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On nonhomogeneous problems involving the (p, q)-Laplacian Operator with critical growth

Tese apresentada ao Corpo Docente do Programa de Pós-Graduação em Matemática - CCEN - UFPB, como requisito parcial para obtenção do título de Doutor em Matemática.
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ATA DE DEFESA DE DOUTORADO JUNTO AO PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA DA UNIVERSIDADE FEDERAL DA PARAÍBA, REALIZADA NO DIA 24 DE JANEIRO DE 2025.

Ao vigésimo quarto dia de janeiro de dois mil e vinte e cinco, às 14:00 horas, na Sala 201 do Departamento de Física da Universidade Federal da Paraíba, foi aberta a sessão pública de Defesa de Tese intitulada "**Sobre problemas não homogêneos envolvendo o operador (p,q)-Laplaciano com crescimento crítico**", do aluno **Hector Alan dos Santos Pereira** que havia cumprido, anteriormente, todos os requisitos para a obtenção do grau de Doutor em Matemática, sob a orientação do Prof. Dr. Bruno Henrique Carvalho Ribeiro. A Banca Examinadora, indicada pelo Colegiado do Programa de Pós-Graduação em Matemática, foi composta pelos professores: Bruno Henrique Carvalho Ribeiro (Orientador), Elisandra de Fátima Gloss de Moraes (Coorientadora), Manassés Xavier de Souza (UFPB), José Anderson Valença Cardoso (UFS), Marco Aurélio Soares Souto (UFCG) e Rayssa Helena Aires de Lima Caju (UCh) . O professor Bruno Henrique Carvalho Ribeiro, em virtude da sua condição de orientador, presidiu os trabalhos e, depois das formalidades de apresentação, convidou o aluno a discorrer sobre o conteúdo da tese. Concluída a explanação, o candidato foi arguido pela banca examinadora que, em seguida, sem a presença do aluno, finalizando os trabalhos, reuniu-se para deliberar tendo concedido a menção: **APROVADO.** E, para constar, foi lavrada a presente ata que será assinada pelos membros da Banca Examinadora.

João Pessoa, 24 de janeiro de 2025.

Bruno Henrique Carvalho Ribeiro ______ Bruno Henrique Carvalho Ribeiro

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"Suba o primeiro degrau com fé. Não é necessário que você veja toda a escada. Apenas dê o primeiro passo."

Martin Luther King

Abstract

Estudamos classes de equações elípticas envolvendo operadores do tipo (p, q)-Laplaciano, considerando não linearidades no intervalo de crescimento crítico dos tipos Sobolev (quando $p < N$, sendo N a dimensão do espaço) e Trudinger-Moser (quando $p = N$). Por meio de métodos variacionais, estabelecemos a multiplicidade de soluções para problemas não homogêneos, considerando tanto casos de crescimento crítico unilateral quanto bilateral. A influência do expoente q é analisada, destacando-se como um fator crucial na obtenção dos resultados, em comparação com os problemas clássicos envolvendo o p-Laplaciano.

Palavras-chave: Métodos variacionais, (p, q)-Laplaciano, Problemas elípticos quasilineares, resultado do tipo Ambrosetti-Prodi, Trudinger-Moser, Expoente crítico de Sobolev.

Abstract

We studied classes of elliptic equations involving (p, q)-Laplacian operators, considering nonlinearities within the critical growth range of the Sobolev type (for $p < N$, where N is the spatial dimension) and the Trudinger-Moser type (for $p = N$). Using variational methods, we established the multiplicity of solutions for nonhomogeneous problems, addressing both cases of unilateral and bilateral critical growth. The influence of the exponent q was analyzed, highlighting its crucial role in obtaining the results, compared to classical problems involving the p-Laplacian.

Keywords: Variational methods, (p, q)-Laplacian, Quasilinear elliptic problems, Ambrosetti-Prodi type result, Trudinger-Moser inequalities, Critical Sobolev exponent spaces.

Contents

Introduction **1**

Motivations 1

Main contributions of this work 7

1 (p, q)-Laplacian equations with critical growth and jumping nonlinearities **12**

1.1 Introduction 12

1.2 Hypotheses and Main Results 12

1.3 Preliminaries 19

1.4 Proof of the Main Theorems 28

1.4.1 Proof of the main theorems 42

2 Critical non-homogeneous problems on the (p, q)-Laplacian **43**

2.1 Introduction 43

2.2 Hypotheses and Main Results 43

2.3 Preliminaries 50

2.4 Proof of the Main Theorems 52

2.4.1 Geometry of the Mountain Pass and estimates for the *minimax* level . . . 52

2.4.2 Proof of the main theorems 57

3 (N, q)-Laplacian equations with critical exponential growth and jumping nonlinearities **58**

3.1 Introduction 58

3.2 Preliminaries 60

3.3 L^∞ estimates and regularity 63

3.3.1 Boundedness of solutions 64

3.3.2 Regularity of solutions 67

3.4 $C_0^1 \times W_0^{1,N}$ topology 68

3.5 A first solution 75

3.6 Proof of the Main Theorems 76

3.6.1 Proof of the main theorem 90

4 Critical nonhomogeneous problems on the (N,q)-Laplacian **91**

4.1 Introduction 91

4.2 Hypotheses and Main Results 91

4.3 Preliminaries 94

4.4 Proof of the Main Theorems 98

4.4.1 Proof of the main theorems 107

Introduction

Motivations

In this study, we investigate the existence and multiplicity of weak solutions for classes of non-homogeneous problems using variational methods that involve partial differential equations associated with the (p, q)-Laplacian operator. The problems are of the type:

$$\begin{cases} -\Delta_p u - \Delta_q u = g(x, u) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω represents a bounded domain in $\mathbb{R}^N$, with the dimensions satisfying $1 < q < p \leq N$, $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a non-linear term and f is a non-homogeneous term. The (p, q)-Laplacian operator is defined by

$$\Delta_p u + \Delta_q u = \operatorname{div}\left((|\nabla u|^{p-2} + |\nabla u|^{q-2})\nabla u\right).$$

This kind of problem plays a crucial role in diverse fields, such as biophysics [32], reaction-diffusion equations [25], and forms a fundamental component in the examination of quasilinear elliptic problems. Problems related to this operator have been extensively explored in existing literature, as demonstrated by studies by [14, 31, 35, 36], and others referenced therein. In this work we address problems where the nonlinearity $g(x, u)$ presents both unilateral critical growth (Chapters 1 and 3) or bilateral growth (Chapters 2 and 4). Chapters 1 and 2 focus on the case $p < N$, addressing critical growth of the polynomial type. In contrast, Chapters 3 and 4 examine the case $p = N$, where the natural critical growth is of the exponential type.

Unilateral problems

Problems involving unilateral critical growth have connections to a seminal problem introduced by A. Ambrosetti and G. Prodi in 1973 (see [1]), which discuss the existence, multiplicity, and absence of solutions for a specific differential equation. This problem has intrigued mathematicians for decades, leading to the exploration of various generalizations arising from the original research. The results of this study are based on the non-homogeneous term and the interplay between nonlinearity and the spectrum of the operator. It was formulated as

$$\begin{cases} -\Delta u = h(u) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with strict conditions on $h(u)$ and $f(x)$. Successive studies have built upon these findings, delving into the diverse dynamics between the data. Defining $h_- = \lim_{s\to-\infty} h(s)/s$ and $h_+ = \lim_{s\to+\infty} h(s)/s$, it was assumed that $0 < h_- < \lambda_1 < h_+ < \lambda_2$, referencing the spectrum under discussion. The unilateral superlinear scenarios, with, $h(u) = 0$ for all $u \leq 0$, $h_- \in (0, \lambda_1)$ and $h_+ = +\infty$, led many researchers to variational approaches and have been analyzed, for instance, in [3] for the case of the Laplacian and [26, 34] for the p-Laplacian case, specifically in the context of subcritical growth for $h(s)$. D. Figueiredo and Y. Jianfu [5] addressed the case where $p = 2$ with $h(u) = \lambda u + u^{2^*-1}$, i.e., they explored the superlinear unilateral behavior of $g(x, u)$ with critical growth (here 2^* concerns the critical Sobolev exponent $2N/(N-2)$). They proved the existence of two solutions for $N > 6$. The insights of [5] were extended by M. Calanchi and B. Ruf [27] to include $N \geq 6$, and by incorporating a subcritical term, they also ventured into discussions for $N = 3, 4$ and 5. Further, [19] explored this problem through the lens of the p-Laplacian operator, achieving results analogous to those of [5, 27]. This framework set the stage for further exploration of the interactions between h_-, h_+, and the spectrum. In Chapter 1 we extend the results of [19] for the (p, q)-Laplacian operator with $1 < q \leq p < N$.

In this unilateral case, Chapter 3 focuses on the scenario $p = N \geq 2$, motivated by the Pohozaev-Trudinger-Moser inequality

$$\sup_{u\in W_0^{1,N}(\Omega), \|\nabla u\|_N \leq 1} \int_\Omega e^{(\alpha_N |u|^{\frac{N}{N-1}})} dx < \infty,$$

where $\alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$ and ω_{N-1} is the unitary $(N-1)$-dimensional sphere volume (see [20, 30]). The two-dimensional scenario, where the critical growth of Sobolev type is substituted with a Trudinger-Moser growth condition, was examined by Calanchi et al. [28], considering the problem given by:

$$\begin{cases} -\Delta u = \lambda u + g(x, u_+) + f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in a bounded domain $\Omega \in \mathbb{R}^2$, where g is a function satisfying a unilateral critical Trudinger-Moser-type growth condition. They showed that for a specific class of functions f and $\lambda_k < \lambda < \lambda_{k+1}, k \geq 1, \lambda_i$ being the eigenvalues of $(-\Delta, H_0^1(\Omega))$, there exist two solutions, one of which is negative. This result was extended by [2] for the cases of the gradient system. One of the primary challenges in dealing with this type of growth condition in g is establishing that the *minimax* level of the functional associated with this problem avoids non-compactness levels. An additional assumption has been introduced to address this issue, ensuring that this level remains below a critical constant. To achieve the appropriate level, the techniques employed require that the Moser functions have support in a ball B_r such that $r > 0$ is selected to be sufficiently small in the various stages of the arguments. The results in Chapter 3 extend this discussion to the (N, q)-Laplacian. Notably, these findings are also novel for the N-Laplacian, as, to the best of our knowledge, the unilateral growth condition in non-homogeneous problems involving this operator has not been previously explored.

It should be noted that, as in the polynomial case, the first solution is obtained employing an argument that relies on the fact that, under established conditions, local minima of the associated functional in the C^1 topology are also local minima in the $W^{1,N}$ topology. However, unlike the polynomial cases addressed in Chapter 1, this result does not appear to be available in the current literature for the exponential case. To address this gap, we had to establish the $C^1 \times W^{1,N}$ topology result ourselves, drawing inspiration from [18], which deals with similar functionals but under slightly different hypotheses.

Bilateral problems

For the bilateral case, it is worth mentioning the work of Tarantello [12], who obtained two distinct solutions of the following problem:

$$\begin{cases} -\Delta u = |u|^{2^*-2}u + f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 3$, $2^* = 2N/(N-2)$ is the critical Sobolev exponent and $f \in H^{-1}(\Omega)$ is non-trivial, satisfying

$$\int_\Omega f u dx < \frac{4}{N-2}\left(\frac{N-2}{N+2}\right)^{(N+2)/4} \|\nabla u\|_2^{(N+2)/2}$$

for all $u \in H_0^1(\Omega)$ and $\|u\|_{2^*} = 1$. Furthermore, we can mention [7] who extended the above problem to the case

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-2}u + f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 < \lambda < \lambda_1$, with $\lambda_1 > 0$ is the first Dirichlet eigenvalue of $-\Delta$ in Ω. Subsequently, Perera [23] demonstrated the existence of $\mu_0 > 0$ such that the aforementioned problem admits two nontrivial solutions for all $f \in L^{2N/(N+2)}(\Omega) \setminus \{0\}$ with $\|f\|_{2N/(N+2)} < \mu_0$ provided that $N = 4$ and λ is not an eigenvalue, or for $N \geq 5$ and $\lambda > 0$.

In the case of p-Laplacian, Chabrowski [16] investigated the following problem:

$$\begin{cases} -\Delta_p u = |u|^{p^*-2}u + \lambda|u|^{q-2}u + f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

considering $\lambda > 0$, $f \neq 0$ under certain conditions, and $1 < p, q < p^*$, $1 < p < N$ and $p^* = Np/(N-p)$ is the critical Sobolev exponent. Chabrowski showed that there exists $\bar{\lambda} > 0$ such that the aforementioned problem has at least two distinct solutions, establishing a negative minimal energy solution using the Variational Principle of Ekeland. In case $q \leq p$, $f \neq 0$ under specific conditions, a solution was obtained for either large $\lambda > 0$ or for a domain Ω of small measure. Furthermore, a second solution using the method employed was inconclusive.

In Perera [23], the problem given by:

$$\begin{cases} -\Delta_p u = \lambda|u|^{p-2}u + \mu|u|^{q-2}u + |u|^{p^*-2}u + f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1 < p < N$,$1 < q < p^*$, $\lambda > 0, \mu \in \mathbb{R}$, was explored. Considering

$$E(u) = \int_\Omega \left(\frac{1}{p}|\nabla u|^p - \frac{\lambda}{p}|u|^p - \frac{\mu}{q}|\nabla u|^q - \frac{1}{p^*}|u|^{p^*} - f(x)u \right) dx, \quad u \in W_0^{1,p}(\Omega)$$

the associated variational functional and

$$S_{N,p} = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|u\|^p}{\|u\|_{p^*}^p}$$

the best Sobolev constant, Perera showed that there exists $\mu_0 > 0$ such that the above problem supports two nontrivial solutions u_1 and u_2 satisfying:

$$E(u_1) < E(u_2),$$

$$0 < E(u_2) < \frac{1}{N} S_{N,p}^{N/p}$$

for all $\mu \in \mathbb{R}$ and $f \in L^{p^{*'}}(\Omega) \setminus \{0\}$, with $p^{*'} = \frac{p^*}{p^*-1}$ and $|\mu| + \|f\|_{p^{*'}} < \mu_0$ in the following cases:

(i) $N \geq p^2$ and $\lambda > 0$ is not a Dirichlet eigenvalue of $-\Delta_p$ in Ω;

(ii) $N(N-p^2) > p^2$ and $\lambda > 0$.

We can continue discussing these non-homogeneous problems with the critical (p,q)-Laplacian operators, given by

$$\begin{cases} -\Delta_p u - \Delta_q u = \mu|u|^{r-2}u + |u|^{p^*-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $1 < q < p < N$, $\mu > 0$, $N \geq 2$ and $1 < r < p^*$. In the present literature, we can mention that Li and Zang [9] showed that this problem possesses infinitely many solutions when $1 < r < q$ and $\mu > 0$ is sufficiently small. Furthermore, Yin and Yang [14] showed that it has a non-trivial solution when $p < r < p^*$ and $\mu > 0$ is sufficiently large. Furthermore, Marano, Candito and Perera [31] have investigated a variant of the problem, which reads:

$$\begin{cases} -\Delta_p u - \Delta_q u = \mu |u|^{q-2}u + \lambda |u|^{p-2}u + |u|^{p^*-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\mu \in \mathbb{R}$ and $\lambda > 0$ are constants that interact with the spectrum of $-\Delta_q$ and $-\Delta_p$, respectively. Assuming $N > p^2$, $\lambda \in (0, \lambda_1)$, and $\mu \leq \mu_1$, where λ_1 and μ_1 are the first eigenvalues of the p-Laplacian and q-Laplacian operators, respectively, they discovered a nontrivial nonnegative solution in the following cases:

(i) $N(p-1)/(N-p) \leq q < (N-p)p/N$,

(ii) $N(p-1)/(N-1) < q < \min\{N(p-1)/(N-p), (N-p)p/N\}$,

(iii) $(1-1/N)p^2 + p < N$ and $q = N(p-1)/(N-1)$,

(iv) $(p-1)p^2/(N-p) < q < N(p-1)/(N-1)$.

These findings establish the existence of nontrivial solutions for the critical (p, q)-Laplacian problem under specific conditions regarding the parameters and dimensions involved.

Recently, Ho, Perera and Sim [22] provided a nontrivial solution to problem (1) for all $\mu > 0$. they demonstrated that the problem (1) possesses a nontrivial weak solution for all $\mu > 0$ in each of the following cases:

(i) $1 < q < N(p-1)/(N-1)$ and $N^2(p-1)/(N-1)(N-p) < r < p^*$,

(ii) $N(p-1)/(N-1) \leq q < p$ and $Nq/(N-p) < r < p^*$.

In particular, problem (1) has a nontrivial weak solution for all $\mu > 0$ when $N^2 - p(p+1)N + p^2 \geq 0, q \leq (N-p)p/N$, and $p < r < p^*$, and when $N^2 - p(p+1)N + p^2 > 0, q < (N-p)p/N$, and $r = p$. Additionally, the non-homogeneous case for the (p, q)-Laplacian operator was investigated by Perera [23], where the problem was formulated as:

$$\begin{cases} -\Delta_p u - \eta \Delta_q u = \lambda |u|^{p-2}u + |u|^{p^*-2}u + f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

Here, Ω is a bounded domain with a smooth boundary in $\mathbb{R}^N$, where $1 < q < p < N$, the parameters λ and η are positive and $f \in L^{p^{*'}}(\Omega)$. The associated variational functional can be expressed as:

$$E(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{p}\int_\Omega |u|^p dx + \frac{\eta}{q}\int_\Omega |\nabla u|^q dx - \frac{1}{p^*}\int_\Omega |u|^{p^*} dx - \int_\Omega fu dx$$

where $u \in W_0^{1,p}(\Omega)$. The author obtained two nontrivial solutions u_1 and u_2 that satisfy $E(u_1) < E(u_2)$ and $0 < E(u_2) < S^{\frac{N}{p}}/N$, for all $\mu > 0$ and $f \in L^{p^{*'}}(\Omega) \setminus \{0\}$ with $\|f\|_{p^{*'}} + \eta < \eta_0$, for some $\eta_0 > 0$, where S is the best Sobolev constant, in the following cases:

(i) $N \geq p^2$ and $\lambda > 0$ is not a Dirichlet eigenvalue of $-\Delta_p$ in Ω,

(ii) $N(N - p^2) > p^2$ and $\lambda > 0$.

In Chapter 2, by introducing the term $f(x)$, where $f \in L^{p^{*'}}(\Omega)$, in problem (2) and considering higher dimensions, we identify two distinct solutions, provided that $\|f\|_{p^{*'}}$ is sufficiently small and $1 < q < \frac{N-p}{p}$. It is worth noting that problem (3) incorporates a parameter η into the q-Laplacian, requiring only the condition $1 < q < p < N$ for the exponent q. Interestingly, this work does not address the case $\eta = 1$, as the parameter plays a crucial role in ensuring the existence of a solution, which is achieved only by adjusting η to be sufficiently small. This adjustment is necessary to manage all possibilities where $1 < q < p$. In our approach, we avoid using this parameter; however, this requires imposing certain restrictions on q.

Concerning the subjects of the Chapter 4, let us begin by mentioning that, in the existing literature, we encounter the homogeneous problem:

$$\begin{cases} -\Delta_N u - \Delta_q u = \mu|u|^{q-2}u + \lambda|u|^{N-2}ue^{|u|^{N'}} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in the studies by Yang and Perera [37]. They were interested in the problem involving $\mu \in \mathbb{R}$ and $\lambda > 0$, assuming $\frac{N}{2} < q < N$ and $\mu < \mu_1$ where μ_1 represents the first eigenvalue of the q-Laplacian. They proved the existence of a nonnegative nontrivial solution for every $\lambda > \lambda^*$, where the existence of such $\lambda^* > 0$ is based on μ. Moreover, if $\mu \geq \mu_1$, then there exists $\lambda^*(\mu)$ such that the aforementioned problem possesses a nontrivial solution for all $\lambda > \lambda^*(\mu)$. Considering the scenario where $1 < q < p = N$ for the problem in (2), in Chapter 4, we replaced the term with the exponent p^* with a nonlinear term $g(x, u)$ that demonstrates critical exponential growth and added a nonhomogeneous term $f(x)$ to the problem. The main motivation was working with problems related to the (N, q)-Laplacian in the same spirit as the ones we addressed in Chapter 2, where the non-linearity was of polynomial growth type. In this context, we were able to identify two solutions to this problem under specific conditions on g and f.

Main contributions of this work

In this work, the structure is divided as follows. Chapter 1 focuses on a unilateral problem involving the (p, q)-Laplacian operator, with the aim of reproducing results similar to those reported in [19]. The specific problem is formulated as:

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda |u|^{p-2}u + u_+^{p^*-1} + g(x, u_+) + f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where Ω is a bounded domain with a smooth boundary in $\mathbb{R}^N$, $1 < q < p < N$, $\lambda \in (0, \lambda_1)$, $f : \Omega \to \mathbb{R}^+$, $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}^+$ is a function with subcritical and p-superlinear growth condition at $+\infty$.This investigation explores all possible scenarios involving the dimension N. The results presented here contribute to the understanding of unilateral superlinear critical growth for the (p, q)-Laplacian operator, based on previous ideas provided in [35] for the subcritical case. A significant challenge was understanding the influence of the q-Laplacian on estimating the upper bounds of *minimax* levels for the associated functionals. This obstacle was overcome by making some L^s estimates of the Talenti functions, following the ideas of [19]. As it was proved in this last paper, we also identify two distinct solutions, one obtained through minimization which is negative, and the other through the Mountain Pass Theorem. It is assumed that $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}^+$ is a continuous function satisfying:

(g_1) $g(x, t) = 0$, $\forall\, t \leq 0$.

(g_2) There exist $p < \theta < p^*$ and $C > 0$ such that $g(x, t) \leq C(1 + t^{\theta-1})$, for all $t > 0$ and $x \in \overline{\Omega}$.

The hypothesis (g_1) concerning $g(x, t)$ corresponds to the unilateral nature of the non-linearity in the problem, while (g_2) illustrates that $g(x, t)$ serves as a subcritical perturbation. This aspect becomes crucial when approaching cases of lower dimensions. For scenarios where $g \equiv 0$, the methodologies employed to identify a second solution are applicable to higher-dimensional cases.

Our first result in this chapter is given by the following:

Theorem 0.0.1. *Suppose $(g_1) - (g_2)$ holds. Assume that $f \in L^\infty(\Omega)$ and $f \leq 0$ nontrivial. Then, for any $p > 1$, there exist two solutions to problem (4) provided that,*

$$N > \begin{cases} \max\{p^2 + p, \frac{p^2}{p-1}\}, & \text{if } 1 < p < 2 \\ (p-1)p^2 + p, & \text{if } p \geq 2. \end{cases}$$

and

$$1 < q < \frac{(N-p)p}{N}.$$

In addressing the problem in lower dimensions, we consider:

(g_3) There exists $\sigma > 0$, such that $K < \sigma < p^*$ and

$$g(x,t) \geq Ct^{\sigma-1} \text{ for all } (x,t) \in \Omega \times \mathbb{R}_+,$$

where $K = K(N,p,q)$ is given by

$$K(N,p) = \begin{cases} \max\left\{p^* - \frac{p}{N-p}, \frac{Nq}{N-p}\right\}, & \text{if } 1 < p \leq \frac{2N}{N+1} \\ \max\left\{p^* - \frac{p}{N-p}, p^* - 1, \frac{Nq}{N-p}\right\}, & \text{if } \frac{2N}{N+1} < p < 2 \\ \max\left\{p^* - \frac{p}{N-p}, p^* - \frac{1}{p-1}, \frac{Nq}{N-p}\right\}, & \text{if } 2 \leq p. \end{cases}$$

Getting our second result of this chapter, that is,

Theorem 0.0.2. *Suppose that g satisfies $(g_1) - (g_3)$. Assume that $f \in L^\infty(\Omega)$ and $f \leq 0$ in Ω. Then, there exist two solutions to the problem (4) for every $1 < q < p < N$.*

Chapter 2 delves into the bilateral case, tackling a similar problem as outlined in [31], with the addition of the non-homogeneous term $f \in L^{p^{*'}}(\Omega)$ in the equation. We consider $\lambda \in (0, \lambda_1)$ and $\mu \in (0, \mu_1)$. In this context, we have identified two solutions. One solution is derived through the application of the Variational Principle of Ekeland, as elucidated in [16]. The second solution is attained by using Talenti concentration functions in the Mountain Pass geometry, following the concepts put forth in [19]. We studied the problem

$$\begin{cases} -\Delta_p u - \Delta_q u = \mu|u|^{q-2}u + \lambda|u|^{p-2}u + |u|^{p^*-2}u + f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

In this chapter, we investigate a scenario in which Ω represents a bounded domain with a smooth boundary in $\mathbb{R}^N$, with $1 < q < p < N$, $\lambda \in (0, \lambda_1)$, $\mu \in (0, \mu_1)$, and $f \in L^\infty(\Omega)$. Unlike the approach in [23], we do not introduce a parameter in the q-Laplacian. However, constraints on the exponent q are still necessary. In this analysis, we assume that f possesses a small norm. Similarly to the methodology used in Chapter 1, we identify a second solution to this problem, contingent upon specific restrictions on the dimension and q. The outcomes of this investigation are summarized as follows:

Theorem 0.0.3. *Suppose $\lambda \in (0, \lambda_1)$, $\mu \in (0, \mu_1]$, $f \in L^{p^{*'}}(\Omega)$ nontrivial. Then, for any $p > 1$, there exists a constant $M > 0$ such that the problem (5) has two solutions provided that $\|f\|_{p^{*'}} \leq M$,*

$$N > \begin{cases} \max\{p^2 + p, p^2/(p-1), (p-1)p^2/(q-1) + p\}, & \textit{if } 1 < p < 2 \\ \max\{(p-1)p^2 + p, (p-1)p^2/(q-1) + p\}, & \textit{if } p \geq 2, \end{cases}$$

and

$$1 < q < (N-p)p/N.$$

Theorem 0.0.4. *Suppose $\lambda \in (0, \lambda_1)$, $\mu = 0$, $f \in L^{p^{*'}}(\Omega)$ is nontrivial. Then, for any $p > 1$, there exists a constant $M > 0$ such that problem (5) has two solutions provided that $\|f\|_{p^{*'}} \leq M$,*

$$N > \begin{cases} \max\{p^2+p, p^2/(p-1)\}, & \text{if } 1 < p < 2 \\ (p-1)p^2 + p, & \text{if } p \geq 2, \end{cases}$$

and

$$1 < q < (N-p)p/N.$$

In Chapter 3, we explore again the unilateral problem, but now in the scenario where $p = N$, drawing inspiration from the concepts presented in [19] and [2]. Our focus lies in establishing the existence of nontrivial solutions for a (N, q)-Laplacian equation defined by:

$$\begin{cases} -\Delta_N u - \Delta_q u = \lambda |u|^{N-2}u + g(x, u_+) + f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where $\Omega \subset \mathbb{R}^N$ represents a bounded domain with a smooth boundary, $\lambda > 0$ is a real parameter, $1 < q < N$ and $g : \overline{\Omega} \times [0, \infty) \to [0, \infty)$ adheres to a Trudinger-Moser growth condition uniformly across $x \in \Omega$. Our study has established the existence of two distinct solutions, one obtained through minimization and being negative, and the second by using the classical Mountain Pass Theorem.

This problem stands as a new result for both the N-Laplacian and the (N, q)-Laplacian. As usual in this kind of problem, the main challenge was understanding which was the best set of assumptions we should provide to allow the mountain pass level to lie below a specific threshold. In this context, g has a critical behavior, implying the existence of a positive constant α_0 such that:

$$\lim_{t \to +\infty} \frac{g(x,t)}{e^{\alpha t^{\frac{N}{N-1}}}} = \begin{cases} 0, & if \quad \alpha > \alpha_0 \\ +\infty, & if \quad \alpha < \alpha_0, \end{cases} \tag{7}$$

uniformly in $x \in \Omega$. Furthermore, the hypotheses put in g are

$(\tilde{g}_1)$ $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $g(x,t) = 0$ for all $x \in \Omega$ and $t \leq 0$;

$(\tilde{g}_2)$ There are $R, M > 0$ such that

$$0 < G(x,t) := \int_0^t g(x,s)ds \leq Mg(x,t), \quad \forall x \in \Omega \text{ and } t \geq R;$$

$(\tilde{g}_3)$ There is $C > 0$ such that

$$sg(x,s) \geq \gamma(s) e^{\alpha_0 s^{\frac{N}{N-1}}}, \qquad \forall x \in \Omega \text{ and } s \geq C,$$

where $\gamma(s)$ is such that

$$\liminf_{s\to+\infty} \frac{\gamma(s)}{e^{\varepsilon_0 s^{\frac{1}{N-1}}}} > 0$$

for some $\varepsilon_0 > 0$.

The main result of this chapter is given below.

Theorem 0.0.5. *Suppose that (7), $(\tilde{g}_1) - (\tilde{g}_3)$ hold and $f \leq 0$ is nontrivial. Then, there exist two solutions to the problem (6).*

Finally, in Chapter 4, inspired by [37] and following the ideas of [16] and [2], we address the bilateral case regarding the (N, q)-Laplacian operator given by the problem:

$$\begin{cases} -\Delta_N u - \Delta_q u = \lambda |u|^{N-2}u + \mu |u|^{q-2}u + g(x,u) + f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8}$$

where Ω denotes a bounded domain with a smooth boundary within $\mathbb{R}^N$, $f \in L^\infty(\Omega)$, $\lambda \in (0, \lambda_1)$, $\mu \in (0, \mu_1]$, and g represents a nonlinearity exhibiting a critical exponential growth. Here we use ideas analogous to Chapters 2 and 3, for the first and the second solutions, respectively. Moreover, when $\mu = 0$, we encounter two solutions for $1 < q < N$ and $N \geq 2$. However, if $0 < \mu \leq \mu_1$, our analysis is limited to the scenario where $2 \leq q < N$. To be more precise, we delved into problem (8) with the following hypotheses: g exhibits critical growth with exponent $\alpha_0 > 0$. This means that there exists a positive constant α_0 such that

$$\lim_{|t|\to+\infty} \frac{g(x,t)}{e^{\alpha |t|^{\frac{N}{N-1}}}} = \begin{cases} 0, & if \quad \alpha > \alpha_0 \\ +\infty, & if \quad \alpha < \alpha_0, \end{cases} \tag{$\bar{\alpha}_0$}$$

uniformly in $x \in \Omega$.

$(\bar{g}_0)$

$$\limsup_{u\to 0^+} \frac{NG(x,u)}{|u|^N} = 0$$

uniformly in $x \in \Omega$;

$(\bar{g}_1)$ $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}^+$ is continuous non-decrasing function and $g(x,0) \equiv 0$ for all $x \in \Omega$, $g(x,u) \geq 0$ in $\Omega \times [0,+\infty)$ and $g(x,u) \leq 0$ in $\Omega \times (-\infty, 0]$;

($\bar{g}_2$) There are $R, M > 0$ such that $\forall\ |u| \geq R$

$$0 < G(x,u) = \int_0^u g(x,s)ds \leq M|g(x,u)|$$

for all $x \in \bar{\Omega}$;

($\bar{g}_3$) There exists $C >> 0$ such that $\forall\ |s| \geq C$

$$sg(x,s) \geq \gamma(s)e^{\alpha_0|s|^{\frac{N}{N-1}}}$$

where $\gamma(s)$ is such that there is $\varepsilon_0 > 0$ with

$$\liminf_{s\to\pm\infty} \frac{\gamma(s)}{e^{\varepsilon_0|s|^{\frac{1}{N-1}}}} > 0$$

for some $\varepsilon_0 > 0$.

The main results of this chapter are:

Theorem 0.0.6. *Suppose* ($\bar{\alpha_0}$), ($\bar{g}_0$) $-$ ($\bar{g}_3$) *hold,* $2 \leq q < N$, $\lambda \in (0, \lambda_1)$, $\mu \in (0, \mu_1]$ *and* $f \in L^\infty(\Omega)$ *is nontrivial. Then there exists* $\eta > 0$ *such that problem* (8) *has two solutions provided that* $\|f\|_\infty \leq \eta$.

The restriction on q is not necessary if we remove the q-linear term from the problem. The last main theorem of this work is as follows.

Theorem 0.0.7. *Suppose* ($\bar{\alpha_0}$),($\bar{g}_0$) $-$ ($\bar{g}_3$) *hold,* $N \geq 2$, $1 \leq q < N$, $\lambda \in (0, \lambda_1)$, $\mu \equiv 0$ *and* $f \in L^\infty(\Omega)$ *is nontrivial. Then there exists* $\eta > 0$ *such that problem* (8) *has two solutions provided that* $\|f\|_\infty \leq \eta$.

Chapter 1

(p, q)-Laplacian equations with critical growth and jumping nonlinearities

1.1 Introduction

In this chapter we will study the problem:

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda |u|^{p-2}u + u_+^{p^*-1} + g(x, u_+) + f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain with a smooth boundary in $\mathbb{R}^N$, $1 < q < p < N$, $\lambda \in (0, \lambda_1)$, $w_+ = \max\{w, 0\}$, $f \in L^\infty(\Omega)$ is non-zero, $g : \Omega \times \mathbb{R} \to \mathbb{R}^+$ is a subcritical function with respect to p^* and p-superlinear. Here we extend the results in [19] to the (p, q)-Laplacian operator. We established the existence of two solutions: one of them negative and the other involving modifications to Talenti's functions. The second solution was derived by imposing stricter conditions on the dimension. Additionally, by introducing a more restrictive hypothesis on the nonlinear term, it became possible to discuss solutions in the lower-dimensional cases.

1.2 Hypotheses and Main Results

Let us begin by assuming that $f \in L^\infty(\Omega)$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}^+$ is continuous such that:

$$g(x, t) = 0, \ \forall \ t \leq 0. \tag{g_1}$$

There exist $p < \theta < p^*$ and $C > 0$ such that $g(x, t) \leq C(1 + t^{\theta-1})$ for all $t > 0$. (g_2)

The hypothesis (g_1) pertaining to $g(x, t)$ aligns with the unidirectional nature of the non-linear component of the problem, while (g_2) illustrates that $g(x, t)$ serves as a subcritical perturbation. This aspect becomes crucial in addressing cases of lower dimensions. In instances where $g \equiv 0$, the methodologies employed to identify a second solution are applicable to higher-dimensional scenarios.

Now, fix $0 < \lambda < \lambda_1$. Then, the associated functional $J : W_0^{1,p}(\Omega) \to \mathbb{R}$ is given by:

$$J(u) = \frac{1}{p}\int_\Omega (|\nabla u|^p - \lambda |u|^p)\,dx + \frac{1}{q}\int_\Omega |\nabla u|^q dx - \frac{1}{p^*}\int_\Omega u_+^{p^*} dx - \int_\Omega G\left(x, u_+\right) dx \\ - \int_\Omega f(x) u dx$$

where

$$G(x, s) = \int_0^s g(x, t) dt.$$

From these first conditions, we obtain that the associated functional J to problem (1.1) is of class C^1 with derivative given by

$$J'(u)v = \int_\Omega (|\nabla u|^{p-2}\nabla u \nabla v - \lambda |u|^{p-2} uv) dx + \int_\Omega |\nabla u|^{q-2}\nabla u \nabla v dx - \int_\Omega u_+^{p^*-1} v dx \\ - \int_\Omega g(x, u_+) v dx - \int_\Omega f(x) v dx$$

for all u and v in $W_0^{1,p}(\Omega)$. By definition, weak solutions of the main problem are exactly the critical points of this functional.

In $W_0^{1,p}(\Omega)$, $1 < p < \infty$ we work with its usual norm

$$\|u\|_{W_0^{1,p}(\Omega)} = \left(\int_\Omega |\nabla u|^p dx\right)^{\frac{1}{p}}.$$

Given that our main framework lies in $W_0^{1,p}(\Omega)$, we simplify the notation by letting $\|\cdot\|_{W_0^{1,p}(\Omega)}$ be represented as $\|\cdot\|$. Furthermore, we denote the standard norm in the $L^p(\Omega)$ spaces as $\|\cdot\|_p$. Furthermore, since $\lambda < \lambda_1$, we introduce

$$\|u\|_\lambda = \left(\|u\|^p - \lambda \|u\|_p^p\right)^{\frac{1}{p}},$$

which establishes a norm equivalent to $\|\cdot\|$ within $W_0^{1,p}(\Omega)$.

The objective of this chapter is to identify two solutions to the problem delineated in (1.1). Initially, a negative solution is sought, which requires $f(x)$ to meet specific criteria. The formulation of this negative solution is as follows:

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda |u|^{p-2} u + f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where the functional associated with this problem is expressed as

$$I(u) = \frac{1}{p}\int_\Omega (|\nabla u|^p - \lambda|u|^p)\,dx + \frac{1}{q}\int_\Omega |\nabla u|^q dx - \int_\Omega f u dx.$$

It is important to note that if a negative solution exists for (1.2), then it similarly constitutes a negative solution to our primary problem. Consequently, the forthcoming results are oriented towards identifying such a negative solution to (1.2).

Lemma 1.2.1. *If $f \in L^{p'}(\Omega)$, then the functional I is coercive and sequentially lower semicontinuous (s.c.i) in the weak sense. Consequently, there exists a global minimum $w \in W_0^{1,p}(\Omega)$ for I.*

Proof. By Hölder's inequality and the equivalence of norms in $W_0^{1,p}(\Omega)$, we have,

$$\begin{aligned} I(u) &= \frac{1}{p}\|u\|_\lambda^p + \frac{1}{q}\|u\|^q_{W_0^{1,q}(\Omega)} - \int_\Omega f(x) u dx \\ &\geq \frac{1}{p}\|u\|_\lambda^p - \|f\|_{p'}\|u\|_p \\ &\geq \frac{1}{p}\|u\|_\lambda^p - C\|u\| \to +\infty, \end{aligned}$$

when $\|u\| \to +\infty$.

Let $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$. So,

$$\|u\|_\lambda^p \leq \liminf \|u_n\|_\lambda^p$$

and

$$\|u\|^q_{W_0^{1,q}(\Omega)} \leq \liminf \|u_n\|^q_{W_0^{1,q}(\Omega)}.$$

Then,

$$\begin{aligned} I(u) &= \frac{1}{p}\|u\|_\lambda^p + \frac{1}{q}\|u\|^q_{W_0^{1,q}(\Omega)} - \int_\Omega f(x) u dx \\ &\leq \liminf \left(\frac{1}{p}\|u_n\|_\lambda^p + \frac{1}{q}\|u_n\|^q_{W_0^{1,q}(\Omega)} - \int_\Omega f(x) u_n dx\right) \\ &= \liminf I(u_n), \end{aligned}$$

which proves that I is sequentially lower semicontinuous in the weak topology. It follows by the Direct Method of the Calculus of Variations that there exists a global minimum for the functional I, denoted throughout this work by $w \in W_0^{1,p}(\Omega)$. □

Lemma 1.2.2. *If f is nontrivial, then the global minimum $w \in W_0^{1,p}(\Omega)$ of I is nontrivial. Moreover, if $f \leq 0$, then $w \leq 0$.*

Proof. The proof is obvious, since $f \neq 0$ implies that $u = 0$ cannot be a critical point to I.

Moreover, to prove that $w \leq 0$, use $v = w^+$ as a test function in the equation $I'(w)v = 0$, which is true for all $v \in W_0^{1,p}(\Omega)$. □

Note that w is also a critical point for the functional J, what we will seek throughout this chapter is to show that w is a local minimum of J in the topology of $W_0^{1,p}(\Omega)$ to obtain the geometry of the mountain pass and then find a second solution to the problem.

To obtain $w < 0$, we will need a maximum principle, and for this we will use a more general result that can be found in Pucci [33].

Consider the equation

$$\text{div}(A(|\nabla u|)\nabla u) - \beta(u) \leq 0, \quad u \geq 0,$$

in $\Omega \subset \mathbb{R}^N$, a possibly unbounded domain with $N \geq 2$.

Furthermore, consider

(*i*) $A \in C(0, \infty)$;

(*ii*) $t \to tA(t)$ strictly increasing in $(0, \infty)$ and $tA(t) \to 0$ as $t \to 0$;

(*iii*) β continuous on $[0, \infty)$;

(*iv*) $\beta(0) = 0$ and β is non-decreasing on some interval $[0, \delta)$, where $\delta > 0$.

Let $h(t) = tA(t)$ for $t > 0$ and $h(0) = 0$, and define

$$H(t) = th(t) - \int_0^t h(s)ds, \ t \geq 0.$$

With these conditions, the following result holds.

Proposition 1.2.3. *Suppose*

$$\liminf_{t \to 0} \frac{H(t)}{th(t)} > 0$$

and either $\beta(s) \equiv 0$ *for* $s \in [0, \tau)$, $\tau > 0$, *or*

$$\int_0^\delta \frac{ds}{H^{-1}(B(s))} = \infty$$

where $B(s) = \int_0^s \beta(\xi)d\xi$, $s > 0$. *If* u *is a solution of*

$$div(A(|\nabla u|)\nabla u) - \beta(u) \leq 0, \quad u \geq 0 \ in \ \Omega,$$

with $u(x_0) = 0$ *for some* $x_0 \in \Omega$, *then* $u \equiv 0$ *in* Ω.

Notice that, by taking

$$A(t) = t^{p-2} + t^{q-2}$$

$t > 0$ and $\beta \equiv 0$, assumptions $(i) - (iv)$ are satisfied. Furthermore,

$$h(t) = t^{p-1} + t^{q-1},$$

and $t > 0$. So,

$$H(t) = t^p + t^q - \frac{t^p}{p} - \frac{t^q}{q} = \left(\frac{p-1}{p}\right) t^p + \left(\frac{q-1}{q}\right) t^q,$$

and,

$$\frac{H(t)}{th(t)} = \left(\frac{p-1}{p}\right)(1 + t^{q-p})^{-1} + \left(\frac{q-1}{q}\right)(1 + t^{p-q})^{-1}.$$

Then,

$$\liminf_{t\to 0} \frac{H(t)}{th(t)} = \left(\frac{q-1}{q}\right) > 0.$$

Considering, $\beta \equiv 0$, $u = -w \geq 0$ and $f(x) \leq 0$ nontrivial, we have

$$\begin{aligned} \operatorname{div}(A(|\nabla(-w)|)\nabla(-w)) &= -\operatorname{div}(A(|\nabla(w)|)\nabla(w)) \\ &= \lambda |w|^{p-2} w + f(x) \\ &\leq 0 \end{aligned}$$

and it follows from the above Proposition that $w < 0$ in Ω. We will now present some crucial results to demonstrate that the global minimum point w of the functional I within the topology of $W_0^{1,p}(\Omega)$ is also a local minimum point of the functional J within the same topology. Propositions 1.2.4 and 1.2.5 are documented in [35], while the first is an adaptation of [4, Theorem 2]. Furthermore, the Proposition 1.2.6 is sourced from [36], and Proposition 1.2.7 is referenced in [21].

Proposition 1.2.4. *Let $u \in W_0^{1,p}(\Omega)$ be a solution of*

$$\begin{cases} -\Delta_p u - \mu \Delta_q u = g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mu > 0$ is a real parameter and $g(x,s)$ is continuous in $s \in \mathbb{R}$ for almost every $x \in \Omega$ and for each $s \in \mathbb{R}$, $g(x,s)$ is Lebesgue measurable with respect to $x \in \Omega$. If

$$|g(x,s)| \leq M(1 + |s|^r),$$

for some $1 \leq r \leq p^*$, where $p^* = Np/(N-p)$, then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$, and $\|u\|_{C_0^{1,\alpha}(\Omega)} \leq C = C(M,p,q,\mu,\Omega)$.

Proposition 1.2.5. *For all $g \in L^\infty(\Omega)$ there exists a unique solution $u \in W_0^{1,p}(\Omega)$ of*

$$\begin{cases} -\Delta_p u - \mu\Delta_q u = g & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In addition, the mapping $Q : L^\infty(\Omega) \to C_0^1(\overline{\Omega})$ defined as $Q(g) = u$ is continuous and compact.

For the next proposition, consider $H(x,\xi) = \int_0^\xi h(x,s)ds$ such that $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with growth

$$|h(x,t)| \leq a(x)(1 + |t|^{s-1}) \quad \text{in} \quad \Omega \times \mathbb{R},$$

where $a \in L^\infty(\mathbb{R})$ and $1 < s \leq p^*$. Consider $\phi : W_0^{1,p}(\Omega) \to \mathbb{R}$ as the C^1 functional defined by

$$\phi(u) := \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega H(x,u(x))dx, \quad u \in W_0^{1,p}(\Omega).$$

Proposition 1.2.6. *If $u_0 \in W_0^{1,p}(\Omega)$ is a local minimum of ϕ in the $C_0^1(\Omega)$ topology, then $u_0 \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$, and u_0 is a local minimum of ϕ in the $W_0^{1,p}(\Omega)$ topology.*

In our case, taking $h(x,t) = \left(\frac{p^*-1}{p^*}\right) t_+^{p^*-2} + g(x,t_+) + 2f(x)$, the result above holds.

Proposition 1.2.7. *Let Ω be a bounded domain with a smooth boundary. If $u \in C^1(\overline{\Omega}) \cap W_0^{1,p}(\Omega)$ and*

$$\begin{cases} -\Delta_p u - \Delta_q u \geq 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

then $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$, where ν is the outward unit normal vector to $\partial\Omega$.

Remark 1. Note that if $u = -w$, then

$$\begin{aligned} -\Delta_p(-w) - \Delta_q(-w) &= -div(A(|\nabla(-w)|)\nabla(-w)) \\ &= div(A(|\nabla(w)|))\nabla(w) \\ &= -\{\lambda|w|^{p-2}w + f(x)\} \\ &\geq 0. \end{aligned}$$

Therefore, the result holds for $u = -w$, where $A(t) = t^{p-2} + t^{q-2}$.

We now have sufficient conditions to state and prove the following result:

Proposition 1.2.8. *Suppose $f \in L^\infty(\Omega)$ is such that the global minimum of I, denoted by w, is negative in Ω. Then, w is a local minimum of J and consequently a negative solution to problem (1.1).*

Proof. We have that $w \in W_0^{1,p}(\Omega)$ is a minimum of I in the $C_0^1(\overline{\Omega})$ topology. By Proposition 1.2.4, $w \in C_0^{1,\alpha}(\overline{\Omega})$. Moreover, by Proposition 1.2.7, there exists $r > 0$ such that $u < 0$ for all $u \in C_0^1(\overline{\Omega})$ with $\|u - w\|_{C^1(\Omega)} < r$. Hence,

$$J(w) = I(w) \leq I(u) = J(u) \quad \text{where} \quad \|u - w\|_{C^1(\Omega)} < r.$$

Thus, w is also a local minimum point of J in the $C_0^1(\Omega)$ topology, and by Proposition 1.2.6, w is a local minimum of J in the $W_0^{1,p}(\Omega)$ topology. □

To conclude this section, let us outline the main theorems of this chapter. The subsequent sections will be dedicated to proving these theorems.

Theorem 1.2.9. *Suppose that* (g_1) *and* (g_2) *hold and that* $f \leq 0$ *is nontrivial. Then, for any* $p > 1$*, there exist two solutions to problem* (1.1) *provided that,*

$$N > \begin{cases} \max\{p^2 + p, \frac{p^2}{p-1}\}, & \text{if } 1 < p < 2 \\ (p-1)p^2 + p, & \text{if } p \geq 2. \end{cases} \tag{1.3}$$

and

$$1 < q < \frac{(N-p)p}{N}.$$

Remark 2. It should be noted that for $p = q = 2$, we encounter the restriction $N > 6$, similar to the restriction imposed on the Laplacian in [5]. Furthermore, when $p = q$, the (p, q)-Laplacian simplifies to the p-Laplacian, as addressed in [19]. The hypotheses here are the same of those presented in the aforementioned paper.

It is important to emphasize that the natural constraints should be $N > p$ and $q < p$. Therefore, in studying the problem in lower dimensions, we follow the concepts presented in [19, 27], where a p-superlinear growth condition is introduced to diminish the *minimax* levels of the functional J. This adjustment enables us to fill the gap in dimension N highlighted in the preceding theorem. What is novel here is that the exponent q now plays a significant role, rendering the problem distinct from its p-Laplacian counterpart. Consider the following additional hypothesis:

There exists $\sigma > 0$, such that $K < \sigma < p^*$ and

$$g(x, t) \geq Ct^{\sigma - 1} \text{ for all } t > 0,$$

where $K = K(N,p,q)$ is given by

$$K(N,p,q) = \begin{cases} \max\left\{p^* - \frac{p}{N-p}, \frac{Nq}{N-p}\right\}, & \text{if } 1 < p \le \frac{2N}{N+1} \\ \max\left\{p^* - \frac{p}{N-p}, p^* - 1, \frac{Nq}{N-p}\right\}, & \text{if } \frac{2N}{N+1} < p < 2 \\ \max\left\{p^* - \frac{p}{N-p}, p^* - \frac{1}{p-1}, \frac{Nq}{N-p}\right\}, & \text{if } 2 \le p. \end{cases} \qquad (g_3)$$

Theorem 1.2.10. *Suppose g satisfies (g_1), (g_2) and (g_3). Assume that $f \in L^\infty(\Omega)$ and $f \le 0$ in Ω. Then, there exist two solutions to the problem (1.1) for every $1 < q < p < N$.*

Remark 3. Condition $f \le 0$ is imposed to ensure that there exists a negative solution which is a local minimum for the functional, so the second solution can be found but it is not a necessary condition. What we only need is that there exists a negative local minimum to the associated functional, and this is possible even if f changes sign in some cases (for $p = 2$ it is sufficient to ensure that $f(x) = h(x) + t\phi_1(x)$ for sufficiently large t, $h \in L^r(\Omega), r > N/2$ and ϕ_1 being a positive eigenfunction associated with λ_1).

We will dedicate the next section to revisiting some preliminary results that have been discussed in previous works.

1.3 Preliminaries

The objective of this section is to present certain results concerning norm estimates for truncations of the Talenti functions, which can be found in [19] and for this reason the proofs will be omitted. These inequalities are related to the norms of L^s of these functions and are included here for the sake of independent understanding.

Consider the best embedding constant from $W^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$ given by

$$S = \inf_{u \in \mathfrak{D}^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|^p}{\|u\|_{p^*}^p}.$$

For $\varepsilon > 0$, we obtain a minimizing function for S given by

$$U_\varepsilon(x) = \frac{C\varepsilon^{\frac{N-p}{p(p-1)}}}{\left[\varepsilon^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}}\right]^{\frac{(N-p)}{p}}}$$

where C is chosen such that

$$-\Delta_p U_\varepsilon = U_\varepsilon^{p^*-1} \text{ in } \mathbb{R}^N.$$

Then we have,

$$\int_{\mathbb{R}^N} |\nabla U_\varepsilon|^p dx = \int_{\mathbb{R}^N} |U_\varepsilon|^{p^*} dx.$$

Fixing $r \in (0,1)$ so that $B(0,2r) \subset \Omega$, let $\phi \in C_c^\infty(\mathbb{R}^N;[0,1])$ satisfy $\phi(x) = 1$ for all $x \in B(0,r)$ and $\phi(x) = 0$ for all $x \in \mathbb{R}^N \backslash B(0,2r)$. Now, define

$$u_\varepsilon = \phi U_\varepsilon. \tag{1.4}$$

These are the truncations of the Talenti functions that require estimation. They will serve as directions for the functional to ensure that their associated *minimax* levels remain below a certain threshold.

Lemma 1.3.1. *Taking into account* $0 < \varepsilon < r$ *and* $1 < p < N$*, the following holds:*

$$c\varepsilon^p \le \int_\Omega |u_\varepsilon|^p \, dx \le \begin{cases} C\varepsilon^p + O\left(\varepsilon^{\frac{(N-p)}{(p-1)}}\right) & \text{if } p^2 < N, \\ C\varepsilon^p \log\left(\frac{1}{\varepsilon}\right) & \text{if } p^2 = N, \\ C\varepsilon^{\frac{(N-p)}{(p-1)}} & \text{if } p^2 > N, \end{cases}$$

$$\int_\Omega |u_\varepsilon|^{p-1} \, dx \le C\varepsilon^{\frac{(N-p)}{p}},$$

$$\int_\Omega |u_\varepsilon|^{p^*-1} \, dx \le C\varepsilon^{\frac{(N-p)}{p}},$$

$$\int_\Omega |u_\varepsilon| \, dx \le \begin{cases} C\varepsilon^{\frac{(N-p)}{p(p-1)}} & \text{if } p > \frac{2N}{N+1}, \\ C\varepsilon^{N-\frac{(N-p)}{p}} \log(\frac{1}{\varepsilon}) & \text{if } p = \frac{2N}{N+1}, \\ C\varepsilon^{N-\frac{(N-p)}{p}} & \text{if } 1 < p < \frac{2N}{N+1}, \end{cases}$$

$$c\varepsilon^{N-\frac{(N-p)s}{p}} \le \int_\Omega |u_\varepsilon|^s \, dx \le C\varepsilon^{N-\frac{(N-p)s}{p}}, \text{ if } p^* - \frac{N}{N-p} < s < p^*.$$

For $1 < \gamma < p$

$$\|\nabla u_\varepsilon\|_\gamma^\gamma \le \begin{cases} C\varepsilon^\beta & \text{if } \gamma \ne \frac{N(p-1)}{N-1} \\ C\varepsilon^\beta \log(\frac{1}{\varepsilon}) & \text{if } \gamma = \frac{N(p-1)}{N-1} \end{cases} \tag{1.5}$$

where,

$$\beta = \beta(\gamma) = \begin{cases} \frac{N-p}{p(p-1)}\gamma & \text{if } 1 < \gamma \le \frac{N(p-1)}{N-1} \\ N - \frac{N}{p}\gamma & \text{if } \frac{N(p-1)}{N-1} < \gamma < p. \end{cases} \tag{1.6}$$

Remark 4. In the case where $\gamma = \frac{N(p-1)}{(N-1)}$, we have that:

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^\gamma dx &= \int_{\mathbb{R}^N} |\nabla \phi U_\varepsilon + \phi \nabla U_\varepsilon|^\gamma dx \\ &\leq C \int_{B(0,2r)} |\nabla U_\varepsilon|^\gamma dx + C \int_{B(0,2r)\setminus B(0,r)} |U_\varepsilon|^\gamma dx \\ &\leq C\varepsilon^{(N-p)\gamma/p(p-1)} \int_{B(0,2r)} \frac{|x|^{[1+(2-p)/(p-1)]\gamma}}{[\varepsilon^{p(p-1)} + |x|^{p/(p-1)}]^{N\gamma/p}} dx \\ &+ C\varepsilon^{(N-p)\gamma/p(p-1)} \int_{B(0,2r)\setminus B(0,r)} \frac{1}{[\varepsilon^{p(p-1)} + |x|^{p/(p-1)}]^{(N-p)\gamma/p}} dx.\end{aligned}$$

Since the second integral is uniformly bounded for $\varepsilon > 0$, we will analyze the first integral.

$$\begin{aligned}\int_{B(0,2r)} \frac{|x|^{[1+(2-p)/(p-1)]\gamma}}{[\varepsilon^{p(p-1)} + |x|^{p/(p-1)}]^{N\gamma/p}} dx &\leq \varepsilon^{-N\gamma/(p-1)} \int_{B(0,2r/\varepsilon)} \frac{\varepsilon^N |\varepsilon y|^{[1/(p-1)]\gamma}}{[1 + |y|^{p/(p-1)}]^{N\gamma/p}} dy \\ &\leq \varepsilon^{N-(N-1)\gamma/(p-1)} \\ &\times \left(|B_1| + \int_{B(0,2r/\varepsilon)\setminus B(0,1)} |y|^{-\gamma(N-1)/(p-1)} dy \right) \\ &\leq C \log\left(\frac{1}{\varepsilon}\right).\end{aligned}$$

It follows that

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^\gamma dx &\leq C\varepsilon^{(N-p)\gamma/p(p-1)} \int_{B(0,2r)} \frac{|x|^{[1+(2-p)/(p-1)]\gamma}}{[\varepsilon^{p(p-1)} + |x|^{p/(p-1)}]^{N\gamma/p}} \\ &+ C\varepsilon^{(N-p)\gamma/p(p-1)} \int_{B(0,2r)\setminus B(0,r)} \frac{1}{[\varepsilon^{p(p-1)} + |x|^{p/(p-1)}]^{(N-p)\gamma/p}} \\ &\leq C\varepsilon^{N(N-p)/p(N-1)} \log\left(\frac{1}{\varepsilon}\right).\end{aligned}$$

Remark 5. Note that if $\beta = \beta(\gamma)$ is as in (1.6) and $C\varepsilon^\beta < C\varepsilon^p$ for $\gamma < \frac{N(p-1)}{N-1}$, then $C\varepsilon^\beta \log\left(\frac{1}{\varepsilon}\right) < C\varepsilon^p$ for $\gamma = \frac{N(p-1)}{N-1}$. In other words, the term $\log\left(\frac{1}{\varepsilon}\right)$ does not hinder because $\beta > p$ implies that,

$$-\varepsilon^p + \varepsilon^\beta \log\left(\frac{1}{\varepsilon}\right) = \varepsilon^p \left(-1 + \varepsilon^{\beta-p} \log\left(\frac{1}{\varepsilon}\right)\right)$$

and since the exponent $\beta - p > 0$, we have that the term within parentheses is negative for $\varepsilon > 0$ sufficiently small.

Lemma 1.3.2. *Consider $0 < \varepsilon < r$ and $1 < p < N$. Then,*

$$\|\nabla u_\varepsilon\|_p^p = S^{\frac{N}{p}} + O\left(\varepsilon^{\frac{(N-p)}{(p-1)}}\right),$$

$$\|u_\varepsilon\|_{p^*}^{p^*} = S^{\frac{N}{p}} + O\left(\varepsilon^{\frac{N}{(p-1)}}\right).$$

Lemma 1.3.3. *Consider* $1 < p < N$. *Given* $p^* - \frac{N}{N-p} < s \le p^*$ *and* $K > 0$, *we have*

$$\int_{\Omega_\varepsilon} |u_\varepsilon|^s dx = \int_{\Omega} |u_\varepsilon|^s dx - O\left(\varepsilon^{\frac{N}{p}}\right),$$

with $\varepsilon \to 0$, $\Omega_\varepsilon = \{x \in \Omega : u_\varepsilon(x) \ge K\}$.

The next result can be found in [15, Lemma A4], and it will be used in the next section.

Proposition 1.3.4. *Consider* $1 < p < N$.

a) *If* $p \ge 3$, *then there exists a constant* $C > 0$ *such that*

$$\left(1 + t^2 + 2tcos(\alpha)\right)^{\frac{p}{2}} \le 1 + t^p + ptcos(\alpha) + C(t^2 + t^{p-1}),$$

for all $t > 0$ *uniformly in* $\alpha \in [0, 2\pi]$.

b) *If* $p \in [2, 3)$ *and* $\gamma \in [p-1, 2]$, *then there exists* $C > 0$ *such that*

$$\left(1 + t^2 + 2t\cos(\alpha)\right)^{\frac{p}{2}} \le 1 + t^p + pt\cos(\alpha) + Ct^\gamma,$$

for all $t > 0$ *uniformly in* $\alpha \in [0, 2\pi]$.

c) *If* $p \in (1, 2)$ *and* $\gamma \in (1, p)$, *then there exists* $C > 0$ *such that*

$$\left(1 + t^2 + 2t\cos(\alpha)\right)^{\frac{p}{2}} \le 1 + t^p + pt\cos(\alpha) + Ct^\gamma,$$

for all $t > 0$ *uniformly in* $\alpha \in [0, 2\pi]$.

The next results given in the next two remarks are facts extracted directly from the Lemma 1.3.1. They will be useful in the proof of the lemmas immediately below them.

Remark 6. Let $1 < \gamma < p$ and $\beta = \beta(\gamma)$ as in (1.6), then,

$$\beta = \frac{N-p}{p(p-1)}\gamma > p$$

if and only if,

$$\gamma > \frac{p^2(p-1)}{N-p},$$

where $1 < \gamma \le \frac{N(p-1)}{N-1}$. Moreover,

$$\beta = N - \frac{N}{p}\gamma > p$$

if and only if,

$$\gamma < \frac{(N-p)p}{N},$$

where $\frac{N(p-1)}{N-1} < \gamma < p$.

Remark 7.

$$\frac{N(p-1)}{N-1} > 1$$

if, and only if,

$$p > 2 - \frac{1}{N}.$$

The following lemma will be of great importance for the next results.

Lemma 1.3.5. *According to hypothesis (1.3), it is possible to choose $1 < \gamma = \gamma_p < p$ in (1.5) such that $\beta_p = \beta(\gamma_p)$ as in (1.6) satisfies*

$$\beta_p > p$$

Proof. Under the hypotheses of Theorem 1.2.9 we must have $N > p^2$. Recalling (1.6) and by Proposition 1.3.4, we can make the following choices:

$$\begin{cases} \gamma_p \in (1,p) \text{ and } \beta_p = N - \frac{N}{p}\gamma_p & \text{if } 1 < p \leq 2 - \frac{1}{N}, \\ \gamma_p \in \left(1, \frac{N(p-1)}{N-1}\right) \text{ and } \beta_p = \frac{N-p}{p(p-1)}\gamma_p & \text{if } 2 - \frac{1}{N} < p < 2, \\ p-1 < \gamma_p < \min\left(2, \frac{N(p-1)}{N-1}\right) \text{ and } \beta_p = \frac{N-p}{p(p-1)}\gamma_p & \text{if } 2 \leq p < 3, \\ \gamma_p = p-1 \text{ and } \beta_p = \frac{N-p}{p(p-1)}\gamma_p = \frac{N-p}{p} & \text{if } p \geq 3, \end{cases} \tag{1.7}$$

furthermore, straightforward calculations and by Remarks 6 and 7, show that it is possible to choose $\gamma_p \in (1,p)$ such that

$$\begin{cases} \beta_p > p & \text{if } 1 < p \leq \frac{2N}{N+1}, \text{ and } N > \frac{p^2}{(p-1)}, \\ \beta_p > \frac{N-p}{p} & \text{if } \frac{2N}{N+1} < p < 3, \\ \beta_p > \frac{N-p}{p(p-1)} & \text{if } p \geq 3. \end{cases}$$

In fact, if $1 < p \leq \frac{2N}{N+1}$, so by Remarks 6 and 7 we can choose γ such that

$$\frac{N(p-1)}{N-1} \leq 1 < \gamma < \frac{(N-p)p}{N} < p,$$

because

$$\frac{(N-p)p}{N} > 1,$$

if and only if

$$N > \frac{p^2}{p-1}.$$

Then,

$$\beta_p = N - \frac{N}{p}\gamma > p.$$

However, if $\frac{2N}{N+1} < p < 2 - \frac{1}{N}$, then

$$\beta_p = N - \frac{N}{p}\gamma_p > \frac{N-p}{p}$$

if, and only if,

$$p - 1 + \frac{p}{N} > \gamma_p$$

which happens because,

$$p - 1 + \frac{p}{N} > 1,$$

if and only if,

$$p + \frac{p}{N} > 2,$$

but

$$p + \frac{p}{N} > \frac{2N}{N+1} + \frac{2N}{N(N+1)} = 2,$$

so, consider, $\gamma_p \in \left(1, p - 1 + \frac{p}{N}\right)$ and we will get the result.
If, $2 - \frac{1}{N} < p \leq 2$, so,

$$\frac{N(p-1)}{N-1} > 1$$

and

$$\beta_p = \frac{N-p}{p(p-1)}\gamma_p > \frac{N-p}{p}$$

if, and only if, $\gamma_p > p - 1$, which happens because, $p \leq 2$ and $\gamma_p > 1$ so,

$$\beta_p > \frac{N-p}{p} > \frac{p^2 + p - p}{p} = p.$$

In analogous way, if $2 < p < 3$, since

$$1 < p - 1 < \frac{N(p-1)}{N-1}$$

we have

$$\beta_p > \frac{N-p}{p} > \frac{p(p^2 - p + 1) - p}{p} = p(p-1) > p,$$

where $\gamma_p \in \left(p - 1, \frac{N(p-1)}{N-1}\right)$. Finally, if $p \geq 3$, since $\gamma_p > 1$, we have

$$\beta_p > \frac{N-p}{p(p-1)} > \frac{p(p^2 - p + 1) - p}{p(p-1)} = p$$

This finishes the proof. □

As evident from this last lemma and the subsequent one, significant effort is being invested in making precise selections of constants γ within certain estimates to ensure that their corresponding β exceeds the values of p. These selections will be pivotal in the estimates, facilitating the choice of ε to achieve the necessary boundedness of the *minimax* level. Consequently, we now must examine the constraints on q to ensure that we reach $\beta_q = \beta(q) > p$ and $\beta_{\gamma_q} > p$.

Remark 8. Under the hypothesis (1.3), we have

$$1 < \frac{(N-p)p}{N},$$

and

$$\frac{(N-p)p}{N} > \frac{N(p-1)}{N-1}.$$

Indeed, the first inequality is obvious if $1 < p < 2$, due to the fact that $N > \frac{p^2}{p-1}$. Furthermore, note that,

$$(p-1)p^2 + p \geq \frac{p^2}{(p-1)}$$

if, and only if,

$$(p-1)^2 p > 1.$$

This is obvious if $p \geq 2$, therefore

$$N > \frac{p^2}{p-1}$$

for all $p \geq 2$ also, which leads to $1 < \frac{(N-p)p}{N}$ again. To see the second inequality of this remark, notice that $N > p^2 + p$ for all $p > 1$ and this implies that

$$N(p^2 + p) - p^2 < N^2 - p^2 < N^2,$$

which is equivalent to,

$$N^2 - N(p^2 + p) + N^2 p - N^2 p + p^2 > 0,$$

thus,

$$Np(N-1) - p^2(N-1) - N^2(p-1) > 0,$$

from which we have,

$$p(N-1)(N-p) > N^2(p-1),$$

therefore,

$$\frac{(N-p)p}{N} > \frac{N(p-1)}{N-1}.$$

Note that from the above observation it makes sense to have $1 < q < \frac{(N-p)p}{N}$ and $\frac{N(p-1)}{N-1} < q <$

$\frac{(N-p)p}{N}$, which are important cases that we need to consider in the following developments.

Lemma 1.3.6. *Under hypothesis (1.3), assume that* $1 < q < \frac{(N-p)p}{N}$*. If* $\beta_q = \beta(q)$ *is defined as in (1.6) with* $\gamma = q$*, then* $\beta_q > p$*. Furthermore, there exists* $\gamma_q \in (1, q)$ *for which* $\beta_{\gamma_q} > p$ *is also satisfied. In this case,* β_{γ_q} *is specified in (1.6) with* $\gamma = \gamma_q$*. Moreover,* γ_q *can be chosen in* $[q-1, 2]$ *if* $q \in [2, 3)$*.*

Proof. Case 1: If $1 < p \leq 2 - \frac{1}{N}$, then $p \in (1, 2)$ and $q \in (1, p) \subset (1, 2)$. Thus, in this case, let us fix $\gamma_q \in (1, q)$. By Remark 7, we have $\frac{N(p-1)}{N-1} \leq 1$. Therefore, $\beta_{\gamma_q} = N - \frac{N}{p}\gamma_q$. It follows that $\beta_{\gamma_q} > p$ if and only if,

$$\gamma_q < \frac{(N-p)p}{N}.$$

Moreover, $\beta_q = N - \frac{N}{p}q > p$ if and only if,

$$q < \frac{(N-p)p}{N}.$$

Case 2: If $2 - \frac{1}{N} < p \leq 2$, once again we have $q \in (1, p) \subset (1, 2)$, and in this case consider $\gamma_q \in (1, q)$. Furthermore, $1 < \frac{N(p-1)}{N-1} < p \leq 2$. Let's assume $1 < q \leq \frac{N(p-1)}{N-1}$, then

$$\beta_q = \frac{N-p}{p(p-1)}q > p$$

if and only if $q > \frac{p^2(p-1)}{N-p}$. Moreover, $\gamma_q < q \leq \frac{N(p-1)}{N-1}$ gives us $\beta_{\gamma_q} = \frac{N-p}{p(p-1)}\gamma_q > p$ if and only if $\gamma_q > \frac{p^2(p-1)}{N-p}$.

By hypothesis, $N > p^2 + p$ if $1 < p < 2$, and $N > p(p^2 - p + 1)$ if $p \geq 2$. Note that $p(p^2 - p + 1) \geq p^2 + p$ if and only if $p \geq 2$. Therefore, in the case $p = 2$, we also have $N > p^2 + p$. Hence,

$$\frac{p^2}{N-p} < \frac{p^2}{p^2 + p - p} = 1.$$

Since $p \in \left(2 - \frac{1}{N}, 2\right]$, we have $p - 1 \leq 1$, and

$$\frac{p^2(p-1)}{N-p} < 1.$$

Therefore, if $q \leq \frac{N(p-1)}{N-1}$, then $\beta_q, \beta_{\gamma_q} > p$ for $\gamma_q \in (1, q)$ and $q \in \left(1, \frac{N(p-1)}{N-1}\right]$.

In the case where $q > \frac{N(p-1)}{N-1}$, we have $\beta_q = N - \frac{N}{p}q > p$ if and only if $q < \frac{(N-p)p}{N}$, that is,

$\beta_q > p$ for $\frac{N(p-1)}{N-1} < q < \frac{(N-p)p}{N}$. Since $\gamma_q \in (1, q)$, we only need to consider $\gamma_q \in (\frac{N(p-1)}{N-1}, q)$, obtaining

$$\begin{aligned}\beta_{\gamma_q} &= N - \frac{N}{p}\gamma_q \\ &> N - \frac{N}{p}\left[\frac{(N-p)p}{N}\right] \\ &= p.\end{aligned}$$

Case 3: If $2 < p < 3 - \frac{2}{N}$, then $1 < \frac{N(p-1)}{N-1} < 2$ because

$$\frac{N(p-1)}{N-1} < 2$$

if and only if

$$p < 3 - \frac{2}{N}.$$

Let us suppose $N > \frac{p^2}{p-2}$, so we have that $\frac{(N-p)p}{N} > 2$. Now assume that $1 < q \leq \frac{N(p-1)}{N-1}$. Since $1 < \gamma_q < \frac{N(p-1)}{N-1}$, it follows from the definition of β_{γ_q} that,

$$\beta_{\gamma_q} = \frac{(N-p)}{p(p-1)}\gamma_q.$$

Moreover, $\beta_{\gamma_q} > p$, if and only if,

$$\gamma_q > \frac{p^2(p-1)}{N-p},$$

but,

$$\frac{p^2(p-1)}{N-p} < 1$$

because $p \geq 2$. Therefore,

$$\beta_{\gamma_q} > p.$$

By the definition of β_q we get,

$$\beta_q = \frac{(N-p)}{p(p-1)}q > \beta_{\gamma_q} > p.$$

Now consider $\frac{N(p-1)}{N-1} < q < 2$, then $\beta_q = N - \frac{N}{p}q > p$ if and only if $q < \frac{(N-p)p}{N}$, thus, $\beta_q > p$ for

$$\frac{N(p-1)}{N-1} < q < 2.$$

Simply take $1 < \gamma_q < \frac{N(p-1)}{N-1}$ to obtain $\beta_{\gamma_q} = \frac{N-p}{p(p-1)} > p$. Now let us assume $q \in [2, p) \subset [2, 3)$. We can simply consider $\gamma_q = q - 1 < \frac{N(p-1)}{N-1}$ to obtain $\beta_{\gamma_q} > p$. For β_q, we have $\frac{N(p-1)}{N-1} < 2 \leq q$ and $\beta_q = N - \frac{N}{p}q > p$, with $q \in \left[2, \frac{(N-p)p}{N}\right)$.

In the case where $N \leq \frac{p^2}{p-2}$, we have $\frac{(N-p)p}{N} \leq 2$, thus for $1 < q < \frac{(N-p)p}{N}$ we obtain $\beta_q, \beta_{\gamma_q} > p$. Therefore, $\beta_q, \beta_{\gamma_q} > p$ where $1 < q < \frac{(N-p)p}{N}$.

Case 4: If $3 - \frac{2}{N} \leq p < 3$, then $\frac{N(p-1)}{N-1} \geq 2$ and $\beta_{\gamma_q} = \frac{(N-p)}{p(p-1)}\gamma_q > p$ and $\beta_q = \frac{(N-p)}{p(p-1)}q > p$ where $q \in (1, 2)$.

If $q \in [2, p)$, then $\beta_q > p$ for $2 \leq q < \frac{N(p-1)}{N-1}$ and for $\frac{N(p-1)}{N-1} \leq q < \frac{(N-p)p}{N}$, and certainly, $\beta_{\gamma_q} > p$.

Case 5: If $p = 3$, we know that $\frac{N(p-1)}{N-1} > 2$. Let's assume $q \in (1, 2)$, then $\gamma_q < q < \frac{N(p-1)}{N-1}$, obtaining $\beta_{\gamma_q} = \frac{N-p}{p(p-1)}\gamma_q > p$ and $\beta_q = \frac{N-p}{p(p-1)}q > p$.

If $q \in [2, p)$, then $q \in [2, 3)$. Take $\gamma_q = q - 1 < \frac{N(p-1)}{N-1}$, consequently, $\beta_{\gamma_q} > p$ and $\beta_q > p$ for $2 \leq q < \frac{N(p-1)}{N-1}$ or $\frac{N(p-1)}{N-1} \leq q < \frac{(N-p)p}{N}$.

Case 6: Finally, if $p > 3$, then $2 < \frac{N(p-1)}{N-1}$, so if $q \in (1, 2)$, we will have $\beta_{\gamma_q}, \beta_q > p$ with $\gamma_q \in (1, q)$. Furthermore, for $N > \frac{p^2}{p-3}$, we have $3 < \frac{(N-p)p}{N}$. Consider $q \in [2, 3)$ and $\gamma_q = q - 1$, it follows that $\beta_{\gamma_q}, \beta_q > p$. However, if $q \in [3, p)$, take $\gamma_q = q - 1$ to obtain $\beta_{\gamma_q} > p$, similar to what was obtained in the case $q \in [2, 3)$, we have $\beta_q > p$ for $3 \leq q < \frac{(N-p)p}{N}$.

If $N \leq \frac{p^2}{p-3}$, we have $2 < \frac{N(p-1)}{N-1} < \frac{(N-p)p}{N} \leq 3$. Consider $q \in [2, 3)$, then $\beta_{\gamma_q} > p$ with $\gamma_q = q-1$, and $\beta_q > p$ where $q \in [2, \frac{(N-p)p}{N})$. In any case, $\beta_q > p$ for $1 < q < \frac{(N-p)p}{N}$.

In cases where $q = \frac{N(p-1)}{N-1}$, we use Remark 5. □

1.4 Proof of the Main Theorems

This first proposition serves a dual purpose: first, it provides a mountain-pass geometry, and secondly, it acts as an auxiliary result to demonstrate that the *minimax* levels can be controlled by strategically selecting directions provided by truncations of the Talenti functions.

Proposition 1.4.1. *Consider $1 < q < p < N$, g satisfying (g_1) and (g_2). Then there exist ε_0, t_0, t_1 positive numbers such that*

$$J(w + tu_\varepsilon) < J(w) + \frac{S^{\frac{N}{p}}}{N}$$

for all $t \in (0, t_0) \cup (t_1, \infty)$ and $\varepsilon \in (0, \varepsilon_0)$.

Proof. We know from Lemma 1.3.2 that,

$$\int_\Omega |\nabla u_\varepsilon|^p dx = S^{\frac{N}{p}} + O\left(\varepsilon^{\frac{(N-p)}{(p-1)}}\right).$$

Therefore, there exists $\varepsilon_0 > 0$ such that $\|u_\varepsilon\| \leq C$ for all $\varepsilon \in (0, \varepsilon_0)$. By the continuity of J at w, we have that for any $\sigma > 0$, there exists $\delta > 0$ such that if $\|v - w\| < \delta$, then

$$|J(v) - J(w)| < \sigma.$$

Taking $v = w + tu_\varepsilon$, we obtain that,

$$J(w + tu_\varepsilon) < J(w) + \frac{S^{\frac{N}{p}}}{N}$$

$\forall\, t \in (0, t_0)$ and $\varepsilon \in (0, \varepsilon_0)$. Now, let us prove that there exists $t_1 > 0$ such that $J(w+tu_\varepsilon) < J(w)$ for all $t \in (t_1, \infty)$ and $\varepsilon \in (0, \varepsilon_0)$.

$$\begin{aligned} J(w + tu_\varepsilon) = &\frac{1}{p}\int_\Omega (|\nabla(w + tu_\varepsilon)|^p - \lambda |w + tu_\varepsilon|^p)\, dx + \frac{1}{q}\int_\Omega |\nabla(w + tu_\varepsilon)|^q dx \\ &- \frac{1}{p^*}\int_\Omega (w + tu_\varepsilon)_+^{p^*} dx - \int_\Omega G\left(x, (w + tu_\varepsilon)_+\right) dx - \int_\Omega f(w + tu_\varepsilon) dx. \end{aligned}$$

Since $g \geq 0$, we have

$$\begin{aligned} J(w + tu_\varepsilon) \leq &\frac{1}{p}\int_\Omega (|\nabla(w + tu_\varepsilon)|^p - \lambda |w + tu_\varepsilon|^p)\, dx + \frac{1}{q}\int_\Omega |\nabla(w + tu_\varepsilon)|^q dx \\ &- \frac{1}{p^*}\int_\Omega (w + tu_\varepsilon)_+^{p^*} dx - \int_\Omega f(w + tu_\varepsilon) dx. \end{aligned}$$

Moreover, we have the equality

$$\begin{aligned} |\nabla(w + tu_\varepsilon)|^p &= \left[|\nabla w|^2 + 2t\nabla w \nabla u_\varepsilon + t^2 |\nabla u_\varepsilon|^2\right]^{\frac{p}{2}} \\ &= |\nabla w|^p \left[1 + 2\left(t\frac{|\nabla u_\varepsilon|}{|\nabla w|}\right)\frac{\nabla w}{|\nabla w|}\frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|} + t^2 \frac{|\nabla u_\varepsilon|^2}{|\nabla w|^2}\right]^{\frac{p}{2}}, \end{aligned}$$

valid for all $x \in \Omega$ where $\nabla w(x) \neq 0$. By Proposition 1.3.4, we have:

a) if $p \geq 3$, then

$$\begin{aligned} &\int_\Omega |\nabla(w + tu_\varepsilon)|^p dx \leq \\ &\int_\Omega |\nabla w|^p \left[1 + t^p \frac{|\nabla u_\varepsilon|^p}{|\nabla w|^p} + pt\frac{\nabla w.\nabla u_\varepsilon}{|\nabla w|^2} + C\left(\left(t\frac{|\nabla u_\varepsilon|}{|\nabla w|}\right)^2 + \left(t\frac{|\nabla u_\varepsilon|}{|\nabla w|}\right)^{p-1}\right)\right] dx \end{aligned}$$

i.e,

$$\int_\Omega |\nabla(w + tu_\varepsilon)|^p dx \leq \int_\Omega |\nabla w|^p \left[1 + t^p \frac{|\nabla u_\varepsilon|^p}{|\nabla w|^p} + pt\frac{\nabla w.\nabla u_\varepsilon}{|\nabla w|^2} + C\left(t\frac{|\nabla u_\varepsilon|}{|\nabla w|}\right)^{\gamma_p}\right] dx,$$

where $\gamma_p \in \{2, p-1\}$.

b) If $p \in (1,2)$ or $p \in [2,3)$, then there exists a constant $C > 0$ such that

$$\int_\Omega |\nabla(w+tu_\varepsilon)|^p dx \leq \int_\Omega |\nabla w|^p \left[1 + t^p \frac{|\nabla u_\varepsilon|^p}{|\nabla w|^p} + pt\frac{\nabla w.\nabla u_\varepsilon}{|\nabla w|^2} + C\left(t\frac{|\nabla u_\varepsilon|}{|\nabla w|}\right)^{\gamma_p}\right] dx,$$

with $\gamma_p \in (1,p)$ or $\gamma_p \in [p-1,2]$ respectively.

c) If $q \geq 3$,

$$\int_\Omega |\nabla(w+tu_\varepsilon)|^q dx \leq \int_\Omega |\nabla w|^q \left[1 + t^q \frac{|\nabla u_\varepsilon|^q}{|\nabla w|^q} + qt\frac{\nabla w.\nabla u_\varepsilon}{|\nabla w|^2} + C\left(t\frac{|\nabla u_\varepsilon|}{|\nabla w|}\right)^{\gamma_q}\right] dx,$$

with $\gamma_q \in \{2, q-1\}$.

d) If $\gamma_q \in (1,q)$ and $q \in (1,2)$ or $q \in [2,3)$, then there exists a constant $C > 0$ such that

$$\int_\Omega |\nabla(w+tu_\varepsilon)|^q dx \leq \int_\Omega |\nabla w|^q \left[1 + t^q \frac{|\nabla u_\varepsilon|^q}{|\nabla w|^q} + qt\frac{\nabla w.\nabla u_\varepsilon}{|\nabla w|^2} + C\left(t\frac{|\nabla u_\varepsilon|}{|\nabla w|}\right)^{\gamma_q}\right] dx.$$

It follows from Lemma 1.3.1 that

$$\int_\Omega |\nabla(w+tu_\varepsilon)|^p dx \leq \int_\Omega |\nabla w|^p \left(1 + t^p \frac{|\nabla u_\varepsilon|^p}{|\nabla w|^p} + pt\frac{\nabla w.\nabla u_\varepsilon}{|\nabla w|^2}\right) dx + Ct^{\gamma_p}\varepsilon^{\beta_p},$$

where $\beta_p = \beta(\gamma_p)$ is given in (1.6). In the same way,

$$\int_\Omega |\nabla(w+tu_\varepsilon)|^q dx \leq \int_\Omega |\nabla w|^q \left(1 + t^q \frac{|\nabla u_\varepsilon|^q}{|\nabla w|^q} + qt\frac{\nabla w.\nabla u_\varepsilon}{|\nabla w|^2}\right) dx + Ct^{\gamma_q}\varepsilon^{\beta_{\gamma_q}}.$$

Here $\beta_{\gamma_q} = \beta(\gamma_q)$ is chosen as in (1.6). So,

$$\begin{aligned} J(w+tu_\varepsilon) \leq & \frac{1}{p}\int_\Omega |\nabla w|^p \left(1 + t^p \frac{|\nabla u_\varepsilon|^p}{|\nabla w|^p} + pt\frac{\nabla w.\nabla u_\varepsilon}{|\nabla w|^2}\right) dx \\ & + \frac{1}{q}\int_\Omega |\nabla w|^q \left(1 + t^q \frac{|\nabla u_\varepsilon|^q}{|\nabla w|^q} + qt\frac{\nabla w.\nabla u_\varepsilon}{|\nabla w|^2}\right) dx + Ct^{\gamma_p}\varepsilon^{\beta_p} + Ct^{\gamma_q}\varepsilon^{\beta_{\gamma_q}} \\ & - \frac{\lambda}{p}\int_\Omega |w+tu_\varepsilon|^p dx - \frac{t^{p^*}}{p^*}\int_\Omega \left(\frac{w}{t} + u_\varepsilon\right)_+^{p^*} dx - \int_\Omega f(w+tu_\varepsilon) dx. \end{aligned}$$

Knowing that $0 = J'(w)u_\varepsilon$, we have that

$$t\int_\Omega |\nabla w|^{p-2}\nabla w \nabla u_\varepsilon dx + t\int_\Omega |\nabla w|^{q-2}\nabla w \nabla u_\varepsilon dx = \lambda t\int_\Omega |w|^{p-2} w u_\varepsilon dx + t\int_\Omega f u_\varepsilon dx,$$

and,

$$
\begin{aligned}
J(w+tu_\varepsilon) \leq & \frac{1}{p}\int_\Omega |\nabla w|^p dx + \frac{1}{q}\int_\Omega |\nabla w|^q dx + \frac{t^p}{p}\int_\Omega |\nabla u_\varepsilon|^p dx + \frac{t^q}{q}\int_\Omega |\nabla u_\varepsilon|^q dx \\
& + \lambda t \int_\Omega |w|^{p-2} w u_\varepsilon dx + t\int_\Omega f u_\varepsilon dx + \frac{C}{p} t^{\gamma_p}\varepsilon^{\beta_p} + \frac{C}{q} t^{\gamma_q}\varepsilon^{\beta\gamma_q} \\
& - \frac{\lambda}{p}\int_\Omega |w+tu_\varepsilon|^p dx - \frac{t^{p^*}}{p^*}\int_\Omega \left(\frac{w}{t}+u_\varepsilon\right)_+^{p^*} dx - \int_\Omega f(w+tu_\varepsilon)dx \\
& + \frac{\lambda}{p}\int_\Omega |w|^p dx + \frac{\lambda t^p}{p}\int_\Omega |u_\varepsilon|^p dx - \frac{\lambda}{p}\int_\Omega |w|^p dx - \frac{\lambda t^p}{p}\int_\Omega |u_\varepsilon|^p dx,
\end{aligned} \tag{1.8}
$$

i.e.,

$$
\begin{aligned}
J(w+tu_\varepsilon) \leq & J(w) + \frac{t^p}{p}\int_\Omega |\nabla u_\varepsilon|^p + \frac{t^q}{q}\int_\Omega |\nabla u_\varepsilon|^q + \lambda t\int_\Omega |w|^{p-2} w u_\varepsilon + C t^{\gamma_p}\varepsilon^{\beta_p} \\
& + C t^{\gamma_q}\varepsilon^{\beta\gamma_q} + \frac{\lambda}{p}\left(\int_\Omega |w|^p + t^p\int_\Omega |u_\varepsilon|^p - \int_\Omega |w+tu_\varepsilon|^p\right) \\
& - \frac{\lambda t^p}{p}\int_\Omega |u_\varepsilon|^p - \frac{t^{p^*}}{p^*}\int_\Omega \left(\frac{w}{t}+u_\varepsilon\right)_+^{p^*} dx.
\end{aligned}
$$

For every $t > 0$ and $p > 1$ we have

$$
\begin{aligned}
\left|\int_\Omega |w+tu_\varepsilon|^p dx - \int_\Omega |w|^p dx - t^p\int_\Omega |u_\varepsilon|^p dx\right| \leq & \, 2^{p-1} p t^{p-1}\int_\Omega u_\varepsilon^{p-1}|w| dx \\
& + 2^{p-1} p t \int_\Omega u_\varepsilon |w|^{p-1} dx.
\end{aligned}
$$

Considering $\Omega_\varepsilon = \{x \in \Omega : u_\varepsilon(x) \geq K_0\}$, where $K_0 = \max_{t\in[t_0,\infty)} \|\frac{w}{t}\|_\infty = \frac{\|w\|_\infty}{t_0}$, we see that

$$
\begin{aligned}
\int_\Omega \left(\frac{w}{t}+u_\varepsilon\right)_+^{p^*} dx \geq \int_{\Omega_\varepsilon} \left(\frac{w}{t}+u_\varepsilon\right)_+^{p^*} dx \geq & \int_{\Omega_\varepsilon} u_\varepsilon^{p^*} dx + \int_{\Omega_\varepsilon} \left|\frac{w}{t}\right|^{p^*} dx \\
& - C\left(\int_{\Omega_\varepsilon} u_\varepsilon^{p^*-1}\left|\frac{w}{t}\right| dx + \int_{\Omega_\varepsilon} u_\varepsilon \left|\frac{w}{t}\right|^{p^*-1} dx\right).
\end{aligned} \tag{1.9}
$$

From Lemma 1.3.3,

$$
\int_{\Omega_\varepsilon} |u_\varepsilon|^{p^*} dx = \int_\Omega |u_\varepsilon|^{p^*} dx - O\left(\varepsilon^{\frac{N}{p}}\right).
$$

Therefore,

$$
\begin{aligned}
J(w+tu_\varepsilon) \leq & J(w) + \frac{t^p}{p}\int_\Omega |\nabla u_\varepsilon|^p dx + \frac{t^q}{q}\int_\Omega |\nabla u_\varepsilon|^q dx - \frac{t^{p^*}}{p^*}\int_\Omega |u_\varepsilon|^{p^*} dx + \frac{t^{p^*}}{p^*} O\left(\varepsilon^{\frac{N}{p}}\right) \\
& + C\left(t\|u_\varepsilon\|_1 + t^{p-1}\|u_\varepsilon\|_{p-1}^{p-1} + t^{p^*-1}\|u_\varepsilon\|_{p^*-1}^{p^*-1}\right) + t^{\gamma_p} O(\varepsilon^{\beta_p}) + t^{\gamma_q} O(\varepsilon^{\beta\gamma_q}) \\
& - \frac{\lambda t^p}{p}\int_\Omega |u_\varepsilon|^p dx,
\end{aligned} \tag{1.10}
$$

Now, take $c_0 > 0$ such that $\|u_\varepsilon\|_{p^*}^{p^*} \geq c_0 > 0$ for all $\varepsilon > 0$ sufficiently small (this can be done due to Lemma 1.3.2). Using the fact that (u_ε) is bounded in $L^s(\Omega)$ for $s = 1, p-1, p^*-1$, and taking $\varepsilon > 0$ sufficiently small, there exist positive constants C and c such that

$$J(w + tu_\varepsilon) \leq J(w) + ct^p + ct^q - t^{p^*}\left(c_0 - O\left(\varepsilon^{\frac{N}{p}}\right)\right) + C\left(t + t^{p-1} + t^{p^*-1} + t^{\gamma_p} + t^{\gamma_q}\right) \qquad (1.11)$$

for all $t \geq t_0$. Take ε sufficiently small such that $c_0 - O\left(\varepsilon^{\frac{N}{p}}\right)$ is positive. The result follows from the fact that $1, \gamma_p, \gamma_q, p-1, q, p, p^*-1 < p^*$. □

Remark 9. It is important to note that the inequalities established in the cases $a)$, $b)$, $c)$ and $d)$ are valid for any γ_p and γ_q within their respective definition intervals. However, we will later use the inequality in (1.10), which require specific selections of γ_p and γ_q to apply the results of Lemmas 1.3.5 and 1.3.6.

Knowing that w is a local minimum for the functional J, we can also see that J exhibits the geometry of a mountain pass.

Proposition 1.4.2. *Suppose* (g_1)-(g_2). *Then the functional* $J \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$ *and the following hold:*

i) There exists $\rho > 0$ *such that* $J(u) \geq J(w)$ *for all* $u \in W_0^{1,p}(\Omega)$ *with* $\|u - w\| = \rho$*;*

ii) There exist $e \in W_0^{1,p}(\Omega)$ *such that* $\|e - w\| > \rho$ *and* $J(e) < J(w)$.

Proof. Take $e = w + tu_\varepsilon$, choosing ε sufficiently small and t sufficiently large. Item ii) will follow from (1.11). □

Now, define

$$\Gamma := \{\gamma \in C([0,1], W_0^{1,p}(\Omega)) : \gamma(0) = w \text{ and } J(\gamma(1)) < J(w)\}.$$

Thus, Γ is non-empty, and the mountain pass level

$$m := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)), \qquad (1.12)$$

is well-defined.

Remark 10. The existence of a (PS) sequence at level m is found in Theorems 2.8 and 2.20 of [29].

Proposition 1.4.3. *Assuming that* (g_1) *and* (g_2) *hold, let* (u_n) *be a (PS) sequence for* J *in* $W_0^{1,p}(\Omega)$*. Then* (u_n) *is bounded.*

Proof. Let (u_n) be a (PS) sequence for $J \in W_0^{1,p}(\Omega)$, that is, a sequence such that $J(u_n)$ is bounded and $\|J'(u_n)\| \to 0$. Thus, for any $\varepsilon > 0$ and n sufficiently large,

$$\sup_{\|u_n\|\neq 0} \frac{|J'(u_n)u_n|}{\|u_n\|} \leq \varepsilon,$$

i.e.,

$$|J'(u_n)u_n| \leq \varepsilon\|u_n\|.$$

Hence, we have

$$J(u_n) - \frac{1}{p}J'(u_n)u_n \leq C + \|u_n\|.$$

Moreover,

$$\begin{aligned} J(u_n) - \frac{1}{p}J'(u_n)u_n = & \frac{1}{p}\int_\Omega (|\nabla u_n|^p - \lambda|u_n|^p)\,dx + \frac{1}{q}\int_\Omega |\nabla u_n|^q dx - \int_\Omega G(x,(u_n)_+)dx \\ & - \frac{1}{p^*}\int_\Omega (u_n)_+^{p^*} dx - \int_\Omega f(x)u_n dx - \frac{1}{p}\int_\Omega (|\nabla u_n|^p - \lambda|u_n|^p)\,dx \\ & - \frac{1}{p}\int_\Omega |\nabla u_n|^q dx + \frac{1}{p}\int_\Omega (u_n)_+^{p^*} dx + \frac{1}{p}\int_\Omega g(x,(u_n)_+)u_n dx \\ & + \frac{1}{p}\int_\Omega f(x)u_n(x)dx. \end{aligned}$$

So,

$$\begin{aligned} J(u_n) - \frac{1}{p}J'(u_n)u_n \geq & \left(\frac{1}{q} - \frac{1}{p}\right)\int_\Omega |\nabla u_n|^q dx + \left(\frac{1}{p} - \frac{1}{p^*}\right)\int_\Omega (u_n)_+^{p^*} dx \\ & - \int_\Omega G(x,(u_n)_+)dx - \left(1 - \frac{1}{p}\right)\int_\Omega f(x)u_n dx. \end{aligned}$$

Since $\frac{1}{p} < \frac{1}{q}$, $f \in L^\infty(\Omega)$, and using (g_1) and (g_2), it follows that,

$$\begin{aligned} \left(\frac{1}{p} - \frac{1}{p^*}\right)\int_\Omega (u_n)_+^{p^*} dx & \leq C + \|u_n\| + \int_\Omega G\left(x,(u_n)_+\right)dx + C\|f\|_\infty\|u_n\| \\ & = C\left(1 + \|u_n\| + \int_\Omega G(x,(u_n)_+)dx\right) \\ & \leq C\left(1 + \|u_n\| + c\left(\int_\Omega (u_n)_+^{p^*} dx\right)^{\frac{\theta}{p^*}}\right) \end{aligned}$$

with $\theta < p^*$, so, we conclude that,

$$\int_\Omega (u_n)_+^{p^*} dx \leq C\left(1 + \|u_n\|\right),$$

for large n. On the other hand, since $L^{p^*}(\Omega) \hookrightarrow L^{\theta}(\Omega)$, we have

$$\int_{\Omega} g(x,(u_n)_+)u_n dx \leq C + \int_{\Omega} (u_n)_+^{p^*} dx.$$

So,

$$\begin{aligned}
\|u_n\|_{\lambda}^{p} &\leq \|u_n\|_{\lambda}^{p} + \|u_n\|_{W^{1,q}(\Omega)}^{q} \\
&= J'(u_n)u_n + \int_{\Omega} (u_n)_+^{p^*} dx + \int_{\Omega} g\left(x,(u_n)_+\right)u_n dx + \int_{\Omega} f(x)u_n dx, \\
&\leq C\left(1 + \|u_n\|\right).
\end{aligned}$$

which implies that (u_n) is bounded. □

Since we cannot guarantee that J satisfies the Palais-Smale (PS) condition, we establish compactness properties for the functional J, provided that m defined in (1.12) remains below a certain threshold.

Proposition 1.4.4. *Assuming* (g_1)-(g_2) *and* $m < J(w) + S^{\frac{N}{p}}/N$*, we conclude that* J *possesses a critical point* $u \neq w$*, obtained as the weak limit of a (PS) sequence at the level* m*.*

Proof. Consider (u_n) as a (PS) sequence for J at the level m. By Proposition 1.4.3, there exists $u \in W_0^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, up to a subsequence. Therefore, using well-known arguments due to Boccardo-Murat [24], we have

$$\begin{aligned}
J'(u_n)v = &\int_{\Omega} \left(|\nabla u_n|\nabla u_n \nabla v - \lambda |u_n|^{p-2}u_n v\right) dx + \int_{\Omega} |\nabla u_n|^{q-2}\nabla u_n \nabla v dx \\
&- \int_{\Omega} u_n^{p^*-1} v dx - \int_{\Omega} g(x,(u_n)_+) v dx - \int_{\Omega} f v dx \\
&\to J'(u)v.
\end{aligned}$$

Thus, $J'(u)v = 0$, i.e., u is a critical point of J. Define $l = \liminf \|u_n - u\|^p$. If $l = 0$, then $u_n \to u$ in $W_0^{1,p}(\Omega)$, and thus if $J(w) = m$, we can choose (u_n) such that $\|u_n - w\| \geq \frac{\rho}{2} > 0$ for all n, where ρ is given in Proposition 1.4.2 (see [29, Theorems 2.8 and 2.20]), so the choice of (u_n) implies that $u \neq w$ and $J(u) = m = J(w)$. However, if $J(w) < m$, then $J(u) = m > J(w)$, which also implies $u \neq w$. So, we can focus in what happens if $l > 0$. It follows from the Brezis-Lieb Lemma that

$$\|u_n\|^p = \|u_n - u\|^p + \|u\|^p + o_n(1),$$

$$\|u_n\|_{W^{1,q}(\Omega)}^{q} = \|u_n - u\|_{W^{1,q}(\Omega)}^{q} + \|u\|_{W^{1,q}(\Omega)}^{q} + o_n(1),$$

$$\|(u_n)_+\|_{p^*}^{p^*} = \|(u_n - u)_+\|_{p^*}^{p^*} + \|u_+\|_{p^*}^{p^*} + o_n(1).$$

Therefore

$$
\begin{aligned}
J(u_n) &= \frac{1}{p}\left(\|u_n-u\|^p+\|u\|^p-\lambda\|u\|_p^p\right)+\frac{1}{q}\left(\|u_n-u\|_{W^{1,q}(\Omega)}^q+\|u\|_{W^{1,q}(\Omega)}^q\right)\\
&-\frac{1}{p^*}\left(\|(u_n-u)_+\|_{p^*}^{p^*}+\|u_+\|_{p^*}^{p^*}\right)-\int_\Omega G\left(x,(u_n)_+\right)dx-\int_\Omega f(x)u_n dx+o_n(1)
\end{aligned}
$$

Since $J(u_n)\to m$, we obtain

$$
m+o_n(1)=J(u)+\frac{1}{p}\|u_n-u\|^p+\frac{1}{q}\|u_n-u\|_{W^{1,q}(\Omega)}^q-\frac{1}{p^*}\|(u_n-u)_+\|_{p^*}^{p^*}.
$$

Similarly, using the fact that $J'(u_n)u_n\to 0$, we get

$$
\begin{aligned}
o_n(1) &= \|u_n-u\|^p+\|u\|^p-\lambda\|u\|_p^p+\|u_n-u\|_{W^{1,q}(\Omega)}^q+\|u\|_{W^{1,q}(\Omega)}^q-\|(u_n-u)_+\|_{p^*}^{p^*}\\
&-\|u_+\|_{p^*}^{p^*}-\int_\Omega g(x,u_+)udx-\int_\Omega f(x)udx.
\end{aligned}
$$

Using the fact that $J'(u)u=0$, we conclude that

$$
o_n(1)=\|u_n-u\|^p+\|u_n-u\|_{W^{1,q}(\Omega)}^q-\|(u_n-u)_+\|_{p^*}^{p^*}.
$$

So,

$$
\begin{aligned}
\|u_n-u\|^p+\|u_n-u\|_{W^{1,q}(\Omega)}^q-o_n(1) &= \|(u_n-u)_+\|_{p^*}^{p^*}\\
&\le \|u_n-u\|_{p^*}^{p^*}\\
&\le S^{\frac{-p^*}{p}}\|u_n-u\|^{p^*}
\end{aligned}
$$

and then,

$$
l+\lim\|u_n-u\|_{W^{1,q}(\Omega)}^q\le S^{\frac{-p^*}{p}}l^{\frac{p^*}{p}},
$$

i.e.,

$$
l\le S^{\frac{-p^*}{p}}l^{\frac{p^*}{p}}.
$$

This implies that

$$
l\ge S^{\frac{N}{p}}.
$$

Now, notice that

$$
\begin{aligned}
m+o_n(1) &= J(u)+\frac{1}{p}\|u_n-u\|^p+\frac{1}{q}\|u_n-u\|_{W^{1,q}(\Omega)}^q-\frac{1}{p^*}\|(u_n-u)_+\|_{p^*}^{p^*}\\
&= J(u)+\left(\frac{1}{p}-\frac{1}{p^*}\right)\|u_n-u\|^p+\left(\frac{1}{q}-\frac{1}{p^*}\right)\|u_n-u\|_{W^{1,q}(\Omega)}^q\\
&\ge J(u)+\frac{1}{N}\|u_n-u\|^p.
\end{aligned}
$$

And so,

$$m \geq J(u) + \frac{l}{N} \geq J(u) + \frac{S^{\frac{N}{p}}}{N}.$$

Using the assumption $m < J(w) + \frac{S^{\frac{N}{p}}}{N}$, we must have $J(u) < J(w)$, which implies that $u \neq w$. $\square$

Now, the remaining task is to demonstrate that the *minimax* level, as defined in (1.12), is below $J(w) + S^{N/p}/N$. This is accomplished in the two final propositions of this chapter.

Proposition 1.4.5. *Consider* m*, as defined in* (1.12)*. Under the conditions outlined in Theorem 1.2.9, it holds that*

$$m < J(w) + \frac{S^{\frac{N}{p}}}{N}.$$

Proof. From Proposition 1.4.1, we know that

$$J(w + tu_\varepsilon) < J(w) + \frac{S^{\frac{N}{p}}}{N},$$

for all $t \in (0, t_0) \cup (t_1, \infty)$ and $\varepsilon \in (0, \varepsilon_0)$. Since $m \leq \max_{t \geq 0} J(w + tu_\varepsilon)$, it suffices to show that there exists sufficiently small $\varepsilon > 0$ such that

$$J(w + tu_\varepsilon) < J(w) + \frac{S^{\frac{N}{p}}}{N},$$

for all $t \in [t_0, t_1]$. Our starting point will be the inequality (1.10), which was derived in the proof of Proposition 1.4.1. Notice that (1.10) if valid for for any $t \geq t_0$ and we choose γ_p and γ_q as in Lemmas 1.3.5 and 1.3.6. From Lemmas 1.3.1 and 1.3.2, using $t \in [t_0, t_1]$, we obtain

$$\begin{aligned} J(w + tu_\varepsilon) \leq & J(w) + \left(\frac{t^p}{p} - \frac{t^{p^*}}{p^*}\right) S^{\frac{N}{p}} + O(\varepsilon^{\beta_q}) - C\varepsilon^p + O\left(\varepsilon^{\frac{N}{p}}\right) + O\left(\varepsilon^{\frac{N}{p-1}}\right) \\ & + C\|u_\varepsilon\|_1 + O(\varepsilon^{\frac{N-p}{p}}) + O\left(\varepsilon^{\frac{N-p}{p-1}}\right) + O\left(\varepsilon^{\beta_p}\right) + O(\varepsilon^{\beta_{\gamma_q}}) \\ \leq & J(w) + \frac{S^{\frac{N}{p}}}{N} - C\varepsilon^p + C\|u_\varepsilon\|_1 + O\left(\varepsilon^{\frac{N-p}{p}}\right) + O\left(\varepsilon^{\beta_p}\right) + O(\varepsilon^{\beta_q}) \\ & + O(\varepsilon^{\beta_{\gamma_q}}), \end{aligned} \tag{1.13}$$

where $\sup_{t \geq 0} \left(\frac{t^p}{p} - \frac{t^{p^*}}{p^*}\right) = \frac{1}{N}$, $\beta_q = \beta(q)$ and $\beta_{\gamma_q} = \beta(\gamma_q)$ is as in (1.6) and satisfy $\beta_q, \beta_{\gamma_q} > p$ due to Lemma 1.3.6. We must now analyze the cases of the estimates of $\|u_\varepsilon\|_1$ and make the appropriate comparisons that depend on the possible values of p. Therefore, we continue by dividing the analysis into several cases:

Case 1: Let us assume $1 < p < \frac{2N}{N+1}$, then $p \in \left(1, 2 - \frac{1}{N}\right)$. Notice that, using the hypothesis

$N > p^2 + p$, we obtain

$$\frac{N-p}{p} \geq N - \frac{N-p}{p} > p, \text{ for } 1 < p \leq \frac{2N}{N+2},$$

$$N - \frac{N-p}{p} > \frac{N-p}{p} > p, \text{ for } \frac{2N}{N+2} < p < 2 - \frac{1}{N}.$$

Thus, drawing from Lemma 1.3.1, we can extend the estimates initiated in (1.13) to derive

$$\begin{aligned} J(w+tu_\varepsilon) \leq & J(w) + \frac{S^{\frac{N}{p}}}{N} - C\varepsilon^p + O\left(\varepsilon^{N-\frac{(N-p)}{p}}\right) + O\left(\varepsilon^{\frac{N-p}{p}}\right) \\ & \leq J(w) + \frac{S^{\frac{N}{p}}}{N} - C\varepsilon^p. \end{aligned}$$

Case 2: If $p = \frac{2N}{N+1}$, then from the Lemma 1.3.1, we obtain practically the same estimate, that is,

$$\begin{aligned} J(w+tu_\varepsilon) \leq & J(w) + \frac{S^{\frac{N}{p}}}{N} - C\varepsilon^p + O\left(\varepsilon^{N-\frac{(N-p)}{p}}\right)\log(\frac{1}{\varepsilon}) + O\left(\varepsilon^{\frac{N-p}{p}}\right) \\ & \leq J(w) + \frac{S^{\frac{N}{p}}}{N} - C\varepsilon^p + o\left(\varepsilon^p\right) \\ & \leq J(w) + \frac{S^{\frac{N}{p}}}{N} - C\varepsilon^p. \end{aligned}$$

Case 3: If $\frac{2N}{N+1} < p < 2$, so $\frac{N-p}{p(p-1)} > \frac{N-p}{p}$ and continue the estimate in (1.13) to reach

$$\begin{aligned} J(w+tu_\varepsilon) \leq & J(w) + \frac{S^{\frac{N}{p}}}{N} - C\varepsilon^p + O\left(\varepsilon^{\frac{(N-p)}{p(p-1)}}\right) + O\left(\varepsilon^{\frac{N-p}{p}}\right) \\ & \leq J(w) + \frac{S^{\frac{N}{p}}}{N} + -C\varepsilon^p + o\left(\varepsilon^p\right) \\ & \leq J(w) + \frac{S^{\frac{N}{p}}}{N} - C\varepsilon^p. \end{aligned}$$

Case 4: If $p \geq 2$ and $N > p(p^2 - p + 1)$, we have

$$2 \leq p < \frac{N-p}{p(p-1)} \leq \frac{N-p}{p}.$$

Now, estimations in (1.13) can be continued in what follows:

$$\begin{aligned} J(w+tu_\varepsilon) \leq & J(w) + \frac{S^{\frac{N}{p}}}{N} - C\varepsilon^p + O\left(\varepsilon^{\frac{(N-p)}{p(p-1)}}\right) + O\left(\varepsilon^{\frac{N-p}{p}}\right) \\ & \leq J(w) + \frac{S^{\frac{N}{p}}}{N} + -C\varepsilon^p + o\left(\varepsilon^p\right) \\ & \leq J(w) + \frac{S^{\frac{N}{p}}}{N} - C\varepsilon^p. \end{aligned}$$

Therefore, in any case we will always have

$$J(w+t_{u_\varepsilon}) < J(w) + \frac{S^{\frac{N}{p}}}{N},$$

for all $t \in [t_0, t_1]$ where $\varepsilon > 0$ is sufficiently small. $\square$

Incorporating hypothesis (g_3) to address the gaps in N and q that were not covered in the previous proposition, we present our final proposition as follows.

Proposition 1.4.6. *Assuming that hypotheses (g_1), (g_2) and (g_3) are satisfied and $1 < q < p < N$, then*

$$m < J(w) + \frac{S^{\frac{N}{p}}}{N}.$$

Proof. Following the same steps as in the previous proposition, but now including g (we ignored it in the proof of the estimates in Proposition 1.4.1, since it was not playing significant role and it is positive), we have

$$\begin{aligned} J(w+tu_\varepsilon) \leq & J(w) + \frac{S^{\frac{N}{p}}}{N} - C\varepsilon^p + C\|u_\varepsilon\|_1 + O\left(\varepsilon^{\frac{N-p}{p}}\right) + O\left(\varepsilon^{\beta_p}\right) + O\left(\varepsilon^{\beta_q}\right) \\ & + O\left(\varepsilon^{\beta\gamma_q}\right) - \int_\Omega G\left(x, (w+tu_\varepsilon)_+\right) dx \end{aligned}$$

where $t \in [t_0, t_1]$. From condition (g_3) and by (1.9) we have

$$\begin{aligned} \int_\Omega G\left(x, (w+tu_\varepsilon)_+\right) dx &\geq C \int_\Omega (w+tu_\varepsilon)_+^\sigma dx \\ &\geq C \int_{\Omega_\varepsilon} \left(u_\varepsilon + \frac{w}{t}\right)_+^\sigma dx \\ &\geq C \int_{\Omega_\varepsilon} u_\varepsilon^\sigma dx - C(\|u_\varepsilon\|_{\sigma-1}^{\sigma-1} + \|u_\varepsilon\|_1). \end{aligned}$$

From Lemma 1.3.3, for $p^* - \frac{N}{N-p} < s < p^*$, we have,

$$\int_{\Omega_\varepsilon} u_\varepsilon^s dx = \int_\Omega u_\varepsilon^s dx - O\left(\varepsilon^{\frac{N}{p}}\right).$$

Taking $s = \sigma$ and $s = \sigma - 1$, note that we can use the Lemma 1.3.1 because we have

$p^* - \frac{N}{N-p} < p^* - \frac{p}{N-p} < \sigma < p^*$ due to (g_3). So it follows from Lemma 1.3.1 that

$$\begin{aligned} J(w+tu_\varepsilon) \leq & J(w) + \frac{S^{\frac{N}{p}}}{N} - C\varepsilon^p + C\|u_\varepsilon\|_1 + O(\varepsilon^{\frac{N-p}{p}}) + O\left(\varepsilon^{\beta_p}\right) + O\left(\varepsilon^{\beta_q}\right) \\ & + O\left(\varepsilon^{\beta_{\gamma_q}}\right) + O\left(\varepsilon^{N-(N-p)(\sigma-1)/p}\right) - C\varepsilon^{N-(N-p)\sigma/p} \\ & \leq J(w) + \frac{S^{\frac{N}{p}}}{N} + C\|u_\varepsilon\|_1 + O\left(\varepsilon^{\frac{N-p}{p}}\right) + O\left(\varepsilon^{\beta_p}\right) + O\left(\varepsilon^{\beta_q}\right) + O\left(\varepsilon^{\beta_{\gamma_q}}\right) \\ & - C\varepsilon^{N-(N-p)\sigma/p}. \end{aligned} \tag{1.14}$$

for sufficiently small $\varepsilon > 0$ and $t \in [t_0, t_1]$. Notice that we disregarded the term ε^p in the estimates (1.14) because it was only beneficial in scenarios with specific dimensional restrictions on N. Currently, by invoking condition (g_3), we can utilize $\varepsilon^{N-(N-p)\sigma/p}$ instead. This approach, however, necessitates a thorough analysis of all the exponents mentioned in (1.14). It is essential to confirm that the exponents $(N-p)/p$, β_p, β_q, β_{γ_q}, and the exponent derived from the $\|u_\varepsilon\|_1$ estimate, are all greater than $N-(N-p)\sigma/p$. Thus, we need to conduct estimates analogous to those in Lemmas 1.3.5 and 1.3.6. This examination will involve studying the possible values of p and q separately.

Case 1: If $1 < p < \frac{2N}{N+1}$, then $q \in (1,2)$ and $\gamma_q \in (1,q)$. Since $\frac{2N}{N+1} < 2 - \frac{1}{N}$, it follows from Remark 7 that $\frac{N(p-1)}{N-1} < 1$, that is, $q > \gamma_q > \frac{N(p-1)}{N-1}$, implying that

$$\beta_q = N - \frac{N}{p}q > N - \frac{(N-p)}{p}\sigma$$

if and only if

$$\sigma > \frac{Nq}{N-p}.$$

By (g_3), we have $\sigma > \max\{p^* - \frac{p}{N-p}, \frac{Nq}{N-p}\}$, hence $\beta_q > N - \frac{(N-p)}{p}\sigma$ for $q \in (1,p)$. Similarly,

$$\beta_{\gamma_q} = N - \frac{N}{p}\gamma_q > N - \frac{(N-p)}{p}\sigma,$$

if and only if

$$\sigma > \frac{N\gamma_q}{N-p}.$$

By (g_3), we have $\sigma > \frac{Nq}{N-p} > \frac{N\gamma_q}{N-p}$, hence $\beta_{\gamma_q} > N - \frac{(N-p)}{p}\sigma$ for $\gamma_q \in (1,q)$. The choice of β_p as in (1.7) gives us $\beta_p > N - \frac{(N-p)}{p}\sigma$. Therefore, from Lemma 1.3.1, we continue the estimates in (1.14) to obtain

$$\begin{aligned}J(w+tu_\varepsilon) \leq & J(w)+\frac{S^{\frac{N}{p}}}{N}+O\left(\varepsilon^{N-\frac{(N-p)}{p}}\right)+O\left(\varepsilon^{\frac{N-p}{p}}\right)+O\left(\varepsilon^{\beta_p}\right)+O\left(\varepsilon^{\beta_q}\right)\\ & +O\left(\varepsilon^{\beta_{\gamma_q}}\right)-C\varepsilon^{N-(N-p)\sigma/p}\\ & \leq J(w)+\frac{S^{\frac{N}{p}}}{N}-C\varepsilon^{N-(N-p)\sigma/p}+o\left(\varepsilon^{N-(N-p)\sigma/p}\right)\\ & < J(w)+\frac{S^{\frac{N}{p}}}{N}\end{aligned}$$

for sufficiently small $\varepsilon>0$ and $t\in[t_0,t_1]$.

Case 2: $p=\frac{2N}{N+1}$. The result is analogous to the previous case, with the only difference being the term $O\left(\varepsilon^{N-\frac{(N-p)}{p}}\right)\log(\frac{1}{\varepsilon})$ instead of $O\left(\varepsilon^{N-\frac{(N-p)}{p}}\right)$.

Case 3: $p>\frac{2N}{N+1}$. This case must be divided also in several steps, actually. First, consider the case $\frac{2N}{N+1}<p\leq 2-\frac{1}{N}$. We have $q\in(1,2)$ and $\gamma_q\in(1,q)$, and also $\frac{N(p-1)}{N-1}\leq 1<\gamma_q<q$. In other words,

$$\beta_q=N-\frac{N}{p}q>N-\frac{(N-p)}{p}\sigma,$$

if and only if $\sigma>\frac{Nq}{N-p}$.

By (g_3), we have $\sigma>\max\{p^*-\frac{p}{N-p},p^*-1,\frac{Nq}{N-p}\}$, hence $\beta_q>N-\frac{(N-p)}{p}\sigma$ for $q\in(1,p)$. Similarly, $\beta_{\gamma_q}>N-\frac{(N-p)}{p}\sigma$ for $\gamma_q\in(1,q)$.

Now, suppose $2-\frac{1}{N}<p<2$. Then $\frac{N(p-1)}{N-1}>1$, $q\in(1,2)$, and $\gamma_q\in(1,q)$. If $q\leq\frac{N(p-1)}{N-1}$, then

$$\beta_q=\frac{(N-p)}{p(p-1)}q>N-\frac{(N-p)}{p}\sigma$$

if and only if,

$$\frac{(N-p)}{p}\sigma>N-\frac{(N-p)}{p(p-1)}q$$

if and only if,

$$\sigma>\frac{Np}{N-p}-\frac{q}{(p-1)}=p^*-\frac{q}{(p-1)}.$$

Note that the above inequality holds because

$$\sigma>p^*-1>p^*-\frac{1}{p-1}>p^*-\frac{q}{p-1},$$

since $p<2$.

Similarly, $\beta_{\gamma_q}>N-\frac{(N-p)}{N}\sigma$ for $\gamma_q\in(1,q)$.

If $q > \frac{N(p-1)}{N-1}$ then

$$\beta_q = N - \frac{N}{p}q > N - \frac{(N-p)}{p}\sigma \text{ for } q \in \left(\frac{N(p-1)}{N-1}, p\right).$$

We just need to take $\gamma_q \in \left(1, \frac{N(p-1)}{N-1}\right)$ and we will have $\beta_{\gamma_q} > N - \frac{(N-p)}{p}\sigma$.

If $2 \leq p \leq 3 - \frac{2}{N}$, then $1 < \frac{N(p-1)}{N-1} \leq 2$. Let us suppose $q \in (1,2)$, then we will have $\gamma_q \in (1,q)$. If $q \leq \frac{N(p-1)}{N-1}$ then

$$\beta_q = \frac{(N-p)}{p(p-1)}q > N - \frac{(N-p)}{p}\sigma$$

because, $\sigma > p^* - \frac{1}{p-1}$ where, $p \geq 2$. In analogous way, we obtain for β_{γ_q}.

If $q > \frac{N(p-1)}{N-1}$, it follows from (g_3) that

$$\beta_q = N - \frac{N}{p}q > N - \frac{(N-p)}{p}\sigma \text{ for } q \in \left(\frac{N(p-1)}{N-1}, p\right).$$

We just need to take $\gamma_q \in \left(1, \frac{N(p-1)}{N-1}\right)$ and we will have $\beta_{\gamma_q} > N - \frac{(N-p)}{p}\sigma$.
Now, suppose $q \in [2,p) \subset [2,3)$, then $\gamma_q \in [q-1,2]$. Note that $q \geq \frac{N(p-1)}{N-1}$, which implies by ($g_3$) that

$$\beta_q > N - \frac{(N-p)}{p}\sigma$$

for $q \in \left[\frac{N(p-1)}{N-1}, p\right)$ and taking $\gamma_q = q-1 < \frac{N(p-1)}{N-1}$, we have $\beta_{\gamma_q} > N - \frac{(N-p)}{p}\sigma$.

If $3 - \frac{2}{N} < p < 3$, then $\frac{N(p-1)}{N-1} > 2$, thus for $q \in (1,2)$ we have $\gamma_q \in (1,q)$ with

$$\beta_q = \frac{(N-p)}{p(p-1)}q > N - \frac{(N-p)}{p}\sigma$$

and

$$\beta_{\gamma_q} = \frac{(N-p)}{p(p-1)}\gamma_q > N - \frac{(N-p)}{p}\sigma.$$

If $q \in [2,p) \subset [2,3)$, then $\gamma_q \in [q-1,2]$, implying that $\gamma_q < \frac{N(p-1)}{N-1}$ and $\beta_{\gamma_q} = \frac{(N-p)}{p(p-1)}\gamma_q > N - \frac{(N-p)}{N}\sigma$.
Suppose $2 \leq q \leq \frac{N(p-1)}{N-1}$ or $\frac{N(p-1)}{N-1} \leq q < p$ from (g_3) in an analogous way we have $\beta_q > N - \frac{(N-p)}{N}\sigma$. Finally, for $p \geq 3$, we have $2 < \frac{N(p-1)}{N-1}$, then, if $q \in (1,2)$, we have no issues because $\gamma_q < q < \frac{N(p-1)}{N-1}$. If $q \in [2,3)$, we take $\gamma_q = q-1$ and thus $\beta_{\gamma_q}, \beta_q > N - \frac{(N-p)}{p}\sigma$ by (g_3). If $q \geq 3$, then $\gamma_q \in \{2, q-1\}$. We simply take $\gamma_q = 2 < \frac{N(p-1)}{N-1}$ and we will have $\beta_{\gamma_q} > N - \frac{(N-p)}{N}\sigma$. Additionally, for $q \in \left[3, \frac{N(p-1)}{N-1}\right)$ or $q \in \left[\frac{N(p-1)}{N-1}, p\right)$, we will have $\beta_q > N - \frac{(N-p)}{N}\sigma$ by (g_3).
The choice of β_p in equation (1.7) gives us $\beta_p > N - \frac{(N-p)}{p}\sigma$ by (g_3). Therefore, from Lemma

1.3.1, we can go back to estimates started in (1.14) to conclude that

$$\begin{aligned} J(w+tu_\varepsilon) \leq & J(w) + \frac{S^{\frac{N}{p}}}{N} + O\left(\varepsilon^{(N-p)/[p(p-1)]}\right) + O(\varepsilon^{\frac{N-p}{p}}) - C\varepsilon^{N-(N-p)\sigma/p} \\ & \leq J(w) + \frac{S^{\frac{N}{p}}}{N} - C\varepsilon^{N-(N-p)\sigma/p} + o\left(\varepsilon^{N-(N-p)\sigma/p}\right) \\ & < J(w) + \frac{S^{\frac{N}{p}}}{N} \end{aligned}$$

for $\varepsilon > 0$ sufficiently small and $t \in [t_0, t_1]$. Here (g_3) was used in the second inequality. □

1.4.1 Proof of the main theorems

Theorem 1.2.9 is proved using Propositions 1.2.8, 1.4.4 and 1.4.5, and Theorem 1.2.10 is proved using Propositions 1.2.8, 1.4.4 and 1.4.6.

Chapter 2

Critical non-homogeneous problems on the (p, q)-Laplacian

2.1 Introduction

In this chapter, we study the problem:

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda |u|^{p-2}u + \mu |u|^{q-2}u + |u|^{p^*-2}u + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded domain with a smooth boundary in $\mathbb{R}^N$, $1 < q < p < N$, $\lambda \in (0, \lambda_1)$, and $0 < \mu \leq \mu_1$ where λ_1 and μ_1 represent the first eigenvalue of the p-Laplacian and the q-Laplacian, respectively. Our work extends the study given in [31] by incorporating the non-homogeneous term $f \in L^{p^{*'}}(\Omega)$ into the equation. Within this framework, we establish the existence of two solutions. The first solution is obtained using Ekeland's Variational Principle, as described in [16], while the second solution is found by evoking Talenti concentration functions in conjunction with the Mountain Pass method, following the approach presented in [19].

In the homogeneous case, [31] identified a nontrivial, nonnegative solution under various restrictions on q and p, using techniques that use Morse theory to compare critical levels and critical points of the associated functional.

In our case, by introducing the inhomogeneous term, we demonstrate the existence of two distinct solutions under very similar restrictions on p and q as the ones given in Chapter 1, provided that the norm of f is sufficiently small.

2.2 Hypotheses and Main Results

Consider $f \in L^{p^{*'}}(\Omega)$ non-zero. Let us define

$$\lambda_1 = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|\nabla u\|_p^p}{\|u\|_p^p}$$

and for the q-Laplacian,

$$\mu_1 = \inf_{u \in W_0^{1,q}(\Omega)\setminus\{0\}} \frac{\|\nabla u\|_q^q}{\|u\|_q^q}$$

which are the first eigenvalues of the problems

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

$$\begin{cases} -\Delta_q u = \mu |u|^{q-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Our main theorems for this chapter are stated below.

Theorem 2.2.1. *Suppose $\lambda \in (0, \lambda_1)$, $0 < \mu \leq \mu_1$, $f \in L^{p^{*'}}(\Omega)$ nontrivial. Then, for any $p > 1$, there exists a constant $M > 0$ such that problem (2.1) has two solutions provided that $\|f\|_{p^{*'}} \leq M$,*

$$N > \begin{cases} \max\{p^2 + p, p^2/(p-1), (p-1)p^2/(q-1) + p\}, & \text{if } 1 < p < 2 \\ \max\{(p-1)p^2 + p, (p-1)p^2/(q-1) + p\}, & \text{if } p \geq 2, \end{cases}$$

and

$$1 < q < (N-p)p/N.$$

Remark 11. Notice that

$$(p-1)p^2 + p \geq p^2(p-1)/(q-1) + p$$

if and only if

$$2 \leq q,$$

which means that, in case $q \geq 2$ we can assume the same restrictions as in Theorem 1.2.9.

Now, in case $\mu = 0$, the situation returns to Theorem 1.2.9 without any further restrictions on q. We have the following theorem.

Theorem 2.2.2. *Suppose $\lambda \in (0, \lambda_1)$, $\mu = 0$, $f \in L^{p^{*'}}(\Omega)$ non-zero. Then, for any $p > 1$, there exists a constant $M > 0$ such that problem (2.1) has two solutions provided that $\|f\|_{p^{*'}} \leq M$,*

$$N > \begin{cases} \max\{p^2 + p, p^2/(p-1)\}, & \text{if } 1 < p < 2 \\ \max\{(p-1)p^2 + p\}, & \text{if } p \geq 2, \end{cases}$$

and

$$1 < q < (N-p)p/N.$$

As usual, we see that weak solutions of the problem (2.1) are critical points of the associated functional given by:

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{p}\int_\Omega |u|^p dx + \frac{1}{q}\int_\Omega |\nabla u|^q dx - \frac{\mu}{q}\int_\Omega |u|^q dx - \frac{1}{p^*}\int_\Omega |u|^{p^*} dx - \int_\Omega fu dx$$

and

$$J'(u)v = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx - \lambda\int_\Omega |u|^{p-2}uv dx + \int_\Omega |\nabla u|^{q-2}\nabla u\nabla v dx - \mu\int_\Omega |u|^{q-2}uv dx - \int_\Omega |u|^{p^*-2}uv dx - \int_\Omega fv dx,$$

for all u and v in $W_0^{1,p}(\Omega)$.

The objective of the remainder of this section is to show that J satisfies $(PS)_c$ for some $c \neq 0$, which will allow us to find $w \in W_0^{1,p}(\Omega)$ as a solution to the problem (2.1) such that $J(w) < 0$. This solution will be obtained using the Variational Principle of Ekeland. Let us first prove a standard fact regarding this scheme. As in Chapter 1, let us consider again

$$S = \inf_{u \in \mathfrak{D}^{1,p}(\mathbb{R}^N)\backslash\{0\}} \frac{\|u\|^p}{\|u\|_{p^*}^p}.$$

the best constant of the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$.

Proposition 2.2.3. *Consider $\mu \leq \mu_1$. If (u_n) is a (PS) sequence for the functional J in $W_0^{1,p}(\Omega)$, then (u_n) is bounded.*

Proof. Consider (u_n) a $(PS)_c$ sequence of the functional J. Then we have

$$\begin{aligned} J(u_n) &= \frac{1}{p}\int_\Omega |\nabla u_n|^p dx + \frac{1}{q}\int_\Omega |\nabla u_n|^q - \frac{\mu}{q}\int_\Omega |u_n|^q dx - \frac{\lambda}{p}\int_\Omega |u_n|^p - \frac{1}{p^*}\int_\Omega |u_n|^{p^*} dx \\ &- \int_\Omega f(x)u_n dx \\ &= c + o(1), \end{aligned}$$

and

$$\begin{aligned} J'(u_n)u_n &= \int_\Omega |\nabla u_n|^p dx + \int_\Omega |\nabla u_n|^q dx - \mu\int_\Omega |u_n|^q - \lambda\int_\Omega |u_n|^p - \int_\Omega |u_n|^{p^*} dx \\ &- \int_\Omega f(x)u_n dx \\ &= o(1)\|u_n\|. \end{aligned}$$

Therefore,

$$\begin{aligned}c+\|u_n\| &\geq J(u_n)-\frac{1}{p}J'(u_n)u_n\\&=\left(\frac{1}{q}-\frac{1}{p}\right)\left(\|\nabla u\|_q^q-\mu\|u_n\|_q^q\right)+\left(\frac{1}{p}-\frac{1}{p^*}\right)\int_\Omega |u_n|^{p^*}dx\\&-\left(1-\frac{1}{p}\right)\int_\Omega f(x)u_n dx.\end{aligned}$$

Since $\mu \leq \mu_1$, we have:

$$\|\nabla u\|_q^q-\mu\|u_n\|_q^q \geq 0,$$

and $q<p$ implies

$$\begin{aligned}\left(\frac{1}{p}-\frac{1}{p^*}\right)\int_\Omega |u_n|^{p^*}dx &\leq c+\|u_n\|+\left(1-\frac{1}{p}\right)\int_\Omega f(x)u_n dx\\&\leq c+\|u_n\|+C\|f\|_{p^{*\prime}}\|u_n\|_{p^*}\\&\leq c+\|u_n\|+S^{\frac{-1}{p}}C\|f\|_{p^{*\prime}}\|u_n\|.\end{aligned}$$

Therefore,

$$\int_\Omega |u_n|^{p^*}dx \leq C\left(1+\|u_n\|\right).$$

Thus,

$$\begin{aligned}\|u_n\|_\lambda^p &\leq \|u_n\|_\lambda^p+\|\nabla u_n\|_q^q-\mu\|u_n\|_q^q+\|u_n\|_{p^*}^{p^*}-\|u_n\|_{p^*}^{p^*}+\int_\Omega fu_n dx-\int_\Omega fu_n dx\\&=J'(u_n)u_n+\|u_n\|_{p^*}^{p^*}+\int_\Omega fu_n dx\\&\leq C(1+\|u_n\|).\end{aligned}$$

Knowing that $\|u_n\|_\lambda^p \leq C\|u_n\|^p$ because they are equivalent norms, we have

$$\|u_n\|^p \leq C(1+\|u_n\|).$$

□

For the next proposition, consider the following:

$$g(t)=-\frac{t^{p^*}}{N}+\left(1-\frac{1}{p}\right)\|f\|_{p^{*\prime}}t.$$

Obviously,

$$\sup_{t\geq 0} g(t)>0.$$

Proposition 2.2.4. *Suppose $c<S^{N/p}/N-\sup_{t\geq 0}g(t)$. Then the functional J satisfies $(PS)_c$.*

Proof. Let (u_n) be a $(PS)_c$ sequence of J in $W_0^{1,p}(\Omega)$, it follows from the previous proposition that (u_n) is bounded in $W_0^{1,p}(\Omega)$. Therefore, passing to a subsequence if necessary, we find that there exists $u \in W_0^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, $u_n \to u$ in $L^s(\Omega)$ where $1 \leq s < p^*$ and $u_n \to u$ a.e. in $x \in \Omega$. It follows from the weak continuity of J' that

$$J'(u)v = \lim_{n\to\infty} J'(u_n)v = 0,$$

for any $v \in W_0^{1,p}(\Omega)$. This means that u is a weak solution of the problem (2.1). Consider

$$l := \liminf \|u_n - u\|^p.$$

If $l = 0$ there is nothing to do. Let us then suppose that $l > 0$. From the Brezis-Lieb Lemma we have that

$$\|u_n\|^p = \|u_n - u\|^p + \|u\|^p + o_n(1),$$

$$\|u_n\|^q_{W^{1,q}(\Omega)} = \|u_n - u\|^q_{W^{1,q}(\Omega)} + \|u\|^q_{W^{1,q}(\Omega)} + o_n(1),$$

$$\|u_n\|^{p^*}_{p^*} = \|u_n - u\|^{p^*}_{p^*} + \|u\|^{p^*}_{p^*} + o_n(1).$$

So, using $J'(u_n)u_n \to 0$ and $J'(u)u = 0$ we have

$$\begin{aligned} o_n(1) &= J'(u_n)u_n \\ &= \left(\|u_n - u\|^p + \|u\|^p - \lambda\|u\|^p_p\right) \\ &+ \left(\|u_n - u\|^q_{W^{1,q}(\Omega)} + \|u\|^q_{W^{1,q}(\Omega)} - \mu\|u\|^q_q\right) \\ &- \left(\|u_n - u\|^{p^*}_{p^*} + \|u\|^{p^*}_{p^*}\right) - \int_\Omega fu dx + o_n(1) \\ &= J'(u)u + \|u_n - u\|^p + \|u_n - u\|^q_{W^{1,q}(\Omega)} - \|u_n - u\|^{p^*}_{p^*} \end{aligned}$$

implying that

$$\|u_n - u\|^{p^*}_{p^*} + o_n(1) = \|u_n - u\|^p + \|u_n - u\|^q_{W^{1,q}(\Omega)}. \tag{2.2}$$

Consequently,

$$\begin{aligned} \|u_n - u\|^p &\leq \|u_n - u\|^p + \|u_n - u\|^q_{W^{1,q}(\Omega)} \\ &= \|u_n - u\|^{p^*}_{p^*} + o_n(1) \\ &\leq S^{\frac{-p^*}{p}} \|u_n - u\|^{p^*} + o_n(1). \end{aligned}$$

Applying the limit on both sides we have

$$l \leq S^{\frac{-p^*}{p}} l^{\frac{p^*}{p}},$$

i.e.,

$$l \geq S^{\frac{N}{p}}.$$

Using Brezis-Lieb, by Equation (2.2) and Hölder's inequality, we get

$$\begin{aligned}
c &= \lim_{n\to\infty} \left[J(u_n) - \frac{1}{p}\langle J'(u_n), u_n \rangle \right] \\
&= \lim_{n\to\infty} \left[\left(\frac{1}{q} - \frac{1}{p}\right) (\|\nabla u_n\|_q^q - \mu\|u_n\|_q^q) + \left(\frac{1}{p} - \frac{1}{p^*}\right) \|u_n\|_{p^*}^{p^*} \right. \\
&\left. - \left(1 - \frac{1}{p}\right) \int_\Omega f u_n dx \right] \\
&\geq \lim_{n\to\infty} \left[\left(\frac{1}{p} - \frac{1}{p^*}\right) \|u_n\|_{p^*}^{p^*} - \left(1 - \frac{1}{p}\right) \int_\Omega f u_n dx \right] \\
&\geq \left(\frac{1}{p} - \frac{1}{p^*}\right) \left(\|u\|_{p^*}^{p^*} + \lim_{n\to\infty} \|u_n - u\|_{p^*}^{p^*} \right) - \left(1 - \frac{1}{p}\right) \int_\Omega f u dx \\
&\geq \frac{1}{N}\|u\|_{p^*}^{p^*} + \frac{1}{N} \lim_{n\to\infty} \Big(\|u_n - u\|^p + \|u_n - u\|_{W^{1,q}(\Omega)}^q + o_n(1) \Big) \\
&- \left(1 - \frac{1}{p}\right) \|f\|_{p^{*\prime}} \|u\|_{p^*} \\
&= \frac{1}{N} \lim_{n\to\infty} \Big(\|u_n - u\|^p + \|u_n - u\|_{W^{1,q}(\Omega)}^q + o_n(1) \Big) - g(\|u\|_{p^*}) \\
&\geq \frac{1}{N} \lim_{n\to\infty} (\|u_n - u\|^p + o_n(1)) - \sup_{t\geq 0} g(t) \\
&= \frac{l}{N} - \sup_{t\geq 0} g(t) \\
&\geq \frac{S^{\frac{N}{p}}}{N} - \sup_{t\geq 0} g(t),
\end{aligned}$$

which is impossible because $c < S^{N/p}/N - \sup_{t\geq 0} g(t)$. Therefore, $l = 0$ and $u_n \to u$ in $W_0^{1,p}(\Omega)$. □

Lemma 2.2.5. *There are constants $\rho_0 = \rho_0(p, N, S) > 0$ and $M = M(p, N, S) > 0$ such that, for all $\|u\| = \rho_0$, we have $J(u) \geq 0$ if $\|f\|_{p^{*\prime}} \leq M$.*

Proof. By Hölder inequality and Sobolev embeddings, we have

$$\begin{aligned}
J(u) &= \frac{1}{p}\left(\int_\Omega |\nabla u|^p dx - \frac{\lambda}{p}\int_\Omega |u|^p \right) + \frac{1}{q}\left(\int_\Omega |\nabla u|^q - \frac{\mu}{q}\int_\Omega |u|^q dx \right) - \frac{1}{p^*}\int_\Omega |u|^{p^*} dx \\
&- \int_\Omega f(x) u dx \\
&\geq \frac{1}{p}\left(\int_\Omega |\nabla u|^p dx - \frac{\lambda}{p}\int_\Omega |u|^p \right) - \frac{1}{p^*}\int_\Omega |u|^{p^*} dx - \int_\Omega f(x) u dx \\
&\geq \frac{1}{p}\left(1 - \frac{\lambda}{\lambda_1}\right) \|u\|^p - \frac{S^{\frac{-p^*}{p}}}{p^*} \|u\|^{p^*} - S^{\frac{-1}{p}} \|f\|_{p^{*\prime}} \|u\|.
\end{aligned}$$

Considering $t = \|u\|$, it follows that

$$J(u) \geq \frac{1}{p}\left(1 - \frac{\lambda}{\lambda_1}\right) t^p - \frac{S^{\frac{-p^*}{p}}}{p^*} t^{p^*} - S^{\frac{-1}{p}} \|f\|_{p^{*\prime}} t$$
$$= t\left\{ t^{p-1}\left(\frac{1}{p}\left(1 - \frac{\lambda}{\lambda_1}\right) - \frac{S^{\frac{-p^*}{p}}}{p^*} t^{p^*-p}\right) - S^{\frac{-1}{p}} \|f\|_{p^{*\prime}} \right\}.$$

Since $p > p^*$ and $\lambda < \lambda_1$, there exists $\rho_0 > 0$ such that

$$h(\rho_0) = \frac{1}{p}\left(1 - \frac{\lambda}{\lambda_1}\right) - \frac{S^{\frac{-p^*}{p}}}{p^*} \rho_0^{p^*-p} \geq 0.$$

Then, choose M such that

$$\|f\|_{p^{*\prime}} \leq S^{\frac{1}{p}} \rho_0^{p-1} h(\rho_0) = M.$$

This implies that

$$\rho_0^{p-1} h(\rho_0) - S^{\frac{-1}{p}} \|f\|_{p^{*\prime}} \geq 0,$$

finishing this proof. □

Lemma 2.2.6. *Let ρ_0 be the constant from Lemma 2.2.5. Then*

$$\inf_{\|u\| < \rho_0} J(u) < 0.$$

Proof. Consider $\phi \in C_0^\infty(\Omega)$ such that $\int_\Omega f(x)\phi dx > 0$. So,

$$J(t\phi) = \frac{t^p}{p}\int_\Omega (|\nabla \phi|^p - \lambda|\phi|^p) dx + \frac{t^q}{q}\int_\Omega (|\nabla \phi|^q - \mu|\phi|^q) dx - \frac{t^{p^*}}{p^*}\int_\Omega |\phi|^{p^*} dx$$
$$- t\int_\Omega f(x)\phi dx$$
$$< 0,$$

for all $t > 0$ sufficiently small. This means that

$$\inf_{\|u\| < \rho_0} J(u) < 0.$$

□

Proposition 2.2.7. *Consider $\mu \leq \mu_1$. Let $M > 0$ and ρ_0 be constants determined by Lemma 2.2.5 and assume $\|f\|_{p^{*\prime}} \leq M$. Then there exists $w \in W_0^{1,p}(\Omega)$, a solution of problem (2.1), such that*

$$J(w) = \inf_{\|u\| < \rho_0} J(u) = c < 0.$$

Proof. It follows from the Lemmas 2.2.5 and 2.2.6 for all $\|u\| = \rho_0$ that

$$c = \inf_{\|v\|<\rho_0} J(v) < 0 \leq J(u).$$

Moreover, from Ekeland Variational Principle, there exists a sequence (u_m) in $\bar{B}(0, \rho_0)$ such that $J(u_m) \to c$ and $J'(u_m)u_m \to 0$ when $m \to \infty$. Furthermore, from Lemma 2.2.5 $M = S^{\frac{1}{p}} \rho_0^{p-1} h(\rho_0)$ so note that without loss of generality, we can decrease ρ_0 to obtain,

$$\frac{S^{\frac{N}{p}}}{N} - \sup_{t\geq 0} g(t) \geq \frac{S^{\frac{N}{p}}}{N} - \sup_{t\geq 0} \left\{ C\rho_0^{p-1} h(\rho_0) S^{\frac{1}{p}} t - \frac{t^{p^*}}{N} \right\} > 0 > c$$

for $\rho_0 > 0$ sufficiently small. Notice that it is immediate that (u_m) is bounded in $W_0^{1,p}(\Omega)$, and by passing to a subsequence if necessary, there exists $w \in W_0^{1,p}(\Omega)$ such that $u_m \rightharpoonup w$ in $W_0^{1,p}(\Omega)$, $u_m \to w$ em $L^s(\Omega)$ where $1 \leq s < p^*$ and $u_m \to w$ a.e. in $x \in \Omega$ a weak solution of the problem (2.1). It follows from Proposition 2.2.4 that $u_m \to w$ in $W_0^{1,p}(\Omega)$, and thus

$$J(w) \longleftarrow J(u_m) \longrightarrow \inf_{\|u\|<\rho_0} J(u).$$

Therefore,

$$J(w) = \inf_{\|u\|<\rho_0} J(u).$$

□

2.3 Preliminaries

In this section, we will state some results that will be very important for this chapter, some of which were already announced in the preliminaries of the previous chapter, but for the reader's convenience, they will also be announced here.

Lemma 2.3.1. *Considering $0 < \varepsilon < r$ and $1 < p < N$. The following estimates hold,*

$$c\varepsilon^p \leq \int_\Omega |\, u_\varepsilon \,|^p \, dx \leq \begin{cases} C\varepsilon^p + O\left(\varepsilon^{(N-p)/(p-1)}\right) & \text{if } p^2 < N, \\ C\varepsilon^p \log(\frac{1}{\varepsilon}) & \text{if } p^2 = N, \\ C\varepsilon^{(N-p)/(p-1)} & \text{if } p^2 > N, \end{cases} \tag{2.3}$$

$$\int_\Omega |\, u_\varepsilon \,|^{p-1} \, dx \leq C\varepsilon^{(N-p)/p}, \tag{2.4}$$

$$\int_\Omega |\, u_\varepsilon \,|^{p^*-1} \, dx \leq C\varepsilon^{(N-p)/p}, \tag{2.5}$$

$$\int_\Omega |\, u_\varepsilon \,|\, dx \le \begin{cases} C\varepsilon^{(N-p)/p(p-1)} & \textit{if } p > 2N/(N+1), \\ C\varepsilon^{N-(N-p)/p} \log(\frac{1}{\varepsilon}) & \textit{if } \; p = 2N/(N+1), \\ C\varepsilon^{N-(N-p)/p} & \textit{if } 1 < p < 2N/(N+1), \end{cases} \tag{2.6}$$

$$c\varepsilon^{N-(N-p)s/p} \le \int_\Omega |\, u_\varepsilon \,|^s \, dx \le C\varepsilon^{N-(N-p)s/p}, \textit{if } p^* - N/(N-p) < s < p^*. \tag{2.7}$$

For $1 < \gamma < p$

$$\|\nabla u_\varepsilon\|_\gamma^\gamma \le \begin{cases} C\varepsilon^\beta & \textit{if } \gamma \ne N(p-1)/(N-1) \\ C\varepsilon^\beta \log(\frac{1}{\varepsilon}) & \textit{if } \gamma = N(p-1)/(N-1) \end{cases} \tag{2.8}$$

where,

$$\beta = \beta(\gamma) = \begin{cases} \frac{N-p}{p(p-1)}\gamma & \textit{if } 1 < \gamma \le \frac{N(p-1)}{N-1} \\ N - \frac{N}{p}\gamma & \textit{if } \frac{N(p-1)}{N-1} < \gamma < p. \end{cases} \tag{2.9}$$

Remark 12. For $s = q-1$ consider, $0 < \varepsilon < r$ so,

$$\begin{aligned} \int_\Omega |u_\varepsilon|^{q-1} dx &= \varepsilon^{(N-p)(q-1)/p(p-1)} \int_{\mathbb{R}^N} \frac{\phi(x)^{q-1}}{\left[\varepsilon^{p/(p-1)} + |x|^{p/(p-1)}\right]^{(N-p)(q-1)/p}} dx \\ &= \varepsilon^{N-[(N-p)(q-1)/p]} \int_{\mathbb{R}^N} \frac{\phi(\varepsilon x)^{q-1}}{\left[1 + |x|^{p/(p-1)}\right]^{(N-p)(q-1)/p}} dx \\ &\le C(N,p,q)\varepsilon^{N-[(N-p)(q-1)/p]} \\ &\quad + C\varepsilon^{N-[(N-p)(q-1)/p]} \int_1^{2r/\varepsilon} R^{N-1-(N-p)(q-1)/(p-1)} dR \\ &\le C\left(\varepsilon^{N-[(N-p)(q-1)/p]} + \varepsilon^{(N-p)(q-1)/p(p-1)}\right). \end{aligned}$$

Then,

$$\int_\Omega |u_\varepsilon|^{q-1} dx = O\left(\varepsilon^{N-(N-p)(q-1)/p}\right) + O\left(\varepsilon^{(N-p)(q-1)/p(p-1)}\right) \tag{2.10}$$

Lemma 2.3.2. *Consider* $0 < \varepsilon < r$ *and* $1 < p < N$. *Then,*

$$\|\nabla u_\varepsilon\|_p^p = S^{\frac{N}{p}} + O\left(\varepsilon^{\frac{(N-p)}{(p-1)}}\right), \tag{2.11}$$

$$\|u_\varepsilon\|_p^p = S^{\frac{N}{p}} + O\left(\varepsilon^{\frac{N}{(p-1)}}\right). \tag{2.12}$$

The following Lemma will be important for estimates in $L^{p^*}(\Omega)$.

Lemma 2.3.3. *Consider* $1 < s < \infty$. *Then there exists a constant* C *(depending on* s*) such that,*

$$\left||\alpha + \beta|^s - |\alpha|^s - |\beta|^s\right| \le C\left(|\alpha|^{s-1}|\beta| + |\alpha||\beta|^{s-1}\right),$$

$\forall\ \alpha, \beta \in \mathbb{R}$.

Remark 13. Consider $\alpha = w$ and $\beta = tu_\varepsilon$ for each $t > 0$, take $s = p^*$ and from the previous Lemma we have,

$$-\|w + tu_\varepsilon\|_{p^*}^{p^*} + \|w\|_{p^*}^{p^*} + t^{p^*}\|u_\varepsilon\|_{p^*}^{p^*} \leq C\left(t\int_\Omega |w|^{p^*-1}u_\varepsilon + t^{p^*-1}\int_\Omega |u_\varepsilon|^{p^*-1}|w|dx\right)$$

2.4 Proof of the Main Theorems

In this section, we will prove the main result of this chapter. However, in order to do so, we will need some auxiliary results.

2.4.1 Geometry of the Mountain Pass and estimates for the *minimax* level

Lemma 2.4.1. *Consider* $1 < q < p < N$, $0 < \mu \leq \mu_1$. *Then there exist positive constants* ε_0, t_0, t_1 *such that*

$$J(w + tu_\varepsilon) < J(w) + \frac{S^{\frac{N}{p}}}{N}$$

for all $t \in (0, t_0) \cup (t_1, \infty)$ *and* $\varepsilon \in (0, \varepsilon_0)$.

Proof. We know from Proposition 1.4.1 taking $v = w + tu_\varepsilon$, that,

$$J(w + tu_\varepsilon) < J(w) + \frac{S^{\frac{N}{p}}}{N}$$

$\forall\, t \in (0, t_0)$ and $\varepsilon \in (0, \varepsilon_0)$. Furthermore, Analogously to what we did to obtain (1.8), we find

$$\begin{aligned} J(w + tu_\varepsilon) \leq & \frac{1}{p}\int_\Omega |\nabla w|^p dx + \frac{1}{q}\int_\Omega |\nabla w|^q dx + \frac{t^p}{p}\int_\Omega |\nabla u_\varepsilon|^p dx + \frac{t^q}{q}\int_\Omega |\nabla u_\varepsilon|^q dx \\ & + \lambda t\int_\Omega |w|^{p-2}w.u_\varepsilon dx + t\mu\int_\Omega |w|^{q-2}w.u_\varepsilon dx + t\int_\Omega |w|^{p^*-2}w.u_\varepsilon dx + t\int_\Omega f u_\varepsilon dx \\ & + \frac{C}{p}t^{\gamma_p}\varepsilon^{\beta_p} + \frac{C}{q}t^{\gamma_q}\varepsilon^{\beta_{\gamma_q}} - \frac{\lambda}{p}\int_\Omega |w + tu_\varepsilon|^p dx - \frac{\mu}{q}\int_\Omega |w + tu_\varepsilon|^q dx \\ & - \frac{1}{p^*}\int_\Omega |w + tu_\varepsilon|^{p^*} dx - \int_\Omega f(w + tu_\varepsilon)dx, \end{aligned}$$

where,

$$\begin{cases} \gamma_s \in (1, s) & if \quad s \in (1, 2), \\ \gamma_s \in [s-1, 2] & if \quad s \in [2, 3), \\ \gamma_s \in \{2, s-1\} & if \quad s \geq 3, \end{cases}$$

with $s \in \{q,p\}$. Moreover, β_r is as in (2.9) since $r \in \{p, \gamma_q\}$. From the Remark 13,

$$\begin{aligned}
J(w+tu_\varepsilon) \leq & \frac{1}{p}\int_\Omega |\nabla w|^p dx + \frac{1}{q}\int_\Omega |\nabla w|^q dx + \frac{t^p}{p}\int_\Omega |\nabla u_\varepsilon|^p dx + \frac{t^q}{q}\int_\Omega |\nabla u_\varepsilon|^q dx \\
& + \lambda t \int_\Omega |w|^{p-2} w u_\varepsilon dx + t\mu \int_\Omega |w|^{q-2} w u_\varepsilon dx + t\int_\Omega |w|^{p^*-2} w u_\varepsilon dx + t\int_\Omega f u_\varepsilon dx \\
& + \frac{C}{p} t^{\gamma_p} \varepsilon^{\beta_p} + \frac{C}{q} t^{\gamma_q} \varepsilon^{\beta_{\gamma_q}} - \frac{\lambda}{p}\int_\Omega |w+tu_\varepsilon|^p dx - \frac{\mu}{q}\int_\Omega |w+tu_\varepsilon|^q dx \\
& - \frac{1}{p^*}\left(\int_\Omega |w|^{p^*} + t^{p^*}\int_\Omega |u_\varepsilon|^{p^*}\right) + C\left(t\int_\Omega |w|^{p^*-1} u_\varepsilon dx + t^{p^*-1}\int_\Omega |w| u_\varepsilon^{p^*-1} dx\right) \\
& - \int_\Omega f(w+tu_\varepsilon) dx.
\end{aligned}$$

Then, adding and subtracting $\frac{\lambda}{p}\int_\Omega |w|^p dx$, $\frac{\mu}{q}\int_\Omega |w|^q dx$, $\frac{\lambda t^p}{p}\int_\Omega |u_\varepsilon|^p dx$ and, $\frac{\mu t^q}{q}\int_\Omega |u_\varepsilon|^q dx$, we obtain

$$\begin{aligned}
J(w+tu_\varepsilon) \leq & \frac{1}{p}\int_\Omega |\nabla w|^p dx + \frac{1}{q}\int_\Omega |\nabla w|^q dx + \frac{t^p}{p}\int_\Omega |\nabla u_\varepsilon|^p dx + \frac{t^q}{q}\int_\Omega |\nabla u_\varepsilon|^q dx \\
& + \lambda t \int_\Omega |w|^{p-2} w u_\varepsilon dx + t\mu \int_\Omega |w|^{q-2} w u_\varepsilon dx + t\int_\Omega |w|^{p^*-2} w u_\varepsilon dx \\
& + \frac{C}{p} t^{\gamma_p} \varepsilon^{\beta_p} + \frac{C}{q} t^{\gamma_q} \varepsilon^{\beta_{\gamma_q}} - \frac{\lambda}{p}\int_\Omega |w+tu_\varepsilon|^p dx - \frac{\mu}{q}\int_\Omega |w+tu_\varepsilon|^q dx \\
& - \frac{1}{p^*}\left(\int_\Omega |w|^{p^*} + t^{p^*}\int_\Omega |u_\varepsilon|^{p^*}\right) + C\left(t\int_\Omega |w|^{p^*-1} u_\varepsilon dx + t^{p^*-1}\int_\Omega |w| u_\varepsilon^{p^*-1} dx\right) \\
& - \int_\Omega f w dx + \frac{\lambda}{p}\int_\Omega |w|^p dx - \frac{\lambda}{p}\int_\Omega |w|^p dx + \frac{\mu}{q}\int_\Omega |w|^q dx - \frac{\mu}{q}\int_\Omega |w|^q dx \\
& + \frac{\lambda t^p}{p}\int_\Omega |u_\varepsilon|^p dx - \frac{\lambda t^p}{p}\int_\Omega |u_\varepsilon|^p dx + \frac{\mu t^q}{q}\int_\Omega |u_\varepsilon|^q dx - \frac{\mu t^q}{q}\int_\Omega |u_\varepsilon|^q dx.
\end{aligned}$$

Arranging the terms of the equation above we obtain

$$\begin{aligned}
J(w+tu_\varepsilon) \leq & \frac{1}{p}\int_\Omega |\nabla w|^p dx + \frac{1}{q}\int_\Omega |\nabla w|^q dx - \frac{\lambda}{p}\int_\Omega |w|^p dx - \frac{\mu}{q}\int_\Omega |w|^q dx - \int_\Omega f w dx \\
& - \frac{1}{p^*}\int_\Omega |w|^{p^*} dx + \frac{t^p}{p}\int_\Omega |\nabla u_\varepsilon|^p dx + \frac{t^q}{q}\int_\Omega |\nabla u_\varepsilon|^q dx + \lambda t\int_\Omega |w|^{p-2} w.u_\varepsilon dx \\
& + t\mu \int_\Omega |w|^{q-2} w.u_\varepsilon dx + t\int_\Omega |w|^{p^*-2} w u_\varepsilon dx + \frac{C}{p} t^{\gamma_p}\varepsilon^{\beta_p} + \frac{C}{q} t^{\gamma_q}\varepsilon^{\beta_{\gamma_q}} \\
& + \frac{\lambda}{p}\left(\int_\Omega |w|^p dx + t^p\int_\Omega |u_\varepsilon|^p dx - \int_\Omega |w+tu_\varepsilon|^p dx\right) - \frac{t^{p^*}}{p^*}\int_\Omega |u_\varepsilon|^{p^*} \\
& - \frac{\lambda t^p}{p}\int_\Omega |u_\varepsilon|^p dx + \frac{\mu}{q}\left(\int_\Omega |w|^q dx + t^q\int_\Omega |u_\varepsilon|^q dx - \int_\Omega |w+tu_\varepsilon|^q dx\right) \\
& - \frac{\mu t^q}{q}\int_\Omega |u_\varepsilon|^q dx + C\left(t\int_\Omega |w|^{p^*-1} u_\varepsilon dx + t^{p^*-1}\int_\Omega |w| u_\varepsilon^{p^*-1} dx\right).
\end{aligned}$$

So,

$$
\begin{aligned}
J(w+tu_\varepsilon) \leq & J(w) + \frac{t^p}{p}\int_\Omega |\nabla u_\varepsilon|^p dx + \frac{t^q}{q}\int_\Omega |\nabla u_\varepsilon|^q dx + Ct\|u_\varepsilon\|_1 \\
& + Ct^{\gamma_p}\varepsilon^{\beta_p} + Ct^{\gamma_q}\varepsilon^{\beta_{\gamma_q}} \\
& + \frac{\lambda}{p}\left(\int_\Omega |w|^p dx + t^p\int_\Omega |u_\varepsilon|^p dx - \int_\Omega |w+tu_\varepsilon|^p dx\right) \\
& - \frac{t^{p^*}}{p^*}\int_\Omega |u_\varepsilon|^{p^*} - \frac{\lambda t^p}{p}\int_\Omega |u_\varepsilon|^p dx - \frac{\mu t^q}{q}\int_\Omega |u_\varepsilon|^q dx \\
& + \frac{\mu}{q}\left(\int_\Omega |w|^q dx + t^q\int_\Omega |u_\varepsilon|^q dx - \int_\Omega |w+tu_\varepsilon|^q dx\right) \\
& + C\left(t\int_\Omega |w|^{p^*-1}u_\varepsilon dx + t^{p^*-1}\int_\Omega |w|u_\varepsilon^{p^*-1}dx\right).
\end{aligned}
$$

Since $\mu, \lambda, t > 0$, $1 < q < p$, and using the estimates provided by

$$
\left|\int_\Omega |w+tu_\varepsilon|^p dx - \int_\Omega |w|^p dx - t^p\int_\Omega |u_\varepsilon|^p dx\right| \leq
$$

$$
2^{p-1}pt^{p-1}\int_\Omega u_\varepsilon^{p-1}|w|dx + 2^{p-1}pt\int_\Omega u_\varepsilon|w|^{p-1}dx
$$

and

$$
\left|\int_\Omega |w+tu_\varepsilon|^q dx - \int_\Omega |w|^q dx - t^q\int_\Omega |u_\varepsilon|^q dx\right| \leq
$$

$$
2^{q-1}qt^{q-1}\int_\Omega u_\varepsilon^{q-1}|w|dx + 2^{q-1}qt\int_\Omega u_\varepsilon|w|^{q-1}dx,
$$

we reach

$$
\begin{aligned}
J(w+tu_\varepsilon) \leq & J(w) + \frac{t^p}{p}\int_\Omega |\nabla u_\varepsilon|^p dx + \frac{t^q}{q}\int_\Omega |\nabla u_\varepsilon|^q dx - \frac{t^{p^*}}{p^*}\int_\Omega |u_\varepsilon|^{p^*}dx \\
& - \frac{\lambda t^p}{p}\int_\Omega |u_\varepsilon|^p dx - \frac{\mu t^q}{q}\int_\Omega |u_\varepsilon|^q dx + t^{\gamma_p}O(\varepsilon^{\beta_p}) + t^{\gamma_q}O(\varepsilon^{\beta_{\gamma_q}}) \\
& + C\left(t\|u_\varepsilon\|_1 + t^{q-1}\|u_\varepsilon\|_{q-1}^{q-1} + t^{p-1}\|u_\varepsilon\|_{p-1}^{p-1} + t^{p^*-1}\|u_\varepsilon\|_{p^*-1}^{p^*-1}\right)
\end{aligned} \tag{2.13}
$$

Using the fact that $(u_\varepsilon)_\varepsilon$ is bounded in $L^s(\Omega)$ for $s = 1, q-1, p-1, p^*-1 < p$, where $\varepsilon > 0$ sufficiently small, and taking $c_0 > 0$ such that $\|u_\varepsilon\|_{p^*}^{p^*} \geq c_0 > 0$ for all $\varepsilon > 0$ sufficiently small, there exist positive constants C and c such that

$$
J(w+tu_\varepsilon) \leq J(w) + ct^q + ct^p - c_0t^{p^*} + C\left(t + t^{q-1} + t^{p-1} + t^{p^*-1} + t^{\gamma_p} + t^{\gamma_q}\right) \tag{2.14}
$$

for all $t \geq t_0$. Knowing that $p-1, p, q, q-1, p^*-1, 1 < p^*$ and taking $\varepsilon > 0$ sufficiently small, the result follows. □

For the geometry of the Mountain Pass, consider that w is a local minimum for the functional J found in Proposition 2.2.7. Notice also that (2.14) shows that one can choose t large enough such that $J(w+tu_\varepsilon)<0$. Moreover, using Lemma 2.2.5, one readily gets

Proposition 2.4.2. *Assume $f \in L^{p^{*'}}(\Omega)$ to be such that $\|f\|_{p^{*'}} \leq M$. Then, the functional $J \in C^1(W_0^{1,p}(\Omega),\mathbb{R})$ and the following hold:*

i) Let w and $\rho_0>0$ be given in Proposition 2.2.7. Then $J(w)<0$, $\|w\|<\rho_0$ and $J(u)\geq 0$ for all $u \in W_0^{1,p}(\Omega)$ with $\|u\|=\rho_0$.

ii) There exists $e \in W_0^{1,p}(\Omega)$ such that $\|e\|>\rho$ and $J(e)<J(w)$.

Now, define

$$\Gamma := \{\theta \in C([0,1],W_0^{1,p}(\Omega)) : \theta(0)=w \text{ and } J(\theta(1))<\alpha\},$$

where $J(w)=\alpha$. Thus, Γ is non-empty and we can define the mountain pass level

$$m := \inf_{\theta \in \Gamma} \max_{t\in[0,1]} J(\gamma(t)).$$

The next result obtains compactness properties for the functional J as long as m is below a specific limit.
Remember that

$$J(w) = \inf_{\|u\|<\rho_0} J(u).$$

Thus we can state the next proposition whose proof is analogous to Proposition 1.4.4 of the previous chapter.

Proposition 2.4.3. *Suppose that $m<J(w)+\frac{S^{\frac{N}{p}}}{N}$. Then J has a critical point $u\neq w$, given by the weak limit of a sequence (PS) at the level m*

To conclude the proof of Theorem 2.2.1, we still need to analyze what happens in the interval $[t_0,t_1]$. For this purpose, we will state and prove that

$$m < J(w)+S^{N/p}/N,$$

for all $t\in[t_0,t_1]$ and sufficiently small ε. We will need $\beta_p,\beta_q,\beta_{\gamma_q}>p$.

Remark 14. Remember that, if, $1<\gamma<p$ and β as in equation (2.8), then,

$$\beta = \frac{(N-p)}{p(p-1)}\gamma > p$$

if and only if,

$$\gamma > p^2(p-1)/(N-p),$$

where $1 < \gamma < N(p-1)/(N-1)$. Moreover,

$$\beta = N - \frac{N}{p}\gamma > p$$

if and only if,

$$\gamma < \frac{(N-p)p}{N},$$

where $\frac{N(p-1)}{(N-1)} < \gamma < p$.

Furthermore,

$$\frac{N(p-1)}{(N-1)} > 1$$

if, and only if,

$$p > 2 - \frac{1}{N}.$$

From Lemma 1.3.6 we have

Lemma 2.4.4. *Let $1 < q < (N-p)p/N$. Then, there exists $\gamma_q \in (1,q)$ such that $\beta_q, \beta_{\gamma_q} > p$ where $N > \max\{p^2+p, p^2/(p-1)\}$ if $1 < p < 2$ and $N > (p-1)p^2 + p$ if $p \geq 2$.*

Remark 15. From the estimate below

$$\int_\Omega |u_\varepsilon|^{q-1} dx = O\left(\varepsilon^{N-(N-p)(q-1)/p}\right) + O\left(\varepsilon^{(N-p)(q-1)/p(p-1)}\right),$$

we have that

$$N - (N-p)(q-1)/p > p.$$

Furthermore, notice also that

$$\frac{(N-p)(q-1)}{p(p-1)} > p$$

if, and only if,

$$N > \frac{p^2(p-1)}{(q-1)} + p.$$

We will need these comparisons to make the correct estimates in the next proposition.

Proposition 2.4.5. *Consider $\lambda \in (0, \lambda_1)$, $0 < \mu \leq \mu_1$, $N \in \mathbb{N}$ such that $1 < q < (N-p)p/N$ and $N > \max\{(p-1)p^2 + p, (p-1)p^2/(q-1) + p\}$ if $p \geq 2$ and $N > \max\{p^2+p, p^2/(p-1), (p-1)p^2/(q-1)+p\}$ if $1 < p < 2$. Then*

$$m < J(w) + \frac{S^{\frac{N}{p}}}{N},$$

for sufficiently small $\varepsilon > 0$.

Proof. Proceeding in a similar way to what was done in Proposition 1.4.5 of Chapter 1 we will have a similar estimate for $J(w+tu_\varepsilon)$. The difference here, for $t \in [t_0, t_1]$, will be given by the term $\int_\Omega |u_\varepsilon|^{q-1}dx$ which was estimated in (2.10) , that is,

$$\begin{aligned} J(w+tu_\varepsilon) \leq & J(w) + S^{N/p}/N + O\left(\varepsilon^{(N-p)/(p-1)}\right) + O(\varepsilon^{N/(p-1)}) + O(\varepsilon^{\beta_q}) - C\varepsilon^p + C\|u_\varepsilon\|_1 \\ & + O(\varepsilon^{(N-p)/p}) + O\left(\varepsilon^{N-(N-p)(q-1)/p}\right) + O\left(\varepsilon^{(N-p)(q-1)/p(p-1)}\right) + O\left(\varepsilon^{\beta_p}\right) + O(\varepsilon^{\beta_{\gamma_q}}) \\ & \leq J(w) + S^{N/p}/N + O(\varepsilon^{\beta_q}) - C\varepsilon^p + C\|u_\varepsilon\|_1 + O(\varepsilon^{(N-p)/p}) + O\left(\varepsilon^{N-(N-p)(q-1)/p}\right) \\ & + O(\varepsilon^{(N-p)(q-1)/p(p-1)}) + O\left(\varepsilon^{\beta_p}\right) + O(\varepsilon^{\beta_{\gamma_q}}) \end{aligned}$$

By Remark 15 and proceeding as in [19], we have

$$J(w+tu_\varepsilon) \leq J(w) + \frac{S^{\frac{N}{p}}}{N} - C\varepsilon^p.$$

for small $\varepsilon > 0$. $\square$

Proposition 2.4.6. *Consider* $\mu = 0$, $N \in \mathbb{N}$ *such that* $N > (p-1)p^2 + p$ *if* $p \geq 2$ *and* $N > \max\{p^2+p, p^2/(p-1)\}$ *if* $1 < p < 2$ *and* $1 < q < (N-p)p/N$. *Then*

$$m < J(w) + \frac{S^{\frac{N}{p}}}{N},$$

for $\varepsilon > 0$ *small sufficiently.*

Proof. It suffices to note that the estimates given in (2.13) imply that

$$\begin{aligned} J(w+tu_\varepsilon) \leq & J(w) + \frac{t^p}{p}\int_\Omega |\nabla u_\varepsilon|^p dx + \frac{t^q}{q}\int_\Omega |\nabla u_\varepsilon|^q dx - \frac{t^{p^*}}{p^*}\int_\Omega |u_\varepsilon|^{p^*}dx \\ & - \frac{\lambda t^p}{p}\int_\Omega |u_\varepsilon|^p dx + C\left(t\|u_\varepsilon\|_1 + t^{p-1}\|u_\varepsilon\|_{p-1}^{p-1} + t^{p^*-1}\|u_\varepsilon\|_{p^*-1}^{p^*-1}\right) \\ & + t^{\gamma_p}O(\varepsilon^{\beta_p}) + t^{\gamma_q}O(\varepsilon^{\beta_{\gamma_q}}). \end{aligned}$$

Similarly to Proposition 2.4.5 the result follows. $\square$

2.4.2 Proof of the main theorems

For the proof of Theorem 2.2.1, we use Propositions 2.2.7, 2.4.3 and 2.4.5, and Theorem 2.2.2 is completed using Propositions 2.2.7, 2.4.3 and 2.4.6.

Chapter 3

(N, q)-Laplacian equations with critical exponential growth and jumping nonlinearities

3.1 Introduction

In this chapter, we establish the existence of nontrivial solutions for a (N, q)-Laplacian equation characterized by:

$$\begin{cases} -\Delta_N u - \Delta_q u = \lambda |u|^{N-2} u + g(x, u_+) + f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^N$ denotes a bounded domain with a smooth boundary, $N \geq 2$, $\lambda > 0$ is a real parameter, $1 < q < N$ and $g : \overline{\Omega} \times \mathbb{R} \to [0, \infty)$ satisfies a Trudinger-Moser growth condition uniformly in $x \in \Omega$. This problem is new for the N-Laplacian and also for the (N, q)-Laplacian. Our objective as in Chapter 1 is to elucidate the effects of incorporating unilateral critical growth and a non-homogeneous term $f \in L^\infty$ into the equation. As is widely acknowledged, one of the primary challenges in dealing with this type of growth condition in g is to demonstrate that the *minimum* level of the associated functional stays below some constant. This requirement is addressed in [5, 27] by introducing an additional hypothesis. To achieve the desired level, the techniques used in [27] for the case $N = 2$ require the Moser functions z_n^r to be supported within a ball B_r, where $r > 0$ needs to be sufficiently small. Here, similar approaches will be used, but more intricate conditions over the choices of these radius are necessary.

Let us begin by assuming that $f \in L^\infty(\Omega)$ and that g exhibits a critical growth with exponent $\alpha_0 > 0$. This means that there exists a positive constant α_0 such that

$$\lim_{t\to+\infty}\frac{g(x,t)}{e^{\alpha t^{\frac{N}{N-1}}}}=\begin{cases}0, & if \quad \alpha>\alpha_0\\ +\infty, & if \quad \alpha<\alpha_0,\end{cases} \tag{α_0}$$

uniformly in $x\in\Omega$. This condition is motivated by the Pohozaev-Trudinger-Moser Inequality (see [20]), which state the existence of a positive constant $C=C(N)$ satisfying

$$\int_\Omega e^{\left(\alpha_N|u|^{\frac{N}{N-1}}\right)}dx\le C(N)|\Omega|, \tag{3.2}$$

for all $u\in W_0^{1,N}(\Omega)$, $N\ge 2$, such that $\|\nabla u\|_N\le 1$, where $\alpha_N=N\omega_{N-1}^{\frac{1}{N-1}}$ and ω_{N-1} is the unitary (N-1)-dimensional sphere volume.

Furthermore, we will consider

($\tilde{g}_1$) $g:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}^+$ is a continuous function and $g(x,t)=0$ for all $x\in\Omega$ and $t\le 0$;

($\tilde{g}_2$) There are $R,M>0$ such that

$$0<G(x,t):=\int_0^t g(x,s)ds\le Mg(x,t),\quad \forall\, x\in\overline{\Omega} \text{ and } t\ge R;$$

($\tilde{g}_3$) There is $C>0$ such that

$$sg(x,s)\ge\gamma(s)e^{\alpha_0 s^{\frac{N}{N-1}}},\qquad \forall x\in\Omega \text{ and } s\ge C,$$

where $\gamma(s)$ is such that

$$\liminf_{s\to+\infty}\frac{\gamma(s)}{e^{\varepsilon_0 s^{\frac{1}{N-1}}}}>0$$

for some $\varepsilon_0>0$.

In this chapter, we will always assume that $0<\lambda<\lambda_1$ and $1<q<N$. The natural space to deal with our equation is $W_0^{1,N}(\Omega)$, where we consider two equivalent norms

$$\|u\|=\left(\int_\Omega|\nabla u|^N dx\right)^{1/N}\quad \text{and}\quad \|u\|_\lambda=\left(\int_\Omega|\nabla u|^N dx-\lambda\int_\Omega|u|^N dx\right)^{1/N}.$$

The functional associated with Equation (3.1) is $J:W_0^{1,N}(\Omega)\to\mathbb{R}$ given by:

$$J(u)=\frac{1}{N}\int_\Omega\left(|\nabla u|^N-\lambda|u|^N\right)dx+\frac{1}{q}\int_\Omega|\nabla u|^q dx-\int_\Omega G(x,u_+)\,dx-\int_\Omega f(x)u dx.$$

By conditions ($\tilde{g}_1$) $-$ ($\tilde{g}_3$) and (α_0), due to (3.2), we obtain that J is of class C^1 with derivative given by

$$J'(u)v = \int_\Omega (|\nabla u|^{N-2}\nabla u \nabla v - \lambda |u|^{N-2} uv)dx + \int_\Omega |\nabla u|^{q-2}\nabla u \nabla v dx$$
$$- \int_\Omega g(x, u_+)v dx - \int_\Omega f(x) v dx,$$

for all u and v in $W_0^{1,N}(\Omega)$. Weak solutions to the problem (3.1) are exactly the critical points of this functional.

The main result of this chapter is given below.

Theorem 3.1.1. *Suppose that* (α_0) *and* ($\tilde{g}_1$) $-$ ($\tilde{g}_3$) *hold and that* $f \in L^\infty(\Omega)$ *is a nontrivial function such that* $f \leq 0$. *Then, there exist two solutions to the problem* (3.1).

As we have done in Chapter 1, we will find two distinct solutions to the problem (3.1). At first, using minimization arguments we get a negative solution, w, for the linear problem

$$\begin{cases} -\Delta_N u - \Delta_q u = \lambda |u|^{N-2}u + f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

which is also a solution for (3.1), since $g(x, w) = 0$. After that, we prove a L^∞ estimate in order to obtain some regularity results, and then show that a minimum for the functional J in the C_0^1 topology is also a minimum in the $W_0^{1,N}$ topology. Finally, we apply the mountain pass theorem without the Palais-Smale condition to prove the existence of a second solution for the main problem of this chapter.

3.2 Preliminaries

The objective of this section is to present some properties that the function g has, which are obtained as a consequence of the conditions ($\tilde{g}_1$) $-$ ($\tilde{g}_3$).

Lemma 3.2.1. *Suppose that* g *satisfies* ($\tilde{g}_1$) $-$ ($\tilde{g}_2$). *Then*

($\tilde{g}_4$) *For* $R > 0$ *and* $M > 0$ *as in condition* ($\tilde{g}_2$), *there is a constant* $C > 0$ *such that*

$$G(x, u) \geq Ce^{\left(\frac{1}{M}u\right)}, \quad \forall\, (x, u) \in \Omega \times [R, +\infty);$$

($\tilde{g}_5$) *There are* $S > 0$ *and* $\sigma > N$ *such that*

$$\sigma G(x, u) \leq u g(x, u), \quad \forall\, (x, u) \in \Omega \times [S, +\infty).$$

($\tilde{g}_6$) *If g also satisfies* (α_0), *for any $\beta > \alpha_0$ there is a constant $C = C(\beta) > 0$ such that*

$$g(x,u) \leq Ce^{\beta|u|^{N'}}, \quad \forall\ (x,u) \in \Omega \times \mathbb{R}.$$

Proof. By ($\tilde{g}_2$), there are $R, M > 0$ such that

$$0 < G(x,t) \leq Mg(x,t), \quad \text{for all } x \in \overline{\Omega} \text{ and } t \geq R.$$

Then,

$$\frac{u-R}{M} = \int_R^u \frac{1}{M}dt \leq \int_R^u \frac{g(x,t)}{G(x,t)}dt = \ln\left(\frac{G(x,u)}{G(x,R)}\right),$$

implying that

$$G(x,u) \geq e^{\frac{(u-R)}{M}}G(x,R) \geq Ce^{\frac{u}{M}}, \quad \text{for all } x \in \overline{\Omega} \text{ and } u \geq R.$$

where $C = e^{\frac{-R}{M}} \min_{x\in\overline{\Omega}} G(x,R) > 0$. Therefore, we have ($\tilde{g}_4$). To prove ($\tilde{g}_5$), using ($\tilde{g}_2$) we obtain,

$$0 < \theta G(x,u) \leq \theta Mg(x,u), \quad \forall u \geq R \text{ and } \theta > 0.$$

Then, for all $x \in \Omega$ and $u \geq S = \max\{R, \theta M\}$, we get

$$ug(x,u) \geq \theta Mg(x,u) \geq \theta G(x,u),$$

which is ($\tilde{g}_5$). For the last item, since g has critical growth, for each $\beta > \alpha_0$ we see that given $\varepsilon = 1$ there exists $R_1 > 0$ such that,

$$g(x,u) \leq e^{\beta u^{N'}} \quad \text{for all, } x \in \Omega \text{ and } u \geq R_1.$$

Considering $(x,u) \in \overline{\Omega} \times [0, R_1]$, since g is continuous in a compact set, there is $K > 0$ such that $0 \leq g(x,u) \leq K$ for all $(x,u) \in \overline{\Omega} \times [0, R_1]$. Recalling that $e^{\beta u^{N'}} \geq 1$ for all $u \geq 0$, we see that

$$g(x,u) \leq \max\{1, K\}e^{\beta|u|^{N'}}, \ \forall (x,u) \in \Omega \times \mathbb{R},$$

and conclude this proof. □

To establish an estimate of g from below, we need the following technical result.

Lemma 3.2.2. *Consider $\sigma > 1$. Then there is $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying*

$$(s+t)^\sigma \leq s^\sigma + \eta(t)s^{\sigma-1}, \quad \textit{for all } s \geq 1 \ \textit{ and } t > 0,$$

such that $\lim_{t\to 0^+}\eta(t) = 0$.

Proof. For $t > 0$, let us denote

$$\eta(t) = \sup_{s\geq 1}\frac{(s+t)^\sigma - s^\sigma}{s^{\sigma-1}}.$$

We need to show that this supremum is finite for any $t > 0$ and $\eta(t) \to 0$ as $t \to 0^+$. For any $t > 0$ and $s \geq 1$ we have

$$\frac{(s+t)^\sigma - s^\sigma}{s^{\sigma-1}} = \frac{s^\sigma}{s^{\sigma-1}}\left[\left(1+\frac{t}{s}\right)^\sigma - 1\right] = \frac{\left[\left(1+\frac{t}{s}\right)^\sigma - 1\right]}{\frac{1}{s}} = t\cdot\frac{[(1+r)^\sigma - 1]}{r},$$

with $r = t/s \in (0, t]$. Now consider

$$\rho(r) = \frac{(1+r)^\sigma - 1}{r},\ r > 0.$$

We see that

$$\rho'(r) = \frac{\sigma(1+r)^{\sigma-1}r - (1+r)^\sigma + 1}{r^2},\ r > 0.$$

Defining $h(r) = \sigma(1+r)^{\sigma-1}r - (1+r)^\sigma + 1$, certainly h is derivable for all $r > 0$, $h(0) = 0$ and

$$h'(r) = \sigma(\sigma-1)(1+r)^{\sigma-2}r > 0,\ \forall\, r > 0.$$

Then h is an increasing function and $h(r) > h(0) = 0$, $\forall\, r > 0$. Thus,

$$\rho'(r) = \frac{h(r)}{r^2} > 0,\ \forall\, r > 0,$$

and ρ is increasing for $r > 0$. Then, for $s \geq 1$ we have $0 < r = t/s \leq t$, which implies

$$\rho(t/s) \leq \rho(t) = \frac{(1+t)^\sigma - 1}{t}.$$

Consequently,

$$\eta(t) = t\cdot\sup_{s\geq 1}\rho(t/s) = t\cdot\rho(t) = (1+t)^\sigma - 1,$$

which means that $\eta(t) \in \mathbb{R}$ and

$$\lim_{t\to 0^+}\eta(t) = 0.$$

From the definition of η we have the result. □

Lemma 3.2.3. *Suppose that* $(\tilde{g}_1)$ *and* $(\tilde{g}_3)$ *hold. So, there are* σ_0, γ_∞ *and* $C_0 > 0$ *such that*

$$sg(x,s) \geq \gamma_\infty e^{\alpha_0(s+\sigma_0)^{N'}}$$

for all $x \in \Omega$ *and* $s \geq C_0$.

Proof. From condition $(\tilde{g}_3)$, there exist $C > 0$ satisfying

$$sg(x,s) \geq \gamma(s)e^{\alpha_0 s^{N'}}, \quad \forall (x,s) \in \Omega \times [C,\infty)$$

and $\varepsilon_0 > 0$ such that

$$\liminf_{s\to+\infty} \frac{\gamma(s)}{e^{\varepsilon_0 s^{\frac{1}{N-1}}}} > 0.$$

So, there are $\gamma_\infty > 0$ and $\tilde{C} \geq 1$ such that

$$\gamma(s) \geq \gamma_\infty e^{\varepsilon_0 s^{\frac{1}{N-1}}}$$

for all $s \geq \tilde{C}$. Thus, for $s \geq C_0 := \max\{C, \tilde{C}\}$ we have

$$sg(x,s) \geq \gamma(s)e^{\alpha_0 s^{N'}} \geq \gamma_\infty e^{\varepsilon_0 s^{\frac{1}{N-1}}} e^{\alpha_0 s^{N'}} = \gamma_\infty e^{\alpha_0\left(s^{N'} + \frac{\varepsilon_0}{\alpha_0} s^{\frac{1}{N-1}}\right)}.$$

From Lemma 3.2.2, for $\varepsilon = \varepsilon_0/\alpha_0$ we can choose $\sigma_0 > 0$ such that $\eta(\sigma_0) < \varepsilon$ and

$$s^{N'} + \varepsilon s^{N'-1} \geq s^{N'} + \eta(\sigma_0)s^{N'-1} \geq (s+\sigma_0)^{N'}$$

for all $s \geq 1$. Therefore,

$$sg(x,s) \geq \gamma_\infty e^{\alpha_0(s^\sigma + \varepsilon s^{\sigma-1})} \geq \gamma_\infty e^{\alpha_0(s+\sigma_0)^{N'}},$$

for all $s \geq C_0$. □

3.3 L^∞ estimates and regularity

As in Chapter 1 we will get the first solution for (3.1) as a negative solution for the problem (3.3), which is obtained by minimization arguments. To properly establish this first existence result, we need to show some regularity results. The issue is that we cannot apply Theorems 1.2.4 and 1.2.6, so it was necessary to overcome these difficulties using Lieberman's regularity results and an adaptation of Theorem 1.1 of [18] for our operator.

In this section, we consider two auxiliary problems, with more general assumptions on the nonlinear terms. The first is

$$\begin{cases} -\Delta_N u - \Delta_q u &= \tilde{f}(x,u) \quad \text{in } \Omega \\ u &= 0 \quad\quad\quad \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

where $\tilde{f} : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function satisfying

$$|\tilde{f}(x,s)| \le \tilde{C}(1+|s|)^{\tilde{p}} e^{\tilde{\beta}|s|^{N'}}, \quad a.e.\ (x,s) \in \Omega \times \mathbb{R}, \tag{3.5}$$

for some $\tilde{C}$, $\tilde{\beta} > 0$ and $\tilde{p} > 1$. For the second problem, we consider $H : \mathbb{R} \to \mathbb{R}$ a C^1 function, which is even and satisfies $H(t) \le C(1+|t|)^{p+1} e^{\beta|t|^{N'}}$ for all $t \in \mathbb{R}$, for some $p > 1$ and C, $\beta > 0$, and denote $h(t) = H'(t)$. For a fixed $u_0 \in L^\infty(\Omega) \cap W_0^{1,N}(\Omega)$ we will deal with

$$\begin{cases} -\Delta_N(u+u_0) - \Delta_q(u+u_0) = \tilde{f}(x, u+u_0) + \mu h(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

A solution for (3.6) is a pair $(\mu, u) \in \mathbb{R} \times W_0^{1,N}(\Omega)$. The purpose of this section is to show that if u is a solution for (3.4) or (μ, u) is a solution for (3.6), then $u \in L^\infty(\Omega)$ and belongs to $C^{1,\gamma}(\overline{\Omega})$, for some $\gamma \in (0,1)$.

3.3.1 Boundedness of solutions

Our first step is to prove that the solutions belong to $L^\infty(\Omega)$. We observe that if $u \in W_0^{1,N}(\Omega)$ is a solution to the problem (3.4), then $(0, u)$ is a solution to (3.6) with $u_0 \equiv 0$. Thus, we will deal with the second problem.

Proposition 3.3.1. *Consider $(\mu, u) \in (-\infty, 0] \times W_0^{1,N}(\Omega)$ a solution to (3.6), where $u_0 \equiv 0$ or $u_0 \in L^\infty(\Omega) \cap W_0^{1,N}(\Omega)$ solves (3.4). Let $\theta > 1$ and $M > 0$ be such that*

$$\|\tilde{f}(\cdot, u_0)\|_{L^\theta(\Omega)} + \|\tilde{f}(\cdot, u+u_0)\|_{L^\theta(\Omega)} \le M.$$

Then, there exists a constant $d = d(N, p, M, \theta, \Omega)$ such that

$$\|u\|_{L^\infty(\Omega)} \le d.$$

Proof. For $k \ge 0$ we consider the truncation functions

$$T_k(s) = \begin{cases} s+k & \text{if } s \le -k \\ s-k & \text{if } s \ge k \\ 0 & \text{if } -k < s < k, \end{cases}$$

and the set

$$\Omega_k = \{x \in \Omega : |u(x)| \ge k\}.$$

Since h is odd, we get $\mu h(s) T_k(s) \le 0$ for all $s \in \mathbb{R}$ if $\mu \le 0$. For $(\mu, u) \in (-\infty, 0] \times W_0^{1,N}(\Omega)$ a solution to (3.6), we will use

$$T_k(u) = (u+k)\chi_{(-\infty,-k]}(u(x)) + (u-k)\chi_{[k,+\infty)}(u(x))$$

as a test function for (3.6). When $u_0 \equiv 0$ we get

$$\begin{aligned} \int_\Omega |\nabla T_k(u)|^N dx &= \int_\Omega |\nabla u|^{N-2}\nabla u \nabla T_k(u)dx \\ &\leq \int_\Omega |\nabla u|^{N-2}\nabla u \nabla T_k(u)dx + \int_\Omega |\nabla u|^{q-2}\nabla u \nabla T_k(u)dx \\ &= \int_\Omega \tilde{f}(x,u)T_k(u)dx. \end{aligned} \tag{3.7}$$

If $u_0 \neq 0$, it is a solution for (3.4). Using $T_k(u)$ as test function for the solutions u and u_0, we see that

$$\begin{aligned} &\int_\Omega \left(|\nabla(u+u_0)|^{N-2}\nabla(u+u_0) - |\nabla u_0|^{N-2}\nabla u_0\right).\nabla T_k(u)dx \\ &+ \int_\Omega \left(|\nabla(u+u_0)|^{q-2}\nabla(u+u_0) - |\nabla u_0|^{q-2}\nabla u_0\right).\nabla T_k(u)dx \\ &\leq \int_\Omega \left(\tilde{f}(x,u+u_0) - \tilde{f}(x,u_0)\right) T_k(u)dx. \end{aligned} \tag{3.8}$$

Taking $\sigma = \frac{N+1}{\theta-1}$ and $r = \theta\sigma$, by the Hölder's inequality we obtain

$$\begin{aligned} &\int_\Omega \left(\tilde{f}(x,u+u_0) - \tilde{f}(x,u_0)\right) T_k(u)dx \\ &\leq \left(\int_\Omega (|\tilde{f}(x,u+u_0)| + |\tilde{f}(x,u_0)|)^\theta dx\right)^{\frac{1}{\theta}} \left(\int_\Omega |T_k(u)|^r dx\right)^{\frac{1}{r}} |\Omega_k|^{\frac{r-1-\sigma}{r}} \\ &\leq M \left(\int_\Omega |T_k(u)|^r dx\right)^{\frac{1}{r}} |\Omega_k|^{\frac{r-1-\sigma}{r}}. \end{aligned} \tag{3.9}$$

On the other hand, since $N \geq 2$ we have a constant $C_1 = C_1(N)$ such that

$$\begin{aligned} &C_1 \int_\Omega |\nabla T_k(u)|^N dx \\ &\leq \int_\Omega \left(|\nabla T_k(u+u_0)|^{N-2}\nabla T_k(u+u_0) - |\nabla T_k(u_0)|^{N-2}\nabla T_k(u_0)\right) \nabla T_k(u)\, dx \\ &= \int_\Omega \left(|\nabla(u+u_0)|^{N-2}\nabla(u+u_0) - |\nabla u_0|^{N-2}\nabla u_0\right).\nabla T_k(u)dx. \end{aligned}$$

Similarly, since $q > 1$, we get

$$\int_\Omega \left(|\nabla(u+u_0)|^{q-2}\nabla(u+u_0) - |\nabla u_0|^{q-2}\nabla u_0\right).\nabla T_k(u)dx$$
$$= \int_\Omega \left(|\nabla T_k(u+u_0)|^{q-2}\nabla T_k(u+u_0) - |\nabla T_k(u_0)|^{q-2}\nabla T_k(u_0)\right)\nabla T_k(u)dx \geq 0.$$

Recalling the Sobolev embedding $W_0^{1,N}(\Omega) \hookrightarrow L^r(\Omega)$, we have

$$\int_\Omega |\nabla T_k(u)|^N dx \geq C_2 \left(\int_\Omega |T_k(u)|^r dx\right)^{\frac{N}{r}}, \tag{3.10}$$

for a constant $C_2 = C_2(N, r, \Omega) > 0$. Joining (3.7)-(3.10) we get,

$$\left(\int_\Omega |T_k(u)|^r dx\right)^{\frac{N}{r}} \leq M' \left(\int_\Omega |T_k(u)|^r dx\right)^{\frac{1}{r}} |\Omega_k|^{\frac{r-1-\sigma}{r}},$$

where $M' = M/(C_1C_2)$. So, by definition of σ and r, we obtain

$$\int_\Omega |T_k(u)|^r dx \leq M'|\Omega_k|^{\frac{r-1-\sigma}{N-1}} = M'|\Omega_k|^{\frac{\theta\sigma-1-\sigma}{N-1}} = M'|\Omega_k|^{\frac{\sigma(\theta-1)-1}{N-1}} = M'|\Omega_k|^{\frac{N}{N-1}}. \tag{3.11}$$

Since $\Omega_m \subset \Omega_k$, for $0 \leq k < m$, and

$$|T_k(s)| = |(s+k)\chi_{(-\infty,-k]}(s) + (s-k)\chi_{[k,+\infty)}(s)| = (|s|-k)(1-\chi_{[-k,k]}(s))$$

for all $s \in \mathbb{R}$, we see that

$$\int_\Omega |T_k(u)|^r dx = \int_{\Omega_k} (|u|-k)^r dx \geq \int_{\Omega_m} (|u|-k)^r dx \geq (m-k)^r |\Omega_m|.$$

Now, substituting the last estimates in (3.11) we have,

$$(m-k)^r|\Omega_m| \leq M'|\Omega_k|^{\frac{N}{N-1}}, \quad \text{for all } 0 \leq k < m.$$

Considering $\varphi(k) = |\Omega_k|$, for $0 \leq k < m$ we have

$$\varphi(m) \leq M'(m-k)^{-r}(\varphi(k))^{\frac{N}{N-1}}. \tag{3.12}$$

Define a sequence $\{k_n\}$ by $k_0 = 0$ and

$$k_n = k_{n-1} + \frac{d}{2^n}, \ n = 1, 2, \cdots \tag{3.13}$$

where $d = 2^N (M')^{\frac{1}{r}} |\Omega|^{\frac{1}{(N-1)r}}$. We see that

$$k_n = d \sum_{j=1}^{n} 2^{-j} \to d \quad \text{as } n \to \infty.$$

We will show by induction that

$$\varphi(k_n) \leq 2^{nr(1-N)} \varphi(0). \tag{3.14}$$

Using $k_n > k_{n-1} \geq 0$ in (3.12), we get

$$\begin{aligned}
\varphi(k_1) &\leq M'(k_1 - k_0)^{-r} (\varphi(k_0))^{\frac{N}{N-1}} = M' \left(\frac{d}{2}\right)^{-r} (\varphi(0))^{\frac{N}{N-1}} \\
&= M' \left[2^{N-1} (M')^{\frac{1}{r}} |\Omega|^{\frac{1}{(N-1)r}}\right]^{-r} |\Omega|^{\frac{N}{N-1}} = 2^{r(1-N)} |\Omega| \\
&= 2^{r(1-N)} \varphi(0).
\end{aligned}$$

Now, we have to assume that the result is valid for m. So

$$\begin{aligned}
\varphi(k_{m+1}) &\leq M'(k_{m+1} - k_m)^{-r} (\varphi(k_m))^{\frac{N}{N-1}} = M' \left(\frac{d}{2^{m+1}}\right)^{-r} (\varphi(k_m))^{\frac{N}{N-1}} \\
&= M' \left(2^{N-(m+1)} (M')^{\frac{1}{r}} |\Omega|^{\frac{1}{(N-1)r}}\right)^{-r} (\varphi(k_m))^{\frac{N}{N-1}} = 2^{-r[N-(m+1)]} |\Omega|^{\frac{-1}{(N-1)}} \varphi(k_m)^{\frac{N}{N-1}} \\
&\leq 2^{-r[N-(m+1)]-mrN} |\Omega| = 2^{r(m+1)(1-N)} \varphi(0),
\end{aligned}$$

which means that it is also valid for $m+1$. Hence, (3.14) holds for any $n \in \mathbb{N}$. Consequently, $\varphi(k_n) \to 0$ as $n \to \infty$. Since $k_n \uparrow d$ and φ is nonincreasing, we obtain

$$0 \leq \varphi(d) = |\Omega_d| \leq \varphi(k_n) \to 0.$$

From the definition of Ω_d, we conclude that $|\Omega_d| = 0$ implies that

$$\|u\|_{L^\infty(\Omega)} \leq d,$$

which finishes this proof. □

3.3.2 Regularity of solutions

From the previous result, we know that solutions for (3.6) are bounded. So we can use Lieberman's regularity results (see [10] and [11]). Let us check the assumptions. We define

$$\tilde{\rho}(t) = t^{N-1} + t^{q-1}, \quad \text{for } t > 0.$$

Notice that $\tilde{\rho}$ is of class C^1 and we have

$$0 < q - 1 \leq \frac{t\tilde{\rho}'(t)}{\tilde{\rho}(t)} \leq N - 1, \quad \forall t > 0.$$

Denoting

$$A(\eta) = \tilde{\rho}(|\eta|)\frac{\eta}{|\eta|} = \left(|\eta|^{N-2} + |\eta|^{q-2}\right)\eta \quad \text{and} \quad a^{ij}(\eta) = \frac{\partial A^i}{\partial \eta_j}(\eta), \quad \text{for } \eta \in \mathbb{R}^N$$

we verify that there is a positive constant Λ such that

$$\sum_{i,j=1}^{N} a^{ij}(\eta)\xi_i\xi_j \geq \frac{\tilde{\rho}(|\eta|)}{|\eta|}|\xi|^2 \quad \text{and} \quad \sum_{i,j=1}^{N} |a^{ij}(\eta)| \leq \Lambda\frac{\tilde{\rho}(|\eta|)}{|\eta|}.$$

for all $\eta, \xi \in \mathbb{R}^N$. Then, [10, Theorem 1] and [11, Theorem 1.7]) give us the following:

Theorem 3.3.2. *Let Ω be a bounded domain in $\mathbb{R}^N$ with $C^{1,\alpha}$ boundary. If $u \in W_0^{1,N}(\Omega) \cap L^\infty(\Omega)$ is a weak solution of*

$$-\mathrm{divA}(|\nabla \mathrm{u}|) = \mathrm{B}(\mathrm{x}, \mathrm{u}) \quad \text{in } \Omega$$

with $|u(x)| \leq M_0$ and $|B(x,t)| \leq \Lambda_1$ for $x \in \Omega$ and $t \in [-M_0, M_0]$, then there is a positive constant $\gamma = \gamma(\alpha, \Lambda, q, N, M_0)$ such that u is in $C^{1,\gamma}(\overline{\Omega})$. Moreover

$$\|u\|_{C^{1,\gamma}(\overline{\Omega})} \leq C(\alpha, \Lambda_1, \Lambda, q, N, M_0, \Omega).$$

We can apply these last results to obtain the boundedness and regularity of w, the global minimum of the functional I. It will be presented in Proposition 3.5.1.

3.4 $C_0^1 \times W_0^{1,N}$ topology

The next proposition will be important to show that a local minimizer of functional J in the C_0^1 topology is also a local minimizer of J in the $W_0^{1,N}$ topology. The proof of this result uses arguments similar to those found in [18], where the author proved a similar result for the N-Laplacian operator and considered a nonlinear term f such that $sf(x,s) \geq 0$ in $\Omega \times \mathbb{R}$.

Firstly, we consider the functional $\tilde{J} : W_0^{1,N}(\Omega) \to \mathbb{R}$ associated with problem (3.4), which is given by

$$\tilde{J}(u) = \frac{1}{N}\int_\Omega |\nabla u|^N dx + \frac{1}{q}\int_\Omega |\nabla u|^q dx - \int_\Omega \tilde{F}(x,u)dx.$$

Proposition 3.4.1. *If $u_0 \in W_0^{1,N}(\Omega)$ is a local minimum of $\tilde{J}$ in the $C_0^1(\overline{\Omega})$ topology, then $u_0 \in C^{1,\gamma}(\overline{\Omega})$ for some $\gamma \in (0,1)$, and u_0 is a local minimum of $\tilde{J}$ in the $W_0^{1,N}(\Omega)$ topology.*

Proof. Let $u_0 \in C_0^1(\overline{\Omega})$ be a local minimum of $\tilde{J}$ on the $C_0^1(\overline{\Omega})$ topology. By the previous sections we know that $u_0 \in C^{1,\gamma}(\overline{\Omega})$ for some $\gamma \in (0,1)$. We have that 0 is a local minimizer of the functional $\tilde{J}(\cdot + u_0)$ in $C_0^1(\overline{\Omega})$ and in this way we will show that 0 is a local minimizer for the functional $\tilde{J}(\cdot + u_0)$ in $W_0^{1,N}(\Omega)$. Let us assume, by contradiction, that 0 is not a local minimizer for the functional $\tilde{J}(\cdot + u_0)$. Then, there is a sequence $\{v_n\}_{n\geq 1} \subset W_0^{1,N}(\Omega)$ such that

$$\|v_n\| \leq \frac{1}{n} \quad \text{and} \quad \tilde{J}(u_0 + v_n) < \tilde{J}(u_0), \ \forall\, n \geq 1. \tag{3.15}$$

Consider, $H(s) = |s|^{p+1}e^{2^{N'}\beta|s|^{N'}}$, for $p = \tilde{p} > 1$ and $\beta = \tilde{\beta} > 0$ as in (3.5). Define, for each $\varepsilon > 0$,

$$\mathcal{C}_\varepsilon := \{u \in W_0^{1,N}(\Omega) : K(u) = \|H(u)\|_{L^1(\Omega)} \leq \varepsilon\}. \tag{3.16}$$

Notice that for each $\varepsilon \in (0,1)$ there is $N_\varepsilon \in \mathbb{N}$ such that $v_n \in \mathcal{C}_\varepsilon$ for $n \geq N_\varepsilon$. In fact, by (3.15) and (3.2) we obtain

$$\int_\Omega H(v_n)dx \leq \left(\int_\Omega |v_n|^{2(p+1)}dx\right)^{\frac{1}{2}} \left(\int_\Omega \exp\left(2^{N'+1}\beta\|v_n\|^{N'}\left|\frac{v_n}{\|v_n\|}\right|^{N'}\right)dx\right)^{\frac{1}{2}} \leq \varepsilon$$

for $n > 1$ large enough. This shows that $\mathcal{C}_\varepsilon$ is not empty, for any $\varepsilon \in (0,1)$. Clearly, K is a convex operator, implying that C_ε is a convex set. Moreover, $\mathcal{C}_\varepsilon$ is closed in $W_0^{1,N}(\Omega)$. Then, $\mathcal{C}_\varepsilon$ is closed in the weak topology of $W_0^{1,N}(\Omega)$. Since $\tilde{f}$ satisfies (3.5), it follows that

$$|\tilde{F}(x,t)| \leq C(1+|t|)^{p+1}e^{\beta|t|^{N'}}, \qquad \text{a.e. } x \in \Omega,\ \forall t \in \mathbb{R}. \tag{3.17}$$

Recalling that $u_0 \in L^\infty(\Omega)$, we get

$$\begin{aligned} |\tilde{F}(x,u+u_0)| &\leq C(1+\|u_0\|_{L^\infty(\Omega)}+|u|)^{p+1}e^{\beta|u+u_0|^{N'}} \\ &\leq C(1+|u|)^{p+1}e^{\beta 2^{N'-1}|u|^{N'}} \\ &\leq C\left(1+H(u(x))\right) \qquad \text{a.e. } x \in \Omega,\ \forall t \in \mathbb{R}, \end{aligned}$$

where $C = C(\|u_0\|_{L^\infty(\Omega)}, p, \beta, N)$. Thus, for any $u \in \mathcal{C}_\varepsilon$ it holds

$$\int_\Omega \tilde{F}(x,u+u_0)dx \leq C\left(|\Omega| + K(u)\right) \leq C\left(|\Omega| + \varepsilon\right) \leq C\left(|\Omega| + 1\right).$$

Consequently, for all $u \in \mathcal{C}_\varepsilon$

$$\begin{aligned} \tilde{J}(u+u_0) &\geq \frac{1}{N}\|u+u_0\|^N - \int_\Omega \tilde{F}(x,u+u_0)dx \\ &\geq \frac{1}{N}\|u+u_0\|^N - C(|\Omega|+1), \end{aligned} \tag{3.18}$$

where C does not depend on $\varepsilon \in (0,1)$ nor $u \in \mathcal{C}_\varepsilon$. This means that $\tilde{J}(\cdot + u_0)$ is bounded from below and coercive in $\mathcal{C}_\varepsilon$. Then, the infimum of $\tilde{J}(\cdot + u_0)$ in $\mathcal{C}_\varepsilon$ is attained. Since (3.15) holds and $v_n \in \mathcal{C}_\varepsilon$ for large n, we can find $u_\varepsilon \in \mathcal{C}_\varepsilon$, $u_\varepsilon \neq 0$, such that

$$\min_{u \in \mathcal{C}_\varepsilon} \tilde{J}(u + u_0) = \tilde{J}(u_\varepsilon + u_0) \leq \tilde{J}(v_n + u_0) < \tilde{J}(u_0), \quad \forall \varepsilon \in (0,1), \quad \forall n \geq N_\varepsilon. \tag{3.19}$$

By (3.18) and (3.19) we have that $\{u_\varepsilon\}_\varepsilon$ is bounded in $W_0^{1,N}(\Omega)$. Since $K(u_\varepsilon) \to 0$ when $\varepsilon \to 0$, in other words, $H(u_\varepsilon) \to 0$ em $L^1(\Omega)$ as $\varepsilon \to 0$, we get $H(u_\varepsilon) \to 0$ a.e. in Ω as $\varepsilon \to 0$. That way, $u_\varepsilon \to 0$ a.e. in Ω when $\varepsilon \to 0$. Then, $u_\varepsilon + u_0 \rightharpoonup u_0$ in $W_0^{1,N}(\Omega)$. By (3.19) and the Lagrange Multiplier Theorem we obtain that, for all $\varepsilon \in (0,1)$ there is $\mu_\varepsilon \in \mathbb{R}$ such that u_ε solves:

$$\tilde{J}'(u_\varepsilon + u_0) = \mu_\varepsilon K'(u_\varepsilon). \tag{3.20}$$

We claim that $\mu_\varepsilon \leq 0$ for all $\varepsilon \in (0,1)$. Let us suppose $\mu_\varepsilon > 0$ for some $\varepsilon > 0$. As the right side of (3.20) is nontrivial, we can choose $\varphi \in W_0^{1,N}(\Omega)$ such that $\tilde{J}'(u_\varepsilon + u_0)\varphi < 0$. So

$$0 > \frac{1}{\mu_\varepsilon}\tilde{J}'(u_\varepsilon + u_0)\varphi = K'(u_\varepsilon)\varphi = \lim_{\tau \to 0^+} \frac{K(u_\varepsilon + \tau\varphi) - K(u_\varepsilon)}{\tau}$$

and then

$$K(u_\varepsilon + \tau\varphi) < K(u_\varepsilon) \leq \varepsilon,$$

for small $\tau > 0$, that is, $u_\varepsilon + \tau\varphi \in \mathcal{C}_\varepsilon$ for $\tau > 0$ small enough. Similarly, $\tilde{J}(u_\varepsilon + \tau\varphi + u_0) < \tilde{J}(u_\varepsilon + u_0)$ for $\tau > 0$ small enough, which contradicts (3.19). Thus, we conclude that $\mu_\varepsilon \leq 0$ for all $\varepsilon \in (0,1)$. Now, considering $h = H'$, equation (3.20) implies that u_ε satisfies

$$-\Delta_N(u_\varepsilon + u_0) - \Delta_q(u_\varepsilon + u_0) = \tilde{f}(x, u_\varepsilon + u_0) + \mu_\varepsilon h(u_\varepsilon) \text{ in } \Omega. \tag{P_ε}$$

We deal now with two cases:

i) $\liminf_{\varepsilon \to 0} \mu_\varepsilon > -\infty$;

ii) $\liminf_{\varepsilon \to 0} \mu_\varepsilon = -\infty$.

If case $i)$ occurs, there are $\varepsilon_0 \in (0,1)$ and $\mu_0 < 0$ such that $\mu_\varepsilon \in (\mu_0, 0]$ for all $\varepsilon \in (0, \varepsilon_0)$. We define

$$\tilde{J}_\varepsilon(u) := \tilde{J}(u + u_0) - \mu_\varepsilon K(u), \ u \in W_0^{1,N}(\Omega),$$

the functional associated with (P_ε). By (3.20) we obtain $\tilde{J}_\varepsilon'(u_\varepsilon) = 0$ for all $\varepsilon \in (0, \varepsilon_0)$. Furthermore by (3.18) and (3.19) we have that $\{\tilde{J}_\varepsilon(u_\varepsilon)\}_\varepsilon$ is bounded in $\mathbb{R}$, so we may choose a subsequence, still denoted by $\{\tilde{J}_\varepsilon(u_\varepsilon)\}_\varepsilon$, such that $\tilde{J}_\varepsilon(u_\varepsilon) \to \rho_0$ as $\varepsilon \to 0$. So,

$$\frac{1}{N}\|u_\varepsilon + u_0\|^N + \frac{1}{q}\|\nabla(u_\varepsilon + u_0)\|^q_{L^q(\Omega)} - \int_\Omega \tilde{F}(x, u_\varepsilon + u_0)dx - \mu_\varepsilon \int_\Omega H(u_\varepsilon)dx \to \rho_0, \tag{3.21}$$

and

$$\|u_\varepsilon + u_0\|^N + \|\nabla(u_\varepsilon + u_0)\|^q_{L^q(\Omega)} - \int_\Omega \tilde{f}(x, u_\varepsilon + u_0)(u_\varepsilon + u_0)dx = \mu_\varepsilon \int_\Omega h(u_\varepsilon)(u_\varepsilon + u_0)dx. \tag{3.22}$$

Moreover, from (3.17), once $p > 1$ and $u_0 \in L^\infty(\Omega)$, for $\theta = (p+1)/p \in (1, 2)$ we get

$$\begin{aligned}\left|\tilde{F}(x, u_\varepsilon + u_0)\right|^\theta &\le C(1 + |u_\varepsilon| + |u_0|)^{\theta(p+1)} \exp\left(\theta\beta 2^{N'-1}(|u_\varepsilon|^{N'} + |u_0|^{N'})\right) \\ &\le C' + C''|u_\varepsilon|^{p+1}e^{2^{N'}\beta|u_\varepsilon|^{N'}}, \quad \text{a.e. } x \in \Omega.\end{aligned}$$

So

$$\int_\Omega \left|\tilde{F}(x, u_\varepsilon + u_0)\right|^\theta dx \le C'|\Omega| + C''K(u_\varepsilon) \le C.$$

Thus, $\tilde{F}(x, u_\varepsilon + u_0) \to \tilde{F}(x, u_0)$ almost everywhere in Ω and it is a bounded sequence in $L^\theta(\Omega)$. Since $|\Omega| < \infty$, it follows by Vitali's Convergence Theorem that

$$\int_\Omega \tilde{F}(x, u_\varepsilon + u_0)dx \to \int_\Omega \tilde{F}(x, u_0)dx, \quad \text{as } \varepsilon \to 0. \tag{3.23}$$

This last convergence, coupled with the fact that $u_\varepsilon + u_0 \rightharpoonup u_0$, $\{(\mu_\varepsilon)\}_\varepsilon$ is bounded and $K(u_\varepsilon) \to 0$ imply that

$$\begin{aligned}\rho_0 &= \liminf_{\varepsilon\to 0}\left[\tilde{J}(u_\varepsilon + u_0) - \mu_\varepsilon K(u_\varepsilon)\right] \\ &= \liminf_{\varepsilon\to 0}\left[\frac{1}{N}\|u_\varepsilon + u_0\|^N + \frac{1}{q}\|u_\varepsilon + u_0\|^q_{W_0^{1,q}(\Omega)} - \int_\Omega \tilde{F}(x, u_\varepsilon + u_0)dx\right] \\ &\ge \frac{1}{N}\|u_0\|^N + \frac{1}{q}\|u_0\|^q_{W_0^{1,q}(\Omega)} - \int_\Omega \tilde{F}(x, u_0)dx \\ &= \tilde{J}(u_0).\end{aligned}$$

On the other hand, by (3.19), $\rho_0 \le \tilde{J}(u_0)$. Then, $\rho_0 = \tilde{J}(u_0)$, which means

$$\lim_{\varepsilon\to 0}\tilde{J}(u_\varepsilon + u_0) = \lim_{\varepsilon\to 0}\tilde{J}_\varepsilon(u_\varepsilon) = \rho_0 = \tilde{J}(u_0),$$

or

$$\begin{aligned}&\frac{1}{N}\|u_\varepsilon + u_0\|^N + \frac{1}{q}\|u_\varepsilon + u_0\|^q_{W_0^{1,q}(\Omega)} - \int_\Omega \tilde{F}(x, u_\varepsilon + u_0)dx \\ &\to \frac{1}{N}\|u_0\|^N + \frac{1}{q}\|u_0\|^q_{W_0^{1,q}(\Omega)} - \int_\Omega \tilde{F}(x, u_0)dx, \quad \text{as } \varepsilon \to 0.\end{aligned}$$

By (3.23) we obtain

$$\frac{1}{N}\|u_\varepsilon+u_0\|^N+\frac{1}{q}\|u_\varepsilon+u_0\|^q_{W_0^{1,q}(\Omega)} \quad\rightarrow\quad \frac{1}{N}\|u_0\|^N+\frac{1}{q}\|u_0\|^q_{W_0^{1,q}(\Omega)}.$$

Since $u_\varepsilon+u_0 \rightharpoonup u_0$ in $W_0^{1,N}(\Omega)$, and so in $W_0^{1,q}(\Omega)$, we get

$$\|u_\varepsilon+u_0\| \to \|u_0\|, \quad \text{as } \varepsilon\to 0.$$

Recalling that $\left(W_0^{1,N}(\Omega),\|\cdot\|\right)$ is uniformly convex, we conclude that $u_\varepsilon+u_0\to u_0$, which means that $u_\varepsilon\to 0$ in $W_0^{1,N}(\Omega)$. Now we intend to apply Proposition 3.3.1 in order to get the uniform boundedness of (u_ε).

From (3.5), for $\theta=(p+1)/p\in(1,2)$, we see that

$$|\tilde{f}(x,u_\varepsilon+u_0)|^\theta \le C(1+|u_0|^{p+1}+|u_\varepsilon|^{p+1})e^{2^{N'}\beta(|u_\varepsilon|^{N'}+|u_0|^{N'})} \le C(1+H(u_\varepsilon)), \quad a.e.\ x\in\Omega,$$

where C does not depend on ε nor u_ε. Thus, there exists $M>0$ satisfying

$$\int_\Omega \left[|\tilde{f}(x,u_0)|^\theta+|\tilde{f}(x,u_\varepsilon+u_0)|^\theta\right]dx \le C\int_\Omega (1+H(u_\varepsilon))dx \le C(|\Omega|+\varepsilon)\le M^\theta.$$

Since u_ε is a solution for (P_ε), we know that $(\mu_\varepsilon,u_\varepsilon)$ is a solution for (3.6). Then, it follows by Proposition 3.3.1 that $\{u_\varepsilon\}_\varepsilon$ is bounded in $L^\infty(\Omega)$, that is

$$\sup_{\varepsilon\in(0,\varepsilon_0)}\|u_\varepsilon\|_{L^\infty(\Omega)}\le C_0,$$

for some $C_0>0$. Now, considering

$$B_\varepsilon(x,t)=\tilde{f}(x,t)+\mu_\varepsilon h(t-u_0)$$

for we have $|B_\varepsilon(x,t)|\le\Lambda_1$ for $(x,t)\in\Omega\times[-C_0-\|u_0\|_{L^\infty(\Omega)},C_0+\|u_0\|_{L^\infty(\Omega)}]$. Thus, by Theorem 3.3.2 we conclude that $u_\varepsilon\in C^{1,\gamma}(\overline{\Omega})$ and

$$\sup_{\varepsilon\in(0,\varepsilon_0)}\|u_\varepsilon\|_{C^{1,\gamma}(\overline{\Omega})}<\infty, \tag{3.24}$$

for some $\gamma\in(0,1)$.

At this moment, let us analyze the situation in Case *ii*), where $\liminf_{\varepsilon\to 0}\mu_\varepsilon=-\infty$. In this case, we consider a sequence $\varepsilon_n\to 0$ such that $\mu_{\varepsilon_n}\to-\infty$. By definition of $h=H'$, we have

$$h(t)=\left[(p+1)|t|^{p-1}t+\beta 2^{N'}N'|t|^{N'-1+p}t\right]e^{\beta 2^{N'}|t|^{N'}}, \quad \forall t\in\mathbb{R}$$

and so

$$sgn(t)h(t) = \left[(p+1)|t|^p + \beta 2^{N'} N' |t|^{N'+p}\right] e^{\beta 2^{N'}|t|^{N'}} \\ \geq C_1 |t|^p e^{\beta 2^{N'}|t|^{N'}}, \quad \forall t \in \mathbb{R} \text{ with } |t| \geq 1. \tag{3.25}$$

Now we fix $M \geq 2\max\{\|u_0\|_{L^\infty}, 1\}$. For $|s| \geq M$ we have $|s - u_0(x)| \geq M/2 \geq |u_0(x)|$ almost everywhere in Ω, so that $|s| \leq |s - u_0(x)| + |u_0(x)| \leq 2|s - u_0(x)|$. Then, using (3.5) and (3.25) we see that

$$|\tilde{f}(x,s)| \leq C(1+|s|)^p e^{\beta |s|^{N'}} \leq C(1+|s-u_0(x)|)^p e^{\beta 2^{N'}|s-u_0(x)|^{N'}} \leq Ch(s-u_0(x)) \cdot sgn(s-u_0(x)),$$

for a positive constant C and for all $s \in \mathbb{R}$ such that $|s| \geq M$. Thus, since $\mu_{\varepsilon_n} \to -\infty$ as $n \to \infty$, there exists n_0 satisfying

$$sgn(s-u_0(x))\left[\tilde{f}(x,s) + \mu_\varepsilon h(s-u_0(x))\right] \leq (C + \mu_{\varepsilon_n})\, h(s-u_0(x)) sgn(s-u_0(x)) \leq 0,$$

a.e. $x \in \Omega$, for all $n \geq n_0$. Now, we consider test functions $\varphi_n = (u_{\varepsilon_n} + u_0 - M)_+$ and $\psi_n = (u_{\varepsilon_n} + u_0 + M)_-$, where $v_-(x) = \min\{v(x), 0\}$. Then, $\tilde{J}'_{\varepsilon_n}(u_{\varepsilon_n})\varphi_n = 0$ gives us

$$\begin{aligned} \int_\Omega |\nabla \varphi_n|^N dx + \int_\Omega |\nabla \varphi_n|^q dx &= \int_\Omega |\nabla(u_{\varepsilon_n} + u_0)|^{N-2} \nabla(u_{\varepsilon_n} + u_0) \nabla \varphi_n dx \\ &\quad + \int_\Omega |\nabla(u_{\varepsilon_n} + u_0)|^{q-2} \nabla(u_{\varepsilon_n} + u_0) \nabla \varphi_n dx \\ &= \int_\Omega \left[\tilde{f}(x, u_{\varepsilon_n} + u_0) + \mu_{\varepsilon_n} h(u_{\varepsilon_n})\right] \varphi_n dx \\ &\leq 0, \quad \forall n \geq n_0, \end{aligned}$$

because $sgn(u_{\varepsilon_n}(x)) = sgn(\varphi_n(x))$ for $\varphi_n(x) > 0$. In an analogous way we obtain

$$\int_\Omega |\nabla \psi_n|^N dx + \int_\Omega |\nabla \psi_n|^q dx \leq 0, \quad \forall n \geq n_0.$$

Then,

$$\|\varphi_n\| = \|\psi_n\| = 0, \quad \forall n \geq n_0,$$

which implies

$$|u_{\varepsilon_n}(x) + u_0(x)| \leq M, \quad \text{a.e. } x \in \Omega. \tag{3.26}$$

Recalling that $\|u_0\|_{L^\infty} \leq M/2$, we obtain the uniform boundedness for u_{ε_n},

$$\|u_{\varepsilon_n}\|_{L^\infty(\Omega)} \leq \frac{3}{2} M, \quad \forall n \geq n_0.$$

Now, let $\alpha > 1$ be an arbitrary real number. We can use $\phi_n = |u_{\varepsilon_n}|^{\alpha-1}u_{\varepsilon_n}$ as a test function. Since $\tilde{J}'_{\varepsilon_n}(u_{\varepsilon_n})\phi_n = 0$ and $\tilde{J}'(u_0)\phi_n = 0$, observing that $\nabla\phi_n = \alpha|u_{\varepsilon_n}|^{\alpha-1}\nabla u_{\varepsilon_n}$, we have

$$
\begin{aligned}
-\mu_{\varepsilon_n}\int_\Omega h(u_{\varepsilon_n})\phi_n dx &= \int_\Omega [\tilde{f}(x,u_{\varepsilon_n}+u_0) - \tilde{f}(x,u_0)]\phi_n dx \\
&\quad - \int_\Omega [|\nabla(u_{\varepsilon_n}+u_0)|^{N-2}\nabla(u_{\varepsilon_n}+u_0) - |\nabla u_0|^{N-2}\nabla u_0]\nabla\phi_n dx \\
&\quad - \int_\Omega [|\nabla(u_{\varepsilon_n}+u_0)|^{q-2}\nabla(u_{\varepsilon_n}+u_0) - |\nabla u_0|^{q-2}\nabla u_0]\nabla\phi_n dx \\
&= \int_\Omega [\tilde{f}(x,u_{\varepsilon_n}+u_0) - \tilde{f}(x,u_0)]|u_{\varepsilon_n}|^{\alpha-1}u_{\varepsilon_n} dx \\
&\quad - \alpha\int_\Omega |u_{\varepsilon_n}|^{\alpha-1}\left(|\nabla(u_{\varepsilon_n}+u_0)|^{N-2}\nabla(u_{\varepsilon_n}+u_0) - |\nabla u_0|^{N-2}\nabla u_0\right)\nabla u_{\varepsilon_n} dx \\
&\quad - \alpha\int_\Omega |u_{\varepsilon_n}|^{\alpha-1}(|\nabla(u_{\varepsilon_n}+u_0)|^{q-2}\nabla(u_{\varepsilon_n}+u_0) - |\nabla u_0|^{q-2}\nabla u_0)\nabla u_{\varepsilon_n} dx,
\end{aligned}
$$

for all $n \geq n_0$. Since $(|z|^{\sigma-2}z - |w|^{\sigma-2}w)(z-w) \geq 0$ for all z, $w \in \mathbb{R}^N$, where $\sigma > 1$, using (3.26) we get

$$
\begin{aligned}
-\mu_{\varepsilon_n}\int_\Omega h(u_{\varepsilon_n})|u_{\varepsilon_n}|^{\alpha-1}u_{\varepsilon_n} dx &\leq \int_\Omega [\tilde{f}(x,u_{\varepsilon_n}+u_0) - \tilde{f}(x,u_0)]|u_{\varepsilon_n}|^{\alpha-1}u_{\varepsilon_n} dx \\
&\leq \sup_{(x,t)\in\Omega\times[-M,M]} |\tilde{f}(x,t)| \int_\Omega |u_\varepsilon|^\alpha dx \\
&\leq C\int_\Omega |u_\varepsilon|^\alpha dx,
\end{aligned}
$$

for all $n \geq n_0$. On the other hand, by (3.25) we see that $h(s)s \geq (p+1)|s|^{p+1}$ for all $s \in \mathbb{R}$. Then, by Hölder's inequality we get

$$
\begin{aligned}
(p+1)(-\mu_{\varepsilon_n})\int_\Omega |u_{\varepsilon_n}|^{\alpha+p} dx &\leq -\mu_{\varepsilon_n}\int_\Omega h(u_{\varepsilon_n})|u_{\varepsilon_n}|^{\alpha-1}u_{\varepsilon_n} dx \\
&\leq C|\Omega|^{\frac{p}{\alpha+p}}\left(\int_\Omega |u_{\varepsilon_n}|^{\alpha+p} dx\right)^{\frac{\alpha}{\alpha+p}},
\end{aligned}
$$

which implies

$$
\|(-\mu_{\varepsilon_n})^{\frac{1}{p}}u_{\varepsilon_n}\|^p_{L^{\alpha+p}(\Omega)} = -\mu_{\varepsilon_n}\|u_{\varepsilon_n}\|^p_{L^{\alpha+p}(\Omega)} \leq C(1+|\Omega|), \quad \forall n \geq n_0,
$$

for $C > 0$ independent of n and α. Recalling that $\alpha > 1$ is arbitrary and letting $\alpha \to +\infty$ we get,

$$
\|(-\mu_{\varepsilon_n})^{\frac{1}{p}}u_{\varepsilon_n}\|^p_{L^\infty(\Omega)} \leq C(1+|\Omega|), \tag{3.27}
$$

for all $n \geq n_0$. Considering

$$B_n(x,t) = \tilde{f}(x,t) + \mu_{\varepsilon_n} h(t - u_0(x)),$$

since $\|u_{\varepsilon_n} + u_0\|_{L^\infty} \leq M$ and $\|u_{\varepsilon_n}\|_{L^\infty} \leq 3M/2$ for all $n \geq n_0$, from (3.25) and (3.27) we see that

$$\begin{aligned}|B_n(x, u_{\varepsilon_n} + u_0)| &= |\tilde{f}(x, u_{\varepsilon_n} + u_0) + \mu_{\varepsilon_n} h(u_{\varepsilon_n})| \\ &\leq \sup_{(x,t)\in\Omega\times[-M,M]} |\tilde{f}(x,t)| + |\mu_{\varepsilon_n} h(u_{\varepsilon_n})| \\ &\leq C'(1 + |\mu_{\varepsilon_n}||u_{\varepsilon_n}|^p) \\ &\leq \Lambda_1, \quad \text{a.e. } x \in \Omega,\end{aligned}$$

for all $n \geq n_0$. So we can apply Theorem 3.3.2 to obtain

$$\sup_{n\geq n_0} \|u_{\varepsilon_n}\|_{C^{1,\gamma}(\overline{\Omega})} < \infty,$$

for some $\gamma \in (0,1)$.

Therefore, in both cases we obtain an uniform bound for a sequence $\{u_{\varepsilon_n}\}_n$ in $C^{1,\gamma}(\overline{\Omega})$. Therefore, there exists a subsequence $\{u_{\varepsilon_n}\}_n$ such that $u_{\varepsilon_n} \to 0$ in $C_0^1(\overline{\Omega})$. Since (3.19) gives us

$$\tilde{J}(u_{\varepsilon_n} + u_0) < \tilde{J}(u_0),$$

we have a contradiction with the local minimality of u_0 for $\tilde{J}$ in the $C_0^1(\overline{\Omega})$ topology. This means that u_0 must be also a local minimum for $\tilde{J}$ in the $W_0^{1,N}(\Omega)$ topology and we conclude this proof. □

3.5 A first solution

In this section we deal with problem (3.3), which is

$$\begin{cases} -\Delta_N u - \Delta_q u = \lambda |u|^{N-2} u + f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

whose associated functional is $I : W_0^{1,N}(\Omega) \to \mathbb{R}$ given by

$$I(u) = \frac{1}{N}\int_\Omega \left(|\nabla u|^N - \lambda |u|^N\right) dx + \frac{1}{q}\int_\Omega |\nabla u|^q dx - \int_\Omega f(x) u dx.$$

We are now in a position to describe our first result of the existence of solutions for (3.1) more completely.

Proposition 3.5.1. *Suppose $f \in L^\infty(\Omega)$ a nontrivial function. Then I has a global minimum, denoted by w, and it satisfies:*

i) $w \in C_0^{1,\gamma}(\overline{\Omega})$, for some $\gamma \in (0,1)$;

ii) If $f \leq 0$ then $w \leq 0$ and it is a solution for problem (3.1);

iii) If $w \leq 0$ then $w < 0$ in Ω and $\frac{\partial w}{\partial \nu} > 0$ on $\partial\Omega$.

iv) w is a local minimum of J in $W_0^{1,N}(\Omega)$.

Proof. As in Chapter 1, we see that I is a C^1 functional, coercive, weakly lower semicontinuous (see Lemma 1.2.1). So, the direct method of the calculus of variations implies the existence of a global minimum $w \in W_0^{1,N}(\Omega)$ for I. This minimum is a critical point for I, which means $I'(w) = 0$, and so is a weak solution to problem (3.3). For i), considering $u \in W_0^{1,N}(\Omega)$ as a critical point for I, or a solution to problem (3.3). We see that $(0,u)$ is a solution for problem (3.6), with $\tilde{f}(x,s) = \lambda|s|^{N-2}s + f(x)$ and $u_0 \equiv 0$. Since

$$|\lambda|s|^{N-2}s + f(x)| \leq \|f\|_{L^\infty(\Omega)} + \lambda|s|^{N-1}, \quad \forall (x,s) \in \Omega \times \mathbb{R}$$

we can apply Proposition 3.3.1, with $\theta = N'$, to obtain $u \in L^\infty(\Omega)$. After that, we can use Theorem 3.3.2 to ensure $u \in C_0^{1,\gamma}(\overline{\Omega})$. In particular, for $u = w$, the minimum of I in $W_0^{1,N}(\Omega)$, which proves i). For ii), if we consider $f \leq 0$, using w_+ as a test function we get

$$0 = I'(w)w_+ = \int_\Omega \left(|\nabla w_+|^N - \lambda|w_+|^N\right) dx + \int_\Omega |\nabla w_+|^q dx - \int_\Omega f(x) w_+ dx \geq \|w_+\|_\lambda^N,$$

which implies that $w_+ \equiv 0$ and $w \leq 0$. Observing that $g(x,w) \equiv 0$, we see that w is also a critical point for the functional J. In other words, w is also a weak solution to problem (3.1). To prove iii), since $w \leq 0$, we can apply the Maximum Principle that can be found in Pucci and Serrin [33] (see also Proposition 1.2.3) to show that $w < 0$ in Ω. After that, we use a version of the Hopf Lemma for the (p,q)-Laplacian operator, which is found in [21] (see also Proposition 1.2.7), to ensure $\frac{\partial w}{\partial \nu} > 0$ on $\partial\Omega$. Finally, since w is a minimum point for I in $C_0^1(\overline{\Omega})$, and ii) implies that $-w$ is in the interior of the cone of positive functions in $C_0^1(\overline{\Omega})$, we have $I(u) = J(u)$ in a neighborhood of w in the $C_0^1(\overline{\Omega})$ topology, which means w is also a local minimum for J in $C_0^1(\overline{\Omega})$. Thus, it follows from Proposition 3.4.1 that w is a local minimum of J in $W_0^{1,N}(\Omega)$. □

3.6 Proof of the Main Theorems

The next lemma provides a mountain-pass geometry and it acts as an auxiliary result to demonstrate that the *minimax* levels can be controlled by strategically selecting directions provided by truncations.

Lemma 3.6.1. *Assume that g satisfies* (α_0) *and* ($\tilde{g}_1$) − ($\tilde{g}_3$) *and consider $\varphi \in W_0^{1,N}(\Omega)\backslash\{0\}$ to be a continuous and nonnegative function. Then $J(w+t\varphi) \to -\infty$ when $t \to +\infty$.*

Proof. In fact, since ($\tilde{g}_4$) is true, there are positive numbers C and R and $\sigma > N$ such that

$$G(x,u) \geq u^{\sigma}, \qquad \forall\ (x,u) \in \overline{\Omega} \times [R, \infty).$$

For a continuous function φ, let $x_1 \in \Omega$ and $r > 0$ be such that $m_\varphi = \min_{B_r(x_1)} \varphi(x) > 0$. So, for all $t > (R + \|w\|_{L^\infty(\Omega)})/m_\varphi$ we get

$$\begin{aligned} J(w+t\varphi) &= \frac{1}{N}\int_\Omega \left(|\nabla(w+t\varphi)|^N - \lambda|w+t\varphi|^N\right)dx + \frac{1}{q}\int_\Omega |\nabla(w+t\varphi)|^q dx \\ &\quad - \int_\Omega G\left(x,(w+t\varphi)_+\right)dx - \int_\Omega f(x)(w+t\varphi)dx \\ &\leq \frac{2^{N-1}}{N}\int_\Omega \left(|\nabla w|^N + t^N|\nabla\varphi|^N\right)dx + \frac{2^{q-1}}{q}\int_\Omega \left(|\nabla w|^q + t^q|\nabla\varphi|^q\right)dx \\ &\quad - C\int_{B_r(x_1)} |w+t\varphi|^\sigma dx + t\|f\|_{L^\infty(\Omega)}\int_\Omega \varphi dx + \|f\|_{L^\infty(\Omega)}\|w\|_{L^1(\Omega)}. \end{aligned}$$

Since $\|t\varphi\|_{L^\sigma(\Omega)} \leq \|w\|_{L^\sigma(\Omega)} + \|w+t\varphi\|_{L^\sigma(\Omega)}$ implies that

$$\|t\varphi\|^\sigma_{L^\sigma(\Omega)} \leq 2^{\sigma-1}\left(\|w\|^\sigma_{L^\sigma(\Omega)} + \|w+t\varphi\|^\sigma_{L^\sigma(\Omega)}\right),$$

we obtain

$$\begin{aligned} J(w+t\varphi) \leq \frac{2^{N-1}t^N}{N}\int_\Omega |\nabla\varphi|^N dx + \frac{2^{q-1}t^q}{q}\int_\Omega |\nabla\varphi|^q dx - \frac{Ct^\sigma}{2^{\sigma-1}}\int_{B_r(x_1)} |\varphi|^\sigma dx + t\|f\|_{L^\infty}\int_\Omega \varphi dx \\ + C_1 \end{aligned}$$

for large $t > 0$, where C_1 depends on w. Since $\sigma > N \geq q$, this implies that $J(w+t\varphi) \to -\infty$ as $t \to +\infty$. □

From this lemma and knowing that w is a local minimum for the functional J, we can also see that J exhibits the mountain pass geometry. In fact, since w is a local minimum for J in $W_0^{1,N}(\Omega)$, there exists $\rho > 0$ such that

$$J(v) \geq J(w), \quad \forall v \in W_0^{1,N}(\Omega) \quad \textit{with} \quad \|v-w\| \leq \rho. \tag{3.28}$$

On the other hand, once $w < 0$, by this previous lemma, we get $J((t-1)(-w)) \to -\infty$ as $t \to \infty$. This means that there is $e \in W_0^{1,N}(\Omega)$ such that $\|e-w\| > \rho$ and $J(e) < J(w)$. So, we can define

$$\Gamma := \{\gamma\ \in C([0,1], W_0^{1,N}(\Omega)) : \gamma(0) = w \text{ and } J(\gamma(1)) < J(w)\}.$$

We see that Γ is non-empty and the mountain pass level,

$$c_0 := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)), \tag{3.29}$$

is well-defined and satisfies $c_0 \geq J(w)$.

Proposition 3.6.2. *Assuming* (α_0) *and* $(\tilde{g}_1) - (\tilde{g}_2)$, *let* $\{u_m\}_m$ *be a (PS) sequence for* J *in* $W_0^{1,N}(\Omega)$. *Then* $\{u_m\}_m$ *is bounded.*

Proof. Let $\{u_m\}_m$ be a (PS) sequence for J in $W_0^{1,N}(\Omega)$, which means a sequence such that $\{J(u_m)\}_m$ is bounded and $\|J'(u_m)\| \to 0$. Thus,

$$|J(u_m)| + |J'(u_m).u_m| \leq C(1 + \|u_m\|).$$

Moreover, it follows from $(\tilde{g}_5)$ in Lemma 3.2.1 that there are $S > 0$ and $\sigma > N$ such that,

$$\sigma G(x,u) \leq ug(x,u), \quad \forall (x,u) \in \Omega \times (S, \infty).$$

So, by Holder's inequality and the embedding $W_0^{1,N}(\Omega) \hookrightarrow L^1(\Omega)$ we obtain

$$\begin{aligned} J(u_m) - \frac{1}{\sigma} J'(u_m).u_m &= \left(\frac{1}{N} - \frac{1}{\sigma}\right) \|u_m\|_\lambda^N + \left(\frac{1}{q} - \frac{1}{\sigma}\right) \|\nabla u_m\|_{L^q}^q \\ &\quad - \int_\Omega \left[G(x,(u_m)_+) - \frac{1}{\sigma} g(x,(u_m)_+)(u_m)_+\right] dx \\ &\quad - \left(1 - \frac{1}{\sigma}\right) \int_\Omega f(x) u_m dx \\ &\geq \left(\frac{\sigma - N}{N\sigma}\right) \|u_m\|_\lambda^N - \left(\frac{\sigma - 1}{\sigma}\right) \int_\Omega f(x) u_m dx \\ &\quad - \int_{\{|u_m(x)| \leq S\}} \left[G(x,(u_m)_+) - \frac{1}{\sigma} g(x,(u_m)_+)(u_m)_+\right] dx \\ &\geq C \left(\frac{\sigma - N}{N\sigma}\right) \|u_m\|^N - C\|u_m\| - C_1. \end{aligned}$$

Then, it follows that

$$\|u_m\| \geq C_2 \|u_m\|^N - C_3$$

which implies that $\{u_m\}$ is bounded in $W_0^{1,N}(\Omega)$. $\square$

Since we cannot guarantee that J satisfies the Palais-Smale condition, we establish some compactness properties provided c_0, defined in (3.29), is below a certain threshold.

Lemma 3.6.3. *Let $\{u_m\}_m$ be a sequence in $L^1(\Omega)$ such that $u_m(x) \to u(x)$ a.e. in Ω, for $u \in L^1(\Omega)$, and let $\tilde{g} : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

$$\sup_{m\in\mathbb{N}} \int_\Omega |\tilde{g}(x, u_m)u_m|dx \le C. \tag{3.30}$$

So $\tilde{g}(x, u_m) \to \tilde{g}(x, u)$ in $L^1(\Omega)$.

Proof. This proof is similar to the one due to Figueiredo, Miyagaki and Ruf (see [6]). At first, we observe that

$$\int_\Omega |\tilde{g}(x, u_m)|dx \le \int_{\{|u_m(x)|\le 1\}} |\tilde{g}(x, u_m)|dx + \int_{\{|u_m(x)|\ge 1\}} |\tilde{g}(x, u_m)u_m|dx \le C',$$

for all $m \in \mathbb{N}$, where $C' = |\Omega| \max_{\overline{\Omega}\times[-1,1]} |\tilde{g}(x, t)| + C$. This means that $\{\tilde{g}(\cdot, u_m)\}_m$ is bounded in $L^1(\Omega)$ and so, Fatou's Lemma implies that $\tilde{g}(\cdot, u) \in L^1(\Omega)$. Now we fix $\varepsilon > 0$. Since $\tilde{g}(\cdot, u) \in L^1(\Omega)$, there is $\delta > 0$ such that

$$\int_A |\tilde{g}(x, u)|dx \le \frac{\varepsilon}{9},$$

if $A \subset \Omega$ is measurable and $|A| \le \delta$. On the other hand, since $u \in L^1(\Omega)$ there is $\tilde{M}$ large enough such that

$$\left|\{x \in \Omega; |u(x)| \ge \tilde{M}\}\right| \le \delta.$$

For $C > 0$ as in (3.30), we consider $M = \max\left\{3C/\varepsilon, \tilde{M}\right\}$. Then, we observe that

$$\left|\int_\Omega |\tilde{g}(x, u_m)|dx - \int_\Omega |\tilde{g}(x, u)|dx\right| \le I_1 + I_2 + I_3$$

where

$$I_1 = \int_{\{|u_m(x)|\ge M\}} |\tilde{g}(x, u_m)|dx,$$

$$I_2 = \left|\int_{\{|u_m(x)|<M\}} |\tilde{g}(x, u_m)|dx - \int_{\{|u(x)|<M\}} |\tilde{g}(x, u)|dx\right|,$$

and

$$I_3 = \int_{\{|u(x)|\ge M\}} |\tilde{g}(x, u)|dx.$$

At first, we notice that

$$I_1 = \int_{\{|u_m(x)|\ge M\}} \left|\frac{\tilde{g}(x, u_m)u_m}{u_m}\right| dx \le \frac{C}{M} \le \frac{\varepsilon}{3}, \quad \forall m.$$

Now,

$$
\begin{aligned}
I_2 &= \left| \int_\Omega |\tilde{g}(x,u_m)| \chi_{\{|u_m(x)|<M\}} dx - \int_\Omega |\tilde{g}(x,u)| \chi_{\{|u(x)|<M\}} dx \right| \\
&\leq \int_\Omega |\tilde{g}(x,u_m) - \tilde{g}(x,u)| \, \chi_{\{|u_m(x)|<M\}} dx + \int_\Omega |\tilde{g}(x,u)| \left| \chi_{\{|u_m(x)|<M\}} - \chi_{\{|u(x)|<M\}} \right| dx.
\end{aligned}
$$

Considering

$$
h_m(x) = |\tilde{g}(x,u_m) - \tilde{g}(x,u)| \chi_{\{|u_m|<M\}},
$$

we see that $h_m(x) \to 0$ a.e. in Ω and,

$$
|h_m(x)| \leq C_1 + |\tilde{g}(x,u)|
$$

where $C_1 = \max\{|\tilde{g}(x,t)|; x \in \overline{\Omega}, |t| \leq M\}$ and $\tilde{g}(\cdot,u) \in L^1(\Omega)$. From the Lebesgue Dominated Convergence Theorem we have $h_m \to 0$ in $L^1(\Omega)$. On the other hand,

$$
\begin{aligned}
\int_\Omega |\tilde{g}(x,u)| \left| \chi_{\{|u_m(x)|<M\}} - \chi_{\{|u(x)|<M\}} \right| dx &\leq \int_{\{|u(x)|<M\}} |\tilde{g}(x,u)| \left| \chi_{\{|u_m(x)|<M\}} - \chi_{\{|u(x)|<M\}} \right| dx \\
&\quad + 2 \int_{\{|u(x)| \geq M\}} |\tilde{g}(x,u)| dx \\
&= \int_\Omega \tilde{h}_m(x) dx + 2I_3,
\end{aligned}
$$

with $\tilde{h}_m(x) := |\tilde{g}(x,u)| \left| \chi_{\{|u_m(x)|<M\}} - \chi_{\{|u(x)|<M\}} \right| \chi_{\{|u(x)|<M\}} \to 0$ a.e. in Ω and satisfying $|\tilde{h}_m(x)| \leq |g(x,u)|$, for all $m \in \mathbb{N}$. Using again the Lebesgue Dominated Convergence Theorem we have $\tilde{h}_m \to 0$ in $L^1(\Omega)$. So, we obtain

$$
I_2 \leq \int_\Omega h_m(x) dx + \int_\Omega \tilde{h}_m(x) dx + 2I_3 \leq \frac{\varepsilon}{3} + 2I_3 \quad \text{for large } m.
$$

Finally, since $M \geq \tilde{M}$,

$$
I_3 = \int_{\{|u(x)| \geq M\}} |\tilde{g}(x,u)| dx \leq \frac{\varepsilon}{9}.
$$

Then

$$
\left| \int_\Omega |\tilde{g}(x,u_m)| dx - \int_\Omega |\tilde{g}(x,u)| dx \right| \leq \frac{2\varepsilon}{3} + 3I_3 \leq \varepsilon, \quad \text{for large } m,
$$

which means that

$$
\int_\Omega |\tilde{g}(x,u_m)| dx \to \int_\Omega |\tilde{g}(x,u)| dx, \quad \text{as } m \to \infty.
$$

Therefore, by the Brezis Lieb Lemma we obtain the desired result. □

Lemma 3.6.4. *Let $\{u_m\}_m \subset W_0^{1,N}(\Omega)$ be a (PS) sequence for J at level C'. Then there exists*

$u \in W_0^{1,N}(\Omega)$ *and a subsequence, still denoted by* $\{u_m\}_m$*, such that*

(i) $g(x,(u_m)_+) \to g(x,u_+)$ *in* $L^1(\Omega)$*;*

(ii) $G(x,(u_m)_+) \to G(x,u_+)$ *in* $L^1(\Omega)$*.*

(iii) $g(x,(u_m)_+)v \to g(x,u_+)v$ *in* $L^1(\Omega)$*, for any* $v \in C_c^\infty(\Omega)$*.*

Proof. Consider $\{u_m\}_m$ as a $(PS)_{C'}$ sequence for J. Proposition 3.6.2 guarantees that $\{u_m\}_m$ is bounded in $W_0^{1,N}(\Omega)$ and is also bounded in $W_0^{1,q}(\Omega)$. So, there exists $u \in W_0^{1,N}(\Omega)$ such that, up to a subsequence, $u_m \rightharpoonup u$ in $W_0^{1,N}(\Omega)$, $u_m \to u$ in $L^1(\Omega)$ and $u_m(x) \to u(x)$ a.e. in Ω, as $m \to \infty$. We need to show that $\{G(\cdot,(u_m)_+)(u_m)_+\}_m$ and $\{g(\cdot,(u_m)_+)(u_m)_+\}_m$ are bounded in $L^1(\Omega)$, so that we could apply the previous lemma. Since $\{u_m\}_m$ is a (PS) sequence at level C', we have $J(u_m) \to C'$, which gives us

$$\frac{1}{N}\|u_m\|_\lambda^N + \frac{1}{q}\|u_m\|^q_{W_0^{1,q}} - \int_\Omega G(x,(u_m)_+)dx - \int_\Omega f(x)u_m dx \to C',$$

and $\|J'(u_m)\| \to 0$, which implies $J'(u_m)u_m \to 0$. This means that

$$\|u_m\|_\lambda^N + \|u_m\|^q_{W_0^{1,q}} - \int_\Omega g(x,(u_m)_+)u_m dx - \int_\Omega f(x)u_m dx \to 0.$$

Because

$$\left|\int_\Omega f(x)u_m dx\right| \le \|f\|_{L^\infty}\|u_m\|_{L^1} \le C\|f\|_{L^\infty}\|u_m\| \le C_1,$$

we see that

$$\int_\Omega G(x,(u_m)_+)dx \le C \quad \text{and} \quad \int_\Omega g(x,(u_m)_+)(u_m)_+ dx \le C.$$

It remains to show that $\{G(\cdot,(u_m)_+)(u_m)_+\}_m$ is bounded in $L^1(\Omega)$. Using $(\tilde{g}_2)$ and the continuity of G we have

$$0 \le G(x,t) \le Mg(x,t) + C_2, \quad \forall (x,t) \in \overline{\Omega} \times \mathbb{R}.$$

Then

$$\int_\Omega G(x,(u_m)_+)(u_m)_+ dx \le C_2 \int_\Omega (u_m)_+ dx + M\int_\Omega g(x,(u_m)_+)(u_m)_+ dx \le C_3.$$

Therefore, by Lemma 3.6.3 we obtain the results of items (i) and (ii). Now, for any $v \in C_c^\infty(\Omega)$ we have

$$\int_\Omega |g(x,(u_m)_+)v - g(x,u_+)v|\,dx \le \|v\|_{L^\infty}\|g(\cdot,(u_m)_+) - g(\cdot,u_+)\|_{L^1} \to 0$$

as $m \to \infty$, and we conclude this proof. □

Proposition 3.6.5. *Assume that* g *satisfies* $(\tilde{g}_1), (\tilde{g}_2)$ *and* (α_0)*. If*

$$c_0 < J(w) + \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1},$$

then J possesses a critical point $u \neq w$.

Proof. By definition of c_0, given in (3.29), we know that $c_0 - J(w) \geq 0$. We can apply the results in [29, Theorems 2.8 and 2.20] in order to obtain a (PS) sequence $\{u_m\}_m$ for J at level c_0, and this sequence can be chosen satisfying $\|u_m - w\| \geq \frac{\rho}{2} > 0$ for all m, if $J(w) = c_0$, where ρ is given in (3.28). We have $\|J'(u_m)\| = o(1)$ and $J(u_m) = c_0 + o(1)$, which implies

$$o(1)\|u_m\| = \int_\Omega \left(|\nabla u_m|^N - \lambda |u_m|^N\right) dx + \int_\Omega |\nabla u_m|^q - \int_\Omega g(x,(u_m)_+)u_m dx - \int_\Omega f(x)u_m dx$$

and

$$c_0 + o(1) = \frac{1}{N}\int_\Omega \left(|\nabla u_m|^N - \lambda |u_m|^N\right) dx + \frac{1}{q}\int_\Omega |\nabla u_m|^q \, dx - \int_\Omega G(x,(u_m)_+) dx - \int_\Omega f(x)u_m dx.$$

By Proposition 3.6.2 we get $\{u_m\}_m$ bounded in $W_0^{1,N}(\Omega)$. So, there exists $u \in W_0^{1,N}(\Omega)$ such that, up to a subsequence, it holds

$$\begin{cases} u_m \rightharpoonup u & \text{in } W_0^{1,N}(\Omega), \\ u_m \to u & \text{in } L^s(\Omega),\ 1 \leq s < \infty, \\ u_m(x) \to u(x) & \text{a.e. in}\, \Omega. \end{cases}$$

Consider $v \in C_c^\infty(\Omega)$. The convergence $u_m \to u$ in $L^N(\Omega)$ implies

$$\int_\Omega |u_m|^{N-2}u_m v dx \to \int_\Omega |u|^{N-2}uv dx,$$

and due to Lemma 3.6.4, item (iii), we have

$$\int_\Omega g(x,(u_m)_+)v dx \to \int_\Omega g(x,(u)_+)v dx.$$

Using a well-known result from [24], it follows that

$$\int_\Omega |\nabla u_m|^{N-2}\nabla u_m \nabla v dx + \int_\Omega |\nabla u_m|^{q-2}\nabla u_m \nabla v dx \to \int_\Omega |\nabla u|^{N-2}\nabla u \nabla v dx + \int_\Omega |\nabla u|^{q-2}\nabla u \nabla v dx$$

as $m \to \infty$. Thus,

$$J'(u_m)v \to J'(u)v.$$

This implies that $J'(u)v = 0$ for all $v \in C_c^\infty(\Omega)$, and so u is a critical point of J. Since we are assuming $f \neq 0$, we get $u \neq 0$. We need to show that $u \neq w$. Let us suppose that $u = w$ and consider $v_m := u_m - w$. It follows that

$$\begin{cases} v_m \rightharpoonup 0 & \text{in } W_0^{1,N}(\Omega), \\ v_m \to 0 & \text{in } L^s(\Omega),\ 1 \le s < \infty, \\ v_m \to 0 & \text{a.e. in}\,\Omega. \end{cases}$$

It follows from the Brezis-Lieb Lemma that

$$\|u_m\|^N + \|u_m\|_{W^{1,q}}^q = \|u_m - u\|^N + \|u\|^N + \|u_m - u\|_{W^{1,q}}^q + \|u\|_{W^{1,q}}^q + o(1). \tag{3.31}$$

Furthermore, from Lemma 3.6.4 we have

$$\int_\Omega G(x,(u_m)_+)dx \to \int_\Omega G(x,u_+)dx = \int_\Omega G(x,w_+)dx = 0.$$

We also have

$$\int_\Omega |f(x)(u_m - u)|dx \le \|f\|_{L^\infty}\|u_m - u\|_{L^1} \to 0, \quad \text{as} \quad m \to \infty.$$

Recalling that we are assuming $u = w$ and w is a negative function, we see that

$$\begin{aligned} J(u_m) - J(w) &= \frac{1}{N}\left(\|u_m\|^N - \|w\|^N\right) + \frac{1}{q}\left(\|u_m\|_{W^{1,q}}^q - \|w\|_{W^{1,q}}^q\right) - \frac{\lambda}{N}\left(\|u_m\|_N^N - \|w\|_N^N\right) \\ &\quad - \int_\Omega G(x,(u_m)_+)dx - \int_\Omega f(x)(u_m - w)dx \\ &= \frac{1}{N}\|v_m\|^N + \frac{1}{q}\|v_m\|_{W^{1,q}}^q + o(1). \end{aligned} \tag{3.32}$$

Now we take $\delta > 0$ such that

$$c_0 + 2\delta < J(w) + \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

Then, from (3.32), it follows that

$$\frac{1}{N}\|v_m\|^N \le \frac{1}{N}\|v_m\|^N + \frac{1}{q}\|v_m\|_{W^{1,q}}^q \le \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1} - \delta,$$

for $m \in \mathbb{N}$ large enough. This implies that

$$\|v_m\|^{\frac{N}{N-1}} \le \left(\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1} - N\delta\right)^{\frac{1}{N-1}} \tag{3.33}$$

for large m. Let $s>1$ and $\varepsilon>0$ be such that

$$s(\alpha_0+\varepsilon)\left(\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}-N\delta\right)^{\frac{1}{N-1}}\leq\alpha_N.$$

So, using $(\tilde{g}_6)$ (see Lemma 3.2.1), the Pohozaev-Trundinger-Moser inequality (3.2) and (3.33) we obtain

$$\begin{aligned}\int_\Omega|g(x,(v_m)_+)|^s dx&\leq C\int_\Omega e^{s(\alpha_0+\varepsilon)|v_m|^{N'}}dx=C\int_\Omega e^{s(\alpha_0+\varepsilon)\|v_m\|^{N'}\left(\frac{v_m}{\|v_m\|}\right)^{N'}}dx\\&\leq C\int_\Omega e^{\alpha_N\left(\frac{v_m}{\|v_m\|}\right)^{N'}}dx\leq C_1.\end{aligned}\tag{3.34}$$

By Hölder's inequality,

$$0\leq\int_\Omega g(x,(v_m)_+)v_m dx\leq\left(\int_\Omega|g(x,(v_m)_+)|^s dx\right)^{\frac{1}{s}}\|v_m\|\leq C\|v_m\|_{s'}\to 0$$

when $m\to\infty$. On the other hand, using again (3.31), we get

$$\begin{aligned}J'(v_m)v_m&=\|v_m\|_\lambda^N+\|v_m\|_{W^{1,q}}^q-\int_\Omega g(x,(v_m)_+)v_m dx-\int_\Omega f(x)v_m dx\\&=\|u_m\|_\lambda^N-\|w\|_\lambda^N+\|u_m\|_{W^{1,q}}^q-\|w\|_{W^{1,q}}^q+o(1)-\int_\Omega g(x,(v_m)_+)v_m dx\\&\quad+\int_\Omega g(x,(u_m)_+)u_m dx-\int_\Omega g(x,(u_m)_+)u_m dx-\int_\Omega f(x)v_m dx\\&=J'(u_m)u_m-J'(w)w-\int_\Omega g(x,(v_m)_+)v_m dx+\int_\Omega g(x,(u_m)_+)u_m dx+o(1)\\&=\int_\Omega g(x,(u_m)_+)u_m dx+o(1).\end{aligned}\tag{3.35}$$

Now, notice that

$$\int_\Omega g(x,(u_m)_+)u_m dx\to 0,\quad\text{as } m\to\infty.$$

In fact, as in (3.34), using $(\tilde{g}_6)$ and the fact of $(u_m)_+=(v_m+w)_+\leq(v_m)_+$, we have

$$\begin{aligned}0\leq\int_\Omega g(x,(u_m)_+)u_m dx\leq C\int_\Omega e^{(\alpha_0+\varepsilon)(u_m)_+^{N'}}(u_m)_+dx\leq C\left(\int_\Omega e^{s(\alpha_0+\varepsilon)(v_m)_+^{N'}}dx\right)^{\frac{1}{s}}\|(v_m)_+\|_{s'}\\\leq C_1\|v_m\|_{s'}\to 0.\end{aligned}$$

Therefore, by (3.35) we have get

$$o(1)=J'(v_m)v_m=\|v_m\|^N+\|v_m\|_{W^{1,q}}^q+o(1)$$

and this implies that

$$\|v_m\|^N + \|v_m\|^q_{W^{1,q}} \to 0, \quad \text{as } m \to +\infty. \tag{3.36}$$

Here we analyze these two cases: $J(w) = c_0$ and $J(w) < c_0$. If $J(w) = c_0$, the sequence $\{u_m\}$ satisfies $\|u_m - w\| \geq \rho/2$. In this case, if $u_m \rightharpoonup u = w$ we get $\|v_m\| = \|u_m - w\| \geq \rho/2$, which contradicts (3.36). So, $J(w) = c_0$ implies that $u \neq w$. For the second case, $J(w) < c_0$, if $u_m \rightharpoonup u = w$ then estimate (3.32) implies that

$$\frac{1}{N}\|v_m\|^N + \frac{1}{q}\|v_m\|^q_{W_0^{1,q}} \to c_0 - J(w) \neq 0,$$

and this also contradicts (3.36). Therefore, we must have $u \neq w$, which means that J has a second critical point, $u \in W_0^{1,N}(\Omega)$. $\square$

Now, the remaining task is to demonstrate that the *minimax* level c_0, as defined in (3.29), is, in fact, below $J(w) + \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}$. In order to obtain this estimate, we introduce the Moser functions

$$\tilde{z}_m(x) = \omega_{N-1}^{\frac{-1}{N}} \begin{cases} (\log m)^{\frac{N-1}{N}} & \text{if } |x| < \frac{1}{m} \\ \frac{\log \frac{1}{|x|}}{(\log m)^{\frac{1}{N}}} & \text{if } \frac{1}{m} \leq |x| \leq 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

For a suitable $x_0 \in \Omega$, to be chosen later, $r \in (0,1)$ such that $B(x_0, r) \subset \Omega$ and $\delta_m \to 0^+$ we denote

$$z_m(x) = \tilde{z}_m\left(\frac{x - x_0}{\delta_m}\right), \quad \forall x \in \overline{\Omega}. \tag{3.37}$$

We see that the following estimates hold:

Lemma 3.6.6. *For any $m \in \mathbb{N}$, the function $z_m \in W_0^{1,N}(\Omega)$ and it holds:*

a) $\|z_m\| = \|\nabla z_m\|_{L^N} = 1$;

b) $\|\nabla z_m\|^s_{L^s} = \delta_m^{N-s} O(\log m)^{\frac{-s}{N}}$, *for $s \in [1, N)$;*

c) $\|z_m\|^s_{L^s} = \delta_m^N O(\log m)^{\frac{-s}{N}}$, *for $s \in [1, \infty)$.*

The control of c_0 will be done in the next proposition.

Proposition 3.6.7. *Suppose that g satisfies (α_0) and $(\tilde{g}_1)-(\tilde{g}_3)$. Consider c_0 as defined in (3.29). Then*

$$c_0 < J(w) + \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

Proof. From Lemma 3.2.3, there are positive constants σ_0, γ_∞ and C_0 such that

$$sg(x,s) \geq \gamma_\infty e^{\alpha_0 (s+2\sigma_0)^{N'}}, \quad \forall (x,s) \in \Omega \times [C_0, \infty) \tag{3.38}$$

Choose and fix $r \in (0,1)$ and a point $x_0 \in \Omega$ close enough to $\partial\Omega$ such that

$$\|w\|_{L^\infty(B_r(x_0))} \leq \sigma_0. \tag{3.39}$$

For this x_0 and $0 < \delta_m < r \leq 1$, we consider z_m as given in (3.37). By Lemma 3.6.1 we know that, for any $m \in \mathbb{N}$, there is $R_m > 0$ satisfying

$$J(w + tz_m) \leq J(w), \quad \forall\, t \geq R_m.$$

Then, there exists $t_m > 0$ such that

$$J(w + t_m z_m) = \max_{t \geq 0} J(w + tz_m).$$

By definition of the mountain pass level c_0 we see that

$$c_0 \leq \max_{t\geq 0} J(w + tz_m) = J(w + t_m z_m), \quad \forall m \in \mathbb{N}.$$

To prove this proposition it is sufficient to show that there is $m \in \mathbb{N}$ such that

$$J(w + t_m z_m) < J(w) + \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}. \tag{3.40}$$

Let us assume, by contradiction, that

$$J(w + t_m z_m) \geq J(w) + \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}, \ \forall m \in \mathbb{N}. \tag{3.41}$$

Using the results in Proposition 1.3.4, as in Proposition 1.4.1 we obtain

$$\begin{aligned}
&\int_\Omega |\nabla(w + tz_m)|^N dx \\
&\leq \int_\Omega |\nabla w|^N \left[1 + t^N \frac{|\nabla z_m|^N}{|\nabla w|^N} + Nt\frac{\nabla w.\nabla z_m}{|\nabla w|^2} + C\xi_N\left(t\frac{|\nabla z_m|}{|\nabla w|}\right)\right] dx \\
&\leq \int_\Omega |\nabla w|^N dx + t^N \int_\Omega |\nabla z_m|^N dx + Nt\int_\Omega |\nabla w|^{N-2}\nabla w.\nabla z_m dx + C\int_\Omega \xi_N\left(t|\nabla z_m|\right) dx
\end{aligned}$$

where

$$\xi_N(s) = \begin{cases} s^2 + s^{N-1} & if \quad N \geq 3, \\ s^{N-1} & if \quad N = 2, \end{cases}$$

and we use the fact that $|\nabla w| \in L^{\infty}(\Omega)$. Similarly,

$$\begin{aligned}
&\int_\Omega |\nabla(w+tz_m)|^q dx \\
&\leq \int_\Omega |\nabla w|^q \left[1 + t^q \frac{|\nabla z_m|^q}{|\nabla w|^q} + qt\frac{\nabla w.\nabla z_m}{|\nabla w|^2} + C\xi_q\left(t\frac{|\nabla z_m|}{|\nabla w|}\right)\right] dx \\
&\leq \int_\Omega |\nabla w|^q dx + t^q \int_\Omega |\nabla z_m|^q dx + qt\int_\Omega |\nabla w|^{q-2}\nabla w.\nabla z_m dx + C\int_\Omega \xi_q\left(t|\nabla z_m|\right) dx
\end{aligned}$$

where

$$\xi_q(s) = \begin{cases} s^2 + s^{q-1} & if \quad N > q \geq 3, \\ s^\gamma & if \quad q \in [2,3), \quad \text{for a fixed } \gamma \in (q-1,2), \\ s^\gamma & if \quad q \in (1,2), \quad \text{for a fixed } \gamma \in (1,q). \end{cases}$$

On the other hand, for every $t > 0$ we have

$$\begin{aligned}
&\left|\int_\Omega |w+tz_m|^N dx - \int_\Omega |w|^N dx - t^N \int_\Omega |z_m|^N dx\right| \\
&\qquad \leq 2^{N-1}Nt^{N-1}\int_\Omega (z_m)^{N-1}|w| dx + 2^{N-1}Nt\int_\Omega z_m|w|^{N-1}dx.
\end{aligned}$$

Thus, since $G(x,s) \geq 0$ and $G(x,w) = 0$, we get

$$\begin{aligned}
J(w+tz_m) - J(w) &\leq \frac{1}{N}\int_\Omega \left(|\nabla(w+tz_m)|^N - |\nabla w|^N\right) dx - \frac{\lambda}{N}\int_\Omega \left(|w+tz_m|^N - |w|^N\right) dx \\
&\quad + \frac{1}{q}\int_\Omega \left(|\nabla(w+tz_m)|^q - |\nabla w|^q\right) dx - \int_\Omega [f(x)(w+tz_m) - f(x)w]\, dx \\
&\leq \frac{t^N}{N}\int_\Omega |\nabla z_m|^N dx + t\int_\Omega |\nabla w|^{N-2}\nabla w.\nabla z_m dx + \frac{t^q}{q}\int_\Omega |\nabla z_m|^q dx \\
&\quad + t\int_\Omega |\nabla w|^{q-2}\nabla w.\nabla z_m dx + C\int_\Omega \xi_N\left(t|\nabla z_m|\right) dx + C\int_\Omega \xi_q\left(t|\nabla z_m|\right) dx \\
&\quad + \lambda 2^{N-1}t^{N-1}\int_\Omega (z_m)^{N-1}|w| dx + \lambda 2^{N-1}t\int_\Omega z_m|w|^{N-1}dx \\
&\quad - t\int_\Omega f(x)z_m dx.
\end{aligned}$$

Recalling that $J'(w)(tz_m) = 0$, we see that

$$\begin{aligned}
0 &= t\int_\Omega |\nabla w|^{N-2}\nabla w.\nabla z_m dx + t\int_\Omega |\nabla w|^{q-2}\nabla w.\nabla z_m dx \\
&\quad + \lambda t\int_\Omega |w|^{N-1}z_m dx - t\int_\Omega f(x)z_m dx, \quad \forall t > 0.
\end{aligned}$$

Then

$$J(w+tz_m)-J(w)\leq\frac{t^N}{N}\int_\Omega|\nabla z_m|^N dx+\frac{t^q}{q}\int_\Omega|\nabla z_m|^q dx+C\int_\Omega\xi_N\left(t|\nabla z_m|\right)dx$$
$$+C\int_\Omega\xi_q\left(t|\nabla z_m|\right)dx+Ct^{N-1}\int_\Omega(z_m)^{N-1}dx+Ct\int_\Omega z_m dx,$$

where we use $w\in L^\infty(\Omega)$. At this point, from Lemma 3.6.6 we obtain

$$J(w+tz_m)-J(w)\leq\frac{t^N}{N}+t^q\delta_m^{N-q}O\left((\log m)^{\frac{-q}{N}}\right)+t^{N-1}\delta_m^{N-(N-1)}O\left((\log m)^{-\frac{(N-1)}{N}}\right)$$
$$+t^2\delta_m^{N-2}O\left((\log m)^{\frac{-2}{N}}\right)+t^\gamma\delta_m^{N-\gamma}O\left((\log m)^{\frac{-\gamma}{N}}\right)$$
$$+t^{N-1}\delta_m^{N}O\left((\log m)^{-\frac{(N-1)}{N}}\right)+t\delta_m^{N}O\left((\log m)^{\frac{-1}{N}}\right),\quad\forall t>0,$$

for a suitable $\gamma\in(1,q)$, where the term with t^2 appears only for $N\geq3$. Since $\delta_m\in(0,1)$ we get

$$J(w+tz_m)-J(w)\leq\frac{t^N}{N}+t^q\delta_m^{N-q}O\left((\log m)^{\frac{-q}{N}}\right)+t^{N-1}\delta_m^{N-(N-1)}O\left((\log m)^{-\frac{(N-1)}{N}}\right)$$
$$+t^2\delta_m^{N-2}O\left((\log m)^{\frac{-2}{N}}\right)+t^\gamma\delta_m^{N-\gamma}O\left((\log m)^{\frac{-\gamma}{N}}\right)+t\delta_m^{N-1}O\left((\log m)^{\frac{-1}{N}}\right).$$

Now, considering $\delta_m=(\log m)^{-1/N}$, since $\delta_m^{N-s}(\log m)^{\frac{-s}{N}}=(\log m)^{-1}$, we see that

$$J(w+tz_m)-J(w)\leq\frac{t^N}{N}+\frac{C}{\log m}\left(t^q+t^{N-1}+t^2+t^\gamma+t\right),\quad\forall t>0,\tag{3.42}$$

for large $m\in\mathbb{N}$. This inequality allows us to show that $\{t_m\}_m$ is bounded from below by a positive constant. Here, again, the term t^2 appears only if $N\geq3$. In fact, considering $t=t_m$ in (3.42), if $t_m\leq1$ it follows from (3.41) that

$$t_m^N\geq\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}-\frac{C}{\log m}\geq\frac{1}{2}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1},$$

for $m\in\mathbb{N}$ sufficiently large. So, $t_m\geq\min\{1,(1/2)\,(\alpha_N/\alpha_0)^{\frac{N-1}{N}}\}$ for large values of m. Thus, there exists $t_0>0$ such that

$$t_m\geq t_0,\quad\forall m\in\mathbb{N}.$$

Since $t_m>0$ is a maximum point for $t\mapsto J(w+tz_m)$, $t>0$, it follows that

$$J'(w+t_mz_m)(t_mz_m)=t_m\frac{d}{dt}J\left(w+tz_m\right)\Big|_{t=t_m}=0,$$

and so

$$\int_\Omega g(x,(w+t_mz_m)_+)(t_mz_m)dx = t_m\int_\Omega |\nabla(w+t_mz_m)|^{N-2}\nabla(w+t_mz_m)\nabla z_m dx \\ + t_m\int_\Omega |\nabla(w+t_mz_m)|^{q-2}\nabla(w+t_mz_m)\nabla z_m dx \\ -\lambda t_m\int_\Omega |w+t_mz_m|^{N-2}(w+t_mz_m)z_m dx - t_m\int_\Omega f z_m dx.$$

Since $\{z_m\}_m$ is bounded in $W_0^{1,N}(\Omega)$ we see that

$$\int_\Omega |\nabla(w+t_mz_m)|^{N-2}\nabla(w+t_mz_m)\nabla(t_mz_m)dx \le \|\nabla(w+t_mz_m)\|_{L^N}^{N-1}\|\nabla(t_mz_m)\|_{L^N} \\ \le C\left(\|w\|^{N-1}+t_m^{N-1}\|z_m\|^{N-1}\right)\|t_mz_m\| \\ \le C\left(t_m+t_m^N\right),$$

and, in a similar way, we get

$$\int_\Omega |\nabla(w+t_mz_m)|^{q-2}\nabla(w+t_mz_m)\nabla(t_mz_m)dx \le C\left(t_m+t_m^q\right),$$

$$\int_\Omega |w+t_mz_m|^{N-2}(w+t_mz_m)(t_mz_m)dx \le C\left(t_m+t_m^N\right),$$

and

$$\int_\Omega |f(x)|(t_mz_m)dx \le Ct_m,$$

for all $m\in\mathbb{N}$. Then,

$$\int_\Omega g(x,(w+t_mz_m)_+)(t_mz_m+w)dx \le C(t_m+t_m^q+t_m^N) \le \tilde{C}t_m^N, \quad \forall m. \tag{3.43}$$

In the last inequality we use that $t_m \ge t_0 > 0$ for all $m\in\mathbb{N}$. Now, for $C_0>0$ as in (3.38) we get

$$(t_mz_m+w)(x) \ge t_0\omega_{N-1}^{\frac{-1}{N}}(\log m)^{\frac{N-1}{N}} - \|w\|_\infty \ge C_0, \quad \forall x\in B(x_0,\delta_m/m),$$

for m sufficiently large. By (3.38) and (3.39), we obtain that

$$\int_{B(x_0,\frac{\delta_m}{m})} g(x,w+t_mz_m)(t_mz_m+w)dx \ge \gamma_\infty\int_{B(x_0,\frac{\delta_m}{m})} e^{\alpha_0(w+t_mz_m+2\sigma_0)^{N'}} \\ \ge \gamma_\infty\left(\frac{\delta_m}{m}\right)^N\frac{\omega_{N-1}}{N}e^{\alpha_0\left(t_m\omega_{N-1}^{-1/N}(\log m)^{1/N'}+\sigma_0\right)^{N'}}. \tag{3.44}$$

Here we observe that $\{t_m\}_m$ is bounded from above. Otherwise, if $t_m\to+\infty$ for some subsequence, then we would have $Nt_m^{N'}(\alpha_0/\alpha_N)-(N+1)\ge \eta t_m^{N'}$ for some $\eta>0$ and, from (3.43) and (3.44) we would obtain

$$C_1 t_m^N \geq \left(\frac{\delta_m}{m}\right)^N e^{t_m^{N'}(\alpha_0/\alpha_N)N\log m} \geq \frac{m^{N+1}}{m^N(\log m)^N} e^{\eta t_m^{N'}\log m} \geq e^{\eta t_m^{N'}},$$

for large values of m, which is an absurd for $t_m \to \infty$. Thus, we obtain the boundedness of $\{t_m\}_m$. Going back to (3.41) and (3.42) we get

$$\left(\frac{\alpha_0}{\alpha_N}\right)^{N-1} t_m^N \geq 1 - \frac{c}{\log m}, \tag{3.45}$$

for some positive constant c, for large m. Now, using the Taylor expansion, we see that for any $a, b > 0$ there exists $\theta \in (0,1)$ satisfying

$$(a+b)^{N'} = a^{N'} + N'a^{N'-1}b + N'(N'-1)(a+\theta b)^{N'-2} \geq a^{N'} + N'a^{N'-1}b.$$

So, recalling that $t_m \geq t_0$ and using (3.45), for some $k_0 > 0$ we get

$$\begin{aligned}
\alpha_0\left(t_m\omega_{N-1}^{-1/N}(\log m)^{1/N'} + \sigma_0\right)^{N'} &\geq \alpha_0\left(t_m^{N'}\omega_{N-1}^{-1/N-1}\log m + N'\sigma_0\left(t_m\omega_{N-1}^{-1/N}(\log m)^{1/N'}\right)^{(N'-1)}\right)\\
&\geq \left[\left(t_m^N(\alpha_0/\alpha_N)^{N-1}\right)^{1/(N-1)} N\log m + k_0(\log m)^{1/N}\right]\\
&\geq \left(1 - c(\log m)^{-1})N\log m + k_0(\log m)^{1/N}\right)\\
&\geq N\log m + k_0(\log m)^{1/N} - cN,
\end{aligned}$$

for m sufficiently large. Then, by (3.43) and (3.44) it follows that

$$C_2 t_m^N \geq \frac{\delta_m^N}{m^N} e^{N\log m + k_0(\log m)^{1/N} - cN} = e^{-cN}\delta_m^N e^{k_0(\log m)^{1/N}} = e^{-cN}\frac{e^{k_0(\log m)^{1/N}}}{\log m},$$

for large m, where the right side tends to $+\infty$ but the left side is bounded. This contradiction ensures (3.40) and concludes this proof. □

3.6.1 Proof of the main theorem

The main result of this chapter, Theorem 3.1.1, has been proved in some steps. Initially, Proposition 3.5.1 ensures the existence of a first solution w for problem (3.1) and shows that w is a local minimum for J in $W_0^{1,N}(\Omega)$. Then we obtain a mountain pass geometry for this functional and Propositions 3.6.5 and 3.6.7 imply the existence of critical point $u \neq w$ for J, which means a second solution for problem (3.1).

Chapter 4

Critical nonhomogeneous problems on the (N, q)-Laplacian

4.1 Introduction

In this chapter, we establish the existence of nontrivial solutions for a (N, q)-Laplacian equation characterized by:

$$\begin{cases} -\Delta_N u - \Delta_q u = \lambda |u|^{N-2}u + \mu |u|^{q-2}u + g(x,u) + f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where Ω denotes a bounded domain with a smooth boundary within $\mathbb{R}^N$, the parameters satisfy $1 < q < N$, and g is a $C^1(\Omega)$ function in $[0, \infty) \times [0, \infty)$ satisfying a Trudinger-Moser growth condition uniformly in $x \in \Omega$. Analogously to chapters 2 and 3 we find two distinct solutions, one by the Ekeland Variational Principle as long as f has a sufficiently small norm and the other by using cuts in the Moser functions so the *minimax* level of the associated functional is a critical level. This problem was inspired by [37].

4.2 Hypotheses and Main Results

Let us begin by assuming that $f \in L^\infty(\Omega)$ and g exhibits critical growth with exponent $\alpha_0 > 0$. This means that there exists a positive constant α_0 such that

$$\lim_{|t| \to +\infty} \frac{|g(x,t)|}{e^{\alpha |t|^{\frac{N}{N-1}}}} = \begin{cases} 0, & if \quad \alpha > \alpha_0 \\ +\infty, & if \quad \alpha < \alpha_0, \end{cases} \tag{$\bar{\alpha}_0$}$$

uniformly in $x \in \Omega$. This condition is motivated by the Pohozaev-Trudinger-Moser Inequality (see [20]), which state the existence of a positive constant $C = C(N)$ satisfying

$$\int_\Omega e^{\left(\alpha_N |u|^{\frac{N}{N-1}}\right)} dx \leq C(N)|\Omega|, \tag{4.2}$$

for all $u \in W_0^{1,N}(\Omega)$, $N \geq 2$, such that $\|\nabla u\|_N \leq 1$, where $\alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$ and ω_{N-1} is the unitary $(N-1)$-dimensional sphere volume.

($\bar{g}_0$)

$$\limsup_{u \to 0^+} \frac{NG(x,u)}{|u|^N} = 0$$

uniformly in $x \in \Omega$;

($\bar{g}_1$) $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous non-decrasing function and $g(x,0) \equiv 0$ for all $x \in \Omega$, $g(x,u) \geq 0$ in $\Omega \times [0, +\infty)$ and $g(x,u) \leq 0$ in $\Omega \times (-\infty, 0]$;

($\bar{g}_2$) There are $R, M > 0$ such that $\forall\ |u| \geq R$

$$0 < G(x,u) = \int_0^u g(x,s)ds \leq M|g(x,u)|$$

for all $x \in \bar{\Omega}$;

($\bar{g}_3$) There is $C > 0$ such that $\forall\ |s| \geq C$

$$sg(x,s) \geq \gamma(s) e^{\alpha_0 |s|^{\frac{N}{N-1}}}$$

where $\gamma(s)$ is such that there is $\varepsilon_0 > 0$ with

$$\liminf_{s \to \pm\infty} \frac{\gamma(s)}{e^{\varepsilon_0 |s|^{\frac{1}{N-1}}}} > 0$$

for some $\varepsilon_0 > 0$.

The next lemma can be proved following the same steps as in Lemma 3.2.3.

Lemma 4.2.1. *Suppose that ($\bar{g}_1$) and ($\bar{g}_3$) hold. So, there are σ_0, γ_∞ and $C_0 > 0$ such that*

$$sg(x,s) \geq \gamma_\infty e^{\alpha_0 (|s| + \sigma_0)^{N'}}$$

for all $x \in \Omega$ and $|s| \geq C_0$.

Now, let us fix that

$$\lambda_1 = \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\|\nabla u\|_p^p}{\|u\|_p^p}$$

and

$$\mu_1 = \inf_{u\in W_0^{1,q}(\Omega)\setminus\{0\}} \frac{\|\nabla u\|_q^q}{\|u\|_q^q}$$

the first eigenvalues of the problems

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

$$\begin{cases} -\Delta_q u = \mu |u|^{q-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

respectively. Consider now the associated functional $J : W_0^{1,N}(\Omega) \to \mathbb{R}$, which is given by:

$$J(u) = \frac{1}{N}\int_\Omega \left(|\nabla u|^N - \lambda |u|^N\right) dx + \frac{1}{q}\int_\Omega (|\nabla u|^q - \mu|u|^q)dx \\ - \int_\Omega G\left(x,u\right) dx - \int_\Omega f(x)udx$$

where

$$G(x,s) = \int_0^s g(x,t)dt.$$

From these first conditions, we obtain that the associated functional J to problem (4.1) is of class C^1 with derivative given by

$$J'(u)v = \int_\Omega (|\nabla u|^{N-2}\nabla u \nabla v - \lambda|u|^{N-2}uv)dx \\ + \int_\Omega (|\nabla u|^{q-2}\nabla u\nabla v - \mu|u|^{q-2}uv)dx - \int_\Omega g(x,u)vdx - \int_\Omega f(x)vdx$$

for all u and v in $W_0^{1,N}(\Omega)$. By definition, weak solutions of the main problem are critical points of this functional.

In $W_0^{1,s}(\Omega)$, we work with its usual norm

$$\|u\|_{W_0^{1,s}(\Omega)} = \left(\int_\Omega |\nabla u|^s dx\right)^{\frac{1}{s}}.$$

Given that our main framework resides in $W_0^{1,N}(\Omega)$, we simplify notation by letting $\|\cdot\|_{W_0^{1,N}(\Omega)}$ be represented as $\|\cdot\|$. The main results of this chapter are stated below.

Theorem 4.2.2. *Suppose that* $(\bar{\alpha}_0)$, $(\bar{g}_0) - (\bar{g}_3)$ *hold,* $2 \le q < N$, $\lambda \in (0,\lambda_1)$, $\mu \in (0,\mu_1]$ *and* $f \in L^\infty(\Omega)$ *is non-trivial. Then there exists* $\eta > 0$ *such that problem* (4.1) *has two solutions provided that* $\|f\|_\infty \le \eta$.

This restriction $q \ge 2$ can be suppressed if we avoid working with the q-linear term $|u|^{q-2}u$ in the problem, which means, if we put $\mu = 0$. This is the subject of the next theorem.

Theorem 4.2.3. *Suppose that* $(\bar{\alpha}_0)$,$(\bar{g}_0) - (\bar{g}_3)$ *hold,* $N \geq 2$, $1 \leq q < N$, $\lambda \in (0, \lambda_1)$, $\mu \equiv 0$ *and* $f \in L^\infty(\Omega)$ *is non-trivial. Then there exists* $\eta > 0$ *such that problem* (4.1) *has two solutions provided that* $\|f\|_\infty \leq \eta$.

4.3 Preliminaries

The objective of this section is to find a solution for problem (4.1) by Ekeland's Variational Principle and display some important properties such as norm estimates for truncations of the Moser functions. First we will recall some consequences of the properties placed in g, proceeding as in Chapter 3 we see that if g has critical growth with the exponent α_0 (see hypothesis ($\bar{\alpha}_0$)), then $\forall\ \beta > \alpha_0$, there is a constant $C > 0$ such that

$$|g(x, u)| \leq Ce^{(\beta|u|^{\frac{N}{N-1}})}$$

$\forall\ (x, u) \in \Omega \times \mathbb{R}$.

The next lemma has already been demonstrated in Chapter 3 and for convenience we will state it, as these are properties that help with calculations throughout this chapter.

Lemma 4.3.1. $(\bar{g}_0) - (\bar{g}_3)$ *implies that:*

($\bar{g}_4$) *Theres is a* $C > 0$ *constant such that* $\forall\ x \in \Omega$ *and* $|u| \geq R$

$$G(x, u) \geq Ce^{\left(\frac{1}{M}u\right)}.$$

($\bar{g}_5$) *There are* $S > 0$ *and* $\sigma > N$ *such that* $\forall\ x \in \Omega$ *and* $|u| \geq S$ *we have*

$$\sigma G(x, u) \leq ug(x, u).$$

($\bar{g}_6$) *There are* $K > 0$ *and* $r > N$ *constants such that*

$$\int_\Omega G(x, u)dx \leq \frac{\varepsilon}{N} \int_\Omega |u|^N dx + K \int_\Omega |u|^r e^{\beta|u|^{\frac{N}{N-1}}} dx$$

for all $\beta > \alpha_0$ *and* $\varepsilon > 0$.

The next proposition is a standard argument, which was also established in all previous chapters, with their respective frameworks.

Proposition 4.3.2. *Assuming* $(\bar{g}_0) - (\bar{g}_2)$ *hold, let* (u_m) *be a (PS) sequence for* J *in* $W_0^{1,N}(\Omega)$. *Then* (u_m) *is bounded.*

Proof. Let $(u_m) \subset W_0^{1,N}(\Omega)$ a PS sequence at a certain level c, so,

$$\frac{1}{N}\left(\int_\Omega |\nabla u_m|^N - \lambda|u_m|^N\right)dx + \frac{1}{q}\left(\int_\Omega |\nabla u_m|^q - \lambda|u_m|^q\right)dx - \int_\Omega G(x,u_m)dx - \int_\Omega f(x)u_m dx \to c$$

and

$$\left(\int_\Omega |\nabla u_m|^N - \lambda|u_m|^N\right)dx \left(\int_\Omega |\nabla u_m|^q - \lambda|u_m|^q\right)dx - \int_\Omega g(x,u_m)u_m dx - \int_\Omega f(x)u_m dx \le \varepsilon\|u_m\|.$$

Consider σ from $(\bar{g}_5)$, then,

$$\begin{aligned}
C + \varepsilon\|u_m\| &\ge \sigma J(u_m) - J'(u_m)u_m \\
&= \left(\frac{\sigma}{N} - 1\right)\int_\Omega \left(|\nabla u_m|^N - \lambda|u_m|^N\right)dx + \left(\frac{\sigma}{q} - 1\right)\int_\Omega \left(|\nabla u_m|^q - \lambda|u_m|^q\right)dx \\
&- \int_\Omega \left(\sigma G(x,u_m) - g(x,u_m)u_m\right)dx - (\sigma - 1)\int_\Omega f(x)u_m dx \\
&\ge C_1\|u_m\|^N - C_2\|u_m\|.
\end{aligned}$$

This last inequality shows that (u_m) is bounded. □

The next two lemmas will give the necessary geometry to find the first solution, which will be a local minimum for the functional.

Lemma 4.3.3. *Suppose that $(\bar{g}_0) - (\bar{g}_2)$ holds. Then, there are constants $\eta > 0$ and $\rho_0 > 0$ such that, for all $u \in W_0^{1,N}(\Omega)$ with $\|u\| = \rho_0$, we have $J(u) \ge 0$, provided $\|f\|_\infty \le \eta$.*

Proof. Let $\beta > \alpha_0$ and $r > N$. So by $(\bar{g}_6)$ and using Lemma 4.3.1 and Sobolev embeddings, we obtain

$$\begin{aligned}
J(u) &= \frac{1}{N}\int_\Omega \left(|\nabla u|^N - \lambda|u|^N\right)dx + \frac{1}{q}\int_\Omega (|\nabla u|^q - \mu|u|^q)dx - \int_\Omega G(x,u)\,dx - \int_\Omega f(x)u dx \\
&\ge \frac{1}{N}\left(1 - \frac{\lambda}{\lambda_1}\right)\|u\|^N - \frac{\varepsilon}{N}\int_\Omega |u|^N dx - K\int_\Omega |u|^r e^{\beta|u|^{\frac{N}{N-1}}}dx - C\|f\|_\infty\|u\|
\end{aligned}$$

By Hölder inequality,

$$\int_\Omega |u|^r e^{\beta|u|^{\frac{N}{N-1}}}dx \le \left(\int_\Omega e^{[p\beta\|u\|^{\frac{N}{N-1}}\left(\frac{|u|}{\|u\|}\right)^{\frac{N}{N-1}}]}dx\right)^{\frac{1}{p}}\left(\int_\Omega |u|^{rs}dx\right)^{\frac{1}{s}}$$

where $\frac{1}{p} + \frac{1}{s} = 1$. Take $p > 1$ such that

$$p\beta\|u\|^{\frac{N}{N-1}} \le \alpha_N. \tag{4.3}$$

Then, by Pohozaev-Trudinger-Moser inequality we get

$$\int_\Omega |u|^r e^{\beta|u|^{\frac{N}{N-1}}} dx \leq C\|u\|_{rs}^r.$$

Therefore,

$$\begin{aligned} J(u) &\geq \frac{1}{N}\left(1-\frac{\lambda}{\lambda_1}\right)\|u\|^N - \frac{\varepsilon}{N}\int_\Omega |u|^N dx - C\|u\|^r - C\|f\|_\infty\|u\| \\ &= \|u\|\left\{\frac{1}{N}\left(1-\frac{\lambda+\varepsilon}{\lambda_1}\right)\|u\|^{N-1} - C\|u\|^{r-1} - C\|f\|_\infty\right\}. \end{aligned}$$

Consider ρ_0 such that

$$h(\rho_0) = \frac{1}{N}\left(1-\frac{\lambda}{\lambda_1}\right)\rho_0^{N-1} - C\frac{\varepsilon}{N}\rho_0^{N-1} - C\rho_0^{r-1} > 0$$

for $\varepsilon > 0$ small enough. Then choose $\eta \geq \|f\|_\infty$ so that $J(u) \geq 0$ for all $\|u\| = \rho_0$. $\square$

Lemma 4.3.4. *Suppose that $(\bar{g}_0) - (\bar{g}_2)$ hold. Then*

$$\inf_{\|u\|<\rho} J(u) < 0$$

where $\rho > 0$ is small enough.

Proof. Consider $\varphi \in C_0^\infty(\Omega)$ such that $\|\varphi\| = 1$ and $\int_\Omega f\varphi dx > 0$. So,

$$\begin{aligned} J(t\varphi) &= \frac{t^N}{N}\int_\Omega \left(|\nabla\varphi|^N - \lambda|\varphi|^N\right) dx + \frac{t^q}{q}\int_\Omega (|\nabla\varphi|^q - \mu|\varphi|^q) dx \\ &\quad - \int_\Omega G(x, t\varphi)\, dx - t\int_\Omega f(x)\varphi dx \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt}J(t\varphi) &= t^{N-1} - \lambda t^{N-1}\|\varphi\|_N^N + t^{q-1}\int_\Omega \left(|\nabla\varphi|^q - \mu|\varphi|^q\right) dx \\ &\quad - \int_\Omega g(x, t\varphi)\varphi dx - \int_\Omega f(x)\varphi dx \\ &\leq t^{N-1} + t^{q-1}\int_\Omega |\nabla\varphi|^q dx - \int_\Omega g(x, t\varphi)\varphi dx - \int_\Omega f(x)\varphi dx. \end{aligned}$$

Since $g(x, 0) = 0$ and $g(x, .)$ is continuous, there is $\rho > 0$ small enough such that for $0 < t < \rho$ we have

$$\frac{d}{dt}J(t\varphi) < 0.$$

Furthermore, $J(0) = 0$ implies that

$$J(t\varphi) < 0$$

for $0 < t < \rho$. $\square$

The last two propositions of this section will give the first solution of the problem, as a consequence of Ekeland's Variational Principle.

Proposition 4.3.5. *Let (u_m) be a (PS) sequence for J in $W_0^{1,N}(\Omega)$ such that*

$$\liminf_{n\to\infty} \|u_m\| < \left(\frac{\alpha_N}{\alpha_0}\right)^{\frac{N-1}{N}}.$$

Then there exists a subsequence of (u_m) that converges to a solution of (4.1).

Proof. Let (u_m) be a sequence (PS) for J in $W_0^{1,N}(\Omega)$, so (u_m) is bounded and using the Lemma 3.6.4 we find that there is $w \in W_0^{1,N}(\Omega)$ a solution of (4.1) such that $u_m \rightharpoonup w$ in $W_0^{1,N}(\Omega)$. Now, consider

$$w_m = w - u_m.$$

Then, $w_m \rightharpoonup 0$ in $W_0^{1,N}(\Omega)$, $w_m \to 0$ in $L^s(\Omega)$ with $1 \le s < \infty$, furthermore, by Brezis-Lieb Lemma and Lemma 3.6.3 we obtain

$$\begin{aligned} o(1) &= J'(u_m)u_m \\ &= \left(\|w\|^N + \|w_m\|^N - \lambda\|w\|_N^N\right) + \left(\|w\|_{W_0^{1,q}}^q + \|w_m\|_{W_0^{1,q}}^q - \mu\|w\|_q^q\right) \\ &\quad - \int_\Omega g(x,u_m)u_m dx + \int_\Omega g(x,u_m)w dx - \int_\Omega g(x,u_m)w dx - \int_\Omega f w dx + o(1) \\ &= J'(w)w + \|w_m\|^N + \|w_m\|_{W_0^{1,q}}^q + \int_\Omega g(x,u_m)w_m dx + o(1). \end{aligned}$$

Since

$$\liminf_{m\to\infty} \|u_m\| < \left(\frac{\alpha_N}{\alpha_0}\right)^{\frac{N-1}{N}},$$

we can choose a subsequence still denoted as (u_m) and a $\sigma > 1$ such that

$$\lim_{m\to\infty} \sigma\alpha_0\|u_m\|^{\frac{N}{N-1}} < \alpha_N.$$

From Hölder and Pohozaev-Trudinger-Moser inequalities we have

$$\begin{aligned} \int_\Omega g(x,u_m)w_m dx &\le \left(\int_\Omega |g(x,u_m)|^\sigma dx\right)^{\frac{1}{\sigma}} \left(\int_\Omega |w_m|^{\sigma'} dx\right)^{\frac{1}{\sigma'}} \\ &\le \left(\int_\Omega e^{\sigma\alpha_0\|u_m\|^{\frac{N}{N-1}}\left(\frac{|u_m|}{\|u_m\|}\right)^{\frac{N}{N-1}}}\right)^{\frac{1}{\sigma}} \left(\int_\Omega |w_m|^{\sigma'} dx\right)^{\frac{1}{\sigma'}} \\ &\le K\|w_m\|_{\sigma'} \to 0, \end{aligned}$$

with $K > 0$ being a constant. Therefore,

$$0 = \lim_{m\to\infty} \left(\|w_m\|^N + \|w_m\|^q_{W_0^{1,q}} \right)$$

i.e.,

$$\|w_m\| \to 0$$

when $m \to \infty$. □

By Lemmas 4.3.3 and 4.3.4 and knowing that $J(u)$ is continuous, let us define

$$-\infty < c_0 \equiv \inf\{J(u); u \in W_0^{1,N}(\Omega), \|u\| \le \rho_0\} < 0. \tag{4.4}$$

Proposition 4.3.6. *Consider* $(\bar{g}_0) - (\bar{g}_3)$. *Let* $\eta > 0$ *and* ρ_0 *be constants determined by Lemma 4.3.3 and assume* $\|f\|_\infty \le \eta$. *Then there exists* $w \in W_0^{1,N}(\Omega)$ *solution of* (4.1) *at level* c_0.

Proof. Consider ρ_0 from (4.3) in Lemma 4.3.3. Notice that it was chosen in such a way that

$$\rho_0 < \left(\frac{\alpha_N}{\alpha_0} \right)^{\frac{N-1}{N}}.$$

From Lemma 4.3.4 we can apply the Ekeland variational principle to find that there exists a sequence (u_m) in $\bar{B}(0, \rho_0)$ such that $J(u_m) \to c_0$ and $J'(u_m)u_m \to 0$ when $m \to \infty$. Therefore, using Proposition 4.3.5, we have a solution. □

4.4 Proof of the Main Theorems

This section provides a mountain-pass geometry and acts as an auxiliary result to demonstrate that *minimax* levels can be controlled by strategically selecting directions provided by truncations of Moser functions. The first lemma, together with the fact that w is a local minimum to J, provides the required geometric properties.

Lemma 4.4.1. *Assume that* g *satisfies* $(\bar{\alpha}_0)$ *and* $(\bar{g}_0) - (\bar{g}_2)$ *and consider* $\varphi \in W_0^{1,N}(\Omega)\backslash\{0\}$ *as a continuous, nontrivial and nonnegative function. Then* $J(w + t\varphi) \to -\infty$ *when* $t \to +\infty$.

Proof. Since $(\bar{g}_4)$ holds, there are positive numbers C and R and $\sigma > N$ such that

$$G(x, u) \ge u^\sigma, \qquad \forall\ (x, u) \in \overline{\Omega} \times [R, \infty).$$

Let $x_1 \in \Omega$ and $r > 0$ be such that $m_\varphi = \min_{B_r(x_1)} \varphi(x) > 0$. So, for all $t > (R + \|w\|_{L^\infty})/m_\varphi$, we get

$$\begin{aligned}
J(w+t\varphi) = & \frac{1}{N}\int_\Omega \left(|\nabla(w+t\varphi)|^N - \lambda|w+t\varphi|^N\right)dx \\
& + \frac{1}{q}\int_\Omega (|\nabla(w+t\varphi)|^q - \mu|w+t\varphi|^q)dx \\
& - \int_\Omega G(x, w+t\varphi)\,dx - \int_\Omega f(x)(w+t\varphi)dx \\
\leq & \frac{2^{N-1}}{N}\int_\Omega \left(|\nabla w|^N + t^N|\nabla\varphi|^N\right)dx + \frac{2^{q-1}}{q}\int_\Omega (|\nabla w|^q + t^q|\nabla\varphi|^q)\,dx \\
& - C\int_{B_r(x_1)} |w+t\varphi|^\sigma dx + t\|f\|_{L^\infty}\int_\Omega \varphi dx + \|f\|_{L^\infty}\|w\|_{L^1}.
\end{aligned}$$

Since $\|t\varphi\|_{L^\sigma} \leq \|w\|_{L^\sigma} + \|w+t\varphi\|_{L^\sigma}$ implies that

$$\|t\varphi\|^\sigma_{L^\sigma} \leq 2^{\sigma-1}\left(\|w\|^\sigma_{L^\sigma} + \|w+t\varphi\|^\sigma_{L^\sigma}\right),$$

we obtain

$$\begin{aligned}
J(w+t\varphi) \leq & \frac{2^{N-1}t^N}{N}\int_\Omega |\nabla\varphi|^N dx + \frac{2^{q-1}t^q}{q}\int_\Omega |\nabla\varphi|^q dx - \frac{Ct^\sigma}{2^{\sigma-1}}\int_{B_r(x_1)} |\varphi|^\sigma dx + t\|f\|_{L^\infty}\int_\Omega \varphi dx \\
& + C_1
\end{aligned}$$

for large $t > 0$, where C_1 depends on w. Since $\sigma > N \geq q \geq 1$, we must have $J(w+t\varphi) \to -\infty$ as $t \to +\infty$. □

The following definition and subsequent lemma are identical to those presented in Chapter 3. However, we repeat them here to ensure a smoother reading experience throughout this chapter. Let us define

$$\tilde{z}_m(x) = \omega_{N-1}^{\frac{-1}{N}}\begin{cases} (\log m)^{\frac{N-1}{N}} & \text{if } |x| < \frac{1}{m} \\ \frac{\log\frac{1}{|x|}}{(\log m)^{\frac{1}{N}}} & \text{if } \frac{1}{m} \leq |x| \leq 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

For suitable $x_0 \in \Omega$, $\delta_m \leq r$ and $r > 0$ satisfying $B(x_r, r) \subset \Omega$, all of them to be chosen later, we denote

$$z_m(x) = \tilde{z}_m\left(\frac{x - x_0}{\delta_m}\right), \quad \forall x \in \overline{\Omega}. \tag{4.5}$$

Following the work in [17], we obtain the estimates:

Lemma 4.4.2. *For any $m \in \mathbb{N}$, the functions $z_m \in W^{1,N}_0(\Omega)$ and it holds:*

a) $\|z_m\| = \|\nabla z_m\|_{L^N} = 1$;

b) $\|\nabla z_m\|_{L^s}^s = \delta_m^{N-s} O(\log m)^{\frac{-s}{N}}$, *for* $s \in (1, N)$;

c) $\|z_m\|_{L^s}^s = \delta_m^N O(\log m)^{\frac{-s}{N}}$, *for* $s \in [1, \infty)$.

Now, being w is a local minimum for J and considering $\varphi = z_m$ in Lemma 4.4.1, it follows that J has the Mountain Pass Geometry:

Proposition 4.4.3. *Suppose* $(\bar{\alpha}_0)$, $(\bar{g}_0) - (\bar{g}_2)$. *Then the functional* $J \in C^1(W_0^{1,N}(\Omega), \mathbb{R})$ *and the following hold:*

i) There exists $\rho > 0$ *such that* $J(u) \geq J(w)$ *for all* $u \in W_0^{1,p}(\Omega)$ *with* $\|u - w\| = \rho$;

ii) There exist $e \in W_0^{1,p}(\Omega)$ *such that* $\|e - w\| > \rho$ *and* $J(e) < J(w)$.

Define

$$\Gamma := \{\gamma \in C([0,1], W_0^{1,N}(\Omega)) : \gamma(0) = w \text{ and } J(\gamma(1)) < J(w)\},$$

and

$$c_1 := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)). \tag{4.6}$$

Proposition 4.4.4. *Assuming* $(\bar{g}_0) - (\bar{g}_2)$ *and* $c_1 < c_0 + \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}$, *we conclude that* J *possesses a critical point* $u \neq w$, *obtained as the weak limit of a sequence (PS) at the level* c_1.

Proof. Let us begin by recalling that $J(w) = c_0$. By Proposition 4.4.3 we know that is possible to obtain a (PS) sequence $\{u_m\}_m$ for J at the level c_1. By definition of c_1, we have $c_1 \geq c_0$. Notice that, in case $c_1 = c_0$, this sequence can be chosen satisfying $\|u_m - w\| \geq \frac{\rho_0}{2}$ for all m, where ρ_0 is given in (4.4). It follows that

$$\begin{aligned} o(1)\|v\| &= J'(u_m)v \\ &= \int_\Omega \left(|\nabla u_m|^{N-2}\nabla u_m \nabla v - \lambda |u_m|^{N-2} u_m v\right) dx \\ &+ \int_\Omega \left(|\nabla u_m|^{q-2}\nabla u_m \nabla v - \mu |u_m|^{q-2} u_m v\right) dx \\ &- \int_\Omega g(x, u_m) v dx - \int_\Omega f v dx \end{aligned}$$

$\forall\, v \in C_0^\infty(\Omega)$. In particular,

$$\begin{aligned} o(1)\|u_m\| &= J'(u_m)u_m \\ &= \int_\Omega \left(|\nabla u_m|^N - \lambda |u_m|^N\right) dx + \int_\Omega \left(|\nabla u_m|^q - \mu |u_m|^q\right) dx \\ &- \int_\Omega g(x, u_m) u_m dx - \int_\Omega f u_m dx \end{aligned}$$

and

$$\begin{aligned}
c_1 + o(1) &= J(u_m) \\
&= \frac{1}{N}\int_\Omega \left(|\nabla u_m|^N - \lambda |u_m|^N\right) dx + \frac{1}{q}\int_\Omega \left(|\nabla u_m|^q - \mu |u_m|^q\right) dx \\
&- \int_\Omega G(x, u_m)dx - \int_\Omega f u_m dx.
\end{aligned}$$

By Proposition 4.3.2, (u_m) is bounded. Then, there exists $u \in W_0^{1,N}(\Omega)$ such that, up to a subsequence,

$$\begin{cases} u_m \rightharpoonup u & \text{in } W_0^{1,N}(\Omega), \\ u_m \to u & \text{in } L^s(\Omega), 1 \le s < \infty, \\ u_m \to u & \text{a.e. in } x \in \Omega. \end{cases}$$

By Lemma 3.6.3 and following the same steps in the proof of Proposition 3.6.5 we see that u is a critical point of J. Notice that $f \neq 0$ gives $u \neq 0$.

Now, consider $v_m = u_m - w$. So,

$$\begin{cases} v_m \rightharpoonup 0 & \text{in } W_0^{1,N}(\Omega), \\ v_m \to 0 & \text{in } L^s(\Omega), 1 \le s < \infty, \\ v_m \to 0 & \text{a.e. in } x \in \Omega. \end{cases}$$

Furthermore,

$$\int_\Omega G(x, u_m)dx \to \int_\Omega G(x, u)dx = \int_\Omega G(x, w)dx.$$

Then,

$$\begin{aligned}
J(u_m) - J(w) &= \frac{1}{N}\left(\|u_m\|^N - \|w\|^N - \lambda\|u_m\|_N^N + \lambda\|w\|_N^N\right) \\
&+ \frac{1}{q}\left(\|u_m\|^q_{W^{1,q}(\Omega)} - \|w\|^q_{W^{1,q}(\Omega)} - \mu\|u_m\|_q^q + \mu\|w\|_q^q\right) \\
&- \int_\Omega G(x, u_m)dx + \int_\Omega G(x, w)dx \\
&- \int_\Omega f(x)(u_m - w)dx.
\end{aligned} \tag{4.7}$$

It follows from Brezis-Lieb Lemma that

$$\begin{aligned}
\frac{1}{N}\|v_m\|^N + \frac{1}{q}\|v_m\|^q_{W_0^{1,q}(\Omega)} + o(1) &= J(u_m) - J(w) \\
&\to c_1 - c_0.
\end{aligned} \tag{4.8}$$

On the other hand, in a similar way to what was done in Proposition 3.6.5, we get

$$\begin{aligned} J'(v_m)v_m &= \|v_m\|^N - \lambda\|v_m\|_N^N + \|v_m\|_{W^{1,q}(\Omega)}^q - \mu\|v_m\|_q^q - \int_\Omega g(x,v_m)v_m dx - \int_\Omega f(x)v_m dx \\ &= J'(u_m)u_m - J'(w)w - \int_\Omega g(x,v_m)v_m dx + \int_\Omega g(x,u_m)u_m dx - \int_\Omega g(x,w)w dx + o(1) \\ &= \int_\Omega g(x,v_m)v_m dx + \int_\Omega g(x,u_m)v_m dx + o(1) \\ &= o(1). \end{aligned} \tag{4.9}$$

Then, we obtain

$$\|v_m\|^N + \|v_m\|_{W_{1,q}(\Omega)}^q \to 0 \tag{4.10}$$

as $m \to +\infty$. Now supose that $c_1 = J(w)$, then the sequence $\{u_m\}_m$ is such that $\|u_m - w\| \geq \frac{\rho_0}{2}$, that is, $\|v_m\| \geq \frac{\rho_0}{2}$ contradicting (4.10). In the case where $c_1 > c_0$, then by (4.8) we get

$$\frac{1}{N}\|v_m\|^N + \frac{1}{q}\|v_m\|_{W_0^{1,q}(\Omega)}^q + o(1) \to c_1 - c_0 > 0$$

contradicting (4.10) again. This finishes the proof of this proposition. □

Now, the remaining task is to demonstrate that the *minimax* level c_1, as defined in (4.6), is in fact below $J(w) + \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}$.

Proposition 4.4.5. *Suppose that g satisfies $(\bar{\alpha}_0)$ and $(\tilde{g}_1) - (\tilde{g}_3)$. Consider c_1 as defined in (4.6), $2 \leq q < N$ and $\mu \in (0, \mu_1]$. Then*

$$c_1 < J(w) + \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

Proof. It suffices to show that there exists sufficiently large m such that

$$J(w + tz_m) < J(w) + \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1},$$

for all $t \geq 0$. From Lemma 4.2.1 for each $k > 0$, there are $\sigma_0 > 0$ and $C_k > 0$ such that

$$sg(x,s) \geq ke^{\alpha_0(|s|+2\sigma_0)^{N'}}, \tag{4.11}$$

for all $x \in \Omega$ and $|s| \geq C_k$. Choose and fix $r > 0$ and $x_0 \in \Omega$ close enough to $\partial\Omega$ such that

$$\|w\|_{L^\infty(B_r(x_0))} \leq \sigma_0. \tag{4.12}$$

Now, for some $0 < \delta_m < r \leq 1$ (to be chosen later) and x_0, we consider z_m as given in (4.5). By

Lemma 4.4.1 we know that, for any $m \in \mathbb{N}$, there is $R_m > 0$ satisfying

$$J(w + tz_m) \leq 0, \quad \forall\, t \geq R_m.$$

Then, there exists $t_m > 0$ such that

$$J(w + t_m z_m) = \max_{t\geq 0} J(w + tz_m).$$

Let us assume, by contadiction, that

$$J(w + t_m z_m) \geq J(w) + \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1} \quad \forall m \in \mathbb{N}. \tag{4.13}$$

Using the results in Proposition 1.3.4, as in Proposition 1.4.1 we obtain

$$\begin{aligned}
&\int_\Omega |\nabla(w + tz_m)|^N dx \\
&\leq \int_\Omega |\nabla w|^N \left[1 + t^N \frac{|\nabla z_m|^N}{|\nabla w|^N} + Nt\frac{\nabla w.\nabla z_m}{|\nabla w|^2} + C\xi_N\left(t\frac{|\nabla z_m|}{|\nabla w|}\right)\right] dx \\
&\leq \int_\Omega |\nabla w|^N dx + t^N \int_\Omega |\nabla z_m|^N dx + Nt\int_\Omega |\nabla w|^{N-2}\nabla w.\nabla z_m dx + C\int_\Omega \xi_N\left(t|\nabla z_m|\right) dx
\end{aligned}$$

where

$$\xi_N(s) = \begin{cases} s^2 + s^{N-1} & if \quad N \geq 3, \\ s^\gamma & if \quad N = 2, \quad \text{for a fixed } \gamma \in (1,2) \end{cases}$$

and we use the fact that $|\nabla w| \in L^\infty(\Omega)$. Similarly,

$$\begin{aligned}
&\int_\Omega |\nabla(w + tz_m)|^q dx \\
&\leq \int_\Omega |\nabla w|^q \left[1 + t^q \frac{|\nabla z_m|^q}{|\nabla w|^q} + qt\frac{\nabla w.\nabla z_m}{|\nabla w|^2} + C\xi_q\left(t\frac{|\nabla z_m|}{|\nabla w|}\right)\right] dx \\
&\leq \int_\Omega |\nabla w|^q dx + t^q \int_\Omega |\nabla z_m|^q dx + qt\int_\Omega |\nabla w|^{q-2}\nabla w.\nabla z_m dx + C\int_\Omega \xi_q\left(t|\nabla z_m|\right) dx
\end{aligned}$$

where

$$\xi_q(s) = \begin{cases} s^2 + s^{q-1} & if \quad N > q \geq 3, \\ s^\gamma & if \quad q \in [2,3), \quad \text{for a fixed } \gamma \in (q-1,2). \end{cases}$$

On the other hand, for every $t > 0$ we have

$$\begin{aligned}
&\left|\int_\Omega |w + tz_m|^N dx - \int_\Omega |w|^N dx - t^N \int_\Omega |z_m|^N dx\right| \\
&\qquad \leq 2^{N-1}Nt^{N-1}\int_\Omega (z_m)^{N-1}|w| dx + 2^{N-1}Nt\int_\Omega z_m|w|^{N-1}dx.
\end{aligned}$$

Similarly, for every $t > 0$

$$\left| \int_{\Omega} |w + tz_m|^q dx - \int_{\Omega} |w|^q dx - t^N \int_{\Omega} |z_m|^q dx \right| \\ \leq 2^{q-1} q t^{q-1} \int_{\Omega} (z_m)^{q-1} |w| dx + 2^{q-1} q t \int_{\Omega} z_m |w|^{q-1} dx.$$

We know that

$$\begin{aligned} J(w + tz_m) = &\int_{\Omega} \left(|\nabla (w + tz_m)|^N - \lambda |w + tz_m|^N \right) dx \\ &+ \frac{1}{q} \int_{\Omega} \left(|\nabla (w + tz_m)|^q - \mu |w + tz_m|^q \right) dx \\ &- \int_{\Omega} G\left(x, w + tz_m\right) dx - \int_{\Omega} f(w + tz_m) dx. \end{aligned}$$

Thus, we get

$$\begin{aligned} J(w + tz_m) - J(w) = &\frac{1}{N} \int_{\Omega} \left(|\nabla (w + tz_m)|^N - |\nabla w|^N \right) dx - \frac{\lambda}{N} \int_{\Omega} \left(|w + tz_m|^N - |w|^N \right) dx \\ &+ \frac{1}{q} \int_{\Omega} \left(|\nabla (w + tz_m)|^q - |\nabla w|^q \right) dx - \frac{\mu}{q} \int_{\Omega} (|w + tz_m|^q - |w|^q) dx \\ &- \int_{\Omega} \left[G(x, w + tz_m) - G(x, w) \right] dx - \int_{\Omega} \left[f(x)(w + tz_m) - f(x) w \right] dx \\ \leq &\frac{t^N}{N} \int_{\Omega} |\nabla z_m|^N dx + t \int_{\Omega} |\nabla w|^{N-2} \nabla w . \nabla z_m dx + \frac{t^q}{q} \int_{\Omega} |\nabla z_m|^q dx \\ &+ t \int_{\Omega} |\nabla w|^{q-2} \nabla w . \nabla z_m dx + C \int_{\Omega} \xi_N \left(t |\nabla z_m| \right) dx + C \int_{\Omega} \xi_q \left(t |\nabla z_m| \right) dx \\ &+ \lambda 2^{N-1} t^{N-1} \int_{\Omega} (z_m)^{N-1} |w| dx + \lambda 2^{N-1} t \int_{\Omega} z_m |w|^{N-1} dx \\ &+ \mu 2^{q-1} t^{q-1} \int_{\Omega} (z_m)^{q-1} |w| dx + \mu 2^{q-1} t \int_{\Omega} z_m |w|^{q-1} dx \\ &- \int_{\Omega} \left[G(x, w + tz_m) - G(x, w) \right] dx - t \int_{\Omega} f(x) z_m dx. \end{aligned}$$

Knowing that $0 = J'(w) t z_m$, we see that

$$t \int_{\Omega} |\nabla w|^{N-2} \nabla w . \nabla z_m dx + t \int_{\Omega} |\nabla w|^{q-2} \nabla w . \nabla z_m dx - t \int_{\Omega} f z_m dx = \\ \lambda t \int_{\Omega} |w|^{N-2} w . z_m dx + \mu t \int_{\Omega} |w|^{q-2} w . z_m dx + t \int_{\Omega} g(x, w) z_m dx.$$

It follows from the above equation and the fact that $w \in L^\infty(\Omega)$ that,

$$\begin{aligned}
J(w+tz_m) - J(w) \leq & \frac{t^N}{N}\int_\Omega |\nabla z_m|^N dx + \frac{t^q}{q}\int_\Omega |\nabla z_m|^q dx + C\int_\Omega \xi_N\left(t|\nabla z_m|\right)dx \\
& + C\int_\Omega \xi_q\left(t|\nabla z_m|\right)dx + Ct^{q-1}\int_\Omega (z_m)^{q-1}|w|dx + Ct^{N-1}\int_\Omega (z_m)^{N-1}dx \\
& - \int_\Omega \left[G(x,w+tz_m) - G(x,w) - g(x,w)tz_m\right]dx \\
& + Ct\int_\Omega z_m dx.
\end{aligned}$$

Using that g is non-decreasing by $(\bar{g}_1)$, it follows from Mean Value Theorem that

$$\begin{aligned}
J(w+tz_m) - J(w) \leq & \frac{t^N}{N}\int_\Omega |\nabla z_m|^N dx + \frac{t^q}{q}\int_\Omega |\nabla z_m|^q dx + C\int_\Omega \xi_N\left(t|\nabla z_m|\right)dx \\
& + C\int_\Omega \xi_q\left(t|\nabla z_m|\right)dx + Ct^{q-1}\int_\Omega (z_m)^{q-1}|w|dx + Ct^{N-1}\int_\Omega (z_m)^{N-1}dx \\
& + Ct\int_\Omega z_m dx, \quad \forall t > 0.
\end{aligned}$$

It follows from Lemma 3.6.6 that

$$\begin{aligned}
J(w+t_m z_m) - J(w) \leq & \frac{t^N}{N} + t^q C\delta_m^{N-q}(\log m)^{\frac{-q}{N}} + t^2 C\delta_m^{N-2}(\log m)^{\frac{-2}{N}} \\
& + t^{N-1}C\delta_m^{N-(N-1)}(\log m)^{\frac{-(N-1)}{N}} + t^\gamma C\delta_m^{N-\gamma}(\log m)^{\frac{-\gamma}{N}} \\
& + t^{q-1}C\delta_m^{N-(q-1)}(\log m)^{\frac{-(q-1)}{N}} + tC\delta_m^{N-1}(\log m)^{\frac{-1}{N}}, \quad \forall t > 0,
\end{aligned}$$

since $\gamma \in (1,q)$ suitable and $\gamma \in (0,1)$. Here, the exponent t^2 appears in case $N \geq 3$. Now, considering $\delta_m = (\log m)^{-1/N}$, since $\delta_m^{N-s}(\log m)^{\frac{-s}{N}} = (\log m)^{-1}$, we see that

$$J(w+tz_m) - J(w) \leq \frac{t^N}{N} + \frac{C}{\log m}\left(t^q + t^{q-1} + t^{N-1} + t^2 + t^\gamma + t\right), \quad \forall t > 0, \tag{4.14}$$

for large $m \in \mathbb{N}$, where the term t^2 appears only if $N \geq 3$. This inequality allows us to show that $\{t_m\}_m$ is bounded from below by a positive constant. In fact, considering $t = t_m$ in (4.14), if $t_m \leq 1$ it follows from (4.13) that

$$\begin{aligned}
t_m^N &\geq \left(\frac{\alpha_N}{\alpha_0}\right)^{N-1} - \frac{C}{\log m} \\
&\geq \frac{1}{2}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1},
\end{aligned}$$

for $m \in \mathbb{N}$ sufficiently large. So, $t_m \geq \min\{1, (1/2)\left(\alpha_N/\alpha_0\right)^{N-1}\}$ for large values of m. Thus, there exists $t_0 > 0$ such that

$$t_m \geq t_0, \quad \forall m \in \mathbb{N}.$$

Since $t_m > 0$ is a maximum point for $t \mapsto J(w + tz_m)$, it follows that

$$J'(w + t_m z_m)(t_m z_m) = t_m \frac{d}{dt} J\left(w + tz_m\right) \Bigg|_{t=t_m} = 0,$$

and so

$$\begin{aligned}\int_\Omega g(x, w + t_m z_m)(t_m z_m)dx = t_m & \int_\Omega |\nabla(w + t_m z_m)|^{N-2}\nabla(w + t_m z_m)\nabla z_m dx \\ & + t_m \int_\Omega |\nabla(w + t_m z_m)|^{q-2}\nabla(w + t_m z_m)\nabla z_m dx \\ & - \lambda t_m \int_\Omega (w + t_m z_m)|^{N-2}(w + t_m z_m) z_m dx \\ & - \mu t_m \int_\Omega (w + t_m z_m)|^{q-2}(w + t_m z_m) z_m dx - t_m \int_\Omega f z_m dx.\end{aligned}$$

Since $\{z_m\}_m$ is bounded in $W_0^{1,N}(\Omega)$ we see that

$$\begin{aligned}\int_\Omega |\nabla(w + t_m z_m)|^{N-2}\nabla(w + t_m z_m)\nabla(t_m z_m)dx &\leq \|\nabla(w + t_m z_m)\|_{L^N}^{N-1}\|\nabla(t_m z_m)\|_{L^N} \\ &\leq C\left(\|w\|^{N-1} + t_m^{N-1}\|z_m\|^{N-1}\right)\|t_m z_m\| \\ &\leq C\left(t_m + t_m^N\right),\end{aligned}$$

and, in a similar way, we get

$$\int_\Omega |\nabla(w + t_m z_m)|^{q-2}\nabla(w + t_m z_m)\nabla(t_m z_m)dx \leq C\left(t_m + t_m^q\right),$$

$$\int_\Omega |w + t_m z_m|^{N-2}(w + t_m z_m)(t_m z_m)dx \leq C\left(t_m + t_m^N\right),$$

$$\int_\Omega |w + t_m z_m|^{q-2}(w + t_m z_m)(t_m z_m)dx \leq C\left(t_m + t_m^q\right),$$

and

$$\int_\Omega |f(x)|(t_m z_m)dx \leq C t_m,$$

for all $m \in \mathbb{N}$. Then,

$$\int_\Omega g(x, w + t_m z_m)(t_m z_m + w)dx \leq C(t_m + t_m^q + t_m^N) \leq \tilde{C} t_m^N, \quad \forall m, \tag{4.15}$$

because $0 < t_0 \leq t_m$ for all $m \in \mathbb{N}$. Then, from this point, we follow the exact same steps as given in Proposition 3.6.7 after inequality (3.45) to arrive at a contradiction, finishing this proof. □

The last case is given when $\mu = 0$, which becomes simpler since there is no interference of the q-linear part.

Proposition 4.4.6. *Suppose that g satisfies ($\bar{\alpha}_0$) and $(\tilde{g}_1) - (\tilde{g}_3)$. Consider c_1 as defined in (4.6), $1 < q < N$ and $\mu \equiv 0$. Then*

$$c_1 < J(w) + \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

Proof. Reproducing the proofs of the previous proposition, we observe that the term with exponent $q-1$ in (4.14) will not appear. So, we do not need to worry about the estimates on z_m^{q-1} when $q \in (1,2)$. That is, we will obtain the estimate:

$$J(w + tz_m) - J(w) \leq \frac{t^N}{N} + \frac{C}{\log m}\left(t^q + t^{q-1} + t^{N-1} + t^2 + t^\gamma + t\right), \quad \forall t > 0,$$

for large $m \in \mathbb{N}$ (noticing that t^2 will only appear if $N \geq 3$). The result follows in a similar way to the previous proposition. □

4.4.1 Proof of the main theorems

Theorem 4.2.2 is proved using Propositions 4.3.6, 4.4.4 and 4.4.5. Moreover, Theorem 4.2.3 is proved using Propositions 4.3.6, 4.4.4 and 4.4.6.

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