UNIVERSIDADE FEDERAL DA PARAÍBA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA
DOUTORADO EM MATEMÁTICA

STRONGLY INDEFINITE PROBLEMS WITH EXPONENTIAL GROWTH IN THE PLANE

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Fevereiro de 2024

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Tese apresentada ao Corpo Docente do Programa de Pós-Graduação em Matemática da UFPB, como requisito parcial para a obtenção do título de Doutor em Matemática.

Área de concentração: Análise

Orientador: Prof. Dr. Uberlandio Batista Severo
Coorientador: Prof. Dr. Manassés Xavier de Souza

João Pessoa - PB
Fevereiro de 2024

Catalogação na publicação
Seção de Catalogação e Classificação

M543s Menezes, Marta Nascimento.
Strongly indefinite problems with exponential growth in the plane / Marta Nascimento Menezes. - João Pessoa, 2024.
136 f. : il.

Orientação: Uberlandio Batista Severo.
Coorientação: Manassés Xavier de Souza.
Tese (Doutorado) - UFPB/CCEN.

1. Matemática - Sistemas Hamiltonianos. 2. Equações de Schrödinger. 3. Crescimento exponential. 4. Desigualdade de Trudinger-Moser. I. Severo, Uberlandio Batista. II. Souza, Manassés Xavier de. III. Título.

UFPB/BC CDU 51(043)

Elaborado por WALQUELINE DA SILVA ARAUJO - CRB-15/514

Universidade Federal da Paraíba

Programa de Pós-Graduação em Matemática

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Aprovada em: 29 de fevereiro de 2024

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Agradecimentos

Ao Professor Uberlandio Severo pela orientação, paciência e respeito.

Ao Professor Manassés de Souza pela coorientação e contribuições sempre essenciais.

Aos professores membros da banca por terem aceitado participar da defesa e pelas importantes observações para a melhoria desse trabalho.

Aos professores do departamento de matemática da Universidade Federal da Paraíba, que foram fundamentais para minha formação acadêmica.

Aos companheiros de disciplina Aelson, GG, Hector, Joyce, Lorena, Ranieri, Ramon e Renato. Pelas sessões de estudo via meet, pelos bizus compartilhados e pelo suporte durante todo o sufoco que passamos juntos.

À Railane e Pedro por nunca terem me deixado entrar em decomposição dentro do meu cativeiro.

À Daynara e André. Foi um privilégio participar um pouco da história de vocês.

Aos meninos do mestrado (hoje quase todos doutores): Antônio, Diego, JP, Manoel e Yasmin, pelos rolês superfaturados na orla e pelos (longos) cafés na copa.

À dona Cris pelo carinho.

Aos meus amigos de UFMG Genilson e Henrique. Aos físicos Dinorah, Marcos e Marcus.

À Maria Francisca e Jorge, por mais motivos do que eu sou capaz de enumerar.

À CAPES pelo apoio financeiro.

The struggle itself towards the heights is enough to fill a man's heart.

One must imagine Sisyphus happy.

Albert Camus

Abstract

In this work, we study questions related to the existence of ground state and nontrivial solution for some classes of strongly indefinite problems with exponential growth in the plane. Firstly, we study Hamiltonian systems, which have been widely addressed in the last years in the mathematical study of standing wave solutions in nonlinear optics. Secondly, we deal with a class of periodic Schrödinger equations involving exponential critical growth, in which we do not use the classic Ambrosetti-Rabinowitz condition. In order to obtain our results, we use variational methods, namely, a reduction method and linking theorems.

Keywords: Hamiltonian systems, Schrödinger equations, Exponential growth, Trudinger-Moser inequality

Resumo

Neste trabalho, estudamos questões relacionadas à existência de soluções não-triviais e de energia mínima para algumas classes de problemas fortemente indefinidos com crescimento exponencial no plano. Primeiramente, estudamos sistemas Hamiltonianos, os quais tem sido amplamente abordados nos últimos anos no estudo de soluções do tipo ondas estacionárias em óptica não-linear. Em seguida, analisamos uma classe de equações de Schrödinger periódicas envolvendo crescimento crítico exponencial e sem considerar a condição clássica de Ambrosetti-Rabinowitz. A fim de obter nossos resultados, usamos métodos variacionais, mais especificamente, um método de redução e teoremas de linking.

Palavras-chave: Sistemas Hamiltonianos, Equações de Schrödinger, Crescimento Exponential, Desigualdade de Trudinger-Moser

Contents

Introduction

In this work, we study the existence of solutions for two classes of strongly indefinite problems. Firstly, we deal with classes of Hamiltonian systems in the plane, namely

$$\begin{cases} -\Delta u + V(x)u = H_v(x,u,v), & x \in \mathbb{R}^2, \\ -\Delta v + V(x)v = H_u(x,u,v), & x \in \mathbb{R}^2, \end{cases} \tag{1}$$

where $V \in C(\mathbb{R}^2, (0,\infty))$, $H \in C^1(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R})$, H_u, H_v denote the partial derivatives of H with respect to the variables u and v, respectively, $z = (u,v)$ and $H_z = (H_u, H_v)$.

Secondly, we study the existence of solutions for the following Schrödinger equation:

$$\begin{cases} -\Delta u + V(x)u = f(x,u), & x \in \mathbb{R}^2, \\ u \in H^1(\mathbb{R}^2), \end{cases} \tag{2}$$

where $V \in C(\mathbb{R}^2, \mathbb{R})$ and $f \in C(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R})$. Throughout the work, we will have additional hypotheses about the functions V, H and f.

Hamiltonian systems have been widely addressed in the last years, for instance, in the mathematical study of standing wave solutions and in nonlinear optics (see [21]), in models of population dynamics (see [51]) and in the study of Bose–Einstein condensates, see [19], etc.

In view of its applications, many researchers have investigated the existence of solution for systems of type (1). In bounded domains of $\mathbb{R}^N$, with $N \geq 3$, see for example [12, 22, 23, 40] and the papers [10, 15, 33, 65] in the whole space. We can cite the work [14] for a broad survey about Hamiltonian systems.

Since the energy functional associated to (1) is strongly indefinite and there is a lack of compactness in unbounded domains, various techniques and methods are employed to treat with those problems, for example, the Strauss' lemma in the radially symmetric function space [31], the concentration compactness principle of Lions [49], the dual variational method [56], the Orlicz space approach [27] and a reduction method, see [13].

In the particular situation when $H_v(x,u,v) = |v|^{p-1}v$, $(p \geq 1)$ and $H_u(x,u,v) = |u|^{q-1}u$, $(q \geq 1)$, all the results mentioned above impose that the exponents p and q must be below the *critical hyperbola*, that is,

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N}{N+2}.$$

This fact is a consequence from the Pohožaev identity [63] and Sobolev embedding theorem. In dimension two, we can get more information on the growth range of the exponents p and q, once we know that $H^1(\mathbb{R}^2) \hookrightarrow L^s(\mathbb{R}^2)$ for all $s \in [2,\infty)$. Nonetheless, $H^1(\mathbb{R}^2) \not\hookrightarrow L^\infty(\mathbb{R}^2)$. In this situation, the maximal growth which allows us to treat (1) variationally in $H^1(\mathbb{R}^2)$ is motivated by the Trudinger–Moser inequality. When Ω is a bounded domain of $\mathbb{R}^2$, this inequality asserts that $e^{\alpha u^2} \in L^1(\Omega)$, for all $\alpha > 0$ and $u \in H_0^1(\Omega)$. Moreover, it is known that

$$\sup_{u \in H_0^1(\Omega)\,:\,\|\nabla u\|_2 \leq 1} \int_\Omega e^{\alpha u^2} \leq C\mu(\Omega), \qquad \text{for all} \quad \alpha \leq 4\pi,$$

for some constant $C = C(\Omega) > 0$, where $\mu(\Omega)$ denotes the Lebesgue measure of Ω. The above inequality is optimal in the following sense: for any growth $e^{\alpha u^2}$, with $\alpha > 4\pi$, the correspondent supremum is infinite (see [50, 62]). For a version of the Trudinger-Moser inequality in $H^1(\mathbb{R}^2)$, see Lemma 1.11.

Inspired by (), de Figueiredo et al. [26] introduced the notion of subcritical and critical exponential growth. More precisely, a function $f : \mathbb{R} \to \mathbb{R}$ has *subcritical exponential growth* at $+\infty$ if for all $\alpha > 0$ it holds

$$\lim_{s \to +\infty} \frac{f(s)}{e^{\alpha s^2}} = 0$$

and $f(s)$ has *critical exponential growth* at $+\infty$ if there exists $\alpha_0 > 0$ such that

$$\lim_{s \to +\infty} \frac{f(s)}{e^{\alpha s^2}} = \begin{cases} 0, & \text{for all} \quad \alpha > \alpha_0, \\ +\infty, & \text{for all} \quad \alpha < \alpha_0. \end{cases}$$

Currently, many researchers have addressed elliptic systems involving exponential growth. For example, in [29], the authors obtained results on the existence of solution for semilinear equations and systems. Moreover, de Figueiredo et al. [28] have studied the existence of nontrivial weak solution for Hamiltonian systems of the form

$$\begin{cases} -\Delta u = g(v), & \text{in} \quad \Omega, \\ -\Delta v = f(u), & \text{in} \quad \Omega, \\ \quad u = v = 0, & \text{on} \quad \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain and the functions $f, g : \mathbb{R} \to \mathbb{R}$ have subcritical or critical exponential growth.

In [38], the authors established the existence of nontrivial solution for Hamiltonian systems of the form

$$\begin{cases} -\Delta u + V(x)u = g(x,v), & x \in \mathbb{R}^2, \\ -\Delta v + V(x)v = f(x,u), & x \in \mathbb{R}^2, \end{cases} \tag{3}$$

when the potential $V(x)$ is neither bounded away from zero, nor bounded from above. The nonlinear terms $f(x,u)$ and $g(x,v)$ have subcritical or critical exponential growth and it was imposed that the potential $V(x)$ satisfies the assumption

$$\lim_{R\to+\infty} \nu_s(\mathbb{R}^2 \setminus \overline{B}_R) = +\infty, \text{ for some } s \in [2,\infty),$$

where ν_s is defined by

$$\nu_s(\Omega) = \inf_{u\in H_0^1(\Omega)\,:\,\|u\|_s=1} \int_\Omega (|\nabla u|^2 + V(x)u^2), \quad \text{for an open } \ \Omega \subset \mathbb{R}^2 \quad \text{and} \quad \nu_s(\emptyset) = +\infty.$$

Generally, the conditions imposed on the potential $V(x)$ are in order to overcome the loss of compactness of the Sobolev embedding $H^1(\mathbb{R}^2) \hookrightarrow L^s(\mathbb{R}^2)$ for $s \in [2,\infty)$.

In the paper [32], the authors studied the Hamiltonian system

$$\begin{cases} -\Delta u + u = g(v), & x \in \mathbb{R}^2, \\ -\Delta v + v = f(u), & x \in \mathbb{R}^2, \end{cases}$$

where $f, g : \mathbb{R} \to \mathbb{R}$ have critical exponential growth. By using a suitable variational framework based on the generalized Nehari manifold method and the concentration compactness principle of Lions, they established the existence of a ground state solution.

More recently, in the paper [52], the authors consider system (3) with $V(x)$, $f(x,u)$ and $g(x,v)$ periodic in x_1, x_2 and f, g with critical exponential growth. They established similar results to ones obtained in [37] without the Ambrosetti-Rabinowitz condition.

In *Chapter 1*, we study the existence of ground state solutions for the Hamiltonian system (1). We consider the case where V and H are periodic or asymptotically periodic. Our main assumption on V is the following:

(V_0) $V \in C(\mathbb{R}^2, \mathbb{R})$, $V(x) = V(x_1, x_2)$ is positive in $\mathbb{R}^2$ and 1-periodic in the variables x_1, x_2.

With respect to the function H, we assume the following conditions:

(H_0) $H \in C^1(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R})$ and $H(x_1, x_2, z)$ is 1-periodic in the variables x_1, x_2;

(H_1) for any $\alpha > 0$, it holds

$$\lim_{|z|\to\infty} \frac{|H_z(x,z)|}{e^{\alpha|z|^2}} = 0, \quad \text{uniformly in } x \in \mathbb{R}^2;$$

(H_2) $H_z(x,z) = o(|z|)$ as $|z| \to 0$, uniformly in $x \in \mathbb{R}^2$;

(H_3) $H(x,z)/|z|^2 \to +\infty$ as $|z| \to \infty$, uniformly in $x \in \mathbb{R}^2$;

(H_4) there exists $g : \mathbb{R}^2 \times \mathbb{R}_+ \to [0, \infty)$ increasing in the second variable such that

$$H_z(x,z) = g(x,|z|)z, \quad \text{for each} \quad (x,z) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

Our first result is stated as follows.

Theorem 0.1. *Suppose that conditions* (V_0) *and* $(H_0) - (H_4)$ *are satisfied. Then, system* (1) *possesses a ground state solution.*

For the *nonperiodic* case, that is, when the functions V and H are not necessarily periodic in x_1, x_2, the function

$$\widehat{H}(x,z) := \frac{1}{2}H_z(x,z) \cdot z - H(x,z), \quad \text{for} \quad (x,z) \in \mathbb{R}^2 \times \mathbb{R}^2,$$

will play an important role. In order to make precise the meaning of being asymptotically periodic, we are going to introduce the class of functions

$$\mathcal{F} := \{\varphi \in C(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) : \text{ for any } \varepsilon > 0, \quad \mu(\{x \in \mathbb{R}^2 : |\varphi(x)| \geq \varepsilon\}) < \infty\},$$

where $\mu(A)$ denotes the Lebesgue measure of a mensurable subset $A \subset \mathbb{R}^2$. Now, we require the following hypotheses:

(V_1) There exist a constant $a_1 > 0$ and a function $V_\infty \in C(\mathbb{R}^2, \mathbb{R})$, 1-periodic in x_1, x_2, such that $V_\infty - V \in \mathcal{F}$ and

$$V_\infty(x) \geq V(x) \geq a_1, \quad \text{for all } \ x \in \mathbb{R}^2;$$

(H_5) There exists $r_0 > 0$ such that

$$\inf\{\widehat{H}(x, z) : x \in \mathbb{R}^2, |z| \geq r\} =: q(r) > 0, \quad \text{for any } \ r \in (0, r_0);$$

(H_6) $\displaystyle\limsup_{|z|\to\infty} \frac{|z||H_z(x, z)|}{\widehat{H}(x, z)} < +\infty$, uniformly in $x \in \mathbb{R}^2$;

(H_7) There exists a function $h : \mathbb{R}^2 \to \mathbb{R}$ with subcritical exponential growth, $\varphi \in \mathcal{F}$, $R_0 > 0$ and $H_\infty \in C^1(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R})$ such that

(i) H_∞ satisfies $(H_0) - (H_4)$;

(ii) $H(x, z) \geq H_\infty(x, z)$ for each $(x, z) \in \mathbb{R}^2 \times \mathbb{R}^2$;

(iii) $|H_z(x, z) - H_{\infty,z}(x, z)| \leq \varphi(x)h(z)$ for each $x \in \mathbb{R}^2$ and $|z| \geq R_0$.

Condition (H_6) is technical and is important to show that Cerami sequences associated to the energy functional are bounded.

The main theorem in the asymptotically period case is the following:

Theorem 0.2. *Assume conditions (V_1), (H_2) and $(H_5) - (H_7)$. Then, system (1) has a nonzero solution.*

For our approach, it was necessary to prove a vector version (see Lemma 1.12) for the Trudinger-Moser inequality presented in Lemma 1.11. We emphasize that all the works mentioned above, involving Hamiltonian systems in dimension two and exponential growth, only consider nonlinear terms of the form $f(x,u)$ and $g(x,v)$. The hypotheses of exponential growth are required separately on f and g. As far as the authors know, this the first work that treats Hamiltonian systems in the plane, with nonlinear terms having exponential growth and depending on u and v at the same time. Moreover, we also consider the case that V and H are asymptotically periodic, which has not been yet studied in the literature for this type of systems in the plane involving exponential growth, even in the case where the system is of the form (3). Thereby, we complemented all the works involving Hamiltonian systems in dimension two. This chapter was published in [57].

In *Chapter 2*, we continue to study the existence of ground state solutions for System (1) with V satisfying condition (V_0) from *Chapter 1*, but now we assume that the nonlinearity has an exponential growth of the critical type.

With respect to the function H, we assume the following conditions:

(H_0) $H \in C^1(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R})$ and $H(x_1, x_2, z)$ is 1-periodic in the variables x_1, x_2;

(H_1) (Critical exponential growth) there exists $\alpha_0 > 0$ such that

$$\lim_{|z|\to\infty} \frac{|H_z(x,z)|}{e^{\alpha|z|^2}} = \begin{cases} 0, & \text{if } \alpha > \alpha_0, \\ +\infty, & \text{if } \alpha < \alpha_0, \end{cases}$$

uniformly in $x \in \mathbb{R}^2$;

(H_2) $H_z(x,z) = o(|z|)$ as $|z| \to 0$, uniformly in $x \in \mathbb{R}^2$;

(H_3) there exists $g : \mathbb{R}^2 \times \mathbb{R}_+ \to [0,\infty)$ increasing in the second variable such that

$$H_z(x,z) = g(x,|z|)z, \quad \text{for each} \quad (x,z) \in \mathbb{R}^2 \times \mathbb{R}^2;$$

(H_4) there exists $R_0, M_0 > 0$ such that

$$0 < H(x,z) \le M_0|H_z(x,z)|, \quad \text{for all} \quad x \in \mathbb{R}^2 \quad \text{and} \quad |z| \ge R_0;$$

(H_5) $\limsup_{|z|\to\infty} \dfrac{|z||H_z(x,z)|}{\widehat{H}(x,z)} =: \beta < +\infty$, uniformly in $x \in \mathbb{R}^2$, where

$$\widehat{H}(x,z) := \frac{1}{2}H_z(x,z)\cdot z - H(x,z), \quad \text{for} \quad (x,z) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

Denoting $\|V\|_\infty = \max_{x\in\mathbb{R}^2} V(x)$, the main result of this chapter is stated as follows:

Theorem 0.3. *Suppose that conditions* (V_0) *and* $(H_0) - (H_5)$ *are satisfied. In addition, we assume that there exists* $p > 2$ *such that*

$$H(x,z) \geq \lambda_0 |z|^p, \quad \textit{for all} \ \ (x,z) \in \mathbb{R}^2 \times \mathbb{R}^2,$$

where

$$\lambda_0 \geq \frac{8(p-2)^{\frac{p-2}{2}}(\beta\alpha_0)^{\frac{p-2}{2}}(4+\|V\|_\infty)^{\frac{p}{2}}}{p^{\frac{p}{2}}} \quad \textit{if} \ \ \beta > 0 \quad \textit{and} \quad \lambda_0 > 0 \ \ \textit{if} \ \ \beta = 0.$$

Then, system (1) *has a ground state solution.*

This chapter is a continuation of the work in Chapter 1. As said before, the existence of nontrivial solutions in the critical growth range in $\mathbb{R}^2$ has been studied before (see [32, 42]). However, as far as we know, these works usually require the hypothesis of critical exponential growth separately on f and g. The main novelty here is that H depends on u and v simultaneously and does not satisfy the Ambrosetti-Rabinowitz condition. In our arguments, we continue to use the reduction method by Szulkin-Weth [59, 60], which allows us to prove that minimizers of the energy functional, on the generalized Nehari manifold, are critical points of the unconstrained functional. The new hypothesis (H_5) and (2.1) will be crucial to our analysis of the Palais-Smale sequences.

Chapter 3 is devoted to the study of the existence of nontrivial solutions for the following class of Hamiltonian systems:

$$\begin{cases} -\Delta u + V(x)u = Q(x)g(v), & x \in \mathbb{R}^2, \\ -\Delta v + V(x)v = Q(x)f(u), & x \in \mathbb{R}^2, \end{cases} \tag{4}$$

where V and Q decay to zero at infinity as $(1+|x|^\alpha)^{-1}$ with $\alpha \in (0,2)$, and $(1+|x|^\beta)^{-1}$ with $\beta \in [2,+\infty)$, respectively.

In the paper [58], motivated by a version of the Trudinger-Moser inequality in $\mathbb{R}^2$ due to Cao [16] (see Lemma 1.11), the author has considered the existence of solution for singular Hamiltonian systems of the form

$$\begin{cases} -\Delta u + V(x)u = \dfrac{g(v)}{|x|^a}, & x \in \mathbb{R}^2, \\ -\Delta v + V(x)v = \dfrac{f(u)}{|x|^a}, & x \in \mathbb{R}^2, \end{cases}$$

where $a \in [0, 2)$, $V : \mathbb{R}^2 \to \mathbb{R}$ is a positive continuous potential, which is bounded away from zero and can be "large" at infinity. Precisely, it was assumed that $1/V \in L^1(\mathbb{R}^2)$ and the functions $f, g : \mathbb{R} \to \mathbb{R}$ have subcritical or critical exponential growth.

In [3], Albuquerque et al. proved that the system

$$\begin{cases} -\Delta u + V(|x|)u = Q(|x|)g(v), & x \in \mathbb{R}^2, \\ -\Delta v + V(|x|)v = Q(|x|)f(u), & x \in \mathbb{R}^2, \end{cases}$$

has a nontrivial solution, by supposing appropriate conditions on the radial potentials $V(r)$ and $Q(r)$ at the origin and at infinity. Under these conditions, they used a version of the Trudinger-Moser inequality and certain compact embedding in weighted Lebesque spaces.

In [44], the authors were interested in studying the system

$$\begin{cases} -\Delta u + V(x)u = g(v), & x \in \mathbb{R}^2, \\ -\Delta v + V(x)v = f(u), & x \in \mathbb{R}^2, \end{cases}$$

for the critical exponential case, when the potential V is a radially symmetric positive function and can vanish at infinity.

We also refer to [17, 43, 45] for some related papers in the context of Lorentz–Sobolev spaces and [46, 64] where the authors also considered the existence and asymptotic behavior of solutions for Hamiltonian systems and planar elliptic equations.

In this chapter, we assume that for some α and β in the range

$$\alpha \in (0, 2) \quad \text{and} \quad \beta \in [2, \infty) \tag{5}$$

the following decay conditions hold:

(V) $V \in C(\mathbb{R}^2)$, there exist $\alpha, a > 0$ such that

$$\frac{a}{1+|x|^\alpha} \leq V(x),$$

and $V(x) \sim |x|^{-\alpha}$ as $|x| \to \infty$;

(Q) $Q \in C(\mathbb{R}^2)$, there exist $\beta, b > 0$ such that

$$0 < Q(x) \leq \frac{b}{1+|x|^\beta},$$

and $Q(x) \sim |x|^{-\beta}$ as $|x| \to \infty$;

For the functions f and g we assume the following:

(h_0) $f, g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous;

(h_1) $f(s) = o(|s|)$ and $g(s) = o(|s|)$ at the origin;

(h_2) there exists $\theta > 0$ such that

$$\begin{cases} 0 < \theta F(s) := \theta \displaystyle\int_0^s f(t)tdt \leq sf(s) \\ 0 < \theta G(s) := \theta \displaystyle\int_0^s g(t)tdt \leq sg(s) \end{cases} \text{for all } s \in (0,\infty);$$

(h_3) there exists constants $M_0 > 0$ and $s_1 > 0$ such that

$$\begin{cases} 0 < \theta F(s) \leq M_0 f(s) \\ 0 < \theta G(s) \leq M_0 g(s) \end{cases} \text{for all } s \in [s_1,\infty).$$

We denote by $L^p_w(\mathbb{R}^2)$ the weighted L^p-space consisting of all measurable functions $u : \mathbb{R}^2 \to \mathbb{R}$ satisfying $\int_{\mathbb{R}^2} w(x)|u|^p dx < \infty$, and introduce the weighted Sobolev space

$$H^1_V(\mathbb{R}^2) := \left\{ u \in L^2_V(\mathbb{R}^2) : |\nabla u| \in L^2(\mathbb{R}^2) \right\},$$

with norm $\|u\|^2 := \|\nabla u\|_2^2 + \int_{\mathbb{R}^2} V(x)u^2$.

First, we prove a Trudinger-Moser-type inequality with an alternative proof to the ones presented in [4, 35, 37].

Theorem 0.4. *Suppose that* (V) *and* (Q) *hold with* α *and* β *satisfying* (5). *For any* $\gamma > 0$ *and* $u \in H^1_V(\mathbb{R}^2)$, *we have*

$$Q(\cdot)(e^{\gamma u^2} - 1) \in L^1(\mathbb{R}^2).$$

Moreover, for any $0 < \gamma < 4\pi$,

$$\sup_{u \in H^1_V(\mathbb{R}^2), \|u\| \leq 1} \int_{\mathbb{R}^2} Q(x)(e^{\gamma u^2} - 1)dx < \infty.$$

Moreover, we can prove that the embedding $H^1_V(\mathbb{R}^2) \hookrightarrow L^p_Q(\mathbb{R}^2)$, with α and β in the range (5), is compact for any $p \in [2, \infty)$. Equipped with this and inspired by [28, 38], our first main result is the subcritical case:

Theorem 0.5 (Subcritical case)**.** *Suppose* $f(s)$ *has subcritical or critical exponential growth,* $g(s)$ *has subcritical exponential growth,* (V), (Q) *and* $(h_0) - (h_3)$ *are satisfied. Then* (4) *possesses a nontrivial weak solution* $(u, v) \in H^1_V(\mathbb{R}^2) \times H^1_V(\mathbb{R}^2)$.

For the next result, we assume that there exists $\gamma_0 > 0$ such that the functions f, g satisfy

$$\liminf_{|s| \to \infty} \frac{sf(s)}{e^{\gamma_0 s^2}}, \quad \liminf_{|s| \to \infty} \frac{sg(s)}{e^{\gamma_0 s^2}} =: \beta_0 > \mathcal{M}, \tag{6}$$

where $\mathcal{M} := \inf_{r>0} \dfrac{4e^{1/2r^2 V_{\max,r}}}{\gamma_0 r^2 Q_{\min,r}}$, $V_{\max,r} := \max_{|x| \leq r} V(x) > 0$ and $Q_{\min,r} := \min_{|x| \leq r} Q(x) > 0$.

Theorem 0.6 (Critical case)**.** *Suppose* $f(s)$ *and* $g(s)$ *has critical exponential growth,* (V), (Q), $(h_0) - (h_3)$ *and* (6) *are satisfied. Then* (4) *possesses a nontrivial weak solution* $(u, v) \in H^1_V(\mathbb{R}^2) \times H^1_V(\mathbb{R}^2)$.

The vanishing behavior of the potential V makes exponential integrability impossible unless we introduce some suitable weight in the target space, that is, for a certain weight Q we need to prove that $H^1_V(\mathbb{R}^2) \hookrightarrow L^p_Q(\mathbb{R}^2)$ for at least some $p \geq 1$. Exploiting the behavior of the functions V and Q, we have showed a Trudinger-Moser-type inequality with an distinct proof to the ones presented in [35, 37], in which we do not use Besicovitch's Covering Lemma. We were inspired by the result presented in [4], where Albuquerque et al. have shown such Trudinger-Moser-type inequality for $\gamma > 0$ in the range $(0, \gamma_*)$ where $\gamma_* \in (0, 4\pi)$. Another problem when dealing with this system is that, since the associated energy functional is strongly indefinite and defined in an infinite-dimensional space, no suitable linking theorem is available. Inspired by the works [28, 38, 58], we use a Galerkin method, that is, we approximate problem (4) with a sequence of finite dimensional problems. Such approximation-type argument was first used by Rabinowitz [54]. Moreover, we believe that the originality of this chapter comes from the the fact that potentials V and Q are not radially symmetric and in the use of hypothesis (6) to estimate the minimax level.

In *Chapter 4*, we study the existence of ground state solutions for the following Schrödinger equation:

$$\begin{cases} -\Delta u + V(x)u = f(x,u), & x \in \mathbb{R}^2, \\ u \in H^1(\mathbb{R}^2), \end{cases} \tag{2}$$

where V is a 1-periodic function with respect to x, 0 lies in the gap of the spectrum of $-\Delta + V$ and the nonlinear term $f(x,s)$ has critical exponential growth.

When $f(x,u) = f(u)$, do Ó-Ruf [36] show the existence of a nontrivial solution, assuming the Ambrosetti-Rabinowitz condition, that is, there exists $\mu > 0$ such that

$$0 \leq \mu F(s) := \mu \int_0^s f(t)dt \leq sf(s), \quad s \in \mathbb{R}.$$

Their approach is to prove that for each $k \in \mathbb{N}$ sufficiently large, there is a nontrivial solution u_k which is k-periodic in x_1 and x_2. The existence of u_k follows from a version of an generalized mountain-pass theorem without the Palais-Smale condition. Then, they prove that, up to a subsequence, the limit of u_k as $k \to \infty$ converges to a solution u.

Also using an approximation argument and the Ambrosetti-Rabinowitz condition, Chen-Tang [20] are able to find a nontrivial solution for (2).

In [60], the authors study the equation $\Delta u + V(x)u = f(x,u)$ in $\mathbb{R}^N$, where f is a superlinear, subcritical nonlinearity, and V and f are periodic in x. They obtained a ground state solution using a method that consists of the reduction of the indefinite variational problem to a definite one.

Using a similar approach, Alves et al. [7] study the problem

$$\begin{cases} -\Delta u + (V(x) - W(x))u = f(x,u), & x \in \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

where $N \geq 2$, $V, W : \mathbb{R}^N \to \mathbb{R}$ are continuous functions verifying some technical conditions and f has critical exponential growth. They proved that the problem has a ground state solution, if the condition

$$F(x,t) := \int_0^t f(x,s)ds \geq \lambda |t|^{q_0}, \quad q_0 > 2,$$

has $\lambda > 0$ sufficiently large.

In this chapter, our main assumption on V is the following:

(V) $V(x) = V(x_1, x_2)$ is continuous, 1-periodic in the variables x_1, x_2 and $0 \notin \sigma(-\Delta + V)$, the spectrum of $\sigma(-\Delta + V)$.

With respect to the function f, we assume the following conditions:

(F_0) $f \in C^1(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R})$ and $f(x_1, x_2, u)$ is 1-periodic in the variables x_1, x_2;

(F_1) (Critical exponential growth) there exists $\alpha_0 > 0$ such that

$$\lim_{|u| \to \infty} \frac{|f(x,u)|}{e^{\alpha |u|^2}} = \begin{cases} 0, & \text{if } \alpha > \alpha_0, \\ +\infty, & \text{if } \alpha < \alpha_0, \end{cases}$$

uniformly in $x \in \mathbb{R}^2$;

(F_2) $f(x,u) = o(u)$ as $|u| \to 0$, uniformly in $x \in \mathbb{R}^2$;

(F_3) $f(x,t)/|t|$ is stricly increasing in $(-\infty, 0)$ and $(0, \infty)$ for every $x \in \mathbb{R}^2$;

(F_4) there exist $R_0, M_0 > 0$ such that

$$0 < F(x,u) \leq M_0 |f(x,u)|, \quad \text{for all} \ \ x \in \mathbb{R}^2 \quad \text{and} \quad |u| \geq R_0;$$

(F_5) $\limsup_{|u| \to \infty} \dfrac{|u||f(x,u)|}{\widehat{F}(x,u)} =: \beta \leq 2$, uniformly in $x \in \mathbb{R}^2$, where

$$\widehat{F}(x,u) := \frac{1}{2} f(x,u)u - F(x,u), \quad \text{for} \quad (x,u) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

For the main result of this chapter, we need the following hypothesis, presented by Chen-Tang in [20]. We assume that there exists $\gamma_0 > 0$ such that the function f satisfy

$$\liminf_{|s|\to\infty} \frac{sf(x,s)}{e^{\gamma_0 s^2}} =: \beta_0 > \mathcal{M}, \text{ uniformly on } x \in \mathbb{R}^2, \tag{7}$$

where $\mathcal{M} =: \dfrac{4}{\gamma_0 \rho^2} e^{16\pi \mathcal{C}_0^2}$ and $\rho > 0$ satisfies $4\pi(4+\rho)\rho\mathcal{C}_0^2 < 1$ and $\mathcal{C}_0 > 0$ is an embedding constant in (4.46).

Theorem 0.7. *Suppose that conditions (V) and $(F_0) - (F_5)$ are satisfied. In addition, we assume that there exists $\gamma_0 > 0$ such that the function f satisfy (7). Then, equation (2) has a ground state solution.*

When (V) holds, the associated functional on $H^1(\mathbb{R}^2)$ for problem (2) is strongly indefinite near the origin. Moreover, by the fact that f has a critical exponential growth, we have a lack of compactness of the embedding $H^1(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2)$ and it becomes difficult to apply the standard methods to prove that $\mathcal{J}(u) = \int_{\mathbb{R}^2} F(x,u)$ is weakly sequentially continuous on $H^1(\mathbb{R}^2)$, and thus use a linking theorem. We try to avoid this problem by applying the reduction method by Szulkin-Weth [59]. The main contribuition in this chapter is that the authors are not aware of any work that proves the existence of ground state solutions for this class of problems and do not use the classic Ambrosseti-Rabinowitz condition.

In order not to resort to the *Introduction* and to make the chapters independent, we will state again, in each chapter, the main results, as well as the hypotheses about the potentials and nonlinearities.

Chapter 1

Hamiltonian systems involving subcritical exponential growth in $\mathbb{R}^2$

The main objective of this chapter is to study ground state solution for the class of Hamiltonian systems

$$\begin{cases} -\Delta u + V(x)u = H_v(x,u,v), & x \in \mathbb{R}^2, \\ -\Delta v + V(x)v = H_u(x,u,v), & x \in \mathbb{R}^2, \end{cases} \tag{1}$$

where $V \in C(\mathbb{R}^2, (0,\infty))$, $H \in C^1(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R})$, H_u, H_v denote the partial derivatives of H with respect to the variables u and v, respectively, $z = (u,v)$ and $H_z = (H_u, H_v)$.

Here, $|y|$ will denote the Euclidian norm of $y \in \mathbb{R}^2$ or the absolute value of $y \in \mathbb{R}$. The dot $\cdot$ will denote the canonical inner product in $\mathbb{R}^2$. In our first result, we consider the *periodic* problem. In this case, our main assumption on V is the following:

(V_0) $V \in C(\mathbb{R}^2, \mathbb{R})$, $V(x) = V(x_1, x_2)$ is positive in $\mathbb{R}^2$ and 1-periodic in the variables x_1, x_2.

With respect to the function H, we assume the following conditions:

(H_0) $H \in C^1(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R})$ and $H(x_1, x_2, z)$ is 1-periodic in the variables x_1, x_2;

(H_1) for any $\alpha > 0$, it holds

$$\lim_{|z|\to\infty} \frac{|H_z(x,z)|}{e^{\alpha|z|^2}} = 0, \quad \text{uniformly in } x \in \mathbb{R}^2;$$

(H_2) $H_z(x,z) = o(|z|)$ as $|z| \to 0$, uniformly in $x \in \mathbb{R}^2$;

(H_3) $H(x,z)/|z|^2 \to +\infty$ as $|z| \to \infty$, uniformly in $x \in \mathbb{R}^2$;

(H_4) there exists $g : \mathbb{R}^2 \times \mathbb{R}_+ \to [0,\infty)$ increasing in the second variable such that

$$H_z(x,z) = g(x,|z|)z, \quad \text{for each} \ \ (x,z) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

The condition (H_1) is motivated by a version of the Trudinger-Moser type inequality, see Lemma 1.12. Note that if $H(x,z)$ behaves as $e^{|z|^\sigma}$, $\sigma \in (0,2)$, for $|z|$ large, then conditions (H_1) and (H_3) are satisfied. The superlinear condition (H_3) is more general than the well-known Ambrosetti-Rabinowitz condition introduced in the paper [2]. Hypothesis (H_4) is a kind of monotonicity condition, which is generally used in the Nehari approach.

The weak solutions of system (1) will be seen as critical points of the energy functional

$$I(z) = \int_{\mathbb{R}^2} (\nabla u \cdot \nabla v + V(x)uv)dx - \int_{\mathbb{R}^2} H(x,z)dx$$

defined on a convenient Banach space E, which will be introduced later. The function $w_0 \in E$ is a ground state solution of (1) whenever

$$I(w_0) = \inf\{I(w) : w \in E\backslash\{0\} \text{ is a weak solution of (1)}\}.$$

Hence, our first result is stated as follows.

Theorem 1.1. *Suppose that conditions (V_0) and $(H_0)-(H_4)$ are satisfied. Then, system (1) possesses a ground state solution.*

Since the functional I is strongly indefinite, for the proof of this theorem, we adapt some ideas contained in [33] and we have applied the treatment developed in Szulkin and Weth [59], which is based on a reduction method. For this, we had to get a version of the Trundinger-Moser inequality for the working space E defined in Section 2.

For the *nonperiodic* case, that is, when the functions V and H are not necessarily periodic in x_1, x_2, the function

$$\widehat{H}(x,z) := \frac{1}{2}H_z(x,z)\cdot z - H(x,z), \quad \text{for} \ \ (x,z) \in \mathbb{R}^2 \times \mathbb{R}^2,$$

will play an important role. In order to make precise the meaning of being asymptotically periodic, we are going to introduce the class of functions

$$\mathcal{F} := \{\varphi \in C(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) : \text{ for any } \varepsilon > 0, \quad \mu(\{x \in \mathbb{R}^2 : |\varphi(x)| \geq \varepsilon\}) < \infty\},$$

where $\mu(A)$ denotes the Lebesgue measure of a mensurable subset $A \subset \mathbb{R}^2$. Now, we require the following hypotheses:

(V_1) There exist a constant $a_1 > 0$ and a function $V_\infty \in C(\mathbb{R}^2, \mathbb{R})$, 1-periodic in x_1, x_2, such that $V_\infty - V \in \mathcal{F}$ and

$$V_\infty(x) \geq V(x) \geq a_1, \quad \text{for all } \ x \in \mathbb{R}^2;$$

(H_5) There exists $r_0 > 0$ such that

$$\inf\{\widehat{H}(x, z) : x \in \mathbb{R}^2, |z| \geq r\} =: q(r) > 0, \quad \text{for any } \ r \in (0, r_0);$$

(H_6) $\limsup_{|z| \to \infty} \dfrac{|z||H_z(x, z)|}{\widehat{H}(x, z)} < +\infty$, uniformly in $x \in \mathbb{R}^2$;

(H_7) There exists a function $h : \mathbb{R}^2 \to \mathbb{R}$ with subcritical exponential growth, $\varphi \in \mathcal{F}$, $R_0 > 0$ and $H_\infty \in C^1(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R})$ such that

(i) H_∞ satisfies $(H_0) - (H_4)$;

(ii) $H(x, z) \geq H_\infty(x, z)$ for each $(x, z) \in \mathbb{R}^2 \times \mathbb{R}^2$;

(iii) $|H_z(x, z) - H_{\infty,z}(x, z)| \leq \varphi(x)h(z)$ for each $x \in \mathbb{R}^2$ and $|z| \geq R_0$.

Condition (H_6) is technical and is important to show that Cerami sequences associated to the energy functional are bounded.

The main theorem in the asymptotically period case is the following:

Theorem 1.2. *Assume conditions (V_1), (H_2) and $(H_5) - (H_7)$. Then, system (1) has a nonzero solution.*

For the proof of this result, in order to obtain a Cerami sequence for the associated functional, we invoke a linking theorem due to Li and Szulkin [47]. We highlighted that as in [52] the Ambrosetti-Rabinowitz condition is not used in our arguments and this makes the task of proving the boundedness of the Cerami sequence more delicate and the demonstration requires a careful analysis in the face of this new scenario with nonlinearities $H_u(x,u,v)$ and $H_u(x,u,v)$ having exponential growth.

After proving that the Cerami sequence is bounded, we deduce that its weak limit is a solution of (1) and the main difficulty is to conclude that this weak limit is nontrivial. For this, by using similar arguments as in [33], we exploit a local version of the linking theorem (see [33, Theorem 2.3]). It is clear that, in our case, the situation is more delicate due to the exponential growth of the nonlinearity and some new difficulties in our analysis must be overcome, for example, the verification of that the Fréchet derivative of the functional $\mathcal{J}(z) = \int_{\mathbb{R}^2} H(x,z)$ is weakly sequentially continuous.

Remark 1.3. *The existence of nontrivial solution for system (1) when $|H_z(x,z)|$ has an critical exponential growth, that is, behaves like $e^{\alpha_0|z|^2}$ at infinite, for some $\alpha_0 > 0$, is an open problem and very interesting, mainly in the asymptotically periodic case. We believe that the main difficult is in the control of the minimax levels of the functional in order to restore the compactness, and to show that the derivative of $\mathcal{J}(z) := \int_{\mathbb{R}^2} H(x,z)$ is weakly sequentially continuous.*

Example 1.4. *Let $a \in C(\mathbb{R}^2,\mathbb{R})$ positive and 1-periodic in x_1, x_2 and define the function H_∞ by*

$$H_\infty(x,z) := a(x)|z|^3(e^{|z|}-1), \quad \text{for } (x,z) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

It is not difficult to see that H_∞ satisfies $(H_0)-(H_4)$. Moreover, considering

$$H(x,z) = a(x)(e^{-|x|^2}+1)|z|^3(e^{|z|}-1), \quad \text{for } (x,z) \in \mathbb{R}^2 \times \mathbb{R}^2$$

we can see H satisfies $(H_5)-(H_7)$, with $\varphi(x) = a(x)e^{-|x|^2} \in \mathcal{F}$ and $h(z) = |z|^2[3(e^{|z|}-1)+|z|e^{|z|}]$.

Throughout this chapter $o_n(1)$ denotes a sequence that converges to 0 as $n \to \infty$. The norm in $L^p(\mathbb{R}^2)$ $(1 \le p < \infty)$ and $L^\infty(\mathbb{R}^2)$ will be denoted respectively by $\|\cdot\|_p$ and $\|\cdot\|_\infty$. We shall use $C, C_0, C_1, C_2, \ldots$ to denote positive constants possibly different.

This chapter is organized as follows: in the forthcoming section, we establish some notations and definitions, and we present the abstract theorems that are used to prove our main results. In Section 1.2, we treat the periodic case, including the proof of a vector version of the Trudinger-Moser inequality. Section 1.3 is devoted to the asymptotically periodic case, where we prove Theorem 1.2.

1.1 Preliminaries

In order to make the text more explanatory, in this section we introduce the abstract theorems that will be applied to prove Theorems 1.1 and 1.2. From now on we will use the following notations and terminologies: E is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and its correspondent norm is $\| \cdot \|$. We are going to suppose that there exists an *orthogonal* decomposition $E = E^+ \oplus E^-$ so that each $z \in E$ is uniquely decomposed, that is, $z = z^+ + z^-$ with $z^\pm \in E^\pm$. For $r > 0$, we consider the following sets:

- $N_r = \{z \in E^+ : \|z\| = r\}$ and $S^+ = \{z \in E^+ : \|z\| = 1\}$;
- for $z \in E$, $E(z) = \mathbb{R}z \oplus E^- \equiv \mathbb{R}z^+ \oplus E^-$ and $\widehat{E}(z) = (\mathbb{R}_+)z \oplus E^- \equiv (\mathbb{R}_+)z^+ \oplus E^-$.

Assume that $I \in C^1(E, \mathbb{R})$ is a functional fulfilling the conditions

(N_1) I has the following form

$$I(z) = \frac{1}{2}\|z^+\|^2 - \frac{1}{2}\|z^-\|^2 - \mathcal{J}(z), \tag{1.1}$$

with $\mathcal{J} \in C^1(E, \mathbb{R})$ weakly lower semicontinuous, $\mathcal{J}(0) = 0$ and, for each $z \neq 0$, there holds

$$\mathcal{J}'(z)z > 2\mathcal{J}(z) > 0;$$

(N_2) for each $z \in E \backslash E^- = \{z \in E : z^+ \neq 0\}$, $I|_{\widehat{E}(z)}$ has a unique nonzero critical point $\widehat{m}(z)$, which is a global maximum point of $I|_{\widehat{E}(z)}$.

Now, we define the *generalized Nehari manifold* associated to I by

$$\mathcal{N} = \{z \in E \backslash E^- : I'(z)z = 0 \text{ and } I'(z)w = 0 \text{ for all } w \in E^-\}.$$

Remark 1.5. *Note that $\mathcal{N}$ contains all nonzero critical points of I, because if $z \neq 0$ is a critical point of I, by (N_1) we have*

$$I(z) = I(z) - \frac{1}{2}I'(z)z = \frac{1}{2}\mathcal{J}'(z)z - \mathcal{J}(z) > 0$$

and therefore we must have $z^+ \neq 0$, that is, $z \in E\backslash E^-$.

In view of this remark, using condition (N_2) and the definition of $\mathcal{N}$, we can define the map $\widehat{m} : E\backslash E^- \to \mathcal{N}$ by

$$\widehat{m}(z) = \{\text{the unique global maximum point of } I|_{\widehat{E}(z)}\}.$$

With these concepts, we require a third hypothesis (N_3) on I, namely,

(N_3) there exists $\delta > 0$ such that $\|\widehat{m}(z)^+\| \geq \delta$, for each $z \in E\backslash E^-$. Moreover, if $\mathcal{W} \subset E\backslash E^-$ is compact, then there exists $c_{\mathcal{W}}$ such that $\|\widehat{m}(z)\| \leq c_{\mathcal{W}}$, for all $z \in \mathcal{W}$.

Moreover, we are going to denote by m the restriction of the map $\widehat{m}$ to S^+, that is, $m = \widehat{m}|_{S^+}$.

The following result presents the main properties related to I, $\widehat{m}$ and m (see proof in [59]).

Lemma 1.6. *If $I \in C^1(E, \mathbb{R})$ satisfies $(N_1) - (N_3)$, then*

(i) *$\widehat{m}$ is continuous and $m : S^+ \to \mathcal{N}$ is a homeomorphism;*

(ii) *the application $\widehat{\Psi} : E^+\backslash\{0\} \to \mathbb{R}$ defined by $\widehat{\Psi}(z) = I(\widehat{m}(z))$ is of class C^1. Moreover, $\Psi := \widehat{\Psi}|_{S^+}$ is also of class C^1 and it holds the equality*

$$\Psi'(z)w = \|m(z)^+\|I'(m(z))w, \text{ for all } w \in T_z(S^+) = \{v \in E^+ : \langle z, v\rangle = 0\};$$

(iii) $\inf_{S^+} \Psi = \inf_{\mathcal{N}} I$;

(iv) *if $(z_n) \subset S^+$ is a Palais-Smale sequence[1] ((PS) sequence for short) for Ψ , then $(m(z_n)) \subset \mathcal{N}$ is a (PS) sequence for I. If $(w_n) \subset \mathcal{N}$ is a bounded (PS) sequence for I, then $(m^{-1}(w_n)) \subset S^+$ is a (PS) sequence for Ψ.*

[1] $(z_n) \subset S^+$ is Palais-Smale sequence for Ψ, if $\Psi(z_n)$ is bounded and $\Psi'(z_n) \to 0$.

In order to better understand the linking theorem used in the asymptotically periodic case, it is introduced a new topology in the space E. For this, let $(e_k)_k \subset E^-$ be a total orthonormal sequence and define the following norm on E:

$$\|z\|_\tau = \max\left\{\|z^+\|, \sum_{k=1}^{\infty} \frac{1}{2}|\langle z^-, e_k\rangle|\right\}, \ z \in E.$$

We call τ-topology the topology induced by this norm.

Remark 1.7. *In bounded sets, the τ-topology coincides with the usual topology of E that is weak on E^- and strong on E^+. Thus, for a bounded sequence $(z_n) \subset (E, \tau)$, we have $z_n \xrightarrow{\tau} z$ in E if, and only if $z_n^+ \to z^+$ and $z_n^- \rightharpoonup z^-$ weakly in E. For details, see for instance [41].*

Definition 1.8. *Given a set $M \subset E$, a homotopy $h : [0,1] \times M \to E$ is said to be admissible if*

(i) h is τ-continuous, that is, if $t_n \to t$ and $z_n \xrightarrow{\tau} z$ then $h(t_n, z_n) \xrightarrow{\tau} h(t, z)$;

(ii) for each $(t, z) \in [0,1] \times M$, there exists a neighborhood U of (t, z) in the product topology $[0,1] \times (E, \tau)$ such that the set $\{w - h(t, w) : (t, w) \in U \cap ([0,1] \times M)\}$ is contained in a finite dimensional subspace of E.

Now, we define the following class of admissible applications:

$$\Gamma := \{h \in C([0,1] \times M, E) : h \text{ is admissible}, \ h(0, \cdot) = Id_M,$$
$$I(h(t, z)) \leq \max\{I(z), -1\} \text{ for all, } (t, z) \in [0,1] \times M\}.$$

The next result was proved in [47, Theorem 2.1]:

Theorem 1.9 (Linking Theorem). *Assume that $I \in C^1(E, \mathbb{R})$ fulfills the hypotheses*

(L_1) *the functional I can be given as in (1.1) with $\mathcal{J}$ being bounded from below, weakly sequentially lower semicontinuous and $\mathcal{J}'$ weakly sequentially continuous;*

(L_2) *there exist $z_0 \in E^+ \backslash \{0\}$, $\rho > 0$ and $R > r > 0$ such that*

$$\inf_{z \in N_r} I(z) \geq \rho, \quad \sup_{z \in \partial M} I(z) \leq 0,$$

where $M = M(z_0, R)$ *stands for* $M := \{z = tz_0 + z^- : z^- \in E^-, \|z\| \le R, t \ge 0\}$ *and* ∂M *denotes the boundary of* $\mathbb{R}z_0 \oplus E^-$. *If we define*

$$c_* := \inf_{h\in\Gamma} \sup_{z\in M} I(h(1, z)), \tag{1.2}$$

then there exists $(z_n) \subset E$ *such that*

$$I(z_n) \to c_* \ge \rho \quad \textit{and} \quad (1 + \|z_n\|)\|I'(z_n)\| \to 0,$$

that is, $(z_n) \subset E$ *is a Cerami sequence for* I *at level* c_* *(*$(Ce)_{c_*}$ *sequence for short).*

We can not directly invoke Theorem 1.9 to prove Theorem 1.2. For this, we shall use the following local version, as proved in [33, Theorem 2.3]:

Theorem 1.10. *Assuming the same hypotheses of Theorem 1.9 and additionally that there exists* $h_0 \in \Gamma$ *such that*

$$c = \sup I(h_0(1, M)),$$

then the functional I *has a nonzero critical point* $z \in h_0(1, M)$ *satisfying* $I(z) = c_*$.

1.2 Proof of Theorem 1.1

In this section, via a minimization argument and Lemma 1.6 we shall obtain a ground state for problem (1). According to condition (V_0), there exist constants $a_0, b_0 > 0$ such that

$$a_0 \le V(x) \le b_0, \quad \text{for all } x \in \mathbb{R}^2. \tag{1.3}$$

Hereafter, we are going to consider $H^1(\mathbb{R}^2)$ endowed with the norm

$$\|u\|_V := \left(\int_{\mathbb{R}^2} [|\nabla u|^2 + V(x)u^2] \right)^{1/2},$$

which is equivalent to its usual norm in view of (1.3). Now, we introduce the Hilbert space $E = H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ endowed with the inner product

$$\langle (u, v), (\varphi, \psi) \rangle = \int_{\mathbb{R}^2} (\nabla u \cdot \nabla \varphi + V(x)u\varphi) + \int_{\mathbb{R}^2} (\nabla v \cdot \nabla \psi + V(x)v\psi),$$

whose correspondent norm is given by

$$\|(u,v)\| = (\|u\|_V^2 + \|v\|_V^2)^{1/2}.$$

Next, we establish a Trudinger–Moser type inequality on the space $E = H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$, where E is endowed with the above norm. For this, we will use the following version of the Trundinger-Moser inequality in $H^1(\mathbb{R}^2)$ (see [16, 34]):

Lemma 1.11. *If $\alpha > 0$ and $u \in H^1(\mathbb{R}^2)$, then*

$$\int_{\mathbb{R}^2} (e^{\alpha u^2} - 1) < \infty.$$

Moreover, if $0 < \alpha < 4\pi$ and $M > 0$, then there exists a constant $C = C(\alpha, M) > 0$ such that

$$\int_{\mathbb{R}^2} (e^{\alpha u^2} - 1) \leq C(\alpha, M),$$

for all $u \in H^1(\mathbb{R}^2)$ with $\|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1$ and $\|u\|_{L^2(\mathbb{R}^2)} \leq M$.

Hence, we can prove the following result:

Lemma 1.12 (Trudinger-Moser inequality)**.** *If $\alpha > 0$ and $(u, v) \in E$, then*

$$\int_{\mathbb{R}^2} (e^{\alpha|(u,v)|^2} - 1) < \infty.$$

Moreover, if $0 < \alpha < 4\pi$ then

$$\sup_{(u,v)\in E,\, \|(u,v)\|\leq 1} \int_{\mathbb{R}^2} (e^{\alpha|(u,v)|^2} - 1) = C(\alpha, a_0) < \infty. \tag{1.4}$$

Proof. Let $\alpha > 0$ and $(u, v) \in E$. If $p, q > 1$ and $1/p + 1/q = 1$ then Young's inequality provides

$$xy - 1 \leq \frac{1}{p}(x^p - 1) + \frac{1}{q}(y^q - 1), \quad \text{for all } \; x, y \geq 0. \tag{1.5}$$

Thus, for $p = q = 2$ and by applying Lemma 1.11 one has

$$\int_{\mathbb{R}^2} (e^{\alpha|(u,v)|^2} - 1) \leq \frac{1}{2} \int_{\mathbb{R}^2} (e^{2\alpha u^2} - 1) + \frac{1}{2} \int_{\mathbb{R}^2} (e^{2\alpha v^2} - 1) < \infty$$

and the first part is proved. To prove (1.4), it is sufficient to show that

$$\sup_{(u,v)\in E,\, \|(u,v)\|=1} \int_{\mathbb{R}^2} (e^{\alpha|(u,v)|^2} - 1) =: C < \infty, \tag{1.6}$$

because if $0 < \alpha < 4\pi$ and $0 < \|(u,v)\| < 1$ then

$$\int_{\mathbb{R}^2} (e^{\alpha|(u,v)|^2} - 1) = \int_{\mathbb{R}^2} \left(e^{\alpha\|(u,v)\|^2 \left|\frac{(u,v)}{\|(u,v)\|}\right|^2} - 1 \right) \leq \int_{\mathbb{R}^2} \left(e^{\alpha\left|\frac{(u,v)}{\|(u,v)\|}\right|^2} - 1 \right) \leq C,$$

since $\|(u,v)/\|(u,v)\|\| = 1$. Thus, let $0 < \alpha < 4\pi$ and $(u,v) \in E$ with $\|(u,v)\|^2 = \|u\|_V^2 + \|v\|_V^2 = 1$. Note that if $\|u\|_V = 0$ then $\|v\|_V^2 = 1$ and therefore

$$\|\nabla v\|_{L^2(\mathbb{R}^2)} \leq 1 \quad \text{and} \quad \int_{\mathbb{R}^2} V(x)v^2 \leq 1$$

and the second inequality implies that $\|v\|_{L^2(\mathbb{R}^2)}^2 \leq 1/a_0$. Hence, by invoking Lemma 1.11, we obtain (1.6). The same conclusion holds if $\|v\|_V = 0$. Now, suppose that $\|u\|_V \neq 0$ and $\|v\|_V \neq 0$. By considering $p = 1/\|u\|_V^2 > 1$ and $q = 1/\|v\|_V^2 > 1$, we have $1/p + 1/q = 1$ and by virtue of (1.5) and Lemma 1.11 we conclude

$$\begin{aligned}
\int_{\mathbb{R}^2} (e^{\alpha|(u,v)|^2} - 1) &= \int_{\mathbb{R}^2} (e^{\alpha u^2} e^{\alpha v^2} - 1) \\
&\leq \frac{1}{p} \int_{\mathbb{R}^2} [e^{\alpha\left(\frac{u}{\|u\|_V}\right)^2} - 1] + \frac{1}{q} \int_{\mathbb{R}^2} [e^{\alpha\left(\frac{v}{\|v\|_V}\right)^2} - 1] \\
&\leq \frac{1}{p} C(\alpha, a_0) + \frac{1}{q} C(\alpha, a_0) = C(\alpha, a_0)
\end{aligned}$$

and the lemma is proved. ■

In order to exploit Lemma 1.6, we consider the following subspaces of E:

$$E^+ := \{(u,u) : u \in H^1(\mathbb{R}^2)\} \quad \text{and} \quad E^- := \{(u,-u) : u \in H^1(\mathbb{R}^2)\}.$$

Note that E^+ and E^- are orthogonal in E and for $z = (u,v) \in E$, if

$$z^+ := \left(\frac{u+v}{2}, \frac{u+v}{2} \right) \quad \text{and} \quad z^- := \left(\frac{u-v}{2}, \frac{v-u}{2} \right),$$

then $z^\pm \in E^\pm$ and we have $z = z^+ + z^-$. Thus, $E = E^+ \oplus E^-$ and a simple computation shows that

$$\int_{\mathbb{R}^2} (\nabla u \cdot \nabla v + V(x)uv) = \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2). \tag{1.7}$$

By using (H_1) and (H_2) we can see that, given $\varepsilon > 0$, $\alpha > 0$ and $q \geq 0$, there exists $C = C(\varepsilon, \alpha, q) > 0$ such that

$$\max\{|H(x,z)|, |H_z(x,z) \cdot z|\} \leq \varepsilon|z|^2 + C|z|^q(e^{\alpha|z|^2} - 1), \quad \text{for all } (x,z) \in \mathbb{R}^2 \times \mathbb{R}^2. \tag{1.8}$$

In view of (1.8) and Lemma 1.12, the energy functional associated to (1), given by

$$I(z) = \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2) - \mathcal{J}(z) \tag{1.9}$$

where $\mathcal{J}(z) := \int_{\mathbb{R}^2} H(x,z)$, is well defined. Furthermore, $I \in C^1(E, \mathbb{R})$ with

$$I'(z)w = \langle z^+, w^+ \rangle - \langle z^-, w^- \rangle - \int_{\mathbb{R}^2} H_z(x,z) \cdot w, \quad \text{for } z, w \in E.$$

Hence, the critical points of I correspond to weak solutions of problem (1).

In order to apply Lemma 1.6, we are going to show that conditions $(N_1) - (N_3)$ are satisfied by I. For this, initially we present four lemmas, whose proofs are similar to ones in [33]. We include them for completeness.

Lemma 1.13. *Suppose that H satisfies (H_2) and (H_4). Then, for each $z \neq 0$ we have*

$$\frac{1}{2}H_z(x,z) \cdot z > H(x,z) > 0.$$

Moreover, $\mathcal{J}(0) = 0$ and $\mathcal{J}$ is weakly lower semicontinuous.

Proof. By (H_2) we get $H(x,0) = 0$ and therefore $\mathcal{J}(0) = 0$. Given $z \neq 0$, it follows from (H_4) that

$$H(x,z) = \int_0^t \frac{d}{dt}[H(x,tz)]dt \; = \int_0^1 H_z(x,tz) \cdot zdt \; = |z|^2 \int_0^1 g(x,t|z|)tdt > 0. \tag{1.10}$$

This identity, (H_4) and the monotonicity of $g(x,\cdot)$ imply that

$$\frac{1}{2}H_z(x,z) \cdot z - H(x,z) = |z|^2 \left(\int_0^1 [g(x,|z|) - g(x,t|z|)]tdt \right) > 0$$

Let's verify that $\mathcal{J}$ is weakly lower semicontinous. Let $(z_n) \subset E$ such that $z_n \rightharpoonup z$ in

E. Up to a subsequence, $z_n(x) \to z(x)$ a.e. in $\mathbb{R}^2$. Since H is nonnegative, Fatou's lemma provides

$$\liminf \mathcal{J}(z_n) = \liminf \int H(x, z_n) \geq \int H(x, z) = \mathcal{J}(z).$$

■

The next lemmas are necessary to show condition (N_2).

Lemma 1.14. *Assume $(H_2) - (H_4)$. Let $s \geq -1$ and $v, z \in \mathbb{R}^2$ with $w = sz + v \neq 0$. Then, for all $x \in \mathbb{R}^2$, we have*

$$H_z(x, z) \cdot \left[s\left(\frac{s}{2} + 1\right) z + (s+1)v \right] + H(x, z) - H(x, z+w) < 0.$$

Proof. Let $y = y(s) := w + z = (1+s)z + v$ and define, for $s \geq -1$,

$$\beta(s) = H_z(x, z) \cdot \left(s\left(\frac{s}{2} + 1\right) z + (s+1)v \right) + H(x, z) - H(x, z+w).$$

If $z = 0$, it follows from (H_2) and lemma 1.13 that $\beta(s) = -H(x, y) < 0$. Hence, we can suppose $z \neq 0$ and consider the following distinct cases:

Case 1: $z \cdot y \leq 0$.

Notice that, by (H_4), $H_z(x, z)y = g(x, |z|)z \cdot y \leq 0$. Thus, recaling that $v = y - (1+s)z$, using Lemma 1.13 and $s \geq -1$, we obtain

$$\begin{aligned} \beta(s) &= -\left(\frac{s^2}{2} + s + 1\right) H_z(x, z) \cdot z + (s+1)H_z(x, z) \cdot y + H(x, z) - H(x, y) \qquad (1.11) \\ &< -\frac{1}{2}(s+1)^2 H_z(x, z) \cdot z + (s+1)H_z(x, y) \cdot y - H(x, y) < 0. \end{aligned}$$

Case 2: $z \cdot y > 0$.

By Lemma 1.13,

$$\beta(-1) = -\frac{1}{2} H_z(x, z) \cdot z + H(x, z) - H(x, y) < -H(x, y) < 0.$$

It follows from (H_4) that $H_z(x, z) \cdot z = g(x, |z|)|z|^2 > 0$. Hence, using (1.11), we have $\lim_{s \to \infty} \beta(s) = -\infty$. Therefore, β attains its maximum at some point $s_0 \in [-1, \infty)$. If

$s_0 = -1$ the result follows from the above inequality. If $s_0 > -1$,

$$0 = \beta'(s_0) = H_z(x, z) \cdot y - H_z(x, z) \cdot z.$$

By (H_4), $g(x, |z|)z \cdot y = g(x, |z|)z \cdot z$, hence $|z| = |y|$. It follows from (1.10) that $H(x, z) = H(x, y)$. Moreover,

$$H_z(x, z) \cdot y = g(x, |z|)z \cdot y \leq g(x, |z|)|z|^2 = H_z(x, z) \cdot z,$$

hence

$$\begin{aligned}\beta(s) &= -\frac{s^2}{2} H_z(x, z) \cdot z + (s + 1)(H_z(x, z) \cdot y - H_z(x, z) \cdot z) \\ &\leq -\frac{s^2}{2} H_z(x, z) \cdot z < 0.\end{aligned}$$

■

Lemma 1.15. *Assume conditions $(H_1) - (H_2)$. There exists $R > 0$ such that $I(w) \leq 0$ for $w \in \widehat{E}(z) \backslash \mathcal{B}_R(0)$, where $\mathcal{B}_R(0) := \{z \in E : \|z\| < R\}$.*

Proof. We argue by contradiction. Suppose there exists a sequence $(w_n) \subset \widehat{E}(z)$ such that $\|w_n\| \to \infty$ and $I(w_n) > 0$. Setting $z_n := \dfrac{w_n}{\|w_n\|}$, we can assume that $z_n \rightharpoonup z_0$ in E. If $z_0 \neq 0$, by Fatou's Lemma and (H_3), we have

$$\begin{aligned}0 \leq \frac{I(w_n)}{\|w_n\|^2} &= \frac{1}{2}\|z_n^+\|^2 - \frac{1}{2}\|z^-\| - \int_{\mathbb{R}^2} \frac{H(x, w_n)}{|w_n|^2}|z_n|^2 \\ &\leq \frac{1}{2}\|z_n^+\|^2 - \frac{1}{2}\|z^-\|^2 - \int_{B_R} \frac{H(x, w_n)}{|w_n|^2}|z_n|^2 \to \infty,\end{aligned}$$

which is an absurdo, thus $z_0 = 0$.

Since $H \geq 0$, by the above estimate, we can obtain $\|z_n^+\| \geq \|z_n^-\|$. Therefore, recalling that $\|z_n\| = 1$, we can conclude that $\|z_n^+\| \geq \dfrac{1}{\sqrt{2}}$.

Since $z \in S^+$, we can use the last inequality to write $z_n^+ = s_n z$, with $\dfrac{1}{\sqrt{2}} \leq s_n \leq 1$. Up to a subsequence, $z_n^+ \to sz$, with $s > 0$, which contradicts $z_n \rightharpoonup 0$. ■

Lemma 1.16. *Suppose that H satisfies $(H_1) - (H_4)$. If $z \in \mathcal{N}$ then for any $w \neq 0$ such that $z + w \in \widehat{E}(z)$, we have $I(z + w) < I(z)$.*

Proof. Let $z \in N$ and $w \neq 0$ with $z + w \in \widehat{E}(z)$. By definition, we can write

$$z + w = (1 + s)z + v \quad \text{with} \quad s \geq -1 \quad \text{e} \quad v \in E^-.$$

Since $z \in \mathcal{N}$, we define $\varphi := s\left(\frac{s}{2} + 1\right) z + (s + 1)v \in E(z)$, then we have

$$0 = I'(z)\varphi = s\left(\frac{s}{2} + 1\right)(\|z^+\|^2 - \|z^-\|^2) - (s + 1)\langle z^-, v\rangle - \int H_z(x, z) \cdot \varphi.$$

Therefore,

$$\begin{aligned}
&I(z + w) - I(z) \\
&\quad = s\left(\frac{s}{2} + 1\right)(||z^+||^2 - \|z^-\|^2) - (s + 1)\langle z^-, v\rangle - \frac{1}{2}\|v\|^2 + \int [H(x, z) - H(x, z + w)] \\
&\quad = -\frac{1}{2}\|v\|^2 + \int [H_z(x, z) \cdot \left(s\left(\frac{s}{2} + 1\right) z + (s + 1)v\right) H(x, z) - H(x, z + w)].
\end{aligned}$$

Since $w \neq 0$, it follows from Lemma 1.14 that $I(z + w) < I(z)$. ■

Lemma 1.17. *Suppose that $(H_1) - (H_4)$ are satisfied. Then,*

(i) for any $z \in \mathcal{N}$, $I|_{\widehat{E}(z)}$ admits a unique maximum point that is precisely z;

(ii) for any $z \in E\backslash E^-$, the set $\widehat{E}(z)$ intersects $\mathcal{N}$ at exactly one point $\widehat{m}(z)$, which is the unique global maximum point of $I|_{\widehat{E}(z)}$.

Proof. Firstly we will show (i). Given $tz + y \in \widehat{E}(z)\backslash\{z\}$, it is enough to consider $w = (t - 1)z + y$ to obtain $tz + y = z + w$. Note that, if $w = 0$ then $t = 1$ and $y = 0$, but this can not occur because $tz + y \neq z$. Hence, $w \neq 0$ and by Lemma 1.16 we conclude that $I(tz + y) < I(z)$.

To prove (ii), by the previous item, it is sufficient to show that $\mathcal{N} \cap \widehat{E}(z) \neq \emptyset$ for each $z \in E\backslash E^-$. Moreover, since $\widehat{E}(z) = \widehat{E}\left(z^+/\|z^+\|\right)$, we can assume $z = (u, u) \in E^+$ and $\|z\| = 1$. By Lemma 1.15, there exists $R > 0$ such that $I(w) \leq 0$ if $w \in \widehat{E}(z)\backslash\mathcal{B}_R(0)$ and if $\|w\| \leq R$ then $I(w) \leq R^2/2$ and so $\sup_{\widehat{E}(z)} I < \infty$.

On the other hand, by using (1.8), for any $\varepsilon > 0$ and $\alpha > 0$ there exists $C > 0$ such that

$$H(x, z) \leq \varepsilon|z|^2 + C|z|^3(e^{\alpha|z|^2} - 1), \quad z \in \mathbb{R}^2. \tag{1.12}$$

From (1.12), Lemma (1.12) and by choosing $0 < \varepsilon < 1/(4\int_{\mathbb{R}^2} |z|^2)$, we have

$$
\begin{aligned}
I(tz) = \frac{t^2}{2}\|z\|^2 - \int_{\mathbb{R}^2} H(x,tz) &\geq \frac{t^2}{2} - \varepsilon t^2 \int_{\mathbb{R}^2} |z|^2 - C_\varepsilon t^3 \int_{\mathbb{R}^2} |z|^3 (e^{\alpha t^2|z|^2} - 1) \\
&\geq \frac{t^2}{4} - Ct^3 \left(\int_{\mathbb{R}^2} |z|^6 \right)^{1/2} \left[\int_{\mathbb{R}^2} (e^{2\alpha t^2 |z|^2} - 1) \right]^{1/2} > 0,
\end{aligned}
$$

for $t > 0$ sufficiently small. Consequently, $\sup_{\widehat{E}(z)} I > 0$.

Now, let $w_n = (w_n^1, w_n^2) = t_n z + (h_n, -h_n) \in \widehat{E}(z)$ be a maximizing sequence for $\sup_{\widehat{E}(z)} I$. Since $\sup_{\widehat{E}(z)} I > 0$, we can suppose that $\|w_n\| \leq R$. Hence, up to a subsequence, $w_n \rightharpoonup w_0$ weakly in E. We can see, up to a subsequence, that $t_n \to t_0 \geq 0$ and $h_n \rightharpoonup h_0$, $w_n^1 \rightharpoonup w_0^1$, $w_n^2 \rightharpoonup w_0^2$ weakly in $H^1(\mathbb{R}^2)$ and therefore $w_0 = (w_0^1, w_0^2) = t_0 z + (h_0, -h_0) \in \widehat{E}(z)$. Recalling that $z \in E^+$, we can write

$$
I(w_n) = \frac{t_n^2}{2}\|z\|^2 - \frac{1}{2}\|h_n\|^2 - \int_{\mathbb{R}^2} H(x, w_n^1, w_n^2),
$$

from where it follows, according to the weak lower semicontinuous of the norm and Fatou's lemma, that

$$
\sup_{\widehat{E}(z)} I = \lim_{n\to\infty} I(w_n) = \frac{t_0^2}{2}\|z\|^2 + \limsup_{n\to\infty} \left[-\frac{1}{2}\|h_n\|^2 - \int_{\mathbb{R}^2} H(x, w_n^1, w_n^2) \right] \leq I(w_0).
$$

Thus, $I(w_0) = \sup_{\widehat{E}(z)} I$ and therefore w_0 is a critical point of $I|_{\widehat{E}(z)}$, showing that $w_0 \in \mathcal{N} \cap \widehat{E}(z)$ and the proof is complete. ■

As an immediate consequence of this lemma, we obtain the following equality:

Corollary 1.18.

$$
\inf_{\eta \in \mathcal{N}} I(\eta) = \inf_{z \in E \setminus E^-} \max_{w \in \widehat{E}(z)} I(w).
$$

Lemma 1.19. *Assume* $(H_1) - (H_2)$ *and let*

$$
c = \inf_{z \in \mathcal{N}} I(z).
$$

Then, $c > 0$ *and* $\|z^+\| \geq \max\{\|z^-\|, \sqrt{2c}\}$ *for all* $z \in \mathcal{N}$.

Proof. For $a > 0$ we recall that $S_a^+ := \{z \in E^+ : \|z\| = a\}$ and $(\mathbb{R}_+)z = \{tz : t \geq 0\}$.

Since $\widehat{E}(z) = \widehat{E}(z^+)$ for any $z \in E\backslash E^-$, from Corollary 1.18, for any $a > 0$, it follows that

$$c = \inf_{z\in E\backslash E^-} \max_{w\in \widehat{E}(z)} I(w) = \inf_{z\in E^+\backslash\{0\}} \max_{w\in \widehat{E}(z)} I(w) = \inf_{z\in S_a^+} \max_{w\in \widehat{E}(z)} I(w) \geq \inf_{z\in S_a^+} \max_{w\in(\mathbb{R}_+)z} I(w).$$

Let $z = (u, u)$ be in S_a^+ and $\alpha > 0$ such that $2\alpha a^2 < 4\pi$. By Lemma 1.12, there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^2} (e^{2\alpha|z|^2} - 1) = \int_{\mathbb{R}^2} (e^{2\alpha a^2(|z|/\|z\|)^2} - 1) \leq C.$$

Then, by using (1.12) we get

$$\begin{aligned} \max_{w\in(\mathbb{R}_+)z} I(w) \geq I(z) = \|u\|_V^2 - \int_{\mathbb{R}^2} H(x,u,u) &\geq \|u\|_V^2 - \varepsilon\int_{\mathbb{R}^2}|z|^2 - C_\varepsilon\int_{\mathbb{R}^2}|z|^3(e^{\alpha|z|^2}-1) \\ &\geq (1-2\varepsilon)\|u\|_V^2 - C_\varepsilon\left(\int_{\mathbb{R}^2}|z|^6\right)^{1/2} \\ &\geq (1-2\varepsilon)\|u\|_V^2 - C_1\|u\|_V^3, \end{aligned}$$

where we have used the continuous embedding $H^1(\mathbb{R}^2) \hookrightarrow L^6(\mathbb{R}^2)$. Hence, taking $\varepsilon = 1/4$ and $a > 0$ sufficiently small so that $1/2 - C_1\|u\|_V = 1/2 - C_1 a/\sqrt{2} \geq 1/4$, we conclude that

$$\max_{w\in(\mathbb{R}_+)z} I(w) \geq \|u\|_V^2\left(\frac{1}{2} - C_1\|u\|_V\right) \geq \frac{a^2}{8} > 0, \quad \text{for all} \;\; z = (u,u) \in S_a^+$$

and consequently $c > 0$. Next, for any $z \in \mathcal{N}$ we have

$$c \leq \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2) - \int_{\mathbb{R}^2} H(x,z) \leq \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2),$$

which implies that $\|z^+\| \geq \max\{\|z^-\|, \sqrt{2c}\}$ and the proof is done. ■

Lemma 1.20. *Suppose that $(H_1)-(H_2)$ are satisfied. If $\mathcal{W} \subset E\backslash E^-$ is a compact subset, then there exists $C_{\mathcal{W}} > 0$ such that $\|\widehat{m}(z)\| \leq C_{\mathcal{W}}$, for all $z \in \mathcal{W}$.*

Proof. Defining $\delta = \sqrt{2c}$, by Lemma 1.19 and noting that $\widehat{m}(z) \in \mathcal{N}$ for any $z \in E\backslash E^-$, we have $\|\widehat{m}(z)^+\| \geq \delta$. Moreover, since $\widehat{m}(z) = \widehat{m}\left(z^+/\|z^+\|\right)$ for any $z \in E\backslash E^-$, without loss of generality, we can assume that $\mathcal{W} \subset S^+$. It follows from Lemma 1.17 that there exists $C_{\mathcal{W}} > 0$ such that

$$I \le 0 \ \text{ on } \ \widehat{E}(z) \backslash \mathcal{B}_{C_{\mathcal{W}}}(0) \ \text{ for all } \ z \in \mathcal{W},$$

where $\mathcal{B}_{C_{\mathcal{W}}}(0) = \{w \in E : \|z\| \le C_{\mathcal{W}}\}$. Recalling that $I(\widehat{m}(z)) \ge c > 0$ for all $z \in E \backslash E^-$, we get

$$\|\widehat{m}(z)\| = \left\| \widehat{m}\left(\frac{z^+}{\|z^+\|}\right)\right\| \le C_{\mathcal{W}} \quad \text{for any } \ z \in \mathcal{W}.$$

■

Lemma 1.21. *If $(H_1) - (H_3)$ are satisfied, then I is coercive on $\mathcal{N}$. In particular, any $(PS)_c$ sequence $(z_n) \subset \mathcal{N}$ for I is bounded.*

Proof. Suppose, by contradiction, that there exists $(z_n) \subset \mathcal{N}$ satisfying $\lim_{n\to\infty} \|z_n\| = +\infty$ with $I(z_n) \le d$ for all $n \in \mathbb{N}$, for some $d > 0$. Setting $w_n = z_n/\|z_n\|$, up to a subsequence, we have $w_n \rightharpoonup w$ weakly in E and $w_n(x) \to w(x)$ a.e. in $x \in \mathbb{R}^2$.

Firstly, we claim that $w = 0$. Indeed, if $w \ne 0$ and $\Omega := \{x \in \mathbb{R}^2 : w(x) \ne 0\}$ then $\mu(\Omega) > 0$ and $|z_n(x)| = \|w_n\||w_n(x)| \to +\infty$ a.e. in $x \in \Omega$. Since

$$0 < \frac{c}{\|z_n\|^2} \le \frac{I(z_n)}{\|z_n\|^2} = \frac{1}{\|z_n\|^2}\left[\frac{1}{2}\|z_n^+\|^2 - \frac{1}{2}\|z_n^-\|^2 - \int_{\mathbb{R}^2} H(x, z_n)\right] \le \frac{1}{2} - \int_{\Omega} \frac{H(x,z_n)}{|z_n|^2}|w_n|^2,$$

in view of (H_3) and Fatou's lemma, we get a contradiction and therefore we must have $w = 0$.

Next, we claim that

$$w_n^+ \not\to 0 \ \text{ in } \ L^p(\mathbb{R}^2) \times L^p(\mathbb{R}^2) \ \text{ for some } p \in (2, \infty). \tag{1.13}$$

In fact, arguing by contradiction, suppose that $w_n^+ \to 0$ in $L^p(\mathbb{R}^2) \times L^p(\mathbb{R}^2)$ for all $p \in (2, \infty)$. By (1.8), for $\varepsilon > 0$, $s > 0$ and $\alpha > 0$ we have

$$\begin{aligned} \int_{\mathbb{R}^2} H(x, sw_n^+) &\le \varepsilon s^2 \|w_n^+\|_2^2 + C_\varepsilon s \int_{\mathbb{R}^2} |w_n^+|(e^{s^2\alpha|w_n^+|^2} - 1) \\ &\le \varepsilon s^2 \|w_n^+\|_2^2 + C_\varepsilon s \left(\int_{\mathbb{R}^2} |w_n^+|^q\right)^{1/q} \left[\int_{\mathbb{R}^2} (e^{s^2\alpha q'|w_n^+|^2} - 1)\right]^{1/q'}, \end{aligned} \tag{1.14}$$

where $q \in (2, \infty)$ and $q' = q/(q-1)$. Fixing $s > 2\sqrt{d}$ and taking $\alpha > 0$ so that $s^2\alpha q' < 4\pi$, by Lemma 1.12 we conclude that

$$\int_{\mathbb{R}^2} (e^{s^2\alpha q'|w_n^+|^2} - 1) \leq C,$$

for some constant $C > 0$. Once $\varepsilon > 0$ is arbitrary, from (1.14) it follows that

$$\int_{\mathbb{R}^2} H(x, sw_n^+) \to 0.$$

Since $sw_n^+ \in \widehat{E}(z_n)$, by Lemma 1.16 we have

$$d \geq I(z_n) \geq I(sw_n^+) \geq \frac{1}{4}s^2 - \int_{\mathbb{R}^2} H(x, sw_n^+) = \frac{1}{4}s^2 + o_n(1)$$

and passing to the limit as $n \to \infty$ we reach $d \geq s^2/4$ which is an absurd and (1.13) is proved. Thus, we can use Lions' lemma [49, Lemma I.1] to obtain $\beta > 0$ and a sequence $(y_n) \subset \mathbb{R}^2$ satisfying

$$\int_{B_1(y_n)} |w_n^+|^2 \geq \beta. \tag{1.15}$$

We may assume, without lost of generality, that $(y_n) \subset \mathbb{Z}^2$. Moreover, doing a translation, if necessary, we can suppose that (y_n) is bounded. Hence, there exists $R > 0$ such that $B_1(y_n) \subset B_R$ and by (1.15) we get

$$\int_{B_R} |w_n^+|^2 \geq \beta.$$

Taking the limit we conclude that $w^+ \neq 0$, which is a contradiction and the proof of the lemma is done. ■

Now, we are ready to prove our main theorem of this section.

Proof of Theorem 1.1. By the previous lemmas, the functional I fulfills conditions $(N_1)-(N_3)$. Thus, according to Lemma 1.6, let $(w_n) \subset S^+$ be such that $\Psi(w_n) \to \inf_{S^+} \Psi$. By Ekeland's Variational Principle, we can suppose $\Psi'(w_n) \to 0$. Thus, by (iii) of Lemma 1.6 we have $I'(z_n) \to 0$ where $z_n = m(w_n) \in \mathcal{N}$. By Lemma 1.21, up to a subsequence, $z_n \rightharpoonup z$ in E and $z_n(x) \to z(x)$ a.e. in $x \in \mathbb{R}^2$. Next, we have the following claim:

Claim 1: z is a critical point of I.

Indeed, by the density of $C_0^\infty(\mathbb{R}^2) \times C_0^\infty(\mathbb{R}^2)$ in E, just to conclude that $I'(z)\eta = 0$

for all $\eta \in C_0^\infty(\mathbb{R}^2) \times C_0^\infty(\mathbb{R}^2)$. Then, let $\eta \in C_0^\infty(\mathbb{R}^2) \times C_0^\infty(\mathbb{R}^2)$, $K = \operatorname{supp}\eta$ and $M > 0$ is a constant to be chosen later. Defining $\Omega_1 = K \cap \{x;\ |z_n(x)| \leq M\}$ and $\Omega_2 = K \cap \{x;\ |z_n(x)| > M\}$, by (H_1) and (H_2), given $\alpha > 0$ we obtain $C > 0$ such that

$$|H_z(x,w)| \leq |w| + C(e^{\alpha|w|^2} - 1) \quad \text{for all} \quad (x,w) \in \mathbb{R}^2 \times \mathbb{R}^2$$

and hence

$$\begin{aligned}|H_z(x,z_n(x)) \cdot \eta(x)| &\leq \|\eta\|_\infty |z_n(x)| + C_1(e^{\alpha|z_n(x)|^2} - 1)\\ &\leq \|\eta\|_\infty M + C_1(e^{\alpha M^2} - 1) \quad \text{a.e. in} \quad x \in \Omega_1.\end{aligned}$$

Since $H_z(x,z_n) \cdot \eta \to H_z(x,z) \cdot \eta$ a.e. in Ω_1, it follows from the Lebesgue Dominated Convergence Theorem that

$$\int_{\Omega_1} [H_z(x,z_n) - H_z(x,z)] \cdot \eta \to 0 \tag{1.16}$$

Now, we take $\alpha > 0$ so that $2\alpha\|z_n\|^2 < 4\pi$. Thus, by Lemma 1.12 and Hölder's inequality, we reach

$$\begin{aligned}&\left|\int_{\Omega_2} [H_z(x,z_n) - H_z(x,z)] \cdot \eta\right|\\ &\leq \frac{\|\eta\|_\infty}{M} \int_{\Omega_2} |H_z(x,z_n) - H_z(x,z)||z_n|\\ &\leq \frac{\|\eta\|_\infty}{M}\left[\|z_n\|_2^2 + \|z_n\|_2\|z\|_2 + C_1\left(\int_{\mathbb{R}^2} [e^{2\alpha|z_n|^2} - 1]\right)^{\frac{1}{2}} \|z_n\|_2\right.\\ &\quad \left. + C_1\left(\int_{\mathbb{R}^2} [e^{2\alpha|z|^2} - 1]\right)^{\frac{1}{2}} \|z_n\|_2\right]\\ &\leq \frac{\|\eta\|_\infty C}{M}\end{aligned} \tag{1.17}$$

for some $C > 0$ independent of M. Hence, given $\delta > 0$ we can consider $M > 0$ so that

$$\left|\int_{\Omega_2} [H_z(x,z_n) - H_z(x,z)] \cdot \eta\right| < \delta, \quad \text{for all} \quad n \in \mathbb{N}. \tag{1.18}$$

Therefore, by (1.16) and (1.18) we obtain

$$\limsup_{n\to\infty} \left|\int_K [H_z(x,z_n) - H_z(x,z)] \cdot \eta\right| \leq \limsup_{n\to\infty} \left|\int_{\Omega_1} [H_z(x,z_n) - H_z(x,z)] \cdot \eta\right| + \delta = \delta$$

and once $\delta > 0$ is arbitrary, we conclude that

$$\int_{\mathbb{R}^2} [H_z(x, z_n) - H_z(x, z)] \cdot \eta \to 0. \tag{1.19}$$

Once

$$I'(z_n)\eta - I'(z)\eta = \langle z_n - z, \eta \rangle - \int_{\mathbb{R}^2} [H_z(x, z_n) - H_z(x, z)] \cdot \eta$$

and according to (1.19) we reach

$$I'(z)\eta = \lim_{n\to\infty} \left[I'(z_n)\eta - \langle z_n - z, \eta \rangle + \int_{\mathbb{R}^2} [H_z(x, z_n) - H_z(x, z)] \cdot \eta \right] = 0,$$

and Claim 1 is proved.

Now, let us suppose that $z_n \to 0$ in $L^p(\mathbb{R}^2) \times L^p(\mathbb{R}^2)$ for some $p \in (2, \infty)$. In view of (1.8), given $\varepsilon > 0$ and $\alpha > 0$, there exists $C_1 > 0$ such that

$$\begin{aligned} \left| \int_{\mathbb{R}^2} H_z(x, z_n) \cdot z_n \right| &\leq \varepsilon \int_{\mathbb{R}^2} |z_n|^2 + C_1 \int_{\mathbb{R}^2} |z_n| (e^{\alpha |z_n|^2} - 1) \\ &\leq \varepsilon \|z_n\|_2^2 + C_1 \left(\int_{\mathbb{R}^2} |z_n|^p \right)^{1/p} \left[\int_{\mathbb{R}^2} (e^{\alpha p' \|z_n\|^2 |z_n / \|z_n\||^2} - 1) \right]^{1/p'}. \end{aligned}$$

Taking $\alpha > 0$ so that $\alpha p' \|z_n\|^2 < 4\pi$ for all $n \in \mathbb{N}$, by Lemma 1.12 the last integral is bounded. Moreover, we know that $\int_{\mathbb{R}^2} |z_n|^p \to 0$ and therefore

$$\int_{\mathbb{R}^2} H_z(x, z_n) \cdot z_n \to 0. \tag{1.20}$$

Similarly, we also conclude that $\int_{\mathbb{R}^2} H_z(x, z_n) \cdot z_n^- \to 0$. From this and since $z_n \in \mathcal{N}$ we have

$$0 = I'(z_n) z_n^- = -\|z_n^-\|^2 - \int_{\mathbb{R}^2} H_z(x, z_n) \cdot z_n^-,$$

which shows that $\|z_n^-\|^2 \to 0$. On the other hand, we also have

$$0 = I'(z_n) z_n = \|z_n^+\|^2 - \|z_n^-\|^2 - \int_{\mathbb{R}^2} H_z(x, z_n) \cdot z_n$$

and by using (1.20) it follows that $z_n^+ \to 0$. But this contradicts Lemma 1.19 and therefore

$z_n \not\to 0$ in $L^p(\mathbb{R}^2) \times L^p(\mathbb{R}^2)$. Again by Lions' lemma

$$\int_{B_1(y_n)} |z_n|^2 \geq \beta > 0,$$

for some $\beta > 0$ and $(y_n) \subset \mathbb{Z}^2$. Once I is invariant by integer translations, the sequence $\tilde{z}_n := z_n(\cdot - y_n) \rightharpoonup \tilde{z}$ in E and again we can show that $\tilde{z}$ is a critical point of I. We observe that by the last inequality $\tilde{z} \neq 0$ and since I does not have nonzero critical points in E^-, we have $\tilde{z} \in \mathcal{N}$ and so $I(\tilde{z}) \geq c = \inf_{\mathcal{N}} I$. Now, let us to prove that $I(\tilde{z}) \leq c$ and thus we can conclude that $c = I(\tilde{z}) = \inf_{\mathcal{N}} I$. In fact, we know that

$$I(\tilde{z}_n) \to c, \quad I'(\tilde{z}_n) \to 0 \quad \text{and} \quad \tilde{z}_n \to \tilde{z} \ \text{ a.e. in } \ \mathbb{R}^2.$$

Hence, it follows from Lemma 1.13 and Fatou's lemma that

$$c + o_n(1) = I(\tilde{z}_n) - \frac{1}{2} I'(\tilde{z}_n)\tilde{z}_n = \int_{\mathbb{R}^2} \widehat{H}(x, \tilde{z}_n) \geq I(\tilde{z}) - \frac{1}{2} I'(\tilde{z})\tilde{z} + o_n(1) = I(\tilde{z}) + o_n(1),$$

which implies that $I(\tilde{z}) \leq c$ and the proof of Theorem 1.1 is concluded. ■

1.3 Proof of Theorem 1.2

In this section, we study the situation when V and H are asymptotically periodic. First, we defined the functional $I_\infty : E \to \mathbb{R}$ by

$$I_\infty(z) := \frac{1}{2}\|z^+\|^2 - \frac{1}{2}\|z^-\|^2 - \int_{\mathbb{R}^2} H_\infty(x, z),$$

where H_∞ is the approximation of the function H according to hypothesis (H_7). Once H_∞ fulfills $(H_0) - (H_4)$ and V_∞ satisfies (V_0), we can invoke Theorem 1.1 to get a least energy solution $z_\infty \in E$ of the periodic system

$$\begin{cases} -\Delta u + V_\infty(x)u = H_{\infty,v}(x, u, v), & x \in \mathbb{R}^2, \\ -\Delta v + V_\infty(x)v = H_{\infty,u}(x, u, v), & x \in \mathbb{R}^2. \end{cases}$$

Therefore, taking the solution z_∞, we define our link set in the following way:

$$M_{R,z_0} := \{z = tz_0 + z^- : z^- \in E^-, \|z\| \leq R, \ t \geq 0\},$$

where $R > 0$ and $z_0 := z_\infty^+$. By virtue of $M_{R,z_0} \subset \widehat{E}(z_0) = \widehat{E}(z_\infty)$, it follows from Lemma 1.17 that

$$\sup_{z \in M_{R,z_0}} I_\infty(z) \leq I_\infty(z_0). \tag{1.21}$$

With the aim of finding a solution for problem (1), it suffices to get a nonzero critical point for the functional I defined in (1.9). Since we do not require an exponential growth for H, we must guarantee that the functional I is well defined.

Lemma 1.22. *Suppose that H satisfies (H_2) and (H_7). Then, for any $\varepsilon > 0$, $q \geq 1$ and $\alpha > 0$, there exists $C = C(\varepsilon, q, \alpha) > 0$ such that*

$$|H_z(x,z)| \leq \varepsilon|z| + C|z|^{q-1}(e^{\alpha|z|^2} - 1) \quad \textit{and} \quad |H(x,z)| \leq \varepsilon|z|^2 + C|z|^q(e^{\alpha|z|^2} - 1), \tag{1.22}$$

for each $(x,z) \in \mathbb{R}^2 \times \mathbb{R}^2$.

Proof. Given $\varepsilon > 0$, we can use (H_2) to obtain $\delta > 0$ such that

$$|H_z(x,z)| \leq \varepsilon|z|, \quad \text{for all} \ \ x \in \mathbb{R}^2, \ |z| \leq \delta. \tag{1.23}$$

Now, if $|z| \geq \delta$ then, by using (H_7), given $q \geq 1$ and $\alpha > 0$ we obtain

$$|H_z(x,z)| \leq |H_{\infty,z}(x,z)| + C_1|\varphi(x)|(e^{\alpha|z|^2} - 1) \leq C|z|^{q-1}(e^{\alpha|z|^2} - 1), \tag{1.24}$$

for some $C = C(\varepsilon, q, \alpha) > 0$. Hence, (1.23) and (1.24) prove the first inequality in (1.22). The second inequality follows from the Mean Value Theorem. ■

1.3.1 Linking geometry

In this subsection, we are going to guarantee that the functional I satisfies the linking structure of Theorem 1.9 (condition (L_2)).

According to condition (V_1), we can work with the same space E. The linking geometry is proved in the next lemma.

Lemma 1.23. *Suppose that H satisfies (H_2), $(H_5)-(H_7)$. Then,*

(i) there exist $r,\rho>0$ such that $I|_{N_r}\geq\rho$;

(ii) there exists $R>r$ such that $I|_{\partial M_{R,z_0}}\leq 0$.

Proof. The item (i) is a consequence of (H_2) and Lemma 1.22. Indeed, given $z=(u,u)\in N_r$, $\alpha>0$, $\varepsilon>0$ and $q>2$, by Lemma 1.22 we get

$$|H(x,z)|\leq\varepsilon|z|^2+C_\varepsilon|z|^q(e^{\alpha|z|^2}-1). \tag{1.25}$$

By using Hölder's inequality, the continuous embedding $E\hookrightarrow L^t(\mathbb{R}^2)\times L^t(\mathbb{R}^2)$ for any $t\geq 2$ and Lemma 1.12, we get

$$\int_{\mathbb{R}^2}|z|^q(e^{\alpha|z|^2}-1)\leq\left(\int_{\mathbb{R}^2}|z|^{2q}\right)^{1/2}\left[\int_{\mathbb{R}^2}(e^{2r^2\alpha|z/r|^2}-1)\right]^{1/2}\leq C\|z\|^q=Cr^q,$$

where we have considered $r>0$ so that $2r^2\alpha<4\pi$. Hence, according to (1.25) we reach

$$I(z)\geq\frac{1}{2}r^2-\varepsilon\int_{\mathbb{R}^2}|z|^2-Cr^q\geq\left(\frac{1}{2}-\varepsilon C_1\right)r^2-Cr^q.$$

Taking $0<\varepsilon<1/(2C_1)$, we can obtain $r>0$ so that $(1/2-\varepsilon C_1)r^2-Cr^q=:\rho>0$. Therefore, there exist $r,\rho>0$ such that $I(z)\geq\rho$ whenever $\|z\|=r$.

For item (ii), let $z=tz_0+z^-\in\partial M_{R,z_0}$. If $\|z\|\leq R$ and $t=0$ then $z=z^-\in E^-$ and we can use (H_7) to obtain

$$I(z)=I(z^-)=-\frac{1}{2}\|z^-\|^2-\int_{\mathbb{R}^2}H(x,z^-)\leq 0.$$

Now, we consider the case $\|z\|=R$ and $t>0$. We argue by contradiction. Suppose that there exists a sequence (z_n) such that $z_n=t_nz_0+z_n^-$, with $t_n>0$, $\|z_n\|=R_n\to\infty$ and $I(z_n)>0$. Thus,

$$0<\frac{I(z_n)}{\|z_n\|^2}=\frac{1}{2}\left(\frac{t_n^2\|z_0\|^2}{\|z_n\|^2}-\frac{\|z_n^-\|^2}{\|z_n\|^2}\right)-\int_{\mathbb{R}^2}\frac{H(x,z_n)}{|z_n|^2}\frac{|z_n|^2}{\|z_n\|^2}. \tag{1.26}$$

Once H is nonnegative, we must have $t_n\|z_0\|>\|z_n^-\|$. Observing that

$$\frac{t_n^2\|z_0\|^2}{\|z_n\|^2}+\frac{\|z_n^-\|^2}{\|z_n\|^2}=1,$$

we can see that

$$\frac{1}{\sqrt{2}\|z_0\|} \leq \frac{t_n}{\|z_n\|} \leq \frac{1}{\|z_0\|} \quad \text{and} \quad \frac{z_n^-}{\|z_n\|} \quad \text{is bounded.}$$

Hence, since E^- is weakly closed, up to a subsequence, we can assume that

$$\frac{t_n}{\|z_n\|} \to \rho_0 > 0, \quad \frac{z_n^-}{\|z_n\|} \rightharpoonup w \in E^- \quad \text{and} \quad \frac{z_n^-(x)}{\|z_n\|} \to w(x) \;\; \text{a.e. in} \;\; x \in \mathbb{R}^2,$$

As $\|z_n\| \to \infty$ we have $t_n \to \infty$ and therefore

$$\lim_{n\to\infty} |z_n(x)| = \infty \;\; \text{a.e. in} \;\; \Omega := \{x \in \mathbb{R}^2 : \rho_0 z_0(x) + w(x) \neq 0\}.$$

In view of $\rho_0 > 0$ and $w \in E^-$, Ω has positive Lebesgue measure. Hence, taking the limit in (1.26), using Fatou's lemma and (H_7), we reach

$$0 \leq \frac{1}{2}(\rho_0^2\|z_0\|^2 - \|w\|^2) - \int_\Omega \liminf_{n\to\infty} \frac{H(x,z_n)}{|z_n|^2}\frac{|z_n|^2}{\|z_n\|^2} = -\infty,$$

which is an absurd and this concludes the proof. ∎

1.3.2 Behavior of Cerami Sequences

In this subsection, we get some crucial results related to Cerami sequences for the functional I.

In the proof of the next lemma, we shall use the following inequality, whose proof can be found in [28, Lemma 2.4]:

$$st \leq \begin{cases} (e^{t^2}-1) + |s|(\log|s|)^{1/2}, & t \in \mathbb{R} \;\text{ and }\; |s| \geq e^{1/4}; \\ (e^{t^2}-1) + \frac{1}{2}s^2, & t \in \mathbb{R} \;\text{ and }\; |s| \leq e^{1/4}. \end{cases} \tag{1.27}$$

Lemma 1.24. *Assume conditions (H_2) and $(H_5)-(H_7)$. If $(z_n) \subset E$ is a $(Ce)_c$ sequence for I, then it is bounded in E.*

Proof. If $(z_n) \subset E$ is a $(Ce)_c$ sequence for I, then

$$c + o_n(1) = I(z_n) - \frac{1}{2}I'(z_n)z_n = \int_{\mathbb{R}^2} \widehat{H}(x, z_n). \tag{1.28}$$

Suppose by contradiction that, up to a subsequence, $\|z_n\| \to +\infty$. Thus,

$$o_n(1) = \frac{I'(z_n)(z_n^+ - z_n^-)}{\|z_n\|^2} = 1 - \int_{\mathbb{R}^2} \frac{H_z(x, z_n) \cdot (z_n^+ - z_n^-)}{\|z_n\|^2}.$$

Setting $w_n := z_n/\|z_n\|$, we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} \frac{H_z(x, z_n) \cdot (w_n^+ - w_n^-)}{\|z_n\|} = 1. \tag{1.29}$$

Recalling that $q(r) = \inf\{\widehat{H}(x, z) : x \in \mathbb{R}^2,\ |z| \geq r\}$ and noting that q is nondecreasing in $r > 0$, we can use (H_5) to conclude that $q(r) > 0$ for all $r > 0$. Moreover, in view of (H_6), $(H_7)(i)$ and $(H_7)(ii)$, it follows that $\widehat{H}(x, z) \to +\infty$ as $|z| \to \infty$ uniformly in $x \in \mathbb{R}^2$. Consequently, $q(r) \to \infty$ as $r \to \infty$.

For $0 \leq a < b \leq \infty$ and $n \in \mathbb{N}$, we set

$$\Omega_n(a, b) := \{x \in \mathbb{R}^2 : a \leq |z_n(x)| < b\}.$$

Using (1.28) and for $0 < r < R < \infty$, we obtain

$$\begin{aligned} c + o_n(1) &= \int_{\Omega_n(0,r)} \widehat{H}(x, z_n) + \int_{\Omega_n(r,R)} \frac{\widehat{H}(x, z_n)}{|z_n|^2}|z_n|^2 + \int_{\Omega_n(R,\infty)} \widehat{H}(x, z_n) \\ &\geq \int_{\Omega_n(0,r)} \widehat{H}(x, z_n) + \frac{q(r)}{R^2} \int_{\Omega_n(r,R)} |z_n|^2 + q(R)\mu(\Omega_n(R, \infty)) \end{aligned}$$

and therefore there exists $C_1 > 0$ such that

$$\max\left\{\int_{\Omega_n(0,r)} \widehat{H}(x, z_n),\ \frac{q(r)}{R^2} \int_{\Omega_n(r,R)} |z_n|^2,\ q(R)\mu(\Omega_n(R, \infty))\right\} \leq C_1. \tag{1.30}$$

In particular,

$$\mu(\Omega_n(R, \infty)) \leq C_1/q(R) \quad \text{for each} \quad n \in \mathbb{N}. \tag{1.31}$$

Now, let $C_3 > 0$ be such that $\|z\|_2^2 \leq C_3\|z\|^2$ for each $z \in E$ and consider $\varepsilon > 0$. By (H_2), there exists $r_\varepsilon > 0$ such that $|H_z(x, z)| \leq \varepsilon|z|/C_3$ for each $|z| \leq r_\varepsilon$. By the definition of

w_n and since $|w_n^+ - w_n^-| = |w_n|$, for any $n \in \mathbb{N}$ we get

$$\begin{aligned}\int_{\Omega_n(0,r_\varepsilon)} \frac{H_z(x,z_n)\cdot(w_n^+ - w_n^-)}{\|z_n\|} &\leq \int_{\Omega_n(0,r_\varepsilon)} \frac{|H_z(x,z_n)|}{|z_n|}|w_n^+ - w_n^-||w_n| \\ &\leq \frac{\varepsilon}{C_3}\int_{\Omega_n(0,r_\varepsilon)} |w_n|^2 \leq \varepsilon\|w_n\|^2 = \varepsilon.\end{aligned} \tag{1.32}$$

Now, let $R_\varepsilon > r_\varepsilon$ to be chosen later. Define $\mathcal{A}_1 = \Omega_n(R_\varepsilon,\infty)\cap\{x\in\mathbb{R}^2 : |H_z(x,z_n)| \leq e^{1/4}\}$ and $\mathcal{A}_2 = \Omega_n(R_\varepsilon,\infty)\cap\{x\in\mathbb{R}^2 : |H_z(x,z_n)| \geq e^{1/4}\}$. Taking $t = |w_n|$ and $s = |H(x,z_n)|$ in (1.27), we get

$$\begin{aligned}\int_{\Omega_n(R_\varepsilon,\infty)} \frac{H_z(x,z_n)\cdot(w_n^+ - w_n^-)}{\|z_n\|} &\leq \frac{1}{\|z_n\|}\int_{\Omega_n(R_\varepsilon,\infty)} |H_z(x,z_n)||w_n| \\ &\leq \frac{1}{\|z_n\|}\int_{\mathcal{A}_2} |H_z(x,z_n)|(\log|H_z(x,z_n)|)^{1/2} \\ &\quad + \frac{1}{\|z_n\|}\int_{\mathcal{A}_1}\frac{1}{2}|H_z(x,z_n)|^2 + \frac{2}{\|z_n\|}\int_{\mathbb{R}^2}(e^{|w_n|^2}-1).\end{aligned} \tag{1.33}$$

By (1.31), we have $\mu(\mathcal{A}_1) \leq \mu(\Omega_n(R_\varepsilon,\infty)) \leq C_1/q(R_\varepsilon)$ and consequently

$$\int_{\mathcal{A}_1} |H_z(x,z_n)|^2 \leq \frac{C_1 e^{1/2}}{q(R_\varepsilon)}, \quad \text{for all} \quad n\in\mathbb{N}.$$

By Lemma 1.12, the integral $\int_{\mathbb{R}^2}(e^{|w_n|^2}-1)$ is bounded. Moreover, given $\alpha > 0$, by (H_1) there exists $R_1 > 0$ such that

$$|H_z(x,z)| \leq e^{\alpha|z|^2} \quad \text{for all} \quad x\in\mathbb{R}^2 \quad \text{and} \quad |z|\geq R_1.$$

In view of (H_6), there exists $c_1 > 0$ such that

$$|H_z(x,z)||z| \leq c_1\widehat{H}(x,z) \quad \text{for all} \quad x\in\mathbb{R}^2 \quad \text{and} \quad |z|\geq R_1.$$

Taking $R_\varepsilon > R_1$, from the two last estimates and (1.28), it follows that

$$\int_{\mathcal{A}_2} |H_z(x,z_n)|(\log|H_z(x,z_n)|)^{1/2} \leq \alpha^{1/2}\int_{\mathcal{A}_2}|H_z(x,z_n)||z_n| \leq \alpha^{1/2}c_1\int_{\mathbb{R}^2}\widehat{H}(x,z_n) \leq C,$$

for some constant $C > 0$. Thus, in view of (1.33), we find $n_0 \in \mathbb{N}$ satisfying

$$\int_{\Omega_n(R_\varepsilon,\infty)} \frac{H_z(x,z_n)\cdot(w_n^+ - w_n^-)}{\|z_n\|} < \varepsilon, \quad \text{for all} \;\; n \geq n_0. \tag{1.34}$$

According to (H_7), given $\alpha > 0$, we obtain $C_\alpha > 0$ such that, for all $x \in \Omega_n(r_\varepsilon, R_\varepsilon)$, we have

$$\begin{aligned}|H_z(x,z_n)| &\leq |H_{z,\infty}(x,z_n)| + C_\alpha|\varphi(x)||z_n|(e^{\alpha|z_n|^2}-1)\\ &\leq C_\varepsilon|z_n| + C_\alpha\|\varphi\|_\infty(e^{\alpha R_\varepsilon^2}-1)|z_n|.\end{aligned}$$

From this estimate and (1.30), recalling the definition of w_n and $|w_n^+ - w_n^-| = |w_n|$, we reach

$$\begin{aligned}&\int_{\Omega_n(r_\varepsilon,R_\varepsilon)} \frac{H_z(x,z_n)\cdot(w_n^+ - w_n^-)}{\|z_n\|}\\ &\quad\leq C_\varepsilon\int_{\Omega_n(r_\varepsilon,R_\varepsilon)} \frac{|z_n||w_n^+ - w_n^-|}{\|z_n\|} + C_\alpha\|\varphi\|_\infty(e^{\alpha R_\varepsilon^2}-1)\int_{\Omega_n(r_\varepsilon,R_\varepsilon)} \frac{|z_n||w_n^+ - w_n^-|}{\|z_n\|}\\ &\quad\leq \frac{C_6}{\|z_n\|^2}\int_{\Omega_n(r_\varepsilon,R_\varepsilon)} |z_n|^2\\ &\quad\leq \frac{C_7}{\|z_n\|^2}\frac{R_\varepsilon^2}{q(r_\varepsilon)} < \varepsilon, \quad \text{for all} \;\; n \geq n_1,\end{aligned}$$

for some $n_1 \in \mathbb{N}$. This estimate, (1.32) and (1.34) show that

$$\int_{\mathbb{R}^2} \frac{H_z(x,z_n)\cdot(w_n^+ - w_n^-)}{\|z_n\|}dx \leq 3\varepsilon, \quad \text{for all} \;\; n \geq \max\{n_0, n_1\}.$$

But this contradicts (1.29) since $\varepsilon > 0$ is arbitrary, and the proof is done. ■

Lemma 1.25. *Suppose that (H_2), (H_6) and (H_7) are fulfilled. Let $c > 0$ and $(z_n) \subset E$ be a $(Ce)_c$ sequence for I. If $z_n \rightharpoonup 0$ weakly in E, then there exists a sequence $(y_n) \subset \mathbb{R}^2$, $R > 0$ and $\beta > 0$ such that $|y_n| \to \infty$ and*

$$\limsup_{n\to\infty} \int_{B_R(y_n)} |z_n|^2 \geq \beta > 0.$$

Proof. Suppose by contradiction that the result is not valid. Thus, for each $R > 0$, one has

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_R(y)} |z_n|^2 = 0,$$

and by invoking Lions' lemma $z_n \to 0$ in $L^s(\mathbb{R}^2) \times L^s(\mathbb{R}^2)$ for each $s > 2$. In view of

(1.22), we have $\int_{\mathbb{R}^2} H(x, z_n) \to 0$ as $n \to \infty$. Similarly, $\int_{\mathbb{R}^2} H_z(x, z_n) \cdot z_n \to 0$. On the other hand,

$$c = \lim_{n\to\infty} \left[I(z_n) - \frac{1}{2} I'(z_n) z_n \right] = \lim_{n\to\infty} \int_{\mathbb{R}^2} \left(\frac{1}{2} H_z(x, z_n) \cdot z_n - H(x, z_n) \right) = 0,$$

and this contradicts the fact that $c > 0$, concluding the proof. ■

For $\Psi \in \mathcal{F}$, $\varepsilon > 0$ and $R > 0$, we set $D_\varepsilon(R) = \{x \in \mathbb{R}^2 : |\Psi(x)| \geq \varepsilon,\ |x| \geq R\}$. Fixed $\Psi \in \mathcal{F}$ and $\varepsilon > 0$, by definition of the class $\mathcal{F}$, a simple argument guarantees that

$$\mu(D_\varepsilon(R)) \to 0 \quad \text{as} \quad R \to \infty. \tag{1.35}$$

In fact, since $\Psi \in \mathcal{F}$, we have $\mu(D_\varepsilon(R)) \to 0 < \infty$ for all $\varepsilon > 0$. To prove the lemma, we have to verify that

$$\lim_{m\to\infty} \mu(D_\varepsilon \cap (\mathbb{R}^2 \backslash B_{R_m})) = 0$$

for each sequence $(R_m) \subset \mathbb{R}$ such that $R_m \to \infty$. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x) = \chi_{D_\varepsilon}(x)$, that is,

$$f(x) = \begin{cases} 1, & \text{for } x \in D_\varepsilon, \\ 0, & \text{for } x \notin D_\varepsilon. \end{cases}$$

Then $f \in L^1(\mathbb{R}^2)$ and $|f|_1 = \int_{\mathbb{R}^2} |f| = \mu(D_\varepsilon)$. Defining the sequence of functions $f_m : \mathbb{R}^2 \to \mathbb{R}$ by $f_m(x) = \chi_{D_\varepsilon \cap (\mathbb{R}^2 \backslash B_{R_m})}(x)$, it follows that $|f_m| \leq |f|$. Since $f_m \to 0$ almost everywhere in $\mathbb{R}^2$ as $m \to \infty$, our claim follows from Lebesgue's Dominated Convergence Theorem. (for details, see Lemma 2.6 of [48]).

By conditions (H_2) and $(H_7)(iii)$, given $\sigma > 0$ and $\alpha > 0$ there exist $C = C(\sigma, \alpha) > 0$ such

$$|H_{\infty,z}(x, z) - H_z(x, z)| \leq \sigma |z| + C|\varphi(x)|(e^{\alpha |z|^2} - 1) \quad \text{for all} \ \ (x, z) \in \mathbb{R}^2 \times \mathbb{R}^2. \tag{1.36}$$

The next two results are technical and the proofs can be done arguing along the same lines of [48, Lemmas 5.1 and 5.2], respectively.

Lemma 1.26. *Assume that (H_2), (H_7) and (V_1) are satisfied. Let $(z_n) \subset E$ be a bounded* *sequence and* $w_n(x) = w(x - y_n)$, *with* $w \in E$ *and* $(y_n) \subset \mathbb{R}^2$. *If* $|y_n| \to \infty$ *then*

$$\int_{\mathbb{R}^2} |[H_{\infty,z}(x,z_n) - H_z(x,z_n)] \cdot w_n| \to 0 \quad \text{and} \quad \int_{\mathbb{R}^2} |[V_\infty(x) - V(x)]z_n \cdot w_n| \to 0.$$

Proof. We know that $w \in L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$. Thus, given $\delta > 0$, we can find $0 < \varepsilon < \delta$ such that for each measurable subset $A \subset \mathbb{R}^2$ satisfying $\mu(A) < \varepsilon$, we have $\int_A |w|^2 < \delta$. Once $\varphi \in \mathcal{F}$, for this $\varepsilon > 0$ we define

$$D_\varepsilon(R) = \{x \in \mathbb{R}^2 : |\varphi(x)| \geq \varepsilon, \ |x| \geq R\}$$

and by virtue of (1.35) there exists $R_\varepsilon > 0$ such that $\mu(D_\varepsilon(R_\varepsilon)) < \varepsilon$. Therefore, $\int_{D_\varepsilon(R_\varepsilon)} |w|^2 < \delta$. Taking $\alpha > 0$ so that $2\alpha\|z_n\|^2 < 4\pi$ for all $n \in \mathbb{N}$ and $\sigma = 1$ in (1.36), by Lemma 1.12 we get

$$\begin{aligned}
&\int_{\mathbb{R}^2 \setminus B_{R_\varepsilon}(0)} |H_{\infty,z}(x,z_n) - H_z(x,z_n)||w_n| \\
&\quad \leq \int_{D_\varepsilon(R_\varepsilon)} |z_n||w_n| + C_1 \int_{(\mathbb{R}^2 \setminus B_{R_\varepsilon}(0)) \cap \{|\varphi(x)| < \varepsilon\}} |\varphi(x)|(e^{\alpha|z_n|^2} - 1)|w_n| \\
&\quad \leq \|z_n\|_2 \left(\int_{D_\varepsilon(R_\varepsilon) - y_n} |w|^2 \right)^{1/2} + \varepsilon C_1 \left(\int_{\mathbb{R}^2} (e^{2\alpha\|z_n\|^2 |z_n/\|z_n\||^2} - 1) \right)^{1/2} \left(\int_{\mathbb{R}^2} |w|^2 \right)^{1/2} \\
&\quad \leq C_2 \delta^{1/2} + C_3 \delta.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_{B_{R_\varepsilon}(0)} |H_{\infty,z}(x,z_n) - H_z(x,z_n)||w_n| \\
&\quad \leq \int_{B_{R_\varepsilon}(0)} |z_n||w_n| + C_1 \int_{B_{R_\varepsilon}(0)} |\varphi(x)|(e^{\alpha|z_n|^2} - 1)|w_n| \\
&\quad \leq \|z_n\|_2^2 \left(\int_{B_{R_\varepsilon}(-y_n)} |w|^2 \right)^{1/2} + C_1\|\varphi\|_\infty \left(\int_{\mathbb{R}^2} (e^{2\alpha|z_n|^2} - 1) \right)^{1/2} \left(\int_{B_{R_\varepsilon}(-y_n)} |w|^2 \right)^{1/2} \\
&\quad \leq C_4 \left(\int_{B_{R_\varepsilon}(-y_n)} |w|^2 \right)^{1/2}
\end{aligned}$$

and since $w \in L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ e $|y_n| \to \infty$, there exists $n_0 \in \mathbb{N}$ such that

$$\int_{B_{R_\varepsilon}(0)} |H_{\infty,z}(x,z_n) - H_z(x,z_n)||w_n| \leq \delta^{1/2}, \quad \text{for all } \ n \geq n_0.$$

Therefore, once $\delta > 0$ is arbitrary, the first convergence is proved. The second one is more simple and similar. Thus, the lemma is proved. ■

We finalize this subsection presenting the following result, which will be useful in the sequel.

Lemma 1.27. *Suppose $\Psi \in \mathcal{F}$ and $s \geq 2$. If $w_n \rightharpoonup w$ in E, then*

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} \Psi |w_n|^s = \int_{\mathbb{R}^2} \Psi |w|^s.$$

Proof. It is analogous to the proof of Lemma 2.6 of [48] and we omit it. ■

1.3.3 Conclusion of the proof of Theorem 1.2

Here, we complete the proof of the second main result of our paper. For this, we shall need of the next lemma.

Lemma 1.28. *Suppose that H satisfies (H_2) and (H_7). Then, the functional $\mathcal{J} : E \to \mathbb{R}$ given by*

$$\mathcal{J}(z) = \int_{\mathbb{R}^2} H(x, z)$$

is weakly sequentially lower semicontinuous and $\mathcal{J}'$ is weakly sequentially continuous.

Proof. Let (z_n) be in E such that $z_n \rightharpoonup z$ in E. Thus, (z_n) is bounded in E and by virtue of the continuous embedding $E \hookrightarrow L^t(\mathbb{R}^2) \times L^t(\mathbb{R}^2)$ for any $t \geq 2$, $(\|z_n\|_t)$ is also bounded and, up to a subsequence, $z_n \to z$ a.e. in $\mathbb{R}^2$. According to (H_7) and Lemma 1.13, we have $H(x, z) \geq H_\infty(x, z) \geq 0$ and therefore by Fatou's lemma we reach

$$\liminf_{n\to\infty} \mathcal{J}(z_n) = \liminf_{n\to\infty} \int_{\mathbb{R}^2} H(x, z_n) \geq \int_{\mathbb{R}^2} H(x, z) = \mathcal{J}(z)$$

and the first part is done. For the second part, once E is reflexive we must prove that if $z_n \rightharpoonup z$ in E then

$$\mathcal{J}'(z_n)w = \int_{\mathbb{R}^2} H_z(x, z_n) \cdot w \to \int_{\mathbb{R}^2} H_z(x, z) \cdot w = \mathcal{J}'(z)w, \quad \text{for each} \;\; w \in E. \tag{1.37}$$

Initially, from the proof of Claim 1 in the periodic case, we knows that (1.37) is valid for $w \in C_0^\infty(\mathbb{R}^2) \times C_0^\infty(\mathbb{R}^2)$.

Now, given $w \in E$, by the density of $C_0^\infty(\mathbb{R}^2) \times C_0^\infty(\mathbb{R}^2)$ in E, given $\delta > 0$ there exists $\eta \in C_0^\infty(\mathbb{R}^2) \times C_0^\infty(\mathbb{R}^2)$ such that $\|w - \eta\| < \delta$. By exploiting the same arguments used in (1.17), we can conclude that

$$\|H_z(x, z_n) - H_z(x, z)\|_2^2 \leq C, \quad \text{for all} \;\; n \in \mathbb{N},$$

for some constant $C > 0$. Thus,

$$\begin{aligned}\left|\int_{\mathbb{R}^2}[H_z(x, z_n) - H_z(x, z)] \cdot w\right| &\leq \|H_z(x, z_n) - H_z(x, z)\|_2\|w - \eta\|_2 \\ &\quad + \left|\int_{\mathbb{R}^2}[H_z(x, z_n) - H_z(x, z)] \cdot \eta\right| \\ &\leq C_1\|w - \eta\| + \left|\int_{\mathbb{R}^2}[H_z(x, z_n) - H_z(x, z)] \cdot \eta\right|\end{aligned}$$

and consequently

$$\limsup_{n\to\infty}\left|\int_{\mathbb{R}^2}[H_z(x, z_n) - H_z(x, z)] \cdot w\right| \leq C_1\delta,$$

which proves the lemma. ■

In view of Lemma 1.28 and Lemma 1.23, the conditions (L_1) and (L_2) of Theorem 1.9 are satisfied. Thus, by invoking Theorem 1.9, we obtain a $(Ce)_c$ sequence $(z_n) \subset E$ for I at level $c \geq \rho > 0$. By Lemma 1.24, up to a subsequence, $z_n \rightharpoonup z$ weakly in E. Now, for $\eta \in E$ we have

$$I'(z_n)\eta - I'(z)\eta = \langle z_n - z, \eta\rangle - \int_{\mathbb{R}^2}[H_z(x, z_n) - H_z(x, z)] \cdot \eta$$

and according to (1.37) we reach

$$I'(z)\eta = \lim_{n\to\infty}\left[I'(z_n)\eta - \langle z_n - z, \eta\rangle + \int_{\mathbb{R}^2}[H_z(x, z_n) - H_z(x, z)] \cdot \eta\right] = 0,$$

that is, z is a critical point of I. If $z \neq 0$, the proof is finished. Thus, suppose that $z = 0$. By Lemma 1.25, there exists a sequence $(y_n) \subset \mathbb{R}^2$, $R > 0$ and $\beta > 0$ such that $|y_n| \to \infty$ as $n \to \infty$ and

$$\limsup_{n\to\infty}\int_{B_R(y_n)}|z_n|^2 \geq \beta > 0. \tag{1.38}$$

Without lost of generality, we can suppose $(y_n) \subset \mathbb{Z}^2$. Setting $\tilde{z}_n(x) := z_n(x + y_n)$ and noticing that $\|\tilde{z}_n\| = \|z_n\|$, up to a subsequence, we have $\tilde{z}_n \rightharpoonup \tilde{z}$ weakly in E. By estimate (1.38), we have $\tilde{z} \neq 0$.

Next, we claim that $I'_\infty(\tilde{z}) = 0$. Indeed, for $w \in E$ let $w_n(x) = w(x - y_n)$. By using change of variable, the periodicity of H_∞ and arguing as above, we can see that

$$I'_\infty(z_n)w_n = I'_\infty(\tilde{z}_n)w = I'_\infty(\tilde{z})w + o_n(1).$$

On the other hand, by Lemma 1.26, we have

$$\begin{aligned} I'_\infty(z_n)w_n &= I'(z_n)w_n + \int_{\mathbb{R}^2} [V_\infty(x) - V(x)]z_n \cdot w_n - \int_{\mathbb{R}^2} [H_z(x, z_n) - H_{\infty,z}(x, z)] \cdot w_n \\ &= I'(z_n)w_n + o_n(1) \end{aligned}$$

and the claim follows from the fact that (z_n) is a Cerami sequence for I.

Defining

$$\widehat{H}_\infty(x, z) = \frac{1}{2} H_{\infty,z}(x, z) \cdot z - H_\infty(x, z),$$

in view of (1.36), given $\sigma > 0$ and $\alpha > 0$ so that $2\alpha\|z_n\|^2 < 4\pi$, we obtain

$$\begin{aligned} |\widehat{H}(x, z_n) - \widehat{H}_\infty(x, z_n)| &\le \frac{1}{2}|H_z(x, z_n) - H_{\infty,z}(x, z_n)||z_n| \\ &\quad + \int_0^1 |H_z(x, tz_n) - H_{\infty,z}(x, tz_n)||z_n| dt \\ &\le \frac{\sigma}{2}|z_n|^2 + \frac{C}{2}|\varphi(x)||z_n|(e^{\alpha|z_n|^2} - 1) + \frac{\sigma}{2}|z_n|^2 \\ &\quad + C\int_0^1 |\varphi(x)|(e^{\alpha t^2|z_n|^2} - 1)|z_n| dt \\ &\le \sigma|z_n|^2 + C_1|\varphi(x)||z_n|(e^{\alpha|z_n|^2} - 1). \end{aligned}$$

This estimate, Lemma 1.12 and Lemma 1.27 ensure that

$$\begin{aligned} \liminf_{n\to\infty} \int_{\mathbb{R}^2} \widehat{H}(x, z_n) &\ge \liminf_{n\to\infty} \left[\int_{\mathbb{R}^2} \widehat{H}_\infty(x, z_n) - \sigma \int_{\mathbb{R}^2} |z_n|^2 - C_1 \int_{\mathbb{R}^2} |\varphi(x)||z_n|(e^{\alpha|z_n|^2} - 1) \right] \\ &\ge \liminf_{n\to\infty} \left[\int_{\mathbb{R}^2} \widehat{H}_\infty(x, z_n) - \sigma C \right. \\ &\qquad \left. - C_1 \left(\int_{\mathbb{R}^2} |\varphi(x)||z_n|^2 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} (e^{2\alpha|z_n|^2} - 1) \right)^{\frac{1}{2}} \right] \\ &\ge \liminf_{n\to\infty} \int_{\mathbb{R}^2} \widehat{H}_\infty(x, z_n) - \sigma C \end{aligned}$$

and consequently the arbitrarity of $\sigma > 0$ and Fatou's lemma show that

$$\liminf_{n\to\infty} \int_{\mathbb{R}^2} \widehat{H}(x, z_n) \geq \int_{\mathbb{R}^2} \widehat{H}_\infty(x, z).$$

Thus,

$$\begin{aligned} c_* = \lim_{n\to\infty} \left[I(z_n) - \frac{1}{2} I'(z_n) z_n \right] &= \liminf_{n\to\infty} \int_{\mathbb{R}^2} \widehat{H}(x, z_n) \\ &\geq \int_{\mathbb{R}^2} \widehat{H}_\infty(x, \tilde{z}) = I_\infty(\tilde{z}) - I'_\infty(\tilde{z})\tilde{z} = I_\infty(\tilde{z}). \end{aligned}$$

By the definition of c_*, (V_1), $(H_7)(ii)$ and (1.21), we obtain

$$c_* \leq \sup_{z\in M_{R,z_0}} I(z) \leq \sup_{z\in M_{R,z_0}} I_\infty(z) \leq I_\infty(z_0) \leq I_\infty(\tilde{z}) \leq c_*.$$

Hence, if we define $h_0 : [0, 1] \times M_{R,z_0} \to E$ by $h_0(t, z) = z$ for any $(t, z) \in [0, 1] \times M_{R,z_0}$, the above inequality implies that

$$\sup_{z\in M_{R,z_0}} I(h_0(1, z)) = c_* > 0.$$

It follows from Theorem 1.10 that I has a nonzero critical point and the proof is finished.

Chapter 2

Hamiltonian systems involving critical exponential growth in $\mathbb{R}^2$ with general nonlinearities

This chapter is a continuation of the work on *Chapter 1*. We study the existence of ground state solution for the same class of Hamiltonian systems:

$$\begin{cases} -\Delta u + V(x)u = H_v(x,u,v), & x \in \mathbb{R}^2, \\ -\Delta v + V(x)v = H_u(x,u,v), & x \in \mathbb{R}^2, \end{cases} \tag{1}$$

where $V \in C(\mathbb{R}^2, (0,\infty))$, $H \in C^1(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R})$. Here, our intention is to deal with system (1) when V and H are periodic in $x = (x_1, x_2)$. Moreover, H is allowed to have a critical exponential growth and depends on u and v simultaneously.

Our main assumption on V is the following:

(V_0) $V(x) = V(x_1, x_2)$ is positive and 1-periodic in the variables x_1, x_2.

With respect to the function H, we assume the following conditions:

(H_0) $H \in C^1(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R})$ and $H(x_1, x_2, z)$ is 1-periodic in the variables x_1, x_2;

(H_1) (Critical exponential growth) there exists $\alpha_0 > 0$ such that

$$\lim_{|z|\to\infty} \frac{|H_z(x,z)|}{e^{\alpha|z|^2}} = \begin{cases} 0, & \text{if } \alpha > \alpha_0, \\ +\infty, & \text{if } \alpha < \alpha_0, \end{cases}$$

uniformly in $x \in \mathbb{R}^2$;

(H_2) $H_z(x,z) = o(|z|)$ as $|z| \to 0$, uniformly in $x \in \mathbb{R}^2$;

(H_3) there exists $g : \mathbb{R}^2 \times \mathbb{R}_+ \to [0,\infty)$ increasing in the second variable such that

$$H_z(x,z) = g(x,|z|)z, \quad \text{for each} \quad (x,z) \in \mathbb{R}^2 \times \mathbb{R}^2;$$

(H_4) there exists $R_0, M_0 > 0$ such that

$$0 < H(x,z) \leq M_0 |H_z(x,z)|, \quad \text{for all} \quad x \in \mathbb{R}^2 \quad \text{and} \quad |z| \geq R_0;$$

(H_5) $\displaystyle\limsup_{|z|\to\infty} \frac{|z||H_z(x,z)|}{\widehat{H}(x,z)} =: \beta < +\infty$, uniformly in $x \in \mathbb{R}^2$, where

$$\widehat{H}(x,z) := \frac{1}{2} H_z(x,z) \cdot z - H(x,z), \quad \text{for} \quad (x,z) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

Remark 2.1. *We observe that we do not require the well-known Ambrosetti-Rabinowitz condition:*

(AR) *there exists $\theta > 2$ such that, for each $x \in \mathbb{R}^2$ and $z \in \mathbb{R}^2 \backslash \{0\}$, there holds*

$$0 < \theta H(x,z) \leq H_z(x,z) \cdot z,$$

which is often used in Hamiltonian systems involving critical exponential growth.

Next, let us consider the Banach space $E := H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$. We say a pair $(u,v) \in E$ is a weak solution of system (1) if the equality

$$\int_{\mathbb{R}^2} (\nabla u \cdot \nabla \varphi + V(x) u \varphi) + \int_{\mathbb{R}^2} (\nabla v \cdot \nabla \psi + V(x) v \psi) = \int_{\mathbb{R}^2} (H_v(x,u,v)\varphi + H_u(x,u,v)\psi)$$

is valid for all $(\varphi, \psi) \in E$. The energy functional associated to (1) is given by

$$I(z) = \int_{\mathbb{R}^2} (\nabla u \cdot \nabla v + V(x) uv) dx - \int_{\mathbb{R}^2} H(x,z) dx, \quad z = (u,v) \in E.$$

As we will see in Section 2.1, this functional is of class C^1 and critical points of I corresponds to weak solutions of Problem (1).

We say that $z_0 \in E$ is a ground state solution of (1) if

$$I(z_0) = \inf\{I(z) : z \in E\backslash\{0\} \text{ is a weak solution of (1)}\}.$$

Denoting $\|V\|_\infty = \max_{x\in\mathbb{R}^2} V(x)$, the main result of this article is stated as follows:

Theorem 2.2. *Suppose that conditions* (V_0) *and* $(H_0) - (H_5)$ *are satisfied. In addition, we assume that there exists* $p > 2$ *such that*

$$H(x, z) \geq \lambda_0 |z|^p, \quad \textit{for all} \ \ (x, z) \in \mathbb{R}^2 \times \mathbb{R}^2, \tag{2.1}$$

where

$$\lambda_0 \geq \frac{8(p-2)^{\frac{p-2}{2}}(\beta\alpha_0)^{\frac{p-2}{2}}(4 + \|V\|_\infty)^{\frac{p}{2}}}{p^{\frac{p}{2}}} \ \ \textit{if} \ \ \beta > 0 \quad \textit{and} \quad \lambda_0 > 0 \ \ \textit{if} \ \ \beta = 0. \tag{2.2}$$

Then, system (1) *has a ground state solution.*

We observe that as a consequence of condition (2.1), we have

$$\lim_{|z|\to\infty} \frac{H(x, z)}{|z|^2} = +\infty, \quad \text{uniformly in} \ \ x \in \mathbb{R}^2. \tag{2.3}$$

As said before, an obstacle in studying Hamiltonian elliptic systems like (1) in dimension two through variational methods is dealing with the lack of compactness, stemming from the unboundedness of the domain and the noncompactness of the Trudinger–Moser functional. Furthermore, here the Nehari manifold is not of class C^1 and we cannot apply the Ekeland Variational Principle. To overcome this difficulty we used arguments introduced by Szulkin and Weth in [59, 60] (see also [32, 33, 52, 57]), which relates this minimizing process on the Nehari manifold with a similar one on a C^1-manifold (see Lemma 1.6). As we are not using the Ambrosetti-Rabinowitz condition and by imposing a critical exponential growth on a general nonlinearity, our argument requires a delicate analysis of the Palais–Smale sequences that converge to the infimum of the functional on the Nehari manifold.

Example 2.3. *It is not difficult to check that the function* $H : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ *given by*

$$H(x, z) = |z|^3(e^{|z|^2} - 1) + \lambda_0 |z|^3,$$

with λ_0 fulfilling (2.2), *satisfies conditions* $(H_0)-(H_5)$ *and* (2.1).

In the forthcoming section, our objective is to apply Lemma 1.6, so we restate some definitions and results, and prove the ones that need some modification due to the critical exponential growth. In Section 2.2, we prove our main result.

2.1 Preliminaries

In this section we present some results that will be used to prove Theorem 2.2. In order to exploit Lemma 1.6, we consider the following subspaces of E:

$$E^+ := \{(u,u) : u \in H^1(\mathbb{R}^2)\} \quad \text{and} \quad E^- := \{(u,-u) : u \in H^1(\mathbb{R}^2)\}.$$

Note that E^+ and E^- are orthogonal in E and for $z=(u,v)\in E$, if

$$z^+ := \left(\frac{u+v}{2}, \frac{u+v}{2}\right) \quad \text{and} \quad z^- := \left(\frac{u-v}{2}, \frac{v-u}{2}\right),$$

then $z^\pm \in E^\pm$ and we have $z = z^+ + z^-$. Thus, $E = E^+ \oplus E^-$ and a simple computation shows that

$$\int_{\mathbb{R}^2} (\nabla u \cdot \nabla v + V(x)uv) = \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2). \tag{2.4}$$

By using (H_1) and (H_2) we can see that, given $\varepsilon>0$, $\alpha>\alpha_0$ and $q\geq 1$, there exists $C=C(\varepsilon,\alpha,q)>0$ such that

$$\max\{|H(x,z)|, |H_z(x,z)||z|\} \leq \varepsilon|z|^2 + C|z|^q(e^{\alpha|z|^2}-1), \quad \text{for all } (x,z)\in\mathbb{R}^2\times\mathbb{R}^2. \tag{2.5}$$

In view of (2.5) and Lemma 1.12, the energy functional associated to (1), given by

$$I(z) = \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2) - \mathcal{J}(z), \tag{2.6}$$

where $\mathcal{J}(z) := \int_{\mathbb{R}^2} H(x,z)$, is well defined. Furthermore, it is not difficult to show that $I\in C^1(E,\mathbb{R})$ with

$$I'(z)w = \langle z^+, w^+\rangle - \langle z^-, w^-\rangle - \int_{\mathbb{R}^2} H_z(x,z)\cdot w, \quad \text{for } z,w\in E.$$

Hence, the critical points of I correspond to weak solutions of Problem (1).

In order to apply Lemma 1.6, we need to show that I satisfies conditions $(N_1)-(N_3)$. Some of these lemmas are already in *Chapter 1*, we restate them and prove the ones where the critical growth hypothesis is used.

Lemma 2.4. *Assuming conditions $(H_1)-(H_3)$, for each $z \neq 0$ we have*

$$\frac{1}{2}H_z(x,z)\cdot z > H(x,z) > 0.$$

Furthermore, $\mathcal{J}(0)=0$ and $\mathcal{J}$ is weakly lower semicontinuous.

By the previous lemma, condition (N_1) is satisfied. The next result is necessary in order to show condition (N_2).

Lemma 2.5. *Suppose that H satisfies $(H_1)-(H_3)$. If $z \in \mathcal{N}$ then for any $w \neq 0$ such that $z+w \in \widehat{E}(z)$, we have $I(z+w) < I(z)$.*

Lemma 2.6. *Assume conditions $(H_1)-(H_3)$ and (2.1). For each $z \in E\backslash E^-$, there exists $R=R(z)>0$ such that $I(w) \leq 0$ for all $w \in \widehat{E}(z)\backslash \mathcal{B}_R(0)$, where $\mathcal{B}_R(0) := \{z \in E : \|z\| < R\}$.*

The next lemma guarantees that condition (N_2) is valid.

Lemma 2.7. *Suppose that H satisfies $(H_1)-(H_3)$ and (2.1).*

(i) for any $z \in \mathcal{N}$, $I|_{\widehat{E}(z)}$ admits a unique maximum point which is precisely at z.

(ii) for any $z \in E\backslash E^-$, the set $\widehat{E}(z)$ intersects $\mathcal{N}$ at exactly one point $\widehat{m}(z)$, which is the unique global maximum point of $I|_{\widehat{E}(z)}$.

Proof. Firstly we will show (i). Given $tz+y \in \widehat{E}(z)\backslash\{z\}$, it is enough to consider $w=(t-1)z+y$ to obtain $tz+y=z+w$. Note that, if $w=0$ then $t=1$ and $y=0$, but this can not occur because $tz+y\neq z$. Hence, $w \neq 0$ and by Lemma 1.16 we conclude that $I(tz+y) < I(z)$.

To prove (ii), by the previous item, it is sufficient to show that $\mathcal{N}\cap \widehat{E}(z) \neq \emptyset$ for each $z \in E\backslash E^-$. Moreover, since $\widehat{E}(z) = \widehat{E}\left(z^+/\|z^+\|\right)$, we can assume $z=(u,u) \in E^+$ and $\|z\|=1$. By Lemma 1.15, there exists $R>0$ such that $I(w) \leq 0$ if $w \in \widehat{E}(z)\backslash\mathcal{B}_R(0)$ and if $\|w\| \leq R$ then $I(w) \leq R^2/2$ and so $\sup_{\widehat{E}(z)} I < \infty$. On the other hand, by using (1.8), for $\varepsilon > 0$ and $\alpha' > \alpha_0$ there exists $C_\varepsilon > 0$ such that

$$H(x,z) \leq \varepsilon |z|^2 + C_\varepsilon |z|^3 (e^{\alpha' |z|^2} - 1), \quad z \in \mathbb{R}^2. \tag{2.7}$$

From (1.12), Lemma 1.12 and by choosing $0 < \varepsilon < 1/(4\int_{\mathbb{R}^2} |z|^2)$, we have

$$\begin{aligned} I(tz) = \frac{t^2}{2}\|z\|^2 - \int_{\mathbb{R}^2} H(x,tz) &\geq \frac{t^2}{2} - \varepsilon t^2 \int_{\mathbb{R}^2} |z|^2 - C_\varepsilon t^3 \int_{\mathbb{R}^2} |z|^3 (e^{\alpha' t^2 |z|^2} - 1) \\ &\geq \frac{t^2}{4} - Ct^3 \left(\int_{\mathbb{R}^2} |z|^6 \right)^{1/2} \left[\int_{\mathbb{R}^2} (e^{2\alpha' t^2 |z|^2} - 1) \right]^{1/2} > 0, \end{aligned}$$

for all $0 < t < \sqrt{2\pi/\alpha'}$ sufficiently small. Consequently, $\sup_{\widehat{E}(z)} I > 0$.

Now, let $w_n = (w_n^1, w_n^2) = t_n z + (h_n, -h_n) \in \widehat{E}(z)$ be a maximizing sequence for $\sup_{\widehat{E}(z)} I$. Since $\sup_{\widehat{E}(z)} I > 0$, we can suppose that $\|w_n\| \leq R$. Hence, up to a subsequence, $w_n \rightharpoonup w_0$ weakly in E. We can see, up to a subsequence, that $t_n \to t_0 \geq 0$ and $h_n \rightharpoonup h_0$, $w_n^1 \rightharpoonup w_0^1$, $w_n^2 \rightharpoonup w_0^2$ weakly in $H^1(\mathbb{R}^2)$. Therefore, $w_0 = (w_0^1, w_0^2) = t_0 z + (h_0, -h_0) \in \widehat{E}(z)$. Recalling that $z \in E^+$ and $\|z\| = 1$, we can write

$$I(w_n) = \frac{t_n^2}{2} - \frac{1}{2}\|h_n\|^2 - \int_{\mathbb{R}^2} H(x, w_n^1, w_n^2),$$

from where it follows, according to the weak lower semicontinuous of the norm and Fatou's lemma, that

$$\sup_{\widehat{E}(z)} I = \lim_{n\to\infty} I(w_n) = \frac{t_0^2}{2} + \limsup_{n\to\infty} \left[-\frac{1}{2}\|h_n\|^2 - \int_{\mathbb{R}^2} H(x, w_n^1, w_n^2) \right] \leq I(w_0).$$

Thus, $I(w_0) = \sup_{\widehat{E}(z)} I$ and therefore w_0 is a critical point of $I|_{\widehat{E}(z)}$, showing that $w_0 \in \mathcal{N} \cap \widehat{E}(z)$ and the proof is complete. ■

As an immediate consequence of this lemma, we obtain the following characterization to the minimal level for I on the generalized Nehari manifold $\mathcal{N}$:

Corollary 2.8.

$$c_* := \inf_{\eta \in \mathcal{N}} I(\eta) = \inf_{z \in E \setminus E^-} \max_{w \in \widehat{E}(z)} I(w).$$

Lemma 2.9. *Suppose that H satisfies $(H_1)-(H_3)$ and (2.1). Then, $c_* > 0$ and*

$$\|z^+\| \geq \max\{\|z^-\|, \sqrt{2c_*}\} \quad \textit{for all} \quad z \in \mathcal{N}.$$

Proof. For $a > 0$, we recall that $S_a^+ := \{z \in E^+ : \|z\| = a\}$ and $(\mathbb{R}_+)z = \{tz : t \geq 0\}$. Since $\widehat{E}(z) = \widehat{E}(z^+)$ for any $z \in E \backslash E^-$, from Corollary 2.8, for any $a > 0$, it follows that

$$c_* = \inf_{z \in E \backslash E^-} \max_{w \in \widehat{E}(z)} I(w) = \inf_{z \in E^+ \backslash \{0\}} \max_{w \in \widehat{E}(z)} I(w) = \inf_{z \in S_a^+} \max_{w \in \widehat{E}(z)} I(w) \geq \inf_{z \in S_a^+} \max_{w \in (\mathbb{R}_+)z} I(w).$$

Let $z = (u, u)$ be in S_a^+ and $\alpha > \alpha_0$. We take $a > 0$ so that $2\alpha a^2 < 4\pi$. By virtue of Lemma 1.12, we reach

$$\int_{\mathbb{R}^2} (e^{2\alpha|z|^2} - 1) = \int_{\mathbb{R}^2} (e^{2\alpha a^2(|z|/\|z\|)^2} - 1) \leq C,$$

for some $C > 0$. Thus, by using (1.12) we get

$$\begin{aligned}
\max_{w \in (\mathbb{R}_+)z} I(w) \geq I(z) = \|u\|_V^2 - \int_{\mathbb{R}^2} H(x, u, u) &\geq \|u\|_V^2 - \varepsilon \int_{\mathbb{R}^2} |z|^2 - C_\varepsilon \int_{\mathbb{R}^2} |z|^3 (e^{\alpha|z|^2} - 1) \\
&\geq \left(1 - \frac{2\varepsilon}{a_0}\right) \|u\|_V^2 - C_\varepsilon \left(\int_{\mathbb{R}^2} |z|^6\right)^{1/2} \\
&\geq \left(1 - \frac{2\varepsilon}{a_0}\right) \|u\|_V^2 - C_1 \|u\|_V^3,
\end{aligned}$$

where we have used the continuous embedding $H^1(\mathbb{R}^2) \hookrightarrow L^6(\mathbb{R}^2)$. Hence, taking $\varepsilon = a_0/4$ and $0 < a < \sqrt{2\pi/\alpha}$ sufficiently small so that $1/2 - C_1\|u\|_V = 1/2 - C_1 a/\sqrt{2} \geq 1/4$, we conclude that

$$\max_{w \in (\mathbb{R}_+)z} I(w) \geq \|u\|_V^2 \left(\frac{1}{2} - C_1 \|u\|_V\right) \geq \frac{a^2}{8} > 0, \quad \text{for all} \quad z = (u, u) \in S_a^+$$

and consequently $c_* > 0$. Now, for any $z \in \mathcal{N}$ we have

$$c_* \leq \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2) - \int_{\mathbb{R}^2} H(x, z) \leq \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2),$$

which implies that $\|z^+\| \geq \max\{\|z^-\|, \sqrt{2c_*}\}$ and the proof is done. ■

Condition (N_3) can be now proved in the following lemma:

Lemma 2.10. *Suppose that H satisfies $(H_1)-(H_3)$ and (2.1). If $\mathcal{W} \subset E \backslash E^-$ is compact,* *then there exists $C_{\mathcal{W}} > 0$ such that $\|\widehat{m}(z)\| \leq C_{\mathcal{W}}$, for all $z \in \mathcal{W}$.*

2.2 Proof of Theorem 2.2

The generalized manifold $\mathcal{N}$ is not necessarily of class C^1 and thus the Ekeland Variational Principle can not be applied directly to obtain a Palais-Smale sequence for I on $\mathcal{N}$. However, in view of item (iv) of Lemma 1.6 and Lemma 2.9, one has $\inf_{S^+} \Psi = \inf_{\mathcal{N}} I = c_* > 0$. Since S^+ is a submanifold of class C^1 of E^+, it follows from the Ekeland Variational Principle that there exists $(w_n) \subset S^+$ such that

$$\Psi(w_n) \to c_* \quad \text{and} \quad \|\Psi'(w_n)\|_* \to 0 \quad \text{as} \quad n \to \infty.$$

Let $z_n = m(w_n) \in \mathcal{N}$. By items (ii)-(iv) of Lemma 1.6, we reach

$$I(z_n) \to c_* \quad \text{and} \quad \|I'(z_n)\| \to 0 \quad \text{as} \quad n \to \infty. \tag{2.8}$$

In the next lemma, we shall guarantee that the above sequence $(z_n) \subset \mathcal{N}$ is bounded. For this, we shall use the following inequality, whose proof can be found in [28, Lemma 2.4]:

$$st \leq \begin{cases} (e^{t^2} - 1) + |s|(\log|s|)^{1/2}, & t \in \mathbb{R} \text{ and } |s| \geq e^{1/4}; \\ (e^{t^2} - 1) + \dfrac{1}{2}s^2, & t \in \mathbb{R} \text{ and } |s| \leq e^{1/4}. \end{cases} \tag{1.27}$$

Lemma 2.11. *Suppose that $(H_0) - (H_5)$ are satisfied. Any Palais-Smale sequence $(z_n) \subset \mathcal{N}$ for I is bounded.*

Proof. Let $(z_n) \subset \mathcal{N}$ be a Palais-Smale sequence for I at level c. Hence,

$$c + o_n(1) = I(z_n) - \frac{1}{2}I'(z_n)z_n = \int_{\mathbb{R}^2} \widehat{H}(x, z_n). \tag{2.9}$$

Suppose by contradiction that, up to a subsequence, $\|z_n\| \to \infty$. Thus,

$$o_n(1) = \frac{I'(z_n)(z_n^+ - z_n^-)}{\|z_n\|^2} = 1 - \int_{\mathbb{R}^2} \frac{H_z(x, z_n) \cdot (z_n^+ - z_n^-)}{\|z_n\|^2}.$$

Setting $w_n := z_n/\|z_n\|$, it follows that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}\frac{H_z(x,z_n)\cdot(w_n^+-w_n^-)}{\|z_n\|}=1. \tag{2.10}$$

In what follows, we claim that the function $Q:(0,\infty)\to\mathbb{R}$ defined by

$$Q(r)=\inf_{x\in\mathbb{R}^2,\,|z|\geq r}\widehat{H}(x,z)$$

has the following properties:

$$Q(r)>0 \ \text{ for all } \ r>0 \quad \text{and} \quad Q(r)\to+\infty \ \text{ as } \ r\to\infty. \tag{2.11}$$

Indeed, by Lemma 2.4 we have $\widehat{H}(x,z)>0$ for all $x\in\mathbb{R}^2$ and $z\neq 0$. Moreover, by using (H_1), with $0<\alpha'<\alpha_0$, and (H_5) we can deduce that $\widehat{H}(x,z)\to+\infty$ as $|z|\to\infty$, uniformly in $x\in\mathbb{R}^2$. Therefore, by the definition of Q, one has $Q(r)\to+\infty$ as $r\to\infty$. From this fact and by the periodicity of $\widehat{H}(x,z)$ in x, for each $r>0$ there exists $M_r>r$ such that

$$Q(r)=\inf_{x\in\mathbb{R}^2,\,|z|\geq r}\widehat{H}(x,z)=\min_{x\in[0,1]\times[0,1],\,r\leq|z|\leq M_r}\widehat{H}(x,z)=\widehat{H}(x_0,z_0)>0$$

and the claim is proved.

For $0\leq a<b\leq\infty$ and $n\in\mathbb{N}$, we set

$$\Omega_n(a,b):=\{x\in\mathbb{R}^2 : a\leq|z_n(x)|<b\}.$$

Using (2.9) and for $0<r<R<\infty$, we obtain

$$\begin{aligned} c+o_n(1) &= \int_{\Omega_n(0,r)}\widehat{H}(x,z_n)+\int_{\Omega_n(r,R)}\frac{\widehat{H}(x,z_n)}{|z_n|^2}|z_n|^2+\int_{\Omega_n(R,\infty)}\widehat{H}(x,z_n)\\ &\geq \int_{\Omega_n(0,r)}\widehat{H}(x,z_n)+\frac{Q(r)}{R^2}\int_{\Omega_n(r,R)}|z_n|^2+Q(R)|\Omega_n(R,\infty)| \end{aligned}$$

and therefore there exists $C_1>0$ such that

$$\max\left\{\int_{\Omega_n(0,r)}\widehat{H}(x,z_n),\ \frac{Q(r)}{R^2}\int_{\Omega_n(r,R)}|z_n|^2,\ Q(R)|\Omega_n(R,\infty)|\right\}\leq C_1. \tag{2.12}$$

In particular,

$$|\Omega_n(R,\infty)| \leq C_1/Q(R) \quad \text{for each} \quad n \in \mathbb{N}. \tag{2.13}$$

Now, let $C_3 > 0$ be such that $\|z\|_2^2 \leq C_3\|z\|^2$ for each $z \in E$ and consider $\varepsilon > 0$. By (H_2), there exists $r_\varepsilon > 0$ such that $|H_z(x,z)| \leq \varepsilon|z|/C_3$ for each $|z| \leq r_\varepsilon$ and $x \in \mathbb{R}^2$. By the definition of w_n and since $|w_n^+ - w_n^-| = |w_n|$, from (2.12) with $r = r_\varepsilon$, for any $n \in \mathbb{N}$ we get

$$\begin{aligned}\int_{\Omega_n(0,r_\varepsilon)} \frac{H_z(x,z_n)\cdot(w_n^+ - w_n^-)}{\|z_n\|} &\leq \int_{\Omega_n(0,r_\varepsilon)} \frac{|H_z(x,z_n)|}{|z_n|}|w_n^+ - w_n^-||w_n| \\ &\leq \frac{\varepsilon}{C_3}\int_{\Omega_n(0,r_\varepsilon)} |w_n|^2 \leq \varepsilon\|w_n\|^2 = \varepsilon.\end{aligned} \tag{2.14}$$

Now, let $R = R_\varepsilon > r_\varepsilon$ to be chosen later. Define $\mathcal{A}_1 = \Omega_n(R_\varepsilon,\infty) \cap \{x \in \mathbb{R}^2 : |H_z(x,z_n)| \leq e^{1/4}\}$ and $\mathcal{A}_2 = \Omega_n(R_\varepsilon,\infty) \cap \{x \in \mathbb{R}^2 : |H_z(x,z_n)| \geq e^{1/4}\}$. Taking $t = |w_n|$ and $s = |H_z(x,z_n)|$ in (1.27), we get

$$\begin{aligned}\int_{\Omega_n(R_\varepsilon,\infty)} \frac{H_z(x,z_n)\cdot(w_n^+ - w_n^-)}{\|z_n\|} &\leq \frac{1}{\|z_n\|}\int_{\Omega_n(R_\varepsilon,\infty)} |H_z(x,z_n)||w_n| \\ &\leq \frac{1}{\|z_n\|}\int_{\mathcal{A}_2} |H_z(x,z_n)|(\log|H_z(x,z_n)|)^{1/2} \\ &\quad + \frac{1}{\|z_n\|}\int_{\mathcal{A}_1} \frac{1}{2}|H_z(x,z_n)|^2 + \frac{2}{\|z_n\|}\int_{\mathbb{R}^2}(e^{|w_n|^2} - 1).\end{aligned} \tag{2.15}$$

By (2.13), we have $|\mathcal{A}_1| \leq |\Omega_n(R_\varepsilon,\infty)| \leq C_1/Q(R_\varepsilon)$ and consequently

$$\int_{\mathcal{A}_1} |H_z(x,z_n)|^2 \leq \frac{C_1 e^{1/2}}{Q(R_\varepsilon)}, \quad \text{for all} \quad n \in \mathbb{N}.$$

By Lemma 1.12, the integral $\int_{\mathbb{R}^2}(e^{|w_n|^2} - 1)$ is bounded. Moreover, given $\alpha > \alpha_0$ by (H_1) there exists $R_1 > 0$ such that

$$|H_z(x,z)| \leq e^{\alpha|z|^2} \quad \text{for all} \quad x \in \mathbb{R}^2 \quad \text{and} \quad |z| \geq R_1.$$

In view of (H_5), there exists $c_1 > 0$ such that

$$|H_z(x,z)||z| \leq c_1\widehat{H}(x,z) \quad \text{for all} \quad x \in \mathbb{R}^2 \quad \text{and} \quad |z| \geq R_1$$

Taking $R_\varepsilon > R_1$, from the two last estimates and (2.9) it follows that

$$\int_{\mathcal{A}_2} |H_z(x,z_n)|(\log|H_z(x,z_n)|)^{1/2} \leq \alpha^{1/2}\int_{\mathcal{A}_2} |H_z(x,z_n)||z_n| \leq \alpha^{1/2}c_1\int_{\mathbb{R}^2} \widehat{H}(x,z_n) \leq C,$$

for some constant $C > 0$. Thus, in view of (2.15), we find $n_0 \in \mathbb{N}$ satisfying

$$\int_{\Omega_n(R_\varepsilon,\infty)} \frac{H_z(x,z_n)\cdot(w_n^+ - w_n^-)}{\|z_n\|} < \varepsilon, \quad \text{for all } \ n \geq n_0. \tag{2.16}$$

According to (H_1) and (H_2), given $\alpha > \alpha_0$ we obtain $C_\alpha > 0$ such that, for all $x \in \Omega_n(r_\varepsilon, R_\varepsilon)$, we have

$$|H_z(x,z_n)| \leq |z_n| + C_\alpha(e^{\alpha R_\varepsilon^2} - 1)|z_n|$$

From this estimate and (2.12), recalling the definition of w_n and $|w_n^+ - w_n^-| = |w_n|$, we reach

$$\begin{aligned}\int_{\Omega_n(r_\varepsilon,R_\varepsilon)} \frac{H_z(x,z_n)\cdot(w_n^+ - w_n^-)}{\|z_n\|} &\leq C_2\int_{\Omega_n(r_\varepsilon,R_\varepsilon)} \frac{|z_n||w_n^+ - w_n^-|}{\|z_n\|} \\ &\leq \frac{C_2}{\|z_n\|^2}\int_{\Omega_n(r_\varepsilon,R_\varepsilon)} |z_n|^2 \\ &\leq \frac{C_3}{\|z_n\|^2}\frac{R_\varepsilon^2}{Q(r_\varepsilon)} < \varepsilon, \quad \text{for all } \ n \geq n_1,\end{aligned}$$

for some $n_1 \in \mathbb{N}$. This estimate, (2.14) and (2.16) show that

$$\int_{\mathbb{R}^2} \frac{H_z(x,z_n)\cdot(w_n^+ - w_n^-)}{\|z_n\|}dx \leq 3\varepsilon, \quad \text{for all } \ n \geq \max\{n_0, n_1\}.$$

But this contradicts (2.10) because $\varepsilon > 0$ is arbitrary and the proof is done.

■

Once the sequence $(z_n) \subset \mathcal{N}$ satisfying (2.8) is bounded, it follows that there exists $z_0 = (u_0, v_0) \in E$ such that, up to a subsequence, $z_n \rightharpoonup z_0$ in E and $z_n \to z_0$ a.e. in $\mathbb{R}^2$ as $n \to \infty$. Our intention is to prove that z_0 is a nonzero critical point of I and to conclude that z_0 is a ground state solution. This will be done in the next propositions and lemmas.

Proposition 2.12. *The weak limit z_0 of the sequence $(z_n) \subset \mathcal{N}$ is a critical point of I.*

Proof. By the density of $C_0^\infty(\mathbb{R}^2) \times C_0^\infty(\mathbb{R}^2)$ in E, it is enough just to deduce that

$I'(z_0)\eta = 0$ for all $\eta \in C_0^\infty(\mathbb{R}^2) \times C_0^\infty(\mathbb{R}^2)$. Then, let $\eta \in C_0^\infty(\mathbb{R}^2) \times C_0^\infty(\mathbb{R}^2)$ and $K := \operatorname{supp} \eta$. By (H_5), there exist $M > 0$ and $c_1 > 0$ such that

$$|H_z(x, z_n)||z_n| \leq c_1 \widehat{H}(x, z_n), \quad \text{for all} \quad |z| \geq M.$$

Defining $\Omega_1 = K \cap \{x;\ |z_n(x)| \leq M\}$ and $\Omega_2 = K \cap \{x;\ |z_n(x)| > M\}$, by (H_1) and (H_2), given $\alpha > \alpha_0$ we obtain $C > 0$ such that

$$|H_z(x, w)| \leq |w| + C(e^{\alpha|w|^2} - 1) \quad \text{for all} \quad (x, w) \in \mathbb{R}^2 \times \mathbb{R}^2$$

and hence

$$\begin{aligned} |H_z(x, z_n(x)) \cdot \eta(x)| &\leq \|\eta\|_\infty |z_n(x)| + C_1 \|\eta\|_\infty (e^{\alpha|z_n(x)|^2} - 1) \\ &\leq \|\eta\|_\infty [M + C_1(e^{\alpha M^2} - 1)] \quad \text{a.e. in} \quad x \in \Omega_1. \end{aligned}$$

Since $H_z(x, z_n) \cdot \eta \to H_z(x, z_0) \cdot \eta$ a.e. in $x \in \Omega_1$, it follows from the Lebesgue Dominated Convergence Theorem that

$$\int_{\Omega_1} [H_z(x, z_n) - H_z(x, z_0)] \cdot \eta \to 0 \tag{2.17}$$

Now, observe that by (H_5), (2.9) and Hölder's inequality, we reach

$$\begin{aligned} &\left| \int_{\Omega_2} [H_z(x, z_n) - H_z(x, z_0)] \cdot \eta \right| \\ &\quad \leq \frac{\|\eta\|_\infty}{M} \int_{\Omega_2} |H_z(x, z_n) - H_z(x, z_0)||z_n| \\ &\quad \leq \frac{\|\eta\|_\infty}{M} \int_{\Omega_2} [|H_z(x, z_n)||z_n| + |H_z(x, z_0)||z_n|] \\ &\quad \leq \frac{\|\eta\|_\infty}{M} \left[c_1 \int_{\mathbb{R}^2} \widehat{H}(x, z_n) + C_1 \left(\int_{\mathbb{R}^2} [e^{2\alpha|z_0|^2} - 1] \right)^{\frac{1}{2}} \|z_n\|_2 \right] \\ &\quad \leq \frac{\|\eta\|_\infty C}{M}, \end{aligned} \tag{2.18}$$

for some $C > 0$ independent of M. Hence, given $\delta > 0$ we can take $M > 0$ so that

$$\left| \int_{\Omega_2} [H_z(x, z_n) - H_z(x, z_0)] \cdot \eta \right| < \delta, \quad \text{for all} \quad n \in \mathbb{N}. \tag{2.19}$$

Therefore, by (2.17) and (2.19) we obtain

$$\limsup_{n\to\infty}\left|\int_K [H_z(x,z_n)-H_z(x,z_0)]\cdot\eta\right| \le \limsup_{n\to\infty}\left|\int_{\Omega_1}[H_z(x,z_n)-H_z(x,z_0)]\cdot\eta\right| + \delta = \delta$$

and once $\delta > 0$ is arbitrary, we conclude that

$$\int_{\mathbb{R}^2}[H_z(x,z_n)-H_z(x,z_0)]\cdot\eta \to 0. \tag{2.20}$$

Since

$$I'(z_n)\eta - I'(z_0)\eta = \langle z_n - z_0, \eta\rangle - \int_{\mathbb{R}^2}[H_z(x,z_n)-H_z(x,z_0)]\cdot\eta$$

and according to (2.20) we reach

$$I'(z_0)\eta = \lim_{n\to\infty}\left[I'(z_n)\eta - \langle z_n - z_0,\eta\rangle + \int_{\mathbb{R}^2}[H_z(x,z_n)-H_z(x,z_0)]\cdot\eta\right] = 0,$$

and the claim is proved.

■

A key result in the proof that the weak limit z_0 of the sequence $(z_n)\subset\mathcal{N}$ is nontrivial is the following estimate:

Proposition 2.13. *The minimal energy level c_* defined in Lemma 2.9 satisfies*

$$\beta c_* < 4\pi/\alpha_0,$$

where $\beta \ge 0$ is given in (H_5).

Proof. If $\beta = 0$ then the above inequality is automatically valid. Assume now $\beta > 0$. Let $\varphi \in C_0^\infty(\mathbb{R}^2)$ be such that $0 \le \varphi \le 1$ in $\mathbb{R}^2$, $\mathrm{supp}(\varphi) = \overline{B}_2$, $\varphi \equiv 1$ in B_1 and $|\nabla\varphi| \le 2$ in $\mathbb{R}^2$. By item (ii) of Lemma 1.17, $\widehat{E}((\varphi,\varphi))$ intersects $\mathcal{N}$ at exactly one point, which is the unique global maximum point of $I|_{\widehat{E}((\varphi,\varphi))}$, that is, there exists an unique $t_0 > 0$ and $(v,-v)\in E^-$ such that $\eta := t_0(\varphi,\varphi)+(v,-v) = (t_0\varphi + v, t_0\varphi - v)\in\mathcal{N}$. Hence,

$$\begin{aligned} I(\eta) &= t_0^2\|\varphi\|_V^2 - \|v\|_V^2 - \int_{\mathbb{R}^2} H(x, t_0\varphi+v, t_0\varphi - v) \\ &\le t_0^2\|\varphi\|_V^2 - \int_{C_1(v)} H(x,t_0\varphi+v,t_0\varphi-v) - \int_{C_2(v)} H(x,t_0\varphi+v,t_0\varphi-v), \end{aligned}$$

where $C_1(v) = \{x \in \mathbb{R}^2 : v(x) \geq 0\}$ and $C_2(v) = \{x \in \mathbb{R}^2 : v(x) < 0\}$. Consequently, by (2.1),

$$
\begin{aligned}
I(\eta) &\leq t_0^2\|\varphi\|_V^2 - \lambda_0\left[\int_{C_1(v)} |(t_0\varphi + v, t_0\varphi - v)|^p\right] - \lambda_0\left[\int_{C_2(v)} |(t_0\varphi + v, t_0\varphi - v)|^p\right] \\
&\leq t_0^2\|\varphi\|_V^2 - \lambda_0\left[\int_{C_1(v)} [\max\{t_0\varphi + v, t_0\varphi - v)\}]^p\right] - \lambda_0\left[\int_{C_2(v)} [\max\{t_0\varphi + v, t_0\varphi - v\}]^p\right] \\
&\leq t_0^2\|\varphi\|_V^2 - \lambda_0 \int_{C_1(v)} |t_0\varphi|^p - \lambda_0 \int_{C_2(v)} |t_0\varphi|^p \\
&< t_0^2\|\varphi\|_V^2 - t_0^p\lambda_0 \int_{B_1} |\varphi|^p \\
&\leq t_0^2(4 + \|V\|_\infty)4\pi - t_0^p\lambda_0\pi \\
&\leq \pi \max_{t\geq 0}\left[4(4 + \|V\|_\infty)t^2 - \lambda_0 t^p\right].
\end{aligned}
$$

Calculating the maximum and according to (2.2) we reach

$$
\beta c_* = \beta \inf_{\mathcal{N}} I \leq \beta I(\eta) < \beta\pi \frac{(p-2)2^{\frac{2}{p-2}}[4(4 + \|V\|_\infty)]^{\frac{p}{p-2}}}{p^{\frac{p}{p-2}}\lambda_0^{\frac{2}{p-2}}} \leq \frac{4\pi}{\alpha_0},
$$

and the proof is done. ■

For the next lemma, we will exploit the inequality

$$
st \leq t^2(e^{t^2} - 1) + s(\log s)^{1/2}, \quad \text{for } \ t \geq 0 \ \text{ and } \ s \geq e^{1/\sqrt[3]{4}}. \tag{2.21}
$$

The proof of (2.21) can be seen in [38, Lemma 4.1].

Lemma 2.14. *Let $z_n = (u_n, v_n) \in \mathcal{N}$ be a sequence satisfying (2.8). Then, there exists $R > 0$ such that*

$$
\lim_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_R(y)} (u_n^2 + v_n^2) > 0. \tag{2.22}
$$

Proof. First, by Proposition 2.13, we can choose $\delta > 0$ so that $\beta c_* \in [0, 4\pi/\alpha_0 - \delta)$. Assume by contradiction that (2.22) does not occur, that is,

$$
\lim_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_R(y)} (u_n^2 + v_n^2) = 0 \quad \text{for all} \quad R > 0, \tag{2.23}
$$

which implies by Lions' lemma (see [49]) that

$$u_n \to 0 \text{ and } v_n \to 0 \text{ strongly in } L^s(\mathbb{R}^2) \text{ for any } s > 2. \tag{2.24}$$

Now, we claim that

$$\int_{\mathbb{R}^2} H(x, z_n) = \int_{\mathbb{R}^2} H(x, u_n, v_n) \to 0. \tag{2.25}$$

In view of (H_4) and (H_5), there exists $c_1 > 0$ such that

$$H(x,z) \le M_0 |H_z(x,z)| \text{ and } |H_z(x,z)||z| \le c_1 \widehat{H}(x,z) \text{ for all } x \in \mathbb{R}^2 \text{ and } |z| \ge R_0.$$

On the other hand, by (H_5), (2.9) and for any $K > R_0$, we deduce that

$$\begin{aligned}\int_{\{|z_n|>K\}} H(x, z_n) &\le M_0 \int_{\{|z_n|>K\}} |H_z(x, z_n)| \\ &\le \frac{M_0}{K} \int_{\{|z_n|>K\}} |H_z(x, z_n)||z_n| \le \frac{M_0 c_1}{K} \int_{\mathbb{R}^2} \widehat{H}(x, z_n) \le \frac{M_0 C}{K},\end{aligned}$$

where $C > 0$ does not depend on K. Given any $\varepsilon > 0$, we can take $K > 0$ large enough so that

$$\int_{\{|z_n|>K\}} H(x, z_n) \le \varepsilon. \tag{2.26}$$

By (2.12), for $\alpha > \alpha_0$ we know that

$$H(x, z_n) \le \varepsilon |z_n|^2 + C_\varepsilon |z_n|^4 (e^{\alpha |z_n|^2} - 1)$$

and thus

$$\begin{aligned}\int_{\{|z_n|\le K\}} H(x, z_n) &\le \varepsilon \int_{\{|z_n|\le K\}} |z_n|^2 + \int_{\{|z_n|\le K\}} |z_n|^4 (e^{\alpha |z_n|^2} - 1) \\ &\le \varepsilon C + 2(e^{\alpha K^2} - 1)(\|u_n\|_4^4 + \|v_n\|_4^4).\end{aligned} \tag{2.27}$$

Therefore, from this inequality, (2.24) and (2.26), it follows that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^2} H(x, z_n) = \limsup_{n\to\infty} \left[\int_{\{|z_n|\le K\}} H(x, z_n) + \int_{\{|z_n|>K\}} H(x, z_n) \right] \le (1 + C)\varepsilon$$

and convergence (2.25) is proved.

Since $I(z_n) \to c_*$, by convergence (2.25) we reach

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} (\nabla u_n \nabla v_n + V(x) u_n v_n) dx = c_*. \tag{2.28}$$

If $z_n \to 0$ strongly in E as $n \to \infty$, then by (2.25) and (2.28) we get that $c_* = 0$, which is not possible. Therefore, we can assume that $\|z_n\| \geq b > 0$ for all $n \in \mathbb{N}$. Once $\langle I'(z_n), z_n \rangle = o_n(1)\|z_n\|$, we have

$$\|z_n\|^2 = \int_{\mathbb{R}^2} H_z(x, z_n) \cdot z_n + o_n(1)\|z_n\|.$$

In view of $(H_0) - (H_1)$, given $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|H_z(x,z)| \leq C_\varepsilon e^{(\alpha_0+\varepsilon)|z|^2} \quad \text{for all} \quad (x,z) \in \mathbb{R}^2 \times \mathbb{R}^2. \tag{2.29}$$

Setting

$$\overline{z}_n = (4\pi/\alpha_0 - \delta)^{1/2} \frac{z_n}{\|z_n\|},$$

where $\delta > 0$ was chosen at the beginning of the proof, we can write

$$\begin{aligned} (4\pi/\alpha_0 - \delta)^{1/2}\|z_n\| &\leq \int_{\mathbb{R}^2} |H_z(x,u_n,v_n)||\overline{z}_n| + o_n(1) \\ &= \frac{C_\varepsilon}{\sqrt{\alpha_0}} \int_{\mathbb{R}^2} \frac{|H_z(x,u_n,v_n)|}{C_\varepsilon} \sqrt{\alpha_0}|\overline{z}_n| + o_n(1) =: I_n. \end{aligned} \tag{2.30}$$

Defining

$$\Gamma_n = \left\{ x \in \mathbb{R}^2 : |H_z(x,u_n,v_n)|/C_\varepsilon \geq e^{1/\sqrt[3]{4}} \right\} \quad \text{and} \quad \Lambda_n = \mathbb{R}^2 \backslash \Gamma_n,$$

by using inequality (2.21) with $s = |H_z(x,u_n,v_n)|/C_\varepsilon$ and $t = \sqrt{\alpha_0}|\overline{z}_n|$ we can estimate

$$\begin{aligned} I_n \leq & \frac{C_\varepsilon}{\sqrt{\alpha_0}} \int_{\Gamma_n} \frac{|H_z(x,u_n,v_n)|}{C_\varepsilon} \left[\log\left(\frac{|H_z(x,u_n,v_n)|}{C_\varepsilon} \right) \right]^{1/2} + o_n(1) \\ & + \int_{\Lambda_n} |H_z(x,u_n,v_n)||\overline{z}_n| + C_\varepsilon \sqrt{\alpha_0} \int_{\mathbb{R}^2} |\overline{z}_n|^2 \left(e^{\alpha_0 |\overline{z}_n|^2} - 1 \right). \end{aligned}$$

Thus, by (2.29) we get

$$
\begin{aligned}
I_n &\leq \sqrt{\frac{\alpha_0+\varepsilon}{\alpha_0}} \int_{\Gamma_n} |H_z(x,u_n,v_n)||z_n| + C_\varepsilon \sqrt{\alpha_0} \int_{\mathbb{R}^2} |\overline{z}_n|^2 \left(e^{\alpha_0 |\overline{z}_n|^2} - 1\right) \\
&\quad + \int_{\Lambda_n} |H_z(x,u_n,v_n)||\overline{z}_n| + o_n(1) \qquad (2.31)\\
&\leq \sqrt{\frac{\alpha_0+\varepsilon}{\alpha_0}} \int_{\mathbb{R}^2} |H_z(x,u_n,v_n)||z_n| + I_{1,n} + I_{2,n} + o_n(1),
\end{aligned}
$$

where

$$
I_{1,n} := C_\varepsilon \sqrt{\alpha_0} \int_{\mathbb{R}^2} |\overline{z}_n|^2 \left(e^{\alpha_0 |\overline{z}_n|^2} - 1\right) \quad \text{and} \quad I_{2,n} := \int_{\Lambda_n} |H_z(x,u_n,v_n)||\overline{z}_n|.
$$

Now, we can take $p>1$ close to 1 such that $p\alpha_0(4\pi/\alpha_0 - \delta) < 4\pi$. Since $\|\overline{z}_n\|^2 = 4\pi/\alpha_0 - \delta$, Lemma 1.12 and (2.24) imply that

$$
\begin{aligned}
I_{1,n} &\leq C_1 \sqrt{\alpha_0} \left[\int_{\mathbb{R}^2} |\overline{z}_n|^{2q}\right]^{1/q} \left[\int_{\mathbb{R}^2} \left(e^{p\alpha_0 |\overline{z}_n|^2} - 1\right)\right]^{1/p} \\
&\leq C_2 \sqrt{\alpha_0} \frac{(4\pi/\alpha_0 - \delta)}{b^2} \left[\int_{\mathbb{R}^2} |z_n|^{2q}\right]^{1/q} \longrightarrow 0,
\end{aligned}
$$

where $1/p + 1/q = 1$. Next, according to $(H_0)-(H_2)$, for any $\rho>0$, there exists $C_{\rho,\varepsilon}>0$ such that

$$
|H_z(x, z_n(x))| \leq \rho |z_n(x)| + C_{\rho,\varepsilon} |z_n(x)|^2, \quad \text{for all} \ \ x \in \Lambda_n. \qquad (2.32)
$$

Hence,

$$
I_{2,n} \leq \int_{\Lambda_n} (\rho|z_n| + C_{\rho,\varepsilon}|z_n|^2)|\overline{z}_n| \leq \left[\rho\left(\int_{\mathbb{R}^2}|z_n|^2\right)^{1/2} + C_{\rho,\varepsilon}\left(\int_{\mathbb{R}^2}|z_n|^4\right)^{1/2}\right]\left(\int_{\mathbb{R}^2}|\overline{z}_n|^2\right)^{1/2}.
$$

Once $\|\overline{z}_n\|$ is bounded and $z_n \to 0$ strongly in $L^4(\mathbb{R}^2)\times L^4(\mathbb{R}^2)$, we reach

$$
\limsup_{n\to\infty} I_{2,n} \leq C\rho,
$$

for some $C>0$ independent of ρ. Thus, we conclude that $I_{1,n} = o_n(1)$ and $I_{2,n} = o_n(1)$. Therefore, (2.30) and (2.31) provide

$$
(4\pi/\alpha_0 - \delta)^{1/2} \|z_n\| \leq o_n(1) + \left(1 + \frac{\varepsilon}{\alpha_0}\right)^{1/2} \int_{\mathbb{R}^2} |H_z(x,z_n)||z_n|. \qquad (2.33)
$$

By virtue of (H_5), given $\delta > 0$ there exists $M > 0$ such that $|H_z(x,z)||z| \leq (\beta+\delta)\widehat{H}(x,z)$ for all $|z| > M$. Hence, by using (2.9) with $c = c_*$ one has

$$\int_{\mathbb{R}^2} |H_z(x,z_n)||z_n| \leq (\beta+\delta)\int_{\mathbb{R}^2} \widehat{H}(x,z_n) + \int_{\{|z_n|\leq M\}} |H_z(x,z_n)||z_n|$$
$$\leq \beta c_* + o_n(1) + \int_{\{|z_n|\leq M\}} |H_z(x,z_n)||z_n|.$$

By virtue of (2.5) and arguing as in (2.27), we can show that

$$\int_{\{|z_n|\leq M\}} |H_z(x,z_n)||z_n| \to 0.$$

and consequently

$$\int_{\mathbb{R}^2} |H_z(x,z_n)||z_n| \leq \beta c_* + o_n(1).$$

Hence, since $\beta c_* \in [0, 4\pi/\alpha_0 - \delta)$, according to (2.33) we have

$$\|z_n\| \leq o_n(1) + \left(1 + \frac{\varepsilon}{\alpha_0}\right)^{1/2}\left(\frac{4\pi}{\alpha_0} - \delta\right)^{-1/2}\beta c_* \leq \left(\frac{4\pi}{\alpha_0} - \hat{\delta}\right)^{1/2}, \tag{2.34}$$

for some $0 < \hat{\delta} < \delta$ and for n sufficiently large. On the other hand, choose $p_1 > 1$ close to 1 and $\alpha > \alpha_0$ close to α_0 satisfying $p_1\alpha(4\pi/\alpha_0 - \hat{\delta}) < 4\pi$, from (2.34) it follows that $p_1\alpha\|u_n\|^2 < 4\pi$ for n sufficiently large. Once $z_n \to 0$ strongly in $L^{q_1}(\mathbb{R}^2) \times L^{q_1}(\mathbb{R}^2)$, where $1/p_1 + 1/q_1 = 1$, by invoking Lemma 1.12 we obtain

$$\int_{\mathbb{R}^2}\left(e^{\alpha|z_n|^2} - 1\right)|z_n| \leq C_1\left(\int_{\mathbb{R}^2}|z_n|^{q_1}\right)^{\frac{1}{q_1}}\left[\int_{\mathbb{R}^2}\left(e^{p_1\alpha|z_n|^2} - 1\right)\right]^{\frac{1}{p_1}} \leq C_2\left(\int_{\mathbb{R}^2}|z_n|^{q_1}\right)^{\frac{1}{q_1}} \to 0.$$

From this convergence and again by using (1.8), we get

$$\int_{\mathbb{R}^2} |H_z(x,z_n)||z_n| \to 0,$$

which together with (2.33) implies that $z_n \to 0$ strongly in $H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$. Thus, again it follows from (2.28) that $c_* = 0$, which is a contradiction and proof of the lemma is complete. ∎

Finalizing Proof of Theorem 2.2. Once (2.22) is valid, we can get a sequence $(y_n) \subset \mathbb{R}^2$

and $\nu > 0$ satisfying

$$\int_{B_R(y_n)} (u_n^2 + v_n^2) \geq \nu.$$

Now, we introduce the following sequences:

$$\tilde{u}_n(\cdot) := u_n(\cdot + y_n) \quad \text{and} \quad \tilde{v}_n(\cdot) := v_n(\cdot + y_n).$$

In view of (V_0) and (H_0), we can see that $\tilde{z}_n := (\tilde{u}_n, \tilde{v}_n) \in \mathcal{N}$ and also satisfies $I(\tilde{z}_n) \to c_*$, $I'(\tilde{z}_n) \to 0$ and

$$\liminf_{n\to\infty} \int_{B_{R_0}(0)} (\tilde{u}_n^2 + \tilde{v}_n^2) dx \geq \nu, \tag{2.35}$$

Observe that, by Lemma 1.21, $(\tilde{z}_n)$ is also bounded in E and, up to a subsequence, we assume $\tilde{z}_n \rightharpoonup \tilde{z}$ weakly in E for some $\tilde{z} = (\tilde{u}, \tilde{v})$ and $\tilde{z}_n \to \tilde{z}$ a.e. in $\mathbb{R}^2$. From Proposition 2.12 it follows that $I'(\tilde{z}) = 0$. Moreover, in view of (2.35) we have $\tilde{z} \neq 0$ and Remark 1.5 guarantee that $\tilde{z} \in \mathcal{N}$, implying that $c_* \leq I(\tilde{z})$. By invoking Fatou's Lemma, we deduce that

$$\begin{aligned} c_* = \liminf_{n\to\infty} I(\tilde{z}_n) = \liminf_{n\to\infty} \left[I(\tilde{z}_n) - \frac{1}{2} I'(\tilde{z}_n)\tilde{z}_n \right] &= \liminf_{n\to\infty} \int_{\mathbb{R}^2} \left[\frac{1}{2} H_z(x, \tilde{z}_n) \cdot \tilde{z}_n - H(x, \tilde{z}_n) \right] \\ &\geq \int_{\mathbb{R}^2} \left[\frac{1}{2} H_z(x, \tilde{z}) \cdot \tilde{z} - H(x, \tilde{z}) \right] \\ &= I(\tilde{z}) - \frac{1}{2} I'(\tilde{z})\tilde{z} \\ &= I(\tilde{z}). \end{aligned}$$

Therefore, $I(\tilde{z}) = c_*$, which shows that $\tilde{z}$ is a ground state for (1) and the theorem is proved.

■

Chapter 3

Hamiltonian systems in the plane involving vanishing potentials

In this chapter, we establish the existence of nontrivial solutions for the class of Hamiltonian systems

$$\begin{cases} -\Delta u + V(x)u = Q(x)g(v), & x \in \mathbb{R}^2, \\ -\Delta v + V(x)v = Q(x)f(u), & x \in \mathbb{R}^2, \end{cases} \tag{4}$$

where V and Q decay to zero at infinity as $(1+|x|^\alpha)^{-1}$ with $\alpha \in (0,2)$, and $(1+|x|^\beta)^{-1}$ with $\beta \in [2,+\infty)$, respectively. The nonlinear terms $f(s)$ and $g(s)$ have subcritical or critical exponential growth. We show an alternative proof of a weighted Trudinger-Moser-type inequality and combine with a Galerkin approximation method and a linking theorem.

Hamiltonian systems where the nonlinearity $H(x,u,v)$ has polynomial growth at infinity have been considered in the literature before, see for example [22, 25, 40]. For problems with a exponential growth in dimension two, see [3, 38, 44].

Here, we shall consider the following assumptions:

(V) $V \in C(\mathbb{R}^2)$, there exist $\alpha, a > 0$ such that

$$\frac{a}{1+|x|^\alpha} \leq V(x),$$

and $V(x) \sim |x|^{-\alpha}$ as $|x| \to \infty$;

(Q) $Q \in C(\mathbb{R}^2)$, there exist $\beta, b > 0$ such that

$$0 < Q(x) \leq \frac{b}{1 + |x|^\beta},$$

and $Q(x) \sim |x|^{-\beta}$ as $|x| \to \infty$.

In particular, we restrict our attention to the case when α and β satisfy

$$\alpha \in (0, 2) \quad \text{and} \quad \beta \in [2, \infty]. \tag{5}$$

With respect to the functions f, g, we assume the following conditions:

(h_0) $f, g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous;

(h_1) $f(s) = o(|s|)$ and $g(s) = o(|s|)$ at the origin;

(h_2) there exists $\theta > 0$ such that

$$\begin{cases} 0 < \theta F(s) := \theta \int_0^s f(t) t dt \leq s f(s) \\ 0 < \theta G(s) := \theta \int_0^s g(t) t dt \leq s g(s) \end{cases} \text{for all } s \in (0, \infty);$$

(h_3) there exists constants $M_0 > 0$ and $s_1 > 0$ such that

$$\begin{cases} 0 < \theta F(s) \leq M_0 f(s) \\ 0 < \theta G(s) \leq M_0 g(s) \end{cases} \text{for all } s \in [s_1, \infty).$$

We say $f(s)$ and $g(s)$ have *subcritical growth* if

$$\lim_{s \to +\infty} \frac{f(s)}{e^{\mu s^2}} = 0,$$
$$\lim_{s \to +\infty} \frac{g(s)}{e^{\nu s^2}} = 0$$

for all $\mu, \nu > 0$.

We say $f(s)$ and $g(s)$ have *critical growth* if there exists *critical exponents* $\mu_0, \nu_0 > 0$

such that

$$\lim_{s\to+\infty}\frac{f(s)}{e^{\mu s^2}}=\begin{cases}0, & \text{for all } \mu>\mu_0;\\ +\infty, & \text{for all } \mu<\mu_0,\end{cases}$$

$$\lim_{s\to+\infty}\frac{g(s)}{e^{\nu s^2}}=\begin{cases}0, & \text{for all } \nu>\nu_0;\\ +\infty, & \text{for all } \nu<\nu_0.\end{cases}$$

Let $L^p_w(\mathbb{R}^2)$ denote the weighted L^p-space consisting of all measurable functions $u:\mathbb{R}^2\to\mathbb{R}$ satisfying

$$\int_{\mathbb{R}^2} w(x)|u|^p dx<\infty.$$

We introduce the weighted Sobolev space

$$H^1_V(\mathbb{R}^2):=\left\{u\in L^2_V(\mathbb{R}^2):|\nabla u|\in L^2(\mathbb{R}^2)\right\},$$

with norm

$$\|u\|^2:=\|\nabla u\|_2^2+\int_{\mathbb{R}^2}V(x)u^2.$$

Remark 3.1. *By the continuous embedding*

$$H^1_V(\mathbb{R}^2)\hookrightarrow H^1_{loc}(\mathbb{R}^2),$$

the definition of Cauchy sequences and Fatou Lemma, we are able to show that $(H^1_V(\mathbb{R}^2),\|\cdot\|)$ *is complete.*

Remark 3.2. *The space* $\mathcal{C}^\infty_0(\mathbb{R}^2)$ *of smooth compactly supported functions is dense in* $(H^1_V(\mathbb{R}^2),\|\cdot\|)$.

Let us denote by $E:=H^1_V(\mathbb{R}^2)\times H^1_V(\mathbb{R}^2)$ equipped with the inner product

$$\langle(u,v),(\varphi,\psi)\rangle_E=\int_{\mathbb{R}^2}[\nabla u\nabla v+V(x)u\varphi+\nabla u\nabla\psi+V(x)v\psi]dx,$$

for all $(u,v),(\varphi,\psi)\in E$, to which corresponds the norm

$$\|(u,v)\|_E=\langle(u,v),(u,v)\rangle_E^{1/2}.$$

We say that weak solutions of (4) are functions $(u,v) \in E$ such that

$$\int_{\mathbb{R}^2} [\nabla u \nabla v + V(x)u\varphi + \nabla u \nabla \psi + V(x)v\psi]dx - \int_{\mathbb{R}^2} Q(x)f(u)\varphi dx - \int_{\mathbb{R}^2} Q(x)g(v)\psi dx = 0$$

for all $(\psi, \varphi) \in E$.

Now, we state our existence results.

Theorem 3.3 (Subcritical case)**.** *Suppose $f(s)$ has subcritical or critical exponential growth, $g(s)$ has subcritical exponential growth, (V), (Q) and $(h_0)-(h_3)$ are satisfied. Then (4) possesses a nontrivial weak solution $(u,v) \in E$.*

For the next result, we assume that there exists $\gamma_0 > 0$ such that the functions f, g satisfy

$$\liminf_{|s|\to\infty} \frac{sf(s)}{e^{\gamma_0 s^2}}, \quad \liminf_{|s|\to\infty} \frac{sg(s)}{e^{\gamma_0 s^2}} =: \beta_0 > \mathcal{M}, \tag{3.1}$$

where

$$\mathcal{M} =: \inf_{r>0} \frac{4e^{1/2r^2 V_{\max,r}}}{\gamma_0 r^2 Q_{\min,r}}, \quad V_{\max,r} := \max_{|x|\le r} V(x) > 0 \quad \text{and} \quad Q_{\min,r} := \min_{|x|\le r} Q(x) > 0.$$

Theorem 3.4 (Critical case)**.** *Suppose $f(s)$ and $g(s)$ has critical exponential growth, (V), (Q), $(h_0)-(h_3)$ and (3.1) are satisfied. Then (4) possesses a nontrivial weak solution $(u,v) \in E$.*

The original characteristics of the class of problems we study in this chapter are that both V and Q vanish at infinity and are not necessarily radially symmetric. Some of the difficulties to study these problems continue to be the lack of compactness of the Sobolev embedding since the domain $\mathbb{R}^2$ is unbounded and the nonlinearities f and g have critical growth. We also know that the energy functional associated with system (4) is strongly indefinite and the nonlinear term $\Psi(u,v) = \int_{\mathbb{R}^2} Q(x)[F(u) + G(v)]dx$ is not weakly lower semi-continuous. Here, we adapt some arguments presented in [58].

This chapter is organized as follows: first, we show a alternative proof to a weighted Trudinger-Moser-type inequality. On Section 3.2, we check the variational framework and the linking geometry for the associated functional. We prove the boundness of the Palais-Smale sequence on Section 3.3. Lastly, we show the main results for the subcritical and critical cases on Section 3.5 and Section 3.4, respectively.

3.1 Preliminaries

In this section we prove a Trudinger-Moser-type inequality with an distinct proof to the ones presented in [35, 37], in which we do not use Besicovitch's Covering Lemma. We were inspired by the proof presented in [4], where Albuquerque et al. have shown a Trudinger-Moser-type inequality for $\gamma > 0$ in the range $(0, \gamma_*)$ where $\gamma_* \in (0, 4\pi)$.

Theorem 3.5. *Suppose that (V) and (Q) are satisfied with $\alpha \in (0, 2)$ and $\beta \in [2, \infty)$. For any $\gamma > 0$ and $u \in E$, we have*

$$Q(\cdot)(e^{\gamma u^2} - 1) \in L^1(\mathbb{R}^2).$$

Moreover, for any $0 < \gamma < 4\pi$,

$$\sup_{u \in E, \|u\| \leq 1} \int_{\mathbb{R}^2} Q(x)(e^{\gamma u^2} - 1)dx < \infty.$$

The proof of this Trudinger-Moser inequality is a direct consequence of the following proposition.

Theorem 3.6. *There exists an constant $C_\gamma > 0$ such that*

$$\int_{\mathbb{R}^2} Q(x)(e^{\gamma u^2} - 1)dx \leq C_\gamma \|u\|^2,$$

for all $0 < \gamma < 4\pi$ and $\|u\| \leq 1$.

Proof. Let $0 < \gamma < 4\pi$ and $u \in H^1_V(\mathbb{R}^2)$ with $\|u\| \leq 1$. First, take $\delta > 0$ such that $(1 + \delta)\gamma < 4\pi$. Since $\alpha < 2$ and $\beta \geq 2$, let us fix $R > 0$ sufficiently large such that

$$\frac{64}{R^2}\left(\frac{1 + \left(\frac{5}{2}R\right)^\alpha}{a}\right) \leq \delta \quad \text{and} \quad \frac{b}{a}\left(\frac{1 + 2^\alpha R^\alpha}{R^\beta}\right) \leq 1 + \delta. \tag{3.2}$$

Now, set $\varphi \in C_0^\infty(B_{2R})$ satisfying

$$\varphi \equiv 1 \text{ in } B_R, \quad \varphi \leq 1 \text{ in } \mathbb{R}^2 \quad \text{and} \quad |\nabla\varphi| \leq \frac{4}{R} \text{ in } B_{2R}.$$

Hence $\varphi u \in W_0^{1,2}(B_R)$ and

$$\begin{aligned}\int_{B_{2R}} |\nabla(\varphi u)|^2 dx &= \int_{B_{2R}} |\nabla \varphi u + \varphi \nabla u|^2 dx \\ &= \int_{B_{2R}} u^2 |\nabla u|^2 dx + 2\int_{B_{2R}} u\varphi \nabla u \nabla \varphi dx + \int_{B_{2R}} \varphi |\nabla u|^2 dx \\ &\le (1+\delta)\int_{B_{2R}} |\nabla u|^2 dx + \left(1+\frac{1}{\delta}\right)\frac{16}{R^2}\int_{B_{2R}} u^2 dx \\ &\le (1+\delta)\int_{B_{2R}} |\nabla u|^2 dx + \left(1+\frac{1}{\delta}\right)\frac{16}{R^2}\frac{1+(2R)^\alpha}{a}\int_{B_{2R}} V(x)u^2 dx.\end{aligned}$$

Considering (3.2), we have that

$$\int_{B_{2R}} |\nabla(\varphi u)|^2 dx \le (1+\delta)\int_{B_{2R}} (|\nabla u|^2 + V(x)u^2) dx. \tag{3.3}$$

Let $v := \dfrac{\varphi u}{\sqrt{1+\delta}}$. By (3.3), it follows that

$$\int_{B_{2R}} |\nabla v|^2 dx \le \|u\|^2 \le 1.$$

Therefore, applying a Trudinger-Moser type inequality by Adachi-Tanaka [1], we obtain

$$\begin{aligned}\int_{B_{2R}} (e^{\gamma(1+\delta)v^2} - 1) dx &\le C_1 \int_{B_{2R}} v^2 dx \\ &\le C_1 \frac{1+(2R)^\alpha}{(1+\delta)a}\int_{B_{2R}} V(x)(\varphi u)^2 dx \\ &\le C_1 \frac{1+(2R)^\alpha}{a}\int_{B_{2R}} (|\nabla u|^2 + V(x)u^2) dx.\end{aligned}$$

By the above inequality

$$\begin{aligned}\int_{B_R} (e^{\gamma u^2} - 1) dx &= \int_{B_R} \left(e^{\gamma(1+\delta)\left(\frac{\varphi u}{\sqrt{1+\delta}}\right)^2} - 1\right) dx \\ &\le \int_{B_{2R}} (e^{\gamma(1+\delta)v^2} - 1) dx \\ &\le C_1 \frac{1+(2R)^\alpha}{a}\int_{B_{2R}} (|\nabla u|^2 + V(x)u^2) dx.\end{aligned} \tag{3.4}$$

For the next step, we introduce the sets

$$A_{j,R} := \{x \in \mathbb{R}^2 : 2^j R < |x| < 2^{j+1}R\}, \quad j = 0, 1, 2, ...$$

and

$$\widehat{A_{j,R}} := \left\{x \in \mathbb{R}^2 : 2^j \frac{R}{2} < |x| < 2^{j+1}\frac{5R}{2}\right\}, \quad j = 0, 1, 2, ...$$

Our objective is to estimate the integral $\int_{A_{j,R}} Q(x)(e^{\gamma}u^2 - 1)$. Let us fix $j \in \mathbb{N} \cup \{0\}$ and let $y := 2^{-j}x$. Thus, if $u_j(y) := u(2^j y) = u(x)$, we obtain

$$\begin{aligned}\int_{A_{j,R}} Q(x)(e^{\gamma u^2} - 1)dx &\leq \int_{A_{j,R}} \frac{b}{1 + |x|^\beta}(e^{\gamma u^2} - 1)dx \\ &\leq \frac{b}{1 + 2^{j\beta}R^\beta} \int_{A_{j,R}} (e^{\gamma u^2} - 1)dx \\ &\leq \frac{b}{1 + 2^{j\beta}R^\beta} 2^{2j} \int_{A_{0,R}} (e^{\gamma u_j^2} - 1)dy. \end{aligned} \tag{3.5}$$

Consider $\varphi_R \in C_0^\infty(\widehat{A_{0,R}})$ such that

$$0 \leq \varphi_R(x) \leq 1, \text{ for all } x \in \widehat{A_{0,R}}, \quad \varphi_R \equiv 1 \text{ in } A_{0,R} \quad \text{and} \quad |\nabla \varphi_R| \leq \frac{8}{R} \text{ in } \widehat{A_{0,R}}.$$

Using the Adachi-Tanaka inequality once more, we obtain the following

$$\begin{aligned}\int_{A_{0,R}} (e^{\gamma u_j^2} - 1)dy &= \int_{A_{0,R}} (e^{\gamma u_j^2} - 1)dy \\ &\leq \int_{\widehat{A_{0,R}}} (e^{\gamma(1+\delta)\left(\frac{\varphi u_j}{\sqrt{1+\delta}}\right)^2} - 1)dy \\ &\leq C_1 \int_{\widehat{A_{0,R}}} \left(\frac{\varphi u_j}{\sqrt{1+\delta}}\right)^2 dy \leq C_1 \int_{\widehat{A_{0,R}}} u_j^2 dy = \frac{C_1}{2^{2j}} \int_{\widehat{A_{j,R}}} u^2 dx \\ &\leq \frac{C_1}{2^{2j}} \left(\frac{1 + \left(2^j \frac{5}{2} R\right)^2}{a}\right) \int_{\widehat{A_{j,R}}} V(x)u^2 dx \\ &\leq \frac{2C_1 R^\alpha \left(\frac{5}{2}\right)^\alpha}{a} \int_{\widehat{A_{j,R}}} V(x)u^2 dx.\end{aligned}$$

Therefore, using (3.5), we have

$$\int_{A_{j,R}} Q(x)(e^{\gamma u^2} - 1)dx \leq \frac{2^{2j+1} C_1 R^\alpha \left(\frac{5}{2}\right)^\alpha b}{(1 + 2^{j\beta}R^\beta)a} \int_{\widehat{Aj,R}} (V(x)u^2 + |\nabla u|^2)dx.$$

Observe that

$$\int_{B_R^c} Q(x)(e^{\gamma u^2}-1)dx \leq \sum_{j=0}^{\infty}\int_{A_{j,R}} Q(x)(e^{\gamma u^2}-1)dx$$
$$\leq \sum_{j=0}^{\infty} \frac{2^{2j+1}C_1R^\alpha\left(\frac{5}{2}\right)^\alpha b}{(1+2^{j\beta}R^\beta)a}\int_{\widehat{Aj,R}}(|\nabla u|^2+V(x)u^2)dx.$$

Now, we affirm that $\sum_{j=0}^{\infty}\chi_{\widehat{A_{j,R}}}(x) \leq 5$, for all $x \in \mathbb{R}^2$. Indeed, let $x \in \widehat{A_{j,R}}$. This implies that $x \notin \widehat{A_{j+3,R}}$ and $x \notin \widehat{A_{j-3,R}}$, for all $j = 0,1,2,3....$ In fact, if $x \in \widehat{A_{j,R}}$, then

$$2^j\frac{R}{2} < |x| < 2^j\frac{5R}{2} < 2^{j+3}\frac{R}{2},$$

thus $x \notin \widehat{A_{j-3,R}}$. The same way we get that $x \notin \widehat{A_{j+3,R}}$.

Therefore,

$$\int_{B_R^c} Q(x)(e^{\gamma u^2}-1)dx \leq \frac{8C_1R^\alpha\left(\frac{5}{2}\right)^\alpha b}{(2^\beta R^\beta)a}\sum_{j=0}^{\infty}\int_{\mathbb{R}^2}(|\nabla u|^2+V(x)u^2)\chi_{\widehat{Aj,R}}dx$$
$$\leq \frac{40C_1R^\alpha\left(\frac{5}{2}\right)^\alpha b}{(2^\beta R^\beta)a}\|u\|^2. \tag{3.6}$$

Since $\beta > \alpha$,

$$\lim_{R\to\infty}\frac{40C_1R^\alpha\left(\frac{5}{2}\right)^\alpha b}{(2^\beta R^\beta)a} = 0.$$

Thus, from (3.4) and (3.6), we conclude the proof.

■

A important consequence is the following compactness result:

Proposition 3.7. *If (V) and (Q) hold with $\alpha \in (0,2)$ and $\beta \in [2,+\infty)$. Then, for all $p \in [2,+\infty)$, the embedding*

$$H_V^1(\mathbb{R}^2) \hookrightarrow L_Q^p(\mathbb{R}^2) \tag{3.7}$$

is continuous. In particular, since $\beta > \alpha$, the above embedding is compact.

Proof. Set $u \in H_V^1(\mathbb{R}^2)$ and observe that by condition (Q) we have

$$\left(\int_{B_R} Q(x)|u|^p dx\right)^{1/q} \leq \left(\int_{B_R}\frac{b}{1+|x|^\beta}|u|^p dx\right)^{1/q} \leq b^{1/q}\|u\|_{L^p(B_R)}. \tag{3.8}$$

By the embedding $H^1(B_R) \hookrightarrow L^p(B_R)$ for all $p \in [1, +\infty)$, we get

$$\begin{aligned} \|u\|_{L^p(B_R)} \le C_1 \|u\|_{H^1(B_R)} &= C_1 \left[\int_{B_R} (|\nabla u|^2 + u^2) dx \right]^{1/2} \\ &\le C_1 \left[\int_{B_R} \left(|\nabla u|^2 + \left(\frac{1+R^\alpha}{a} \right) V(x) u^2 \right) dx \right]^{1/2} \\ &\le C_R \left[\int_{B_R} (|\nabla u|^2 + V(x) u^2) dx \right]^{1/2}, \end{aligned} \tag{3.9}$$

because $V(x) \ge \dfrac{a}{1+|x|^\alpha} \ge \dfrac{a}{1+R^\alpha}$. Thus, for each $R > 0$, it follows from (3.8) and (3.9) that

$$\int_{B_R} Q(x)|u|^p dx \le bC_R^p \left[\int_{B_R} (|\nabla u|^2 + V(x)u^2) dx \right]^{p/2} \le bC_R^p \|u\|^p. \tag{3.10}$$

For each $p \in [2, +\infty)$, there is $C_p > 0$ such that

$$|s|^p \le C_p(e^{s^2} - 1), \text{ for all } s \in \mathbb{R}.$$

Proceeding as in the proof of Theorem 3.5, we have

$$\int_{B_R^c} Q(x)|u|^p dx \le C_p \int_{B_R^c} Q(x)(e^{\gamma u^2} - 1) dx \le \frac{8C_1 R^\alpha \left(\frac{5}{2}\right)^\alpha b}{(2^\beta R^\beta) a} \sum_{j=0}^{\infty} \int_{\mathbb{R}^2} V(x) u^2 \chi_{\widehat{Aj,R}} dx.$$

We conclude

$$\int_{B_R^c} Q(x)|u|^p dx \le \frac{40C_1 R^\alpha \left(\frac{5}{2}\right)^\alpha b}{(2^\beta R^\beta) a} \|u\|^2. \tag{3.11}$$

Now, if $(u_m) \subset E$ is such that $u_m \to 0$ in E, then by (3.10) and (3.11), we get

$$\int_{\mathbb{R}^2} Q(x)|u_m|^p dx = \int_{B_R} Q(x)|u_m|^p dx + \int_{B_R^c} Q(x)|u_m|^p dx \to 0 \text{ as } m \to \infty$$

and the continuity of the embedding is proved for all $p \in [2, +\infty)$.

Suppose that $(u_m) \subset E$ such that $u_m \rightharpoonup 0$ in E. Since $\beta > \alpha$,

$$\lim_{R \to \infty} \frac{40C_1 R^\alpha \left(\frac{5}{2}\right)^\alpha b}{(2^\beta R^\beta) a} = 0.$$

Thus, for all $\varepsilon > 0$, there exists $m_0 > 0$ such that

$$\int_{B_R^c} Q(x)|u_m|^p dx \le \varepsilon,$$

for all $m \geq m_0$, that is $u_m \to 0$ in $L^p_Q(B^c_R)$.

■

3.2 Linking geometry

If $f(x,s)$ and $g(x,s)$ have subcritical growth, then for each $\mu > 0$ and $\nu > 0$, given $\varepsilon > 0$ and $q \geq 1$, there exists a constant $C = C(\varepsilon, q) > 0$ such that for all $s \in [0, \infty)$,

$$
\begin{aligned}
|f(s)| &\leq \frac{\varepsilon}{2}|s| + C|s|^{q-1}(e^{\mu s^2} - 1), \\
|g(s)| &\leq \frac{\varepsilon}{2}|s| + C|s|^{q-1}(e^{\nu s^2} - 1),
\end{aligned}
$$

for any $s \in \mathbb{R}$. Hence, by the Ambrosetti-Rabinowitz condition (h_3) we have

$$
|F(s)| \leq \frac{\varepsilon}{2}|s|^2 + C|s|^q(e^{\mu s^2} - 1), \tag{3.12}
$$

$$
|G(s)| \leq \frac{\varepsilon}{2}|s|^2 + C|s|^q(e^{\nu s^2} - 1), \tag{3.13}
$$

for any $s \in \mathbb{R}$.

Therefore, given $u \in H^1_V(\mathbb{R}^2)$, we obtain the estimate

$$
\begin{aligned}
\int_{\mathbb{R}^2} Q(x)F(u)dx &\leq \frac{\varepsilon}{2}\int_{\mathbb{R}^2} Q(x)|u|^2dx + C\int_{\mathbb{R}^2} Q(x)|u|^q(e^{\mu|u|^2} - 1)dx \\
&\leq \frac{\varepsilon}{2}\int_{\mathbb{R}^2} Q(x)|u|^2dx \\
&\quad + C\left(\int_{\mathbb{R}^2} Q(x)|u|^{qp}dx\right)^{1/p}\left(\int_{\mathbb{R}^2} Q(x)(e^{\mu p'|u|^2} - 1)dx\right)^{1/p'},
\end{aligned} \tag{3.14}
$$

where we used Hölder's inequality with $p > 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$. By the continuous embedding (3.7) and the Trudinger-Moser inequality, we have

$$
\int_{\mathbb{R}^2} Q(x)F(u)dx < +\infty \text{ for any } u \in H^1_V(\mathbb{R}^2).
$$

Therefore, it follows that the functional

$$
I(u,v) := \int_{\mathbb{R}^2} [\nabla u \nabla v + V(x)uv]dx - \int_{\mathbb{R}^2} Q(x)F(u)dx - \int_{\mathbb{R}^2} Q(x)G(v)dx
$$

is well defined on $(E, \|\cdot\|)$. We also can see that $I \in C^1(E, \mathbb{R})$ with

$$I'(u,v)(\varphi,\psi) = \int_{\mathbb{R}^2} [\nabla u \nabla v + V(x)u\varphi + \nabla u \nabla \psi + V(x)v\psi]dx \\ - \int_{\mathbb{R}^2} Q(x)f(u)\varphi dx - \int_{\mathbb{R}^2} Q(x)g(v)\psi dx$$

for all $(\varphi, \psi) \in E$.

Using the notation

$$E^+ = \{(u,u) \in E\} \text{ and } E^- = \{(v,-v) \in E\},$$

we can see that $E = E^+ \oplus E^-$.

In the following lemmas, we establish the geometry of the Linking Theorem.

Lemma 3.8. *There exists $\rho, \sigma > 0$ such that $I(z) \geq \sigma$, for $z \in S := \partial B_\rho \cap E^+$.*

Proof. Using (3.14), the continuous embedding (3.7) and the Trudinger-Moser inequality, we have that for any $\varepsilon > 0$,

$$\int_{\mathbb{R}^2} Q(x)F(u)dx \leq C\frac{\varepsilon}{2}\|u\|^2 + C(\mu, q, \varepsilon)\|u\|^q,$$

for any $u \in H^1_V(\mathbb{R}^2)$ with $\|u\| = \widetilde{\rho}$, where $\widetilde{\rho} > 0$ satisfies $\mu p'\widetilde{\rho}^2 < 4\pi$.

By the same argument,

$$\int_{\mathbb{R}^2} Q(x)G(u)dx \leq C\frac{\varepsilon}{2}\|u\|^2 + C(\nu, q, \varepsilon)\|u\|^q,$$

for any $u \in H^1_V(\mathbb{R}^2)$ with $\|u\| = \overline{\rho}$, where $\overline{\rho} > 0$ satisfies $\mu p'\overline{\rho}^2 < 4\pi$.

Thus, choosing $\rho > 0$ conveniently,

$$I(u,u) = \int_{\mathbb{R}^2} \left[|\nabla u|^2 + V(x)u^2\right] - \int_{\mathbb{R}^2} Q(x)F(u) - \int_{\mathbb{R}^2} Q(x)G(u) \\ \geq \|u\|^2 - \varepsilon \int_{\mathbb{R}^2} Q(x)|u|^2 - C_1\|u\|^q.$$

By (3.7), there exists $C_\varepsilon > 0$ such that

$$I(u,u) \geq (1 - C_\varepsilon)\|u\|^2 - C_1\|u\|^q.$$

Therefore, for $\varepsilon > 0$ sufficiently small, we can find $\rho, \sigma > 0$ such that $I(u,u) \geq \sigma > 0$, whenever $\|u\| = \rho$.

■

Let $y \in H^1_V \backslash \{0\}$ be a fixed nonnegative function and

$$Q_y = \left\{ r(y,y) + w : \quad w \in E^-, \|w\| \leq R_0 \text{ and } 0 \leq r \leq R_1 \right\},$$

where the constants R_0 and R_1 will be chosen in the following lemma.

Lemma 3.9. *There exists positive constants R_0, R_1, which depends on y, such that $I(z) \leq 0$ for all $z \in \partial Q_y$.*

Proof. The boundary of the set Q_y in $\mathbb{R}(y,y) \oplus E^-$ is composed of three parts:

- $z \in \partial Q_y \cap E^-$. For all $z = (u,-u) \in E^-$, we have

$$I(z) = -\|u\|^2 - \int_{\mathbb{R}^2} Q(x)[F(u) + G(-u)]dx \leq 0.$$

- $z \in B_1 := \{R_1(y,y) + (u,-u) \in \partial Q_y \text{ with } \|(u,-u)\|_E \leq R_0\}$. We have

$$I(z) = R_1^2\|y\|^2 - \|u\|^2 - \int_{\mathbb{R}^2} Q(x)[F(x, R_1 y + u) + G(x, R_1 y - u)]dx.$$

  It follows from (h_3) that for all $\delta > 0$, there exists $C_1 = C_1(\delta) > 0$ such that

$$F(x,s), G(x,s) \geq C_1 s^\theta - \delta s^2,$$

  for all $(x,s) \in \mathbb{R}^2 \times [0,\infty)$. Thus, there exists $C_1 = C_1(y) > 0$ such that

$$I(z) \leq R_1^2\|y\|^2 + C_1(R_1^2 - R_1^\theta).$$

  Taking $R_1 = R_1(y)$ sufficiently large, we have $I(z) \leq 0$.

- $z \in B_2 := \{r(y,y) + (u,-u) \in \partial Q_y \text{ with } \|(u,-u)\|_E = R_0 \text{ and } 0 \leq r \leq R_1\}$. We have

$$
\begin{aligned}
I(z) &= r^2\|y\|^2 - \|u\|^2 - \int_{\mathbb{R}^2} Q(x)[F(ry+u) + G(ry-u)]dx \\
&\leq R_1^2\|y\|^2 - \frac{1}{2}R_0^2.
\end{aligned}
$$

If $\sqrt{2}R_1\|y\| \leq R_0$, we have $I(z) \leq 0$.

■

3.3 Behaviour of Palais-Smale sequences

For this section, we shall use the inequality (1.27), whose proof can be found in [28, Lemma 2.4]:

$$
st \leq \begin{cases} (e^{t^2} - 1) + |s|(\log|s|)^{1/2}, & t \in \mathbb{R} \text{ and } |s| \geq e^{1/4}; \\ (e^{t^2} - 1) + \dfrac{1}{2}s^2, & t \in \mathbb{R} \text{ and } |s| \leq e^{1/4}. \end{cases} \tag{1.27}
$$

Proposition 3.10. *Supose (V), (Q) and $(h_0) - (h_3)$ are satisfied. Let $(u_m, v_m) \in E$ such that*

(I_1) $I(u_m, v_m) = c + \delta_m$, where $\delta_m \to 0$ as $m \to \infty$;

(I_2) $|I'(u_m, v_m)(\varphi, \psi)| \leq \varepsilon_m\|(\varphi, \psi)\|_E$, for $\varphi, \psi \in E$, where $\varepsilon_m \to 0$ as $m \to \infty$.

Then

$$
\begin{aligned}
&\|u_m\| \leq C, \|v_m\| \leq C, \\
&\int_{\mathbb{R}^2} Q(x)f(u_m)u_m dx \leq C, \quad \int_{\mathbb{R}^2} Q(x)g(v_m)v_m dx \leq C \\
&\int_{\mathbb{R}^2} Q(x)F(u_m)dx \leq C, \quad \int_{\mathbb{R}^2} Q(x)G(u_m)dx \leq C.
\end{aligned}
$$

Proof. Choosing $(\varphi, \psi) = (u_m, v_m)$ in (I_2) yields

$$
\left| 2\langle u_m, v_m\rangle_E - \int_{\mathbb{R}^2} Q(x)f(u_m)u_m dx - \int_{\mathbb{R}^2} Q(x)g(v_m)v_m dx \right| \leq \varepsilon_m\|(u_m, v_m)\|.
$$

By (I_1) and (h_3), we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^2} Q(x)[f(u_m)u_m + g(v_m)v_m]dx \\
&\leq 2\int_{\mathbb{R}^2} Q(x)[F(u_m) + G(v_m)]dx + 2c + 2\delta_m + \varepsilon_m\|(u_m, v_m)\| \\
&\leq \frac{2}{\theta}\int_{\mathbb{R}^2} Q(x)[f(u_m)u_m + g(v_m)v_m]dx + 2c + 2\delta_m + \varepsilon_m\|(u_m, v_m)\|,
\end{aligned}
$$

which implies

$$
\int_{\mathbb{R}^2} Q(x)[f(u_m)u_m + g(v_m)v_m]dx \leq \frac{\theta}{\theta - 2}(2c + 2\delta_m + \varepsilon_m\|(u_m, v_m)\|).
$$

Taking $(\varphi, \psi) = (v_m, 0)$ and $(\varphi, \psi) = (0, u_m)$ in (I_2),

$$
\begin{aligned}
\|v_m\|^2 - \varepsilon_m\|v_m\| &\leq \int_{\mathbb{R}^2} Q(x)f(u_m)v_m dx \\
\|u_m\|^2 - \varepsilon_m\|u_m\| &\leq \int_{\mathbb{R}^2} Q(x)g(v_m)u_m dx.
\end{aligned}
$$

Setting $U_m = \dfrac{u_m}{\|u_m\|}$ and $V_m = \dfrac{v_m}{\|v_m\|}$, we have

$$
\|v_m\|^2 \leq \int_{\mathbb{R}^2} Q(x)f(u_m)V_m dx + \varepsilon_m \tag{3.15}
$$

$$
\|u_m\|^2 \leq \int_{\mathbb{R}^2} Q(x)g(v_m)U_m dx + \varepsilon_m. \tag{3.16}
$$

Now we estimate (3.15). Let us define

$$
\begin{aligned}
I_m &= \int_{\left\{x\in\mathbb{R}^2 : f(u_m)(x) \leq e^{1/4}\right\}} Q(x)f(u_m)V_m dx, \\
J_m &= \int_{\left\{x\in\mathbb{R}^2 : f(u_m)(x) \geq e^{1/4}\right\}} Q(x)f(u_m)V_m dx,
\end{aligned}
$$

such that

$$
\int_{\mathbb{R}^2} Q(x)f(u_m)V_m dx \leq I_m + J_m.
$$

By Young's Inequality,

$$\begin{aligned} I_m &\le \frac{1}{2}\int_{\{x\in\mathbb{R}^2: f(u_m)(x)\le e^{1/4}\}} Q(x)\left((f(u_m))^2+V_m^2\right)dx \\ &= \frac{1}{2}\int_{\{x\in\mathbb{R}^2: f(u_m)(x)\le e^{1/4}\}} Q(x)(f(u_m))^2 dx \\ &\quad+\frac{1}{2}\int_{\{x\in\mathbb{R}^2: f(u_m)(x)\le e^{1/4}\}} Q(x)V_m^2 dx. \end{aligned} \tag{3.17}$$

Since $\|V_m\|=1$, by the continuous embedding (3.7),

$$\int_{\{x\in\mathbb{R}^2: f(u_m)(x)\le e^{1/4}\}} Q(x)V_m^2 \le C. \tag{3.18}$$

By $(h_0)-(h_1)$, there exists constants $c>0$ and $s_0>0$ such that

$$|f(s)|\le c|s| \text{ for } s\in[0,s_0].$$

We also have

$$|f(s)|\le \frac{e^{1/4}}{s_0}s, \text{ for } s\in(s_0,\infty).$$

Therefore,

$$\int_{\{x\in\mathbb{R}^2: f(u_m)(x)\le e^{1/4}\}} Q(x)(f(u_m))^2 dx \le C\int_{\{x\in\mathbb{R}^2: f(u_m)(x)\le e^{1/4}\}} Q(x)f(u_m)u_m dx. \tag{3.19}$$

From (3.17), (3.18) and (3.19), it follows that

$$I_m \le C+\int_{\mathbb{R}^2} Q(x)f(u_m)u_m dx. \tag{3.20}$$

Observe that if $f(s)$ has subcritical growth, for each $\mu>0$, there exists a constant $C>0$ such that for all $s\in[0,\infty)$,

$$f(s)\le Ce^{\mu s^2}. \tag{3.21}$$

Similarly, if $f(s)$ has critical growth with critical exponent $\mu_0>0$, for each $\mu>\mu_0$, there exists $C>0$ such that we obtain the same estimate.

Choosing $\delta > 0$ small enough such that $\delta < 4\pi$, by Lemma 3.30, we obtain

$$\begin{aligned} J_m &= \frac{C}{\delta} \int_{\{x\in\mathbb{R}^2: f(u_m)(x)\geq e^{1/4}\}} Q(x)\frac{f(u_m)}{C}\delta V_m dx \\ &\leq \delta C \int_{\mathbb{R}^2} Q(x)(e^{\delta^2 V_m^2}-1)dx \\ &\quad + \frac{1}{\delta}\int_{\{x\in\mathbb{R}^2: f(u_m)(x)\geq e^{1/4}\}} Q(x)f(u_m)\left[\log\left(\frac{f(u_m)}{C}\right)\right]^{1/2} dx. \end{aligned} \tag{3.22}$$

First, we have that

$$\int_{\mathbb{R}^2} Q(x)(e^{\delta^2 V_m^2}-1)dx \leq C. \tag{3.23}$$

By (3.21),

$$\int_{\{x\in\mathbb{R}^2: f(u_m)(x)\geq e^{1/4}\}} Q(x)f(u_m)\left[\log\left(\frac{f(u_m)}{C}\right)\right]^{1/2} dx \leq C\int_{\mathbb{R}^2} Q(x)f(u_m)u_m dx.$$

By this estimate, together with (3.22) and (3.23), we have

$$J_m \leq C + \int_{\mathbb{R}^2} Q(x)f(u_m)u_m dx. \tag{3.24}$$

Thus, by (3.20) and (3.24), we get

$$\int_{\mathbb{R}^2} Q(x)f(u_m)V_m dx \leq C_1 + C_2\int_{\mathbb{R}^2} Q(x)f(u_m)u_m dx.$$

Combining this estimate with (3.15) implies

$$\|v_m\| \leq C_1 + C_2\int_{\mathbb{R}^2} Q(x)f(u_m)u_m dx.$$

By the same argument above, we reach

$$\|u_m\| \leq C_1 + C_2\int_{\mathbb{R}^2} Q(x)g(v_m)v_m dx.$$

We obtain

$$\|(u_m, v_m)\| \leq C(C_1 + \delta_m + \varepsilon_m\|(u_m, v_m)\| + \varepsilon_m),$$

which implies that $\|(u_m, v_m)\| \leq C$.

■

By the boundedness of $(u_n, v_n)_n$ in $(E, \|\cdot\|)$, we may assume that $u_n \rightharpoonup u_0$ and $v_n \rightharpoonup v_0$ in $H^1_V(\mathbb{R}^2)$.

In view of Proposition 3.10, we may apply [26, Lemma 2.1]:

$$Q(x)f(u_n) \to Q(x)f(u_0),$$
$$Q(x)g(v_n) \to Q(x)g(v_0),$$

in $L^1_{loc}(\mathbb{R}^2)$.

Using this, we can prove that (u_0, v_0) is a weak solution of (4). To prove that this solution is nontrivial we will prove the following result:

Lemma 3.11. *If $(u_m, v_m) \subset E$ is a sequence such that $I(u_m, v_m) \to c$, $I'(u_m, v_m) \to 0$ and (u_0, v_0) is its weak limit, then*

$$\int_{\mathbb{R}^2} Q(x)F(u_m) \to \int_{\mathbb{R}^2} Q(x)F(u_0),$$
$$\int_{\mathbb{R}^2} Q(x)G(v_m) \to \int_{\mathbb{R}^2} Q(x)G(v_0).$$

Proof. From (h_3) and (h_4), it follows

$$0 \leq \lim_{|s|\to\infty} \frac{F(s)}{sf(s)} \leq \lim_{|s|\to\infty} \frac{M_0}{|s|} = 0,$$

thus, for any $\varepsilon > 0$ there exists $s_0 = s_0(\varepsilon) > 0$ such that

$$F(x, s) \leq \varepsilon s f(s) \text{ for any } s \in [s_0, \infty].$$

Since $(u_0, v_0) \in E$ and recalling Proposition 3.10, we obtain

$$\int_{\mathbb{R}^2} Q(x)f(u_0)u_0 dx \leq C \text{ and } \int_{\mathbb{R}^2} Q(x)g(v_n)v_n dx \leq C,$$

for any $n \geq 1$, for some constant $C > 0$.

Therefore, for a fixed $\varepsilon > 0$, we have

$$\int_{\{|u_0|\geq s_0\}} Q(x)F(u_0)dx \leq \varepsilon \int_{\{|u_0|\geq s_0\}} Q(x)f(u_0)u_0 dx \leq C\varepsilon$$

and

$$\int_{\{|u_n|\geq s_0\}} Q(x)F(u_n)dx \leq \varepsilon \int_{\{|u_n|\geq s_0\}} Q(x)f(u_n)u_n dx \leq C\varepsilon.$$

Now, we define

$$h_n(x) := Q(x)\chi_{\{|u_n|<s_0\}}F(u_n) \text{ and } h(x) := Q(x)\chi_{\{|u_0|<s_0\}}F(u_0).$$

Then (h_n) is a sequence of measurable functions and

$$h_n(x) \to h(x) \text{ a.e. for } x \in \mathbb{R}^2.$$

Using (3.12) with $\mu > \mu_0$, $q = 2$ and $\varepsilon > 0$, we obtain for any $|s| \leq s_0$

$$|F(s)| \leq \varepsilon|s|^2 + C(\mu,\varepsilon)|s|^2(e^{\mu|s|^2} - 1) \leq C(\mu,\varepsilon,s_0)|s|^2.$$

Now, we let

$$\zeta_n(x) := C(\mu,\varepsilon,s_0)Q(x)u_n^2 \text{ and } \zeta(x) := C(\mu,\varepsilon,s_0)Q(x)u_0^2.$$

This yields $0 \leq h_n(x) \leq \zeta_n(x)$, $x \in \mathbb{R}^2$. Note that $(\zeta_n)_n$ is a sequence of measurable functions, $\zeta_n(x) \to \zeta(x)$ a.e. in $\mathbb{R}^2$, and by the compact embedding 3.7,

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} \zeta_n(x)dx = \int_{\mathbb{R}^2} \zeta(x).$$

Thus, applying the generalized Lebesgue dominated convergence theorem, we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} h_n(x)dx = \int_{\mathbb{R}^2} h(x).$$

Therefore, for any fixed $\varepsilon > 0$, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} Q(x)F(u_n)dx - \int_{\mathbb{R}^2} Q(x)F(u_0)dx\right| \\
&\leq \int_{\{|u_n|\geq s_0\}} Q(x)F(u_n)dx + \int_{\{|u_0|\geq s_0\}} Q(x)F(u_0)dx \\
&\quad + \left|\int_{\{|u_n|<s_0\}} Q(x)F(u_n)dx - \int_{\{|u_0|<\overline{s}\}} Q(x)F(x,u_0)dx\right| \\
&\leq 2C\varepsilon + \left|\int_{\mathbb{R}^2} h_n(x)dx - \int_{\mathbb{R}^2} h(x)dx\right|
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we complete the proof.

■

3.4 Finite dimensional problem

Associated with the eigenvalues $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_j \to \infty$ of $(-\Delta + V(x), H^1_V(\mathbb{R}^2))$, there exists an orthonormal basis $\{\varphi_1, \varphi_2, \dots\}$ corresponding eigenfunctions in $H^1_V(\mathbb{R}^2)$. We set

$$\begin{aligned}
E_n^+ &= \operatorname{span}\{(\varphi_i, \varphi_i) : i = 1, \dots, n\}, \\
E_n^- &= \operatorname{span}\{(\varphi_i, -\varphi_i) : i = 1, \dots, n\}, \\
E_n &= E_n^+ \oplus E_n^-.
\end{aligned}$$

Let $y \in H^1_V(\mathbb{R}^2)$ be a fixed nonnegative function and

$$Q_{n,y} = \{r(y,y) + w : w \in E_n^-, \|w\|_E \leq R_0 \text{ and } 0 \leq r \leq R_1\},$$

where R_0 and R_1 are given in Lemma 3.9.

We establish the following notation

$$H_{n,y} = \mathbb{R}(y,y) \oplus E_n, \quad H_{n,y}^+ = \mathbb{R}(y,y) \oplus E_n^+, \quad H_{n,y}^- = \mathbb{R}(y,y) \oplus E_n^-,$$

define the class of mappings

$$\Gamma = \{h \in C(Q_{n,y}, H_{n,y}) : h(z) = z \text{ on } \partial Q_{n,y}\}$$

and set

$$c_{n,y} = \inf_{h\in\Gamma_{n,y}} \max_{z\in Q_{n,y}} I(h(z)).$$

Using the intersection theorem due [53, Proposition 5.9], we have

$$h(Q_{n,y} \cap (\partial B_\rho \cap E_n^+) \neq \emptyset, \text{ for all } h \in \Gamma_{n,y},$$

which in combination with Lemma 3.9, implies that $c_{n,y} \geq \sigma > 0$. An upper bound for the min-max level $c_{n,y}$ comes from the fact that since the identity mapping $I : Q_{n,y} \to H_{n,y}$ belongs to $\Gamma_{n,y}$, we have for $z = r(y, y) + (u, -u) \in Q_{n,y}$ that

$$I(z) = R_1^2\|y\|^2 - \|u\|^2 - \int_{\mathbb{R}^2} Q(x)[F(R_1 y + u) + G(R_1 y - u)]dx \leq R_1^2\|y\|^2.$$

Thus, we have $0 < \sigma \leq c_{n,y} \leq R_1^2\|y\|^2$.

We denote by $I_{n,y}$ the functional I restricted to the finite-dimensional subspace $H_{n,y}$. By the lemmas 3.8 and 3.9, the geometry of the linking theorem holds, thus applying the linking theorem for $I_{n,y}$ yields a (PS)-sequence, which is bound by Proposition 3.10. Since $H_{n,y}$ is finite-dimensional, we get the following result.

Proposition 3.12. *For each $n \in \mathbb{N}$ and for each $y \in H_V^1$ a fixed nonnegative function, the functional $I_{n,y}$ has a critical point at level $c_{n,y}$. More precisely, there is a $z_{n,y} \in H_{n,y}$ such that*

$$I_{n,y}(z_{n,y}) = c_{n,y} \in [\sigma, R_1^2\|y\|^2] \text{ and } (I_{n,y})'(z_{n,y}) = 0.$$

Furthermore, $\|z_{n,y}\| \leq C$.

3.5 The subcritical case

Theorem 3.13. *Suppose $f(s)$ and $g(s)$ have subcritical exponential growth. Then (4) possesses a nontrivial weak solution $(u, v) \in E$.*

Proof. Let $y \in H_V^1(\mathbb{R}^2)$ be a nonnegative function. By Proposition 3.12, we have a sequence $z_{n,y} = (u_{n,y}, v_{n,y}) \in H_{n,y}$ such that $\|z_{n,y}\| \leq C$ and

$$\begin{aligned} & I_{n,y}(z_{n,y}) = c_{n,y} \in [\sigma, R_1^2\|y\|^2], \\ & (I_{n,y})'(z_{n,y}) = 0, \\ & (u_{n,y}, v_{n,y}) \rightharpoonup (u_0, v_0) \text{ in } E. \end{aligned} \tag{3.25}$$

By Proposition 3.10, we get

$$\begin{aligned} & \int_{\mathbb{R}^2} Q(x)f(u_{n,y})u_{n,y}dx \leq C \text{ and } \int_{\mathbb{R}^2} Q(x)g(v_{n,y})v_{n,y}dx \leq C \\ & \int_{\mathbb{R}^2} Q(x)F(u_{n,y})dx \leq C \text{ and } \int_{\mathbb{R}^2} Q(x)G(u_{n,y})dx \leq C. \end{aligned}$$

Substituting the test function $(0, \psi)$ and $(\varphi, 0)$ in (3.25), where $\psi, \varphi \in C_0^\infty(\mathbb{R}^2)$ are arbitrary functions, we reach

$$\begin{aligned} & \int_{\mathbb{R}^2} (\nabla u_{n,y} \nabla \psi + V(x)u_{n,y}\psi)dx = \int_{\mathbb{R}^2} Q(x)g(v_{n,y})\psi dx \\ & \int_{\mathbb{R}^2} (\nabla v_{n,y} \nabla \varphi + V(x)v_{n,y}\varphi)dx = \int_{\mathbb{R}^2} Q(x)f(u_{n,y})\varphi dx. \end{aligned}$$

Taking the limit and using the fact that $C_0^\infty(\mathbb{R}^2)$ is dense in $H_V^1(\mathbb{R}^2)$, we have

$$\begin{aligned} & \int_{\mathbb{R}^2} (\nabla u_0 \nabla \psi + V(x)u_0\psi)dx = \int_{\mathbb{R}^2} Q(x)g(v_0)\psi dx \text{ for all } \psi \in H_V^1(\mathbb{R}^2), \\ & \int_{\mathbb{R}^2} (\nabla v_0 \nabla \varphi + V(x)v_0\varphi)dx = \int_{\mathbb{R}^2} Q(x)f(u_0)\varphi dx \text{ for all } \varphi \in H_V^1(\mathbb{R}^2). \end{aligned}$$

Now we prove u_0 and v_0 are nontrivial. Assume by contradiction that $u_0 \equiv 0$, which implies that $v_0 \equiv 0$. Since g has subcritical growth, we see that for all $\nu > 0$, there exists $C_1, C_2 > 0$ such that

$$g(x, s) \leq C_1 s + C_2 s^2(e^{\nu s^2} - 1) \text{ for all } s \in [0, \infty).$$

Combining Holder inequality and Lemma 3.5, and choosing $\nu > 0$ and $q > 1$ sufficiently close to 1 such that $\nu q\|v_{n,y}\|^2 < 4\pi$ and $s > 2$,

$$\begin{aligned}\int_{\mathbb{R}^2} Q(x)g(v_{n,y})u_{n,y}dx &\leq \int_{\mathbb{R}^2} Q(x)v_{n,y}u_{n,y} + C_2Q(x)v_{n,y}^2u_{n,y}(e^{\nu v_{n,y}^2}-1)\\ &\leq C_1\|v_{n,y}\|_{L^2_Q(\mathbb{R}^2)}\|u_{n,y}\|_{L^2_Q(\mathbb{R}^2)}\\ &\quad + C_2\left(\int_{\mathbb{R}^2} Q(x)(e^{q\nu\|v_{n,y}\|^2(v_{n,y}/\|v_{n,y}\|)^2}-1)dx\right)^{1/q}\|u_{n,y}\|_{L^{2s}_Q(\mathbb{R}^2)}\|v_{n,y}\|^2_{L^{2s}_Q(\mathbb{R}^2)}\\ &\leq C_1\|v_{n,y}\|_{L^2_Q(\mathbb{R}^2)}\|u_{n,y}\|_{L^2_Q(\mathbb{R}^2)} + C\|u_{n,y}\|_{L^{2s}_Q(\mathbb{R}^2)}\|v_{n,y}\|^2_{L^{2s}_Q(\mathbb{R}^2)}.\end{aligned}$$

Thus,

$$\int_{\mathbb{R}^2}(|\nabla u_{n,y}|^2+V(x)|u_{n,y}|^2)dx = \int_{\mathbb{R}^2} Q(x)g(v_{n,y})u_{n,y} \to 0,$$

since $v_{n,y}\to 0$ and $u_{n,y}\to 0$ in $L^p_Q(\mathbb{R}^2)$ for all $p\geq 2$. Consequently, $u_{n,y}\to 0$ in $H^1_V(\mathbb{R}^2)$. This implies that

$$\int_{\mathbb{R}^2}(\nabla u_{n,y}\nabla v_{n,y}+V(x)u_{n,y}v_{n,y})dx \to 0.$$

Using this and the fact that $(I_{n,y})'(u_{n,y},u_{n,y})=0$, we reach that

$$\int_{\mathbb{R}^2} Q(x)f(u_{n,y})u_{n,y}\to 0 \text{ and } \int_{\mathbb{R}^2} Q(x)g(v_{n,y})v_{n,y}\to 0.$$

By these limits and (h_3), we get

$$\int_{\mathbb{R}^2} Q(x)F(u_{n,y})dx\to 0 \text{ and } \int_{\mathbb{R}^2} Q(x)G(v_{n,y})dx\to 0.$$

Therefore, we see that $c_{n,y}=0$, which is a contradiction.

■

3.6 The critical case

3.6.1 Proof of Theorem 3.4

Next, we assume that f and g have critical growth with critical exponent μ_0. By Lemma 3.14, there is $\delta>0$ such that

$$c_n := c_{n,\overline{u}} \leq \frac{4\pi}{\mu_0}-\delta,$$

where $c_{n,\overline{u}}$ is defined in Proposition 3.12.

By Propositions 3.10 and 3.12, we obtain a sequence $z_n := z_{n,\overline{u}} = (u_n, v_n) \in H_{n,\overline{u}}$ such that $\|(u_n, v_n)\| \leq C$ and

$$I_{n,\overline{u}}(u_n, v_n) = c_n \in \left[\sigma, \frac{4\pi}{\mu_0} - \delta\right), \tag{3.26}$$

$$(I_{n,\overline{u}})'(u_n, v_n) = 0, \tag{3.27}$$

$$(u_n, v_n) \rightharpoonup (u_0, v_0) \text{ weakly in } E.$$

By Proposition 3.10 and arguing as in the subcritical case

$$\int_{\mathbb{R}^2} (\nabla u_0 \nabla \psi + V(x) u_0 \psi) dx = \int_{\mathbb{R}^2} Q(x) g(v_0) \psi dx \text{ for all } \psi \in H^1_V,$$
$$\int_{\mathbb{R}^2} (\nabla v_0 \nabla \varphi + V(x) v_0 \varphi) dx = \int_{\mathbb{R}^2} Q(x) f(u_0) \varphi dx \text{ for all } \varphi \in H^1_V.$$

Now, we prove by contradiction that u_0 and v_0 are nontrivial. Assume that $u_0 \equiv 0$, which implies that $v_0 \equiv 0$. If $\|u_n\| \to 0$, we get $\langle u_n, v_n \rangle \to 0$, a contradiction. Thus, we assume that $u_n \geq b > 0$ for all $n \in \mathbb{N}$ and consider

$$\|u_n\|^2 = \int_{\mathbb{R}^2} Q(x) g(v_n) u_n dx. \tag{3.28}$$

By (h_0), for any fixed $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|g(t)| \leq C_\varepsilon e^{(\mu_0 + \varepsilon)t^2}, \text{ for } t \in \mathbb{R}^2. \tag{3.29}$$

Define $\overline{u}_n = \left(\frac{4\pi}{\mu_0} - \delta\right)^{1/2} \frac{u_n}{\|u_n\|}$. Applying the following inequality, which was proved in [38, Lemma 4.1],

$$st \leq t^2(e^{t^2} - 1) + |s|(\log)^{1/2}, \text{ for all } t \geq 0 \text{ and } s \geq e^{1/\sqrt[3]{4}}. \tag{3.30}$$

Applying (3.30) with $s = g(v_n)/\mu_0$ and $t = \sqrt{\mu_0}\overline{u}_n$ and arguing as in Proposition 3.10, we get

$$\left(\frac{4\pi}{\mu_0}-\delta\right)^{1/2}\|u_n\| = \int_{\mathbb{R}^2} g(v_n)\overline{u}_n dx \tag{3.31}$$
$$= \frac{1}{\sqrt{\mu_0}}\int_{\mathbb{R}^2} Q(x)g(v_n)\sqrt{\mu_0}\overline{u}_n dx$$
$$\le \int_{\left\{x\in\mathbb{R}^2: g(v_n(x))/\mu_0\le e^{1/4}\right\}} Q(x)\left[(e^{\mu_0\overline{u}_n^2}-1)+\frac{1}{2}\frac{[g(v_n)]^{1/2}}{\mu_0}\right]dx$$
$$+\int_{\left\{x\in\mathbb{R}^2: g(v_n(x))/\mu_0\ge e^{1/4}\right\}} Q(x)\left[(e^{\mu_0\overline{u}_n^2}-1)+\frac{g(v_n)}{\sqrt{\mu_0}}\left[\log\left(\frac{g(v_n)}{\sqrt{\mu_0}}\right)\right]^{1/2}\right]dx$$
$$= J_{1,n}+J_{2,n}+J_{3,n},$$

where

$$J_{1,n} = \int_{\mathbb{R}^2} Q(x)(e^{\mu_0\overline{u}_n^2}-1)dx,$$
$$J_{2,n} = \frac{1}{2}\int_{\left\{x\in\mathbb{R}^2: g(v_n(x))/\mu_0\le e^{1/4}\right\}} Q(x)\frac{[g(v_n)]^{1/2}}{\mu_0}dx$$
$$J_{3,n} = \int_{\left\{x\in\mathbb{R}^2: g(v_n(x))/\mu_0\ge e^{1/4}\right\}} Q(x)\frac{g(v_n)}{\sqrt{\mu_0}}\left[\log\left(\frac{g(v_n)}{\sqrt{\mu_0}}\right)\right]^{1/2}dx.$$

Let us estimate these integrals. Since $\|\overline{u}_n\|^2 = \frac{4\pi}{\mu_0}-\delta$, by Theorem 3.6, we obtain that $J_{1,n}\to 0$.

Using $(h_1)-(h_3)$, there exists a constant $C>0$ such that

$$(g(v_n))^2 \le C|v_n|^2 \text{ in } \left\{x\in\mathbb{R}^2 : g(v_n(x))\le e^{1/4}\right\}.$$

By the compact embedding, we get $J_{2,n}\to 0$.

Using the critical growth hypothesis, we can estimate $J_{3,n}$. Observe that

$$J_{3,n} \le \frac{1}{\sqrt{\mu_0}}\int_{\mathbb{R}^2} Q(x)g(v_n)(\mu_0+\varepsilon)^{1/2}v_n dx$$
$$\le \left(1+\frac{\varepsilon}{\mu_0}\right)^{1/2}\int_{\mathbb{R}^2} Q(x)g(v_n)v_n dx$$

Substituting this in (3.31), we get

$$\left(\frac{4\pi}{\mu_0}-\delta\right)^{1/2}\|u_n\| \le o_n(1)+\left(1+\frac{\varepsilon}{\mu_0}\right)^{1/2}\int_{\mathbb{R}^2} Q(x)g(v_n)v_n dx. \tag{3.32}$$

Using the same argument

$$\|v_n\|^2 = \int_{\mathbb{R}^2} Q(x)f(u_n)v_n dx,$$

we see that

$$\left(\frac{4\pi}{\mu_0} - \delta\right)^{1/2} \|v_n\| \leq o_n(1) + \left(1 + \frac{\varepsilon}{\mu_0}\right)^{1/2} \int_{\mathbb{R}^2} Q(x)f(u_n)u_n dx. \tag{3.33}$$

By (3.26),

$$\int_{\mathbb{R}^2} [\nabla u_n \nabla v_n + V(x)u_n v_n] - \int_{\mathbb{R}^2} Q(x)[F(u_n) + G(v_n)] \leq \frac{4\pi}{\mu_0} - \delta,$$

and since

$$\int_{\mathbb{R}^2} Q(x)F(u_n) \to 0 \quad \text{and} \quad \int_{\mathbb{R}^2} Q(x)G(v_n) \to 0,$$

we obtain

$$|\langle u_n, v_n\rangle| \leq o_n(1) + \frac{4\pi}{\mu_0} - \delta,$$

By (3.27),

$$\int_{\mathbb{R}^2} [\nabla u_n \nabla v_n + V(x)u_n v_n] - \int_{\mathbb{R}^2} Q(x)[F(u_n) + G(v_n)] = 0,$$

and this implies

$$\int_{\mathbb{R}^2} Q(x)f(u_n)u_n dx + \int_{\mathbb{R}^2} Q(x)g(v_n)v_n dx \leq o_n(1) + 2\left(\frac{4\pi}{\mu_0} - \delta\right)$$

So, combining (3.32) and (3.33), we get

$$\|u_n\| + \|v_n\| \leq o_n(1) + 2\left(1 + \frac{\varepsilon}{\mu_0}\right)^{1/2} \left(\frac{4\pi}{\mu_0} - \delta\right)$$

for $\varepsilon > 0$ sufficiently small and n sufficiently large.

Therefore, without loss of generality, we can assume that there is a subsequence of (v_n) such that

$$\|v_n\| \leq \left(\frac{4\pi}{\mu_0} - \delta\right)^{1/2}.$$

Consequently,

$$\int_{\mathbb{R}^2} Q(x)g(v_n)v_n dx \to 0. \tag{3.34}$$

Hence,

$$\langle u_n, v_n \rangle \to 0,$$

which implies $c_n = 0$, which is a contradiction. This completes the proof.

3.6.2 Level estimate

For the next result, we assume that there exists $\gamma_0 > 0$ such that the functions f, g satisfy

$$\liminf_{|s|\to\infty} \frac{sf(s)}{e^{\gamma_0 s^2}}, \quad \liminf_{|s|\to\infty} \frac{sg(s)}{e^{\gamma_0 s^2}} =: \beta_0 > \mathcal{M},$$

where $\mathcal{M} =: \inf_{r>0} \frac{4e^{1/2r^2 V_{\max,r}}}{\gamma_0 r^2 Q_{\min,r}}$, $V_{\max,r} := \max_{|x|\le r} V(x) > 0$ and $Q_{\min,r} := \min_{|x|\le r} Q(x) > 0$.

For fixed $r > 0$, we consider the Moser sequence

$$\widetilde{w}_n(x) := \frac{1}{\sqrt{2\pi}} \begin{cases} \sqrt{\log n}, & \text{if } |x| \le r/n, \\ \dfrac{\log r/|x|}{\sqrt{\log n}}, & \text{if } r/n \le |x| \le r, \\ 0, & \text{if } |x| \ge r. \end{cases}$$

It is well known that $\widetilde{w}_n \in H_0^1(B_r) \subset H_V^1(\mathbb{R}^2)$ and it is possible to prove that

$$1 \le \|\widetilde{w}_n\|^2 \le 1 + \frac{d_n(r)}{\log n} V_{\max,r},$$

where $d_n(r) := r^2/4 + o_n(1)$ and $o_n(1) \to 0$ as $n \to \infty$.

Let $w_n := \widetilde{w}_n/\|\widetilde{w}_n\| \in H_0^1(B_r) \subset H_V^1(\mathbb{R}^2)$. We have $\|w_n\| = 1$ and for $|x| \le r/n$

$$w_n^2(x) = \frac{1}{2\pi} \log n \left(\frac{1}{\|\widetilde{w}_n\|^2} \right) \ge \frac{1}{2\pi} \left(\log n - \frac{d_n(r) V_{\max,r}}{\|\widetilde{w}_n\|^2} \right) \ge \frac{1}{2\pi} \left(\log n - d_n(r) V_{\max,r} \right) \tag{3.35}$$

Lemma 3.14. *There exists a nonnegative function $\overline{u} \in H_V^1$ such that*

$$\sup_{\eta \in \mathbb{R}_+(\overline{u},\overline{u}) \oplus E^-} I(\eta) < \frac{4\pi}{\gamma_0}. \tag{3.36}$$

We prove by contradiction. Suppose that for all $k \in \mathbb{N}$

$$\sup_{\eta \in \mathbb{R}_+(\overline{u},\overline{u})\oplus E^-} I(\eta) \geq \frac{4\pi}{\gamma_0}. \tag{3.37}$$

Thus, for all fixed $k \geq 1$, there exists a nonnegative sequence $\zeta_n^k \to 0$ as $n \to \infty$ and a sequence

$$\eta_{n,k} = \tau_{n,k}(w_k, w_k) + (u_{n,k}, -u_{n,k}), \quad u_{n,k} \in H_V^1(\mathbb{R}^2),$$

such that $I(\eta) \geq \dfrac{4\pi}{\gamma_0} - \zeta_n^k$.

Observe that if $h : [0, \infty) \to \mathbb{R}$ is a function such that $h(t) = I(t\eta)$. Since $h(0) = 0$ and $\lim_{t\to 0} h(t) = -\infty$, there exists a maximum point $t_{n,k}$, that is,

$$I(t_{n,k}\eta_{n,k}) \geq I(\eta_{n,k}) \geq \frac{4\pi}{\gamma_0} - \zeta_n^k \quad \text{and} \quad I'(t_{n,k}\eta_{n,k}) = 0.$$

Renaming $t_{n,k}\eta_{n,k} := \eta_{n,k}$, we may assume, without loss of generality, that $t_{n,k} = 1$.

This means that

$$\begin{aligned}
&\tau_{n,k}^2\|w_k\|^2 - \|u_{n,k}\|^2 - \int_{\mathbb{R}^2} Q(x)F(\tau_{n,k}w_k + u_{n,k})dx - \int_{\mathbb{R}^2} Q(x)G(\tau_{n,k}w_k - u_{n,k})dx \\
&\geq \frac{4\pi}{\gamma_0} - \zeta_n^k
\end{aligned} \tag{3.38}$$

and

$$\begin{aligned}
\tau_{n,k}^2\|w_k\|^2 - \|u_{n,k}\|^2 = &\int_{\mathbb{R}^2} Q(x)f(\tau_{n,k}w_k + u_{n,k})(\tau_{n,k}w_k + u_{n,k})dx \\
&- \int_{\mathbb{R}^2} Q(x)g(\tau_{n,k}w_k - u_{n,k})(\tau_{n,k}w_k - u_{n,k})dx.
\end{aligned} \tag{3.39}$$

Hence,

$$\tau_{n,k}^2 \geq \frac{4\pi}{\gamma_0} - \zeta_n^k \tag{3.40}$$

and

$$\begin{aligned}
\tau_{n,k}^2 \geq &\int_{\mathbb{R}^2} Q(x)f(\tau_{n,k}w_k + u_{n,k})(\tau_{n,k}w_k + u_{n,k})dx \\
&- \int_{\mathbb{R}^2} Q(x)g(\tau_{n,k}w_k - u_{n,k})(\tau_{n,k}w_k - u_{n,k})dx.
\end{aligned} \tag{3.41}$$

Set $\beta_0 > 0$ such that the condition (3.1) is satisfied. Thus, given $\varepsilon > 0$, there exists

$R_\varepsilon > 0$ such that

$$tf(t), tg(t) \geq (\beta_0 - \varepsilon)e^{\gamma_0 t^2}, \quad \text{for all} \quad t \geq R_\varepsilon. \tag{3.42}$$

Choosing k sufficiently large, there exists $k_0 > 0$ such that

$$\tau_{n,k}\sqrt{\frac{\log k}{2\pi}} \geq R_\varepsilon, \quad \text{for all} k \geq k_0.$$

Since

$$w_k(x) = \sqrt{\frac{\log k}{2\pi}} \quad \text{for all} \quad x \in B_{r/k},$$

we get

$$\max\{\tau_{n,k}w_k + u_k, \tau_{n,k}w_k - u_k\} \geq \tau_{n,k}w_k \geq R_\varepsilon, \quad \text{for all} \quad x \in B_{r/k}.$$

Let $x_k \in B_{r/k}$ be the minimum point of the weight Q on $B_{r/k}$, that is,

$$Q(x_k) := \min_{|x|\leq r/k} Q(x) \geq Q_{\min,r}. \tag{3.43}$$

Observe that $\lim_{k\to\infty} Q(x_k) = Q(0) > 0$. Therefore, using (3.41), (3.42) and (3.35), we get

$$\begin{aligned}
\tau_{n,k}^2 \geq & \int_{\mathbb{R}^2} Q(x)f(\tau_{n,k}w_k + u_{n,k})(\tau_{n,k}w_k + u_{n,k})dx \\
& - \int_{\mathbb{R}^2} Q(x)g(\tau_{n,k}w_k - u_{n,k})(\tau_{n,k}w_k - u_{n,k})dx \\
& \geq (\beta_0 - \varepsilon)\int_{B_{r/k}} Q(x)e^{\gamma_0(\tau_{n,k}w_k)^2}dx \\
& \geq (\beta_0 - \varepsilon)Q_{\min,r}\left(\frac{r}{k}\right)^2 e^{\frac{\gamma_0}{2\pi}\tau_{n,k}^2[\log k - d_k(r)V_{\max,r}]},
\end{aligned}$$

when $|x| \leq r/k$.

Setting $s_{n,k} := \tau_{n,k}^2 - \dfrac{4\pi}{\gamma_0}$, we have

$$s_{n,k} + \frac{4\pi}{\gamma_0} \geq (\beta_0 - \varepsilon)Q_{\min,r}r^2 e^{\frac{\gamma_0}{2\pi}s_{n,k}[\log k - d_k(r)V_{\max,r}]}e^{-2d_k(r)V_{\max,r}}. \tag{3.44}$$

This inequality implies that $\{s_{n,k}\}_n$ is bounded for each $k \geq k_0$. Thus, there exists $s_k \in \mathbb{R}^2$ such that $\limsup_{n\to\infty} = s_k$. By (3.40), we have that $s_k \geq 0$. Observe that $s_k \to 0$

as $k \to \infty$. In fact, taking the $\limsup_{n\to\infty}$, we have

$$\frac{s_k}{e^{\frac{\gamma_0}{2\pi}s_k[\log k - d_k(r)V_{\max,r}]}} + \frac{4\pi}{\gamma_0}\frac{1}{e^{\frac{\gamma_0}{2\pi}s_k[\log k - d_k(r)V_{\max,r}]}} \geq (\beta_0 - \varepsilon)Q_{\min,r}r^2 e^{-2d_k(r)V_{\max,r}}.$$

If $s_k \to \rho > 0$, as $k \to \infty$, we have $(\beta_0 - \varepsilon)Q_{\min,r}r^2 e^{-\frac{r^2}{2}V_{\max,r}} \leq 0$, which is an absurd. Now, we observe two cases:

$$s_k \log k \to +\infty \quad \text{or} \quad s_k \log k \to \rho > 0 \quad \text{as} \quad k \to +\infty.$$

If $s_k \log k \to +\infty$, since (3.44), we get a contradiction. If $s_k \log k \to \rho > 0$, we have

$$\frac{4\pi}{\gamma_0} \geq (\beta_0 - \varepsilon)Q_{\min,r}r^2 e^{\frac{\gamma_0}{2\pi}\rho} e^{-\frac{r^2}{2}V_{\max,r}} \geq (\beta_0 - \varepsilon)Q_{\min,r}r^2 e^{-\frac{r^2}{2}V_{\max,r}}$$

and this contradicts (3.1), since $\varepsilon > 0$ is arbitrary.

Chapter 4

Periodic Schrödinger equations with exponential growth without the Ambrosetti-Rabinowitz condition

In this chapter, we establish the existence of nontrivial solutions for the following Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = f(x,u), \\ u \in H^1(\mathbb{R}^2), \end{cases} \tag{2}$$

where V is a 1-periodic function with respect to x, 0 lies in the gap of the spectrum of $-\Delta + V$ and the nonlinear term $f(x,s)$ has critical exponential growth.

In the two dimensional case, when $f(x,u)$ has exponential growth on u and $V(x)$ is a positive potential bounded away from zero (i.e. the definite case), motivated by the Moser–Trudinger inequality, the existence of nontrivial solutions to has been studied by many authors (see, [8, 9, 16, 26, 28, 30, 39, 42]).

However, let $\sigma(\mathcal{S})$ be the spectrum of the operator $-\Delta + V$ defined in $L^2(\mathbb{R}^2)$. When V is continuous and periodic, it is known that $\sigma(\mathcal{S})$ is purely continuous, bounded from below and is the union of disjoint closed intervals (see [55, Theorem XIIL 100]). In this chapter, we study the case where 0 lies in the spectral gap, precisely,

(V) $V(x) = V(x_1, x_2)$ is continuous, 1-periodic in the variables x_1, x_2 and

$$\lambda := \sup[\sigma(\mathcal{S}) \cap (-\infty, 0)] < 0 < \Lambda := \inf[\sigma(\mathcal{S}) \cap (0, \infty)].$$

This way, when (V) holds and the operator $-\Delta+V$ has a purely continuous spectrum consisting of closed disjoint intervals, we have the indefinite case. We can reference [6] for a study with a subcritical exponential growth and [7, 20, 36] concerning the existence of nontrivial solutions for (2) with a exponential growth of the critical type.

With respect to the function f, we assume the following conditions:

(F_0) $f \in C^1(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R})$ and $f(x_1, x_2, u)$ is 1-periodic in the variables x_1, x_2;

(F_1) (Critical exponential growth) there exists $\alpha_0 > 0$ such that

$$\lim_{|u|\to\infty} \frac{|f(x,u)|}{e^{\alpha|u|^2}} = \begin{cases} 0, & \text{if } \alpha > \alpha_0, \\ +\infty, & \text{if } \alpha < \alpha_0, \end{cases}$$

uniformly in $x \in \mathbb{R}^2$;

(F_2) $f(x,u) = o(u)$ as $|u| \to 0$, uniformly in $x \in \mathbb{R}^2$;

(F_3) $f(x,t)/|t|$ is stricly increasing in $(-\infty, 0)$ and $(0, \infty)$ for every $x \in \mathbb{R}^2$;

(F_4) there exists $R_0, M_0 > 0$ such that

$$0 < F(x,u) \leq M_0|f(x,u)|, \quad \text{for all} \quad x \in \mathbb{R}^2 \quad \text{and} \quad |u| \geq R_0;$$

(F_5) $\limsup_{|u|\to\infty} \dfrac{|u||f(x,u)|}{\widehat{F}(x,u)} =: \beta \leq 2$, uniformly in $x \in \mathbb{R}^2$, where

$$\widehat{F}(x,u) := \frac{1}{2} f(x,u)u - F(x,u), \quad \text{for} \quad (x,u) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

Remark 4.1. *We observe that we do not require the well-known Ambrosetti-Rabinowitz condition:*

(AR) *there exists $\theta > 2$ such that, for each $x \in \mathbb{R}^2$ and $u \in \mathbb{R}^2 \backslash \{0\}$, there holds*

$$0 < \theta F(x,u) \leq f(x,u)u.$$

Also, the critical exponential growth condition implies the following

$$\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^2}, \quad \textit{uniformly in} \quad x \in \mathbb{R}^2. \tag{4.1}$$

For the main result of this chapter, we need the following hypothesis, presented by Chen-Tang in [20]. We assume that there exists $\gamma_0 > 0$ such that the function f satisfy

$$\liminf_{|s|\to\infty} \frac{sf(x,s)}{e^{\gamma_0 s^2}} =: \beta_0 > \mathcal{M}, \text{ uniformly on } x \in \mathbb{R}^2 \tag{4.2}$$

where $\mathcal{M} := \dfrac{4}{\gamma_0 \rho^2} e^{16\pi \mathcal{C}_0^2}$ and $\rho > 0$ satisfies $4\pi(4+\rho)\rho\mathcal{C}_0^2 < 1$ and $\mathcal{C}_0 > 0$ is the embedding constant in (4.3).

Theorem 4.2. *Suppose that conditions (V) and $(F_0) - (F_5)$ are satisfied. In addition, we assume that there exists $\gamma_0 > 0$ such that the function f satisfy (4.2). Then, equation (2) has a ground state solution.*

Example 4.3. *Let $a \in C(\mathbb{R}^2, \mathbb{R})$ positive and 1-periodic in x_1, x_2 and define the function f by*

$$F(x,t) := a(x)t^3(e^{t^2} - 1).$$

Observe that

$$f(x,t) := a(x)t^2(e^{t^2}(2t^2+3) - 3).$$

It is not difficult to see that f satisfies $(F_0) - (F_5)$ and (4.2).

The present chapter is organized as follows: in the forthcoming section, we introduce some notations and results regarding the function space setting in which we work. In Section 4.2, we prove our main theorem and in Section 4.3 we present the proof of a level estimate established by [20].

4.1 Preliminaries

In this section we present the function space setting in which we develop our variational approach and some results about it.

We denote by $\mathcal{S}$ the selfadjoint operator $-\Delta + V$ acting on $L^2(\mathbb{R}^2)$ with domain $\mathcal{D}(\mathcal{S}) := H^2(\mathbb{R}^2)$. Let $\{\mathcal{E}(\lambda) : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)\}_{\lambda \in \mathbb{R}}$ be the spectral family of $\mathcal{S}$ and

$$E^- = \mathcal{E}(0^-)E, \quad E^+ = [id - \mathcal{E}(0)]E.$$

By (V), we have $E = E^- \oplus E^+$. For any $u \in E$, we can write $u = u^- + u^+$, where

$$u^- := \mathcal{E}(0-)u \in E^-, \quad u^+ := [id - \mathcal{E}(0)]u \in E^+, \tag{4.3}$$

and

$$\mathcal{S}u^- = -|\mathcal{S}|u^-, \quad \mathcal{S}u^+ = |\mathcal{S}|u^+, \quad \forall u \in E \cap \mathcal{D}(\mathcal{S}). \tag{4.4}$$

The spaces E^- and E^+ are $\mathcal{S}$-invariants. Moreover, if $u \in E^-$ and $v \in E^+$, then u and v are both orthogonal with respect to $\langle \cdot, \cdot \rangle_2$ and $\langle \cdot, \cdot \rangle$. By definition, $|\mathcal{S}|u = \mathcal{S}u$ if $u \in E^+$ and $|\mathcal{S}|u = -\mathcal{S}u$ if $u \in E^-$. Thus, $|\mathcal{S}| : E \to E$ is a positive self-adjoint operator. Therefore, we can define the square root of $|\mathcal{S}|$, which is also a self-adjoint operator. We can veirfy that

$$\left(|\mathcal{S}|^{1/2}\right)^2 u = |\mathcal{S}|u, \forall u \in D(\mathcal{S}).$$

If we consider the Hilbert space $E := D(|\mathcal{S}|^{1/2})$ with the inner product

$$\langle u, v \rangle := \langle |\mathcal{S}|^{1/2}u, |\mathcal{S}|^{1/2}v \rangle_{L^2}, \tag{4.5}$$

and the corresponding norm

$$\|u\| := ||\mathcal{S}|^{1/2}u|_2, \tag{4.6}$$

The next lemma proves the equivalence between norms of E and $H^1(\mathbb{R}^2)$. For the proof, see [11, 61].

Lemma 4.4. *Assume V. Then $E = E^- \oplus E^+$, $\|\cdot\|$ is equivalent to $\|\cdot\|_{H^1}$ on E and for any $u = u^- + u^+ \in E$, it holds $\langle u^+, u^- \rangle = \langle u^+, u^- \rangle_2 = 0$. Moreover,*

$$\langle \mathcal{S}u, u \rangle_2 = -\|u\|^2 \leq \lambda \|u\|_2^2, \quad u \in E^- \tag{4.7}$$

and

$$\langle \mathcal{S}u, u \rangle_2 = \|u\|^2 \geq \lambda \|u\|_2^2, \quad u \in E^+. \tag{4.8}$$

Remark 4.5. *It follows from Lemma 4.4 that $\|u\|^2 = \|u^-\|^2 + \|u^+\|^2$ and for any $p \in [2, \infty)$ the embedding $E \hookrightarrow L^p(\mathbb{R}^2)$ is continuous.*

Now, consider the bilinear form $B : E \times E \to \mathbb{R}$,

$$B(u,v) = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)dx.$$

We obtain

$$\begin{aligned}
B(u,v) &= \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)dx \\
&= \langle \mathcal{S}u, v\rangle_2 \\
&= \langle \mathcal{S}u^- + \mathcal{S}u^+, v\rangle_2 \\
&= \langle \mathcal{S}u^+, v\rangle_2 + \langle \mathcal{S}u^-, v\rangle_2 \\
&= \langle |\mathcal{S}|u^+, v^+\rangle_2 + (|\mathcal{S}|u^-, v^-\rangle_2 \\
&= \langle |\mathcal{S}|^{1/2}u^+, |\mathcal{S}|^{1/2}v^+\rangle_2 + \langle |\mathcal{S}|^{1/2}u^-, |\mathcal{S}|^{1/2}v^-\rangle_2 \\
&= \langle u^+, v^+\rangle - \langle u^-, v^-\rangle \\
&= \langle u^+, v\rangle - \langle u^-, v\rangle.
\end{aligned}$$

In particular,

$$B(u,u) = \|u^+\|^2 - \|u^-\|^2. \tag{4.9}$$

We know the energy functional associated to (2) is given by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^2} (|\nabla u|^2 + V(x)u^2)dx - \int_{\mathbb{R}^2} F(x,u)dx, \quad u \in H^1(\mathbb{R}^2),$$

where F is the primitive of f. This functional is of class C^1 and critical points of I corresponds to weak solutions of Problem (2).

In view of (4.9), we have

$$I(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \int_{\mathbb{R}^2} F(x,u)dx, \quad u = u^- + u^+ \in E^- \oplus E^+. \tag{4.10}$$

4.2 Proof of Theorem 4.2

In this section, we prove our main theorem. Our intention is to apply Lemma 1.6 and a minimization argument to obtain a ground state for problem (2). Some of these results were already proved in *Chapter 1*, *Chapter 2* and by Szulkin-Weth [60], so we present them without proof.

In order to exploit Lemma 1.6, we consider the spectral decomposition of $-\Delta + V$ with respect to the positive and negative part of the spectrum, given by $E = E^+ \oplus E^-$, where $u = u^+ + u^- \in E^+ \oplus E^-$.

By using (F_1) and (F_2) we can see that, given $\varepsilon > 0$, $\alpha > \alpha_0$ and $q \geq 1$, there exists $C = C(\varepsilon, \alpha, q) > 0$ such that

$$\max\{|F(x,u)|, |f(x,u)||u|\} \leq \varepsilon|u|^2 + C|u|^q(e^{\alpha|u|^2} - 1), \quad \text{for all } (x,u) \in \mathbb{R}^2 \times \mathbb{R}^2. \quad (4.11)$$

In view of (4.11) and Lemma 1.12, the energy functional associated to (2), given by

$$I(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \mathcal{J}(u), \quad (4.12)$$

where $\mathcal{J}(u) := \int_{\mathbb{R}^2} F(x,u)$, is well defined. Furthermore, it is not difficult to show that $I \in C^1(E, \mathbb{R})$ with

$$I'(u)w = \langle u^+, w^+ \rangle - \langle u^-, w^- \rangle - \int_{\mathbb{R}^2} f(x,u)w, \quad \text{for } u, w \in E.$$

Hence, the critical points of I correspond to weak solutions of Problem (2).

In order to apply Lemma 1.6, we are going to show that I satisfies conditions $(N_1) - (N_3)$. For this, initially we present some lemmas.

Lemma 4.6. *Assuming conditions $(F_1) - (F_3)$, for each $u \neq 0$ we have*

$$\frac{1}{2}f(x,u)u > F(x,u) > 0.$$

Furthermore, $\mathcal{J}(0) = 0$ and $\mathcal{J}$ is weakly lower semicontinuous.

By the previous lemma, condition (N_1) is satisfied. The next result is necessary in order to show condition (N_2) and its proof is contained in Lemma 2.3 of [60] and therefore we omit it.

Lemma 4.7. *Suppose that H satisfies $(F_1) - (F_3)$. If $z \in \mathcal{N}$ then for any $w \neq 0$ such that $u + w \in \widehat{E}(u)$, we have $I(u+w) < I(u)$.*

Lemma 4.8. *Assume conditions* $(F_1)-(F_3)$ *and* (4.2). *For each* $u \in E\backslash E^-$, *there exists* $R = R(u) > 0$ *such that* $I(w) \leq 0$ *for all* $w \in \widehat{E}(u)\backslash\mathcal{B}_R(0)$, *where* $\mathcal{B}_R(0) := \{u \in E : \|u\| < R\}$.

Proof. Suppose that the result is not valid. Then, for some $u_0 \in E\backslash E^-$ there exists a sequence $(w_n) \subset \widehat{E}(u_0)$ such that $\|w_n\| \to \infty$ and $I(w_n) > 0$. Defining $u_n = w_n/\|w_n\|$, we can assume that $u_n \rightharpoonup w_0$ in E. If $w_0 \neq 0$ and $\Omega_0 := \{x \in \mathbb{R}^2 : |w_0(x)| > 0\}$, then by Fatou's Lemma and (4.1), we have

$$\begin{aligned} 0 < \frac{I(w_n)}{\|w_n\|^2} &= \frac{1}{2}\|u_n^+\|^2 - \frac{1}{2}\|u_n^-\|^2 - \int_{\mathbb{R}^2} \frac{F(x,w_n)}{|w_n|^2}|u_n|^2 \\ &\leq \frac{1}{2}\|u_n^+\|^2 - \frac{1}{2}\|u_n^-\|^2 - \int_{\Omega_0} \frac{F(x,w_n)}{|w_n|^2}|u_n|^2 \to -\infty, \end{aligned}$$

which is a contradiction and thus $w_0 = 0$. Moreover, since H is nonnegative by the previous inequality we also have $\|u_n^+\| \geq \|u_n^-\|$. Once $\|u_n\| = 1$ we deduce that $\|u_n^+\| \geq 1/\sqrt{2}$.

Now, as $\widehat{E}(u_0) = \widehat{E}(u_0^+/\|u_0^+\|)$ we can assume $u_0 \in S^+$, that is, $\|u_0\| = 1$. Thus, we can write $u_n^+ = s_n u_0$ with $1/\sqrt{2} \leq s_n \leq 1$ for all $n \in \mathbb{N}$. Indeed, one has $w_n = r_n u_0 + w_n^-$ with $r_n \geq 0$ and $w_n^- \in E^-$. Hence,

$$\frac{1}{\sqrt{2}} \leq \|u_n^+\| = \frac{\|w_n^+\|}{\|w_n\|} = \frac{r_n}{\|w_n\|}\|u_0\| = \frac{r_n}{\|w_n\|} =: s_n$$

and $s_n = r_n/\sqrt{r_n^2 + \|w_n^-\|^2} \leq 1$. Therefore, up to a subsequence, $s_n \to s$ for some $s > 0$. From this, we deduce that $u_n^+ \to su_0$ strongly in E. On the other hand, we have $\|u_n^-\| \leq 1$ and so, up to a subsequence, $u_n^- \rightharpoonup w$ with $w \in E^-$. Thus, $u_n = s_n u_0 + u_n^- \rightharpoonup su_0 + w$. By the uniqueness of the weak limit, it follows that $su_0 + w = 0$, which implies that $u_0 = 0$. This is a contradiction and the lemma is proved.

■

The next lemma guarantees that condition (N_2) is valid.

Lemma 4.9. *Suppose that* F *satisfies* $(F_1)-(F_3)$ *and* (4.2).

(i) for any $u \in \mathcal{N}$, $I|_{\widehat{E}(u)}$ *admits a unique maximum point which is precisely at* u.

(ii) for any $u \in E\backslash E^-$, the set $\widehat{E}(u)$ intersects $\mathcal{N}$ at exactly one point $\widehat{m}(u)$, which is the unique global maximum point of $I|_{\widehat{E}(u)}$.

Proof. Firstly we will show (i). Given $tu + y \in \widehat{E}(u)\backslash\{u\}$, it is enough to consider $w = (t-1)u + y$ to obtain $tu + y = u + w$.

Note that, if $w = 0$ then $t = 1$ and $y = 0$, but this can not occur because $tz + y \neq u$. Hence, $w \neq 0$ and by Lemma 1.16 we conclude that $I(tu + y) < I(u)$.

To prove (ii), by the previous item, it is sufficient to show that $\mathcal{N} \cap \widehat{E}(u) \neq \emptyset$ for each $u \in E\backslash E^-$. Moreover, since $\widehat{E}(u) = \widehat{E}\left(u^+/\|u^+\|\right)$. By Lemma 1.15, there exists $R > 0$ such that $I(w) \leq 0$ if $w \in \widehat{E}(u)\backslash\mathcal{B}_R(0)$ and if $\|w\| \leq R$ then $I(w) \leq R^2/2$ and so $\sup_{\widehat{E}(u)} I < \infty$.

On the other hand, by using (1.8), for $\varepsilon > 0$ and $\alpha' > \alpha_0$ there exists $C_\varepsilon > 0$ such that

$$F(x,u) \leq \varepsilon|u|^2 + C_\varepsilon|u|^3(e^{\alpha'|u|^2} - 1), \quad u \in \mathbb{R}^2. \tag{4.13}$$

From (1.12), Lemma 1.12 and by choosing $0 < \varepsilon < 1/(4\int_{\mathbb{R}^2}|u|^2)$, we have

$$\begin{aligned} I(tu) = \frac{t^2}{2}\|u\|^2 - \int_{\mathbb{R}^2} F(x,tu) &\geq \frac{t^2}{2} - \varepsilon t^2 \int_{\mathbb{R}^2}|u|^2 - C_\varepsilon t^3 \int_{\mathbb{R}^2}|u|^3(e^{\alpha' t^2|u|^2} - 1) \\ &\geq \frac{t^2}{4} - Ct^3\left(\int_{\mathbb{R}^2}|u|^6\right)^{1/2}\left[\int_{\mathbb{R}^2}(e^{2\alpha' t^2|u|^2} - 1)\right]^{1/2} > 0, \end{aligned}$$

for all $0 < t < \sqrt{2\pi/\alpha'}$ sufficiently small. Consequently, $\sup_{\widehat{E}(u)} I > 0$.

Now, let $w_n == t_n u + h_n, \in \widehat{E}(u)$ be a maximizing sequence for $\sup_{\widehat{E}(u)} I$. Since $\sup_{\widehat{E}(u)} I > 0$, we can suppose that $\|w_n\| \leq R$. Hence, up to a subsequence, $w_n \rightharpoonup w_0$ weakly in E. We can see, up to a subsequence, that $t_n \to t_0 \geq 0$ and $h_n \rightharpoonup h_0$, $w_n \rightharpoonup w_0$ weakly in $H^1(\mathbb{R}^2)$. Therefore, $w_0 = t_0 u + h_0 \in \widehat{E}(u)$. Recalling that $u \in E^+$ and $\|u\| = 1$, we can write

$$I(w_n) = \frac{t_n^2}{2} - \frac{1}{2}\|h_n\|^2 - \int_{\mathbb{R}^2} F(x,w_n),$$

from where it follows, according to the weak lower semicontinuous of the norm and Fatou's lemma, that

$$\sup_{\widehat{E}(u)} I = \lim_{n\to\infty} I(w_n) = \frac{t_0^2}{2} + \limsup_{n\to\infty}\left[-\frac{1}{2}\|h_n\|^2 - \int_{\mathbb{R}^2} F(x,w_n)\right] \leq I(w_0).$$

Thus, $I(w_0) = \sup_{\widehat{E}(u)} I$ and therefore w_0 is a critical point of $I|_{\widehat{E}(u)}$, showing that

$w_0 \in \mathcal{N} \cap \widehat{E}(u)$ and the proof is complete.

■

As an immediate consequence of this lemma, we obtain the following characterization to the minimal level for I on the generalized Nehari manifold $\mathcal{N}$:

Corollary 4.10.

$$c_* := \inf_{\eta \in \mathcal{N}} I(\eta) = \inf_{u \in E \setminus E^-} \max_{w \in \widehat{E}(u)} I(w).$$

Lemma 4.11. *Suppose that F satisfies $(F_1) - (F_3)$. Then, $c_* > 0$ and*

$$\|u^+\| \geq \max\{\|u^-\|, \sqrt{2c_*}\} \quad \textit{for all} \quad u \in \mathcal{N}.$$

Proof. For $a > 0$, we recall that $S_a^+ := \{u \in E^+ : \|u\| = a\}$ and $(\mathbb{R}_+)u = \{tu : t \geq 0\}$. Since $\widehat{E}(u) = \widehat{E}(u^+)$ for any $u \in E \setminus E^-$, from Corollary 1.18, for any $a > 0$, it follows that

$$c_* = \inf_{u \in E \setminus E^-} \max_{w \in \widehat{E}(u)} I(w) = \inf_{u \in E^+ \setminus \{0\}} \max_{w \in \widehat{E}(u)} I(w) = \inf_{u \in S_a^+} \max_{w \in \widehat{E}(u)} I(w) \geq \inf_{u \in S_a^+} \max_{w \in (\mathbb{R}_+)u} I(w).$$

Let $u \in S_a^+$ and $\alpha > \alpha_0$. We take $a > 0$ so that $2\alpha a^2 < 4\pi$. By virtue of Lemma 1.12, we reach

$$\int_{\mathbb{R}^2} (e^{2\alpha |u|^2} - 1) = \int_{\mathbb{R}^2} (e^{2\alpha a^2 (|u|/\|u\|)^2} - 1) \leq C,$$

for some $C > 0$. Thus, by using (1.12) we get

$$\begin{aligned}
\max_{w \in (\mathbb{R}_+)u} I(w) \geq I(u) = \|u\|_V^2 - \int_{\mathbb{R}^2} F(x, u) &\geq \|u\|_V^2 - \varepsilon \int_{\mathbb{R}^2} |u|^2 - C_\varepsilon \int_{\mathbb{R}^2} |u|^3 (e^{\alpha |u|^2} - 1) \\
&\geq \left(1 - \frac{2\varepsilon}{a_0}\right) \|u\|_V^2 - C_\varepsilon \left(\int_{\mathbb{R}^2} |u|^6\right)^{1/2} \\
&\geq \left(1 - \frac{2\varepsilon}{a_0}\right) \|u\|_V^2 - C_1 \|u\|_V^3,
\end{aligned}$$

where we have used the continuous embedding $H^1(\mathbb{R}^2) \hookrightarrow L^6(\mathbb{R}^2)$. Hence, taking $\varepsilon = a_0/4$ and $0 < a < \sqrt{2\pi/\alpha}$ sufficiently small so that $1/2 - C_1\|u\|_V = 1/2 - C_1 a/\sqrt{2} \geq 1/4$, we conclude that

$$\max_{w \in (\mathbb{R}_+)u} I(w) \geq \|u\|_V^2 \left(\frac{1}{2} - C_1 \|u\|_V\right) \geq \frac{a^2}{8} > 0, \quad \text{for all} \quad u \in S_a^+$$

and consequently $c_* > 0$. Now, for any $u \in \mathcal{N}$ we have

$$c_* \leq \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \int_{\mathbb{R}^2} F(x,u) \leq \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2),$$

which implies that $\|u^+\| \geq \max\{\|u^-\|, \sqrt{2c_*}\}$ and the proof is done.

■

Condition (N_3) can be now proved in the following lemma:

Lemma 4.12. *Suppose that F satisfies $(F_1)-(F_3)$. If $\mathcal{W} \subset E\backslash E^-$ is compact, then there exists $C_{\mathcal{W}} > 0$ such that $\|\widehat{m}(u)\| \leq C_{\mathcal{W}}$, for all $z \in \mathcal{W}$.*

Proof. Setting $\delta = \sqrt{2c_*}$, by Lemma 1.19 and noting that $\widehat{m}(u) \in \mathcal{N}$ for any $u \in E\backslash E^-$, we have $\|\widehat{m}(u)^+\| \geq \delta$. Once $\widehat{m}(u) = \widehat{m}\,(u^+/\|u^+\|)$ for any $u \in E\backslash E^-$, without loss of generality, we can assume that $\mathcal{W} \subset S^+$. By the compactness of $\mathcal{W}$ and from Lemma 1.15, we can see that there exists $C_{\mathcal{W}} > 0$ such that

$$I \leq 0 \;\text{ on }\; \widehat{E}(u)\backslash B_{C_{\mathcal{W}}}(0) \;\text{ for all } u \in \mathcal{W},$$

where $B_{C_{\mathcal{W}}}(0) = \{w \in E : \|w\| \leq C_{\mathcal{W}}\}$. Recalling that $I(\widehat{m}(u)) \geq c_* > 0$ for all $u \in E\backslash E^-$, we get that $\|\widehat{m}(u)\| = \|\widehat{m}\,(u^+/\|u^+\|)\| \leq C_{\mathcal{W}}$ for any $u \in \mathcal{W}$ and the result is proved.

■

The generalized manifold $\mathcal{N}$ is not necessarily of class C^1 and thus the Ekeland Variational Principle can not be applied directly to obtain a Palais-Smale sequence for I on $\mathcal{N}$. However, in view of item (iv) of Lemma 1.6 and Lemma 1.19, one has $\inf_{S^+} \Psi = \inf_{\mathcal{N}} I = c_* > 0$. Since S^+ is a submanifold of class C^1 of E^+, it follows from the Ekeland Variational Principle that there exists $(w_n) \subset S^+$ such that

$$\Psi(w_n) \to c_* \quad \text{and} \quad \|\Psi'(w_n)\|_* \to 0 \quad \text{as} \quad n \to \infty.$$

Let $u_n = m(w_n) \in \mathcal{N}$. By items (ii)-(iv) of Lemma 1.6, we reach

$$I(u_n) \to c_* \quad \text{and} \quad \|I'(u_n)\| \to 0 \quad \text{as} \quad n \to \infty. \tag{4.14}$$

In the next lemma, we shall guarantee that the above sequence $(u_n) \subset \mathcal{N}$ is bounded. For this, we shall use the following inequality, whose proof can be found in [28, Lemma 2.4]:

$$st \leq \begin{cases} (e^{t^2}-1)+|s|(\log|s|)^{1/2}, & t\in\mathbb{R} \text{ and } |s|\geq e^{1/4}; \\ (e^{t^2}-1)+\frac{1}{2}s^2, & t\in\mathbb{R} \text{ and } |s|\leq e^{1/4}. \end{cases} \tag{4.15}$$

Lemma 4.13. *Suppose that $(F_0)-(F_5)$ are satisfied. Any Palais-Smale sequence $(u_n)\subset\mathcal{N}$ for I is bounded.*

Proof. Let $(u_n)\subset\mathcal{N}$ be a Palais-Smale sequence for I at level c. Hence,

$$c+o_n(1)=I(u_n)-\frac{1}{2}I'(u_n)u_n=\int_{\mathbb{R}^2}\widehat{F}(x,u_n). \tag{4.16}$$

Suppose by contradiction that, up to a subsequence, $\|u_n\|\to\infty$. Thus,

$$o_n(1)=\frac{I'(u_n)(u_n^+-u_n^-)}{\|u_n\|^2}=1-\int_{\mathbb{R}^2}\frac{f(x,u_n)\cdot(u_n^+-u_n^-)}{\|u_n\|^2}.$$

Setting $w_n:=u_n/\|u_n\|$, it follows that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}\frac{f(x,u_n)\cdot(w_n^+-w_n^-)}{\|u_n\|}=1. \tag{4.17}$$

In what follows, we claim that the function $Q:(0,\infty)\to\mathbb{R}$ defined by

$$Q(r)=\inf_{x\in\mathbb{R}^2,\,|u|\geq r}\widehat{F}(x,u)$$

has the following properties:

$$Q(r)>0 \text{ for all } r>0 \quad\text{and}\quad Q(r)\to+\infty \text{ as } r\to\infty. \tag{4.18}$$

Indeed, by Lemma 1.13 we have $\widehat{F}(x,u)>0$ for all $x\in\mathbb{R}^2$ and $u\neq 0$. Moreover, by using (F_1), with $0<\alpha'<\alpha_0$, and (4.1) we can deduce that $\widehat{F}(x,u)\to+\infty$ as $|u|\to\infty$, uniformly in $x\in\mathbb{R}^2$. Therefore, by the definition of Q, one has $Q(r)\to+\infty$ as $r\to\infty$. From this fact and by the periodicity of $\widehat{F}(x,u)$ in x, for each $r>0$ there exists $M_r>r$ such that

$$Q(r)=\inf_{x\in\mathbb{R}^2,\,|u|\geq r}\widehat{F}(x,u)=\min_{x\in[0,1]\times[0,1],\,r\leq|u|\leq M_r}\widehat{F}(x,u)=\widehat{F}(x_0,z_0)>0$$

and the claim is proved.

For $0 \le a < b \le \infty$ and $n \in \mathbb{N}$, we set

$$\Omega_n(a,b) := \{x \in \mathbb{R}^2 : a \le |u_n(x)| < b\}.$$

Using (4.16) and for $0 < r < R < \infty$, we obtain

$$\begin{aligned} c + o_n(1) &= \int_{\Omega_n(0,r)} \widehat{F}(x,u_n) + \int_{\Omega_n(r,R)} \frac{\widehat{F}(x,u_n)}{|u_n|^2}|u_n|^2 + \int_{\Omega_n(R,\infty)} \widehat{F}(x,u_n) \\ &\ge \int_{\Omega_n(0,r)} \widehat{F}(x,u_n) + \frac{Q(r)}{R^2}\int_{\Omega_n(r,R)} |u_n|^2 + Q(R)|\Omega_n(R,\infty)| \end{aligned}$$

and therefore there exists $C_1 > 0$ such that

$$\max\left\{\int_{\Omega_n(0,r)} \widehat{F}(x,u_n),\ \frac{Q(r)}{R^2}\int_{\Omega_n(r,R)} |u_n|^2,\ Q(R)|\Omega_n(R,\infty)|\right\} \le C_1. \tag{4.19}$$

In particular,

$$|\Omega_n(R,\infty)| \le C_1/Q(R) \quad \text{for each} \quad n \in \mathbb{N}. \tag{4.20}$$

Next, let $C_3 > 0$ be such that $\|u\|_2^2 \le C_3\|u\|^2$ for each $u \in E$ and consider $\varepsilon > 0$. By (F_2), there exists $r_\varepsilon > 0$ such that $|f(x,u)| \le \varepsilon|u|/C_3$ for each $|u| \le r_\varepsilon$ and $x \in \mathbb{R}^2$. By the definition of w_n and since $|w_n^+ - w_n^-| = |w_n|$, from (4.19) with $r = r_\varepsilon$, for any $n \in \mathbb{N}$ we get

$$\begin{aligned} \int_{\Omega_n(0,r_\varepsilon)} \frac{f(x,u_n)\cdot(w_n^+ - w_n^-)}{\|z_n\|} &\le \int_{\Omega_n(0,r_\varepsilon)} \frac{|f(x,u_n)|}{|u_n|}|w_n^+ - w_n^-||w_n| \\ &\le \frac{\varepsilon}{C_3}\int_{\Omega_n(0,r_\varepsilon)} |w_n|^2 \le \varepsilon\|w_n\|^2 = \varepsilon. \end{aligned} \tag{4.21}$$

Now, let $R = R_\varepsilon > r_\varepsilon$ to be chosen later. Define $\mathcal{A}_1 = \Omega_n(R_\varepsilon,\infty) \cap \{x \in \mathbb{R}^2 : |f(x,u_n)| \le e^{1/4}\}$ and $\mathcal{A}_2 = \Omega_n(R_\varepsilon,\infty) \cap \{x \in \mathbb{R}^2 : |f(x,u_n)| \ge e^{1/4}\}$. Taking $t = |w_n|$ and $s = |f(x,u_n)|$ in (4.29), we get

$$\begin{aligned} \int_{\Omega_n(R_\varepsilon,\infty)} \frac{f(x,u_n)\cdot(w_n^+ - w_n^-)}{\|u_n\|} &\le \frac{1}{\|u_n\|}\int_{\Omega_n(R_\varepsilon,\infty)} |f(x,u_n)||w_n| \\ &\le \frac{1}{\|u_n\|}\int_{\mathcal{A}_2} |f(x,u_n)|(\log|f(x,u_n)|)^{1/2} \\ &\quad + \frac{1}{\|u_n\|}\int_{\mathcal{A}_1} \frac{1}{2}|f(x,u_n)|^2 + \frac{2}{\|u_n\|}\int_{\mathbb{R}^2} (e^{|w_n|^2} - 1). \end{aligned} \tag{4.22}$$

By (4.20), we have $|\mathcal{A}_1| \leq |\Omega_n(R_\varepsilon, \infty)| \leq C_1/Q(R_\varepsilon)$ and consequently

$$\int_{\mathcal{A}_1} |f(x,u_n)|^2 \leq \frac{C_1 e^{1/2}}{Q(R_\varepsilon)}, \quad \text{for all} \quad n \in \mathbb{N}.$$

By Lemma 1.12, the integral $\int_{\mathbb{R}^2}(e^{|w_n|^2}-1)$ is bounded. Moreover, given $\alpha > \alpha_0$ by (F_1) there exists $R_1 > 0$ such that

$$|f(x,u)| \leq e^{\alpha |u|^2} \quad \text{for all} \quad x \in \mathbb{R}^2 \quad \text{and} \quad |u| \geq R_1.$$

In view of (F_5), there exists $c_1 > 0$ such that

$$|f(x,u)||u| \leq c_1 \widehat{F}(x,u) \quad \text{for all} \quad x \in \mathbb{R}^2 \quad \text{and} \quad |u| \geq R_1$$

Taking $R_\varepsilon > R_1$, from the two last estimates and (4.16) it follows that

$$\int_{\mathcal{A}_2} |f(x,u_n)|(\log|f(x,u_n)|)^{1/2} \leq \alpha^{1/2}\int_{\mathcal{A}_2}|f(x,u_n)||u_n| \leq \alpha^{1/2}c_1\int_{\mathbb{R}^2}\widehat{F}(x,u_n) \leq C,$$

for some constant $C > 0$. Thus, in view of (4.22), we find $n_0 \in \mathbb{N}$ satisfying

$$\int_{\Omega_n(R_\varepsilon,\infty)} \frac{f(x,u_n)\cdot(w_n^+ - w_n^-)}{\|u_n\|} < \varepsilon, \quad \text{for all} \quad n \geq n_0. \tag{4.23}$$

According to (F_1) and (F_2), given $\alpha > \alpha_0$ we obtain $C_\alpha > 0$ such that, for all $x \in \Omega_n(r_\varepsilon, R_\varepsilon)$, we have

$$|f(x,u_n)| \leq |u_n| + C_\alpha(e^{\alpha R_\varepsilon^2}-1)|u_n|$$

From this estimate and (4.11), recalling the definition of w_n and $|w_n^+ - w_n^-| = |w_n|$, we reach

$$\begin{aligned}\int_{\Omega_n(r_\varepsilon,R_\varepsilon)} \frac{f(x,u_n)\cdot(w_n^+ - w_n^-)}{\|u_n\|} &\leq C_2\int_{\Omega_n(r_\varepsilon,R_\varepsilon)}\frac{|u_n||w_n^+ - w_n^-|}{\|u_n\|}\\ &\leq \frac{C_2}{\|u_n\|^2}\int_{\Omega_n(r_\varepsilon,R_\varepsilon)}|u_n|^2\\ &\leq \frac{C_3}{\|u_n\|^2}\frac{R_\varepsilon^2}{Q(r_\varepsilon)} < \varepsilon, \quad \text{for all} \quad n \geq n_1,\end{aligned}$$

for some $n_1 \in \mathbb{N}$. This estimate, (4.21) and (4.23) show that

$$\int_{\mathbb{R}^2} \frac{f(x,u_n)\cdot(w_n^+ - w_n^-)}{\|u_n\|}dx \leq 3\varepsilon, \quad \text{for all} \quad n \geq \max\{n_0, n_1\}.$$

But this contradicts (4.17) because $\varepsilon > 0$ is arbitrary and the proof is done.

■

Once the sequence $(u_n) \subset \mathcal{N}$ satisfying (4.14) is bounded, it follows that there exists $u_0 \in E$ such that, up to a subsequence, $u_n \rightharpoonup u_0$ in E and $u_n \to u_0$ a.e. in $\mathbb{R}^2$ as $n \to \infty$. Our intention is to prove that u_0 is a nonzero critical point of I and to conclude that u_0 is a ground state solution. This will be done in the next propositions and lemmas.

Proposition 4.14. *The weak limit u_0 of the sequence $(u_n) \subset \mathcal{N}$ is a critical point of I.*

Proof. By the density of $C_0^\infty(\mathbb{R}^2)$ in E, it is enough just to deduce that $I'(u_0)\eta = 0$ for all $\eta \in C_0^\infty(\mathbb{R}^2)$. Then, let $\eta \in C_0^\infty(\mathbb{R}^2)$ and $K := \operatorname{supp}\eta$.

By (F_6), there exist $M > 0$ and $c_1 > 0$ such that

$$|f(x,u_n)||u_n| \leq c_1\widehat{F}(x,u_n), \quad \text{for all} \quad |u| \geq M.$$

Defining $\Omega_1 = K \cap \{x;\ |u_n(x)| \leq M\}$ and $\Omega_2 = K \cap \{x;\ |u_n(x)| > M\}$, by (F_1) and (F_2), given $\alpha > \alpha_0$ we obtain $C > 0$ such that

$$|f(x,w)| \leq |w| + C(e^{\alpha|w|^2} - 1) \quad \text{for all} \quad (x,w) \in \mathbb{R}^2 \times \mathbb{R}^2$$

and hence

$$\begin{aligned} |f(x,u_n(x))\cdot\eta(x)| &\leq \|\eta\|_\infty |u_n(x)| + C_1\|\eta\|_\infty(e^{\alpha|u_n(x)|^2} - 1) \\ &\leq \|\eta\|_\infty[M + C_1(e^{\alpha M^2} - 1)] \quad \text{a.e. in} \quad x \in \Omega_1. \end{aligned}$$

Since $f(x,u_n)\cdot\eta \to f(x,u_0)\cdot\eta$ a.e. in $x \in \Omega_1$, it follows from the Lebesgue Dominated Convergence Theorem that

$$\int_{\Omega_1} [f(x,u_n) - f(x,u_0)]\cdot\eta \to 0 \tag{4.24}$$

Now, observe that by (F_6), (4.16) and Hölder's inequality, we reach

$$\begin{aligned}
&\left|\int_{\Omega_2}[f(x,u_n)-f(x,u_0)]\cdot\eta\right| \\
&\quad\leq \frac{\|\eta\|_\infty}{M}\int_{\Omega_2}|f(x,u_n)-f(x,u_0)||u_n| \\
&\quad\leq \frac{\|\eta\|_\infty}{M}\int_{\Omega_2}[|f(x,u_n)||u_n|+|f(x,u_0)||u_n|] \\
&\quad\leq \frac{\|\eta\|_\infty}{M}\left[c_1\int_{\mathbb{R}^2}\widehat{F}(x,u_n)+C_1\left(\int_{\mathbb{R}^2}[e^{2\alpha|u_0|^2}-1]\right)^{\frac{1}{2}}\|u_n\|_2\right] \\
&\quad\leq \frac{\|\eta\|_\infty C}{M},
\end{aligned} \tag{4.25}$$

for some $C>0$ independent of M. Hence, given $\delta>0$ we can take $M>0$ so that

$$\left|\int_{\Omega_2}[f(x,u_n)-f(x,u_0)]\cdot\eta\right|<\delta, \quad \text{for all} \quad n\in\mathbb{N}. \tag{4.26}$$

Therefore, by (1.16) and (1.18) we obtain

$$\limsup_{n\to\infty}\left|\int_K[f(x,u_n)-f(x,u_0)]\cdot\eta\right|\leq\limsup_{n\to\infty}\left|\int_{\Omega_1}[f(x,u_n)-f(x,u_0)]\cdot\eta\right|+\delta=\delta$$

and once $\delta>0$ is arbitrary, we conclude that

$$\int_{\mathbb{R}^2}[f(x,u_n)-f(x,u_0)]\cdot\eta\to 0. \tag{4.27}$$

Since

$$I'(u_n)\eta-I'(u_0)\eta=\langle u_n-u_0,\eta\rangle-\int_{\mathbb{R}^2}[f(x,u_n)-f(x,u_0)]\cdot\eta$$

and according to (4.27) we reach

$$I'(u_0)\eta=\lim_{n\to\infty}\left[I'(u_n)\eta-\langle u_n-u_0,\eta\rangle+\int_{\mathbb{R}^2}[f(x,u_n)-f(x,u_0)]\cdot\eta\right]=0,$$

and the claim is proved.

■

The next lemma have been proved by [20]. The proof is presented in Section 4.3.

Lemma 4.15. *There exists $\overline{n} \in \mathbb{N}$ such that*

$$\max_{s\geq 0, v\in E^-} I(v + sw_{\overline{n}}) < \frac{2\pi}{\gamma_0}. \tag{4.28}$$

Now, our objective is to prove that the weak limit u_0 of the sequence $(u_n) \subset \mathcal{N}$ is nontrivial.

For the next lemma, we will exploit the inequality

$$st \leq t^2(e^{t^2} - 1) + s(\log s)^{1/2}, \quad \text{for} \;\; t \geq 0 \;\; \text{and} \;\; s \geq e^{1/\sqrt[3]{4}}. \tag{4.29}$$

The proof of (4.29) can be seen in [38, Lemma 4.1].

Lemma 4.16. *Let $(u_n) \in \mathcal{N}$ be a sequence satisfying (4.14). Then, there exists $R > 0$ such that*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_R(y)} u_n^2 > 0. \tag{4.30}$$

Proof. First, by Proposition 4.15, we can choose $\delta > 0$ so that $c_* \in [0, 2\pi/\alpha_0 - \delta/2)$. Assume by contradiction that (4.30) does not occur, that is,

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_R(y)} u_n^2 = 0 \quad \text{for all} \quad R > 0, \tag{4.31}$$

which implies by Lions' lemma (see [49]) that

$$u_n \to 0 \;\; \text{strongly in} \;\; L^s(\mathbb{R}^2) \;\; \text{for any} \;\; s > 2. \tag{4.32}$$

Now, we claim that

$$\int_{\mathbb{R}^2} F(x, u_n) \to 0. \tag{4.33}$$

In view of (F_4) and (F_5), there exists $c_1 > 0$ such that

$$F(x,u) \leq M_0|f(x,u)| \;\; \text{and} \;\; |f(x,u)||u| \leq c_1\widehat{F}(x,u) \;\; \text{for all} \;\; x \in \mathbb{R}^2 \;\; \text{and} \;\; |u| \geq R_0.$$

On the other hand, by (F_5), (4.16) and for any $K > R_0$, we deduce that

$$\begin{aligned}\int_{\{|u_n|>K\}} F(x,u_n) &\le M_0 \int_{\{|u_n|>K\}} |f(x,u_n)| \\ &\le \frac{M_0}{K}\int_{\{|u_n|>K\}} |f(x,u_n)||u_n| \le \frac{M_0 c_1}{K}\int_{\mathbb{R}^2} \widehat{F}(x,u_n) \le \frac{M_0 C}{K},\end{aligned}$$

where $C > 0$ does not depend on K. Given any $\varepsilon > 0$, we can take $K > 0$ large enough so that

$$\int_{\{|u_n|>K\}} F(x,u_n) \le \varepsilon. \tag{4.34}$$

By (1.8), for $\alpha > \alpha_0$ we know that

$$F(x,u_n) \le \varepsilon|u_n|^2 + C_\varepsilon |u_n|^4 (e^{\alpha|u_n|^2} - 1)$$

and thus

$$\begin{aligned}\int_{\{|u_n|\le K\}} F(x,u_n) &\le \varepsilon \int_{\{|u_n|\le K\}} |u_n|^2 + \int_{\{|u_n|\le K\}} |u_n|^4 (e^{\alpha|u_n|^2} - 1) \\ &\le \varepsilon C + 2(e^{\alpha K^2} - 1)\|u_n\|_4^4.\end{aligned} \tag{4.35}$$

Therefore, from this inequality, (4.32) and (4.34), it follows that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^2} F(x,u_n) = \limsup_{n\to\infty} \left[\int_{\{|u_n|\le K\}} F(x,u_n) + \int_{\{|u_n|>K\}} F(x,u_n)\right] \le (1+C)\varepsilon$$

and convergence (4.33) is proved. Since $I(u_n) \to c_*$, by convergence (4.33) we reach

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} (|\nabla u_n|^2 + V(x)u_n^2)dx = 2c_*. \tag{4.36}$$

If $u_n \to 0$ strongly in $H^1(\mathbb{R}^2)$ as $n \to \infty$, then by (4.33) and (4.36) we get that $c_* = 0$, which is not possible. Therefore, we can assume that $\|u_n\| \ge b > 0$ for all $n \in \mathbb{N}$.

Once $\langle I'(u_n), u_n\rangle = o_n(1)\|u_n\|$, we have

$$\|u_n\|^2 = \int_{\mathbb{R}^2} f(x,u_n)u_n + o_n(1)\|u_n\|.$$

In view of $(F_0)-(F_1)$, given $\varepsilon>0$ there exists $C_\varepsilon>0$ such that

$$|f(x,u)| \le C_\varepsilon e^{(\alpha_0+\varepsilon)|u|^2} \quad \text{for all} \quad (x,u)\in\mathbb{R}^2\times\mathbb{R}^2. \tag{4.37}$$

Setting

$$\overline{u}_n = (4\pi/\alpha_0-\delta)^{1/2}\frac{u_n}{\|u_n\|},$$

where $\delta>0$ was chosen at the beginning of the proof, we can write

$$\begin{aligned}(4\pi/\alpha_0-\delta)^{1/2}\|u_n\| &\le \int_{\mathbb{R}^2}|f(x,u_n)||\overline{u}_n| + o_n(1)\\ &= \frac{C_\varepsilon}{\sqrt{\alpha_0}}\int_{\mathbb{R}^2}\frac{|f(x,u_n)|}{C_\varepsilon}\sqrt{\alpha_0}|\overline{u}_n| + o_n(1) =: I_n.\end{aligned} \tag{4.38}$$

Defining

$$\Gamma_n = \left\{x\in\mathbb{R}^2 : |f(x,u_n)|/C_\varepsilon \ge e^{1/\sqrt[3]{4}}\right\} \quad \text{and} \quad \Lambda_n = \mathbb{R}^2\backslash\Gamma_n,$$

by using inequality (4.29) with $s=|f(x,u_n)|/C_\varepsilon$ and $t=\sqrt{\alpha_0}|\overline{u}_n|$ we can estimate

$$\begin{aligned}I_n \le& \frac{C_\varepsilon}{\sqrt{\alpha_0}}\int_{\Gamma_n}\frac{|f(x,u_n)|}{C_\varepsilon}\left[\log\left(\frac{|f(x,u_n)|}{C_\varepsilon}\right)\right]^{1/2} + o_n(1)\\ &+\int_{\Lambda_n}|f(x,u_n)||\overline{u}_n| + C_\varepsilon\sqrt{\alpha_0}\int_{\mathbb{R}^2}|\overline{u}_n|^2\left(e^{\alpha_0|\overline{u}_n|^2}-1\right).\end{aligned}$$

Thus, by (4.37) we get

$$\begin{aligned}I_n &\le \sqrt{\frac{\alpha_0+\varepsilon}{\alpha_0}}\int_{\Gamma_n}|f(x,u_n)||u_n| + C_\varepsilon\sqrt{\alpha_0}\int_{\mathbb{R}^2}|\overline{u}_n|^2\left(e^{\alpha_0|\overline{u}_n|^2}-1\right)\\ &\quad+\int_{\Lambda_n}|f(x,u_n)||\overline{u}_n| + o_n(1)\\ &\le \sqrt{\frac{\alpha_0+\varepsilon}{\alpha_0}}\int_{\mathbb{R}^2}|f(x,u_n)||u_n| + I_{1,n}+I_{2,n}+o_n(1),\end{aligned} \tag{4.39}$$

where

$$I_{1,n} := C_\varepsilon\sqrt{\alpha_0}\int_{\mathbb{R}^2}|\overline{u}_n|^2\left(e^{\alpha_0|\overline{u}_n|^2}-1\right) \quad \text{and} \quad I_{2,n} := \int_{\Lambda_n}|f(x,u_n)||\overline{u}_n|.$$

Now, we can take $p>1$ close to 1 such that $p\alpha_0(4\pi/\alpha_0-\delta)<4\pi$. Since

$\|\overline{z}_n\|^2 = 4\pi/\alpha_0 - \delta$, Lemma 1.12 and (4.32) imply that

$$I_{1,n} \leq C_1\sqrt{\alpha_0}\left[\int_{\mathbb{R}^2} |\overline{u}_n|^{2q}\right]^{1/q}\left[\int_{\mathbb{R}^2}\left(e^{p\alpha_0|\overline{u}_n|^2}-1\right)\right]^{1/p}$$
$$\leq C_2\sqrt{\alpha_0}\frac{(4\pi/\alpha_0-\delta)}{b^2}\left[\int_{\mathbb{R}^2}|u_n|^{2q}\right]^{1/q} \longrightarrow 0,$$

where $1/p + 1/q = 1$. Next, according to $(F_0)-(F_2)$, for any $\rho > 0$, there exists $C_{\rho,\varepsilon} > 0$ such that

$$|f(x,u_n(x))| \leq \rho|u_n(x)| + C_{\rho,\varepsilon}|u_n(x)|^2, \quad \text{for all } \ x \in \Lambda_n. \tag{4.40}$$

Hence,

$$I_{2,n} \leq \int_{\Lambda_n}(\rho|u_n|+C_{\rho,\varepsilon}|u_n|^2)|\overline{u}_n| \leq \left[\rho\left(\int_{\mathbb{R}^2}|u_n|^2\right)^{1/2}+C_{\rho,\varepsilon}\left(\int_{\mathbb{R}^2}|u_n|^4\right)^{1/2}\right]\left(\int_{\mathbb{R}^2}|\overline{u}_n|^2\right)^{1/2}.$$

Once $\|\overline{u}_n\|$ is bounded and $u_n \to 0$ strongly in $L^4(\mathbb{R}^2)$, we reach

$$\limsup_{n\to\infty} I_{2,n} \leq C\rho,$$

for some $C > 0$ independent of ρ. Thus, we conclude that $I_{1,n} = o_n(1)$ and $I_{2,n} = o_n(1)$. Therefore, (4.38) and (4.39) provide

$$(4\pi/\alpha_0-\delta)^{1/2}\|u_n\| \leq o_n(1) + \left(1+\frac{\varepsilon}{\alpha_0}\right)^{1/2}\int_{\mathbb{R}^2}|f(x,u_n)||u_n|. \tag{4.41}$$

By virtue of (F_5), given $\delta > 0$ there exists $M > 0$ such that $|f(x,u)||u| \leq (\beta+\delta)\widehat{F}(x,u)$ for all $|u| > M$. Hence, by using (4.16) with $c = c_*$ one has

$$\int_{\mathbb{R}^2}|f(x,u_n)||u_n| \leq (\beta+\delta)\int_{\mathbb{R}^2}\widehat{F}(x,u_n) + \int_{\{|u_n|\leq M\}}|f(x,u_n)||u_n|$$
$$\leq \beta c_* + o_n(1) + \int_{\{|u_n|\leq M\}}|f(x,u_n)||u_n|.$$

By virtue of (4.19) and arguing as in (4.35), we can show that

$$\int_{\{|u_n|\leq M\}}|f(x,u_n)||u_n| \to 0.$$

and consequently

$$\int_{\mathbb{R}^2} |f(x,u_n)||u_n| \leq \beta c_* + o_n(1).$$

Hence, since $\beta c_* \in [0, 4\pi/\alpha_0 - \delta)$, according to (4.41) we have

$$\|u_n\| \leq o_n(1) + \left(1 + \frac{\varepsilon}{\alpha_0}\right)^{1/2} \left(\frac{4\pi}{\alpha_0} - \delta\right)^{-1/2} \beta c_* \leq \left(\frac{4\pi}{\alpha_0} - \hat{\delta}\right)^{1/2}, \tag{4.42}$$

for some $0 < \hat{\delta} < \delta$ and for n sufficiently large. On the other hand, choose $p_1 > 1$ close to 1 and $\alpha > \alpha_0$ close to α_0 satisfying $p_1\alpha(4\pi/\alpha_0 - \hat{\delta}) < 4\pi$, from (4.42) it follows that $p_1\alpha\|u_n\|^2 < 4\pi$ for n sufficiently large. Once $u_n \to 0$ strongly in $L^{q_1}(\mathbb{R}^2)$, where $1/p_1 + 1/q_1 = 1$, by invoking Lemma 1.12 we obtain

$$\int_{\mathbb{R}^2} \left(e^{\alpha|u_n|^2} - 1\right)|u_n| \leq C_1 \left(\int_{\mathbb{R}^2} |u_n|^{q_1}\right)^{\frac{1}{q_1}} \left[\int_{\mathbb{R}^2} \left(e^{p_1\alpha|u_n|^2} - 1\right)\right]^{\frac{1}{p_1}} \leq C_2 \left(\int_{\mathbb{R}^2} |u_n|^{q_1}\right)^{\frac{1}{q_1}} \to 0.$$

From this convergence and again by using (1.8), we get

$$\int_{\mathbb{R}^2} |f(x,u_n)||u_n| \to 0,$$

which together with (4.41) implies that $u_n \to 0$ strongly in $H^1(\mathbb{R}^2)$. Thus, again it follows from (4.36) that $c_* = 0$, which is a contradiction and proof of the lemma is complete.

■

Finalizing Proof of Theorem 4.2. Once (4.30) is valid, we can get a sequence $(y_n) \subset \mathbb{R}^2$ and $\nu > 0$ satisfying

$$\int_{B_R(y_n)} u_n^2 \geq \nu.$$

Now, we introduce the following sequence:

$$\tilde{u}_n(\cdot) := u_n(\cdot + y_n).$$

In view of (V) and (F_0), we can see that $\tilde{u}_n \in \mathcal{N}$ and also satisfies $I(\tilde{u}_n) \to c_*$, $I'(\tilde{u}_n) \to 0$ and

$$\liminf_{n\to\infty} \int_{B_{R_0}(0)} \tilde{u}_n^2 dx \geq \nu. \tag{4.43}$$

Observe that, by Lemma 1.21, $(\tilde{u}_n)$ is also bounded in E and, up to a subsequence, we assume $\tilde{u}_n \rightharpoonup \tilde{u}$ weakly in E for some $\tilde{u}$ and $\tilde{u}_n \to \tilde{u}$ a.e. in $\mathbb{R}^2$. From Proposition 4.14 it follows that $I'(\tilde{u}) = 0$. Moreover, in view of (4.43) we have $\tilde{u} \neq 0$ and Remark 1.5 guarantee that $\tilde{u} \in \mathcal{N}$, implying that $c_* \leq I(\tilde{u})$. By invoking Fatou's Lemma, we deduce that

$$\begin{aligned} c_* = \liminf_{n\to\infty} I(\tilde{u}_n) = \liminf_{n\to\infty} \left[I(\tilde{u}_n) - \frac{1}{2} I'(\tilde{u}_n)\tilde{u}_n \right] &= \liminf_{n\to\infty} \int_{\mathbb{R}^2} \left[\frac{1}{2} f(x, \tilde{u}_n) \cdot \tilde{u}_n - F(x, \tilde{u}_n) \right] \\ &\geq \int_{\mathbb{R}^2} \left[\frac{1}{2} f(x, \tilde{u}) \cdot \tilde{u} - F(x, \tilde{u}) \right] \\ &= I(\tilde{u}) - \frac{1}{2} I'(\tilde{u})\tilde{u} \\ &= I(\tilde{u}). \end{aligned}$$

Therefore, $I(\tilde{u}) = c_*$, which shows that $\tilde{u}$ is a ground state for (2) and the theorem is proved.

■

4.3 Proof of Lemma 4.15

Here, for completeness, we present the proof made by Chen-Tang [20]. The following lemma is very important and crucial, which has been proved in [18, 20].

Lemma 4.17. *Assume that $V \in L^\infty(\mathbb{R}^2)$. Then for any $\mu > 0$ there exists two constant $\mathcal{K}_0 > 0$ and $\mathcal{K}_\mu > 0$ such that*

$$\|\nabla u\|_\infty + \|u\|_\infty \leq \mathcal{K}_0 \|u\|_2, \quad \forall u \in \mathcal{E}(0)E = E^- \tag{4.44}$$

and

$$\|u\|_\infty \leq \mathcal{K}_\mu \|u\|_2, \quad \forall u \in \mathcal{E}(\mu)E. \tag{4.45}$$

Applying Lemma 4.17, we deduce that

$$\|\nabla u\|_\infty + \|v\|_\infty \leq \mathcal{C}_0 \|v\|, \quad \forall v \in E^-. \tag{4.46}$$

We may assume that $V(0) < 0$. By (V), we can choose a constant $\rho \in (0, 1/2) \cap$

$(0, 4/\|V\|_\infty)$ such that

$$4\pi\mathcal{C}_0(4+\rho)\rho < 1 \quad \text{and} \quad V(x) \geq 0, \quad |x| \leq \rho. \tag{4.47}$$

Now, we consider Moser's sequence of functions

$$\widetilde{w}_n(x) := \frac{1}{\sqrt{2\pi}} \begin{cases} \sqrt{\log n}, \text{ if } |x| \leq \rho/n, \\ \dfrac{\log \rho/|x|}{\sqrt{\log n}}, \text{ if } \rho/n \leq |x| \leq \rho, \\ 0, \text{ if } |x| \geq \rho. \end{cases} \tag{4.48}$$

We have

$$\|w_n^+\|^2 - \|w_n^-\|^2 = \int_{\mathbb{R}^2} (|\nabla w_n|^2 + V(x)w_n^2)dx \leq \int_{B_\rho} |\nabla w_n|^2 dx = 1. \tag{4.49}$$

Proof. Assume by contradiction that the lemma is false, that is,

$$\max_{s\geq 0, v\in E^-} I(v + sw_{\overline{n}}) \geq \frac{2\pi}{\gamma_0}, \quad n \in \mathbb{N}. \tag{4.50}$$

Let $v_n \in E^-$ and $s_n > 0$ such that $I(v_n + s_n w_n) = \max_{s\geq 0, v\in E^-} I(v + sw_n)$. Then we have

$$I(v_n + s_n w_n) \geq 2\pi/\alpha_0 \quad \text{and} \quad \langle I'(v_n + s_n w_n), v_n + s_n w_n\rangle = 0,$$

that is,

$$\frac{1}{2}(s_n^2\|w_n^+\|^2 - \|v_n + s_n w_n^+\|^2) - \int_{\mathbb{R}^2} F(x, v_n + s_n w_n)dx \geq \frac{2\pi}{\alpha_0} \tag{4.51}$$

and

$$s_n^2\|w_n^+\|^2 - \|v_n + s_n w_n^-\|^2 = \int_{\mathbb{R}^2} f(x, v_n + s_n w_n)(v_n + s_n w_n)dx. \tag{4.52}$$

From (4.3), (4.5), (4.46) and (4.48), we have

$$\begin{aligned} |\langle w_n^-, v_n\rangle| = |\langle w_n, v_n\rangle| &= \left| \int_{\mathbb{R}^2} [\nabla w_n \nabla v_n + V(x) w_n v_n] dx \right| \\ &\leq \|\nabla v_n\|_\infty \int_{\mathbb{R}^2} |\nabla w_n| dx + \|V\|_\infty \|v_n\|_\infty \int_{\mathbb{R}^2} |w_n| dx \\ &\leq \frac{\sqrt{2\pi}\mathcal{C}_0\rho}{\sqrt{\log n}} \|v_n\|. \end{aligned} \tag{4.53}$$

Hence, it follows from (4.3), (4.5), (4.6), (4.49) and (4.53) that

$$\begin{aligned} s_n^2\|w_n^+\|^2 - \|v_n + s_n w_n^-\|^2 &= s_n^2(\|w_n^+\|^2 - \|w_n^+\|^2) - \|v_n\|^2 - 2s_n\langle v_n, w_n^-\rangle \\ &\leq s_n^2 - \|v_n\|^2 + \frac{2\sqrt{2\pi}\mathcal{C}_0\rho s_n}{\sqrt{\log n}}\|v_n\|. \end{aligned} \tag{4.54}$$

Combining (4.51), (4.52) with (4.54), we have

$$\frac{4\pi}{\alpha_0} \leq s_n^2 - \|v_n\|^2 + \frac{2\sqrt{2\pi}\mathcal{C}_0\rho s_n}{\sqrt{\log n}}\|v_n\| \leq s_n^2\left(1 + \frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right) \tag{4.55}$$

and

$$\begin{aligned} s_n^2\left(1 + \frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right) &\geq s_n^2 - \|v_n\|^2 + \frac{2\sqrt{2\pi}\mathcal{C}_0\rho s_n}{\sqrt{\log n}}\|v_n\| \\ &\geq \int_{\mathbb{R}^2} f(x, v_n + s_n w_n)(v_n + s_n w_n)dx. \end{aligned} \tag{4.56}$$

Moreover, (4.55) implies

$$s_n^2 \geq \frac{4\pi}{\alpha_0}\left(1 + \frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right), \quad \frac{\|v_n\|}{s_n} \leq 1 + \frac{2\sqrt{2\pi}\mathcal{C}_0\rho}{\sqrt{\log n}}. \tag{4.57}$$

Let $M_n = \dfrac{1}{\sqrt{2\pi}}\sqrt{\log n}$. By (4.46), (4.48) and (4.57), we have

$$\begin{aligned} v_n(x) + s_n w_n(x) &\geq -\|v_n\|_\infty + s_n M_n \\ &\geq -\mathcal{C}_0\|v_n\| + s_n M_n \\ &\geq (1 - 2\mathcal{C}_0/M_n)s_n M_n, \quad x \in B_\rho/n. \end{aligned} \tag{4.58}$$

By (4.2), we can choose $\varepsilon > 0$ such that

$$\frac{\kappa - \varepsilon}{1 + \varepsilon} >> \frac{4e^{16\pi\mathcal{C}_0^2}}{\alpha_0\rho^2}. \tag{4.59}$$

Note that

$$\int_0^t s^2 f(x,s)ds = s^2 F(x,s)\Big|_0^t - 2\int_0^t sF(x,s)ds$$

Then,

$$\liminf_{|t|\to\infty} \frac{t^2 F(x,t)}{e^{\alpha_0 t^2}} \geq \liminf_{|t|\to\infty} \frac{\int_0^t s^2 f(x,s)ds}{e^{\alpha_0 t^2}} = \liminf_{|t|\to\infty} \frac{tf(x,t)}{2\alpha_0} e^{\alpha_0 t^2} = \frac{\kappa}{2\alpha_0}. \tag{4.60}$$

It follows from (4.2) and (4.60) that there exists $t_\varepsilon > 0$ such that

$$tf(x,t) \geq (\kappa - \varepsilon) e^{\alpha_0 t^2}, \quad t^2 F(x,t) \geq \frac{\kappa - \varepsilon}{2\alpha_0} e^{\alpha_0 t^2}, \quad \forall x \in \mathbb{R}^2, \quad |t| \geq t_\varepsilon. \tag{4.61}$$

From now on, in the sequel, all inequalities hold for large $n \in \mathbb{N}$. By (4.56), (4.58) and (4.61), we have

$$\begin{aligned} s_n^2 \left(1 + \frac{2\pi \mathcal{C}_0^2 \rho^2}{\log n}\right) &\geq \int_{\mathbb{R}^2} f(x, v_n + s_n w_n)(v_n + s_n w_n) dx \\ &\geq (\kappa - \varepsilon) \int_{B_{\rho/n}} e^{\alpha_0 (v_n + s_n w_n)^2} \\ &\geq \frac{\pi(\kappa - \varepsilon)\rho^2}{n^2} e^{\alpha_0 s_n^2 M_n^2 (1 - 2\mathcal{C}_0/M_n)^2} \\ &\geq \frac{\pi(\kappa - \varepsilon)\rho^2}{n^2} \exp\left[\frac{\alpha_0 s_n^2 \log n}{2\pi}\left(1 - \frac{4\mathcal{C}_0}{M_n}\right)\right] \\ &\geq \pi(\kappa - \varepsilon)\rho^2 \exp\left\{2\log n \left[\frac{\alpha_0 s_n^2}{4\pi}\left(1 - \frac{4\mathcal{C}_0}{M_n}\right) - 1\right]\right\}, \end{aligned}$$

which implies that there exists a constant

$$2\log n \left[\frac{\alpha_0 s_n^2}{4\pi}\left(1 - \frac{4\mathcal{C}_0}{M_n}\right) - 1\right] \leq A,$$

that is,

$$s_n^2 \leq \frac{4\pi}{\alpha_0}\left(1 - \frac{4\mathcal{C}_0}{M_n}\right)^{-1}\left(1 + \frac{A}{\log n}\right). \tag{4.62}$$

Hence, from (4.10), (4.48), (4.54), (4.58) and (4.61), we obtain

$$\begin{aligned}
& I(v_n + s_n w_n) \\
& = \frac{1}{2}(s_n^2 \|w_n^+\|^2 - \|v_n + s_n w_n^-\|^2 - \int_{\mathbb{R}^2} F(x, v_n + s_n w_n) dx \\
& \leq \frac{s_n^2}{2} - \frac{1}{2}\|v_n\|^2 + \frac{\sqrt{2\pi}\mathcal{C}_0 \rho s_n}{\sqrt{\log n}}\|v_n\| - \int_{\mathbb{R}^2} F(x, v_n + s_n w_n) dx \\
& \leq \frac{s_n^2}{2} - \frac{1}{2}\|v_n\|^2 + \frac{\sqrt{2\pi}\mathcal{C}_0 \rho s_n}{\sqrt{\log n}}\|v_n\| - \frac{\kappa - \varepsilon}{2\alpha_0} \int_{B_{\rho/n}} \frac{e^{\alpha_0 (v_n + s_n w_n)^2}}{(v_n + s_n w_n)^2} \\
& \leq \frac{s_n^2}{2} - \frac{1}{2}\|v_n\|^2 + \frac{\sqrt{2\pi}\mathcal{C}_0 \rho s_n}{\sqrt{\log n}}\|v_n\| - \frac{(\kappa - \varepsilon)\pi \rho^2 e^{\alpha_0 (-\mathcal{C}_0 \|v_n\| + s_n M_n)^2}}{2\alpha_0 n^2 (-\mathcal{C}_0 \|v_n\| + s_n M_n)^2}.
\end{aligned} \tag{4.63}$$

Both (4.57) and (4.62) show that $\frac{4\pi}{\alpha_0}(1 - \varepsilon) \leq s_n^2 \leq \frac{4\pi}{\alpha_0}(1 + \varepsilon)$.

There are three cases to distinguish.

Case i) $\frac{4\pi}{\alpha_0}(1-\varepsilon) \leq s_n^2 \leq \frac{4\pi}{\alpha_0}$. It follows from (4.55) that $\|v_n\| \leq 2\pi \mathcal{C}_0 s_n M_n / \log n$. Then (4.63) leads to

$$\begin{aligned}
& I(v_n + s_n w_n) \\
& \leq \frac{s_n^2}{2} - \frac{1}{2}\|v_n\|^2 + \frac{\sqrt{2\pi}\mathcal{C}_0 \rho s_n}{\sqrt{\log n}}\|v_n\| - \frac{(\kappa - \varepsilon)\pi \rho^2 e^{\alpha_0 (-\mathcal{C}_0 \|v_n\| + s_n M_n)^2}}{2\alpha_0 n^2 (-\mathcal{C}_0 \|v_n\| + s_n M_n)^2} \\
& \leq \frac{s_n^2}{2}\left(1 + \frac{2\pi \mathcal{C}_0^2 \rho^2}{\log n}\right) - \frac{(\kappa - \varepsilon)\rho^2 e^{\alpha_0 s_n^2 M_n^2 (1 - 2\mathcal{C}_0 \|v_n\| / s_n M_n)}}{8 n^2 (1 + \varepsilon) M_n^2} \\
& \leq \frac{s_n^2}{2}\left(1 + \frac{2\pi \mathcal{C}_0^2 \rho^2}{\log n}\right) - \frac{(\kappa - \varepsilon)\pi \rho^2 e^{\frac{\alpha_0 s_n^2}{2\pi}(\log n - 4\pi \mathcal{C}_0^2)}}{4 n^2 (1 + \varepsilon) M_n^2}
\end{aligned} \tag{4.64}$$

Let us define a function $\varphi_n(s)$ as follows:

$$\varphi_n(s) = \frac{s^2}{2}\left(1 + \frac{2\pi \mathcal{C}_0^2 \rho^2}{\log n}\right) - \frac{(\kappa - \varepsilon)\pi \rho^2 e^{\frac{\alpha_0 s_n^2}{2\pi}(\log n - 4\pi \mathcal{C}_0^2)}}{4 n^2 (1 + \varepsilon) M_n^2}$$

Set $\widehat{s}_n > 0$ such that $\varphi_n'(\widehat{s}_n) = 0$. Then

$$\widehat{s}_n = \frac{4\pi}{\alpha_0}\left[1 + \frac{8\pi \mathcal{C}_0^2 + \log 4(1 + \varepsilon) - \log(\alpha_0 (\kappa - \varepsilon)\rho^2)}{2(\log n - 4\pi \mathcal{C}_0^2)}\right] + O\left(\frac{1}{\log^2 n}\right) \tag{4.65}$$

and

$$\varphi_n(s_n) \leq \varphi_n(\widehat{s}_n) = \frac{1 + \frac{2\pi \mathcal{C}_0^2 \rho^2}{\log n}}{2}\widehat{s}_n^2 - \frac{\pi\left(1 + \frac{2\pi \mathcal{C}_0^2 \rho^2}{\log n}\right)}{\alpha_0 (\log n - 4\pi \mathcal{C}_0^2)}. \tag{4.66}$$

Using (4.65), we have

$$
\begin{aligned}
&\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)\widehat{s}_n^2\\
&=\frac{4\pi}{\alpha_0}\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)\left[1+\frac{8\pi\mathcal{C}_0^2+\log 4(1+\varepsilon)-\log(\alpha_0(\kappa-\varepsilon)\rho^2)}{2(\log n-4\pi\mathcal{C}_0^2)}\right]+O\left(\frac{1}{\log^2 n}\right)\\
&\le\frac{4\pi}{\alpha_0}\left[1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}+\frac{8\pi\mathcal{C}_0^2+\log 4(1+\varepsilon)-\log(\alpha_0(\kappa-\varepsilon)\rho^2)}{2(\log n-4\pi\mathcal{C}_0^2)}\right]+O\left(\frac{1}{\log^2 n}\right) \qquad (4.67)
\end{aligned}
$$

Hence, from (4.47), (4.59), (4.64), (4.66) and (4.67), we derive

$$
\begin{aligned}
I(v_n+s_nw_n)\le\varphi_n(s_n)&\\
&=\frac{1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}}{2}\widehat{s}_n^2-\frac{\pi\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)}{\alpha_0(\log n-4\pi\mathcal{C}_0^2)}\\
&\le\frac{4\pi}{\alpha_0}\left[\frac{1}{2}-\frac{1-4\pi\mathcal{C}_0^2\rho^2}{4\log n}+\frac{8\pi\mathcal{C}_0^2+\log 4(1+\varepsilon)-\log(\alpha_0(\kappa-\varepsilon)\rho^2)}{4(\log n-4\pi\mathcal{C}_0^2)}\right]\\
&\quad+O\left(\frac{1}{\log^2 n}\right)\\
&\le\frac{4\pi}{\alpha_0}\left[\frac{1}{2}-\frac{1-4\pi\mathcal{C}_0^2\rho^2}{4\log n}\right]+O\left(\frac{1}{\log^2 n}\right).
\end{aligned}
$$

This contradicts with (4.50) due to (4.47).

Case ii) $\frac{4\pi}{\alpha_0}(1+2\pi\mathcal{C}_0\|v_n\|/s_nM_n)\le s_n^2\le\frac{4\pi}{\alpha_0}(1+\varepsilon)$.

Then (4.55), (4.56), (4.58), (4.59), (4.61) and (4.62) yield

$$
\begin{aligned}
\frac{4\pi}{\alpha_0}(1+\varepsilon)\ge s_n^2\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)&\\
&\ge\int_{\mathbb{R}^2}f(x,v_n+s_nw_n)(v_n+s_nw_n)dx\\
&\ge(\kappa-\varepsilon)\int_{B_{\rho/n}}e^{\alpha_0(v_n+s_nw_n)^2}dx\\
&\ge\frac{\pi(\kappa-\varepsilon)\rho^2}{n^2}e^{\alpha_0(-\mathcal{C}_0\|v_n\|+s_nM_n)^2}\\
&\ge\frac{\pi(\kappa-\varepsilon)\rho^2}{n^2}e^{\alpha_0s_n^2M_n^2(1-2\mathcal{C}_0\|v_n\|/s_nM_n)}\\
&\ge\frac{\pi(\kappa-\varepsilon)\rho^2}{n^2}e^{2\log n(1-\mathcal{C}_0^2\|v_n\|^2/s_n^2M_n^2)}\\
&\ge\pi(\kappa-\varepsilon)\rho^2e^{-16\pi\mathcal{C}_0^2\|v_n\|^2/s_n^2}\\
&\ge\frac{4\pi}{\alpha_0}(1+\varepsilon)e^{15\pi\mathcal{C}_0^2},
\end{aligned}
$$

which yields a contradiction.

Case iii) $\frac{4\pi}{\alpha_0} \le s_n^2 \le \frac{4\pi}{\alpha_0}(1 + 2\mathcal{C}_0\|v_n\|/s_nM_n)$. Then it follows from (4.55) that

$$\|v_n\|^2 - \frac{2\sqrt{2\pi}\mathcal{C}_0\rho s_n}{\log n}\|v_n\| \le \frac{8\pi\mathcal{C}_0\|v_n\|}{\alpha_0 s_n M_n} = \frac{8\pi\sqrt{2\pi}\mathcal{C}_0}{\alpha_0 s_n\sqrt{\log n}}\|v_n\|,$$

which, together with (4.57) and (4.62), implies that

$$\frac{\|v_n\|}{s_n} \le \frac{2\sqrt{2\pi}\mathcal{C}_0(1+\rho)}{\sqrt{\log n}}. \tag{4.68}$$

It follows from (4.63) and (4.68) that

$$\begin{aligned} & I(v_n + s_n w_n) \\ & \le \frac{s_n^2}{2} - \frac{1}{2}\|v_n\|^2 + \frac{\sqrt{2\pi}\mathcal{C}_0\rho s_n}{\sqrt{\log n}}\|v_n\| - \frac{(\kappa-\varepsilon)\pi\rho^2 e^{\alpha_0(-\mathcal{C}_0\|v_n\|+s_nM_n)^2}}{2\alpha_0 n^2(-\mathcal{C}_0\|v_n\| + s_nM_n)^2} \\ & \le \frac{s_n^2}{2}\left(1 + \frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right) - \frac{(\kappa-\varepsilon)\rho^2 e^{\alpha_0(-\mathcal{C}_0\|v_n\|+s_nM_n)^2}}{8n^2(1+\varepsilon)M_n^2} \\ & \le \frac{s_n^2}{2}\left(1 + \frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right) - \frac{(\kappa-\varepsilon)\rho^2 e^{\alpha_0 s_n^2 M_n^2(1-2\mathcal{C}_0\|v_n\|/s_nM_n)}}{8n^2(1+\varepsilon)M_n^2} \\ & \le \frac{s_n^2}{2}\left(1 + \frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right) - \frac{(\kappa-\varepsilon)\pi\rho^2 e^{\frac{\alpha_0 s_n^2}{2\pi}[\log n - 8\pi(1+\rho)\mathcal{C}_0^2]}}{4n^2(1+\varepsilon)\log n}. \end{aligned} \tag{4.69}$$

Setting

$$\psi_n(s) = \frac{s^2}{2}\left(1 + \frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right) - \frac{(\kappa-\varepsilon)\pi\rho^2 e^{\frac{\alpha_0 s_n^2}{2\pi}[\log n - 8\pi(1+\rho)\mathcal{C}_0^2]}}{4n^2(1+\varepsilon)\log n}.$$

Let $\tilde{s}_n > 0$ such that $\psi_n'(\tilde{s}_n) = 0$. Then

$$\tilde{s}_n^2 = \frac{4\pi}{\alpha_0}\left[1 + \frac{16\pi\mathcal{C}_0^2(1+\rho) + \log 4(1+\varepsilon) - \log(\alpha_0(\kappa-\varepsilon)\rho^2)}{2[\log n - 8\pi(1+\rho)\mathcal{C}_0^2]}\right] + O\left(\frac{1}{\log^2 n}\right) \tag{4.70}$$

and

$$\psi_n(s) = \psi_n(\tilde{s}_n) = \frac{1 + \frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}}{2}\tilde{s}_n^2 - \frac{\pi\left(1 + \frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)}{\alpha_0[\log n - 8\pi(1+\rho)\mathcal{C}_0^2]}. \tag{4.71}$$

Combining (4.70) with (4.71), we have

$$
\begin{aligned}
&\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)\tilde{s}_n^2 \\
&=\frac{4\pi}{\alpha_0}\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)\left[1+\frac{16\pi\mathcal{C}_0^2(1+\rho)+\log 4(1+\varepsilon)-\log(\alpha_0(\kappa-\varepsilon)\rho^2)}{2[\log n-8\pi(1+\rho)\mathcal{C}_0^2]}\right] \qquad (4.72)\\
&\quad+O\left(\frac{1}{\log^2 n}\right)\\
&\le\frac{4\pi}{\alpha_0}\left\{1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}+\frac{16\pi\mathcal{C}_0^2(1+\rho)+\log 4(1+\varepsilon)-\log(\alpha_0(\kappa-\varepsilon)\rho^2)}{2[\log n-8\pi(1+\rho)\mathcal{C}_0^2]}\right\} \qquad (4.73)\\
&\quad+O\left(\frac{1}{\log^2 n}\right) \qquad (4.74)
\end{aligned}
$$

Hence, from (4.47), (4.71) and (4.72), we deduce

$$
\begin{aligned}
\psi_n(s)&\le\frac{1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}}{2}\tilde{s}_n^2-\frac{\pi\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)}{\alpha_0[\log n-8\pi(1+\rho)\mathcal{C}_0^2]}\\
&\le\frac{4\pi}{\alpha_0}\left\{\frac{1}{2}-\frac{1-4\pi\mathcal{C}_0^2\rho^2}{4\log n}+\frac{16\pi\mathcal{C}_0^2(1+\rho)+\log 4(1+\varepsilon)-\log(\alpha_0(\kappa-\varepsilon)\rho^2)}{4[\log n-8\pi(1+\rho)\mathcal{C}_0^2]}\right\}\\
&\quad+O\left(\frac{1}{\log^2 n}\right)\\
&\le\frac{4\pi}{\alpha_0}\left[\frac{1}{2}-\frac{1-4\pi\mathcal{C}_0^2\rho(4+\rho)}{4\log n}\right]+O\left(\frac{1}{\log^2 n}\right)
\end{aligned}
$$

It follows from (4.69) that

$$
I(v_n+s_nw_n)\le\psi_n(s_n)\le\frac{4\pi}{\alpha_0}\left[\frac{1}{2}-\frac{1-4\pi\mathcal{C}_0^2\rho(4+\rho)}{4\log n}\right]+O\left(\frac{1}{\log^2 n}\right).
$$

and this contradicts (4.50) in view of (4.47). The above three cases show that there exists $\tilde{n}\in\mathbb{N}$ such that (4.28) holds and the proof is done.

■

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