Federal University of Paraíba
Graduate Program of the Departament of Mathematics
Ph.D in Mathematics

Applications of geometric identities to rigidity problems

by
Murilo Chavedar de Souza Araújo

João Pessoa - PB
2025

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under supervision of:

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and co-supervision of:

Prof. Dr. Marcio Silva Santos

Thesis submitted to the Graduate Program of the Departament of Mathematics of the Federal University of Paraíba (UFPB), in partial fulfillment of the necessary requirements for the degree of Ph.D in Mathematics .

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Resumo

Nesta tese, exploramos as aplicações de identidades integrais, como por exemplo identidade do tipo Reilly e do tipo Pohozaev, em diversos contextos geométricos, destacando seus papéis na obtenção de desigualdades e resultados de rigidez para classes específicas de variedades Riemannianas.

Primeiramente, consideramos o contexto de variedades V-estáticas, que são variedades Riemannianas com bordo, curvatura escalar constante e uma métrica que é um ponto crítico do funcional de volume com uma métrica fixa no bordo. Nesse contexto, empregamos a nossa identidade do tipo Reilly para estabelecer desigualdades de Heintze-Karcher e Minkowski para domínios limitados. Além disso, examinamos os fenômenos de rigidez associados a essas desigualdades, especialmente nos casos em que a igualdade é atingida, iluminando a estrutura geométrica dessas variedades. Além disso, obtemos uma desigualdade para domínios em variedades m-quasi Einstein junto a uma caracterização de rigidez. Tal desigualdade é motivada pela estabilidade da energia de Wang-Yau.

Por fim, direcionamos nossa atenção para problemas sobredeterminados ponderados em variedades Riemannianas com densidade. Ao estudar um problema de Poisson associado ao Laplaciano ponderado, obtemos uma desigualdade de Heintze-Karcher e um teorema do tipo "Soap Bubble" que caracterizam bolas geodésicas nesses espaços ponderados. Ao impor condições de fronteira de Dirichlet e Neumann, estabelecemos ainda um resultado do tipo Serrin em cones generalizados e cones convexos do espaço Euclidiano, identificando bolas métricas como as unicas soluções para o problema sobredeterminado subjacente.

Palavras-chave: Problemas de rigidez, identidades integrais, identidade de Reilly, problemas sobre-determinados

Abstract

In this thesis, we explore the applications of integral identities, such as the Reilly-type and Pohozaev-type identities, in various geometric contexts, highlighting their roles in obtaining inequalities and rigidity results for specific classes of Riemannian manifolds.

First, we consider the context of V-static manifolds, which are Riemannian manifolds with boundary, constant scalar curvature, and a metric that is a critical point of the volume functional with a fixed boundary metric. In this context, we employ our Reilly-type identity to establish Heintze-Karcher and Minkowski inequalities for bounded domains. Furthermore, we examine the rigidity phenomena associated with these inequalities, especially in cases where equality is achieved, shedding light on the geometric structure of these manifolds. Additionally, we obtain an inequality for domains in m-quasi Einstein manifolds along with a rigidity characterization. This inequality is motivated by the stability of the Wang-Yau energy.

Finally, we turn our attention to weighted overdetermined problems on Riemannian manifolds with density. By studying a Poisson problem associated with the weighted Laplacian, we derive a Heintze-Karcher inequality and a Soap Bubble-type theorem that characterize geodesic balls in these weighted spaces. By imposing Dirichlet and Neumann boundary conditions, we also establish a Serrin-type result in generalized cones and convex cones of Euclidean space, identifying metric balls as the unique solutions to the underlying overdetermined problem.

Keywords: Rigidity problem, integral identities, Reilly's identity, Overdetermined Problem

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1 Introduction

Integral identities in Riemannian geometry have been an important tool for studying various analytical and geometric problems in geometric analysis and mathematical physics, especially those that involve the so-called rigidity results, this is, classifying solutions of some elliptic system from one known model. Despite the simplicity of such an idea, this approach yields many relevant results.

The Reilly identity is an important geometric identity that has been used to study rigidity problems, such as the well-known Alexandrov (Soap Bubble) theorem. The Pohozaev identity is another example of a geometric identity that has been applied to analyze rigidity problems, including the Serrin problem. These examples of integral identities and rigidty problems will be detailed in the following, are motivating for the work developed in this thesis.

In this scenario, this thesis investigates the role of geometric identities, such as those involving the Laplacian, Hessian, and divergence, in understanding the structure of Riemannian manifolds and solving rigidity problems. Integral identities like Reilly's identity and Pohozaev identity, in a generalized format, are used to reveal critical relationships between intrinsic and extrinsic properties of manifolds, providing tools to address classical and modern rigidity problems by imposing constraints that stabilize geometric structures, which encapsulate key information about curvature, volume, and boundary behavior, as well as their applications.

Our contributions are presented essentially in two parts. The first of these parts, presented in chapter 3, is motivated by classical results such as Alexandrov's theorem. The second part of our contributions, presented in Chapter 4, is motivated by the study of overdetermined problems in weight manifolds and it is motivated by Serrin's problem.

Alexandrov's (Soap Bubble) theorem [1], a widely recognized rigidity result in Riemannian geometry, asserts that:

> *Let M^n be a closed and embedded hypersurface with constant mean curvature in Euclidean space $\mathbb{R}^{n+1}$, then M^n is a round sphere.*

Although the initial proof by Alexandrov employed a reflection method in partial differential equations, numerous alternative proofs have emerged over time. Notably, Reilly-Ros provided an elegant proof of Alexandrov's theorem, by using Reilly's identity, which is a central component of the work presented here.

Reilly's identity, obtained in [2], says that if (M^n, g) is a smooth Riemannian manifold and $\Omega \subset M$ is a smooth domain with boundary $\partial\Omega$, then for any $f \in C^\infty(\overline{\Omega})$, the following integral identity is verified:

$$\int_\Omega \left[(\Delta f)^2 - \left|\nabla^2 f\right|^2\right] dv = \int_{\partial\Omega} \left(h(\overline{\nabla} z, \overline{\nabla} z) + 2u\overline{\Delta} z + Hu^2\right) da + \int_\Omega Ric(\nabla f, \nabla f) dv, \qquad (1.1)$$

where Δ and ∇^2 are the Laplacian and the Hessian in g, $z = f|_{\partial\Omega}, u = \langle \nabla f, \nu \rangle$, and ν denotes the outward-pointing unit normal vector. $\overline{\nabla}$ and $\overline{\Delta}$ indicate the gradient and the Laplacian of the induced metric $\overline{g}$ in $\partial\Omega$. Furthermore, h, H are the second fundamental form and the mean curvature of $\partial\Omega$ respectively, and Ric is the Ricci curvature of (M, g). A sample examination in the proof shows this identity is obtained by integrating another relevant geometric identity (Bochner's formula) plus the divergence theorem.

Ros' proof of Alexandrov's theorem [3] relies crucially on the Reilly [2] and Hsiung-Minkowski [4] identities.

Starting from Reilly's identity (1.1), [3] chooses a suitable function f, which is the solution of a Dirichlet's problem

$$\Delta f \quad = \quad -1 \quad \text{in} \quad \Omega, \quad f = 0 \quad \text{on} \quad \partial\Omega, \tag{1.2}$$

and proves the named Heintze-Karcher inequality: if Ω is a compact domain in $\mathbb{R}^{n+1}$, then

$$\frac{n-1}{n} \int_{\partial\Omega} \frac{1}{H} \, da \geq Vol(\Omega). \tag{1.3}$$

Moreover, equality in (1.3) holds if and only if Ω is isometric to a Euclidean ball. If, in particular, H is constant, Hsiung-Minkowski's identity ensures that (1.3) is an equality, which implies that $\partial\Omega$ is an n-dimensional sphere.

Furthermore, Reilly's identity can be applied to obtain a sharp lower bound for the first eigenvalue of the Laplacian and demonstrate an Obata-type theorem for manifolds with a boundary (see [5], [2], [6], [7], and [3] for these and further applications). The study of Heintze-Karcher inequalities and related rigidity results in more general Riemannian (and pseudo-Riemannian) spaces are intriguing and have motivated significant efforts see, for example, [8], [9], [10], [11], and [12].

The versatility of Reilly's identity serves as a motivating force to pursue extensions of this foundational work into more general ambient spaces. Following the paradigm set by Reilly-Ros, our aim is to derive new geometric inequalities and Alexandrov-type results for domains situated in broader contexts. A notable avenue of exploration involves extending these ideas to domains within static or substatic manifolds, a class of manifolds with wide-ranging applications in General Relativity. These manifolds naturally arise in the study of static spacetimes satisfying the Null Energy Condition, as observed in prior work such as [13], [14], and [10], and further discussed in [15].

We remember that a Riemannian triple (M, g, f) is said to be substatic if

$$f\text{Ric} + (\Delta f)g - \nabla^2 f \geq 0, \tag{1.4}$$

and, in this case, the function $f \in C^\infty(M)$ is the potential function. When the left-hand side of (1.4) vanishes, the triple is said static. Noteworthy contributions in this direction include the work by Li and Xia in [14], where a generalized Reilly's identity is established for general two-tensors. This formula, after a suitable choice of tensors, plays a crucial role in addressing the case of substatic manifolds with multiple boundary components, as outlined in Theorem 3.1 of [14] or Theorem 3.1.6 here.

Despite the resemblance of the last definition to (1.4) with a reversed inequality, it holds distinct importance as a variational problem, as elucidated below and rooted in the seminal work [16]. Let γ be a smooth metric on ∂M, and define $\mathcal{M}_\gamma$ as the set of all metrics in M such that $g\Big|_{T(\partial M)} = \gamma$. Fixing a constant K, consider the set $\mathcal{M}_\gamma^K$ comprising all metrics $g \in \mathcal{M}_\gamma$ with a scalar curvature R equal to K. In [16], it is demonstrated that a metric $g \in \mathcal{M}_\gamma^K$ possessing the property that the first Dirichlet eigenvalue of $(n-1)\Delta_g + K$ is positive serves as a critical point for the volume functional $Vol : \mathcal{M}_\gamma^K \to \mathbb{R}$. Importantly, this criticality condition is met if and only if there exists a smooth function V on M that satisfies Definition (3.1.1). Moreover, the existence of such a V-static potential yields significant geometric behaviors on the manifold and its boundary, such as constant scalar curvature and the regular level set $V^{-1}(0)$ being a totally umbilical hypersurface, as detailed in [16, Theorem 3.2]. This equation and its geometric consequences will be invoked multiple times in the course of this work. Classical examples of V-static manifolds include geodesic balls in space forms, see [16, Theorem 6], and certain warped products, see [17, Section 3].

The exploration of the V-static ambient is particularly intriguing, driven by its close connection to Besse's conjecture, a significant and as-yet-unsolved problem known as the Critical Point Equation (CPE) problem, see [18, page 128]. The conjecture regards to critical points of the Hilbert-Einstein functional subject to the constraints of unit volume and constant scalar curvature. The interplay between the CPE problem and V-static metrics is detailed in works such as [19] and [20].

In short, integral identities obtained through the classical divergence theorem, together with the derivative properties of geometric tensors, have served as a powerful tool for deriving inequalities and rigidity results in this setting. With this main idea and motivated by the works highlighted above, the first objective of this thesis is to derive a Reilly-type identity applicable to V-static ambient. To achieve this, following what was presented in [21], we adopt a tensorial framework from the prior work of Li and Xia, to our context (cf. Theorem 3.1.8). This tensorial choice is instrumental in establishing a Heintze-Karcher inequality for domains featuring multiple boundary components within V-static ambient (see Theorem 3.1.16). Notably, this result holds for domains properly contained in V-static manifolds as follows.

Theorem 1.0.1 *Let (M^n, g, V) be an n-dimensional V-static Riemannian triple. Consider $\Omega \subset M^n$ a bounded domain properly contained in $int(M)$ such that the smooth boundary $\partial\Omega$ is strictly mean convex. Then we have*

$$n\int_\Omega \Big(V - \frac{n}{n-1}f\Big)\, dv \le (n-1)\int_{\partial\Omega} \frac{V}{H}\, da, \tag{1.5}$$

where f is a solution of the Dirichlet's problem

$$\begin{cases} \Delta f + \frac{R}{n-1}f = -1 & in\ \Omega \\ f = 0 & on\ \partial\Omega. \end{cases} \tag{1.6}$$

Furthermore, if the equality is verified in (1.5), *then $\partial\Omega$ is umbilic.*

This same problem leads to another noteworthy application of Reilly's identity in order to derive a generalized form of the classical Minkowski inequality. This is achieved by chosing a solution f for a suitable Neumann problem, specifically:

$$\begin{cases} \Delta f = -1 & \text{in } \Omega, \\ f_\nu = c & \text{on } \partial\Omega. \end{cases}$$

where $c = -\frac{Vol(\Omega)}{Area(\partial\Omega)}$, Reilly [22] applies his formula (1.1) to establish an inequality for domains $\Omega \subset (M^n, g)$, where (M^n, g) is a manifold with non-negative Ricci curvature. To be precise, if $\partial\Omega$ is convex (indicating that the second fundamental form is positive definite, i.e., $h \geq 0$), a straightforward computation in (1.1) yields the following isoperimetric inequality:

$$Area(\partial\Omega)^2 \geq \frac{n}{n-1} Vol(\Omega) \int_{\partial\Omega} H \, da,$$

where equality holds if and only if Ω is isometric to a Euclidean ball. Generalizations of this inequality for domains in more general ambient spaces have been an active area of research, with advancements made by considering (M^n, g) as a space form [23], a warped product [8], a substatic manifold [14], and in the context of spacetime [15].

Using our Reilly-type identity, we derive the following Minkowski-type inequality for domains within V-static manifolds, see [21]:

Theorem 1.0.2 *Let (M^n, g, V) be an n-dimensional V-static Riemannian triple and $\Omega \subset M^n$ be a bounded domain with connected smooth boundary $\partial\Omega$ where $V > 0$. Suppose the second fundamental form of $\partial\Omega$ satisfies*

$$h_{\alpha\beta} - \frac{V_{,\nu}}{V} g_{\alpha\beta} \geq 0. \tag{1.7}$$

The following inequality then holds

$$\frac{n-1}{n}\Big(\int_{\partial\Omega} V \, da\Big)^2 \quad \geq \quad \int_{\Omega} \Big(V - \frac{n}{n-1} f\Big) \, dv \int_{\partial\Omega} VH \, da + \int_{\partial\Omega} V \, dv \int_{\partial\Omega} f \, dv. \tag{1.8}$$

Here f is a solution of the PDE

$$\begin{cases} \Delta f + \frac{R}{n-1} f &= -1 \quad \text{in } \Omega \\ V f_{,\nu} - V_{,\nu} f &= cV \quad \text{on } \partial\Omega, \end{cases}$$

where c is a suitable constant[1]*. Furthermore, if there is a point in $\partial\Omega$ at which occurs the strict inequality in* (1.7) *and the equality in* (1.8) *is verified, then $\partial\Omega$ is umbilic and of constant mean curvature.*

Furthermore, as discussed in section 3.2, Reilly's identity serves as a powerful tool for studying problems applied to General Relativity and quasi-local masses. In this direction, Miao, Tam, and Xie [24] looked at the relation between two quasi-local masses: the Brown-York and Wang-Yau masses. In [24, Theorem 1.1], they find conditions under which the Brown-York mass of a compact, time-symmetric, spacelike hypersurface Ω in a time-oriented spacetime N^4 be a strict local minimum for the Wang–Yau quasi-local energy. For this purpose, it is crucial to develop the second variation for Wang-Yau Energy [24, Proposition 2.1]. This goal, for example, leads the authors to demonstrate the following integral inequality for compact Riemannian manifolds with non-negative Ricci curvature, see [24, Proposition 3.1], a consequence of Reilly's identity (1.1):

[1] The choice of c is linked with the existence of solution for the underlying Neumann Problem.

Theorem 1.0.3 *[24] Let (Ω, g) be a compact Riemannian manifold of dimension $n \geq 3$. Suppose Ω has smooth boundary Σ which has positive mean curvature H. Let H and h be the mean curvature and the second fundamental form of Σ with respect to the outward normal, respectively. If g has non-negative Ricci curvature, then*

$$\int_\Sigma \left[\frac{(\overline{\Delta}\eta)^2}{H} - h(\overline{\nabla}\eta, \overline{\nabla}\eta) \right] da \geq 0 \tag{1.9}$$

for any smooth function η defined in Σ. $\overline{\nabla}$ and $\overline{\Delta}$ are the gradient and Laplacian intrinsic to Σ, respectively. Moreover, equality in (1.9) holds for some η if and only if η is the boundary value of some smooth function u which satisfies $\nabla^2 u = 0$ and $Ric(\nabla u, \nabla u) = 0$ in Ω.

Besides being an interesting integral inequality with a rigidity statement, this result has a link with the abovementioned discussion. When Ω is a bounded domain in $\mathbb{R}^3$ and Σ is convex, the left side of (1.9) is the second variation along η of the Wang-Yau quasi-local energy at the 2-surface Σ in a time-symmetric slice of the Minkowski spacetime. In this case, the above result can be viewed as a stability result for this energy. We also observe that the equality case of the Theorem produces a strong rigidity result because a particular coordinate function is the only case when this occurs. We also remark that a crucial step to obtaining this inequality is Reilly's identity, cited at the beginning.

In order to study this type of inequality in more general ambient spaces, Kwong and Miao [25] generalized Theorem 1.0.3 to the case of hypersurfaces that are the boundary of a bounded domain contained in a static manifold. Just to give a physical motivation for the study of these manifolds, it is well-known that the existence of such a solution of

$$(\Delta V)g + V\mathrm{Ric} - \nabla^2 V = 0$$

in a 3-dimensional manifold allows to construct a space-time satisfying the vacuum Einstein equations (with cosmological constant), whose properties, physically interpreted, justify the terminology *static*, see, for example, [19, Proposition 2.7]. In this scenario, they proved the following result:

Theorem 1.0.4 *[25, Theorem 3] Let (M^n, g, V) be a compact n-dimensional static Riemannian manifold with boundary Σ and static potential $V > 0$ on M. Denote by h and H the second fundamental form and the mean curvature respectively of Σ on M. If $H > 0$, then*

$$\int_\Sigma V \Big[\frac{(\overline{\Delta}\eta + (n-1)k\eta)^2}{H} - h(\overline{\nabla}\eta, \overline{\nabla}\eta) \Big] da \geq \int_\Sigma \overline{\nabla}_\nu V \Big[|\overline{\nabla}\eta|^2 - (n-1)k\eta^2 \Big] da$$

for any function η in Σ. Here $k \leq 0$ is a non-positive constant such that the Ricci tensor $Ric \geq (n-1)kg$. Moreover, equality holds only if

1. *$k = 0$ and η is the boundary value of the function u at (Ω, g) satisfying $\nabla^2 u = 0$,*

   or

2. *$k < 0$, g is Einstein, i.e., $Ric = (n-1)kg$, and η is the boundary value of the function u at (Ω, g) satisfying $\nabla^2 u + kug = 0$.*

An interesting point to note in this theorem also use a generalization of Reilly's identity in its proof, see [13, Theorem 1.1].

The purpose of Section 3.2 is to extend this last result in the context of m-quasi-Einstein manifolds (possibly with a non-empty boundary), additionally exploring the characterization of the equality case. More precisely, we will present our contribution for this topic by means of Theorem 1.0.5, stated below, which is part of the paper [26].

In the same direction as said for static metrics, one crucial motivation to approach the m-quasi-Einstein metrics is studying Einstein manifolds that have a structure of warped product. In fact, if $m > 1$ is an integer, (M^n, g, V) is an m-quasi-Einstein triple if and only if there is a smooth $(n+m)$-dimensional warped product Einstein metric having M as the base space (see [27] and [28]). From this observation, the study of warped product Einstein manifolds reduces to the study of the m-quasi-Einstein equation on the lower-dimensional base space.

In [27], the authors treat the non-empty boundary case and consider, for the case $m > 1$, the following quantity:

$$\rho(x) = \frac{1}{m-1}\left[(n-1)\lambda - R\right],$$

where R is the scalar curvature of (M^n, g). With this notation, we may present our result.

Theorem 1.0.5 *Let (M^n, g, V) be a m-quasi Einstein triple, $m > 1$, and $\Omega \subset M$ be a compact domain, with boundary Σ and m-quasi Einstein potential $V > 0$ on Ω. Denote by h, H and R the second fundamental form, the mean curvature and the scalar curvature of Σ in Ω, respectively. If $H > 0$ and $(n-1)k \leq Ric \leq \rho g$, where k is a non-positive constant, then*

$$\int_\Sigma V\Big[\frac{(\overline{\Delta}\eta + (n-1)k\eta)^2}{H} - h(\overline{\nabla}\eta, \overline{\nabla}\eta)\Big]\, da \quad \geq \quad \int_\Sigma \nabla_\nu V\Big[|\overline{\nabla}\eta|^2 - (n-1)k\eta^2\Big]\, da \quad (1.10)$$

where η is any function in Σ.

Moreover, equality holds only if

1. *$k = 0$ and η is the boundary value of a function f at (Ω, g) satisfying $\nabla^2 f = 0$, and them (M^n, g, V) is the Riemannian product $(\overline{M}, \overline{g}) \times (R, g_0)$ of a complete Riemannian manifold $(\overline{M}, \overline{g})$ and the real line (R, g_0), where g_0 denotes the canonical metric of R.*

   or

2. *$k < 0$, g is Einstein, i.e., $Ric = (n-1)kg$, and η is the boundary value of a function f at (Ω, g) satisfying $\nabla^2 f + kfg = 0$. In addition, if ∂M is not empty then M is isometric to*

$$\Big([0,\infty) \times N, dt^2 + \sqrt{-k}\cosh^2(\sqrt{k}t)g_{\mathbb{S}^{n-1}}, C\sinh\sqrt{-k}t\Big)$$

   where N is an Einstein metric with negative Ricci curvature, and C is an arbitrary positive constant. If ∂M is empty, then M is isometric to either

$$\Big(\mathbb{H}^n, dt^2 + \sqrt{-k}\sinh^2(\sqrt{-k}t)g_{\mathbb{S}^{n-1}}, C\cosh(\sqrt{-k}t)\Big)$$

   or

$$\Big(\mathbb{R} \times F, dt^2 + e^{2\sqrt{-k}t}g_F, Ce^{2\sqrt{-k}t}\Big),$$

   where F is Ricci flat and C is an arbitrary positive constant.

The second portion of our contributions, detailed in Chapter 4, is driven by the investigation of overdetermined problems in weighted manifolds.

An interesting connection between the first part of the work, presented in Chapter 3, and the second part, examined in Chapter 4, lies in the study of the existence of a solution for (1.6), which is discussed in Lemmas 3.1.13 and 3.1.14. We emphasize that the investigation of domains satisfying the Dirichlet problem (1.6) is closely related to a Serrin-type problem in the Riemannian setting, similar to those studied in [29] and [30], for example. In fact, in those papers, the authors have studied the rigidity for the problem

$$\begin{cases} \Delta u + nku & = & -1 & \text{in} & \Omega \\ u & = & 0 & \text{on} & \partial\Omega \\ u_\nu & = & -c & \text{on} & \partial\Omega \end{cases}$$

where Ω is a domain in a space form, or more generally, in a manifold with Ricci bounded below. Furthermore, for a specific problem in a V-static ambient, we refer to [31, Theorem 1 and 2] for further discussions.

An important and famous type of overdetermined problem is presented in [32] when Serrin proved the following celebrated result:

If there exists a positive solution $u \in C^2(\overline{\Omega})$ to the overdetermined problem:

$$\begin{cases} \Delta u & = & -1 & \textit{in} & \Omega \\ u & = & 0 & \textit{on} & \partial\Omega \\ u_\nu & = & -c & \textit{on} & \partial\Omega \end{cases} \tag{1.11}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary $\partial\Omega$ of class C^2, ν denotes the unit normal to $\partial\Omega$ and u_ν denote the normal derivative of u, then Ω must be a ball and u is radially symmetric.

The classical radial solution of the Serrin's problem (1.11) on the Euclidean ball centered at the origin O with radius R, denoted by $B_R(O)$, is given by

$$u(x) = \frac{R^2 - |x|^2}{2n}, \tag{1.12}$$

where $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^n$ and $R = cn$.

An "overdetermined problem" refers to a partial differential equation (PDE) in which "too many" boundary conditions are assigned, such as both the Dirichlet and Neumann conditions, as indicated in (1.11). This means that the system of conditions is "overdetermined" relative to the degrees of freedom in the PDE. Also, it is remarkable that there exists an interesting connection between the mean curvature of $\partial\Omega$ and the normal derivative u_ν, see [33]; this link (1.2) with overdetermined Serrin's problem, see [32] and [30].

The existence of "too many" boundary conditions are common in physical contexts, like fluid dynamics (see also [34] and [35, Section 2]), where the overdetermined nature of the problem can significantly impact the behavior of the system. In this context, (1.11) has its roots in the following physical problem (proposed by Fosdick, as stated by Serrin in [32, Page 1]):

"Consider a viscous incompressible fluid moving in straight parallel streamlines through a straight pipe of given cross sectional form Ω*. If we fix rectangular coordinates in space with the* z *axis directed along the pipe, it is well known that the flow velocity* u *is then a function of* x, y *alone satisfying the Poisson differential equation*

$$\Delta u = -A,$$

where A *is a constant related to the viscosity and density of the fluid and to the rate of change of pressure per unit length along the pipe. Supplementary to the differential equation one has the adherence condition*

$$u = 0 \quad on \quad \partial\Omega.$$

Finally, the tangential stress per unit area on the pipe wall is given by the quantity μu_ν *where* μ *is the viscosity. Our result states that the tangential stress on the pipe wall is the same at all points of the wall if and only if the pipe has a circular cross section."*

For more rigorous physical interpretations of Serrin's problem, we refer to [36] and [35, Section 2]. Serrin proved his result using a method developed by Alexandrov [1]: the moving plane method. In this direction, Serrin's proof indicate the intrinsic relation of this problem with one of the most celebrated result in Differential geometry, Alexandrov's Soap Bubble Theorem.

Within these parameters, Alexandrov's theorem offers a significant physical interpretation: constant mean curvature (CMC) hypersurfaces are intrinsically associated with the geometry of liquid drops and "soap bubbles". For a comprehensive review of this topic, refer to [37]. Additionally, there is a variational characterization of CMC surfaces and their connection to the isoperimetric problem, see, for example, [37, Section 2.1]. To further understand the relationship between Serrin and Alexandrov's symmetry problems, see [38], [39], see also [40, Appendix D].

An additional point of convergence between Alexandrov's problem and Serrin's problem is the existence of an intriguing alternative proof that employs integral techniques. In the same edition of Serrin's paper, Weinberger [41] provided a simpler proof based on a method that is now known as the *P-function approach*. Briefly, Weinberger defines a sub-harmonic function $P(u)$ associated with the solution of (1.11) and applies the classical strong maximum principle to demonstrate that this function is constant, thereby proving the rigidity result. A key step in proving this constancy is the classical *Pohozaev identity* for Euclidean domains (for further details, see the comprehensive survey [42, Section 1.2]).

Continuing the study of the Serrin's problem, we investigate these types of results within the framework of weighted manifolds, defined in Chapter 4 and presented here for convenience of the reader. A weighted manifold is defined as a triple $(M, \langle\ ,\ \rangle, w\, dv)$, where $(M, \langle\ ,\ \rangle =: g)$ is a Riemannian manifold with Riemannian measure dv, $w : M \to \mathbb{R}$ is a smooth function, and $dv_w = wdv$ is the weighted measure. The geometry of weighted manifolds is characterized by the weighted metric structure, which affects the measures of intrinsic metric objects (e.g., weighted length of curves and weighted volume of metric balls).

Weighted manifolds provide a natural framework for understanding and generalizing classical results in geometric probability, including isoperimetric inequalities and concentration of measure phenomena. These results have significant implications across various fields, such as statistical mechanics, information theory, diffusion processes, and optimal transport. For an extensive overview see the comprehensive survey by Morgan [43], see also [44], [45] and [46].

In the direction of proving Alexandrov-type results using integral identities, a crucial step is to obtain a Heintze-Karcher inequality. In [47], the authors derived such an inequality (linked with a rigidity statement) for spaces with nonnegative α-Bakry-Émery-Ricci tensor. In this context, we consider the following weighted Poisson problem:

$$\begin{cases} \Delta_w u + k(n+\alpha)u & = & -1 & \text{in} & \Omega \\ u & > & 0 & \text{in} & \Omega \\ u & = & 0 & \text{on} & \partial\Omega \end{cases} \tag{1.13}$$

to establish a Heintze-Karcher inequality for manifolds with α-Bakry-Émery-Ricci tensor bounded below (see Theorem 4.0.7). For the existence and positivity of solutions to (1.13), we refer to [48, Lemma 3.10]. See also the recent discussion in the setting of weighted manifolds and the relation between solutions and the first eigenvalue in [49].

A crucial step in direction to obtain the Heintze-Karcher inequality in Theorem 4.0.7 is getting a Reilly-type identity that can be adapted to our context (see Proposition 4.0.3). Within this goal, we also establish the Soap Bubble's Theorem 1.0.6. In the following result, we consider the value

$$c = \frac{1}{\mathrm{Area}_w(\partial\Omega)}\left[\mathrm{Vol}_w(\Omega) + (n+\alpha)k\int_\Omega u\, dv_w\right]. \tag{1.14}$$

While, in general, it is expected that c depends on the solution u, the case $k=0$ is particularly meaningful and provides a parallel with [33, Theorem 2.2], see Remark 4.0.10 and [26]:

Theorem 1.0.6 *Let Ω be a bounded domain in an n-dimensional Riemannian manifold $(M, \langle\cdot,\cdot\rangle, w\, dvol)$ with $Ric_w^\alpha \geq k(n+\alpha-1)g$ for some $k \in \mathbb{R}$, $\alpha > 0$ and a smooth positive function w. Suppose u is a solution of the PDE* (1.13)*. Then,*

$$\int_\Omega (H_0 - H_w)u_\nu^2\, dv_w \geq 0,$$

where equality holds if and only if Ω is a metric ball and u is a radial function. Here, H_w denotes the weighted mean curvature of $\partial\Omega$ with respect to ν, $H_0 = \frac{n+\alpha-1}{(n+\alpha)c}$, and c is the constant given by (1.14)*. In particular, if $H_w \geq H_0$, then Ω is a metric ball.*

In Section 4.0.3, we study the rigidity problem (1.13) which is motivated by an overdetermined condition. More specifically, we deal with the problem

$$\begin{cases} \Delta_w u + k(n+\alpha)u & = & -1 & \text{in} & \Omega \\ u & = & 0 & \text{on} & \partial\Omega \\ |\nabla u| & = & c & \text{on} & \partial\Omega, \end{cases} \tag{1.15}$$

We observe that the use of the same notation for the constant c in (1.14) and (1.15) is not a coincidence: the overdetermined condition $|\nabla u| = -u_\nu = c$ on $\partial\Omega$ establishes this equivalence; see (4.14). In [50], the authors study this problem for domains in Euclidean space (in this case, $k=0$) and a homogeneous weight w of degree $\alpha > 0$, which means that $\langle \nabla w, x\rangle = \alpha w$, where x is the position vector. Following Weinberger's approach, we define a suitable P-function associated with (1.15) and establish a Pohozaev-type identity for manifolds endowed with a closed conformal vector field, which is of interest in its own right (see Proposition 4.0.13 and Section 4.0.2 for underlying definitions).

In order to extend previous results established in the Euclidean setting, we work with the following structure. Let (N, g_N) be a connected $(n-1)$-dimensional Riemannian manifold. Consider $(0,\infty)\times N$ a product manifold endowed with a warped metric given by

$$g = dr\otimes dr + r^2 g_N.$$

We denote such warped product as $M = (0,\infty)\times_r N$ and call it by a generalized cone. When $N = \mathbb{S}^{n-1}$, $M = \mathbb{R}^n\setminus\{0\}$, and, more generally, when $\mathcal{D}$ is a smooth region of $\mathbb{S}^{n-1}$, M is a solid conical region. Beyond the work of [50] in the Euclidean space, a large class of rigidity results associated with overdetermined problems in convex cones has emerged (see [51], [52], [53], [54], for example). In the following result, we consider the such warped structure in order to obtain a rigidity result related with the overdetermined problem (1.15):

Theorem 1.0.7 *Let $(M^n = (0,\infty)\times_r N^{n-1}, \langle\ ,\ \rangle, w\ dvol)$ be a generalized cone such that $Ric_w^\alpha \geq k(n+\alpha-1)g$, where k is a nonnegative constant, and w is a homogeneous smooth function of degree $\alpha > 0$. If u is a positive solution of* (1.15)*, then Ω is a metric ball and u is a radial function.*

In Section 4.0.4 we focus our attention on the weighted Serrin's problem for convex cones of the Euclidean space. Again work [50] is an important motivation.

Under this context, considering the open cone $\Sigma = \{tx; x\in\omega, t\in(0,\infty)\}$ in $\mathbb{R}^n$, where ω is an open connected domain on the unit sphere $\mathbb{S}^{n-1}$, it is worth to note (as observed in [55]) that (1.12) also is a solution for the following partially overdetermined problem

$$\begin{cases} \Delta u &= -1 \quad \text{in } B_R(o)\cap\Sigma,\\ u &= 0 \quad \text{on } \partial B_R(o)\cap\Sigma,\\ u_\nu &= -c \quad \text{on } \partial B_R(o)\cap\Sigma,\\ u_\nu &= 0 \quad \text{on } B_R(o)\cap\partial\Sigma.\end{cases}$$

As in Serrin's classical result (1.11), it is natural to investigate symmetry characterizations, both for the domains and the solutions, in the sense of *sector-like domains*, which, in this specific characterization, can be viewed as intersections between a ball and a cone. These domains will be defined below in a quite general way (see Definition 1.0.8).

In this setting, Pacella and Tralli [56], as well as Ciraolo and Roncoroni [55], have extended the study of rigidity characterizations for partially overdetermined problems by employing integral methods. While [56] focus on the partially overdetermined problem in a sector-like domain in the Euclidean space, [55] consider more general operators than the Laplacian in the Euclidean space, including possibly degenerate operators, and also addressing analogous problems in space forms, the hyperbolic space and the (hemi)sphere. See also [57] for related overdetermined problems concerning cones in spheres.

Under these circumstances and motivations, if we denote O the pole of the model, we define an *open cone* $\Sigma \subset \mathbb{R}^n$ with vertex at $\{O\}$ as the set $\Sigma := \{tx; x \in \omega, t \in I\}$, for some open domain $\omega \subset \mathbb{S}^{n-1}$ and $I \subset (0, \infty)$ an interval.

Moreover, we say that a cone Σ is *convex* if its second fundamental form is nonnegative at every point $x \in \partial\Sigma$. Here, if ν denotes the outward unit normal vector field, the second fundamental form is given by $A(X, Y) = g(\nabla_X \nu, Y)$, where X and Y are tangent vector fields to $\partial\Sigma$. In the same way as [55] and [56] we have the concept of sector-like domain that we define below and illustrated in figure (1).

Definition 1.0.8 *Let Σ be an open cone as described above, such that $\partial\Sigma \setminus \{O\}$ is smooth. A domain $\Omega \subset \Sigma$ is called a sector-like domain if*

$$\Gamma = \partial\Omega \cap \Sigma \quad \textit{and} \quad \Gamma_1 = \partial\Omega \setminus \overline{\Gamma},$$

are such that the $(n-1)$-Hausdorff measures satisfy $\mathcal{H}_{n-1}(\Gamma) > 0$, $\mathcal{H}_{n-1}(\Gamma_1) > 0$, and Γ is a smooth $(n-1)$-dimensional manifold, while $\partial\Gamma = \partial\Gamma_1 \subset \partial\Omega \setminus \{O\}$ is a smooth $(n-2)$-dimensional manifold.

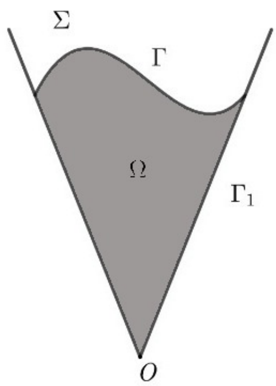

Figure 1 – A sector-like domain Ω inside Σ

Motivated by the findings in [50], we study the overdetermined problem

$$\begin{cases} \Delta_w u & = & -1 & \text{in} & \Omega \\ u & = & 0 & \text{on} & \Gamma \\ u_\nu & = & -c & \text{on} & \Gamma \\ u_\nu & = & 0 & \text{on} & \Gamma_1 \setminus \{O\} \\ \nabla^2 \log w(\nabla u, \nabla u) + \dfrac{\langle \nabla \log w, \nabla u \rangle^2}{\alpha} & \leq & 0 & \text{in} & \Omega \end{cases} \tag{1.16}$$

in a sector-like domain Ω contained in a convex cone $\Sigma \subset \mathbb{R}^n$ and obtaining the following result.

Theorem 1.0.9 *Let Σ a convex open cone in $\mathbb{R}^n$ such that $\Sigma \setminus \{O\}$ is smooth and $\Omega \subset \Sigma$ a sector-like domain. If f is a homogeneous weight of degree $\alpha > 0$ and there exists a solution $u \in C^1(\Omega \cup \Gamma \cup \Gamma_1 \setminus \{O\}) \cap W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$ to the problem (1.16), then $\Omega = \Sigma \cap B_r(x_0)$ where $B_r(x_0)$ is a ball and $u = \frac{r^2 - |x - x_0|^2}{2(n+\alpha)}$.*

This thesis is systematically organized and next we present a overview of the text. Chapter 2 introduces foundational concepts of Riemannian geometry, including the Levi-Civita connection, curvature and operators like the gradient and Laplacian and some integrals identities related to these operators. In the following chapters we present our contributions to solving the problems mentioned above. More specifically, Chapter 3 delves into Reilly's generalized identity and its applications, beginning with V-static manifolds in Section 3.1, which contribute a generalized Reilly-type identity and extensions of Heintze-Karcher and Minkowski inequalities [21]. Section 3.2 explores m-quasi-Einstein manifolds, generalizing Einstein metrics, with results from [26]. Chapter 4 examines overdetermined problems in weighted manifolds and convex cones, presenting the Soap Bubble Theorem 1.0.6, Theorem 4.0.7, Theorem 1.0.7 and Theorem 1.0.9 as extensions of classical rigidity results, supported by [58] and [59].

2 Notations, basic concepts and classical results

This chapter provides a brief review of some basic concepts in Riemannian Geometry, as well as some fundamental results that will be used in this text. In section 2.1 we introduce the concepts of connections, Riemannian metrics, and curvatures: Riemannian curvature (Rm), sectional curvature (K), Ricci curvature (Ric), and scalar curvature (R). Most results will not be proven here; the reader may find more comprehensive material in the references [60], [61]and [62]. The section 2.2 introduces the key operators like the gradient, divergence, and Laplacian.

For a better understanding and ease of reading we organize below the main notations used throughout the thesis. The gradient, the Laplacian, and the Hessian on $(M, g = \langle\ ,\ \rangle)$ will be denoted by ∇, Δ, and ∇^2, respectively. Taking $\Omega \subset M$ a bouded domain with smooth boundary $\partial\Omega$ the induced metric, gradient, Laplacian, and Hessian on $\partial\Omega$ will be denoted by $\overline{g}$, $\overline{\nabla}$, $\overline{\Delta}$, and $\overline{\nabla}^2$, respectively.

Let ν denote the normal unit outward vector on $\partial\Omega$. We define $h(X,Y) = g(\nabla_X \nu, Y)$ as the second fundamental form and $H = \mathrm{tr}_g h$ as the mean curvature (with respect to $-\nu$) of $\partial\Omega$.

Let dv and da denote the canonical measures on (M, g) and $\partial\Omega$, respectively. The Ricci curvature and the scalar curvature of (M, g) will be denoted by Ric and R, respectively.

We will use a subscript index to denote covariant derivatives with respect to the metric $\overline{g}$, and a comma together with a subscript index to denote covariant derivatives with respect to the metric g. For example, $f_\alpha = \overline{\nabla}_\alpha f$ and $R_{,i} = \nabla_i R$.

We will adopt Einstein's summation convention, which omits explicit summation symbols for repeated indices within the same term, along with the convention that indices written in Latin letters range over the set $\{1, \ldots, n\}$, and indices written in Greek letters range over the set $\{1, \ldots, n-1\}$.

2.1 Fundamentals of Riemannian geometry

Unless stated otherwise, (M, g) will denote a smooth Riemannian manifold of dimension n with a smooth metric g. At a point $p \in M$, we denote by T_pM the tangent space of M at p, and by $X(M)$ the space of all smooth vector fields on M.

Recall that a differentiable manifold M of dimension n is a topological manifold (a Hausdorff topological space locally homeomorphic to an open subset of $\mathbb{R}^n$) with a connected structure endowed with a maximal C^∞ atlas.

If M is a differentiable manifold of dimension n, a Riemannian metric g on M is a differentiable mapping g, such that for every $p \in M$, $g(p)$ is an inner product on T_pM. A Riemannian manifold of dimension n is a pair (M, g), where M is a differentiable manifold of dimension n, and g is a Riemannian metric on M. Conveniently, we will denote $g_p(X_p, Y_p)$ by $\langle X_p, Y_p\rangle_p$, and we will omit the subscripts when no confusion arises.

If (U, ϕ) is a local chart of M, then in this local chart, the Riemannian metric g is represented by a positive-definite symmetric matrix $(g_{ij}(p))_{1\leq i,j\leq n}$, satisfying $g_{ij} = g_{ji}$ for all $i, j = 1, \ldots, n$, and

$$\sum_{i,j=1}^{n} g_{ij}x_i x_j > 0 \quad \text{for all } x = (x_1, \ldots, x_n) \neq 0,$$

where $g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle$ depends smoothly on $p \in M$.

Levi-Civita Connection

An affine connection on a differentiable manifold M is a mapping

$$D : X(M) \times X(M) \to X(M),$$
$$(X, Y) \mapsto D_X Y,$$

that satisfies the following properties:

- $D_{fX+gY}Z = fD_X Z + gD_Y Z$ (linearity in the first variable);
- $D_X(Y + Z) = D_X Y + D_X Z$ (additivity in the second variable);
- $D_X(fY) = fD_X Y + X(f)Y$ (Leibniz rule),

where $X, Y, Z \in X(M)$ and $f, g \in C^\infty(M)$. The term $D_X Y$ is read as the "covariant derivative of Y in the direction of X."

If the affine connection additionally satisfies:

- $X\langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle$ (compatibility with the metric);
- $D_X Y - D_Y X = [X, Y]$ (symmetry),

then it is called a Riemannian connection (or Levi-Civita connection). Note that an affine connection is $\mathbb{R}$-linear in the second variable, i.e., if $a, b \in \mathbb{R}$, then

$$D_X(aY + bZ) = aD_X Y + bD_X Z.$$

Recall that $[X, Y] = XY - YX$ is a vector field called the Lie bracket of X and Y. It satisfies the following properties:

1. $[X, Y] = -[Y, X]$,
2. $[aX + bY, Z] = a[X, Z] + b[Y, Z]$,
3. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (Jacobi identity),
4. $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$.

From [60, Theorem 3.6] we have that, given a Riemannian manifold M, there exists a unique affine connection D on M that is symmetric and compatible with the Riemannian metric.

Intrinsic Geometry

We define the Riemannian curvature of (M, g), denoted by Rm, as the correspondence that associates to each pair $X, Y \in X(M)$ an operator

$$\mathrm{Rm}(X, Y) : X(M) \to X(M),$$

given by

$$\mathrm{Rm}(X, Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z,$$

where $Z \in X(M)$ and D denotes the Riemannian connection of M. The Riemannian curvature tensor, a rank-4 tensor also denoted by Rm, is defined as

$$\mathrm{Rm}(X, Y, Z, W) = g(\mathrm{Rm}(X, Y)Z, W),$$

for $X, Y, Z, W \in X(M)$.

The Riemannian curvature field Rm has the following properties:

1. Rm is trilinear;
2. $\mathrm{Rm}(X, Y)Z = -\mathrm{Rm}(Y, X)Z$;
3. $\mathrm{Rm}(X, Y)Z + \mathrm{Rm}(Y, Z)X + \mathrm{Rm}(Z, X)Y = 0$ (first Bianchi identity).

From a tensorial perspective, these properties are expressed as:

1. $\mathrm{Rm}(X, Y, Z, W) + \mathrm{Rm}(Y, Z, X, W) + \mathrm{Rm}(Z, X, Y, W) = 0$;
2. $\mathrm{Rm}(X, Y, Z, W) = -\mathrm{Rm}(Y, X, Z, W)$;
3. $\mathrm{Rm}(X, Y, Z, W) = -\mathrm{Rm}(X, Y, W, Z)$;
4. $\mathrm{Rm}(X, Y, Z, W) = \mathrm{Rm}(Z, W, X, Y)$.

It is convenient to express the above in a coordinate system (U, x) in a neighborhood of $p \in M$. Let $\{X_1, \ldots, X_n\}$ be an orthonormal basis in T_pM. Then,

$$\mathrm{Rm}(X_i, X_j)X_k = \sum_{l=1}^{n} R_{ijk}^l X_l,$$

and

$$\mathrm{Rm}(X_i, X_j, X_k, X_l) = g(\mathrm{Rm}(X_i, X_j)X_k, X_l) = R_{ijkl},$$

where

$$R_{ijkl} = \sum_{m=1}^{n} g_{lm} \mathrm{Rm}(X_l, X_m) R_{ijk}^m.$$

Given $p \in M$ and $\pi \subset T_pM$, a two-dimensional plane, the **sectional curvature** of π at p is defined as

$$K(\pi) = \frac{\mathrm{Rm}(X, Y, X, Y)}{g(X, X)^2 g(Y, Y)^2 - g(X, Y)^2},$$

where $\{X, Y\}$ is a basis for π. The definition of sectional curvature is independent of the chosen basis.

Since rank-4 tensors are complex, simpler tensors that summarize the information contained in the Riemannian curvature tensor are often considered. The most significant of these is the Ricci curvature tensor.

At a fixed $p \in M$, let $\{e_1, \dots, e_n\}$ be an orthonormal basis of T_pM. The Ricci curvature of (M, g), denoted by Ric, is a rank-2 tensor defined as:

$$\begin{aligned} \mathrm{Ric}(X, Y) &:= \sum_{k=1}^{n} \mathrm{Rm}(X, e_k, Y, e_k) \\ &= -\sum_{k=1}^{n} \mathrm{Rm}(e_k, X, Y, e_k) \\ &= -\sum_{k=1}^{n} g(\mathrm{Rm}(e_k, X)Y, e_k) \\ &= -\mathrm{trace}_g(\mathrm{Rm}(e_k, X)Y, e_k). \end{aligned}$$

The scalar curvature, denoted by R, is a smooth function on M defined by

$$R(x) = \sum_{k=1}^{n} \mathrm{Ric}(e_k, e_k)(x) = \mathrm{trace}(\mathrm{Ric}(e_k, e_k)(x)).$$

Extrinsic Geometry

Let (M^n, g_M) and $(\bar{M}^{n+k}, g_{\bar{M}})$ be Riemannian manifolds of dimension n and $n + k$, respectively. Suppose M is an oriented manifold, possibly with boundary.

Consider an immersion $\varphi : M \to \bar{M}$, i.e., φ is a differentiable mapping such that for each $p \in M$, the differential $d\varphi_p : T_pM \to T_{\varphi(p)}\bar{M}$ is injective. In other words, φ is a map of constant rank $n = \dim M$ for all $p \in M$.

The metric of $\bar{M}$ naturally induces a metric on M as follows: for $v_1, v_2 \in T_pM$, define

$$g_M(v_1, v_2) = g_{\bar{M}}(d\varphi_p(v_1), d\varphi_p(v_2)).$$

Thus, φ defines an isometric immersion of M into $\bar{M}$.

By the rank theorem, since φ is an immersion, φ is locally an inclusion modulo coordinate changes. More precisely, for each $p \in M$, there exists a neighborhood $U \subset M$ of p such that $\varphi(U) \subset \bar{M}$ is a submanifold of $\bar{M}$.

At each $p \in M$, the inner product in $T_p\bar{M}$ decomposes $T_p\bar{M}$ as the direct sum

$$T_p\bar{M} = T_pM \oplus (T_pM)^{\perp},$$

where $(T_pM)^{\perp}$ is the orthogonal complement of T_pM in $T_p\bar{M}$. We write

$$v = v_T + v_\perp \in T_pM \oplus (T_pM)^\perp.$$

Thus, we can consider the tangent and normal bundles of M, denoted by $T(M)$ and $N(M)$, respectively.

Let $\bar{D}$ be the Riemannian connection on $\bar{M}$. For $X, Y \in X(M)$, we can consider local extensions $X, Y \in X(\bar{M})$ of X and Y, respectively, and define

$$D_XY = (\bar{D}_XY)_T,$$

as the Riemannian connection relative to the induced metric of M.

In particular, we can define the second fundamental form of the immersion $\varphi : M \to \bar{M}$,

$$h : T(M) \times T(M) \to N(M),$$

as the symmetric bilinear map given by

$$h(X,Y) = \bar{D}_XY - D_XY = (\bar{D}_XY)^\perp,$$

where X, Y are local extensions of X, Y to $\bar{M}$.

The mean curvature vector of φ at $p \in M$ is the normal vector defined by

$$H(p) = \sum_{j=1}^{n} h(X_j, X_j),$$

where $\{X_1, \ldots, X_n\}$ is an orthonormal basis of T_pM.

We associate the symmetric bilinear map h with a self-adjoint linear operator $S_\eta : T_pM \to T_pM$, called the shape operator of φ at $p \in M$ with respect to $\eta \in (T_pM)^\perp$, defined by

$$g_M(S_\eta X, Y) = g_{\bar{M}}(h(X,Y), \eta),$$

for all $X, Y \in T_pM$.

Moreover, we have

$$S_\eta(X) = (\bar{D}_XN)_T,$$

where N is a local extension of $\eta \in (T_pM)^\perp$ to $\bar{M}$, and $X \in T_pM$.

Therefore,

$$g_{\bar{M}}(H, \eta) = \mathrm{tr}(S_\eta),$$

for any $\eta \in (T_pM)^\perp$. In particular, the mean curvature vector H above does not depend on the choice of the orthonormal basis.

Some Important Identities

Below are some important identities used throughout the text. In all cases, we consider (M^n, g) as a Riemannian manifold with Riemannian curvature tensor Rm. Additionally, $X, Y, Z, W \in X(M)$.

First Bianchi Identity

$$\mathrm{Rm}(X,Y)Z + \mathrm{Rm}(Z,X)Y + \mathrm{Rm}(Y,Z)X = 0.$$

For a proof of this result, see [60, Proposition IV.2.4]. Similarly, the first Bianchi identity can be expressed as:

$$\mathrm{Rm}(X,Y,W,Z) + \mathrm{Rm}(Y,W,X,Z) + \mathrm{Rm}(W,X,Y,Z) = 0.$$

In coordinates:

$$R_{ijkl} + R_{jkil} + R_{kijl} = 0.$$

Second Bianchi Identity

$$(D_Z\mathrm{Rm})(X,Y)W + (D_X\mathrm{Rm})(Y,Z)W + (D_Y\mathrm{Rm})(Z,X)W = 0.$$

For a proof of this result, see [62, Proposition 3.1.1]. Similarly, the second Bianchi identity can be written as:

$$(D_U\mathrm{Rm})(V,W,X,Y) + (D_V\mathrm{Rm})(W,U,X,Y) + (D_W\mathrm{Rm})(U,V,X,Y) = 0.$$

In coordinates:

$$D_pR_{ijkl} + D_iR_{jpkl} + D_jR_{pikl} = 0.$$

Contracted Second Bianchi Identity

$$g^{im}D_mR_{ijkl} = (\mathrm{div}_g\mathrm{Rm})_{jkl} = D_kR_{jl} - D_lR_{jk}.$$

For a proof of this result, see [62, Proposition 3.1.5].

Twice Contracted Second Bianchi Identity

$$g^{im}D_mR_{ik} = D_mR_{mk} = \frac{1}{2}D_kR.$$

Gauss Equation Let (M^n, g) be a Riemannian manifold and $\Sigma^{n-1} \subset M$ a hypersurface. Then:

$$\mathrm{Rm}(X,Y,Z,W) = \mathrm{Rm}_\Sigma(X,Y,Z,W) - h(X,W)h(Y,Z) + h(X,Z)h(Y,W),$$

where h denotes the second fundamental form. For a proof of this result, see [62, Proposition 3.2.4 (Tangential Curvature Equation)] or [60, Proposition VI.3.1].

Twice Contracted Gauss Equation

Taking the double trace of the Gauss Equation, we have:

$$R - 2\mathrm{Ric}(\nu,\nu) = R_\Sigma - H^2 + \|h\|^2,$$

where $\mathrm{Ric}(\nu,\nu)$ is the Ricci curvature evaluated in the normal direction to Σ, H is the mean curvature, and $\|h\|$ denotes the norm of the second fundamental form.

Ricci Identity

$$f_{,ijj} = (\Delta f)_{,i} + R_{ij}f_{,j},$$

where f is a smooth function on M.

2.2 Differentiable operators in Riemannian manifolds

Next we introduces fundamental concepts in the analysis of smooth functions and vector fields on Riemannian manifolds. It explores operators such as the gradient, divergence, Laplacian, and Hessian, which are essential tools for studying the interplay between geometry and analysis also delves into key identities and properties of these operators, such as the divergence theorem and relations between the Hessian and Laplacian. These results provide a foundation for applications in geometric analysis, including integral formulas, curvature relations, and boundary behavior, which are central to the study of Riemannian geometry.

Definition 2.2.1 *Let $f : M^n \to \mathbb{R}$ be a smooth function. The gradient of f is the smooth vector field ∇f, defined on M, such that:*

$$\langle \nabla f, X \rangle = X(f),$$

for all $X \in X(M)$. For each $p \in M$, we can also write:

$$\langle \nabla f(p), X_p \rangle = X_p(f) = df_p(X_p).$$

From the definition, it follows that the gradient of a smooth function is uniquely determined by the condition above. Let $f, g : M^n \to \mathbb{R}$ be smooth functions. Two importants propriety are

(i) $\nabla(f + g) = \nabla f + \nabla g$,

(ii) $\nabla(fg) = g\nabla f + f\nabla g$.

Definition 2.2.2 *Let X be a smooth vector field on M^n. The divergence of X is the smooth function $div : M^n \to \mathbb{R}$, defined for each $p \in M$ by*

$$(divX)(p) = tr\{v \mapsto (\nabla_v X)(p)\},$$

where $v \in T_pM$, and tr denotes the trace of the linear operator within braces.

If X, Y are smooth vector fields on M^n, and $f : M^n \to \mathbb{R}$ is a smooth function, we have the following two importants propriety

(i) $\mathrm{div}(X + Y) = \mathrm{div}X + \mathrm{div}Y$.

(ii) $\mathrm{div}(fX) = f\mathrm{div}X + \langle \nabla f, X \rangle$.

Definition 2.2.3 *Let $f : M^n \to \mathbb{R}$ be a smooth function. The Laplacian of f is the function $\Delta f : M^n \to \mathbb{R}$ given by*

$$\Delta f = div(\nabla f).$$

In this context, given smooth functions $f, g : M^n \to \mathbb{R}$, we have

$$\Delta(fg) = g\Delta f + f\Delta g + 2\langle \nabla f, \nabla g \rangle.$$

In particular,

$$\frac{1}{2}\Delta(f^2) = f\Delta f + |\nabla f|^2.$$

Definition 2.2.4 *Let $f : M^n \to \mathbb{R}$ be a smooth function. The Hessian of f is the field of linear operators $(Hess\ f)_p : T_pM \to T_pM$, defined for $v \in T_pM$ by $(Hess\ f)_p(v) = \nabla_v \nabla f$.*

It follows from the properties of the Riemannian connection that if X is any extension of v to a neighborhood of p in M, then $(\text{Hess } f)_p(v) = (\nabla_X \nabla f)(p)$. We also use the notation $\nabla^2 f$ to denote the Hessian of the smooth function $f : M^n \to \mathbb{R}$ when there is no need to specify the point $p \in M$ or the vector $v \in T_pM$. In this sense, if $f : M^n \to \mathbb{R}$ is a smooth function and $p \in M$, then $(\text{Hess } f)_p : T_pM \to T_pM$ is a self-adjoint linear operator and $\Delta f = tr(\text{Hess } f)$.

From the work done so far, given $f \in C^\infty(M)$, we can consider the operator Hess f : $\mathfrak{X}(M) \to \mathfrak{X}(M)$. Associated with this operator, we can consider the Hessian form of f, which is a bilinear, symmetric, self-adjoint form denoted by Hess $f : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$ and given by

$$\begin{aligned}(X,Y) \mapsto \text{Hess } f(X,Y) &= \langle \text{Hess } f(X), Y\rangle \quad \text{(by definition of the Hessian form)}\\ &= \langle \nabla_X \nabla f, Y\rangle \quad \text{(by definition of the Hessian operator)}\\ &= X(Y(f)) - (\nabla_X Y)f \quad \text{(connection is compatible with the metric)}\end{aligned}$$

The following are some classic results that will be used throughout the text.

Theorem 2.2.5 *(Divergence Theorem) Let M^n be a compact oriented Riemannian manifold and $X \in \mathfrak{X}(M)$. If the boundary of M is equipped with the orientation and metric induced by the inclusion $j : \partial M \to M$ and ν denotes the outward unit normal to M along ∂M, then*

$$\int_M (divX) dv = \int_{\partial M} \langle X, \nu\rangle \, da,$$

where the term on the right-hand side should be interpreted as zero if $\partial M = \varnothing$.

Corollary 2.2.6 *Let (M^n, g) be a compact Riemannian manifold with boundary ∂M. Denote by $\overline{g}$ the metric induced on ∂M and ν the outward normal vector along ∂M. Given $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$, we have the following "integration by parts formula":*

$$\int_M \langle \nabla f, X\rangle \ dv = \int_{\partial M} f \langle X, \nu\rangle_g \ da - \int_M (f\, divX) \ dv.$$

Corollary 2.2.7 *Let M^n be a compact and oriented Riemannian manifold with boundary ∂M equipped with the orientation and metric induced by the inclusion $j : \partial M \to M$ (possibly $\partial M = \varnothing$). If $f : M \to \mathbb{R}$ is a smooth function and ν denotes the unit outward normal to M along ∂M, then:*

(a)

$$\int_M \Delta f \ dv \ = \ \int_{\partial M} \frac{\partial f}{\partial \nu} \ da,$$

(b) (1st Green's identity)

$$\int_M \Big(\langle \nabla f, \nabla g\rangle + f\Delta g\Big) \ dv = \int_{\partial M} f \frac{\partial g}{\partial \nu} \ da.$$

(c) (2nd Green's identity)

$$\int_M \left(f\Delta g - g\Delta f\right)\, dv = \int_{\partial M} \left(f\frac{\partial g}{\partial \nu} - g\frac{\partial f}{\partial \nu}\right)\, da.$$

where $\dfrac{\partial f}{\partial \nu} = \langle \nabla f, \nu\rangle$ *is the normal derivative of* f *along* ∂M, *and the integral over the boundary is to be interpreted as zero if* $\partial M = \varnothing$.

We end this section with a result that will be used repeatedly throughout the text.

Lemma 2.2.8 *Let* V *be an* n*-dimensional real vector space equipped with an inner product. If* $T : V \to V$ *is a self-adjoint linear operator, then*

$$|T|^2 \geq \frac{1}{n}(tr\ T)^2,$$

with equality if and only if T *is a multiple of the identity operator.*

3 Reilly's generalized identity and applications

This chapter focuses on Reilly's generalized identity and its applications, exploring its versatility in deriving inequalities and rigidity results within various geometric contexts. The chapter begins by analyzing V-static manifolds, defined by a specific criticality condition for the volume functional under fixed boundary metrics. Using a tensorial framework, the work establishes new integral identities, including generalized versions of the Heintze-Karcher and Minkowski inequalities. These results highlight the conditions under which boundary components exhibit geometric rigidity, such as being umbilic or possessing constant mean curvature.

The chapter further extends the analysis to m-quasi-Einstein manifolds. Rigidity results and integral inequalities are derived, emphasizing their implications for warped product structures. The use of Reilly-type identities underpins the chapter's contributions, offering a unified approach to addressing classical and modern geometric problems.

3.1 V-static case

The results presented here are part of [21]. This paper explores the extension and applications of a generalized Reilly-type identity within the context of V-static manifolds. The study builds upon existing integral formulas, particularly those developed by [14], and adapts then to the V-static framework. This adaptation allows the derivation of new geometric inequalities and rigidity results that generalize classical results in differential geometry.

The Reilly-type identity specifically adapted for V-static manifolds allows us to study the domain properties, boundary conditions, mean curvature, and Ricci curvature, providing a comprehensive framework for analyzing geometric inequalities. In the chapter we use this identity to extend the Heintze-Karcher and Minkowski inequalities to bounded domains within V-static manifolds, in Sections (3.1.3) and (3.1.4), respectively.

The generalized Heintze-Karcher inequality relates the volume of a domain to surface integrals over its boundary's curvatures. In this setting, the inequality holds for domains with mean convex boundaries. When equality is achieved, it is shown that the boundary must be umbilic. Similarly, the generalized Minkowski inequality establishes a connection between the integrals of the second fundamental form, mean curvature, and domain volume. The equality case in this inequality also leads to rigidity results, requiring that the boundary exhibit constant mean curvature.

For a clear and precise display of the results will be presented a detailed analysis of the conditions under which equality is achieved in these inequalities. The rigidity results demonstrate that specific geometric constraints, such as umbilicity or constant boundary mean curvature, are necessary for equality to hold. These findings are supported by a robust mathematical framework that incorporates extensions of Bochner's formula, variational principles, and integral techniques leveraging divergence theorems and eigenvalue problems.

3.1.1 V-static metrics

Most results will not be proven here; the reader may find more comprehensive material in the references [16], [17] e [19].

Definition 3.1.1 *Let (M^n, g) be an n-dimensional Riemannian manifold $(n \geq 3)$ that is complete and connected, with boundary ∂M (possibly disconnected). We say that g is a V-static metric if there exists a smooth (non-trivial) solution V on M^n that satisfies the overdetermined elliptic system:*

$$\begin{cases} L_g^*(V) &= & -(\Delta V)g + \nabla^2 V - VRic = g & \text{in} & M \\ V & > & 0 & \text{on} & int(M) \\ V & = & 0 & \text{on} & \partial M \end{cases}$$

where $int(M)$ denotes the interior of M, and Ric, Δ, ∇^2 represent the Ricci tensor, the Laplace operator, and the Hessian on (M^n, g), respectively. Equivalently, we refer to the metric g as V-static with potential V-static V. The triple (M^n, g, V) is called a V-static triple.

It is worth noting that L_g^* is the formal [1] L^2-adjoint of the linearization of the scalar curvature operator L_g, that is, if h is a symmetric $(0, 2)$ tensor on M, the linearization L_g of the scalar curvature map $R : \mathcal{M} \to \mathbb{R}$ is given by:

$$L_g(h) = -\Delta_g(\text{tr}_g h) + \text{div}_g \text{div}_g h - hRic(g)$$

and by convention, $\Delta_g f = \text{tr}_g(\nabla_g^2 f)$, and we denote that $\mathcal{M}$ is the cone of the Riemannian metric on M. For further details, see [18, cf. Eq. (1.183), page 64] ad [19].

Let γ be a smooth metric on ∂M, and let $\mathcal{M}_\gamma$ be the set of all metrics on M such that $g\Big|_{T(\partial M)} = \gamma$. We define the set $\mathcal{M}_\gamma \subset \mathcal{M}$. By [16, Lemma 2.1], the scalar curvature map R is smooth. Let $g_0 \in \mathcal{M}_\gamma$, and suppose the scalar curvature of g_0 is constant and equal to K. If 0 is not an eigenvalue of the Dirichlet operator $(n-1)\Delta_{g_0} + K$, then, by [16, Lemma 2.2], $M_\gamma^K = \{g \in \mathcal{M}_\gamma \mid R(g) = K\}$ is a submanifold of $\mathcal{M}_\gamma$. Miao and Tam [16] showed that the critical metric defined in (3.1.1) is related to the study of critical points of the volume functional on $\mathcal{M}_\gamma^K$. For details, see Section 2 and Theorem 2.1 in [16]. Another approach is due to [19, Theorem 2.3]. In both approaches, it is assumed that 0 is not an eigenvalue of the Dirichlet operator $(n-1)\Delta_{g_0} + K$, i.e., the first eigenvalue of $(n-1)\Delta_{g_0} + K$ is positive.

Next, we consider $V \geq 0$, $V^{-1}(0) = \partial M$, and the first eigenvalue of $(n-1)\Delta_{g_0} + K$ is positive. In this context, Cruz and Santos [63], in Theorem A, showed an important characterization for the V-static metric in $\mathcal{M}_{c,0} = \{g \in \mathcal{M}^{k,2} \subset W^{k,2} \mid R_g = c \text{ and } H_g = 0\}$, the variety of Riemannian metrics with prescribed curvature. A similar characterization was used by Cruz and Nunes [64] for the study of V-static manifolds in dimension 3.

For examples of V-static manifolds, see [19] (Examples 1.3, 1.4, and 1.5), and for examples of compact V-static manifolds, see [16] in Theorem 3.1 and [17] in Theorems 4.1 and 4.2. See also [63] in Section 2.

[1] For a definition of the formal adjoint, see [18, Appendix E, page 460].

In next, we will present details of some of these examples that will be used later.

Example 3.1.2 *(Geodesic ball in $\mathbb{R}^n$) Let $(\mathbb{R}^n, g)$ be the Euclidean space with the canonical metric g. Consider $B_{\overline{r}} \subset \mathbb{R}^n$ a geodesic ball of radius $\overline{r}$ with the induced metric g. Let*

$$V(x) = \frac{\overline{r}^2}{2(n-1)} - \frac{|x|^2}{2(n-1)}.$$

Under these conditions, we have $Ric = 0$ and

$$\nabla^2 V = -\frac{1}{n-1} g, \quad \Delta f = -\frac{n}{n-1}.$$

Thus,

$$\begin{aligned} -\Delta V g + \nabla^2 V - V Ric &= \frac{n}{n-1} g - \frac{1}{n-1} g \\ &= g. \end{aligned}$$

Note also that $V^{-1}(0) = \partial(B_{\overline{r}})$, because

$$\begin{aligned} 0 &= V(x) \\ &= \frac{\overline{r}^2}{2(n-1)} - \frac{|x|^2}{2(n-1)} \\ &= \overline{r}^2 - |x|^2. \end{aligned}$$

Thus, $|x|^2 = \overline{r}^2$ and $V^{-1}(0) = \partial(B_{\overline{r}})$. Therefore, we conclude that $(B_{\overline{r}}, g, V)$ is a V-static triple.

Example 3.1.3 *(Geodesic ball in $\mathbb{H}^n$) Consider the Minkowski space $\mathbb{R}^{n+1}_1 = (\mathbb{R}^{n+1}, g)$, where*

$$g = dx_1^2 + dx_2^2 + \ldots + dx_n^2 - dt^2$$

and the hyperbolic space given by

$$\mathbb{H}^n = \{(x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^{n+1} \ ; \ x_1^2 + x_2^2 + \ldots + x_n^2 - t^2 = -1, \ t \geq 1\}$$

embedded in $\mathbb{R}^{n+1}_1$ and the induced metric in $\mathbb{H}^n$ by g. Therefore, g is a Riemannian metric. Fixing the point $p = (0, 0, \ldots, 1) \in \mathbb{H}^n$, consider $B_{\overline{r}} \subset \mathbb{H}^n$ a geodesic ball of radius $\overline{r}$. Let

$$V(x_1, x_2, \ldots, x_n, t) = \frac{1}{n-1}\Big(1 - \frac{\cosh r}{\cosh \overline{r}}\Big),$$

where r is the geodesic distance from the point x to p and $t = \cosh r$ is the height function. Therefore,

$$\begin{aligned} \nabla^2 V &= -\frac{\nabla^2(t)}{(n-1)\cosh \overline{r}} \\ &= -\frac{t}{(n-1)\cosh \overline{r}} g \end{aligned}$$

and

$$(\Delta V) g = -\frac{nt}{(n-1)\cosh \overline{r}} g.$$

Since $Ric = -(n-1)g$, we have

$$\begin{aligned} -\Delta V g + \nabla^2 V - V Ric &= \frac{nt}{(n-1)\cosh g} g - \frac{t}{(n-1)\cosh \overline{r}} g + \frac{1}{n-1}\Big(1 - \frac{t}{\cosh \overline{r}}\Big)(n-1)g \\ &= \frac{nt - t + (n-1)\cosh \overline{r} - (n-1)t}{(n-1)\cosh \overline{r}} g \\ &= g. \end{aligned}$$

Moreover, $V^{-1}(0) = \partial(B_{\overline{r}})$. Thus, $(B_{\overline{r}}, g, V)$ is a V-static triple.

Example 3.1.4 *(Geodesic ball in $\mathbb{S}^n$) Consider $\mathbb{S}^n$ the canonical sphere in the Euclidean space $\mathbb{R}^{n+1}$ and the point $p = (0, 0, \ldots, 1) \in \mathbb{S}^n$. Let $B_{\overline{r}} \subset \mathbb{S}^n$ be the geodesic ball centered at p with radius $\overline{r} < \frac{\pi}{2}$, g the induced metric on $B_{\overline{r}}$, and*

$$V(x_1, x_2, \ldots, x_n, t) = \frac{1}{n-1}\Big(\frac{\cos r}{\cos \overline{r}} - 1\Big)$$

where r is the geodesic distance from $(x_1, x_2, \ldots, x_n, t)$ to p. Since $t = \cos r$ is the height function, we have

$$\begin{aligned} \nabla^2 V &= \frac{\nabla^2(t)}{(n-1)\cos \overline{r}} g \\ &= -\frac{t}{(n-1)\cos \overline{r}} g \end{aligned}$$

and

$$\Delta V g = -\frac{nt}{(n-1)\cos \overline{r}} g.$$

Since $Ric = (n-1)g$, we have

$$\begin{aligned} -\Delta V + \nabla^2 V - V Ric &= \frac{nt}{(n-1)\cos \overline{r}} g - \frac{t}{(n-1)\cos \overline{r}} g - \frac{1}{n-1}\Big(\frac{t}{\cos \overline{r}} - 1\Big)(n-1)g \\ &= \frac{nt - t - (n-1)t + (n-1)\cos \overline{r}}{(n-1)\cos \overline{r}} g \\ &= g. \end{aligned}$$

Moreover, $V^{-1}(0) = \partial(B_{\overline{r}})$. Thus, $(B_{\overline{r}}, g, V)$ is a V-static triple.

The following results provide some properties of a Riemannian manifold (M^n, g) with a potential function V.

Theorem 3.1.5 *[16, Theorem 3.2] Consider (M^n, g, V) a V-static triple. Then, (M^n, g, V) has the following properties:*

1. $\Delta V = -\frac{R}{n-1} V - \frac{n}{n-1}$.
2. *The scalar curvature of g is constant.*
3. *We have that $|\nabla_g V|(x) \neq 0$, for all $x \in \partial M$.*
4. *$|\nabla_g V|$ is constant along each connected component of ∂M.*

5. *The boundary ∂M is a totally umbilical hypersurface*

6. *If M is compact, with a smooth boundary (possibly disconnected) Σ such that $V = 0$ on Σ and the first Dirichlet eigenvalue of $(n-1)\Delta + R$ is non-negative, where R is the scalar curvature of g, then along each connected component Σ_i of Σ, the Gauss equation leads us to obtain that at each point of Σ the relation $2Ric(\nu,\nu) + R^{\Sigma} = R + \frac{n-2}{n-1}H^2$, where R^{Σ} is the scalar curvature of Σ and H is the mean curvature of Σ.*

3.1.2 General integral formula

In this section, inspired by the results of [13] and [14] in the context of static and sub-static manifolds, respectively, we obtain an integral formula to be applied in bounded domains contained in V-static manifolds.

In such a work was obtained the following result.

Theorem 3.1.6 *[14, Theorem 3.1] Let (M^n, g) be a smooth Riemannian manifold and $\Omega \subset M$ a smooth domain with a smooth boundary $\partial\Omega$. Let P_{ij} be a smooth symmetric $(0,2)$-tensor in Ω, and let P be the trace of P_{ij} with respect to g. Define $A_{ij}(f) := f_{,ij} + \frac{1}{n-1}Pfg_{ij} - fP_{ij}$, and its trace with respect to g, denoted by $A(f) = \Delta f + \frac{P}{n-1}f$. Then, for every smooth function $V, f \in C^{\infty}(\overline{\Omega}, \mathbb{R})$, we have the following identity:*

$$
\begin{aligned}
& \int_{\Omega} V\Big[A(f)^2 - |A_{ij}(f)|^2\Big]dv \\
= & \int_{\partial\Omega} \Big[Vh(\overline{\nabla}z, \overline{\nabla}z) + 2Vu\overline{\Delta}z + VHu^2 + V_{,\nu}|\overline{\nabla}z|^2\Big]\, da \\
+ & \int_{\partial\Omega} \Big[2Vzf_{,i}P_{i\nu} - z^2V_{,i}P_{i\nu} - z^2VP_{\nu i,i}\Big]\, da \\
+ & \int_{\Omega} \Big[(V_{,ij}) - \Delta V g_{ij} - VP_{ij} + V(R_{ij} - P_{ij})\Big] f_{,i}f_{,j}\, dv \\
+ & \int_{\Omega} \Big[P_{ij}(V_{,ij} + \frac{1}{n-1}PVg_{ij} - VP_{ij}) + VP_{ij,ji} + 2V_iP_{ij,j}\Big] f^2\, dv.
\end{aligned}
$$

Here, ν denotes the unit normal vector pointing outward, $z = f|_{\partial\Omega}$, $u = \nabla_{\nu}f$, $V_{,\nu} = \nabla_{\nu}V$, and h, H are the second fundamental form and the mean curvature (sum of the principal curvatures) of $\partial\Omega$, respectively.

First, we recall that Theorem 3.1.6 is a generalized Reilly formula, i.e., if $P_{ij} = 0$ and $V = 1$, we obtain the classical Reilly's identity [2]. For the proof of this version of Theorem 3.1.6, the central idea is to establish an integral identity for a smooth function V on $\overline{\Omega}$, with the assumption that the ratio $\frac{\nabla^2 V}{V}$ is continuous up to the boundary. This identity involves both integrals in Ω and boundary integrals on $\partial\Omega$. The key points are:

- Choosing the Tensor P_{ij}: The proof starts by choosing an appropriate tensor P_{ij}, specifically designed to satisfy the conditions from a previous theorem. This tensor is expressed in terms of the second derivatives of V, as:

$$
P_{ij} = \frac{1}{V}\left(V_{,ij} - \Delta V\, g_{ij}\right),
$$

which is a symmetric tensor. The choice of this tensor ensures the structure needed for the rest of the proof. The trace of this tensor gives the expression

$$P = -\frac{n-1}{V}\Delta V,$$

which is used later in the proof.

- Computation of $A_{ij}(f)$: Using the chosen tensor P_{ij}, the expression for $A_{ij}(f)$ is derived, which involves the second derivatives of f and V. It turns out that this computation leads to:

$$A_{ij}(f) = f_{,ij} - \frac{V_{,ij}}{V}f,$$

  and the trace of $A_{ij}(f)$ gives the expression:

$$A(f) = \Delta f - \frac{\Delta V}{V}f.$$

  This expression plays a key role in simplifying the integrals that appear in the identity.

- Boundary Integrals and use of the Ricci Identity: Boundary integrals involve terms that can be written in terms of the second fundamental form h, the mean curvature H, and other boundary data. A key step is to simplify the boundary terms by using the identity of Ricci curvature and the second fundamental form. In particular, it is shown that the boundary terms can be written as:

$$\int_{\partial\Omega}\Big[2z\nabla^2 V(\overline{\nabla}z,\nu) - 2zu(\overline{\Delta}V + HV_\nu)\Big]\,da,$$

  where $z = f|_{\partial\Omega}$, $u = \nabla_\nu f$, and ν is the outward normal vector. These terms are carefully derived using integration by parts, the properties of the normal vector, and the decomposition of the Ricci tensor.

The next result is a particlar case of Theorem 3.1.6 and will be important in Section 3.2.1 where an application in the context of m-quasi Einstens will be presented. This last application is part of paper [26].

Theorem 3.1.7 *[13, Theorem 1.1] Let (Ω^n, g) be an n-dimensional compact Riemannian manifold with smooth boundary $\partial\Omega$. Consider $V:\Omega\to\mathbb{R}$ and $f:\Omega\to\mathbb{R}$ as smooth functions and K as a constant. The following relation holds:*

$$\begin{aligned}
&\int_\Omega V\Big[(\Delta f + Knf)^2 - |\nabla^2 f + Kfg|^2\Big]\,dv\\
= &\int_\Omega \Big[\nabla^2 V - (\Delta V)g - 2(n-1)KVg + VRic\Big](\nabla f,\nabla f)\,dv\\
+ &\ (n-1)K\int_\Omega(\Delta V + nKV)f^2\,dv + \int_{\partial\Omega}\nabla_\nu V\Big[|\overline{\nabla}z|^2 - (n-1)Kz^2\Big]\,da\\
+ &\int_{\partial\Omega} V\Big[2f_{,\nu}\overline{\Delta}z + H(f_{,\nu})^2 + h(\overline{\nabla}z,\overline{\nabla}z) + 2(n-1)Kf_{,\nu}z\Big]\,da.
\end{aligned}$$

Here, ν denotes the outward-pointing unit normal vector, $z = f|_{\partial\Omega}$, $u = \nabla f$, $V_{,\nu} = \nu V$, $h(\ ,\ )$ represents the second fundamental form, and H is the mean curvature (sum of the principal curvatures) of $\partial\Omega$.

Now, we make a choice of P in Theorem 3.1.6 that works for our purpose to establish a Reilly-type identity for V-static manifolds. In this direction, we consider

$$P_{ij} = \frac{1}{V}\Big(V_{,ij} - (\Delta V + 1)g_{ij}\Big). \tag{3.1}$$

Note that

$$\begin{aligned} Q_{ij} &= R_{ij} - P_{ij} \\ &= \frac{1}{V}\left[VR_{ij} - V_{,ij} + (\Delta V + 1)g_{ij}\right] \end{aligned} \tag{3.2}$$

is a tensor that vanishes in the case of V-static manifolds. The next result and, mainly, its consequences will be the main tool of this work. Its proof follows the same line of Theorem 1.1 in [14].

Theorem 3.1.8 *Let (M^n, g) be an n-dimensional Riemannian manifold and $\Omega \subset M^n$ a bounded domain with smooth boundary $\partial\Omega$. Let $V \in C^\infty(\overline{\Omega})$ be a smooth function such that P defined by* (3.1) *is continuous up to $\partial\Omega$. Then, for all $f \in C^\infty(\overline{\Omega})$, we have*

$$\begin{aligned} &\int_\Omega V\Big[\Big(\Delta f - \frac{1}{V}(\Delta V + \frac{n}{n-1})f\Big)^2 - \Big|\nabla^2 f - \frac{1}{V}(\nabla^2 V + \frac{1}{n-1}g)f\Big|^2\Big]\, dv \\ = &\int_{\partial\Omega}\Big[Vh(\overline{\nabla} z, \overline{\nabla} z) + 2Vu\overline{\Delta} z + VHu^2 + V_{,\nu}|\overline{\nabla} z|^2 + 2z\nabla^2 V(\overline{\nabla} z, \nu)\Big]\, da \\ - &\int_{\partial\Omega}\Big[2zu(\overline{\Delta} V + HV_{,\nu} + 1) + z^2\frac{\nabla^2 V - (\Delta V + 1)g}{V}(\nabla V, \nu)\Big]\, dv \\ + &\int_\Omega\Big[|\nabla f|^2 + \frac{1}{V}\Big(\Delta V + \frac{n}{n-1}\Big)f^2\Big]\, dv \\ + &\int_\Omega VQ_{ij}(Vf_{,i} - V_{,i}f)(Vf_{,j} - V_{,j}f)\, dv. \end{aligned}$$

Here, ν denotes the outward-pointing unit normal vector, $z = f|_{\partial\Omega}, u = \nabla_\nu f, V_{,\nu} = \nabla_\nu V, h(\cdot,\cdot), H$ are the second fundamental form and the mean curvature of $\partial\Omega$, respectively.

Proof: By considering the trace of P_{ij} in the metric g

$$P = -\Big[\frac{(n-1)\Delta V + n}{V}\Big],$$

and returning to the definition of A_{ij} in Theorem 3.1.6, we get

$$\begin{aligned} A_{ij}(f) &= f_{,ij} + \frac{1}{n-1}Pfg_{ij} - fP_{ij} \\ &= f_{,ij} - \frac{1}{n-1}\Big[\frac{(n-1)\Delta V + n}{V}\Big]fg_{ij} - f\frac{1}{V}(V_{,ij} - (\Delta V + 1)g_{ij}) \\ &= f_{,ij} - \frac{1}{V}(V_{,ij} + \frac{1}{n-1}g_{ij})f, \end{aligned}$$

and

$$A(f) = \mathrm{tr}(A_{ij}(f)) = \Delta f - \frac{1}{V}\Big(\Delta V + \frac{n}{n-1}\Big)f.$$

Furthermore, by straightforward calculations

$$V_{,ij} + \frac{1}{n-1}PVg_{ij} + \frac{1}{n-1}g_{ij} - VP_{ij} = 0, \tag{3.3}$$

and

$$\Delta V + \frac{1}{n-1} PV + \frac{n}{n-1} = 0.$$

Next, we are going to obtain two identities that will be used in the sequence of the proof by taking covariant derivatives from (3.2) and (3.3). In the following, the Ricci Identity (namely $f_{,ijj} = (\Delta f)_{,i} + R_{ij} f_{,j}$) and the second (contracted) Bianchi identity (namely, $2R_{ij,i} = R_{,j}$) will be used.

Statement 1:

$$Q_{ij} V_{,ji} + \frac{1}{2} R_j V_{,j} = 2P_{ji,i} V_{,j} + V P_{ij,ji} \tag{3.4}$$

By (3.3), we have

$$\begin{aligned} 0 &= \Big[V_{,ij} + \frac{1}{n-1} PV g_{ij} - V P_{ij} + \frac{1}{n-1} g_{ij}\Big]_{,ji} \\ &= \Big[V_{,ijj} + \frac{1}{n-1} (PV)_{,i} - V_{,j} P_{ij} - V P_{ij,j} + \frac{1}{n-1} g_{ij,j}\Big]_{,i} \\ &= \Big[(\Delta V)_{,i} + R_{ij} V_{,j} + \frac{1}{n-1} (PV)_{,i} - V_{,j} P_{ij} - V P_{ij,j}\Big]_{,i} \\ &= \Big[R_{ij} V_{,j} - V_{,j} P_{ij} - V P_{ij,j}\Big]_{,i} \\ &= (R_{ij} - P_{ij}) V_{,ji} + R_{ij,i} V_{,j} - 2P_{ij,i} V_{,j} - V P_{ij,ji} \\ &= Q_{ij} V_{,ji} + \frac{1}{2} R_{,j} V_{,j} - 2P_{ji,i} V_{,j} - V P_{ij,ji}. \end{aligned}$$

Statement 2:

$$V_{,j} Q_{ij,i} - \frac{1}{2} V_{,j} R_{,j} + \frac{1}{V} Q_{ij} V_{,i} V_{,j} = 0. \tag{3.5}$$

By taking the covariant derivative of (3.2), see that

$$\begin{aligned} V Q_{ij,i} + V_{,i} Q_{ij} &= \Big[V Q_{ij}\Big]_{,i} \\ &= (\Delta V + 1)_{,j} - V_{,iji} + V_{,i} R_{ij} + V R_{ij,i} \\ &= (\Delta V + 1)_{,j} - (\Delta V)_{,j} - R_{ij} V_{,i} + V_{,i} R_{ij} + V R_{ij,i} \\ &= \frac{1}{2} V R_{,j}. \end{aligned}$$

Now, we analyse in Theorem 3.1.6 the second integral on $\partial\Omega$, which is

$$\begin{aligned} & \int_{\partial\Omega} \Big[2V f f_{,i} P_{i\nu} - f^2 V_{,i} P_{i\nu} - f^2 V P_{\nu i,i}\Big]\, da \\ = & \int_{\partial\Omega} \Big[2V f f_{,i} P_{i\nu} - f^2 (V P_{i\nu})_{,i}\Big]\, da \\ = & \int_{\partial\Omega} \Big[2f f_{,i} (V_{,i\nu} - (\Delta V + 1) g_{i\nu}) - f^2 (V_{,i\nu i} - (\Delta V)_{,i} g_{i\nu} - \Delta V g_{i\nu,i})\Big]\, da \\ = & \int_{\partial\Omega} \Big[2f f_{,i} (V_{,i\nu} - (\Delta V + 1) g_{i\nu}) - f^2 \Big((\Delta V)_{,\nu} + R_{i\nu} V_{,i} - (\Delta V)_{,i} g_{i\nu}\Big)\Big]\, da \\ = & \int_{\partial\Omega} \Big[2f f_{,i} (V_{,i\nu} - (\Delta V + 1) g_{i\nu}) - f^2 R_{i\nu} V_{,i}\Big]\, da \\ = & \int_{\partial\Omega} \Big[2z \nabla^2 V(\overline{\nabla} z, \nu) - 2zu(\overline{\Delta} V + H V_\nu + 1) - f^2 R_{i\nu} V_{,i}\Big]\, da. \end{aligned} \tag{3.6}$$

where, in the last identity we have used $\Delta V = \overline{\Delta} V + \nabla^2 V(\nu,\nu) + HV_\nu$.

Returning to Theorem 3.1.6, we analyse the two integrals in Ω. Using (3.3), (3.4), (3.5) and integration by parts, we have

$$
\begin{aligned}
&\int_\Omega \Big[V_{,ij} - \Delta V g_{ij} - VP_{ij} + V(R_{ij} - P_{ij})\Big] f_{,i} f_{,j}\, dv \\
&+ \int_\Omega \Big[P_{ij}(V_{,ij} + \frac{1}{n-1} PV g_{ij} - VP_{ij}) + VP_{ij,ji} + 2V_{,i}P_{ij,j}\Big] f^2\, dv \\
=\ & \int_\Omega \Big[g_{ij} + VQ_{ij}\Big] f_{,i} f_{,j}\, dv + \int_\Omega \Big[P_{ij}(-\frac{1}{n-1} g_{ij}) + Q_{ij}V_{,ij} + \frac{1}{2} R_{,j}V_{,j}\Big] f^2\, dv \\
=\ & \int_\Omega \Big[|\nabla f|^2 + VQ_{ij} f_{,i} f_{,j}\Big]\, dv + \int_\Omega \Big[\frac{1}{2} R_{,j} V_{,j} f^2 - \frac{1}{n-1} P f^2\Big]\, dv + \int_\Omega Q_{ij} V_{,ji} f^2\, dv \\
=\ & \int_\Omega \Big[|\nabla f|^2 + VQ_{ij} f_{,i} f_{,j} + \frac{1}{2} R_{,j} V_{,j} f^2 + \frac{1}{V}(\Delta V + \frac{n}{n-1}) f^2\Big]\, dv + \int_{\partial\Omega} V_{,j} f^2 Q_{j\nu}\, da \\
&- \int_\Omega \Big[V_{,j} f^2 Q_{ij,i} + 2V_{,j} Q_{ij} f f_{,i}\Big]\, dv \\
=\ & \int_\Omega \Big[|\nabla f|^2 + VQ_{ij} f_{,i} f_{,j} + \frac{1}{V}(\Delta V + \frac{n}{n-1}) f^2\Big]\, dv \\
&+ \int_{\partial\Omega} V_{,j} f^2 Q_{j\nu}\, da + \int_\Omega \Big[\frac{1}{V} Q_{ij} V_{,i} V_{,j} f^2 - 2V_{,j} Q_{ij} f f_{,i}\Big]\, dv
\end{aligned}
\tag{3.7}
$$

Replacing (3.6) and (3.7) in Theorem 3.1.6, we obteined that

$$
\begin{aligned}
&\int_\Omega V\Big[\Big(\Delta f - \frac{1}{V}(\Delta V + \frac{n}{n-1}) f\Big)^2 - \Big|\nabla^2 f - \frac{1}{V}(\nabla^2 V + \frac{1}{n-1} g) f\Big|^2\Big]\, dv \\
=\ & \int_{\partial\Omega} \Big[V h(\overline{\nabla} z, \overline{\nabla} z) + 2Vu\overline{\Delta} z + VHu^2 + V_{,\nu} |\overline{\nabla} z|^2\Big]\, da \\
+\ & \int_{\partial\Omega} \Big[2z\nabla^2 V(\overline{\nabla} z, \nu) - 2zu(\overline{\Delta} V + HV_{,\nu} + 1) + z^2(V_{,j} Q_{j\nu} - R_{i\nu} V_{,i})\Big]\, da \\
+\ & \int_\Omega \Big[|\nabla f|^2 + \frac{1}{V}\Big(\Delta V + \frac{n}{n-1}\Big) f^2\Big]\, dv \\
+\ & \int_\Omega \frac{1}{V} Q_{ij} (V f_{,i} - V_{,i} f)(V f_{,j} - V_{,j} f)\, dv
\end{aligned}
$$

$$
\begin{aligned}
=\ & \int_{\partial\Omega} \Big[V h(\nabla z, \nabla z) + 2Vu\Delta z + VHu^2 + V_{,\nu} |\nabla z|^2\Big]\, da \\
+\ & \int_{\partial\Omega} \Big[2z\nabla^2 V(\overline{\nabla} z, \nu) - 2zu(\overline{\Delta} V + HV_{,\nu} + 1) - z^2 \frac{\nabla^2 V - (\Delta V + 1) g}{V} (\nabla V, \nu)\Big]\, da \\
+\ & \int_\Omega \Big[|\nabla f|^2 + \frac{1}{V}\Big(\Delta V + \frac{n}{n-1}\Big) f^2\Big]\, dv \\
+\ & \int_\Omega \frac{1}{V} Q_{ij} (V f_{,i} - V_{,i} f)(V f_{,j} - V_{,j} f)\, dv
\end{aligned}
$$

$$
\begin{aligned}
=\ & \int_{\partial\Omega} \Big[V h(\overline{\nabla} z, \overline{\nabla} z) + 2Vu\overline{\Delta} z + VHu^2 + V_{,\nu} |\overline{\nabla} z|^2\Big]\, da \\
+\ & \int_{\partial\Omega} \Big[2z\nabla^2 V(\overline{\nabla} z, \nu) - 2zu(\overline{\Delta} V + HV_{,\nu} + 1) - z^2 \frac{\nabla^2 V - (\Delta V + 1) g}{V} (\nabla V, \nu)\Big]\, da \\
+\ & \int_\Omega \Big[|\nabla f|^2 + \frac{1}{V}\Big(\Delta V + \frac{n}{n-1}\Big) f^2\Big]\, dv \\
+\ & \int_\Omega V \cdot Q\Big(\nabla f - \frac{\nabla V}{V} f, \nabla f - \frac{\nabla V}{V} f\Big)\, dv,
\end{aligned}
$$

which concludes our result. □

Suppose that (M^n, g, V) is a V-static triple. Therefore, there is $V \in C^\infty(M)$ such that V is nonnegative and $V \cdot Q = 0$. Furthermore, the tensor $P_{ij} = R_{ij}$ is clearly continuous up to $\partial\Omega$. This allows the following consequence

Corollary 3.1.9 *Let (M^n, g, V) be a V-static triple and $\Omega \subset M$ a bounded domain. Then, for all $f \in C^\infty(\overline{\Omega})$,*

$$\begin{aligned} & \int_\Omega V\Big[\Big(\Delta f - \frac{1}{V}(\Delta V + \frac{n}{n-1})f\Big)^2 - \Big|\nabla^2 f - \frac{1}{V}(\nabla^2 V + \frac{1}{n-1}g)f\Big|^2\Big]\, dv \\ = & \int_{\partial\Omega}\Big[Vh(\overline{\nabla}z, \overline{\nabla}z) + 2Vu\overline{\Delta}z + VHu^2 + V_{,\nu}|\overline{\nabla}z|^2 + 2z\nabla^2 V(\overline{\nabla}z, \nu)\Big]\, da \\ + & \int_{\partial\Omega} 2zuV Ric(\nu,\nu)\, da - \int_{\partial\Omega} z^2 Ric(\nabla V, \nu)\, da \\ + & \int_\Omega \Big[|\nabla f|^2 + \frac{1}{V}\Big(\Delta V + \frac{n}{n-1}\Big)f^2\Big]\, dv. \end{aligned}$$

Here ν denotes the outward-pointing unit normal vector, $z = f|_{\partial\Omega}, u = \nabla_\nu f, V_{,\nu} = \nabla_\nu V, h(\cdot,\cdot), H$ are the second fundamental form and the mean curvature of $\partial\Omega$ respectively.

Proof: The proof is the direct application of Theorem 3.1.8 with the observation that

$$(\overline{\Delta}V + 1)\overline{g}(\nu,\nu) + HV_{,\nu} = (\Delta V + 1)g(\nu,\nu) - \nabla^2 V(\nu,\nu) = -V Ric(\nu,\nu).$$

□

Remark 3.1.10 *V-static Equation* (3.1.1) *could be used to give a more geometric point of view of the right-hand side of Corollary 3.1.9. Indeed,*

- $\frac{1}{V}(\Delta V + \frac{n}{n-1}) = -\frac{R}{n-1}$
- $\frac{1}{V}(\nabla^2 V + \frac{1}{n-1}g) = Ric - \frac{R}{n-1}g.$

3.1.3 Heintze-Karcher type inequality

This section is devoted to proving a version of the Heintze-Karcher type inequality (1.5) to domains on V-static manifolds. First, we establish some previous conditions in treating domains with multiple boundaries, including connected components that intersect ∂M. This hypothesis was inspired by the sub-static case, see [14].

Definition 3.1.11 *Let (M^n, g, V) be a V-static triple and $\Omega \subset M^n$ be a domain with smooth boundary $\partial\Omega = \Sigma \bigcup \Big(\bigcup_{l=1}^{\tau} N_l\Big)$. We say that Ω satisfies V-static boundary condition if the following conditions hold:*

(H1) the components N_l are contained in ∂M;

(H2) $\Sigma \subset int(M)$ is strictly mean convex (this means $H > 0$);

(H3) In each N_l,

$$R^{N_l} \geq R + \frac{n-2}{n-1} H^2, \tag{3.8}$$

where R^{N_l} denote the scalar curvature in N_l, and H its mean curvature.

Remark 3.1.12 *1. Condition $(H1)$ implies that all the boundary components N_l are umbilical, smooth, closed $(n-1)$-dimensional Riemannian submanifolds in M, see [16, Theorem 3.2]. This result also implies that the right-hand side of* (3.8) *is a constant.*

2. A motivation for the condition $(H3)$ comes from the $V-$static equation that could control one term in the integral identity of Corollary 3.1.9. In fact, by (3.1.1) *and Gauss equation*

$$\frac{-\nabla^2 V(\nu,\nu) + \Delta V + 1}{V} = -Ric(\nu,\nu) = \frac{1}{2}\Big(R^{N_l} - R - \frac{n-2}{n-1}H^2\Big).$$

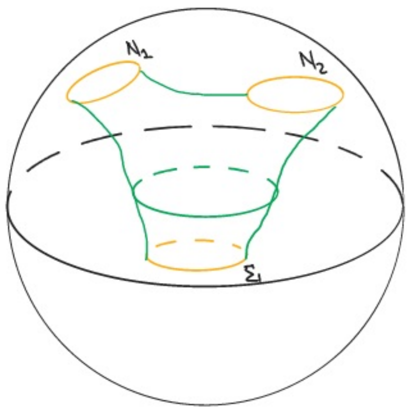

Figure 2 – Illustration of the definition (3.1.11).

Next, we choose a special solution for the Schrodinger operator, which appears on the left-hand side of Corollary 3.1.9. In this direction, it is important to prove that the first Dirichlet eigenvalue of the operator $\Delta - q$ in Ω is positive where $q := \frac{1}{V}\Big(\Delta V + \frac{n}{n-1}\Big)$. Note that in case that M^n is a V-static manifold

$$\frac{1}{V}\Big(\Delta V + \frac{n}{n-1}\Big) = -\frac{R}{n-1},$$

where R denote the scalar curvature. Recall the first eigenvalue for a bounded domain Ω is defined as, see [7, Section 3.2]

$$\lambda_1(\Delta - q, \Omega) = \inf_{f \in H_0^1(\Omega),\, ||f||^2_{L_2(\Omega)}=1} \int_\Omega \Big(|\nabla f|^2 + qf^2\Big)\, dv.$$

In the following two lemmas we will study the signal of the first eigenvalue $\lambda_1(\Delta - q, \Omega)$ for a bounded domain Ω of the operator $\Delta - q$ and the applications of this signal in the existence of a solution to our problem.

Lemma 3.1.13 *Suppose $V \in C^\infty(int(M))$ and $V > 0$ in $int(M)$. Let $q := \frac{1}{V}\Big(\Delta V + \frac{n}{n-1}\Big)$. If $\Omega \subset int(M)$ is a bounded domain, then $\lambda_1(\Delta - q, \Omega) > 0$.*

Proof: To prove the first statement, we only need to show that

$$\int_{\Omega}(|\nabla f|^2 + qf^2)\, dv \geq 0,$$

for any $f \in C_c^{\infty}(\Omega)$ with $||f||^2_{L^2(\Omega)} = 1$. By integration-by-parts and using the relation

$$\frac{\Delta V}{V} f^2 = div\Big(\frac{\nabla V}{V} f^2\Big) - 2\frac{f}{V}\langle \nabla f, \nabla V\rangle + \Big|\frac{\nabla V}{V} f\Big|^2,$$

we obtain

$$\begin{aligned}
\int_{\Omega}\Big(|\nabla f|^2 + \frac{1}{V}(\Delta V + \frac{n}{n-1})f^2\Big)\, dv &= \int_{\Omega}\Big(|\nabla f|^2 + \frac{\Delta V}{V} f^2\Big)\, dv + \frac{n}{n-1}\int_{\Omega}\frac{f^2}{V}\, dv \\
&> \int_{\Omega}\Big(|\nabla f|^2 + \frac{\Delta V}{V} f^2\Big)\, dv \\
&= \int_{\Omega}\Big|\nabla f - \frac{\nabla V}{V} f\Big|^2\, dv \geq 0
\end{aligned}$$

with this we conclude that $\lambda_1(\Delta - q, \Omega) \geq 0$.

To prove the last statement, we recall the fact that if one bounded domain is strictly contained in another one, e.g., $\Omega \subsetneq \Omega'$, then, by [65, Lemma 1], $\lambda_1(\Omega) > \lambda_1(\Omega')$. If $\Omega \neq \mathrm{int}(M)$, then there exists a bounded domain $\Omega' \subset \mathrm{int}(M)$ which strictly contains Ω, i.e., $\Omega \subsetneq \Omega' \subset$ M, then combining with the initial statement, we have $\lambda_1(\Omega) > \lambda_1(\Omega') \geq 0$. Therefore, since $\Omega \subset \mathrm{int}(M)$, $\lambda_1(\overline{\Delta} - q, \Omega) > 0$. □

The following result discusses the signal of such a solution; see [10, Theorem 2.3] for a discussion in the sub-static case. A slight change in the proof of this result gives the following existence result.

Lemma 3.1.14 *Let (M, g, V) be a V-static triple, $q = \frac{1}{V}\Big(\Delta V + \frac{n}{n-1}\Big)$ and $\Omega \subset M$ be a bounded domain with smooth boundary such that $\partial\Omega = \Sigma \cup \bigcup_{l=1}^{\tau} N_l$ and satisfies the condition $(H1)$ and $(H2)$ of Definition 3.1.11. Then, there exists only one smooth solution f to*

$$\begin{cases} \overline{\Delta} f - qf &= -1 \quad \text{in } \Omega \\ \quad f &= 0 \quad \text{on } \Sigma, \\ \quad f &= c_l \quad \text{on } N_l, \quad l = 1, \dots, \tau, \end{cases}$$

given $c_l > 0$ constants. Moreover, $f > 0$ in Ω and $\Sigma = \{f = 0\}$ is a regular level set of f. Furthermore, $\frac{\partial f}{\partial \nu}(x) < 0$ at any $x \in \Sigma$, and ν is unit normal to Σ in x pointing outside of Ω. In particular, $\nu = -\frac{\nabla f}{|\nabla f|}$.

Next, we choose some suitable constants c_l to treat with the Schrodinger operator described above. Let

$$\tilde{c}_l = \max\Big\{\frac{(\Delta V - \nabla^2 V(\nu, \nu))(z) + 1}{V(z)} \ ; \ z \in N_l\Big\}, \quad c_l = \begin{cases} \frac{n-1}{n\tilde{c}_l}, & \text{if } \tilde{c}_l \neq 0 \\ 0 & \text{if } \tilde{c}_l = 0. \end{cases} \tag{3.9}$$

By $(H3)$ of Definition 3.1.11 we have $c_l \geq 0$ for all $l = 1, \dots, \tau$.

Remark 3.1.15 *We observe if we have $c_l = 0$ in a non empty component N_l then $Ric(\nu,\nu) \geq 0$ in this component. With additional condiction (H3) of the Definition 3.1.11 we have $Ric(\nu,\nu) = 0$.*

Theorem 3.1.16 *Let (M^n, g, V) be an n-dimensional V-static Riemannian triple and consider $\Omega \subset M^n$ a bounded domain such that $\partial\Omega = \Sigma \bigcup \Big(\bigcup_{l=1}^{\tau} N_l \Big)$ satisfies $(H1) - (H3)$. Then, we have*

$$\Big(\int_\Omega \Big(V - \frac{n}{n-1} f \Big) \, dv - \sum_{l=1}^{\tau} c_l \int_{N_l} V_{,\nu_{N_l}} \, da \Big)^2 \leq \frac{n-1}{n} \Big[\int_\Omega \Big(V - \frac{n}{n-1} f \Big) \, dv - \sum_{l=1}^{\tau} c_l \int_{N_l} \Big(V_{,\nu_{N_l}} + \frac{n}{n-1} u \Big) \, da \Big] \int_\Sigma \frac{V}{H} \, da, \tag{3.10}$$

where f is a solution of the PDE

$$\begin{cases} \Delta f + \frac{R}{n-1} f &= -1 \quad in\ \Omega \\ f &= 0 \quad on\ \Sigma, \\ f &= c_l \quad on\ N_l, \quad l = 1, \ldots, \tau. \end{cases} \tag{3.11}$$

If the equality is verified in (3.10)*, then Σ is umbilic.*

Proof: Consider a solution of (3.11), where c_l are given by (3.9) and whose existence is verified by Lemma 3.1.14. We use the integral identity obtained in Corollary 3.1.9 applied such solution f.

First, by Cauchy-Schwarz inequality

$$\Big| \nabla^2 f - \frac{1}{V} (\nabla^2 V + \frac{1}{n-1} g) f \Big|^2 \geq \Big(\Delta f - \frac{1}{V} (\Delta V - \frac{n}{n-1}) f \Big)^2 \frac{1}{n} \tag{3.12}$$

and then,

$$\begin{aligned} &\int_\Omega V \Big[\Big(\Delta f - \frac{1}{V} (\Delta V + \frac{n}{n-1}) f \Big)^2 - \Big| \nabla^2 f - \frac{1}{V} (\nabla^2 V + \frac{1}{n-1} g) f \Big|^2 \Big] \, dv \\ \leq &\int_\Omega V \Big[\Big(\Delta f - \frac{1}{V} (\Delta V + \frac{n}{n-1}) f \Big)^2 - \frac{1}{n} \Big(\Delta f - \frac{1}{V} (\Delta V + \frac{n}{n-1}) f \Big)^2 \Big] \, dv \\ = &\int_\Omega V \Big[1 - \frac{1}{n} \Big] \, dv \\ = &\frac{n-1}{n} \int_\Omega V \, dv. \end{aligned} \tag{3.13}$$

Since $f = 0$ in Σ, $V = 0$ in N_l and $f = c_l$ in N_l, the restriction $z = f|_{\partial\Omega}$ is such that $\nabla z = 0$ on $\partial\Omega$. Then,

$$\int_{\partial\Omega} \Big[Vh(\overline{\nabla} z, \overline{\nabla} z) + 2Vu\overline{\Delta} z + VHu^2 + V_{,\nu} |\overline{\nabla} z|^2 + 2z\nabla^2 V(\overline{\nabla} z, \nu) \Big] \, da = \int_{\partial\Omega} VHu^2 \, da. \tag{3.14}$$

In addition, on the components N_l,

$$\begin{aligned} \overline{\Delta} V + 1 + HV_{,\nu} &= \Delta V + 1 - \nabla^2 V(\nu,\nu) \\ &= -VRic(\nu,\nu) \\ &= 0, \end{aligned}$$

what implies

$$\begin{aligned}
&\int_{\partial\Omega}\Big[2zu(\overline{\Delta}V+HV_{,\nu}+1)+z^2\frac{\nabla^2V-(\Delta V+1)g}{V}(\overline{\nabla}V,\nu)\Big]\,da\\
=\;&\int_{\Sigma}\Big[2zu(\overline{\Delta}V+HV_{,\nu}+1)+z^2\frac{\nabla^2V-(\Delta V+1)g}{V}(\nabla V,\nu)\Big]\,da\\
+\;&\int_{N_l}\Big[2zu(\overline{\Delta}V+HV_{,\nu}+1)+z^2\frac{\nabla^2V-(\Delta V+1)g}{V}(\nabla V,\nu)\Big]\,da\\
=\;&-c_l^2\int_{N_l}V_{,\nu_{N_l}}\frac{[(\Delta V+1)g-\nabla^2V](\nu_{N_l},\nu_{N_l})}{V}\,da\\
\leq\;&-c_l^2\int_{N_l}V_{,\nu_{N_l}}\max_{N_l}\frac{[(\Delta V+1)g-\nabla^2V](\nu_{N_l},\nu_{N_l})}{V}\,da\\
\leq\;&-\frac{n-1}{n}c_l\int_{N_l}V_{,\nu_{N_l}}\,da
\end{aligned}\tag{3.15}$$

since (3.11), the definition of c_l and the fact that $V_{,\nu_{N_l}}<0$, which follows from the fact that $V=0$ in each N_l while $V>0$ in Ω.

Since

$$f\Delta f+\frac{R}{n-1}f^2=-f,$$

we use integration by parts

$$\begin{aligned}
-\int_\Omega f\,dv &= \int_\Omega f\Delta f\,dv+\frac{R}{n-1}\int_\Omega f^2\,dv\\
&= \int_{\partial\Omega}fu\,da-\int_\Omega|\nabla f|^2\,dv+\frac{R}{n-1}\int_\Omega f^2\,dv\\
&= \int_\Sigma fu\,da+\sum_{l=1}^{\tau}\int_{N_l}fu\,da-\int_\Omega|\nabla f|^2\,dv+\frac{R}{n-1}\int_\Omega f^2\,dv\\
&= \sum_{l=1}^{\tau}c_l\int_{N_l}u\,da-\int_\Omega|\nabla f|^2\,dv+\frac{R}{n-1}\int_\Omega f^2\,dv\\
&= \sum_{l=1}^{\tau}c_l\int_{N_l}u\,da-\int_\Omega\Big[|\nabla f|^2+\frac{1}{V}\Big(\Delta V+\frac{n}{n-1}\Big)f^2\Big]\,dv,
\end{aligned}$$

and, then

$$\int_\Omega\Big[|\nabla f|^2+\frac{1}{V}\Big(\Delta V+\frac{n}{n-1}\Big)f^2\Big]\,dv=\int_\Omega f\,dv-\sum_{l=1}^{\tau}c_l\int_{N_l}u\,da.\tag{3.16}$$

Replacing (3.13), (3.14), (3.15) and (3.16) into Corollary 3.1.9, we have

$$\frac{n-1}{n}\Big[\int_\Omega\Big(V-\frac{n}{n-1}f\Big)\,dv-\sum_{l=1}^{\tau}c_l\int_{N_l}\Big(V_{,\nu_{N_l}}-\frac{n}{n-1}u\Big)\,da\Big]\geq\int_\Sigma VHu^2\,da.\tag{3.17}$$

Using integration by parts again and (3.11), we see thar

$$\begin{aligned}
-\int_\Omega V\,d\Omega &= \int_\Omega\Big(V\Delta f-f\Delta V-\frac{n}{n-1}f\Big)\,dv\\
&= \int_\Omega\Big(V\Delta f-f\Delta V\Big)\,dv-\frac{n}{n-1}\int_\Omega f\,dv\\
&= \int_\Sigma Vu\,da-\sum_{l=1}^{\tau}c_l\int_{N_l}V_{,\nu_{N_l}}\,da-\frac{n}{n-1}\int_\Omega f\,dv,
\end{aligned}$$

what implies

$$\int_{\Omega}\Big(V-\frac{n}{n-1}f\Big)\,dv-\sum_{l=1}^{\tau}c_l\int_{N_l}V_{,\nu_{N_l}}\,da \;=\; -\int_{\Sigma}Vu\,da. \tag{3.18}$$

Combining (3.17), (3.18) and using Hölder's inequality

$$\begin{aligned}&\Big(\int_{\Omega}\Big(V-\frac{n}{n-1}f\Big)\,dv-\sum_{l=1}^{\tau}c_l\int_{N_l}V_{,\nu_{N_l}}\,da\Big)^2=\Big(-\int_{\Sigma}Vu\,da\Big)^2\\ \leq\;&\int_{\Sigma}VHu^2\,da\int_{\Sigma}\frac{V}{H}\,da\\ \leq\;&\frac{n-1}{n}\Big[\int_{\Omega}\Big(V-\frac{n}{n-1}f\Big)\,dv-\sum_{l=1}^{\tau}c_l\int_{N_l}\Big(V_{,\nu_{N_l}}-\frac{n}{n-1}u\Big)\,da\Big]\int_{\Sigma}\frac{V}{H}\,da,\end{aligned}$$

which proves (3.10).

Equality in (3.10) implies that (3.12) is a equality, and then

$$\nabla^2 f-\frac{1}{V}(\nabla^2 V+\frac{1}{n-1}g)f=-\frac{1}{n}g. \tag{3.19}$$

Constraining (3.19) to (the level set) Σ and remembering that $f|_{\Sigma}=0$ we conclude that Σ is umbilic. □

The last result has an interesting application to the particular case in which the boundary of the domain is properly contained in int(M). In this case, $\partial\Omega=\Sigma$ and $N_l=\emptyset$.

Proof of Theorem 1.0.1:

Since Ω is properly contained in int(M), we have that $\partial\Omega=\Sigma$, this is, $\cup N_l=\emptyset$. Repeating the steps of the proof of Theorem 3.1.16, we have the integral inequality

$$\Big[\int_{\Omega}\Big(V-\frac{n}{n-1}f\Big)\,dv\Big]^2\leq\frac{n-1}{n}\Big[\int_{\Omega}\Big(V-\frac{n}{n-1}f\Big)\,dv\Big]\int_{\Sigma}\frac{V}{H}\,da. \tag{3.20}$$

Lemma 3.1.14 applied to (3.18) implies $\int_{\Omega}\Big(V-\frac{n}{n-1}f\Big)\,dv>0$. We simplify the relation (3.20) to conclude the inequality. Equality case is done as the early result.

□

3.1.4 Minkowski-type inequality

This section is devoted to proving a Minkowski-type inequality for domains in V-static manifolds. First, we will rearrange the boundary terms in the integral formula of Corollary 3.1.9 that deals with our purpose.

Corollary 3.1.17 *Let (M, Ω, V) and f as in Corollary 3.1.9. If Σ is a connected component of $\partial\Omega$ such that $V > 0$ then*

$$
\begin{aligned}
&\int_\Omega \Big[V\Big(\Delta f - \frac{1}{V}\Big(\Delta V + \frac{n}{n-1}\Big)f\Big)^2 - V\Big|\nabla^2 f - \frac{1}{V}\Big(\nabla^2 V + \frac{1}{n-1}g\Big)f\Big|^2\Big]\, dv \\
= &\int_{\partial\Omega} \Big[V\Big[h - \frac{V_{,\nu}}{V}g\Big]\Big(\overline{\nabla} z - \frac{\overline{\nabla} V}{V}z, \overline{\nabla} z - \frac{\overline{\nabla} V}{V}z\Big)\Big]\, da \\
+ &\int_{\partial\Omega} \Big[VH\Big(u - \frac{V_{,\nu}}{V}z\Big)^2 + 2V\Big(u - \frac{V_{,\nu}}{V}z\Big)\Big(\overline{\Delta} z - \frac{(\overline{\Delta} V + 1)}{V}z\Big) - \frac{z^2}{V}V_{,\nu}\Big]\, da \\
+ &\int_{\partial\Omega\setminus\Sigma} \Big[Vh(\overline{\nabla} z, \overline{\nabla} z) + 2Vu\overline{\Delta} z + VHu^2 + V_{,\nu}|\overline{\nabla} z|^2\Big]\, da \\
+ &\int_{\partial\Omega\setminus\Sigma} \Big[2z\nabla^2 V(\overline{\nabla} z, \nu) - 2zu(\overline{\Delta} V + 1 + HV_{,\nu})\Big]\, da \\
+ &\int_{\partial\Omega\setminus\Sigma} z^2 \frac{\Delta V + 1)g - \nabla^2 V}{V}(\nabla V, \nu)\, da \\
+ &\int_\Omega \Big(|\nabla f|^2 + \frac{1}{V}(\Delta V + \frac{n}{n-1})f^2\Big)\, dv.
\end{aligned}
$$

Proof: Using Gauss-Weingarten formula on $\partial\Omega$, this is

$$V_{,\alpha\nu} = \nabla_\alpha V_{,\nu} - h_{\alpha\beta} V_{,\beta},$$

we focus our attention only on the boundary integrals in Corollary 3.1.9. Since

$$(\overline{\Delta} V + 1)\overline{g}(\nu, \nu) + HV_{,\nu} = (\Delta V + 1)g(\nu, \nu) - \nabla^2 V(\nu, \nu) = -V Ric(\nu, \nu),$$

we have

$$
\begin{aligned}
&\int_\Sigma \Big(Vh(\overline{\nabla} z, \overline{\nabla} z) + 2Vu\overline{\Delta} z + VHu^2 + V_{,\nu}|\overline{\nabla} z|^2\Big)\, da \\
+ &\int_\Sigma \Big(2z\nabla^2 V(\overline{\nabla} z, \nu) - 2zu(\overline{\Delta} V + 1 + HV_{,\nu}) - z^2 \frac{\nabla^2 V - (\Delta V + 1)g}{V}(\nabla V, \nu)\Big)\, da \\
= &\int_\Sigma \Big(Vh(\overline{\nabla} z, \overline{\nabla} z) + 2Vu\overline{\Delta} z + VHu^2 + V_{,\nu}|\overline{\nabla} z|^2\Big)\, da \\
+ &\int_\Sigma \Big(2zu - z^2\frac{V_{,\nu}}{V}\Big)\big(-(\overline{\Delta} V + 1) - HV_{,\nu}\big)\, da \\
+ &\int_\Sigma \Big(2zz_\alpha - z^2\frac{V_\alpha}{V}\Big)(\overline{\nabla}_\alpha V_{,\nu} - h_{\alpha\beta}V_\beta)\, da
\end{aligned}
\tag{3.21}
$$

Furthermore, since $\partial\Sigma = \emptyset$, we use integration by parts to get

$$
\begin{aligned}
\int_\Sigma \Big(2zz_\alpha - z^2\frac{V_\alpha}{V}\Big)\overline{\nabla}_\alpha V_{,\nu}\, da &= \int_\Sigma \Big\langle 2z\overline{\nabla} z - \frac{z^2}{V}\overline{\nabla} V, \overline{\nabla} V_{,\nu}\Big\rangle\, da \\
&= \int_\Sigma \Big\langle \overline{\nabla} z^2, \overline{\nabla} V_{,\nu}\Big\rangle\, da - \int_\Sigma \Big\langle \frac{z^2}{V}\overline{\nabla} V, \overline{\nabla} V_{,\nu}\Big\rangle\, da \\
&= -\int_\Sigma V_{,\nu}\mathrm{div}(\overline{\nabla} z^2)\, da \\
&+ \int_\Sigma V_{,\nu}\mathrm{div}\Big(\frac{z^2}{V}\overline{\nabla} V\Big)\, da \\
&= -\int_\Sigma V_{,\nu} 2(z\overline{\Delta} z + |\overline{\nabla} z|^2)\, da \\
&+ \int_\Sigma \Big(\frac{z^2}{V}\overline{\Delta} V + \frac{2z}{V}\Big\langle \overline{\nabla} z, \overline{\nabla} V\Big\rangle - \frac{z^2}{V^2}|\overline{\nabla} V|^2\Big)V_\nu\, da
\end{aligned}
\tag{3.22}
$$

We replace (3.22) in (3.21), and complete squares in the above identity to get the result. In more detail, first we have

$$\begin{aligned}
& \int_\Sigma \Big(Vh(\overline{\nabla} z,\overline{\nabla} z)+2Vu\overline{\Delta} z+VHu^2+V_{,\nu}|\overline{\nabla} z|^2\Big)\,da \\
+ & \int_\Sigma \Big(z^2\frac{V_{,\nu}}{V}-2zu\Big)(\overline{\Delta} V+1+HV_{,\nu}\Big)\,da \\
+ & \int_\Sigma \Big(2zz_\alpha - z^2\frac{V_\alpha}{V}\Big)(\overline{\nabla}_\alpha V_{,\nu}-h_{\alpha\beta}V_\beta)\,da \\
= & \int_\Sigma \Big(Vh(\overline{\nabla} z,\overline{\nabla} z)+2Vu\overline{\Delta} z+VHu^2+V_{,\nu}|\overline{\nabla} z|^2\Big)\,da \\
+ & \int_\Sigma \Big(z^2\frac{V_{,\nu}}{V}-2zu\Big)\Big(\overline{\Delta} V+1+HV_{,\nu}\Big)\,da \\
- & \int_\Sigma h_{\alpha\beta}V_\beta\Big(2zz_\alpha - z^2\frac{V_\alpha}{V}\Big)\,da \\
+ & \int_\Sigma V_{,\nu}\Big(\frac{z^2}{V}\overline{\Delta} V+\frac{2z}{V}z_\alpha V_\alpha-\frac{z^2}{V^2}|\overline{\nabla} V|^2-2z\overline{\Delta} z-2|\overline{\nabla} z|^2\Big)\,da.
\end{aligned}$$

We notice that,

$$\begin{aligned}
Vh(\overline{\nabla} z,\overline{\nabla} z)-h_{\alpha\beta}V_\beta(2zz_\alpha-z^2\frac{V_\alpha}{V}) &= Vh(\overline{\nabla} z,\overline{\nabla} z)-2zh_{\alpha\beta}z_\alpha V_\beta+\frac{z^2}{V}h_{\alpha\beta}V_\alpha V_\beta \\
&= Vh(\overline{\nabla} z,\overline{\nabla} z)-2zh(\overline{\nabla} z,\overline{\nabla} V)+\frac{z^2}{V}h(\overline{\nabla} V,\overline{\nabla} V) \\
&= V\Big(h(\overline{\nabla} z,\overline{\nabla} z)-\frac{2z}{V}h(\overline{\nabla} z,\overline{\nabla} V)+\frac{z^2}{V^2}h(\overline{\nabla} V,\overline{\nabla} V)\Big) \\
&= V\Big(h(\overline{\nabla} z,\overline{\nabla} z)-2h(\overline{\nabla} z,\frac{z}{V}\overline{\nabla} V)+h(\frac{z}{V}\overline{\nabla} V,\frac{z}{V}\overline{\nabla} V)\Big) \\
&= Vh\Big(\overline{\nabla} z-\frac{z}{V}\overline{\nabla} V,\overline{\nabla} z-\frac{z}{V}\overline{\nabla} V\Big),
\end{aligned}$$

and

$$\begin{aligned}
V_{,\nu}\Big(-|\overline{\nabla} z|^2+\frac{2z}{V}z_\alpha V_\alpha-\frac{z^2}{V^2}|\overline{\nabla} V|^2\Big) &= V_{,\nu}\Big(-|\overline{\nabla} z|^2+\frac{2z}{V}\Big\langle\overline{\nabla} z,\overline{\nabla} V\Big\rangle-\frac{z^2}{V^2}|\overline{\nabla} V|^2\Big) \\
&= -V_{,\nu}g\Big(\overline{\nabla} z-\frac{z}{V}\overline{\nabla} V,\overline{\nabla} z-\frac{z}{V}\overline{\nabla} V\Big).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
& Vh(\overline{\nabla} z,\overline{\nabla} z)-h_{\alpha\beta}V_\beta\Big(2zz_\alpha-\frac{z^2}{V}V_\alpha\Big)+V_{,\nu}\Big(-|\overline{\nabla} z|^2+\frac{2z}{V}z_\alpha V_\alpha-\frac{z^2}{V^2}|\overline{\nabla} V|^2\Big) \\
= & Vh\Big(\overline{\nabla} z-\frac{z}{V}\overline{\nabla} V,\overline{\nabla} z-\frac{z}{V}\overline{\nabla} V\Big)-V_{,\nu}g\Big(\overline{\nabla} z-\frac{z}{V}\overline{\nabla} V,\overline{\nabla} z-\frac{z}{V}\overline{\nabla} V\Big) \\
= & V\Big(h-\frac{V_{,\nu}}{V}g\Big)\Big(\overline{\nabla} z-\frac{z}{V}\overline{\nabla} V,\overline{\nabla} z-\frac{z}{V}\overline{\nabla} V\Big). \qquad (3.23)
\end{aligned}$$

Since

$$\begin{aligned}
VHu^2+\Big(z^2\frac{V_{,\nu}}{V}-2zu\Big)HV_{,\nu} &= VH\Big(u^2+\frac{z^2}{V^2}V_{,\nu}^2-2\frac{z}{V}uV_{,\nu}\Big) \\
&= VH\Big(u-\frac{z}{V}V_{,\nu}\Big)^2,
\end{aligned}$$

and

$$\begin{aligned} & 2Vu\overline{\Delta}z + \Big(z^2\frac{V_{,\nu}}{V} - 2zu\Big)(\overline{\Delta}V + 1) - 2zV_{,\nu}\overline{\Delta}z + \frac{z^2}{V}V_{,\nu}\overline{\Delta}V \\ = \;& 2Vu\overline{\Delta}z - 2zu\overline{\Delta}V + 2\frac{z^2}{V}V_{,\nu}\overline{\Delta}V - 2zV_{,\nu}\overline{\Delta}z + \frac{z^2}{V}V_{,\nu} - 2zu \\ = \;& 2V\Big(u\overline{\Delta}z - \frac{z}{V}u\overline{\Delta}V + \frac{z^2}{V^2}V_{,\nu}\overline{\Delta}V - \frac{z}{V}V_{,\nu}\overline{\Delta}z\Big) + \frac{z^2}{V}V_{,\nu} - 2zu \\ = \;& 2V\Big(\overline{\Delta}z(u - \frac{z}{V}V_{,\nu}) - \frac{\overline{\Delta}V}{V}z(u - \frac{z}{V}V_{,\nu})\Big) + \frac{z^2}{V}V_{,\nu} - 2zu \\ = \;& 2V\Big(\overline{\Delta}z - \frac{\overline{\Delta}V}{V}z\Big)\Big(u - \frac{z}{V}V_{,\nu}\Big) + \frac{z^2}{V}V_{,\nu} - 2zu, \end{aligned}$$

we have

$$\begin{aligned} & VHu^2 + \Big(z^2\frac{V_{,\nu}}{V} - 2zu\Big)HV_{,\nu} + 2Vu\overline{\Delta}z + \Big(z^2\frac{V_{,\nu}}{V} - 2zu\Big)(\overline{\Delta}V + 1) - 2zV_{,\nu}\overline{\Delta}z + \frac{z^2}{V}V_{,\nu}\overline{\Delta}V \\ = \;& VH\Big(u - \frac{z}{V}V_{,\nu}\Big)^2 + 2V\Big(\overline{\Delta}z - \frac{\overline{\Delta}V}{V}z\Big)\Big(u - \frac{z}{V}V_{,\nu}\Big) + \frac{z^2}{V}V_{,\nu} - 2zu \\ = \;& VH\Big(u - \frac{z}{V}V_{,\nu}\Big)^2 + 2V\Big(\overline{\Delta}z - \frac{\overline{\Delta}V}{V}z\Big)\Big(u - \frac{z}{V}V_{,\nu}\Big) + \frac{2z^2}{V}V_{,\nu} - 2zu - \frac{z^2}{V}V_{,\nu} \\ = \;& VH\Big(u - \frac{z}{V}V_{,\nu}\Big)^2 + 2V\Big(\overline{\Delta}z - \frac{\overline{\Delta}V}{V}z\Big)\Big(u - \frac{z}{V}V_{,\nu}\Big) - \frac{z}{V}2\Big(u - \frac{z}{V}V_{,\nu}\Big) - \frac{z^2}{V}V_{,\nu} \\ = \;& VH\Big(u - \frac{z}{V}V_{,\nu}\Big)^2 + 2V\Big(\overline{\Delta}z - \frac{(\overline{\Delta}V + 1)}{V}z\Big)\Big(u - \frac{z}{V}V_{,\nu}\Big) - \frac{z^2}{V}V_{,\nu}. \end{aligned} \tag{3.24}$$

We get the desired result by integrating relations (3.23) and (3.24). □

We can demonstrate the main result of this section.

Proof of Theorem 1.0.2: Consider Neumann's problem

$$\begin{cases} \operatorname{div}(V^2\nabla w) = \Big(\frac{n}{n-1}f - V\Big) & \text{in } \Omega \\ V^2\nabla_\nu w = cV & \text{on } \partial\Omega, \end{cases} \tag{3.25}$$

where f is a smooth function on M and $c = \dfrac{\int_\Omega \Big(\frac{n}{n-1}f - V\Big)\,dv}{\int_\Sigma V\,da}$.

By direct calculation, it is easy to see that (3.25) is equivalent to

$$\begin{cases} \Delta f + \frac{R}{n-1}f = -1 & \text{in } \Omega \\ Vf_{,\nu} - V_{,\nu}f = cV & \text{on } \Sigma, \end{cases} \tag{3.26}$$

by the correspondence $f = wV$.

The existence and uniqueness (up to an additive αV) of the solution to (3.26) is due to the Fredholm alternative. For a version of Fredholm's alternative helpful in this situation, see, for example, [66, Theorem 5.1].

Now, we consider the integral formula of Corollary 3.1.17 applied to f solution of (3.26). First, by Cauchy-Schwarz inequality

$$\Big|\nabla^2 f - \frac{1}{V}\Big(\nabla^2 V + \frac{1}{n-1}g\Big)f\Big|^2 \geq \Big[\Delta f - \frac{1}{V}\Big(\Delta V + \frac{n}{n-1}\Big)\Big]^2\frac{1}{n} \tag{3.27}$$

what implies

$$\int_\Omega V\Big[\Big(\Delta f-\frac{1}{V}\Big(\Delta V+\frac{n}{n-1}\Big)f\Big)^2-\Big|\nabla^2 f-\frac{1}{V}\Big(\nabla^2 V+\frac{1}{n-1}g\Big)f\Big|^2\Big]\,dv\le\frac{n-1}{n}\int_\Omega V\,dv$$

Then, by Corollary 3.1.17, we have

$$\begin{aligned}
\frac{n-1}{n}\int_\Omega V\,dv &\ge \int_\Sigma\Big[V\Big[h-\frac{V_{,\nu}}{V}\overline{g}\Big]\Big(\overline{\nabla}z-\frac{\overline{\nabla}V}{V}z,\overline{\nabla}z-\frac{\overline{\nabla}V}{V}z\Big)\Big]\,da\\
&+\int_\Sigma\Big[VH\Big(u-\frac{V_{,\nu}}{V}z\Big)^2+2V\Big(u-\frac{V_{,\nu}}{V}z\Big)\Big(\overline{\Delta}z-\frac{(\overline{\Delta}V+1)}{V}z\Big)-\frac{z^2}{V}V_{,\nu}\Big]\,da\\
&+\int_\Omega\Big(|\nabla f|^2+\frac{1}{V}(\Delta V+\frac{n}{n-1})f^2\Big)\,dv\\
&\ge \int_\Omega\Big(|\nabla f|^2+\frac{1}{V}(\Delta V+\frac{n}{n-1})f^2\Big)\,dv \qquad (3.28)\\
&+\int_\Sigma\Big[VH\Big(u-\frac{V_{,\nu}}{V}z\Big)^2+2V\Big(u-\frac{V_{,\nu}}{V}z\Big)\Big(\overline{\Delta}z-\frac{(\overline{\Delta}V+1)}{V}z\Big)-\frac{z^2}{V}V_{,\nu}\Big]\,da\\
&\quad\text{(by hypothesis (1.7))}\\
&=\int_\Omega\Big(|\nabla f|^2+\frac{1}{V}(\Delta V+\frac{n}{n-1})f^2\Big)\,dv\\
&+\int_\Sigma VHc^2\,da+\int_\Sigma\Big[2c(V\overline{\Delta}z-z(\overline{\Delta}V+1))-\frac{z^2}{V}V_{,\nu}\Big]\,da\quad\text{(by (3.26))}\\
&=\int_\Omega\Big(|\nabla f|^2+\frac{1}{V}(\Delta V+\frac{n}{n-1})f^2\Big)\,dv+c^2\int_\Sigma VH\,da\\
&-2c\int_\Sigma z\,da-\int_\Sigma\frac{z^2}{V}V_{,\nu}\,da\quad\text{(by Divergence Theorem)}
\end{aligned}$$

Using a similar argument to one done in (3.16), we have

$$\int_\Omega\Big[|\nabla f|^2+\frac{1}{V}\Big(\Delta V+\frac{n}{n-1}\Big)f^2\Big]\,dv = \int_\Omega f\,dv+\int_\Sigma zu\,da.$$

Therefore

$$\begin{aligned}
\frac{n-1}{n}\int_\Omega V\,dv &\ge \int_\Omega f\,dv+\int_\Sigma zu\,da+c^2\int_\Sigma VH\,da\\
&-2c\int_\Sigma z\,da-\int_\Sigma\frac{z^2}{V}V_{,\nu}\,da\\
&=\int_\Omega f\,dv+c^2\int_\Sigma VH\,da-2c\int_\Sigma z\,da+\int_\Sigma\frac{z}{V}(Vf_{,\nu}-fV_{,\nu})\,da\\
&=\int_\Omega f\,dv+c^2\int_\Sigma VH\,da-2c\int_\Sigma z\,da+c\int_\Sigma z\,da\\
&=\int_\Omega f\,dv+c^2\int_\Sigma VH\,da-c\int_\Sigma z\,da.
\end{aligned}$$

So,

$$\begin{aligned}
\frac{n-1}{n}\int_\Omega\Big(V-\frac{n}{n-1}f\Big)\,dv &\ge c^2\int_\Sigma VH\,da-c\int_\Sigma z\,da\\
&=\frac{\Big(\int_\Omega\Big(\frac{n}{n-1}f-V\Big)\,dv\Big)^2}{\Big(\int_\Sigma V\,da\Big)^2}\int_\Sigma VH\,da-\frac{\int_\Omega\Big(\frac{n}{n-1}f-V\Big)\,dv}{\int_\Sigma V\,da}\int_\Sigma z\,da,
\end{aligned}$$

what implies

$$\begin{aligned}\frac{n-1}{n}\Big(\int_\Sigma V\,da\Big)^2\int_\Omega\Big(V-\frac{n}{n-1}f\Big)\,dv \quad &\geq \quad \Big(\int_\Omega\Big(V-\frac{n}{n-1}f\Big)\,dv\Big)^2\int_\Sigma VH\,da\\ &+ \quad \int_\Sigma V\,da\int_\Omega\Big(V-\frac{n}{n-1}f\Big)\,dv\int_\Sigma z\,da.\end{aligned} \tag{3.29}$$

To simplify this expression, we intend to control the signal of $\int_\Omega\Big(V-\frac{n}{n-1}f\Big)\,dv$. We use the minimum principle in this direction as in [10, Theorem 2.2]. Indeed, the function $w=\frac{f}{V}$, solution of (3.25), satisfies

$$\mathcal{L}w=\Delta w+\frac{2}{V}\langle\nabla w,\nabla V\rangle-\frac{1}{V}\Big(\frac{n}{n-1}\Big)w=-\frac{1}{V}<0,$$

where $h=-\frac{n}{(n-1)V}<0$. In particular, w is subject to a Hopf's principle in Ω (see, for example, [67, Theorem 3.71]) what implies

$$0>\frac{\partial w}{\partial\nu}(x)=\frac{c}{V},$$

for any $x\in\Sigma$. Since $V>0$ in $\overline{\Omega}$, it follows that $c<0$ and

$$\int_\Omega\Big(V-\frac{n}{n-1}f\Big)\,dv>0,$$

what permits simplify (3.29) and proves (1.8).

Treating the equality case is similar to one done in [14], and we make some comments here just to completeness. Supposing that there is a point in Σ at which inequality in (1.7) is strict, we consider the set $S:=\{x\in\Sigma\ ;\ (h_{\alpha\beta}-\frac{V_{,\nu}}{V}g_{\alpha\beta})(x)>0\}$ that is non-empty. This is clearly an open set and, for each $p\in S$, there exists a connected neighborhood $N_p\subset\Sigma$ of p such that $(h_{\alpha\beta}-\frac{V_{,\nu}}{V}g_{\alpha\beta})(x)>0$. Since, by equality in (3.28),

$$\Big(h-\frac{V_\nu}{V}g\Big)\Big(\nabla\Big(\frac{f}{V}\Big),\nabla\Big(\frac{f}{V}\Big)\Big)=0\quad\text{in}\quad\Sigma,$$

from which $f=\alpha V$ in N_p for some $\alpha\in\mathbb{R}$.

Furthermore, equality in (1.8) implies that (3.27) is a equality, and then,

$$f_{,ij}-\frac{1}{V}\Big(V_{,ij}+\frac{1}{n-1}g_{ij}\Big)f=\frac{1}{n}g_{ij}\text{ in }\Omega.$$

The function $\tilde{f}:=f-\alpha V$ satisfies

$$\begin{aligned}\tilde{f}_{,ij}-\frac{1}{V}\Big(V_{,ij}+\frac{1}{n-1}g_{ij}\Big)\tilde{f} \quad &= \quad \Big(f-\alpha V\Big)_{,ij}-\frac{1}{V}\Big(V_{,ij}+\frac{1}{n-1}g_{ij}\Big)(f-\alpha V)\\ &= \quad f_{,ij}-\alpha V_{,ij}-\frac{1}{V}\Big(V_{,ij}+\frac{1}{n-1}g_{ij}\Big)f+\alpha\Big(V_{,ij}+\frac{1}{n-1}g_{ij}\Big)\\ &= \quad f_{,ij}-\frac{1}{V}(V_{,ij}+\frac{1}{n-1}g_{ij})f+\frac{\alpha}{n-1}g_{ij}\\ &= \quad \frac{1}{n}g_{ij}+\frac{\alpha}{n-1}g_{ij}\\ &= \quad \frac{n(\alpha+1)-1}{n(n-1)}g_{ij}.\end{aligned}$$

in Ω.

Then $\tilde{f}_{,ij} = \lambda g_{ij}$ on N_p, where $\lambda = \frac{n(\alpha+1)-1}{n(n-1)}$. Since $\tilde{f} = 0$ on N_p, we conclude $h_{\alpha\beta} = -\frac{\lambda}{|\nabla f|} g_{\alpha\beta}$ in N_p. Therefore, S is closed by connectedness $S = \Sigma$. By the same argument applied in Σ, we can prove that Σ is umbilic of constant mean curvature.

□

Remark 3.1.18 *Considering one additional boundary condition that becomes the last one in an overdetermined problem (strongly related to Serrin's problem, see [32]),*

$$\begin{cases} \Delta f + \frac{R}{n-1} f & = -1 \quad in\ \Omega \\ f_{,\nu} & = c \quad on\ \Sigma \\ f & = 0 \quad on\ \Sigma \end{cases}$$

we obtain the following weighted inequality

$$\frac{n-1}{n}\Big(\int_\Sigma V\, da\Big)^2 \geq \int_\Omega \Big(V - \frac{n}{n-1} f\Big)\, dv \int_\Sigma VH\, da, \tag{3.30}$$

and when the equality holding in (3.30)*, then* Σ *is umbilical.*

3.2 m-quasi Einstein case

This section focuses on the study of m-quasi-Einstein manifolds, a generalization of Einstein metrics incorporating a potential function V, an additional parameter $m > 0$, and a scalar constant λ. We now remember this concept.

Definition 3.2.1 *Let* (M^n, g) *be an* $n-$*dimensional* $(n \geq 3)$*, complete and connected Riemannian manifold. We say that* g *is a m-quasi-Einstein metric if there is a smooth (nontrivial) solution* V *on* M^n *satisfying the overdetermined system*

$$m\nabla^2 V - V Ric + \lambda V g = 0 \quad in\ M \tag{3.31}$$
$$V > 0 \quad on\ int(M)$$

In the case where the manifold M *has a (possibly disconnected) boundary* ∂M*, the additional condition* $V = 0$ *on* ∂M *is considered. Here,* $0 < m < \infty$ *is a integer constant and* λ *is a smooth real function on* M*. We call* V *of m-quasi-Einstein potential and the triple* (M^n, g, V) *of a m-quasi-Einstein triple.*

This framework arises naturally in mathematical physics, particularly in the context of warped product structures and static spacetimes.

Here we derive integral inequalities and rigidity results using a generalized Reilly-type identity, emphasizing the connection between analytic properties and geometric structures. The main contribution, Theorem 1.0.5, establishes constraints on curvature and boundary geometry, leading to rigidity in certain cases. These results, rooted in the work of [26], illustrate how m-quasi-Einstein metrics extend classical results and offer new insights into geometric analysis.

3.2.1 An integral inequality

The results presented here are part of the [26] and it's focuses on generalizing an integral inequality previously established for static manifolds by [25] to the context of m-quasi-Einstein manifolds. It also examines the rigidity of these manifolds under equality conditions.

Integral identities, such as Reilly's identity, play a crucial role in geometric analysis by providing tools to derive inequalities and rigidity results. Our technique extend these methods to the broader class of m-quasi-Einstein metrics, which are closely related to warped product Einstein metrics.

The main result here is an integral inequality for a compact domain Ω within an m-quasi-Einstein manifold (M^n, g, V). Assuming certain conditions on the Ricci curvature and boundary properties, the inequality links the geometry of Ω to its boundary. Specifically, for a smooth function η defined on Ω, the inequality involves terms like the mean curvature H, the second fundamental form h, and the gradient $\nabla\eta$. It also incorporates the Ricci curvature and a non-positive constant k.

In the equality case, the inequality provides a rigidity characterization of the manifold. Two scenarios emerge: if $k = 0$, the boundary function η corresponds to a harmonic function, while for $k < 0$, the manifold is Einstein, with $Ric = (n-1)kg$, and η corresponds to a function satisfying a specific elliptic equation. In both cases, the geometry of the manifold is determined, with precise characterizations of its isometry to warped product spaces or hyperbolic spaces depending on whether the boundary is empty.

Our contribution related to this theme and presented here, see also [26], extends prior results on static manifolds and provides new insights into m-quasi-Einstein manifolds, which are important in the study of Einstein metrics and mathematical physics. Applications include understanding stability in quasi-local energy, such as Wang–Yau energy, and contributing to the classification of Einstein and quasi-Einstein manifolds. The results demonstrate the strong interplay between analytic properties (integral identities) and geometric structures (curvature and boundary conditions).

In the same direction as said for static metrics, one crucial motivation to approach the m-quasi-Einstein metrics is studying Einstein manifolds that have a structure of warped product. In fact, if $m > 1$ is an integer, (M^n, g, V) is an m-quasi-Einstein triple if and only if there is a smooth $(n+m)$-dimensional warped product Einstein metric having M as the base space, see [27] and [28]. From this observation, the study of warped product Einstein manifolds reduces to the study of the m-quasi-Einstein equation on the lower-dimensional base space.

In [27] the authors treat the non-empty boundary case and consider, for the case $m > 1$, the following quantity

$$\rho(x) \quad = \quad \frac{1}{m-1}[(n-1)\lambda - R], \tag{3.32}$$

where R is the scalar curvature of (M^n, g).

Next, inspired by the aforementioned works, we carry out the same scope of ideas and applications of general Reilly's Identity as a way to obtain new results for the case of m-quasi Einstein manifolds. Along the proof, we use the following version of Reilly's Identity due Qiu and Xia, see [13, Theorem 1.1] and Theorem 3.1.7.

From now on, we aim to demonstrate Theorem 1.0.5 by analysing the integrals mention above.

Proof of Theorem 1.0.5: First, by considering a domain Ω contained in an m-quasi-Einstein manifold, we introduce the following 2-tensor

$$P = Ric - \rho g. \tag{3.33}$$

From (3.31), (3.32) and (3.33),

$$\begin{aligned}
\nabla^2 V - \Delta V g - 2(n-1)kVg + VRic &= [\nabla^2 V - \Delta V g - VRic] + 2V[Ric - (n-1)kg] \\
&= \frac{V}{m}\Big((1-m)Ric - (\lambda + R - n\lambda)g\Big) + 2V[Ric - (n-1)kg] \\
&= (1-m)\frac{V}{m}\Big(Ric + \frac{R - \lambda(n-1)}{m-1}g\Big) + 2V[Ric - (n-1)kg] \\
&= (1-m)\frac{V}{m}\Big(Ric - \rho g\Big) + 2V[Ric - (n-1)kg] \\
&= -(m-1)\frac{V}{m}P + 2V[Ric - (n-1)kg]. \qquad (3.34)
\end{aligned}$$

Furthermore, a straightforward calculation shows that

$$\begin{aligned}
(n-1)k(\Delta V + nkV) &= (n-1)k\Big(\frac{V}{m}(R - n\lambda) + nkV\Big) \\
&= (n-1)k\frac{V}{m}(R - n\lambda + mnk) \\
&= (n-1)k\frac{V}{m}[R - n(\lambda - mk)]. \qquad (3.35)
\end{aligned}$$

Now, since $k \leq 0$, given η a non-trivial function on Σ, the problem

$$\begin{cases} \Delta f + nkf = 0 & \text{in } \Omega \\ f = \eta & \text{on } \Sigma \end{cases}$$

has a unique solution.

Thus, from (3.1.7), (3.34) and (3.35)

$$\begin{aligned}
&\int_\Omega V\Big[(\Delta f + Knf)^2 - |\nabla^2 f + Kfg|^2\Big]\, dv \\
= &\int_\Omega \Big(-(m-1)\frac{V}{m}P + 2V[Ric - (n-1)kg]\Big)(\nabla f, \nabla f)\, dv \\
+ &\ (n-1)k\int_\Omega \frac{V}{m}[R - n(\lambda - mk)]f^2\, dv + \int_\Sigma \nabla_\nu V\Big[|\overline{\nabla} z|^2 - (n-1)Kz^2\Big]\, da \\
+ &\int_\Sigma V\Big[2f_{,\nu}\overline{\Delta} z + H(f_{,\nu})^2 + h(\overline{\nabla} z, \overline{\nabla} z) + 2(n-1)Kf_{,\nu}z\Big]\, da.
\end{aligned}$$

Since, by hypothesis, $P \leq 0$ and $Ric \geq (n-1)kg$, i.e.,

$$(n-1)kg \leq Ric \leq \rho g = \frac{1}{m-1}\Big((n-1)\lambda - R\Big)g,$$

from which follows

$$n(n-1)k \leq R \leq \frac{n(n-1)}{m+n-1}\lambda,$$

and then $\lambda \geq (m+n-1)k$. We also have, $R - n(\lambda - mk) \leq 0$.

Therefore

$$\begin{aligned} &- \int_\Sigma V\Big[2z_{,\nu}\overline{\Delta}z + H(z_{,\nu})^2 + h(\overline{\nabla}z, \overline{\nabla}z) + 2(n-1)kz_{,\nu}z\Big]\, da \\ &\geq \int_\Omega V|\nabla^2 f + kfg|^2\, dv + \int_\Sigma \nabla_\nu V\Big[|\overline{\nabla}z|^2 - (n-1)kz^2\Big]\, da. \end{aligned} \tag{3.36}$$

Since by hypothesis $H > 0$, we have

$$V\Big[\sqrt{H}z_{,\nu} + \frac{\overline{\Delta}z + (n-1)kz}{\sqrt{H}}\Big]^2 = V\Big[H(z_{,\nu})^2 + 2z_{,\nu}\overline{\Delta}z + 2(n-1)kzz_{,\nu} + \frac{(\overline{\Delta}z + (n-1)kz)^2}{H}\Big]$$

and then

$$\begin{aligned} &V\Big[\frac{(\overline{\Delta}z + (n-1)kz)^2}{H} - h(\overline{\nabla}f, \overline{\nabla}f)\Big] \\ &= V\Big[\sqrt{H}z_{,\nu} + \frac{\overline{\Delta}z + (n-1)kz}{\sqrt{H}}\Big]^2 - V\Big[2z_{,\nu}\overline{\Delta}z + H(z_{,\nu})^2 + h(\overline{\nabla}z, \overline{\nabla}z) + 2(n-1)kz_{,\nu}z\Big]. \end{aligned} \tag{3.37}$$

Therefore, using (3.37), we can rewrite the relation (3.36) as follows

$$\begin{aligned} &\int_\Sigma V\Big[\frac{(\overline{\Delta}z + (n-1)kz)^2}{H} - h(\overline{\nabla}f, \overline{\nabla}f)\Big]\, da \\ &= \int_\Sigma V\Big[\sqrt{H}z_{,\nu} + \frac{\overline{\Delta}z + (n-1)kz}{\sqrt{H}}\Big]^2\, da \\ &- \int_\Sigma V\Big[2f_{,\nu}\overline{\Delta}z + H(z_{,\nu})^2 + h(\overline{\nabla}z, \overline{\nabla}z) + 2(n-1)kz_{,\nu}z\Big]\, da \\ &\geq \int_\Omega V|\nabla^2 f + kfg|^2\, dv + \int_\Sigma \nabla_\nu V\Big[|\overline{\nabla}z|^2 - (n-1)kz^2\Big]\, da \end{aligned}$$

and then

$$\int_\Sigma V\Big[\frac{(\overline{\Delta}z + (n-1)kz)^2}{H} - h(\overline{\nabla}f, \overline{\nabla}f)\Big]\, da \geq \int_\Sigma \overline{\nabla}_\nu V\Big[|\overline{\nabla}z|^2 - (n-1)kz^2\Big]\, da$$

Now, let us assume that the equality holds in (1.10). Thus,

$$\begin{aligned} &k[R - n(\lambda - mk)] = 0 \\ &\nabla^2 f + kfg = 0 \\ &Hz_{,\nu} + \overline{\Delta}z + (n-1)kz = 0 \end{aligned} \tag{3.38}$$

From the relation (3.38) we have $k = 0$ or $R - n(\lambda - mk) = 0$.

If $k = 0$ we have $\nabla^2 f = 0$ and the conclusion follows by [68, Theorem B].

In the later case, since $R = n(\lambda - mk)$ and $R \leq \frac{n(n-1)}{m+n-1}\lambda$, we have $\lambda = (m+n-1)k$ and it follows that $R = n(n-1)k$. Since $Ric \geq (n-1)kg$, we have $Ric = (n-1)kg$ and (Ω, g) is Einstein. Since g is analytic, see [27, Proposition 2.4], we conclude that (M, g, V) is Einstein.

In addition, since $\lambda < 0$, we use [27, Proposition 3.1] and [69, Proposition 4.2] to conclude that

- If ∂M is not empty then M is isometric to

$$\Big([0,\infty)\times N, dt^2+\sqrt{-k}\cosh^2(\sqrt{k}t)g_{\mathbb{S}^{n-1}}, C\sinh\sqrt{-k}t\Big)$$

  where N is is an Einstein metric with negative Ricci curvature, and C is an arbitrary positive constant.

- If ∂M is empty then M is isometric to either

$$\Big(\mathbb{H}^n, dt^2+\sqrt{-k}\sinh^2(\sqrt{-k}t)g_{\mathbb{S}^{n-1}}, C\cosh(\sqrt{-k}t)\Big)$$

  or

$$\Big(\mathbb{R}\times F, dt^2+e^{2\sqrt{-k}t}g_F, Ce^{2\sqrt{-k}t}\Big)$$

  where F is Ricci flat and C is an arbitrary positive constant.

This concludes the proof. $\square$

4 Overdetermined problems in weighted manifolds

As presented in the introduction 1, the classical Serrin's problem (1.11) has important connections in different contexts. Next we explores Serrin's problem within the context of weighted Riemannian manifolds, focusing on generalizations of classical symmetry and rigidity results.

Utilizing tools such as the weighted Laplacian and weighted mean curvature, we establish rigidity results in generalized cones, showing that under specific curvature conditions, solutions to overdetermined problems exhibit unique symmetry properties. we derive a weighted version of the Heintze-Karcher inequality and establish the Soap Bubble Theorem, demonstrating that solutions are symmetric under specific curvature conditions. Moreover, we develop Pohozaev-type identities for weighted manifolds, which serve as critical analytical tools in this framework. In this context we findings generalize well-known geometric and analytical results to weighted spaces, providing new insights into symmetry and rigidity in non-Euclidean settings. The chapter ends with a study of the weighted Serrin's problem for convex cones of the Euclidean space. The results presented here can be found in [58] and [59].

4.0.1 Heintze-Karcher type inequality and Soap Bubble Theorem

In this section, we consider Σ^{n-1} as a hypersurface immersed in a weighted Riemannian manifold $(M^n, g, w\,dv)$, where ∇ and Δ denote the gradient and the laplacian with respect to the metric of M^n, and $\overline{\nabla}$ and $\overline{\Delta}$ indicate the gradient and the Laplacian of the induced metric $\overline{g}$ on Σ. Let $\Omega \subseteq M$ be a domain such that $\partial\Omega = \Sigma$. Furthermore, ν is the unit outward normal with respect to Σ, and $f_\nu = g(\nabla f, \nu)$. Also, let $h(X,Y) = g(\nabla_X \nu, Y)$ and $H = \mathrm{tr}_g h$ be the second fundamental form and the mean curvature (with respect to $-\nu$) of Σ, respectively.

The *weighted volume* of a bounded domain $\Omega \subset M$ and the *weighted area* of a smooth hypersurface $\Sigma \subset M$ are defined as

$$\mathrm{Vol}_w(\Omega) := \int_\Omega dv_w = \int_\Omega w\,dv, \qquad \mathrm{Area}_w(\Sigma) := \int_\Sigma da_w = \int_\Sigma w\,da,$$

where dv and da represent the Riemannian volume and area elements, respectively.

Associated to a weighted manifold (M, g, dv_w) there is a natural divergence form second-order diffusion operator: the weighted Laplacian (or w-Laplacian). This operator is defined by

$$\Delta_w u = \Delta u + \langle \nabla \log w, \nabla u \rangle\,, \tag{4.1}$$

where $u \in C^2(M)$. The α-Bakry-Émery-Ricci tensor is given by

$$Ric_w^\alpha = Ric - \nabla^2 \log w - \frac{1}{\alpha} d\log w \otimes d\log w,$$

where $\alpha \in \mathbb{R} \setminus \{0\}$ and Ric is the Ricci tensor of $(M, \langle\ ,\ \rangle)$.

In this setting, we define a notion of mean curvature in the weighted context as follows:

$$H_w = H + \langle \nu, \nabla \log w \rangle. \tag{4.2}$$

This is known as the weighted mean curvature.

Similar to the standard case, there is a relationship between the intrinsic and extrinsic drift Laplacians, as stated in the following result, which we include here for the sake of completeness.

Lemma 4.0.1 *Let Σ^{n-1} be an hypersurface immersed in M^n and $f \in C^\infty(M)$ a smooth function. Then*

$$\Delta_w f = \overline{\Delta}_w f + H_w f_\nu + \nabla^2 f(\nu, \nu),$$

where w is a smooth positive function.

Proof: First, we have the classical formula

$$\Delta f = \overline{\Delta} f + H f_\nu + \nabla^2 f(\nu, \nu).$$

Now, we simply use the formulas for the weighted Laplacian (4.1) and the weighted mean curvature (4.2):

$$\begin{aligned} \Delta_w f &= \Delta f + \langle \nabla \log w, \overline{\nabla} f + f_\nu \nu \rangle \\ &= \overline{\Delta} f + H f_\nu + \nabla^2 f(\nu, \nu) + \langle \overline{\nabla} \log w, \overline{\nabla} f \rangle + f_\nu \langle \nabla \log w, \nu \rangle \\ &= \overline{\Delta}_w f + H_w f_\nu + \nabla^2 f(\nu, \nu). \end{aligned}$$

□

Additionally, a combination of the classical Bochner identity, the Cauchy-Schwarz inequality, and the Bergström inequality yields the following Bochner-type inequality for such weighted operators, as stated in [70, Appendix A] (see [71, Section 1.5] for an elementary proof of this result).

Lemma 4.0.2 *Let $(M^n, g, w\, dvol)$ be a Riemanninan manifold $u \in C^\infty(M)$, $\alpha > 0$ is real and w a smooth positive function. Then*

$$\frac{1}{2}\Delta_w |\nabla u|^2 \geq \frac{(\Delta_w u)^2}{n+\alpha} + \langle \nabla u, \nabla \Delta_w u \rangle + Ric_w^\alpha(\nabla u, \nabla u)$$

where equality holds if and only if

$$\begin{aligned} \nabla^2 u &= \frac{\Delta u}{n} g \\ \frac{\Delta u}{n} &= \frac{\langle \nabla \log w, \nabla u \rangle}{\alpha} = \frac{\Delta_w u}{n+\alpha}. \end{aligned}$$

The next result provides a useful Reilly's type theorem for weighted manifolds (see [2] for the classical case). Let us denote dv_w and da_w the weighted measures related with M and its boundary ∂M, respectively.

Proposition 4.0.3 *Let $(M, g, w\, dvol)$ be an n-dimensional compact Riemannian manifold with boundary ∂M, $u \in C^\infty(M)$, $\alpha > 0$ and w a smooth positive function. Then*

$$\Big(\frac{n+\alpha-1}{n+\alpha}\Big)\int_M (\Delta_w u)^2\, dv_w \;\geq\; \int_M Ric_w^\alpha(\nabla u, \nabla u)\, dv_w + \int_{\partial M}\Big[2u_\nu \overline{\Delta}_w z + H_w u_\nu^2 + h(\overline{\nabla} z, \overline{\nabla} z)\Big]\, da_w, \tag{4.3}$$

where $z = f|_{\partial\Omega}$ and ν is the unit outward normal vector field to ∂M. Furthermore the equality holds in (4.3) *if and only if*

$$\nabla^2 u = \frac{\Delta u}{n} g, \quad and$$
$$\frac{\Delta u}{n} = \frac{\langle \nabla \log w, \nabla u\rangle}{\alpha} = \frac{\Delta_w u}{n+\alpha}.$$

Proof: Integrating the inequality of Lemma 4.0.2 we have

$$\frac{1}{2}\int_M \Delta_w |\nabla u|^2\, dv_w \geq \int_M \Big[\frac{(\Delta_w u)^2}{n+\alpha} + \langle \nabla u, \nabla \Delta_w u\rangle + Ric_w^\alpha(\nabla u, \nabla u)\Big]\, dv_w. \tag{4.4}$$

Using the divergence theorem

$$\frac{1}{2}\int_M \Delta_w |\nabla u|^2\, dv_w = \frac{1}{2}\int_{\partial M}\Big\langle \nabla |\nabla u|^2, \nu\Big\rangle\, da_w = \int_{\partial M} \nabla^2 u(\nu, \nabla u)\, da_w. \tag{4.5}$$

$$\langle \nabla(\Delta_w u), \nabla u\rangle = div_w(\Delta_w u \nabla u) - (\Delta_w u)^2,$$

we have

$$\int_M \langle \nabla(\Delta_w u), \nabla u\rangle\, dv_w = \int_{\partial M} u_\nu \Delta_w u\, da_w - \int_M (\Delta_w u)^2\, dv_w. \tag{4.6}$$

By replacing (4.5) and (4.6) in (4.4) follows

$$\Big(\frac{n+\alpha-1}{n+\alpha}\Big)\int_M (\Delta_w u)^2\, dv_w \geq \int_M Ric_w^\alpha(\nabla u, \nabla u)\, dv_w + \int_{\partial M}\Big[u_\nu \Delta_w u - \nabla^2 u(\nu, \nabla u)\Big]\, da_w. \tag{4.7}$$

Applying Lemma 4.0.1, in the immersion $\partial M \hookrightarrow M$, we have

$$u_\nu(\Delta_w u) = u_\nu\Big(\overline{\Delta}_w u + H_w u_\nu + \nabla^2 u(\nu, \nu)\Big). \tag{4.8}$$

In addition, since $\nabla u = \overline{\nabla} u + u_\nu \nu$, we conclude

$$\begin{aligned}\nabla^2 u(\nu, \nabla u) &= \nabla^2 u(\nu, \overline{\nabla} u + u_\nu \nu)\\ &= \Big\langle \nabla_{\overline{\nabla} u}(\overline{\nabla} u + u_\nu \nu), \nu\Big\rangle + u_\nu \nabla^2 u(\nu, \nu)\\ &= -h(\overline{\nabla} u, \overline{\nabla} u) + \Big\langle \overline{\nabla} u, \overline{\nabla} u_\nu\Big\rangle + u_\nu \langle \nabla_{\overline{\nabla} u}\nu, \nu\rangle + u_\nu \nabla^2 u(\nu, \nu)\\ &= -h(\overline{\nabla} u, \overline{\nabla} u) + \Big\langle \overline{\nabla} u, \overline{\nabla} u_\nu\Big\rangle + u_\nu \nabla^2 u(\nu, \nu)\end{aligned} \tag{4.9}$$

where in the last equalities we use the definition of the second fundamental form and that $\langle \nu, \nu\rangle = 1$. From (4.8) and (4.9) we have

$$\begin{aligned}&u_\nu \Delta_w u - \nabla^2 u(\nu, \nabla u)\\ =\;& 2u_\nu \overline{\Delta}_w u - \overline{div}_w(u_\nu \overline{\nabla} u) + H_w u_\nu^2 + h(\overline{\nabla} u, \overline{\nabla} u).\end{aligned} \tag{4.10}$$

Replacing (4.10) in (4.7), using again the divergence theorem and since $\partial(\partial M) = \emptyset$ the result follows. The equality in (4.4) implies rigidity as established by Lemma 4.0.2. □

From now on, we consider a smooth function u as a solution to (PDE) (1.13). For such a class of functions, we can apply the Reilly's-type inequality obtained earlier to derive an inequality that characterizes rigidity for metric balls. Before presenting this result, for the convenience of the reader, we recall an important Obata-type result due to Farina-Roncoroni:

Lemma 4.0.4 *Let $(M^n, g, w\ dvol)$ be an n-dimensional Riemannian manifold (not necessarily complete) and let Ω be a domain in M such that $\overline{\Omega}$ is compact. Assume that there exists a function $u : \Omega \to \mathbb{R}$ such that $u \in C^0(\overline{\Omega}) \cup C^2(\Omega)$ and is a solution to*

$$\begin{cases} \nabla^2 u &= -\left(\frac{1}{n+\alpha} + ku\right) g & in \quad \Omega \\ u &> 0 & in \quad \Omega \\ u &= 0 & on \quad \partial\Omega \end{cases}$$

where $k \in \mathbb{R}$ and $\alpha > 0$ is real. Then Ω is a metric ball and u is a radial function, i.e. u depends only on the distance from the center of the ball.

The proof of the Lemma 4.0.4 is an adaptation of the argument presented in [72, Lemma 6] to the setting of weighted manifolds.

In this setting, we have the following result.

Proposition 4.0.5 *Let $(M, g, w\ dvol)$ be an n-dimensional Riemannian manifold such that w is a smooth positive function. Let $\Omega \subset M$ be a bounded domain, $k \in \mathbb{R}$, $\alpha > 0$ and $u \in C^\infty(\Omega)$ a solution of (1.13), then*

$$-\int_{\partial\Omega} u_\nu \left[H_w u_\nu + \left(\frac{n+\alpha-1}{n+\alpha}\right)\right] da_w \geq \int_\Omega \left(Ric_w^\alpha - k(n+\alpha-1)g\right)(\nabla u, \nabla u) dv_w$$

where equality holds if and only if Ω is a metric ball and u is a radial function.

Proof: Since $u = 0$ on $\partial\Omega$, then $\overline{\nabla} u = 0$ and $\overline{\Delta}_w u = 0$ on $\partial\Omega$.

Furthermore, since u is a solution of (1.13), we can use the divergence theorem to obtain

$$\begin{aligned} \int_\Omega (\Delta_w u)^2\, dv_w &= \int_\Omega \Delta_w u \Big[-1 - k(n+\alpha)u \Big]\, dv_w \\ &= -\int_\Omega \Delta_w u\, dv_w - k(n+\alpha) \int_\Omega u \Delta_w u\, dv_w \\ &= -\int_{\partial\Omega} u_\nu\, da_w - k(n+\alpha) \Big[\int_{\partial\Omega} u u_\nu\, da_w - \int_\Omega \langle \nabla u, \nabla u \rangle\, dv_w \Big] \\ &= -\int_{\partial\Omega} u_\nu\, da_w + k(n+\alpha) \int_\Omega \langle \nabla u, \nabla u \rangle\, dv_w, \end{aligned}$$

where in the last equality we use that $u = 0$ at the boundary. Thus, the result follows directly from Proposition 4.0.3 and Lemma 4.0.4. □

Corollary 4.0.6 *Let $(M, g, w\, dvol)$ be an n-dimensional Riemannian manifold and w a smooth positive function. Let $\Omega \subset M$ be a bounded domain and $u \in C^\infty(\Omega)$ a solution of the PDE* (1.13). *Assume that $H_w = \frac{1}{u_\nu} \cdot \frac{n+\alpha-1}{(n+\alpha)(n-1)}$, $Ric_w^\alpha \geq k(n+\alpha-1)g$, where $k \in \mathbb{R}$ and $\alpha > 0$. Then Ω is a metric ball and u is a radial function.*

Proof: Since $H_w = \frac{1}{u_\nu} \cdot \frac{n+\alpha-1}{(n+\alpha)(n-1)}$ and $Ric_w^\alpha \geq k(n+\alpha-1)g$, we obtain that equality holds in Proposition 4.0.5 and, therefore, we have the desired result. □

The next result is a Heintze-Karcher type inequality for the weighted mean curvature and Ric_w^α.

Theorem 4.0.7 *(Heintze-Karcher type inequality) Let $(M, g, w\, dvol)$ be an n-dimensional Riemannian manifold and w a smooth positive function. Let $\Omega \subset M$ be a bounded domain where weighted mean curvature H_w of $\partial\Omega$ is positive and $u \in C^\infty(\Omega)$ a solution of the PDE* (1.13). *If $Ric_w^\alpha \geq k(n+\alpha-1)g$, where $\alpha > 0$ and $k \in \mathbb{R}$, then*

$$\left(\frac{n+\alpha-1}{n+\alpha}\right)^2 \int_{\partial\Omega} \frac{1}{H_w}\, da_w \geq \frac{n+\alpha-1}{n+\alpha} Vol_w(\Omega) + (n+\alpha-1)k \int_\Omega u\, dv_w$$

where equality holds if and only if Ω is a metric ball and u is a radial function.

Proof: Since

$$-u_\nu \left[H_w u_\nu + \frac{n+\alpha-1}{n+\alpha}\right] = -\frac{1}{H_w}\left[H_w u_\nu + \frac{n+\alpha-1}{n+\alpha}\right]^2 + \left(\frac{n+\alpha-1}{n+\alpha}\right) u_\nu + \frac{1}{H_w}\left(\frac{n+\alpha-1}{n+\alpha}\right)^2 \quad (4.11)$$

and

$$\begin{aligned} \int_{\partial\Omega} u_\nu\, da_w &= \int_\Omega \Delta_w u\, dv_w \\ &= -Vol_w(\Omega) - (n+\alpha)k \int_\Omega u\, dv_w, \end{aligned} \quad (4.12)$$

we replace (4.11) and (4.12) in Proposition 4.0.5 to get

$$\begin{aligned} \left(\frac{n+\alpha-1}{n+\alpha}\right)^2 \int_{\partial\Omega} \frac{1}{H_w} da_w \;\geq\; & \int_\Omega \left(Ric_w^\alpha - k(n+\alpha-1)g\right)(\nabla u, \nabla u) dv_w \\ & + \int_{\partial\Omega} \frac{1}{H_w}\left[H_w u_\nu + \frac{n+\alpha-1}{n+\alpha}\right]^2 da_w \\ & + \left(\frac{n+\alpha-1}{n+\alpha}\right)\left(Vol_w(\Omega) + (n+\alpha)k \int_\Omega u\, dv_w\right). \end{aligned}$$

Since $Ric_w^\alpha \geq k(n+\alpha-1)g$ and $H_w > 0$, the result follows. The equality case follows again from Lemma 4.0.4. □

As an application of the last result, we can obtain an Alexandrov-type result for the weighted mean curvature in the space of solid cones. Following [73], we define a solid cone in $\mathbb{R}^n$ as the manifold

$$\mathcal{C} := \{tp; \quad t \geq 0, p \in \mathcal{D}\},$$

where $\mathcal{D}$ is a smooth region of the unit sphere $\mathbb{S}^{n-1}$. When $\mathcal{C}$ is endowed with a α-homogeneous density w (which means that $w(tp) = t^\alpha w(p)$, for any $t > 0$ and $p \in \mathcal{C} \setminus \{0\}$), it is possible to prove the following Minkowski's identity (see [73, Proposition 5.1]):

$$\int_\Sigma [(n+\alpha-1) - H_w\langle x, \nu\rangle] da_w = 0,$$

where Σ is a smooth, closed, orientable hypersurface embedded in $\mathcal{C} \setminus \{0\}$, ν is the exterior unit normal along the hypersurface, and x is the position vector in $\mathbb{R}^n$. A consequence of the same [73, Proposition 5.1] is the following: when H_w is constant and $\alpha \neq -n$,

$$(n+\alpha-1)Area_w(\Sigma) = (n+\alpha)H_w Vol_w(\Omega), \tag{4.13}$$

where Ω is the bounded domain such that $\partial\Omega = \Sigma$.

Remark 4.0.8 *We emphasize that [73, Proposition 5.1] is stated in a more general way, allowing the free-boundary case, where $\partial\Sigma = \Sigma \cap \partial M \neq \emptyset$, and Σ meets ∂M orthogonally at the points of $\partial\Sigma$. However, the divergence theorem used in that proof could be easily applied in our situation, where $\partial\Sigma = \emptyset$.*

Corollary 4.0.9 *Let $\mathcal{C} \subset \mathbb{R}^n$ be a solid cone endowed with a α-homogeneous density w, $\alpha > 0$. Suppose that $Ric_w^\alpha \geq 0$ in $\mathcal{C}$. If Σ is a smooth, compact, embedded, orientable hypersurface in $\mathcal{C} \setminus \{0\}$ with H_w constant, then Σ is a $(n-1)$-sphere.*

Proof: Let Ω be a bounded domain such that $\partial\Omega = \Sigma$. Let us consider a solution of the problem

$$\begin{cases} \Delta_w u &= -1 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

If H_w is constant, we can use (4.13) to see that

$$\left(\frac{n+\alpha-1}{n+\alpha}\right)\int_{\partial\Omega} \frac{1}{H_w}\, da_w = \left(\frac{n+\alpha-1}{n+\alpha}\right)\frac{1}{H_w} Area_w(\partial\Omega) = Vol_w(\Omega).$$

Since equality holds in Theorem 4.0.7, we conclude the result. □

For more general ambient spaces, we can prove a limitation for the weighted mean curvature in order to get the Soap Bubble type stated in Theorem 1.0.6.

Proof of Theorem 1.0.6:

First note that, since $H_0 = \frac{n+\alpha-1}{(n+\alpha)c}$,

$$\begin{aligned} \int_{\partial\Omega} H_w u_\nu^2\, da_w &= H_0 \int_{\partial\Omega} u_\nu^2\, da_w + \int_{\partial\Omega} (H_w - H_0)u_\nu^2\, da_w \\ &= H_0 \int_{\partial\Omega} (u_\nu + c)^2\, da_w - \frac{2(n+\alpha-1)}{n+\alpha}\int_{\partial\Omega} u_\nu\, da_w \\ &\quad - \frac{(n+\alpha-1)c}{n+\alpha} Area_w(\partial\Omega) + \int_{\partial\Omega} (H_w - H_0)u_\nu^2\, da_w. \end{aligned}$$

Then,

$$\begin{aligned} &-\int_{\partial\Omega} u_\nu \left[H_w u_\nu - \frac{n+\alpha-1}{n+\alpha}\right] da_w \\ &= -H_0 \int_{\partial\Omega} (u_\nu + c)^2 da_w + \frac{n+\alpha-1}{n+\alpha}\left[\int_{\partial\Omega} u_\nu\, da_w + cArea_w(\partial\Omega)\right] + \int_{\partial\Omega} (H_0 - H_w)u_\nu^2 da_w \end{aligned}$$

We observe that the constant c defined in (1.14) satisfies

$$c = -\frac{\int_{\partial\Omega} u_\nu \, da_w}{Area_w(\partial\Omega)}. \tag{4.14}$$

Then, we use Proposition 4.0.5 to conclude that

$$\int_{\partial\Omega}(H_0 - H_w)u_\nu^2 da_w \geq H_0 \int_{\partial\Omega}(u_\nu + c)^2 da_w + \int_\Omega (Ric_w^\alpha - k(n+\alpha-1)g)(\nabla u, \nabla u) dv_w.$$

Since $Ric_w^\alpha \geq k(n+\alpha-1)g$ we conclude the result. Again, the equality case is approached by using Lemma 4.0.4. □

Remark 4.0.10 *We note that, in comparison with [33], the results presented in this section may depend on u. For instance, the constant H_0 in Theorem 1.0.6 depends on c, which, in turn, depends on u. This dependency is expected in problems involving the genesis of curvature (see analogous results in the Riemannian setting in [30, Theorem 1, Theorem 4]).*

In any case, we highlight that the case $k = 0$, where there is no dependence on u, is particularly intriguing. Under the hypothesis $Ric_w^\alpha \geq 0$, an inequality similar to the one in Theorem 1.0.6 holds, and the constant H_0 is given explicitly by

$$H_0 = \left(\frac{n+\alpha-1}{n+\alpha}\right)\frac{\mathrm{Area}_w(\partial\Omega)}{\mathrm{Vol}_w(\Omega)}, \tag{4.15}$$

which is linked to the constant R in [33, Theorem 2.2]. Moreover, the above quantity is related to the well-known Cheeger constant; see [74].

Remark 4.0.11 *In the particular situation described in Corollary 4.0.9, if H_w is constant, formula* (4.13) *guarantees that $H_w = H_0$, see* (4.15)*. Consequently, an alternative proof for the Corollary 4.0.9 is given by Theorem 1.0.6. We mention that an approach to obtain a more general Alexandrov's theorem within this framework suggests the potential for analyzing appropriate Minkowski-type identities.*

4.0.2 A Pohozaev type identity for weighted manifolds

In this section, we provide a Pohozaev type identity addressed to weighted manifolds. We remember that a vector field $X \in \mathfrak{X}(M)$ is called *closed conformal* if

$$\nabla_Y X = \varphi Y, \quad \text{for all } Y \in \mathfrak{X}(M),$$

for some smooth function φ called the *conformal factor*. Firstly, we present some auxiliary lemmas as follows.

Lemma 4.0.12 *Let f be a smooth function on $(M, g, w\ dvol)$ where w is a smooth positive function. Then,*

$$\Delta_w \langle X, \nabla f\rangle = \langle \nabla\varphi, \nabla f\rangle (2-n) + 2\varphi\Delta_w f + \langle \nabla\Delta_w f, X\rangle - \nabla f \langle \nabla \log w, X\rangle$$

where X is a closed conformal vector field with conformal factor φ.

Proof: Firstly, since X is closed conformal vector field we conclude that

$$\Delta \langle X, \nabla f\rangle = \langle \nabla\varphi, \nabla f\rangle (2-n) + 2\varphi\Delta f + \langle \nabla\Delta f, X\rangle,$$

see [30, Lemma 2]. Thus, using the above equation and taking into account the definition of weighted Laplacian we have

$$\begin{aligned}
\Delta_w \langle X, \nabla f\rangle &= \Delta \langle X, \nabla f\rangle + \langle \nabla \log w, \nabla \langle X, \nabla f\rangle\rangle \\
&= \langle \nabla\varphi, \nabla f\rangle (2-n) + 2\varphi\Delta f + \langle \nabla\Delta f, X\rangle + \langle \nabla \log w, \varphi\nabla f + \nabla_X \nabla f\rangle . \\
&= \langle \nabla\varphi, \nabla f\rangle (2-n) + 2\varphi(\Delta_w f - \langle \nabla \log w, \nabla f\rangle) + \langle \nabla\Delta_w f, X\rangle \\
&- \langle \nabla_X \nabla \log w, \nabla f\rangle - \langle \nabla \log w, \nabla_X \nabla f\rangle + \langle \nabla \log w, \varphi\nabla f + \nabla_X \nabla f\rangle .
\end{aligned}$$

On another hand, since the vector field X is closed and conformal we get the following

$$\begin{aligned}
\varphi \langle \nabla \log w, \nabla f\rangle &= \langle \nabla \log w, \nabla_{\nabla f} X\rangle \\
&= \nabla f \langle \nabla \log w, X\rangle - \langle \nabla_{\nabla f} \nabla \log w, X\rangle \\
&= \nabla f \langle \nabla \log w, X\rangle - \langle \nabla_X \nabla \log w, \nabla f\rangle
\end{aligned}$$

From above equations we obtain the desired result. □

Now, we present a new version of the Pohozaev-type identity for weighted manifolds. This version is based on the following problem

$$\begin{cases}
\Delta_w u + k(n+\alpha)u &= -1 \quad \text{in} \quad \Omega \\
u &> 0 \quad \text{in} \quad \Omega \\
u &= 0 \quad \text{on} \quad \partial\Omega \\
|\nabla u| &= c \quad \text{on} \quad \partial\Omega
\end{cases} \tag{4.16}$$

where Ω is a bounded domain in $(M, g, dv = w\ dvol)$ and $\alpha > 0$ is constant.

Proposition 4.0.13 *Let $(M, g, dv_w = w\ dvol)$ be a Riemannian manifold where w is a smooth positive function, and $X \in \mathfrak{X}(M)$ be a closed conformal vector field with conformal factor φ. If u is solution of* (4.16), *then*

$$\begin{aligned}
2\int_\Omega \varphi u\ dv_w &= \int_\Omega (c^2 - u) div_w X\ dv_w + \int_\Omega u^2 \Big(\frac{n-2}{2}\Delta_w\varphi - 2k(n+\alpha)\varphi\Big)\ dv_w \\
&- \int_\Omega u\nabla u \langle \nabla \log w, X\rangle\ dv_w.
\end{aligned}$$

Proof: From Bochner-type identity in Lemma 4.0.12 and (4.16) we have

$$\Delta_w \langle X, \nabla u\rangle = \langle \nabla\varphi, \nabla u\rangle (2-n) + 2\varphi(-1 - k(n+\alpha)u) - k(n+\alpha)\langle \nabla u, X\rangle - \nabla u \langle \nabla \log w, X\rangle .$$

Thus, multiplying this identity by u and using again (4.16), we conclude that

$$\begin{aligned}
u\Delta_w \langle X, \nabla u\rangle - \langle X, \nabla u\rangle \Delta_w u &= u \langle \nabla\varphi, \nabla u\rangle (2-n) + 2\varphi u(-1 - k(n+\alpha)u) + \langle X, \nabla u\rangle \\
&- u\nabla u \langle \nabla \log w, X\rangle
\end{aligned} \tag{4.17}$$

Now, note that

$$u\Delta_w \langle X, \nabla u\rangle - \langle X, \nabla u\rangle \Delta_w u = div_w(u\nabla \langle X, \nabla u\rangle - \langle X, \nabla u\rangle \nabla u).$$

Since $u = 0$ along of the boundary and $\nu = -\frac{\nabla u}{|\nabla u|}$, we conclude from divergence theorem (see [73, Lemma 2.1])

$$\begin{aligned}
\int_\Omega \Big(u\Delta_w \langle X, \nabla u\rangle - \langle X, \nabla u\rangle \Delta_w u\Big)\, dv_w &= \int_\Omega div_w(u\nabla \langle X, \nabla u\rangle - \langle X, \nabla u\rangle \nabla u)\, dv_w \\
&= \int_{\partial\Omega} \langle u\nabla \langle X, \nabla u\rangle - \langle X, \nabla u\rangle \nabla u, \nu\rangle \; da_w \quad \text{(by div. theorem)} \\
&= c\int_{\partial\Omega} \langle X, \nabla u\rangle \; da_w \quad \text{(by PDE (4.16))} \\
&= -c^2 \int_{\partial\Omega} \langle X, \nu\rangle \; da_w \\
&= -c^2 \int_\Omega div_w X \; dv_w \quad \text{(by div. theorem)} \qquad (4.18)
\end{aligned}$$

We intend to study the integral of the right side of (4.17). First, since $u = 0$ on $\partial\Omega$ and $divX = n\varphi$, we have

$$\begin{aligned}
\int_\Omega \langle X, \nabla u\rangle \; dv_w &= \int_\Omega \Big(div_w(uX) - u div_w X\Big)\, dv_w \\
&= \int_{\partial\Omega} u\langle X, \nu\rangle \; da_w - \int_\Omega u div_w X \; dv_w \quad \text{(by div. theorem)} \\
&= -\int_\Omega u div_w X \; dv_w \qquad (4.19)
\end{aligned}$$

Again, since $u = 0$ on $\partial\Omega$, from the divergence theorem we guarantee that

$$0 = \int_\Omega div_w(u^2\nabla\varphi)\, dv_w = \int_\Omega u^2\Delta_w\varphi \; dv_w + 2\int_\Omega u\langle \nabla u, \nabla\varphi\rangle \; dv_w.$$

Thus,

$$\int_\Omega u\langle \nabla\varphi, \nabla u\rangle \; dv_w = -\frac{1}{2}\int_\Omega u^2\Delta_w\varphi \; dv_w. \qquad (4.20)$$

Integrating (4.17) and from (4.18),(4.19) and (4.20) we get the desired result. □

Remark 4.0.14 *A very important class where the last identity holds is the warped products with one-dimensional fiber. Recall that if $M = I\times_\rho N$ is a warped product, the vector field $X = \rho\partial_r$ is a closed conformal vector field with conformal factor $\varphi = \rho'$. Moreover, manifolds endowed with a nontrivial closed conformal vector field are locally isometric to a warped product with a one-dimensional factor (see [75, Section 3]). In the following section, we choose a specific warping function to describe a cone-type ambient where the rigidity for Serrin's problem will hold.*

4.0.3 A rigidity result for domains in solid cones

Firstly, let us introduce the following P-function

$$P(u) = |\nabla u|^2 + \frac{2}{n+\alpha}u + ku^2, \qquad (4.21)$$

where $u \in C^2(\Omega)$ and $\Omega \subset M$ is a bounded domain with smooth boundary. Motivated by [72], we can guarantee under geometric hypothesis that P-function is subharmonic in the weighted sense.

Lemma 4.0.15 *Let $(M^n, g, w\ dvol)$ be an $n-$dimensional Riemannian manifold such that $Ric_w^\alpha \geq k(n+\alpha-1)g$, where $k \in \mathbb{R}$, $\alpha > 0$ and w is a smooth positive function. Let $\Omega \subset M$ be a domain and let $u \in C^2(\Omega)$ be a solution to $\Delta_w u + k(n+\alpha)u = -1$ in Ω. Then $\Delta_w P(u) \geq 0$ in Ω. Moreover, $\Delta P(u) = 0$ if and only if $\nabla^2 u = -\left(\frac{1}{n+\alpha}+ku\right)g$ in Ω and $Ric_w^\alpha(\nabla u, \nabla u) = k(n+\alpha-1)|\nabla u|^2$ in Ω.*

Proof: Notice that,

$$\begin{aligned}
\Delta_w|\nabla u|^2 &\geq \frac{2}{n+\alpha}(\Delta_w u)^2 + 2\langle \nabla u, \nabla \Delta_w u\rangle + 2Ric_w^\alpha(\nabla u, \nabla u)\\
&\geq \frac{2}{n+\alpha}\Delta_w u(-1-k(n+\alpha)u) + 2\langle \nabla u, \nabla(-1-k(n+\alpha)u)\rangle + 2k(n+\alpha-1)|\nabla u|^2\\
&= -\frac{2}{n+\alpha}\Delta_w u - 2ku\Delta_w u - 2k|\nabla u|^2\\
&= -\frac{2}{n+\alpha}\Delta_w u - k\Delta_w(u^2).
\end{aligned}$$

Hence $\Delta_w P(u) \geq 0$. Finally, taking into account the Lemma 4.0.2, it is clear that $\Delta P(u) = 0$ if and only if $Ric_w^\alpha(\nabla u, \nabla u) = k(n+\alpha-1)|\nabla u|^2$ in Ω and

$$\begin{aligned}
\nabla^2 u &= \frac{\Delta u}{n} g\\
\frac{\Delta u}{n} &= \frac{\langle \nabla w, \nabla u\rangle}{\alpha} = \frac{\Delta_w u}{n+\alpha}.
\end{aligned}$$

Thus, since u is a solution of $\Delta_w u + k(n+\alpha)u = -1$ we conclude the desired resut. □

Let (N, g_N) be a connected $(n-1)$-Riemannian manifold. Let us consider $M = (0,\infty)\times N$ a product manifold endowed with a warped metric given by

$$g_M = dr \otimes dr + r^2 g_N.$$

From now on, let us denote $M = (0,\infty) \times_r N$. Now, we are able to state and proof our main rigidity result. We remember that, in M, a function w is homogeneous of degree $\alpha > 0$ if $\langle \nabla w, X\rangle = \alpha w$, where $X = r\partial_r$.

Proof of Theorem 1.0.7: Firstly, taking the P-function given by (4.21), we can conclude that $P(u) = c^2$ on $\partial\Omega$. Since this function is subharmonic in the weighted sense, we have that $P(u)$ attains its maximum in $\partial\Omega$ or $P(u)$ is a constant function (for the strong maximum principle applied to elliptic operators, we refer to [76, Theorem 2.4]). Now, suppose by contradiction that $P(u) < c^2$ in Ω. Thus

$$|\nabla u|^2 + \frac{2}{n+\alpha}u + ku^2 < c^2. \tag{4.22}$$

Since $u = 0$ along the boundary, a straightforward calculation shows that

$$\begin{aligned} 0 &= \int_\Omega div_w(u\nabla u)\, dv_w \\ &= \int_\Omega u\Delta_w u\, dv_w + \int_\Omega |\nabla u|^2\, dv_w \\ &= \int_\Omega \Big[|\nabla u|^2 - u(1+k(n+\alpha)u)\Big]\, dv_w. \end{aligned}$$

Thus,

$$\int_\Omega |\nabla u|^2\, dv_w = \int_\Omega u(1+k(n+\alpha)u)\, dv_w. \tag{4.23}$$

Notice that, since $X = r\partial_r \in \mathfrak{X}(M)$ is a closed conformal vector field with conformal factor $\varphi = 1$ and w is homogeneous, we have

$$\begin{aligned} div_w X &= div X + \langle \nabla \log w, X\rangle \\ &= n + \langle \nabla \log w, X\rangle \\ &= n+\alpha. \end{aligned} \tag{4.24}$$

Integrating (4.22) we have

$$\int_\Omega |\nabla u|^2\, dv_w + \frac{2}{n+\alpha}\int_\Omega u\, dv_w + k\int_\Omega u^2\, dv_w < c^2 Vol_w(\Omega). \tag{4.25}$$

On the other hand, since $\varphi = 1$ and (4.24), we use Proposition 4.0.13 to see that

$$\frac{(n+\alpha+2)}{n+\alpha}\int_\Omega u\, dv_w + 2k\int_\Omega u^2\, dv_w = c^2 Vol_w(\Omega) \tag{4.26}$$

Using (4.23), and replacing (4.26) in (4.25), we get

$$\frac{n+\alpha+2}{n+\alpha}\int_\Omega u\, dv_w + k(n+\alpha+1)\int_\Omega u^2\, dv_w < \frac{(n+\alpha+2)}{n+\alpha}\int_\Omega u\, dv_w + 2k\int_\Omega u^2\, dv_w,$$

which implies

$$k(n+\alpha-1)\int_\Omega u^2\, dv_w < 0.$$

Since k is nonnegative we reach a contradiction.

Thus, using maximum principle, $P(u) \equiv c^2$ and, consequently, $\Delta_w P(u) = 0$. The rigidity conclusion follows from Lemma 4.0.15 and Lemma 4.0.4. □

4.0.4 The weighted Serrin's problem for convex cones of the Euclidean space

This section addresses Serrin's problem in the context of convex cones within the Euclidean space, focusing on the weighted Laplacian. We analyze sector-like domains within convex cones, utilizing the weighted Laplacian to investigate rigidity results under specific boundary and weight conditions. Building upon prior work, they emphasize the role of weighted geometric properties and the α-Bakry-Émery Ricci tensor in our analysis.

Our key findings include characterizations of symmetry and rigidity for solutions to the weighted Serrin-type problem. By employing techniques such as Bochner-type inequalities and the properties of conformal vector fields, we demonstrate how the weight function constrains the geometry of the domain. These results, can be find in [59], highlight the interaction between the geometry of convex cones, boundary conditions, and the influence of the weight function in determining domain symmetry.

Throughout this section, we consider $M = \mathbb{R}^n$. Recall the definition of an open cone Σ in $\mathbb{R}^n$, $n \geq 2$, with vertex at the origin O: denoting by ω an open connected domain on the unit sphere $\mathbb{S}^n$, we define

$$\Sigma = \{tx \,:\, x \in \omega,\ t \in (0, +\infty)\}\,.$$

Furthermore, a sector-like domain Ω is a domain where the boundary components are denoted by

$$\Gamma = \partial\Omega \cap \Sigma \quad \text{and} \quad \Gamma_1 = \partial\Omega \setminus \overline{\Gamma},$$

with suitable properties (recall Definition 1.0.8).

Given a positive smooth function w, the Euclidean space can modify its natural measure according to the following rule: the new volume element and surface area are given by $dv_w = w\, dv$ and $da_w = w\, da$, where dv and da represent the Euclidean volume and surface area elements for the canonical metric in $\mathbb{R}^n$ and $(\mathbb{R}^n, \langle\ ,\ \rangle, dv_w)$ is a *weighted manifold.* In this setting, the *weighted Laplacian* is the operator defined by (4.1) and repeated here for convenience of the reader

$$\Delta_w = \Delta + \langle \nabla \log w, \nabla \cdot \rangle.$$

In this section, we study the problem (1.16)

$$\begin{cases} \Delta_w u & = & -1 & \text{in} & \Omega \\ u & = & 0 & \text{on} & \Gamma \\ u_\nu & = & -c & \text{on} & \Gamma \\ u_\nu & = & 0 & \text{on} & \Gamma_1 \setminus \{O\} \\ \nabla^2 \log w(\nabla u, \nabla u) + \dfrac{\langle \nabla \log w, \nabla u\rangle^2}{\alpha} & \leq & 0 & \text{in} & \Omega \end{cases}$$

where ν denote the exterior unit normal to $\partial\Omega$ and w is a homogeneous function of degree $\alpha > 0$, i.e. $\langle \nabla w, x\rangle = \alpha w$, where x is the position vector field. As a notable example we can take $w(x) = |x|^\alpha$, where $\alpha = 2$ correspond to the Gaussian model. More than merely studying a new extension of the cone problem for a weighted operator, this section extends the analysis conducted in [50]. This, for example, explains the hypothesis imposed on the function w and the final condition in (1.16).

For the next, we remember that a vector field $X \in \mathfrak{X}(M)$ is called *closed conformal* if $\nabla_Y X = \varphi Y$, for all $Y \in \mathfrak{X}(M)$, for some smooth function φ called the *conformal factor.* When $M = \mathbb{R}^n = (0, \infty) \times_t \mathbb{S}^{n-1}$ we have $\varphi = t' = 1$.

Lemma 4.0.12 plays an important role in the main result of this section. As an important particular case, we obtain the following consequence.

Corollary 4.0.16 *If u is a solution of* (1.16)*, where $\Omega \subset \mathbb{R}^n$ and w is homogeneous of degree α, then $\Delta_w \langle x, \nabla u\rangle = -2$, where x is the position vector field.*

Proof: The proof is completed by applying Lemma 4.0.12, observing that the position vector field x is a closed conformal field with $\varphi = 1$. Moreover, since w is α-homogeneous, $\langle \nabla \log w, x\rangle = \alpha$. $\square$

Applying Lemma 4.0.2 to our specific problem (1.16), we obtain the following rigidity characterization.

Corollary 4.0.17 *If u is a solution of* (1.16)*, where $\Omega \subset \mathbb{R}^n$, then*

$$\frac{1}{2}\Delta_w|\nabla u|^2 \geq \frac{1}{n+\alpha}.$$

Furthermore, the equality holds if and only if $\Omega = \Sigma \cap B_r(x_0)$ where $B_r(x_0)$ is a ball.

Proof: Since $Ric = 0$ in $\mathbb{R}^n$, we use (1.16) to conclude

$$Ric_w^\alpha(\nabla u, \nabla u) = -\nabla^2 \log w(\nabla u, \nabla u) - \frac{\langle \nabla \log w, \nabla u\rangle^2}{\alpha} \geq 0.$$

To complete the proof we just replace the solution u in Lemma 4.0.2. The conclusion of the equality case follows from Lemma 4.0.4. $\square$

We finish this section with the proof of its main result, Theorem 1.0.9.

Proof of Theorem 1.0.9: Since u a solution of (1.16), repeating the process of integration by parts multiple times leads to

$$\begin{aligned}
\int_\Omega u\, dv_w &= -\int_\Omega u\Delta_w u\, dv_w \\
&= -\int_\Gamma u u_\nu\, da_w - \int_{\Gamma_1} u u_\nu\, da_w + \int_\Omega |\nabla u|^2\, dv_w \\
&= -\int_\Omega |\nabla u|^2 \Delta_w u\, dv_w \\
&= -\int_{\Gamma\cup\Gamma_1} |\nabla u|^2 u_\nu da_w + \int_\Omega \left\langle \nabla|\nabla u|^2, \nabla u\right\rangle\, dv_w \\
&= -\int_\Gamma |\nabla u|^2 u_\nu\, da_w + \int_\Omega \left\langle \nabla|\nabla u|^2, \nabla u\right\rangle\, dv_w. \\
&= -c^2\int_\Gamma u_\nu\, da_w + \int_\Omega \left\langle \nabla|\nabla u|^2, \nabla u\right\rangle\, dv_w. \\
&= -c^2\int_{\partial\Omega} u_\nu\, da_w + \int_\Omega \left\langle \nabla|\nabla u|^2, \nabla u\right\rangle\, dv_w.
\end{aligned}$$

Then,

$$\int_\Omega u\, dv_w = c^2 Vol_w(\Omega) + \int_\Omega \left\langle \nabla|\nabla u|^2, \nabla u\right\rangle\, dv_w. \tag{4.27}$$

On other hand, using the divergence theorem,

$$\begin{aligned}
\int_\Omega \left\langle \nabla|\nabla u|^2, \nabla u\right\rangle\, dv_w &= -\int_\Omega u\Delta_w|\nabla u|^2\, dv_w + \int_{\partial\Omega} u\langle \nabla|\nabla u|^2, \nu\rangle\, da_w \\
&\leq -\frac{2}{n+\alpha}\int_\Omega u\, dv_w + \int_{\Gamma_1} u\langle \nabla|\nabla u|^2, \nu\rangle\, da_w \\
&\leq -\frac{2}{n+\alpha}\int_\Omega u\, dv_w,
\end{aligned}$$

where, in the first inequality, we use Corollary 4.0.17, and in the second inequality, we use the convexity of Σ. Then, (4.27) becomes

$$\Big(1+\frac{2}{n+\alpha}\Big)\int_\Omega u\,dv_w \le c^2 Vol_w(\Omega), \tag{4.28}$$

being the equality characterized by Corollary 4.0.17.

We will demonstrate that the equality always holds. Indeed, using the α-homogeneity of w, we have that

$$\begin{aligned}
\mathrm{div}_w(ux) &= u\mathrm{div}_w(x) + \langle \nabla u, x\rangle \\
&= u(\mathrm{div}(x) + \langle x, \nabla \log w\rangle) + \langle \nabla u, x\rangle \\
&= un + u\langle x, \nabla \log w\rangle + \langle \nabla u, x\rangle \\
&= (n+\alpha)u + \langle \nabla u, x\rangle\,,
\end{aligned}$$

therefore

$$\int_\Omega (n+\alpha)u\,dv_w = \int_\Omega \mathrm{div}_w(ux)\,dv_w - \int_\Omega \langle \nabla u, x\rangle\ dv_w.$$

Since x is the position vector field we have $\langle x, \nu\rangle = 0$, so

$$\begin{aligned}
\int_\Omega \mathrm{div}_w(ux)\ dv_w &= \int_{\partial\Omega} u\langle x,\nu\rangle\ da_w \\
&= \int_\Gamma u\langle x,\nu\rangle\ da_w + \int_{\Gamma_1} u\langle x,\nu\rangle\ da_w \\
&= 0,
\end{aligned}$$

and we get

$$(n+\alpha)\int_\Omega u\,dv_w = -\int_\Omega \langle \nabla u, x\rangle\ dv_w. \tag{4.29}$$

Furthermore we observe that $\nabla^2 u(\nabla u, x) = 0$ on Γ_1. Indeed, note that

$$\begin{aligned}
0 &= x\langle \nabla u, \nu\rangle \\
&= \langle \nabla_x \nabla u, \nu\rangle + \langle \nabla u, \nabla_x \nu\rangle,
\end{aligned} \tag{4.30}$$

on Γ_1. Now, given a vector field $Y \in \mathfrak{X}(\Gamma_1)$, note that

$$\begin{aligned}
\langle \nabla_x \nu, Y\rangle &= -\langle \nu, \nabla_x Y\rangle \\
&= -\langle \nu, \nabla_Y x + [x, Y]\rangle \\
&= -\langle \nu, Y + [x, Y]\rangle \\
&= 0.
\end{aligned}$$

Thus, from above equation, we deduce that $\nabla_x \nu = 0$ and from (4.30) we conclude that $\nabla^2 u(\nabla u, x) = 0$. Since $\nabla^2 u(\nabla u, x) = 0$ on Γ_1, from integration by parts, (4.29) and using

Corollary 4.0.16, we have

$$\begin{aligned}
\int_\Omega u\, dv_w &= \frac{1}{n+\alpha}\Big(-\int_\Omega \langle x, \nabla u\rangle\ dv_w\Big)\\
&= \frac{1}{n+\alpha}\Big(\int_\Omega \langle x, \nabla u\rangle\, \Delta_w u\, dv_w\Big)\\
&= \frac{1}{n+\alpha}\Big(\int_{\partial\Omega} \langle x, \nabla u\rangle\, u_\nu\ da_w + \int_\Omega u\Delta_w(\langle x, \nabla u\rangle)\, dv_w - \int_{\partial\Omega} u\langle \nu, \nabla\langle x, \nabla u\rangle\rangle\ da_w\Big)\\
&= \frac{1}{n+\alpha}\Big(\int_{\Gamma} \langle x, \nabla u\rangle\, u_\nu\ da_w - \int_{\Gamma_1} u\langle \nu, \nabla\langle x, \nabla u\rangle\rangle\ da_w - 2\int_\Omega u\ dv_w\Big)\\
&= \frac{1}{n+\alpha}\Big(c^2\int_{\partial\Omega} \langle x, \nu\rangle\ da_w - 2\int_\Omega u\ dv_w\Big)\\
&= \frac{1}{n+\alpha}\Big(c^2\int_\Omega \mathrm{div}_w(x)\, dv_w - 2\int_\Omega u\, dv_w\Big)\\
&= c^2 Vol_w(\Omega) - \frac{2}{n+\alpha}\int_\Omega u\, dv_w
\end{aligned}$$

therefore occurs the equality in (4.28). This implies, by Corollary 4.0.17 that $\Omega = \Sigma \cap B_r(x_0)$ where $B_r(x_0)$ is a ball. Now, we observe that from Lemma 4.0.2, we get $\nabla^2 u = \frac{\Delta u}{n} g = -\frac{1}{n+\alpha} g$. Thus, after some routine calculation one arrives at $u = \frac{r^2 - |x-x_0|^2}{2(n+\alpha)}$. □

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