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Programa de Pós-Graduação em Matemática
Doutorado em Matemática

Nested Hilbert schemes on Hirzebruch surfaces and quiver varieties

by

Pedro Henrique dos Santos

João Pessoa - PB
February/2024

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under the supervision of

Prof. Dr. Ugo Bruzzo

and the co-supervision of

Prof. Dr. Valeriano Lanza

Tese apresentada ao Corpo Docente do Programa de Pós-Graduação em Matemática - UFPB, como requisito parcial para obtenção do título de Doutor em Matemática.

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Resumo

Esquemas de Hilbert foram introduzidos por Grothendieck. Eles são um exemplo fundamental da noção de espaços de moduli de estruturas geométricas. O trabalho de Nakajima a respeito das propriedades do esquema de Hilbert de pontos no plano complexo tem sido base para diversos trabalhos que visam entender as propriedades dos esquemas de Hilbert de outras superfícies, assim como em dimensões mais altas. Além disso, o esquema de Hilbert aninhado de pontos no plano complexo foi estudado por von Flach, Jardim e Lanza. Ademais, Bartocci, Bruzzo, Lanza e Rava obtiveram uma descrição quiver para o esquema de Hilbert de pontos do espaço total Ξ_n de fibrados de linha sobre a linha projetiva apropriados. Neste trabalho, nós mostramos que o esquema de Hilbert aninhado de pontos na variedade que acabamos de mencionar, parametrizando pares de 0-ciclos aninhados, é uma variedade de quiver associada a um quiver com relações adequado, generalizando trabalhos prévios sobre esquemas de Hilbert aninhados de pontos no plano complexo, em uma direção, e sobre os esquemas de Hilbert de pontos em Ξ_n em outra.

Palavras-chave: Bandeiras de feixes *framed*, esquemas de Hilbert, superfícies de Hirzebruch, espaços de moduli de representações de quivers.

Abstract

Hilbert schemes were introduced by Grothendieck. They are a fundamental example of the notion of moduli spaces of geometric structures. The work of Nakajima on the properties of the Hilbert schemes of points of the complex plane has been the basis of many works that try to understand the properties of Hilbert schemes of other 2-dimensional varieties and also for higher dimensions. Furthermore, the nested Hilbert scheme of points on the complex plane was studied by von Flach, Jardim and Lanza. Moreover, Bartocci, Bruzzo, Lanza and Rava obtained a quiver description to the Hilbert scheme of points of the total space Ξ_n of appropriate line bundles over the projective line. In this work we show that the nested Hilbert scheme of points on the last varieties, parameterizing pairs of nested 0-cycles, is the quiver variety associated with a suitable quiver with relations, generalizing previous work about nested Hilbert schemes on the complex plane, in one direction, and about the Hilbert schemes of points of Ξ_n in another direction.

Keywords: Framed flags of sheaves, Hilbert schemes, Hirzebruch surfaces, Moduli spaces of quiver representations.

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"Estou convicto de que a Matemática pode e deve ser ensinada de forma espontânea, leve, humana e, em alguns casos, até mesmo alegre, para que se torne fonte de prazer intelectual e conquiste um número cada vez maior de adeptos."

Gilberto Geraldo Garbi

Dedicatory

A minha avó...

Contents

Introduction

Hilbert schemes are quite important in modern Algebraic Geometry, as they have a pivotal role in many areas, such as the construction of moduli spaces and deformation theory. Introduced by Grothendieck in [20], Hilbert schemes are a fundamental example of the notion of moduli spaces of geometric structures. They are schemes, in the sense of Algebraic Geometry, which parameterize the closed subschemes of a scheme X (usually, a regular quasi-projective variety) that have fixed invariants; for example, the zero-dimensional subschemes, also known as 0-cycles, in which case they are called Hilbert schemes of points.

When working with Hilbert schemes of points, it is quite clear that the difficulty of the problem increases quite fast with the dimension of the space and the number of points, so that any kind of generalization may be quite hard. For instance, even for small values of the dimension of X and the number of points, such issues as the characterization of the irreducible components, their smoothness, reducedness etc., may be very challenging.

In the case where $\dim(X) = 1$, the Hilbert scheme of n points on X is isomorphic to the n-th symmetric product of X:

$$\mathrm{Hilb}^n(X) \simeq \frac{X^n}{S_n},$$

where S_n is the n-th group of permutations; and if X is nonsingular, then $\mathrm{Hilb}^n(X)$ is also nonsingular, cf. [34].

The case $\dim(X) = 2$ was studied by such authors as Briançon in [6], Iarrobino in [27] and Nakajima in [34], starting with the case $X = \mathbb{C}^2$. Nakajima in particular obtained a description of the Hilbert scheme in terms of linear data, called ADHM data.[3]

[3] ADHM stands for the name of the following scientists: Atiyah, Drinfeld, Hitchin and Manin, because of their paper [1] **Construction of instantons** from 1978.

Theorem (Nakajima, Theorem 1.9 in [34]). *Let*

$$\widetilde{H} := \{(B_1, B_2, I);\ [B_1, B_2] = 0,\ \nexists\ S \subsetneq \mathbb{C}^c; B_1(S), B_2(S) \subseteq S \text{ and } \operatorname{Im}(I) \subseteq S\},$$

where $B_1, B_2 \in \operatorname{End}(\mathbb{C}^c)$ *and* $I \in \operatorname{Hom}(\mathbb{C}, \mathbb{C}^c)$. *The group* $\mathrm{GL}_c(\mathbb{C})$ *acts on* $\widetilde{H}$ *by*

$$g \cdot (B_1, B_2, I) := (gB_1g^{-1}, gB_2g^{-1}, gI).$$

The quotient $H := \dfrac{\widetilde{H}}{\mathrm{GL}_c(\mathbb{C})}$ *is a nonsingular variety that represents the Hilbert functor of* 0*-cycles on* $X = \mathbb{C}^2$ *of length* c.

This was generalized by Henni and Jardim [24] who proved that $\mathrm{Hilb}^c(\mathbb{C}^n)$ may be also realized as a GIT quotient. For all $n \in \mathbb{N}$, if V and W are complex vector spaces of dimension c and 1, respectively, then

$$\mathrm{Hilb}^c(\mathbb{C}^n) \simeq (\mathcal{C}(n, c) \times \operatorname{Hom}(V, W))/\!\!/\,\mathrm{GL}(V),$$

where $\mathcal{C}(n, c)$ is the variety of n-uples of commuting $c \times c$ matrices.

In many ways, quivers play a basic role in the study and description of Hilbert schemes, and, more generally, of moduli spaces of framed sheaves. They are a powerful tool for encoding by simple algebraic data the complicated structure of these spaces. Starting from an often very simple directed graph one first constructs an associative algebra, then constructs a moduli space of its representations (called a *quiver variety*), and eventually discovers that this is also the moduli space of nontrivial geometric structures. For instance, if we consider instantons, i.e., anti-self-dual connections on the 4-sphere, including degenerate configurations, we have a singular moduli space, whose resolution of singularities is on the one hand the moduli space of framed torsion-free sheaves on the complex projective plane; and on the other hand, thanks to Nakajima's work [34], can be regarded as the moduli space of representations of the path algebra of a quiver with relations — the ADHM quiver. Since a rank one torsion-free sheaf on $\mathbb{P}^2$, framed on a line, may be identified with the ideal sheaf of a 0-cycle on $\mathbb{C}^2$, also the Hilbert scheme of points of $\mathbb{C}^2$ is a quiver variety. A geometric description of some Nakajima quiver varieties was studied by Kuznetsov in [30]. Other examples of this correspondence are

- moduli spaces of instantons on ALE (Asymptotically Locally Euclidean) spaces [29];[4]

[4]ALE spaces are resolutions of singularities of quotients $\mathbb{C}^2/G$, where G is a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$.

- (equivariant) Hilbert schemes of points of ALE spaces ([30] and references therein);
- the crepant resolutions of singularities $\mathbb{C}^3/G$, where G is a finite subgroup of $\mathrm{SL}_3(\mathbb{C})$, are moduli spaces of structures called G-constellations (a generalization of the G-Hilbert schemes), and are moduli spaces of representations of the McKay quivers [14, 13]. Related constructions and results can also be found in [11, 12] and other papers.

Another construction, very relevant to the present thesis, is described in [42, 43]. One consider *framed flags* on $\mathbb{P}^2$, i.e., pairs (E, F), where E and F are torsion-free sheaves of the same rank on $\mathbb{P}^2$, such that $E \subset F$, F is framed on a line, and the quotient F/E has dimension zero and is supported away from the line. The moduli space of these pairs turns out to be a quiver variety associated with an *enhanced ADHM quiver.* When $\operatorname{rk} E = \operatorname{rk} F = 1$ the moduli space is the *nested Hilbert scheme* of $\mathbb{C}^2$, and parameterizes pairs of nested 0-cycles.

Now we shift our attention to *Hirzebruch surfaces.* By removing a suitable rational curve ℓ_∞ from the n-th Hirzebruch surface Σ_n, with $n \geq 1$, one obtains the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-n)$, that we denote Ξ_n. In this case, rank 1 torsion-free sheaves on Σ_n framed on ℓ_∞ are ideal sheaves of 0-cycles in Ξ_n. Building on the description of the moduli spaces of framed sheaves on Σ_n that was performed in [4], in [2, 3] it was shown that the Hilbert schemes of points $\mathrm{Hilb}^c(\Xi_n)$ are quiver varieties associated with a suitable quiver, denoted Q_n. The aim of the present work is to obtain a similar description in the case of the *nested Hilbert schemes of points* $\mathrm{Hilb}^{c',c}(\Xi_n)$.

It is quite natural to phrase our central problem in categorical language, so that we aim to establish an isomorphism between the functor of families of representations of a certain quiver, and the functor of families of nested 0-cycles on Ξ_n), we start by rephrasing the results of both [4, 2] and [42, 43] in full categorical language. Our main result:

Theorem. *Let $\mathcal{M}^{n,\mathrm{fr},s}_{\mathbf{v},\Theta}$ be the moduli space of framed representations of the quiver Q^{enh}_n with dimension vector $\mathbf{v} = (c, c, c - c', c - c', 1)$, stable with respect to the stability parameter $\Theta = (\theta_1, \theta_2, \theta_3, \theta_4)$, where*

$$\theta_1 > 0, \quad \theta_3, \theta_4 < 0, \quad \theta_1 + \theta_2 + (\theta_3 + \theta_4)(c - c') > 0, \quad -\theta_1 < \theta_2 < -\frac{c-1}{c}\theta_1.$$

$\mathcal{M}^{n,\mathrm{fr},s}_{\mathbf{v},\Theta}$ is isomorphic to the nested Hilbert scheme $\mathrm{Hilb}^{c',c}(\Xi_n)$.

is proved in Chapter 4; the trick for doing that is similar to the case of the projective plane, i.e., to regard the spaces of representations of an "enhanced" quiver as a space of morphisms between two copies of the quiver Q_n, although the present case is more complicated and technically more involved.

Our work is divided in 5 chapters. The first chapter is dedicated to displaying all our objects of study: quivers, Hilbert schemes, moduli spaces and framed flags of sheaves. As it was said before, our work generalizes both works on nested Hilbert schemes of points on the projective plane and on Hilbert schemes of points on the total space Ξ_n of a suitable line bundle over the projective line, so that the second and third chapter are dedicated to obtain a sound understanding of these works and develop their categorical approaches.

In the fourth and final chapter we prove our main theorem and some auxiliary results, such as: a characterization for the stable representations of the quiver Q_n^{enh} in terms of the stability of the datum corresponding to Q_n and an additional hypothesis.

Further developments. There are some interesting topics that we expect to explore in future works. For instance, differently from [2, 3, 43], we did not succeed yet to identify the chamber in the space of stability parameters of the quivers Q_n^{enh} that contains our parameter (i.e., the one that corresponds to the nested Hilbert scheme of points of Ξ_n); by doing so we would open also the possibility of studying variation of stability and wall-crossing phenomena. The double nested Hilbert scheme of $\mathbb{C}^n$ or Ξ_n, which became an object of interest in recent works as [19] and [32], does not have a description in terms of representations of a quiver so far, but we expect that one can achieve this goal by suitably adapting the techniques used in this thesis. Finally, one might also wonder if these nested Hilbert schemes are connected and irreducible; a similar question is open (in full generality) also for $\mathrm{Hilb}^c(\mathbb{C}^t)$ and $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^t}^{\oplus r}, c)$, see [24, 23, 21] and [16]; we do think that a quiver description may help to answer these kind of problems.

Notation and Terminology

- Unless the contrary is mentioned, all vector spaces are $\mathbb{C}$-vector spaces.
- All schemes will be connected noetherian schemes of finite type over $\mathbb{C}$.
- All locally free sheaves will have finite rank.
- $\triangle$ denotes the end of a definition.
- $\square$ denotes the end of the proof of a lemma, proposition or theorem.
- We will denote $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, with $\mathbb{N} = \{1, 2, 3, ...\}$.

Chapter 1

Preliminaries

In this chapter we present the main subjects of our work. In the first section we explore the quiver world: examples of quivers, operations, representations and sub-representations and morphisms. In the second section, we talk about Hilbert schemes and nested Hilbert schemes. The third section is dedicated to explore some ideas on the stability conditions for quiver representations. The fourth section is about framed flags of sheaves, where we define one of our main objects of study for the Chapters 2 and 4. The fifth and final section is devoted to explore the notion of monads, to make our work self-contained.

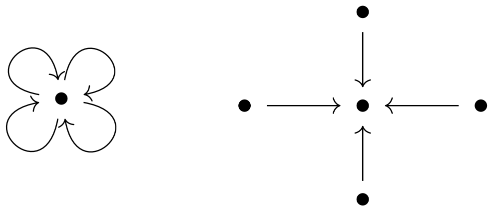
1.1 Quivers

In many ways, quivers play a basic role in the study and description of moduli spaces of framed sheaves. They are a powerful tool for encoding by simple algebraic data the complicated structure of these spaces. This section has the aim of presenting some quiver generalities that will be useful for this work.

1.1.1 Definitions and examples

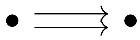
Definition 1.1.1. A (finite) quiver Q is a quadruple $Q = (Q_0, A, s, t)$, where Q_0 and A are finite sets (the set of vertices and the set of arrows, respectively) and s and t are maps from A to Q_0 that associate to each arrow its source and target, respectively. $\triangle$

Example 1.1.2. The quiver composed by a single vertex and r arrows is called the r-loop and it is denoted by L_r. The quiver composed by $r + 1$ vertices and r arrows with a commom target is denoted by S_r. One can see the quivers L_4 (left) and S_4 (right) below:



(1.1)

Example 1.1.3. The quiver K_n is the quiver with two vertices and n arrows with the same sources and targets. One can see K_2 below.



Definition 1.1.4.

1. A representation X of a quiver Q consists of a family of vector spaces of finite dimension $\{V_i\}_{i\in Q_0}$ and a family of linear maps $\{f_a : V_{s(a)} \to V_{t(a)}\}_{a\in A}$; we shall write $X = (\{V_i\}_{i\in Q_0}, \{f_a\}_{a\in A})$.
2. A subrepresentation Y of X is a collection of subspaces $\{W_i \subseteq V_i\}_{i\in Q_0}$ such that $f_a(W_{s(a)}) \subseteq W_{t(a)}$, $\forall a \in A$, i.e., the family of linear maps of Y is the restriction of the family of X to the corresponding subspaces. △

Example 1.1.5. For the quiver $Q = L_{13}$, a representation of Q is composed by a finite dimensional vector space V and 13 elements in $\mathrm{End}(V)$.

Definition 1.1.6. A morphism $u : X \to Y$ between two representations

$$X = (\{V_i\}_{i\in Q_0}, \{f_a\}_{a\in A}) \quad \text{and} \quad Y = (\{W_i\}_{i\in Q_0}, \{g_a\}_{a\in A})$$

is a collection of linear maps $(u_i : V_i \to W_i)_{i\in Q_0}$ such that the following diagram commutes for every $a \in A$:

$$\begin{array}{ccc} V_{s(a)} & \xrightarrow{f_a} & V_{t(a)} \\ {\scriptstyle u_{s(a)}}\downarrow & & \downarrow{\scriptstyle u_{t(a)}} \\ W_{s(a)} & \xrightarrow[g_a]{} & W_{t(a)} \end{array}$$

An isomorphism of representations is a morphism u where every u_i is an isomorphism of vector spaces. △

Definition 1.1.7. The dimension vector of a representation $X = (\{V_i\}_{i\in Q_0}, \{f_a\}_{a\in A})$ is the vector $\mathbf{v} = (\dim(V_i))_{i\in Q_0} \in \mathbb{N}_0^{Q_0}$. △

Remark 1.1.8. Two isomorphic representations share the same dimension vector.

Definition 1.1.9. Let Q be a quiver.

1. A path in Q is a sequence $\alpha = a_1 a_2 \cdots a_m$ of arrows such that $s(a_i) = t(a_{i+1})$ for $i \in \{1, \ldots, m-1\}$;

2. The $\mathbb{C}$-vector space $\mathbb{C}Q$, whose basis is the set of all paths in Q, can be made into a $\mathbb{C}$-algebra, called the path algebra of Q, by defining the multiplication $\alpha\alpha'$ as the concatenation of α and α', if the target of α' is the source of α, and zero, otherwise.

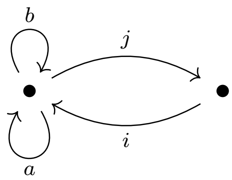
△

Remark 1.1.10. The notation $\alpha\alpha'$ in the path algebra means α' followed by α.

Definition 1.1.11. A relation is a formal sum of paths that start and end at the same vertex. A quiver with relations is a quiver together with a collection of relations. △

Remark 1.1.12. A set of relations clearly define an ideal of the path algebra.

Example 1.1.13. The ADHM quiver is the quiver



with the relation $ab - ba + ij$.

Definition 1.1.14. Given a quiver $Q = (Q_0, A, s, t)$ and a dimension vector $\mathbf{v} = (v_i)_{i \in Q_0}$, the space of representations of Q with dimension vector $\mathbf{v}$ is

$$\mathrm{Rep}(Q, \mathbf{v}) := \bigoplus_{a \in A} \mathrm{Hom}(\mathbb{C}^{v_{s(a)}}, \mathbb{C}^{v_{t(a)}}) = \bigoplus_{a \in A} \mathrm{Mat}_{v_{t(a)} \times v_{s(a)}}(\mathbb{C}). \tag{1.2}$$

△

Remark 1.1.15. $\mathrm{Rep}(Q, \mathbf{v})$ is a vector space of dimension $d = \sum_{a: i \to j} v_i v_j$.

Remark 1.1.16. Even though we have used $\mathbb{C}$-vector spaces in this subsection, the constructions can be done with k-vector spaces for any field k.

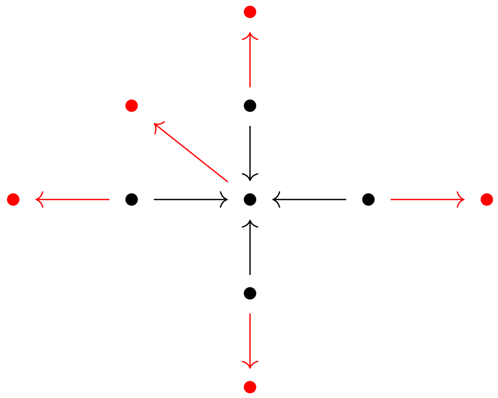
1.1.2 Operations

Throughout this subsection, we fix a quiver $Q = (Q_0, A, s, t)$.

Definition 1.1.17 (Framing)**.** A framing of the quiver Q is a new quiver Q^{fr}; its set of vertices is given by $Q_0 \sqcup Q_0'$, where Q_0' is a copy of Q_0 with a fixed bijection $Q_0 \xrightarrow{\simeq} Q_0'$, $i \mapsto i'$.; the set of arrows is the disjoint union between A and an additional set of arrows given by: $a_i : i \to i'$, from the vertex i to the corresponding vertex i', for each vertex $i \in Q_0$. The maps $\bar{s}$ and $\bar{t}$ of the quiver Q^{fr} are defined in the natural way $\bar{t}(a) = t(a), \bar{s}(a) = s(a)$ when $a \in A$, $\bar{s}(a_i) = i$ and $\bar{t}(a_i) = i'$. $\triangle$

Example 1.1.18. For the quiver $Q = S_4$, one has that

S_4^{fr} :

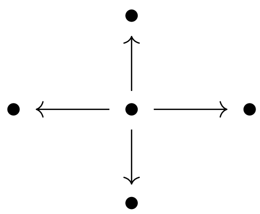


Definition 1.1.19 (Opposite quiver)**.** The quiver Q^{op} is obtained from Q by inverting the direction of the arrows from A. (The set of vertices remains Q_0.)

$\triangle$

Example 1.1.20. Again, when $Q = S_4$, one has

S_4^{op} :



Moreover, $L_r^{\text{op}} = L_r$ for all $r \in \mathbb{N}$, since all the arrows of L_r are loops.

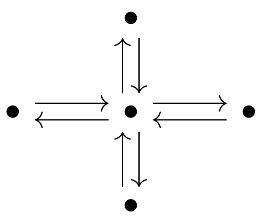
Definition 1.1.21 (Double)**.** The quiver Q^{double} is derived from Q through the operation

$$Q^{\text{double}} := Q \sqcup Q^{\text{op}}.$$

$\triangle$

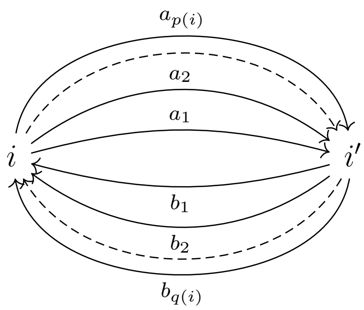
Example 1.1.22. For the quiver S_4, it follows from Examples 1.1.2 and 1.1.20 that

S_4^{double} :



Example 1.1.23. For the r-loop quiver, one has $L_r^{\text{double}} = L_{2r}$.

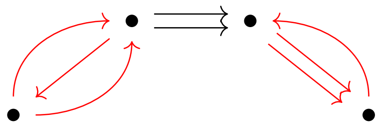
Definition 1.1.24 (Generalized framing - [5])**.** A generalized framing of Q is a quiver Q^{gfr} whose set of vertices is $Q_0 \sqcup Q_0'$, where Q_0' is a copy of Q_0 via a fixed bijection $i \mapsto i'$, and whose set of arrows is composed by A plus a set of new ones:



with $p(i) > 0$ e $q(i) \geq 0$, for all $i \in Q_0$. △

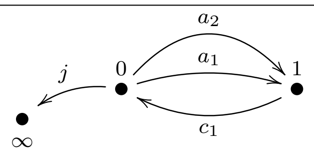
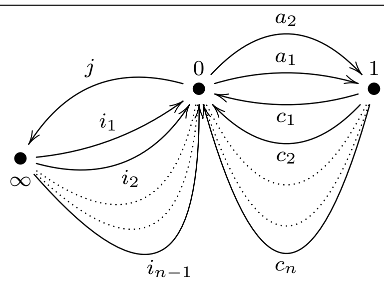
Example 1.1.25. If we consider the quiver $Q = K_2$, i.e., the quiver with two vertices and two arrows with the same source and target, then a possible generalized framing for Q is the following:

K_2^{gfr} :



Remark 1.1.26. In the definition of Q^{gfr}, when $p(i) = 1$ and $q(i) = 0$ for all $i \in Q_0$, we recover the standard definition of Q^{fr}.

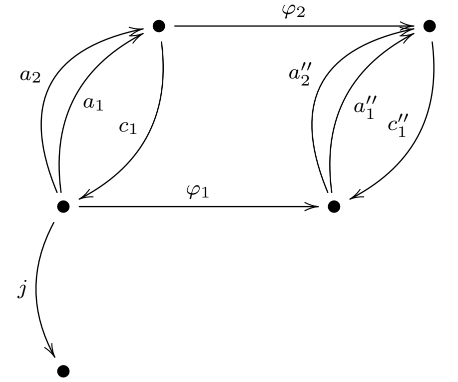
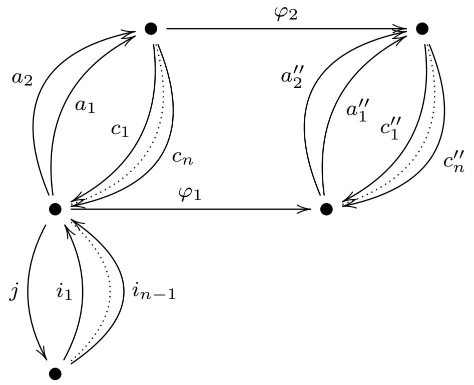
Example 1.1.27. For $n \geq 1$ we define the quivers Q_n as the quivers

Values of n	Quiver Q_n
$n = 1$	
$n \geq 2$	

with the relations

$$\begin{cases} a_1c_1a_2 = a_2c_1a_1 & \text{when } n = 1, \\ \\ a_1c_q = a_2c_{q+1} & \\ & \text{for} \quad q = 1, \ldots, n-1 \qquad \text{when } n \geq 2. \\ c_qa_1 - c_{q+1}a_2 = i_qj & \end{cases} \tag{1.3}$$

Example 1.1.28. For $n \geq 1$, we define the quivers Q_n^{enh} as the quivers

<table>
<tr><th>Values of n</th><th>Quiver Q_n^{enh}</th></tr>
<tr><td>$n = 1$</td><td>
</td></tr>
<tr><td>$n \geq 2$</td><td>
</td></tr>
</table>

with the relations

$$\begin{cases} a_1c_1a_2 = a_2c_1a_1 & \\ a''_1c''_1a''_2 = a''_2c''_1a''_1 & \\ \varphi_1c_1 = c''_1\varphi_2 & \text{when } n = 1, \\ \varphi_2a_1 = a''_1\varphi_1 & \\ \varphi_2a_2 = a''_2\varphi_1 & \\ \hline a_1c_1 = a_2c_2 & \\ c_1a_1 + i_1j = c_2a_2 & \\ a''_1c''_1 = a''_2c''_2 & \\ c''_1a''_1 = c''_2a''_2 & \\ \varphi_1i_1 = 0 & \text{when } n = 2, \\ \varphi_2a_1 = a''_1\varphi_1 & \\ \varphi_2a_2 = a''_2\varphi_1 & \\ \varphi_1c_1 = c''_1\varphi_2 & \\ \varphi_1c_2 = c''_2\varphi_2 & \\ \hline a_1c_q = a_2c_{q+1} \quad \text{for} \quad q = 1, \ldots, n-1 & \\ c_qa_1 + i_qj = c_{q+1}a_2 \quad \text{for} \quad q = 1, \ldots, n-1 & \\ a''_1c''_q = a''_2c''_{q+1} \quad \text{for} \quad q = 1, \ldots, n-1 & \\ c''_qa''_1 = c''_{q+1}a''_2 \quad \text{for} \quad q = 1, \ldots, n-1 & \text{when } n \geq 3. \\ \varphi_1i_q = 0 \quad \text{for} \quad q = 1, \ldots, n-1 & \\ \varphi_2a_q = a''_q\varphi_1 \quad \text{for} \quad q = 1, 2 & \\ \varphi_1c_q = c''_q\varphi_2 \quad \text{for} \quad q = 1, \ldots, n & \end{cases} \tag{1.4}$$

1.2 Hilbert schemes

Given a quasi-projective scheme X over an algebraically closed field k, we can define a contravariant functor from the category of Noetherian schemes over k to the category of sets

$$\mathcal{H}ilb_X : \mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{Sets}$$

that, for each scheme U, is given by:

$$\mathcal{H}ilb_X(U) = \left\{ Z \subseteq X \times U; \quad \begin{array}{c} Z \text{ is a closed subscheme,} \\ \begin{array}{ccc} Z & \xrightarrow{i} & X \times U \\ \pi \downarrow & & \downarrow p_2 \\ U & \xrightarrow[=]{} & U \end{array} \\ \pi = p_2 \circ i \text{ is flat} \end{array} \right\} \tag{1.5}$$

and acts on morphisms as follows. If $f : U \to V$ is a morphism of schemes, then:

$$\begin{array}{rccc} \mathcal{H}ilb_X(f): & \mathcal{H}ilb_X(V) & \to & \mathcal{H}ilb_X(U) \\ & Z & \mapsto & f^*Z \subset V \times Z \end{array}$$

In order for this definition to make sense, we have to ensure that

1. f^*Z is a closed subscheme of $X \times U$; and
2. the morphism $\pi_U = p_U \circ \widetilde{i} : f^*Z \to U$ is flat.

In fact, (1) follows from [39, Lemma 29.2.4] and (2) is a consequence of [22, Proposition 9.2(b)] or [39, Lemma 29.25.7].

In other words, such functor associates to a scheme U the set of families of closed subschemes of X parameterized by U.

When we consider the projection $\pi : Z \to U$, $\mathcal{O}_X(1)$ an ample line bundle over X and an element $u \in U$, the Hilbert polynomial at u is defined by the formula

$$P_u(m) = \chi(\mathcal{O}_{Z_u} \otimes O_X(m)),$$

where $Z_u = \pi^{-1}(u)$. Given a polynomial P, we define $\mathcal{H}ilb_X^P$ as the sub-functor of $\mathcal{H}ilb_X$ that associates to U the set of families of closed subschemes of X parameterized by U that have P as their Hilbert polynomial.

In 1960/61 Grothendieck proved that $\mathcal{H}ilb_X^P$ is representable, i.e., there exists a scheme Hilb_X^P such that

$$\mathcal{H}ilb_X^P(U) \simeq \mathrm{Hom}(U, \mathrm{Hilb}_X^P)$$

for any scheme U.

Suppose now that P is a constant polynomial. If $P(m) = n$, for all $m \in \mathbb{Z}$, we denote by $X^{[n]}$ the corresponding scheme Hilb_X^P.

Definition 1.2.1. The scheme $X^{[n]}$ is the so-called Hilbert scheme of n points on X. $\triangle$

When X is smooth and $\dim(X) = 1$, it is known that $X^{[n]}$ is isomorphic to the n-th symmetric product of X, explicitly:

$$X^{[n]} \simeq \frac{X^n}{S_n},$$

where S_n is the group of permutations of the set $\{1, \ldots, n\}$.

The case where $\dim(X) = 2$ was widely studied by authors such as (Briançon, 1977, [6]) and (Iarrobino, 1977, [27]). The particular case $X = \mathbb{C}^2$ was studied by Nakajima (1999) em [34], where he obtained a description in terms of linear maps, called ADHM data.

1.2.1 Nested Hilbert schemes

The nested Hilbert schemes are, in general terms, schemes that parameterize pairs of 0-cycles (Z, Z') in such way that $Z \subseteq Z'$, i.e., if X is a scheme, the scheme $\mathrm{Hilb}^{n,n'}(X) \subseteq \mathrm{Hilb}^n(X) \times \mathrm{Hilb}^{n'}(X)$ parameterizes the pairs of 0-cycles of length n and n', respectively, with $n \leq n'$, that satisfy

$$Z \subseteq Z'.$$

These schemes appear in works as [37, 42, 43] and [9].

We can also consider the description from Eq. 1.5 for the nested case. In general, we consider a quasi-projective scheme X over $\mathbb{C}$, $r \in \mathbb{N}$ and a r-tuple of integer polynomials

$$\mathbf{P}(m) = (P_1(m), \ldots, P_r(m)). \tag{1.6}$$

For every scheme U, we define $\mathcal{H}ilb_X^{\mathbf{P}(m)}(U)$ as

$$\left\{ (Z_1, \cdots, Z_r) \subseteq X \times U; \quad \begin{array}{c} Z_1 \subseteq \ldots \subseteq Z_r \subset X \times U \\ \text{are } U-\text{flat closed subeschemes} \\ \text{and } Z_i \text{ has } P_i(m) \text{ as its Hilbert polynomial} \end{array} \right\}.$$

Therefore, we get a contravariant functor $\mathcal{H}ilb_X^{\mathbf{P}(m)} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Set}$, called the nested Hilbert functor.

Theorem 1.2.2. *For any* $r \geq 1$ *and* $\mathbf{P}(m)$ *as in* (1.6)*, the nested Hilbert functor* $\mathcal{H}ilb_X^{\mathbf{P}(m)}$ *is represented by a projective scheme* $\mathrm{Hilb}^{\mathbf{P}(m)}(X)$*, called the nested Hilbert scheme of* X *relative to* $\mathbf{P}(m)$*.*

Proof. See [38, Theorem 4.5.1] for the representability of $\mathcal{H}ilb_{\mathbb{P}^t}^{\mathbf{P}(m)}$ and comments after its proof for the general case. □

Remark 1.2.3. Sometimes the nested Hilbert functor can be found in literature under the name 'flag Hilbert functor'; in this case, the scheme that represents it is called the flag Hilbert scheme.

If we let the polynomial $\mathbf{P}(m)$ be a r-tuple of natural numbers $(k_1, \ldots, k_r)$ it is usual to write $\mathcal{H}ilb_X^{k_1,\ldots,k_r}$ instead of $\mathcal{H}ilb_X^{\mathbf{P}(m)}$, so we shall write $\mathrm{Hilb}^{k_1,\ldots,k_r}(X)$ for the scheme that represents that functor.

Remark 1.2.4. Notice that if we do not make the assumption of

$$k_1, \leq k_2 \leq \ldots \leq k_r,$$

we have the possibility of the nested Hilbert scheme/functor to be empty.

1.3 Moduli spaces

The notion of what means for points in a space to represent geometric objects is formalized by the idea of moduli spaces. Here we explore the idea of moduli spaces of quiver representations based on Geometric Invariant Theory, cf. [33].

1.3.1 Stability conditions, categorial and GIT quotients.

When we have a quiver $Q = (Q_0, A, s, t)$, a stability parameter for Q may be regarded as an element $\theta \in \mathbb{R}^{\#Q_0}$. Some choices for the stability parameter give rise to interesting moduli spaces, for instance the stability condition that appears in the work [34] of Nakajima is equivalent to a quite simple choice of a stability parameter. The goal of this subsection is to develop the theory concerning the stability conditions, based on works as [28] (the first about this topic), [4], [34], [17], etc.

Imagine that you want to consider a space of isomorphism classes of representations of a quiver Q with a fixed dimension vector $\mathbf{v} = (v_i)_{i \in Q_0}$. Geometrically, this space should be the orbit space

$$\frac{\mathrm{Rep}(Q, \mathbf{v})}{\mathrm{GL}(\mathbf{v})},$$

where $\mathrm{GL}(\mathbf{v}) = \bigoplus_{i \in Q_0} \mathrm{GL}(v_i)$. However, such an orbit space, in most cases, is not well behaved, for instance, it may not have reasonable separation properties, see [17, Section 2.1]. In order to avoid this issue, as a first option, we may use the space

$$\mathrm{Rep}(Q, \mathbf{v}) /\!/ \mathrm{GL}(\mathbf{v}) := \mathrm{Spec}\left(\mathbb{C}[\mathrm{Rep}(Q, \mathbf{v})]^{\mathrm{GL}(\mathbf{v})}\right), \tag{1.7}$$

i.e., the spectrum of the algebra of $\mathrm{GL}(\mathbf{v})$-invariant polynomials on the vector space $\mathrm{Rep}(Q, \mathbf{v})$. This is an affine algebraic variety by definition, and a categorical quotient in the sense of the definition below.

Definition 1.3.1 (Categorical quotients)**.** Given a category $\mathcal{C}$, a categorical quotient of an object $X \in \mathrm{Obj}(\mathcal{C})$ with a G-action $* : G \times X \to X$ is a morphism $\pi : X \to Y$ that satisfies the following conditions:

(1) π is invariant, i.e., $\pi \circ * = \pi \circ p_2$, where $p_2 : G \times X \to X$ is the natural projection; and

(2) for any morphism $\pi' : X \to Z$ satisfying the condition (1) there is a unique morphism $h : Y \to Z$ such that $\pi' = h \circ \pi$.

Another notion, sometimes used as a substitute of the usual quotient as an orbit space, the geometric quotient, is defined as:

Definition 1.3.2 (Geometric quotient)**.** A geometric quotient of an algebraic variety X with the action of an algebraic group G is a morphism of varieties $\pi : X \to Y$ such that

1. $\pi^{-1}(y)$ is an orbit of G for all $y \in Y$;
2. The topology of Y is the quotient topology, i.e., $U \subseteq Y$ is open if, and only if, $\pi^{-1}(U)$ is open;
3. For any open subset $U \subseteq Y$, the map $\pi^{\#} : \mathbb{C}[U] \to \mathbb{C}[\pi^{-1}(U)]^G$ is an isomorphism.

△

Example 1.3.3. The canonical map $\pi : \mathbb{C}^{r+1} \setminus \{0\} \to \mathbb{P}^r$ is a geometric quotient.

Example 1.3.4. If G is an algebraic group, for any closed subgroup $H \subseteq G$, the canonical morphism $\pi : G \to G/H$ is a geometric quotient.

Remark 1.3.5. Every geometric quotient is a categorical quotient, cf. [33, Prop. 0.1, §2].

When Q is a quiver with no oriented cycles, one has

$$\mathbb{C}[\mathrm{Rep}(Q,\mathbf{v})]^{\mathrm{GL}(\mathbf{v})} = \mathbb{C}.$$

Therefore, $\mathrm{Rep}(Q,\mathbf{v})/\!/\mathrm{GL}(\mathbf{v})$ is just a point, cf. [17, Corollary 2.1.2]. This shows that the categorical quotient may cause the loss of a lot of geometric information. With a view to avoiding this kind of issue when studying moduli problems, one usually introduces the notion of a stability parameter and replaces the orbit space $\mathrm{Rep}(Q,\mathbf{v})/\mathrm{GL}(\mathbf{v})$ or the categorical quotient $\mathrm{Rep}(Q,\mathbf{v})/\!/\mathrm{GL}(\mathbf{v})$ by a moduli space of (semi)stable representations. This approach has been widely used and explored in literature. In general, though, moduli spaces built in this way depend on the stability parameter.

The theory behind this construction, i.e., quotients by a reductive group action via stability conditions, is called Geometric Invariant Theory and was first explored by Mumford in [33]. Here, we are going to exhibit some constructions, definitions and results, to make this work as self-contained as possible.

First, we consider an affine algebraic G-variety[1] X, not necessarily irreducible, where G is a reductive algebraic group. The G-action on X will be denoted by $*$. Given a rational character, i.e., an algebraic group homomorphism $\chi : G \to \mathbb{C}^*$, Mumford defines a scheme $X/\!/_{\chi}G$ as follows. Let G act on the product $X \times \mathbb{C}$ through the formula:

$$g \cdot (x, z) := (g * x, \chi(g)^{-1} \cdot z).$$

The coordinate ring of $X \times \mathbb{C}$ is the algebra $\mathbb{C}[X \times \mathbb{C}] \simeq \mathbb{C}[X][z]$ of polynomials in the variable z with coefficients in the coordinate ring of X. One can see that this algebra has a natural grading, given by the degree of a polynomial. Let $A_{\chi} := \mathbb{C}[X \times \mathbb{C}]^G$ be the sub-algebra of G-invariants. This algebra is a finitely generated graded algebra, and a polynomial $f(z) = \sum_{m=0}^{M} f_m \cdot z^m \in \mathbb{C}[X][z]$ is G-invariant if and only if f_m is χ^m-semi-invariant for all $m \in \{0, \ldots, M\}$, i.e.,

$$f_m(g^{-1} * x) = \chi(g)^m \cdot f_m(x),\ \forall g \in G, x \in X \text{ and } m \in \{0, \ldots, M\}.$$

[1] An algebraic group is a variety equiped with a structure of group such that the multiplication and inversion maps are morphisms of varieties. A G-variety is a variety X equiped with an action of the algebraic group G $\begin{array}{rccc} *: & G \times X & \to & X \\ & (g,x) & \mapsto & g * x \end{array}$ which is also a morphism of varieties.

Let $\mathbb{C}[X]^{\chi^m}$ be the vector space of χ^m-semi-invariant functions. Clearly one has:

$$A_\chi = \mathbb{C}[X \times \mathbb{C}]^G = \bigoplus_{m \geq 0} \mathbb{C}[X]^{\chi^m},$$

which provides a grading for the algebra A_χ. Let

$$X/\!/_\chi G := \mathrm{Proj}(A_\chi)$$

be the projective spectrum of the graded algebra A_χ. The scheme $X/\!/_\chi G$ is quasi-projective and it is called the GIT quotient of X by the given G-action. In addition, if X is reduced/irreducible, so is $X/\!/_\chi G$.

Set $A_\chi^{>0} := \bigoplus_{m>0} \mathbb{C}[X]^{\chi^m}$ and let $\mathcal{I}$ be the set of proper homogeneous ideals $I \subset A_\chi$ and such that $A_\chi^{>0} \nsubseteq I$. An element $I \in \mathcal{I}$ is called a maximal homogeneous ideal if it is not contained in another element $I' \in \mathcal{I}$. The closed points of the scheme $X/\!/_\chi G$ correspond to maximal homogeneous ideals.

Given a nonzero homogeneous semi-invariant $f \in A_\chi$, we define

$$X_f := \{x \in X \big| f(x) \neq 0\}.$$

To obtain a sound understanding of the GIT quotient $X/\!/_\chi G$ we recall the following definition from [33].

Definition 1.3.6.

(1) A point $x \in X$ is said to be χ-semistable if there exist $m \in \mathbb{N}$ and a χ^m-semi-invariant $f \in \mathbb{C}[X]^{\chi^m}$ such that $x \in X_f$;

(2) A point $x \in X$ is said to be χ-stable if there exist $m \in \mathbb{N}$ and a χ^n-semi-invariant $f \in \mathbb{C}[X]^{\chi^m}$ such that $x \in X_f$ and, moreover, one has: (i) the action $G \times X_f \to X_f$ is a closed morphism and (ii) the isotropy group[2] G_x of the point x is finite;

(3) Two χ-semistable points x and x' are called equivalent if, and only if,

$$\overline{G * x} \cap \overline{G * x'} \neq \emptyset.$$

△

Remark 1.3.7. Usually one writes X_χ^{ss} and X_χ^s for the sets of semistable and stable points, respectively. Of course, one has:

$$X_\chi^s \subseteq X_\chi^{ss} \subseteq X.$$

[2]The isotropy group of an element x is the subgroup of G whose elements fix the point x, i.e., $G_x = \{g \in G|\ g * x = x\}$.

Example 1.3.8. When we consider the trivial character $\chi = 1$, we have

$$A_\chi = \mathbb{C}[X]^G[z].$$

The regular function $z \in A_\chi$ is a homogenenous regular function of degree one, that does not vanish on X. Thus, we have that $X_z = X$ and any point $x \in X$ is χ-stable. Therefore, for this choice of χ:

$$X/\!/_\chi G = \mathrm{Proj}(A_\chi) = \mathrm{Proj}\mathbb{C}[X]^G[z] = \mathrm{Spec}(\mathbb{C}[X]^G) = X/\!/G.$$

In 1994, A. King [28] established a completely algebraic notion of stability for representations of algebras. Furthermore, when working with quiver representations, he also showed that his notion of stability is actually equivalent to the one proposed by Mumford in 1.3.6; here we shall exhibit some of the constructions of King's work.

We begin with a quiver $Q = (Q_0, A, s, t)$ and fix an element $\theta \in \mathbb{R}^{\#Q_0}$. We shall see that this choice of θ, the so-called stability parameter, is equivalent to a choice of a character $\chi : G \to \mathbb{C}^*$. Let $0 \neq V = \bigoplus_{i \in Q_0} V_i$ be a finite dimensional representation of Q of dimension vector $\mathbf{v} = \dim_{Q_0}(V) \in \mathbb{N}_0^{\#Q_0}$. The slope of V is defined by the formula

$$\mathrm{slope}_\theta(V) := \frac{\theta \cdot \dim_{Q_0}(V)}{\dim_{\mathbb{C}}(V)}, \tag{1.8}$$

where $\dim_{\mathbb{C}}(V) = \sum_{i \in Q_0} \dim(V_i)$. Alternatively, one can write $\dim_{\mathbb{C}}(V) = \theta^+ \cdot \dim_{Q_0}(V)$, with $\theta^+ := (1, \ldots, 1) \in \mathbb{Z}^{\#Q_0}$.

Definition 1.3.9. A nonzero representation $V \in \mathrm{Rep}(Q, \mathbf{v})$ is said to be θ-semistable if for any subrepresentation $V' \subseteq V$ we have

$$\mathrm{slope}_\theta(V') \leq \mathrm{slope}_\theta(V).$$

A nonzero representation is called θ-stable if the strict inequality holds for any nonzero proper subrepresentation $V' \subset V$. △

Example 1.3.10. Fix $\theta = 0$. Then any representation V of Q is θ-semistable, and X is θ-stable if, and only if, it is simple as an $\mathbb{C}Q$-module, i.e., it does not admit a nonzero proper sub-module.

Remark 1.3.11. Consider $\theta \in \mathbb{R}^{\#Q_0}$ and define $\theta' = \theta - c \cdot \theta^+$, where $c \in \mathbb{R}$. One can check that a representation of Q is θ-semistable if, and only if, it is θ'-semistable. Furthermore, given a representation V, it is always possible to find $c \in \mathbb{R}$ such that $\theta' \cdot \dim_{Q_0}(V) = 0$. See [36, Proposition 3.4].

Definition 1.3.9 is a very special case of a more general notion, proposed by King in [28], which applies to associative $\mathbb{C}$-algebras. Consider such an algebra L and let $K(L)$ be the Grothendieck group[3] of finite dimensional L-modules. One can see that the map $V \mapsto \dim_{\mathbb{C}}(V)$ can be extended to a group homomorphism

$$K(L) \longrightarrow \mathbb{R}.$$

When we consider any other additive group homomorphism $\phi : K(L) \to \mathbb{R}$ and a nonzero finite dimensional A-module V, one defines:

$$\mathrm{slope}_\phi(V) := \frac{\phi([V])}{\dim_{\mathbb{C}}(V)},$$

where $[V]$ means the class of V in $K(L)$. The definition proposed by King reads as follows.

Definition 1.3.12. A finite dimensional L-module V is ϕ-semistable if for any nonzero L-submodule V' of V one has

$$\mathrm{slope}_\phi(V') \leq \mathrm{slope}_\phi(V). \qquad \triangle$$

To see that this definition indeed generalizes Definition 1.3.9, let us consider $L := \mathbb{C}Q$. In this case, the map $[V] \mapsto \dim_{Q_0}(V)$ is a group homomorphism

$$\dim_{Q_0} : K(\mathbb{C}Q) \to \mathbb{Z}^{\#Q_0}.$$

Moreover, for any $\theta \in \mathbb{R}^{\#Q_0}$ one can define another group homomorphism

$$\begin{array}{rccc} \phi_\theta : & \mathbb{Z}^{\#Q_0} & \longrightarrow & \mathbb{R} \\ & x = (x_i)_{i \in Q_0} & \longmapsto & \sum_{i \in Q_0} \theta_i x_i \,, \end{array}$$

and this allows one to identify $\mathbb{R}^{\#Q_0}$ with $\mathrm{Hom}(\mathbb{Z}^{\#Q_0}, \mathbb{R})$.

Summing up, given $\theta \in \mathbb{R}^{\#Q_0}$, the composition $\phi = \phi_\theta \circ \dim_{Q_0}$ is a group homomorphism

$$\phi : K(\mathbb{C}Q) \to \mathbb{R};$$

with this choice, Definition 1.3.12 of ϕ-semistability for $\mathbb{C}Q$-modules turns out to be the same as the one established in Definition 1.3.9.

[3]The group $K(L)$ is the abelian group generated by the set $\{[X],\ X \in L - \mathrm{mod}\}$ of isomorphism classes of finite dimensional L-modules with the relations: $[M_1] - [M_2] + [M_3] = 0$ for all short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of finite dimensional L-modules.

Proposition 1.3.13. *Given a group homomorphism $\phi : K(L) \to \mathbb{R}$, the finite dimensional ϕ-semistable L-modules form an abelian category. An L-module V is ϕ-stable if, and only if, V is a simple object of this category.*

Proof. See [17, Proposition 2.3.5]. □

As a consequence of Proposition 1.3.13, any θ-semistable representation V of a quiver Q has a Jordan-Hölder filtration

$$0 = V_0 \subset V_1 \ldots \subset V_m = V,$$

i.e., $V_1, \ldots, V_{m-1}$ are subrepresentations such that $\frac{V_k}{V_{k-1}}$ is a θ-stable representation (i.e., a simple $\mathbb{C}Q$-module) for any $k \in \{1, \ldots, m\}$. Finally, we get a graded representation

$$\mathrm{gr}^s(V) := \bigoplus_{k=0}^{m} \frac{V_k}{V_{k-1}},$$

that does not depend (up to isomorphism) on the choice of such a filtration.

Given a vector $\theta = (\theta_i)_{i \in Q_0} \in \mathbb{Z}^{\# Q_0}$, we can build a character

$$\begin{array}{cccc} \chi_\theta : & \mathrm{GL}(\mathbf{v}) & \longrightarrow & \mathbb{C}^* \\ & g = (g_i)_{i \in Q_0} & \longmapsto & \prod_{i \in Q_0} \det(g_i)^{-\theta_i}. \end{array} \tag{1.9}$$

Remark 1.3.14. If we fix a representation V of the quiver Q of dimension vector $\mathbf{v} = \dim_{Q_0}(V)$, then the character χ_θ is zero on the subgroup $\mathbb{C}^* \subseteq \mathrm{GL}(\mathbf{v})$ if, and only if, $\theta \cdot \mathbf{v} = 0$.

The relation between King's and Mumford's notions of stability comes from the following theorem, whose proof we omit, but one can find in [17, Theorem 2.3.7].

Theorem 1.3.15. *For any dimension vector $\mathbf{v} \in \mathbb{N}_0^{\# Q_0}$ and any $\theta \in \mathbb{Z}^{\# Q_0}$ such that $\theta \cdot \mathbf{v} = 0$, one has:*

1. *A representation $X \in \mathrm{Rep}(Q, \mathbf{v})$ is χ_θ-semistable (χ_θ-stable), according to Definition 1.3.9 if, and only if, it is θ-semistable (θ-stable), according to Definition 1.3.6.*

2. *A pair (V, V') of χ_θ-semistable representations are equivalent in the sense of Definition 1.3.6 if, and only if, $\mathrm{gr}^s(V) \simeq \mathrm{gr}^s(V')$.*

We denote by $\mathrm{Rep}_\theta^{ss}(Q, \mathbf{v})$ and $\mathrm{Rep}_\theta^{s}(Q, \mathbf{v})$ the set of semistable and stable (respec.) representations of the quiver Q with dimension vector $\mathbf{v}$. The quasi-projective variety given by the GIT quotient $\mathrm{Rep}_\theta^{ss}(Q, \mathbf{v}) /\!/_{\chi_\theta} \mathrm{GL}(\mathbf{v})$ will be denoted by $\mathcal{R}_\theta(Q, \mathbf{v})$.

Remark 1.3.16. A representation V of Q is θ-semistable if, anf only if, V^*, the dual representation of Q^{op}, is $(-\theta)$-semistable. So, we have the following canonical isomorphisms:

$$\mathrm{Rep}_\theta^{ss}(Q,\mathbf{v}) \simeq \mathrm{Rep}_{-\theta}^{ss}(Q^{\mathrm{op}},\mathbf{v}) \quad \text{and} \quad \mathcal{R}_\theta(Q,\mathbf{v}) \simeq \mathcal{R}_{-\theta}(Q^{\mathrm{op}},\mathbf{v}).$$

1.3.2 Stability for framed representations

According to Definitions 1.1.17 and 1.1.24, to provide a representation of Q^{fr}, where $Q = (Q_0, A, s, t)$ is a fixed quiver, is equivalent to providing a representation of the quiver Q, say $V = \bigoplus_{i\in Q_0} V_i$, together with a collection of linear maps $j_i : V_i \to W_i$, for $i \in Q_0$, where W_i is the vector space put at the vertex $i' \in Q_0'$; we set also $W = \bigoplus_{i\in Q_0} W_i$, and we let $\mathbf{w} := \dim_{Q_0}(W) \in \mathbb{Z}^{\#Q_0}$ denote the dimension vector of W. In other words, a representation of Q^{fr} may be regarded as a pair $(V,\,(j_i)_{i\in Q_0})$, where V is a representation of Q and the collection $(j_i)_{i\in Q_0}$ is an additional collection of linear maps $j_i : V_i \to W_i$ for $i \in Q_0$, as before.

The dimension vector of a representation of Q^{fr} can be seen as an element $\mathbf{v}\times\mathbf{w} \in \mathbb{Z}^{\#Q_0} \times \mathbb{Z}^{\#Q_0'} = \mathbb{Z}^{\#(Q_0 \sqcup Q_0')}$ and we write

$$\mathrm{Rep}(Q^{\mathrm{fr}},\mathbf{v},\mathbf{w}) := \mathrm{Rep}(Q^{\mathrm{fr}},\mathbf{v}\times\mathbf{w})$$

to denote the space of representations $(V,\,(j_i)_{i\in Q_0})$ of the quiver Q^{fr} with dimension vector $\mathbf{v} := \dim_{Q_0}(V)$ and $\mathbf{w} := \dim_{Q_0}(W)$.

Consider an element $g \in \mathrm{GL}(\mathbf{v})$ and write $g = (g_i)_{i\in Q_0}$. One defines an action

$$* : \mathrm{GL}(\mathbf{v}) \times \mathrm{Rep}(Q^{\mathrm{fr}},\mathbf{v},\mathbf{w}) \longrightarrow \mathrm{Rep}(Q^{\mathrm{fr}},\mathbf{v},\mathbf{w})$$

by letting

$$g * ((f_a)_{a\in A}, (j_i)_{i\in Q_0}) \longmapsto \Big((g_{t(a)} \circ f_a \circ g_{s(a)}^{-1})_{a\in A}, (j_i \circ g_i^{-1})_{i\in Q_0} \Big).$$

So, $\mathrm{Rep}(Q^{\mathrm{fr}},\mathbf{v},\mathbf{w})$ is a $\mathrm{GL}(\mathbf{v})$-variety and since $\mathrm{GL}(\mathbf{v})$ can be seen as a subgroup of

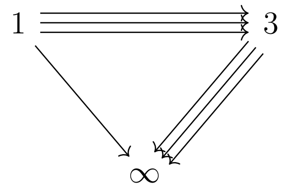
$$\mathrm{GL}(\mathbf{v}) \times \mathrm{GL}(\mathbf{w}) = \prod_{i\in Q_0} \mathrm{GL}(V_i) \times \prod_{i\in Q_0} \mathrm{GL}(W_i),$$

the action of $\mathrm{GL}(\mathbf{v})$ turns out to be the restriction of the change-of-basis action of $\mathrm{GL}(\mathbf{v}) \times \mathrm{GL}(\mathbf{w})$ on $\mathrm{Rep}(Q^{\mathrm{fr}},\mathbf{v},\mathbf{w})$.

Now, we are going to explore an idea due to Crawley-Boevey [15] that naturally appears when working with the notions of stability for framed and generalized framed representations. Suppose that we have a quiver Q and a dimension vector $\mathbf{w} = (w_i)_{i\in Q_0} \in \mathbb{N}_0^{\#Q_0}$. Crawley-Boevey introduces a quiver $Q^{\mathbf{w}}$ whose set of vertices is $Q_0 \sqcup \{\infty\}$, where ∞ is a new additional vertex, and the set of edges of $Q^{\mathbf{w}}$ is obtained by adding to the set of arrows of Q precisely w_i new arrows from the vertex i to the vertex ∞.

Example 1.3.17. When we consider the quiver K_3 with the dimension vector $\mathbf{w} = (1, 3)$, one has:

$K_3^{\mathbf{w}} = K_3^{(1,3)}$:



After that, taking any dimension vector $\mathbf{v} = (v_i)_{i\in Q_0} \in \mathbb{N}_0^{\#Q_0}$, by putting $\widetilde{v_i} = v_i$ for $i \in Q_0$ and $v_\infty = 1$ we obtain a dimension vector $\widetilde{\mathbf{v}} \in \mathbb{N}_0^{\#Q_0+1}$ for the quiver Q^{w}. Clearly we have a embedding

$$\begin{array}{rcl} \mathrm{GL}(\mathbf{v}) & \hookrightarrow & \mathrm{GL}(\widetilde{\mathbf{v}}) \\ g = (g_i)_{i\in Q_0} & \longmapsto & \widetilde{g} = ((g_i)_{i\in Q_0}, \mathrm{Id}_{\mathbb{C}}) \end{array}$$

Consequently, $\mathrm{GL}(\mathbf{v}) \xrightarrow{\simeq} \dfrac{\mathrm{GL}(\widetilde{\mathbf{v}})}{\mathbb{C}^*}$, and then $\mathrm{Rep}(Q^{\mathbf{w}}, \widetilde{\mathbf{v}})$ is a $\mathrm{GL}(\mathbf{v})$-variety. If we let $\widetilde{V} := \bigoplus_{i\in Q_0\sqcup\{\infty\}} V_i$ and $W = \bigoplus_{i\in Q_0} W_i$ be a pair of vector spaces such that

$$\dim_{Q_0\sqcup\{\infty\}}(\widetilde{V}) = \widetilde{\mathbf{v}} \quad \text{and} \quad \dim_{Q_0}(\widetilde{W}) = \widetilde{\mathbf{w}}.$$

We first identify V_∞ with $\mathbb{C}$, a one-dimensional vector space with a fixed basis; then, for each $i \in Q_0$, we fix a basis of the vector space W_i. In this way, given any collection of w_i linear maps from V_i to V_∞, one can use the basis of W_i to fabricate a single linear map $j_i : V_i \to W_i$.

Hence, we see that any element of $\mathrm{Rep}(Q^{\mathbf{w}}, \widetilde{\mathbf{v}})$ supported by the vector space $\widetilde{V}$ generates an element of $\mathrm{Rep}(Q^{\mathrm{fr}}, \mathbf{v}, \mathbf{w})$ and this assignment produces a map

$$\mathrm{Rep}(Q^{\mathbf{w}}, \widetilde{\mathbf{v}}) \longrightarrow \mathrm{Rep}(Q^{\mathrm{fr}}, \mathbf{v}, \mathbf{w}). \tag{1.10}$$

This map turns out to be a $\mathrm{GL}(\mathbf{v})$-equivariant vector space isomorphism that depends on the choice of basis for W. More details can be found in [5] and [17].

In order to apply Definition 1.3.6 to the case of the $\mathrm{GL}(\mathbf{v})$-action on $\mathrm{Rep}(Q^{\mathrm{fr}}, \mathbf{v}, \mathbf{w})$ and a character $\chi_\theta : \mathrm{GL}(\mathbf{v}) \to \mathbb{C}^*$, we will use precisely the isomorphism in eq.(1.10) to build a relation between the GIT stability of elements of $\mathrm{Rep}(Q^{\mathrm{fr}}, \mathbf{v}, \mathbf{w})$ and the stability proposed by King , cf. Def. 1.3.9, of elements in $\mathrm{Rep}(Q^{\mathbf{w}}, \widetilde{\mathbf{v}})$.

Consider a parameter $\theta = (\theta_i)_{i \in Q_0} \in \mathbb{R}^{\# Q_0}$ and define

$$\widetilde{\theta} := \left((\theta_i)_{i \in Q_0}, \theta_\infty := - \sum_{i \in Q_0} \theta_i \cdot v_i \right).$$

The isomorphism in eq.(1.10) allows us to understand that the $\widetilde{\theta}$-stability in the sense of King for the $\mathrm{GL}(\widetilde{\mathbf{v}})$-action on $\mathrm{Rep}(Q^{\mathbf{w}}, \widetilde{\mathbf{v}})$ may be reinterpreted as the GIT notion of stability of the $\mathrm{GL}(\mathbf{v})$-action on $\mathrm{Rep}(Q^{\mathrm{fr}}, \mathbf{v}, \mathbf{w})$.

Remark 1.3.18. The definition of $\widetilde{\theta}$ implies that $\widetilde{\theta} \cdot \widetilde{\mathbf{v}} = 0$, as required by Theorem 1.3.15.

As an important result of [17], we have the following characterization of the θ^+-semistable representations of the quiver Q^{fr}.

Proposition 1.3.19. *An element of* $\mathrm{Rep}(Q^{\mathrm{fr}}, \mathbf{v}, \mathbf{w})$ *is* θ^+*-semistable with respect to the* $\mathrm{GL}(\mathbf{v})$*-action if, and only if, there is a nontrivial subrepresentation of the quiver* Q *such that every vector subspace is contained in the kernel of the correspoding map* j_i*. In addition, the* $\mathrm{GL}(\mathbf{v})$*-action is free on* $\mathrm{Rep}^{ss}(Q^{\mathrm{fr}}, \mathbf{v}, \mathbf{w})$ *and any* θ^+*-semistable representation is actually* θ^+*-stable.*

Proof. See [17, Lemma 3.2.3] □

When we consider the case of a generalized framing, the construction of the space of representations is similar to the framed one. In fact, a representation of the quiver Q^{gfr} amounts to a representation of the quiver Q and, for any $i \in Q_0$, we have two collections of linear maps $j_1, \ldots, j_{p(i)} : V_i \to W_i$ and $h_1, \ldots, h_{q(i)} : W_i \to V_i$. By considering a dimension vector $\mathbf{v}$ corresponding to the vertices of Q and a dimension vector $\mathbf{w}$ corresponding to the new ones added to Q_0, for an element $X \in \mathrm{Rep}(Q^{\mathrm{gfr}}, \mathbf{v} \times \mathbf{w})$ one can write:

$$X = \left(V = \bigoplus_{i \in Q_0} V_i, (f_a)_{a \in Q_1}, W = \bigoplus_{i \in Q_0} W_i, \{j_1, \ldots, j_{p(i)}\}_{i \in Q_0}, \{h_1, \ldots, h_{q(i)}\}_{i \in Q_0} \right).$$

The following result from [5, Lemma 3.3] authorizes us to define stability for gfr-quivers in terms of the notion proposed in 1.3.9 and the modifications due to Crawley-Boevey, described before Remark 1.3.18.

Lemma 1.3.20. *For all dimension vectors* $(\mathbf{v}, \mathbf{w}) \in \mathbb{N}_0^{\#(Q_0 \sqcup Q_0')}$ *for the quiver* Q^{gfr}, *there exists a quiver* $Q^{\mathbf{w}}$ *with vertex set equal to* $Q_0 \sqcup \{\infty\}$ *such that there is a* $\mathrm{GL}(\mathbf{v})$*-equivariant isomorphism*

$$\mathrm{Rep}(Q^{\mathrm{gfr}}, \mathbf{v}, \mathrm{w}) \simeq \mathrm{Rep}(Q^{\mathbf{w}}, \mathbf{v}, 1). \tag{1.11}$$

Remark 1.3.21. The quiver $Q^{\mathbf{w}}$ is very similar to the one defined in [15], described before Ex. 1.3.17. In this case, the set of vertices of $Q^{\mathbf{w}}$ is $Q_0 \sqcup \{\infty\}$ and we build its set of arrows adding to A, for any $i \in Q_0$, $w_i p(i)$ arrows from i to ∞ and $w_i q(i)$ arrows from ∞ to i. It is easy to see that $\mathrm{GL}(\mathbf{v})$ acts on $\mathrm{Rep}(Q^{\mathbf{w}}, \mathbf{v}, 1)$ by change of basis and that any stability parameter $\theta \in \mathbb{R}^{\#Q_0}$ for $(\mathbf{v}, \mathbf{w})$-dimensional representations of the quiver Q^{gfr} gives rise to a stability parameter $\widetilde{\theta}$ for the quiver $Q^{\mathbf{w}}$ just by adding the entry $\theta_\infty = -\theta \cdot \mathbf{v}$ (again, as in the usual framed case).

With the previous lemma in mind, we define stability for representations of the quiver Q^{gfr} based on the stability of the corresponding quiver $Q^{\mathbf{w}}$.

Definition 1.3.22. A $(\mathbf{v}, \mathbf{w})$-dimensional representation of Q^{gfr} is θ-semistable (resp. θ-stable) if, and only if, its image through the isomorphism (1.11) is $\widetilde{\theta}$-semistable (resp. $\widetilde{\theta}$-stable).

△

When we consider the quiver Q_n from Example 1.1.27, we can characterize the $(2c, 1-2c)$-semistability in a more operational way; this will appear later in this work.

1.3.3 Working with families

Let Q be a quiver; Q_0 will denote the set of vertices, and A the set of arrows. Remember that a stability parameter θ for Q may be regarded as an element in $\mathbb{R}^{\#Q_0}$. In this subsection, we introduce the notion of families of representations of quiver and some related topics, useful in the Chapter 4.

Definition 1.3.23. 1. A family of representations of Q parameterized by a scheme T is, for every $v \in Q_0$, a locally free sheaf $\mathcal{W}_v$ on T, and for any arrow $a \in A$, a sheaf morphism $\phi_a \colon \mathcal{W}_{s(a)} \to \mathcal{W}_{t(a)}$. Note that for every closed point $t \in T$ by taking fibers this induces a representation of Q in the usual sense.
2. A morphism between two families of representations $(T, \mathcal{W}_v, \phi_a)$ and $(S, \mathcal{U}_v, \psi_a)$ is a scheme morphism $f \colon T \to S$ and a collection of sheaf morphisms

$$\{F_v \colon \mathcal{W}_v \to f^*\mathcal{U}_v,\ v \in Q_0\}$$

such that for every arrow $a \in A$ the diagram

$$\begin{array}{ccc} \mathcal{W}_{s(a)} & \xrightarrow{F_{s(a)}} & f^*\mathcal{U}_{s(a)} \\ \phi_a \downarrow & & \downarrow f^*\psi_a \\ \mathcal{W}_{t(a)} & \xrightarrow{F_{t(a)}} & f^*\mathcal{U}_{t(a)} \end{array}$$

commutes. A morphism is an isomorphism when f and all morphisms F_v are isomorphisms.

3. A family of representations is θ-stable if for every closed point $t \in T$ the representation corresponding to t is θ-stable. △

Remark 1.3.24. Note that if $(S, \mathcal{U}_v, \psi_a)$ is a family of representations parameterized by S, and $f\colon T \to S$ is a scheme morphism, then $(T, f^*\mathcal{U}_v, f^*\psi_a)$ is a family of representations parameterized by T.

Definition 1.3.25 (The representation moduli functor)**.** The functor of families of representations of Q is the functor

$$\begin{aligned} \mathfrak{R}^Q\colon \quad \mathrm{Sch}^{\mathrm{op}} &\to \mathrm{Set} \\ T &\mapsto \left\{ \begin{array}{l} \text{isomorphism classes of families of} \\ \text{representations of } Q \text{ parameter-} \\ \text{ized by } T \end{array} \right\} \end{aligned} \tag{1.12}$$

The action of this functor on morphisms is by pullbacks: if $f \in \mathrm{Hom}(T, S)$, then

$$\mathcal{W}_{s(a)} \xrightarrow{\phi_a} W_{t(a)} \quad \text{is sent to} \quad f^*\mathcal{W}_{s(a)} \xrightarrow{f^*\phi_a} f^*W_{t(a)}.$$ △

After fixing a dimension vector $\mathbf{v}$ and a stability parameter θ, one can also introduce the subfunctor $\mathfrak{R}^{Q,s}_{\mathbf{v},\theta}$ of θ-stable $\mathbf{v}$-dimensional representations. If $\mathbf{v}$ is primitive[4], this functor is representable by a fine moduli space $\mathcal{M}^s_{\mathbf{v},\theta}$ [28, Proposition. 5.3].

1.4 Framed flags of sheaves

We introduce now the notion of framed flag of sheaves (of length 1). Let X be an irreducible projective smooth surface, and D a divisor in it (for the moment we only establish some notation, and at this level of generality we do not need to make any additional assumptions on X and D). A framed flag of length 1 and type (r, γ, c, ℓ) on (X, D) is a triple (E, F, ϕ), where

[4]A vector $\mathbf{v} \in \mathbb{Z}^r$ is said to primitive if it is not a non-trivial multiple of another vector $\mathbf{u} \in \mathbb{Z}^r$.

- E and F are torsion-free sheaves on X, with $E \subset F$, $r = \operatorname{rk} E = \operatorname{rk} F$;
- the support of F/E is 0-dimensional and is disjoint from D;
- ϕ is an isomorphism of $F_{|D}$ with $\mathcal{O}_D^{\oplus r}$;
- $c_1(F) = \gamma \in \mathrm{NS}(X)$; $c = c_2(F)$, $\ell = c_2(E) - c_2(F) = h^0(X, F/E)$.

As a consequence, ϕ also provides an isomorphism $E_{|D} \simeq \mathcal{O}_D^{\oplus r}$. Note that $c_1(E) = \gamma$ and $\gamma \cdot D = 0$.

We define the functor $\mathfrak{F}^{X,D}_{r,\gamma,c,\ell}$ of families of framed flags of length 1 on (X, D) as

$$\mathfrak{F}^{X,D}_{r,\gamma,c,\ell}(T) = \{\text{isomorphism classes of triples } (E, F, \psi)\}$$

where

- E, F are rank r torsion-free sheaves on $X \times T$, flat on T, with $E \subset F$;
- for all closed points $t \in T$, the support of $(F/E)_t$ is 0-dimensional and is disjoint from D;
- for all closed $t \in T$, $c_2(F_t) = c$, $c_2(E_t) - c_2(F_t) = \ell$, $c_1(F_t) = \gamma$;
- ψ is an isomorphism $\psi\colon E_{|D\times T} \to \mathcal{O}^{\oplus r}_{D\times T}$;
- morphisms of families of framed flags are defined in the obvious way;
- the functor acts on scheme morphisms by pullback.

This functor was defined in [42] for $X = \mathbb{P}^2$ and D a line, that we denote as usual ℓ_∞ (note that necessarily $\gamma = 0$ in that case). Again in [42], it was proved that in that case this functor is representable. This may be generalized as follow.

Theorem 1.4.1. *Let X be a smooth, irreducible projective surface, and let D be a smooth, irreducible, big and nef divisor in X. Then for every choice of (r, γ, c, ℓ), the functor $\mathfrak{F}^{X,D}_{r,\gamma,c,\ell}$ is representable.*

Proof. According to Corollary 3.3 of [10] there exists a fine moduli space of torsion-free sheaves F on X, with invariants $\operatorname{rk} F = r$, $c_1(F) = \gamma$, $c_2(F) = c$, framed on D to the trivial sheaf. Then the proof of Proposition 1 in [42] applies verbatim. □

Remark 1.4.2. This theorem can be further generalized by replacing the trivial sheaf on D with any semistable vector bundle of rank r.

We denote by $\mathcal{F}^{X,D}_{r,\gamma,c,\ell}$ the scheme representing the functor $\mathfrak{F}^{X,D}_{r,\gamma,c,\ell}$.

1.5 Monads

For completeness sake, we now present the basic definitions and some general results about monads. A monad M on a scheme T is a three-term complex of locally free sheaves of $\mathcal{O}_T$-modules, having nontrivial cohomology only in the middle term:

$$M: \qquad 0 \to \mathcal{U} \xrightarrow{a} \mathcal{V} \xrightarrow{b} \mathcal{W} \to 0. \tag{1.13}$$

The cohomology of the monad will be denoted by

$$\mathcal{E}(M) = \frac{\ker(b)}{\operatorname{Im}(a)}.$$

$\mathcal{E}(M)$ is a coherent sheaf of $\mathcal{O}_T$-modules. A morphism (or isomorphism) of monads is just a morphism (isomorphism) of complexes. The display of the monad (1.13) is the commutative diagram below, with exact rows and columns, where the maps are naturally induced:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{U} & \xrightarrow{a} & \ker(b) & \longrightarrow & \mathcal{E}(M) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle i} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{U} & \xrightarrow{a} & \mathcal{V} & \longrightarrow & \operatorname{coker}(a) & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle b} & & \downarrow & & \\
 & & & & \mathcal{W} & = & \mathcal{W} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Let $\widetilde{S} = X \times S$, where X is a smooth connected projective variety over $\mathbb{C}$ and S is a scheme. If $\mathcal{F}$ is an $\mathcal{O}_{\widetilde{S}}$-module, we denote by $\mathcal{F}_s$ its restriction to the fibre of $\widetilde{S}$ over $s \in S$.

Lemma 1.5.1. *Let*

$$M: \qquad 0 \to \mathcal{U} \xrightarrow{a} \mathcal{V} \xrightarrow{b} \mathcal{W} \to 0$$

be a monad on $\widetilde{S}$, whose cohomology sheaf we denote by $\mathcal{E}$. If $\mathcal{E}$ is flat on S, then for all $s \in S$ the restricted complex M_s is a monad, whose cohomology is isomorphic to $\mathcal{E}_s$.

Proof. This is Lemma 2.2 from [4]. □

Lemma 1.5.2. *Consider* $T, S \in \mathbf{Sch}$ *and* $f : S \to T$. *Let* $\mathcal{E}^\bullet$ *be a family of monads on* $\widetilde{T}$ *whose cohomology sheaves are flat over* T. *Then*

$$\mathcal{H}^0\left((\mathrm{Id} \times f)^* \mathcal{E}^\bullet\right) \simeq (\mathrm{Id} \times f)^* \, \mathcal{H}^0(\mathcal{E}^\bullet).$$

This lemma is a consequence of the following results:

Lemma 1.5.3. *If* $f : S \to T$ *is a morphism of schemes and*

$$M : \quad 0 \to \mathcal{U} \xrightarrow{\alpha} \mathcal{V} \xrightarrow{\beta} \mathcal{W} \to 0$$

is a monad on T, *then there is a morphism from* $\mathcal{H}^0(f^*M)$ *to* $f^*\mathcal{H}^0(M)$.

Proof. The sequence

$$0 \to \ker(\beta) \to \mathcal{V} \xrightarrow{\beta} \mathcal{W} \to 0$$

is exact and as $\mathcal{W}$ is locally free, the sequence

$$0 \to f^* \ker(\beta) \to f^*\mathcal{V} \xrightarrow{f^*\beta} f^*\mathcal{W} \to 0$$

is also exact. The morphism $f^* \ker(\beta) \to f^*\mathcal{W}$ is the zero morphism, since f^* is right-exact; by the universal property of the kernel[5], we can obtain the dotted morphism on the commutative diagram below:

$$\begin{array}{ccccccccc} & & & & f^*\ker(\beta) & & & & \\ & & & \swarrow & \downarrow & \searrow^{0} & & & \\ 0 & \longrightarrow & \ker(f^*\beta) & \longrightarrow & f^*\mathcal{V} & \longrightarrow & f^*\mathcal{W} & \longrightarrow & 0. \end{array}$$

This morphism turns out to be an isomorphism since we can build the commutative diagram below as $\mathcal{W}$ is locally free

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & f^*\ker(\beta) & = & f^*\ker(\beta) & \longrightarrow & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \ker(f^*\beta) & \longrightarrow & f^*\mathcal{V} & \longrightarrow & f^*\mathcal{W} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ & & 0 & \longrightarrow & f^*\mathcal{W} & = & f^*\mathcal{W} & \longrightarrow & 0. \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

[5] Check [40], p. 53-54 for more details on the universality of kernels, cokernels and images.

Analogously, we can consider the short exact sequence

$$0 \to \mathrm{Im}(\alpha) \to \mathcal{V} \to \mathcal{Q} := \frac{\mathcal{V}}{\mathrm{Im}(\alpha)} \to 0$$

and get a exact sequence

$$0 \to \mathcal{T} \to f^*\mathrm{Im}(\alpha) \to f^*\mathcal{V} \to f^*\mathcal{Q} \to 0.$$

Define $\mathcal{I}' := \dfrac{f^*\mathrm{Im}(\alpha)}{\mathcal{T}}$ and notice that

$$0 \to \mathcal{I}' \xrightarrow{m'} f^*\mathcal{V} \to f^*\mathcal{Q} \to 0$$

is also exact. By the universal property of the image[5], there exists a unique monomorphism $\mathrm{Im}(f^*\alpha) \xrightarrow{\nu} \mathcal{I}'$ such that the diagram

$$\begin{array}{ccc} \mathrm{Im}(f^*\alpha) & \xrightarrow{\nu} & \mathcal{I}' \\ \downarrow & \swarrow_{m'} & \\ f^*\mathcal{V} & & \end{array}$$

commutes. Therefore, we get a morphism

$$\mathcal{H}^0(f^*M) = \frac{\ker(f^*\beta)}{\mathrm{Im}(f^*\alpha)} \to \frac{f^*\ker(\beta)}{\mathcal{I}'}$$

Since we have that the following exact sequences,

$$0 \to \mathrm{Im}(\alpha) \to \ker(\beta) \to \mathcal{H}^0(M) \to 0$$

$$0 \to \mathcal{T}' \to f^*\mathrm{Im}(\alpha) \to f^*\ker(\beta) \to f^*\mathcal{H}^0(M) \to 0$$

and the map $f^*\alpha : f^*\mathcal{U} \to f^*\mathcal{V}$ takes values in $f^*\ker(\beta) = \ker(f^*\beta)$ we may identify $\mathcal{T}'$ with $\mathcal{T}$. Hence,

$$f^*\mathcal{H}^0(M) = \frac{f^*\ker(\beta)}{\mathcal{I}'}.$$

So that we have a morphism

$$\mathcal{H}^0(f^*M) \to f^*\mathcal{H}^0(M).$$

□

Lemma 1.5.4. *If $\mathcal{E}^\bullet$ is family of monads on $\widetilde{T}$ whose cohomology sheaves are flat over T and $f : S \to T$ is a morphism of schemes, then $(\mathrm{Id} \times f)^*\mathcal{E}^\bullet$ is a monad and there is a morphism*

$$\mathcal{H}^0\left((\mathrm{Id} \times f)^*\mathcal{E}^\bullet\right) \to (\mathrm{Id} \times f)^*\,\mathcal{H}^0(\mathcal{E}^\bullet),$$

which is an isomorphism because of Lemma 1.5.1.

Proof. In fact, the existence of the morphism follows from Lemma 1.5.3 applied to the morphism $\mathrm{Id} \times f : \widetilde{S} \to \widetilde{T}$. In addition, for every $s \in S$ we get from Lemma 1.5.1 that

$$[\mathcal{H}^0((\mathrm{Id}\times f)^*\mathcal{E}^\bullet)]_s = \mathcal{H}^0([(\mathrm{Id}\times f)^*\mathcal{E}^\bullet]_s) = \mathcal{H}^0(\mathcal{E}^\bullet_{f(s)}) = [\mathcal{H}^0(\mathcal{E}^\bullet)]_{f(s)} = [(\mathrm{Id}\times f)^*\mathcal{H}^0(\mathcal{E}^\bullet)]_s,$$

so that

$$\mathcal{H}^0\left((\mathrm{Id}\times f)^*\mathcal{E}^\bullet\right) \simeq (\mathrm{Id}\times f)^*\mathcal{H}^0(\mathcal{E}^\bullet).$$

□

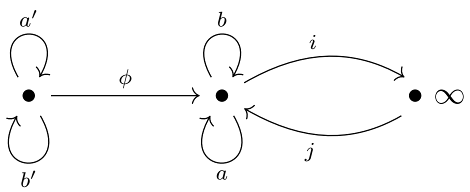
Chapter 2

The case of the projective plane

In this chapter, we recall some results and constructions from the papers [42, 43], recasting them in a full categorical setting, and providing additional details. This will also provide a setting for our subsequent work on the nested Hilbert schemes of points on the total space of the line bundles $\mathcal{O}_{\mathbb{P}^1}(-n)$.

2.1 ADHM construction of framed flags of sheaves on the projective plane

The enhanced ADHM quiver $\bar{Q}$ is the quiver



(2.1)

with the relations

$$ab - ba + ij = 0; \quad a\phi - \phi a' = 0; \quad b\phi - \phi b' = 0; \quad j\phi = 0; \quad a'b' - b'a' = 0. \tag{2.2}$$

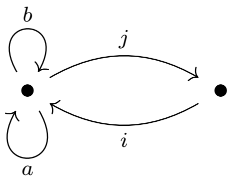
Within the setup of Section 1.4, Jardim and von Flach in [42] prove that in the case $(X, D) = (\mathbb{P}^2, \ell_\infty)$, for $\mathbf{v} = (\ell, c + \ell, r)$, and with a suitable choice of the stability parameter Θ, the functors of families of stable framed representations $\mathfrak{R}^{\bar{Q},\mathrm{fr},s}_{\mathbf{v},\Theta}$ and the functor of families of framed flags $\mathfrak{F}_{r,c,\ell} = \mathfrak{F}^{\mathbb{P}^2,\ell_\infty}_{r,0,c,\ell}$ are isomorphic. We review here their proof, providing some more details, especially about the categorical formalization of the problem.

The first step will be to represent $\mathfrak{F}_{r,c,\ell}$ as a functor of families of representations of $\bar{Q}$. The components of the dimension vector of this quiver list the dimensions of the vector spaces attached to the vertexes from left to right.

The crux of the above mentioned result is the following theorem.

Theorem 2.1.1. [42, 43] *Let $\mathbf{v} = (\ell, c+\ell, r)$, and let $\Theta = (\theta, \theta', \theta_\infty) \in \mathbb{R}^3$ with $\theta' > 0$ and $\theta + \theta' < 0$. Let $\mathfrak{R}^{\bar{Q},\mathrm{fr},s}_{\mathbf{v},\Theta}$ be the functor of families of framed representations of the enhanced ADHM quiver $\bar{Q}$ depicted in equation* (2.1) *with the relations* (2.2)*, framed at the vertex ∞.*[1] *There exists a natural transformation $\eta\colon \mathfrak{R}^{\bar{Q},\mathrm{fr},s}_{\mathbf{v},\Theta} \to \mathfrak{F}_{r,c,\ell}$ which is an isomorphism of functors.*

We shall also need to consider the standard ADHM quiver, which we shall denote by Q:



(2.3)

with the relation

$$ab - ba + ij = 0. \tag{2.4}$$

We develop now some theory which will be needed to prove Theorem 2.1.1. We introduce the following categories:

- the category $\mathcal{A}_Q$ of families of representations of the ADHM quiver Q with the relations (2.4). An object in $\mathcal{A}_Q$ is a collection $(T, \mathcal{V}, \mathcal{W}, A, B, I, J)$, where T is a scheme, $\mathcal{V}$ and $\mathcal{W}$ are vector bundles on T, and

  $$A, B \in \mathrm{End}(\mathcal{V}), \quad I \in \mathrm{Hom}(\mathcal{W}, \mathcal{V}), \quad J \in \mathrm{Hom}(\mathcal{V}, \mathcal{W})$$

  satisfying the condition

  $$AB - BA + IJ = 0.$$

  Let $\mathcal{A}^s_Q$ be the full subcategory of families of representations that are stable with respect to the standard Nakajima's stability condition from [34].

- The category $\mathbf{Kom}_{\mathbb{P}^2}$ of families of complexes of coherent sheaves on $\mathbb{P}^2$. Objects are given by a scheme T and a complex of coherent sheaves on $T \times \mathbb{P}^2$; the morphisms are the obvious ones. $\mathbf{Kom}^{\mathrm{flat}}_{\mathbb{P}^2}$ is the full subcategory of families of complexes whose cohomology sheaves are flat on T.

[1] Note that the vector space W corresponding to the framing vertex has dimension r.

$\mathcal{A}_Q$ and $\mathbf{Kom}_{\mathbb{P}^2}$ are categories over the category **Sch** of schemes.[2] Their fiber categories over $T = \operatorname{Spec}\mathbb{C}$ are the category of representations of the ADHM quiver Q (and then $\mathcal{V}$, $\mathcal{W}$ are just vector spaces) and the category of complexes of coherent sheaves over $\mathbb{P}^2$, respectively. If T is a scheme, we denote by $\mathcal{A}_Q(T)$ the fiber of $\mathcal{A}_Q$ over T, i.e., the category of families of representations of Q parameterized by T, with a similar meaning for $\mathbf{Kom}_{\mathbb{P}^2}(T)$.

Remark 2.1.2. By Nakajima's work we know that, fixing the dimension vector $\mathbf{v} = (r, c)$, the corresponding functor of families of stable representations of the quiver Q is represented by a scheme which is isomorphic to the moduli space $\mathcal{M}(r, c)$ of isomorphism classes of torsion-free sheaves on $\mathbb{P}^2$, of rank r and second Chern class c, with a framing to the trivial sheaf on a fixed line.

We introduce a functor

$$\mathfrak{K}_Q \colon \mathcal{A}_Q \to \mathbf{Kom}_{\mathbb{P}^2}$$

of categories over **Sch**; this is a relative version of the "absolute" standard functor which associates a complex with a representation of the ADHM quiver. The functor $\mathfrak{K}_Q$ associates with a family of representations of Q parameterized by a scheme T the corresponding family of 3-term complexes on $\mathbb{P}^2 \times T$. Note that as we are not requiring the representations to be stable the 3-term complex may have nontrivial cohomology in every degree, i.e., it may not be a monad. If $\mathcal{X} = (T, \mathcal{V}, \mathcal{W}, A, B, I, J)$ is an object in $\mathcal{A}_Q$, then $\mathfrak{K}_Q(\mathcal{X})$ is the following complex supported in degree -1, 0 and 1, whose terms are sheaves on $T \times \mathbb{P}^2$:

$$\mathcal{V} \boxtimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\alpha} (\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{W}) \boxtimes \mathcal{O}_{\mathbb{P}^2} \xrightarrow{\beta} \mathcal{V} \boxtimes \mathcal{O}_{\mathbb{P}^2}(1)$$

where the morphisms α, β are given by

$$\alpha = \begin{pmatrix} zA + x1_{\mathcal{V}} \\ zB + y1_{\mathcal{V}} \\ zJ \end{pmatrix}, \qquad \beta = \begin{pmatrix} -zB - y1_{\mathcal{V}}, & zA + x1_{\mathcal{V}}, & zI \end{pmatrix}$$

with (x, y, z) homogeneous coordinates in $\mathbb{P}^2$. Note that $\beta \circ \alpha = 0$ automatically.

A morphism $\xi = (f, \xi_1, \xi_2)$ of families of representations

$$\mathcal{X} = (S, \mathcal{V}, \mathcal{W}, A, B, I, J) \xrightarrow{\xi} \tilde{\mathcal{X}} = (T, \tilde{\mathcal{V}}, \tilde{\mathcal{W}}, \tilde{A}, \tilde{B}, \tilde{I}, \tilde{J})$$

[2] Actually, since they admit pullbacks, both categories are fibered categories over **Sch**. See [41], Definition 3.5, or [39], Section 4.33.

is a morphism $f\colon S \to T$ and a pair of morphisms $\xi_1\colon \mathcal{V} \to f^*\tilde{\mathcal{V}}$, $\xi_2\colon \mathcal{W} \to f^*\tilde{\mathcal{W}}$ satisfying

$$\xi_1 \circ A = f^*\tilde{A} \circ \xi_1, \qquad \xi_1 \circ B = f^*\tilde{B} \circ \xi_1, \qquad \xi_2 \circ J = f^*\tilde{J} \circ \xi_1, \qquad \xi_1 \circ I = f^*\tilde{I} \circ \xi_2.$$

The morphism $\mathfrak{K}_Q(\xi)\colon \mathfrak{K}_Q(\mathcal{X}) \to \mathfrak{K}_Q(\tilde{\mathcal{X}})$ between the corresponding monads is given by the diagram

$$\begin{array}{ccccc} \mathcal{V} \boxtimes \mathcal{O}_{\mathbb{P}^2}(-1) & \xrightarrow{\alpha} & (\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{W}) \boxtimes \mathcal{O}_{\mathbb{P}^2} & \xrightarrow{\beta} & \mathcal{V} \boxtimes \mathcal{O}_{\mathbb{P}^2}(1) \\ \downarrow{\scriptstyle \xi_1\times\mathrm{Id}} & & \downarrow{\scriptstyle (\xi_1\oplus\xi_1\oplus\xi_2)\times\mathrm{Id}} & & \downarrow{\scriptstyle \xi_1\times\mathrm{Id}} \\ f^*\tilde{\mathcal{V}} \boxtimes \mathcal{O}_{\mathbb{P}^2}(-1) & \xrightarrow{f^*\tilde{\alpha}} & f^*(\tilde{\mathcal{V}} \oplus \tilde{\mathcal{V}} \oplus \tilde{\mathcal{W}}) \boxtimes \mathcal{O}_{\mathbb{P}^2} & \xrightarrow{f^*\tilde{\beta}} & f^*\tilde{\mathcal{V}} \boxtimes \mathcal{O}_{\mathbb{P}^2}(1) \end{array} \tag{2.5}$$

Proposition 2.1.3. *For every scheme T, the functor $\mathfrak{K}_Q(T)\colon \mathcal{A}_Q(T) \to \mathbf{Kom}_{\mathbb{P}^2}(T)$ is exact and faithful.*

Proof. The proof Proposition 2.3.5 in [37] applies verbatim. □

The next result requires that the representations we consider are framed and stable. So we define $\mathcal{A}_Q^{\mathrm{frs}}$ as the subcategory of $\mathcal{A}_Q$ whose objects are family of framed representations of Q, stable with respect to the standard stability condition. Note that this category is not additive, and neither it is a full subcategory as the morphisms at the framing vertex are restricted.

Proposition 2.1.4. *$\mathfrak{K}_Q$ maps the subcategory $\mathcal{A}_Q^{\mathrm{frs}}$ to the subcategory $\mathbf{Kom}_{\mathbb{P}^2}^{\mathrm{flat}}$.*

Proof. The stability of the family of representations on which we act by $\mathfrak{K}_Q$ implies that the morphism α is injective and β is surjective. Then we may reduce to prove the following fact: if

$$0 \to \mathcal{E}' \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathcal{E}'' \to 0$$

is a complex of families of locally free coherent sheaves on $T \times \mathbb{P}^2$, with α injective and β surjective, then the cohomology sheaf $\mathcal{H} = \ker\beta / \operatorname{im}\alpha$ is flat over T. To prove this we first consider the exact sequence

$$0 \to \ker\beta \to \mathcal{E} \to \mathcal{E}'' \to 0,$$

where $\mathcal{E}$ and $\mathcal{E}''$ are flat over T, so that $\ker\beta$ is flat as well. Then one applies Lemma 2.1.4 in [26] to the exact sequence

$$0 \to \operatorname{im}\alpha \to \ker\beta \to \mathcal{H} \to 0.$$

□

Remark 2.1.5. The image $\mathfrak{K}_Q(\mathcal{A}_Q^s)$ is the subcategory of $\mathbf{Kom}_{\mathbb{P}^2}^{\text{flat}}$ whose objects are families of monads for the ADHM quiver (in particular their cohomology is flat over T).

The following lemma will be very useful in the construction of the natural transformation betwenn the suitable functors that will be defined later, since it provides a characterization for the stable representations of the enhanced ADHM quiver.

Lemma 2.1.6 (Lemma 4 of [42] / Lemma 2 of [43])**.** *Fix a triple $(r, c, c') \in \mathbb{Z}^3_{>0}$, write $\Theta = (\theta, \theta', \theta_\infty)$ and assume that $\theta' > 0$ and $\theta + c'\theta' < 0$. Let*

$$X = (W, V, V', A, B, I, J, A', B', F, G)$$

be a representation of the enhanced ADHM quiver with dimension vector (r, c, c'). The following are equivalent:

(i) X is Θ-stable;

(ii) X is Θ-semistable;

(iii) X satisfies the following conditions:

(S1) $F \in \mathrm{Hom}(V', V)$ is injective;

(S2) The ADHM datum (W, V, A, B, I, J) is stable, i.e., there is no proper subspace $0 \subset S \subsetneq V$ such that $A(S),\ B(S),\ \mathrm{Im}(I) \subset S$.

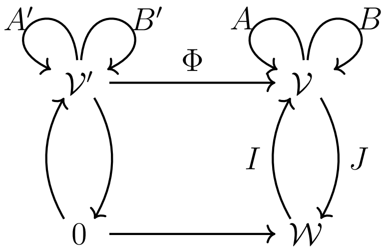
Now we construct the natural transformation $\eta\colon \mathfrak{R}^{\bar{Q}\text{fr}s}_{\mathbf{v},\Theta} \to \mathfrak{F}_{r,c,\ell}$. The trick for doing that is to regard a representation of the enhanced ADHM quiver as a morphism of representations of the standard ADHM quiver. Let $(T, \mathcal{V}', \mathcal{V}, \mathcal{W}, A', B', A, B, I, J, \Phi)$ be a family of framed representations of the enhanced ADHM quiver, framed at the vertex 0. So T a scheme, and $\mathcal{V}'$ and $\mathcal{V}$ are vector bundles on T of rank n and $n+\ell$, respectively. $\mathcal{W}$ is the trivial bundle $W \otimes \mathcal{O}_T$ for some fixed vector space W of dimension r. Moreover,

$$A', B' \in \mathrm{End}(\mathcal{V}'), \quad A, B \in \mathrm{End}(\mathcal{V}), \quad I \in \mathrm{Hom}(\mathcal{W}, \mathcal{V}),$$

$$J \in \mathrm{Hom}(\mathcal{V}, \mathcal{W}), \quad \Phi \in \mathrm{Hom}(\mathcal{V}', \mathcal{V}).$$

Assume that this representation is stable as in Theorem 2.1.1. Lemma 2.1.6 implies that Φ is injective. This defines a morphism of families of representations of the standard

ADHM quiver described by the following diagram

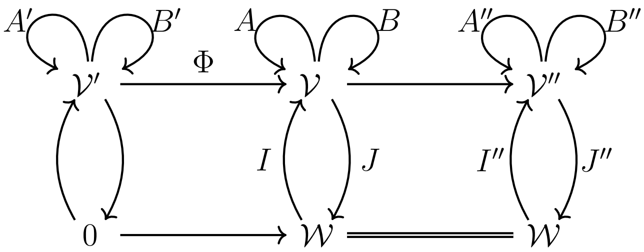


(2.6)

Let $\mathcal{V}'' = \mathcal{V}/\Phi(\mathcal{V}')$; note that $\mathcal{V}''$ is locally free (of rank ℓ) as Φ is injective on every fiber of $\mathcal{V}'$. The morphisms A, B, A', B', I, J induce morphisms

$$A'', B'' \in \mathrm{End}(\mathcal{V}''), \quad I'' \in \mathrm{Hom}(\mathcal{W}, \mathcal{V}''), \quad J'' \in \mathrm{Hom}(\mathcal{V}'', \mathcal{W})$$

which define a quotient family of representations of the ADHM quiver. This is represented in the diagram



(2.7)

i.e., we have an exact sequence of families of representations of the standard ADHM quiver

$$0 \to \mathcal{X}' \to \mathcal{X} \to \mathcal{X}'' \to 0.$$

Here $\mathcal{X}$ and $\mathcal{X}''$ are families of stable representations. Applying the exact functor $\mathfrak{K}_Q$ we obtain an exact sequence of complexes of coherent sheaves on $T \times \mathbb{P}^2$

$$0 \to E_{\mathcal{X}'} \to E_{\mathcal{X}} \to E_{\mathcal{X}''} \to 0$$

whose nonzero terms are in degree -1, 0 and 1. This exact sequence of complexes makes up the following commutative diagram with exact rows, whose columns are the complexes corresponding to $\mathcal{X}'$, $\mathcal{X}$, $\mathcal{X}''$, respectively:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{V}' \boxtimes \mathcal{O}_{\mathbb{P}^2}(-1) & \to & \mathcal{V} \boxtimes \mathcal{O}_{\mathbb{P}^2}(-1) & \to & \mathcal{V}'' \boxtimes \mathcal{O}_{\mathbb{P}^2}(-1) & \to & 0 \\
 & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha''} & & \\
0 & \to & (\mathcal{V}' \oplus \mathcal{V}') \boxtimes \mathcal{O}_{\mathbb{P}^2} & \to & (\mathcal{V} \oplus \mathcal{V} \oplus \mathcal{W}) \boxtimes \mathcal{O}_{\mathbb{P}^2} & \to & (\mathcal{V}'' \oplus \mathcal{V}'' \oplus \mathcal{W}) \boxtimes \mathcal{O}_{\mathbb{P}^2} & \to & 0 \\
 & & \downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta''} & & \\
0 & \to & \mathcal{V}' \boxtimes \mathcal{O}_{\mathbb{P}^2}(1) & \to & \mathcal{V} \boxtimes \mathcal{O}_{\mathbb{P}^2}(1) & \to & \mathcal{V}'' \boxtimes \mathcal{O}_{\mathbb{P}^2}(1) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Since $\mathcal{X}$ and $\mathcal{X}''$ are stable, the associated long exact cohomology sequence reduces to

$$0 \to \mathcal{H}^0(E_{\mathcal{X}'}) \to \mathcal{H}^0(E_{\mathcal{X}}) \to \mathcal{H}^0(E_{\mathcal{X}''}) \to \mathcal{H}^1(E_{\mathcal{X}'}) \to 0$$

(note that $\mathcal{H}^{-1}(E_{\mathcal{X}''}) = 0$ as α'' is fiberwise injective, hence injective due to flatness). We show that $\mathcal{H}^0(E_{\mathcal{X}'}) = \ker \beta' / \operatorname{im} \alpha' = 0$. First we note that, thinking of ℓ_∞ as the line $z = 0$ is $\mathbb{P}^2$, we may write α', β' restricted to $T \times \ell_\infty$ as

$$\alpha'_{|T\times\ell_\infty} = \begin{pmatrix} x1_{\mathcal{V}'} \\ y1_{\mathcal{V}'} \end{pmatrix}, \qquad \beta'_{|T\times\ell_\infty} = \begin{pmatrix} -y1_{\mathcal{V}'}, & x1_{\mathcal{V}'} \end{pmatrix}.$$

As a simple computation shows, one has $\operatorname{im} \alpha' = \ker \beta'$ on $T \times \ell_\infty$ so that $\mathcal{H}^0(E_{\mathcal{X}'})$ is zero on $T \times \ell_\infty$, hence it has rank 0. Then it must be zero as it injects into $\mathcal{H}^0(E_{\mathcal{X}})$ which is torsion-free.

Moreover one has:

- $F = \mathcal{H}^0(E_{\mathcal{X}''})$ is a torsion-free sheaf on $T \times \mathbb{P}^2$, with a framing ϕ to the trivial sheaf on $T \times \ell_\infty$, where ℓ_∞ is a line in $\mathbb{P}^2$. Moveover, for every closed point $t \in T$, the second Chern class of $F_{|\{t\}\times\mathbb{P}^2}$ is n.
- F and $E = \mathcal{H}^0(E_{\mathcal{X}})$ are flat over T by Proposition 2.1.4 as $\mathcal{X}$ and $\mathcal{X}''$ are stable.
- $\mathcal{H}^1(E_{\mathcal{X}'})$ is a rank 0 coherent sheaf on $T \times \mathbb{P}^2$, supported away from $T \times \ell_\infty$. For every closed point $t \in T$, the restriction of the schematic support of $\mathcal{H}^1(E_{\mathcal{X}'})$ to the fiber over t is a length ℓ 0-cycle in $\mathbb{P}^2$.
- $\mathcal{H}^1(E_{\mathcal{X}'})$ is flat over T as it is a quotient of flat sheaves. (One can also prove this directly as in Proposition 2.1.4.)

Thus the triple (E, F, ϕ) is a flat family of framed flags on $\mathbb{P}^2$ parameterized by the scheme T. This defines the natural transformation η. One can indeed show that for any scheme morphism $f\colon T \to S$ the diagram

$$\begin{array}{ccc} \mathfrak{R}^{\bar{Q}\mathrm{frs}}_{\mathbf{v},\Theta}(S) & \xrightarrow{\mathfrak{R}^{\bar{Q}\mathrm{frs}}_{\mathbf{v},\Theta}(f)} & \mathfrak{R}^{\bar{Q},\mathrm{frs}}_{\mathbf{v},\Theta}(T) \\ \downarrow{\scriptstyle \eta_S} & & \downarrow{\scriptstyle \eta_T} \\ \mathfrak{F}_{r,c,\ell}(S) & \xrightarrow{\mathfrak{F}_{r,c,\ell}(f)} & \mathfrak{F}_{r,c,\ell}(T) \end{array}$$

commutes. In fact, first notice that the diagram

$$\begin{array}{ccc} \mathfrak{R}^{Q,\mathrm{fr},s}_{\mathrm{v}}(S) & \xrightarrow{f^*} & \mathfrak{R}^{Q,\mathrm{fr},s}_{\mathrm{v}}(T) \\ \downarrow{\scriptstyle \mathfrak{K}_Q} & & \downarrow{\scriptstyle \mathfrak{K}_T} \\ \mathrm{Kom}(\mathbb{P}^2 \times S) & \xrightarrow{(\mathrm{Id}\times f)^*} & \mathrm{Kom}(\mathbb{P}^2 \times T) \end{array}$$

commutes, i.e.,

$$(\mathrm{Id} \times f)^* \circ \mathfrak{K}_Q = \mathfrak{K}_Q \circ f^*. \tag{2.8}$$

So that, if we consider an element in $\mathfrak{R}^{\bar{Q}\mathrm{frs}}_{\mathbf{v},\Theta}(S)$, say $[S, X]$. We have that

$$\begin{aligned} [\mathfrak{F}_{r,c,\ell}(f) \circ \eta_S]([S, X]) &= [(\mathrm{Id} \times f)^* \circ \eta_S]([S, X]) \\ &= (\mathrm{Id} \times f)^* \eta_S([S, X]) \\ &= (\mathrm{Id} \times f)^* (\mathcal{H}^\bullet(\mathfrak{K}_Q([S, X]))) \\ &\overset{(1.5.2)}{=} \mathcal{H}^\bullet((\mathrm{Id} \times f)^* \mathfrak{K}_Q([S, X])) \\ &\overset{(2.8)}{=} \mathcal{H}^\bullet(\mathfrak{K}_Q(f^*([S, X]))) \\ &= \eta_T(f^*([S, X])) \\ &= (\eta_T \circ f^*)([S, X]) \\ &= (\eta_T \circ \mathfrak{R}^{\bar{Q}\mathrm{frs}}_{\mathbf{v},\Theta}(f))([S, X]). \end{aligned} \tag{2.9}$$

Remark 2.1.7. Actually, the notation $\mathcal{H}^\bullet(\mathfrak{K}_Q([S, X]))$ does not make much sense, but it is written in this way to simplify the definition of η; we do not explicit the morphisms because they are completely determined by the functoriality requirement.

To show that η is actually an isomorphism one constructs a natural transformation going the opposite direction which is both a right and a left inverse to η. This is accomplished by tracing back the steps that led to the definition of η. Thus, given a family of framed flags on $\mathbb{P}^2$ with the required numerical invariants, one defines families of representations $\mathcal{X}$, $\mathcal{X}''$ of the standard ADHM quiver, with a surjection $\Psi\colon \mathcal{X} \to \mathcal{X}''$. Then one defines $\mathcal{X}'$ as the kernel of Ψ; the families $\mathcal{X}'$ and $\mathcal{X}$ now combine to yield a family of representations of the enhanced ADHM quiver $\bar{Q}$. This concludes the proof of Theorem 2.1.1.

As we recalled in Section 1.4, the functor $\mathfrak{F}_{r,c,\ell}$ is representable, so that there is a fine moduli scheme $\mathcal{F}_{r,c,\ell}$ for framed flags on $\mathbb{P}^2$ with numerical invariants r, n, ℓ. So we have:

Corollary 2.1.8. *The moduli scheme $\mathcal{M}^{\bar{Q}\mathrm{frs}}_{\mathbf{v},\Theta}$ representing the functor $\mathfrak{R}^{\bar{Q}\mathrm{frs}}_{\mathbf{v},\Theta}$ is isomorphic to the moduli scheme $\mathcal{F}_{r,c,\ell}$.*

As expected, the space of stability parameters of the enhanced ADHM quiver has a chamber structure, i.e., it can be divided into polyhedral cones whose interiors have the property that stability and semistability are equivalent, whilst for parameters on the faces there exist strictly semistable representations. This is a typical phenomenon when one deals with variation of stability conditions in algebraic geometry. In literature, the cones are called chambers, their faces are called walls, and a parameter inside a chamber is said to be generic. The chamber considered in Theorem 2.1.1 is depicted in Fig. 2.1 below.

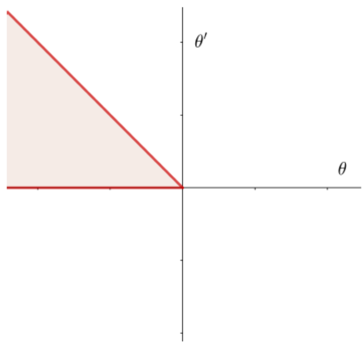


Figure 2.1: Stability chamber in the space of parameters for the enhanced ADHM quiver. The chamber is the region in pink and the walls are the rays in red.

We shall see in the next chapter a different chamber structure for the stability parameters of the quiver Q_n of Example 1.1.27.

Chapter 3

The case of the Hirzebruch surfaces

3.1 Hirzebruch surfaces

The Hirzebruch surfaces Σ_n, with $n \in \mathbb{N}_0$, are fibrations over the complex projective line $\mathbb{P}^1$ whose fibers are copies of $\mathbb{P}^1$. The formal definition is

$$\Sigma_n = \mathbb{P}([\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)] \to \mathbb{P}^1).$$

We can list some properties of these surfaces:

1. All of them, except Σ_1, are minimal surfaces, i.e., it is not possible to contract a curve on them without creating a singularity;
2. Σ_0, $\mathbb{P}^2$ and Σ_n for $n \geq 2$ are the only minimal projective surfaces that have a dense open set isomorphic to $\mathbb{C}^2$, i.e., they are rational;
3. $\Sigma_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$;
4. Σ_1 is isomorphic to the blow-up of $\mathbb{P}^2$ at a point;
5. Σ_m and Σ_n are not isomorphic unless $m = n$.

We denote by $\mathfrak{e}$ and $\mathfrak{h}$ the cohomology classes of the sections of the fibration $\Sigma_n \to \mathbb{P}^1$ that square to $-n$ and n, respectively, and by $\mathfrak{f}$ the class of the fiber. We shall use $(\mathfrak{h}, \mathfrak{f})$ as a basis of $\mathrm{Pic}(\Sigma_n)$ over $\mathbb{Z}$, i.e.,

$$\mathrm{Pic}(\Sigma_n) \simeq \mathfrak{h} \cdot \mathbb{Z} \oplus \mathfrak{f} \cdot \mathbb{Z}.$$

One has:

$$\mathfrak{h}^2 = n; \quad \mathfrak{e} = \mathfrak{h} - n\mathfrak{f}; \quad \mathfrak{h} \cdot \mathfrak{f} = 1; \quad \mathfrak{f}^2 = 0; \quad \mathfrak{e}^2 = -n.$$

We fix a curve $\ell_\infty \in \mathfrak{h}$ and call it the line at infinity. We shall write $\mathcal{O}_{\Sigma_n}(p,q)$ for $\mathcal{O}_{\Sigma_n}(p\mathfrak{h}+q\mathfrak{f})$, and for any sheaf $\mathcal{E}$ of $\mathcal{O}_{\Sigma_n}$-modules we shall write

$$\mathcal{E}(p,q) := \mathcal{E} \otimes \mathcal{O}_{\Sigma_n}(p,q), \quad \text{for } p,q \in \mathbb{Z}.$$

We can see the n-th Hirzebruch surface immersed in $\mathbb{P}^1 \times \mathbb{P}^2$ as a hypersurface:

$$\Sigma_n = \left\{([y_1;y_2],[x_1;x_2;x_3]) \in \mathbb{P}^1 \times \mathbb{P}^2 | x_1 y_1^n = x_2 y_2^n\right\}. \tag{3.1}$$

In this case, the line at infinity (ℓ_∞) is cut by the equation $x_3 = 0$.

We also let

Definition 3.1.1. A framed sheaf on Σ_n is a pair $(\mathcal{E},\varphi)$, where

1. $\mathcal{E}$ is a torsion-free sheaf on Σ_n such that

$$\mathcal{E}\big|_{\ell_\infty} \simeq \mathcal{O}_{\ell_\infty}^{\oplus \mathrm{rk}(\mathcal{E})}, \tag{3.2}$$

2. φ is a fixed isomorphism $\varphi : \mathcal{E}\big|_{\ell_\infty} \xrightarrow{\simeq} \mathcal{O}_{\ell_\infty}^{\oplus \mathrm{rk}(\mathcal{E})}$. △

Remark 3.1.2. The isomorphism φ is the so-called "framing at infinity". Sometimes we shall use the expression "sheaf trivial at infinity" meaning a sheaf that satisfies (3.2), but it does not have a designated framing.

Remark 3.1.3. The condition (3.2) implies that $c_1(\mathcal{E}) \propto \mathfrak{e}$. Indeed, write

$$c_1(\mathcal{E}) = p\mathfrak{h} + q\mathfrak{f},$$

for some $p,q \in \mathbb{Z}$. Since $\mathcal{E}$ is trivial at infinity, we have

$$c_1(\mathcal{E}\big|_{\ell_\infty}) = 0.$$

Therefore one has

$$(p\mathfrak{h}+q\mathfrak{f})\cdot\mathfrak{h} = 0 \iff p\mathfrak{h}^2 + q\mathfrak{f}\cdot\mathfrak{h} = 0 \iff pn + q = 0 \iff q = -pn.$$

So,

$$c_1(\mathcal{E}) = p\mathfrak{h} + q\mathfrak{f} = p\mathfrak{h} - pn\mathfrak{f} = p(\mathfrak{h} - n\mathfrak{f}) = p\mathfrak{e}.$$

Definition 3.1.4. An isomorphism $\Upsilon : (\mathcal{E},\varphi) \to (\mathcal{E}',\varphi')$ between two framed sheaves is an isomorphism of sheaves $\Upsilon : \mathcal{E} \xrightarrow{\sim} \mathcal{E}'$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{E}\big|_{\ell_\infty} & \xrightarrow{\varphi} & \mathcal{O}_{\ell_\infty}^{\oplus \mathrm{rk}(\mathcal{E})} \\ {\scriptstyle \Upsilon|_{\ell_\infty}}\downarrow & \nearrow{\scriptstyle \varphi'} & \\ \mathcal{E}'\big|_{\ell_\infty} & & \end{array}$$

△

3.2 ADHM data and Hilbert schemes as quiver varieties

One defines $\mathcal{M}^n(r,a,c)$ as the moduli space parameterizing isomorphism classes of framed sheaves $(\mathcal{E},\varphi)$ on Σ_n with Chern character

$$\mathrm{ch}(\mathcal{E}) = \left(r, a\mathfrak{e}, -c - \frac{na^2}{2}\right), \tag{3.3}$$

where $r,a,c \in \mathbb{N}$, $r \geq 1$ and $0 \leq a \leq r-1$. In particular, if $r=1$, then $a=0$ and

$$\mathcal{M}^n(1,0,c) \simeq \mathrm{Hilb}^c(\Sigma_n \setminus \ell_\infty) = \mathrm{Hilb}^c(\mathrm{tot}(\mathcal{O}_{\mathbb{P}^1}(-n))), \tag{3.4}$$

where the first isomorphism is given by associating to $\mathcal{E}$ the schematic support of $\dfrac{\mathcal{E}^{**}}{\mathcal{E}}$. Some properties of $\mathcal{M}^n(r,a,c)$ can be found in [4]. For instance:

- It has a structure of smooth algebraic variety;
- It is irreducible;
- It is nonempty if, and only if, $c + \dfrac{na(a-1)}{2} \geq 0$;
- It is a fine[1] moduli space of framed sheaves on Σ_n whose universal framed sheaf is constructed as the cohomology of a universal monad.
- $\dim(\mathcal{M}^n(r,a,c)) = 2rc + (r-1)na^2$. In particular, $\dim(\mathcal{M}^n(1,0,c)) = 2c$ does not depend on n.[2]

Consider the definition of Σ_n as in (3.1) and choose $c+1$ fibers $\mathfrak{f}_0, \ldots, \mathfrak{f}_c \in \mathfrak{f}$ such that for any $[(\mathcal{E},\varphi)] \in \mathrm{Hilb}^c(\mathrm{tot}(\mathcal{O}_{\mathbb{P}^1}(-n))$

$$\mathcal{E}\Big|_{\mathfrak{f}_m} = \mathcal{O}_{\mathfrak{f}_m}.$$

Indeed, we choose the fibers as closed subvarieties cut in Σ_n by the equations

$$\mathfrak{f}_m = \left\{[y_1,y_2] = \left[\cos\left(\pi\frac{m}{c+1}\right), \sin\left(\pi\frac{m}{c+1}\right)\right]\right\}, \quad m = 0,\ldots,c. \tag{3.5}$$

[1] A fine moduli **space** is a scheme that represents the corresponding functor of moduli. For more details on this, cf. Section 6 of [4].

[2] This is compatible with [38, Theorem 4.6.9], since we have the isomorphism (3.4).

So, we obtain an open cover $\{U_m^{nc}\}_{m=0,\ldots,c}$ of $\mathrm{Hilb}^c(\mathrm{tot}(\mathcal{O}_{\mathbb{P}^1}(-n))$ by putting

$$U_m^{nc} = \left\{ [(\mathcal{E},\varphi)] \in \mathrm{Hilb}^c(\mathrm{tot}(\mathcal{O}_{\mathbb{P}^1}(-n)) \;\middle|\; \mathcal{E}\big|_{\mathfrak{f}_m} = \mathcal{O}_{\mathfrak{f}_m} \right\}. \tag{3.6}$$

Each of these open sets is isomorphic to $\mathrm{Hilb}^c(\mathbb{C}^2)$ [2], so that it admits a Nakajima's ADHM description as in [34, Theorem 1.9] in terms of three matrices: $b_1,\ b_2 \in \mathrm{End}(\mathbb{C}^c)$ and $e \in \mathrm{Hom}(\mathbb{C}^c, \mathbb{C})$, satisfying the conditions

(i) $[b_1, b_2] = 0$;

(ii) For all $(z,w) \in \mathbb{C}^2$ there is no nonzero vector $v \in \mathbb{C}^c$ such that

$$\begin{cases} b_1(v) = zv \\ b_2(v) = wv \\ v \in \ker(e). \end{cases}$$

Remark 3.2.1. The maps b_1, b_2 and e are the transposed of the ones in Nakajima's ADHM description. Here, the condition (ii) is the so-called co-stability condition, thus the stability condition used by Nakajima is satisfied by the matrices $({}^t b_1, {}^t b_2, {}^t e)$. We stress that we are considering transposed matrices. The ADHM data for the open set U_m^{nc} will be denoted by (b_{1m}, b_{2m}, e_m).

Definition 3.2.2 (Stability for Q_n)**.** Consider $\theta \in \mathbb{R}^2$. A $(\mathbf{v}, w)$-dimensional representation (V_0, V_1, W) of the quiver Q_n is said to be θ-semistable if, for any subrepresentation $S = (S_0, S_1) \subseteq (V_0, V_1)$, one has:

$$\text{if } S_0 \subseteq \ker(e), \text{ then } \theta \cdot (\dim(S_0), \dim(S_1)) \leq 0; \tag{3.7}$$

$$\text{if } S_0 \supseteq \mathrm{Im}(f_i) \text{ for } i = 1, \ldots, n-1, \text{ then } \theta \cdot (\dim(S_0), \dim(S_1)) \leq \theta \cdot \mathbf{v}. \tag{3.8}$$

For $n = 1$ the condition in (3.8) must hold for any subrepresentation. A θ-semistable representation is θ-stable if strict inequality holds in (3.7) whenever $S \neq 0$ and in (3.8) whenever $S \neq (V_0, V_1)$.

△

We are interested in the stability parameter $\vartheta_c = (2c, -2c+1)$. The following lemma from [31], which we report here with its full proof for future use, characterizes the representations which are ϑ_c-(semi)stable.

Lemma 3.2.3. *An element $X = (A_1, A_2; C_1, \ldots, C_n; e; f_1, \ldots, f_{n-1}) \in \mathrm{Rep}(Q_n, \vec{v}_c, 1)$ is θ_c-semistable if and only if*

(Q2) *for all sub-representations* $S = (S_0, S_1)$ *such that* $S_0 \subseteq \ker(e)$*, one has*

$$\dim(S_0) \leq \dim(S_1),$$

and if $\dim(S_0) = \dim(S_1)$*, then* $S = 0$*;*

(Q3) *for all sub-representations* $S = (S_0, S_1)$ *such that* $S_0 \supseteq \mathrm{Im}(f_i)$*,* $i \in \{1, \dots, n-1\}$*, one has* $\dim(S_0) \leq \dim(S_1)$*; for* $n = 1$*, this must hold for any sub-representation.*

Furthermore, θ_c*-semistability and* θ_c*-stability are equivalent.*

Proof. Suppose that X is θ_c-semistable. We shall prove that (Q2) and (Q3) hold. In order to do that, consider a subrepresentation $S = (S_0, S_1)$ of X such that $S_0 \subseteq \ker(e)$. By the θ_c-semistability of X, we must have:

$$\theta_c \cdot (\dim(S_0), \dim(S_1)) \leq 0. \tag{3.9}$$

Equivalently, one has:

$$(2c, 1-2c) \cdot (\dim(S_0), \dim(S_1)) \leq 0 \iff 2c(\dim(S_0) - \dim(S_1)) + \dim(S_1) \leq 0. \tag{3.10}$$

Thus,

$$\dim(S_0) - \dim(S_1) \leq 0 \iff \dim(S_0) \leq \dim(S_1).$$

If $\dim(S_0) = \dim(S_1)$, (3.10) provides us that $\dim(S_0) = \dim(S_1) \leq 0$. Therefore,

$$S_0 = S_1 = 0 \Longrightarrow S = 0.$$

For the statement (Q3), start by considering a representation $S = (S_0, S_1)$ whose vector space S_0 contains $\mathrm{Im}(f_i)$ for $i \in \{1, \dots, n-1\}$. Since X is θ_c-semistable, one has

$$(2c, -2c+1) \cdot (\dim(S_0), \dim(S_1)) \leq (2c, -2c+1) \cdot (c, c).$$

So,

$$2c \cdot (\dim(S_0) - \dim(S_1)) + \dim(S_1) \leq c. \tag{3.11}$$

If $\dim(S_0) > \dim(S_1)$, (3.11) gives us

$$\underbrace{2c}_{>0} \cdot \underbrace{(\dim(S_0) - \dim(S_1))}_{\geq 1} + \underbrace{\dim(S_1)}_{\geq 0} \leq c.$$

This is a contradition, therefore we must have $\dim(S_0) \leq \dim(S_1)$.

On the other hand, assume that (Q2) and (Q3) hold. Consider a subrepresentation $S = (S_0, S_1)$ such that $S_0 \subseteq \ker(e)$. By (Q2) one has $\dim(S_0) - \dim(S_1) \leq 0$. If $\dim(S_0) - \dim(S_1) = 0$, we have that $S_1 = S_2 = 0$. In this case, we trivially obtain

$$\theta_c \cdot (\dim(S_0), \dim(S_1)) \leq 0.$$

So we may assume that $\dim(S_0) - \dim(S_1) \leq -1$. So that,

$$\begin{aligned}\theta_c \cdot (\dim(S_0), \dim(S_1)) &= 2c(\dim(S_0) - \dim(S_1)) + \dim(S_1) \\ &\leq 2c \cdot (-1) + c \\ &= -2c + c \\ &\leq 0,\end{aligned}$$

as wanted. When the subrepresentation is such that $S_0 \supseteq \operatorname{Im}(f_i)$ for $i \in \{1, \ldots, n-1\}$. By (Q3), we have $\dim(S_0) \leq \dim(S_1)$. Since $2c - 1 \geq 0$, one obtains:

$$(2c-1)(\dim(S_0) - \dim(S_1)) \leq 0.$$

Thus

$$\begin{aligned}\theta_c \cdot (\dim(S_0), \dim(S_1)) - \theta_c \cdot (c, c) &= 2c \dim(S_0) + (1-2c)\dim(S_1) - c \\ &= 2c \dim(S_0) + (1-2c)\dim(S_1) - c \\ &= (2c-1)(\dim(S_0) - \dim(S_1)) + \dim(S_0) - c \\ &\leq 0\end{aligned}$$

Therefore,

$$\theta_c \cdot (\dim(S_0), \dim(S_1)) \leq \theta_c \cdot (c, c). \tag{3.12}$$

It is obvious that stability implies semistability. We show the converse. Suppose that X is θ_c-semistable. From (Q2) we have that $\dim(S_0) < \dim(S_1)$ and $\dim(S_0) = \dim(S_1)$ implies $S = 0$. In other words, in this case, the inequality (3.9) is strict if $S \neq 0$. Additionally, if

$$(2c, 1-2c) \cdot (\dim(S_0), \dim(S_1)) = \theta_c \cdot (\dim(S_0), \dim(S_1)) = \theta_c(c, c) = c. \tag{3.13}$$

Then

$$\underbrace{2c(\dim(S_0) - \dim(S_1))}_{\leq 0} + \underbrace{\dim(S_1)}_{\leq c} = c \tag{3.14}$$

This means that we must have

$$\dim(S_0) = \dim(S_1) = c.$$

Thus, if S is proper, we must have the strict inequality in (3.12), i.e., θ_c-semistability also implies the θ_c-stability, as wanted. □

Remark 3.2.4. The space of stability parameters for the quiver Q_n, as usual, has a chamber structure. In particular, the notion of semistability corresponding to the parameter θ_c is the same which is obtained by letting θ vary in the cone

$$\Gamma_c = \left\{ \theta = (\theta_0, \theta_1) \in \mathbb{R}^2 | \theta_0 > 0, -\theta_0 < \theta_1 < -\frac{c-1}{c}\theta_0 \right\}. \tag{3.15}$$

Notice that Γ_c is actually a chamber, meaning that one can define strictly semistable representations for parameters on the walls

$$R_1 \;=\; \left\{\theta = (\theta_0, \theta_1) \in \mathbb{R}^2 | \theta_0 > 0, \theta_0 + \theta_1 = 0\right\}$$

and

$$R_2 \;=\; \left\{\theta = (\theta_0, \theta_1) \in \mathbb{R}^2 | \theta_0 > 0, (c-1)\theta_0 + c\theta_1 = 0\right\}.$$

We denote by $\mathrm{Rep}(\mathcal{B}_n, \mathbf{v})^s_\Theta$ the space of representations of the algebra $\mathcal{B}_n$ with dimension vector $\mathbf{v}$, stable with respect to the stability parameter Θ. The main theorem in [2] and Theorem 3.8 in [3] yield

Theorem 3.2.5. *For every $n \geq 1$ and $c \geq 1$ the Hilbert scheme $\mathrm{Hilb}^c(\Xi_n)$ is isomorphic to the GIT quotient*

$$\mathrm{Rep}(\mathcal{B}_n, \mathbf{v})^{\mathrm{fr},s}_\Theta /\!\!/_\Theta \mathrm{GL}_c(\mathbb{C}) \times \mathrm{GL}_c(\mathbb{C}).$$

Chapter 4

Framed flags on Hirzebruch surfaces and nested Hilbert schemes

The main goal of this chapter is to prove the following theorem:

Theorem. *Let* $\mathcal{M}^{n,\mathrm{fr},s}_{\mathbf{v},\Theta}$ *be the moduli space of framed representations of the quiver* Q^{enh}_n *with dimension vector* $\mathbf{v} = (c, c, c - c', c - c', 1)$*, stable with respect to the stability parameter* $\Theta = (\theta_1, \theta_2, \theta_3, \theta_4)$*, where*

$$\theta_1 > 0, \quad \theta_3, \theta_4 < 0, \quad \theta_1 + \theta_2 + (\theta_3 + \theta_4)(c - c') > 0, \quad -\theta_1 < \theta_2 < -\frac{c-1}{c}\theta_1.$$

$\mathcal{M}^{n,\mathrm{fr},s}_{\mathbf{v},\Theta}$ *is isomorphic to the nested Hilbert scheme* $\mathrm{Hilb}^{c',c}(\Xi_n)$.

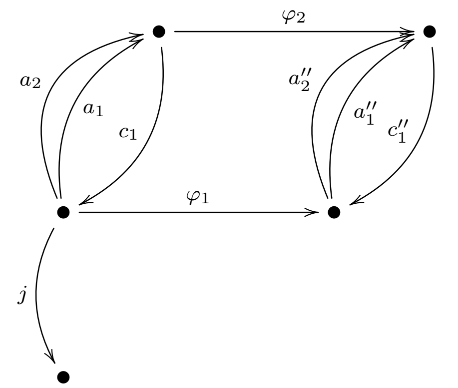
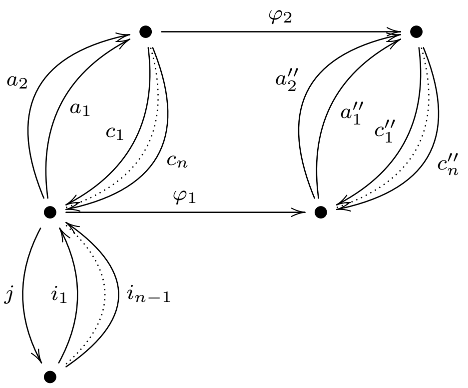
This result establishes a relation between two, at first, unrelated worlds: the quiver world and the framed flags world. In the first section, we define and characterize the $\underline{\theta}$-stable framed representations of the quiver Q^{enh}_n. The second is dedicated to the "absolute" case of our main result, which means that there we prove a crucial part of the isomorphism aforementioned. In the third section, we recall the notion of framed flags on Hirzebruch surfaces and after that we discuss the categorical approach of the isomorphism in Theorem 3.2.5 and finally we set the categorical approach for the nested case, which allow us to finalize the proof of our main result.

4.1 Stability for the quiver Q^{enh}_n

The goal of this section is to define and characterize the stable representations of the quiver Q^{enh}_n, fitting in some essential properties.

4.1.1 Representations of the quiver Q_n^{enh}

We start recalling the picture of the quiver Q_n^{enh}.

<table>
<tr><th>Values of n</th><th>Quiver Q_n^{enh}</th></tr>
<tr><td>$n = 1$</td><td>
</td></tr>
<tr><td>$n \geq 2$</td><td>
</td></tr>
</table>

If we fix a vector $\mathbf{v} = (v_1, v_2, v_1'', v_2'', r) \in \mathbb{N}_0^5$. A representation X of the quiver Q_n^{enh} with dimension vector $\mathbf{v}$ can be written as

$$X = (V_1, V_2, V_1'', V_2'', W, f_1, f_2, g_1, \ldots, g_n; f_1'', f_2'', g_1'', \ldots, g_n'', \ell, h_1, \ldots, h_{n-1}, F_1, F_2),$$

where:

- V_1, V_2, V_1'', V_2'' and W are $\mathbb{C}$-vector spaces such that

$$\dim(V_1) = v_1,$$

$$\dim(V_2) = v_2,$$

$$\dim(V_1'') = v_1'',$$

$$\dim(V_2'') = v_2'',$$

$$\dim(W) = r.$$

- $f_1, f_2 \in \mathrm{Hom}(V_1, V_2)$; $g_1, \ldots, g_n \in \mathrm{Hom}(V_2, V_1)$; $f_1'', f_2'' \in \mathrm{Hom}(V_1'', V_2')$; $g_1'', \ldots, g_n'' \in \mathrm{Hom}(V_2'', V_1'')$; $\ell \in \mathrm{Hom}(V_1, W)$; $h_1, \ldots, h_{n-1} \in \mathrm{Hom}(W, V_1)$; $F_1 \in \mathrm{Hom}(V_1, V_1'')$; $F_2 \in \mathrm{Hom}(V_2, V_2'')$.
- The relations of Q_n^{enh} force these maps to satisfy:

$$f_1 g_1 f_2 = f_2 g_1 f_1, \;\; f_1'' g_1'' f_2'' = f_2'' g_1'' f_1'',$$
$$F_1 g_1 = g_1'' F_2, \;\; F_2 f_1 = f_1'' F_1,$$
$$F_2 f_2 = f_2'' F_1,$$

for $n = 1$.

$$f_1 g_1 = f_2 g_2, \;\; g_1 f_1 + h_1 \ell = g_2 f_2, \;\; F_1 h_1 = 0,$$
$$F_2 f_1 = f_1'' F_1, \;\; F_2 f_2 = f_2'' F_1, \;\; F_1 g_1 = g_1'' F_2,$$
$$F_1 g_2 = g_2'' F_2, \;\; f_1'' g_1'' = f_2'' g_2'', \;\; g_1'' f_1'' = g_2'' f_2'',$$

for $n = 2$. And for $n \geq 3$, they must satisfy

$$f_1 g_t = f_2 g_{t+1}, \; t \in \{1, \ldots, n-1\}; \;\; g_t f_1 + h_t \ell = g_{t+1} f_2, \; t \in \{1, \ldots, n-1\}$$
$$f_1'' g_t'' = f_2'' g_{t+1}'', \; t \in \{1, \ldots, n-1\}; \;\; g_t'' f_1'' = g_{t+1}'' f_2'', \; t \in \{1, \ldots, n-1\}$$
$$F_2 f_t = f_t'' F_1, \; t \in \{1, 2\}; \;\; F_1 g_t = g_t'' F_2, \; t \in \{1, \ldots, n\}$$
$$F_1 h_t = 0, \; t \in \{1, \ldots, n-1\}$$

Remark 4.1.1. For $n = 1$ it will be understood that there are no arrows h_t.

4.1.2 Characterizing stable representations of the quiver Q_n^{enh}

Based on Subsection 1.3.1, we can express the stability condition for as follows. Consider a stability parameter $\underline{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4) \in \mathbb{R}^4$ Note that the representation X is $\underline{\theta}$-semistable if the following conditions hold.

(1) For all subrepresentations $S = (S_1, S_2, S_1'', S_2'')$ such that $S_1 \subseteq \ker(\ell)$, one has

$$\underline{\theta} \cdot \dim(S) := \theta_1 s_1 + \theta_2 s_2 + \theta_3 s_1'' + \theta_4 s_2'' \leq 0.$$

(2) For all subrepresentations $S = (S_1, S_2, S_1'', S_2'')$ such that $S_1 \supseteq \mathrm{Im}(h_i)$, for $i \in \{1, \ldots, n-1\}$, one has

$$\underline{\theta} \cdot \dim(S) \leq \underline{\theta} \cdot \dim(X).$$

X is $\underline{\theta}$-stable if the inequalities are strict for $0 \neq S \subsetneq X$. $\triangle$

Now, we want to prove a very important lemma that allow us to characterize the stable representations of the quiver Q_n^{enh} and it is very useful to the proof of the main theorem of this chapter.

Lemma 4.1.2 (Characterization Lemma)**.** *Let*

$$X = (V_1, V_2, V_1'', V_2'', W, f_1, f_2, g_1, \dots, g_n, f_1'', f_2'', g_1'', \dots, g_n'', \ell, h_1, \dots, h_{n-1}, F_1, F_2)$$

be a representation with dimension vector $(c, c, c - c', c - c', 1)$ *of* Q_n^{enh} *and consider* $\underline{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4) \in \mathbb{R}^4$*. Suppose that*

- $\theta_1 > 0,\ \theta_3,\ \theta_4 < 0$;
- $\theta_1 + \theta_2 + (\theta_3 + \theta_4)(c - c') > 0$;
- $(\theta_1, \theta_2) \in \Gamma_c$*, that is,* $-\theta_1 < \theta_2 < -\dfrac{c-1}{c}\theta_1$.

Then the following statements are equivalent:

(i) X *is* $\underline{\theta}$*-stable;*

(ii) X *is* $\underline{\theta}$*-semistable;*

(iii) X *satisfies the following conditions:*

(C1) $F_1 \in \mathrm{Hom}(V_1, V_1')$ *and* $F_2 \in \mathrm{Hom}(V_2, V_2')$ *are surjective;*

(C2) $\overline{X} := (V_1, V_2, W, f_1, f_2, g_1, \dots, g_{,}\ell, h_1, \dots, h_{n-1})$ *is a* $\theta_c = (2c, 1 - 2c)$*-stable representation of the quiver* Q_n*.*

Proof. If X is $\underline{\theta}$-stable, then X is obviously $\underline{\theta}$-semistable. We assume that X is $\underline{\theta}$-semistable in order to prove (C1) and (C2). Notice that $\widetilde{X} := (\mathbb{C}^c, \mathbb{C}^c, \mathrm{Im}(F_1), \mathrm{Im}(F_2))$ is a subrepresentation of X such that $\mathbb{C}^c = S_1 \supseteq \mathrm{Im}(h_i)$, for all $i \in \{1, \dots, n-1\}$. By the $\underline{\theta}$-semistability of X, we obtain:

$$\theta_1 c + \theta_2 c + \theta_3 \dim(\mathrm{Im}(F_1)) + \theta_4 \dim(\mathrm{Im}(F_2)) \leq \theta_1 c + \theta_2 c + \theta_3 (c - c') + \theta_4 (c - c').$$

This inequality turns out to be

$$\theta_3 \dim(\mathrm{Im}(F_1)) + \theta_4 \dim(\mathrm{Im}(F_2)) \leq \theta_3 (c - c') + \theta_4 (c - c').$$

Since we have $\mathrm{Im}(F_i) \subseteq \mathbb{C}^{c-c'}$ for $i = 1, 2$ and θ_3 and θ_4 are less than zero, we can also obtain

$$\theta_3 \dim(\mathrm{Im}(F_1)) + \theta_4 \dim(\mathrm{Im}(F_2)) \geq \theta_3 (c - c') + \theta_4 (c - c').$$

By combining these inequalities, we actually have the following:

$$\theta_3 \dim(\mathrm{Im}(F_1)) + \theta_4 \dim(\mathrm{Im}(F_2)) = \theta_3(c - c') + \theta_4(c - c').$$

Consequently, we must have

$$\dim(\mathrm{Im}(F_1)) = \dim(\mathrm{Im}(F_2)) = c - c'.$$

In other words, F_1 and F_2 are surjective. Now, we want to prove $(C2)$ still under the hyphotesis that X is $\underline{\theta}$-semistable. Consider a nonzero subrepresentation $S = (S_1, S_2)$ of $\overline{X}$ such that $S_1 \subseteq \ker(\ell)$. Then we have that

$$\widetilde{X} := (S_1, S_2, \mathbb{C}^{c-c'}, \mathbb{C}^{c-c'})$$

is a subrepresentation of X such that $S_1 \subseteq \ker(\ell)$. By the $\underline{\theta}$-semistability of X we have

$$\theta_1 \dim(S_1) + \theta_2 \dim(S_2) + (\theta_3 + \theta_4)(c - c') = \theta_1 s_1 + \theta_2 s_2 + (\theta_3 + \theta_4)(c - c') \leq 0.$$

If $s_1 \geq s_2$, we have that $\theta_2 s_2 \geq \theta_2 s_1$, since $\theta_2 < 0$. Thus,

$$\begin{aligned}
\theta_1 s_1 + \theta_2 s_2 + (\theta_3 + \theta_4)(c - c') &\geq \theta_1 s_1 + \theta_2 s_1 + (\theta_3 + \theta_4)(c - c') \\
&= (\theta_1 + \theta_2) s_1 + (\theta_3 + \theta_4)(c - c') \\
&\geq \theta_1 + \theta_2 + (\theta_3 + \theta_4)(c - c') \\
&> 0,
\end{aligned}$$

since we can assume $s_1 \geq 1$, because $s_1 = 0$ implies $s_2 = 0$ and this means $S = 0$. So, $s_1 < s_2$, as wanted.
On the other hand, suppose that $S = (S_1, S_2)$ is proper and $S_1 \supseteq \mathrm{Im}(h_t)$ $\forall t \in \{1, \ldots, n-1\}$. As before, we can consider $\widetilde{X} := (S_1, S_2, \mathbb{C}^{c-c'}, \mathbb{C}^{c-c'})$ as a representation of X, since we have the relations between the maps inherit from the path algebra. Using the $\underline{\theta}$-semistability of X one more time, we obtain the following inequality:

$$\theta_1 s_1 + \theta_2 s_2 + (\theta_3 + \theta_4)(c - c') \leq (\theta_1 + \theta_2)c + (\theta_3 + \theta_4)(c - c').$$

Then,

$$\theta_1 s_1 + \theta_2 s_2 \leq (\theta_1 + \theta_2)c.$$

As $(\theta_1, \theta_2) \in \Gamma_c$, we know that

$$\theta_1 + \theta_2 < \frac{\theta_1}{c}.$$

Then, we conclude that

$$\theta_1 s_1 + \theta_2 s_2 < \theta_1 \iff \theta_1(s_1 - 1) + \theta_2 s_2 < 0.$$

If $s_1 > s_2$, which means that $s_2 \leq s_1 - 1$, we obtain:

$$(\theta_1 + \theta_2) s_2 \leq \theta_1(s_1 - 1) + \theta_2 s_2 < 0.$$

However, this cannot happen, since $\theta_1 + \theta_2 > 0$ and $s_2 \geq 0$. So, $s_1 \leq s_2$, as wanted.

Finally, we want to prove that (iii) implies (i). For this, consider a nonzero representation $\widetilde{X} = (S_1, S_2, S_1'', S_2'')$ of X, such that $S_1 \subseteq \ker(\ell)$. If (S_1, S_2) is a nonzero subrepresentation of $\overline{X}$, we obtain that $s_1 < s_2$. As we are under the hyphotesis that $(\theta_1, \theta_2) \in \Gamma_c$, we see that

$$\frac{c-1}{c} < -\frac{\theta_2}{\theta_1} < 1.$$

However, since $s_1 < s_2 \leq c$, we can assume $s_1 \leq c-1$ and $s_2 \leq c$. In such a way that

$$\frac{s_1}{s_2} \leq \frac{c-1}{c} < -\frac{\theta_2}{\theta_1}.$$

Consequently, we must have

$$\theta_1 s_1 + \theta_2 s_2 < 0. \tag{4.1}$$

As θ_3 and θ_4 are both negative, one has

$$\theta_3 s_1'' + \theta_4 s_2'' \leq 0. \tag{4.2}$$

By adding (4.1) and (4.2) we obtain the desired inequality. In the case where $s_1 = s_2 = 0$, it must hold that $s_i'' > 0$ for some $i \in \{1, 2\}$. By using one more time the fact that θ_3 and θ_4 are both negative, it is valid that

$$\theta_1 s_1 + \theta_2 s_2 + \theta_3 s_1'' + \theta_4 s_2'' = \theta_3 s_1'' + \theta_4 s_2'' < 0.$$

So, in both cases, we conclude that

$$\theta_1 s_1 + \theta_2 s_2 + \theta_3 s_1'' + \theta_4 s_2'' < 0.$$

Since we have the (θ_1, θ_2)-stability for $\overline{X}$, the maps $h_1, \ldots, h_{n-1}$ are zero, therefore we have to prove that for any proper subrepresentation, we have

$$\underline{\theta} \cdot (s_1, s_2, s_1'', s_2'') < \underline{\theta} \cdot (c, c, c - c', c - c').$$

Using one more time the condition $(C2)$, we obtain that $s_1 \leq s_2$ and then $\theta_2 s_2 \leq \theta_2 s_1$, since $\theta_2 < 0$. Therefore,

$$\theta_1 s_1 + \theta_2 s_2 + \theta_3 s_1'' + \theta_4 s_2'' \leq (\theta_1 + \theta_2) s_1 + \theta_3 s_1'' + \theta_4 s_2''.$$

If $s_1 = c$, we also have $s_2 = c$ and then we would have $S = X$, since F_1 and F_2 are surjective. Thus, we can assume that $s_1 \leq c - 1$. So,

$$\theta_1 s_1 + \theta_2 s_2 + \theta_3 s_1'' + \theta_4 s_2'' \leq (\theta_1 + \theta_2)(c-1) + \theta_3 s_1'' + \theta_4 s_2''. \tag{4.3}$$

We already know that

$$\theta_1 + \theta_2 + (\theta_3 + \theta_4)(c - c') > 0.$$

Thus

$$\theta_3 s_1'' + \theta_4 s_2'' < \theta_1 + \theta_2 + (\theta_3 + \theta_4)(c - c'),$$

since $\theta_3 s_1'' + \theta_4 s_2'' \leq 0$. By adding $(\theta_1 + \theta_2)c$ to both sides of this inequality, we get

$$(\theta_1 + \theta_2)(c - 1) + \theta_3 s_1'' + \theta_4 s_2'' < (\theta_1 + \theta_2)c + (\theta_3 + \theta_4)(c - c'). \tag{4.4}$$

By combining (4.3) and (4.4), we obtain the result:

$$\theta_1 s_1 + \theta_2 s_2 + \theta_3 s_1'' + \theta_4 s_2'' < (\theta_1 + \theta_2)c + (\theta_3 + \theta_4)(c - c').$$

□

Lemma 4.1.3. *Given $\underline{\theta}$ as in Lemma 4.1.2 and a $\underline{\theta}$-stable representation X of numerical type $(c, c, c - c', c - c', 1)$ of Q_n^{enh}, we can construct a $\theta_{c'}$-stable representation $\overline{\overline{X}}$ of Q_n of numerical type $(c', c', 1)$ in the following way:*

$$\overline{\overline{X}} := (\overline{V_1}, \overline{V_2}, W, f_1', f_2', g_1', \ldots, g_n', \ell', h_1', \ldots, h_{n-1}'),$$

where $V_1' = \ker(F_1)$, $V_2' = \ker(F_2)$ and the maps $f_1', f_2' \in \operatorname{Hom}(V_1', V_2')$, $g_1', \ldots, g_n' \in \operatorname{Hom}(V_2', V_1')$, $\ell' \in \operatorname{Hom}(V_1', W)$ and $h_1', \ldots, h_{n-1}' \in \operatorname{Hom}(W, V_2')$ are defined by:

$$\begin{aligned} f_i' &= f_i|_{\ker(F_1)}, \quad \textit{for } i = 1, 2; \\ g_i' &= g_i|_{\ker(F_2)}, \quad \textit{for } i = 1, \ldots, n; \\ h_i' &= h_i, \quad \textit{for } i = 1, \ldots, n-1; \\ \ell' &= \ell|_{\ker(F_1)}. \end{aligned}$$

Proof. We start by noticing that these maps are well defined. In fact, we have that $\operatorname{Im}(f_i') \subseteq \ker(F_2)$ and $\operatorname{Im}(g_i') \subseteq \ker(F_1)$, since $F_2 f_i = f_i'' F_1$ for $i \in \{1, 2\}$ and $F_1 g_i = g_i'' F_2$ for $i \in \{1, \ldots, n\}$. In addition, they also satisfy all the necessary relations in a straightforward way, for instance: take $x \in \ker(F_2)$ and observe that

$$f_1'(g_i'(x)) = f_1'(g_1(x)) = f_1(g_1(x)) = f_2(g_{i+1}(x)) = f_2'(g_{i+1}'(x)).$$

Hence, $f_1' g_i' = f_2' g_{i+1}'$ and the other relations can be proved in a completely analogous way. For the $\theta_{c'}$-stability, consider a subrepresentation $S = (S_1, S_2)$ of $\overline{\overline{X}}$ such that $S_1 \subseteq \ker(\ell')$, one can see S as a subrepresentation of $\overline{X}$ (notation of Lemma 4.1.2), since we have the natural inclusions $i_1 : V_1' = \ker(F_1) \to V_1$ and $i_2 : V_2' = \ker(F_2) \to V_2$, and we also know that

$$S_1 \subseteq \ker(\ell') = \ker(\ell|_{\ker(F_1)}) = \ker(\ell) \cap \ker(F_1) \subseteq \ker(\ell).$$

The θ_c-stability of $\overline{X}$ implies that $\dim(S_1) := s_1 < s_2 =: \dim(S_2)$ or $s_1 = s_2 = 0$. In a completely analogous way, we use that $h_i' = h_i$ to show that if $S = (S_1, S_2)$ is a subrepresentation of $\overline{\overline{X}}$ such that $S_1 \supseteq \operatorname{Im}(h_i') = \operatorname{Im}(h_i)$, it holds that $s_1 \leq s_2$, by using the θ_c-stability of $\overline{X}$ one more time. Therefore, $\overline{\overline{X}}$ is $\theta_{c'}$-stable by Lemma 3.2.3. □

If we set

$$\mathbb{X} := \mathrm{Hom}(V_1, V_2)^{\oplus 2} \oplus \mathrm{Hom}(V_2, V_1)^{\oplus n} \oplus \mathrm{Hom}(W, V_1)^{\oplus(n-1)} \oplus \mathrm{Hom}(V_1, W) \oplus \mathrm{Hom}(V_1'', V_2'')^{\oplus 2} \oplus \mathrm{Hom}(V_2'', V_1'')^{\oplus n} \oplus \mathrm{Hom}(V_1, V_1'') \oplus \mathrm{Hom}(V_2, V_2''),$$

the group $G := \mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \times \mathrm{GL}(V_1'') \times \mathrm{GL}(V_2'')$ acts on $\mathbb{X}$ by the rule

$$(g_1, g_2, g_3, g_4) * (A_i, C_i, h_i, \ell, A_i'', C_i'', F_1, F_2) = (g_2 A_i g_1^{-1}, g_1 C_i g_2^{-1}, g_1 h_i, \ell g_1^{-1}, g_4 A_i'' g_3^{-1}, g_3 C_i'' g_4^{-1}, g_3 F_1 g_1^{-1}, g_4 F_2 g_2^{-1}). \quad (4.5)$$

Proposition 4.1.4. *The action* (4.5) *is free on the* $\underline{\theta}$*-stable points of* $\mathrm{Rep}(Q_n^{\mathrm{enh}}, \mathbf{v})$.

Proof. In fact, if $(g_1, g_2, g_3, g_4) * X = X$, one has the following equations:

$$g_2 A_i g_1^{-1} = A_i; \qquad g_1 C_i g_2^{-1} = C_i; \quad (4.6)$$

$$g_2 h_i = h_i; \qquad \ell g_1^{-1} = \ell; \quad (4.7)$$

$$g_4 A_i'' g_3^{-1} = A_i''; \qquad g_3 C_i'' g_4^{-1} = C_i''; \quad (4.8)$$

$$g_3 F_1 g_1^{-1} = F_1; \qquad g_4 F_2 g_2^{-1} = F_2. \quad (4.9)$$

Assume that X is $\underline{\theta}$-stable and notice that the equation $\ell g_1^{-1} - \ell = 0$ is equivalent to $\ell(g_1 - \mathrm{Id}_{V_1}) = 0$, which is equivalent to

$$\mathrm{Im}(g_1 - \mathrm{Id}_{V_1}) \subseteq \ker(\ell), \quad (4.10)$$

while $(g_2 - \mathrm{Id}_{V_2})h_i = 0$ correspond to

$$\mathrm{Im}(h_i) \subseteq \ker(g_2 - \mathrm{Id}_{V_2}). \quad (4.11)$$

By the Characterization Lemma, we have that F_1 and F_2 are surjective and the representation

$$\widetilde{X} = (A_1, A_2, C_1, \ldots, C_n, \ell, h_1, \ldots, h_{n-1})$$

of the quiver Q_n is $(2c, 1 - 2c)$-stable. We have that

$$(S_1, S_2) := (\mathrm{Im}(g_1 - \mathrm{Id}_{V_1}), \mathrm{Im}(g_2 - \mathrm{Id}_{V_2})) \subset (V_1, V_2)$$

is a subrepresentation of $\widetilde{X}$ that satisfies $S_1 \subseteq \ker(\ell)$. By Lemma 3.2.3, we must have $\dim(S_1) \leq \dim(S_2)$. In fact, we actually have that $\dim(S_1) = \dim(S_2)$, since $\dim(S_1) < \dim(S_2)$ implies that

$$\begin{aligned} \dim(\ker(g_2 - \mathrm{Id}_{V_2})) &= c - \dim(S_2) \\ &< c - \dim(S_1) \\ &= \dim(\ker(g_1 - \mathrm{Id}_{V_1})), \end{aligned}$$

which contradicts Lemma 3.2.3, since

$$(\overline{S_1}, \overline{S_2}) := (\ker(g_1 - \mathrm{Id}_{V_1}), \ker(g_2 - \mathrm{Id}_{V_2})) \subseteq (V_1, V_2)$$

is a subrepresentation of $\widetilde{X}$ satisfying $\mathrm{Im}(h_i) \subseteq \overline{S_1}$ for all $i \in \{1, \ldots, n-1\}$, so that we must have $\dim(\overline{S_1}) \leq \dim(\overline{S_2})$. Therefore, we have that $\dim(S_1) = \dim(S_2)$ and then $S_1 = S_2 = 0$, again by Lemma 3.2.3. This means that $g_1 = \mathrm{Id}_{V_1}$ and $g_2 = \mathrm{Id}_{V_2}$. By using (4.9) and the fact that F_1 and F_2 are surjective, we obtain that $g_3 = \mathrm{Id}_{V_3}$ and $g_4 = \mathrm{Id}_{V_4}$, as wanted. □

Lemma 4.1.5. *Let $\mathbb{X}_0 \subset \mathbb{X}$ be subscheme defined by the relations in* (1.4). *Then $\mathbb{X}_0$ is preserved by the G-action defined in Eq.* (4.5).

Proof. Consider $X = (A_i, C_i, h_i, \ell, A_i'', C_i'', F_1, F_2) \in \mathbb{X}_0$ and $(g_1, g_2, g_3, g_4) \in G$. One has that

$$\begin{aligned}(g_1, g_2, g_3, g_4) * X &= \\ &= (g_2 A_i g_1^{-1}, g_1 C_i g_2^{-1}, g_1 h_i, \ell g_1^{-1}, g_4 A_i'' g_3^{-1}, g_3 C_i'' g_4^{-1}, g_3 F_1 g_1^{-1}, g_4 F_2 g_2^{-1}).\end{aligned} \tag{4.12}$$

Since X satisfies (1.4), it follows that $(g_1, g_2, g_3, g_4) * X$ satisfies as well. For instance,

$$\begin{aligned}(g_3 F_1 g_1^{-1})(g_1 C_i g_2^{-1}) &= g_3 F_1 C_i g_2^{-1} \\ &\overset{(1.4)}{=} g_3 C_i'' F_2 g_2^{-1} \\ &= (g_3 C_i'' g_4^{-1})(g_4 F_2 g_2^{-1}).\end{aligned}$$

Thus, $(g_1, g_2, g_3, g_4) * X \in \mathbb{X}_0$. Since X is arbitrary, $\mathbb{X}_0$ is preserved by the G-action. □

Remark 4.1.6. Each representation $X = (A_i, C_i, h_i, \ell, A_i'', C_i'', F_1, F_2)$ of the quiver Q_n^{enh} corresponds to a point $X \in \mathbb{X}_0$. Furthermore, two framed representations X and Y are isomorphic if, and only if, the corresponding points in $\mathbb{X}_0$ are in the same orbit. In fact, one can write $X = g * Y$, for some $g = (g_1, g_2, g_3, g_4) \in G$ and g straightforwardly defines the isomorphism $X \xrightarrow{\sim} Y$. Conversely, one just builds g with the corresponding maps out of the isomorphism in hands.

4.2 Framed flags on Hirzebruch surfaces

Here, we recall the notion explored in Section 1.4 to remind the reader of the concept and to fix a better notation for the functor of framed flags on Σ_n. With the notation of Section 1.4, we are considering $X = \Sigma_n$, $D = \ell_\infty$ and framed flags of length 1 and type $(1, 0, c', c - c')$, where $c > c' \in \mathbb{N}$, on (Σ_n, ℓ_∞). A framed flag with these properties is a triple (E, F, φ), where

- E and F are torsion-free sheaves on Σ_n, with $E \subset F$ and $1 = \mathrm{rk}(E) = \mathrm{rk}(F)$;

- the schematic support of F/E is 0-dimensional and does not intersect ℓ_∞;

- φ is an isomorphism between $F\Big|_{\ell_\infty}$ and $\mathcal{O}_{\ell_\infty}$;

- $c_1(F) = 0$; $c_2(F) = c'$;

- $c - c' = c_2(E) - c_2(F) = h^0(\Sigma_n, F/E) = \dim\left(H^0(\Sigma_n, F/E)\right)$.

Remark 4.2.1. φ also provides an isomorphism $E\Big|_{\ell_\infty} \simeq \mathcal{O}_{\ell_\infty}$ and it holds that $c_1(E) = 0$.

The functor $\mathfrak{F}^{\Sigma_n,\ell_\infty}_{1,0,c',c-c'}$ of families of framed flags of length 1 on (Σ_n, ℓ_∞) is defined, for $T \in \mathrm{Sch}$, as

$$\mathfrak{F}^{\Sigma_n,\ell_\infty}_{1,0,c',c-c'}(T) = \{\text{isomorphism classes of triples } (E, F, \varphi)\},$$

where

- E and F are rank 1 torsion-free sheaves on $\Sigma_n \times T$, flat on T, with $E \subset F$;

- for all closed points $t \in T$, $c_2(F_t) = c'$, $c_2(E_t) - c_2(F_t) = c - c'$ and $c_1(F_t) = 0$;

- for all closed points $t \in T$, the schematic support of $(F/E)_t$ is 0-dimensional and does not intersect $\{t\} \times \ell_\infty$;

- φ is an isomorphism between $F\Big|_{\ell_\infty \times T}$ and $\mathcal{O}_{\ell_\infty \times T}$.

Remark 4.2.2. Remember that morphisms of framed flags are defined in the obvious way and the functor acts on scheme morphisms by pullback.

We denote the functor $\mathfrak{F}^{\Sigma_n,\ell_\infty}_{1,0,c',c-c'}$ by $\mathfrak{F}_{c',c}$ to simplify the notation. By Theorem 1.4.1 $\mathfrak{F}_{c',c}$ is representable by a scheme that we will denote by $\mathcal{F}_{c',c}$.

4.3 The main result: a key point

Theorem 4.3.1. *There is a set-theoretical bijection between the space of framed representations of the quiver Q_n^{enh} that are stable as in Lemma 4.1.2 and the nested Hilbert scheme of points on Ξ_n (and the moduli space of framed flags of sheaves on Σ_n as well)*

$$\Psi : \text{Rep}^{\text{st}}(Q_n^{\text{enh}}, (c, c, c-c', c-c', 1)) \to \text{Hilb}^{c',c}(\Xi_n).$$

The proof of this result is long and requires some intermediate steps. We start by considering an element in $\text{Rep}^{\text{st}}(Q_n^{\text{enh}}, (c, c, c-c', c-c', 1))$, where c and c' are positive integers and $c' < c$. One can write

$$X = (V_1, V_2, V_1'', V_2'', W, f_1, f_2, g_1, \ldots, g_n, f_1'', f_2'', g_1'', \ldots, g_n'', \ell, h_1, \ldots, h_{n-1}, F_1, F_2).$$

By Lemma 4.1.2, the maps F_1 and F_2 are surjective and the datum referring to the quiver Q_n is stable. The maps F_1 and F_2 can be seen as a morphism of representations of the quiver Q_n and therefore the Lemma 4.1.3 actually produces a short exact sequence in the category of representations of the quiver Q_n that is given by:

$$0 \to X_{c'} \xrightarrow{i} X_c \xrightarrow{F} X_{c-c'} \to 0,$$

where $X_{c'}$ is $\theta_{c'}$-stable, X_c is θ_c-stable and the maps i and F are given by

$$i = (i_1, i_2, \text{Id}_W) \quad \text{and} \quad F = (F_1, F_2, \mathbf{0}).$$

When we consider the open cover of $\text{Hilb}^c(\Xi_n)$ given by the open sets exhibited in (3.6), namely: (U_m^{nc}), $m = 0, \ldots, c$, we can provide an open cover $\left(U_j^{nc'}\right)$ $j = 0, \ldots, c$ to $\text{Hilb}^{c'}(\Xi_n)$. In fact, as X_c is θ_c-stable, there exists an $m \in \{0, 1, \ldots, c\}$ such that $X_c \in U_m^{nc}$ and then the matrix

$$A_{2m} := s_m f_1 + c_m f_2, \quad \text{where } s_m = \sin\left(\tfrac{m\pi}{c+1}\right) \text{ and } c_m = \cos\left(\tfrac{m\pi}{c+1}\right) \tag{4.13}$$

is invertible by [2, Proposition 3.3]. One has

$$A'_{2m} := s_m f'_1 + c_m f'_2 = s_m f_1|_{\ker(F_1)} + c_m f_2|_{\ker(F_1)} = A_{2m}|_{\ker(F_1)}; \tag{4.14}$$

then A'_{2m} is invertible as well and we obtain an open cover of $\text{Hilb}^{c'}(\Xi_n)$ with $c+1$ open sets, instead of $c'+1$ which is the cardinality of the one proposed by Bartocci, Bruzzo, Lanza and Rava in [2]. Moreover, one also has that

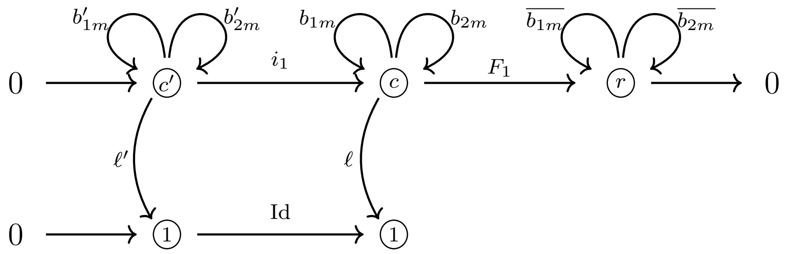
$$U_j^{nc'} \simeq \text{Hilb}^{c'}(\mathbb{C}^2),\ \forall j \in \{0, \ldots, c\},\ \text{and} \tag{4.15}$$

$$U_m^{nc} \simeq \text{Hilb}^{c}(\mathbb{C}^2),\ \forall m \in \{0, \ldots, c\}. \tag{4.16}$$

By [2, Proposition 3.3], we can build a short exact sequence on the category of representations of the transposed ADHM quiver:

$$0 \to X_{c'}^{\mathrm{ADHM}} \to X_{c}^{\mathrm{ADHM}} \to X_{c-c'}^{\mathrm{ADHM}} \to 0. \tag{4.17}$$

In this case, $X_{c'}^{\mathrm{ADHM}}$ and X_{c}^{ADHM} are co-stable in the sense of Nakajima. In other words, one can write:



(4.18)

where $r = c - c'$ and the endomorphisms $\overline{b_{1m}}$ and $\overline{b_{2m}}$ of the quotient vector space are straightforwardly defined in the following lemma.

Lemma 4.3.2. *One can see that the maps*

$$\overline{b_{jm}}(\overline{x}) := \overline{b_{jm}(x)}, \quad j = 1, 2$$

are well defined and $[\overline{b_{1m}}, \overline{b_{2m}}] = 0$.

Proof. In fact, if $\overline{x} = \overline{y}$, then $x - y \in \mathrm{Im}(f)$, i.e. $x - y = f(z)$. Therefore,

$$b_{jm}(x) - b_{jm}(y) = b_{jm}(x - y) = b_{jm}(f(z)) = f(b'_{jm}(z)) \in \mathrm{Im}(f),$$

and this means that

$$\overline{b_{jm}}(\overline{x}) = \overline{b_{jm}(x)} = \overline{b_{jm}(y)} = \overline{b_{jm}}(\overline{y}).$$

On the other hand, $[\overline{b_{1m}}, \overline{b_{2m}}] = 0$ follows from $[b_{1m}, b_{2m}] = 0$. □

Consider an element $X = (b_1, b_2, e) \in \mathrm{End}(\mathbb{C}^c)^{\oplus 2} \oplus \mathrm{Hom}(\mathbb{C}^c, \mathbb{C})$. The complex on Σ_n associated with X is given by [2, p. 2151] and we write it here again:

$$M_X : \quad 0 \to \mathcal{O}_{\Sigma_n}(0,-1)^{\oplus c} \xrightarrow{\alpha} \mathcal{O}_{\Sigma_n}(1,-1)^{\oplus c} \oplus \mathcal{O}_{\Sigma_n}^{\oplus(c+1)} \xrightarrow{\beta} \mathcal{O}_{\Sigma_n}(1,0)^{\oplus c} \to 0, \tag{4.19}$$

where

$$\alpha = \begin{bmatrix} \mathrm{Id}_c(y_{2m}^n s_{\mathfrak{e}}) + {}^t b_2 s_\infty \\ \mathrm{Id}_c y_{1m} + {}^t b_1 y_{2m} \\ 0 \end{bmatrix} \tag{4.20}$$

and

$$\beta = \left[\begin{array}{ccc} \mathrm{Id}_c y_{1m} + {}^t b_1 y_{2m}, & -\left(\mathrm{Id}_c(y_{2m}^n s_{\mathfrak{e}}) + {}^t b_2 s_\infty\right), & {}^t e s_\infty \end{array}\right]. \tag{4.21}$$

Moreover, given a morphism of ADHM representations $\varphi : X \to \widetilde{X}$, the corresponding morphism of complexes is given by the vertical arrows in the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\Sigma_n}(0,-1)^{\oplus \widetilde{c}} & \xrightarrow{\widetilde{\alpha}} & \mathcal{O}_{\Sigma_n}(1,-1)^{\oplus \widetilde{c}} \oplus \mathcal{O}_{\Sigma_n}^{\oplus(\widetilde{c}+1)} & \xrightarrow{\widetilde{\beta}} & \mathcal{O}_{\Sigma_n}(1,0)^{\oplus \widetilde{c}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_{\mathcal{U}}} & & \downarrow{\scriptstyle \varphi_{\mathcal{V}}} & & \downarrow{\scriptstyle \varphi_{\mathcal{W}}} & & \\ 0 & \longrightarrow & \mathcal{O}_{\Sigma_n}(0,-1)^{\oplus c} & \xrightarrow{\alpha} & \mathcal{O}_{\Sigma_n}(1,-1)^{\oplus c} \oplus \mathcal{O}_{\Sigma_n}^{\oplus(c+1)} & \xrightarrow{\beta} & \mathcal{O}_{\Sigma_n}(1,0)^{\oplus c} & \longrightarrow & 0 \end{array} \tag{4.22}$$

Taking into account the structure of the maps α, β, $\widetilde{\alpha}$ and $\widetilde{\beta}$, it will be good to express the vertical maps in a similar structure. In order to do that, we note that:

$$\begin{aligned} \mathrm{Hom}\left(\mathcal{O}_{\Sigma_n}(0,-1)^{\oplus\widetilde{c}}, \mathcal{O}_{\Sigma_n}(0,-1)^{\oplus c}\right) &\simeq \mathrm{Hom}\left(\mathcal{O}_{\Sigma_n}, \mathcal{O}_{\Sigma_n}(0,1)^{\oplus\widetilde{c}} \otimes \mathcal{O}_{\Sigma_n}(0,-1)^{\oplus c}\right) \\ &\simeq \mathrm{Mat}(\widetilde{c} \times c, \mathbb{C}), \end{aligned}$$

i.e., the map $\varphi_{\mathcal{U}}$ can be seen as a $\widetilde{c} \times c$ matrix. In a complete similar way, the morphism $\varphi_{\mathcal{W}}$ can be seen as a $\widetilde{c} \times c$ matrix. When we try to do the same process for the map $\varphi_{\mathcal{V}}$, it is a litte bit more complicated, since this map does not have a matrix structure. However, we can write $\varphi_{\mathcal{V}}$ as a matrix in blocks. The morphism $\varphi_{\mathcal{V}}$ can be written as:

$$\varphi_{\mathcal{V}} = \left[\begin{array}{cc} \varphi_{\mathcal{V}}^{11} & \varphi_{\mathcal{V}}^{12} \\ \varphi_{\mathcal{V}}^{21} & \varphi_{\mathcal{V}}^{22} \end{array}\right],$$

where:

$$\begin{aligned} \varphi_{\mathcal{V}}^{11} &\in \mathrm{Hom}\left(\mathcal{O}_{\Sigma_n}(1,-1)^{\oplus\widetilde{c}}, \mathcal{O}_{\Sigma_n}(1,-1)^{\oplus c}\right); \\ \varphi_{\mathcal{V}}^{12} &\in \mathrm{Hom}\left(\mathcal{O}_{\Sigma_n}(1,-1)^{\oplus\widetilde{c}}, \mathcal{O}_{\Sigma_n}^{\oplus(c+1)}\right); \\ \varphi_{\mathcal{V}}^{21} &\in \mathrm{Hom}\left(\mathcal{O}_{\Sigma_n}^{\oplus(\widetilde{c}+1)}, \mathcal{O}_{\Sigma_n}(1,-1)^{\oplus c}\right); \\ \varphi_{\mathcal{V}}^{22} &\in \mathrm{Hom}\left(\mathcal{O}_{\Sigma_n}^{\oplus(\widetilde{c}+1)}, \mathcal{O}_{\Sigma_n}^{\oplus(c+1)}\right). \end{aligned}$$

Now, observe that

$$\begin{aligned} \mathrm{Hom}\left(\mathcal{O}_{\Sigma_n}^{\oplus(\widetilde{c}+1)}, \mathcal{O}_{\Sigma_n}(1,-1)^{\oplus c}\right) &= \mathrm{Hom}\left(\mathcal{O}_{\Sigma_n}, \mathcal{O}_{\Sigma_n}(1,-1)\right)^{\oplus(\widetilde{c}+1)c} \\ &= \mathrm{Hom}\left(\mathcal{O}_{\Sigma_n}(-1,1), \mathcal{O}_{\Sigma_n}\right)^{\oplus(\widetilde{c}+1)c} \\ &= \left[H^0(\mathcal{O}_{\Sigma_n}(-1,1))\right]^{\oplus(\widetilde{c}+1)c} \end{aligned}$$

and $H^0(\mathcal{O}_{\Sigma_n}(-1,1))$ is zero, from [4, Lemma 3.1], so that $\varphi_{\mathcal{V}}^{21} = 0$. As before, one can see that $\varphi_{\mathcal{V}}^{11} \in \mathrm{Mat}(\widetilde{c} \times c, \mathbb{C})$ and $\varphi_{\mathcal{V}}^{22} \in \mathrm{Mat}((\widetilde{c}+1) \times (c+1), \mathbb{C})$.
We make the choice $\varphi_{\mathcal{V}}^{12} = 0$, so that we can write

$$\varphi_{\mathcal{V}} = \begin{bmatrix} P & 0 \\ 0 & \overline{T} \end{bmatrix},$$

where $P \in \mathrm{Mat}(\widetilde{c} \times c, \mathbb{C})$ and $\overline{T} \in \mathrm{Mat}((\widetilde{c}+1) \times (c+1), \mathbb{C})$. We also specialize the choice of $\overline{T}$ by writing

$$\varphi_{\mathcal{V}} = \begin{bmatrix} P & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $P \in \mathrm{Mat}(\widetilde{c} \times c, \mathbb{C})$ and $T \in \mathrm{Mat}(\widetilde{c} \times c, \mathbb{C})$. For the diagram (4.22) to commute, we must look for matrices P and T that satisfy:

$$\alpha \circ \varphi_{\mathcal{U}} = \varphi_{\mathcal{V}} \circ \widetilde{\alpha} \tag{4.23}$$

$$\beta \circ \varphi_{\mathcal{V}} = \varphi_{\mathcal{W}} \circ \widetilde{\beta} \tag{4.24}$$

Concerning the equation (4.23), we have:

$$\varphi_{\mathcal{V}} \circ \widetilde{\alpha} = \begin{bmatrix} P & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \mathrm{Id}_{\widetilde{c}}(y_{2m}^n s_{\mathfrak{e}}) + {}^t\widetilde{b_2} s_\infty \\ \mathrm{Id}_{\widetilde{c}} y_{1m} + {}^t\widetilde{b_1} y_{2m} \\ 0 \end{bmatrix} \tag{4.25}$$

$$= \begin{bmatrix} P \cdot \mathrm{Id}_{\widetilde{c}}(y_{2m}^n s_{\mathfrak{e}}) + P \cdot {}^t\widetilde{b_2} s_\infty \\ T \cdot \mathrm{Id}_{\widetilde{c}} y_{1m} + T \cdot {}^t\widetilde{b_1} y_{2m} \\ 0 \end{bmatrix} \tag{4.26}$$

and

$$\alpha \circ \varphi_{\mathcal{U}} = \begin{bmatrix} \mathrm{Id}_c(y_{2m}^n s_{\mathfrak{e}}) + {}^t b_2 s_\infty \\ \mathrm{Id}_c y_{1m} + {}^t b_1 y_{2m} \\ 0 \end{bmatrix} \cdot A \tag{4.27}$$

$$= \begin{bmatrix} \mathrm{Id}_c A(y_{2m}^n s_{\mathfrak{e}}) + {}^t b_2 A s_\infty \\ \mathrm{Id}_c A y_{1m} + {}^t b_1 A y_{2m} \\ 0 \end{bmatrix} \tag{4.28}$$

Thus,

$$\varphi_{\mathcal{V}} \circ \widetilde{\alpha} - \alpha \circ \varphi_{\mathcal{U}} = \begin{bmatrix} P \cdot \mathrm{Id}_{\widetilde{c}}(y_{2m}^n s_{\mathfrak{e}}) + P \cdot {}^t\widetilde{b_2} s_\infty \\ T \cdot \mathrm{Id}_{\widetilde{c}} y_{1m} + T \cdot {}^t\widetilde{b_1} y_{2m} \\ 0 \end{bmatrix} - \begin{bmatrix} \mathrm{Id}_c A(y_{2m}^n s_{\mathfrak{e}}) + {}^t b_2 A s_\infty \\ \mathrm{Id}_c A y_{1m} + {}^t b_1 A y_{2m} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} (P \cdot \mathrm{Id}_{\widetilde{c}} - \mathrm{Id}_c A)(y_{2m}^n s_{\mathfrak{e}}) + (P \cdot {}^t\widetilde{b_2} - {}^t b_2 A) s_\infty \\ (T \cdot \mathrm{Id}_{\widetilde{c}} - \mathrm{Id}_c A) y_{1m} + (T \cdot {}^t\widetilde{b_1} - {}^t b_1 A) y_{2m} \\ 0 \end{bmatrix}. \tag{4.29}$$

Then, $\varphi_{\mathcal{V}} \circ \widetilde{\alpha} - \alpha \circ \varphi_{\mathcal{U}} = 0$ is equivalent to

$$P \cdot \mathrm{Id}_{\widetilde{c}} - \mathrm{Id}_c A = 0 \tag{4.30}$$

$$P \cdot {}^t\widetilde{b_2} - {}^t b_2 A = 0 \tag{4.31}$$

$$T \cdot \mathrm{Id}_{\widetilde{c}} - \mathrm{Id}_c A = 0 \tag{4.32}$$

$$T \cdot {}^t\widetilde{b_1} - {}^t b_1 A = 0. \tag{4.33}$$

On the other hand, from (4.24), we get:

$$\beta \circ \varphi_{\mathcal{V}} = \left[\mathrm{Id}_c y_{1m} + {}^t b_1 y_{2m}, \quad -\left(\mathrm{Id}_c(y_{2m}^n s_{\mathfrak{e}}) + {}^t b_2 s_\infty\right), \quad {}^t e s_\infty \right] \cdot \begin{bmatrix} P & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \left[\mathrm{Id}_c \cdot P y_{1m} + {}^t b_1 \cdot P y_{2m}, \quad -\left(\mathrm{Id}_c \cdot T(y_{2m}^n s_{\mathfrak{e}}) + {}^t b_2 \cdot T s_\infty\right), \quad {}^t e s_\infty \right]$$

and

$$\varphi_{\mathcal{W}} \circ \widetilde{\beta} = B \cdot \left[\mathrm{Id}_{\widetilde{c}} y_{1m} + {}^t\widetilde{b_1} y_{2m}, \quad -\left(\mathrm{Id}_{\widetilde{c}}(y_{2m}^n s_{\mathfrak{e}}) + {}^t\widetilde{b_2} s_\infty\right), \quad {}^t\widetilde{e} s_\infty \right]$$

$$= \left[B \cdot \mathrm{Id}_{\widetilde{c}} y_{1m} + B \cdot {}^t\widetilde{b_1} y_{2m}, \quad -\left(B \cdot \mathrm{Id}_{\widetilde{c}}(y_{2m}^n s_{\mathfrak{e}}) + B \cdot {}^t\widetilde{b_2} s_\infty\right), \quad B \cdot {}^t\widetilde{e} s_\infty \right].$$

Again, $\beta \circ \varphi_{\mathcal{V}} - \varphi_{\mathcal{W}} \circ \widetilde{\beta} = 0$ is equivalent to

$$\mathrm{Id}_c \cdot P - B \cdot \mathrm{Id}_{\widetilde{c}} = 0 \tag{4.34}$$

$${}^t b_1 \cdot P - B \cdot {}^t\widetilde{b_1} = 0 \tag{4.35}$$

$$B \cdot \mathrm{Id}_{\widetilde{c}} - \mathrm{Id}_c \cdot T = 0 \tag{4.36}$$

$$B \cdot {}^t\widetilde{b_2} - {}^t b_2 \cdot T = 0 \tag{4.37}$$

$${}^t e - B \cdot {}^t\widetilde{e} = 0. \tag{4.38}$$

Putting the equations from (4.30) to (4.33) and from (4.34) to (4.38) together, one can see that the matrices

$$A = B = P = T = {}^t\varphi$$

are a solution for all the equations, since $\varphi : X \to \widetilde{X}$ is a morphism of representations, and this provides the morphism

$$M_{\widetilde{X}} \to M_X \tag{4.39}$$

between the corresponding complexes.

Now we construct the complex $M_{X_{c-c'}^{\mathrm{ADHM}}}$ with the datum $(\overline{b_{1m}}, \overline{b_{2m}}, \mathbf{0}_{(c-c')\times 1})$, and we have the following vanishing for the cohomology sheaves.

Lemma 4.3.3. *The complex $M_{X_{c-c'}^{\mathrm{ADHM}}}$ given by*

$$0 \to \mathcal{O}_{\Sigma_n}(0,-1)^{\oplus(c-c')} \xrightarrow{\alpha''} \mathcal{O}_{\Sigma_n}(1,-1)^{\oplus(c-c')} \oplus \mathcal{O}_{\Sigma_n}^{\oplus(c-c')} \xrightarrow{\beta''} \mathcal{O}_{\Sigma_n}(1,0)^{\oplus(c-c')} \to 0,$$

where

$$\alpha'' = \begin{bmatrix} \mathrm{Id}_{c'-c}(y_{2m}^n s_{\mathfrak{e}}) + {}^t\overline{b_{2m}} s_\infty \\ \mathrm{Id}_{c'-c} y_{1m} + {}^t\overline{b_{1m}} y_{2m} \end{bmatrix} \tag{4.40}$$

and

$$\beta'' = \left[\ \mathrm{Id}_{c'-c} y_{1m} + {}^t\overline{b_{1m}} y_{2m}, \quad -\left(\mathrm{Id}_{c'-c}(y_{2m}^n s_{\mathfrak{e}}) + {}^t\overline{b_{2m}} s_\infty\right) \ \right], \tag{4.41}$$

satisfies

$$\mathcal{H}^{-1}(M_{X_{c-c'}^{\mathrm{ADHM}}}) = \mathcal{H}^0(M_{X_{c-c'}^{\mathrm{ADHM}}}) = 0.$$

Proof. In fact, the restrictions y_1, y_2 of y_{1m}, y_{2m} to ℓ_∞ may be regarded as homogeneous coordinates on ℓ_∞;[1] moreover, the section $s_{\mathfrak{e}}$ has no zeroes on ℓ_∞ (actually $\mathcal{O}_{\Sigma_n}(1,-n)\Big|_{\ell_\infty}$ is trivial as $\mathfrak{e}\cdot\mathfrak{h} = 0$). Omitting to write the restriction to ℓ_∞, we have

$$\alpha'' = \begin{pmatrix} \mathrm{Id}_{c'-c}(y_2^n s_{\mathfrak{e}}) \\ \mathrm{Id}_{c'-c} y_1 + {}^t\overline{b_{1m}} y_2 \end{pmatrix}$$

and

$$\beta'' = \left(-(\mathrm{Id}_{c'-c} y_1 + {}^t\overline{b_{1m}} y_2),\ \mathrm{Id}_{c'-c}(y_2^n s_{\mathfrak{e}})\right).$$

So $(v_1, v_2) \in \ker \beta''$ if and only if

$$(y_1 + y_2\, {}^t\overline{b_{1m}}) v_1 = y_2^n s_{\mathfrak{e}} v_2. \tag{4.42}$$

We show that $\operatorname{Im}\alpha'' = \ker\beta''$; of course we only have to check that $\operatorname{Im}\alpha'' \supset \ker\beta''$. If $y_2 \neq 0$, let (v_1, v_2) satisfy (4.42), and set

$$v = \frac{v_1}{y_2^n s_{\mathfrak{e}}}.$$

[1]Note that (y_{1m}, y_{2m}) are sections of $\mathcal{O}_{\Sigma_n}(0,1)$, which restricted to $\ell_\infty \simeq \mathbb{P}^1$ is $\mathcal{O}_{\mathbb{P}^1}(1)$ as $\mathfrak{h}\cdot\mathfrak{f} = 1$.

Then, taking (4.42) into account, one has $\alpha''(v) = (v_1, v_2)$.

In the patch $y_1 \neq 0$ the morphism $M = y_1 + y_2\,{}^t b_1$ is invertible at $y_2 = 0$, hence it is invertible in a neighborhood of that point. Then setting $v = M^{-1}v_2$ we again have $\alpha''(v) = (v_1, v_2)$ in that neighborhood. As this neighborhood and the neighborhood $y_2 \neq 0$ cover ℓ_∞ the claim follows.
On the other hand, $\mathcal{H}^{-1}(M_{X^{\mathrm{ADHM}}_{c-c'}}) = \ker(\alpha'')$ and α'' is injective by [2, Statement (i), p. 2151] and this finalizes the proof. $\square$

Therefore, we can use (4.18) and the morphism (4.39) to get a short exact sequence

$$0 \to M_{X^{\mathrm{ADHM}}_{c-c'}} \to M_{X^{\mathrm{ADHM}}_{c}} \to M_{X^{\mathrm{ADHM}}_{c'}} \to 0.$$

Explicitly, this exact sequence is the following diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\Sigma_n}(0,-1)^{\oplus(c-c')} & \xrightarrow{\varphi'_{\mathcal{U}}} & \mathcal{O}_{\Sigma_n}(0,-1)^{\oplus c} & \xrightarrow{\varphi_{\mathcal{U}}} & \mathcal{O}_{\Sigma_n}(0,-1)^{\oplus c'} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha''} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha'} & & \\
0 & \longrightarrow & \mathcal{O}_{\Sigma_n}(1,-1)^{\oplus(c-c')} \oplus \mathcal{O}_{\Sigma_n}^{\oplus(c-c')} & \xrightarrow{\varphi'_{\mathcal{V}}} & \mathcal{O}_{\Sigma_n}(1,-1)^{\oplus c} \oplus \mathcal{O}_{\Sigma_n}^{\oplus(c+1)} & \xrightarrow{\varphi_{\mathcal{V}}} & \mathcal{O}_{\Sigma_n}(1,-1)^{\oplus c'} \oplus \mathcal{O}_{\Sigma_n}^{\oplus(c'+1)} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta''} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta'} & & \\
0 & \longrightarrow & \mathcal{O}_{\Sigma_n}(1,0)^{\oplus(c-c')} & \xrightarrow{\varphi'_{\mathcal{W}}} & \mathcal{O}_{\Sigma_n}(1,0)^{\oplus c} & \xrightarrow{\varphi_{\mathcal{W}}} & \mathcal{O}_{\Sigma_n}(1,0)^{\oplus c'} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

By calculating cohomology we have:

$$\mathcal{H}^0(M_{X^{\mathrm{ADHM}}_{c-c'}}) \to \mathcal{H}^0(M_{X^{\mathrm{ADHM}}_{c}}) \to \mathcal{H}^0(M_{X^{\mathrm{ADHM}}_{c'}}) \to \mathcal{H}^1(M_{X^{\mathrm{ADHM}}_{c-c'}}) \to \mathcal{H}^1(M_{X^{\mathrm{ADHM}}_{c}}).$$

Since X^{ADHM}_c is co-stable, $M_{X^{\mathrm{ADHM}}_c}$ is a monad, so that

$$\mathcal{H}^{-1}(M_{X^{\mathrm{ADHM}}_c}) = \mathcal{H}^1(M_{X^{\mathrm{ADHM}}_c}) = 0.$$

By Lemma 4.3.3, $\mathcal{H}^0(M_{X^{\mathrm{ADHM}}_{c-c'}})$ is also zero, then this exact sequence turns out to be short:

$$0 \to \mathcal{H}^0(M_{X^{\mathrm{ADHM}}_{c}}) \to \mathcal{H}^0(M_{X^{\mathrm{ADHM}}_{c'}}) \to \mathcal{H}^1(M_{X^{\mathrm{ADHM}}_{c-c'}}) \to 0 \tag{4.43}$$

The sheaves $E = \mathcal{H}^0(M_{X^{\mathrm{ADHM}}_c})$ and $F = \mathcal{H}^0(M_{X^{\mathrm{ADHM}}_{c'}})$ are rank 1 framed torsion-free sheaves with Chern character $(1, 0, -c)$ and $(1, 0, -c')$, respectively. Moreover,

$$\mathrm{coker}(\beta'') = \mathcal{H}^1(M_{X^{\mathrm{ADHM}}_{c-c'}}) \simeq \frac{\mathcal{H}^0(M_{X^{\mathrm{ADHM}}_{c}})}{\mathcal{H}^0(M_{X^{\mathrm{ADHM}}_{c'}})} \tag{4.44}$$

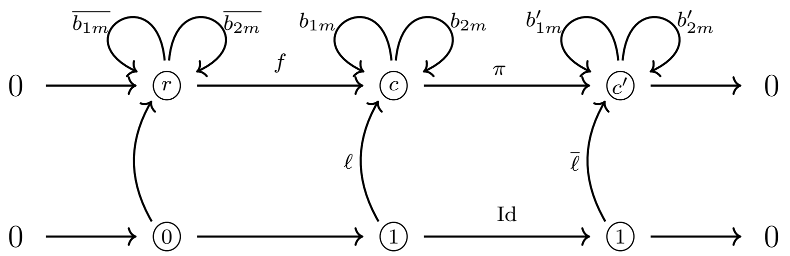
is a rank 0 sheaf of length $c - c'$ supported outside ℓ_∞, since the ranks of $\mathcal{H}^0(M_{X_c^{\text{ADHM}}})$ and $\mathcal{H}^0(M_{X_{c'}^{\text{ADHM}}})$ are equal and they are framed. Therefore, we get a framed flag of sheaves on Σ_n, which corresponds to a point in $\text{Hilb}^{c',c}(\Xi_n)$, that we denote by (E, F, φ).

Now, we are going to build the correspondence in the opposite direction, by tracing back the steps of the first map. We start by considering $S \in \text{Hilb}^{c',c}(\Xi_n)$. One can write $S = (S^{c'}, S^c)$, where $S^{c'}$ and S^c are 0-cycles of length c' and c, respectively. When we consider the open cover of $\text{Hilb}^c(\Xi_n)$ given by the open sets given by (3.6), namely: (U_m^{nc}) $m = 0, \ldots, c$, we already know that we can provide an open cover $\left(U_j^{nc'}\right)$ $j = 0, \ldots, c$ to $\text{Hilb}^{c'}(\Xi_n)$. Moreover, one also has

$$U_j^{nc'} \simeq \text{Hilb}^{c'}(\mathbb{C}^2), \ \forall j \in \{0, \ldots, c\}, \text{ and} \tag{4.45}$$

$$U_m^{nc} \simeq \text{Hilb}^c(\mathbb{C}^2), \ \forall m \in \{0, \ldots, c\}. \tag{4.46}$$

So we obtain an element $\widetilde{S} \in \text{Hilb}^{c',c}(\mathbb{C}^2)$. By Theorem 2.1.1, there is a stable representation of the enhanced ADHM quiver with dimension vector $(r = c - c', c, 1)$ fitting in the following short exact sequence in the category of ADHM representations, where the maps ℓ', b'_{1m} and b'_{2m} are inherited by the quotient:



(4.47)

One can rewrite this sequence as

$$0 \to L \xrightarrow{f} M \xrightarrow{\pi} N \to 0, \tag{4.48}$$

where M and N are stable and $\dim(M) = (c, 1)$ and $\dim(N) = (c', 1)$. The morphism $\pi : M \to N$ is surjective and $f : L \to M$ is injective. One may use [2, Equations 3.13], getting

$$A_1 = c_m {}^t b_{1m} + s_m \mathrm{Id}_c, \qquad A_2 = -s_m {}^t b_{1m} + c_m \mathrm{Id}_c, \tag{4.49}$$

$$\overline{A}_1 = c_m {}^t \overline{b}_{1m} + s_m \mathrm{Id}_{c-c'}, \qquad \overline{A}_2 = -s_m {}^t \overline{b}_{1m} + c_m \mathrm{Id}_{c-c'}, \tag{4.50}$$

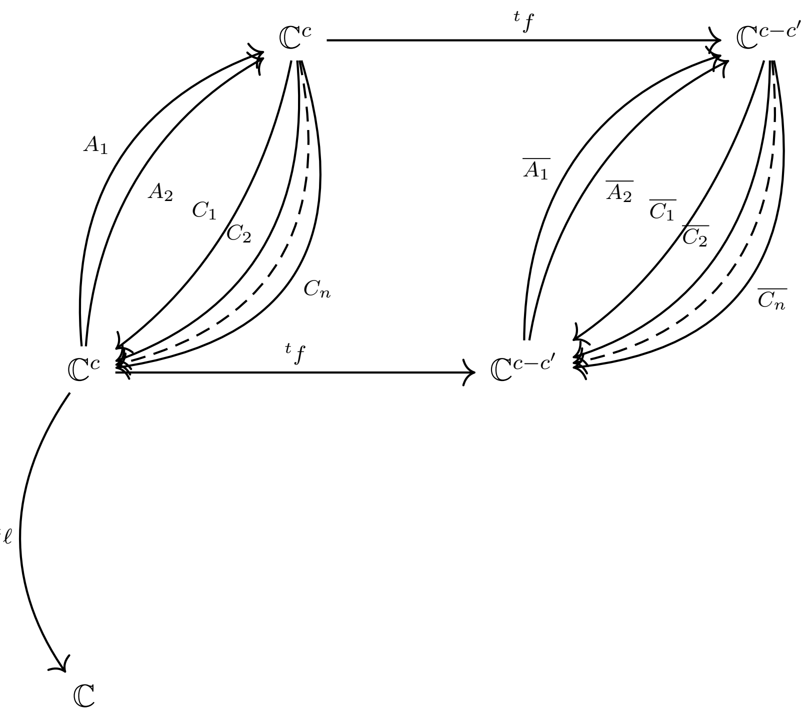
$$\begin{bmatrix} C_1 \\ \vdots \\ \vdots \\ C_n \end{bmatrix} = (\sigma_m^{n-1} \otimes \mathrm{Id}_c) \begin{bmatrix} \mathrm{Id}_c \\ {}^t b_{1m} \\ \vdots \\ {}^t b_{1m}^{n-1} \end{bmatrix} {}^t b_{2m} \tag{4.51}$$

$$\begin{bmatrix} \overline{C}_1 \\ \vdots \\ \vdots \\ \overline{C}_n \end{bmatrix} = (\sigma_m^{n-1} \otimes \mathrm{Id}_{c-c'}) \begin{bmatrix} \mathrm{Id}_{c-c'} \\ {}^t \overline{b}_{1m} \\ \vdots \\ \left({}^t \overline{b}_{1m}\right)^{n-1} \end{bmatrix} {}^t \overline{b}_{2m}, \tag{4.52}$$

where σ_m^{n-1} is the matrix defined by the condition

$$(s_m z_1 + c_m z_2)^p (c_m z_1 - s_m z_2)^{n-1-p} = \sum_{q=0}^{n-1} (\sigma_m^{n-1})_{pq} z_1^{n-1-q} z_2^q,$$

for $(z_1, z_2) \in \mathbb{C}^2$, where s_m, c_m are the numbers defined in equation (4.13) and therefore we obtain a representation of the quiver Q_n^{enh} with dimension vector $(c, c, c-c', c-c', 1)$, let us say X, and we write:



Remark 4.3.4. One can check by substitution that the representation (written without the vector spaces)

$$X = (A_1, A_2, C_1, \ldots, C_n, {}^t\ell, \overline{A}_1, \overline{A}_2, \overline{C}_1, \ldots, \overline{C}_n, {}^tf, {}^tf)$$

satisfies all the necessary relations exhibited right before Remark 4.1.1.

This representation turns out to be stable. In fact, the map ${}^tf$ is surjective and the Q_n datum is stable, by construction. Then, the stability of X follows from Lemma 4.1.2 and we get a point in $\mathrm{Rep}^{\mathrm{st}}(Q_n^{\mathrm{enh}}, (c, c, c-c', c-c', 1))$, as required, and this finalizes the proof of the existence of a set-theoretical bijection between the suitable sets. This is the "absolute" case, and now we are able to explore the categorical approach of the problem with the aim of showing that this bijection actually yields an isomorphism of schemes.

Remark 4.3.5. In both ways of the definition of the bijection from Theorem 4.3.1 there is a dependence on $m \in \{0, \ldots, c\}$. However, the maps are still well defined, since the dependence on m is fixed in one way by the item (2) of Proposition 4.4.3 and in the other way, by the action (4.5) of the group that defines the moduli space.

4.4 The categorical approach of the problem

We begin this section by revising and stating some facts about the categorical techniques behind Theorem 3.2.5 with the purpose of proving the following result.

Theorem 4.4.1. *The bijection from Theorem 4.3.1 yields an isomorphism of schemes*

$$\mathrm{Rep}^{\mathrm{st}}(Q_n^{\mathrm{enh}}, (c, c, c-c', c-c', 1)) \simeq \mathrm{Hilb}^{c',c}(\Xi_n).$$

This section is divided in two subsections. The goal of the first subsection is provide a sound understanding to the categorical approach of Theorem 3.2.5 and in the second one we prove the main theorem of the present work.

4.4.1 The base case: approaching $\mathrm{Hilb}^c(\Xi_n)$ categorically

In order to show that the set-theoretical bijection from Theorem 4.3.1 yields an isomorphism of schemes, we shall work with families of representations of the quiver Q_n parameterized by schemes and, as it was done before, we shall prove the existence of a natural transformation between the underlying functors. We shall see later on that this scheme represents the functor of families of stable representations of the quiver Q_n. To that end we introduce:

- the category $\mathcal{A}_n$ of families of representations of the quiver Q_n with the relations (1.3). For $n \geq 2$, an object of $\mathcal{A}_n$ is a collection

  $$(T, \mathcal{V}_0, \mathcal{V}_1, \mathcal{W}, A_1, A_2, B_1, \ldots, B_n, I_1, \ldots, I_{n-1}, J)$$

  where

  - T is a scheme;
  - $\mathcal{W}, \mathcal{V}_0, \mathcal{V}_1$ are vector bundles on T;
  - $A_1, A_2 \in \mathrm{Hom}(\mathcal{V}_0, \mathcal{V}_1)$, $B_1, \ldots, B_n \in \mathrm{Hom}(\mathcal{V}_1, \mathcal{V}_0)$, $I_1, \ldots, I_{n-1} \in \mathrm{Hom}(\mathcal{W}, \mathcal{V}_0)$, $J \in \mathrm{Hom}(\mathcal{V}_0, \mathcal{W})$ satisfying the conditions

  $$A_1 B_q = A_2 B_{q+1}, \quad B_q A_1 - B_{q+1} A_2 = I_q J, \quad q = 1, \ldots, n-1.$$

  For $n = 1$ the objects are collections (T, A_1, A_2, B_1, J) with $A_1 B_1 A_2 = A_2 B_1 A_1$.

- For a fixed $\mathbf{v} = (r, c_0, c_1)$, $\mathcal{A}_n(\mathbf{v})$ is the full subcategory of $\mathcal{A}_n$ of families of representations of Q^n with dimension vector $\mathbf{v}$, i.e., $\operatorname{rk}\mathcal{W} = r$, $\operatorname{rk}\mathcal{V}_0 = c_0$, $\operatorname{rk}\mathcal{V}_1 = c_1$.
- For a fixed stability parameter Θ, $\mathcal{A}_n(\mathbf{v})^s_\Theta$ is the full subcategory of $\mathcal{A}_n(\mathbf{v})$ whose objects are framed representations that are stable with respect to Θ.
- The category $\mathbf{Kom}_n$ of families of complexes of cohererent sheaves on the variety Σ_n.
- Its full subcategory $\mathbf{Kom}_n^{\text{flat}}$ whose objects are families of complexes of cohererent sheaves on Σ_n whose cohomology sheaves are flat on the base scheme.

Morphisms in these categories are defined as in the previous Section in the case of $\mathbb{P}^2$.

The next step would be to define a functor $\mathcal{A}_n \to \mathbf{Kom}_n$. However we are unable to do that in full generality, and we need to restrict to representations satisfying a kind of nondegeneracy condition, corresponding to the regularity of the pencil $\nu_1 A_1 + \nu_2 A_2$, where $\boldsymbol{\nu} = [\nu_1, \nu_2] \in \mathbb{P}^1$ (see condition (P2) in [2], p. 2137). We consider a full subcategory $\mathcal{A}_{n,\boldsymbol{\nu}}$ characterized by the condition that the homomorphism

$$A_{\boldsymbol{\nu}} = \nu_2 A_1 + \nu_1 A_2$$

is an isomorphism. Of course this fixes the second and third components of the dimension vector to be equal.

We want to define a functor

$$\mathfrak{K}_{n,\boldsymbol{\nu}} \colon \mathcal{A}_{n,\boldsymbol{\nu}} \to \mathbf{Kom}_n$$

of categories over $\mathbf{Sch}$.

We recall that we may represent the n-th Hirzebruch surface Σ_n as

$$\Sigma_n = \left\{([y_1, y_2], [x_1, x_2, x_3]) \in \mathbb{P}^1 \times \mathbb{P}^2 \mid x_1 y_1^n = x_2 y_2^n\right\} , \tag{4.53}$$

and for every $\boldsymbol{\nu} = [\nu_1, \nu_2] \in \mathbb{P}^1$ we introduce the additional pair of coordinates

$$[y_{1,\boldsymbol{\nu}}, y_{2,\boldsymbol{\nu}}] = [\nu_1 y_1 + \nu_2 y_2, -\nu_2 y_1 + \nu_1 y_2] \,.$$

The set $\left\{ y_{2,\boldsymbol{\nu}}^{q} y_{1,\boldsymbol{\nu}}^{h-q} \right\}_{q=0}^{h}$ is a basis for $H^0\left(\mathcal{O}_{\Sigma_n}(0,h)\right) = H^0\left(\pi^*\mathcal{O}_{\mathbb{P}^1}(h)\right)$ for all $h \geq 1$, where $\pi\colon \Sigma_n \longrightarrow \mathbb{P}^1$ is the canonical projection. Furthermore the (unique up to homotheties) global section $s_{\mathfrak{e}}$ of $\mathcal{O}_{\Sigma_n}(\mathfrak{e})$ induces an injection $\mathcal{O}_{\Sigma_n}(0,n) \rightarrowtail \mathcal{O}_{\Sigma_n}(1,0)$, so that the set

$$\left\{ (y_{2,\boldsymbol{\nu}}^{q} y_{1,\boldsymbol{\nu}}^{n-q}) s_{\mathfrak{e}} \right\}_{q=0}^{n} \cup \{s_\infty\}$$

is a basis for $H^0\left(\mathcal{O}_{\Sigma_n}(1,0)\right)$, where s_∞ is a section whose vanishing locus is ℓ_∞.

We define the functor $\mathfrak{K}_{n,\boldsymbol{\nu}}$ on objects. If

$$X = (T, \mathcal{V}_0, \mathcal{V}_1, \mathcal{W}, A_1, A_2, B_1, \ldots, B_n, I_1, \ldots, I_{n-1}, J)$$

is an object in $\mathcal{A}_{n,\boldsymbol{\nu}}$, then $\mathfrak{K}_{n,\boldsymbol{\nu}}(X)$ is the complex

$$0 \to \mathcal{V}_0^* \boxtimes \mathcal{O}_{\Sigma_n}(0,-1) \xrightarrow{\alpha_{\boldsymbol{\nu}}} \mathcal{V}_0^* \boxtimes \mathcal{O}_{\Sigma_n}(1,-1) \oplus (\mathcal{V}_0^* \oplus \mathcal{W}^*) \boxtimes \mathcal{O}_{\Sigma_n} \xrightarrow{\beta_{\boldsymbol{\nu}}} \mathcal{V}_0^* \boxtimes \mathcal{O}_{\Sigma_n}(1,0) \to 0 \tag{4.54}$$

with the morphisms $\alpha_{\boldsymbol{\nu}}$, $\beta_{\boldsymbol{\nu}}$ given by

$$\alpha_{\boldsymbol{\nu}} = \begin{pmatrix} \mathrm{id} \otimes (y_{2,\boldsymbol{\nu}}^{n} s_{\mathfrak{e}}) + A_{\boldsymbol{\nu}}^* B_{\boldsymbol{\nu}}^* \otimes s_\infty \\ \mathrm{id} \otimes y_{1,\boldsymbol{\nu}} + C_{\boldsymbol{\nu}}^* (A_{\boldsymbol{\nu}}^*)^{-1} \otimes y_{2,\boldsymbol{\nu}} \\ -I_{\boldsymbol{\nu}} \otimes y_{2\boldsymbol{\nu}} \end{pmatrix},$$

$$\beta_{\boldsymbol{\nu}} = \left(\mathrm{id} \otimes y_{1,\boldsymbol{\nu}} + C_{\boldsymbol{\nu}}^* (A_{\boldsymbol{\nu}}^*)^{-1} \otimes y_{2,\boldsymbol{\nu}}, \quad - \left(\mathrm{id} \otimes (y_{2,\boldsymbol{\nu}}^{n} s_{\mathfrak{e}}) + A_{\boldsymbol{\nu}}^* B_{\boldsymbol{\nu}}^* \otimes s_\infty \right), \quad J^* \otimes s_\infty \right),$$

where we have set

$$B_{\boldsymbol{\nu}}^* = \sum_{q=1}^{n} \binom{n-1}{q-1} \nu_1^{n-q} \nu_2^{q-1} B_q^*, \quad C_{\boldsymbol{\nu}} = \nu_1 A_1 - \nu_2 A_2, \quad I_{\boldsymbol{\nu}} = (\nu_1^2 + \nu_2^2) \sum_{q=1}^{n-1} \binom{n-2}{q-1} \nu_1^{n-q-1} \nu_2^{q-1} I_q^*$$

(for $n = 1$ we understand that $B_{\boldsymbol{\nu}} = 1$ and $I_{\boldsymbol{\nu}} = 0$).

The action of $\mathfrak{K}_{n,\boldsymbol{\nu}}$ on morphisms is defined as in the case of $\mathbb{P}^2$, see (2.5). We omit the cumbersome but trivial details.

We see now some properties of the functor $\mathfrak{K}_{n,\boldsymbol{\nu}}$. Let $\mathcal{H}^\bullet$ denote the cohomology sheaves of a complex on $T \times \Sigma_n$.

Proposition 4.4.2. *If $\mathcal{X} \in \mathcal{A}_{n,\boldsymbol{\nu}}$ is a family of framed representations of Q_n, then $\mathcal{H}^0(\mathfrak{K}_{n,\boldsymbol{\nu}}(\mathcal{X}))$ is torsion-free and $\mathcal{H}^{-1}(\mathfrak{K}_{n,\boldsymbol{\nu}}(\mathcal{X})) = 0$. If the dimension vector of $\mathcal{X}$ is $(1, c, c)$ for some c, and $\mathcal{X}$ is stable with respect to the stability parameter $(2c, -2c+1)$, then $\mathcal{H}^1(\mathfrak{K}_{n,\boldsymbol{\nu}}(\mathcal{X})) = 0$.*

Proof. This follows from the special case $T = \operatorname{Spec}\mathbb{C}$, in which case it is proved in the Section A.1 of [2]. $\square$

Let $\mathcal{A}_{n,\boldsymbol{\nu}}(c)^{\mathrm{fr}s}$ be subcategory of $\mathcal{A}_{n,\boldsymbol{\nu}}$ whose objects are families of framed representations with $\operatorname{rk}\mathcal{V}_0 = \operatorname{rk}\mathcal{V}_1 = c$, stable with the respect to the stability parameter $\Theta = (2c, -2c+1)$.

Proposition 4.4.3. *1. $\mathfrak{K}_{n,\boldsymbol{\nu}}$ maps $\mathcal{A}_{n,\boldsymbol{\nu}}(c)^{\mathrm{fr}s}$ into $\mathbf{Kom}_n^{\mathrm{flat}}$.*

2. If $\mathcal{X} \in \mathcal{A}_{n,\boldsymbol{\nu}}(c)^{\mathrm{fr}s} \cap \mathcal{A}_{n,\boldsymbol{\nu}'}(c)^{\mathrm{fr}s}$ then the complexes $\mathfrak{K}_{n,\boldsymbol{\nu}}(\mathcal{X})$ and $\mathfrak{K}_{n,\boldsymbol{\nu}'}(\mathcal{X})$ are quasi-isomorphic.

Let $\mathcal{X} \to \mathcal{X}''$ be a surjective morphism in $\mathcal{A}^{\mathrm{fr}}_{n,\boldsymbol{\nu}}$ for a scheme T, where the dimensional vectors of $\mathcal{X}$ and $\mathcal{X}''$ are $(1,c,c)$ and $(0, c-c', c-c')$, respectively. Assume that $\mathcal{X}$ is stable with respect to the stability parameter $(2c, -2c+1)$. Let $\mathcal{X}'$ be the corresponding kernel. Then:

3. $\mathcal{X}' \in \mathcal{A}_{n,\boldsymbol{\nu}}(c')^{\mathrm{fr}s}$, with respect to the stability parameter $(2c', -2c'+1)$.

4. The sequence of morphisms of complexes of coherent sheaves on $T \times \Sigma_n$

$$0 \to \mathfrak{K}_{n,\boldsymbol{\nu}}(\mathcal{X}'') \to \mathfrak{K}_{n,\boldsymbol{\nu}}(\mathcal{X}) \to \mathfrak{K}_{n,\boldsymbol{\nu}}(\mathcal{X}') \to 0 \tag{4.55}$$

is exact.

Proof. 1. This goes exactly as in Proposition 2.1.4.

2. This is essentially proved in [2], albeit in a different language.

3. It follows from a direct computation.

4. The sequence (4.55) can be written as a diagram with three rows and three columns; the second and third column are complexes as in (4.54), and the first column too, but with $\mathcal{W} = 0$. The exactness of the rows is equivalent to the exactness of the sequence $0 \to \mathcal{X}' \to \mathcal{X} \to \mathcal{X}'' \to 0$. $\square$

We conclude this section by stating and briefly discussing the correspondence between the functor of families of representations of the quivers Q^n, and the Hilbert scheme functor for the varieties Ξ_n; that is, we categorize Theorem 3.2.5.

Theorem 4.4.4. *Let $\mathfrak{R}^{n\mathrm{fr}s}_{c,\Theta}$ be the functor of families of framed representations of the quiver with relations Q^n, with dimension vector $(1,c,c)$, stable with respect to stability parameter $\Theta = (2c, -2c+1)$. Let $\mathfrak{Hilb}^c_{\Xi_n}$ be the functor of isomorphism classes of families of length c 0-cycles on the variety Ξ_n. There is a natural transformation $\eta_n \colon \mathfrak{R}^{n\mathrm{fr}s}_{c,\Theta} \to \mathfrak{Hilb}^c_{\Xi_n}$ which is an isomorphism of functors.*

Proof. The natural transformation η_n is defined by means of the functors $\mathfrak{K}_{n,\boldsymbol{\nu}}$, also in view of part 1 of Proposition 4.4.3: if $\mathcal{X}$ is a family of representation of Q^n, with dimension vector $(1, c, c)$, and Θ-stable, it is in $\mathcal{A}^1_{n,\boldsymbol{\nu}}(c)^s_\Theta$ for some m; then $\mathcal{H}^0(\mathfrak{K}_{n,\boldsymbol{\nu}}(\mathcal{X}))$ is a family of length c 0-cycles on Ξ_n. That η_n is an isomorphism of functors is just the categorical way of stating Theorem 3.2.5, and ultimately is the main content of [2]. □

The version of Remark 2.1.2 in the present context is that the Hilbert scheme $\mathrm{Hilb}^c(\Xi_n)$ represents the functor $\mathfrak{R}^{n,\mathrm{fr},s}_{c,\Theta}$.

4.4.2 The nested case: a complete proof of Theorem 4.4.1

In this subsection we prove the following theorem:

Theorem 4.4.5. *The bijection from Theorem 4.3.1 yields an isomorphism of schemes*

$$\mathrm{Rep}^{\mathrm{st}}(Q_n^{\mathrm{enh}}, (c, c, c-c', c-c', 1)) \simeq \mathrm{Hilb}^{c',c}(\Xi_n) \simeq \mathcal{F}_{c',c}.$$

Proposition 4.4.6. *Let $\overline{\mathfrak{R}}^{n,\mathrm{fr},s}_{c,c',\underline{\theta}}$ be the functor of families of framed $\underline{\theta}$-stable representations of the quiver Q_n^{enh} of dimension vector $(c, c, c-c', c-c', 1)$. For any $n \geq 1$, there exists a natural transformation*

$$\eta_n : \overline{\mathfrak{R}}^{n,\mathrm{fr},s}_{c,c',\underline{\theta}} \to \mathcal{H}ilb^{c',c}_{\Xi_n},$$

which is an isomorphism of functors.

The key for the construction of the natural transformation $\eta_n : \overline{\mathfrak{R}}^{n,\mathrm{fr},s}_{c,c',\underline{\theta}} \to \mathcal{H}ilb^{c',c}_{\Xi_n}$ is to regard a representation of the quiver Q_n^{enh} as a morphism of representations of the standard quiver Q_n. Let

$$X = (T, \mathcal{V}_0, \mathcal{V}_1, \mathcal{W}, \mathcal{V}'_0, \mathcal{V}'_1, A_1, A_2, B_1, \ldots, B_n, I_1, \ldots, I_{n-1}, J, A'_1, A'_2, B'_1, \ldots, B'_n, F_1, F_2)$$

be a family of representations of the quiver Q_n^{enh}, with T a scheme, $\mathcal{V}_0, \mathcal{V}_1, \mathcal{W}, \mathcal{V}'_0, \mathcal{V}'_1$

vector bundles on T of rank $c, c, 1, c-c', c-c'$, respectively, and

$$\begin{aligned}
A_1, A_2 &\in \mathrm{Hom}(\mathcal{V}_0, \mathcal{V}_1), \\
B_1, \ldots, B_n &\in \mathrm{Hom}(\mathcal{V}_1, \mathcal{V}_0), \\
I_1, \ldots, I_{n-1} &\in \mathrm{Hom}(\mathcal{W}, \mathcal{V}_0), \\
J &\in \mathrm{Hom}(\mathcal{V}_0, \mathcal{W}), \\
A'_1, A'_2 &\in \mathrm{Hom}(\mathcal{V}'_0, \mathcal{V}'_1), \\
B'_1, \ldots, B'_n &\in \mathrm{Hom}(\mathcal{V}'_1, \mathcal{V}'_0), \\
F_1 &\in \mathrm{Hom}(\mathcal{V}_0, \mathcal{V}'_0), \\
F_2 &\in \mathrm{Hom}(\mathcal{V}_1, \mathcal{V}'_1).
\end{aligned}$$

If we assume that X is stable as in Lemma 4.1.2, one has that F_1 and F_2 are surjective. This defines a surjective morphism of families of representations of the quiver Q_n. Define $\mathcal{V}''_0 := \ker(F_1)$ and $\mathcal{V}''_1 := \ker(F_2)$; note that they are vector bundles on T of rank c'. The morphisms $A_1, A_2, B_1, \ldots, B_n, J$ induce morphisms

$$A''_1, A''_2 \in \mathrm{Hom}(\mathcal{V}''_0, \mathcal{V}''_1);\ B''_1, \ldots, B''_n \in \mathrm{Hom}(\mathcal{V}''_1, \mathcal{V}''_0);\ J'' \in \mathrm{Hom}(\mathcal{V}''_0, \mathcal{W});$$

and that defines a "kernel" family of representations of the quiver Q_n.

As we have natural inclusions $i_0 : \mathcal{V}''_0 \to \mathcal{V}_0$ and $i_1 : \mathcal{V}''_1 \to \mathcal{V}_1$ and the isomorphism $\mathrm{Id}_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}$, we can actually build a short exact sequence of families of representations of the quiver Q_n parameterized by T

$$0 \to \mathcal{X}'' \to \mathcal{X} \to \mathcal{X}' \to 0.$$

By Lemma 4.1.2 and Lemma 4.1.3, $\mathcal{X}''$ and $\mathcal{X}$ are families of stable framed representations, so that Proposition 4.4.3 implies that

$$0 \to \mathfrak{K}_{n,\boldsymbol{\nu}}(\mathcal{X}') \to \mathfrak{K}_{n,\boldsymbol{\nu}}(\mathcal{X}) \to \mathfrak{K}_{n,\boldsymbol{\nu}}(\mathcal{X}'') \to 0$$

is exact and by taking cohomology, one has that

$$\mathcal{H}^0(\mathfrak{K}_{n,\boldsymbol{\nu}}(\mathcal{X}')) \to \mathcal{H}^0(\mathfrak{K}_{n,\boldsymbol{\nu}}(\mathcal{X})) \to \mathcal{H}^0(\mathfrak{K}_{n,\boldsymbol{\nu}}(\mathcal{X}'')) \to \mathcal{H}^1(\mathfrak{K}_{n,\boldsymbol{\nu}}(\mathcal{X}')) \to \mathcal{H}^1(\mathfrak{K}_{n,\boldsymbol{\nu}}(\mathcal{X})) \tag{4.56}$$

is exact as well.

On the other hand, $\mathcal{H}^0(\mathfrak{K}_{n,\boldsymbol{\nu}}(\mathcal{X}')) = 0$, by Lemma 4.3.3 and $\mathcal{H}^1(\mathfrak{K}_{n,\boldsymbol{\nu}}(\mathcal{X})) = 0$, since $\mathcal{X}$ is stable. Thus, (4.56) reduces to

$$0 \to \mathcal{H}^0(\mathfrak{K}_{n,\boldsymbol{\nu}}(\mathcal{X})) \to \mathcal{H}^0(\mathfrak{K}_{n,\boldsymbol{\nu}}(\mathcal{X}'')) \to \mathcal{H}^1(\mathfrak{K}_{n,\boldsymbol{\nu}}(\mathcal{X}')) \to 0 \tag{4.57}$$

Additionally, one has:

- $F := \mathcal{H}^0(\mathfrak{K}_{n,\boldsymbol{\nu}}(\mathcal{X}''))$ is a torsion-free coherent sheaf on $T \times \Sigma_n$, with a framing φ to the trivial sheaf on $T \times \ell_\infty$. Moreover, the second Chern class of $F\Big|_{\{t\}\times\Sigma_n}$ is c' for every closed point $t \in T$.
- F and $E := \mathcal{H}^0(\mathfrak{K}_{n,\boldsymbol{\nu}}(\mathcal{X}))$ are flat over T, by Proposition 4.4.3, since $\mathcal{X}$ and $\mathcal{X}'$ are stable.
- $\mathcal{H}^1(\mathfrak{K}_{n,m}(\mathcal{X}'))$ is a rank 0 coherent sheaf on $T \times \Sigma_n$, supported away from $T \times \ell_\infty$. For every closed point $t \in T$, the restriction of the schematic support of $\mathcal{H}^1(\mathfrak{K}_{n,\boldsymbol{\nu}}(\mathcal{X}'))$ to the fiber over t is a 0-cycle on Σ_n of length $c - c'$.
- $\mathcal{H}^1(\mathfrak{K}_{n,\boldsymbol{\nu}}(\mathcal{X}'))$ is flat over T, as it is a quotient of flat sheaves.

Therefore, the triple (E, F, φ) is a flat family of framed flags of sheaves on Σ_n parameterized by the scheme T. This defines the natural transformation

$$\eta_n : \overline{\mathfrak{R}}^{n,\mathrm{fr},s}_{c,c',\underline{\theta}} \to \mathcal{H}ilb^{c',c}_{\Xi_n}.$$

To prove that η_n is indeed a natural transformation, we need to show that for any scheme morphism $f : S \to T$ the diagram

$$\begin{array}{ccc}
\overline{\mathfrak{R}}^{n,\mathrm{fr},s}_{c,c',\underline{\theta}}(T) & \xrightarrow{\overline{\mathfrak{R}}^{n,\mathrm{fr},s}_{c,c',\underline{\theta}}(f)} & \overline{\mathfrak{R}}^{n,\mathrm{fr},s}_{c,c',\underline{\theta}}(S) \\
\Big\downarrow{\scriptstyle \eta_{n,T}} & & \Big\downarrow{\scriptstyle \eta_{n,S}} \\
\mathfrak{F}_{c',c}(T) & \xrightarrow{\mathfrak{F}_{c',c}(f)} & \mathfrak{F}_{c',c}(S)
\end{array}$$

commutes.

Indeed, if we consider an element in $\overline{\mathfrak{R}}^{n,\mathrm{fr},s}_{c,c',\underline{\theta}}(T)$, say $[T,X]$. We have that

$$\begin{aligned}
[\mathfrak{F}_{c',c}(f)\circ\eta_{n,T}]([T,X]) &= [(\mathrm{Id}\times f)^*\circ\eta_{n,T}]([T,X]) \\
&= (\mathrm{Id}\times f)^*(\mathcal{H}^\bullet(\mathfrak{K}_{n,\boldsymbol{\nu}}([T,X]))) \\
&\overset{(1.5.2)}{=} \mathcal{H}^\bullet((\mathrm{Id}\times f)^*\mathfrak{K}_{n,\boldsymbol{\nu}}([T,X])) \\
&= \mathcal{H}^\bullet(\mathfrak{K}_{n,\boldsymbol{\nu}}(f^*([T,X]))) \\
&= \eta_{n,S}(f^*([T,X])) \\
&= (\eta_{n,S}\circ\overline{\mathfrak{R}}^{n,\mathrm{fr},s}_{c,c',\underline{\theta}}(f))([T,X]).
\end{aligned}$$

Notice that the same considerations as in Remark 2.1.7 work here. To show that η_n is actually a natural isomorphism, we must construct another natural transformation

$$\tau_n : \mathcal{H}ilb^{c',c}_{\Xi_n} \to \overline{\mathfrak{R}}^{n,\mathrm{fr},s}_{c,c',\underline{\theta}}$$

which is both a right and left inverse to η_n. This can be done just by tracing back the steps that conduct to the definition of η_n, i.e., given a family of framed flags on Σ_n with the required numerical invariants, define two families of representations $\mathcal{X}''_{\mathrm{ADHM}}$ and $\mathcal{X}_{\mathrm{ADHM}}$ of the transposed ADHM quiver with an injection $\phi : \mathcal{X}''_{\mathrm{ADHM}} \to \mathcal{X}_{\mathrm{ADHM}}$. Then, one constructs $\mathcal{X}'_{\mathrm{ADHM}}$ as the quotient and we use [2, Equations 3.13] to obtain a stable family of representations of the quiver Q_n^{enh} with the required dimension vector and relations, and this finalizes the proof of Theorem 4.4.1.

Corollary 4.4.7. *For any $n \geq 1$, the schemes $\mathrm{Rep}^{\mathrm{st}}(Q_n^{\mathrm{enh}},(c,c,c-c',c-c',1))$ and $\mathrm{Hilb}^{c',c}(\Xi_n)$ are isomorphic and one can write*

$$\mathrm{Rep}^{\mathrm{st}}(Q_n^{\mathrm{enh}},(c,c,c-c',c-c',1)) \simeq \mathrm{Hilb}^{c',c}(\Xi_n) \simeq \mathcal{F}_{c',c}$$

i.e., the nested Hilbert scheme of points in Ξ_n can be seen as a moduli space of stable framed representations of a suitable quiver with relations.

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