43] that the Riemannian fiber M^{n+p} has constant sectional curvature k and the warping function f is a solution of the following differential equations

$$
\frac{f''(h)}{f(h)} = \frac{k}{f^2(h)} + \frac{f'^2(h)}{f^2(h)} = \bar{k}. \tag{4.1.17}
$$

In this case, we obtain the following consequence of Proposition C.

Corollary 4.1.2 *Suppose that $-I\times_f M^{n+p}$ is a GRW spacetime with constant sectional curvature $\bar{k}$ and let $x:\Sigma^n \looparrowright -I\times_f M^{n+p}$ be a spacelike mean curvature flow soliton with respect to $\mathcal{K}=f(t)\partial_t$ and with soliton constant $c\in\mathbb{R}$. Let $\vec{H}$ be its mean curvature vector. Then,*

$$\frac{1}{2}\Delta_{-cu}|\vec{H}|^2 = cf'|\vec{H}|^2 + \langle A_{\vec{H}}, A_{\vec{H}}\rangle - \langle(\bar{\nabla}_{(\cdot)}\vec{H})^\perp, (\bar{\nabla}_{(\cdot)}\vec{H})^\perp\rangle.$$

4.2 Rigidity results under the strong null convergence condition

As in [59, 77], in our next results we will assume that the second fundamental form A of $x:\Sigma^n \looparrowright -I\times_f M^{n+p}$ is locally timelike, which means that $A(U,V)$ is locally timelike, for all vector fields $U,V\in\mathfrak{X}(\Sigma)$. We note that this hypothesis is automatically satisfied in the case of codimension one.

Thus, considering the strong null convergence condition (SNCC)

$$K_M \geq \sup_I (ff'' - f'^2), \tag{4.2.1}$$

which was introduced by Alías and Colares in [7], where K_M stands for the sectional curvature of the Riemannian fiber M^{n+p}, from Proposition C we get the following rigidity result.

Theorem 4.2.1 *Let $-I\times_f M^{n+p}$ be a GRW spacetime whose Riemannian fiber M^{n+p} has constant sectional curvature k and obeying the SNCC (4.2.1), and let $x:\Sigma^n \looparrowright -I\times_f M^{n+p}$ be a compact spacelike mean curvature flow soliton with respect to $\mathcal{K}=f(t)\partial_t$ and with soliton constant $c\in\mathbb{R}$, whose second fundamental form A is locally timelike. If $cf'|\vec{H}|^2+\langle A_{\vec{H}},A_{\vec{H}}\rangle\geq 0$ on Σ^n, then $-I\times_f M^{n+p}$ has constant sectional curvature, the mean curvature vector $\vec{H}$ is parallel and $cf'|\vec{H}|^2+\langle A_{\vec{H}},A_{\vec{H}}\rangle = 0$ on Σ^n. Furthermore, if $f''\leq 0$ on $[h_*,h^*]$, where h_* and h^* are the minimum and maximum of the height function h on Σ^n, then $x(\Sigma^n)$ is contained in a slice $\{t\}\times M^{n+p}$.*

Proof. If the second fundamental form A is locally timelike, we have that

$$-\langle(\bar{\nabla}_{(\cdot)}\vec{H})^\perp, (\bar{\nabla}_{(\cdot)}\vec{H})^\perp\rangle = -c^2\langle A(\mathcal{K}^\top,\cdot), A(\mathcal{K}^\top,\cdot)\rangle \geq 0.$$

Then, from Proposition C, jointly with (4.2.1), $cf'|\vec{H}|^2 + \langle A_{\vec{H}}, A_{\vec{H}}\rangle \geq 0$ and using that the second fundamental form A is locally timelike, we have that

$$\frac{1}{2}\Delta_{-cu}|\vec{H}|^2 = cf'|\vec{H}|^2 + \langle A_{\vec{H}}, A_{\vec{H}}\rangle - \langle(\bar{\nabla}_{(\cdot)}\vec{H})^\perp, (\bar{\nabla}_{(\cdot)}\vec{H})^\perp\rangle$$
$$+ (n-1)\left(\frac{k}{f^2} - \frac{1}{f^2}(f''f - f'^2)\right)|\vec{H}|^2\left(-1 + \frac{|\vec{H}|^2}{c^2f^2}\right) \geq 0.$$

Since $x(\Sigma^n)$ is compact, we get that $|\vec{H}|^2$ is constant. Then, $\Delta_{-cu}|\vec{H}|^2 = 0$ and we conclude that $cf'|\vec{H}|^2 + \langle A_{\vec{H}}, A_{\vec{H}}\rangle = 0$, $-\langle(\bar{\nabla}_{(\cdot)}\vec{H})^\perp, (\bar{\nabla}_{(\cdot)}\vec{H})^\perp\rangle = 0$ and $\frac{k}{f^2} - \frac{1}{f^2}(f''f - f'^2) = 0$. Then,

$$\frac{k}{f^2} + \left(\frac{f'}{f}\right)^2 = \frac{f''}{f}.$$

Therefore, from (4.1.17), $-I \times_f M^{n+p}$ has constant sectional curvature, and the mean curvature vector $\vec{H}$ is parallel.

Furthermore, since $|\vec{H}|^2$ is constant, if $f'' \leq 0$, from Corollary 3.0.2 we conclude that $x(\Sigma^n)$ is contained in a slice $\{t\} \times M^{n+p}$. ■

Given an oriented Riemannian manifold Σ^n and $q > 0$, we can consider the following space of integrable functions

$$\mathcal{L}^q_{-cu}(\Sigma^n) = \{\varphi \in C^\infty(\Sigma^n) : |\varphi|^q \in \mathcal{L}^1_{-cu}(\Sigma^n)\}.$$

As an application of [78, Theorem 3], we obtain the following criterion of integrability.

Lemma 4.2.1 *Let Σ^n be an n-dimensional complete oriented Riemannian manifold. If $\varphi \in C^\infty(\Sigma^n)$ is a nonnegative $(-cu)$-subharmonic function on Σ^n and $\varphi \in \mathcal{L}^q_{-cu}(\Sigma^n)$, for some $q > 1$, then φ is constant.*

Next, we apply Lemma 4.2.1 to prove the following result.

Theorem 4.2.2 *Let $-I \times_f M^{n+p}$ be a GRW spacetime whose Riemannian fiber M^{n+p} has constant sectional curvature k and obeying the SNCC (4.2.1), and let $x : \Sigma^n \looparrowright -I \times_f M^{n+p}$ be a complete spacelike mean curvature flow soliton with respect to $\mathcal{K} = f(t)\partial_t$ and with soliton constant $c \in \mathbb{R}$, whose second fundamental form A is locally timelike. If $cf' \geq 0$ on Σ^n and $|\vec{H}|^2 \in \mathcal{L}^q_{-cu}(\Sigma^n)$, for some $q > 1$, then $-I \times_f M^{n+p}$ has constant sectional curvature, the mean curvature vector $\vec{H}$ is parallel, $\langle A_{\vec{H}}, A_{\vec{H}}\rangle = 0$*

and $cf' = 0$ on Σ. Furthermore, Σ^n is either contained in a slice $\{t\} \times M^{n+p}$ or isometric to a direct product $V^{n-1} \times J$ of an $(n-1)$-dimensional complete Riemannian manifold V^{n-1} with a straight line J.

Proof. We can reason as in the proof of Theorem 4.2.1 to get that $\Delta_{-cu}|\vec{H}|^2 \geq 0$. Since $|\vec{H}|^2 \in \mathcal{L}^q_{-cu}(\Sigma^n)$, from Lemma 4.2.1 we have that $|\vec{H}|^2$ is constant. Following the same ideas of the proof of Theorem 4.2.1 we also obtain that $-I \times_f M^{n+p}$ has constant sectional curvature, the mean curvature vector $\vec{H}$ is parallel, $\langle A_{\vec{H}}, A_{\vec{H}} \rangle = 0$ and $cf' = 0$ on Σ^n. Moreover, from equation (1.2.17) we conclude that $\nabla^2 u = 0$. Consequently, assuming that u is not constant on Σ^n, [76, Theorem 2] guarantees that Σ^n must be isometric to a direct product $V^{n-1} \times J$ of an $(n-1)$-dimensional complete Riemannian manifold V^{n-1} with a straight line J. ∎

4.3 Nonexistence results via Omori-Yau's maximum principle

First, we establish a suitable version of the Omori-Yau's maximum principle for the drift Laplacian which can be regarded as a sort of extension of [19, Theorem 3.2] and [35, Proposition 3.15].

Proposition D *Let $-I \times_f M^{n+p}$ be a GRW spacetime obeying the SNCC (4.2.1) and let $x : \Sigma^n \looparrowright -I \times_f M^{n+p}$ be a complete spacelike mean curvature flow soliton with respect to $\mathcal{K} = f(t)\partial_t$ and with soliton constant $c \neq 0$, whose second fundamental form A is locally timelike. If the function $\frac{(n-1)f''(h)+cf(h)f'(h)}{f(h)}$ is bounded from below on Σ^n, then the Omori-Yau's maximum principle holds for the drift Laplacian Δ_{-cu}, that is, for $\varphi \in C^2(\Sigma^n)$ with $\sup_\Sigma \varphi < +\infty$, there exists a sequence of points $\{p_k\}_{k\geq 1}$ in Σ^n such that*

$$\lim_k \varphi(p_k) = \sup_k \varphi, \ \lim_k |\nabla \varphi(p_k)| = 0 \ \textit{and} \ \lim_k \Delta_{-cu}\varphi(p_k) \leq 0.$$

Proof. From Gauss' equation, we have that

$$\langle R(X,Y)Z,W\rangle = \langle \bar{R}(X,Y)Z,W\rangle + \langle A(X,Z),A(Y,W)\rangle - \langle A(X,W),A(Y,Z)\rangle, \tag{4.3.1}$$

for every tangent vector fields $X, Y, Z, W \in \mathfrak{X}(\Sigma^n)$, where R and $\bar{R}$ denote the curvature tensor of Σ^n and $-I \times_f M^{n+p}$ respectively.

Let us consider $X \in \mathfrak{X}(\Sigma^n)$ and take a (local) orthonormal frame $\{E_i\}_{i=1}^n$ in Σ^n. It follows from (4.3.1) that the Ricci curvature Ric of Σ^n satisfies

$$\mathrm{Ric}(X,X) = \sum_{i=1}^{n} \langle \bar{R}(X,E_i)X,E_i\rangle - \sum_{i=1}^{n}\langle A(X,E_i),A(X,E_i)\rangle - \langle A(X,X),\vec{H}\rangle.$$

Since the second fundamental form A is locally timelike, we have that

$$\langle A(X,E_i),A(X,E_i)\rangle \leq 0,$$

for every $i = 1,\dots,n$. Then,

$$\mathrm{Ric}(X,X) \geq \sum_{i=1}^{n}\langle \bar{R}(X,E_i)X,E_i\rangle - \langle A_{\vec{H}}X,X\rangle. \tag{4.3.2}$$

Hence, from Lemma 1.2.1 and (4.3.2) we get

$$\mathrm{Ric}(X,X) - c\nabla^2 u(X,X) \geq \sum_{i=1}^{n}\langle \bar{R}(X,E_i)X,E_i\rangle + cf'(h)\langle X,X\rangle. \tag{4.3.3}$$

To estimate the first summand on the right-hand side of inequality (4.3.3), let us consider $X^* = (\pi_M)_*(X)$ and $E_i^* = (\pi_M)_*(E_i)$. Using the properties of the curvature tensor, similar to what was done in Proposition C, and from (1.2.3) and [66, Proposition 7.42] we have

$$\begin{aligned}\sum_i\langle \overline{R}(X,E_i)X,E_i\rangle &= \sum_i\langle R_M(X^*,E_i^*)X^*,E_i^*\rangle + (n-1)((\ln f)'(h))^2|X|^2\\ &\quad - (n-2)(\ln f)''(h)\langle X,\nabla h\rangle^2 - (\ln f)''(h)|\nabla h|^2|X|^2,\end{aligned} \tag{4.3.4}$$

where R_M denotes the curvature tensor of the Riemannian fiber M^{n+p}. Writing $X^* = X + \langle X,\partial_t\rangle\partial_t$, we can estimate the first summand on the right-hand side of (4.3.4) as follows

$$\begin{aligned}\sum_i\langle R_M(X^*,E_i^*)X^*,E_i^*\rangle &= f^2(h)(|X^*|_M^2|E_i^*|_M^2 - \langle X^*,E_i^*\rangle_M^2)K_M(X^*,E_i^*)\\ &\geq \frac{1}{f^2(h)}((n-1)|X|^2 + |\nabla h^2||X|^2\\ &+ (n-2)\langle X,\nabla h\rangle^2)\min_i K_M(X^*,E_i^*).\end{aligned} \tag{4.3.5}$$

Consequently, from (4.2.1) and (4.3.5) we have that

$$\sum_i \langle R_M(X^*, E_i^*)X^*, E_i^*\rangle \geq ((n-1)|X|^2 + |\nabla h|^2|X|^2 + (n-2)\langle X, \nabla h\rangle^2)(\ln f)''(h). \tag{4.3.6}$$

Substituting (4.3.6) into (4.3.4), we get

$$\begin{aligned} \sum_i \langle \overline{R}(X, E_i)X, E_i\rangle &\geq ((n-1)|X|^2 + |\nabla h|^2|X|^2 + (n-2)\langle X, \nabla h\rangle^2)(\ln f)''(h) \\ &+ (n-1)((\ln f)'(h))^2|X|^2 - (n-2)(\ln f)''(h)\langle X, \nabla h\rangle^2 \\ &- (\ln f)''(h)|\nabla h|^2|X|^2 \\ &= (n-1)\frac{f''(h)}{f(h)}|X|^2. \end{aligned} \tag{4.3.7}$$

Hence, from (4.3.3) and (4.3.7) we obtain

$$\mathrm{Ric} - c\nabla^2 u \geq \left((n-1)\frac{f''(h)}{f(h)} + cf'(h)\right)\langle\,,\rangle.$$

Therefore, since the right-hand side of the above inequality is bounded from bellow, we conclude our proof by applying Theorem 1.3.1. ■

Proposition D allows us to the following nonexistence result that extends Theorem 4.2 in [19] and Theorem 1.2 in [35].

Theorem 4.3.1 *Let $-I \times_f M^{n+p}$ be a GRW spacetime obeying the SNCC (4.2.1), whose Riemannian fiber M^{n+p} has constant sectional curvature k. There does not exist complete spacelike mean curvature flow soliton $x : \Sigma^n \looparrowright -I \times_f M^{n+p}$ with respect to $\mathcal{K} = f(t)\partial_t$ and with soliton constant $c \neq 0$, whose second fundamental form A is locally timelike, and such that $cf'(h) \geq 0$ and the function $\frac{(n-1)f''(h)+cf(h)f'(h)}{f(h)}$ is bounded from below.*

Proof. Let us suppose for the sake of contradiction the existence of such a spacelike mean curvature flow soliton. Since we are assuming that $cf'(h) \geq 0$, taking a (local) orthonormal frame $\{E_i\}_{i=1}^n$ in Σ^n, from Proposition C, jointly with (4.2.1) and using the hypothesis that the second fundamental form A is locally timelike, we obtain

$$\begin{aligned} \frac{1}{2}\Delta_{-cu}|\vec{H}|^2 &\geq \langle A_{\vec{H}}, A_{\vec{H}}\rangle = \sum_{i,l}\langle A(E_i, E_l), \vec{H}\rangle^2 \geq \sum_i \langle A(E_i, E_i), \vec{H}\rangle^2 \\ &\geq \frac{1}{n}\left(\sum_i \langle A(E_i, E_i), \vec{H}\rangle\right)^2 = \frac{1}{n}\left\langle \sum_i A(E_i, E_i), \vec{H}\right\rangle^2 = \frac{1}{n}|\vec{H}|^4. \end{aligned}$$

The third inequality follows from the Cauchy-Schwarz's inequality. Then, observe that

$$\begin{aligned}\Delta_{-cu}\left(-\frac{1}{\sqrt{1+|\vec{H}|^2}}\right) &= \left(-\frac{1}{\sqrt{1+|\vec{H}|^2}}\right)'\Delta_{-cu}|\vec{H}|^2 + \left(-\frac{1}{\sqrt{1+|\vec{H}|^2}}\right)''|\nabla|\vec{H}|^2|^2 \\ &= \frac{\Delta_{-cu}|\vec{H}|^2}{2(1+|\vec{H}^2)^{\frac{3}{2}}} - \frac{3}{4}\frac{|\nabla|\vec{H}|^2|^2}{(1+|\vec{H}|^2)^{\frac{5}{2}}} \\ &\geq \frac{|\vec{H}|^4}{n(1+|\vec{H}|^2)^{\frac{3}{2}}} - \frac{3}{4}\frac{|\nabla|\vec{H}|^2|^2}{(1+|\vec{H}|^2)^{\frac{5}{2}}}. \end{aligned} \tag{4.3.8}$$

Proceeding as in the proof of [28, Theorem 2], dividing both sides of (4.3.8) by $\sqrt{1+|\vec{H}|^2}$ we get

$$\frac{|\vec{H}|^4}{n(1+|\vec{H}|^2)^2} \leq \frac{1}{\sqrt{1+|\vec{H}|^2}}\Delta_{-cu}\left(-\frac{1}{\sqrt{1+|\vec{H}|^2}}\right) + \frac{3}{4}\frac{|\nabla|\vec{H}|^2|^2}{(1+|\vec{H}|^2)^3}. \tag{4.3.9}$$

Since the second fundamental form A is locally timelike and $\frac{(n-1)f''(h)+cf(h)f'(h)}{f(h)}$ is bounded from below, Proposition D enable us to apply the Omori-Yau's maximum principle to the function $-\frac{1}{\sqrt{1+|\vec{H}|^2}}$ and conclude that for j sufficiently large, there exists a sequence of points $\{p_j\} \subset \Sigma^n$ such that

$$\frac{1}{\sqrt{1+|\vec{H}|^2}}(p_j) < \inf\left(\frac{1}{\sqrt{1+|\vec{H}|^2}}\right) + \frac{1}{j},$$

$$\frac{|\nabla|\vec{H}|^2|^2}{4(1+|\vec{H}|^2)^3}(p_j) < \frac{1}{j},$$

$$\Delta_{-cu}\left(-\frac{1}{\sqrt{1+|\vec{H}|^2}}\right)(p_j) < \frac{1}{j}.$$

Combining these with (4.3.9), it follows that

$$\frac{|\vec{H}|^4}{n(1+|\vec{H}|^2)^2}(p_j) < \frac{1}{j}\left(\inf\left(\frac{1}{\sqrt{1+|\vec{H}|^2}}\right) + \frac{1}{j}\right) + \frac{3}{j}.$$

When $j \to \infty$, $\frac{1}{\sqrt{1+|\vec{H}|^2}}(p_j)$ goes to its infimum and $|\vec{H}|^2(p_j)$ goes to its supremum. Therefore,

$$\frac{(\sup_\Sigma |\vec{H}|^2)^2}{(1+\sup_\Sigma |\vec{H}|^2)^2} \leq 0.$$

Then, we have $\sup_\Sigma |\vec{H}|^2 < \infty$. In fact, if $\sup_\Sigma |\vec{H}|^2 = \infty$, we obtain that

$$\frac{(\sup_\Sigma |\vec{H}|^2)^2}{(1+\sup_\Sigma |\vec{H}|^2)^2} = \frac{1}{(1+\frac{1}{\sup_\Sigma |\vec{H}|^2})^2} = 1,$$

which cannot occur. Thus, $\sup_\Sigma |\vec{H}|^2 < \infty$ and it follows that $|\vec{H}| \equiv 0$, leading us to a contradiction with the fact that Σ^n is a trapped submanifold from Proposition A. ∎

Remark 4.3.1 *In the case where the codimension is* 1 *and* $f = 0$, *the SNCC hypothesis in the above theorem means that the sectional curvature of the Riemannian fiber satisfies* $K_M \geq 0$. *However, in [19, Theorem 4.2], it is only required that the sectional curvature is bounded from below, which is a weaker assumption than the one required here. Therefore, from Theorem 4.2 in [19], we get that: There is no complete spacelike translating soliton into* $\mathbb{R}_1 \times M^n$ *for* M^n *non-negatively curved. Furthermore, if we consider the surface*

$$\Sigma = \{(c \ln y, x, y) : y > 0\} \subset \mathbb{R}_1 \times \mathbb{H}^2,$$

where the constant $c \in \mathbb{R}$ *is such that* $0 < |c| < 1$ *and* $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ *stands for the* 2*-dimensional hyperbolic space endowed with the complete metric* $\langle , \rangle_{\mathbb{H}^2} = \frac{1}{y^2}(dx^2 + dy^2)$. *This is an example of a complete spacelike translating soliton with constant men curvature* $H = \frac{c}{\sqrt{1-c^2}}$. *However, this example does not fall within the hypotheses of the theorem 4.2 from [19]; the hypothesis that the Riemannian base has nonnegative Ricci curvature is necessary to ensure the nonexistence result.*

We note that the boundedness of the function $\frac{(n-1)f''(h)+cf(h)f'(h)}{f(h)}$ is automatically satisfied if we assume that the spacelike mean curvature flow soliton lies in a slab of the ambient spacetime. Then, from Theorem 4.3.1 we obtain the following consequence.

Corollary 4.3.1 *Let* $-I \times_f M^{n+p}$ *be a GRW spacetime obeying the SNCC (4.2.1) and whose Riemannian fiber* M^{n+p} *has constant sectional curvature* k. *There does not exist complete spacelike mean curvature flow soliton* $x : \Sigma^n \looparrowright -I \times_f M^{n+p}$ *with respect to* $\mathcal{K} = f(t)\partial_t$ *and with soliton constant* $c \neq 0$, *whose the second fundamental form* A *is locally timelike, lying in a slab of* $-I \times_f M^{n+p}$ *with* $cf'(h) \geq 0$.

4.4 Nonexistence results via polynomial volume growth

We need to quote a suitable maximum principle that will be used to prove our nonexistence results in this subsection. Then, let Σ^n be a connected, oriented, complete noncompact Riemannian manifold. We denote by $B(p,t)$ the geodesic ball centered at p and with radius t. Given a polynomial function $\sigma:(0,+\infty)\to(0,+\infty)$, we say that Σ^n has *polynomial volume growth* like $\sigma(t)$ if there exists $p\in\Sigma^n$ such that

$$\mathrm{vol}(B(p,t))=\mathcal{O}(\sigma(t)),$$

as $t\to+\infty$, where vol denotes the standard Riemannian volume related to the metric of Σ^n. As it was already observed in the beginning of [6, Section 2], if $p,q\in\Sigma^n$ are at distance d from each other, we can verify that

$$\frac{\mathrm{vol}(B(p,t))}{\sigma(t)}\geq\frac{\mathrm{vol}(B(q,t-d))}{\sigma(t-d)}.\frac{\sigma(t-d)}{\sigma(t)}.$$

Thus, the choice of p in the notion of volume growth is immaterial. For this reason, we will just say that Σ^n has polynomial volume growth.

Keeping in mind this previous digression, we have the following key lemma, which corresponds to a particular case of a new maximum principle due to Alías, Caminha, and do Nascimento (see [6, Theorem 2.1]).

Lemma 4.4.1 *Let Σ^n be a connected, oriented, complete noncompact Riemannian manifold, and $X\in\mathfrak{X}(\Sigma^n)$ be a bounded vector field on Σ^n, with $|X|\leq c<+\infty$. Assume that $f\in\mathcal{C}^\infty(\Sigma)$ is such that $\langle\nabla f,X\rangle\geq 0$ on Σ^n and $div X\geq af$ on Σ^n, for some constant $a>0$. If Σ^n has polynomial volume growth, then $f\leq 0$ on Σ^n.*

Returning to the context of spacelike mean curvature flow soliton in a GRW spacetime, we obtain the following nonexistence result.

Theorem 4.4.1 *There does not exist complete noncompact spacelike mean curvature flow soliton $x:\Sigma^n\looparrowright -I\times_f M^{n+p}$ with respect to $\mathcal{K}=f(t)\partial_t$ and with soliton constant $c>0$ ($c<0$), having polynomial volume growth and satisfying:*

(a) $\sup_\Sigma\zeta_c(h)<0$ *(*$\inf_\Sigma\zeta_c(h)>0$*);*

(b) $u=g(h)>0$*, where g is a primitive of f;*

(c) $X=e^{cu}\nabla u$ *is a bounded vector field on Σ^n.*

Proof. Let us suppose for the sake of contradiction the existence of such complete noncompact spacelike mean curvature flow soliton $x : \Sigma^n \looparrowright -I \times_f M^{n+p}$. Considering the vector field $X = e^{cu}\nabla u$, we observe that

$$\langle X, \nabla u\rangle = \langle e^{cu}\nabla u, \nabla u\rangle = e^{cu}|\nabla u|^2 \geq 0.$$

Furthermore, using (1.2.14) and (2.1.2), we have that

$$e^{-cu}\mathrm{div}_\Sigma X = e^{-cu}\mathrm{div}_\Sigma(e^{cu}\nabla u) = \Delta_{-cu}u = -\zeta(h).$$

Now, let us assume that $c > 0$ and $\sup \zeta_c(h) < 0$. Since $e^{cu} \geq cu$, we get

$$\mathrm{div}_\Sigma X = -e^{cu}\zeta_c(h) \geq -cu\zeta_c(h) \geq -\sup_\Sigma \zeta_c(h)u.$$

Then, since Σ^n has polynomial volume growth and $X = e^{cu}\nabla u$ is a bounded vector field on Σ^n, from Lemma 4.4.1, we have that $u \leq 0$ on Σ^n, which corresponds to a contradiction. Similarly, when $c < 0$ and $\inf \zeta_c(h) > 0$ we have the same result. ■

Next, we derive a second nonexistence result from Lemma 4.4.1.

Theorem 4.4.2 *Let $-I \times_f M^n$ be a GRW spacetime obeying the NCC (2.2.9). There does not exist complete noncompact spacelike mean curvature flow soliton $x : \Sigma^n \looparrowright -I \times_f M^n$ with respect to $\mathcal{K} = f(t)\partial_t$ and with soliton constant $c \neq 0$, having polynomial volume growth and satisfying:*

(a) the second soliton function is such that $\inf_\Sigma \tilde{\zeta}_c(h) = \inf_\Sigma\{cf'(h) + |A|^2\} > 0$;

(b) $\inf_\Sigma\{cu\} > 0$, where $u = g(h)$ and g is a primitive of f;

(c) $X = e^{cu}\nabla(-\frac{H}{c})$ is a bounded vector field on Σ^n.

Proof. Let us suppose for the sake of contradiction the existence of such a complete noncompact spacelike mean curvature flow soliton $x : \Sigma^n \looparrowright -I \times_f M^n$. Considering the vector field $X = e^{cu}\nabla(-\frac{H}{c})$, we note that

$$\left\langle X, \nabla\left(-\frac{H}{c}\right)\right\rangle = \left\langle e^{cu}\nabla\left(-\frac{H}{c}\right), \nabla\left(-\frac{H}{c}\right)\right\rangle = e^{cu}\left|\nabla\left(-\frac{H}{c}\right)\right|^2 \geq 0.$$

Furthermore, from (2.2.9) we obtain

$$\mathrm{Ric}_M(N^*, N^*) - (n-1)(\ln f)''(h)|\nabla h|^2 \geq 0.$$

Then, taking into account that $-\frac{H}{c} > 0$, from (2.2.8) we have that

$$\Delta_{-cu}\left(-\frac{H}{c}\right) = \left(\tilde{\zeta}_c + \mathrm{Ric}_M(N^*, N^*) - (n-1)(\ln f)''(h)|\nabla h|^2\right)\left(-\frac{H}{c}\right) \geq \tilde{\zeta}_c\left(-\frac{H}{c}\right).$$

On the other hand, since

$$\Delta_{-cu}\left(-\frac{H}{c}\right) = e^{-cu}\mathrm{div}_\Sigma\left(e^{cu}\nabla\left(-\frac{H}{c}\right)\right),$$

we concluded that

$$\begin{aligned}\mathrm{div}_\Sigma X = \mathrm{div}_\Sigma\left(e^{cu}\nabla\left(-\frac{H}{c}\right)\right) &\geq e^{cu}\tilde{\zeta}_c\left(-\frac{H}{c}\right)\\ &\geq (cu)\inf_\Sigma \tilde{\zeta}_c\left(-\frac{H}{c}\right) \geq \inf_\Sigma\{cu\}\inf_\Sigma \tilde{\zeta}_c\left(-\frac{H}{c}\right).\end{aligned}$$

Since Σ^n has polynomial volume growth and $X = e^{cu}\nabla\left(-\frac{H}{c}\right)$ is a bounded vector field, from Lemma 4.4.1, we have that $-\frac{H}{c} \leq 0$ on Σ^n, which corresponds to a contradiction. ■

When the ambient spacetime is a Lorentzian product space, Theorem 4.4.2 reads as follows.

Corollary 4.4.1 *Let $-I \times M^n$ be a Lorentzian product space whose Riemannian fiber M^n has nonnegative Ricci curvature. There does not exist complete noncompact spacelike translation soliton $x : \Sigma^n \looparrowright -I \times M^n$ with respect to ∂_t and with soliton constant $c \neq 0$, having polynomial volume growth, $\inf_\Sigma |A|^2 > 0$, $\inf_\Sigma\{ch\} > 0$ and such that $X = e^{ch}\nabla(-\frac{H}{c})$ is a bounded vector field.*

Proceeding, we also obtain the another nonexistence result for higher codimension.

Theorem 4.4.3 *Let $-I \times_f M^{n+p}$ be a GRW spacetime obeying the SNCC (4.2.1) and whose Riemannian fiber M^{n+p} has a constant seccional curvature k. There does not exist complete noncompact spacelike mean curvature flow soliton $x : \Sigma^n \looparrowright -I \times_f M^{n+p}$ with respect to $\mathcal{K} = f(t)\partial_t$ and with soliton constant $c \neq 0$, whose second fundamental form A is locally timelike, having polynomial volume growth and satisfying: $\inf_\Sigma\{cf'(h)\} > 0$, $\inf_\Sigma\{cu\} > 0$, and such that $X = e^{cu}\nabla|\vec{H}|^2$ is a bounded vector field.*

Proof. Let us suppose for the sake of contradiction the existence of such a complete noncompact spacelike mean curvature flow soliton $x : \Sigma^n \looparrowright -I \times_f M^{n+p}$. Considering the vector field $X = e^{cu}\nabla|\vec{H}|^2$, we have that

$$\langle X, \nabla|\vec{H}|^2\rangle = \langle e^{cu}\nabla|\vec{H}|^2, \nabla|\vec{H}|^2\rangle = e^{cu}|\nabla|\vec{H}|^2|^2 \geq 0.$$

Furthermore, from Proposition C, (4.2.1), and using the hypotheses that the second fundamental form A is locally timelike and $\inf_\Sigma\{cf'\} > 0$, we get

$$\frac{1}{2}\Delta_{-cu}|\vec{H}|^2 \geq \inf_\Sigma\{cf'(h)\}|\vec{H}|^2.$$

On the other hand, since

$$\Delta_{-cu}|\vec{H}|^2 = e^{-cu}\mathrm{div}_\Sigma(e^{cu}\nabla|\vec{H}|^2),$$

we concluded that

$$\mathrm{div}_\Sigma X = \mathrm{div}_\Sigma(e^{cu}\nabla|\vec{H}|^2) \geq e^{cu}2\inf_\Sigma\{cf'(h)\}|\vec{H}|^2 \geq 2\inf_\Sigma\{cf'(h)\}cu|\vec{H}|^2 \geq a|\vec{H}|^2,$$

where $a = 2\inf_\Sigma\{cu\}\inf_\Sigma\{cf'(h)\} > 0$.

Therefore, since Σ^n has polynomial volume growth, and $X = e^{cu}\nabla|\vec{H}|^2$ is a bounded vector field, from Lemma 4.4.1, we conclude that $|\vec{H}|^2 \leq 0$ on Σ^n, leading us to a contradiction. ∎

Chapter 5

The mean curvature flow soliton equation in entire graphs

Let $\Omega \subseteq M^n$ be a connected domain, and consider $z \in C^\infty(\Omega)$ as a smooth function such that $z(\Omega) \subseteq I$. Then, $\Sigma^n(z)$ will denote the *(vertical) graph over* Ω determined by z, defined as follows

$$\Sigma^n(z) = \{(z(p), p) : p \in \Omega\} \subset \bar{M}^{n+1} = -I \times_f M^n.$$

We say that the graph is *entire* if $\Omega = M^n$. Note that $h(z(p), p) = z(p), p \in \Omega$. Hence, h and z can be identified naturally. The metric induced on Ω from the Lorentzian metric of the ambient GRW spacetime via $\Sigma^n(z)$ is

$$g_z = -dz^2 + f(z)^2 \langle\, , \rangle_M. \tag{5.0.1}$$

The Gauss map of a graph $\Sigma^n(z)$ over Ω is given by the vector field

$$N(p) = \frac{f(z(p))}{\sqrt{f(z(p))^2 - |Dz(p)|_M^2}} \left(\partial_t|_{(z(p),p)} + \frac{Dz(p)}{f(z(p))^2} \right), \quad p \in \Omega, \tag{5.0.2}$$

where Dz stands for the gradient of z in M^n and $|Dz|_M$ its norm, both with respect to the metric $\langle\, , \rangle_M$. In fact, let us consider the function $G : \bar{M}^{n+1} \to \mathbb{R}$ given by $G(t, p) = t - z(p)$. Note that $G \equiv 0$ on $\Sigma^n(z)$, so consider a curve $\alpha : (-\varepsilon, \varepsilon) \to \Sigma^n(z)$ such that $\alpha(0) = p$ and $\alpha'(0) = v \in T\Sigma^n(z)$. Then

$$0 = \frac{d}{dt}(G \circ \alpha)(t)|_{t=0} = \alpha'(0)(G) = v(G) = \langle \bar{\nabla} G, v \rangle.$$

Since $v \in T\Sigma^n(z)$ was chosen arbitrarily, $\bar{\nabla} G$ is orthogonal to $T\Sigma^n(z)$. On the other hand, let us consider $w = -\langle w, \partial_t \rangle \partial_t + w^* \in T_{(t,p)}\bar{M}^{n+1}$, where $w^* \in \mathfrak{X}(M^n)$. Then, we obtain that

$$\begin{aligned} w(G) &= (-\langle w, \partial_t \rangle + w^*)(t - z(p)) \\ &= -\langle w, \partial_t \rangle \partial_t (t - z(p)) + \frac{w^*(t - z(p))}{f(z)^2} \\ &= -\langle w, \partial_t \rangle - \frac{1}{f(z(p))^2} \langle Dz(p), w \rangle \\ &= \left\langle w, -\partial_t - \frac{1}{f(z(p))^2} Dz(p) \right\rangle . \end{aligned}$$

Therefore,

$$\bar{\nabla} G = -\partial_t - \frac{1}{f(z(p))^2} Dz(p).$$

Taking $\bar{N} = -\bar{\nabla} G$, we get that

$$\begin{aligned} \langle \bar{N}, \bar{N} \rangle &= \langle \partial_t, \partial_t \rangle + 2 \left\langle \partial_t, \frac{1}{f(z(p))^2} Dz(p) \right\rangle + \left\langle \frac{1}{f(z(p))^2} Dz(p), \frac{1}{f(z(p))^2} Dz(p) \right\rangle \\ &= -1 + \frac{1}{f(z(p))^2} |Dz(p)|^2_M . \end{aligned}$$

Finally, taking $N = \frac{\bar{N}}{|\bar{N}|}$ we get the Gauss map given before. Furthermore, we have that

$$\langle \bar{N}, \partial_t \rangle = \left\langle \partial_t + \frac{1}{f(z(p))^2} Dz(p), \partial_t \right\rangle = \langle \partial_t, \partial_t \rangle = -1 < 0,$$

that is, the Gauss map N is future-pointing.

Moreover, we know that a hypersurface is spacelike if and only if its normal field is timelike. Thus, a graph $\Sigma^n(z)$ is a spacelike hypersurface if and only if N is timelike, in other words, if and only if $\bar{N}$ is timelike, that is, if and only if

$$\langle \bar{N}, \bar{N} \rangle = -1 + \frac{1}{f(z(p))^2} |Dz(p)|^2_M < 0.$$

Then, we get that a graph $\Sigma^n(z)$ is a spacelike hypersurface if and only if $|Dz|_M < f(z)$. On the other hand, from [10, Lemma 3.1], regarding the scenario where M^n is a simply connected manifold, we obtain that every complete spacelike hypersurface $x : \Sigma^n \looparrowright -I \times_f M^n$ such that the warping function f is bounded on Σ^n is an entire spacelike graph over M^n. In particular, the same occurs for complete spacelike hypersurfaces lying in a slab of $-I \times_f M^n$. It is noteworthy to mention that, unlike the scenario with graphs into a Riemannian space, an entire spacelike graph $\Sigma^n(z)$ in a GRW spacetime may not be complete. This implies that the induced Riemannian metric (5.0.1) on M^n is not necessarily complete on M^n. For instance, Albujer [2, Section 3] constructed explicit examples of noncomplete entire maximal spacelike graphs (that is, whose mean curvature is identically zero) in the Lorentzian product space $-\mathbb{R} \times \mathbb{H}^2$.

From (5.0.2), we have that the Weingarten operator related to the future-pointing Gauss map (5.0.2) is given by

$$\begin{aligned} AX = &- \frac{1}{f(z)\sqrt{f(z)^2 - |Dz|_M^2}} D_X Dz - \frac{f'(z)}{\sqrt{f(z)^2 - |Dz|_M^2}} X \\ &+ \left(\frac{-\langle D_X Dz, Dz\rangle_M}{f(z)\left(f(z)^2 - |Dz|_M^2\right)^{3/2}} + \frac{f'(z)\langle Dz, X\rangle_M}{\left(f(z)^2 - |Dz|_M^2\right)^{3/2}} \right) Dz, \end{aligned} \tag{5.0.3}$$

for any vector field X tangent to Ω, where D denotes the Levi-Civita connection of (M^n, g_M). Consequently, if $\Sigma^n(z)$ is a spacelike graph defined over a domain $\Omega \subseteq M^n$, it is not difficult to verify from (5.0.3) that the future mean curvature function $H(z)$ of $\Sigma^n(z)$ is given by

$$H(z) = \operatorname{div}_M \left(\frac{Dz}{n f(z)\sqrt{f(z)^2 - |Dz|_M^2}} \right) + \frac{f'(z)}{n\sqrt{f(z)^2 - |Dz|_M^2}} \left(n + \frac{|Dz|_M^2}{f(z)^2} \right), \tag{5.0.4}$$

where div_M stands for the divergence operator computed in the metric $\langle\, , \rangle_M$.

Hence, from (1.2.1) and (5.0.4) we have that $\Sigma^n(z)$ is a spacelike mean curvature flow soliton with respect to $\mathcal{K} = f(t)\partial_t$ and with soliton constant c if, and only if, $|Dz|_M < f(z)$ and z is a solution of the following nonlinear differential equation

$$\operatorname{div}_M \left(\frac{Dz}{f(z)\sqrt{f(z)^2 - |Dz|_M^2}} \right) = -\frac{1}{\sqrt{f(z)^2 - |Dz|_M^2}} \left\{ c f(z)^2 + f'(z) \left(n + \frac{|Dz|_M^2}{f(z)^2} \right) \right\}. \tag{5.0.5}$$

In particular, when the ambient spacetime is a Lorentzian product space $-I \times M^n$, equation (5.0.5) reads as follows

$$\operatorname{div}_M \left(\frac{Dz}{\sqrt{1 - |Dz|_M^2}} \right) = -\frac{c}{\sqrt{1 - |Dz|_M^2}}. \tag{5.0.6}$$

Before we proceed to our results, we need an auxiliary results.

Proposition E *[3, Proposition 1] Let M^n be a complete Riemannian manifold and $\Sigma^n(z)$ an entire spacelike vertical graph in $-I \times_f M^n$. If*

$$|Dz|_M^2 \leq f(z)^2 - \beta$$

for certain positive constant $\beta > 0$, then $\Sigma^n(z)$ is complete.

In this context, we obtain the following Calabi-Bernstein type result.

Theorem 5.0.1 *Let $-I \times_f M^n$ be a GRW spacetime whose Riemannian fiber M^n is complete noncompact. Let $z \in C^\infty(M)$ be a bounded entire solution of equation (5.0.5) for $c \neq 0$, with finite C^2 norm, such that $|Dz|_M \leq \alpha f(z)$, for some constant $0 < \alpha < 1$. Suppose that $\Sigma^n(z)$ is above the slice $\{t_0\} \times M^n$ and converges asymptotically to it at infinity, for some $t_0 \in I$. If the soliton function $\zeta_c(z)$ is nonpositive, then $z \equiv t_0$.*

Proof. Let $z \in C^\infty(M)$ be such a solution of equation (5.0.5). Since z is bounded, we have that $\Sigma^n(z)$ is contained in a slab of $\overline{M}^{n+1}$. Consequently, since we are also assuming that $|Dz|_M \leq \alpha f(z)$, for some constant $0 < \alpha < 1$, we get that

$$|Dz|_M^2 \leq f(z)^2 - \beta,$$

for $\beta = (1-\alpha^2)\inf_{\Sigma(z)} f(z)^2$. Thus, we can apply Proposition E jointly with [10, Proposition 3.2] to conclude that $\Sigma^n(z)$ is complete noncompact. We finish the proof by applying Theorem 2.2.1. ∎

From Theorem 5.0.1 we obtain the following nonexistence result.

Corollary 5.0.1 *Let $-I \times M^n$ be a Lorentzian product space whose Riemannian fiber M^n is complete noncompact. There does not exist bounded entire solution $z \in C^\infty(M)$ of equation (5.0.6) for $c \neq 0$, with $|Dz|_M \leq \alpha$, for some constant $0 < \alpha < 1$, $\zeta_c(z) \leq 0$ and such that $\Sigma^n(z)$ is above the slice $\{t_0\} \times M^n$ and converges asymptotically to it at infinity, for some $t_0 \in I$.*

Proceeding, we obtain another nonexistence result from Theorem 4.4.1.

Theorem 5.0.2 *Let $-I \times_f M^n$ be a GRW spacetime whose Riemannian fiber M^n is complete noncompact with polynomial volume growth. For any constant $c > 0$ ($c < 0$), there does not exist bounded entire solution $z \in C^\infty(M)$ of equation (5.0.5) with $|Dz|_M \leq \alpha f(z)$, for some constant $0 < \alpha < 1$, and such that $\sup_\Sigma \zeta_c(z) < 0$ ($\inf_\Sigma \zeta_c(z) > 0$).*

Proof. Reasoning as in the proof of [8, Corollary 5.1], it follows from (5.0.1) that $d\Sigma = \sqrt{|G|}dM$, where dM and $d\Sigma^n$ stand for the Riemannian volume elements of $(M^n, \langle\,,\rangle_M)$ and $(\Sigma^n(z), g_z)$, respectively, and $|G| = det(g_{ij})$ with

$$g_{ij} = g_z(E_i, E_j) = f(z)^2\delta_{ij} - E_i(z)E_j(z).$$

Here, $\{E_1, \ldots, E_n\}$ denotes a local orthonormal frame with respect to the metric $\langle\, ,\rangle_M$. So, it is not difficult to verify that

$$|G| = f^{2(n-1)}(z)(f(z)^2 - |Dz|_M^2).$$

Consequently,

$$d\Sigma = f^{n-1}(z)\sqrt{f(z)^2 - |Dz|_M^2}\, dM. \tag{5.0.7}$$

Hence, since we are supposing that $(M^n, \langle\, ,\rangle_M)$ has polynomial volume growth and taking into account that z is bounded, relation (5.0.7) guarantees that $(\Sigma^n(z), g_z)$ also has polynomial volume growth.

On the other hand, the boundedness of z guarantees that we can take a primitive g of f such that $g(z) > 0$. Moreover, since $|Dz|_M \leq \alpha f(z)$, the vector field $X = e^{cg(z)}\nabla g(z)$ is also bounded. Therefore, since $\Sigma^n(z)$ is complete noncompact, we are in a position to apply Theorem 4.4.1 and conclude the proof. ■

Part II

Gap results and existence of CMC free boundary hypersurfaces in rotational domains

Chapter 6

Preliminaries II

Throughout this chapter and the next ones, we will consider $\Omega \subset \mathbb{R}^{n+1}$, with $n \geq 2$, be a rotation domain with smooth boundary $\partial\Omega \subset F^{-1}(1)$ where $F : \mathbb{R}^n \times I \to \mathbb{R}$ is a smooth function for some interval $I \subset \mathbb{R}$. We denote by $\bar{N} := \frac{\nabla F}{|\nabla F|}$ the outward unit normal to $\partial\Omega$. Let $\Sigma^n \hookrightarrow \Omega$ a hypersurface with boundary such that $\partial\Sigma \subset \partial\Omega$. We denote N the outward unit normal to Σ and ν the outward conormal along $\partial\Sigma$ in Σ. Then, $\langle N, \nu \rangle = 0$. In this setting, the shape operator of Σ with respect to N at p is the self-adjoint linear operator $A : T_pM \longrightarrow T_pM$ given by $A(X) = -\nabla_X N$, where ∇ stands for the Levi-Civita connection on M. The eigenvalues of A are the principal curvatures of Σ in M and the mean curvature is given by $H = \frac{\mathrm{tr} A}{n}$. In this scope, we have an important definition, as follows.

Definition 6.0.1 *A hypersurface Σ, as above, is called* free boundary *if Σ meets $\partial\Omega$ orthogonally, there is, $\nu = \bar{N}$ along $\partial\Sigma$ or, equivalently, $\langle N, \bar{N} \rangle = 0$ along $\partial\Sigma$.*

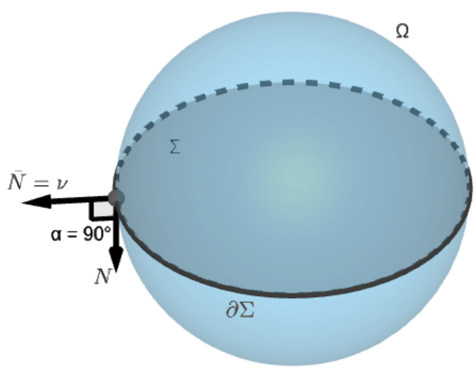

Figure 6.1: A free boundary surface in the ball

More specifically, for $n = 2$, let us consider a rotational hypersurface in the following sense. Let $\alpha(t) = (f(t), t)$ be a plane curve α that is the graph of a positive real valued smooth function $f : I \to \mathbb{R}$ in the x_1x_3-plane. Let Θ be a parametrization of the unit circle $\mathbb{S}^1$ in the plane $x_3 = 0$. The surface of revolution with generatrix α can be parametrized by

$$X(\Theta, t) = (\Theta f(t), t) = (\cos\theta f(t), \sin\theta f(t), t).$$

In this scope, we study free boundary surfaces Σ in domains Ω whose boundary is a hypersurface of revolution given above.

Let us also consider $F : \mathbb{R}^2 \times I \to \mathbb{R}$ be the smooth function defined by

$$F(x, y) = \frac{1}{2}\left(|x|^2 - f(y)^2\right) + 1, \tag{6.0.1}$$

where $x = (x_1, x_2)$ and $y = x_3$, we have that $\Omega \subset F^{-1}(1)$. Notice that 1 is a regular value of F.

Observe that

$$\nabla F(x, y) = (x, -f(y)f'(y)) = (x, y) + (0, -y - f(y)f'(y)),$$

where $y = \langle (x, y), E_3\rangle$. Then,

$$\begin{aligned} D^2F &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -(f'(y))^2 - f(y)f''(y) \end{pmatrix} \\ &= \mathrm{Id}_{3\times 3} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -(f'(y))^2 - f(y)f''(y) - 1 \end{pmatrix}. \end{aligned}$$

Therefore, for all $X, Y \in T(\Sigma)$ we have

$$\begin{aligned} \mathrm{Hess}_\Sigma F(X, Y) &= \langle \bar\nabla_X (\bar\nabla F)^\top, Y\rangle \\ &= \langle \bar\nabla_X (\bar\nabla F - \langle \bar\nabla F, N\rangle N), Y\rangle \\ &= \langle \bar\nabla_X \bar\nabla F, Y\rangle - \langle \bar\nabla F, N\rangle \langle \bar\nabla_X N, Y\rangle \\ &= D^2F(X, Y) + \langle \bar\nabla F, N\rangle \langle A_N X, Y\rangle \\ &= \langle X, Y\rangle + g(x, y)\langle A_N X, Y\rangle - ((f'(y))^2 + f(y)f''(y) + 1)\langle TX, Y\rangle, \end{aligned} \tag{6.0.2}$$

where $T : T_{(x,y)})\Sigma \to T_{(x,y)}\Sigma$ is given by $TX = \langle X, E_3^{\top}\rangle E_3^{\top}$ and

$$g(x,y) := \langle \bar{\nabla} F, N\rangle = \langle (x,y), N\rangle + \langle N, E_3\rangle(-y - f(y)f'(y)).$$

Remark 6.0.1 The free boundary condition give us that $\langle \bar{\nabla} F, N\rangle = 0$ along $\partial\Sigma$. Then, $g(x,y) = 0$ for all $(x,y) \in \partial\Sigma$.

It is easy to check that T is a self-adjoint operator whose $E_3^{\top}$ is an eigenvector associated with the eigenvalue $|E_3^{\top}|^2$. Besides, we can take any nonzero vector in $T\Sigma$, orthogonal to $E_3^{\top}$, to verify that zero is also an eigenvalue of T. Therefore

$$0 \le \langle TX, X\rangle \le |E_3^{\top}|^2 |X|^2, \ \forall X \in T_{(x,y)}\Sigma. \tag{6.0.3}$$

From now one, we use the following condition for the function f of the profile curve

$$(f')^2 + ff'' + 1 \le 0. \tag{6.0.4}$$

Remark 6.0.2 We note that the condition (6.0.4) has an intriguing interpretation in terms of the principal curvatures of the meridian and parallels of the profile curve. Indeed, in dimension 3, the principal curvatures of the profile curve of $\partial\Omega$ are

$$-\frac{f''}{(1+(f')^2)^{\frac{3}{2}}} \text{ and } \frac{1}{f\sqrt{1+(f')^2}},$$

for the meridians and the parallels, respectively. Then, the inequality $(f')^2 + ff'' + 1 \le 0$ means that $\kappa_1 \ge \kappa_2$ in Ω, where κ_1 and κ_2 are the principal curvatures of the meridians and parallels, respectively. Furthermore, this makes clear that $\kappa_1 = \kappa_2$ for all s and therefore, $\partial\Omega$ is a sphere, precisely in the equality case.

Remark 6.0.3 In an alternative way of the last remark, we observe that if $(f')^2 + ff'' + 1 = 0$, we get

$$0 = (f'(t))^2 + f(t)f''(t) + 1 = (t + f(t)f'(t))'.$$

Then,

$$t + f(t)f'(t) = c_1,$$

where c_1 is a constant. Thus,

$$(f(t)^2)' = 2f(t)f'(t) = 2(c_1 - t) = (2c_1 t - t^2)'.$$

Therefore,

$$f(t)^2 = 2c_1 t - t^2 + c_2,$$

where c_2 is a constant. It implies that

$$F(x, y) = \frac{1}{2}(|x|^2 + y^2 - 2c_1 y - c_2) + 1.$$

Then, the set $F^{-1}(1)$ is the sphere

$$x_1^2 + x_2^2 + (y - c_1)^2 = c_2 + c_1^2.$$

Now we present an auxiliary lemma, which gives us important information about the eigenvalues of $\mathrm{Hess}_\Sigma F(x, y)$.

Lemma 6.0.1 *Suppose that $(f')^2 + ff'' + 1 \leq 0$. Then for each $(x, y) \in \Sigma$, the eigenvalues of $Hess_\Sigma F(x, y)$ are greater of equal to*

$$1 + k_1 g(x, y) \text{ and } 1 + k_2 g(x, y),$$

where $k_1 \leq k_2$ are the principal curvatures of Σ with respect to the normal vector N.

Proof. Suppose that $(f')^2 + ff'' + 1 \leq 0$, then using (6.0.2) and (6.0.3), we have that

$$\begin{aligned} \mathrm{Hess}_\Sigma F(X, X) &= \langle X, X\rangle + g(x, y)\langle A_N X, X\rangle - ((f'(y))^2 + f(y)f''(y) + 1)\langle TX, X\rangle \\ &\geq \langle X + g(x, y)AX, X\rangle. \end{aligned}$$

But, the eigenvalues of $X \to X + g(x, y)AX$ are

$$1 + k_1 g(x, y) \text{ and } 1 + k_2 g(x, y),$$

where $k_1 \leq k_2$ are the eigenvalues of A. Then, k_1 and k_2 are the principal curvature of Σ and the eigenvalues $\lambda_1 \leq \lambda_2$ of $\mathrm{Hess}_\Sigma F(x, y)$ satisfy that

$$\lambda_1 \geq 1 + k_1 g(x, y) \text{ and } \lambda_2 \geq 1 + k_2 g(x, y).$$

■

Chapter 7

Gap results

This chapter aims to give a topological classification of CMC free boundary surfaces and minimal free boundary hypersurfaces in the rotational domains, as defined earlier. We employ a gap condition in the umbilicity tensor and the graph function whose rotation generates the boundary domain. We subdivide our analysis into three-dimensional and higher-dimensional cases.

7.1 CMC Free Boundary Surfaces in 3-dimensional rotational domains

In this section, we get a topological characterization for CMC Free Boundary Surfaces in 3-dimensional rotational domains.

The next proposition shows that the gap condition given below implies the convexity of F on Σ, and the proof of the result follows the same steps as in [13, Lemma 2.1].

Proposition F *Let Σ be a compact free boundary CMC surface in Ω. Assume that $(f')^2 + ff'' + 1 \leq 0$ and for all points (x,y) in Σ,*

$$|\phi|^2 g(x,y)^2 \leq \frac{1}{2}(2 + 2Hg(x,y))^2, \tag{7.1.1}$$

where $\phi = A - H\langle \cdot, \cdot \rangle$ is the umbilicity tensor. Then,

$$Hess_\Sigma F(X,X) \geq 0,$$

for all $(x,y) \in \Sigma$ and $X \in T_{(x,y)}\Sigma$.

Proof. By Lemma 6.0.1, we have that the eigenvalues $\lambda_1 \leq \lambda_2$ of $\text{Hess}_\Sigma F(x,y)$ satisfy that

$$\lambda_1 \geq 1 + k_1 g(x,y) := \tilde{\lambda_1} \text{ and } \lambda_2 \geq 1 + k_2 g(x,y) := \tilde{\lambda_2},$$

where k_1 and k_2 are the principal curvatures of Σ. In order to prove $\text{Hess}_\Sigma F(X,X) \geq 0$, we need to show that λ_1 and λ_2 are nonnegative. Using condition (7.1.1) we have

$$\begin{aligned} 4\tilde{\lambda_1}\tilde{\lambda_2} &= 4(1 + k_1 g(x,y))(1 + k_2 g(x,y)) \\ &= 4 + 4k_2 g(x,y) + 4k_1 g(x,y) + 4k_1 k_2 g(x,y)^2 \\ &= 4 + 8Hg(x,y) + 2(4H^2 - |A|^2) g(x,y)^2 \\ &= (2 + 2Hg(x,y))^2 - 2|\phi|^2 g(x,y)^2 \geq 0. \end{aligned} \tag{7.1.2}$$

Then, $\tilde{\lambda_1}$ and $\tilde{\lambda_2}$ have the same sing. Therefore, we need to show that at least one $\tilde{\lambda_i}$ is non-negative. For this, we will to show that function v defined on Σ and given by

$$v := \tilde{\lambda_1} + \tilde{\lambda_2} = 2 + 2Hg(x,y)$$

is nonnegative. Note that we can assume that Σ is not totally umbilical; otherwise, it is obvious to check. Let us suppose that $v(p) < 0$ at some point $p \in \Sigma$. The free boundary condition ensures that $g(x,y) = 0$ for all $(x,y) \in \partial\Sigma$ (see Remark 6.0.1), then

$$v = 2 + 2Hg(x,y) = 2$$

along $\partial\Sigma$. Choose $q \in \partial\Sigma$ and let $\alpha : [0,1] \to \Sigma$ be a continuous curve such that $\alpha(0) = p$ and $\alpha(1) = q$ (see Figure 7.1). Since v changes the signal along α, there is a point $p_0 = \alpha(t_0)$, $t_0 \in (0,1)$ such that $v(p_0) = 0$. In particular $g(x,y)(p_0) \neq 0$. Therefore, the condition (7.1.1) implies that

$$|\phi|^2(p_0) = 0,$$

and hence p_0 is an umbilical point. Since Σ is not a totally umbilical surface, we have that p_0 is an isolated point. Furthermore, there is $\varepsilon > 0$ such that $v(\alpha(t)) < 0$, if $t \in [t_0 - \varepsilon, t_0)$ and $v(\alpha(t)) > 0$, if $t \in (t_0, t_0 + \varepsilon]$, or vice-versa.

Let $D_{r_0}(p_0)$ be a geodesic disk with radius r_0 centered at p_0 such that p_0 is the only umbilical point of Σ on $D_{r_0}(p_0)$. We can choose r_0 and ε in such way that $\alpha(t) \in D_{r_0}(p_0)$ for all $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$. Choose $\bar{r_0} < r_0$ such that $\alpha(t_0 - \varepsilon),\ \alpha(t_0 + \varepsilon) \notin D_{\bar{r_0}}(p_0)$. Let

$\mathcal{A} = D_{r_0}(p_0) \setminus D_{\bar{r}_0}(p_0)$ be the annulus determined by these two discs and denote by β a path in $\mathcal{A}$ joining the points $\alpha(t_0 - \varepsilon)$ and $\alpha(t_0 + \varepsilon)$. Again, v changes the signal along of β, and therefore there is a point $\tilde{q} \in D_{r_0}(p_0)$ such that $v(\tilde{q}) = 0$. But, as above, it implies that $\tilde{q}$ is another umbilical point in $D_{r_0}(p_0)$ which is a contradiction and we conclude that $v \geq 0$ as desired.

Then, $\lambda_i \geq \tilde{\lambda}_i \geq 0$ for all i. Therefore $Hess_\Sigma F(X, X) \geq 0$.

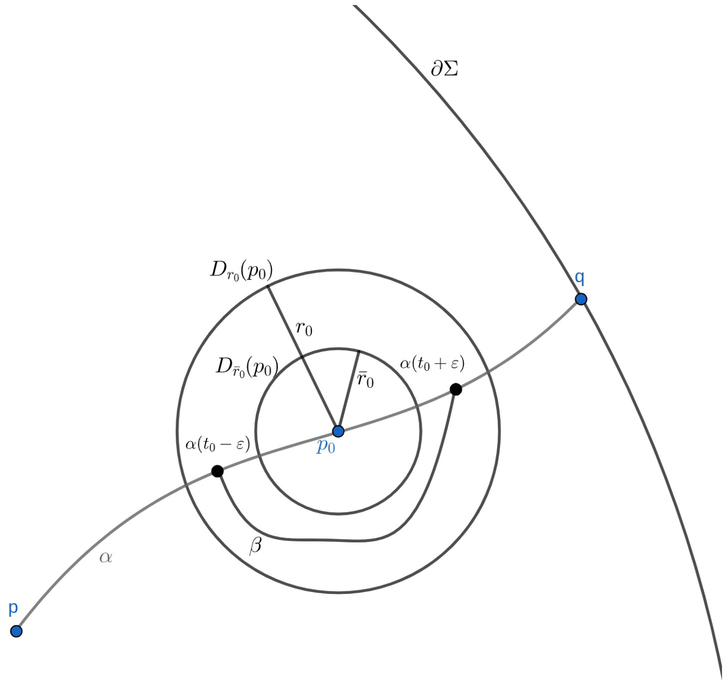

Figure 7.1: Analysis of the sign of v

■

Remark 7.1.1 Observe that, from (7.1.2) if we want to prove that the gap (7.1.1) is valid, it is enough to show that $\tilde{\lambda}_i$ is non-negative for $i = 1, 2$.

Lemma 7.1.1 *Suppose that* $(f')^2 + ff'' + 1 \leq 0$. *Then the Weingarten operator* $A^{\mathbb{R}^3}_{\partial\Omega}$ *of* $F^{-1}(1) = \partial\Omega$ *in* $\mathbb{R}^3$ *with respect to inward unit normal satisfies*

$$\langle A^{\mathbb{R}^3}_{\partial\Omega} X, X\rangle \geq k_1 |X|^2 > 0, \ \forall X \in T\partial\Omega, \ x \neq 0.$$

Proof. We claim that both eigenvalues $k_1 \leq k_2$ of $A^{\mathbb{R}^3}_{\partial\Omega}$ are positive. Let $U \subset \mathbb{R}^2$ be an open set and $x : U \subset \mathbb{R}^2 \to V \subset \partial\Omega$ the immersion

$$x(\theta, t) = (\cos\theta f(t), \sin\theta f(t), t), \ (\theta, t) \in U.$$

A straight forward calculation shows that the inward unit normal is given by

$$\bar{N} = \frac{1}{\sqrt{1+(f')^2}}(-\cos\theta, -\sin\theta, f')$$

and the coefficients of the first and second fundamental forms are given by

$$E = f^2,\ \ F = 0 \text{ and } G = (f')^2 + 1$$

and

$$e = \frac{f}{\sqrt{1+(f')^2}},\ \ f = 0 \text{ and } g = -\frac{f''}{\sqrt{1+(f')^2}}.$$

Therefore, the Gaussian curvature of $\partial\Omega$ at $x(\theta, t)$ is

$$K(\theta, t) = \frac{eg - f^2}{EG - F^2} = -\frac{ff''}{(1+(f')^2)^2 f^2} > 0.$$

Hence, K is strictly positive on $\partial\Omega$. In particular, k_1 and k_2 have the same sign. Furthermore, a simple calculation gives us

$$H = \frac{eG + gE}{2EG} = \frac{1 + (f')^2 - ff''}{2f(1+(f')^2)^{\frac{3}{2}}} > 0.$$

Therefore $k_2 > 0$ and $k_1 > 0$. Thus, for all $X \in T\partial\Omega$ with $X \neq 0$,

$$\langle A^{\mathbb{R}^3}_{\partial\Omega} X, X\rangle \geq k_1|X|^2 > 0.$$

■

Now, we are in conditions to prove the following gap result for CMC surfaces in 3-dimensional rotational domains. The argument follows the ideas of [11].

Theorem 7.1.1 *Let Σ^2 be a compact CMC surface with a free boundary in $F^{-1}(1)$. If $(f')^2 + ff'' + 1 \leq 0$ and*

$$|\phi|^2 g(x,y)^2 \leq \frac{1}{2}(2 + 2Hg(x,y))^2$$

on Σ, then Σ is homeomorphic to a disk or an annulus.

Proof. vFirst, we claim that the geodesic curvature k_g of $\partial\Sigma$ in Σ is positive. In fact, given $X, Y \in T\partial\Sigma$, we have on $\partial\Sigma$ that

$$\nabla^{\mathbb{R}^3}_X Y = \nabla^{\partial\Omega}_X Y + \langle A^{\mathbb{R}^3}_{\partial\Omega} X, Y\rangle \bar{N} = \nabla^{\partial\Sigma}_X Y + \langle A^{\partial\Omega}_{\partial\Sigma} X, Y\rangle N + \langle A^{\mathbb{R}^3}_{\partial\Omega} X, Y\rangle \bar{N}$$

and

$$\nabla_X^{\mathbb{R}^3} Y = \nabla_X^{\Sigma} Y + \langle A_{\Sigma}^{\mathbb{R}^3} X, Y\rangle N = \nabla_X^{\Sigma} Y + \langle A_{\partial\Sigma}^{\Sigma} X, Y\rangle \bar{N} + \langle A_{\Sigma}^{\mathbb{R}^3} X, Y\rangle N.$$

Then, we will have $A_{\partial\Sigma}^{\Sigma} = A_{\partial\Omega}^{\mathbb{R}^3}$ on $\partial\Sigma$, where $\partial\Omega = F^{-1}(1)$. Hence, if $X \in T\partial\Sigma$ is unitary, it follows from Lemma 7.1.1 that

$$k_g = \langle A_{\partial\Sigma}^{\Sigma} X, X\rangle = \langle A_{\partial\Omega}^{\mathbb{R}^3} X, X\rangle > 0. \tag{7.1.3}$$

Now, observe that if either Σ is totally umbilical or Σ has nonnegative Gaussian curvature everywhere, then Σ is homeomorphic to a disk. In fact, if Σ is totally umbilical, we have that the Gaussian curvature K_Σ of Σ satisfies

$$K_\Sigma = H^2 \geq 0.$$

Then, in any case, Σ has nonnegative Gaussian curvature everywhere. From the Gauss-Bonnet theorem and (7.1.3) , it follows

$$\int_\Sigma K_\Sigma + \int_{\partial\Sigma} k_g = 2\pi\mathcal{X}(\Sigma) > 0,$$

which shows that

$$\mathcal{X}(\Sigma) = 2 - 2\hat{g} - r > 0,$$

where $\hat{g}$ and r are respectively the genus and quantity connected components of Σ. Then, $\hat{g} = 0$ and $r = 1$. Therefore, $\mathcal{X}(\Sigma) = 1$, Σ is orientable and has exactly one boundary component. Thus, Σ is homeomorphic to a disk.

Therefore, from now on, let us assume that Σ is not a totally umbilical surface and has negative Gaussian curvature at some point of Σ.

Consider

$$\mathcal{C} = \{p \in \Sigma; F(p) = \min_{x\in\Sigma} F(x)\}.$$

Given $p, q \in \mathcal{C}$, let $\gamma : [0,1] \to \Sigma$ be a geodesic such that $\gamma(0) = p$ and $\gamma(1) = q$. It follows from Proposition F $\text{Hess}_\Sigma F \geq 0$ on Σ. Then,

$$\frac{d^2}{dt^2}(F\circ\gamma) = \text{Hess}_\Sigma F\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right) \geq 0$$

for all $t \in [0,1]$. Since $p, q \in \mathcal{C}$, we have

$$\frac{d}{dt}(F\circ\gamma)(0) = \frac{d}{dt}(F\circ\gamma)(1) = 0,$$

which implies that F is constant on γ by the maximum principle. Then, we conclude that $(F\circ\gamma)(t) \equiv \min_\Sigma F$. Therefore, $\gamma([0,1]) \subset \mathcal{C}$ and $\mathcal{C}$ must be a totally convex subset of Σ. In particular, totally convex property of $\mathcal{C}$ also assures that $\gamma([0,1]) \subset \mathcal{C}$ for all geodesic loop $\gamma : [0,1] \to \Sigma$, based at a point $p \in \mathcal{C}$. Moreover, we assure that each geodesic γ which connect two points in $\mathcal{C}$ is completely inside of Σ, that is, the trace of γ does not have points of $\partial\Sigma$. In fact, if the trace of γ has a point in $\partial\Sigma$, from (7.1.3), we have $k_g(\gamma) > 0$ in a neighborhood of this point which is absurd, because geodesics have zero geodesic curvature. Hence, $\mathcal{C}$ is contained in the interior of Σ.

Finally, we claim that Σ is homeomorphic to either a disk or an annulus. To see this, we divide into two cases:

Case 1: $\mathcal{C}$ consists of a single point.

Case 2: $\mathcal{C}$ contains more than one point.

For Case 1, let $p \in \Sigma\backslash\partial\Sigma$ be the only point of $\mathcal{C}$. Suppose that there is a non-trivial homotopy class $[\alpha] \in \pi_1(\Sigma, p)$, then we can find a geodesic loop $\gamma : [0,1] \to \Sigma$, $\gamma(0) = \gamma(1) = p$ with $\gamma \in [\alpha]$. But, since $\mathcal{C}$ is totally convex, $\gamma([0,1]) \subset \mathcal{C}$ and, in particular, $\mathcal{C}$ has more than one point, which is a contradiction. This implies that $\pi_1(\Sigma, p)$ is trivial. Thus, Σ is simply connected, and we conclude that Σ is homeomorphic to a disk.

For Case 2, we may assume that Σ is not homeomorphic to a disk. Given $p \in \mathcal{C}$ we can find a geodesic loop $\gamma : [0,1] \to \Sigma$, $\gamma(0) = \gamma(1) = p$ belonging to a non-trivial homotopy class $[\alpha] \in \pi_1(\Sigma, p)$. The totally convexity of $\mathcal{C}$ ensures that $\gamma([0,1]) \subset \mathcal{C}$. We claim that γ is a regular curve. Indeed, if $\gamma'(0) \neq \gamma'(1)$, we can choose $\varepsilon_0 > 0$ small and for each $\varepsilon < \varepsilon_0$ consider the minimizing geodesic $\tilde{\gamma}_\varepsilon$ joining $\gamma(1-\varepsilon)$ and $\gamma(0+\varepsilon)$. Since $\mathcal{C}$ is totally convex and $\gamma \subset \mathcal{C}$, we conclude that $\tilde{\gamma}_\varepsilon \subset \mathcal{C}$. Now, we can choose an nonempty open set $U \subset \{\tilde{\gamma}_\varepsilon\}_{\varepsilon<\varepsilon_0}$ of $\mathcal{C}$. Thus, for any geodesic $\beta(t) \in U$,

$$0 = \frac{d^2}{dt^2}(F\circ\beta) = \mathrm{Hess}_\Sigma F\left(\frac{d\beta}{dt}, \frac{d\beta}{dt}\right) \geq 0,$$

Therefore, $\mathrm{Hess}_\Sigma F\left(\frac{d\beta}{dt}, \frac{d\beta}{dt}\right) = 0$ in U. In particular, if β is such that $\beta'(0) = e_i$ by the proof of Lemma 6.0.1 and the proof of Proposition F

$$0 = \mathrm{Hess}_\Sigma F(e_i, e_i) \geq 1 + \langle\bar{\nabla} F, N\rangle k_i = \tilde{\lambda}_i \geq 0.$$

Then,

$$1 + \langle \bar{\nabla} F, N \rangle k_1 = 1 + \langle \bar{\nabla} F, N \rangle k_2 = 0,$$

and we get that $k_1 = k_2$ in U. Thus, the open subset U is totally umbilical, which shows that Σ must be totally umbilical which is a contradiction. Therefore $\mathcal{C}$ has to be equal to the unique closed geodesic γ. Since $[\alpha]$ was chosen to be arbitrary, this implies that $\pi_1(\Sigma, p) \approx \mathbb{Z}$, and hence Σ is homeomorphic to an annulus. ∎

Remark 7.1.2 We note that Theorem 7.1.1 is equivalent to [11] and [13, Theorem 1.3] when Ω is a ball (in this case, we have $g(x,y) = \langle (x,y), N \rangle$). In fact, the rigidity statement in these last ones is expected: the free boundary CMC disks in the ball are totally umbilical (by Nitsche's result [65]). However, not all free boundary CMC annuli in the ball are catenoids or Delaunay surfaces: this is proved in recent papers by Fernandez-Hauwirth-Mira ([40]) and Cerezo-Fernandez-Mira ([26]). In this sense, in the ball case, it shows that there exist free boundary minimal and CMC annuli that do not satisfy the gap inequality.

7.2 Minimal Free Boundary hypersurfaces in $(n+1)$-dimensional rotational domains

In this section, let us consider a rotational hypersurface in the following sense. Let $\alpha(t) = (f(t), t)$ be a plane curve α that is the graph of a positive real valued smooth function $f : I \to \mathbb{R}$ in the $x_1 x_{n+1}$-plane. Let Θ be a parametrization of the $n-$dimensional unit sphere in the hyperplane $x_{n+1} = 0$. The hypersurface of revolution with generatrix α can be parametrized by

$$X(\Theta, t) = (\Theta f(t), t).$$

In this scope, we study minimal free boundary surfaces in domains Ω whose boundary is a hypersurface of revolution given above. Let us denote $x = (x_1, x_2, ..., x_n)$ and $y = x_{n+1}$. Let $F : \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be the smooth function defined by

$$F(x,y) = \frac{1}{2}\left(|x|^2 - f(y)^2\right) + 1,$$

we have that $\partial\Omega \subset F^{-1}(1)$. Observe that, analogous to what was done in the previous section for dimension 3, denoted by Σ a minimal free boundary surface in Ω, we have

$$\nabla F(x,y) = (x, -f(y)f'(y)) = (x,y) + (0, -y - f(y)f'(y)),$$

where $y = \langle (x,y), E_{n+1}\rangle$. Then, for all $X, Y \in T(\Sigma)$ we have

$$\mathrm{Hess}_\Sigma F(X,Y) = \langle X,Y\rangle + g(x,y)\langle A_N X, Y\rangle - ((f'(y))^2 + f(y)f''(y) + 1)\langle TX, Y\rangle,$$

where $T : T_{(x,y)})\Sigma \to T_{(x,y)}\Sigma$ is given by $TX = \langle X, E_{n+1}^\top\rangle E_{n+1}^\top$ and

$$g(x,y) = \langle \bar{\nabla} F, N\rangle = \langle (x,y), N\rangle + \langle N, E_{n+1}\rangle(-y - f(y)f'(y)).$$

We can write

$$\mathrm{Hess}_\Sigma F(X,X) = \langle X,X\rangle + \langle A(X,X), (\nabla F)^\perp\rangle - ((f'(y))^2 + f(y)f''(y) + 1)\langle TX, Y\rangle, \tag{7.2.1}$$

It is easy to check that T is a self-adjoint operator whose $E_{n+1}^\top$ is an eigenvector associated with the eigenvalue $|E_{n+1}^\top|^2$. Besides, we can take any nonzero vector in $T\Sigma$, orthogonal to $E_{n+1}^\top$, to verify that zero is also an eigenvalue of T. Therefore

$$0 \leq \langle TX, X\rangle \leq |E_{n+1}^\top|^2|X|^2, \ \forall X \in T_{(x,y)}\Sigma.$$

Before presenting the results of this section, let us introduce an auxiliary lemma given by Chen in [27].

Lemma 7.2.1 *[27, Lemma 4.1] Let $a_1, ..., a_n$ and b be real numbers. If*

$$\sum_{i=1}^n a_i^2 \leq \frac{(\sum_{i=1}^n a_i)^2}{n-1} - \frac{b}{n-1}, \tag{7.2.2}$$

then $2a_ia_j \geq \frac{b}{n-1}$ for every $i, j \in \{1, ..., n\}$.

Proof. Let $a_1, ..., a_n$ be a sequence of numbers satisfying

$$\sum_{i=1}^n a_i^2 \leq \frac{(\sum_{i=1}^n a_i)^2}{n-1} - \frac{b}{n-1}.$$

It follows that

$$(n-2)a_n^2 - 2(\sum_{i=1}^{n-1} a_i)a_n + n\sum_{i=1}^{n-1} a_i^2 - \left(\sum_{i=1}^{n-1} a_i\right)^2 + b \leq 0.$$

This is a quadratic inequality on a_n. Therefore, its discriminant is non-negative, that is:

$$4\left(\sum_{i=1}^{n-1} a_i\right)^2 - 4(n-2)(n-1)\sum_{i=1}^{n-1} a_i^2 + 4(n-2)\left(\sum_{i=1}^{n-1} a_i\right)^2 - 4(n-2)b \geq 0.$$

Equivalently,

$$\sum_{i=1}^{n-1} a_i^2 \leq \frac{(\sum_{i=1}^{n-1} a_i)^2}{n-2} - \frac{b}{n-1}.$$

This inequality is of the same type as (7.2.2). Continuing the same process $(n-1)$ times, we obtain that $2a_ia_j \geq \frac{b}{n-1}$ for every $i, j \in \{1, ..., n\}$. Then, we conclude the poof of the lemma. ■

The next proposition shows that the gap condition given bellow implies the convexity of F on Σ.

Proposition G *Let Σ^n be a minimal free boundary hypersurface n-dimensional in Ω, with $n \geq 3$. Assume that $(f')^2 + ff'' + 1 \leq 0$. If*

$$|A(x,y)|^2 g(x,y)^2 \leq \frac{n}{n-1}, \tag{7.2.3}$$

for every $(x,y) \in \Sigma^n$. Then,

$$Hess_\Sigma F(X,X) \geq 0,$$

for all $(x,y) \in \Sigma$ and $X \in T_{(x,y)}\Sigma$.

Proof. Suppose that $(f')^2 + ff'' + 1 \leq 0$, then using (7.2.1) we get

$$\mathrm{Hess}_\Sigma F(X,X) \geq \langle X, X\rangle + \langle A(X,X), (\nabla F)^\perp\rangle. \tag{7.2.4}$$

Let $\{e_1, ..., e_n\}$ be an orthonormal basis of eigenvectors of $\mathrm{Hess}_\Sigma F$ at $(x,y) \in \Sigma$ with respective eigenvalues $\lambda_1, ..., \lambda_n$. We want to show that $\lambda_i \geq 0$ for every i. By (7.2.4), $\lambda_i \geq \tilde{\lambda_i} := 1 + \langle A(e_i,e_i), (\nabla F)^\perp\rangle$ and joining with (7.2.3) we get

$$\begin{aligned}\sum_{i=1}^{n} \tilde{\lambda_i}^2 &= n + 2\sum_{i=1}^{n}\langle A(e_i,e_i), (\nabla F)^\perp\rangle + \sum_{i=1}^{n}\langle A(e_i,e_i), (\nabla F)^\perp\rangle^2 \\ &= n + \sum_{i=1}^{n}\langle A(e_i,e_i), (\nabla F)^\perp\rangle^2 \\ &\leq n + |\nabla F^\perp|^2 \sum_{i=1}^{n} |A(e_i,e_i)|^2\end{aligned}$$

$$\leq n + |\nabla F^{\perp}|^2|A|^2$$
$$\leq n + \frac{n}{n-1} = \frac{n^2}{n-1}.$$

On the other hand, we have that $(\sum_{i=1}^{n} \tilde{\lambda}_i)^2 = n^2$ since Σ^n is minimal. Then

$$\sum_{i=1}^{n} \tilde{\lambda}_i^2 \leq \frac{(\sum_{i=1}^{n} \tilde{\lambda}_i)^2}{n-1}.$$

By Lemma 7.2.1, where $\tilde{\lambda}_i = a_i$ and $b = 0$, we get that $2\tilde{\lambda}_i\tilde{\lambda}_j \geq 0$. Consequently, the eigenvalues $\tilde{\lambda}_i$, $i = 1, ..., n$, have all the same sign. Since $\sum_{i=1}^{n} \tilde{\lambda}_i = n$, we conclude that $\tilde{\lambda}_i \geq 0$ for every i. Therefore, $\lambda_i \geq \tilde{\lambda}_i \geq 0$ for every i. Then

$$Hess_{\Sigma}F(X, X) \geq 0,$$

for all $(x, y) \in \Sigma$ and $X \in T_{(x,y)}\Sigma$. ■

Now, we can prove the following result for minimal free boundary hypersurfaces.

Theorem 7.2.1 *Let Σ^n be a n-dimensional free boundary minimal hypersurface in a domain Ω with boundary $\partial\Omega \subset F^{-1}(1)$. Assume that $(f')^2 + ff'' + 1 \leq 0$. If*

$$|A|^2 g(x, y)^2 \leq \frac{n}{n-1},$$

for every $(x, y) \in \Sigma^n$, then one of the following is true:

1. Σ^n is diffeomorphic to a disk $\mathbb{D}^n$.

2. Σ^n is diffeomorphic to $\mathbb{S}^1 \times \mathbb{D}^{n-1}$ and $C(\Sigma^n)$ is a closed geodesic.

Proof. Firstly, let us define $\mathcal{C} = \{p \in \Sigma : F(p) = \min_{\Sigma} F\}$. From Proposition G,

$$Hess_{\Sigma}F(X, X) \geq 0,$$

for all $p \in \Sigma$ and $X \in T_p\Sigma$. The convexity of $Hess_{\Sigma}F$ strongly restricts the set $\mathcal{C}$ and the topology of Σ. As in the proof of Theorem 7.1.1, the convexity of Hess$F$ restricted to Σ implies that the set $\mathcal{C}$ is a totally convex set of Σ.

From now on, the proof follows the same line as in [18, Theorem 3.7] that uses standard Morse's theory. We divide into two cases:

Case 1: $\mathcal{C}$ consists of a single point.

Case 2: $\mathcal{C}$ contains more than one point.

If $\mathcal{C} = \{p_0\}$, for some $p_0 \in \Sigma$, then F has only one critical point $p_0 \in \Sigma$ and by standard Morse theory (see Milnor [62]), we conclude that Σ is diffeomorphic to a disk $\mathbb{D}^n$.

If $\mathcal{C}$ contains more than one point we can show that $\dim(\mathcal{C}) = 1$ and $\mathcal{C}$ is a geodesic. In fact, let us consider p_1 and p_2 two distinct points in $\mathcal{C}$. Since $\mathcal{C}$ is totally convex, we can find a minimizing geodesic joining p_1 and p_2 contained in $\mathcal{C}$. Let γ be the maximal geodesic extending this minimizing geodesic segment and still contained in $\mathcal{C}$. If there exists a point $q \in \mathcal{C} \setminus \gamma$, we conclude that $\mathcal{C}$ contains the cone obtained by the union of all geodesic segments with extremities in q and in γ. Then, we get that $\dim(\mathrm{Ker}(\mathrm{Hess}_\Sigma F) \geq 2$ for every point in this cone. Proceeding again as in the proof of Theorem 7.1.1, we get that

$$1 + \langle A(e_1, e_1), \nabla F^\perp \rangle = 1 + \langle A(e_2, e_2), \nabla F^\perp \rangle = 0.$$

Therefore, by the Cauchy-Schwarz inequality we obtain

$$|A(e_i, e_i)|^2 |\nabla F^\perp|^2 \geq 1.$$

Then, from (7.2.3)

$$\frac{n}{n-1} \geq |\nabla F^\perp|^2 |A|^2 \geq \left(|A(e_1, e_1)|^2 + |A(e_2, e_2)|^2\right) |\nabla F^\perp|^2 \geq 2.$$

As this is a contradiction when $n \geq 3$, we conclude that $\mathcal{C}$ has to be equal to the geodesic γ and $\dim(\mathcal{C}) = 1$. In this case, $\mathcal{C}$ is not a closed geodesic (what would imply that Σ is diffeomorphic to a disk) or is a closed geodesic (what would force Σ to be diffeomorphic to $\mathbb{S}^1 \times \mathbb{D}^{n-1}$, from standard Morse theory).

■

Chapter 8

Examples of CMC free boundary surfaces in the rotational ellipsoid

In this section, we show that there are a catenoid and some portions of Delaunay surfaces that are free boundary surfaces on the rotational ellipsoid

$$a^2x^2 + a^2y^2 + b^2z^2 = R^2, \tag{8.0.1}$$

with $a^2 \leq b^2$ and some constant R^2, and satisfy the pinching condition (7.1.1).

Remark 8.0.1 Let us consider

$$f(y) = \frac{b}{a}\sqrt{\left(\frac{R}{b}\right)^2 - y^2},$$

in (6.0.1), where $a^2 \leq b^2$ and R is a constant. Then, we obtain the rotational ellipsoid given by (8.0.1). In this case, the hypothesis $(f')^2 + ff'' + 1 \leq 0$ is automatically satisfied. In fact, we have

$$f'(y) = -\frac{yb}{a\sqrt{\left(\frac{R}{b}\right)^2 - y^2}}$$

and

$$f''(y) = -\frac{R^2}{ab\left(\left(\frac{R}{b}\right)^2 - b^2\right)^{\frac{3}{2}}}.$$

Therefore,

$$(f')^2 + ff'' + 1 = -\frac{y^2b^2}{a^2\left(\left(\frac{R}{b}\right)^2 - y^2\right)} + \frac{b\sqrt{\left(\frac{R}{b}\right)^2 - y^2}}{a}\left(-\frac{R^2}{ab\left(\left(\frac{R}{b}\right)^2 - b^2\right)^{\frac{3}{2}}}\right) + 1$$

$$= \frac{(a^2-b^2)\left(\left(\frac{R}{b}\right)^2-y^2\right)}{a^2\left(\left(\frac{R}{b}\right)^2-y^2\right)} = \frac{a^2-b^2}{a^2} \leq 0.$$

First, let us consider a smooth curve parametrized by arc length in the xz-plane $\beta(s) = (x(s), 0, z(s))$, with $x(s) > 0$ and denote by Σ the surface obtained by rotation of β around the z-axis.

We start presenting a lemma with sufficient conditions for a general rotational surface to satisfy the pinching condition (7.1.1) in the rotational ellipsoid.

Lemma 8.0.1 *Suppose that the curve β satisfies the following conditions*

$$-1 \leq x''(s)\left(x(s) - \frac{x'(s)}{z'(s)}z(s)\frac{b^2}{a^2}\right), \text{ if } z'(s) \neq 0, \tag{8.0.2}$$

$$-1 \leq z(s)z''(s)\frac{b^2}{a^2}, \text{ if } z'(s) = 0, \tag{8.0.3}$$

$$-x(s)x'(s)^2 \leq z'(s)x'(s)z(s)\frac{b^2}{a^2}, \tag{8.0.4}$$

with $a^2 \leq b^2$. Then, Σ satisfies the pinching condition

$$|\phi|^2 g(x,y)^2 \leq \frac{1}{2}(2 + 2Hg(x,y))^2,$$

on the rotational ellipsoid given in (8.0.1).

Proof. From Remark 7.1.1 , it suffices to show that

$$\tilde{\lambda}_1 = 1 + k_1 g(x,y) \geq 0 \text{ and } \tilde{\lambda}_2 = 1 + k_2 g(x,y) \geq 0.$$

Let us consider $X : [s_1, s_2] \times \mathbb{S}^1 \to \mathbb{R}^3$ given by

$$X(s,\theta) = (x(s)\cos(\theta), x(s)\sin(\theta), z(s)),$$

obtained by rotation of β around the z-axis. Therefore,

$$X_s(s,\theta) = (x'(s)\cos(\theta), x'(s)\sin(\theta), z'(s))$$

and

$$X_\theta(s,\theta) = (-x(s)\sin(\theta), x(s)\cos(\theta), 0).$$

Then, a straight forward computation shows that

$$N = (-z'(s)\cos(\theta), -z'(s)\sin(\theta), x'(s)).$$

Thus,

$$\begin{aligned}\langle (x,y), N\rangle &= -x(s)z'(s)\cos^2(\theta) - x(s)z'(s)\sin^2(\theta) + x'(s)z(s) \\ &= -x(s)z'(s) + x'(s)z(s). \end{aligned} \tag{8.0.5}$$

From (8.0.5) and Remark 8.0.1, we get

$$\begin{aligned} g(x,y) &= \langle \bar{\nabla} F, N\rangle \\ &= \langle (x,y), N\rangle - \langle N, E_3\rangle (y + f(y)f'(y)) \\ &= -x(s)z'(s) + x'(s)z(s) - x'(s)\left(z(s) - \frac{b^2}{a^2} z(s)\right) \\ &= -x(s)z'(s) + x'(s)z(s)\frac{b^2}{a^2}. \end{aligned} \tag{8.0.6}$$

A straight forward computation shows that the coefficients of the first and second fundamental forms are given by

$$E = 1, \text{ and } G = x(s)^2$$

and

$$e = -x''(s)z'(s) + z''(s)x'(s) \text{ and } g = x(s)z'(s).$$

Then, we get that

$$k_1 = \frac{e}{E} = x'(s)z''(s) - x''(s)z'(s) \text{ and } k_2 = \frac{g}{G} = \frac{z'(s)}{x(s)}. \tag{8.0.7}$$

Since β is parametrized by arc length, we have $\langle \beta'(s), \beta''(s)\rangle = 0$. Thus, we get that

$$z'(s)z''(s) = -x'(s)x''(s). \tag{8.0.8}$$

If $z'(s) \neq 0$, using (8.0.8) we can write

$$\begin{aligned} k_1(s) &= (x'(s)z''(s) - x''(s)z'(s))\frac{z'(s)}{z'(s)} \\ &= \frac{x'(s)z''(s)z'(s) - x''(s)z'(s)^2}{z'(s)} \\ &= \frac{-x''(s)x'(s)^2 - x''(s)z'(s)^2}{z'(s)} \end{aligned}$$

$$= -\frac{x''(s)(x'(s)^2 + z'(s)^2)}{z'(s)}$$

$$= -\frac{x''(s)}{z'(s)}. \tag{8.0.9}$$

Then, using (8.0.6) and (8.0.2)

$$\begin{aligned}\tilde{\lambda}_1 &= 1 + k_1 g(x, y) \\ &= 1 - \frac{x''(s)}{z'(s)} \left(-x(s)z'(s) + x'(s)z(s)\frac{b^2}{a^2} \right) \\ &= 1 + x''(s) \left(x(s) - \frac{x'(s)}{z'(s)} z(s) \frac{b^2}{a^2} \right) \geq 0.\end{aligned}$$

If $z'(s) = 0$, since the curve is parameterized by the arc length, then $x'(s)^2 = 1$. Using (8.0.6) and (8.0.3), we get

$$\begin{aligned}\tilde{\lambda}_1 &= 1 + k_1 g(x, y) \\ &= 1 + (x'(s)z''(s) - x''(s)z'(s)) \left(-x(s)z'(s) + x'(s)z(s)\frac{b^2}{a^2} \right) \\ &= 1 + z''(s)z(s)\frac{b^2}{a^2} \geq 0.\end{aligned}$$

Finally, using again that the curve is parameterized by the arc length, together with (8.0.6) and (8.0.4), we obtain

$$\begin{aligned}\tilde{\lambda}_2 &= 1 + k_2 g(x, y) \\ &= 1 + \frac{z'(s)}{x(s)} \left(-x(s)z'(s) + x'(s)z(s)\frac{b^2}{a^2} \right) \\ &= \frac{x(s) - x(s)z'(s)^2 + z'(s)x'(s)z(s)\frac{b^2}{a^2}}{x(s)} \\ &= \frac{x(s)x'(s)^2 + z'(s)x'(s)z(s)\frac{b^2}{a^2}}{x(s)} \geq 0.\end{aligned}$$

Therefore, $\tilde{\lambda}_1(s) \geq 0$ and $\tilde{\lambda}_2(s) \geq 0$ as desired. ■

The function

$$\rho(s) = x(s) - \frac{x'(s)}{z'(s)} z(s) \frac{b^2}{a^2}. \tag{8.0.10}$$

that appears in (8.0.2) has an important geometric meaning. In fact, if $\rho(s_0) = 0$, then we can proof that Σ is orthogonal to the rotational ellipsoid E given by

$$a^2x^2 + a^2y^2 + b^2z^2 = R^2,$$

where $R^2 := a^2x(s_0)^2 + b^2z(s_0)^2$. In particular we have the following lemma.

Lemma 8.0.2 *Assume that $\beta(s)$ is defined for $s \in [c,d]$ and consider $\mathcal{Z} = \{s \in [c,d]; z'(s) = 0\}$. Let us consider a and b positive real numbers, such that $a^2 \le b^2$ and define the function $\rho : [c,d] \setminus \mathcal{Z} \to \mathbb{R}$ by*

$$\rho(s) = x(s) - \frac{x'(s)}{z'(s)} z(s) \frac{b^2}{a^2}.$$

Let $s_1 < s_2$ be two values in $[c,d]$ such that:

(i) $\rho(s_1) = \rho(s_2) = 0$,

(ii) $a^2 x(s_1)^2 + b^2 z(s_1)^2 = a^2 x(s_2)^2 + b^2 z(s_2)^2 := R^2$ and

(iii) $a^2 x(s)^2 + b^2 z(s)^2 < R^2$ for all $s \in (s_1, s_2)$.

Then, the rotation of $\beta_{|[s_1,s_2]}$ produces a free boundary surface Σ inside the rotational ellipsoid E given by

$$a^2x^2 + a^2y^2 + b^2z^2 = R^2. \tag{8.0.11}$$

Proof. The ellipsoid given in (8.0.11) can be parametrized by

$$\bar{X}(s,\theta) = \left(\frac{R}{a}\cos(s)\cos(\theta), \frac{R}{a}\cos(s)\sin(\theta), \frac{R}{b}\sin(s)\right).$$

A straight calculation show that

$$\bar{N} = \frac{(\frac{1}{b}\cos(s)\cos(\theta), \frac{1}{b}\cos(s)\sin(\theta), \frac{1}{a}\sin(s))}{\sqrt{\frac{cos^2(s)}{b^2} + \frac{\sin^2(s)}{a^2}}}.$$

Now, observe that if $\rho(s_1) = \rho(s_2) = 0$, then

$$0 = \rho(s_i) = x(s_i) - \frac{x'(s_i)}{z'(s_i)} z(s_i) \frac{b^2}{a^2}.$$

We have $z(s_i) \neq 0$. In fact, if $z(s_i) = 0$, we conclude that $x(s_i) = 0$, what does not happen. Thus, we can write

$$x'(s_i) = \frac{a^2}{b^2}\frac{x(s_i)}{z(s_i)} z'(s_i),$$

$i = 1, 2$. Therefore,

$$\beta'(s_i) = \left(\frac{a^2}{b^2}\frac{x(s_i)}{z(s_i)} z'(s_i), 0, z'(s_i)\right)$$

$$= \frac{z'(s_i)}{z(s_i)} \frac{a}{b} \left(\frac{a}{b} x(s_i), 0, \frac{b}{a} z(s_i) \right)$$

On the other hand, using (ii) we have that the curve β intersects the ellipsoid at the points $\beta(s_i)$. The normal at these points is given by

$$\bar{N}(\beta(s_i)) = \frac{\left(\frac{a}{b} x(s_i), 0, \frac{b}{a} z(s_i)\right)}{R\sqrt{\frac{cos^2(s)}{b^2} + \frac{\sin^2(s)}{a^2}}}.$$

Then,

$$\beta'(s_i) = \frac{z'(s_i)}{z(s_i)} \frac{a}{b} R \sqrt{\frac{cos^2(s)}{b^2} + \frac{\sin^2(s)}{a^2}} \bar{N}(\beta(s_i)).$$

Thus, the rotation of $\beta_{|[s_1,s_2]}$ is orthogonal to the ellipsoid in (8.0.11). As by hypothesis we have $a^2x(s)^2 + b^2z(s)^2 < R^2$ for all $s \in (s_1, s_2)$, we get that $\Sigma \subset E$. ■

Before presenting examples of CMC free boundary surfaces, let us introduce an example in the case where $H = 0$, that is, a minimal free boundary surface in the rotational ellipsoid.

Example 4 *Consider Σ the catenoid obtained by revolving the curve*

$$\beta(s) = (\cosh(s), 0, s)$$

around the z-axis. Parameterizing by arc length we obtain the curve

$$\bar{\beta}(s) = (\cosh(\sinh^{-1}(s)), 0, \sinh^{-1}(s)).$$

Taking $a^2 = 1$ and $b^2 = 2$ in (8.0.10), we get that $\rho(s) = 0$ if and only if

$$\frac{1}{2\sinh^{-1}(s)} = \tanh(\sinh^{-1}(s)).$$

Solving the equation we get that $s_1 = -0,755...$ and $s_2 = 0,755...$ are such that $\rho(s_i) = 0$ for $i = 1, 2$. The parity of the functions $\cosh(s)$ and $\sinh^{-1}(s)$ ensures that

$$(\cosh(\sinh^{-1}(s_1))^2 + 2(\sinh^{-1}(s_1))^2 = (\cosh(\sinh^{-1}(s_2)))^2 + 2(\sinh^{-1}(s_2))^2,$$

once $s_1 = -s_2$. Then, let us define

$$R^2 := (\cosh(\sinh^{-1}(s_1))^2 + 2(\sinh^{-1}(s_1))^2 = (\cosh(\sinh^{-1}(s_2)))^2 + 2(\sinh^{-1}(s_2))^2.$$

This way, the degrowth and growth of $\cosh(s)$ in $(s_1, 0)$ and $(0, s_2)$, respectively, and the fact that $\sinh^{-1}(s)$ is increasing guarantee that $(\cosh(\sinh^{-1}(s))^2 + 2(\sinh^{-1}(s))^2 < R^2$ for all $s \in (s_1, s_2)$. Then, Σ is a free boundary surface in the ellipsoid E given by

$$x^2 + y^2 + 2z^2 = R^2.$$

Furthermore, with some calculations we get that

$$-1 - x''(s)\left(x(s) - \frac{x'(s)}{z'(s)}z(s)\frac{b^2}{a^2}\right) = -1 - \frac{1}{(1+s^2)^{\frac{3}{2}}}\left(\cosh(\sinh^{-1}(s)) - 2s\sinh^{-1}(s)\right) \le 0$$

and

$$-x(s)x'(s)^2 - z'(s)x'(s)z(s)\frac{b^2}{a^2} = -\frac{s}{1+s^2}\left(s\cosh(\sinh^{-1}(s)) + 2\sinh^{-1}(s)\right) \le 0,$$

for all $s \in [s_1, s_2]$. Then, from Lemma 8.0.1, Σ satisfies the condition

$$|\phi|^2 g(x,y)^2 \le \frac{1}{2}(2 + 2Hg(x,y))^2.$$

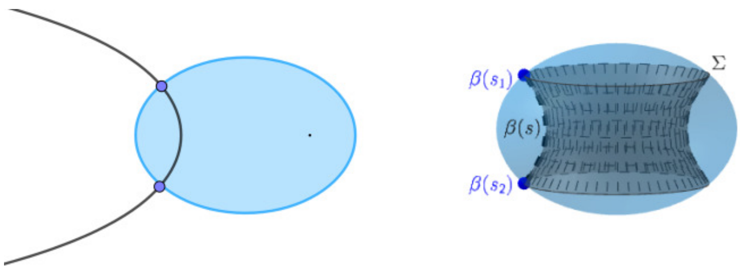

Figure 8.1: Catenoid free boundary in the elipsoid

Now, let us consider a smooth curve parametrized by arc length in the xz-plane $\beta(s) = (x(s), 0, z(s))$, with $x(s) > 0$, where

$$x(s) = \frac{1}{H}\sqrt{1 + B^2 + 2B\sin\left(Hs + \frac{3\pi}{2}\right)} \tag{8.0.12}$$

and

$$z(s) = \int_{\frac{3\pi}{2H}}^{s+\frac{3\pi}{2H}} \frac{1 + B\sin(Ht)}{\sqrt{1 + B^2 + 2B\sin(Ht)}}dt \tag{8.0.13}$$

are given by the solution of Kenmotsu [51, Section 2, Equation (11)], where $B, H \in \mathbb{R}$, with $H > 0$, $B \ge 0$ and $B \ne 1$. Let denote by Σ the surface obtained by rotation of β around the z-axis.

From Delaunay's Theorem, we know that any complete surface of revolution with constant mean curvature is a sphere, a catenoid, or a surface whose generating curve is given by β.

A surface whose generating curve is given by β is called a Delaunay surface, with parameters H and B, which can be of different types. If $B=0$ we get right cylinders. If $0<B<1$, Delaunay surfaces are embedded and they are called unduloids. If $B>1$ they are only immersed and called nodoids.

Observe that the components of the velocity vector of the curve $\beta(s)$ in the xz-plane are given by

$$x'(s)=\frac{B\cos(Hs+\frac{3\pi}{2})}{\sqrt{1+B^2+2B\sin(Hs+\frac{3\pi}{2})}} \text{ and } z'(s)=\frac{1+B\sin(Hs+\frac{3\pi}{2})}{\sqrt{1+B^2+2B\sin(Hs+\frac{3\pi}{2})}}.$$

And the acceleration components are given by

$$x''(s)=\frac{-BH(B+\sin(Hs+\frac{3\pi}{2}))(B\sin(Hs+\frac{3\pi}{2})+1)}{(1+B^2+2B\sin(Hs+\frac{3\pi}{2}))^{\frac{3}{2}}} \tag{8.0.14}$$

and

$$z''(s)=\frac{HB^2\cos(Hs+\frac{3\pi}{2})(B+\sin(Hs+\frac{3\pi}{2}))}{(1+B^2+2B\sin(Hs+\frac{3\pi}{2}))^{\frac{3}{2}}}.$$

Let us assume that $0<B<1$. The key observation in this case is that the function z satisfies $z'(s)>0$ for all s. In fact,

$$\begin{aligned} z'(s) &= \frac{1+B\sin(Hs+\frac{3\pi}{2})}{\sqrt{1+B^2+2B\sin(Hs+\frac{3\pi}{2})}} \\ &= \frac{1+B(\sin(Hs)\cos\frac{3\pi}{2}+\sin\frac{3\pi}{2}\cos(Hs))}{\sqrt{1+B^2+2B\sin(Hs+\frac{3\pi}{2})}} \\ &= \frac{1-B\cos(Hs)}{\sqrt{1+B^2+2B\sin(Hs+\frac{3\pi}{2})}}>0 \end{aligned}$$

since $1-B\cos(Hs)\geq 1-B>0$.

Let s_0 be the smaller positive value such that $x''(s_0)=0$. One can easily check that $s_0=s_0(H,B)=\frac{1}{H}\sin^{-1}(-B)+\frac{\pi}{2H}$, where $\sin^{-1}:[-1,1]\to[-\frac{\pi}{2},\frac{\pi}{2}]$. Indeed, if $s_0=\frac{1}{H}\sin^{-1}(-B)+\frac{\pi}{2H}$, we get

$$\sin\left(Hs_0+\frac{3\pi}{2}\right)=-B \text{ and } \cos\left(Hs_0+\frac{3\pi}{2}\right)=\sqrt{1+B^2}.$$

Then, from (8.0.14) we get

$$x''(s_0)=\frac{-BH(B-B)(-B^2+1)}{(1-B^2)^{\frac{3}{2}}}=0.$$

Furthermore, we have $x''(s) > 0$ for all $s \in (-s_0, s_0)$. In fact, we have $B + \sin\left(Hs + \frac{3\pi}{2}\right) < 0$ for all $s \in (-s_0, s_0)$ and $1 + B\sin\left(Hs + \frac{3\pi}{2}\right) > 0$. Then,

$$x''(s) = \frac{-BH(B + \sin(Hs + \frac{3\pi}{2}))(B\sin(Hs + \frac{3\pi}{2}) + 1)}{(1 + B^2 + 2B\sin(Hs + \frac{3\pi}{2}))^{\frac{3}{2}}} > 0$$

for all $s \in (-s_0, s_0)$.

Thus, given $s \in (-s_0, s_0)$ we have $z'(s) > 0$ and $x''(s) > 0$.

Remark 8.0.2 In this case, we only have $x' = 0$ at point 0, so the tangent is only vertical at this point. Therefore, we only have one wave of the unduloid inside the ellipsoid.

Now, let us see some properties of the function ρ that we will need later.

Lemma 8.0.3 *Fix* $0 < B < 1$, $H > 0$, *and consider the function* $\rho : [-s_0, s_0] \to \mathbb{R}$ *given by (8.0.10). Then,*

i) $\rho(0) > 0$.

ii) $\rho'(0) = 0$ *and* $\rho'(s_0) \leq 0$.

iii) ρ *is increasing in* $(-s_0, 0)$ *and decreasing in* $(0, s_0)$.

Proof. Observe that i) follows directly. In fact, we have

$$\rho(0) = x(0) - \frac{x'(0)}{z'(0)} z(0) \frac{b^2}{a^2} = \frac{1 - B}{H} > 0$$

since $x'(0) = 0$. To proof ii), we observe that, since β is parametrized by arc length, we get

$$\begin{aligned}
\rho'(s) &= x'(s) - \frac{b^2}{a^2} \frac{((x'(s)z(s))'z'(s) - x'(s)z(s)z''(s))}{z'(s)^2} \\
&= x'(s) + \frac{b^2}{a^2} \frac{x'(s)z(s)z''(s) - x''(s)z(s)z'(s) - x'(s)z'(s)^2}{z'(s)^2} \\
&= \left(1 - \frac{b^2}{a^2}\right) x'(s) + \frac{b^2}{a^2} z(s) \left(\frac{x'(s)z''(s) - x''(s)z'(s)}{z'(s)^2}\right) \\
&= \frac{(a^2 - b^2)}{a^2} x'(s) - \frac{b^2}{a^2} z(s) \left(\frac{x'(s)}{z'(s)}\right)'.
\end{aligned}$$

As $x'(0) = 0$ and $z(0) = 0$ it follows that $\rho'(0) = 0$. On the other hand, using the expressions for k_1 given in (8.0.7) and (8.0.9) we get

$$\rho'(s) = \frac{(a^2 - b^2)}{a^2} x'(s) - \left(\frac{-x'(s)z''(s) + x''(s)z'(s)}{z'(s)^2}\right) z(s) \frac{b^2}{a^2}$$

$$= \frac{(a^2-b^2)}{a^2}x'(s) + \frac{k_1(s)}{z'(s)^2}z(s)\frac{b^2}{a^2}$$
$$= \frac{(a^2-b^2)}{a^2}x'(s) - \frac{x''(s)}{z'(s)^3}z(s)\frac{b^2}{a^2}.$$

Then, since $x''(s_0) = 0$ we have that

$$\begin{aligned}\rho'(s_0) &= (a^2-b^2)x'(s_0)\\ &= \frac{(a^2-b^2)}{a^2}\frac{B\cos(Hs_0+\frac{3\pi}{2})}{\sqrt{1+B^2+2\sin(Hs_0+\frac{3\pi}{2})}}\\ &= \frac{(a^2-b^2)}{a^2}\frac{B\sqrt{1-B^2}}{\sqrt{1-B^2}}\\ &= \frac{(a^2-b^2)}{a^2}B \leq 0\end{aligned}$$

because $a^2 \leq b^2$ and $B > 0$. Finally, as $x''(s) > 0$ and $x'(0) = 0$ we get that $x'(s) > 0$ for all $s \in (0, s_0)$ and $x'(s) < 0$ for all $s \in (-s_0, 0)$. In the same way, we have $z(s) > 0$ in $(0, s_0)$ and $z(s) < 0$ in $(-s_0, 0)$, then we obtain

$$\rho'(s) = \frac{(a^2-b^2)}{a^2}x'(s) - \frac{x''(s)}{z'(s)^3}z(s)\frac{b^2}{a^2} > 0$$

in $(-s_0, 0)$, and

$$\rho'(s) = \frac{(a^2-b^2)}{a^2}x'(s) - \frac{x''(s)}{z'(s)^3}z(s)\frac{b^2}{a^2} < 0$$

in $(0, s_0)$. Therefore, ρ is increasing in $(-s_0, 0)$ and decreasing in $(0, s_0)$. ■

The next lemma gives us conditions to have an unduloid that is a free boundary surface on the rotational ellipsoid.

Lemma 8.0.4 *Fix $0 < B < 1$, $H > 0$, and set $z_0 = \frac{1-B^2}{HB}$. If $z(s_0) \geq z_0$, then $\rho(\bar{s}) = 0$ for some $\bar{s} \in (0, s_0]$. In particular, the surface obtained by rotation of $\beta|_{[-\bar{s},\bar{s}]}$ is a free boundary CMC surface inside the rotational ellipsoid E given by*

$$a^2x^2 + a^2y^2 + b^2z^2 = \bar{R}^2,$$

where $\bar{R}^2 := a^2x(\bar{s})^2 + b^2z(\bar{s})^2$.

Proof. If $z(s_0) \geq z_0$, then we get

$$\rho(s_0) = x(s_0) - \frac{x'(s_0)}{z'(s_0)}z(s_0)\frac{b^2}{a^2}$$

$$\le x(s_0) - \frac{x'(s_0)}{z'(s_0)} z_0 \frac{b^2}{a^2}$$
$$= \frac{(a^2-b^2)}{a^2} \frac{\sqrt{1-B^2}}{H} \le 0.$$

By assertion i) of Lemma 8.0.3, $\rho(0) > 0$, and then by continuity there is $\bar{s} \in (0\ s_0]$ such that $\rho(\bar{s}) = 0$. Using the parity of functions $\sin(Ht + \frac{3\pi}{2})$ and $\sin(Ht)$, we get

$$x(-s) = x(s),\ x'(-s) = -x'(s),\ z(-s) = -z(s) \text{ and } z'(-s) = z'(s),$$

and thus,

$$\rho(-\bar{s}) = \rho(\bar{s}) = 0.$$

Moreover, $x'(0) = 0$ and $x''(s) > 0$ imply that $x'(s) > 0$ for all $s \in (0, \bar{s}]$. Therefore, $x'(s) > 0$ and $z'(s) > 0$ in $(0, \bar{s})$, and it ensures $a^2x^2(s) + b^2z^2(s) < \bar{R}^2 := a^2x^2(\bar{s}) + b^2z^2(\bar{s})$ for all $s \in (0, \bar{s}]$. Because the curve β is symmetric with respect to x-axis we get $a^2x^2(s) + b^2z^2(s) \le \bar{R}^2$ for all $s \in [-\bar{s}, \bar{s}]$ and we conclude that the surface is free boundary by Lemma 8.0.2. ■

Example 5 *Fix $B = 0,9$ and $H = 0,1$, so we have $z_0 = \frac{1-B^2}{HB} = 2,111...$ and $s_0 = 10\sin^{-1}(-0,9) + 5\pi \approx 4,51026$. Then, we get*

$$z_0(s_0) = \int_{15\pi}^{4,51026+15\pi} \left(\frac{1 + (0,9)\sin(0,1t)}{\sqrt{1 + (0,9)^2 + (1,8)\sin(0,1t)}} \right) dt \approx 2,71697.$$

Therefore, $z(s_0) \ge z_0$. From Lemma 8.0.4, there is $\bar{s} \in (0, s_0]$ such that the surface obtained by rotation of $\beta|_{[-\bar{s},\bar{s}]}$ is a free boundary CMC surface inside the rotational ellipsoid E given by

$$a^2x^2 + a^2y^2 + b^2z^2 = \bar{R}^2, \qquad (8.0.15)$$

where $\bar{R}^2 := a^2x(\bar{s})^2 + b^2z(\bar{s})^2$.

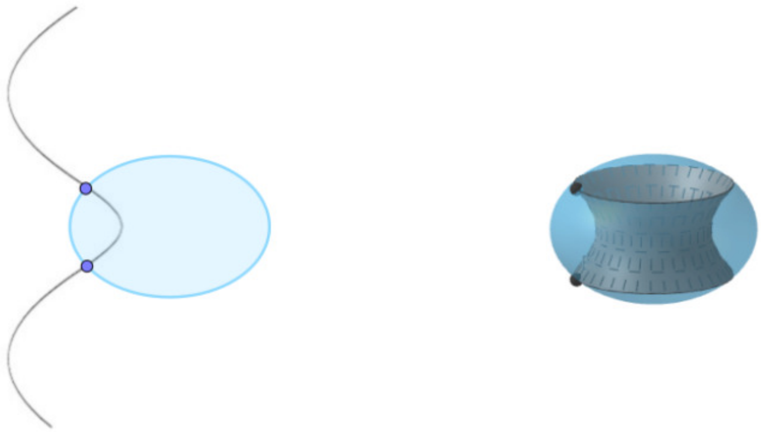

Figure 8.2: Unduloid free boundary in the elipsoid

The next example says essentially that there are portions of unduloids that are free boundary in the ellipsoid given by (8.0.15) and satisfy the conditions of Lemma 8.0.1, there is, satisfy

$$|\phi|^2 g(x,y)^2 \leq \frac{1}{2}(2 + 2Hg(x,y))^2,$$

on Σ.

Example 6 *Fix $0 < B < 1$ and $H > 0$ and consider $\beta(s) = (x(s), 0, z(s))$ as above and set $z_0 = \frac{1-B^2}{HB}$. Let s_0 be the smaller positive value such that $x''(s_0) = 0$, in other words, $s_0 = \frac{1}{H}\sin^{-1}(-B) + \frac{\pi}{2H}$. Suppose $z(s_0) \geq z_0$. From Lemma (8.0.4), the surface Σ obtained by rotation of $\beta|_{[-\bar{s},\bar{s}]}$, for some $\bar{s} \in (0, s_0]$, is a free boundary CMC surface inside the rotational ellipsoid E given by (8.0.11). Moreover, in this case, for all $s \in [-\bar{s}, \bar{s}]$ we have*

(i) $x''(s) \geq 0$. In fact, we have $[-\bar{s}, \bar{s}] \subset [-s_0, s_0]$, where s_0 was chosen to be the largest neighborhood of 0 where $x''(s) \geq 0$.

(ii) $\rho(s) := x(s) - \frac{x'(s)}{z'(s)} z(s) \frac{b^2}{a^2} \geq 0$. Indeed, from Lemma 8.0.4, $\rho(\bar{s}) = 0$. From Lemma 8.0.3, ρ is increasing in $(-s_0, 0)$ and decreasing in $(0, s_0)$. Therefore, $\rho(s) \geq 0$.

(iii) $z(s)x'(s) \geq 0$. In fact, since $z'(s) > 0$ and $x''(s) > 0$ in $(-s_0, s_0)$, we get that z and x' are both growing in $(-s_0, s_0)$. Since $z(0) = x'(0) = 0$, we conclude that x' and z has the same sing.

The items (i), (ii) and (iii) guarantee that the inequalities in Lemma 8.0.1 are satisfied. In fact, from (i) and (ii) we get (8.0.2). Since $z' > 0$*, we do not need to show the validity of (8.0.3). Using that* $x > 0$ *and (iii) we get (8.0.4). Therefore,*

$$|\phi|^2 g(x,y)^2 \leq \frac{1}{2}(2 + 2Hg(x,y))^2,$$

on Σ.

Now, let us assume that $B > 1$. Let r_0 be the smaller positive value such that $z'(r_0) = 0$. We can check that $r_0 = r_0(H,B) = \frac{1}{H}\sin^{-1}\left(-\frac{1}{B}\right) + \frac{\pi}{2H}$, where $\sin^{-1} : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$. Indeed, if $r_0 = \frac{1}{H}\sin^{-1}\left(-\frac{1}{B}\right) + \frac{\pi}{2H}$, we get that $\sin\left(Hr_0 + \frac{3\pi}{2}\right) = -\frac{1}{B}$. Then,

$$z'(r_0) = \frac{1 + B\left(-\frac{1}{B}\right)}{\sqrt{B^2 - 1}} = 0.$$

In this case, we have $z'(r) < 0$ and $x''(r) > 0$ for all $r \in (-r_0, r_0)$. In fact, we have $1 - B\cos(Hr) \leq 1 - B < 0$. Then,

$$\begin{aligned} z'(r) &= \frac{1 + B\sin(Hr + \frac{3\pi}{2})}{\sqrt{1 + B^2 + 2B\sin(Hr + \frac{3\pi}{2})}} \\ &= \frac{1 + B(\sin(Hr)\cos\frac{3\pi}{2} + \sin\frac{3\pi}{2}\cos(Hr))}{\sqrt{1 + B^2 + 2B\sin(Hr + \frac{3\pi}{2})}} \\ &= \frac{1 - B\cos(Hs)}{\sqrt{1 + B^2 + 2B\sin(Hs + \frac{3\pi}{2})}} < 0. \end{aligned}$$

Furthermore, we have $B + \sin\left(B + \sin(Hr + \frac{3\pi}{2}\right) \geq B - 1 > 0$. Therefore,

$$x''(r) = \frac{-BH(B + \sin(Hr + \frac{3\pi}{2}))(B\sin(Hr + \frac{3\pi}{2}) + 1)}{(1 + B^2 + 2B\sin(Hr + \frac{3\pi}{2}))^{\frac{3}{2}}} > 0$$

for all $r \in (-r_0, r_0)$.

Remark 8.0.3 In this case, since $z' \neq 0$ for all $r \in (-r_0, r_0)$, we do not have horizontal tangents. Therefore, the node of the nodoids does not lie inside the ellipsoid.

In the next Lemma we are going to show that there are portions of nodoids that are free boundary in the ellipsoid given by (8.0.15) and satisfy the conditions of Lemma 8.0.1, there is, satisfy

$$|\phi|^2 g(x,y)^2 \leq \frac{1}{2}(2 + 2Hg(x,y))^2,$$

on Σ.

Lemma 8.0.5 *Fix $B > 1$ and $H > 0$ and consider $\beta(r) = (x(r), 0, z(r))$, with x and z given in (8.0.12) and (8.0.13), respectively. Let r_0 as above, then, there is $\bar{r} \in (-r_0, r_0)$ such that $\rho(\bar{r}) = 0$ and the surface obtained by rotation of $\beta|_{[-\bar{r},\bar{r}]}$ is a free boundary CMC surface inside the rotational ellipsoid E given by*

$$a^2x^2 + a^2y^2 + b^2z^2 = \bar{R}^2,$$

where $\bar{R}^2 := a^2x(\bar{s})^2 + b^2z(\bar{s})^2$. Furthermore, we have

$$|\phi|^2 g(x,y)^2 \leq \frac{1}{2}(2 + 2Hg(x,y))^2$$

on Σ.

Proof. In fact, we have that

$$\rho(0) = x(0) - \frac{x'(0)}{z'(0)} z(0) \frac{b^2}{a^2} = \frac{|1-B|}{H} > 0,$$

and $\rho(r) \to -\infty$ when $r \to r_0$. Then, by continuity there is $\bar{r} \in (0, r_0)$ such that $\rho(\bar{r}) = 0$. Using the parity of function ρ we have

$$\rho(\bar{r}) = \rho(-\bar{r}) = 0.$$

Moreover, $x'(0) = 0$ and $x''(r) > 0$ imply that $x'(r) < 0$ for all $r \in (-\bar{r}, 0)$. Therefore, $x'(r) < 0$ and $z'(r) < 0$ in $(-\bar{r}, 0)$, and it ensures $a^2x^2(r) + b^2z^2(r) < \bar{R}^2 := a^2x^2(-\bar{r}) + b^2z^2(-\bar{r})$ for all $r \in (-\bar{r}, 0)$. Because the curve β is symmetric with respect to x-axis we get $a^2x^2(r) + b^2z^2(r) \leq \bar{R}^2$ for all $r \in [-\bar{r}, \bar{r}]$ and we conclude that the surface is free boundary by Lemma 8.0.2. Furthermore, in this case, for all $r \in [-\bar{r}, \bar{r}]$ we have

(i) $\rho(r) \geq 0$. Indeed, as already calculated in Lemma 8.0.3, we have

$$\rho'(r) = \frac{(a^2 - b^2)}{a^2} x'(r) - \frac{x''(r)}{z'(r)^3} z(r) \frac{b^2}{a^2}.$$

Since $x''(r) > 0$ and $x'(0) = 0$ we get that $x'(r) < 0$ for all $r \in (-r_0, 0)$ and $x'(r) > 0$ for all $r \in (0, r_0)$. Similarly, we have $z(r) > 0$ in $(-r_0, 0)$ and $z(r) < 0$ in $(0, r_0)$, then we obtain $\rho'(r) > 0$, $\forall r \in (-\bar{r}, 0)$, and $\rho'(r) < 0$, $\forall r \in (0, \bar{r})$. Therefore, ρ is increasing in $(-r_0, 0)$ and decreasing in $(0, r_0)$. Since $\rho(0) > 0$, we conclude that $\rho(r) \geq 0$, for all $r \in [-\bar{r}, \bar{r}]$.

(ii) $x'(r)z(r) \leq 0$. In fact, since $x''(r) > 0$ and $z'(r) < 0$ in $(-r_0, r_0)$, we get that x' is growing in $(-r_0, r_0)$ and z is descending in $(-r_0, r_0)$. Since $z(0) = x'(0) = 0$, we conclude that x' and z have opposite signs.

The items (i), (ii) and (iii) guarantee that the inequalities in Lemma 8.0.1 are satisfied. In fact, from $x''(r) > 0$ and (i) we get (8.0.2). Since $z' < 0$, we do not need to show the validity of (8.0.3). Using that $x > 0$ and (ii) we get (8.0.4). Therefore,

$$|\phi|^2 g(x,y)^2 \leq \frac{1}{2}(2 + 2Hg(x,y))^2,$$

on Σ.

■

Example 7 *Fix $B = 1,1$ and $H = 0,1$. Then, we have $r_0 \approx 10\sin^{-1}(-0.91) + 5\pi \approx 4,297....$ Therefore, $z'(r_0) = 0$ and from Lemma 8.0.5, there is $\bar{r} \in (0, r_0]$ such that the surface obtained by rotation of $\beta|_{[-\bar{r},\bar{r}]}$ is a free boundary CMC surface inside the rotational ellipsoid E given by*

$$a^2x^2 + a^2y^2 + b^2z^2 = \bar{R}^2,$$

where $\bar{R}^2 := a^2x(\bar{r})^2 + b^2z(\bar{r})^2$.

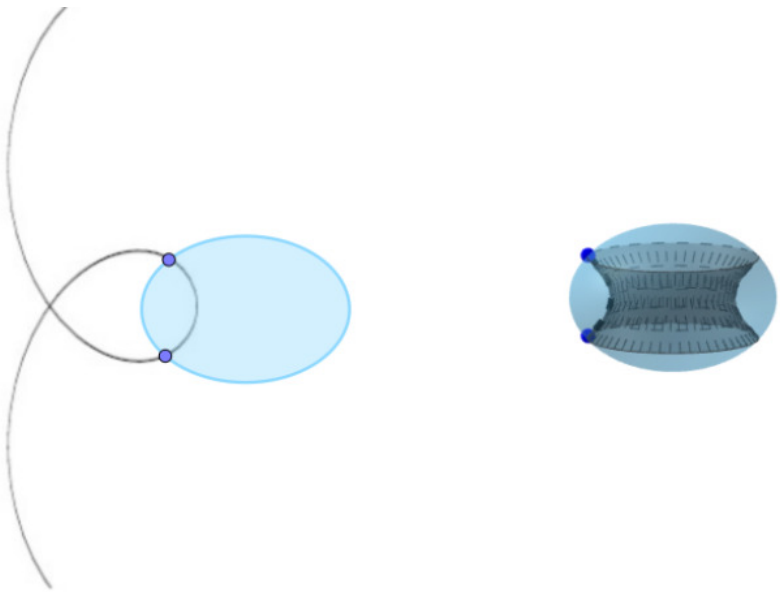

Figure 8.3: Nodoid free boundary in the elipsoid

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