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Weighted Hardy-Sobolev type inequalities and applications to indefinite elliptic problems

by

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Ranieri de França Freire

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Abstract

In this work, we proved some Hardy-Sobolev type inequalities and, as a consequence, obtained weighted Sobolev embeddings in the upper half-space. As applications, we addressed some Liouville-type results for two classes of indefinite quasilinear elliptic problems in the upper half-space. Additionally, for these classes of problems, we obtained existence results using the Fibering Method.

Keywords: Hardy-Sobolev type inequality; Weighted Sobolev spaces; Sobolev trace embedding; Liouville type results; Fibering method.

Resumo

Neste trabalho, provamos algumas desigualdades do tipo Hardy-Sobolev e, como consequência, obtivemos imersões de Sobolev com peso no semi-espaço superior. Como aplicações, abordamos alguns resultados do tipo Liouville para duas classes de problemas elípticos quasilineares indefinidos no semi-espaço superior. Além disso, para essas classes de problemas, obtivemos resultados de existência utilizando o Fibering method.

Palavras-chave: Desigualdade do tipo Hardy-Sobolev; Espaços de Sobolev com peso; Imersões de Sobolev no traço; Teoremas do tipo Liouville; Fibering method.

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Notation

Throughout this work, we will use the following list of mathematical symbols and notations:

- $\mathbb{R}^N_+ := \{x = (x', x_N) \in \mathbb{R}^N : x' \in \mathbb{R}^{N-1}, x_N > 0\}$ denotes the upper half-space;
- B_r denotes the ball centered at 0 with radius $r > 0$ in $\mathbb{R}^N$;
- $B^+_r = B_r \cap \mathbb{R}^N_+$ denotes the half ball;
- $C^\infty_0(\mathbb{R}^N)$ denotes the space of infinitely differentiable real functions whose support is compact in $\mathbb{R}^N$;
- $\hookrightarrow$ denotes continuous embedding;
- $\to$ denotes strong convergence;
- $\rightharpoonup$ denotes weak convergence.

Chapter 0

Introduction

The Hardy inequality has its origins in the work of the english mathematician Godfrey Harold Hardy in 1920 (see [32]). Precisely, if $p > 1$ and f is a nonegative function $p-$ integrable then

$$\left(\frac{p-1}{p}\right)^p \int_0^\infty \left(\frac{1}{x}\int_0^x f(s)ds\right)^p dx \leq \int_0^\infty f(x)^p dx.$$

As is well-known in nowadays, the Hardy inequality is a fundamental tool in analysis, particularly in the study of differential equations and mathematical physics (see for instance [36] and references therein).

The Hardy inequality has since been generalized in various ways, including extensions to weighted inequalities and settings involving different domains or operators. It has become a powerful tool in mathematical analysis, particularly in the study of partial differential equations, where it provides insights into the regularity and integrability of solutions. The inequality is named after Hardy, but its influence extends through many areas of mathematics, including functional analysis, potential theory, and geometric analysis. Necas in [45] established that if Ω is a bounded domain with $\partial\Omega$ sufficiently smooth, then

$$\int_\Omega d(x)^{\alpha-p}|u|^p dx \leq C_0 \int_\Omega d(x)^\alpha |\nabla u|^p dx, \quad u \in C_0^\infty(\Omega), \tag{1}$$

where $d(x)$ denote the distance of x to $\partial\Omega$, $p \geq 1$ and $\alpha \leq p-1$.

Matskewich and Sobolevskii in [42] prove that when Ω is a open convex with $\partial\Omega \in C^1$, it holds

$$\int_\Omega \frac{|u|^p}{d(x)^p}\mathrm{d}x \leq \left(\frac{p}{p-1}\right)^p \int_\Omega |\nabla u|^p \mathrm{d}x, \quad \forall u \in C_0^\infty(\Omega).$$

Moreover, they proved that the constant $(\frac{p}{p-1})^p$ is optimal. In the hole space $\mathbb{R}^N$, for $1 \leq p < N$, the N- dimensional classical Hardy inequality is proved by using a symmetrization argument(see [12]) and is states as follows

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx \leq \left(\frac{p}{N-p} \right)^p \int_{\mathbb{R}^N} |\nabla u|^p dx, \quad \forall u \in C_0^\infty(\mathbb{R}^N).$$

The constant $(\frac{p}{N-p})^p$ is optimal and is never achieved. This inequality for $p = 2$ is known as uncertain principle(see [27]). It illustrates a relationship between the norms of a function and its gradient, establishing a connection between the function's behavior near the origin and its smoothness.

Hardy type inequalities in the half-space have been studied in many papers, including [18, 21, 23, 28, 32, 50]. We emphasize that in all of the mentioned papers, the study of Hardy inequalities in the half-space typically focuses on functions within $C_0^\infty(\mathbb{R}_+^N)$, which provides a useful framework for studying elliptic problems with Dirichlet boundary conditions.

To explore partial differential equations with different types of boundary conditions, it is useful to examine inequalities for restrictions of functions in $C_0^\infty(\mathbb{R}^N)$. In [47], Pfluger proved that for general unbounded domains the following inequality holds: Assume that $1 < p < N$ and let $\Omega \subset \mathbb{R}^N$ be an unbounded exterior domain with noncompact boundary. Then, there exists $C > 0$ such that

$$\int_\Omega \frac{|u|^p}{(1+|x|)^p} dx \leq C \left(\int_\Omega |\nabla u|^p dx + \int_{\partial\Omega} \frac{|x \cdot \nu|}{(1+|x|)^p} |u|^p d\sigma \right), \tag{2}$$

for all for $u \in C_0^\infty(\mathbb{R}^N)$, where ν is the unit outer normal to the boundary. See also [6, 10, 15, 29, 34, 46, 52] for related results.

In the recent work [4], Felix et. al. proved inequalities in $\mathbb{R}_+^N$ for restrictions of functions in $C_0^\infty(\mathbb{R}^N)$. Specifically, they proved the following Hardy-Sobolev type inequality: If $N \geq 2$ and $1 < p < \infty$, then for any $u \in C_0^\infty(\mathbb{R}^N)$, the following inequality holds

$$\left(\frac{p-1}{p} \right)^p \int_{\mathbb{R}_+^N} \frac{|u|^p}{(1+x_N)^p} dx \leq \int_{\mathbb{R}_+^N} |\nabla u|^p dx + \int_{\mathbb{R}^{N-1}} |u|^p dx'.$$

These inequalities allow us to apply a variational framework to derive results on the existence and nonexistence of solutions for quasilinear elliptic problems with Robin boundary conditions in the zero-mass case of the form

$$
\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f(x,u), & \mathbb{R}^N_+ \\ |\nabla u|^{p-2}\nabla u\cdot\nu + |u|^{p-2}u = 0, & \mathbb{R}^{N-1}, \end{cases} \tag{3}
$$

with $1 < p \le N$. However, it cannot be applied to study problems with Neumann boundary conditions in the zero mass case. For related results see also [19, 41] and references therein.

In Chapter 1, with the aim of addressing problems with Neumann boundary conditions, we will initially establish a weighted Hardy-Sobolev type inequality in the upper half-space $\mathbb{R}^N_+$. Specifically, for $N \ge 2$ and $0 < p-1 < \gamma$, we will prove the inequality

$$
C^p_{p,\gamma}\int_{\mathbb{R}^N_+}\frac{|u|^p}{(1+x_N)^{p-\gamma}}\,\mathrm{d}x + C^{p-1}_{p,\gamma}\int_{\mathbb{R}^{N-1}}|u|^p\,\mathrm{d}x' \le \int_{\mathbb{R}^N_+}(1+x_N)^\gamma|\nabla u|^p\,\mathrm{d}x, \tag{4}
$$

for all $u \in C_0^\infty(\mathbb{R}^N)$ where $C_{p,\gamma} = \frac{\gamma-p+1}{p}$. We also observe that this inequality was proven in [3] using a different approach.

In addition to producing an inequality for functions in $C_0^\infty(\mathbb{R}^N)$, we also emphasize that we have determined the associated constants for these inequalities. Similar to the classical Hardy and Sobolev inequalities in $\mathbb{R}^N$, we believe that we have obtained the optimal values of the associated constants, in contrast to the results in [41], where the exact constants are unknown. With this result, we obtained more precise a priori estimates for eventual solutions for elliptical problems with Neumann, and Robin boundary conditions that we use the inequality, to obtain Liouville-type results.

The inequality (4) plays a central role in the whole work, allowing us to obtain Sobolev and trace embedding results for a weighted Sobolev space defined on $\mathbb{R}^N_+$. An important application of inequality (4) is the following Sobolev inequality:

$$
\left(\int_{\mathbb{R}^N_+}\frac{|u|^q}{(1+x_N)^{p-\gamma}}\,\mathrm{d}x\right)^{p/q} \le C_0\int_{\mathbb{R}^N_+}(1+x_N)^\gamma|\nabla u|^p\,\mathrm{d}x, \quad \forall u \in C_0^\infty(\mathbb{R}^N),
$$

whenever $q \in [p, p^*]$ if $1 < p < N$ and $q \in [p, \infty)$ if $N = p$.

In Chapter 2, we will use this embedding results as a tools for studying problems with Neumann boundary conditions in the upper half-space. Precisely, we will address results of existence and nonexistence for the following class of quasilinear elliptic problems with indefinite nonlinearity :

$$\begin{cases} -\mathrm{div}(\rho(x)|\nabla u|^{p-2}\nabla u) = a(x)|u|^{q-2}u - b(x)|u|^{s-2}u, & \mathbb{R}^N_+, \\ \qquad |\nabla u|^{p-2}\nabla u \cdot \nu = 0, & \mathbb{R}^{N-1}, \end{cases} \tag{5}$$

where $1 < p \leq N$ and $1 < s, q \leq p^*$ and the weight function ρ, a, b satisfy certain growth conditions. As mentioned in [47], the operator $-\mathrm{div}(\rho(x)|\nabla u|^{p-2}\nabla u)$ appears in many nonlinear diffusion problems. Equations of the form

$$-\mathrm{div}(\rho(x)|\nabla u|^{p-2}\nabla u) = f(x, u)$$

appears in several areas such as differential geometry[38], astrophysics[40], population genetics[11] and elsewhere. Problems of this type have been studied for many authors with different boundary conditions, for instante, in [30], the authors approach the problem

$$\begin{cases} -\mathrm{div}(\rho(x)|\nabla u|^{p-2}\nabla u) = \lambda a(x)|u|^{q-2}u - |u|^{s-2}u & \text{in } \Omega, \\ \rho(x', 0)|\nabla u|^{p-2}\nabla u \cdot \nu + b(x')|u|^{p-2} = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a exterior domain and $p < s < q < p^*$, and show the existence of a parameter λ^* such that there exists weak solutions for $\lambda^* \leq \lambda$. Similar results were obtained in [5] for the problem in the half-space

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \lambda a(x)|u|^{q-2}u - b(x)|u|^{s-2}u & \text{in } \mathbb{R}^N_+, \\ |\nabla u|^{p-2}\nabla u \cdot \nu + |u|^{p-2}u = 0, & \text{on } \mathbb{R}^{N-1}, \end{cases}$$

where the authors consider the supercritical case for s. The critical case, $s = p^*$, with Neumann boundary condition was considered in [53]. The semilinear case with a perturbation was considered in [22]. For classical references on problems involving indefinite nonlinearity, we refer the reader to the papers [7, 9, 13, 35, 41] and references therein. Indefinite elliptic equations appears in the study of prescribing sign-changing scalar curvature problem, see, for instance, [17].

Our approach is inspired by the papers [35, 41], where the authors obtained results on the existence and nonexistence of solutions for problem (5) using the Fibering method (see [25, 48]). We emphasize that in the mentioned works the authors deal only with the case $1 < p < N$ and we also consider the case $p = N$.

In the trace sense, another consequence of inequality (4) we will establish a Sobolev trace inequality.

$$\left(\int_{\mathbb{R}^{N-1}} |u|^q \,\mathrm{d}x'\right)^{p/q} \leq C_0 \int_{\mathbb{R}^N_+} (1+x_N)^\gamma |\nabla u|^p \,\mathrm{d}x, \quad \forall u \in C_0^\infty(\mathbb{R}^N), \tag{6}$$

whenever $q \in [p, p_*]$ if $1 < p < N$ and $q \in [p, \infty)$ if $N = p$.

In Chapter 3, as an application of our trace embeddings obtained in Chapter 1, we will address results of existence and nonexistence for the following class of quasilinear elliptic problems in the upper half-space:

$$\begin{cases} -\mathrm{div}(\rho(x)|\nabla u|^{p-2}\nabla u) = 0 & \text{in } \mathbb{R}^N_+, \\ \rho(x',0)|\nabla u|^{p-2}\nabla u \cdot \nu = h(x')|u|^{q-2}u - m(x')|u|^{s-2}u, & \text{on } \mathbb{R}^{N-1}. \end{cases} \tag{7}$$

For references in the study of elliptical problems with indefinite boundary conditions we refer [51, 54].

We establish Liouville-type results proving that, under certain conditions, the size of the constant $C_{p,\gamma}$, given in the Hardy-Sobolev inequality (4) along with the ratio h/m can lead to a nonexistence scenario. We employ a similar approach to that in [35, 41]. To this, we use the Fibering method in combination with the trace inequality (6).

The contents of the thesis are divided as follows: In Chapter 1, we prove a Hardy type inequality and as consequence, we obtain some weighted Sobolev and trace inequalities. In the second chapter we apply theses results to establish existence and nonexistence results for problem (5) via the Fibering methods. Finally, in Chapter 3, we use our Sobolev trace embedding to address results of existence and nonexistence for problem (7).

Chapter 1

Some weighted Hardy-Sobolev embeddings

In this chapter, we will present some results that are essential for the development of the upcoming chapters. First, we will prove a Hardy-Sobolev type inequality (4) and introduce new weighted Sobolev spaces. As a consequence, we will derive embeddings of these spaces into weighted Lebesgue spaces.

It is noteworthy that our embedding results, as presented in the paper [24], can effectively be applied to investigate the existence and nonexistence of solutions via a variational framework for a broad range of elliptic problems in a zero mass scenario with Neumann boundary conditions in the upper half-space. Additionally, we will establish a Sobolev trace embedding that enables us to address existence and nonexistence results for a class of quasilinear elliptic problems with indefinite nonlinear boundary conditions in the upper half-space.

1.1 The weighted Sobolev space

Definition 1.1 *Let $C_\delta^\infty(\mathbb{R}_+^N)$ the set of the functions in $C_0^\infty(\mathbb{R}^N)$ restricted to $\mathbb{R}_+^N$. For $p > 1$ and $\gamma > p-1$, let us consider the weighted Sobolev space $\mathcal{D}_\gamma^{1,p}(\mathbb{R}_+^N)$ defined as the completion of the space $C_\delta^\infty(\mathbb{R}_+^N)$ with respect to the norm*

$$\|u\|_{\mathcal{D}_\gamma^{1,p}(\mathbb{R}_+^N)} = \left(\int_{\mathbb{R}_+^N} (1+x_N)^\gamma |\nabla u|^p \, \mathrm{d}x + \int_{\mathbb{R}_+^N} \frac{|u|^p}{(1+x_N)^{p-\gamma}} \, \mathrm{d}x \right)^{1/p}.$$

Our initial purpose is to establish weighted Sobolev embeddings of the form

$$\mathcal{D}^{1,p}_{\gamma}(\mathbb{R}^N_+) \hookrightarrow L^q(\mathbb{R}^N_+, (1+x_N)^{\gamma-p}),$$

for some $p, q > 1$ and $\gamma \in \mathbb{R}$. To this, we introduce the weighted Lebesgue space defined by

$$L^q(\mathbb{R}^N_+, (1+x_N)^{\gamma-p}) := \left\{ u : \mathbb{R}^N_+ \to \mathbb{R} \text{ measurable} : \int_{\mathbb{R}^N_+} \frac{|u|^q}{(1+x_N)^{p-\gamma}} \, \mathrm{d}x < \infty \right\}, \tag{1.1}$$

equipped with the norm

$$\|u\|_{q,p,\gamma} := \left(\int_{\mathbb{R}^N_+} \frac{|u|^q}{(1+x_N)^{p-\gamma}} \, \mathrm{d}x \right)^{1/q}. \tag{1.2}$$

1.2 A Hardy-Sobolev type inequality

We begin by stating a Hardy-Sobolev type inequality, which will play a central role in proving our main results in this chapter and throughout all this work. For related Hardy-type inequalities on the upper half-space, we refer to the works [18, 21, 23, 28, 32, 50]. In the mentioned context, we are considering functions in $C^\infty_0(\mathbb{R}^N)$, while in the aforementioned work, the inequalities are established for functions in $C^\infty_0(\mathbb{R}^N_+)$, as seen, for instance, in [21, Theorem 6.9].

Our Hardy-Sobolev type inequality is state as follows.

Theorem 1.1 (Hardy) *Let $N \geq 2$ and $0 < p-1 < \gamma$. Then, for every $u \in C^\infty_0(\mathbb{R}^N)$ it holds*

$$C^p_{p,\gamma} \int_{\mathbb{R}^N_+} \frac{|u|^p}{(1+x_N)^{p-\gamma}} \, \mathrm{d}x + C^{p-1}_{p,\gamma} \int_{\mathbb{R}^{N-1}} |u|^p \, \mathrm{d}x' \leq \int_{\mathbb{R}^N_+} (1+x_N)^{\gamma} |\nabla u|^p \, \mathrm{d}x, \tag{1.3}$$

where

$$C_{p,\gamma} = \frac{\gamma - p + 1}{p}. \tag{1.4}$$

Proof. Let $p > 1$ and σ be a real number to be chosen later. For any $u \in C^\infty_0(\mathbb{R}^N)$, by the Fundamental theorem of calculus we obtain

$$(\sigma+1) \int_{\mathbb{R}^N_+} (1+x_N)^{\sigma} |u|^p \, \mathrm{d}x + \int_{\mathbb{R}^{N-1}} |u|^p \, \mathrm{d}x' = -\int_{\mathbb{R}^N_+} (1+x_N)^{\sigma+1} (|u|^p)_{x_N} \, \mathrm{d}x.$$

On the other hand,

$$\left| - \int_{\mathbb{R}^N_+} (1+x_N)^{\sigma+1} (|u|^p)_{x_N} \, \mathrm{d}x \right| \leq p \int_{\mathbb{R}^N_+} (1+x_N)^{\sigma+1} |u|^{p-1} |\nabla u| \, \mathrm{d}x.$$

For $a, b \geq 0$, real numbers and $\varepsilon > 0$, we can use the Young inequality to get

$$pab \leq \varepsilon(\sigma+1) a^{\frac{p}{p-1}} + \left(\frac{p-1}{\varepsilon(\sigma+1)} \right)^{p-1} b^p.$$

Taking into account that

$$p(1+x_N)^{\sigma+1} |u|^{p-1} |\nabla u| = p(1+x_N)^{(\sigma+1-\frac{\gamma}{p})} |u|^{p-1} (1+x_N)^{\frac{\gamma}{p}} |\nabla u|,$$

we obtain

$$p \int_{\mathbb{R}^N_+} (1+x_N)^{(\sigma+1} |u|^{p-1} |\nabla u| \, \mathrm{d}x \leq \varepsilon(\sigma+1) \int_{\mathbb{R}^N_+} (1+x_N)^{(\sigma+1-\gamma/p)\frac{p}{p-1}} |u|^p \, \mathrm{d}x + \left(\frac{p-1}{\varepsilon(\sigma+1)} \right)^{p-1} \int_{\mathbb{R}^{N_+}} (1+x_N)^{\gamma} |\nabla u|^p \, \mathrm{d}x.$$

Combining the above inequalities, we get

$$(\sigma+1) \int_{\mathbb{R}^N_+} (1+x_N)^{\sigma} |u|^p \, \mathrm{d}x + \int_{\mathbb{R}^{N-1}} |u|^p \, \mathrm{d}x' \leq \varepsilon(\sigma+1) \int_{\mathbb{R}^N_+} (1+x_N)^{(\sigma+1-\gamma/p)\frac{p}{p-1}} |u|^p \, \mathrm{d}x + \left(\frac{p-1}{\varepsilon(\sigma+1)} \right)^{p-1} \int_{\mathbb{R}^N_+} (1+x_N)^{\gamma} |\nabla u|^p \, \mathrm{d}x.$$

Next, choosing σ such that

$$\left(\sigma + 1 - \frac{\gamma}{p} \right) \frac{p}{p-1} = \sigma,$$

we have $\sigma = \gamma - p$ and $\sigma + 1 = \gamma - p + 1 > 0$. Thus, one has

$$(\sigma+1)(1-\varepsilon) \int_{\mathbb{R}^N_+} (1+x_N)^{\sigma} |u|^p \, \mathrm{d}x + \int_{\mathbb{R}^{N-1}} |u|^p \, \mathrm{d}x' \leq \left(\frac{p-1}{\varepsilon(\sigma+1)} \right)^{p-1} \int_{\mathbb{R}^{N_+}} (1+x_N)^{\gamma} |\nabla u|^p \, \mathrm{d}x,$$

which implies

$$(\sigma+1)^p (\varepsilon^{p-1} - \varepsilon^p) \int_{\mathbb{R}^N_+} (1+x_N)^{\sigma} |u|^p \, \mathrm{d}x + (\sigma+1)^{p-1} \varepsilon^{p-1} \int_{\mathbb{R}^{N-1}} |u|^p \, \mathrm{d}x' \leq (p-1)^{p-1} \int_{\mathbb{R}^{N_+}} (1+x_N)^{\gamma} |\nabla u|^p \, \mathrm{d}x.$$

Since the function $f(\varepsilon) = \varepsilon^{p-1} - \varepsilon^p$, for $\varepsilon > 0$ has it maximum at

$$\varepsilon_0 = \frac{p-1}{p} \quad \text{and} \quad f(\varepsilon_0) = (p-1)^{p-1}\frac{1}{p^p},$$

a simple computation shows that

$$\left(\frac{\sigma+1}{p}\right)^p \int_{\mathbb{R}^N_+} (1+x_N)^\sigma |u|^p \,\mathrm{d}x + \left(\frac{\sigma+1}{p}\right)^{p-1} \int_{\mathbb{R}^{N-1}} |u|^p \,\mathrm{d}x' \leq \int_{\mathbb{R}^{N}+} (1+x_N)^\gamma |\nabla u|^p \,\mathrm{d}x,$$

and this concludes the proof. ■

As a consequence of Theorem 1.1, if we consider the seminorm defined by

$$\|u\|_\gamma = \left(\int_{\mathbb{R}^N_+} (1+x_N)^\gamma |\nabla u|^p \,\mathrm{d}x\right)^{1/p},$$

we immediately obtained the following result that will be crucial for our purpose.

Corollary 1.2 *For $0 < p-1 < \gamma$, $\|\cdot\|_\gamma$ define a norm that is equivalent to $\|.\|_{\mathcal{D}^{1,p}_\gamma(\mathbb{R}^N_+)}$ in the Sobolev space $\mathcal{D}^{1,p}_\gamma(\mathbb{R}^N_+)$.*

Proof. From the definition of $\mathcal{D}^{1,p}_\gamma(\mathbb{R}^N_+)$ and a density argument we see that

$$C^p_{p,\gamma} \int_{\mathbb{R}^N_+} \frac{|u|^p}{(1+x_N)^{p-\gamma}} \,\mathrm{d}x \leq \int_{\mathbb{R}^N_+} (1+x_N)^\gamma |\nabla u|^p \,\mathrm{d}x, \quad \forall u \in \mathcal{D}^{1,p}_\gamma(\mathbb{R}^N_+). \tag{1.5}$$

In fact, if $u \in \mathcal{D}^{1,p}_\gamma(\mathbb{R}^N_+)$ there exists $(u_n) \subset C^\infty_\delta(\mathbb{R}^N_+)$ such that $u_n \to u$ in $\mathcal{D}^{1,p}_\gamma(\mathbb{R}^N_+)$. By Theorem 1.1 we have

$$C^p_{p,\gamma} \int_{\mathbb{R}^N_+} \frac{|u_n|^p}{(1+x_N)^{p-\gamma}} \,\mathrm{d}x \leq \int_{\mathbb{R}^N_+} (1+x_N)^\gamma |\nabla u_n|^p \,\mathrm{d}x.$$

Then, passing to the limit we obtain (1.5).

Clearly we have $\|u\|_\gamma \leq \|u\|_{\mathcal{D}^{1,p}_\gamma(\mathbb{R}^N_+)}$. On the other hand, from (1.5) we get

$$\begin{aligned}\|u\|^p_{\mathcal{D}^{1,p}_\gamma(\mathbb{R}^N_+)} =& \int_{\mathbb{R}^N_+} (1+x_N)^\gamma |\nabla u|^p \,\mathrm{d}x + \int_{\mathbb{R}^N_+} \frac{|u|^p}{(1+x_N)^{p-\gamma}} \,\mathrm{d}x \\ \leq& (1 + C^{-p}_{p,\gamma}) \int_{\mathbb{R}^N_+} (1+x_N)^\gamma |\nabla u|^p \,\mathrm{d}x \\ =& (1 + C^{-p}_{p,\gamma}) \|u\|^p_\gamma,\end{aligned}$$

and this completes the proof. ■

1.3 The weighted Sobolev embedding

In this section we present some weighted Sobolev embedding, derived from the Hardy inequality (1.3), that will be usefully in our applications.

Theorem 1.3 (Sobolev inequality) *Let $N \geq 2$ and $0 < p-1 < \gamma$. Then, there exists a constant $C_0 > 0$ such that*

$$\left(\int_{\mathbb{R}^N_+} \frac{|u|^q}{(1+x_N)^{p-\gamma}}\,\mathrm{d}x\right)^{p/q} \leq C\int_{\mathbb{R}^N_+}(1+x_N)^\gamma|\nabla u|^p\,\mathrm{d}x, \quad \forall u \in \mathcal{D}^{1,p}_\gamma(\mathbb{R}^N_+),$$

that is

$$\mathcal{D}^{1,p}_\gamma(\mathbb{R}^N_+) \hookrightarrow L^q(\mathbb{R}^N_+,(1+x_N)^{\gamma-p}),$$

whenever $q \in [p,p^*]$ if $1 < p < N$ and $q \in [p,\infty)$ if $N = p$.

Proof. First assume that $u \in C_0^\infty(\mathbb{R}^N)$ and we shall proceed with the proof in several steps.

Step 1: Assume $1 < p < N$ and $p-1 < \gamma \leq p$.

By the classical Gagliardo-Nirenberg-Sobolev inequality(see [53, 31, 44]) we have

$$\left(\int_{\mathbb{R}^N_+}|v|^{p^*}\,\mathrm{d}x\right)^{(N-p)/N} \leq C\int_{\mathbb{R}^N_+}|\nabla v|^p\,\mathrm{d}x,\ \forall v \in C_0^\infty(\mathbb{R}^N), \tag{1.6}$$

which holds for every $1 \leq p < N$. Thus, for $q \in [p,p^*]$, by the interpolation inequality (A.6), there exists $\alpha \in [0,1]$ such that $q = (1-\alpha)p + \alpha p^*$ and

$$\begin{aligned}\left(\int_{\mathbb{R}^N_+}\frac{|u|^q}{(1+x_N)^{p-\gamma}}\,\mathrm{d}x\right)^{p/q} &\leq \left(\int_{\mathbb{R}^N_+}\frac{|u|^p}{(1+x_N)^{p-\gamma}}\,\mathrm{d}x\right)^{(1-\alpha)\frac{p}{q}}\left(\int_{\mathbb{R}^N_+}\frac{|u|^{p^*}}{(1+x_N)^{p-\gamma}}\,\mathrm{d}x\right)^{\alpha\frac{p}{q}}\\ &\leq \left(\int_{\mathbb{R}^N_+}\frac{|u|^p}{(1+x_N)^{p-\gamma}}\,\mathrm{d}x\right)^{(1-\alpha)\frac{p}{q}}\left(\int_{\mathbb{R}^N_+}|u|^{p^*}\,\mathrm{d}x\right)^{\alpha\frac{p}{q}}.\end{aligned}$$

Then by Theorem 1.1 and (1.6)

$$\begin{aligned}\left(\int_{\mathbb{R}^N_+}\frac{|u|^q}{(1+x_N)^{p-\gamma}}\,\mathrm{d}x\right)^{p/q} \leq& C\left(\int_{\mathbb{R}^N_+}(1+x_N)^\gamma|\nabla u|^p\,\mathrm{d}x\right)^{(1-\alpha)\frac{p}{q}}\left(\int_{\mathbb{R}^N_+}|\nabla u|^p\,\mathrm{d}x\right)^{\alpha\frac{p^*}{q}}\\ \leq& C\left(\int_{\mathbb{R}^N_+}(1+x_N)^\gamma|\nabla u|^p\,\mathrm{d}x\right)^{\frac{(1-\alpha)p+\alpha p^*}{q}}\\ =& C\int_{\mathbb{R}^N_+}(1+x_N)^\gamma|\nabla u|^p\mathrm{d}x.\end{aligned}$$

Step 2: Next, assume $1 < p < N$ and $\gamma > p$.

Once again, by interpolation, it is sufficient to prove that

$$\left(\int_{\mathbb{R}^N_+} \frac{|v|^{p^*}}{(1+x_N)^{p-\gamma}}\,\mathrm{d}x\right)^{(N-p)/N} \le C\int_{\mathbb{R}^N_+}(1+x_N)^\gamma|\nabla v|^p\,\mathrm{d}x,\ \forall v\in C_0^\infty(\mathbb{R}^N). \tag{1.7}$$

Defining $v =: u/(1+x_N)^\sigma$ with $u \in C_0^\infty(\mathbb{R}^N)$ and using a simple computation we see that

$$\nabla v = \frac{1}{(1+x_N)^\sigma}\left(\nabla u - (0', \frac{\sigma u}{(1+x_N)})\right)$$

and consequently, there exists a constant $C = C(p,\sigma) > 0$ such that

$$|\nabla v|^p \le C\left(\frac{|\nabla u|^p}{(1+x_N)^{p\sigma}} + \frac{|u|^p}{(1+x_N)^{(\sigma+1)p}}\right).$$

This, together with (1.6), implies that

$$\left(\int_{\mathbb{R}^N_+} \frac{|u|^{p^*}}{(1+x_N)^{\sigma p^*}}\,\mathrm{d}x\right)^{(N-p)/N} \le C\int_{\mathbb{R}^N_+}\left(\frac{|\nabla u|^p}{(1+x_N)^{p\sigma}} + \frac{|u|^p}{(1+x_N)^{(\sigma+1)p}}\right)\mathrm{d}x. \tag{1.8}$$

Choosing $\sigma < 0$ such that $\sigma p^* = p - \gamma$ we deduce that

$$-\sigma p = \frac{(\gamma-p)(N-p)}{N} < \gamma - p < \gamma$$

and hence $(\sigma+1)p > p - \gamma$. Thus, from (1.8) we get

$$\begin{aligned}\left(\int_{\mathbb{R}^N_+} \frac{|u|^{p^*}}{(1+x_N)^{p-\gamma}}\,\mathrm{d}x\right)^{(N-p)/N} &\le C\int_{\mathbb{R}^N_+}\left(\frac{|\nabla u|^p}{(1+x_N)^{p\sigma}} + \frac{|u|^p}{(1+x_N)^{(\sigma+1)p}}\right)\mathrm{d}x\\ &\le C\int_{\mathbb{R}^N_+}\left((1+x_N)^\gamma|\nabla u|^p + \frac{|u|^p}{(1+x_N)^{p-\gamma}}\right)\mathrm{d}x\end{aligned}$$

and by the Hardy inequality (1.3) we conclude that (1.7) holds.

Step 3: Assume that $p = N$.

For $u \in C_0^\infty(\mathbb{R}^N)$, applying inequality (1.6) with $p = 1$ and $v = \frac{|u|^N}{(1+x_N)^\sigma}$ we get

$$\left(\int_{\mathbb{R}^N_+} \frac{|u|^{N^2/(N-1)}}{(1+x_N)^{\frac{\sigma N}{N-1}}}\,\mathrm{d}x\right)^{(N-1)/N} \le C|\sigma|\int_{\mathbb{R}^N_+}\frac{|u|^N}{(1+x_N)^{\sigma+1}}\,\mathrm{d}x + CN\int_{\mathbb{R}^N_+}\frac{|u|^{N-1}|\nabla u|}{(1+x_N)^\sigma}\,\mathrm{d}x.$$

Choosing $\sigma + 1 = N - \gamma$ and using Young's inequality, we obtain

$$\int_{\mathbb{R}^N_+}\frac{|u|^{N-1}|\nabla u|}{(1+x_N)^\sigma}\,\mathrm{d}x = \int_{\mathbb{R}^N_+}\frac{|u|^{N-1}(1+x_N)^{\gamma/N}|\nabla u|}{(1+x_N)^{\sigma+\gamma/N}}\,\mathrm{d}x$$

$$\leq C \int_{\mathbb{R}^N_+} \left(\frac{|u|^N}{(1+x_N)^{N-\gamma}} + (1+x_N)^\gamma |\nabla u|^N \right) \mathrm{d}x,$$

where we used that

$$\left(\sigma + \frac{\gamma}{N}\right) \frac{N}{N-1} = N - \gamma. \tag{1.9}$$

Since $\frac{\sigma N}{N-1} \leq N - \gamma$, using Theorem 1.1 we get

$$\left(\int_{\mathbb{R}^N_+} \frac{|u|^{N^2/(N-1)}}{(1+x_N)^{N-\gamma}} \mathrm{d}x \right)^{(N-1)/N} \leq C \int_{\mathbb{R}^N_+} (1+x_N)^\gamma |\nabla u|^N \mathrm{d}x. \tag{1.10}$$

Thus, interpolation inequality (A.6), with $\theta \in [0,1]$ such that $q = (1-\theta)N + \theta N^2/(N-1)$, implies

$$\begin{aligned}
\left(\int_{\mathbb{R}^N_+} \frac{|u|^q}{(1+x_N)^{N-\gamma}} \mathrm{d}x \right)^{N/q} &\leq \left(\int_{\mathbb{R}^N_+} \frac{|u|^N}{(1+x_N)^{N-\gamma}} \mathrm{d}x \right)^{(1-\theta)\frac{N}{q}} \left(\int_{\mathbb{R}^N_+} \frac{|u|^{N^2/(N-1)}}{(1+x_N)^{N-\gamma}} \mathrm{d}x \right)^{\theta \frac{N}{q}} \\
&\leq C \left(\int_{\mathbb{R}^N_+} (1+x_N)^\gamma |\nabla u|^N \mathrm{d}x \right)^{(1-\theta)\frac{N}{q} + \frac{N^2}{N-1}\frac{\theta}{q}} \\
&= C \int_{\mathbb{R}^N_+} (1+x_N)^\gamma |\nabla u|^N \mathrm{d}x
\end{aligned}$$

for any $q \in [N, N^2/(N-1)]$. Since $N < N+1 < N^2/(N-1)$, in particular we get

$$\left(\int_{\mathbb{R}^N_+} \frac{|u|^{N+1}}{(1+x_N)^{N-\gamma}} \mathrm{d}x \right)^{N/(N+1)} \leq C \int_{\mathbb{R}^N_+} (1+x_N)^\gamma |\nabla u|^N \mathrm{d}x. \tag{1.11}$$

Once again, applying (1.6) with $p = 1$ and $v = \frac{|u|^{N+1}}{(1+x_N)^\sigma}$ and using Young's inequality we have

$$\left(\int_{\mathbb{R}^N_+} \frac{|u|^{\frac{N(N+1)}{N-1}}}{(1+x_N)^{\frac{\sigma N}{N-1}}} \mathrm{d}x \right)^{(N-1)/N} \leq C \left(\int_{\mathbb{R}^N_+} \frac{|u|^{N+1}}{(1+x_N)^{\sigma+1}} \mathrm{d}x + \int_{\mathbb{R}^N_+} \frac{|u|^N |\nabla u|}{(1+x_N)^\sigma} \mathrm{d}x \right).$$

Using Hölder's inequality and (1.9), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N_+} \frac{|u|^N |\nabla u|}{(1+x_N)^\sigma} \mathrm{d}x &= \int_{\mathbb{R}^N_+} \frac{|u|^N (1+x_N)^{\gamma/N} |\nabla u|}{(1+x_N)^{\sigma+\gamma/N}} \mathrm{d}x \\
&\leq \left(\int_{\mathbb{R}^N_+} \frac{|u|^{\frac{N^2}{N-1}}}{(1+x_N)^{N-\gamma}} \mathrm{d}x \right)^{(N-1)/N} \left(\int_{\mathbb{R}^N_+} (1+x_N)^\gamma |\nabla u|^N \mathrm{d}x \right)^{1/N}.
\end{aligned}$$

Thus, from (1.10) and (1.11) one has

$$\left(\int_{\mathbb{R}^N_+} \frac{|u|^{\frac{N(N+1)}{N-1}}}{(1+x_N)^{N-\gamma}} \mathrm{d}x \right)^{\frac{N-1}{N+1}} \leq C \int_{\mathbb{R}^N_+} (1+x_N)^\gamma |\nabla u|^N \mathrm{d}x$$

and interpolation implies

$$\left(\int_{\mathbb{R}^N_+} \frac{|u|^q}{(1+x_N)^{N-\gamma}}\,\mathrm{d}x\right)^{N/q} \leq C\int_{\mathbb{R}^N_+}(1+x_N)^\gamma|\nabla u|^N\,\mathrm{d}x,$$

for any $q\in[N,\frac{N(N+1)}{N-1}]$. Reiterating this argument with $k=N+2,N+3,\dots$, we get

$$\left(\int_{\mathbb{R}^N_+} \frac{|u|^{Nk/(N-1)}}{(1+x_N)^{N-\gamma}}\,\mathrm{d}x\right)^{(N-1)/k} \leq C\int_{\mathbb{R}^N_+}(1+x_N)^\gamma|\nabla u|^N\,\mathrm{d}x.$$

Now, given $q\in(N,\infty)$, we can choose $k\geq N$ such that $q\in(N,Nk/(N-1))$ and by interpolation, we can conclude the desired inequality for smooth functions.

Let $u\in\mathcal{D}^{1,p}_\gamma(\mathbb{R}^N_+)$ and $(u_n)\subset C_0^\infty(\mathbb{R}^N)$ such that $u_n\to u$ in $\mathcal{D}^{1,p}_\gamma(\mathbb{R}^N_+)$. From the smooth functions case, we see that

$$\left(\int_{\mathbb{R}^N_+} \frac{|u_n-u_m|^q}{(1+x_N)^{p-\gamma}}\,\mathrm{d}x\right)^{p/q} \leq C\int_{\mathbb{R}^N_+}(1+x_N)^\gamma|\nabla u_n-\nabla u_m|^p\,\mathrm{d}x\longrightarrow 0,$$

Thus, (u_n) is a Cauchy sequence in $L^q(\mathbb{R}^N_+,(1+x_N)^{\gamma-p})$, which is a Banach space in view of Theorem A.1. Therefore, there exists $w\in L^q(\mathbb{R}^N_+,(1+x_N)^{\gamma-p})$ such that

$$u_n\to w\quad\text{in}\quad L^q(\mathbb{R}^N_+,(1+x_N)^{\gamma-p}).$$

Given that $u_n\to u$ in $L^p(\mathbb{R}^N_+,(1+x_N)^{\gamma-p})$, we have $w=u$. Thus, passing to the limit in the inequality

$$\left(\int_{\mathbb{R}^N_+} \frac{|u_n|^q}{(1+x_N)^{p-\gamma}}\,\mathrm{d}x\right)^{p/q} \leq C\int_{\mathbb{R}^N_+}(1+x_N)^\gamma|\nabla u_n|^p\,\mathrm{d}x,$$

and we obtain the desired result. ∎

1.4 A Sobolev trace embedding

In this section, we present the trace embedding results that enable the treatment of problems with nonlinear boundary conditions (see Chapter 3). To this, we will introduce here a new weighted Sobolev space.

Definition 1.2 *For $p>1$ and $\gamma>p-1$, we consider the weighted Sobolev space $\mathcal{E}^{1,p}_\gamma(\mathbb{R}^N_+)$ defined as the completion of the space $C_\delta^\infty(\mathbb{R}^N_+)$ with respect to the norm*

$$\|u\|_{\mathcal{E}^{1,p}_\gamma(\mathbb{R}^N_+)}=\left(\int_{\mathbb{R}^N_+}(1+x_N)^\gamma|\nabla u|^p\,\mathrm{d}x+\int_{\mathbb{R}^{N-1}}|u|^p\,\mathrm{d}x\right)^{1/p}.$$

As a consequence of Theorem 1.1 we have the following result.

Corollary 1.4 *Let $0 < p-1 < \gamma$. In the space $\mathcal{E}^{1,p}_\gamma(\mathbb{R}^N_+)$, the norm $\|\cdot\|_{\mathcal{E}^{1,p}_\gamma(\mathbb{R}^N_+)}$ is equivalent to $\|\cdot\|_\gamma$. In particular we have*

$$\mathcal{E}^{1,p}_\gamma(\mathbb{R}^N_+) = \mathcal{D}^{1,p}_\gamma(\mathbb{R}^N_+).$$

Proof. If $0 < p-1 < \gamma$, by inequality (1.3), the definition of $\mathcal{E}^{1,p}_\gamma(\mathbb{R}^N_+)$ and a density argument we see that

$$C^{p-1}_{p,\gamma} \int_{\mathbb{R}^{N-1}} |u|^p \, \mathrm{d}x \le \int_{\mathbb{R}^N_+} (1+x_N)^\gamma |\nabla u|^p \, \mathrm{d}x, \quad \forall u \in \mathcal{E}^{1,p}_\gamma(\mathbb{R}^N_+). \tag{1.12}$$

In fact, if $u \in \mathcal{E}^{1,p}_\gamma(\mathbb{R}^N_+)$, there exists $(u_n) \subset C^\infty_\delta(\mathbb{R}^{\mathbb{N}}_+)$ such that $u_n \to u$ in $\mathcal{E}^{1,p}_\gamma(\mathbb{R}^N_+)$. By Theorem 1.1 we have

$$C^{p-1}_{p,\gamma} \int_{\mathbb{R}^{N-1}} |u_n|^p \, \mathrm{d}x \le \int_{\mathbb{R}^N_+} (1+x_N)^\gamma |\nabla u_n|^p \, \mathrm{d}x, \quad \forall u \in \mathcal{E}^{1,p}_\gamma(\mathbb{R}^N_+).$$

Thus, taking the limit we obtain (1.12). Then, for all $u \in \mathcal{E}^{1,p}_\gamma(\mathbb{R}^N_+)$ we have

$$\|u\|_\gamma \le \|u\|_{\mathcal{E}^{1,p}_\gamma(\mathbb{R}^N_+)} \quad \text{and} \quad \|u\|_{\mathcal{E}^{1,p}_\gamma(\mathbb{R}^N_+)} \le C\|u\|_\gamma.$$

Since $\|.\|_\gamma$ is equivalent to $\|.\|_{\mathcal{D}^{1,p}_\gamma(\mathbb{R}^N_+)}$, by Theorem 1.1, we have

$$\mathcal{E}^{1,p}_\gamma(\mathbb{R}^N_+) = \overline{C^\infty_\delta(\mathbb{R}^{\mathbb{N}}_+)}^{\|.\|_{\mathcal{E}^{1,p}_\gamma(\mathbb{R}^N_+)}} = \overline{C^\infty_\delta(\mathbb{R}^N_+)}^{\|.\|_\gamma} = \overline{C^\infty_\delta(\mathbb{R}^N_+)}^{\|.\|_{\mathcal{D}^{1,p}_\gamma(\mathbb{R}^N_+)}} = \mathcal{D}^{1,p}_\gamma(\mathbb{R}^N_+).$$

■

This fact allows us to obtain the following Sobolev trace embedding which plays a fundamental role in our applications:

Theorem 1.5 (Trace embedding) *Let $N \ge 2$ and $0 < p-1 < \gamma$. Then, there exists a constant $C_0 > 0$ such that*

$$\left(\int_{\mathbb{R}^{N-1}} |u|^q \, \mathrm{d}x' \right)^{p/q} \le C_0 \int_{\mathbb{R}^N_+} (1+x_N)^\gamma |\nabla u|^p \, \mathrm{d}x, \quad \forall u \in \mathcal{D}^{1,p}_\gamma(\mathbb{R}^N_+), \tag{1.13}$$

which is equivalent to

$$\mathcal{D}^{1,p}_\gamma(\mathbb{R}^N_+) \hookrightarrow L^q(\mathbb{R}^{N-1}),$$

for all $q \in [p, p_*]$ if $1 < p < N$ and $q \in [N, \infty)$ if $p = N$.

Proof. First, assume $u \in C_\delta^\infty(\mathbb{R}_+^N)$. If $1 < p < N$, we can use the trace embedding (see [26, 43]) and obtain

$$\left(\int_{\mathbb{R}^{N-1}} |u|^{p_*}\,\mathrm{d}x'\right)^{\frac{p}{p_*}} \le C\int_{\mathbb{R}_+^N} |\nabla u|^p\,\mathrm{d}x \le C\int_{\mathbb{R}_+^N}(1+x_N)^\gamma|\nabla u|^p\,\mathrm{d}x, \quad \forall u \in C_\delta^\infty(\mathbb{R}_+^N).$$

Given $q \in [p, p_*]$, by the classical interpolation inequality we have

$$\|u\|_{L^q(\mathbb{R}^{N-1})} \le \|u\|_{L^p(\mathbb{R}^{N-1})}^\alpha \|u\|_{L^{p_*}(\mathbb{R}^{N-1})}^{1-\alpha},$$

with $\alpha \in [0,1]$. Then, by using the Hardy inequality (1.3) we obtain

$$\begin{aligned}
\left(\int_{\mathbb{R}^{N-1}} |u|^q \mathrm{d}x'\right)^{p/q} &\le \left(\int_{\mathbb{R}^{N-1}} |u|^p \mathrm{d}x'\right)^\alpha \left(\int_{\mathbb{R}^{N-1}} |u|^{p_*} \mathrm{d}x'\right)^{\frac{p(1-\alpha)}{p_*}} \\
&\le \left(\int_{\mathbb{R}_+^N}(1+x_N)^\gamma|\nabla u|^p\mathrm{d}x\right)^\alpha \left(\int_{\mathbb{R}_+^N}(1+x_N)^\gamma|\nabla u|^p\mathrm{d}x\right)^{1-\alpha} \\
&= \int_{\mathbb{R}_+^N}(1+x_N)^\gamma|\nabla u|^p\mathrm{d}x.
\end{aligned}$$

Now, suppose that $p = N$, let $q \ge N$ and $\sigma > 0$ to be chosen later. First, we observe that

$$\begin{aligned}
|u(x',0)|^q &= -\int_0^\infty \frac{\partial}{\partial x_N}\left(\frac{|u|^q}{(1+x_N)^\sigma}\right)\mathrm{d}x_N \\
&\le q\int_0^\infty \frac{|u|^{q-1}|\nabla u|}{(1+x_N)^\sigma}\,\mathrm{d}x_N + \sigma\int_0^\infty \frac{|u|^q}{(1+x_N)^{\sigma+1}}\,\mathrm{d}x_N.
\end{aligned}$$

By integration, we get

$$\int_{\mathbb{R}^{N-1}} |u|^q\,\mathrm{d}x' \le q\int_{\mathbb{R}_+^N}\frac{|u|^{q-1}|\nabla u|}{(1+x_N)^\sigma}\,\mathrm{d}x + \sigma\int_{\mathbb{R}_+^N}\frac{|u|^q}{(1+x_N)^{\sigma+1}}\,\mathrm{d}x.$$

Choosing $\sigma > 0$ such that $\frac{\sigma N}{N-1} \ge N-\gamma$, we have that $\sigma+1 \ge N-\gamma$ whenever $\gamma \ge 0$. Then, we can apply Theorem 1.3 to obtain a constant $C_0 > 0$ such that

$$\int_{\mathbb{R}_+^N}\frac{|u|^q}{(1+x_N)^{\sigma+1}}\,\mathrm{d}x \le \int_{\mathbb{R}_+^N}\frac{|u|^q}{(1+x_N)^{N-\gamma}}\,\mathrm{d}x \le C_0\left(\int_{\mathbb{R}_+^N}(1+x_N)^\gamma|\nabla u|^N\,\mathrm{d}x\right)^{q/N}.$$

On the other hand, by Hölder's inequality, we have

$$\int_{\mathbb{R}_+^N}\frac{|u|^{q-1}|\nabla u|}{(1+x_N)^\sigma}\,\mathrm{d}x \le \left(\int_{\mathbb{R}_+^N}\frac{|u|^{\frac{(q-1)N}{N-1}}}{(1+x_N)^{\frac{\sigma N}{N-1}}}\,\mathrm{d}x\right)^{(N-1)N}\left(\int_{\mathbb{R}_+^N}|\nabla u|^N\,\mathrm{d}x\right)^{1/N}.$$

Since $(q-1)N/(N-1) \ge N$, once again by applying Theorem 1.3, we get

$$\int_{\mathbb{R}_+^N}\frac{|u|^{q-1}|\nabla u|}{(1+x_N)^\sigma}\,\mathrm{d}x \le \left(\int_{\mathbb{R}_+^N}\frac{|u|^{\frac{(q-1)N}{N-1}}}{(1+x_N)^{N-\gamma}}\,\mathrm{d}x\right)^{(N-1)N}\left(\int_{\mathbb{R}_+^N}(1+x_N)^\gamma|\nabla u|^N\,\mathrm{d}x\right)^{1/N}$$

$$\leq C_1 \left(\int_{\mathbb{R}^N_+} (1+x_N)^\gamma |\nabla u|^N \, \mathrm{d}x \right)^{q/N},$$

for some constant $C_1 > 0$. Combining the above inequalities, we conclude the proof of the inequality for functions in $C^\infty_\delta(\mathbb{R}^{\mathbb{N}}_+)$.

Now, let $u \in \mathcal{D}^{1,p}_\gamma(\mathbb{R}^N_+)$ and consider a sequence $(u_n) \subset C^\infty_\delta(\mathbb{R}^N_+)$ such that

$$u_n \to u \quad \text{in} \quad \mathcal{D}^{1,p}_\gamma(\mathbb{R}^N_+).$$

By Corollary 1.4, we see that

$$u_n \to u \quad \text{in} \quad \mathcal{E}^{1,p}_\gamma(\mathbb{R}^N_+).$$

By inequality (1.13) for functions in $C^\infty_\delta(\mathbb{R}^N_+)$ we have

$$\left(\int_{\mathbb{R}^{N-1}} |u_n - u_m|^q \, \mathrm{d}x' \right)^{p/q} \leq C \int_{\mathbb{R}^N_+} (1+x_N)^\gamma |\nabla u_n - \nabla u_m|^p \, \mathrm{d}x \longrightarrow 0.$$

Then, there exists $w \in L^q(\mathbb{R}^{N-1})$ such that

$$u_n \to w \quad \text{in} \quad L^q(\mathbb{R}^{N-1}).$$

By the convergence in $\mathcal{E}^{1,p}_\gamma(\mathbb{R}^N_+)$ we have

$$u_n \to u \quad \text{in} \quad L^p(\mathbb{R}^{N-1}),$$

and therefore $u = w$. Thus, taking the limit in the inequality

$$\left(\int_{\mathbb{R}^{N-1}} |u_n|^q \, \mathrm{d}x' \right)^{p/q} \leq C \int_{\mathbb{R}^N_+} (1+x_N)^\gamma |\nabla u_n|^p \, \mathrm{d}x$$

we conclude the proof of Theorem 1.5. ■

1.5 Some comments

Some questions remains open with respect to the $\mathcal{D}^{1,p}_\gamma(\mathbb{R}^N_+)$ theory:

- We observe that the condition $\gamma > p-1$ is sufficient to prove the Hardy type inequality (1.3), which aligns with the sufficient condition proven in [39]. However, determining a necessary condition on γ for (1.3) remains an open question.

- If $\gamma \geq p$, we observe that

$$\mathcal{D}_{\gamma}^{1,p}(\mathbb{R}_{+}^{N}) \hookrightarrow W^{1,p}(\mathbb{R}_{+}^{N}),$$

resulting in bounded state solutions. Therefore, a natural question arises: what happens in the case when $p-1 < \gamma < p$?

Chapter 2

Application 1: Liouville type and Existence results for a quasilinear elliptic problem via Fibering method

In this chapter, we present the results obtained in the paper [24]. Our discussion focuses on Liouville-type results and the existence of solutions for the following model of quasilinear elliptic problems:

$$\begin{cases} -\text{div}(\rho(x)|\nabla u|^{p-2}\nabla u) = a(x)|u|^{q-2}u - b(x)|u|^{s-2}u, & \mathbb{R}^N_+, \\ |\nabla u|^{p-2}\nabla u \cdot \nu = 0, & \mathbb{R}^{N-1}, \end{cases} \tag{$\mathcal{P}_1$}$$

where ν is the unit outer normal to the boundary $\partial\mathbb{R}^N_+ := \mathbb{R}^{N-1}$, $1 < q, s \leq p^*$ if $1 < p < N$ and $1 < q, s < \infty$ if $p = N$.

Throughout this chapter, we assume that ρ, a, $b \in L^1_{\text{loc}}(\mathbb{R}^N_+)$ and are positive functions. For $1 < p < N$, we denote by $p^* := Np/(N-p)$ the critical exponent for the Sobolev embedding and $p^* = \infty$ if $p = N$.

From a mathematical perspective, the nature of problem ($\mathcal{P}_1$) is described according to the behavior of the competing terms $a(x)|u|^{q-2}u$ and $b(x)|u|^{s-2}u$ as determined by the integrability properties of the ratio $a(x)^{1/p}/b(x)^{1/s}$ (as discussed in Alama-Tarantello [7, 9]). The interplay between the weight functions $a(x)$ and $b(x)$ significantly impacts the existence and nonexistence of solutions to ($\mathcal{P}_1$) and has garnered substantial attention among researchers, see, for instance, [13, 30]. We mention that the weight functions are not necessarily spherically symmetric. Thus, we are motivated to pursue new weighted Sobolev embeddings to enable variational frameworks in diverse settings.

In the works [41, 35], based on a Hardy-type inequality due to K. Pflüger, [47] (see also [34]) and the Fibering Method, it was established the existence and Liouville-type results for a similar class of quasilinear elliptic problems with Robin boundary condition in an unbounded domain $\Omega \subset \mathbb{R}^N$ with noncompact smooth boundary, $1 < p < N$, $q, s \in (1, p^*)$, $\rho \in L^\infty(\Omega) \cap L^\infty(\partial\Omega)$ and $0 < \rho_0 < \rho(x)$, where the potentials a and b vanish at infinity.

It is important to mention that the approach in [41] used to treat a problem with Robin boundary conditions cannot be used to study problems with Neumann boundary conditions because their argument is based on K. Pflüger's inequality, which does not allow one to eliminate the boundary term. We also highlight that based on a Hardy-type inequality in [2], the existence and nonexistence results for semilinear elliptic problems with Robin boundary conditions were addressed using a variational approach. For related results, see also [30].

Our approach here is based on a new class of Hardy-type inequalities, which allows us to consider problems with Neumann boundary conditions. We also emphasize that we have determined the associated constants for these inequalities. Similar to the classical Hardy and Sobolev inequalities in $\mathbb{R}^N$, we believe that we have obtained the optimal values of the associated constants, in contrast to the results in [41], where the exact constants are unknown. Hence, we gave a partial answer to a question raised in [41]. With these results, we obtained more precise a priori estimates for eventual solutions of $(\mathcal{P}_1)$ to obtain Liouville-type results. Moreover, we have incorporated the extreme scenario where $p = N$ into our analysis.

Henceforth, we presume that the weight function ρ adheres to the following technical hypothesis:

(H_0) there are constants $\rho_0 > 0$ and $\gamma > p - 1$ such that

$$\rho(x) \geq \rho_0(1 + x_N)^\gamma \quad \text{a.e. in } \mathbb{R}^N_+.$$

First, we must introduce our variational setting to describe our results for $(\mathcal{P}_1)$. Let $C^\infty_\delta(\mathbb{R}^N_+)$ be the set of all functions $u \in C^\infty_0(\mathbb{R}^N)$ restricted to $\mathbb{R}^N_+$

Definition 2.1 *Assume assumption* (H_0). *Let us consider the weighted space* E *defined as the closure of* $C_\delta^\infty(\mathbb{R}_+^N)$ *with respect to the norm*

$$\|u\| := \left(\int_{\mathbb{R}_+^N} \rho(x)|\nabla u|^p \,\mathrm{d}x\right)^{1/p}.$$

Clearly, from (H_0) we have the continuous embedding

$$(E, \|\cdot\|) \hookrightarrow \left(\mathcal{D}_\gamma^{1,p}(\mathbb{R}_+^N), \|\cdot\|_\gamma\right).$$

Here, by a weak solution of $(\mathcal{P}_1)$ we mean a function $u \in E$ such that

$$\int_{\mathbb{R}_+^N} \rho(x)|\nabla u|^{p-2}\nabla u \nabla\varphi \,\mathrm{d}x = \int_{\mathbb{R}_+^N} (a(x)|u|^{q-2}u - b(x)|u|^{s-2}u)\varphi \,\mathrm{d}x, \tag{2.1}$$

holds for every $\varphi \in C_\delta^\infty(\mathbb{R}_+^N)$.

2.1 Main results

Before establishing our main results, let us start by introducing some notation that will be used throughout. We denote by

$$C_{p,\gamma} := \frac{\gamma - p + 1}{p}$$

the constant given in the inequality (1.3) and

$$\eta(r,q,t) := \frac{(t-r)^{t-r}}{(q-r)^{q-r}(t-q)^{t-q}} \quad \text{if} \quad r < q < t. \tag{2.2}$$

2.1.1 Liouville-type results

To state our nonexistence results, we shall introduce the following class of functions:

$$\mathcal{K}_0 := \left\{k \in C(\overline{\mathbb{R}_+^N}, (0,\infty)) : k(x)(1+x_N)^{p-\gamma} \in L^\infty(\mathbb{R}_+^N)\right\}.$$

Our first concern is to assert nonexistence when $s < q < p$.

Theorem 2.1 (p-sublinear case) *Assume* (H_0) *and suppose that* $b \in \mathcal{K}_0 \cap L^1(\mathbb{R}_+^N)$. *If* $1 < s < q < p \leq N$ *and* $a/b \in L^\infty(\mathbb{R}_+^N)$ *with*

$$\left\|\frac{a}{b}\right\|_\infty^{p-s} \left(\frac{b_0 C_{p,\gamma}^{-p}}{\rho_0}\right)^{q-s} < \eta(s,q,p), \tag{2.3}$$

then ($\mathcal{P}_1$) *possesses only the trivial weak solution. Hereafter,* $b_0 > 0$ *denotes a constant such that*

$$b(x)(1 + x_N)^{p-\gamma} \le b_0 \quad \textit{in} \quad \mathbb{R}^N_+.$$

Remark 2.1 *Straightforward computation shows that, for* $\lambda > 0$ *sufficiently small, the functions,*

$$a(x) = \frac{\lambda(1 + x_N)^{\gamma-p}}{(1 + |x|)^{\theta_1}} \quad \textit{and} \quad b(x) = \frac{(1 + x_N)^{\gamma-p}}{(1 + |x|)^{\theta_2}},$$

satisfy the assumptions of Theorem 2.1 whenever $\max\{N, N + \gamma - p\} < \theta_2 \le \theta_1$.

In our second nonexistence result, we address the case where $p < q < s$.

Theorem 2.2 (p-superlinear case) *Assume* (H_0) *and suppose that* $b \in \mathcal{K}_0$. *If* $1 < p < N$, $p < q < s \le p^*$ *and* $a/b \in L^\infty(\mathbb{R}^N_+)$ *with*

$$\left\|\frac{a}{b}\right\|_\infty^{s-p} \left(\frac{b_0 C_{p,\gamma}^{-p}}{\rho_0}\right)^{s-q} < \eta(p, q, s), \tag{2.4}$$

then ($\mathcal{P}_1$) *possesses only the trivial weak solution. Moreover, the same result holds if* $p = N$ *and* $p < q < s < \infty$.

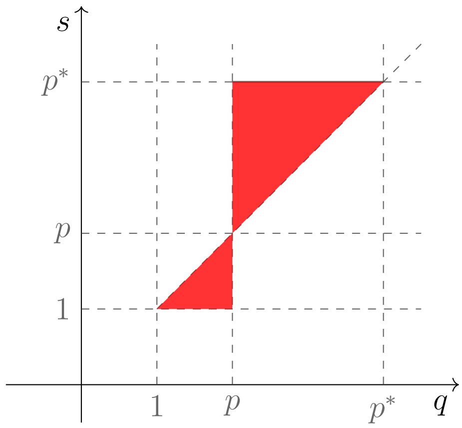

Figure 2.1: Nonexistence of solutions for ($\mathcal{P}_1$)

Remark 2.2 *Typical examples of functions satisfying the assumptions of Theorem 2.2 are*

$$a(x) = \lambda(1 + x_N)^{\theta_1} \quad \textit{and} \quad b(x) = \mu(1 + x_N)^{\theta_2},$$

with $\theta_1 \le \theta_2 \le \gamma - p$ *and* $\lambda > 0$ *sufficiently small or* $\mu > 0$ *sufficiently large.*

The basic idea to prove Theorems 2.1 and 2.2 relies on refining the arguments presented in [41] by using a specific key estimate.

2.1.2 Existence results

To establish our existence results, it is necessary to impose additional hypotheses on the weight functions a and b to ensure the compactness of the Sobolev embedding, thereby enabling the application of the Fibering Method as demonstrated in papers [35, 41]. To this end, we introduce the following class of functions:

$$\mathcal{K} := \left\{ k \in C(\overline{\mathbb{R}^N_+}, (0,\infty)) \text{ such that } \lim_{|x|\to\infty} k(x)(1+x_N)^{p-\gamma} = 0 \right\}.$$

We assume $1 < p \leq N$ to state our existence results. Our first result considers $p-$superlinear case $p < s < q$ or when $s < p < q$.

Theorem 2.3 *If* (H_0) *holds, then* $(\mathcal{P}_1)$ *has a nontrivial and nonnegative weak solution when one of the following conditions occurs:*

i) $p < s < q < p^*$ *and* $a, b \in \mathcal{K}$*;*

ii) $1 < s < p < q < p^*$, $a \in \mathcal{K}$, *and* $b \in L^1(\mathbb{R}^N_+) \cap \mathcal{K}_0$.

On our second existence result, we treat the $p-$sublinear case $s < q < p$ or when $q < p < s$.

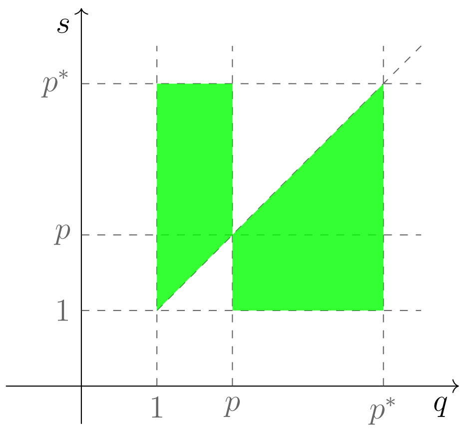

Figure 2.2: Existence of nontrivial solutions for $(\mathcal{P}_1)$

Theorem 2.4 *If* (H_0) *holds, then* $(\mathcal{P}_1)$ *has a nontrivial and nonnegative weak solution when one of the following conditions occurs:*

i) $1 < q < s < p$ *and* $a, b \in L^1(\mathbb{R}^N_+) \cap \mathcal{K}_0$*;*

ii) $1 < q < p < s < p^*$, $a \in L^1(\mathbb{R}^N_+) \cap \mathcal{K}_0$ *and* $b \in \mathcal{K}$.

To present our third existence result, we consider the functionals defined on E by

$$A(u) = \int_{\mathbb{R}^N_+} a(x)|u|^q \,\mathrm{d}x \quad \text{and} \quad B(u) = \int_{\mathbb{R}^N_+} b(x)|u|^s \,\mathrm{d}x. \tag{2.5}$$

We note that under the assumptions of Theorem 2.2, for $v \in E$, the following key inequality holds true (see Lemma 2.9)

$$A(v)^{s-p} < \eta(p,q,s)B(v)^{q-p}\|v\|^{p(s-q)} < \eta(p,q,s)\left(\frac{q}{p}\right)^{s-q} B(v)^{q-p}\|v\|^{p(s-q)}.$$

We consider the existence of solutions for ($\mathcal{P}_1$) in a subset of the complementary case of this inequality. Precisely, considering the set

$$\mathcal{D}_1 := \left\{ u \in E : A(u)^{s-p} > \left(\frac{q}{p}\right)^{s-q} \eta(p,q,s)B(u)^{q-p}\|u\|^{p(s-q)} \right\}, \tag{2.6}$$

a counterpart of Theorem 2.2 reads as follows.

Theorem 2.5 *Assume that* (H_0) *holds. If* $a, b \in \mathcal{K}$, $p < q < s < p^*$, $\mathcal{D}_1 \neq \emptyset$ *and*

$$\frac{a^{1/q}}{b^{1/s}} \in L^{\frac{sq}{s-q}}(\mathbb{R}^N_+), \tag{2.7}$$

then, ($\mathcal{P}_1$) *has a nontrivial and nonnegative weak solution.*

Remark 2.3 *The functions* $a = b = \lambda k$, *with* k *given by*

$$k(x) = (1 + x_N)^{\gamma-p}(1 + |x|)^{-\theta},$$

satisfy the assumptions of Theorem 2.5 for $\theta > \max\{N, N + \gamma - p\}$ *and* λ *sufficiently large. In fact, first, we observe that*

$$\left[\frac{a(x)^{1/q}}{b(x)^{1/s}}\right]^{\frac{sq}{s-q}} = a(x) = \lambda(1 + x_N)^{\gamma-p}(1 + |x|)^{-\theta} \in L^1(\mathbb{R}^N_+),$$

whenever $\theta > N + \gamma - p$. *For* $u \in E \backslash \{0\}$ *fixed, one has*

$$A(u)^{s-p}/B(u)^{q-p} = \lambda^{s-q}\|u\|^{q(s-p)}_{L^q(\mathbb{R}^N_+,k)}\|u\|^{s(p-q)}_{L^s(\mathbb{R}^N_+,k)}.$$

Since $s > q$, *for* λ *sufficiently large we see that*

$$A(u)^{s-p} > \left(\frac{q}{p}\right)^{s-q} \eta(p,q,s)B(u)^{q-p}\|u\|^{p(s-q)},$$

and hence $\mathcal{D}_1 \neq \emptyset$.

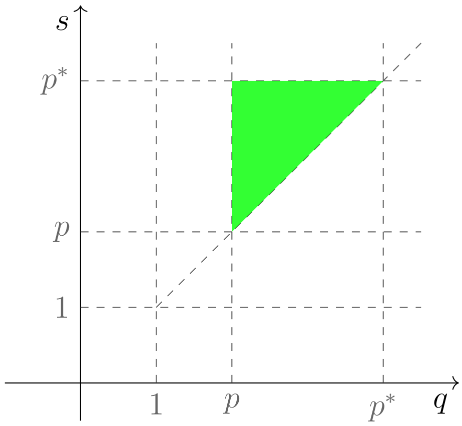

Figure 2.3: Existence of nontrivial solutions for ($\mathcal{P}_1$)

Finally, we obtain a existence result for the case $s < q < p$. Similar to the previous existence, we assume that the set

$$\mathcal{C}_1 := \left\{ u \in E : A(u)^{p-s} > \left(\frac{q}{s}\right)^{p-s} \eta(s,q,p) B(u)^{p-q} \|u\|^{p(q-s)} \right\} \tag{2.8}$$

is nonempty. This condition, similar to the case when $\mathcal{D}_1 \neq \emptyset$, indicates that $a(x)$ is sufficiently "large" relative to $b(x)$, in contrast with the nonexistence case in Theorem (2.1) when occurs $A(u)^{p-s} < \eta(s,q,p)B(u)^{p-q}\|u\|^{p(q-s)}$ for all $u \in E\backslash\{0\}$. Furthermore this combination of the exponents s, q, p we are able to prove the existence without an extra hypothesis of integrability for the quotient $a(x)^{1/q}/b(x)^{1/s}$ as (2.7) in the case $p < q < s$. Our fourth, and last existence result for ($\mathcal{P}_1$), is the following:

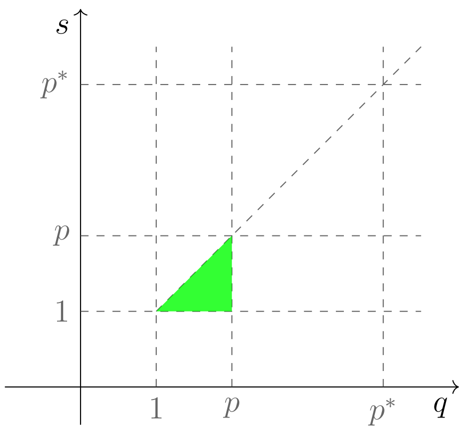

Figure 2.4: Existence of nontrivial solutions for ($\mathcal{P}_1$)

Theorem 2.6 *Assume* (H_0), $s < q < p \leq N$ *and* $a, b \in L^1(\mathbb{R}^N_+) \cap \mathcal{K}_0$. *If* $\mathcal{C}_1 \neq \emptyset$, *then* ($\mathcal{P}_1$) *has a nontrivial and nonnegative weak solution.*

The proofs of Theorems 2.3, 2.4, 2.5 and 2.6 are based on the classical Fibering Method; for great references in this theory, see [25, 35, 41, 49, 48].

Remark 2.4 *We finally highlight that our results for* $(\mathcal{P}_1)$ *can be extended to a more general class of problems of the form*

$$\begin{cases} -\mathrm{div}(\rho(x)|\nabla u|^{p-2}\nabla u) + \lambda \dfrac{|u|^{p-2}u}{(1+x_N)^{p-\gamma}} = a(x)|u|^{q-2}u - b(x)|u|^{s-2}u & \text{in } \mathbb{R}^N_+, \\ \rho(x)|\nabla u|^{p-2}\nabla u \cdot \nu + \mu |u|^{p-2}u = 0, & \text{on } \mathbb{R}^{N-1}. \end{cases}$$

In fact, considering the norm

$$\|u\|^p_{\lambda,\mu} = \int_{\mathbb{R}^N_+} \rho(x)|\nabla u|^p \,\mathrm{d}x + \lambda \int_{\mathbb{R}^N_+} \frac{|u|^p}{(1+x_N)^{p-\gamma}} \,\mathrm{d}x + \mu \int_{\mathbb{R}^{N-1}} |u|^p \,\mathrm{d}x',$$

and invoking Theorem 1.1, we can see that, for certain conditions on the parameters λ, μ *depending on* ρ_0 *and* $C_{p,\gamma}$*, the norms* $\|\cdot\|_{\lambda,\mu}$ *is equivalent to* $\|\cdot\|$*. Therefore, the same approach can treat this more general class of problems.*

It is worth mentioning that if $q < s = p$, the Direct Methods in the Calculus of Variations ensure the existence of solutions to $(\mathcal{P}_1)$. In the case that $s = p < q$, the mountain-pass approach can be applied to establish the existence of solutions to $(\mathcal{P}_1)$.

2.2 Proof of our Liouville-type Results

In this section, we shall focus on proving our Liouville-type results. The following estimate is fundamental in our analysis.

Lemma 2.7 (key estimate) *Assume condition* (H_0) *and* $1 < s < q < p$. *If* $b \in L^1(\mathbb{R}^N_+) \cap \mathcal{K}_0$, *then* B *is well defined in* E. *In addition, if* $a/b \in L^\infty(\mathbb{R}^N_+)$, *then*

$$A(v)^{p-s} \leq \left(\left\| \frac{a}{b} \right\|_\infty^{(p-s)} \left[\frac{b_0 C_{p,\gamma}^{-p}}{\rho_0} \right]^{q-s} \right) B(v)^{p-q} \|v\|^{p(q-s)}, \quad \forall v \in E. \tag{2.9}$$

In particular, A *is well-defined. Furthermore, if* (2.3) *holds, then*

$$A(v)^{p-s} < \eta(s,q,p) B(v)^{p-q} \|v\|^{p(q-s)}, \quad \forall v \in E \backslash \{0\}. \tag{2.10}$$

Proof. Since $b \in \mathcal{K}_0$, we have $b(x) \leq b_0/(1+x_N)^{p-\gamma}$. Thus, by Hölder's inequality,

$$\begin{aligned} B(v) = \int_{\mathbb{R}^N_+} b^{\frac{p-s}{p}} b^{s/p} |v|^s \, \mathrm{d}x \leq & \|b\|_{L^1(\mathbb{R}^N_+)}^{(p-s)/s} \left(\int_{\mathbb{R}^N_+} b|v|^p \, \mathrm{d}x \right)^{s/p} \\ \leq & b_0^{s/p} \|b\|_{L^1(\mathbb{R}^N_+)}^{(p-s)/s} \left(\int_{\mathbb{R}^N_+} \frac{|v|^p}{(1+x_N)^{p-\gamma}} \, \mathrm{d}x \right)^{s/p}, \end{aligned}$$

which in finite by Theorem 1.1 and assumption (H_0). Since a is nonnegative, we get $a(1+x_N)^{p-\gamma} \leq b_0 a/b$, which implies

$$\|a(1+x_N)^{p-\gamma}\|_\infty \leq b_0 \left\| \frac{a}{b} \right\|_\infty. \tag{2.11}$$

If $s < q < p$ we can write $q = (1-\alpha)s + \alpha p$ with $\alpha = (q-s)/(p-s) \in (0,1)$. Thus, by Hölder's inequality,

$$A(v) = \int_{\mathbb{R}^N_+} a|v|^q \, \mathrm{d}x = \int_{\mathbb{R}^N_+} (a|v|^s)^{1-\alpha} (a|v|^p)^\alpha \, \mathrm{d}x \leq \left(\int_{\mathbb{R}^N_+} a|v|^s \, \mathrm{d}x \right)^{1-\alpha} \left(\int_{\mathbb{R}^N_+} a|v|^p \, \mathrm{d}x \right)^\alpha.$$

Using that $1-\alpha = (p-q)/(p-s)$ we obtain

$$A(v)^{p-s} \leq \left(\int_{\mathbb{R}^N_+} a|v|^s \, \mathrm{d}x \right)^{p-q} \left(\int_{\mathbb{R}^N_+} a|v|^p \, \mathrm{d}x \right)^{q-s}. \tag{2.12}$$

Now, observe that

$$\int_{\mathbb{R}^N_+} a|v|^s \, \mathrm{d}x = \int_{\mathbb{R}^N_+} \frac{a}{b} (b|v|^s) \, \mathrm{d}x \leq \left\| \frac{a}{b} \right\|_\infty B(v).$$

Thus, (2.11) and Theorem 1.1 gives

$$\int_{\mathbb{R}^N_+} a|v|^p \,\mathrm{d}x \le \|a(1+x_N)^{p-\gamma}\|_\infty \int_{\mathbb{R}^N_+} \frac{|v|^p}{(1+x_N)^{p-\gamma}} \,\mathrm{d}x \le b_0 \left\|\frac{a}{b}\right\|_\infty \frac{C_{p,\gamma}^{-p}}{\rho_0}\|v\|^p.$$

Therefore, plugging the last two inequalities into (2.12), we estimate (2.9). ■

Throughout this chapter, we will consider the following auxiliary function:

$$G_1(r,v) = A(v)r^{q-p} - B(v)r^{s-p}, \quad r>0 \quad \text{and} \quad v \in E. \tag{2.13}$$

Lemma 2.8 *Assume the assumptions in Theorem 2.1. For each fixed $v \in E\backslash\{0\}$ the function $G_1(.,v)$ has a unique critical point which is a maximum and is given by*

$$\bar{r}(v) = \left(\frac{B(v)(p-s)}{A(v)(p-q)}\right)^{1/(q-s)}. \tag{2.14}$$

Moreover,

$$G_1(\bar{r}(v),v) = \max_{r>0} G_1(r,v) = \left(\frac{A(v)^{p-s}}{\eta(s,q,p)B(v)^{p-q}}\right)^{1/(q-s)} > 0, \tag{2.15}$$

where $\eta(s,q,p)$ was defined in (2.2).

Proof. Note that for each $v \in E\backslash\{0\}$, we have

$$\frac{\partial G_1}{\partial r}(r,v) = (q-p)A(v)r^{q-p-1} - (s-p)B(v)r^{s-p-1}.$$

Thus,

$$\frac{\partial G_1}{\partial r}(r,v) = 0 \iff r = \bar{r}(v) = \left(\frac{B(v)(p-s)}{A(v)(p-q)}\right)^{1/(q-s)}.$$

Moreover, we can see that $\lim_{r\to+\infty} G_1(r,v) = 0$, $\lim_{r\to 0^+} G_1(r,v) = -\infty$ and due to (2.14),

$$A(v)\bar{r}(v)^{q-p} = \frac{p-s}{p-q}B(v)\bar{r}(v)^{s-p}.$$

Since,

$$\begin{aligned} G_1(\bar{r}(v),v) &= A(v)\bar{r}(v)^{q-p} - B(v)\bar{r}(v)^{s-p} \\ &= \left(\frac{q-s}{p-q}\right) B(v)\bar{r}(v)^{s-p} \\ &= \left(\frac{q-s}{p-q}\right) B(v) \left(\frac{B(v)(p-s)}{A(v)(p-q)}\right)^{(s-p)/(q-s)} \\ &= \left(\frac{A(v)^{p-s}}{\eta(s,q,p)B(v)^{p-q}}\right)^{1/(q-s)} > 0 \end{aligned}$$

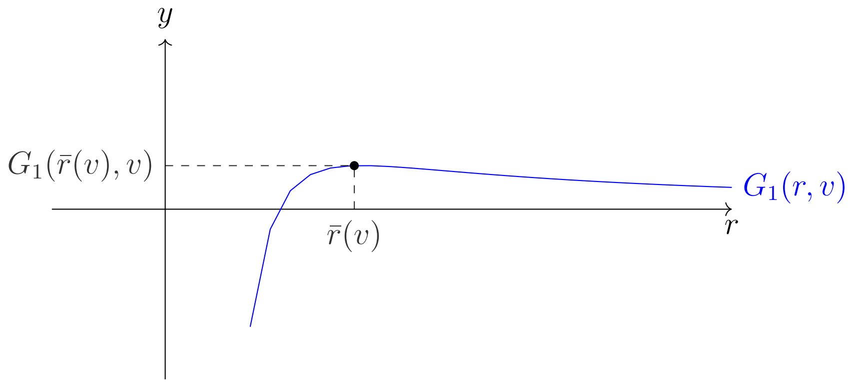

Figure 2.5: Maximum of $G_1(.,v)$ for $s<q<p$.

we can conclude that $G_1(.,v)$ has a unique global maximum at $r=\bar r(v)>0$. ■

Now we are ready to present the proof of Theorem 2.1.

Proof of Theorem 2.1. Assume by contradiction that $(\mathcal{P}_1)$ has a nontrivial weak solution $u_0\in E$. Then, from the definition (2.1), Lemma 2.7 and a density argument imply

$$\|u_0\|^p = A(u_0)-B(u_0)=G_1(1,u_0).$$

On the other hand, by estimate (2.10), we have

$$A(u_0)^{p-s}<\eta(s,q,p)B(u_0)^{p-q}\|u_0\|^{p(q-s)},$$

which combined with (2.15) gives $G_1(\bar r(u_0),u_0)<\|u_0\|^p$. Thus, we get

$$G_1(\bar r(u_0),u_0)<\|u_0\|^p=G_1(1,u_0),$$

which contradicts the fact that $\bar r(u_0)$ is the maximum of $G_1(.,u_0)$ and this concludes the proof of Theorem 2.1. ■

Next, we shall focus on the proof of our second Liouville-type result.

Lemma 2.9 *Assume condition (H_0), $p<q<s\le p^*$ for $p<N$ and $p<q<s<\infty$ for $p=N$. If $b\in\mathcal{K}_0$, then B is well defined. In addition, if $a/b\in L^\infty(\mathbb{R}^N_+)$, then*

$$A(v)^{s-p}\le\left(\left\|\frac{a}{b}\right\|_\infty^{s-p}\left[\frac{b_0C_{p,\gamma}^{-p}}{\rho_0}\right]^{s-q}\right)B(v)^{q-p}\|v\|^{p(s-q)},\quad\forall v\in E. \tag{2.16}$$

In particular, A is well-defined. Furthermore, if (2.4) holds then

$$A(v)^{s-p}<\eta(p,q,s)B(v)^{q-p}\|v\|^{p(s-q)},\quad\forall v\in E\backslash\{0\}. \tag{2.17}$$

Proof. If $b \in \mathcal{K}_0$ and $v \in E$ we see that

$$B(v) = \int_{\mathbb{R}^N_+} b|v|^s \, \mathrm{d}x \le b_0 \int_{\mathbb{R}^N_+} \frac{|v|^s}{(1+x_N)^{p-\gamma}} \, \mathrm{d}x,$$

which is finite thanks to assumption (H_0) and Theorem 1.3. Using again that $b \in \mathcal{K}_0$ and a is nonnegative we have $a(1+x_N)^{p-\gamma} \le b_0 a/b$, which implies

$$\|a(1+x_N)^{p-\gamma}\|_\infty \le b_0 \left\| \frac{a}{b} \right\|_\infty .$$

Since $p < q < s$ we can write $q = (1-\alpha)p + \alpha s$ with $\alpha = (q-p)/(s-p) \in (0,1)$. Thus, by Hölder's inequality we get

$$A(v) = \int_{\mathbb{R}^N_+} a|v|^q \, \mathrm{d}x = \int_{\mathbb{R}^N_+} (a|v|^p)^{1-\alpha}(a|v|^s)^\alpha \, \mathrm{d}x \le \left(\int_{\mathbb{R}^N_+} a|v|^p \, \mathrm{d}x \right)^{1-\alpha} \left(\int_{\mathbb{R}^N_+} a|v|^s \, \mathrm{d}x \right)^{\alpha} .$$

Taking into account that $1-\alpha = (s-q)/(s-p)$ we obtain

$$A(v)^{s-p} \le \left(\int_{\mathbb{R}^N_+} a|v|^p \, \mathrm{d}x \right)^{s-q} \left(\int_{\mathbb{R}^N_+} a|v|^s \, \mathrm{d}x \right)^{q-p} . \tag{2.18}$$

Now, thanks to Theorem 1.1 and (2.11) we get

$$\int_{\mathbb{R}^N_+} a|v|^p \, \mathrm{d}x \le \|a(1+x_N)^{p-\gamma}\|_\infty \int_{\mathbb{R}^N_+} \frac{|v|^p}{(1+x_N)^{p-\gamma}} \, \mathrm{d}x \le b_0 \left\| \frac{a}{b} \right\|_\infty \frac{C_{p,\gamma}^{-p}}{\rho_0} \|v\|^p$$

and notice that

$$\int_{\mathbb{R}^N_+} a|v|^s \, \mathrm{d}x = \int_{\mathbb{R}^N_+} \frac{a}{b} \left(b|v|^s\right) \, \mathrm{d}x \le \left\| \frac{a}{b} \right\|_\infty B(v).$$

Therefore, plugging the last two inequalities into (2.18), we obtain estimate (2.16). ∎

Arguing along the same lines as in the proof of Lemma 2.8, we can obtain the following result:

Lemma 2.10 *Assume condition* (H_0), $p < q < s \le p^*$ *for* $p < N$ *and* $p < q < s < \infty$ *for* $p = N$. *If* $b \in \mathcal{K}_0$ *and* $a/b \in L^\infty(\mathbb{R}^N_+)$, *then for each* $v \in E \backslash \{0\}$ *the function* $G_1(.,v)$ *defined by* (2.13) *has a unique critical point at*

$$\bar{r}(v) = \left(\frac{A(v)(q-p)}{B(v)(s-p)} \right)^{1/(s-q)} . \tag{2.19}$$

Moreover,

$$G_1(\bar{r}(v), v) = \max_{r>0} G_1(r,v) = \left(\frac{A(v)^{s-p}}{\eta(p,q,s) B(v)^{q-p}} \right)^{1/(s-q)} > 0. \tag{2.20}$$

Proof. With a direct calculation we have

$$\frac{\partial G_1}{\partial r}(r,v) = (q-p)r^{q-p-1}A(v) - (s-p)r^{s-p-1}B(v).$$

Then, we observe that $0 = \frac{\partial G_1}{\partial r}(r,v)$ if, and only if,

$$r^{s-q} = \frac{q-p}{s-p}\frac{A(v)}{B(v)},$$

which implies that the only critical point of $G_1(.,v)$ is given by (2.19).

Now we observe that $\lim_{r\to+\infty} G_1(r,v) = -\infty$, $\lim_{r\to 0^+} G_1(r,v) = 0$ and by (2.19)

$$B(v)\bar{r}(v)^{s-p} = \frac{q-p}{s-p}A(v)\bar{r}(v)^{q-p}.$$

Thus, since

$$\begin{aligned}
G_1(\bar{r}(v),v) &= A(v)\bar{r}(v)^{q-p} - B(v)\bar{r}(v)^{s-p} \\
&= \left(\frac{s-q}{s-p}\right) A(v)\bar{r}(v)^{q-p} \\
&= \left(\frac{s-q}{s-p}\right) A(v) \left(\frac{A(v)(q-p)}{B(v)(s-p)}\right)^{(q-p)/(s-q)} \\
&= \frac{(s-q)(q-p)^{\frac{q-p}{s-q}}}{(s-p)^{\frac{s-p}{s-q}}}\frac{A(v)^{\frac{s-p}{s-q}}}{B(v)^{\frac{q-p}{s-q}}} \\
&= \left(\frac{A(v)^{s-p}}{\eta(p,q,s)B(v)^{q-p}}\right)^{1/(s-q)} > 0,
\end{aligned}$$

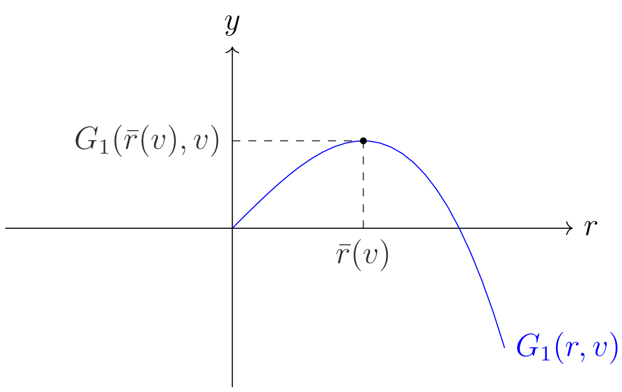

Figure 2.6: Maximum of $G_1(.,v)$ for $p < q < s$.

we can conclude that $G_1(.,v)$ has a unique global maximum at $r = \bar{r}(v) > 0$. ■

Now we are ready to present the proof of Theorem 2.2.

Proof of Theorem 2.2:. Arguing by contradiction, suppose that $(\mathcal{P}_1)$ has a non-trivial weak solution $u_0 \in E$. From the definition (2.1), Lemma 2.9 and a density argument we have

$$\|u_0\|^p = A(u_0) - B(u_0) = G_1(1, u_0). \tag{2.21}$$

On the other hand, by estimate (2.17), we have

$$A(u_0)^{s-p} < \eta(p,q,s) B(u_0)^{q-p} \|u_0\|^{p(s-q)}.$$

This, together with (2.20) implies that $G_1(\bar{r}(u_0), u_0) < \|u_0\|^p$. Therefore, we obtain

$$G_1(\bar{r}(u_0), u_0) < \|u_0\|^p = G_1(1, u_0),$$

contradicting the fact that $\bar{r}(u_0)$ is the maximum of $G_1(., u_0)$ and this concludes the proof. ∎

2.3 Proof of our Existence Results

This section is devoted to proving Theorems 2.3, 2.4, 2.5 and 2.6. To this purpose, we shall first prove a compactness result.

Lemma 2.11 *Assume condition* (H_0) *and* $1 < p \leq N$.

1. *If* $k \in L^1(\mathbb{R}^N_+) \cap \mathcal{K}_0$, *then the embedding*

$$E \hookrightarrow L^q\left(\mathbb{R}^N_+, k(x)\right) \tag{2.22}$$

is compact for all $1 < q < p \leq N$.

2. *If* $k \in \mathcal{K}$ *and* $p < N$, *then the embedding (2.22) is continuous for* $q \in [p, p^*]$ *and compact for* $q \in [p, p^*)$. *If* $p = N$, *the embedding is compact for all* $q \in [p, \infty)$.

Proof. If $k \in L^1(\mathbb{R}^N_+)$, by Hölder's inequality,

$$\int_{\mathbb{R}^N_+} k|u|^q \,\mathrm{d}x = \int_{\mathbb{R}^N_+} k^{\frac{p-q}{p}} k^{q/p} |u|^q \,\mathrm{d}x \leq \|k\|_1^{\frac{p-q}{p}} \left(\int_{\mathbb{R}^N_+} k|u|^p \,\mathrm{d}x \right)^{q/p}.$$

Also, since $k \in \mathcal{K}_0$ we have $k(x) \leq k_0(1 + x_N)^{\gamma - p}$ and by Theorem 1.1 and assumption (H_0) we obtain

$$\int_{\mathbb{R}^N_+} k|u|^q \,\mathrm{d}x \leq C\|k\|_1^{\frac{p-q}{p}} \left(\int_{\mathbb{R}^N_+} \frac{|u|^p}{(1+x_N)^{p-\gamma}} \right)^{q/p} \leq C\|k\|_1^{\frac{p-q}{p}} \|u\|^q.$$

Now, if $(u_n) \subset E$ is a bounded sequence, up to a subsequence, we can assume that $u_n \rightharpoonup 0$ in E. Given $\varepsilon > 0$ there exists $R = R(\varepsilon) > 0$ such that $\|k\|_{L^1(B_R^c(0)\cap\mathbb{R}_+^N)} \leq \varepsilon$ and hence

$$\int_{B_R^c\cap\mathbb{R}_+^N} k|u_n|^q \,\mathrm{d}x \leq C\varepsilon^{\frac{p-q}{p}}\|u_n\|^q \leq C_1\varepsilon^{\frac{p-q}{p}}.$$

To complete the proof for the first case, it is enough to use the classical Sobolev compact embedding to obtain the compact embedding $E \hookrightarrow W^{1,p}(B_R^+) \hookrightarrow L^q(B_R^+)$.

Assuming $k \in \mathcal{K}$, we have that $k(x) \leq C_0(1+x_N)^{\gamma-p}$ for some constant $C_0 > 0$, which implies that the embedding is continuous by Theorem 1.3 and the assumption (H_0) if $1 < p < N$ and $q \in [p, p^*]$ or $q \in [p, \infty)$ if $p = N$. For $R > 0$, we can write

$$\int_{\mathbb{R}_+^N} k(x)|u|^q \,\mathrm{d}x = \int_{B_R^+} k(x)|u|^q \,\mathrm{d}x + \int_{(B_R^+)^c\cap\mathbb{R}_+^N} k(x)|u|^q \,\mathrm{d}x.$$

If $(u_n) \subset E$ is a bounded sequence, up to a subsequence, $u_n \rightharpoonup 0$ in E. Since the embedding $E \hookrightarrow W^{1,p}(B_R^+) \hookrightarrow L^q(B_R^+)$ is compact for all $q \in [p, p^*)$ if $1 < p < N$ or $q \in [p, \infty)$ if $p = N$, it holds

$$\int_{B_R^+} k(x)|u_n|^q \,\mathrm{d}x \leq C\int_{B_R^+} |u_n|^q \,\mathrm{d}x \longrightarrow 0. \tag{2.23}$$

Given $\varepsilon > 0$, since $k \in \mathcal{K}$ we can choose $R = R(\varepsilon) > 0$ large enough such that $k(x)(1+x_N)^{p-\gamma} < \varepsilon$ for any $x \in B_R^c \cap \mathbb{R}_+^N$, which implies

$$\int_{(B_R^+)^c\cap\mathbb{R}_+^N} k(x)|u_n|^q \,\mathrm{d}x < \varepsilon\int_{(B_R^+)^c\cap\mathbb{R}_+^N} \frac{|u_n|^q}{(1+x_N)^{p-\gamma}} \,\mathrm{d}x \leq C\varepsilon\|u_n\|^q. \tag{2.24}$$

The proof of the second case follows from (2.23)-(2.24). ■

To prove our existence results, let us consider the functional $I : E \to \mathbb{R}$ associated with $(\mathcal{P}_1)$, defined as follows:

$$I(u) = \frac{1}{p}\|u\|^p - \frac{1}{q}A(u) + \frac{1}{s}B(u), \tag{2.25}$$

where A and B are defined in (2.5).

Straightforward computation shows that $I \in C^1(E, \mathbb{R})$ and critical points of I are weak solutions of $(\mathcal{P}_1)$(see [20]).

To prove that I has a critical point, we shall use the Fibering Method [25, 48]. To this end, we proceed with some basic results.

Lemma 2.12 *Let $1 < p \leq N$ and $q < \min\{s,p\}$ or $q > \max\{s,p\}$. Then, for each $v \in E\backslash\{0\}$ there exits a unique real number $r(v) > 0$ such that the pair $(r(v), v)$ satisfies the equation*

$$\|v\|^p = r(v)^{q-p}A(v) - r(v)^{s-p}B(v) = G_1(r(v), v). \tag{2.26}$$

Furthermore, the map $r : E\backslash\{0\} \to \mathbb{R}$ belongs to $C^1(E\backslash\{0\}, \mathbb{R})$ and $\mu r(\mu v) = r(v)$ for all $\mu > 0$ and $v \in E\backslash\{0\}$.

Proof. *Existence:* Consider the function $f : (0, \infty) \times E \to \mathbb{R}$ defined by

$$f(r, v) = \|v\|^p r^{p-q} + B(v)r^{s-q} - A(v),$$

and note that $f(r, v) = 0$ if and only if (2.26) holds. If $v \in E\backslash\{0\}$ and $q > \max\{s,p\}$ we have $\lim\limits_{r\to 0^+} f(r, v) = \infty$ and $\lim\limits_{r\to +\infty} f(r, v) = -A(v) < 0$.

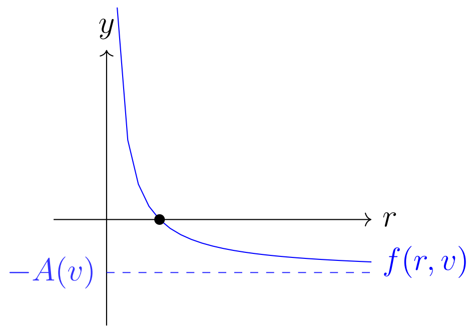

Figure 2.7: Graph of $f(., v)$ for $q > \max\{s, p\}$.

In the case $q < \min\{s,p\}$, it holds $\lim_{r\to 0^+} f(r, v) = -A(v) < 0$ and $\lim_{r\to\infty} f(r, v) = \infty$.

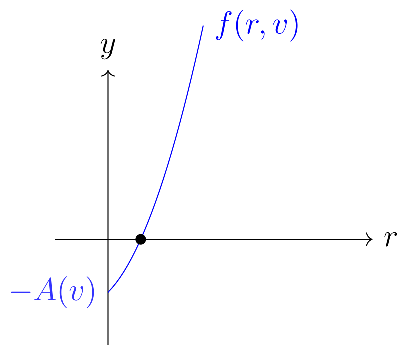

Figure 2.8: Graph of $f(., v)$ for $q < \min\{s, p\}$.

Thus, in any case, by the Intermediate Value Theorem, there exists $r(v) > 0$ such that $f(r(v), v) = 0$.

Uniqueness: Fixed $v \in E\backslash\{0\}$, suppose that there are $r_1, r_2 > 0$ satisfying (2.26). Consequently,

$$\|v\|^p r_1^{p-q} + B(v) r_1^{s-q} = A(v) = \|v\|^p r_2^{p-q} + B(v) r_2^{s-q},$$

which is equivalent to

$$\|v\|^p (r_1^{p-q} - r_2^{p-q}) + B(v)(r_1^{s-q} - r_2^{s-q}) = 0.$$

Therefore, $r_1 = r_2$ and so the map $r : E\backslash\{0\} \to \mathbb{R}$ satisfying (2.26) is well defined.
Regularity: To prove that r belongs to class C^1, we observe that

$$\frac{\partial f}{\partial r}(r, v) = (p-q) r^{p-q-1} \|v\|^p + (s-q) B(v) r^{s-q-1} \neq 0, \quad \text{in} \quad (0, \infty) \times E\backslash\{0\}.$$

Given $v \in E\backslash\{0\}$, using the implicit function theorem(see [16]), we obtain open sets $J \subset \mathbb{R}$ and $V \subset E\backslash\{0\}$ containing $r(v)$ and v respectively, and a C^1-function $\tau : V \to J$ satisfying

$$\tau(v) = r(v) \quad \text{and} \quad f(\tau(w), w) = 0, \quad \forall w \in V.$$

By the uniqueness $r \equiv \tau$ in V and therefore r is a C^1-function in V. Since $v \in E\backslash\{0\}$ is arbitrary, we obtain $r \in C^1(E\backslash\{0\}, \mathbb{R})$.

Finally, given $\mu > 0$ and $v \in E\backslash\{0\}$ we have that $f(r(\mu v), \mu v) = 0$, that is,

$$A(v) = \mu^{p-q} r(\mu v)^{p-q} \|v\|^p + \mu^{s-q} r(\mu v)^{q-s} B(v). \tag{2.27}$$

Since $f(r(v), v) = 0$, we have

$$A(v) = r(v)^{p-q} \|v\|^p + r(v)^{q-s} B(v),$$

which combined with (2.27) implies

$$0 = (\mu^{p-q} r(\mu v)^{p-q} - r(v)^{p-q}) \|v\|^p + (\mu^{s-q} r(\mu v)^{s-q} - r(v)^{s-q}) B(v).$$

Thus, $(\mu r(\mu v))^{p-q} = r(v)^{p-q}$ and this concludes the proof. ■

Remark 2.5 *Suppose that there exists an open $\Omega \subset E\backslash\{0\}$ and $r \in C^1(\Omega, \mathbb{R})$ such that $(r(v), v)$ satisfies (2.26) for each $v \in \Omega$ with $r(v) \neq 0$ in Ω, that is,*

$$\|v\|^p = r(v)^{q-p} A(v) - r(v)^{s-p} B(v). \tag{2.28}$$

Then, we have

$$\begin{aligned}I(r(v)v) =& \frac{r(v)^p}{p}\|v\|^p - \frac{r(v)^q}{q}A(v) + \frac{r(v)^s}{s}B(v)\\ =& \left(\frac{1}{p} - \frac{1}{q}\right) A(v)r(v)^q + \left(\frac{1}{s} - \frac{1}{p}\right) B(v)r(v)^s.\end{aligned}$$

In particular, if $r > 0$ and rv is a critical point of I, it holds

$$\langle I'(rv), v\rangle = 0,$$

which is equivalent to (2.28).

The above remark motivates us to consider the *reduced functional* defined by

$$\mathcal{I}(v) := I(r(v)v) = \left(\frac{1}{p} - \frac{1}{q}\right) A(v)r(v)^q + \left(\frac{1}{s} - \frac{1}{p}\right) B(v)r(v)^s. \tag{2.29}$$

Next, we shall need the following result to characterize the fibering method.

Lemma 2.13 *Let $H \in C^1(E\backslash\{0\}, \mathbb{R})$ such that $\langle H'(v), v\rangle \neq 0$ if $H(v) = 1$. If $v_c \in \Omega$ is a critical point of $\mathcal{I}$ under the constraint $H(v) = 1$, then $u = r(v_c)v_c$ is a critical point of I.*

Proof. Let $r \in C^1(\Omega, \mathbb{R})$ as in Remark 2.5, that is, for each $v \in \Omega_1 \subset E\backslash\{0\}$ the pair $(r(v), v)$ satisfies (2.26), more specifically

$$\|v\|^p = r(v)^{q-p}A(v) - r(v)^{s-p}B(v).$$

Then we can define $\mathcal{I} : \Omega \to \mathbb{R}$ as in (2.29) and

$$\langle I'(r(v)v), v\rangle = 0, \forall v \in \Omega_1. \tag{2.30}$$

In fact,

$$\begin{aligned}\langle I'(r(v)v), v\rangle =& r(v)^{p-1}\|v\|^p - r(v)^{q-1}A(v) + r(v)^{s-1}B(v)\\ =& r(v)^{p-1}[\|v\|^p - r(v)^{q-p}A(v) + r(v)^{s-p}B(v)] = 0\end{aligned}$$

If v_c is a critical point of $\mathcal{I}$ under the constraint $H(v) = 1$, by the Lagrange Multiplier Theorem(see [37, Proposition 14.3]), there exists $\lambda \in \mathbb{R}$ such that

$$\mathcal{I}'(v_c) = \lambda H'(v_c). \tag{2.31}$$

On the other hand, by the definition of $\mathcal{I}$ and (2.30) we have

$$\langle \mathcal{I}'(v), w\rangle = r(v)\langle I'(r(v)v), w\rangle + \langle r'(v), w\rangle\langle I'(r(v)v), v\rangle = r(v)\langle I'(r(v)v), w\rangle \quad (2.32)$$

for all $w \in E$. Then by (2.30) and (2.31)

$$0 = r(v_c)\langle I'(r(v_c)v_c), v_c\rangle = \langle \mathcal{I}'(v_c), v_c\rangle = \lambda\langle H'(v_c), v_c\rangle.$$

Since $\langle H'(v_c), v_c\rangle \neq 0$ we have that $\lambda = 0$ and hence, by (2.31) and (2.32),

$$0 = \mathcal{I}'(v_c) = r(v_c)I'(r(v_c)v_c).$$

Therefore, $r(v_c)v_c$ is a critical point of I. ∎

Remark 2.6 *We shall the spherical fibering method where we consider the constrait $S^1 = \{v \in E := \|v\|^p = H(v) = 1\}$ and study the minimization problem*

$$\inf_{v\in S^1} \mathcal{I}(v) \quad (2.33)$$

It is clear that the condition $\langle H(v), v\rangle \neq 0$ is satisfied. It is noteworthy that if v_0 is a minimum of $\mathcal{I}$ on the sphere, then $|v_0|$ also attains this minimum. In fact, first we observe that

$$\||v|\|^p = r(|v|)^{q-p}A(|v|) - r(|v|)^{s-p}B(|v|) \Leftrightarrow \|v\|^p = r(|v|)^{q-p}A(v) - r(|v|)^{s-p}B(v).$$

By Lemma 2.12, for $q < \min\{s, p\}$ or $q > \max\{s, p\}$, there exists a unique $r > 0$ such that $\|v\|^p = r^{q-p}A(v) - r^{s-p}B(v)$. Then we have $r(v) = r(|v|)$ and consequently

$$\begin{aligned}\mathcal{I}(|v|) &= \left(\frac{1}{p} - \frac{1}{q}\right) A(|v|)r(|v|)^q + \left(\frac{1}{s} - \frac{1}{p}\right) B(|v|)r(|v|)^s\\ &= \left(\frac{1}{p} - \frac{1}{q}\right) A(v)r(v)^q + \left(\frac{1}{s} - \frac{1}{p}\right) B(v)r(v)^s\\ &=\mathcal{I}(v).\end{aligned}$$

Thus, if v_0 attains (2.33), $|v_0|$ also attains. This implies, by using Lemma 2.13, that solutions for ($\mathcal{P}_1$), can be taken as nonnegative without loss of generality.

Now, we are ready to proceed with the proof of Theorems 2.3 and 2.4.

Proof of Theorem 2.3:. For each fixed $v \in E\backslash\{0\}$, by Lemma 2.12, there exists $r(v) > 0$ such that the pair $(r(v), v)$ satisfies (2.26) and hence

$$\|v\|^p r(v)^{p-q} + B(v)r(v)^{s-q} = A(v). \quad (2.34)$$

As a consequence, we can consider the reduced functional $\mathcal{I}$ as

$$\mathcal{I}(v) = \left(\frac{1}{s} - \frac{1}{q}\right) B(v)r(v)^s + \left(\frac{1}{p} - \frac{1}{q}\right) \|v\|^p r(v)^p > 0.$$

If S^1 denotes the unity sphere in E, we can define

$$M := \inf_{v \in S^1} \mathcal{I}(v).$$

Now, consider a sequence (v_n) such that $\|v_n\| = 1$ and $M = \lim \mathcal{I}(v_n)$. Going if necessary to a subsequence, we may assume that $v_n \rightharpoonup v_0$ in E with $\|v_0\| \leq 1$ and by Lemma 2.11

$$A(v_n) \to A(v_0) \geq 0 \quad \text{and} \quad B(v_n) \to B(v_0) \geq 0.$$

We claim that $v_0 \neq 0$. Indeed, suppose that $v_0 = 0$. By Lemma 2.12, there exists a sequence $r(v_n) > 0$ such that

$$\|v_n\|^p = r(v_n)^{q-p} A(v_n) - r(v_n)^{s-p} B(v_n). \tag{2.35}$$

Using that $\|v_n\| = 1$, we get

$$1 = r(v_n)^{q-p} A(v_n) - B(v_n) r(v_n)^{s-p} \leq r(v_n)^{q-p} A(v_n).$$

Since $q > p$ and $A(v_n) \to 0$, we obtain $r(v_n) \to \infty$. On the other hand, we have

$$\mathcal{I}(v_n) = \left(\frac{1}{s} - \frac{1}{q}\right) B(v_n) r(v_n)^s + \left(\frac{1}{p} - \frac{1}{q}\right) r(v_n)^p \geq \left(\frac{1}{p} - \frac{1}{q}\right) r(v_n)^p.$$

Taking the limit above, we obtain a contradiction and hence $v_0 \neq 0$. From the last inequality, up to a subsequence, we can assume that $r(v_n) \to r_0 \geq 0$ and taking to the limit in (2.35) we obtain

$$r_0^{p-q} + B(v_0) r_0^{s-q} = A(v_0), \tag{2.36}$$

which implies that $r_0 > 0$.

Next, we shall prove that $\|v_0\| = 1$. Otherwise, there exists $\mu > 1$ such that $\|\mu v_0\| = 1$. From Lemma 2.12, there are $r(v_0) > 0$ such that

$$\|v_0\|^p r(v_0)^{p-q} + B(v_0) r(v_0)^{s-q} = A(v_0).$$

This, combined with (2.36) and the fact that $\mu > 1$ implies

$$r_0^{p-q} + B(v_0) r_0^{s-q} < r(v_0)^{p-q} + B(v_0) r(v_0)^{s-q},$$

equivalently

$$r(v_0)^{p-q}\left[\left(\frac{r(v_0)}{r_0}\right)^{q-p}-1\right]+B(v_0)r(v_0)^{s-q}\left[\left(\frac{r(v_0)}{r_0}\right)^{q-s}-1\right]<0.$$

Since $\max\{s,p\}<q$, we have that $r_0>r(v_0)$. Now, consider the function

$$\psi(t)=\left(\frac{1}{s}-\frac{1}{q}\right)B(v_0)t^s+\left(\frac{1}{p}-\frac{1}{q}\right)\|v_0\|^p t^p,\quad t>0$$

and observe that ψ is strictly increasing. Thus,

$$M=\liminf_{n\to\infty}\mathcal{I}(v_n)\geq\left(\frac{1}{s}-\frac{1}{q}\right)B(v_0)r_0^s+\left(\frac{1}{p}-\frac{1}{q}\right)r_0^p\|v_0\|^p=\psi(r_0).$$

On the other hand, we have

$$\psi(r_0)>\psi(r(v_0))=I(r(v_0)v_0)=I(\mu r(\mu v_0)v_0)=\mathcal{I}(\mu v_0),$$

which contradicts the definition of M because $\|\mu v_0\|=1$ and hence we concluded that $\|v_0\|=1$. From (2.36) and the uniqueness of the solution $r(v_0)$ we have $r_0=r(v_0)$ and

$$\begin{aligned}M=\lim_{n\to\infty}\mathcal{I}(v_n)&=\lim_{n\to\infty}\left(\frac{1}{s}-\frac{1}{q}\right)B(v_n)r(v_n)^s+\left(\frac{1}{p}-\frac{1}{q}\right)r(v_n)^p\\&=\left(\frac{1}{s}-\frac{1}{q}\right)B(v_0)r_0^s+\left(\frac{1}{p}-\frac{1}{q}\right)r_0^p\\&=\mathcal{I}(v_0).\end{aligned}$$

Since v_0 is a critical point of $\mathcal{I}$ under S^1 so is $|v_0|$ and we can assume $v_0\geq 0$. Applying Lemma 2.13 with $H(v)=\|v\|^p$, we conclude that $u=r_0v_0$ is a critical point of I, and this completes the proof. ■

Proof of Theorem 2.4. For each fixed $v\in E\backslash\{0\}$, by Lemma 2.12 there exist $r(v)$ such that

$$\|v\|^p r(v)^{p-q}+B(v)r(v)^{s-q}=A(v), \tag{2.37}$$

and hence, we can write the reduced functional $\mathcal{I}$ as

$$\mathcal{I}(v)=\left(\frac{1}{s}-\frac{1}{q}\right)B(v)r(v)^s+\left(\frac{1}{p}-\frac{1}{q}\right)\|v\|^p r(v)^p<0.$$

If $\|v\|=1$, we see that A and B are bounded because of our embedding results. From (2.37), it follows that

$$0<r(v)^{p-q}\leq r(v)^{p-q}+B(v)r(v)^{s-q}=A(v),$$

which implies that r is bounded because $p > q$. Therefore, we can consider the minimization problem

$$-\infty < M := \inf_{v \in S^1} \mathcal{I}(v) < 0.$$

Let $(v_n) \subset S^1$ be a minimizing sequence. Up to a subsequence, we can assume that $v_n \rightharpoonup v_0$ weakly in E with $\|v_0\| \leq 1$. Furthemore, by Lemma 2.11

$$A(v_n) \to A(v_0) \quad \text{and} \quad B(v_n) \to B(v_0).$$

Since $(r(v_n))$ is bounded, up to a subsequence, we can assume that $r(v_n) \to r_0 \geq 0$.

Now observe that I is weakly lower semicontinuous and $r(v_n)v_n \rightharpoonup r_0 v_0$, then

$$I(r_0 v_0) \leq \liminf I(r(v_n)v_n) = \liminf \mathcal{I}(v_n) = M < 0$$

and so $r_0 v_0 \neq 0$. From (2.37), we have

$$\|v_n\|^p r(v_n)^{p-q} + B(v_n) r(v_n)^{s-q} = A(v_n).$$

Passing to the limit and observing that $\|v_0\| \leq 1$ we obtain

$$\|v_0\|^p r_0^{p-q} + B(v_0) r_0^{s-q} \leq A(v_0).$$

On the other hand, applying Lemma 2.12 for v_0, we have

$$\|v_0\|^p r(v_0)^{p-q} + B(v_0) r(v_0)^{s-q} = A(v_0),$$

which implies that $r_0 \leq r(v_0)$. Now, suppose by contradiction that $r_0 < r(v_0)$ and consider the function

$$\psi(t) := I(tv_0) = \frac{t^p}{p}\|v_0\|^p - \frac{t^q}{q} A(v_0) + \frac{t^s}{s} B(v_0), \quad t \in [0, r(v_0)]$$

and observe that ψ is strictly decreasing. Indeed, first note that $\psi(0) = 0$ and $\psi(r(v_0)) = \mathcal{I}(v_0) < 0$. In addition, we observe that $\psi'(0) = 0$ and for $t \neq 0$,

$$0 = \psi'(t) = t^{p-1}\|v_0\|^p - t^{q-1} A(v_0) + t^{s-1} B(v_0) \Leftrightarrow \|v_0\|^p = t^{q-p} A(v_0) - t^{s-p} B(v_0) \Leftrightarrow t = r(v_0).$$

Consequently, ψ must be strictly decreasing on $[0, r(v_0)]$. Thus,

$$M = \liminf I(r(v_n)v_n) \geq I(r_0 v_0) > I(r(v_0)v_0) = \mathcal{I}(v_0).$$

By Lemma 2.12 we have $\mu r(\mu v) = r(v)$ for all $v \neq 0$ and taking $\mu = \|v_0\|^{-1}$ we have $\mu v_0 \in S^1$ and

$$\mathcal{I}(\mu v_0) = I(\mu r(\mu v_0) v_0) = I(r(v_0)v_0) = \mathcal{I}(v_0) < M,$$

which is a contradiction and therefore $r(v_0) = r_0$. Then,

$$1 = \lim_{n\to\infty} \|v_n\|^p = r_0^{q-p} A(v_0) - r_0^{s-p} B(v_0) = \|v_0\|^p.$$

and

$$\begin{aligned} M = \lim_{n\to\infty} \mathcal{I}(v_n) &= \lim_{n\to\infty} \left(\frac{1}{s} - \frac{1}{q}\right) B(v_n) r(v_n)^s + \left(\frac{1}{p} - \frac{1}{q}\right) \|v_n\|^p r(v_n)^p \\ &= \left(\frac{1}{s} - \frac{1}{q}\right) B(v_0) r_0^s + \left(\frac{1}{p} - \frac{1}{q}\right) \|v_0\|^p r_0^p \\ &= \mathcal{I}(v_0). \end{aligned}$$

Since v_0 is a critical point of $\mathcal{I}$ under S^1, it follows that $|v_0|$ is also a critical point. Thus, we can assume without loss of generality that $v_0 \geq 0$. Applying Lemma 2.13, we conclude that $u = r_0 v_0$ is a critical point of I, which completes the proof. ■

Moving forward, we are proceeding to prove our third existence result.

Lemma 2.14 *Assume the assumption of Theorem 2.5. Then for each $v \in E\backslash\{0\}$ the function $G_1(.,v)$ defined by (2.13) has a unique critical point at*

$$\bar{r}(v) = \left(\frac{A(v)(q-p)}{B(v)(s-p)}\right)^{1/(s-q)}. \tag{2.38}$$

Moreover,

$$G_1(\bar{r}(v), v) = \max_{r>0} G_1(r, v) = \left(\frac{A(v)^{s-p}}{\eta(p,q,s) B(v)^{q-p}}\right)^{1/(s-q)} > 0. \tag{2.39}$$

Proof. By a direct calculation we have

$$\frac{\partial G_1}{\partial r}(r, v) = (q-p) r^{q-p-1} A(v) - (s-p) r^{s-p-1} B(v).$$

Then, we observe that $0 = \frac{\partial G_1}{\partial r}(r, v)$ if, and only if,

$$r^{s-q} = \frac{q-p}{s-p} \frac{A(v)}{B(v)},$$

which implies that the only critical point of $G_1(.,v)$ is given by (2.38).

Note that $\lim_{r\to+\infty} G_1(r,v) = -\infty$, $\lim_{r\to 0^+} G_1(r,v) = 0$ and by (2.38)

$$B(v)\bar{r}(v)^{s-p} = \frac{q-p}{s-p}A(v)\bar{r}(v)^{q-p}.$$

Thus, since

$$\begin{aligned}
G_1(\bar{r}(v),v) &= A(v)\bar{r}(v)^{q-p} - B(v)\bar{r}(v)^{s-p}\\
&= \left(\frac{s-q}{s-p}\right) A(v)\bar{r}(v)^{q-p}\\
&= \left(\frac{s-q}{s-p}\right) A(v) \left(\frac{A(v)(q-p)}{B(v)(s-p)}\right)^{(q-p)/(s-q)}\\
&= \frac{(s-q)(q-p)^{\frac{q-p}{s-q}}}{(s-p)^{\frac{s-p}{s-q}}} \frac{A(v)^{\frac{s-p}{s-q}}}{B(v)^{\frac{q-p}{s-q}}}\\
&= \left(\frac{A(v)^{s-p}}{\eta(p,q,s)B(v)^{q-p}}\right)^{1/(s-q)} > 0,
\end{aligned}$$

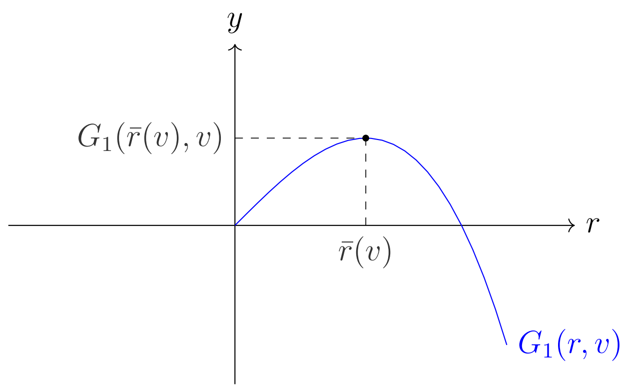

Figure 2.9: Maximum of $G_1(.,v)$ for $p<q<s$.

we can conclude that $G_1(.,v)$ has a unique global maximum at $r = \bar{r}(v) > 0$. ■

Under the assumptions in Theorem 2.5, we introduce the set

$$\Omega_1 := \{v \in E\backslash\{0\} : \|v\|^p < G_1(\bar{r}(v),v)\},$$

where $\bar{r}(v)$ is given by (2.38).

Remark 2.7 *If $\mathcal{D}_1$ is the set defined in (2.6), then $\mathcal{D}_1 \subset \Omega_1$ and hence $\Omega_1 \neq \emptyset$. Indeed, first we observe that by (2.38) we have*

$$B(v) = \left(\frac{q-p}{s-p}\right)\bar{r}(v)^{q-s}A(v), \tag{2.40}$$

and from the definition of G, we obtain

$$G_1(\bar{r}(v), v) = \bar{r}(v)^{q-p}A(v) - \left(\frac{q-p}{s-p}\right)\bar{r}(v)^{q-p}A(v) = \left(\frac{s-q}{s-p}\right)A(v)\bar{r}(v)^{q-p}. \quad (2.41)$$

If $v \in \mathcal{D}_1$, we see that

$$\|v\|^p < \left(\frac{p}{q}\right)\eta(p,q,s)^{\frac{1}{q-s}}A(v)^{\frac{s-p}{s-q}}B(v)^{\frac{p-q}{s-q}}.$$

Thus, from (2.40) and (2.41), it follows

$$\begin{aligned}\|v\|^p &< \left(\frac{p}{q}\right)\frac{(s-q)(q-p)^{\frac{q-p}{s-q}}}{(s-p)^{\frac{s-p}{s-q}}}A(v)^{\frac{s-p}{s-q}}\left[\left(\frac{q-p}{s-p}\right)\bar{r}(v)^{q-s}A(v)\right]^{\frac{p-q}{s-q}} \\ &= \frac{p}{q}\left(\frac{s-q}{s-p}\right)\bar{r}(v)^{q-p}A(v) \\ &< G_1(\bar{r}(v), v),\end{aligned}$$

and so we conclude that $\mathcal{D}_1 \subset \Omega_1$.

Next, we will prove some technical properties of Ω_1 that play an important role in proving Theorem 2.5.

Lemma 2.15 *If $p < q < s < p^*$, for each $v \in \Omega_1$ there exists a unique real number $r(v) > \bar{r}(v)$ such that the pair $(r(v), v)$ satisfies*

$$\|v\|^p = r(v)^{q-p}A(v) - r(v)^{s-p}B(v) = G_1(r(v), v),$$

and $r \in C^1(\Omega_1, \mathbb{R})$. Furthermore, for any $v \in \Omega_1$ and $\mu > 0$, it holds $\mu v \in \Omega_1$, and as a consequence, $\Omega_1 \cap S^1 \neq \emptyset$.

Proof. If $v \in \Omega_1$ we have $\|v\|^p < G_1(\bar{r}(v), v)$. Since $G_1(r, v) = r^{q-p}\left(A(v) - B(v)r^{s-q}\right)$ and $p < q < s$, it follows that

$$\lim_{r\to\infty} G_1(r, v) = -\infty,$$

and so by the Intermediate Value Theorem, there exists a real number $r(v) > \bar{r}(v)$ such that the pair $(r(v), v)$ verifies $\|v\|^p = G_1(r(v), v)$. To prove that $r(v)$ is unique, we observe $(q-p)\bar{r}(v)^{q-p}A(v) = (s-p)\bar{r}(v)^{s-p}B(v)$ and hence we can write

$$G_1(r, v) = A(v)\left(r^{q-p} - \frac{q-p}{s-p}\bar{r}(v)^{q-s}r^{s-p}\right).$$

Consequently,

$$\frac{\partial G_1}{\partial r}(r,v) = (q-p)r^{s-p-1}A(v)(r^{q-s} - \bar{r}(v)^{q-s})$$
$$= (q-p)r^{s-p-1}A(v)\left(\frac{1}{r^{s-q}} - \frac{1}{\bar{r}(v)^{s-q}}\right) < 0,$$

for all $r > \bar{r}(v)$, thereby implying the uniqueness of $r(v)$.

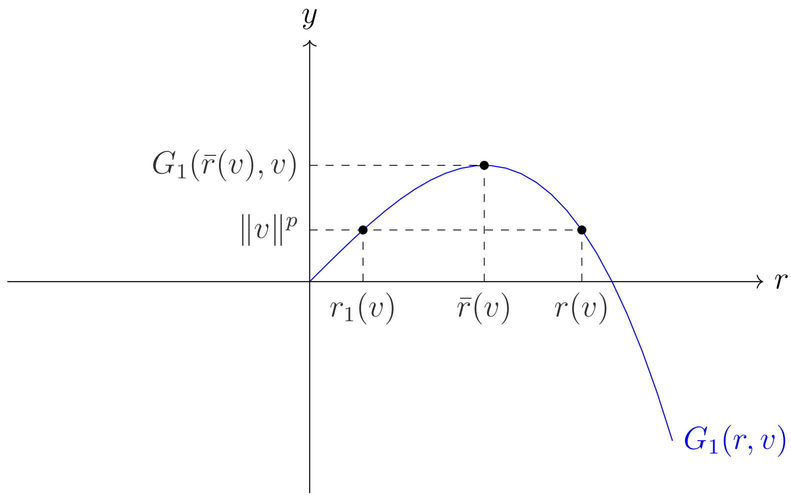

Figure 2.10: Existence of $r(v) > \bar{r}(v)$ for $p < q < s$.

To verify that the map r is a C^1, by setting $r = r(v)$, we obtain

$$\frac{\partial G_1}{\partial r}(r(v),v) = (q-p)r(v)^{p-s+1}A(v)(r(v)^{q-s} - \bar{r}(v)^{q-s}) < 0. \tag{2.42}$$

Now, considering the function $f : (0,\infty) \times \Omega_1 \to \mathbb{R}$ given by

$$f(r,v) = G_1(r,v) - \|v\|^p,$$

by (2.42) we see that given $v \in \Omega_1$, $\frac{\partial f}{\partial r}(r(v),v) < 0$. Using the implicit function theorem, we obtain open sets $J \subset \mathbb{R}$ and $V \subset \Omega_1$ containing $r(v)$ and v respectively, and a C^1-function $\tau : V \to J$ satisfying

$$\tau(v) = r(v) > \bar{r}(v) \quad \text{and} \quad f(\tau(w),w) = 0, \quad \forall w \in V.$$

Then, there exists a neighborhood $U \subset V$ containing v such that $\tau = r > \bar{r}$ in U. By the uniqueness, $r \equiv \tau$ in U and therefore r is a C^1-function in U. Since v is arbitrary, we have $r \in C^1(\Omega_1, \mathbb{R})$.

Now, suppose that $v \in \Omega_1$, that is

$$\|v\|^p < G_1(\bar{r}(v),v) = \frac{s-q}{s-p}\bar{r}(v)^{q-p}A(v).$$

Since, $\bar{r}(v) = \mu\bar{r}(\mu v)$ for all $\mu > 0$, we get

$$\|\mu v\|^p < \frac{s-q}{s-p}(\bar{r}(\mu v))^{q-p}A(\mu v) = G_1(\bar{r}(\mu v), \mu v),$$

which implies that $\mu v \in \Omega_1$ and this completes the proof. ■

Lemma 2.16 *The following statement holds*

$$\inf_{v\in\Omega_1\cap S^1} B(v) > 0.$$

Proof. For any $v \in \Omega_1 \cap S^1$, from (2.38) we get

$$\begin{aligned} 1 = \|v\|^p < G_1(\bar{r}(v), v) &= \left(\frac{s-q}{s-p}\right) A(v)\bar{r}(v)^{q-p} \\ &= \left(\frac{s-q}{s-p}\right) A(v) \left(\frac{A(v)(q-p)}{B(v)(s-p)}\right)^{(q-p)/(s-q)}, \end{aligned}$$

which implies

$$\eta(p,q,s)B(v)^{q-p} < A(v)^{s-p}.$$

By using Hölder's inequality, it follows

$$\begin{aligned} A(v) = \int_{\mathbb{R}^N_+} a|v|^q \,\mathrm{d}x &= \int_{\mathbb{R}^N_+} \frac{a}{b^{q/s}} b^{q/s}|v|^q \,\mathrm{d}x \\ &\leq \left(\int_{\mathbb{R}^N_+} \left[\frac{a}{b^{q/s}}\right]^{\frac{s}{s-q}} \mathrm{d}x\right)^{(s-q)/s} \left(\int_{\mathbb{R}^N_+} b|v|^s \,\mathrm{d}x\right)^{q/s}. \end{aligned}$$

Thus, we get $A(v) \leq C_{a,b}B(v)^{q/s}$ with

$$0 < C_{a,b} = \left(\int_{\mathbb{R}^N_+} \left[\frac{a^{1/q}}{b^{1/s}}\right]^{\frac{sq}{s-q}} \mathrm{d}x\right)^{(s-q)/s},$$

which is finite due to assumption (2.7). By combining the above inequalities, we obtain

$$\eta(p,q,s)B(v)^{q-p} < A(v)^{s-p} \leq C_{a,b}^{s-p} B(v)^{(s-p)q/s},$$

and hence $0 < \eta(p,q,s)C_{a,b}^{p-s} < B(v)^{(s-q)p/s}$, thereby yielding the desired result. ■

Lemma 2.17 *If $\mathcal{D}_1$ is the set defined in (2.6), then $\mathcal{D}_1 \cap S^1 \neq \varnothing$, where S^1 is the unit sphere in E. Moreover,*

$$\mathcal{I}(v) < 0, \quad \forall v \in \mathcal{D}_1. \tag{2.43}$$

Proof. If $v \in \mathcal{D}_1$, the computation in Remark 2.7 shows that

$$\|v\|^p < \frac{p}{q}\left(\frac{s-q}{s-p}\right)\bar{r}(v)^{q-p}A(v), \tag{2.44}$$

and for $\mu > 0$, by (2.38) we easily obtain

$$\mu\bar{r}(\mu v) = \bar{r}(v), \quad \forall v \in E\backslash\{0\}.$$

Thus,

$$\|\mu v\|^p < \frac{p}{q}\left(\frac{s-q}{s-p}\right)\mu^{p-q}\bar{r}(v)^{q-p}A(\mu v) = \frac{p}{q}\left(\frac{s-q}{s-p}\right)\bar{r}(\mu v)^{q-p}A(\mu v),$$

which implies that $\mu v \in \mathcal{D}_1$. In particular, choosing $\mu = \|v\|^{-1}$ we conclude that $\mathcal{D}_1 \cap S^1 \neq \varnothing$.

To verify (2.43), since the pair $(r(v), v)$ satisfies

$$B(v)r(v)^s = A(v)r(v)^q - \|v\|^p r(v)^p,$$

from (2.29), the fact that $\bar{r}(v) < r(v)$ for each $v \in \mathcal{D}_1 \subset \Omega_1$ and inequality (2.44), we get

$$\begin{aligned}\mathcal{I}(v) &= \left(\frac{1}{s}-\frac{1}{q}\right)A(v)r(v)^q + \left(\frac{1}{p}-\frac{1}{s}\right)\|v\|^p r(v)^p \\ &< \left(\frac{1}{s}-\frac{1}{q}\right)A(v)r(v)^q + \left(\frac{1}{p}-\frac{1}{s}\right)\frac{p}{q}\left(\frac{s-q}{s-p}\right)A(v)r(v)^q.\end{aligned}$$

Since the last term of the inequality above is zero, this completes the proof.

■

Remark 2.8 *If $1 < p < q < s$, by Lemma 2.15, for each $v \in \Omega_1$ there is only a real value $r(v)$ such that*

$$G_1(r(v), v) = \|v\|^p \quad \textit{and} \quad r(v) > \bar{r}(v).$$

Since $r(|v|) > \bar{r}(|v|) = \bar{r}(v)$ with

$$G_1(r(|v|), v) = \|v\|^p$$

we must to have $r(v) = r(|v|)$ and hence $\mathcal{I}(v) = \mathcal{I}(|v|)$. Therefore, if v_0 is a minimum of $\mathcal{I}$ under $\Omega_1 \cap S^1$, so is $|v_0|$. This implies that solutions to the minimization problem,

$$\inf_{v\in\Omega_1\cap S^1}\mathcal{I}(v),$$

and hence, by Lemma 2.13, solutions for ($\mathcal{P}_1$) in the case $1 < p < q < s$, can be taken as nonnegative without loss of generality.

Now we are ready to prove the Theorem 2.5.

Proof of Theorem 2.5:. Let $r \in C^1(\Omega_1, \mathbb{R})$ be the function given by the Lemma 2.15. For $v \in S^1$, we have

$$1 = r(v)^{q-p}A(v) - r(v)^{s-p}B(v),$$

which implies that

$$r(v) < \left(\frac{A(v)}{B(v)}\right)^{1/(s-q)}, \quad v \in S^1. \tag{2.45}$$

Since A is bounded in S^1, by Lemma 2.16 we have that r is bounded in $\Omega_1 \cap S^1$. Hence, $\mathcal{I}$ is lower bounded in $\Omega_1 \cap S^1$ and hence in view of Lemma 2.17

$$M = \inf_{v \in \Omega_1 \cap S^1} \mathcal{I}(v) < 0. \tag{2.46}$$

Let $(v_n) \subset \Omega_1 \cap S^1$ be a minimizing sequence. Up to a subsequence, $v_n \rightharpoonup v_0$ weakly in E with $\|v_0\| \leq 1$. Lemmas 2.11 then imply

$$A(v_n) \to A(v_0) \quad \text{and} \quad B(v_n) \to B(v_0).$$

By Lemma 2.15, the sequence $(r(v_n))$ satisfies $r(v_n) > \bar{r}(v_n)$. Moreover, from (2.45), the sequence $(r(v_n))$ is bounded and, up to a subsequence, we can assume that $r(v_n) \to r_0 \geq 0$. Thus, we obtain

$$0 > M = \liminf \mathcal{I}(v_n) \geq \left(\frac{1}{p} - \frac{1}{q}\right) A(v_0)r_0^q + \left(\frac{1}{s} - \frac{1}{p}\right) B(v_0)r_0^s.$$

Considering that $p < q < s$, we concluded that $r_0 > 0$. Furthermore, from (2.38), it follows

$$\lim_{n\to+\infty} \bar{r}(v_n) = \lim_{n\to+\infty} \left(\frac{A(v_n)(q-p)}{B(v_n)(s-p)}\right)^{1/(s-q)} = \left(\frac{A(v_0)(q-p)}{B(v_0)(s-p)}\right)^{1/(s-q)} = \bar{r}(v_0),$$

and hence $r_0 \geq \bar{r}(v_0)$. Furthermore,

$$\lim_{n\to+\infty} G_1(\bar{r}(v_n), v_n) = G_1(\bar{r}(v_0), v_0).$$

Since $v_n \in \Omega_1$, we get

$$\|v_0\|^p \leq \liminf_{n\to\infty} \|v_n\|^p \leq \liminf_{n\to\infty} G_1(\bar{r}(v_n), v_n) = G_1(\bar{r}(v_0), v_0).$$

Assume by contradiction that $v_0 \notin \Omega_1$, that is, $\|v_0\|^p = G_1(\bar{r}(v_0), v_0)$. Since $\|v_n\|^p = G_1(r(v_n), v_n)$, taking to the limit we get

$$G_1(\bar{r}(v_0), v_0) = \|v_0\|^p \leq \liminf_{n\to\infty} \|v_n\|^p = \liminf_{n\to\infty} G_1(r(v_n), v_n) = G_1(r_0, v_0),$$

which implies that $\bar{r}(v_0) = r_0$ because $\bar{r}(v_0)$ is the global maximum of $G_1(.,v_0)$. Then, $r(v_n) \to \bar{r}(v_0)$ and from the definition of $\mathcal{I}$ and (2.38) we obtain

$$M = \lim_{n\to\infty} \mathcal{I}(v_n) = \left(\frac{1}{p} - \frac{1}{q}\right) A(v_0)\bar{r}(v_0)^q + \left(\frac{1}{s} - \frac{1}{p}\right) B(v_0)\bar{r}(v_0)^s$$

and $(s-p)B(v_0)\bar{r}(v_0)^s = A(v_0)(q-p)\bar{r}(v_0)^q$. As a consequence, we infer that

$$\begin{aligned} M &= A(v_0)\bar{r}(v_0)^q \left[\left(\frac{1}{p} - \frac{1}{q}\right) + \left(\frac{1}{s} - \frac{1}{p}\right)\frac{(q-p)}{s-p}\right] \\ &= A(v_0)\bar{r}(v_0)^q \frac{(q-p)}{p}\left(\frac{1}{q} - \frac{1}{s}\right) > 0 \end{aligned}$$

because $p < q < s$, which contradicts (2.46) and hence we conclude that $v_0 \in \Omega_1$.

Claim: $r_0 = r(v_0)$.

Assuming that the claim is true, we can take the limit at

$$1 = \|v_n\|^p = G_1(r(v_n), v_n),$$

to obtain

$$1 = A(v_0)r(v_0)^{q-p} - B(v_0)r(v_0)^{s-p} = G_1(r(v_0), v_0) = \|v_0\|^p.$$

Thus, we conclude that $v_0 \in \Omega_1 \cap S^1$ and we also have

$$M = \lim_{n\to\infty} \mathcal{I}(v_n) = \left(\frac{1}{p} - \frac{1}{q}\right) A(v_0)r(v_0)^q + \left(\frac{1}{s} - \frac{1}{p}\right) B(v_0)r(v_0)^s = \mathcal{I}(v_0).$$

Therefore, by Lemma 2.13, $r(v_0)v_0$ is a nonnegative and nontrivial critical point if I in E. This completes the proof of Theorem 2.5.

It remains to prove $r_0 = r(v_0)$. Since $v_0 \in \Omega_1$, by Lemma 2.15, we can choose $\mu_0 > 0$ such that $\mu_0 v_0 \in \Omega_1 \cap S^1$. By Lemma 2.15 we know that $r(v_0) > \bar{r}(v_0)$ and $G_1(r(v_0), v_0) = \|v_0\|^p$. Taking the limit at $\|v_n\|^p = G_1(r(v_n), v_n)$, we get $\|v_0\|^p \leq G_1(r_0, v_0)$. Consequently,

$$G_1(r(v_0), v_0) = \|v_0\|^p \leq G_1(r_0, v_0).$$

Since $G_1(r, v_0)$ is decreasing for $r \geq \bar{r}(v_0)$ and $r(v_0) > \bar{r}(v_0)$, it follows that $r_0 \leq r(v_0)$. In fact, we have

$$\bar{r}(v_0) \leq r_0 \leq r(v_0).$$

Suppose by contradiction that $r_0 < r(v_0)$. Since $G_1(r, v_0)$ is strictly decreasing for all $r \in (r_0, r(v_0))$, we see that

$$\|v_0\|^p = G_1(r(v_0), v_0) < G_1(r, v_0), \quad \forall r \in [r_0, r(v_0)).$$

Considering the function

$$h(r) = I(rv_0), \quad r \in (r_0, r(v_0)),$$

a straightforward computation shows that

$$h'(r) = r^{p-1} \left(\|v_0\|^p - G_1(r, v_0) \right) < 0,$$

which implies that h is strictly decreasing. Thus, we get

$$M = \liminf_{n \to \infty} I(r(v_n)v_n) \geq I(r_0 v_0) > I(r(v_0)v_0) = I(r(\mu_0 v_0)\mu_0 v_0) = \mathcal{I}(\mu_0 v_0),$$

with $\mu_0 v_0 \in \Omega_1 \cap S^1$. This contradicts the definition of M and hence $r_0 = r(v_0)$. ■

To end this section we focus on the proof of Theorem 2.6.

Lemma 2.18 *Assume the assumptions in Theorem 2.6. For each fixed $v \in E \backslash \{0\}$ the function $G_1(., v)$ has a unique critical point which is a maximum and is given by*

$$\bar{r}(v) = \left(\frac{B(v)(p-s)}{A(v)(p-q)} \right)^{1/(q-s)}. \tag{2.47}$$

Moreover,

$$G_1(\bar{r}(v), v) = \max_{r>0} G_1(r, v) = \left(\frac{A(v)^{p-s}}{\eta(s,q,p) B(v)^{p-q}} \right)^{1/(q-s)} > 0, \tag{2.48}$$

where $\eta(s, q, p)$ was defined in (2.2).

Proof. By a straighfoward calculation we see that

$$\frac{\partial G_1}{\partial r}(r, v) = 0 \iff r = \bar{r}(v) = \left(\frac{B(v)(p-s)}{A(v)(p-q)} \right)^{1/(q-s)}.$$

Furthermore, we observe that $\lim_{r \to +\infty} G_1(r, v) = 0$, $\lim_{r \to 0^+} G_1(r, v) = -\infty$ and due to (2.47),

$$A(v)\bar{r}(v)^{q-p} = \left(\frac{p-s}{p-q} \right) B(v)\bar{r}(v)^{s-p}.$$

Since,

$$
\begin{aligned}
G_1(\bar{r}(v), v) &= A(v)\bar{r}(v)^{q-p} - B(v)\bar{r}(v)^{s-p} \\
&= \left(\frac{q-s}{p-q}\right) B(v)\bar{r}(v)^{s-p} \\
&= \left(\frac{q-s}{p-q}\right) B(v) \left(\frac{B(v)(p-s)}{A(v)(p-q)}\right)^{(s-p)/(q-s)} \\
&= \left(\frac{A(v)^{p-s}}{\eta(s,q,p)B(v)^{p-q}}\right)^{1/(q-s)} > 0,
\end{aligned}
$$

we can conclude that $G_1(.,v)$ has a unique global maximum at $r = \bar{r}(v) > 0$. ■

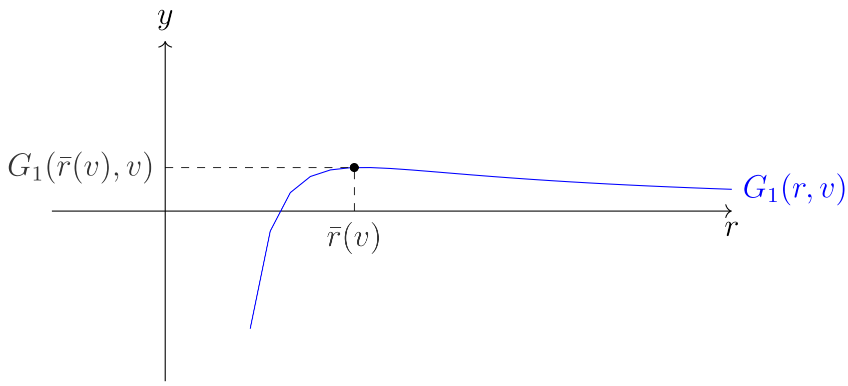

Figure 2.11: Maximum of G_1 for $s < q < p$.

To establish Theorem 2.6, we will explore once more the properties of the set

$$\Omega_1 = \{v \in E\backslash\{0\} : \|v\|^p < G_1(\bar{r}(v), v)\}.$$

Remark 2.9 *Let $s < q < p$, if $\mathcal{C}_1$ is the set defined in (2.8), then $\mathcal{C}_1 \subset \Omega_1$ and hence $\Omega_1 \neq \emptyset$, if $\mathcal{C}_1 \neq \emptyset$. Indeed, first we observe that $v \in \mathcal{C}_1$ if, and only if,*

$$\|v\|^p < \left(\frac{s}{q}\right)^{\frac{p-s}{q-s}} \left(\frac{A(v)^{p-s}}{\eta(s,q,p)B(v)^{p-q}}\right)^{1/(q-s)} = \left(\frac{s}{q}\right)^{\frac{p-s}{q-s}} G_1(\bar{r}(v), v).$$

Then, since $s < q < p$, we have $\|v\|^p < G_1(\bar{r}(v), v)$ and $v \in \Omega_1$.

Lemma 2.19 *If $s < q < p$, for each $v \in \Omega_1$ there exists a unique positive real number $r(v) > \bar{r}(v)$ such that the pair $(r(v), v)$ satisfies*

$$\|v\|^p = r(v)^{q-p}A(v) - r(v)^{s-p}B(v) = G_1(r(v), v),$$

and $r \in C^1(\Omega_1, \mathbb{R})$. Furthermore, for any $v \in \Omega_1$ and $\mu > 0$, it holds $\mu v \in \Omega_1$, and as a consequence, $\Omega_1 \cap S^1 \neq \emptyset$.

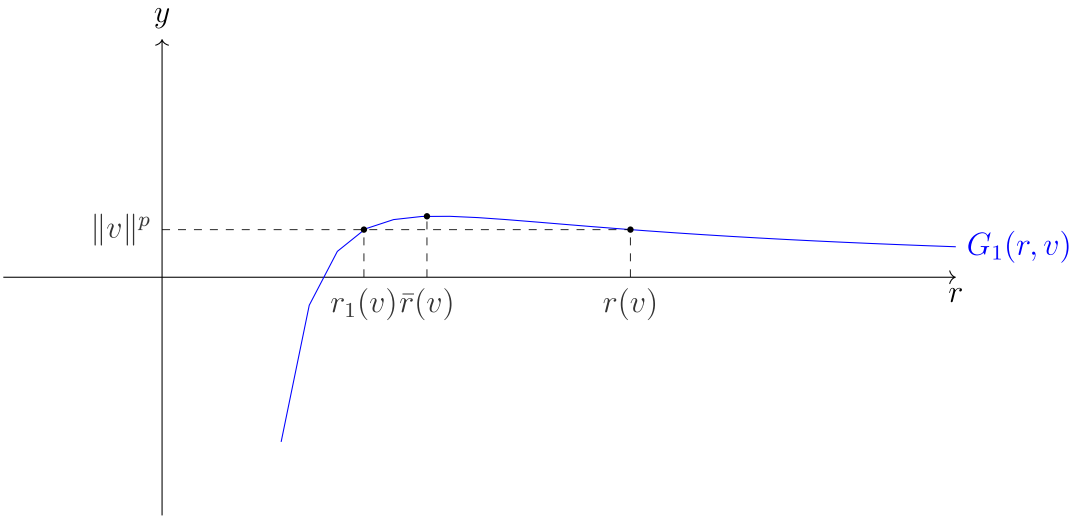

Figure 2.12: Existence of $r(v) > \bar{r}(v)$ for $s < q < p$.

Proof. If $v \in \Omega_1$ we have $\|v\|^p < G_1(\bar{r}(v), v)$. Since $s < q < p$, we have

$$\lim_{r\to\infty} G_1(r, v) = 0,$$

and so by the Intermediate Value Theorem, there exists a real number $r(v) > \bar{r}(v)$ such that the pair $(r(v), v)$ verifies $\|v\|^p = G_1(r(v), v)$. To prove that $r(v)$ is unique, we observe that $\frac{\partial G_1}{\partial r} < 0$ for all $r > \bar{r}(v)$. In fact, first note that by (2.47)

$$(q - p)A(v) = (s - p)\bar{r}(v)^{s-q}B(v).$$

Consequently,

$$\begin{aligned}\frac{\partial G_1}{\partial r}(r, v) &= (q - p)r^{q-p-1}A(v) - (s - p)r^{s-p-1}B(v)\\ &= (s - p)B(v)r^{q-p-1}\bar{r}(v)^{s-q} - (s - p)r^{s-p-1}B(v)\\ &= (s - p)r^{q-p-1}B(v)\left(\frac{1}{\bar{r}(v)^{q-s}} - \frac{1}{r^{q-s}}\right) < 0,\end{aligned}$$

for all $r > \bar{r}(v)$, thereby implying the uniqueness of $r(v)$.

To verify that the map r is a C^1, by setting $r = r(v)$, we obtain

$$\frac{\partial G_1}{\partial r}(r(v), v) = (s - p)r^{q-p-1}B(v)\left(\frac{1}{\bar{r}(v)^{q-s}} - \frac{1}{r(v)^{q-s}}\right) < 0. \tag{2.49}$$

Now, considering the function $f : (0, \infty) \times \Omega_1 \to \mathbb{R}$ given by

$$f(r, v) = G_1(r, v) - \|v\|^p,$$

by (2.49) we see that given $v \in \Omega_1$, $\frac{\partial f}{\partial r}(r(v), v) < 0$. Using the implicit function theorem, we obtain open sets $J \subset \mathbb{R}$ and $V \subset \Omega_1$ containing $r(v)$ and v respectively, and a C^1-function $\tau : V \to J$ satisfying

$$\tau(v) = r(v) > \bar{r}(v) \quad \text{and} \quad f(\tau(w), w) = 0, \quad \forall w \in V.$$

Then, there exists a neighborhood $U \subset V$ containing v such that $\tau = r > \bar{r}$ in U. By the uniqueness $r \equiv \tau$ in U and therefore r is a C^1-function in U. Since v is arbitrary, we have $r \in C^1(\Omega_1, \mathbb{R})$.

Now, observe that given $\mu > 0$

$$\mu \bar{r}(\mu v) = \bar{r}(v), \quad \forall v \in E \backslash \{0\}. \tag{2.50}$$

In fact, by (2.47) we have

$$\bar{r}(\mu v) = \left(\frac{B(\mu v)(p-s)}{A(\mu v)(p-q)} \right)^{1/(q-s)} = \frac{\mu^{\frac{s}{q-s}}}{\mu^{\frac{q}{q-s}}} \left(\frac{B(v)(p-s)}{A(v)(p-q)} \right)^{1/(q-s)} = \frac{1}{\mu} \bar{r}(v).$$

By (2.47) we can write

$$A(v) = \left(\frac{p-s}{p-q} \right) B(v) \bar{r}(v)^{s-q}$$

and therefore

$$G_1(\bar{r}(v), v) = \bar{r}(v)^{q-p} A(v) - \bar{r}(v)^{s-p} B(v) = \frac{q-s}{p-q} \bar{r}(v)^{s-p} B(v).$$

Suppose that $v \in \Omega_1$, that is

$$\|v\|^p < G_1(\bar{r}(v), v) = \frac{q-s}{p-q} \bar{r}(v)^{s-p} B(v).$$

Then, using (2.50) we obtain

$$\|\mu v\|^p < \mu^{p-s+s} \frac{q-s}{p-q} \bar{r}(v)^{s-p} B(v) = \frac{q-s}{p-q} \bar{r}(\mu v)^{s-p} B(\mu v) = G_1(\bar{r}(\mu v), \mu v),$$

which implies that $\mu v \in \Omega_1$. Taking $\mu = \|v\|^{-1}$ we have that $\Omega_1 \cap S^1 \neq \emptyset$. ■

Lemma 2.20 *Let $s < q < p$. If $v \in \mathcal{C}_1$, where $\mathcal{C}_1$ is the set given by (2.8), we have*

$$\bar{r}(v) < \left(\frac{q}{s} \right)^{\frac{1}{q-s}} \bar{r}(v) < r(v), \tag{2.51}$$

where r is the function given in Lemma 2.19. Moreover,

$$\mathcal{I}(v) < 0, \quad \forall v \in \mathcal{C}_1$$

and $\mathcal{C}_1 \cap S^1 \neq \emptyset$.

Proof. Since $\bar{r}(v) < r(v)$ and $G_1(r, v)$ is decreasing for $r > \bar{r}(v)$, as shown in the proof of Lemma 2.19, to establish (2.51), it suffices to demonstrate that if $v \in \mathcal{C}_1$, then

$$G_1\left(\left(\frac{q}{s}\right)^{\frac{1}{q-s}} \bar{r}(v), v\right) > G_1(r(v), v).$$

As verified in Remark 2.9, $v \in \mathcal{C}_1$ if, and only if, satisfies

$$G_1(r(v), v) = \|v\|^p < \left(\frac{s}{q}\right)^{\frac{p-s}{q-s}} G_1(\bar{r}(v), v). \tag{2.52}$$

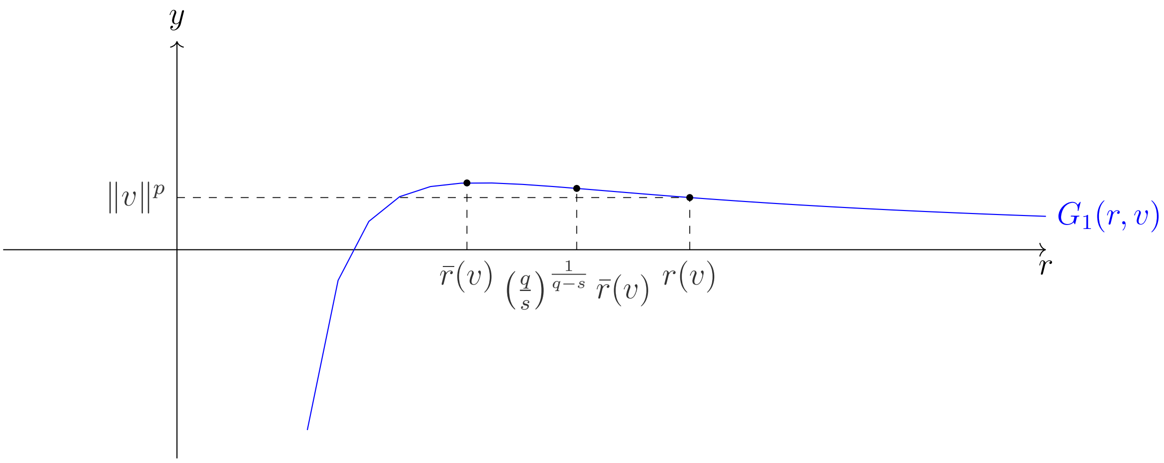

Figure 2.13: $\left(\frac{q}{s}\right)^{\frac{1}{q-s}} \bar{r}(v) < r(v)$

Thus, since $s < q < p$, we have

$$\begin{aligned} G_1\left(\left(\frac{q}{s}\right)^{\frac{1}{q-s}} \bar{r}(v), v\right) &= \left(\frac{q}{s}\right)^{\frac{q-p}{q-s}} \bar{r}(v)^{q-p} A(v) - \left(\frac{q}{s}\right)^{\frac{s-p}{q-s}} \bar{r}(v)^{s-p} B(v) \\ &> \left(\frac{q}{s}\right)^{\frac{s-p}{q-s}} \bar{r}(v)^{q-p} A(v) - \left(\frac{q}{s}\right)^{\frac{s-p}{q-s}} \bar{r}(v)^{s-p} B(v) \\ &= \left(\frac{s}{q}\right)^{\frac{p-s}{q-s}} G_1(\bar{r}(v), v) \\ &> G_1(r(v), v), \end{aligned}$$

which conclude (2.51).

To verify that $\mathcal{I} < 0$ in $\mathcal{C}_1$, by (2.47) we write

$$B(v) = \frac{p-q}{p-s} \bar{r}(v)^{q-s} A(v).$$

and by (2.51)

$$\bar{r}(v)^{q-s} < \frac{s}{q} r(v)^{q-s}.$$

Therefore,

$$\begin{aligned}\mathcal{I}(v) &= \left(\frac{1}{p}-\frac{1}{q}\right)A(v)r(v)^q + \left(\frac{1}{s}-\frac{1}{p}\right)B(v)r(v)^s\\ &= \left(\frac{q-p}{pq}\right)A(v)r(v)^q + \left(\frac{p-s}{sp}\right)\left(\frac{p-q}{p-s}\right)\bar{r}(v)^{q-s}A(v)r(v)^s\\ &< \left(\frac{q-p}{pq}\right)A(v)r(v)^q + \left(\frac{p-q}{sp}\right)A(v)\frac{s}{q}r(v)^q\\ &=\frac{1}{p}A(v)r(v)^q\left(\frac{q-p}{q}+\frac{p-q}{q}\right)\\ &=0.\end{aligned}$$

Let $\mu > 0$, then, by (2.52) and (2.50),

$$\begin{aligned}\|\mu v\|^p &<\mu^p\left(\frac{s}{q}\right)^{\frac{p-s}{q-s}} G_1(\bar{r}(v),v)\\ &=\mu^p\left(\frac{s}{q}\right)^{\frac{p-s}{q-s}}(\bar{r}(v)^{q-p}A(v)-\bar{r}(v)^{s-p}B(v))\\ &=\left(\frac{s}{q}\right)^{\frac{p-s}{q-s}}(\mu^{p-q}\mu^q\bar{r}(v)^{q-p}A(v)-\mu^{p-s}\mu^s\bar{r}(v)^{s-p}B(v))\\ &=\left(\frac{s}{q}\right)^{\frac{p-s}{q-s}}(\bar{r}(\mu v)^{q-p}A(\mu v)-\bar{r}(\mu v)^{s-p}B(\mu v))\\ &=\left(\frac{s}{q}\right)^{\frac{p-s}{q-s}} G_1(\bar{r}(\mu v),\mu v),\end{aligned}$$

which implies that $\mu v \in \mathcal{C}_1$. Taking $\mu = \|v\|^{-1}$ we conclude that $\mathcal{C}_1 \cap S^1 \neq \emptyset$. ■

Remark 2.10 *If $1 < s < q < p$, by Lemma 2.19, for each $v \in \Omega_1$ there is only a real value $r(v)$ such that*

$$G_1(r(v),v) = \|v\|^p \quad \text{and} \quad r(v) > \bar{r}(v).$$

Since $r(|v|) > \bar{r}(|v|) = \bar{r}(v)$ with $G_1(r(|v|),v) = \|v\|^p$, we have $r(v) = r(|v|)$ and hence $\mathcal{I}(v) = \mathcal{I}(|v|)$. Therefore, if v_0 is a minimum of $\mathcal{I}$ under $\Omega_1 \cap S^1$, so is $|v_0|$. This implies that solutions to the minimization problem,

$$\inf_{v\in\Omega_1\cap S^1}\mathcal{I}(v),$$

and hence, by Lemma 2.13, the solutions for $(\mathcal{P}_1)$ for $1 < s < q < p$, can be taken as nonnegative without loss of generality.

Now we proceed with the proof of Theorem 2.6.

Proof of Theorem 2.6. Observe that r, given by Lemma 2.19 is bounded in S^1. In fact, if $v \in S^1$ we have

$$1 = r(v)^{q-p}A(v) - r(v)^{s-p}B(v) \leq r(v)^{q-p}A(v),$$

then $r(v) \leq A(v)^{\frac{1}{p-q}}$. Since $q < p$ and A is bounded in S^1 due to Lemma 2.11, we conclude that r is bounded in S^1. Thus, $\mathcal{I}$ is bounded in S^1 and due to Remark 2.9 and Lemma 2.20 we have

$$-\infty < M = \inf_{v \in \Omega_1 \cap S^1} \mathcal{I}(v) < 0.$$

Let $(v_n) \subset \Omega_1 \cap S^1$ a minimizing sequence. There exist $v_0 \in E$ such that $v_n \rightharpoonup v_0$ in E and by Lemma 2.11,

$$A(v_n) \to A(v_0) \quad \text{and} \quad B(v_n) \to B(v_0). \tag{2.53}$$

Moreover, if needed, passing to a subsequence, we have $r(v_n) \to r_0 \geq 0$. In particular, observe that $r_0 > 0$, otherwise

$$0 > M = \lim_{n\to\infty} \mathcal{I}(v_n) = \left(\frac{1}{p} - \frac{1}{q}\right) A(v_0) r_0^q + \left(\frac{1}{s} - \frac{1}{p}\right) B(v_0) r_0^s = 0,$$

which is impossible. Furthermore, we observe that $v_0 \neq 0$ since

$$1 = r(v_n)^{q-p}A(v_n) - r(v_n)^{s-p}B(v_n) \leq r(v_n)^{q-p}A(v_n),$$

which implies that $0 < r_0 \leq A(v_0)^{\frac{1}{p-q}}$.

Now, lets prove that $v_0 \in \Omega_1$. First observe that by (2.47) and (2.53) we have

$$\bar{r}(v_n) = \left(\frac{B(v_n)(p-s)}{A(v_n)(p-q)}\right)^{1/(q-s)} \longrightarrow \left(\frac{B(v_0)(p-s)}{A(v_0)(p-q)}\right)^{1/(q-s)} = \bar{r}(v_0)$$

and

$$G_1(\bar{r}(v_n), v_n) \longrightarrow G_1(\bar{r}(v_0), v_0).$$

Then, we get

$$\|v_0\|^p \leq \liminf_{n\to\infty} \|v_n\|^p \leq \liminf_{n\to\infty} G_1(\bar{r}(v_n), v_n) = G_1(\bar{r}(v_0), v_0).$$

Suppose that $v_0 \notin \Omega_1$, by the inequality we obtain

$$\|v_0\|^p = G_1(\bar{r}(v_0), v_0).$$

Conversely,

$$\|v_0\|^p \leq \liminf_{n\to\infty} \|v_n\|^p = \liminf_{n\to\infty} G_1(r(v_n), v_n) = G_1(r_0, v_0) \tag{2.54}$$

and therefore $G_1(\bar{r}(v_0), v_0) = \|v_0\|^p \leq G(r_0, v_0)$. Since by Lemma 2.18 $\bar{r}(v_0)$ is the global maximum of $G_1(., v_0)$, it follows that $\bar{r}(v_0) = r_0$. By (2.47) we can write

$$B(v_0)\bar{r}(v_0)^s = \left(\frac{p-q}{p-s}\right) A(v_0)\bar{r}(v_0)^q.$$

Thus

$$\begin{aligned} M &= \left(\frac{1}{p} - \frac{1}{q}\right) A(v_0)\bar{r}(v_0)^q + \left(\frac{1}{s} - \frac{1}{p}\right) B(v_0)\bar{r}(v_0)^s \\ &= \left(\frac{q-p}{pq}\right) A(v_0)\bar{r}(v_0)^q + \left(\frac{p-s}{sp}\right)\left(\frac{p-q}{p-s}\right) A(v_0)\bar{r}(v_0)^q \\ &= \frac{A(v_0)\bar{r}(v_0)^q}{p}\left(\frac{q-p}{q} + \frac{p-q}{s}\right) \\ &> 0, \end{aligned}$$

which is an absurd and therefore $v_0 \in \Omega_1$. Now, we assert that $r_0 = r(v_0)$. To begin, let $\mu = \|v_0\|^{-1}$. According to Lemma 2.19, it follows that $\mu v_0 \in \Omega_1 \cap S^1$. Observe now that

$$\begin{aligned} G_1(\mu r(\mu v_0), v_0) &= \frac{1}{\mu^p}(r(\mu v_0)^{q-p} A(\mu v_0) - r(\mu v_0)^{s-p} B(\mu v_0)) \\ &= \frac{1}{\mu^p} G_1(r(\mu v_0), \mu v_0) \\ &= \frac{1}{\mu^p} \|\mu v_0\|^p \\ &= \|v_0\|^p \\ &= G_1(r(v_0), v_0). \end{aligned}$$

Then, given that $r(v_0) > \bar{r}(v_0)$ and $\mu r(\mu v_0) > \bar{r}(\mu v_0) = \bar{r}(v_0)$, by Lemma 2.19 we have

$$\mu r(\mu v_0) = r(v_0). \tag{2.55}$$

Furthermore, since $\bar{r}(v_n) < r(v_n)$, taking the limit we get $\bar{r}(v_0) \leq r_0$ and by (2.54) we obtain

$$\bar{r}(v_0) \leq r_0 \leq r(v_0).$$

Suppose by contradiction that $r_0 < r(v_0)$ and consider the function

$$g(t) := \frac{\partial}{\partial t} I(t v_0) = t^{p-1}(\|v_0\|^p - G_1(t, v_0)),$$

for $t \in (r_0, r(v_0))$. Note that $g(t) < 0$, hence $I(tv_0)$ is decreasing on t and by (2.55)

$$M = \liminf_{n\to\infty} I(r(v_n)v_n) \geq I(r_0 v_0) > I(r(v_0)v_0) = I(r(\mu v_0)\mu v_0) = \mathcal{I}(\mu v_0),$$

with $\mu v_0 \in \Omega_1 \cap S^1$, which is impossible. Therefore, $r_0 = r(v_0)$, consequently

$$1 = \lim_{n\to\infty} \|v_n\|^p = \lim_{n\to\infty} G_1(r(v_n), v_n) = G_1(r(v_0), v_0) = \|v_0\|^p,$$

which ensures that $v_0 \in \Omega_1 \cap S^1$, and

$$M = \lim_{n\to\infty} \mathcal{I}(v_n) = \left(\frac{1}{p} - \frac{1}{q}\right) A(v_0) r(v_0)^q + \left(\frac{1}{s} - \frac{1}{p}\right) B(v_0) r(v_0)^s = \mathcal{I}(v_0).$$

Finally, by Lemma 2.13, $r(v_0)v_0$ is a nonnegative and nontrivial critical point if I in E. ■

2.4 Final comments

In this section, we explore potential future developments stemming from the results established in this chapter.

- Our results demonstrate the existence of solutions for problems involving nonlinearities within the subcritical growth range in terms of new Sobolev embedding proved in the present work. It would be interesting to explore the existence of solutions for nonlinearities with corresponding critical growth.
- We address the case $p = N$ with polynomial growth. It is also important to consider scenarios where the nonlinearities exhibit exponential growth in the fashion of Trudinger-Moser-type inequalities.

Chapter 3

Application 2: ρ-harmonic functions with indefinite boundary conditions

As a second application of our Hardy-type inequality (1.1), in this chapter, we will address the existence and nonexistence of solutions for the following class of problems:

$$\begin{cases} -\mathrm{div}(\rho(x)|\nabla u|^{p-2}\nabla u) = 0 & \text{in } \mathbb{R}^N_+, \\ \rho(x',0)|\nabla u|^{p-2}\nabla u \cdot \nu = h(x')|u|^{q-2}u - m(x')|u|^{s-2}u, & \text{on } \mathbb{R}^{N-1}, \end{cases} \tag{$\mathcal{P}_2$}$$

where $\mathbb{R}^N_+ = \{x = (x', x_N) \in \mathbb{R}^N : x' \in \mathbb{R}^{N-1}, x_N > 0\}$ standards for the upper half-space, ν is the unit outer normal to the boundary $\partial\mathbb{R}^N_+ := \mathbb{R}^{N-1}$ and

$$1 < p \leq N \quad \text{and} \quad 1 < q, s \leq p_*.$$

Here and what follows, $p_* := (N-1)p/(N-p)$ for $p < N$ and $p_* = \infty$ when $p = N$ denotes the critical exponent of the Sobolev trace embedding. Throughout this chapter, we will assume that ρ, h, m are positive functions satisfying

$$\rho \in L^1_{\mathrm{loc}}(\mathbb{R}^N_+) \quad \text{and} \quad h, m \in L^1_{\mathrm{loc}}(\mathbb{R}^{N-1}).$$

If we assume that $\rho \equiv 1$, problem ($\mathcal{P}_2$) has been considered by many authors, see for instance, [1, 2, 26]

3.1 Main results

We also will assume that the weight function ρ satisfy the hypothesis:

(H_0) there are constants $\rho_0 > 0$ and $\gamma > p-1$ such that

$$\rho(x) \geq \rho_0(1 + x_N)^\gamma \quad \text{a.e. in } \mathbb{R}^N_+.$$

Here, by a weak solution of $(\mathcal{P}_2)$ we mean a function $u \in E$ such that

$$\int_{\mathbb{R}^N_+} \rho(x)|\nabla u|^{p-2}\nabla u \nabla\varphi \, \mathrm{d}x = \int_{\mathbb{R}^{N-1}} (h(x')|u|^{q-2}u - m(x')|u|^{s-2}u)\varphi \, \mathrm{d}x', \tag{3.1}$$

holds for every $\varphi \in C^\infty_\delta(\mathbb{R}^N_+)$.

3.1.1 Liouville-type results

Our first nonexistence result for $(\mathcal{P}_2)$ is established as follows.

Theorem 3.1 (p-sublinear case) *Assume (H_0) and suppose that $m \in L^1(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$. If $1 < s < q < p \leq N$ and $h/m \in L^\infty(\mathbb{R}^{N-1})$ with*

$$\left\| \frac{h}{m} \right\|_\infty^{p-s} \left(\frac{\|m\|_\infty C_{p,\gamma}^{1-p}}{\rho_0} \right)^{q-s} < \eta(s,q,p), \tag{3.2}$$

then $(\mathcal{P}_2)$ possesses only the trivial weak solution.

Remark 3.1 *Straightforward computation shows that, for $\lambda > 0$ sufficiently small, the functions,*

$$h(x') = \frac{\lambda}{(1+|x'|)^{\theta_1}} \quad \text{and} \quad m(x') = \frac{1}{(1+|x'|)^{\theta_2}},$$

for $x' \in \mathbb{R}^{N-1}$, satisfy the assumptions of Theorem 3.1 whenever $0 < N-1 < \theta_2 \leq \theta_1$.

In our second nonexistence result, we address the case where $p < q < s$.

Theorem 3.2 (p-superlinear case) *Assume (H_0) and suppose that $m \in L^\infty(\mathbb{R}^{N-1})$ and $1 < p < N$. If $1 < p < q < s \leq p_*$ and $h/m \in L^\infty(\mathbb{R}^{N-1})$ with*

$$\left\| \frac{h}{m} \right\|_\infty^{s-p} \left(\frac{\|m\|_\infty C_{p,\gamma}^{1-p}}{\rho_0} \right)^{s-q} < \eta(p,q,s), \tag{3.3}$$

then $(\mathcal{P}_2)$ possesses only the trivial weak solution. Moreover, the same result holds if $p = N$ and $p < q < s < \infty$.

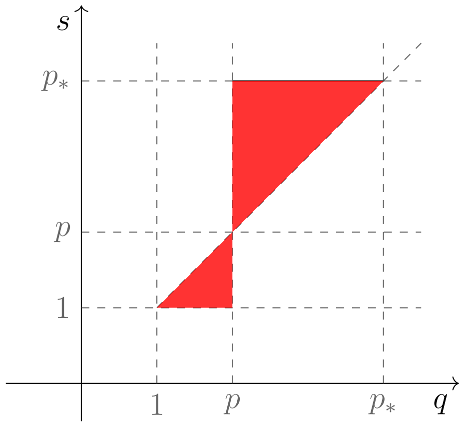

Figure 3.1: Nonexistence of solutions for ($\mathcal{P}_2$)

3.1.2 Existence results

Our existence results are established using the Fibering method. Theorem 1.5, combined with the additional conditions on the potentials, more precisely, $h, m \in L^1(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$, ensure that the embeddings

$$E \hookrightarrow L^q(\mathbb{R}^{N-1}, h(x')) \quad \text{and} \quad E \hookrightarrow L^s(\mathbb{R}^{N-1}, m(x'))$$

are both valid and compact. Thus facilitating the application of the Fibering Method, as shown in [35, 41].

In our first existence result we explore two combinations of the exponents. Precisely, we consider the cases $1 < s < p < q < p_*$ and $1 < p < s < q < p_*$.

Theorem 3.3 *Assume* (H_0) *holds and* $1 < p \leq N$*. If* $\max\{s, p\} < q < p_*$*, and* $h, m \in L^1(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$*, then* ($\mathcal{P}_2$) *has a nontrivial and nonnegative weak solution.*

In our second existence result, we will consider the cases $1 < q < s < p$ or $1 < q < p < s < p_*$.

Theorem 3.4 *If* (H_0) *holds,* $q < \min\{s, p\}$ *with* $s < p_*$*, and* $h, m \in L^1(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$*, then* ($\mathcal{P}_2$) *has a nontrivial and nonnegative weak solution.*

To present our third existence result of this chapter, we consider the functionals defined on E by

$$H(u) = \int_{\mathbb{R}^{N-1}} h|u|^q \, dx \quad \text{and} \quad M(u) = \int_{\mathbb{R}^{N-1}} m|u|^s \, dx. \tag{3.4}$$

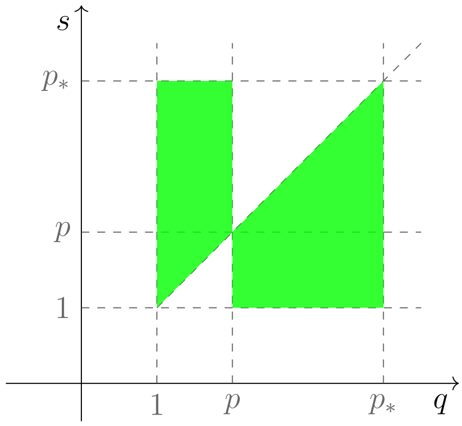

Figure 3.2: Existence of nontrivial solutions for ($\mathcal{P}_2$)

We note that under the assumptions of Theorem 2.2, for $u \in E$, the following key inequality holds true (see Lemma 3.9)

$$H(u)^{s-p} < \eta(p,q,s)M(u)^{q-p}\|u\|^{p(s-q)} < \eta(p,q,s)\left(\frac{q}{p}\right)^{s-q} M(u)^{q-p}\|u\|^{p(s-q)}.$$

We consider the existence of solutions for ($\mathcal{P}_2$) in a subset of the complementary situation of this inequality.

$$\mathcal{D}_2 := \left\{ u \in E : H(u)^{s-p} > \left(\frac{q}{p}\right)^{s-q} \eta(p,q,s)M(u)^{q-p}\|u\|^{p(s-q)} \right\}. \tag{3.5}$$

Theorem 3.5 *Let $1 < p \le N$ and assume that (H_0) holds. If $h, m \in L^1(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$, $p < q < s < p_*$, $\mathcal{D}_2 \neq \emptyset$ and*

$$\frac{h^{1/q}}{m^{1/s}} \in L^{\frac{sq}{s-q}}(\mathbb{R}^{N-1}), \tag{3.6}$$

then, ($\mathcal{P}_2$) has a nontrivial and nonnegative weak solution.

Remark 3.2 *The functions $h = m = \lambda w$, with w given by*

$$w(x') = \frac{1}{(1+|x'|)^\theta},$$

satisfy the assumptions of Theorem 3.5 for $\theta > N-1$ and λ sufficiently large. In fact, first, we observe that

$$\left[\frac{h(x')^{1/q}}{m(x')^{1/s}}\right]^{\frac{sq}{s-q}} = h(x') = \lambda(1+|x'|)^{-\theta} \in L^1(\mathbb{R}^{N-1}),$$

whenever $\theta > N-1$. For $u \in E\backslash\{0\}$ fixed, one has

$$H(u)^{s-p}/M(u)^{q-p} = \lambda^{s-q}\|u\|^{q(s-p)}_{L^q(\mathbb{R}^{N-1},w)}\|u\|^{s(p-q)}_{L^s(\mathbb{R}^{N-1},w)}.$$

Since $s > q$, for λ sufficiently large we see that

$$H(u)^{s-p} > \left(\frac{q}{p}\right)^{s-q} \eta(p,q,s)M(u)^{q-p}\|u\|^{p(s-q)},$$

and hence $\mathcal{D}_2 \neq \emptyset$. Conversely, it is easy to verify that for λ small the potentials satisfies the condition for nonexistence, (3.3). As mentioned in [41], the condition $\mathcal{D}_2 \neq \emptyset$ can be interpreted as saying that h is sufficient "large" with respect to m.

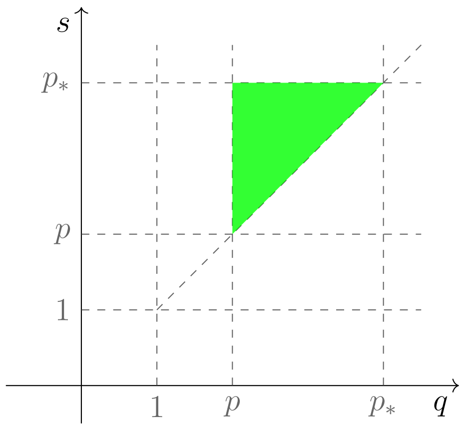

Figure 3.3: Existence of nontrivial solutions for ($\mathcal{P}_2$)

Our last existence result is derived from the consideration of the set

$$\mathcal{C}_2 := \left\{u \in E : H(u)^{p-s} > \left(\frac{q}{s}\right)^{p-s} \eta(s,q,p)M(u)^{p-q}\|u\|^{p(q-s)}\right\}, \tag{3.7}$$

where, similar to the earlier case, we assume that h is sufficiently "large" compared to m, expressed by the condition $\mathcal{C}_2 \neq \emptyset$. Our fourth existence result for ($\mathcal{P}_2$) is stated below:

Theorem 3.6 *Assume (H_0), $s < q < p \leq N$ and $h, m \in L^1(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$. If $\mathcal{C}_2 \neq \emptyset$, then ($\mathcal{P}_2$) has a nontrivial and nonnegative weak solution.*

It is important to note that if $q < s = p$, the Direct Methods in the Calculus of Variations guarantee the existence of solutions to ($\mathcal{P}_2$). Conversely, when $s = p < q$, the mountain-pass theorem can be employed to find solutions.

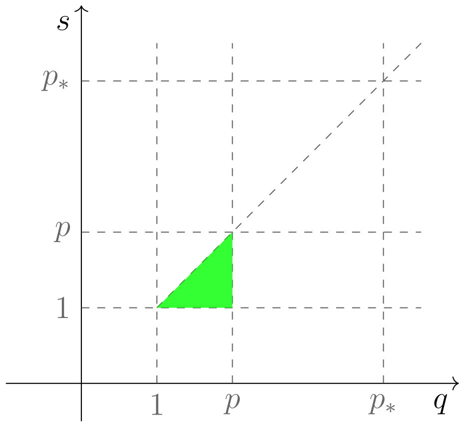

Figure 3.4: Existence of nontrivial solutions for ($\mathcal{P}_2$)

The chapter is organized as follows: In Section 3, we present the proofs of Theorems 1.1 and 1.5. In Section 3.2, we establish our Liouville type results, Theorems 3.1 and 3.2. In Section 3.3 we present the proofs of our existence results by proving Theorems 3.3, 3.4, 3.5 and 3.6.

3.2 Proof of our Liouville type results

In this section, our focus is on proving Liouville-type results. The following estimate plays a crucial role in our analysis.

Lemma 3.7 (key estimate) *Assume condition* (H_0) *and* $1 < s < q < p$. *If* $m \in L^1(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$ *then* M *is well defined in* E. *In addition, if* $h/m \in L^\infty(\mathbb{R}^{N-1})$, *then*

$$H(v)^{p-s} \leq \left(\left\| \frac{h}{m} \right\|_\infty^{(p-s)} \left[\frac{\|m\|_\infty C_{p,\gamma}^{1-p}}{\rho_0} \right]^{q-s} \right) M(v)^{p-q} \|v\|^{p(q-s)}, \quad \forall v \in E. \tag{3.8}$$

In particular, H *is well-defined. Furthermore, if* (2.3) *holds, then*

$$H(v)^{p-s} < \eta(s,q,p) M(v)^{p-q} \|v\|^{p(q-s)}, \quad \forall v \in E \backslash \{0\}. \tag{3.9}$$

Proof. First we observe that

$$\begin{aligned} M(v) := \int_{\mathbb{R}^{N-1}} m|v|^s \, \mathrm{d}x &= \int_{\mathbb{R}^{N-1}} m^{\frac{p-s}{p}} m^{s/p} |v|^s \, \mathrm{d}x \\ &\leq \|m\|_{L^1(\mathbb{R}^N_+)}^{(p-s)/s} \left(\int_{\mathbb{R}^{N-1}} m|v|^p \, \mathrm{d}x \right)^{s/p} \end{aligned}$$

$$\leq \|m\|_{\infty}^{s/p} \|m\|_{L^1(\mathbb{R}^N_+)}^{(p-s)/s} \left(\int_{\mathbb{R}^{N-1}} |v|^p \,\mathrm{d}x \right)^{s/p},$$

which is finite by Theorem 1.1 and assumption (H_0).

If $s < q < p$ we can express $q = (1-\alpha)s + \alpha p$ with $\alpha = (q-s)/(p-s) \in (0,1)$. Thus, by Hölder's inequality,

$$\begin{aligned} H(v) = \int_{\mathbb{R}^{N-1}} h|v|^q \,\mathrm{d}x' &= \int_{\mathbb{R}^{N-1}} (h|v|^s)^{1-\alpha} (h|v|^p)^{\alpha} \,\mathrm{d}x' \\ &\leq \left(\int_{\mathbb{R}^{N-1}} h|v|^s \,\mathrm{d}x' \right)^{1-\alpha} \left(\int_{\mathbb{R}^{N-1}} h|v|^p \,\mathrm{d}x' \right)^{\alpha}. \end{aligned}$$

Using that $1-\alpha = (p-q)/(p-s)$ we obtain

$$H(v)^{p-s} \leq \left(\int_{\mathbb{R}^{N-1}} h|v|^s \,\mathrm{d}x' \right)^{p-q} \left(\int_{\mathbb{R}^{N-1}} h|v|^p \,\mathrm{d}x' \right)^{q-s}. \tag{3.10}$$

Now, observe that

$$\int_{\mathbb{R}^{N-1}} h|v|^s \,\mathrm{d}x' = \int_{\mathbb{R}^{N-1}} \frac{h}{m} (m|v|^s) \,\mathrm{d}x' \leq \left\| \frac{h}{m} \right\|_{\infty} M(v).$$

Thus, (1.12) and (H_0) gives

$$\int_{\mathbb{R}^{N-1}} h|v|^p \,\mathrm{d}x' \leq \|m\|_{\infty} \left\| \frac{h}{m} \right\|_{\infty} \int_{\mathbb{R}^{N-1}} |v|^p \,\mathrm{d}x' \leq \|m\|_{\infty} \left\| \frac{h}{m} \right\|_{\infty} \frac{C_{p,\gamma}^{1-p}}{\rho_0} \|v\|^p.$$

Therefore, plugging the last two inequalities into (3.10), we estimate (3.8). ■

Throughout this chapter, we will consider the following auxiliary function:

$$G_2(r,v) = H(v) r^{q-p} - M(v) r^{s-p}, \quad r > 0 \quad \text{and} \quad v \in E. \tag{3.11}$$

Lemma 3.8 *Assume the assumptions in Theorem 2.1. For each fixed $v \in E\backslash\{0\}$ the function $G_2(.,v)$ has a unique critical point which is a maximum and is given by*

$$\bar{r}(v) = \left(\frac{M(v)(p-s)}{H(v)(p-q)} \right)^{1/(q-s)}.$$

Moreover,

$$G_2(\bar{r}(v), v) = \max_{r>0} G_2(r,v) = \left(\frac{H(v)^{p-s}}{\eta(s,q,p) M(v)^{p-q}} \right)^{1/(q-s)} > 0, \tag{3.12}$$

where $\eta(s,q,p)$ was defined in (2.2).

Proof. For each $v \in E\backslash\{0\}$, we have

$$\frac{\partial G_2}{\partial r}(r,v) = (q-p)H(v)r^{q-p-1} - (s-p)M(v)r^{s-p-1}.$$

Thus,

$$\frac{\partial G_2}{\partial r}(r,v) = 0 \iff r = \bar{r}(v) = \left(\frac{M(v)(p-s)}{H(v)(p-q)}\right)^{1/(q-s)}.$$

Moreover, we can see that $\lim_{r\to+\infty} G_2(r,v) = 0$, $\lim_{r\to 0^+} G_2(r,v) = -\infty$ and by (3.15)

$$H(v)\bar{r}(v)^{q-p} = \frac{p-s}{p-q}M(v)\bar{r}^{s-p}$$

Thus, since

$$\begin{aligned} G_2(\bar{r}(v),v) =& H(v)\bar{r}(v)^{q-p} - M(v)\bar{r}(v)^{s-p} \\ =& \left(\frac{p-s}{q-p} - 1\right) M(v)\bar{r}(v)^{s-p} \\ =& \frac{q-s}{p-q}M(v)\left(\frac{M(v)(p-s)}{H(v)(p-q)}\right)^{\frac{s-p}{q-s}} \\ =& \frac{(q-s)(p-s)^{\frac{s-p}{q-s}}}{(p-q)^{\frac{q-p}{q-s}}}\frac{M(v)^{\frac{q-p}{q-s}}}{H(v)^{\frac{s-p}{q-s}}} \\ =& \left(\frac{H(v)^{p-s}}{\eta(s,q,p)M(v)^{p-q}}\right)^{1/(q-s)} > 0, \end{aligned}$$

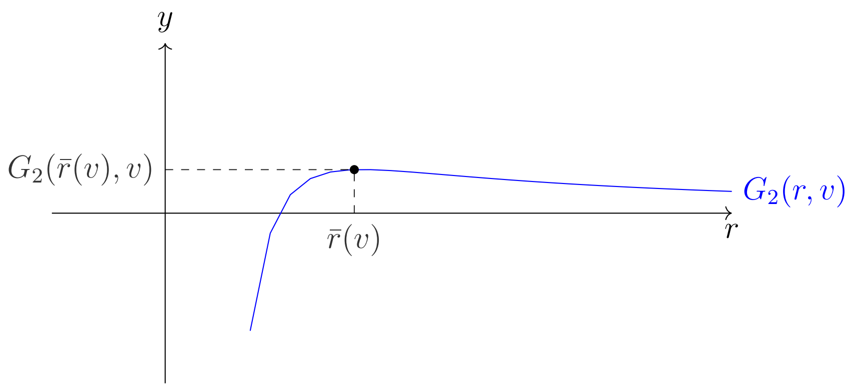

Figure 3.5: Maximum of G_2 for $s < q < p$.

we can conclude that $G_2(.,v)$ has a unique global maximum at $r = \bar{r}(v) > 0$. ■

Now, we proceed to present the proof of Theorem 3.1.

Proof of Theorem 3.1. Assume by contradiction that $(\mathcal{P}_2)$ has a nontrivial weak solution $u_0 \in E$. Then, from the definition (3.1), Lemma 3.7 and a density argument, it follows that

$$\|u_0\|^p = H(u_0) - M(u_0) = G_2(1, u_0).$$

On the other hand, by estimate (3.9), we have

$$H(u_0)^{p-s} < \eta(s,q,p) M(u_0)^{p-q} \|u_0\|^{p(q-s)},$$

which combined with (3.12) gives $G_2(\bar{r}(u_0), u_0) < \|u_0\|^p$. Thus, we get

$$G_2(\bar{r}(u_0), u_0) < \|u_0\|^p = G_2(1, u_0),$$

which contradicts the fact that $\bar{r}(u_0)$ is the maximum of $G_2(., u_0)$ and this concludes the proof of Theorem 3.1. ■

Next, we turn our focus to proving our second Liouville-type result.

Lemma 3.9 *Assume condition* (H_0), $p < q < s \leq p_*$. *If* $m \in L^\infty(\mathbb{R}^{N-1})$, *then* M *is well defined in* E. *In addition, if* $h/m \in L^\infty(\mathbb{R}^{N-1})$, *then*

$$H(v)^{s-p} \leq \left(\left\| \frac{h}{m} \right\|_\infty^{s-p} \left[\frac{\|m\|_\infty C_{p,\gamma}^{1-p}}{\rho_0} \right]^{s-q} \right) M(v)^{q-p} \|v\|^{p(s-q)}, \quad \forall v \in E.$$

In particular, H *is well-defined. Furthermore, if* (3.3) *holds then*

$$H(v)^{s-p} < \eta(p,q,s) M(v)^{q-p} \|v\|^{p(s-q)}, \quad \forall v \in E \backslash \{0\}. \tag{3.13}$$

Proof. By Theorem 1.5 and (H_0) we obtain

$$M(v) = \int_{\mathbb{R}^{N-1}} m|v|^s \, \mathrm{d}x' \leq \|m\|_\infty \int_{\mathbb{R}^{N-1}} |v|^s \, \mathrm{d}x' \leq C\|v\|^s.$$

Since $p < q < s$ we can write $q = (1-\alpha)p + \alpha s$ with $\alpha = (q-p)/(s-p) \in (0,1)$. Thus, by Hölder's inequality we get

$$H(v) = \int_{\mathbb{R}^{N-1}} h|v|^q \, \mathrm{d}x' = \int_{\mathbb{R}^{N-1}} (h|v|^p)^{1-\alpha} (h|v|^s)^\alpha \, \mathrm{d}x' \leq \left(\int_{\mathbb{R}^{N-1}} h|v|^p \, \mathrm{d}x' \right)^{1-\alpha} \left(\int_{\mathbb{R}^{N-1}} h|v|^s \, \mathrm{d}x' \right)^\alpha.$$

Taking into account that $1-\alpha = (s-q)/(s-p)$ we obtain

$$H(v)^{s-p} \leq \left(\int_{\mathbb{R}_+^{N-1}} h|v|^p \, \mathrm{d}x' \right)^{s-q} \left(\int_{\mathbb{R}^{N-1}} h|v|^s \, \mathrm{d}x' \right)^{q-p}. \tag{3.14}$$

Moreover

$$\int_{\mathbb{R}^{N-1}} h|v|^s \, \mathrm{d}x' = \int_{\mathbb{R}^{N-1}} \frac{h}{m} (m|v|^s) \, \mathrm{d}x' \leq \left\| \frac{h}{m} \right\|_\infty M(v)$$

and by (1.12) and (H_0) we have

$$\int_{\mathbb{R}^{N-1}} h|v|^p \, \mathrm{d}x' \le \|m\|_\infty \left\|\frac{h}{m}\right\|_\infty \int_{\mathbb{R}^{N-1}} |v|^p \, \mathrm{d}x' \le \|m\|_\infty \left\|\frac{h}{m}\right\|_\infty \frac{C_{p,\gamma}^{1-p}}{\rho_0}\|v\|^p.$$

Thus, plugging the last two inequalities into (3.14) we obtain

$$\begin{aligned} H(v)^{s-p} &\le \left(\|m\|_\infty \left\|\frac{h}{m}\right\|_\infty \frac{C_{p,\gamma}^{1-p}}{\rho_0}\|v\|^p\right)^{s-q} \left(\left\|\frac{h}{m}\right\|_\infty M(v)\right)^{q-p} \\ &= \left(\left\|\frac{h}{m}\right\|_\infty^{s-p} \left[\frac{\|m\|_\infty C_{p,\gamma}^{1-p}}{\rho_0}\right]^{s-q}\right) M(v)^{q-p}\|v\|^{p(s-q)}, \end{aligned}$$

which conclude the desired inequality. (3.13) can be derived directly by applying (3.3) to the inequality above. ■

Lemma 3.10 *Assume condition* (H_0)*,* $p < q < s \le p_*$*. If* $h, m \in L^\infty(\mathbb{R}^{N-1})$*, then for each* $v \in E\backslash\{0\}$ *the function* $G_2(., v)$ *defined by* (3.11) *has a unique critical point at*

$$\bar{r}(v) = \left(\frac{H(v)(q-p)}{M(v)(s-p)}\right)^{1/(s-q)}. \tag{3.15}$$

Moreover,

$$G_2(\bar{r}(v), v) = \max_{r>0} G_2(r, v) = \left(\frac{H(v)^{s-p}}{\eta(p,q,s)M(v)^{q-p}}\right)^{1/(s-q)} > 0. \tag{3.16}$$

Proof. For each $v \in E\backslash\{0\}$, we have

$$\frac{\partial G_2}{\partial r}(r, v) = (q-p)H(v)r^{q-p-1} - (s-p)M(v)r^{s-p-1}.$$

Thus,

$$\frac{\partial G_2}{\partial r}(r, v) = 0 \iff r = \bar{r}(v) = \left(\frac{H(v)(q-p)}{M(v)(s-p)}\right)^{1/(s-q)}.$$

Now note that $\lim_{r\to+\infty} G_2(r, v) = -\infty$, $\lim_{r\to 0^+} G_2(r, v) = 0$ and by (3.15)

$$M(v)\bar{r}(v)^{s-p} = \frac{q-p}{s-p} H(v)\bar{r}(v)^{q-p}.$$

Thus, since

$$\begin{aligned} G_2(\bar{r}(v), v) &= H(v)\bar{r}(v)^{q-p} - M(v)\bar{r}(v)^{s-p} \\ &= \left(\frac{s-q}{s-p}\right) H(v)\bar{r}(v)^{q-p} \\ &= \left(\frac{s-q}{s-p}\right) H(v) \left(\frac{H(v)(q-p)}{M(v)(s-p)}\right)^{(q-p)/(s-q)} \end{aligned}$$

$$= \frac{(s-q)(q-p)^{\frac{q-p}{s-q}}}{(s-p)^{\frac{s-p}{s-q}}} \frac{H(v)^{\frac{s-p}{s-q}}}{M(v)^{\frac{q-p}{s-q}}}$$
$$= \left(\frac{H(v)^{s-p}}{\eta(p,q,s)M(v)^{q-p}} \right)^{1/(s-q)} > 0,$$

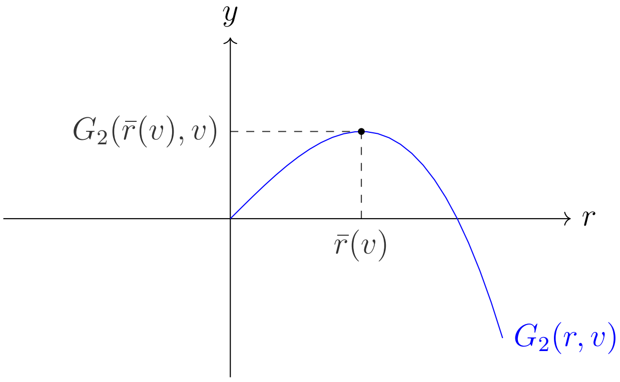

Figure 3.6: Maximum of G_2 for $p < q < s$.

we conclude that the function $G_2(., v)$ has a unique global maximum at $r = \bar{r}(v) > 0$.
■

Now we are ready to complete the proof of Theorem 3.2.

Proof of Theorem 3.2:. Arguing by contradiction, suppose that $(\mathcal{P}_2)$ has a non-trivial weak solution $u_0 \in E$. From the definition (3.1), Lemma 3.9 and a density argument we have

$$\|u_0\|^p = H(u_0) - M(u_0) = G_2(1, u_0).$$

On the other hand, by estimate (3.13), we have

$$H(u_0)^{s-p} < \eta(p,q,s)M(u_0)^{q-p}\|u_0\|^{p(s-q)}.$$

This, together with (3.16) implies that $G_2(\bar{r}(u_0), u_0) < \|u_0\|^p$. Therefore, we obtain

$$G_2(\bar{r}(u_0), u_0) < \|u_0\|^p = G_2(1, u_0),$$

contradicting the fact that $\bar{r}(u_0)$ is the maximum of $G_2(., u_0)$ and this concludes the proof. ■

3.3 Proof of our existence results

To prove our existence results, we must first establish a compactness result.

Lemma 3.11 *Assume condition* (H_0) *and* $1 < p \leq N$. *If* $w \in L^1(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$, *then the embedding*

$$E \hookrightarrow L^q\left(\mathbb{R}^{N-1}, w(x')\right)$$

is continuous for $q \in (1, p_*]$ *and compact for* $q \in (1, p_*)$. *If* $p = N$, *the embedding is compact for all* $q \in (1, \infty)$.

Proof. Consider $p < N$ and let $1 < q < p$. By Hölder's inequality,

$$\int_{\mathbb{R}^{N-1}} w|u|^q \, \mathrm{d}x' = \int_{\mathbb{R}^{N-1}} w^{\frac{p-q}{p}} w^{q/p} |u|^q \, \mathrm{d}x' \leq \|w\|_1^{\frac{p-q}{p}} \left(\int_{\mathbb{R}^{N-1}} w|u|^p \, \mathrm{d}x' \right)^{q/p}.$$

By assumption (H_0) and Theorem 1.1 we obtain

$$\int_{\mathbb{R}^{N-1}} w|u|^q \, \mathrm{d}x' \leq \|w\|_\infty^{\frac{q}{p}} \|w\|_1^{\frac{p-q}{p}} \left(\int_{\mathbb{R}^{N-1}} |u|^p \, \mathrm{d}x' \right)^{q/p} \leq \left(\frac{C_{p,\gamma}^{1-p}}{\rho_0} \right)^{q/p} \|w\|_\infty^{\frac{q}{p}} \|w\|_1^{\frac{p-q}{p}} \|u\|^q.$$

If $q \in [p, p_*]$, by Theorem 1.5 and (H_0) we obtain

$$\int_{\mathbb{R}^{N-1}} w|u|^q \, \mathrm{d}x' \leq \|w\|_\infty \int_{\mathbb{R}^{N-1}} |u|^q \, \mathrm{d}x' \leq C\|u\|^q.$$

Now, if $(u_n) \subset E$ is a bounded sequence, up to a subsequence, we can assume that $u_n \rightharpoonup 0$ in E. By Rellich-Kondrachov Theorem[8, Theorem 6.3]. the embedding

$$W^{1,p}(B_R) \hookrightarrow L^s(B_R \cap \mathbb{R}^{N-1}) \tag{3.17}$$

is compact for all $s \in [1, p_*)$ where B_R denotes the ball of radius R in $\mathbb{R}^N$. Consider then the sequence (v_n) defined by

$$v_n(x, x_N) = \begin{cases} u_n(x', x_N), & \text{if } x_N > 0 \\ u_n(x', -x_N), & \text{if } x_N \leq 0, \end{cases}$$

with $(x', x_N) \in B_R$. Observe that $v_n \in W^{1,p}(B_R)$ and $\|v_n\|_{1,p,B_R} \leq C$. In fact, first we notice that

$$\int_{B_R} |\nabla v_n|^p \, \mathrm{d}x = 2 \int_{B_R^+} |\nabla u_n|^p \, \mathrm{d}x \quad \text{and} \quad \int_{B_R} |v_n|^p \, \mathrm{d}x = 2 \int_{B_R^+} |u_n|^p \, \mathrm{d}x.$$

Applying inequality (1.5) and assumption (H_0), we deduce $E \hookrightarrow W^{1,p}(B_R^+)$, implying

$$\|v_n\|_{1,p,B_R}^p = 2\|u_n\|_{1,p,B_R^+}^p \leq C.$$

Passing to a subsequence, we have $v_n \rightharpoonup 0$ in $W^{1,p}(B_R)$ and by (3.17), it follows that $v_n \to 0$ in $L^q(B_R \cap \mathbb{R}^{N-1})$. Since $u_n = v_n$ in $B_R \cap \mathbb{R}^{N-1}$, we conclude that $u_n \to 0$ in $L^q(B_R \cap \mathbb{R}^{N-1})$. Given $\varepsilon > 0$ there exists $R = R(\varepsilon) > 0$ such that $\|w\|_{L^1(B_R^c(0)\cap\mathbb{R}^{N-1})} \leq \varepsilon$. If $q < p$ we have

$$\int_{B_R\cap\mathbb{R}^{N-1}} w|u_n|^q \, \mathrm{d}x' \leq C \int_{B_R\cap\mathbb{R}^{N-1}} |u_n|^q \, \mathrm{d}x' < C\varepsilon$$

for n large. Conversely, by Hölder's inequality,

$$\int_{B_R^c\cap\mathbb{R}^{N-1}} w|u_n|^q \, \mathrm{d}x' \leq \|w\|_{L^1(B_R^c(0)\cap\mathbb{R}^{N-1})}^{\frac{p-q}{p}} \left(\int_{B_R^c\cap\mathbb{R}^{N-1}} w|u_n|^p \, \mathrm{d}x' \right)^{q/p} \leq C\varepsilon^{\frac{p-q}{p}},$$

concluding the case, $q < p$. Now, if $q \in [p, p_*)$, analogously we have

$$\int_{B_R\cap\mathbb{R}^{N-1}} w|u_n|^q \, \mathrm{d}x' < C\varepsilon,$$

for n large. On the other hand, let $\alpha \in (0,1)$ such that $p < q/(1-\alpha) \leq p_*$, then, by Hölder, Theorem 1.5 and (H_0), we obtain

$$\begin{aligned}\int_{B_R^c\cap\mathbb{R}^{N-1}} w|u_n|^q \, \mathrm{d}x' &= \int_{B_R^c\cap\mathbb{R}^{N-1}} w^\alpha w^{1-\alpha}|u_n|^q \, \mathrm{d}x' \\ &\leq \|w\|_{L^1(B_R^c(0)\cap\mathbb{R}^{N-1})}^{\alpha} \|w\|_\infty^{1-\alpha} \left(\int_{B_R^c\cap\mathbb{R}^{N-1}} |u_n|^{\frac{q}{1-\alpha}} \, \mathrm{d}x' \right)^{1-\alpha} \\ &\leq C\varepsilon^\alpha\end{aligned}$$

and we conclude the case $p < N$. The case $p = N$ is similar. ■

Lemma 3.12 *Let $1 < p < N$ and $q < \min\{s,p\}$ or $q > \max\{s,p\}$. Then, for each $v \in E\backslash\{0\}$ there exits a unique real number $r(v) > 0$ such that the pair $(r(v), v)$ satisfies the equation*

$$\|v\|^p = r(v)^{q-p} H(v) - r(v)^{s-p} M(v) = G_2(r(v), v). \tag{3.18}$$

Furthermore, the map $r : E\backslash\{0\} \to \mathbb{R}$ belongs to $C^1(E\backslash\{0\}, \mathbb{R})$ and $\mu r(\mu v) = r(v)$ for all $\mu > 0$ and $v \in E\backslash\{0\}$.

Proof. *Existence:* Consider the function $f : (0,\infty) \times E \to \mathbb{R}$ defined by

$$f(r, v) = \|v\|^p r^{p-q} + M(v) r^{s-q} - H(v).$$

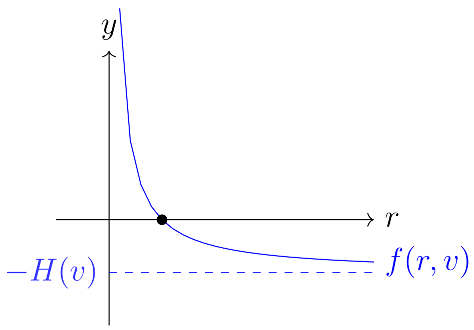

Figure 3.7: Graph of $f(., v)$ for $q > \max\{s, p\}$.

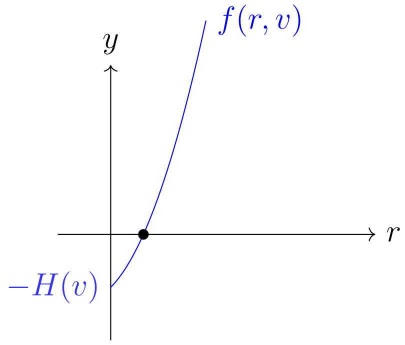

Figure 3.8: Graph of $f(., v)$ for $q < \min\{s, p\}$.

Note that $f(r, v) = 0$ if and only if (3.18) holds. If $v \in E\backslash\{0\}$ and $q > \max\{s, p\}$ we have $\lim_{r\to 0^+} f(r, v) = \infty$ and $\lim_{r\to +\infty} f(r, v) = -H(v) < 0$. In the case $q < \min\{s, p\}$, it holds $\lim_{r\to 0^+} f(r, v) = -H(v) < 0$ and $\lim_{r\to\infty} f(r, v) = \infty$. Thus, in any case, by the Intermediate Value Theorem, there exists $r(v) > 0$ such that $f(r(v), v) = 0$.

Uniqueness: Fixed $v \in E\backslash\{0\}$, suppose that there are $r_1, r_2 > 0$ satisfying (3.18). Consequently,

$$\|v\|^p r_1^{p-q} + M(v) r_1^{s-q} = H(v) = \|v\|^p r_2^{p-q} + M(v) r_2^{s-q},$$

which is equivalent to

$$\|v\|^p (r_1^{p-q} - r_2^{p-q}) + M(v)(r_1^{s-q} - r_2^{s-q}) = 0.$$

Therefore, $r_1 = r_2$ and so the map $r : E\backslash\{0\} \to \mathbb{R}$ satisfying (3.18) is well defined.

Regularity: To prove that r belongs to class C^1, we observe that

$$\frac{\partial f}{\partial r}(r, v) = (p - q) r^{p-q-1} \|v\|^p + (s - q) M(v) r^{s-q-1} \neq 0, \quad \text{in} \quad (0, \infty) \times E\backslash\{0\}.$$

Given $v \in E\backslash\{0\}$, using the implicit function theorem(see [16]), we obtain open sets $I \subset \mathbb{R}$ and $V \subset E\backslash\{0\}$ containing $r(v)$ and v respectively, and a C^1-function $\tau : V \to I$

satisfying

$$\tau(v) = r(v) \quad \text{and} \quad f(\tau(w), w) = 0, \quad \forall w \in V.$$

By the uniqueness $r \equiv \tau$ in V and therefore r is a C^1-function in V. Since v is arbitrary we have $r \in C^1(E\backslash\{0\}, \mathbb{R})$.

Finally, given $\mu > 0$ and $v \in E\backslash\{0\}$ we have that $f(r(\mu v), \mu v) = 0$, that is,

$$H(v) = \mu^{p-q} r(\mu v)^{p-q} \|v\|^p + \mu^{s-q} r(\mu v)^{q-s} M(v). \tag{3.19}$$

Since $f(r(v), v) = 0$, we have

$$H(v) = r(v)^{p-q} \|v\|^p + r(v)^{q-s} M(v),$$

which combined with (3.19) implies

$$0 = (\mu^{p-q} r(\mu v)^{p-q} - r(v)^{p-q}) \|v\|^p + (\mu^{s-q} r(\mu v)^{s-q} - r(v)^{s-q}) M(v).$$

Thus, $(\mu r(\mu v))^{p-q} = r(v)^{p-q}$ and this concludes the proof. ■

To obtain a weak solution in the sense of in(3.1), we will introduction of the functional $J : E \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{p}\|u\|^p - \frac{1}{q} H(u) + \frac{1}{s} M(u),$$

where

$$H(u) = \int_{\mathbb{R}^{N-1}} h|u|^q \, \mathrm{d}x', \quad M(u) = \int_{\mathbb{R}^{N-1}} m|u|^s \, \mathrm{d}x'.$$

Straightforward computation shows that $J \in C^1(E, \mathbb{R})$(see [20]) and critical points of J are weak solutions of $(\mathcal{P}_2)$.

Remark 3.3 *Suppose that there exists an open $\Omega \subset E\backslash\{0\}$ and $r \in C^1(\Omega, \mathbb{R})$ such that $(r(v), v)$ satisfies (3.18) for each $v \in \Omega$ with $r(v) \neq 0$ in Ω, that is,*

$$\|v\|^p = r(v)^{q-p} H(v) - r(v)^{s-p} M(v). \tag{3.20}$$

Then, we have

$$\begin{aligned} J(r(v)v) =& \frac{r(v)^p}{p}\|v\|^p - \frac{r(v)^q}{q} H(v) + \frac{r(v)^s}{s} M(v) \\ =& \left(\frac{1}{p} - \frac{1}{q}\right) H(v) r(v)^q + \left(\frac{1}{s} - \frac{1}{p}\right) M(v) r(v)^s. \end{aligned}$$

In particular, if $r > 0$ and rv is a critical point of J, it holds

$$\langle J'(rv), v\rangle = 0,$$

which is equivalent to (3.20).

The above remark motivates us to consider the *reduced functional* J defined by

$$\mathcal{J}(v) := J(r(v)v) = \left(\frac{1}{p} - \frac{1}{q}\right) H(v)r(v)^q + \left(\frac{1}{s} - \frac{1}{p}\right) M(v)r(v)^s. \tag{3.21}$$

To obtain a critical point of J we will need the next result.

Lemma 3.13 *Let $\Phi \in C^1(E\backslash\{0\}, \mathbb{R})$ such that $\langle \Phi'(v), v\rangle \neq 0$ if $\Phi(v) = 1$. If $v_c \in \Omega$ is a critical point of $\mathcal{J}$ under the constraint $\Phi(v) = 1$, then $u = r(v_c)v_c$ is a critical point of J.*

Proof. Let $r \in C^1(\Omega, \mathbb{R})$ as in Remark 3.3 that is, for each $v \in \Omega \subset E\backslash\{0\}$ the pair $(r(v), v)$ satisfies (2.26), more specifically

$$\|v\|^p = r(v)^{q-p}H(v) - r(v)^{s-p}M(v).$$

Then we can define $\mathcal{J} : \Omega \to \mathbb{R}$ as in (3.21) and

$$\langle J'(r(v)v), v\rangle = 0, \forall v \in \Omega. \tag{3.22}$$

In fact,

$$\begin{aligned}\langle J'(r(v)v), v\rangle =& r(v)^{p-1}\|v\|^p - r(v)^{q-1}H(v) + r(v)^{s-1}M(v)\\ =& r(v)^{p-1}[\|v\|^p - r(v)^{q-p}H(v) + r(v)^{s-p}M(v)] = 0\end{aligned}$$

If v_c is a critical point of $\mathcal{I}$ under the constraint $\Phi(v) = 1$, by the Lagrange Multiplier Theorem(see [37, Proposition 14.3]), there exists $\mu \in \mathbb{R}$ such that

$$\mathcal{J}'(v_c) = \lambda\Phi'(v_c). \tag{3.23}$$

On the other hand, by the definition of $\mathcal{J}$ and (3.22) we have

$$\langle \mathcal{J}'(v), w\rangle = r(v)\langle J'(r(v)v), w\rangle + \langle r'(v), w\rangle\langle J'(r(v)v), v\rangle = r(v)\langle J'(r(v)v), w\rangle \tag{3.24}$$

for all $w \in E$. Then by (3.22) and (3.23)

$$0 = r(v_c)\langle J'(r(v_c)v_c), v_c\rangle = \langle \mathcal{J}'(v_c), v_c\rangle = \lambda\langle \Phi'(v_c), v_c\rangle.$$

Since $\langle \Phi'(v_c), v_c\rangle \neq 0$ we have that $\mu = 0$ and hence, by (3.23) and (3.24),

$$0 = \mathcal{J}'(v_c) = r(v_c)J'(r(v_c)v_c).$$

Therefore, $r(v_c)v_c$ is a critical point of J. ■

Remark 3.4 *We consider the constraint* $S^1 = \{v \in E : \|v\|^p = 1\}$ *and analyze the minimization problem*

$$\inf_{v\in S^1} \mathcal{J}(v). \tag{3.25}$$

It is clear that for $\Phi = \|.\|^p$*, the condition* $\langle \Phi(v), v\rangle \neq 0$ *in Lemma 3.13 is satisfied. It is noteworthy that if* v_0 *is a minimum of* $\mathcal{J}$ *on the sphere, then* $|v_0|$ *also attains this minimum. Indeed, first we observe that*

$$\||v|\|^p = r(|v|)^{q-p}H(|v|) - r(|v|)^{s-p}M(|v|) \Leftrightarrow \|v\|^p = r(|v|)^{q-p}H(v) - r(|v|)^{s-p}M(v).$$

By Lemma 3.12, for $q < \min\{s, p\}$ *or* $q > \max\{s, p\}$*, there exists a unique* $r > 0$ *such that* $\|v\|^p = r^{q-p}H(v) - r^{s-p}M(v)$*. Then we have* $r(v) = r(|v|)$ *and consequently*

$$\begin{aligned}\mathcal{J}(|v|) &= \left(\frac{1}{p} - \frac{1}{q}\right) H(|v|)r(|v|)^q + \left(\frac{1}{s} - \frac{1}{p}\right) M(|v|)r(|v|)^s\\ &= \left(\frac{1}{p} - \frac{1}{q}\right) H(v)r(v)^q + \left(\frac{1}{s} - \frac{1}{p}\right) M(v)r(v)^s\\ &=\mathcal{J}(v).\end{aligned}$$

Thus, if v_0 *attains* (3.25), $|v_0|$ *also attains. This implies, by using Lemma 3.13, that solutions for* ($\mathcal{P}_2$)*, can be taken as nonnegative without loss of generality.*

We can now proceed with the proofs of Theorems 3.3 and 3.4.

Proof of Theorem 3.3. :For each fixed $v \in E\backslash\{0\}$, by Lemma 3.12, there exists $r(v) > 0$ such that the pair $(r(v), v)$ satisfying the equation

$$\|v\|^p r(v)^{p-q} + M(v)r(v)^{s-q} = H(v). \tag{3.26}$$

Then is well defined the reduced functional $\mathcal{J}$ in (3.21) and by the inequality above we can write

$$\mathcal{J}(v) = \left(\frac{1}{s} - \frac{1}{q}\right) M(v)r(v)^s + \left(\frac{1}{p} - \frac{1}{q}\right) \|v\|^p r(v)^p > 0.$$

Define

$$M := \inf_{v \in S^1} \mathcal{J}(v).$$

where S^1 denotes the unity sphere in E.

Now, consider (v_n) a minimizing sequence. Going if necessary to a subsequence, we may assume that $v_n \rightharpoonup v_0$ in E with $\|v_0\| \leq 1$ and by Lemma 3.11

$$H(v_n) \to H(v_0) \quad \text{and} \quad M(v_n) \to M(v_0).$$

We claim that $v_0 \neq 0$. Indeed, suppose that $v_0 = 0$. By Lemma 3.12, we have

$$\|v_n\|^p = r(v_n)^{q-p} H(v_n) - r(v_n)^{s-p} M(v_n).$$

Using that $\|v_n\| = 1$, we get

$$1 = r(v_n)^{q-p} H(v_n) - M(v_n) r(v_n)^{s-p} \leq r(v_n)^{q-p} H(v_n).$$

Since $q > p$ and $H(v_n) \to 0$, we obtain $r(v_n) \to \infty$. On the other hand,

$$\mathcal{J}(v_n) = \left(\frac{1}{s} - \frac{1}{q}\right) M(v_n) r(v_n)^s + \left(\frac{1}{p} - \frac{1}{q}\right) r(v_n)^p \geq \left(\frac{1}{p} - \frac{1}{q}\right) r(v_n)^p.$$

By taking the limit in the previous step, we reach a contradiction, which implies that $v_0 \neq 0$. From the final inequality, we can assume, up to a subsequence, that $r(v_n) \to r_0 \geq 0$ and taking to the limit in (3.26) we obtain

$$r_0^{p-q} + M(v_0) r_0^{s-q} = H(v_0) > 0 \tag{3.27}$$

which implies that $r_0 > 0$.

Next, we shall prove that $v_0 \in S^1$. Otherwise, there exists $\mu > 1$ such that $\mu v \in S^1$. From Lemma 3.12, there are $r(v_0) > 0$ such that

$$\|v_0\|^p r(v_0)^{p-q} + M(v_0) r(v_0)^{s-q} = H(v_0).$$

This, combined with (3.27) and the fact that $\mu > 1$ implies

$$r_0^{p-q} + M(v_0) r_0^{s-q} < r(v_0)^{p-q} + M(v_0) r(v_0)^{s-q}.$$

Since $s, p < q$, we have that $r_0 > r(v_0)$. Now, consider the function

$$\psi(t) = \left(\frac{1}{s} - \frac{1}{q}\right) M(v_0) t^s + \left(\frac{1}{p} - \frac{1}{q}\right) \|v_0\|^p t^p, \quad t > 0$$

and observe that ψ is strictly increasing. Thus,

$$M = \liminf_{n\to\infty} \mathcal{J}(v_n) \geq \left(\frac{1}{s} - \frac{1}{q}\right) M(v_0) r_0^s + \left(\frac{1}{p} - \frac{1}{q}\right) r_0^p \|v_0\|^p = \psi(r_0).$$

However, we also observe that

$$\psi(r_0) > \psi(r(v_0)) = J(r(v_0)v_0) = J(\mu r(\mu v_0)v_0) = \mathcal{J}(\mu v_0),$$

This contradicts the definition of M, given that $\mu v_0 \in S^1$. Hence we concluded that $v_0 \in S^1$. From (3.27) and the uniqueness of the solution $r(v_0)$, it follows that $r_0 = r(v_0)$ and

$$M = \lim_{n\to\infty} \mathcal{J}(v_n) = \mathcal{J}(v_0).$$

Since v_0 is a critical point of $\mathcal{J}$ under S^1 so is $|v_0|$ and we can assume $v_0 \geq 0$. Applying Lemma 3.13 with $\Phi(v) = \|v\|^p$, we conclude that $u = r_0 v_0$ is a critical point of J, and this completes the proof. ∎

Now we present the proof of Theorem 3.4.

Proof of Theorem 3.4. As in the previous result there exist $r(v)$ such that

$$\|v\|^p r(v)^{p-q} + M(v) r(v)^{s-q} = H(v), \tag{3.28}$$

which ensures the reduced functional $\mathcal{J}$ is well-defined and can be expressed as

$$\mathcal{J}(v) = \left(\frac{1}{s} - \frac{1}{q}\right) M(v) r(v)^s + \left(\frac{1}{p} - \frac{1}{q}\right) \|v\|^p r(v)^p < 0.$$

Now observe that from (3.28), we deduce that

$$0 < r(v)^{p-q} \leq r(v)^{p-q} + M(v) r(v)^{s-q} = H(v).$$

By our embedding results, H and M are bounded in S^1. Therefore, r is bounded given that $p > q$. Consequently, we can study the minimization problem

$$-\infty < M := \inf_{v\in S^1} \mathcal{J}(v) < 0.$$

Let $(v_n) \subset S^1$ be a minimizing sequence. By considering a subsequence if necessary, we can assume that $v_n \rightharpoonup v_0$ weakly in E with $\|v_0\| \leq 1$. Furthemore, by Lemma 3.11

$$H(v_n) \to H(v_0) \quad \text{and} \quad M(v_n) \to M(v_0).$$

Up to a subsequence, we can assume that $r(v_n) \to r_0 \geq 0$. Now observe that $r_0 v_0 \neq 0$, since $r(v_n)v_n \rightharpoonup r_0 v_0$ and

$$J(r_0 v_0) \leq \liminf J(r(v_n)v_n) = \liminf \mathcal{J}(v_n) = M < 0.$$

From (3.28), we have

$$\|v_n\|^p r(v_n)^{p-q} + M(v_n) r(v_n)^{s-q} = H(v_n).$$

Passing to the limit and observing that $\|v_0\| \leq 1$ we obtain

$$\|v_0\|^p r_0^{p-q} + M(v_0) r_0^{s-q} \leq H(v_0).$$

On the other hand, applying Lemma 3.12 for v_0, we have

$$\|v_0\|^p r(v_0)^{p-q} + M(v_0) r(v_0)^{s-q} = H(v_0),$$

which implies that $r_0 \leq r(v_0)$. Now, suppose by contradiction that $r_0 < r(v_0)$ and consider the function

$$\psi(t) := J(tv_0) = \frac{t^p}{p}\|v_0\|^p - \frac{t^q}{q} H(v_0) + \frac{t^s}{s} M(v_0), \quad t \in [0, r(v_0)].$$

Observe that ψ is strictly decreasing on $[0, r(v_0)]$. Thus,

$$M = \liminf J(r(v_n)v_n) \geq J(r_0 v_0) > J(r(v_0)v_0) = \mathcal{J}(v_0).$$

Now note that, for all $\mu > 0$ and $v \in E\backslash\{0\}$,

$$\mathcal{J}(\mu v) = \mathcal{J}(v)$$

given that $\mu r(\mu v) = r(v)$ by Lemma 3.12. Setting $\mu = \|v_0\|^{-1}$, we have $\mu v_0 \in S^1$ and

$$\mathcal{J}(\mu v_0) = \mathcal{J}(v_0) < M,$$

which is a contradiction and therefore $r(v_0) = r_0$. Then,

$$1 = \lim_{n\to\infty} \|v_n\|^p = r_0^{q-p} H(v_0) - r_0^{s-p} M(v_0) = \|v_0\|^p.$$

and

$$M = \lim_{n\to\infty} \mathcal{J}(v_n) = \lim_{n\to\infty} \left(\frac{1}{s} - \frac{1}{q}\right) M(v_n) r(v_n)^s + \left(\frac{1}{p} - \frac{1}{q}\right) \|v_n\|^p r(v_n)^p = \mathcal{J}(v_0).$$

Since v_0 is a critical point of $\mathcal{J}$ under S^1, $|v_0|$ must also be a critical point. Therefore, we can assume without loss of generality that $v_0 \geq 0$. Applying Lemma 3.13, we conclude that $u = r_0 v_0$ is a critical point of J, and we complete the proof. ■

Now we aim to establish our third existence result.

Lemma 3.14 *Assume the assumption of Theorem 3.5. Then for each $v \in E\backslash\{0\}$ the function $G_2(.,v)$ defined by (3.11) has a unique critical point at*

$$\bar{r}(v) = \left(\frac{H(v)(q-p)}{M(v)(s-p)}\right)^{1/(s-q)}. \tag{3.29}$$

Moreover,

$$G_2(\bar{r}(v),v) = \max_{r>0} G_2(r,v) = \left(\frac{H(v)^{s-p}}{\eta(p,q,s)M(v)^{q-p}}\right)^{1/(s-q)} > 0. \tag{3.30}$$

Proof. For each $v \in E\backslash\{0\}$, we have

$$\frac{\partial G_2}{\partial r}(r,v) = 0 \iff r = \bar{r}(v) = \left(\frac{H(v)(q-p)}{M(v)(s-p)}\right)^{1/(s-q)}.$$

Now note that $\lim_{r\to+\infty} G_2(r,v) = -\infty$, $\lim_{r\to 0^+} G_2(r,v) = 0$ and by (3.29)

$$M(v)\bar{r}(v)^{s-p} = \frac{q-p}{s-p} H(v)\bar{r}(v)^{q-p}.$$

Thus,

$$\begin{aligned} G_2(\bar{r}(v),v) &= \left(\frac{s-q}{s-p}\right) H(v)\bar{r}(v)^{q-p} \\ &= \left(\frac{s-q}{s-p}\right) H(v) \left(\frac{H(v)(q-p)}{M(v)(s-p)}\right)^{(q-p)/(s-q)} \\ &= \frac{(s-q)(q-p)^{\frac{q-p}{s-q}}}{(s-p)^{\frac{s-p}{s-q}}} \frac{H(v)^{\frac{s-p}{s-q}}}{M(v)^{\frac{q-p}{s-q}}} \\ &= \left(\frac{H(v)^{s-p}}{\eta(p,q,s)M(v)^{q-p}}\right)^{1/(s-q)} > 0, \end{aligned}$$

and we have that $G_2(.,v)$ has a unique global maximum at $r = \bar{r}(v) > 0$. ∎

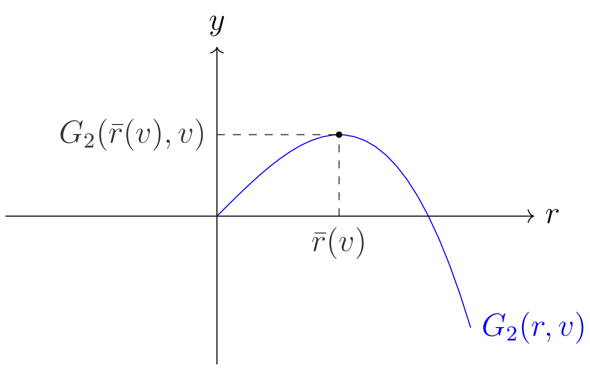

Figure 3.9: Maximum of G_2 for $p < q < s$.

To prove our third existence result for $(\mathcal{P}_2)$ we define the set

$$\Omega_2 := \{v \in E\backslash\{0\} : \|v\|^p < G_2(\bar{r}(v), v)\},$$

where $\bar{r}(v)$ is given by (3.29).

Remark 3.5 *Let $\mathcal{D}_2$ the set defined in (3.5). We observe that $\emptyset \neq \mathcal{D}_2 \subset \Omega_2$. In fact, by (3.15) we have*

$$M(v) = \left(\frac{q-p}{s-p}\right) \bar{r}(v)^{q-s} H(v), \tag{3.31}$$

and therefore

$$G_2(\bar{r}(v), v) = \bar{r}(v)^{q-p} H(v) - \left(\frac{q-p}{s-p}\right) \bar{r}(v)^{q-p} H(v) = \left(\frac{s-q}{s-p}\right) H(v) \bar{r}(v)^{q-p}. \tag{3.32}$$

From the definition, for each $v \in \mathcal{D}_2$, it holds

$$\|v\|^p < \left(\frac{p}{q}\right) \eta(s,q,p)^{\frac{1}{q-s}} H(v)^{\frac{s-p}{s-q}} M(v)^{\frac{p-q}{s-q}}.$$

Thus, from (3.31), (3.32) and the fact that $p < q$, we obtain

$$\begin{aligned} \|v\|^p &< \left(\frac{p}{q}\right) \frac{(s-q)(q-p)^{\frac{q-p}{s-q}}}{(s-p)^{\frac{s-p}{s-q}}} H(v)^{\frac{s-p}{s-q}} \left[\left(\frac{q-p}{s-p}\right) \bar{r}(v)^{q-s} H(v)\right]^{\frac{p-q}{s-q}} \\ &= \frac{p}{q} \left(\frac{s-q}{s-p}\right) \bar{r}(v)^{q-p} H(v) \\ &< G_2(\bar{r}(v), v), \end{aligned}$$

which implies $\mathcal{D}_2 \subset \Omega_2$.

Next, we will establish several technical properties of Ω_2, which are crucial for proving Theorem 3.5.

Lemma 3.15 *If $p < q < s < p_*$, for each $v \in \Omega_2$ there exists a unique real number $r(v) > \bar{r}(v)$ such that the pair $(r(v), v)$ satisfies*

$$\|v\|^p = r(v)^{q-p} H(v) - r(v)^{s-p} M(v) = G_2(r(v), v),$$

and $r \in C^1(\Omega_2, \mathbb{R})$. Furthermore, for any $v \in \Omega_2$ and $\mu > 0$, it holds $\mu v \in \Omega_2$, and as a consequence, $\Omega_2 \cap S^1 \neq \emptyset$.

Proof. If $v \in \Omega_2$ we have $\|v\|^p < G_2(\bar{r}(v), v)$. Since $G_2(r,v) = r^{q-p}\left(H(v) - M(v)r^{s-q}\right)$ and $p < q < s$, it follows that

$$\lim_{r\to\infty} G_2(r,v) = -\infty,$$

and so by the Intermediate Value Theorem, there exists a real number $r(v) > \bar{r}(v)$ such that the pair $(r(v), v)$ verifies $\|v\|^p = G_2(r(v), v)$. To prove that $r(v)$ is unique, we observe $(q-p)\bar{r}(v)^{q-p}H(v) = (s-p)\bar{r}(v)^{s-p}M(v)$ and hence we can write

$$G_2(r,v) = H(v)\left(r^{q-p} - \frac{q-p}{s-p}\bar{r}(v)^{q-s}r^{s-p}\right).$$

Consequently,

$$\frac{\partial G_2}{\partial r}(r,v) = (q-p)r^{s-p-1}H(v)(r^{q-s} - \bar{r}(v)^{q-s}) < 0,$$

for all $r > \bar{r}(v)$, thereby implying the uniqueness of $r(v)$.

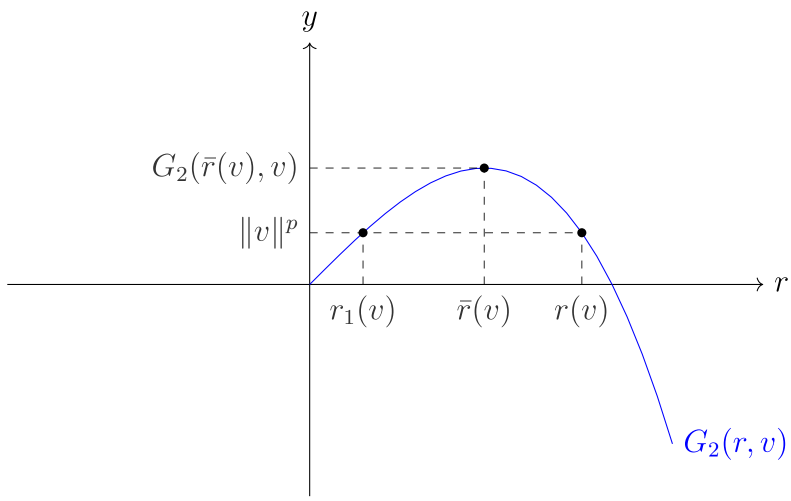

Figure 3.10: Existence of $r(v) > \bar{r}(v)$.

To verify that r is a C^1 function, let $v \in \Omega_2$, set $r = r(v)$ and obtain

$$\frac{\partial G_2}{\partial r}(r(v),v) = (q-p)r(v)^{p-s+1}H(v)(r(v)^{q-s} - \bar{r}(v)^{q-s}) < 0. \tag{3.33}$$

Now, let the function $f : (0,\infty)\times\Omega_2 \to \mathbb{R}$ given by

$$f(r,v) = G_2(r,v) - \|v\|^p.$$

From (3.33) we see that $\frac{\partial f}{\partial r}(r(v),v) < 0$. Therefore, by the implicit function theorem(see [16]), there exists open sets $I \subset \mathbb{R}$ and $V \subset E\backslash\{0\}$ containing $r(v)$ and v respectively, and a C^1-function $\tau : V \to I$ satisfying

$$\tau(v) = r(v) > \bar{r}(v) \quad \text{and} \quad f(\tau(w), w) = 0, \quad \forall w \in V.$$

Then, there exists a neighborhood $U \subset V$ containing v such that $\tau = r > \bar{r}$ in U. By the uniqueness, $r \equiv \tau$ in U hence r is a C^1-function in U and therefore, $r \in C^1(\Omega_2, \mathbb{R})$.

Now, suppose that $v \in \Omega_2$. This means

$$\|v\|^p < G_2(\bar{r}(v), v) = \frac{s-q}{s-p}\bar{r}(v)^{q-p}H(v),$$

where we use (3.32). Since by (3.15) $\mu^{-1}\bar{r}(v) = \bar{r}(\mu v)$ for all $\mu > 0$, we get

$$\|\mu v\|^p < \frac{s-q}{s-p}\bar{r}(\mu v)^{q-p}H(\mu v) = G_2(\bar{r}(\mu v), \mu v),$$

which implies that $\mu v \in \Omega_2$. This completes the proof. ■.

Lemma 3.16 *Under the assumptions of Theorem 2.5 it holds*

$$\inf_{v \in \Omega_2 \cap S^1} M(v) > 0.$$

Proof. By Lemma 3.15 we have $\Omega_2 \cap S^1 \neq \emptyset$. Now, given $v \in \Omega_2 \cap S^1$, from (3.32), (3.15) and the fact that $1 = \|v\|^p < G_2(\bar{r}(v), v)$, we obtain

$$\eta(p,q,s)M(v)^{q-p} < H(v)^{s-p}.$$

By applying Hölder's inequality, we have

$$\begin{aligned} H(v) = \int_{\mathbb{R}^{N-1}} h|v|^q \, \mathrm{d}x' &= \int_{\mathbb{R}^{N-1}} \frac{h}{m^{q/s}} m^{q/s}|v|^q \, \mathrm{d}x' \\ &\le \left(\int_{\mathbb{R}^{N-1}} \left[\frac{h}{m^{q/s}} \right]^{\frac{s}{s-q}} \mathrm{d}x' \right)^{(s-q)/s} \left(\int_{\mathbb{R}^{N-1}} m|v|^s \, \mathrm{d}x' \right)^{q/s}. \\ &\le C_{h,m} M(v)^{q/s} \end{aligned}$$

where

$$0 < C_{h,m} = \left(\int_{\mathbb{R}^{N-1}} \left[\frac{h^{1/q}}{m^{1/s}} \right]^{\frac{sq}{s-q}} \mathrm{d}x' \right)^{(s-q)/s},$$

which is finite by (3.6). Combining the above inequalities, we get

$$\eta(p,q,s)M(v)^{q-p} < H(v)^{s-p} \le C_{h,m}^{s-p} M(v)^{(s-p)q/s}.$$

Hence

$$0 < \eta(p,q,s)C_{h,m}^{p-s} < M(v)^{(s-q)p/s},$$

and we conclude the result. ■

Lemma 3.17 *If $\mathcal{D}_2$ is the set defined in (3.5), then $\mathcal{D}_2 \cap S^1 \neq \varnothing$, where S^1 is the unit sphere in E. Moreover,*

$$\mathcal{J}(v) < 0, \quad \forall v \in \mathcal{D}_2. \tag{3.34}$$

Proof. If $v \in \mathcal{D}_2$, the computation in Remark 3.5 shows that

$$\|v\|^p < \frac{p}{q}\left(\frac{s-q}{s-p}\right)\bar{r}(v)^{q-p}H(v). \tag{3.35}$$

For $\mu > 0$, by (3.15), we have

$$\bar{r}(\mu v) = \mu^{-1}\bar{r}(v), \quad \forall v \in E\backslash\{0\}.$$

Thus,

$$\|\mu v\|^p < \frac{p}{q}\left(\frac{s-q}{s-p}\right)\mu^{p-q}\bar{r}(v)^{q-p}H(\mu v) = \frac{p}{q}\left(\frac{s-q}{s-p}\right)\bar{r}(\mu v)^{q-p}H(\mu v),$$

which implies that $\mu v \in \mathcal{D}_2$. In particular, choosing $\mu = \|v\|^{-1}$ we conclude that $\mathcal{D}_2 \cap S^1 \neq \varnothing$.

To verify (3.34), since the pair $(r(v), v)$ satisfies

$$M(v)r(v)^s = H(v)r(v)^q - \|v\|^p r(v)^p,$$

from (3.21), the fact that $\bar{r}(v) < r(v)$ for each $v \in \mathcal{D}_2 \subset \Omega_2$ and inequality (3.35), we get

$$\begin{aligned}\mathcal{J}(v) &= \left(\frac{1}{s}-\frac{1}{q}\right)H(v)r(v)^q + \left(\frac{1}{p}-\frac{1}{s}\right)\|v\|^p r(v)^p \\ &< \left(\frac{1}{s}-\frac{1}{q}\right)H(v)r(v)^q + \left(\frac{1}{p}-\frac{1}{s}\right)\frac{p}{q}\left(\frac{s-q}{s-p}\right)H(v)r(v)^q.\end{aligned}$$

We observe that the last term of the inequality above is zero and the proof is complete. ∎

Remark 3.6 *If $1 < p < q < s$, by Lemma 3.15, for each $v \in \Omega_2$ there is only a real value $r(v)$ such that*

$$G_2(r(v), v) = \|v\|^p \quad \textit{and} \quad r(v) > \bar{r}(v).$$

Since $r(|v|) > \bar{r}(|v|) = \bar{r}(v)$ with

$$G_2(r(|v|), v) = \|v\|^p$$

we must to have $r(v) = r(|v|)$ and hence $\mathcal{J}(v) = \mathcal{J}(|v|)$. Therefore, if v_0 is a minimum of $\mathcal{J}$ under $\Omega_2 \cap S^1$, so is $|v_0|$. This implies that solutions to the minimization problem,

$$\inf_{v \in \Omega_2 \cap S^1} \mathcal{J}(v),$$

and hence, by Lemma 2.13, solutions to $(\mathcal{P}_2)$ in the case $1 < p < q < s$, can be chosen to be nonnegative without loss of generality.

Now we can proceed with the proof of Theorem 3.5.

Proof of Theorem 3.5:. Let $r \in C^1(\Omega_2, \mathbb{R})$ be the function given by the Lemma 3.15. For $v \in S^1$, we have

$$1 = r(v)^{q-p} H(v) - r(v)^{s-p} M(v),$$

which implies

$$r(v) < \left(\frac{H(v)}{M(v)} \right)^{1/(s-q)}, \quad v \in S^1. \tag{3.36}$$

Since H is bounded in S^1, by Lemma 3.16 we have that r is bounded in $\Omega_2 \cap S^1$. Hence, $\mathcal{J}$ is lower bounded in $\Omega_2 \cap S^1$ and by Lemma 3.17, we have

$$M = \inf_{v \in \Omega_2 \cap S^1} \mathcal{J}(v) < 0. \tag{3.37}$$

Let $(v_n) \subset \Omega_2 \cap S^1$ be a minimizing sequence. Up to a subsequence, $v_n \rightharpoonup v_0$ weakly in E with $\|v_0\| \leq 1$. Lemmas 3.11 then imply

$$H(v_n) \to H(v_0) \quad \text{and} \quad M(v_n) \to M(v_0).$$

By Lemma 3.15, the sequence $(r(v_n))$ satisfies $r(v_n) > \bar{r}(v_n)$. Furthermore, from (3.36), the sequence $(r(v_n))$ is bounded. Up to a subsequence, we can assume that $r(v_n) \to r_0 \geq 0$. Consequently, we obtain

$$0 > M = \liminf \mathcal{J}(v_n) \geq \left(\frac{1}{p} - \frac{1}{q} \right) H(v_0) r_0^q + \left(\frac{1}{s} - \frac{1}{p} \right) M(v_0) r_0^s.$$

which implies that $r_0 > 0$. In addition, from (3.15) and Lemma 3.11, it follows

$$\lim_{n \to +\infty} \bar{r}(v_n) = \bar{r}(v_0).$$

and hence

$$\lim_{n \to +\infty} G_2(\bar{r}(v_n), v_n) = G_2(\bar{r}(v_0), v_0).$$

Since $v_n \in \Omega_2$, we get

$$\|v_0\|^p \leq \liminf_{n\to\infty} \|v_n\|^p \leq \liminf_{n\to\infty} G_2(\bar{r}(v_n), v_n) = G_2(\bar{r}(v_0), v_0).$$

We shall now show that $v_0 \in \Omega_2$. Assume by contradiction that $v_0 \notin \Omega_2$, that is, $\|v_0\|^p = G_2(\bar{r}(v_0), v_0)$. Since $\|v_n\|^p = G_2(r(v_n), v_n)$, taking to the limit we get

$$G_2(\bar{r}(v_0), v_0) = \|v_0\|^p \leq \liminf_{n\to\infty} \|v_n\|^p = \liminf_{n\to\infty} G_2(r(v_n), v_n) = G_2(r_0, v_0),$$

which implies that $\bar{r}(v_0) = r_0$ because $\bar{r}(v_0)$ is the global maximum of $G_2(., v_0)$. Then, $r(v_n) \to \bar{r}(v_0)$ and from the definition of $\mathcal{J}$ and (3.15) we obtain

$$M = \lim_{n\to\infty} \mathcal{J}(v_n) = \left(\frac{1}{p} - \frac{1}{q}\right) H(v_0)\bar{r}(v_0)^q + \left(\frac{1}{s} - \frac{1}{p}\right) M(v_0)\bar{r}(v_0)^s$$

and $(s-p)M(v_0)\bar{r}(v_0)^s = H(v_0)(q-p)\bar{r}(v_0)^q$. As a consequence, we infer that

$$\begin{aligned} M &= M(v_0)\bar{r}(v_0)^s \left[\frac{(s-p)}{q-p}\left(\frac{1}{p} - \frac{1}{q}\right) + \left(\frac{1}{s} - \frac{1}{p}\right)\right] \\ &= M(v_0)\bar{r}(v_0)^s \frac{(s-p)}{p}\left(\frac{1}{q} - \frac{1}{s}\right) > 0 \end{aligned}$$

because $p < q < s$, which contradicts (3.37) and hence we conclude that $v_0 \in \Omega_2$.
Claim: $r_0 = r(v_0)$.

Assuming that the claim is true, we can take the limit at

$$1 = \|v_n\|^p = G_2(r(v_n), v_n),$$

to obtain

$$1 = G_2(r(v_0), v_0) = \|v_0\|^p.$$

Thus, we conclude that $v_0 \in \Omega_2 \cap S^1$ and we also have

$$M = \lim_{n\to\infty} \mathcal{J}(v_n) = \left(\frac{1}{p} - \frac{1}{q}\right) H(v_0)r(v_0)^q + \left(\frac{1}{s} - \frac{1}{p}\right) M(v_0)r(v_0)^s = \mathcal{J}(v_0).$$

Therefore, by Lemma 3.13, $r(v_0)v_0$ is a nonnegative and nontrivial critical point if J in E.

It remains to prove $r_0 = r(v_0)$. Since $v_0 \in \Omega_2$, by Lemma 3.15, we can choose $\mu_0 > 0$ such that $\mu_0 v_0 \in \Omega_2 \cap S^1$. By Lemma 3.15 we know that $r(v_0) > \bar{r}(v_0)$ and $G_2(r(v_0), v_0) = \|v_0\|^p$. Taking the limit at $\|v_n\|^p = G_2(r(v_n), v_n)$, we get $\|v_0\|^p \leq G_2(r_0, v_0)$. Consequently,

$$G_2(r(v_0), v_0) = \|v_0\|^p \leq G_2(r_0, v_0).$$

Since $G_2(r, v_0)$ is decreasing for $r \geq \bar{r}(v_0)$ and $r(v_0) > \bar{r}(v_0)$, it follows that $r_0 \leq r(v_0)$. In fact, we have

$$\bar{r}(v_0) \leq r_0 \leq r(v_0).$$

Suppose by contradiction that $r_0 < r(v_0)$. Since $G_2(r, v_0)$ is strictly decreasing for all $r \in (r_0, r(v_0))$, we see that

$$\|v_0\|^p = G_2(r(v_0), v_0) < G_2(r, v_0), \quad \forall r \in [r_0, r(v_0)).$$

Considering the function

$$z(r) = J(rv_0), \quad r \in (r_0, r(v_0)),$$

a straightforward computation shows that

$$z'(r) = r^{p-1} \left(\|v_0\|^p - G_2(r, v_0)\right) < 0,$$

which implies that z is strictly decreasing. Thus, we get

$$M = \liminf_{n\to\infty} J(r(v_n)v_n) \geq J(r_0 v_0) > J(r(v_0)v_0) = J(r(\mu_0 v_0)\mu_0 v_0) = \mathcal{J}(\mu_0 v_0),$$

with $\mu_0 v_0 \in \Omega_2 \cap S^1$. This contradicts the definition of M and hence $r_0 = r(v_0)$. This completes the proof of Theorem 3.5. ■

Finally, we turn our attention for the proof of Theorem 3.6.

Lemma 3.18 *Assume the assumptions in Theorem 3.6. For each fixed $v \in E\backslash\{0\}$ the function $G_2(., v)$ has a unique critical point which is a maximum and is given by*

$$\bar{r}(v) = \left(\frac{M(v)(p-s)}{H(v)(p-q)}\right)^{1/(q-s)}. \tag{3.38}$$

Moreover,

$$G_2(\bar{r}(v), v) = \max_{r>0} G_2(r, v) = \left(\frac{H(v)^{p-s}}{\eta(s,q,p)M(v)^{p-q}}\right)^{1/(q-s)} > 0, \tag{3.39}$$

where $\eta(s, q, p)$ was defined in (2.2).

Proof. We start by verifying that

$$\frac{\partial G_2}{\partial r}(r, v) = (q-p)r^{q-p-1}H(v) - (s-p)r^{s-p-1}M(v).$$

Then we observe that

$$\frac{\partial G_2}{\partial r}(r,v) = 0 \Leftrightarrow r = \left(\frac{M(v)(p-s)}{H(v)(p-q)}\right)^{1/(q-s)}.$$

Furthermore, we observe that $\lim_{r\to+\infty} G_2(r,v) = 0$, $\lim_{r\to 0^+} G_2(r,v) = -\infty$ and due to (3.38),

$$H(v)\bar{r}(v)^{q-p} = \left(\frac{p-s}{p-q}\right) M(v)\bar{r}(v)^{s-p}.$$

Since,

$$\begin{aligned} G_2(\bar{r}(v),v) &= H(v)\bar{r}(v)^{q-p} - M(v)\bar{r}(v)^{s-p} \\ &= \left(\frac{q-s}{p-q}\right) M(v)\bar{r}(v)^{s-p} \\ &= \left(\frac{q-s}{p-q}\right) M(v) \left(\frac{M(v)(p-s)}{H(v)(p-q)}\right)^{(s-p)/(q-s)} \\ &= \left(\frac{H(v)^{p-s}}{\eta(s,q,p)M(v)^{p-q}}\right)^{1/(q-s)} > 0, \end{aligned}$$

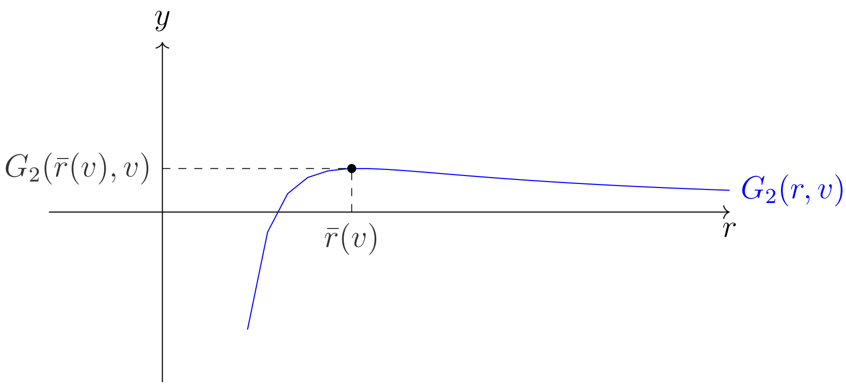

Figure 3.11: Maximum of G_2 for $p < q < s$.

we can conclude that $G_2(.,v)$ attains its unique global maximum at $r = \bar{r}(v) > 0$.

■

As in the previous case, we need to considerate the set

$$\Omega_2 = \{v \in E\backslash\{0\} : \|v\|^p < G_2(\bar{r}(v),v)\}.$$

Remark 3.7 *Consider $s < q < p$ and let $\mathcal{C}_2$ the set defined in (3.7). We observe that $\mathcal{C}_2 \subset \Omega_2$ and hence $\Omega_2 \neq \emptyset$. In fact, note that $v \in \mathcal{C}_2$ if, and only if,*

$$\|v\|^p < \left(\frac{s}{q}\right)^{\frac{p-s}{q-s}} \left(\frac{H(v)^{p-s}}{\eta(s,q,p)M(v)^{p-q}}\right)^{1/(q-s)} = \left(\frac{s}{q}\right)^{\frac{p-s}{q-s}} G_2(\bar{r}(v),v).$$

Then, since $s < q < p$, we have $\|v\|^p < G_2(\bar{r}(v), v)$ and $v \in \Omega_2$.

Lemma 3.19 *If $s < q < p$, for each $v \in \Omega_2$ there exists a unique positive real number $r(v) > \bar{r}(v)$ such that the pair $(r(v), v)$ satisfies*

$$\|v\|^p = r(v)^{q-p} H(v) - r(v)^{s-p} M(v) = G_2(r(v), v),$$

and $r \in C^1(\Omega_2, \mathbb{R})$. Furthermore, for any $v \in \Omega_2$ and $\mu > 0$, it holds $\mu v \in \Omega_2$, and as a consequence, $\Omega_2 \cap S^1 \neq \emptyset$.

Proof. If $v \in \Omega_2$ we have $\|v\|^p < G_2(\bar{r}(v), v)$. Given that $s < q < p$,

$$\lim_{r \to \infty} G_2(r, v) = 0.$$

By the Intermediate Value Theorem, there exists a positive real number $r(v) > \bar{r}(v)$ such that the pair $(r(v), v)$ verifies $\|v\|^p = G_2(r(v), v)$. To prove that $r(v)$ is unique, we observe that G_2 is decreasing for all $r > \bar{r}(v)$. In fact, first note that by (3.38)

$$(q-p)H(v) = (s-p)\bar{r}(v)^{s-q} M(v).$$

Consequently,

$$\begin{aligned} \frac{\partial G_2}{\partial r}(r, v) &= (q-p) r^{q-p-1} H(v) - (s-p) r^{s-p-1} M(v) \\ &= (s-p) M(v) r^{q-p-1} \bar{r}(v)^{s-q} - (s-p) r^{s-p-1} M(v) \\ &= (s-p) r^{q-p-1} M(v) \left(\frac{1}{\bar{r}(v)^{q-s}} - \frac{1}{r^{q-s}} \right) < 0, \end{aligned}$$

for all $r > \bar{r}(v)$, thereby implying the uniqueness of $r(v)$.

Next, we will prove that r is a C^1 function. In fact, given $v \in \Omega_2$, by setting $r = r(v)$, we obtain $\frac{\partial G_2}{\partial r}(r(v), v) < 0$. Now, consider the function $f : (0, \infty) \times \Omega_2 \to \mathbb{R}$ given by

$$f(r, v) = G_2(r, v) - \|v\|^p.$$

Given $v \in \Omega_2$, we observe that $\frac{\partial f}{\partial r}(r(v), v) < 0$. By the implicit function theorem(see [16]), there exists open sets $I \subset \mathbb{R}$ and $V \subset E \backslash \{0\}$ containing $r(v)$ and v respectively, and a C^1-function $\tau : V \to I$ satisfying

$$\tau(v) = r(v) > \bar{r}(v) \quad \text{and} \quad f(\tau(w), w) = 0, \quad \forall w \in V.$$

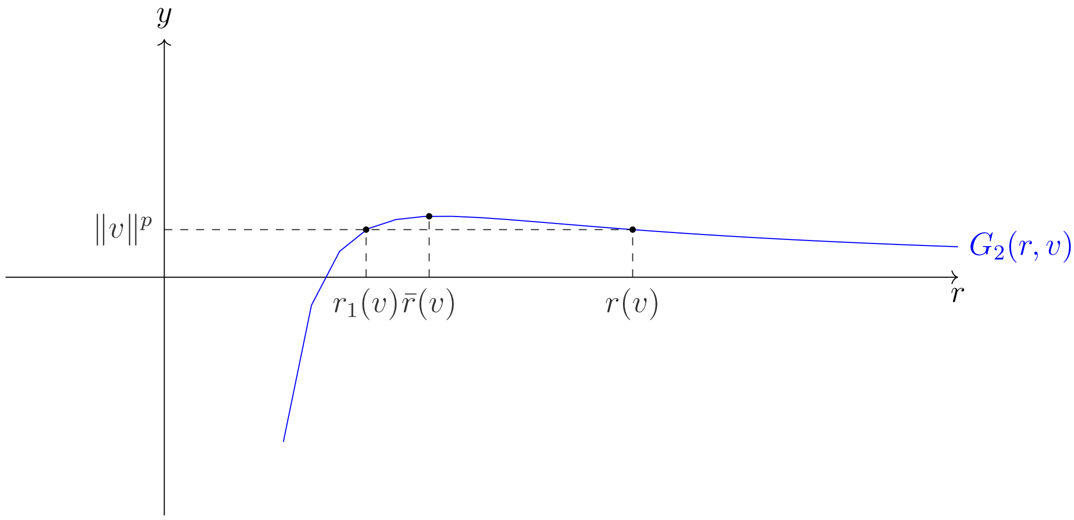

Figure 3.12: Existence of $r(v) > \bar{r}(v)$ for $s < q < p$.

Then, there exists a neighborhood $U \subset V$ containing v such that $\tau = r > \bar{r}$ in U. By the uniqueness, $r \equiv \tau$ in U hence r is a C^1-function in U and therefore, $r \in C^1(\Omega_2, \mathbb{R})$.

Now, observe that given $\mu > 0$

$$\mu\bar{r}(\mu v) = \bar{r}(v), \quad \forall v \in E\backslash\{0\}. \tag{3.40}$$

In fact, by (3.38) we have

$$\bar{r}(\mu v) = \left(\frac{M(\mu v)(p-s)}{H(\mu v)(p-q)}\right)^{1/(q-s)} = \frac{\mu^{\frac{s}{q-s}}}{\mu^{\frac{q}{q-s}}}\left(\frac{M(v)(p-s)}{H(v)(p-q)}\right)^{1/(q-s)} = \frac{1}{\mu}\bar{r}(v).$$

By (3.38) we can write

$$H(v) = \left(\frac{p-s}{p-q}\right) M(v)\bar{r}(v)^{s-q}$$

and hence

$$G_2(\bar{r}(v), v) = \bar{r}(v)^{q-p}H(v) - \bar{r}(v)^{s-p}M(v) = \frac{q-s}{p-q}\bar{r}(v)^{s-p}M(v).$$

Suppose that $v \in \Omega_2$, that is

$$\|v\|^p < G_2(\bar{r}(v), v) = \frac{q-s}{p-q}\bar{r}(v)^{s-p}M(v).$$

Using (3.40) we obtain

$$\|\mu v\|^p < \mu^{p-s+s}\frac{q-s}{p-q}\bar{r}(v)^{s-p}M(v) = \frac{q-s}{p-q}\bar{r}(\mu v)^{s-p}M(\mu v) = G_2(\bar{r}(\mu v), \mu v),$$

which implies that $\mu v \in \Omega_2$. Taking $\mu = \|v\|^{-1}$ we have that $\Omega_2 \cap S^1 \neq \emptyset$. ■

Lemma 3.20 *Let $s < q < p$. If $v \in \mathcal{C}_2$, where $\mathcal{C}_2$ is the set given by (3.7), we have*

$$\bar{r}(v) < \left(\frac{q}{s}\right)^{\frac{1}{q-s}} \bar{r}(v) < r(v), \tag{3.41}$$

where r is the function given in Lemma 3.19. Moreover,

$$\mathcal{J}(v) < 0, \quad \forall v \in \mathcal{C}_2$$

and $\mathcal{C}_2 \cap S^1 \neq \emptyset$.

Proof. As shown in the proof of Lemma 3.19, $G_2(r, v)$ is decreasing for $r > \bar{r}(v)$. Since $\bar{r}(v) < r(v)$, to establish (3.41), it suffices to demonstrate that if $v \in \mathcal{C}_2$, then

$$G_2\left(\left(\frac{q}{s}\right)^{\frac{1}{q-s}} \bar{r}(v), v\right) > G_2(r(v), v).$$

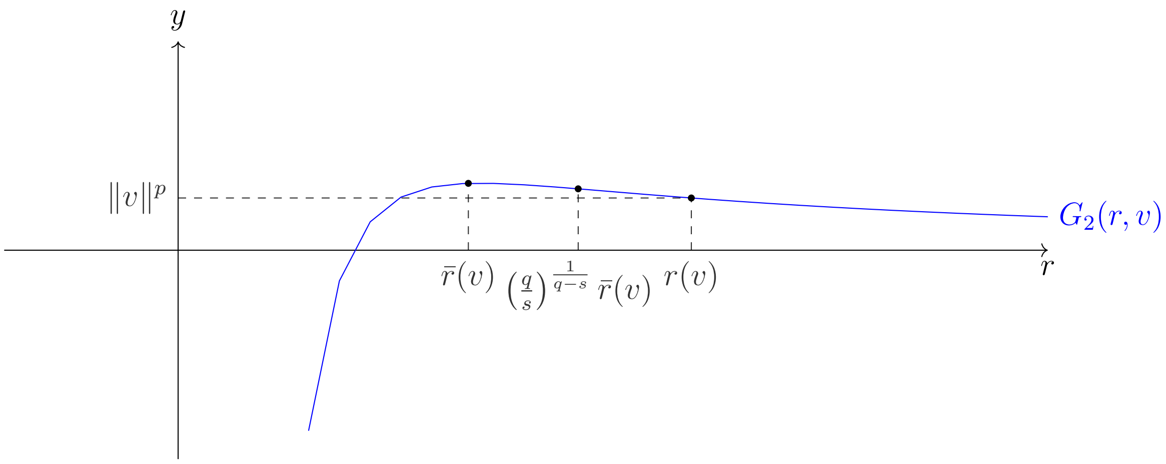

Figure 3.13: $\left(\frac{q}{s}\right)^{\frac{1}{q-s}} \bar{r}(v) < r(v)$.

As observed in Remark 3.7, $v \in \mathcal{C}_2$ if and only if

$$G_2(r(v), v) = \|v\|^p < \left(\frac{s}{q}\right)^{\frac{p-s}{q-s}} G_2(\bar{r}(v), v). \tag{3.42}$$

Thus, since $s < q < p$, we have

$$\begin{aligned} G_2\left(\left(\frac{q}{s}\right)^{\frac{1}{q-s}} \bar{r}(v), v\right) &= \left(\frac{q}{s}\right)^{\frac{q-p}{q-s}} \bar{r}(v)^{q-p} H(v) - \left(\frac{q}{s}\right)^{\frac{s-p}{q-s}} \bar{r}(v)^{s-p} M(v) \\ &> \left(\frac{q}{s}\right)^{\frac{s-p}{q-s}} \bar{r}(v)^{q-p} H(v) - \left(\frac{q}{s}\right)^{\frac{s-p}{q-s}} \bar{r}(v)^{s-p} M(v) \\ &= \left(\frac{s}{q}\right)^{\frac{p-s}{q-s}} G_2(\bar{r}(v), v) \end{aligned}$$

$$>G_2(r(v), v),$$

which conclude (3.41). To verify that $\mathcal{J} < 0$ in $\mathcal{C}_2$, by (3.38) we write

$$M(v) = \frac{p-q}{p-s}\bar{r}(v)^{q-s}H(v).$$

and by (3.41)

$$\bar{r}(v)^{q-s} < \frac{s}{q}r(v)^{q-s}.$$

Therefore,

$$\begin{aligned}
\mathcal{J}(v) &= \left(\frac{1}{p} - \frac{1}{q}\right) H(v)r(v)^q + \left(\frac{1}{s} - \frac{1}{p}\right) M(v)r(v)^s \\
&= \left(\frac{q-p}{pq}\right) H(v)r(v)^q + \left(\frac{p-s}{sp}\right)\left(\frac{p-q}{p-s}\right) \bar{r}(v)^{q-s}H(v)r(v)^s \\
&< \left(\frac{q-p}{pq}\right) H(v)r(v)^q + \left(\frac{p-q}{sp}\right) \frac{s}{q}H(v)r(v)^q \\
&= \frac{1}{p}H(v)r(v)^q \left(\frac{q-p}{q} + \frac{p-q}{q}\right) \\
&= 0.
\end{aligned}$$

Let $v \in \mathcal{C}_2$ and $\mu > 0$, then, by multiplying (3.42) by μ^p and by (3.40) we obtain

$$\begin{aligned}
\|\mu v\|^p &< \mu^p \left(\frac{s}{q}\right)^{\frac{p-s}{q-s}} G_2(\bar{r}(v), v) \\
&= \mu^p \left(\frac{s}{q}\right)^{\frac{p-s}{q-s}} (\bar{r}(v)^{q-p}H(v) - \bar{r}(v)^{s-p}M(v)) \\
&= \left(\frac{s}{q}\right)^{\frac{p-s}{q-s}} (\mu^{p-q}\mu^q\bar{r}(v)^{q-p}H(v) - \mu^{p-s}\mu^s\bar{r}(v)^{s-p}M(v)) \\
&= \left(\frac{s}{q}\right)^{\frac{p-s}{q-s}} (\bar{r}(\mu v)^{q-p}H(\mu v) - \bar{r}(\mu v)^{s-p}M(\mu v)) \\
&= \left(\frac{s}{q}\right)^{\frac{p-s}{q-s}} G_2(\bar{r}(\mu v), \mu v),
\end{aligned}$$

which implies that $\mu v \in \mathcal{C}_2$. Taking $\mu = \|v\|^{-1}$ we conclude that $\mathcal{C}_2 \cap S^1 \neq \emptyset$. ■

Remark 3.8 *If $1 < s < q < p$, by Lemma 3.19, for each $v \in \Omega_2$ there is only a real value $r(v)$ such that*

$$G_2(r(v), v) = \|v\|^p \quad \textit{and} \quad r(v) > \bar{r}(v).$$

Since $r(|v|) > \bar{r}(|v|) = \bar{r}(v)$ with $G_2(r(|v|), v) = \|v\|^p$, we have $r(v) = r(|v|)$ and hence $\mathcal{J}(v) = \mathcal{J}(|v|)$. Therefore, if v_0 is a minimum of $\mathcal{I}$ under $\Omega_1 \cap S^1$, so is $|v_0|$. This implies that solutions to the minimization problem,

$$\inf_{v\in\Omega_2\cap S^1} \mathcal{J}(v),$$

and hence, by Lemma 3.13, the solutions for $(\mathcal{P}_2)$ for $1 < s < q < p$, can be taken as nonnegative without loss of generality.

Now we are ready to present the proof of Theorem 3.6.

Proof of Theorem 3.6. Observe that r, given in Lemma 3.19 is bounded in S^1. In fact, if $v \in S^1$ we have

$$1 = r(v)^{q-p}H(v) - r(v)^{s-p}M(v) \leq r(v)^{q-p}H(v),$$

then $r(v) \leq H(v)^{\frac{1}{p-q}}$. Since $1 < q < p$ and H is bounded in S^1 due to Lemma 3.11, we conclude that r is bounded in S^1. Thus, $\mathcal{J}$ is bounded in S^1 and due to Remark 3.7 and Lemma 3.20 we have

$$-\infty < M = \inf_{v\in\Omega_2\cap S^1} \mathcal{J}(v) < 0.$$

Let $(v_n) \subset \Omega_2 \cap S^1$ be a minimizing sequence. There exist $v_0 \in E$ such that, going if necessary to a subsequence, $v_n \rightharpoonup v_0$ in E and by Lemma 3.11,

$$H(v_n) \to H(v_0) \quad \text{and} \quad M(v_n) \to M(v_0). \tag{3.43}$$

Up to a subsequence, we have $r(v_n) \to r_0 \geq 0$. In particular, we see that $r_0 > 0$, otherwise

$$0 > M = \lim_{n\to\infty} \mathcal{J}(v_n) = \left(\frac{1}{p} - \frac{1}{q}\right) H(v_0)r_0^q + \left(\frac{1}{s} - \frac{1}{p}\right) M(v_0)r_0^s = 0,$$

which is impossible. Furthermore, we observe that $v_0 \neq 0$ because both $H(v_0)$ and $M(v_0)$ cannot be 0 simultaneously. We claim that $v_0 \in \Omega_2$. To do this, from (3.38) and (3.43) we have

$$\bar{r}(v_n) = \left(\frac{M(v_n)(p-s)}{H(v_n)(p-q)}\right)^{1/(q-s)} \longrightarrow \left(\frac{M(v_0)(p-s)}{H(v_0)(p-q)}\right)^{1/(q-s)} = \bar{r}(v_0)$$

and hence

$$G_2(\bar{r}(v_n), v_n) \longrightarrow G_2(\bar{r}(v_0), v_0).$$

Then, we get

$$\|v_0\|^p \leq \liminf_{n\to\infty} \|v_n\|^p \leq \liminf_{n\to\infty} G_2(\bar{r}(v_n), v_n) = G_2(\bar{r}(v_0), v_0).$$

Suppose that $v_0 \notin \Omega_2$, that is,

$$\|v_0\|^p = G_2(\bar{r}(v_0), v_0).$$

Conversely,

$$\|v_0\|^p \leq \liminf_{n\to\infty} \|v_n\|^p = \liminf_{n\to\infty} G_2(r(v_n), v_n) = G_2(r_0, v_0) \tag{3.44}$$

and therefore $G_2(\bar{r}(v_0), v_0) = \|v_0\|^p \leq G(r_0, v_0)$. Since by Lemma 3.18 $\bar{r}(v_0)$ is the global maximum of $G_2(., v_0)$, it follows that $\bar{r}(v_0) = r_0$. By (3.38) we can write

$$M(v_0)\bar{r}(v_0)^s = \left(\frac{p-q}{p-s}\right) H(v_0)\bar{r}(v_0)^q.$$

Thus,

$$\begin{aligned} M =& \left(\frac{1}{p} - \frac{1}{q}\right) H(v_0)\bar{r}(v_0)^q + \left(\frac{1}{s} - \frac{1}{p}\right) M(v_0)\bar{r}(v_0)^s \\ =& \left(\frac{q-p}{pq}\right) H(v_0)\bar{r}(v_0)^q + \left(\frac{p-s}{sp}\right) \left(\frac{p-q}{p-s}\right) H(v_0)\bar{r}(v_0)^q \\ =& \frac{H(v_0)\bar{r}(v_0)^q}{p} \left(\frac{q-p}{q} + \frac{p-q}{s}\right) \\ >& 0, \end{aligned}$$

which is a contradiction and therefore $v_0 \in \Omega_2$. Now, we assert that $r_0 = r(v_0)$. To begin, let $\mu = \|v_0\|^{-1}$. According to Lemma 3.19, it follows that $\mu v_0 \in \Omega_2 \cap S^1$. Note that

$$\begin{aligned} G_2(\mu r(\mu v_0), v_0) =& \frac{1}{\mu^p}(r(\mu v_0)^{q-p} H(\mu v_0) - r(\mu v_0)^{s-p} M(\mu v_0)) \\ =& \frac{1}{\mu^p} G_2(r(\mu v_0), \mu v_0) \\ =& \frac{1}{\mu^p} \|\mu v_0\|^p \\ =& \|v_0\|^p \\ =& G_2(r(v_0), v_0). \end{aligned}$$

Then, given that $r(v_0) > \bar{r}(v_0)$ and $\mu r(\mu v_0) > \bar{r}(\mu v_0) = \bar{r}(v_0)$, by Lemma 3.19 we have

$$\mu r(\mu v_0) = r(v_0). \tag{3.45}$$

Furthermore, since $\bar{r}(v_n) < r(v_n)$, taking the limit we get $\bar{r}(v_0) \leq r_0$ and by (3.44) we obtain

$$\bar{r}(v_0) \leq r_0 \leq r(v_0).$$

Suppose by contradiction that $r_0 < r(v_0)$ and consider the function

$$\sigma(t) := \frac{\partial}{\partial t} J(tv_0) = t^{p-1}(\|v_0\|^p - G_2(t, v_0)),$$

for $t \in (r_0, r(v_0))$. Note that $\sigma(t) < 0$, hence $J(tv_0)$ is decreasing on t and by (3.45)

$$M = \liminf_{n\to\infty} J(r(v_n)v_n) \geq J(r_0 v_0) > J(r(v_0)v_0) = J(r(\mu v_0)\mu v_0) = \mathcal{J}(\mu v_0),$$

with $\mu v_0 \in \Omega_2 \cap S^1$, which is impossible. Therefore, $r_0 = r(v_0)$, consequently

$$1 = \lim_{n\to\infty} \|v_n\|^p = \lim_{n\to\infty} G_2(r(v_n), v_n) = G_2(r(v_0), v_0) = \|v_0\|^p,$$

which implies that $v_0 \in \Omega_2 \cap S^1$, and

$$M = \lim_{n\to\infty} \mathcal{J}(v_n) = \left(\frac{1}{p} - \frac{1}{q}\right) H(v_0) r(v_0)^q + \left(\frac{1}{s} - \frac{1}{p}\right) M(v_0) r(v_0)^s = \mathcal{J}(v_0).$$

Finally, by Lemma 3.13, $r(v_0)v_0$ is a nonnegative and nontrivial critical point if I in E and this completes the proof. ∎

Appendix A

In this Appendix, we present some basic and very important properties of the weighted Sobolev space $\mathcal{D}^{1,p}_\gamma(\mathbb{R}^N_+)$.

A.1 Properties of the weighted Lebesgue space

The Lebesgue spaces play a central role in the Sobolev's space theory and consequently in the study of differential equations. This follows by the fact that some of the main properties of Sobolev spaces, used in the study of this equations, derive from the Lebesgue space theory.

First, let us recall your definition. Given $p, q > 1$ and $\gamma \in \mathbb{R}$, we consider the weighted Lebesgue space defined by

$$L^q(\mathbb{R}^N_+, (1 + x_N)^{\gamma-p}) := \left\{ u : \mathbb{R}^N_+ \to \mathbb{R} \text{ measurable} : \int_{\mathbb{R}^N_+} \frac{|u|^q}{(1 + x_N)^{p-\gamma}} \,\mathrm{d}x < \infty \right\}, \tag{A.1}$$

equipped with the norm

$$\|u\|_{q,p,\gamma} := \left(\int_{\mathbb{R}^N_+} \frac{|u|^q}{(1 + x_N)^{p-\gamma}} \,\mathrm{d}x \right)^{1/q}. \tag{A.2}$$

If $p = q$ we denote

$$L^q(\mathbb{R}^N_+, (1 + x_N)^{\gamma-p}) = L^p_\gamma(\mathbb{R}^N_+) \quad \text{and} \quad \|.\|_{q,p,\gamma} = \|.\|_{p,\gamma}.$$

The proof of the next result is based on some ideas from [14, Theorem 4.8].

Proposition A.1 *Let $p, q > 1$ and $\gamma \in \mathbb{R}$. Then, the weighetd Lebesgue space $L^q(\mathbb{R}^N_+, (1+x_N)^{\gamma-p})$ endowed with the norm $\|.\|_{q,p,\gamma}$ is a Banach space.*

Proof. Let $(u_n) \subset L^q(\mathbb{R}^N_+, (1+x_N)^{\gamma-p})$ be a Cauchy sequence and let (u_{n_k}) a subsequence such that

$$\|u_{n_{k+1}} - u_{n_k}\|_{q,p,\gamma} \leq 2^{-k}.$$

Lets see that (u_{n_k}) converges in $L^q(\mathbb{R}^N_+, (1+x_N)^{\gamma-p})$. We will denote $u_{n_k} = u_k$. Now consider the sequence (g_n) defined by

$$g_n = \sum_{k=1}^{n} |u_{k+1} - u_k|.$$

Note that $\|g_n\|_{q,p,\gamma} \leq 1$ because

$$\|g_n\|_{q,p,\gamma} \leq \sum_{k=1}^{n} \|u_{k+1} - u_k\|_{q,p,\gamma} \leq \sum_{k=1}^{n} 2^{-k} \leq 1.$$

Furthermore, we have $g_1 \leq \cdots \leq g_n \leq \cdots$. Then,

$$\frac{|g_1|^q}{(1+x_N)^{p-\gamma}} \leq \cdots \leq \frac{|g_n|^q}{(1+x_N)^{p-\gamma}} \leq \cdots$$

and by Monotone convergence theorem there exists $g \in L^q(\mathbb{R}^N_+, (1+x_N)^{\gamma-p})$ such that $g_n(x) \to g(x)$ a.e. in $\mathbb{R}^N_+$. Now observe that

$$g_{m-1} = g_{n-1} + |u_{n+1} - u_n| + \cdots + |u_m - u_{m-1}|.$$

Then

$$|u_m - u_n| \leq |u_m - u_{m-1}| + \cdots + |u_{n+1} - u_n| = g_{m-1} - g_{n-1} \leq g - g_{n-1}. \tag{A.3}$$

It follow that $(u_n(x))$ is a Cauchy sequence and converges a.e. in $\mathbb{R}^N_+$ to a limit, say $u(x)$. Moreover, by (A.3), $|u - u_n| \leq g$ which implies that

$$\frac{|u-u_n|^q}{(1+x_n)^{p-\gamma}} \leq \frac{|g|^q}{(1+x_N)^{p-\gamma}} \in L^1(\mathbb{R}^N_+).$$

Therefore, by dominated convergence theorem, we conclude that $u_n \to u$ in $L^q(\mathbb{R}^N_+, (1+x_N)^{\gamma-p})$ which complete the proof. ■

The proof of the next result is based on some ideas from [14, Theorem 4.10].

Proposition A.2 *Let $\gamma \in \mathbb{R}$ and $1 < q, p < \infty$. Then, the weighted Lebesgue space $L^q(\mathbb{R}^N_+, (1+x_N)^{\gamma-p})$ is reflexive.*

Proof. First, we recall that by the Millman-Pettis Theorem (see [14, Theorem 3.31]) every uniformly convex Banach space is reflexive. Then, to conclude the proposition it is sufficient to prove that $L^q(\mathbb{R}^N_+, (1+x_N)^{\gamma-p})$ is uniformly convex.

First we consider $q \geq 2$. Let $\varepsilon > 0$ and $u, v \in L^q(\mathbb{R}^N_+, (1+x_N)^{\gamma-p})$ such that $\|u\|_{q,p,\gamma}, \|v\|_{q,p,\gamma} \leq 1$ and $\|u-v\|_{q,p,\gamma} > \varepsilon$. The first Clarkson inequality tells us that ,

$$\left\|\frac{f+g}{2}\right\|_q^q + \left\|\frac{f-g}{2}\right\|_q^q \leq \frac{1}{2}(\|f\|_q^q + \|g\|_q^q), \quad \forall f, g \in L^q(\mathbb{R}^N_+) \text{ and } q \geq 2. \tag{A.4}$$

Taking $f = u(1+x_N)^{\frac{\gamma-p}{q}}$ and $g = v(1+x_N)^{\frac{\gamma-p}{q}}$ we have

$$\begin{aligned}\left\|\frac{u+v}{2}\right\|_{q,p,\gamma}^q &\leq \frac{1}{2}\|u\|_{q,p,\gamma}^q + \frac{1}{2}\|v\|_{q,p,\gamma}^q - \left\|\frac{u-v}{2}\right\|_{q,p,\gamma}^q \\ &\leq 1 - \frac{1}{2^q}\|u-v\|_{q,p,\gamma}^q \\ &< 1 - \left(\frac{\varepsilon}{2}\right)^q,\end{aligned}$$

that is

$$\left\|\frac{u+v}{2}\right\|_{q,p,\gamma} < \left(1 - \left(\frac{\varepsilon}{2}\right)^q\right)^{1/q}.$$

Therefore, taking $\delta = 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^q\right)^{1/q}$ we obtain

$$\left\|\frac{u+v}{2}\right\|_{q,p,\gamma} < 1 - \delta,$$

which concludes that $L^q(\mathbb{R}^N_+, (1+x_N)^{\gamma-p})$ is reflexive for $q \geq 2$. For $1 < q \leq 2$, consider the second Clarkson inequality, that is, for all $f, g \in L^q(\mathbb{R}^N_+)$ and $1 < q \leq 2$ we have

$$\left\|\frac{f+g}{2}\right\|_q^{q'} + \left\|\frac{f-g}{2}\right\|_q^{q'} \leq \left(\frac{1}{2}\|f\|_q^q + \frac{1}{2}\|g\|_q^q\right)^{1/(q-1)}, \tag{A.5}$$

where $1/q + 1/q' = 1$. Applying this inequality for $f = u(1+x_N)^{\frac{\gamma-p}{q}}$ and $g = v(1+x_N)^{\frac{\gamma-p}{q}}$ we obtain

$$\begin{aligned}\left\|\frac{u+v}{2}\right\|_{q,p,\gamma}^{q'} &\leq \left(\frac{1}{2}\|u\|_{q,p,\gamma}^q + \frac{1}{2}\|v\|_{q,p,\gamma}^q\right)^{1/(q-1)} - \left\|\frac{u-v}{2}\right\|_{q,p,\gamma}^{q'} \\ &\leq 1 - \frac{1}{2^{q'}}\|u-v\|_{q,p,\gamma}^{q'} \\ &< 1 - \left(\frac{\varepsilon}{2}\right)^{q'},\end{aligned}$$

thus

$$\left\|\frac{u+v}{2}\right\|_{q,p,\gamma} < \left(1 - \left(\frac{\varepsilon}{2}\right)^{q'}\right)^{1/q'}.$$

Taking $\delta = 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^{q'}\right)^{1/q'}$ we obtain

$$\left\|\frac{u+v}{2}\right\|_{q,p,\gamma} < 1 - \delta.$$

Therefore, $L^q(\mathbb{R}^N_+, (1+x_N)^{\gamma-p})$ is uniformly convex and we conclude that $L^q(\mathbb{R}^N_+, (1+x_N)^{\gamma-p})$ is reflexive for $1 < q \leq 2$ and we finish the proof. ■

We remark that the Clarkson inequalities, (A.4) and (A.5) can be found in [33] in the itens 15.7 and 15.8.

Proposition A.3 (Interpolation inequality) *If $u \in L^r(\mathbb{R}^N_+, (1+x_N)^{\gamma-p}) \cap L^s(\mathbb{R}^N_+, (1+x_N)^{\gamma-p})$ with $1 \leq r < s < \infty$, then $u \in L^q(\mathbb{R}^N_+, (1+x_N)^{\gamma-p})$ for all, $r \leq q \leq s$, and the following interpolation inequality holds:*

$$\int_{\mathbb{R}^N_+} \frac{|u|^q}{(1+x_N)^{p-\gamma}}\,\mathrm{d}x \leq \left(\int_{\mathbb{R}^N_+} \frac{|u|^r}{(1+x_N)^{p-\gamma}}\,\mathrm{d}x\right)^{1-\alpha} \left(\int_{\mathbb{R}^N_+} \frac{|u|^s}{(1+x_N)^{p-\gamma}}\,\mathrm{d}x\right)^{\alpha}, \qquad \text{(A.6)}$$

where $q = (1-\alpha)r + \alpha s$, with $\alpha \in [0,1]$.

Proof. Since $1 = \alpha + (1-\alpha)$ and $q = (1-\alpha)r + \alpha s$, we apply the Hölder inequality and obtain

$$\begin{aligned}\int_{\mathbb{R}^N_+} \frac{|u|^q}{(1+x_N)^{p-\gamma}}\,\mathrm{d}x &= \int_{\mathbb{R}^N_+} \frac{|u|^{(1-\alpha)r}}{(1+x_N)^{(p-\gamma)(1-\alpha)}} \frac{|u|^{\alpha s}}{(1+x_N)^{(p-\gamma)\alpha}}\mathrm{d}x \\ &\leq \left(\int_{\mathbb{R}^N_+} \frac{|u|^r}{(1+x_N)^{p-\gamma}}\,\mathrm{d}x\right)^{1-\alpha} \left(\int_{\mathbb{R}^N_+} \frac{|u|^s}{(1+x_N)^{p-\gamma}}\,\mathrm{d}x\right)^{\alpha}.\end{aligned}$$

■

A.2 Properties of the weighted Sobolev space

In this section, we shall prove some properties of the weighted Sobolev space $\mathcal{D}^{1,p}_\gamma(\mathbb{R}^N_+)$ that have been used throughout the thesis. The proof of the main properties are based on the ideas from the classical theory of Sobolev spaces.

Theorem A.4 *$\mathcal{D}^{1,p}_\gamma(\mathbb{R}^N_+)$ is a Banach space.*

Proof. Let $(u_n) \subset \mathcal{D}^{1,p}_\gamma(\mathbb{R}^N_+)$ be a Cauchy sequence. In particular, (u_n) and $((1+x_N)\nabla u_n)$ are Cauchy sequences in $L^p_\gamma(\mathbb{R}^N_+)$ and $L^p_\gamma(\mathbb{R}^N_+)^N$, respectively. Since they are Banach spaces there exist $u \in L^p_\gamma(\mathbb{R}^N_+)$ and $\omega \in L^p_\gamma(\mathbb{R}^N_+)^N$ such that

$$u_n \to u \quad \text{in} \quad L^p_\gamma(\mathbb{R}^N_+) \quad \text{and} \quad (1+x_N)\nabla u_n \to \omega \quad \text{in} \quad L^p_\gamma(\mathbb{R}^N_+)^N.$$

Now we claim that $\nabla u = \eta$ where $\eta = \omega/(1+x_N)$. In fact, if $\eta = (\eta_1, \cdots, \eta_N)$ and $\varphi \in C_0^\infty(\mathbb{R}_+^N)$, by Hölder inequality we have

$$\begin{aligned}\left|\int_{\mathbb{R}_+^N} (u\varphi_{x_i} - u_n\varphi_{x_i})\,\mathrm{d}x\right| &\le \int_{\mathbb{R}_+^N} |u-u_n||\varphi_{x_i}|\,\mathrm{d}x \\ &= \int_{\mathbb{R}_+^N} \frac{|u-u_n|}{(1+x_N)^{\frac{p-\gamma}{p}}}|\varphi_{x_i}|(1+x_N)^{\frac{p-\gamma}{p}}\,\mathrm{d}x \\ &\le \left(\int_{\mathbb{R}_+^N} \frac{|u-u_n|^p}{(1+x_N)^{p-\gamma}}\,\mathrm{d}x\right)^{1/p}\left(\int_{\mathbb{R}_+^N} (1+x_N)^{\frac{p-\gamma}{p-1}}|\varphi_{x_i}|^{\frac{p}{p-1}}\,\mathrm{d}x\right)^{\frac{p-1}{p}},\end{aligned}$$

which implies

$$\int_{\mathbb{R}_+^N} u_n\varphi_{x_i}\,\mathrm{d}x \longrightarrow \int_{\mathbb{R}_+^N} u\varphi_{x_i}\,\mathrm{d}x, \quad \forall\varphi\in C_0^\infty(\mathbb{R}_+^N). \tag{A.7}$$

On the other hand,

$$\begin{aligned}\left|\int_{\mathbb{R}_+^N} \eta_i\varphi - (u_m)_{x_i}\varphi\,\mathrm{d}x\right| &\le \int_{\mathbb{R}_+^N} |\eta_i - (u_m)_{x_i}||\varphi|\,\mathrm{d}x \\ &\le \left(\int_{\mathbb{R}_+^N} |\eta_i-(u_m)_{x_i}|^p(1+x_N)^\gamma\,\mathrm{d}x\right)^{1/p} \|\varphi\|_{\frac{p}{p-1}}.\end{aligned}$$

Since $(1+x_N)\nabla u_m \to \omega$ in $L^q(\mathbb{R}_+^N, (1+x_N)^{\gamma-p})^N$ we get

$$\begin{aligned}\int_{\mathbb{R}_+^N} |\eta_i-(u_m)_{x_i}|^p(1+x_N)^\gamma\,\mathrm{d}x &= \int_{\mathbb{R}_+^N} (1+x_N)^\gamma\left|\frac{\omega_i}{(1+x_N)} - (u_m)_{x_i}\right|^p\,\mathrm{d}x \\ &= \int_{\mathbb{R}_+^N} \left|(1+x_N)^{\gamma/p-1}\omega_i - (1+x_N)^{\gamma/p}(u_m)_{x_i}\right|^p\,\mathrm{d}x \\ &= \int_{\mathbb{R}_+^N} |(1+x_N)^{\frac{\gamma-p}{p}}\omega_i - (1+x_N)(u_m)_{x_i}(1+x_N)^{\frac{\gamma-p}{p}}|^p\,\mathrm{d}x \\ &= \int_{\mathbb{R}_+^N} \frac{|\omega_i-(1+x_N)(u_m)_{x_i}|^p}{(1+x_N)^{p-\gamma}}\,\mathrm{d}x \\ &\longrightarrow 0.\end{aligned}$$

Thus, we have

$$\int_{\mathbb{R}_+^N} (u_m)_{x_i}\varphi\,\mathrm{d}x \to \int_{\mathbb{R}_+^N} \eta_i\varphi\,\mathrm{d}x, \quad \forall\varphi\in C_0^\infty(\mathbb{R}_+^N). \tag{A.8}$$

Therefore, by (A.7) and (A.8) we obtain

$$\int_{\mathbb{R}_+^N} u\varphi\,\mathrm{d}x = \lim_{n\to\infty}\int_{\mathbb{R}_+^N} u_m\varphi\,\mathrm{d}x = -\lim_{n\to\infty}(u_m)_{x_i}\varphi\,\mathrm{d}x = -\int_{\mathbb{R}_+^N}\eta_i\varphi\,\mathrm{d}x,$$

which implies that $\nabla u = \eta$. Since $\mathcal{D}_\gamma^{1,p}(\mathbb{R}_+^N)$ is closed, $u \in \mathcal{D}_\gamma^{1,p}(\mathbb{R}_+^N)$ and hence $\mathcal{D}_\gamma^{1,p}(\mathbb{R}_+^N)$is a Banach space. ∎

Theorem A.5 *The space $\mathcal{D}^{1,p}_{\gamma}(\mathbb{R}^N_+)$ is reflexive.*

Proof. c Consider the operator $T : \mathcal{D}^{1,p}_{\gamma}(\mathbb{R}^N_+) \to L^p_{\gamma}(\mathbb{R}^N_+) \times L^p_{\gamma}(\mathbb{R}^N_+)^N$ defined by $T(u) = (u, (1 + x_N)\nabla u)$, where in $L^p_{\gamma}(\mathbb{R}^N_+) \times L^p_{\gamma}(\mathbb{R}^N_+)^N$ we consider the norm

$$\|(u, \omega)\|^p = \|u\|^p_{q,p,\gamma} + \|\omega\|^p_{q,p,\gamma}.$$

It is clear that T is an isometry. Since $\mathcal{D}^{1,p}_{\gamma}(\mathbb{R}^N_+)$ is a Banach space and, by Theorem A.2, $L^q(\mathbb{R}^N_+, (1+x_N)^{\gamma-p})$ is reflexive, it follows that $T(\mathcal{D}^{1,p}_{\gamma}(\mathbb{R}^N_+))$ is a closed subspace of a reflexive space. Therefore, $T(\mathcal{D}^{1,p}_{\gamma}(\mathbb{R}^N_+))$ is reflexive and hence $\mathcal{D}^{1,p}_{\gamma}(\mathbb{R}^N_+)$ is reflexive. ∎

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