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Doutorado em Matemática

Existence and Concentration of Positive Harmonic Functions with Nonlinear Boundary Condition in Expanding Domains

by

Leandro Favacho da Costa

João Pessoa - PB

Abril/2025

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under supervision of

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and co-supervision of

Prof. Dr. José Anderson Valença Cardoso

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ATA DE DEFESA DE DOUTORADO JUNTO AO PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA DA UNIVERSIDADE FEDERAL DA PARAÍBA, REALIZADA NO DIA 04 DE ABRIL DE 2025.

Ao quarto dia de abril de dois mil e vinte e cinco, às 10:00 horas, no Departamento de Matemática da Universidade Federal da Paraíba, foi aberta a sessão pública de Defesa de Tese intitulada "**Existência e Concentração de Funções Harmônicas Positivas com Condição de Fronteira Não Linear em Domínios Expandidos**", do aluno **Leandro Favacho da Costa** que havia cumprido, anteriormente, todos os requisitos para a obtenção do grau de Doutor em Matemática, sob a orientação do Prof. Dr. João Marcos Bezerra do Ó. A Banca Examinadora, indicada pelo Colegiado do Programa de Pós-Graduação em Matemática, foi composta pelos professores: João Marcos Bezerra do Ó (Orientador), Everaldo Souto de Medeiros (UFPB), Uberlandio Batista Severo (UFPB), Giovany de Jesus Malcher Figueiredo (UnB), Edcarlos Domingos da Silva (UFG) e Jonison Lucas dos Santos Carvalho (UFS). O professor João Marcos Bezerra do Ó, em virtude da sua condição de orientador, presidiu os trabalhos e, depois das formalidades de apresentação, convidou o aluno a discorrer sobre o conteúdo da tese. Concluída a explanação, o candidato foi arguido pela banca examinadora que, em seguida, sem a presença do aluno, finalizando os trabalhos, reuniu-se para deliberar tendo concedido a menção: **aprovado.** E, para constar, foi lavrada a presente ata que será assinada pelos membros da Banca Examinadora.

João Pessoa, 04 de abril de 2025.

João Marcos Bezerra do Ó

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Abstract

This thesis investigates the existence of a positive harmonic function u_ϵ defined in the rescaled domain $\Omega_\epsilon = \epsilon^{-1}\Omega$, subject to a nonlinear boundary condition, where $\epsilon > 0$, and Ω is a bounded domain in $\mathbb{R}^n$, with $n \geq 3$. In the case where $\epsilon \to 0$, corresponding to expanding domains, it is established that there exists a constant $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, the principal problem admits a nonconstant, positive least energy solution u_ϵ. Moreover, it is demonstrated that u_ϵ exhibits a power-type decay and an exponential decay in the first $n-1$ variables.

Furthermore, in the case where $\epsilon \to 0$, it is shown that the points where u_ϵ attains its maximum concentrate around a point of maximum for the mean curvature of $\partial\Omega$. In contrast, in the case where $\epsilon \to \infty$, corresponding to contracting domains, it is proven that there exists a constant $\epsilon^* > 0$ such that for each $\epsilon > \epsilon^*$, the unique positive solution u_ϵ of the principal problem is constant.

To conduct the analysis developed in Chapter 2, it is essential to investigate the existence of a ground state solution for the limit problem. Chapter 1 is dedicated to this study, addressing the problem of the existence and nonexistence of harmonic functions in the upper half-space, subject to an indefinite nonlinear boundary condition. The existence of a ground state solution is established, which is radially symmetric and exhibits exponential decay in the first $n-1$ variables.

Furthermore, in Chapter 1, an investigation is carried out on the existence and nonexistence of weak solutions in alternative cases that are not directly related to the problem considered in Chapter 2. In one such case, variational minimization techniques are employed to demonstrate the existence of a nontrivial weak solution. Additionally, a theorem is presented that characterizes the nonexistence of solutions under certain conditions.

Keywords: Least energy solution, Exponential decay, Concentration problem, Existence and Nonexistence of solutions.

Resumo

Esta tese investiga a existência de uma função harmônica positiva u_ϵ definida no domínio $\Omega_\epsilon = \epsilon^{-1}\Omega$, sujeita a uma condição de contorno não linear, onde $\epsilon > 0$ e Ω é um domínio limitado em $\mathbb{R}^n$. No caso em que $\epsilon \to 0$, correspondente a domínios em expansão, estabelece-se que existe uma constante $\epsilon_0 > 0$ tal que, para todo $\epsilon \in (0, \epsilon_0)$, o problema principal admite uma solução de menor energia positiva não constante u_ϵ. Além disso, demonstra-se que u_ϵ exibe decaimento do tipo potência, e um decaimento exponential nas primeiras $n-1$ variáveis.

Além disso, no caso onde $\epsilon \to 0$, mostra-se que os pontos onde u_ϵ atinge seu máximo concentram-se em torno de um ponto de máximo para a curvatura média de $\partial\Omega$. No caso em que $\epsilon \to \infty$, correspondendo a domínios que se contraem, prova-se que existe $\epsilon^* > 0$ tal que para cada $\epsilon > \epsilon^*$ a única solução positiva u_ϵ do problema principal é constante.

Para conduzir a análise desenvolvida no Capítulo 2, é essencial investigar a existência de uma solução do estado fundamental para o problema limite. O Capítulo 1 é dedicado a esse estudo, abordando o problema da existência e inexistência de funções harmônicas no semi-espaço superior, sujeitas a uma condição de contorno não linear e indefinida. A existência de uma solução do estado fundamental é estabelecida, sendo esta radialmente simétrica e exibindo decaimento exponencial nas primeiras $n-1$ variáveis.

Além disso, no Capítulo 1, é realizada uma investigação sobre a existência e inexistência de soluções fracas em casos alternativos que não estão diretamente relacionados ao problema considerado no Capítulo 2. Em um desses casos, técnicas de minimização variacional são empregadas para demonstrar a existência de uma solução fraca não trivial. Adicionalmente, é apresentado um teorema que caracteriza a inexistência de soluções sob determinadas condições.

Palavras-chave: Solução de energia mínima, Decaimento exponential, Problema de concentração, Não existência de soluções.

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"*Let nothing disturb you, let nothing frighten you, all things are passing; God alone is unchanging. Patience attains all that it strives for. He who has God finds he lacks nothing; God alone is enough.*"

Saint Teresa of Avila

Dedicatory

To my beloved wife Mrs.Vanessa Favacho, without whom this work would not have been possible.

"*Idcirco relinquet homo patrem suum et matrem, et adhærēbit uxori suæ, et erunt duo in carne una.*"

Genesis 2:24

Contents

Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, where $n \geq 3$, with a smooth boundary $\partial\Omega$, and let ν denote the unit outward normal to $\partial\Omega$. Consider the following singularly perturbed nonlinear boundary value problem:

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega \\ \epsilon \partial_\nu u + u = |u|^{q-2} u - |u|^{r-2} u & \text{on} \quad \partial\Omega, \end{cases} \tag{SP}$$

where $2 < r < q < 2_* = 2(n-1)/(n-2)$, and $\epsilon > 0$ is a parameter. The primary objective of the present work is to investigate the existence of a nonconstant least-energy solution to Problem (SP), as well as the asymptotic behavior of such a solution. It can be proved through a scaling argument that there exists a one-to-one correspondence between the solutions of Problem (SP) and the solutions of the following problem:

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega_\epsilon \\ \partial_\nu u + u = |u|^{q-2} u - |u|^{r-2} u & \text{on} \quad \partial\Omega_\epsilon, \end{cases} \tag{P_ϵ}$$

where $2 < r < q < 2_*$,

$$\Omega_\epsilon = \epsilon^{-1}\Omega = \left\{\epsilon^{-1} z : \ z = (x,t) \in \Omega\right\},$$

and $\epsilon > 0$ is a parameter. Given the smoothness of the domain Ω_ϵ and the regular character of the boundary nonlinearity, the classical results in elliptic regularity theory (see [20, Theorems 6.30 and 6.31]) guarantee that every weak solution to Problem (P_ϵ) belongs to $C^\infty\left(\overline{\Omega_\epsilon}\right)$. Furthermore, it is asserted that these solutions are strictly one-signed. In fact, suppose, for the sake of contradiction, that u is a nontrivial solution to Problem (P_ϵ) that changes sign in Ω_ϵ; that is, there exist points $x_1, x_2 \in \Omega_\epsilon$ such that $u(x_1) < 0$ and $u(x_2) > 0$. By continuity, there exists $x_0 \in \Omega_\epsilon$ such that $u(x_0) = 0$. Since u is harmonic in Ω_ϵ, the strong maximum principle implies that u cannot attain a nontrivial maximum or minimum in the interior unless it is constant. Therefore, the extremal values of u must be attained on the boundary $\partial\Omega_\epsilon$.

Now consider the boundary condition:

$$\partial_\nu u + u \;=\; |u|^{q-2}\,u - |u|^{r-2}\,u =: f\,(u) \quad \text{on} \quad \partial\Omega_\epsilon,$$

where $2 < r < q < 2_*$. For $u > 0$, the function $f\,(u) = u^{q-1} - u^{r-1}$ satisfies $f\,(u) < 0$ for $0 < u < 1$ and $f\,(u) > 0$ for $u > 1$. Consequently, if u attains a sufficiently large positive maximum on the boundary, then $f\,(u) - u > 0$, implying $\partial_\nu u > 0$. However, this contradicts the classical Hopf boundary point lemma, which asserts that $\partial_\nu u < 0$ at a boundary maximum. A similar contradiction arises if u attains a sufficiently large negative minimum on the boundary.

Hence, a sign-changing solution with large boundary extrema cannot satisfy the boundary condition consistently with the Hopf lemma. This contradiction implies that any nontrivial solution to Problem (P_ϵ) must be strictly one-signed in $\overline{\Omega_\epsilon}$. Therefore, it is assumed throughout this work that the solutions of Problem (P_ϵ) are positive in $\overline{\Omega_\epsilon}$.

It can be verified that for every $\epsilon > 0$, Problem (P_ϵ) admits a unique positive constant solution. To establish the existence of a nonconstant positive solution, this work relies on the study of the associated limit problem, namely, the problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \mathbb{R}^n_+ \\ \partial_\eta u + u = \lambda\,|u|^{q-2}\,u - |u|^{r-2}\,u & \text{on} \quad \partial\mathbb{R}^n_+, \end{cases} \tag{P_λ}$$

where $\mathbb{R}^n_+ := \{z = (x,t) \in \mathbb{R}^n :\; x \in \mathbb{R}^{n-1},\; t > 0\}$, with $n \geq 3$, denotes the upper-half space, $2 < r < q < 2_*$, and η denoting the unit outward normal to the boundary $\partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$. It is observed that, by selecting a point on the boundary $\partial\Omega_\epsilon$ and taking the limit as $\epsilon \to 0$, the domain approaches a half-space, which, after an appropriate rotation and translation, may be assumed to be $\mathbb{R}^n_+$.

In Chapter 1, it is shown that for all $\lambda > 0$ and $2 < r < q < 2_*$, Problem (P_λ) admits a ground state solution u_λ, which is proven to be radially symmetric with respect to the first $n-1$ variables through the use of the classical moving plane method [6,7,32]. Additionally, u_λ and its derivatives exhibit exponential decay with respect to the first $n-1$ variables. These properties are employed to derive an upper bound estimate for the minimax level $c_{q,r}(\Omega_\epsilon)$, which enables the proof that the mountain-pass solution u_ϵ of (P_ϵ) is a positive, nonconstant least energy solution of (P_ϵ), for sufficiently small ϵ. Furthermore, it is shown that u_ϵ exhibits power-type decay and exponential decay in the first $n-1$ variables. To conclude the results in the case $\epsilon \to 0$, it is established that the point at which u_ϵ attains its maximum is located on the boundary $\partial\Omega$, concentrating at the point of maximal mean curvature $\mathcal{H}(z)$ on $\partial\Omega$.

It is also noteworthy that in Chapter 1, the analysis of the existence and nonexistence of solutions to Problem (P_λ) is conducted, considering the interplay between the exponents q and r and the values of λ. As demonstrated in Section 1.2, in the case where $\lambda > 0$ and $2 < r < q < 2_*$, a ground state solution is established through the application of a min-max argument. In Section 1.3, for the case where $\lambda > 0$ and $2 < q < r < 2_*$, the existence of a nontrivial weak solution is ensured by employing a classical variational approach. In Section 1.4, a Pohozaev-type identity, along with an elementary mathematical argument, is utilized to investigate the nonexistence of solutions in some instances involving q and r.

The thesis concludes with the analysis of the case where ϵ is large, corresponding to contracting domains. In this setting, it is demonstrated that there exists $\epsilon^* > 0$ such that Problem (P_ϵ) admits only a positive constant solution for all $\epsilon > \epsilon^*$.

Motivations

An extensive body of literature exists in the field of partial differential equations concerning Neumann problems associated with second-order semilinear elliptic equations. In the present context, those who have made significant contributions to the development of the present work are highlighted. In the works [24, 29], the authors examined the following problem:

$$\begin{cases} d\Delta u - u + u^p = 0 & \text{in} \quad \Omega \\ \partial_\nu u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{P}$$

where d is a positive constant, and p satisfies $1 < p < n/(n-2)$, where $n \geq 3$, with the understanding that $1 < p < +\infty$ if $n = 2$. This problem may be regarded as a model or a reduced version of the activator-inhibitor system proposed by Gierer and Meinhardt [19] as a model of biological pattern formation. Moreover, this problem is related to the simplified model used by Keller and Segel in the study of the chemotactic interaction of amoebas, as described in [22].

From a mathematical perspective, in [29], the authors derived a priori estimates for positive solutions to Problem (P) as functions of d and showed that if d is sufficiently large, no nonconstant positive solution exists for (P). In [27, 28], the same authors applied the mountain pass lemma [8] to establish the existence of a least energy solution u_d for (P). A detailed description of the shape of u_d was provided, and it was proven that when d is sufficiently small, u_d exhibits only a local maximum over $\overline{\Omega}$. Moreover, it was shown that the maximum is achieved at exactly one point P_d on the boundary (u_d exhibits a "point-condensation phenomena" as $d \to 0$). Additionally, it was established that $H(P_d)$, the mean curvature of $\partial\Omega$ at P_d, approaches the maximum of $H(P)$ over $\partial\Omega$ as $d \to 0$. In [14], Del Pino and Felmer showed that the results in [27, 28] hold without some delicate technical nondegeneracy-uniqueness assumptions, thereby significantly broadening the class of nonlinearities under consideration. The following limit problem was utilized throughout the work:

$$\begin{cases} \Delta w - w + f(w) = 0 \quad \text{in} \quad \mathbb{R}^n_+ \\ w > 0 \quad \text{in} \quad \mathbb{R}^n_+, \quad w(0) = \max_{\partial\mathbb{R}^n_+} w, \\ \lim_{|x|\to\infty} w(x) = 0, \quad \partial_\eta u = 0 \quad \text{on} \quad \partial\mathbb{R}^n_+. \end{cases} \tag{Q}$$

Together with the cited works previously, many others have demonstrated the significance of considering the knowledge of a ground state solution of the limiting problem when analyzing the existence of least energy solutions (see for example [5, 9, 15]).

Concerning problems involving more general boundary terms, it can be highlighted that the model studied in 1902 by Steklov in [33] is particularly relevant for Mathematical Physics, namely, the model

$$\Delta v = 0, \quad \text{in} \quad S, \qquad \partial_\eta v = hv + f \quad \text{on} \quad \partial S, \tag{S}$$

where S is a convex surface, f is a given function, continuous on ∂S, and h is a parameter. This problem presents a linear boundary term. Consider a more general situation. As described in [38, Example 5], the temperature $u(x,t)$ of a body at position x and time t satisfies the heat conduction equation

$$c(x)\,\rho(x)\,u_t - \operatorname{div}(k(x,u)\,\nabla u) = F(x,t),$$

where $c(x)$ and $\rho(x)$ denote the specific heat and density of the substance, respectively, $k(x, u)$ is the internal thermal conductivity, and $F(x, t)$ is the heat source density. In the case of a homogeneous and isotropic material, the coefficients $c(x)$, $\rho(x)$, and $k(x, u)$ reduce to constants, so the equation simplifies to

$$u_t = (k/c\rho)\,\Delta u + F(x,t)/c\rho.$$

When a heat flux q is specified on the boundary $\partial\Omega$, the corresponding boundary condition is given by Fourier's law:

$$\partial_\eta u = q/k,$$

where η denotes the outward unit normal vector. On the other hand, Newton's law of cooling models the heat exchange between the body and the surrounding environment. According to this law, the heat flux through the boundary is proportional to the difference between the external and internal temperatures:

$$q = \alpha(u_1 - u),$$

where $\alpha > 0$ is the heat transfer coefficient, and u_1 represents the temperature of the surrounding medium.

Equating the expressions for the heat flux from Fourier's and Newton's laws yields the nonlinear Robin boundary condition:

$$(k/\alpha)\partial_\eta u + u = u_1 \quad \text{on} \quad \partial\Omega.$$

Consider the external temperature $u_1 = |u|^{q-2}\,u - |u|^{r-2}\,u$ on $\partial\Omega$, where $2 < r < q < 2_*$. This leads to the steady-state boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ (k/\alpha)\partial_\eta u + u = |u|^{q-2}\,u - |u|^{r-2}\,u & \text{on} \quad \partial\Omega, \end{cases}$$

where $\epsilon = k/\alpha$ is a small parameter representing the ratio between internal and external thermal conductivities.

As $\epsilon \to 0$, the problem exhibits a *concentration phenomenon*. From a physical perspective, this regime corresponds to highly efficient heat exchange with the surrounding medium. Mathematically, it leads to solutions whose temperature profile concentrates around a unique point on the boundary $\partial\Omega$, which, as will be seen later (see Theorem J), corresponds to the point where the mean curvature of the boundary attains its maximum.

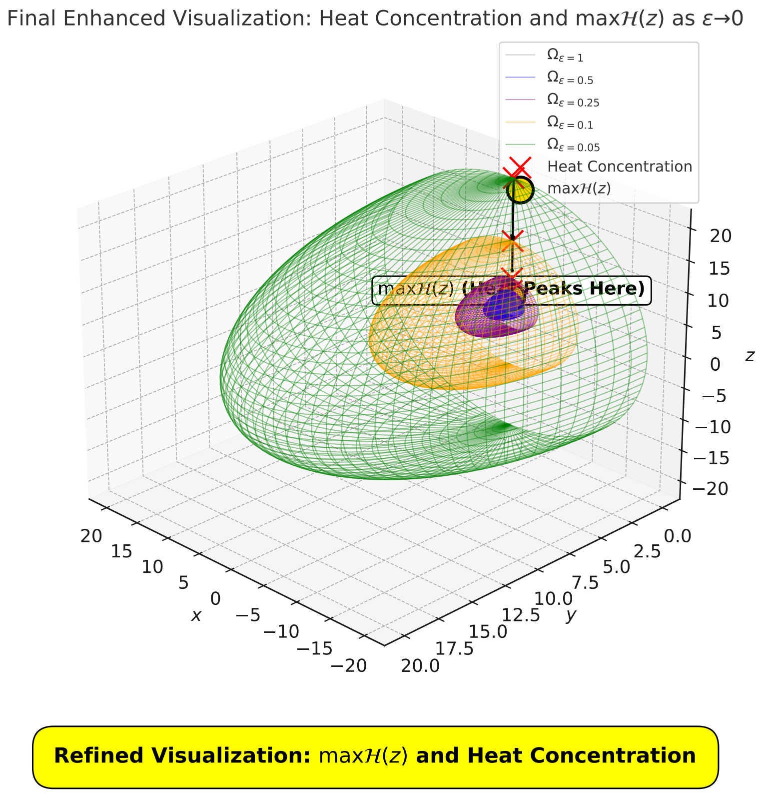


Figure 1: The figure illustrates the concentration phenomenon associated with the physical problem under consideration. As the parameter $\epsilon \to 0$, the rescaled domain Ω_ϵ expands. The points marked with a red X indicate the locations on the boundary $\partial\Omega_\epsilon$ where the temperature u_ϵ attains its maximum value. In the limit as $\epsilon \to 0$, the heat becomes concentrated near a single point on the boundary $\partial\Omega$. It will be shown later (see Theorem J) that this concentration occurs precisely at the point where the mean curvature reaches its maximum on the $\partial\Omega$.

Continuing to consider the works that address a singularly perturbed problem involving a nonlinear boundary condition, the work of Del Pino and Flores [15] is particularly notable. In this work, the authors studied the relationship between the extremals of the best constant for the Sobolev trace embedding of $H^1(\Omega)$ into $L^{p+1}(\partial\Omega)$, where $1 < p < n/(n-2)$, for $n \geq 3$, and the solutions of the Euler-Lagrange equation

$$\begin{cases} \Delta u_\lambda - u_\lambda = 0 & \text{in} \quad \Omega_\lambda \\ \partial_\nu u_\lambda = |u_\lambda|^{p-1} u_\lambda & \text{on} \quad \partial\Omega_\lambda, \end{cases} \tag{DF}$$

where

$$\Omega_\lambda = \lambda\Omega = \{\lambda x : \ x \in \Omega\},$$

and $\lambda \to \infty$. It is mentioned that, after applying a suitable scaling argument, Problem (DF) transforms into

$$\begin{cases} \epsilon^2 \Delta u - u = 0 & \text{in} \quad \Omega \\ \partial_\nu u = u^p & \text{on} \quad \partial\Omega, \end{cases}$$

where $\epsilon = 1/\lambda$. One of the main results of the work states that the extremals u_λ form a single bump at the boundary, whose shape is asymptotically that of an extremal for the half-space embedding. This bump is centered (in the Ω-coordinate) around a point of maximum mean curvature of $\partial\Omega$.

The class of problems described by (P_λ) represents a special case of the broader model:

$$\Delta u = 0, \quad \text{in} \quad \Omega, \qquad \partial_\nu u = g(x,u) \quad \text{on} \quad \partial\Omega, \tag{S}$$

where $\Omega \subset \mathbb{R}^n$ is an open set. This model has been extensively studied in the literature. Steklov's work [33] focused on a linear perturbation $g(x,u)$. Since then, several types of perturbations have been considered [11, 12, 30, 31], some of which are related to Physical, Electrochemistry, Geometrical, and other areas of study [33, 39, 40]. From a mathematical standpoint, Cabré and Solà-Morales [11] considered the perturbation $g(x,u) = f(u)$, where f is a $C^{1,\alpha}$ function defined on the half-space $\mathbb{R}^n_+$. Their work investigates the existence, uniqueness, symmetry, variational properties, and asymptotic behavior of a class of solutions to Problem (S), commonly referred to as Layer solutions. Notably, it has been observed that the problem investigated in the half-space arises naturally through a blow-up process during the analysis of solutions to the following problem:

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega \\ \epsilon\partial_\nu u = f(z,u) & \text{on} \quad \partial\Omega, \end{cases} \tag{CSM}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, $z \in \partial\Omega$, and $\epsilon > 0$ is a parameter. In the case where $\epsilon \to 0$, and under certain conditions imposed on f, the problem naturally leads to the problem in the half-space and the concept of a layer or increasing solution.

More recently, intending to investigate combined concave-convex effects, Furtado et al [18] analyzed the perturbation

$$g(x,u) = a(x)|u|^{p-2}u + \mu b(x)|u|^{q-2}u, \qquad \text{on} \qquad \partial\Omega,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $1 < q < 2 < p < 2_*,\ 2_* = 2(n-1)/(n-2)$, with a and b being weighted functions satisfying certain integrability conditions. Results concerning the multiplicity of solutions were obtained. Taking the perturbation

$$g(x,u) = |u|^{p-2}u - u, \quad 2 < p < 2_*, \quad \text{on} \quad \partial\mathbb{R}^n_+,$$

into consideration, Abreu et al [2] established the existence of a ground state solution $w = w(x_1, \ldots, x_{n-1}, t)$, which is radially symmetric and exhibits exponential decay in the first $n-1$ variables. Additionally, it was shown that w exhibits a power-type decay in the variable t.

The present work examines the Problem (P_λ), where the perturbation is defined as

$$g(x,u) = \lambda\,|u|^{q-2}\,u - |u|^{r-2}\,u - u, \qquad \text{on} \qquad \partial\mathbb{R}^n_+.$$

A significant aspect of this perturbation is the presence of the exponents q and r in conjunction with the parameter λ. From a mathematical standpoint, this allows for examining both the existence and nonexistence of solutions to Problem (P_λ), particularly by exploring the interactions among the exponents q, r, and λ. This enhances the interest in the problem, given the application of several methods and the derivation of distinct properties of the solutions examined in each case.

Main results

The presentation of the primary results commences with the findings outlined in Chapter 1. The first theorem establishes that, for the case where $2 < r < q < 2_* = 2(n-1)/(n-2)$ and for all $\lambda > 0$, a ground state solution to Problem (P_λ) is obtained through the application of a variational approach [8].

Theorem A *Assume $2 < r < q < 2_*$. Then, for all $\lambda > 0$, Problem (P_λ) admits a ground state solution u_λ.*

For all $\lambda > 0$, the Moser iteration method [26] is applied to establish that $u_\lambda \in L^\infty(\mathbb{R}^n_+)$ and, in the trace sense, $u_\lambda|_{\mathbb{R}^{n-1}} \in L^\infty(\mathbb{R}^{n-1})$. Moreover, by employing a Harnack-type inequality, it is shown that u_λ exhibits a power-type decay. Finally, a regularity for u_λ is derived. These findings are summarized in the following theorem.

Theorem B *Let u_λ be a weak solution of Problem (P_λ), where $\lambda > 0$ and $2 < r < q < 2_*$. Then, in the trace sense, $u_\lambda|_{\mathbb{R}^{n-1}} \in L^\infty(\mathbb{R}^{n-1})$ and $u_\lambda \in L^\infty(\mathbb{R}^n_+)$. Furthermore, if u_λ is a nonnegative weak solution of Problem (P_λ), then $u_\lambda \in C^{2,\alpha}_{loc}\left(\overline{\mathbb{R}^n_+}\right) \cap C^\infty\left(\mathbb{R}^n_+\right)$ and it has a power-type decay, more precisely,*

$$u_\lambda(z) = O\left(|z|^{2-n}\right), \quad \text{as} \quad |z| \to \infty.$$

Inspired by [32], the classical moving plane method [6, 7] is applied to establish that u_λ exhibits radial symmetry with respect to the variable x. Furthermore, it is shown that u_λ is monotone decreasing in the ρ-direction, where $\rho = |x|$.

Theorem C *If u_λ is a nonnegative weak solution of Problem (P_λ), where $\lambda > 0$ and $2 < r < q < 2_*$, then u_λ is radially symmetric with respect to variable x, that is,*

$$u_\lambda(x,t) = u_\lambda(\rho,t) \quad \text{if} \quad \rho = |x|.$$

Moreover, $u_\rho(\rho,t) < 0$ in $(0,+\infty) \times [0,+\infty)$.

With the assistance of Theorem C, it can be demonstrated that u_λ exhibits exponential decay with respect to the variable x. Additionally, a lower power-type decay with respect to the variable t is derived. These findings are summarized in the following theorem.

Theorem D *Let u_λ be a nonnegative weak solution of Problem (P_λ), where $\lambda > 0$ and $2 < r < q < 2_*$. Then, there exist positive constants c_1 and c_2 such that*

$$u_\lambda(x,t) \le c_1 \exp\left(-c_2|x|\right) \frac{1}{\left(1+t^2\right)^{(n-2)/2}}, \quad \text{for all} \quad (x,t) \in \overline{\mathbb{R}^n_+}.$$

In the case where $2 < q < r < 2_*$ and $\lambda > 0$, the minimization of lower semicontinuous functionals, as outlined by D.G. De Figueiredo in [13], is applied to demonstrate the following theorem.

Theorem E *Let $2 < q < r < 2_*$ and $\lambda > 0$. Then, there exists $\Lambda > 0$ such that Problem (P_λ) possesses a nontrivial weak solution for all $\lambda > \Lambda$.*

To conclude the results of Chapter 1, motivated by [1, 21], the nonexistence of solutions to Problem (P_λ) is analyzed for specific values of the exponents q and r, as well as the parameter λ. The subsequent theorem is proven by adapting a Pohozaev-type identity from [1, Proposition 5.1], along with the development of an elementary mathematical argument.

Theorem F *Let u_λ be a weak solution of Problem (P_λ). Then $u_\lambda \equiv 0$ if one of the conditions hold:*

1. *If $\lambda \leq 0$ and $r, q \in (2, 2_*) \cup (2_*, +\infty)$;*
2. *If $\lambda > 0$, $q \in (2_*, +\infty)$, and $2 < r < q$.*

Furthermore, if $\lambda > 0$ and $2 < q < r < 2_*$, then $u_\lambda \equiv 0$ for all $\lambda \in (0, \lambda^*)$, where

$$\lambda^* = \frac{1}{\left[\left(\frac{q-2}{r-2}\right)^{\frac{q-2}{r-2}}\left(\frac{r-q}{r-2}\right)\right]^{\frac{r-q}{r-2}}}.$$

The principal results of Chapter 2 are presented herein. Throughout the chapter, it is assumed that $2 < r < q < 2_*$. The first result establishes the existence of a nonconstant positive least energy solution u_ϵ to Problem (P_ϵ) in the case where $\epsilon \to 0$, corresponding to expanding domains.

Theorem G *There exists a constant $\epsilon_0 > 0$ such that Problem (P_ϵ) admits a nonconstant positive least energy solution u_ϵ for all $\epsilon \in (0, \epsilon_0)$.*

By applying the Moser iteration method [26], an L^∞ estimate for the weak solutions of Problem (P_ϵ) is derived. This leads to the proof that u_ϵ exhibits uniform decay at infinity. Furthermore, with respect to decay, the second result establishes that a power-type decay can be derived for u_ϵ.

Theorem H *Let $u_\epsilon \in C^\infty(\Omega_\epsilon) \cap C^{1,\beta}\left(\overline{\Omega_\epsilon}\right)$ be a positive solution of Problem (P_ϵ). Then, there exists a positive constant C_0 independent of ϵ such that*

$$u_\epsilon(z) \leq \frac{C_0}{\left(1 + |z|^2\right)^{(n-2)/2}}, \quad \text{for} \quad \text{all} \quad z \in \overline{\Omega}_\epsilon.$$

Building upon the ideas presented in [3], it is proven that u_ϵ exhibits exponential decay.

Theorem I *There exists $c > 0$ such that*

$$u_\epsilon (x, t) \leq ce^{-\alpha|x|} \left(\frac{1}{1+t^2} \right)^{(n-2)/2},$$

for all $|x| \geq 1$, and $t \geq 0$.

Let $c_{q,r} (\Omega_\epsilon)$ and $c_{q,r} \left(\mathbb{R}^n_+\right)$ denote the least energy levels associated with I_{Ω_ϵ} and the least energy level of the functional associated with Problem (P_λ), respectively. If $\mathcal{H} (z)$ represents the mean curvature of the boundary at the point $z \in \partial\Omega$, the following concentration result can be stated.

Theorem J *Assume that $n \geq 3$, and let u_ϵ be the least energy solution of (P_ϵ), as obtained in Theorem G. If $z_\epsilon \in \partial\Omega_\epsilon$ is a point where u_ϵ attains its maximum value, then*

$$\mathcal{H} (\epsilon z_\epsilon) \to \max_{z\in\partial\Omega} \mathcal{H} (z), \quad as \quad \epsilon \to 0.$$

Moreover, for $\epsilon \to 0$, it follows that

$$c_{q,r} (\Omega_\epsilon) = c_{q,r} \left(\mathbb{R}^n_+\right) - \epsilon\gamma \max_{z\in\partial\Omega} \mathcal{H} (z) + o (\epsilon),$$

where $\gamma = \gamma (q, r, n)$ is a positive constant.

The thesis concludes with the analysis of Problem (P_ϵ) for large values of ϵ. The following nonexistence result is established.

Theorem K *There exists $\epsilon^* > 0$ such that for each $\epsilon > \epsilon^*$, the unique solution of Problem (P_ϵ) is constant.*

Outline

The thesis is organized as follows. Chapter 1 is dedicated to the study of the limit problem (P_λ). Section 1.1 presents the central problem studied in Chapter 1, as well as the appropriate space for the development of the results, inner product, and norm. Furthermore, the crucial definition of weak solution of Problem (P_λ) is provided in this section. This section will play a significant role throughout the entire work. In Section 1.2, the case $2 < r < q < 2_*$, $\lambda > 0$ is considered. This section is divided into the following subsections: Subsection 1.2.1 presents some auxiliary results that will be of great utility throughout the section; Subsection 1.2.2 establishes the existence of a positive ground state solution u_λ; Subsection 1.2.3 presents the important relationship between the best constant for the Sobolev embedding studied in Chapter 1 and the concept of least energy solution. Subsection 1.2.4 focuses on the regularity of u_λ and the derivation of its appropriate power-type decay; In Subsection 1.2.5, it is shown that, for all $\lambda > 0$, u_λ is radially symmetric with respect to the variable x, leading to an exponential decay with respect to x. Section 1.3 addresses the case $2 < q < r < 2_*$, $\lambda > 0$. Section 1.4 concludes the chapter with a nonexistence result for Problem (P_λ).

In Chapter 2, the principal problem in expanding domains is studied. Section 2.1 presents the fundamental results that underpin the analysis throughout the chapter. Section 2.2 summarizes the results related to the limit problem (P_λ), which are utilized in the thesis. In section 2.3, an upper bound estimate is derived, which plays a crucial role in the proof of Theorem G. Section 2.4 is dedicated to the proofs of Theorem H and Theorem I. Section 2.5 provides a lower bound estimate and completes the proof of Theorem J. Finally, Section 2.6 concludes the chapter with the presentation of a nonexistence result for the case where ϵ is large (Theorem K).

Notations

- $\mathbb{R}^n_+ := \{z = (x,t) \in \mathbb{R}^n : \ x \in \mathbb{R}^{n-1}, \ t > 0\}$, $n \geq 3$, is the Euclidean upper half-space;

- $C_0^\infty(\mathbb{R}^n)$ is the space of infinitely differentiable functions with compact support on $\mathbb{R}^n$; the space of test functions;

- $C^1(E;\mathbb{R})$ is the space of continuously differentiable functions from E to $\mathbb{R}$;

- $c_{q,r}\left(\mathbb{R}^n_+\right) = \inf\limits_{\gamma\in\Gamma} \max\limits_{t\in[0,1]} I_\lambda(\gamma(t))$ is the minimax level associated to the functional I_λ;

- $H^1(\Omega) = W^{1,2}(\Omega)$ is a Sobolev space;

- $\mathcal{N} = \left\{u \in E \setminus \{0\} : \ I'_\lambda(u)u = 0\right\}$ is the Nehari manifold associated to I_λ.

- The Big-O notation: $g = \mathcal{O}(f)$, as $x \to x_0$, for $x_0 \in [-\infty, +\infty]$, if $\limsup_{x\to x_0}(g/f)(x) < \infty$;

- The little-o notation: $g = o(f)$, as $x \to x_0$, for $x_0 \in [-\infty, +\infty]$, if $\lim_{x\to x_0}(g/f)(x) = 0$;

- $X \hookrightarrow Y$ denotes that X is continuous embedded in Y;

- $D^j = \frac{\partial}{\partial x_j}$ is the partial derivative with respect to the variable x_j;

- $C^{2,\alpha}_{loc}\left(\overline{\mathbb{R}^n_+}\right)$, where $0 < \alpha < 1$, is a Hölder space.

- $C^\infty\left(\mathbb{R}^n_+\right)$ is the set of all functions defined on the upper half-space $\mathbb{R}^n_+$ that are infinitely differentiable (i.e., have continuous derivatives of all orders) on $\mathbb{R}^n_+$;

- $c_{q,r}(\Omega_\epsilon) = \inf\limits_{\gamma\in\Gamma} \max\limits_{t\in[0,1]} I_{\Omega_\epsilon}(\gamma(t)) > 0$ is the minimax level associated to the functional I_{Ω_ϵ};

- $N_\epsilon = \left\{ v \in H^1(\Omega_\epsilon) \setminus \{0\} : \ \|v\|_\tau^2 = \int_{\partial\Omega_\epsilon} (v^+)^q \ d\sigma - \int_{\partial\Omega_\epsilon} (v^+)^r \ d\sigma \right\}$ is the Nehari manifold associated to I_{Ω_ϵ};

- $B_\rho(z)$ is a ball centered in z with radius ρ;

- $\mathcal{H}(z)$ is the mean curvature of the boundary at the point $z \in \partial\Omega$;

- $D^2G(x)$ is the Hessian matrix of G at x;

- $\overline{u}$ is the average of u over $\partial\Omega$;

Chapter 1

The Problem in the Upper Half-Space

The study of the limit problem (P_λ) is of considerable significance in this work. In this chapter, this problem will be analyzed, not only to yield results that contribute to the development of Chapter 2 but also to investigate the existence and nonexistence of solutions to the problem, independently of Problem (P_ϵ).

The chapter is structured into two primary parts. The first part concentrates on deriving results that will be directly applicable in Chapter 2, placing particular emphasis on radial symmetry and exponential decay in the first $n-1$ variables of the ground-state solution. The second part focuses on the examination of the existence and nonexistence of solutions in contexts that are distinct from Chapter 2.

It is noteworthy that classical methods, including the Moser iteration method, the moving planes method will be employed to achieve these results.

1.1 Preliminaries

Consider the nonlinear boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \mathbb{R}^n_+ \\ \partial_\nu u + u = \lambda \left|u\right|^{q-2} u - \left|u\right|^{r-2} u & \text{on} \quad \partial\mathbb{R}^n_+, \end{cases} \tag{P_λ}$$

where $\mathbb{R}^n_+ := \{z = (x,t) \in \mathbb{R}^n : \; x \in \mathbb{R}^{n-1}, \; t > 0\}$, $n \geq 3$, is the Euclidean upper half-space, λ is a real parameter, $2 < q, r < \infty$, and ν is the unit outward normal to the boundary $\partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$.

Let the subspace of $D^{1,2}\left(\mathbb{R}^n_+\right)$ be defined as follows:

$$E := \left\{u \in D^{1,2}(\mathbb{R}^n_+) : \; u|_{\mathbb{R}^{n-1}} \in L^2(\mathbb{R}^{n-1})\right\},$$

where

$$D^{1,2}(\mathbb{R}^n_+) := \overline{C_0^\infty(\mathbb{R}^n)}^{\|\cdot\|_{D^{1,2}(\mathbb{R}^n_+)}}, \quad \text{with} \quad \|u\|_{D^{1,2}(\mathbb{R}^n_+)} := \left(\int_{\mathbb{R}^n_+} |\nabla u|^2 \, \mathrm{d}z\right)^{1/2},$$

and $u|_{\mathbb{R}^{n-1}}$ is understood in the trace sense. An inner product on the space E is defined by

$$\langle u, v\rangle_E := \int_{\mathbb{R}^n_+} \nabla u \cdot \nabla v \, \mathrm{d}z + \int_{\mathbb{R}^{n-1}} uv \, \mathrm{d}x,$$

and the associated norm is given by

$$\|u\|_E^2 := \int_{\mathbb{R}^n_+} |\nabla u|^2 \, \mathrm{d}z + \int_{\mathbb{R}^{n-1}} |u|^2 \, \mathrm{d}x.$$

It can be verified that E is a Hilbert space. Furthermore, since $C_0^\infty(\mathbb{R}^n)$ is dense in $D^{1,2}(\mathbb{R}^n_+)$, combined with the fact that the trace operator is a bounded linear operator from $D^{1,2}(\mathbb{R}^n_+)$ to $L^2(\mathbb{R}^{n-1})$, it can be concluded that the restrictions to $\mathbb{R}^n_+$ of functions in $C_0^\infty(\mathbb{R}^n)$ are dense in E.

A function $u \in E$ is said to be a weak solution of Problem (P_λ) if

$$\int_{\mathbb{R}^n_+} \nabla u \nabla \varphi \, \mathrm{d}z + \int_{\mathbb{R}^{n-1}} u\varphi \, \mathrm{d}x = \int_{\mathbb{R}^{n-1}} \left(\lambda |u|^{q-2} u - |u|^{r-2} u\right) \varphi \, \mathrm{d}x, \tag{1}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$. It is observed that the functions $u(x,t) = at + b$, where $a, b \in \mathbb{R}$, satisfying the condition $-a + b = \lambda b^{q-1} - b^{r-1}$, are classical solutions to Problem (P_λ), but are not weak solutions to (P_λ) in E. Indeed, if u were a weak solution of (P_λ) in E, the condition $u_{|_{\mathbb{R}^{n-1}}} \in L^2(\mathbb{R}^{n-1})$ would be satisfied. However, the trace of u is simply the constant value b, which does not belong to $L^2(\mathbb{R}^{n-1})$ unless $b = 0$.

In the case where $\lambda > 0$ and $2 < r < q < 2_*$, this work is devoted to the demonstration of the existence of a ground state solution of (P_λ), that is, a nontrivial weak solution $u_\lambda = u_\lambda(x,t)$ defined on E, whose energy is minimal among the energy of all nontrivial weak solutions of (P_λ) in E.

1.2 The Case $2 < r < q < 2_*$, $\lambda > 0$.

1.2.1 Auxiliary Results

To establish the existence of a ground state solution for Problem (P_λ) for all $\lambda > 0$ and $2 < r < q < 2_*$, variational methods, particularly those derived from variants of the minimax theorem, are employed. In this context, it is pertinent to study the corresponding energy functional $I_\lambda : E \to \mathbb{R}$, which is defined by

$$I_\lambda(u) := \frac{1}{2}\int_{\mathbb{R}^n_+} |\nabla u|^2 \,\mathrm{d}z + \frac{1}{2}\int_{\mathbb{R}^{n-1}} |u|^2 \,\mathrm{d}x + \frac{1}{r}\int_{\mathbb{R}^{n-1}} \left(u^+\right)^r \mathrm{d}x - \frac{\lambda}{q}\int_{\mathbb{R}^{n-1}} \left(u^+\right)^q \mathrm{d}x,$$

where $u^+(x) = \max\{u(x), 0\}$.

Lemma 1.2.1 *For all $2 \le q \le 2_*$, the Sobolev embedding*

$$E \hookrightarrow L^q\left(\mathbb{R}^{n-1}\right)$$

is continuous.

Proof. With the aid of the interpolation inequality and the trace embedding theorem, for $0 < \theta < 1$, it follows that

$$\begin{aligned}
\int_{\mathbb{R}^{n-1}} |u|^q \,\mathrm{d}x &\le \left(\int_{\mathbb{R}^{n-1}} |u|^2 \,\mathrm{d}x\right)^{\theta q/2} \left(\int_{\mathbb{R}^{n-1}} |u|^{2_*} \,\mathrm{d}x\right)^{(1-\theta)q/2_*}, \\
&\le C\left(\int_{\mathbb{R}^{n-1}} |u|^2 \,\mathrm{d}x\right)^{\theta q/2} \left(\int_{\mathbb{R}^n_+} |\nabla u|^2 \,\mathrm{d}z\right)^{(1-\theta)q/2} \\
&\le C\,\|u\|_E^{\theta q}\,\|u\|_E^{(1-\theta)q},
\end{aligned}$$

where C is a positive constant. This concludes the proof. ∎

As a consequence of Lemma 1.2.1, it is established that the functional I_λ is well-defined on E. Furthermore, by applying standard arguments, it can be shown that $I_\lambda \in C^1(E, \mathbb{R})$, with

$$I_\lambda'(u)\,\varphi = \int_{\mathbb{R}^n_+} \nabla u \nabla \varphi \,\mathrm{d}z + \int_{\mathbb{R}^{n-1}} u\varphi \,\mathrm{d}x - \int_{\mathbb{R}^{n-1}} \left[\lambda\left(u^+\right)^{q-1} - \left(u^+\right)^{r-1}\right]\varphi \,\mathrm{d}x, \quad (2)$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$. Additionally, it can be observed that $u \in E$ is a critical point of I_λ if and only if it is a weak solution to Problem (P_λ). The following lemma establishes a geometric property of the energy functional I_λ.

Lemma 1.2.2 *If $\lambda > 0$ and $2 < r < q < 2_*$, the functional I_λ possesses the geometrical mountain-pass structure on the space E.*

Proof. To establish the lemma, following the strategy outlined in [16], it suffices to verify the following three assertions:

1. $I_\lambda(0) = 0$.

2. There exist $\rho > 0$ and $\alpha > 0$ such that $I_\lambda(u) \geq \alpha$ for all $u \in \partial B_\rho(0)$.

3. There exists $e \in E$ with $\|e\|_E > \rho$ such that $I_\lambda(e) < 0$.

Item 1 is immediate. Using Lemma 1.2.1, it holds that

$$I_\lambda(u) \geq \frac{1}{2}\|u\|_E^2 - \frac{\lambda}{q}\|u^+\|_{L^q(\mathbb{R}^{n-1})}^q \geq \|u\|_E^2\left(\frac{1}{2} - \frac{\lambda C}{q}\|u\|_E^{q-2}\right),$$

where C is a positive constant. Set $\rho = \|u\|_E$ sufficiently small such that

$$\left(\frac{1}{2} - \frac{\lambda C}{q}\rho^{q-2}\right) =: c(\rho) > 0.$$

Consequently, it can be concluded that

$$I_\lambda(u) \geq \rho^2 c(\rho) > 0,$$

which completes the proof of Item 2. To prove Item 3, let $\varphi \in C_0^\infty(\mathbb{R}^n) \setminus \{0\}$, and let $t > 0$. It can be observed that

$$I_\lambda(t\varphi) = \frac{t^2}{2}\|\varphi\|_E^2 + \frac{t^r}{r}\int_{\mathbb{R}^{n-1}}(\varphi^+)^r \,\mathrm{d}x - \frac{\lambda t^q}{q}\int_{\mathbb{R}^{n-1}}(\varphi^+)^q \,\mathrm{d}x,$$

and it is clear that $I_\lambda(t\varphi) \to -\infty$ as $t \to \infty$, since $2 < r < q$. Therefore, the proof of Item 3 is complete, thereby concluding the proof of Lemma 1.2.2. ■

As a consequence of Lemma 1.2.2, the minimax level $c_{q,r}(\mathbb{R}_+^n)$, defined as

$$c_{q,r}(\mathbb{R}_+^n) = \inf_{\gamma\in\Sigma}\max_{t\in[0,1]} I_\lambda(\gamma(t)),$$

is strictly positive, where

$$\Sigma = \{\gamma \in C([0,1]; E) : \gamma(0) = 0 \quad \text{and} \quad I_\lambda(\gamma(1)) < 0\}.$$

Therefore, there exists a Palais-Smale sequence (in short (PS)) $(u_m) \subset E$ at level $c_{q,r}(\mathbb{R}_+^n)$ for the functional I_λ, that is,

$$I_\lambda(u_m) \to c_{q,r}(\mathbb{R}_+^n) \qquad \text{and} \qquad I_\lambda'(u_m) \to 0.$$

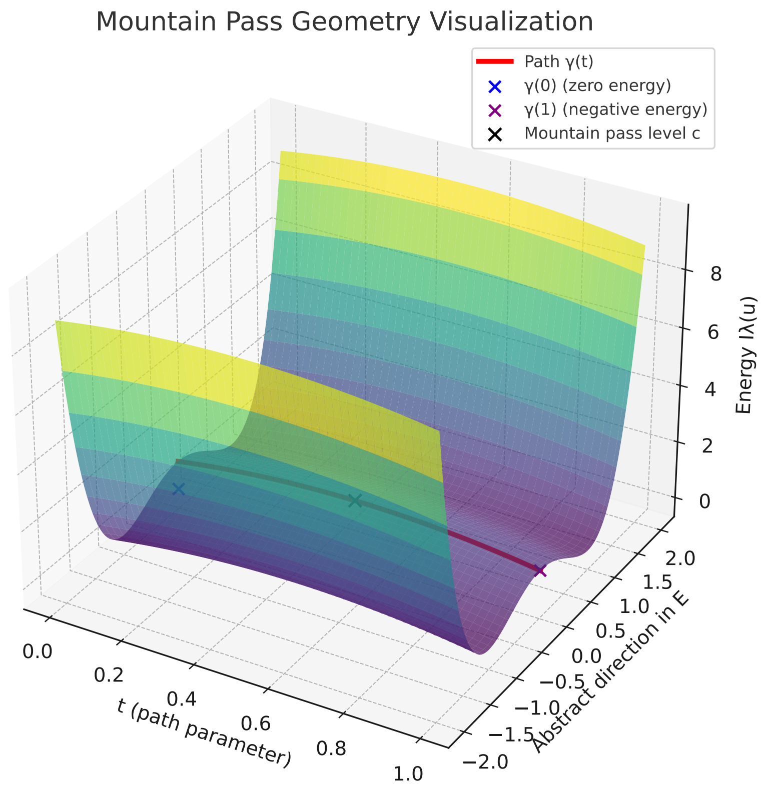


Figure 1.1: Mountain-pass geometry. Generated by AI using Mathplotlib.

The following lemma proves that a (PS) sequence is necessarily bounded in E.

Lemma 1.2.3 *If $(u_m) \subset E$ is a (PS) sequence at the level $c_{q,r}\left(\mathbb{R}^n_+\right)$, then (u_m) is bounded, and there exists a constant $b > 0$ such that*

$$\left\|u_m^+\right\|_{L^q(\mathbb{R}^{n-1})} \geq b > 0,$$

for sufficiently large m.

Proof. Let $(u_m) \subset E$ be a (PS) sequence at the minimax level $c_{q,r}\left(\mathbb{R}^n_+\right)$. For sufficiently large values of m, and for all $\lambda > 0$, it can be verified that

$$\begin{aligned}\left(\frac{1}{2}-\frac{1}{q}\right)\|u_m\|_E^2+\left(\frac{1}{r}-\frac{1}{q}\right)\left\|u_m^+\right\|_{L^r(\mathbb{R}^{n-1})}^r &= I_\lambda\left(u_m\right)-\frac{1}{q}\,I_\lambda'\left(u_m\right)u_m \\ &\leq c_{q,r}\left(\mathbb{R}^n_+\right)+\frac{\epsilon}{q}\left\|u_m\right\|_E.\end{aligned}$$

Since $2 < r < q < 2_*$, this implies that (u_m) is bounded. Furthermore, for sufficiently large values of m, it can be concluded that

$$0 < \frac{c_{q,r}\left(\mathbb{R}^n_+\right)}{2} \leq I_\lambda\left(u_m\right)-\frac{1}{2}\,I_\lambda'\left(u_m\right)u_m < \lambda\left(\frac{q-2}{2q}\right)\left\|u_m^+\right\|_{L^q(\mathbb{R}^{n-1})}^q.$$

Therefore, the proof of Lemma 1.2.3 is complete. ■

1.2.2 Existence of a Ground State Solution

This subsection is dedicated to proving Theorem A. To this end, consider the following definitions of balls centered at $y \in \mathbb{R}^{n-1}$ with radius ρ:

$$B_\rho^+(y) = \left\{z \in \mathbb{R}_+^n : \ |z - (y,0)| < \rho\right\}, \quad \Gamma_\rho(y) = \left\{x \in \mathbb{R}^{n-1} : \ |x - y| < \rho\right\},$$

where $(y,0)$ lies on the boundary of $\mathbb{R}_+^n$.

Lemma 1.2.4 *For each $q \in [2, 2_*]$ and $y \in \mathbb{R}^{n-1}$, there exists a positive constant $C = C(n,q)$ such that*

$$\|u\|_{L^q(\Gamma(y))} \leq C\left(\|\nabla u\|^2_{L^2\left(B_1^+(y)\right)} + \|u\|^2_{L^2(\Gamma_1(y))}\right)^{1/2}, \quad u \in E.$$

Proof. By applying the trace embedding theorem $H^1\left(B_1^+(y)\right) \hookrightarrow L^q\left(\partial B_1^+(y)\right)$ and Friedrich's inequality (see [1]), it follows that for all $u \in E$,

$$\begin{aligned}
\|u\|_{L^q(\Gamma(y))} &\leq C\left(\|\nabla u\|^2_{L^2(B_1^+(y))} + \|u\|^2_{L^2(B_1^+(y))}\right)^{1/2} \\
&\leq C\left(\|\nabla u\|^2_{L^2\left(B_1^+(y)\right)} + \|u\|^2_{L^2(\Gamma_1(y))}\right)^{1/2},
\end{aligned}$$

where C is a positive constant. This completes the proof. ■

Next, before proceeding with the proof of Theorem A, an essential lemma is demonstrated.

Lemma 1.2.5 *If $(u_m) \subset E$ is a (PS) sequence, then there exists a positive constant $C = C(n,r,q)$ such that*

$$\sup_{y \in \mathbb{R}^{n-1}} \int_{\Gamma_1(y)} \left(u_m^+\right)^2 \mathrm{d}x \geq C.$$

Proof. The proof is divided into three distinct cases.

Case 1: Consider $\theta \in (0,1)$ fixed such that $1/r = (1-\theta)/2 + \theta/q$. Interpolation and Lemma 1.2.4 imply that

$$\begin{aligned}
\left\|u_m^+\right\|^r_{L^r(\Gamma_1(y))} &\leq \left\|u_m^+\right\|^{(1-\theta)r}_{L^2(\Gamma_1(y))} \left\|u_m^+\right\|^{\theta r}_{L^q(\Gamma_1(y))} \\
&\leq C\left\|u_m^+\right\|^{(1-\theta)r}_{L^2(\Gamma_1(y))}\left(\left\|\nabla u_m^+\right\|^2_{L^2\left(B_1^+(y)\right)} + \left\|u_m^+\right\|^2_{L^2(\Gamma_1(y))}\right)^{\theta r/2}.
\end{aligned}$$

It is noted that

$$\frac{1}{r} = \frac{1-\theta}{2} + \frac{\theta}{q} \quad \Longleftrightarrow \quad \theta = \frac{q(r-2)}{r(q-2)}.$$

Moreover,

$$2n > \frac{(2-q)}{r-q} r \quad \Rightarrow \quad 2n\,(r-q) \geq r\,(2-q)\,, \quad n \geq 3.$$

Adding the term nrq to the last inequality, and through straightforward calculations, it can be deduced that

$$\frac{2n}{n-1} \geq \frac{2r\,(q-2)}{q\,(r-2)}.$$

By choosing $\overline{r} = 2n/(n-1) \in (2, r)$, it follows that $\theta\overline{r} \geq 2$. Thus, the following inequality holds:

$$\left\|u_m^+\right\|_{L^r(\Gamma_1(y))}^r \leq C \left(\sup_{y \in \mathbb{R}^{n-1}} \int_{\Gamma_1(y)} \left|u_m^+\right|^2 \mathrm{d}x \right)^{(1-\theta)r/2} \left\|u_m^+\right\|_H^{\theta r-2} \left\|u_m^+\right\|_{B_1^+,\Gamma_1,y}^2 ,$$

where

$$\left\|u_m^+\right\|_{B_1^+,\Gamma_1,y} = \left(\left\|\nabla u_m^+\right\|_{L^2\left(B_1^+(y)\right)}^2 + \left\|u_m^+\right\|_{L^2(\Gamma_1(y))}^2 \right)^{1/2} .$$

Covering $\mathbb{R}^{n-1}$ by a family of balls $(\Gamma(y))_{y\in\mathbb{R}^{n-1}}$ such that each point of $\mathbb{R}^{n-1}$ is contained in at most n balls. Summing up this inequality over this family, it can be concluded that

$$0 < b^r < \left\|u_m^+\right\|_{L^r(\mathbb{R}^{n-1})}^r \leq C \left(\sup_{y \in \mathbb{R}^{n-1}} \int_{\Gamma_1(y)} \left|u_m^+\right|^2 \mathrm{d}x \right)^{(1-\theta)r/2} \left\|u_m^+\right\|_E^{\theta r} .$$

Therefore, this inequality, in conjunction with Lemma 1.2.3, completes the proof of case 1.

Case 2: Let $\overline{r} \in (r, q)$ and $\tau \in (0, 1)$ be fixed such that $1/r = (1-\tau)/2 + \tau/\overline{r}$. It is observed that $\tau = \overline{r}\,(r-2)\,/r\,(\overline{r}-2)$. Again, by interpolation and Lemma 1.2.4, it holds that

$$\begin{aligned} \left\|u_m^+\right\|_{L^r(\Gamma_1(y))}^{\overline{r}} &\leq \left\|u_m^+\right\|_{L^2(\Gamma_1(y))}^{(1-\tau)\overline{r}} \left\|u_m^+\right\|_{L^{\overline{r}}(\Gamma_1(y))}^{\tau\overline{r}} \\ &\leq C \left(\sup_{y \in \mathbb{R}^{n-1}} \int_{\Gamma_1(y)} \left|u_m^+\right|^2 \mathrm{d}x \right)^{(1-\tau)\overline{r}/2} \left\|u_m^+\right\|_{B_1^+,\Gamma_1,y}^{\tau\overline{r}/2} , \end{aligned}$$

where C is a positive constant. Through straightforward calculations, it can be concluded that $\tau\overline{r} \geq 2$. Thus, by applying the same reasoning as in Case 1, the conclusion for Case 2 is obtained.

Case 3: Finally, let $\overline{q} \in (q, 2_*)$ and $\beta \in (0,1)$ be such that $1/q = (1-\beta)/r + \beta/\overline{q}$. By interpolation and Lemma 1.2.4, it follows that

$$\begin{aligned} \left\|u_m^+\right\|_{L^q(\Gamma_1(y))}^{\overline{q}} &\leq \left\|u_m^+\right\|_{L^r(\Gamma_1(y))}^{(1-\beta)\overline{q}} \left\|u_m^+\right\|_{L^{\overline{q}}(\Gamma_1(y))}^{\beta\overline{q}} \\ &\leq \left\|u_m^+\right\|_{L^2(\Gamma_1(y))}^{(1-\tau)(1-\beta)\overline{q}} \left\|u_m^+\right\|_{L^{\overline{r}}(\Gamma_1(y))}^{\tau(1-\beta)\overline{q}} \left\|u_m^+\right\|_{L^{\overline{q}}(\Gamma_1(y))}^{\beta\overline{q}} \\ &\leq C\left(\sup_{y\in\mathbb{R}^{n-1}} \int_{\Gamma_1(y)} \left|u_m^+\right|^2 \,\mathrm{d}x\right)^{(1-\tau)(1-\beta)\overline{q}/2} \left\|u_m^+\right\|_{B_1^+,\Gamma_1,y}^{(\tau-\beta\tau+\beta)\overline{q}/2}, \end{aligned}$$

where $\overline{r} \in (r, q)$, $\tau = \overline{r}\,(r-2)\,/r\,(\overline{r}-2) \in (0,1)$ with $\tau\overline{r} > 2$, and $\beta = \overline{q}\,(r-q)\,/q\,(r-\overline{q})$. Through straightforward calculations, it can be inferred that $\overline{q}\,(\tau - \tau\beta + \beta) \geq 2$. Therefore, by applying the reasoning from Case 1, the proof of Case 3 is concluded, thus completing the proof of Lemma 1.2.5. ∎

At this point, the proof of Theorem A can be established.

Proof of Theorem A. Let $(u_m) \subset E$ be a (PS) sequence at the level $c_{q,r}\left(\mathbb{R}_+^n\right)$. In light of Lemma 1.2.5, there exists a sequence of points $(y_m) \subset \mathbb{R}^{n-1}$ such that

$$\int_{\Gamma_1(y_m)} \left(u_m^+\right)^2 \mathrm{d}x \geq \frac{C}{2},$$

where $C = C\,(n,q,r) > 0$. Defining $w_m(x) = u_m\,(x + y_m)$, it can be inferred that

$$\int_{\Gamma_1(0)} \left(w_m^+\right)^2 \mathrm{d}x \geq \frac{C}{2}, \quad I_\lambda(w_m) \to c_{q,r}\left(\mathbb{R}_+^n\right) \quad \text{and} \quad I_\lambda'(w_m) \to 0 \tag{3}$$

(see [25, Lemma I.1]). As a consequence of Lemma 1.2.3, (w_m) is bounded and, up to a subsequence, $w_m \rightharpoonup w$ weakly in E and $w_m \to w$ in $L_{loc}^s\left(\mathbb{R}^{n-1}\right)$ for all $2 \leq s < 2_*$. It is observed that, by (3), w is nontrivial. Furthermore, by taking $\varphi = w^-$ as a test function in (1), it follows that

$$\int_{\mathbb{R}_+^n} \nabla w \nabla w^- \,\mathrm{d}z + \int_{\mathbb{R}^{n-1}} w w^- \,\mathrm{d}x = \int_{\mathbb{R}^{n-1}} \left(\lambda \left(w^+\right)^{q-1} w^- - \left(w^+\right)^{r-1} w^-\right) \,\mathrm{d}x = 0,$$

which implies that $w^- = 0$, thereby ensuring that w is positive. It is hereby claimed that

$$I_\lambda(w) = c_{q,r}\left(\mathbb{R}_+^n\right) \quad \text{and} \quad I_\lambda'(w) = 0.$$

Indeed, given that E is a Hilbert space, the weak convergence $w_m \rightharpoonup w$ implies that

$$\int_{\mathbb{R}_+^n} \nabla w_m \cdot \nabla\varphi \,\mathrm{d}z + \int_{\mathbb{R}^{n-1}} w_m \varphi \,\mathrm{d}x \to \int_{\mathbb{R}_+^n} \nabla w \cdot \nabla\varphi \,\mathrm{d}z + \int_{\mathbb{R}^{n-1}} w\varphi \,\mathrm{d}x,$$

for all $\varphi \in C_0^{\infty}(\mathbb{R}^n)$. Furthermore, since $w_m \to w$ in $L^s_{loc}(\mathbb{R}^{n-1})$ and (w_m) is bounded in E, it can be concluded from Vitali's convergence theorem and the continuity of the map $t \longmapsto (t^+)^{\theta-1}$ that

$$\left(w_m^+\right)^{r-1} \rightharpoonup \left(w^+\right)^{r-1} \quad \text{in} \quad L^{r/r-1}\left(\mathbb{R}^{n-1}\right),$$

and

$$\left(w_m^+\right)^{q-1} \rightharpoonup \left(w^+\right)^{q-1} \quad \text{in} \quad L^{q/q-1}\left(\mathbb{R}^{n-1}\right).$$

As a result, for any $\varphi \in L^s(\mathbb{R}^{n-1})$, the following holds:

$$\int_{\mathbb{R}^{n-1}} \left(w_m^+\right)^{r-1} \varphi \,\mathrm{d}x \longrightarrow \int_{\mathbb{R}^{n-1}} \left(w^+\right)^{r-1} \varphi \,\mathrm{d}x,$$

and

$$\int_{\mathbb{R}^{n-1}} \left(w_m^+\right)^{q-1} \varphi \,\mathrm{d}x \longrightarrow \int_{\mathbb{R}^{n-1}} \left(w^+\right)^{q-1} \varphi \,\mathrm{d}x.$$

Hence, it can be deduced that

$$I_\lambda^{'}(w_m)\,\varphi \to I_\lambda^{'}(w)\,\varphi,$$

for all $\varphi \in E$ and, according to (3), it holds that $I_\lambda^{'}(w) = 0$. Invoking this fact, along with the weak lower semicontinuity of the norm in E, yields the validity of the following estimate:

$$\begin{aligned} I_\lambda(w) &= I_\lambda(w) - \frac{1}{q}\, I_\lambda^{'}(w)w \\ &\leq \varliminf_{m\to\infty} \left(\frac{1}{2} - \frac{1}{q}\right) \|w_m\|_E^2 + \left(\frac{1}{r} - \frac{1}{q}\right) \left\|w_m^+\right\|_{L^r(\mathbb{R}^{n-1})}^r \\ &\leq \varliminf_{m\to\infty} \left(I_\lambda(w_m) - \frac{1}{q} I_\lambda^{'}(w_m)\, w_m\right) \\ &= c_{q,r}\left(\mathbb{R}^n_+\right). \end{aligned}$$

Thus,

$$I_\lambda(w) \leq c_{q,r}\left(\mathbb{R}^n_+\right).$$

Furthermore, by [16, Theorem 1.8], it can be inferred that

$$c_{q,r}\left(\mathbb{R}^n_+\right) = \inf_{v\in\mathcal{N}} I_\lambda(v),$$

where $\mathcal{N} = \left\{v \in E \setminus \{0\} : \ I'_\lambda(v)v = 0\right\}$ is the Nehari manifold associated to I_λ. Since $w \in \mathcal{N}$, it can be concluded that

$$c_{q,r}\left(\mathbb{R}^n_+\right) \leq I_\lambda(w).$$

Therefore, $I_\lambda(w) = c_{q,r}\left(\mathbb{R}^n_+\right)$. Consequently, w is a positive ground state solution of Problem (P_λ) for all $\lambda > 0$, thus completing the proof of Theorem A. ■

1.2.3 The Best Constant for the embedding $E \hookrightarrow L^p\left(\mathbb{R}^{n-1}\right)$

If $n \geq 3$, it was established in Lemma 1.2.1 that, for all $2 \leq p \leq 2_*$, the Sobolev embedding $E \hookrightarrow L^p\left(\mathbb{R}^{n-1}\right)$ is continuous. It was also demonstrated that this embedding is of significant utility, particularly in ensuring the well-definition of the energy functional I_λ. This subsection presents an alternative approach to this embedding, specifically by exploring its connection with the concepts of best constant and extremals.

Initially, it can be observed that, for all $u \in E$ and for all $2 \leq p \leq 2_*$, this embedding implies that the following Sobolev inequality is valid:

$$S\left(\int_{\mathbb{R}^{n-1}} |u|^p \ dx\right)^{2/p} \leq \int_{\mathbb{R}^n_+} |\nabla u|^2 \ dz + \int_{\mathbb{R}^{n-1}} |u|^2 \ dx,$$

for some positive constant S. The best constant for this inequality is defined by

$$S_p\left(\mathbb{R}^n_+\right) = \inf_{u \in E\setminus\{0\}} \frac{\int_{\mathbb{R}^n_+} |\nabla u|^2 \ dz + \int_{\mathbb{R}^{n-1}} |u|^2 \ dx}{\left(\int_{\mathbb{R}^{n-1}} |u|^p \ dx\right)^{2/p}}.$$

An important question concerns the existence of extremals for this embedding, namely, functions u at which the infimum is attained. The Euler-Lagrange associated with inequality (1.2.3) is given by

$$\begin{cases} \Delta w = 0 & \text{in } \mathbb{R}^n_+ \\ \partial_\eta w = |w|^{p-2} w - w & \text{on } \partial\mathbb{R}^n_+, \end{cases} \tag{ADM}$$

where η is the unit outward normal to the boundary $\partial\mathbb{R}^n_+$, and $2 < p < 2_*$ (see [2]). In order to address this question, let the Nehari manifold be defined as follows:

$$\mathcal{N} = \{u \in E \setminus \{0\} : \ J'(u)\, u = 0\},$$

where

$$J(u)=\frac{1}{2}\int_{\mathbb{R}_+^n}|\nabla u|^2\ \mathrm{d}z+\frac{1}{2}\int_{\mathbb{R}^{n-1}}u^2\ \mathrm{d}x-\frac{1}{p}\int_{\mathbb{R}^{n-1}}\left(u^+\right)^p\ \mathrm{d}x$$

is the energy functional associated with (ADM). Let

$$c_p\left(\mathbb{R}_+^n\right)=\inf_{u\in\mathcal{N}}J(u)$$

be the least energy level of J associated with (ADM). The subsequent lemma is established.

Lemma 1.2.6 *The best constant S_p and the least-energy level satisfy the following relation:*

$$c_p\left(\mathbb{R}_+^n\right)=\frac{p-2}{2p}\left(S_p\left(\mathbb{R}_+^n\right)\right)^{p/(p-2)}.$$

Proof. If $u\in\mathcal{N}$, it follows that

$$\int_{\mathbb{R}_+^n}|\nabla u|^2\ \mathrm{d}z+\int_{\mathbb{R}^{n-1}}|u|^2\ \mathrm{d}x=\int_{\mathbb{R}^{n-1}}|u|^p\ \mathrm{d}x,$$

which implies

$$S_p\left(\mathbb{R}_+^n\right)\leq\left(\int_{\mathbb{R}_+^n}|\nabla u|^2\ \mathrm{d}z+\int_{\mathbb{R}^{n-1}}|u|^2\ \mathrm{d}x\right)^{(p-2)/p}.$$

Conversely, it can be observed that

$$J(u)=\frac{p-2}{2p}\left(\int_{\mathbb{R}_+^n}|\nabla u|^2\ \mathrm{d}z+\int_{\mathbb{R}^{n-1}}|u|^2\ \mathrm{d}x\right).$$

Thus, it holds that

$$\frac{p-2}{2p}\left(S_p\left(\mathbb{R}_+^n\right)\right)^{p/(p-2)}\leq J(u).$$

Since the preceding inequality holds for an arbitrary $u\in\mathcal{N}$, and that $c_p\left(\mathbb{R}_+^n\right)$ the least energy level of the functional J satisfies $c_p\left(\mathbb{R}_+^n\right)=\inf_{u\in\mathcal{N}}J(u)$, it can be concluded that

$$\frac{p-2}{2p}\left(S_p\left(\mathbb{R}_+^n\right)\right)^{p/(p-2)}\leq c_p\left(\mathbb{R}_+^n\right). \tag{1.2.1}$$

Now, suppose that $u\in E\setminus\{0\}$ is an extremal for the embedding studied in this section. It is claimed that the function $s\mapsto J(su)$ has a maximum $s=\bar{s}>0$, which is its unique critical point. Indeed, set

$$f(s)=J(su)=\frac{s^2}{2}\left(\int_{\mathbb{R}_+^n}|\nabla u|^2\ \mathrm{d}z+\int_{\mathbb{R}^{n-1}}|u|^2\ \mathrm{d}x\right)-\frac{s^p}{p}\int_{\mathbb{R}^{n-1}}|u|^p\ \mathrm{d}x.$$

Then, it is noted that

$$f'(s)=0 \quad \Longleftrightarrow \quad s^{p-2}=\frac{\int_{\mathbb{R}_+^n}|\nabla u|^2\ \mathrm{d}z+\int_{\mathbb{R}^{n-1}}|u|^2\ \mathrm{d}x}{\int_{\mathbb{R}^{n-1}}|u|^p\ \mathrm{d}x}. \tag{6}$$

One can see that the equation (6) has a solution $s=\overline{s}>0$. Moreover,

$$f''(\overline{s})=\left(\int_{\mathbb{R}_+^n}|\nabla u|^2\ \mathrm{d}z+\int_{\mathbb{R}^{n-1}}|u|^2\ \mathrm{d}x\right)(2-p)<0,$$

since $p>2$. Thus, $f(\overline{s})=J(\overline{s}u)$ is a maximum value, which implies that $\overline{s}u\in\mathcal{N}$. Hence, it follows that

$$S_p\left(\mathbb{R}_+^n\right)=\left(\int_{\mathbb{R}_+^n}|\nabla u|^2\ dz+\int_{\mathbb{R}^{n-1}}|u|^2\ dx\right)^{(p-2)/p}\left(\overline{s}^2\right)^{(p-2)/p}. \tag{1.2.2}$$

Using (1.2.2), it can be concluded that

$$\begin{aligned} c_p\left(\mathbb{R}_+^n\right)\leq J(\overline{s}u) &= \left(\frac{p-2}{2p}\right)\overline{s}^2\left(\int_{\mathbb{R}_+^n}|\nabla u|^2\ \mathrm{d}z+\int_{\mathbb{R}^{n-1}}|u|^2\ \mathrm{d}x\right) \\ &= \frac{p-2}{2p}\left(S_p\left(\mathbb{R}_+^n\right)\right)^{p/(p-2)}, \end{aligned}$$

that is,

$$c_p\left(\mathbb{R}_+^n\right)\leq\frac{p-2}{2p}\left(S_p\left(\mathbb{R}_+^n\right)\right)^{p/(p-2)}. \tag{1.2.3}$$

Finally, from (1.2.1) and (1.2.3), the result can be deduced. ■

1.2.4 Regularity and Power-Type Decay

This subsection is dedicated to the proof of Theorem B. To this end, the following auxiliary nonlinear boundary value problem will be considered:

$$\begin{cases} \Delta v=0 & \text{in} \quad \mathbb{R}_+^n \\ \partial_\eta v+v=\lambda a(x)|v|^{\sigma-1}v-b(x)|v|^{\tau-1}v & \text{on} \quad \mathbb{R}^{n-1}, \end{cases} \tag{AP}$$

where $\lambda>0$, $a,b\in L^\infty\left(\mathbb{R}^{n-1}\right)$, $b>0$, and $1<\tau<\sigma<2_*-1$. Let v be a weak solution of (AP), that is,

$$\int_{\mathbb{R}_+^n}\nabla v\nabla\varphi\ \mathrm{d}z+\int_{\mathbb{R}^{n-1}}v\varphi\ \mathrm{d}x=\int_{\mathbb{R}^{n-1}}\left[\lambda a(x)|v|^{\sigma-1}v-b(x)|v|^{\tau-1}v\right]\varphi\ \mathrm{d}x, \tag{5}$$

for all $\varphi\in C_0^\infty$.

Initially, the Moser iteration method is applied to establish a L^∞ estimate.

Proposition A *Let v be a weak solution of (AP). Then, $v \in L^{\infty}\left(\mathbb{R}^n_+\right)$, and its trace $v|_{\mathbb{R}^{n-1}}$ belongs to $L^{\infty}\left(\mathbb{R}^{n-1}\right)$.*

Proof. Let v be a weak solution of Problem (AP). By modifying the test function, it can be assumed, without loss of generality, that v is nonnegative. For each $k \in \mathbb{N}$, define

$$\varphi_k = v_k^{2(\beta-1)} v \quad \text{and} \quad w_k = v v_k^{\beta-1}, \quad \beta > 1,$$

where $v_k = \min\{v, k\}$, and β to be determined later. It is observed that

$$0 \leq v_k \leq v, \quad \langle \nabla v_k, \nabla v \rangle \geq 0 \quad \text{and} \quad |\nabla v_k| \leq |\nabla v| .$$

The choice of φ_k as a test function in (5) implies that

$$\begin{aligned} & \int_{\mathbb{R}^n_+} v_k^{2(\beta-1)} |\nabla v|^2 \, \mathrm{d}z + 2(\beta-1) \int_{\mathbb{R}^n_+} v v_k^{2(\beta-1)-1} \nabla v_k \nabla v \, \mathrm{d}z + \int_{\mathbb{R}^{n-1}} v_k^{2(\beta-1)} v^2 \, \mathrm{d}x \\ = \; & \lambda \int_{\mathbb{R}^{n-1}} a(x) \, v^{\sigma+1} v_k^{2(\beta-1)} \, \mathrm{d}x - \int_{\mathbb{R}^{n-1}} b(x) \, v^{\tau+1} v_k^{2(\beta-1)} \, \mathrm{d}x, \end{aligned}$$

from which it can be inferred that

$$\begin{aligned} \int_{\mathbb{R}^n_+} v_k^{2(\beta-1)} |\nabla v|^2 \, \mathrm{d}z & \leq \lambda |a|_{\infty} \int_{\mathbb{R}^{n-1}} v_k^{2(\beta-1)} v^{\sigma+1} \mathrm{d}x \\ & \leq C \int_{\mathbb{R}^{n-1}} v^{\sigma-1} w_k^2 \, \mathrm{d}x, \end{aligned}$$

where C is a positive constant. It follows from the classical trace embedding theorem and the preceding inequality that

$$\begin{aligned} \left(\int_{\mathbb{R}^{n-1}} w_k^{2_*} \mathrm{d}x \right)^{2/2_*} & \leq c_1 \int_{\mathbb{R}^n_+} |\nabla w_k|^2 \, \mathrm{d}z \\ & \leq 2c_1 \int_{\mathbb{R}^n_+} v_k^{2(\beta-1)} |\nabla v|^2 + (\beta-1)^2 v^2 v_k^{2(\beta-2)} |\nabla v_k|^2 \, \mathrm{d}z \\ & \leq 2c_1 \int_{\mathbb{R}^n_+} v_k^{2(\beta-1)} |\nabla v|^2 + (\beta-1)^2 v_k^{2(\beta-1)} |\nabla v|^2 \, \mathrm{d}z \\ & \leq c_2 \beta^2 \int_{\mathbb{R}^n_+} v_k^{2(\beta-1)} |\nabla v|^2 \, \mathrm{d}z \\ & \leq c_3 \beta^2 \int_{\mathbb{R}^{n-1}} v^{\sigma-1} w_k^2 \, \mathrm{d}x, \end{aligned}$$

where the inequality $1 + (\beta-1)^2 \leq \beta^2$, with $\beta \geq 1$, has been applied. By applying Hölder's inequality with $p = 2_*/(\sigma-1)$ and $s = 2_*/(2_* - (\sigma-1))$, $1/p + 1/s = 1$, the following inequality holds:

$$\int_{\mathbb{R}^{n-1}} v^{\sigma-1} w_k^2 \,\mathrm{d}x \leq \left(\int_{\mathbb{R}^{n-1}} v^{2_*} \mathrm{d}x\right)^{\frac{\sigma-1}{2_*}} \left(\int_{\mathbb{R}^{n-1}} w_k^{\frac{22_*}{2_*-\sigma+1}} \,\mathrm{d}x\right)^{\frac{2_*-\sigma+1}{2_*}} .$$

Thus, it can be concluded that

$$\begin{aligned} \left(\int_{\mathbb{R}^{n-1}} w_k^{2_*} \,\mathrm{d}x\right)^{2/2_*} &\leq c_3\beta^2 \left(\int_{\mathbb{R}^{n-1}} v^{2_*} \mathrm{d}x\right)^{\frac{\sigma-1}{2_*}} \left(\int_{\mathbb{R}^{n-1}} w_k^{\frac{22_*}{2_*-\sigma+1}} \mathrm{d}x\right)^{\frac{2_*-\sigma+1}{2_*}} \\ &\leq c_3\beta^2 \left(\int_{\mathbb{R}^{n-1}} v^{2_*} \mathrm{d}x\right)^{\frac{\sigma-1}{2_*}} \left(\int_{\mathbb{R}^{n-1}} v^{\frac{\beta 22_*}{2_*-\sigma+1}} \mathrm{d}x\right)^{\frac{2_*-\sigma+1}{2_*}} , \end{aligned}$$

where the inequality $|w_k| \leq |v|^\beta$ has been applied. Let $\beta = (2_* - \sigma + 1)/2$. It is observed that $\beta > 1$. With the application of Lemma 1.2.1, the following holds:

$$\begin{aligned} \left(\int_{\mathbb{R}^{n-1}} w_k^{2_*} \mathrm{d}x\right)^{2/2_*} &\leq c_3\beta^2 \left(\int_{\mathbb{R}^{n-1}} v^{2_*} \mathrm{d}x\right)^{\frac{\sigma-1}{2_*}} \left(\int_{\mathbb{R}^{n-1}} v^{2_*} \mathrm{d}x\right)^{\frac{2\beta}{2_*}} \\ &\leq c_4\beta^2 \|v\|_E^{\sigma-1} \|v\|^{2\beta}_{L^{\beta\alpha_*}(\mathbb{R}^{n-1})} , \end{aligned}$$

where $\alpha_* = (22_*)/(2_* - \sigma + 1)$. Furthermore, the Dominated Convergence Theorem implies that

$$\lim_{k\to 0} \left(\int_{\mathbb{R}^{n-1}} |w_k|^{2_*} \,\mathrm{d}x\right)^{2/2_*} = \left(\int_{\mathbb{R}^{n-1}} |v|^{2_*\beta} \,\mathrm{d}x\right)^{\frac{2\beta}{2_*\beta}} .$$

Consequently, it can be deduced that

$$\|v\|_{L^{2_*\beta}(\mathbb{R}^{n-1})} \leq \left(c_5\beta^2 \|v\|_E^{\sigma-1}\right)^{\frac{1}{2\beta}} \|v\|_{L^{\beta\alpha_*}(\mathbb{R}^{n-1})} .$$

Next, let $\beta_0 = \beta$, and define $\beta_{m+1}\alpha_* = 2_*\beta_m$ inductively for $m = 0, 1, 2, \ldots$. By employing the preceding process for β_1, the inequality above leads to the following:

$$\begin{aligned} \|v\|_{L^{2_*\beta_1}(\mathbb{R}^{n-1})} &\leq \left(c_5\beta_1^2 \|v\|_E^{\sigma-1}\right)^{\frac{1}{2\beta_1}} \|v\|_{L^{\beta_1\alpha_*}(\mathbb{R}^{n-1})} \\ &\leq \left(c_5\beta_1^2 \|v\|_E^{\sigma-1}\right)^{\frac{1}{2\beta_1}} \left(c_5\beta^2 \|v\|_E^{\sigma-1}\right)^{\frac{1}{2\beta}} \|v\|_{L^{\beta\alpha_*}} \\ &\leq \left(c_6 \|v\|_E^{\sigma-1}\right)^{\frac{1}{2\beta_1}+\frac{1}{2\beta}} \beta^{\frac{1}{\beta}} \beta_1^{\frac{1}{\beta_1}} \|v\|_{L^{2_*}(\mathbb{R}^{n-1})} . \end{aligned}$$

Furthermore, it can be noted that $\beta_m = \left(\frac{2_*}{\alpha_*}\right)^m \beta,\ m = 1, 2, \ldots$. Thus, by iteration, it can be inferred that

$$\begin{aligned} \|v\|_{L^{\beta_m 2_*}(\mathbb{R}^{n-1})} &\leq \left(c_6 \|v\|_E^{\sigma-1}\right)^{\frac{1}{2\beta_m}+\ldots+\frac{1}{2\beta_1}+\frac{1}{2\beta}} \beta^{1/\beta}\beta^{1/\beta_1} \ldots \beta_m^{1/\beta_m} \|v\|_{L^{2_*}(\mathbb{R}^{n-1})} \\ &= \left(c_6 \|v\|_E^{\sigma-1}\right)^{\frac{1}{2\beta}\sum_{k=0}^m \left(\frac{\alpha_*}{2_*}\right)^k} \beta^{\frac{1}{\beta}\sum_{k=0}^m \left(\frac{\alpha_*}{2_*}\right)^k} \left(\frac{2_*}{\alpha_*}\right)^{\frac{1}{\beta}\sum_{k=0}^m \left(\frac{\alpha_*}{2_*}\right)^k} \|v\|_{L^{2_*}(\mathbb{R}^{n-1})} . \end{aligned}$$

Taking the limit as $m \to \infty$, and observing that

$$\lim_{m\to\infty} \frac{1}{2\beta} \sum_{k=0}^{m} \left(\frac{\alpha_*}{2_*}\right)^k = \frac{1}{2_* - \sigma - 1} > 0,$$

it can be concluded that

$$\|v\|_{L^\infty(\mathbb{R}^{n-1})} \leq c_7 \left(\|v\|_E^{\sigma-1}\right)^{\frac{1}{2_*-\sigma}} \|v\|_E < \infty.$$

Therefore, $v \in L^\infty(\mathbb{R}^{n-1})$. Now, for each $k \in \mathbb{N}$, define the set

$$\Omega(k) = \left\{z = (x,t) \in \overline{\mathbb{R}^n_+} : \ v(z) > k\right\}.$$

It is observed from the integral formulation of Chebyshev's inequality that $\Omega(k)$ has finite Lebesgue measure, as $v \in L^{2_*}\left(\mathbb{R}^n_+\right)$, and its trace $v|_{\mathbb{R}^{n-1}}$ belongs to $L^2(\mathbb{R}^{n-1})$. Thus, the function

$$\varphi(z) = \begin{cases} (v-k)(z), & \text{if} \quad z \in \Omega(k) \\ 0, & \text{if} \quad z \in \overline{\mathbb{R}^n_+} \setminus \Omega(k), \end{cases}$$

belongs to the space E, and $\nabla\varphi = \nabla v$ in $\Omega(k)$. Given that $v \in L^\infty(\mathbb{R}^{n-1})$, there exists a constant $M > 0$ such that

$$\|v\|_{L^\infty(\mathbb{R}^{n-1})} \leq M.$$

Consequently, by taking $k > M$, it follows that $\varphi(x,0) = 0$ for all $x \in \mathbb{R}^{n-1}$, since $(x,0) \in \overline{\mathbb{R}^n_+} \setminus \Omega(k)$. Hence, by choosing φ as a test function in (5), it can be concluded that

$$\int_{\mathbb{R}^n_+} \nabla v \cdot \nabla\varphi \, \mathrm{d}z + \int_{\mathbb{R}^{n-1}} v\varphi \, \mathrm{d}x = \int_{\mathbb{R}^{n-1}} \left[\lambda a(x) |v|^{\sigma-1} v - b(x) |v|^{\tau-1} v\right] \varphi \, \mathrm{d}x = 0.$$

This implies that

$$\int_{\Omega(k)} |\nabla v|^2 \, \mathrm{d}z = 0,$$

so that v is constant in $\Omega(k)$ or $|\Omega(k)| = 0$. Therefore, $v \in L^\infty\left(\mathbb{R}^n_+\right)$. ■

Remark 1.2.1 *1. Nonnegative weak solutions of Problem* (P_λ) *are positive in* $\mathbb{R}^n_+$ *by Proposition A and Harnack inequality.*

2. *From Proposition A and regularity results established in [23, 36], it follows that weak solutions of Problem* (P_λ) *belong to* $C^{1,\alpha}_{loc}(\overline{\mathbb{R}^n_+})$*, for some* $0<\alpha<1$*. Furthermore, by applying a maximum principle from Vasquez [37], it can be concluded that* $u>0$ *in* $\overline{\mathbb{R}^n_+}$*.*

Next, a Harnack-type inequality is established (see [20, Chapter 8]). To this end, let $y\in\mathbb{R}^{n-1}$ and $0<s<\rho$ be fixed. Consider $B_s^+\subset B_\rho^+$ and $\Gamma_s^+\subset\Gamma_\rho^+$. To simplify the notation, $B_\rho^+(y)$ is abbreviated as B_ρ^+ and $\Gamma_\rho(y)$ is abbreviated as Γ_ρ.

Lemma 1.2.7 *Let* u *be a weak solution of Problem* (P_λ) *satisfying* $0<u\le M$ *in* $B_{3\rho}^+$*. Then, there exist constants* $C=C(n,M)>0$ *and* $\theta_0>1$ *such that*

$$\max_{B_\rho^+} u+\max_{\Gamma_\rho} u\le C\rho^{-(n-1)/\theta_0}\left(\rho^{-1}\|u\|^{\theta_0}_{L^{\theta_0}(B_{2\rho}^+)}+\|u\|^{\theta_0}_{L^{\theta_0}(\Gamma_{2\rho})}\right)^{1/\theta_0}.$$

In particular, it holds

$$\lim_{|z|\to+\infty}u(z)=0,\quad \textit{for}\quad \textit{all}\quad z\in\overline{\mathbb{R}^n_+}.$$

Proof. Let it be given that $u\ge\epsilon>0$ on $\overline{\mathbb{R}^n_+\cap B_{3\rho}^+}$. Define the function φ as

$$\varphi=\eta^2u^\beta,$$

where $\beta>1$, $0\le\eta(z)\le1$, $\eta\in C^1(B_{3\rho})$ and $\sup(\eta)\subset B_\rho^+$. By applying the chain and product rules, the gradient of φ is given by

$$\nabla\varphi=\eta^2\beta u^{\beta-1}\nabla u+2u^\beta\eta\nabla\eta.$$

As a consequence of the choice of φ as a test function in (1), the following holds:

$$\int_{B_\rho^+}\eta^2\beta u^{\beta-1}|\nabla u|^2\,\mathrm{d}z+\int_{B_\rho^+}2u^\beta\eta(\nabla u\cdot\nabla\eta)\,\mathrm{d}z=\lambda\int_{\Gamma_\rho}\eta^2u^\beta u^{q-1}\mathrm{d}x$$
$$-\int_{\Gamma_\rho}\eta^2u^\beta u^{r-1}\mathrm{d}x-\int_{\Gamma_\rho}\eta^2uu^\beta\mathrm{d}x.$$

This implies that

$$\int_{B_\rho^+}\eta^2\beta u^{\beta-1}|\nabla u|^2\,\mathrm{d}z\le\lambda M^{q-2}\int_{\Gamma_\rho}\eta^2u^{\beta+1}\mathrm{d}x+2\int_{B_\rho^+}u^\beta\eta|\nabla u||\nabla\eta|\,\mathrm{d}z,$$

where the inequality $-\nabla u\cdot\nabla\eta\le|\nabla u||\nabla\eta|$ has been used. Considering the Young's inequality

$$cd\le\frac{1}{2}\epsilon^2c^2+\frac{1}{2}\epsilon^{-2}d^2,$$

with

$$c = \eta u^{\frac{\beta-1}{2}} |\nabla u| \quad \text{and} \quad d = u^{\frac{\beta+1}{2}} |\nabla \eta|,$$

the second integral on the right-hand side of the previous inequality can be estimated in the following manner:

$$2\int_{B_\rho^+} u^\beta \eta |\nabla u| |\nabla \eta| \, \mathrm{d}x \leq \epsilon^2 \int_{B_\rho^+} \eta^2 u^{\beta-1} |\nabla u|^2 \, \mathrm{d}x + \epsilon^{-2} \int_{B_\rho^+} u^{\beta+1} |\nabla \eta|^2 \, \mathrm{d}x.$$

Thus, the use of this estimate provides that

$$\left(\beta - \epsilon^2\right) \int_{B_\rho^+} \eta^2 u^{\beta-1} |\nabla u|^2 \, \mathrm{d}z \leq \lambda M^{q-2} \int_{\Gamma_\rho} \eta^2 u^{\beta+1} \mathrm{d}x + \epsilon^{-2} \int_{B_\rho^+} u^{\beta+1} |\nabla \eta|^2 \, \mathrm{d}z,$$

which implies that

$$\begin{aligned} \int_{B_\rho^+} \eta^2 u^{\beta-1} |\nabla u|^2 \, \mathrm{d}z &\leq \frac{\lambda M^{q-2}}{(\beta - \epsilon^2)} \int_{\Gamma_\rho} \eta^2 u^{\beta+1} \mathrm{d}x + \frac{\epsilon^{-2}}{(\beta - \epsilon^2)} \int_{B_\rho^+} u^{\beta+1} |\nabla \eta|^2 \, \mathrm{d}z \\ &= C\left(1 - \frac{\epsilon^2}{\beta}\right)^{-1} \beta^{-1} \left(\epsilon^{-2} \int_{B_\rho^+} u^{\beta+1} |\nabla \eta|^2 \, \mathrm{d}z + \int_{\Gamma_\rho} \eta^2 u^{\beta+1} \mathrm{d}x\right). \end{aligned}$$

By choosing β large enough and defining $v = u^s$, where $2s = \beta + 1$, the following is valid:

$$\begin{aligned} \frac{1}{s^2} \int_{B_\rho^+} (\eta |\nabla v|)^2 \, \mathrm{d}z &= \int_{B_\rho^+} \eta^2 u^{\beta-1} |\nabla u|^2 \, \mathrm{d}z \\ &\leq C\beta^{-1} \left(\int_{B_\rho^+} (v |\nabla \eta|)^2 \, \mathrm{d}z + \int_{\Gamma_\rho} (\eta v)^2 \, \mathrm{d}x \right). \end{aligned}$$

By adding the term $\left(\int_{\Gamma_\rho} (\eta v)^2 \, dx\right)$ to both sides of the preceding inequality, the following inequality is deduced:

$$\left(\frac{1}{s}\right)^2 \|\eta |\nabla v|\|^2_{L^2\left(B_\rho^+\right)} + \|\eta v\|^2_{L^2(\Gamma_\rho)} \leq \left(c\beta^{-1} + 1\right) \|\eta v\|^2_{L^2(\Gamma_\rho)} + c\beta^{-1} \|v |\nabla \eta|\|^2_{L^2\left(B_\rho^+\right)}.$$

This implies that

$$\left(\|\eta |\nabla v|\|^2_{L^2\left(B_\rho^+\right)} + \|\eta v\|^2_{L^2(\Gamma_\rho)}\right)^{1/2} \leq sC\left(1 + \beta^{-1}\right)^{1/2} \left(\|v |\nabla \eta|\|^2_{L^2\left(B_\rho^+\right)} + \|\eta v\|^2_{L^2(\Gamma_\rho)}\right)^{1/2}.$$

Let $\eta(z) = 1$ in B_{r_2} and $\eta(z) = 0$ outside B_{r_1}, where $1 \leq r_2 < \rho \leq r_1 \leq 2$, $|\nabla \eta| \leq \frac{2}{(r_1 - r_2)}$, $2\gamma = 2_*$, and $(1 + \beta^{-1}) < C$. From the preceding inequality, it can be inferred that

$$\left(\|\nabla v\|^2_{L^2\left(B_{r_2}^+\right)} + \|v\|^2_{L^2\left(\Gamma_{r_2}\right)}\right)^{1/2} \leq \left(\|\eta |\nabla v|\|^2_{L^2\left(B_\rho^+\right)} + \|\eta v\|^2_{L^2(\Gamma_\rho)}\right)^{1/2}$$

$$\leq \frac{2sC}{(r_1 - r_2)} \left(\|v\|^2_{L^2(B^+_{r_1})} + \|v\|^2_{L^2(\Gamma_{r_1})} \right)^{1/2}.$$

Through the application of Hölder's inequality, in conjunction with Lemma 1.2.4, it can be concluded that

$$\begin{aligned} \left(\|v\|^{2_*}_{L^{2_*}(B^+_{r_2})} + \|v\|^{2_*}_{L^{2_*}(\Gamma_{r_2})} \right)^{1/2_*} &\leq C \left(\|\nabla v\|^2_{L^2(B^+_{r_2})} + \|v\|^2_{L^2(\Gamma_{r_2})} \right)^{1/2} \\ &\leq \frac{2sC}{(r_1 - r_2)} \left(\|v\|^2_{L^2(B^+_{r_1})} + \|v\|^2_{L^2(\Gamma_{r_1})} \right)^{1/2}. \end{aligned}$$

Given that $v = u^s$, it can be deduced that

$$\left(\int_{B^+_{r_2}} |u|^{s2_*} \, \mathrm{d}z + \int_{\Gamma_{r_2}} |u|^{s2_*} \, \mathrm{d}x \right)^{1/2_*} \leq \frac{2sC}{(r_1 - r_2)} \left(\int_{B^+_{r_1}} |u|^{2s} \, \mathrm{d}z + \int_{\Gamma_{r_1}} |u|^{2s} \, \mathrm{d}x \right)^{1/2}.$$

Now, let

$$\phi(l, t) = \left(\int_{B^+_t(y)} |u|^l \, \mathrm{d}z + \int_{\Gamma_t(y)} |u|^l \, \mathrm{d}x \right)^{1/l},$$

where $l, t > 0$. By choosing the s-th root of the preceding inequality and setting $\theta = 2s$, it can be inferred that

$$\phi(\theta\gamma, r_2) \leq \left(C\theta (r_1 - r_2)^{-1} \right)^{2/\theta} \phi(\theta, r_1).$$

Next, for some $\theta_0 > 0$, define

$$\theta_m = \gamma^m \theta_0, \quad r_m = 1 + 2^{-m}, \quad m = 0, 1, 2, \ldots$$

The choice of θ_0 will be such that $\theta_m \neq 1$. Then, from the preceding inequality, it can be concluded that

$$\begin{aligned} \phi(\theta_{m+1}, r_{m+1}) &\leq \left(C\gamma^m \theta_0 (r_m - r_{m+1})^{-1} \right)^{\frac{2}{(\gamma^m \theta_0)}} \phi(\theta_m, r_m) \\ &\leq \left(C^{2/\theta_0} \right)^{\sum \gamma^{-m}} \left((2\gamma)^{2/\theta_0} \right)^{\sum m\gamma^{-m}} \phi(\theta_0, 2). \end{aligned}$$

Since $\gamma > 1$, by taking the limit in the preceding inequality, it can be deduced that

$$\max_{B^+_1} u + \max_{\Gamma_1} u = \phi(+\infty, 1) \leq C\phi(\theta_0, 2).$$

Finally, by choosing $\theta_0 > 1$ and defining $\overline{z} = \rho z$ with $z \in B^+_2$, and $\overline{x} = \rho x$ with $x \in \Gamma_2$, the proof is concluded. ■

It is now possible to establish a power-type decay for the weak solutions to Problem (P_λ).

Lemma 1.2.8 *If u_λ is a nonnegative weak solution of Problem (P_λ), with $\lambda > 0$, then it has a power-type decay in $\overline{\mathbb{R}^n_+}$, more precisely,*

$$u_\lambda(z) = O\left(|z|^{2-n}\right) \quad as \quad |z| \to \infty.$$

Proof. Let u_λ be a weak solution of Problem (P_λ), and let $A > 0$ be a constant to be chosen later. Consider the function $\phi : \overline{\mathbb{R}^n_+} \to \mathbb{R}$ defined by $\phi = (Au_\lambda - v)_+$, where

$$v(x,t) = \left(\frac{\mu}{(\mu+t)^2 + |x|^2}\right)^{(n-2)/2}, \quad \mu > 0.$$

It is noted that

$$\frac{\partial v}{\partial x_i}(z) = \frac{(2-n)\,\mu^{(n-2)/2} x_i}{\left[(\mu+t)^2 + |x|^2\right]^{n/2}}, \quad i = 1, \ldots, n-1; \quad \frac{\partial v}{\partial t}(z) = \frac{(2-n)\,\mu^{(n-2)/2}(\mu+t)}{\left[(\mu+t)^2 + |x|^2\right]^{n/2}};$$

$$\frac{\partial^2 v}{\partial x_i^2}(z) = \frac{(2-n)\,\mu^{(n-2)/2}}{\left[(\mu+t)^2 + |x|^2\right]^{n/2}} + \frac{n(n-2)\,\mu^{(n-2)/2} x_i^2}{\left[(\mu+t)^2 + |x|^2\right]^{1+n/2}}, \quad i = 1, \ldots, n-1;$$

and

$$\frac{\partial^2 v}{\partial t^2}(z) = \frac{(2-n)\,\mu^{(n-2)/2}}{\left[(\mu+t)^2 + |x|^2\right]^{n/2}} + \frac{n(n-2)\,\mu^{(n-2)/2}(\mu+t)^2}{\left[(\mu+t)^2 + |x|^2\right]^{1+n/2}}.$$

Thus, v is a positive solution to the problem

$$\begin{cases} \Delta v = 0 & \text{in} \quad \mathbb{R}^n_+ \\ \partial_\eta v = (n-2) v^{2_*-1} & \text{on} \quad \mathbb{R}^{n-1}. \end{cases}$$

By applying Lemma 1.2.7, it follows that $u_\lambda(z) \to 0$ as $|z| \to \infty$. Next, choose $A > 0$ such that $\phi \equiv 0$ if $|z| \leq R$. Let

$$\begin{cases} -\Delta(Au - v) = 0 & \text{in} \quad \mathbb{R}^n_+ \\ \partial_\eta(Au - v) = A\left(\lambda u^{q-1} - u^{r-1} - u\right) - (n-2)\, v^{2_*-1} & \text{on} \quad \mathbb{R}^{n-1}. \end{cases}$$

By choosing $\phi = (Au_\lambda - v)_+$ as a test function for this problem, the following holds:

$$\begin{aligned} \int_{|z|\geq R} |\nabla \phi|^2 \, \mathrm{d}z &= \int_{|x|\geq R} \left(\lambda A u^{q-1} - A u^{r-1} - Au\right) \phi \, \mathrm{d}x - (n-2) \int_{|x|\geq R} v^{2_*-1} \phi \, \mathrm{d}x \\ &\leq \int_{|x|\geq R} \left(\lambda A u^{q-1} - A u^{r-1} - Au\right) \phi \, \mathrm{d}x \\ &\leq 0. \end{aligned}$$

Thus, $\phi \equiv 0$ in $\overline{\mathbb{R}^n_+}$. As a result, it follows that $u_\lambda \leq Cv$ in $\overline{\mathbb{R}^n_+}$, which yields the power-type decay as desired. ■

The regularity of solutions to Problem (P_λ) is now examined. First, the following lemma is established.

Lemma 1.2.9 *Let u_λ be a nonnegative weak solution of Problem (P_λ) for $\lambda > 0$. Then, for each $i = 1, \ldots, n$, it holds that $D^i u_\lambda \in H^1\left(\mathbb{R}^n_+\right)$.*

Proof. Consider the i^{th}-difference quotient of size h defined as

$$D_i^h u(z) = \frac{u(x + he_i, t) - u(x, t)}{|h|},$$

for $1 \leq i < n - 1$, and $h \in \mathbb{R} \setminus \{0\}$, where $\{e_1, \ldots, e_{n-1}\}$ is the canonical base of $\mathbb{R}^{n-1}$ (for more details, see [17]). The choice of $\varphi = D_{-h}\left(D_h u\right)$ as a test function in (1) indicates that

$$\begin{aligned} & \int_{\mathbb{R}^n_+} \nabla u \cdot \nabla \left(D_{-h}\left(D_h u\right)\right) \mathrm{d}z + \int_{\mathbb{R}^{n-1}} u \left(D_{-h}\left(D_h u\right)\right) \mathrm{d}x \\ = & \int_{\mathbb{R}^{n-1}} \left(\lambda \left|u\right|^{q-2} u - \left|u\right|^{r-2} u\right) D_{-h}\left(D_h u\right) \mathrm{d}x. \end{aligned}$$

Using the definition of $D_h u$, this expression will be rewritten in a more convenient form as follows. Initially, it can be noted that

$$\begin{aligned} \nabla u \cdot \nabla \left(D_{-h}\left(D_h u\right)\right) &= \frac{1}{|h|} \nabla u \left(x + he_i, t\right) \cdot \nabla \left(D_h u \left(x - he_i, t\right)\right) \\ &- \frac{1}{|h|} \nabla u \left(x + he_i, t\right) \cdot \nabla \left(D_h u\right) \\ &- \nabla \left(D_h u\right) \cdot \nabla \left(D_h u \left(x - he_i, t\right)\right) + \left|\nabla \left(D_h u\right)\right|^2. \end{aligned}$$

Rewriting the first expression on the right-hand side above as

$$\begin{aligned} & \frac{1}{|h|} \nabla u \left(x + he_i, t\right) \cdot \nabla \left(D_h u \left(x - he_i, t\right)\right) \\ = & \nabla \left(D_h u\right) \cdot \nabla \left(D_h u \left(x - he_i, t\right)\right) + \frac{1}{|h|} \nabla u \cdot \nabla \left(D_h u \left(x - he_i, t\right)\right), \end{aligned}$$

it follows that

$$\begin{aligned} \nabla u \cdot \nabla \left(D_{-h}\left(D_h u\right)\right) &= \frac{1}{|h|} \nabla u \cdot \nabla \left(D_h u \left(x - he_i, t\right)\right) - \frac{1}{|h|} \nabla u \left(x + he_i, t\right) \cdot \nabla \left(D_h u\right) \\ &+ \left|\nabla \left(D_h u\right)\right|^2. \end{aligned}$$

Next, it can be observed that

$$\begin{aligned} u \left(D_{-h}\left(D_h u\right)\right) &= \frac{1}{|h|} u \left(x + he_i, t\right) D_h u \left(x - he_i, t\right) \\ &- \frac{1}{|h|} u \left(x + he_i, t\right) D_h u - \left(D_h u\right) \left(D_h u \left(x - he_i, t\right)\right) + \left|D_h u\right|^2. \end{aligned}$$

Rewriting the first expression on the right-hand side above as

$$\begin{aligned}\frac{1}{|h|}u\left(x+he_i,t\right)D_hu\left(x-he_i,t\right) &= \frac{1}{|h|}u\cdot D_hu\left(x-he_i,t\right)\\ &+ \left(D_hu\right)\left(D_hu\left(x-he_i,t\right)\right),\end{aligned}$$

it holds that

$$u\left(D_{-h}\left(D_hu\right)\right)=\frac{1}{|h|}u\cdot D_hu\left(x-he_i,t\right)-\frac{1}{|h|}u\left(x+he_i,t\right)\left(D_hu\right)+\left|D_hu\right|^2$$

Furthermore, it can be observed that

$$\begin{aligned}\left(\lambda u^{q-1}-u^{r-1}\right)D_{-h}\left(D_hu\right) &= \left[\frac{\lambda}{|h|}u^{q-1}\left(D_hu\left(x-he_i,t\right)\right)-\frac{\lambda}{|h|}u^{q-1}D_hu\right]\\ &- \left[\frac{1}{|h|}u^{r-1}\left(D_hu\left(x-he_i,t\right)\right)-\frac{1}{|h|}u^{r-1}D_hu\right].\end{aligned}$$

Hence, the following holds:

$$\begin{aligned}&\frac{1}{|h|}\int_{\mathbb{R}^n_+}\nabla u\cdot\nabla\left(D_hu\left(x-he_i,t\right)\right)\mathrm{d}z-\frac{1}{|h|}\int_{\mathbb{R}^n_+}\nabla u\left(x+he_i,t\right)\cdot\nabla\left(D_hu\right)\mathrm{d}z\\ +&\int_{\mathbb{R}^n_+}\left|\nabla\left(D_hu\right)\right|^2\mathrm{d}z+\frac{1}{|h|}\int_{\mathbb{R}^{n-1}}u\left(D_hu\left(x-he_i,t\right)\right)\mathrm{d}x\\ -&\frac{1}{|h|}\int_{\mathbb{R}^{n-1}}u\left(x+he_i,t\right)\left(D_hu\right)\mathrm{d}x+\int_{\mathbb{R}^{n-1}}\left|D_hu\right|^2\mathrm{d}x\\ =&\frac{\lambda}{|h|}\int_{\mathbb{R}^{n-1}}u^{q-1}\left(D_hu\left(x-he_i,t\right)\right)\mathrm{d}x-\frac{\lambda}{|h|}\int_{\mathbb{R}^{n-1}}u^{q-1}D_hu\ \mathrm{d}x\\ -&\frac{1}{|h|}\int_{\mathbb{R}^{n-1}}u^{r-1}\left(D_hu\left(x-he_i,t\right)\right)\mathrm{d}x+\frac{1}{|h|}\int_{\mathbb{R}^{n-1}}u^{r-1}D_hu\ \mathrm{d}x,\end{aligned}$$

which implies that

$$\begin{aligned}&\int_{\mathbb{R}^n_+}\left|\nabla\left(D_hu\right)\right|^2\mathrm{d}z+\int_{\mathbb{R}^{n-1}}\left|D_hu\right|^2\mathrm{d}x\\ =&\frac{1}{|h|}\int_{\mathbb{R}^n_+}\nabla u\left(x+he_i,t\right)\cdot\nabla\left(D_hu\right)\mathrm{d}z+\frac{1}{|h|}\int_{\mathbb{R}^{n-1}}u\left(x+he_i,t\right)\left(D_hu\right)\mathrm{d}x\\ -&\frac{1}{|h|}\int_{\mathbb{R}^{n-1}}\left(\lambda u^{q-1}-u^{r-1}\right)D_hu\ \mathrm{d}x\\ =&\frac{1}{|h|}\int_{\mathbb{R}^{n-1}}\left(\lambda u^{q-1}\left(x+he_i,t\right)-u^{r-1}\left(x+he_i,t\right)\right)D_hu\ \mathrm{d}x\\ -&\frac{1}{|h|}\int_{\mathbb{R}^{n-1}}\left(\lambda u^{q-1}\left(x,t\right)-u^{r-1}\left(x,t\right)\right)D_hu\ \mathrm{d}x\\ \leq&\ \lambda\int_{\mathbb{R}^{n-1}}\frac{\left|u^{q-1}\left(x+he_i,0\right)-u^{q-1}\left(x,0\right)\right|}{|h|}\left|D_hu\right|\ \mathrm{d}x\\ +&\int_{\mathbb{R}^{n-1}}\frac{\left|u^{r-1}\left(x+he_i,0\right)-u^{r-1}\left(x,0\right)\right|}{|h|}\left|D_hu\right|\mathrm{d}x.\end{aligned}$$

Given that for each $a, b \in (0, +\infty)$ fixed, there exists $\theta \in (0,1)$ such that

$$\left|a^{p-1} - b^{p-1}\right| = (p-1)\left(\theta a + (1-\theta) b\right)^{p-2} |a-b|.$$

This implies that

$$\begin{aligned} \|D_h u\|_E^2 &\leq \lambda (q-1) \int_{\mathbb{R}^{n-1}} \left(\theta u(x+he_i,0) + (1-\theta) u(x,0)\right)^{q-2} |D_h u|^2 \,\mathrm{d}x \\ &+ (r-1) \int_{\mathbb{R}^{n-1}} \left(\tau u(x+he_i,0) + (1-\tau) u(x,0)\right)^{r-2} |D_h u|^2 \,\mathrm{d}x, \end{aligned}$$

where $\tau \in (0,1)$. For fixed $\Gamma_1 := \Gamma_{R_1}(0) \subset \mathbb{R}^{n-1}$ and $\Gamma_2 := \Gamma_{R_2}(0) \subset \mathbb{R}^{n-1}$, it can be observed that

$$\begin{aligned} & \int_{\mathbb{R}^{n-1}} \left(\theta u(x+he_i,0) + (1-\theta) u(x,0)\right)^{q-2} |D_h u|^2 \,\mathrm{d}x \\ + & \int_{\mathbb{R}^{n-1}} \left(\tau u(x+he_i,0) + (1-\tau) u(x,0)\right)^{r-2} |D_h u|^2 \,\mathrm{d}x \\ \leq & \; 2^{q-2} \int_{\mathbb{R}^{n-1}} \left((u(x+he_i,0))^{q-2} + (u(x,0))^{q-2} \right) |D_h u|^2 \,\mathrm{d}x \\ + & \; 2^{r-2} \int_{\mathbb{R}^{n-1}} \left((u(x+he_i,0))^{r-2} + (u(x,0))^{r-2} \right) |D_h u|^2 \,\mathrm{d}x \\ \leq & \; 2^{q-2} \left[\|u\|_{L^\infty(\Gamma_1)}^{q-2} \int_{\Gamma_1} |D_h u|^2 \,\mathrm{d}x + \|u\|_{L^\infty(\Gamma_1^c)}^{q-2} \int_{\Gamma_1^c} |D_h u|^2 \,\mathrm{d}x \right] \\ + & \; 2^{r-2} \left[\|u\|_{L^\infty(\Gamma_2)}^{r-2} \int_{\Gamma_2} |D_h u|^2 \,\mathrm{d}x + \|u\|_{L^\infty(\Gamma_2^c)}^{r-2} \int_{\Gamma_2^c} |D_h u|^2 \,\mathrm{d}x \right], \end{aligned}$$

which implies that

$$\begin{aligned} \|D_h u\|_E^2 &\leq \lambda (q-1) 2^{q-2} \left[\|u\|_{L^\infty(\mathbb{R}^{n-1})}^{q-2} \int_{\Gamma_1} |D_h u|^2 \,\mathrm{d}x + \|u\|_{L^\infty(\Gamma_1^c)}^{q-2} \int_{\mathbb{R}^{n-1}} |D_h u|^2 \,\mathrm{d}x \right] \\ &+ (r-1) 2^{r-2} \left[\|u\|_{L^\infty(\mathbb{R}^{n-1})}^{r-2} \int_{\Gamma_2} |D_h u|^2 \,\mathrm{d}x + \|u\|_{L^\infty(\Gamma_2^c)}^{r-2} \int_{\mathbb{R}^{n-1}} |D_h u|^2 \,\mathrm{d}x \right]. \end{aligned}$$

By Lemma 1.2.9, it is possible to choose Γ_1 and Γ_2 such that

$$\|u\|_{L^\infty(\Gamma_1^c)}^{q-2} \leq \frac{1}{\lambda (q-1) 2^{q-1}} \quad \text{and} \quad \|u\|_{L^\infty(\Gamma_2^c)}^{r-2} \leq \frac{1}{(r-1) 2^{r-1}}.$$

Hence, it can be inferred that

$$\|D_h u\|_E^2 \leq C\left(r, q, \|u\|_{L^\infty(\mathbb{R}^{n-1})}\right) \left(\int_{\Gamma_1} |D_h u|^2 \, dx + \int_{\Gamma_2} |D_h u|^2 \, dx \right).$$

Furthermore, since $u \in C^{1,\alpha}(\Gamma)$, it can be concluded that

$$\int_{\mathbb{R}^n_+} |\nabla (D_h u)|^2 \,\mathrm{d}z + \int_{\mathbb{R}^{n-1}} |D_h u|^2 \,\mathrm{d}x \leq C. \tag{6}$$

Let $D^j = \frac{\partial}{\partial x_j}$ for $1 \le j \le n$. Given any $\varphi \in C_0^\infty(\mathbb{R}^n)$, the definition of weak derivative, in conjunction with (6), implies that

$$\begin{aligned}
\left| \int_{\mathbb{R}_+^n} u \cdot D_{-h}\left(D^j \varphi\right) \mathrm{d}z \right| &= \left| \int_{\mathbb{R}_+^n} u \left(\frac{D^j \varphi(x - he_i, t) - D^j \varphi(x,t)}{|h|} \right) \mathrm{d}z \right| \\
&= \left| \int_{\mathbb{R}_+^n} \left(\frac{D^j u(x,t) - D^j u(x + he_i, t)}{|h|} \right) \varphi(x,t) \ \mathrm{d}z \right| \\
&= \left| \int_{\mathbb{R}_+^n} D_h\left(D^j u\right) \varphi \ \mathrm{d}z \right| \\
&\le \left\| D_h\left(D^j u\right) \right\|_{L^2\left(\mathbb{R}_+^n\right)} \|\varphi\|_{L^2\left(\mathbb{R}_+^n\right)} \\
&\le C \|\varphi\|_{L^2\left(\mathbb{R}_+^n\right)} .
\end{aligned}$$

By taking the limit as $|h| \to 0$, it can be inferred that

$$\left| \int_{\mathbb{R}_+^n} u \cdot D^i\left(D^j \varphi\right) \ \mathrm{d}z \right| \le C \|\varphi\|_{L^2\left(\mathbb{R}_+^n\right)}, \tag{7}$$

for all $1 \le i < n-1$ and $1 \le j \le n$. Finally, selecting $\varphi \in C_0^\infty(\mathbb{R}^n)$ as a test function in (1) and applying the inequality

$$\begin{aligned}
\left| \int_{\mathbb{R}_+^n} u \cdot D^n\left(D^n \varphi\right) \ \mathrm{d}z \right| &= \left| \int_{\mathbb{R}_+^n} D^n u \cdot D^n \varphi \ \mathrm{d}z \right| \\
&\le \sum_{i=1}^{n-1} \left| \int_{\mathbb{R}_+^n} u \cdot D^{i,i} \varphi \ \mathrm{d}z \right| \\
&\le C \|\varphi\|_{L^2\left(\mathbb{R}_+^n\right)},
\end{aligned}$$

the result follows by invoking [10, Proposition 9.3]. ∎

The following lemma states that the derivatives of a nonnegative weak solution u_λ of Problem (P_λ) exhibit decay. This result is fundamental to the development of the present study.

Lemma 1.2.10 *If $u_\lambda \in E$ is a nonnegative weak solution of Problem (P_λ), where $\lambda > 0$, then for each $1 \le i \le n$, it holds that*

$$\lim_{|z| \to \infty} \left| D^i u_\lambda(z) \right| = 0, \quad \textit{for} \quad \textit{all} \quad z \in \mathbb{R}_+^n.$$

Proof. Let $1 \le i \le n-1$. For each $\varphi \in C_0^\infty(\mathbb{R}^n)$, the choice of $D^i \varphi$ as a test function in (1) indicates that

$$\int_{\mathbb{R}_+^n} \nabla u \cdot \nabla\left(D^i \varphi\right) \ \mathrm{d}z + \int_{\mathbb{R}^{n-1}} u \ D^i \varphi \ \mathrm{d}x = \int_{\mathbb{R}^{n-1}} \left(\lambda u^{q-1} - u^{r-1}\right) D^i \varphi \ \mathrm{d}x.$$

As a consequence of the definition of weak derivative, it can be inferred that

$$\begin{aligned}\int_{\mathbb{R}^n_+} \nabla\left(D^i u\right)\cdot\nabla\varphi \,\mathrm{d}z + \int_{\mathbb{R}^{n-1}} \left(D^i u\right)\varphi \,\mathrm{d}x &= \int_{\mathbb{R}^{n-1}} \lambda\,(q-1)\,u^{q-2}\left(D^i u\right)\varphi\,\mathrm{d}x \\ &- \int_{\mathbb{R}^{n-1}} (r-1)\,u^{r-2}\left(D^i u\right)\varphi\,\mathrm{d}x.\end{aligned}$$

Hence, $v = D^i u$ is a weak solution of Problem (AP), where $\sigma = \tau = 1$, $a = (q-1)\,u^{q-2}$, and $b = (r-1)\,u^{r-2}$. Therefore, by Proposition A, $D^i u \in L^\infty\left(\mathbb{R}^n_+\right)$, and its trace belongs to $L^\infty\left(\mathbb{R}^{n-1}\right)$. Now, by choosing $\varphi = \eta\,(D^i u)^\beta_\pm$ as a test function in (1), where $\eta \in C_0^\infty\left(\mathbb{R}^n\right)$ and $\beta > 1$, the argument follows as in the proof of Lemma 1.2.7, completing the case $1 \le i \le n-1$. For the case where $i = n$, it is noted that

$$\frac{\partial}{\partial\nu}\left(u\,(x,0)\right) = -u_t \quad \Rightarrow u_t = -\lambda u^{q-1} + u^{r-1} + u \quad \Rightarrow \Delta u_t = 0,$$

that is, u_t is a harmonic function. Furthermore,

$$\begin{aligned}\frac{\partial}{\partial\nu}\left(u_t\right) &= -\lambda\,(q-1)\,u^{q-2}\,\frac{\partial u}{\partial\nu} + (r-1)\,u^{r-2}\,\frac{\partial u}{\partial\nu} + \frac{\partial u}{\partial\nu} \\ &= \lambda\,(q-1)\,u^{q-2}u_t - (r-1)\,u^{r-2}u_t - u_t,\end{aligned}$$

that is,

$$\frac{\partial}{\partial\nu}\left(u_t\right) + u_t = \lambda\,(q-1)\,u^{q-2}u_t - (r-1)\,u^{r-2}u_t.$$

This implies that

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} \varphi\,\frac{\partial}{\partial\nu}(u_t)\,\mathrm{d}x + \int_{\mathbb{R}^{n-1}} \varphi u_t\,\mathrm{d}x &= \lambda\,(q-1)\int_{\mathbb{R}^{n-1}} \left(u^{q-2}u_t\right)\varphi\,\mathrm{d}x \\ &- (r-1)\int_{\mathbb{R}^{n-1}} \left(u^{r-2}u_t\right)\varphi\,\mathrm{d}x.\end{aligned}$$

Since u_t is harmonic, it can be deduced that

$$\int_{\mathbb{R}^n_+} \nabla\varphi\cdot\nabla u_t\,\mathrm{d}z + \int_{\mathbb{R}^{n-1}} u_t\varphi\,\mathrm{d}x = \int_{\mathbb{R}^{n-1}} \left(\lambda\,(q-1)\,u^{q-2}u_t - (r-1)\,u^{r-2}u_t\right)\varphi\,\mathrm{d}x.$$

Therefore, u_t is a weak solution of Problem (AP), and the proof is concluded. ∎

Corollary 1.2.1 *If u_λ is a nonnegative weak solution of Problem (P_λ), where $\lambda > 0$, then $u_\lambda \in C^{2,\alpha}_{loc}\left(\overline{\mathbb{R}^n_+}\right) \cap C^\infty\left(\mathbb{R}^n_+\right)$.*

Proof. Since u_λ is a harmonic function, it is concluded that $u_\lambda \in C^\infty\left(\mathbb{R}^n_+\right)$. By Proposition A and regularity results established in [23], it can be inferred that weak solutions of Problem (P_λ) belong to $C^{1,\alpha}_{loc}\left(\mathbb{R}^n_+\right)$. According to Lemma 1.2.9, if $v = D^i u$, then v is a weak solution of Problem (AP), which implies that $v = D^i u \in L^\infty\left(\mathbb{R}^n_+\right)$, with its trace belonging to $L^\infty\left(\mathbb{R}^{n-1}\right)$. Moreover, the case $i = n$ was addressed in Lemma 1.2.9. By applying the results from [23], it can be deduced that $D^i u \in C^\infty\left(\mathbb{R}^n_+\right)$, thereby concluding the proof. ■

1.2.5 Symmetry and Exponential Decay

This subsection is dedicated to the proof of Theorem C and Theorem D. The proof of Theorem C relies on the classical moving plane method, as presented in [32] (see also [21, 35, 40] for related results).

Proof of Theorem C. Let $\theta > 0$ be a real number. Define

$$E_\theta = \left\{z \in \overline{\mathbb{R}^n_+} :\ x_1 > \theta\right\}, \quad T_\theta = \left\{z \in \mathbb{R}^n_+ :\ x_1 = \theta\right\},$$

and consider the reflection

$$z = (x_1, \ldots, x_{n-1}, t) \longmapsto z^\theta = (2\theta - x_1, x_2, \ldots, x_{n-1}, t),$$

where $z \in E_\theta$. Let the function w^θ be given by

$$w^\theta(z) = u\left(z^\theta\right) - u(z).$$

It is observed that if $z \in T_\theta$, then $u\left(z^\theta\right) = u(z)$. It is asserted that there exists $\theta > 0$ such that

$$w^\theta(z) > 0 \qquad \text{for} \quad \text{all} \quad z \in E_\theta. \tag{8}$$

Indeed, since $u(z) \to 0$ as $|z| \to \infty$, it is possible to choose a sufficiently large θ such that

$$w^\theta(2\theta, x_2, \ldots, t) = u(0, x_2, \ldots, t) - u(2\theta, x_2, \ldots, t) > 0. \tag{9}$$

It will be established that (8) holds for this choice of θ. Suppose, for the sake of contradiction, that there exists $z_\theta \in E_\theta$ such that $w^\theta(z_\theta) \le 0$. In particular, it is possible to choose

$$w^\theta(z_\theta) = \inf\left\{w^\theta(z) :\ z \in E_\theta\right\} \le 0.$$

It is claimed that $z_\theta \in \mathbb{R}^{n-1} \cap E_\theta$. Otherwise, $z_\theta \in \mathbb{R}^n_+ \cap E_\theta$, and thus $B(z_\theta, 2\delta) \subset \mathbb{R}^n_+ \cap E_\theta$ for some $\delta > 0$ sufficiently small. Defining $v^\theta(z) = w^\theta(z) - w^\theta(z_\theta)$, it follows that $v^\theta(z_\theta) = 0$, and

$$\begin{cases} \Delta v^\theta = 0 & \text{in} \quad B(z_\theta, \delta) \\ v^\theta \geq 0 & \text{in} \quad B(z_\theta, \delta). \end{cases}$$

By applying Harnack's inequality, it is obtained that $\sup_{B(z_\theta,\delta)} v^\theta = 0$. Moreover, since v^θ is harmonic in $B(z_\theta, \delta)$, the strong maximum principle implies that $v^\theta \equiv 0$ in $B(z_\theta, \delta)$. Using unique continuation methods for elliptic equations, it can be concluded that $v^\theta \equiv 0$ in E_θ. Consequently, w^θ is a non-positive constant in E_θ, which contradicts (9). Hence, $w^\theta(z) > 0$ for all $z \in \mathbb{R}^n_+ \cap E_\theta$, implying that

$$w^\theta(z) \geq 0 \quad \forall z \in E_\theta \cap \mathbb{R}^{n-1}.$$

As a result, $z_\theta \in \mathbb{R}^{n-1} \cap E_\theta$ and $w^\theta(z_\theta) = 0$. Furthermore, by selecting a ball $B \subset \mathbb{R}^n_+ \cap E_\theta$ such that $z_\theta \in \partial B$, the following holds:

$$\begin{cases} \Delta w^\theta = 0 & \text{in} \quad B \\ w^\theta > 0 & \text{in} \quad B. \end{cases}$$

This, in conjunction with Hopf's lemma, implies that $\left(\frac{\partial w^\theta}{\partial \nu}\right)(z_\theta) < 0$, which leads to a contradiction with

$$\begin{aligned} 0 > \frac{\partial w^\theta}{\partial \nu}(z_\theta) &= \frac{\partial u}{\partial \nu}(z^\theta) - \frac{\partial u}{\partial \nu}(z_\theta) \\ &= \lambda\left(u^{q-1}(z^\theta) - u^{q-1}(z_\theta)\right) - \left(u^{r-1}(z^\theta) - u^{r-1}(z_\theta)\right) - \left(u(z^\theta) - u(z_\theta)\right) \\ &= 0. \end{aligned}$$

Now, let

$$\theta_0 := \inf\{\theta > 0 \quad \text{such} \quad \text{that} \quad (8) \quad \text{holds}\}.$$

It will be proven that $\theta_0 = 0$. Suppose, for the sake of contradiction, that $\theta_0 > 0$. It is observed that $w^{\theta_0} \equiv 0$ on T_{θ_0} and

$$\begin{cases} \Delta w^{\theta_0} = 0 & \text{in} \quad E_{\theta_0} \\ w^{\theta_0} > 0 & \text{in} \quad E_{\theta_0}. \end{cases}$$

As a consequence of Hopf's lemma, it can be concluded that

$$2u_{x_1}(\theta_0,\overline{x}) = -\left(w^{\theta_0}\right)_{x_1}(\theta_0,\overline{x}) < 0, \tag{10}$$

where $\overline{x} = (x_2,\ldots,t)$. Thus, there exists $\epsilon > 0$ such that

$$2(\theta_0-\epsilon) - x_1 < \theta_0 - \epsilon < x_1 < \theta_0,$$

and

$$w^{\theta_0-\epsilon}(x_1,\overline{x}) = u(2(\theta_0-\epsilon)-x_1,\overline{x}) - u(x_1,\overline{x}) > 0.$$

Consequently, for each $(\theta_0,\overline{x}) \in T_{\theta_0}$, there exists $\delta > 0$ such that

$$w^{\theta_0-\epsilon}(z) > 0 \quad \text{for} \quad \text{all} \quad z \in B((\theta_0,\overline{x}),\delta) \cap \left(\mathbb{R}^n_+ \setminus E_{\theta_0}\right). \tag{11}$$

It is asserted that there exists $\epsilon > 0$ such that

$$w^{\theta_0-\epsilon}(z) > 0 \quad \text{for} \quad \text{all} \quad x \in E_{\theta_0-\epsilon}. \tag{12}$$

If this were not the case, there exist sequences $(\theta_k) \subset \mathbb{R}_+$ and $(z_k) \subset E_{\theta_k}$ such that $\theta_k \to \theta_0$, and

$$w^{\theta_k}(z_k) < 0 \quad \text{with} \quad \operatorname{dist}(z_k, T_{\theta_0}) \to 0.$$

The following two cases are examined:

Case 1: There exists a subsequence (z_{k_l}) such that $z_{k_l} \to z_0 \in T_{\theta_0}$, which is impossible due to (11).

Case 2: (z_k) satisfies $\|z_k\| \to \infty$. In this case, by (11), it can be assumed, without loss of generality, that

$$w^{\theta_k}(z_k) = \inf\left\{w^{\theta_k}(z) : \ z \in E_{\theta_k}\right\}.$$

Given that $v^{\theta_k}(z) := w^{\theta_k}(z) - w^{\theta_k}(z_k)$, it holds that $v^{\theta_k}(z_k) = 0$, and

$$\begin{cases} \Delta v^{\theta_k} = 0 & \text{in} \quad B(z_k,\delta_k) \\ v^{\theta_k} > 0 & \text{in} \quad B(z_k,\delta_k). \end{cases}$$

As a consequence of the Harnack inequality, $v^{\theta_k} \equiv 0$ in $B(z_k,\delta_k)$. Combined with unique continuation methods for elliptic equations, this implies that w^{θ_k} is constant in E_{θ_k}, leading to a contradiction with $w^{\theta_k} \in E$. Consequently, the assertion (12) contradicts the choice of θ_0 for $\theta_0 > 0$. Thus, it must be that $\theta_0 = 0$, and it can be concluded that

$$u(-x_1, \ldots, t) \geq u(x_1, \ldots, t) \qquad \text{in} \qquad \overline{\mathbb{R}^n_+}.$$

An analogous argument establishes that

$$u(-x_1, \ldots, t) \leq u(x_1, \ldots, t) \qquad \text{in} \qquad \overline{\mathbb{R}^n_+}.$$

Thus, u is symmetric with respect to the plane T_0, and $u_{x_1} = 0$ on T_0. This argument remains valid after any rotation of coordinate axes in the variables $x_2, \ldots, x_{n-1}$.

Finally, by defining $u(x,t) = v(\rho, t)$, where $\rho = |x|$, it will be demonstrated that $v_\rho(\rho, t) < 0$ for all $(\rho, t) \in (0, \infty) \times [0, \infty)$. To this end, it is noted that since u is symmetric in $\mathbb{R}^{n-1}$, the reasoning used to derive (10) similarly applies to $x_2, \ldots, x_{n-1}$ and all $\theta > 0$. Let $x_0 \in \mathbb{R}^{n-1}$ be such that $x_0 = (x_{1,0}, \ldots, x_{n-1,0})$ with $x_{i,0} > 0$. Noting that

$$v_\rho(\rho_0, t) = \sum_{i=1}^{n-1} \frac{\partial u}{\partial x_i}(x_0, t) \cdot \frac{x_{i,0}}{|x_0|} < 0, \quad \rho_0 = |x_0|,$$

and given that u is symmetric, it follows that $v_\rho(\rho, t) < 0$ for all $(\rho, t) \in (0, +\infty) \times (0, +\infty)$. To complete the proof, it remains to demonstrate that $v_\rho(\rho, 0) < 0$ for all $\rho > 0$. Suppose, for the sake of contradiction, that $v_\rho(r_0, 0) = 0$ for some $\rho_0 > 0$. Since $u \in C^{2,\alpha}_{loc}\left(\overline{\mathbb{R}^n_+}\right) \cap C^\infty\left(\mathbb{R}^n_+\right)$, it can be verified that

$$\begin{cases} \Delta v_\rho = 0 & \text{in} \quad B^+(\rho_0) \\ v_\rho < 0 & \text{in} \quad B^+(\rho_0), \end{cases}$$

where $B^+(\rho_0) = B_\delta(\rho_0, 0) \cap \overline{\mathbb{R}^n_+}$ for some $\delta > 0$. By applying Hopf's lemma, the following holds:

$$\begin{aligned} 0 < \frac{\partial v_\rho}{\partial \eta}(\rho_0, 0) &= -(v_\rho)_t(\rho_0, 0) \\ &= -(v_t)_\rho(\rho_0, 0) \\ &= -\frac{\partial}{\partial \rho}\left(-\lambda v^{q-1} + v^{r-1} + v\right)(\rho_0, 0) \\ &= v_\rho(\rho_0, 0)\left[\lambda(q-1)v^{q-2} - (r-1)v^{r-2} - 1\right] \\ &= 0, \end{aligned}$$

which leads to a contradiction, thereby concluding the proof. ■

An additional result must be established prior to presenting the demonstration of the exponential decay of u_λ, where $\lambda > 0$.

Lemma 1.2.11 *Let u_λ be a nonnegative weak solution of Problem (P_λ), where $\lambda > 0$. Then, for each $s > 0$, there exists a constant $c_i = c_i(s) > 0$ such that, for every $i = 1, \ldots, n-1$, the following holds:*

$$u_\lambda(x_1, \ldots, x_i, \ldots, t) \leq c_i \left| D^i u_\lambda(x_1, \ldots, x_i, \ldots, t) \right|, \quad |x_i| \geq s.$$

Proof. Let $i \in \{1, \ldots, n-1\}$ be fixed, and let $s > 0$. For each $z \in \overline{\mathbb{R}^n_+}$, define the function

$$D^i_s u(z) := \begin{cases} D^i u(x_1, \ldots, x_i + s, \ldots, t), & x_i > 0 \\ D^i u(x_1, \ldots, -x_i + s, \ldots, t), & x_i \leq 0. \end{cases}$$

As a consequence of Theorem C, $D^i u = u_\rho \cdot \frac{x_i}{\rho} < 0$ for all $x_i > 0$. By combining the results from Theorem B, it is possible to choose $R > 0$ and $A_{i_1} := A_{i_1}(R, s) > 0$ such that

$$u^{q-2}(z) \leq \frac{1}{2\lambda(q-1)}, \quad \text{for all} \quad z \in \overline{\mathbb{R}^n_+}, \quad \text{with} \quad |z| \geq R,$$

and

$$\varphi_i := \left(A_{i_1} u + D^i_s u\right)_+ \equiv 0, \quad \text{for all} \quad z \in \overline{\mathbb{R}^n_+}, \quad \text{with} \quad |z| \leq R.$$

Furthermore, the choice of φ_i as a test function in the problem

$$-\Delta\left(A_{i_1} u + D^i_s u\right) = 0 \quad \text{in} \quad \mathbb{R}^n_+$$

with

$$\begin{aligned} -\frac{\partial}{\partial t}\left(A_{i_1} u + D^i_s u\right) &= \lambda A_{i_1} u^{q-1} + \lambda(q-1) u^{q-2} D^i_s u - \left(A_{i_1} u^{r-1} + (r-1) u^{r-2} D^i_s u\right) \\ &- \left(A_{i_1} u + D^i_s u\right) \quad \text{on} \quad \mathbb{R}^{n-1}, \end{aligned}$$

results in

$$\begin{aligned} \int_{\mathbb{R}^n_+} |\nabla \varphi_i|^2 \, \mathrm{d}z &= \int_{\mathbb{R}^{n-1}} \varphi_i \left[\lambda A_{i_1} u^{q-1} + \lambda(q-1) u^{q-2} D^i_s u\right] \mathrm{d}x \\ &- \int_{\mathbb{R}^{n-1}} \varphi_i \left[A_{i_1} u^{r-1} + (r-1) u^{q-2} D^i_s u\right] \mathrm{d}x - \int_{\mathbb{R}^{n-1}} \left(A_{i_1} u + D^i_s u\right) \varphi_i \, \mathrm{d}x. \end{aligned}$$

This implies that

$$\begin{aligned}\int_{\mathbb{R}^n_+} |\nabla\varphi_i|^2 \,\mathrm{d}z + \int_{\mathbb{R}^{n-1}} \varphi_i^2 \,\mathrm{d}x &= \lambda \int_{\mathbb{R}^{n-1}} \left(A_{i_1} u^{q-1} + (q-1)\, u^{q-2} D^i_s u\right) \varphi_i \,\mathrm{d}x \\ &- \int_{\mathbb{R}^{n-1}} \left(A_{i_1} u^{r-1} + (r-1)\, u^{r-2} D^i_s u\right) \varphi_i \,\mathrm{d}x \\ &\leq \lambda\,(q-1) \int_{\mathbb{R}^{n-1}} u^{q-2} \left(A_{i_1} u + D^i_s u\right) \varphi \,\mathrm{d}x \\ &- \int_{\mathbb{R}^{n-1}} u^{r-2} \left(A_{i_1} u + D^i_s u\right) \varphi_i \,\mathrm{d}x,\end{aligned}$$

from which it is deduced that

$$\int_{\mathbb{R}^n_+} |\nabla\varphi_i|^2 \,\mathrm{d}z + \int_{\mathbb{R}^{n-1}} \varphi_i^2 \,\mathrm{d}x \leq \frac{1}{2} \int_{\mathbb{R}^{n-1}} \varphi_i^2 \,\mathrm{d}x.$$

Thus, it can be inferred that

$$\varphi_i \equiv 0 \quad \text{in} \quad \overline{\mathbb{R}^n_+}.$$

Hence, the following inequality holds:

$$u\,(x_1, \ldots, x_i, \ldots, t) \leq A_{i_1}^{-1} \left(-D^i u\,(x_1, \ldots, x_i + s, \ldots, t)\right).$$

Given that $D^i u\,(x_1, \ldots, x_i, \ldots, t) < 0$ for $x_i > 0$, it can be concluded that

$$u\,(x_1, \ldots, x_i + s, \ldots, t) < u\,(x_1, \ldots, x_i, \ldots, t) \leq A_{i_1}^{-1} \left(-D^i u\,(x_1, \ldots, x_i + s, \ldots, t)\right). \tag{13}$$

Now, for every $z \in \overline{\mathbb{R}^n_+}$, consider the function

$$D^i_{-s} u\,(z) := \begin{cases} D^i u\,(x_1, \ldots, x_i - s, \ldots, t)\,, & x_i < 0 \\ D^i u\,(x_1, \ldots, -x_i - s, \ldots, t)\,, & x_i \geq 0. \end{cases}$$

Again, by Theorem C, it is obtained that $D^i u = u_\rho \cdot \frac{x_i}{\rho} > 0$ for all $x_i < 0$, which, in conjunction with Theorem B, implies the existence of $R > 0$ and $A_{i_2} := A_{i_2}(R, s) > 0$ such that

$$u^{q-2}(z) \leq \frac{1}{2\lambda}, \quad \text{for} \quad \text{all} \quad z \in \overline{\mathbb{R}^n_+}, \quad \text{with} \quad |z| \geq R,$$

and

$$\phi_i := \left(A_{i_2} u - D^i_{-s} u\right)_+ \equiv 0 \quad \text{for} \quad \text{all} \quad z \in \overline{\mathbb{R}^n_+}, \quad \text{with} \quad |z| \leq R.$$

As a consequence of the choice of ϕ_i as a test function in the problem

$$-\Delta\left(A_{i_2}u - D^i_{-s}u\right) = 0 \quad \text{in} \quad \mathbb{R}^n_+,$$

with

$$\begin{aligned} -\frac{\partial}{\partial t}\left(A_{i_2}u - D^i_{-s}u\right) &= \left(\lambda A_{i_2}u^{q-1} - \lambda\left(q-1\right)u^{q-2}D^i_{-s}u\right) \\ &- \left(A_{i_2}u^{r-1} - \left(r-1\right)u^{r-2}D^i_{-s}u\right) \\ &- \left(A_{i_2}u - D^i_{-s}u\right) \quad \text{on} \quad \mathbb{R}^{n-1}, \end{aligned}$$

the following holds:

$$\begin{aligned} \int_{\mathbb{R}^n_+}\left|\nabla\phi_i\right|^2\,\mathrm{d}z + \int_{\mathbb{R}^{n-1}}\phi_i^2\,\mathrm{d}x &= \int_{\mathbb{R}^{n-1}}\left(\lambda A_{i_2}u^{q-1} - \lambda\left(q-1\right)u^{q-2}D^i_{-s}u\right)\phi_i\,\mathrm{d}x \\ &- \int_{\mathbb{R}^{n-1}}\left(A_{i_2}u^{r-1} - \left(r-1\right)u^{r-2}D^i_{-s}u\right)\phi_i\,\mathrm{d}x. \end{aligned}$$

Therefore, by applying the same reasoning as in the preceding argument, it is concluded that $\phi_i \equiv 0$ in $\overline{\mathbb{R}^n_+}$, which implies that

$$u\left(x_1,\ldots,x_i,\ldots,t\right) \leq A_{i_2}^{-1}D^iu\left(x_1,\ldots,x_i-s,\ldots,t\right).$$

Since $D^iu\left(x_1,\ldots,x_i,\ldots,t\right) > 0$ for $x_i < 0$, it can be inferred that

$$u\left(x_1,\ldots,x_i-s,\ldots,t\right) < u\left(x_1,\ldots,x_i,\ldots,t\right) \leq A_{i_2}^{-1}D^iu\left(x_1,\ldots,x_i-s,\ldots,t\right) \quad (14)$$

for $x_i < 0$. Thus, by applying inequalities (13) and (14), the desired conclusion is established. ■

Proof of Theorem D. Let $x_i > 0\,(1 \leq i \leq n-1)$. As a consequence of Lemma 1.2.11, the following holds:

$$\frac{\partial}{\partial x_i}\left(\ln\left(u\left(x_1,\ldots,x_i+s,\ldots,t\right)\right)\right) = \frac{D^iu\left(x_1,\ldots,x_i+s,\ldots,t\right)}{u\left(x_1,\ldots,x_i+s,\ldots,t\right)} \leq -c_i^{-1}.$$

By integrating the preceding expression, the following result is obtained:

$$\ln\left(u\left(x_1,\ldots,x_i+s,\ldots,t\right)\right) - \ln\left(u\left(x_1,\ldots,s,\ldots,t\right)\right) \leq -c_i^{-1}x_i,$$

which leads to the inequality

$$u\left(x_1,\ldots,x_i+s,\ldots,t\right) \leq u\left(x_1,\ldots,s,\ldots,t\right)\exp\left(-c_i^{-1}\left|x_i\right|\right), \quad x_i > 0. \quad (15)$$

Once more, applying Lemma 1.2.11, it is established that

$$\frac{\partial}{\partial x_i}\left(\ln\left(u\left(x_1,\ldots,x_i-s,\ldots,t\right)\right)\right)=\frac{D^i u\left(x_1,\ldots,x_i-s,\ldots,t\right)}{u\left(x_1,\ldots,x_i-s,\ldots,t\right)}\geq c_i^{-1},\quad x_i<0.$$

As a result of integrating the preceding expression, the following holds

$$\ln\left(u\left(x_1,\ldots,-s,\ldots,t\right)\right)-\ln\left(u\left(x_1,\ldots,x_i-s,\ldots,t\right)\right)\geq c_i^{-1}\left(-x_i\right),$$

which implies that

$$u\left(x_1,\ldots,x_i-s,\ldots,t\right)\leq u\left(x_1,\ldots,-s,\ldots,t\right)\exp\left(-c_i^{-1}\left|x_i\right|\right),\quad x_i<0. \tag{16}$$

Therefore, from (15)-(16) and Theorem B, it can be concluded that

$$u\left(x_1,\ldots,x_i,\ldots,t\right)\leq c_1\cdot\frac{1}{\left(1+t^2\right)^{(n-2)/2}}\exp\left(-c_2\left|x_i\right|\right),\quad \left|x_i\right|\geq s>0.$$

Thus, the proof is complete. ■

This section concludes by establishing the lower power-type decay with respect to the variable t.

Proposition B *Let u_λ be a nonnegative weak solution of Problem (P_λ), where $\lambda>0$. For each $x\in\mathbb{R}^{n-1}$ fixed, there exist positive numbers c_1,c_2 and t_0 such that*

$$\frac{c_1}{t^n}\leq u\left(x,t\right)\leq\frac{c_2}{\left(1+t^2\right)^{(n-2)/2}}\quad for\quad all\quad t\geq t_0.$$

Proof. By applying the mean value theorem for harmonic functions, it follows that

$$u\left(x,t\right)=\frac{1}{w_n R^n}\int_{B((x,t),R)}u\left(\overline{z}\right)\ \mathrm{d}\overline{z},\quad \text{for}\quad \text{all}\quad B\left(\left(x,t\right),R\right)\subset\mathbb{R}^n_+,$$

where $\overline{z}=\left(\overline{x},\overline{t}\right)$, and w_n denotes the volume of the unit ball in $\mathbb{R}^n$. It can be assumed that $t>1$. By choosing $R=t$, the following is obtained:

$$u\left(x,t\right)=\frac{1}{w_n t^n}\int_{B((x,t),t)}u\left(\overline{z}\right)\ \mathrm{d}\overline{z}\geq\frac{1}{w_n t^n}\int_{B((x,1),1)}u\left(\overline{z}\right)\ \mathrm{d}\overline{z}=\frac{C(x)}{t^n},$$

for all $t\geq 1$. The proof is completed by combining Theorem D with the preceding results. ■

1.3 The Case $2 < q < r < 2_*$, $\lambda > 0$

In this section, a minimization argument is employed to demonstrate Theorem E. Let F be a subspace of E defined as:

$$F := \left\{ u \in E : \int_{\mathbb{R}^{n-1}} |u|^r \, dx < \infty \right\},$$

endowed by the norm

$$\|u\|_F := \left(\|u\|_E^2 + \|u\|_{L^r(\mathbb{R}^{n-1})}^2 \right)^{1/2}.$$

It can be noted that F is a reflexive Banach space, and that the set of the restrictions to $\mathbb{R}_+^n$ of functions in $C_0^\infty(\mathbb{R}^n)$ is dense in F. These facts facilitate the establishment of a natural correspondence between the definition of weak solutions to Problem (P_λ) in F and the definition provided in (1). In light of Lemma 1.2.1, it can be inferred that the embedding

$$F \hookrightarrow L^s\left(\mathbb{R}^{n-1}\right), \quad 2 \le s \le 2_* \tag{17}$$

is continuous. This implies that the functional I_λ is well-defined in F. Furthermore, it holds that $I_\lambda \in C^1(F; \mathbb{R})$.

The development of this section relies on the classical variational theory presented in [13] (see also [4] for related results). Initially, the following fundamental property of the energy functional I_λ is demonstrated.

Lemma 1.3.1 *Let $2 < q < r < 2_*$ and $\lambda > 0$. Then there exist $\rho > 0$ and $\alpha > 0$ such that*

$$I_\lambda(u) \ge \alpha > 0, \quad \text{for} \quad \|u\|_F = \rho.$$

Proof. Without loss of generality, it can be assumed that

$$\|u\|_E^2 + \|u\|_{L^r(\mathbb{R}^{n-1})}^2 = \|u\|_F^2 = \rho^2 < 1.$$

Using Lemma 1.2.1 and (17), it follows that

$$\begin{aligned} I_\lambda(u) &\ge \frac{1}{2}\|u\|_E^2 - \frac{c_1\lambda}{q}\|u\|_E^q \\ &\ge c_2\|u\|_F^2\left(\frac{1}{2} - \frac{c_1\lambda}{q}\|u\|_F^{q-2}\right) \\ &= c_2\rho^2\left(\frac{1}{2} - \frac{c_1\lambda}{q}\rho^{q-2}\right), \end{aligned}$$

where c_1 and c_2 are positive constants. Hence, it can be concluded that

$$I_\lambda (u) > c(\rho)\rho^2 > 0.$$

This leads to the desired result. ■

The following lemma establishes a significant property associated with the energy functional I_λ.

Lemma 1.3.2 *Let $2 < q < r < 2_*$ and $\lambda > 0$. Then, there exists $\Lambda > 0$ such that*

$$-\infty < \inf_{\|u\|_F \leq \rho} I_\lambda (u) < 0, \quad \forall \lambda > \Lambda. \tag{18}$$

Proof. Let

$$\Lambda := \inf_{\|u\|_F \leq \rho} \left\{ \frac{q}{2} \|u\|_E^2 + \frac{q}{r} \|u\|_{L^r(\mathbb{R}^{n-1})}^r : \int_{\mathbb{R}^{n-1}} |u|^q \, \mathrm{d}x = 1 \right\}.$$

It is stated that $\Lambda > 0$. Indeed, suppose, for the sake of contradiction, that there exists a sequence (u_k) satisfying $\|u_k\|_F \leq \rho$, such that

$$\frac{q}{2} \|u_k\|_E^2 + \frac{q}{r} \|u_k\|_{L^r(\mathbb{R}^{n-1})}^r = o_k (1) \qquad \text{and} \qquad \int_{\mathbb{R}^{n-1}} |u_k|^q \, \mathrm{d}x = 1,$$

where $o_k (1) \to 0$ as $k \to \infty$. Consequently, by applying (17), it can be concluded that

$$1 = \int_{\mathbb{R}^{n-1}} |u_k|^q \, \mathrm{d}x \leq c \|u_k\|_F^q = o_k (1),$$

which leads to a contradiction. On the other hand, if $\lambda > \Lambda$, it follows from the definition of Λ that there exists u_λ, satisfying $\|u_\lambda\|_F \leq \rho$ and $\int_{\mathbb{R}^{n-1}} |u_\lambda|^q \, \mathrm{d}x = 1$, such that

$$\lambda > \frac{q}{2} \|u_\lambda\|_E^2 + \frac{q}{r} \|u_\lambda\|_{L^r(\mathbb{R}^{n-1})}^r.$$

This implies that

$$I_\lambda (u_\lambda) = \frac{1}{2} \|u_\lambda\|_E^2 + \frac{1}{r} \|u_\lambda\|_{L^r(\mathbb{R}^{n-1})}^r - \frac{\lambda}{q} \int_{\mathbb{R}^{n-1}} |u_\lambda|^q \, \mathrm{d}x < 0,$$

and consequently, (18) holds. ■

Lemma 1.3.3 *Let $2 < q < r < 2_*$ and $\lambda > 0$. The energy functional I_λ is weakly lower semicontinuous in F.*

Proof. Consider a sequence $(u_k) \subset F$ such that $u_k \rightharpoonup u$ weakly in F. The objective is to establish that

$$I_\lambda(u) \leq \liminf_{k\to\infty} I_\lambda(u_k).$$

Since the norm $\|\cdot\|_E$ arises from an inner product, the map

$$u \mapsto \frac{1}{2}\|u\|_E^2 = \frac{1}{2}\int_{\mathbb{R}^n_+} |\nabla u|^2 \, dz + \frac{1}{2}\int_{\mathbb{R}^{n-1}} |u|^2 \, \mathrm{d}x$$

is convex and weakly lower semicontinuous in E, and therefore also in F. Hence,

$$\frac{1}{2}\|u\|_E^2 \leq \liminf_{k\to\infty} \frac{1}{2}\|u_k\|_E^2.$$

Regarding the nonlinear positive term, it is observed that the function $s \mapsto (s^+)^r$, where $r \geq 1$, is convex and continuous on $\mathbb{R}$. This implies that the associated functional

$$u \mapsto \int_{\mathbb{R}^{n-1}} \left(u^+\right)^r \; \mathrm{d}x$$

is convex on $L^r\left(\mathbb{R}^{n-1}\right)$. Moreover, due to the continuity of the trace embedding $F \hookrightarrow L^r(\mathbb{R}^{n-1})$, this functional is well-defined and bounded on F, with $u_k \rightharpoonup u$ also weakly in $L^r\left(\mathbb{R}^{n-1}\right)$. Consequently, by the weak lower semicontinuity of convex and lower semicontinuous functionals on reflexive Banach spaces (see [10]), it follows that

$$\frac{1}{r}\int_{\mathbb{R}^{n-1}} (u^+)^r \, \mathrm{d}x \leq \liminf_{k\to\infty} \frac{1}{r}\int_{\mathbb{R}^{n-1}} (u_k^+)^r \, \mathrm{d}x.$$

For the last term, it is observed that the embedding $F \hookrightarrow L^q(\mathbb{R}^{n-1})$ is compact since $q < r < 2_*$, which implies strong convergence in L^q. Therefore, $u_k^+ \to u^+$ strongly in $L^q(\mathbb{R}^{n-1})$, and thus

$$\int_{\mathbb{R}^{n-1}} (u_k^+)^q \, \mathrm{d}x \to \int_{\mathbb{R}^{n-1}} (u^+)^q \, \mathrm{d}x,$$

yielding

$$\lim_{k\to\infty} \frac{\lambda}{q}\int_{\mathbb{R}^{n-1}} (u_k^+)^q \, \mathrm{d}x = \frac{\lambda}{q}\int_{\mathbb{R}^{n-1}} (u^+)^q \, \mathrm{d}x.$$

Combining the above estimates gives

$$\liminf_{k\to\infty} I_\lambda(u_k) \geq I_\lambda(u),$$

which establishes the weak lower semicontinuity of I_λ in F. ∎

Proof of Theorem E. In accordance with Lemmas 1.3.2, 1.3.3, and [13, Theorem 1.1], there exists $u_\lambda \in \overline{B_\rho}(0)$ such that

$$-\infty < \inf_{u\in \overline{B_\rho}(0)} I_\lambda(u) = I_\lambda(u_\lambda), \quad \text{for} \quad \text{all} \quad \lambda > \Lambda.$$

In addition, since Lemma 1.3.1 holds, it follows from [13, Corollary 3.2] that I_λ has a nontrivial weak solution $u_\lambda \in B_\rho(0)$ with $I_\lambda(u_\lambda) < 0$, for all $\lambda > \Lambda$. ■

1.4 Nonexistence of solution

This section is dedicated to the proof of Theorem F. The initial segment of the proof involves adapting a Pohozaev-type identity, as presented in [1, Proposition 5.1], to validate items 1 and 2. To complete the proof of the theorem, an elementary mathematical argument is developed, which specifically establishes, for the case where $\lambda > 0$ and $2 < q < r$, an interval within which the solution to the problem is the trivial solution. Following this, a comparison with Theorem E can be made.

Proof of Theorem F. Let u_λ be a weak solution of Problem (P_λ). By selecting $\phi = u_\lambda$ as a test function in (1), the following expression is obtained:

$$\int_{\mathbb{R}^n_+} |\nabla u_\lambda|^2 \, \mathrm{d}z + \int_{\mathbb{R}^{n-1}} |u_\lambda|^2 \, \mathrm{d}x = \int_{\mathbb{R}^{n-1}} \left(\lambda |u_\lambda|^q - |u_\lambda|^r\right) \, \mathrm{d}x. \tag{19}$$

This implies that the inequality

$$\lambda \|u_\lambda\|^q_{L^q(\mathbb{R}^{n-1})} \geq \|u_\lambda\|^r_{L^r(\mathbb{R}^{n-1})}. \tag{20}$$

is valid. Consequently, if $\lambda \leq 0$, it follows that $u_\lambda \equiv 0$, thereby proving the validity of item 1. Consider $\lambda > 0$. From this point onward, the notation u_λ will be simplified to u. Through the application of [1, Proposition 5.1], with $p = 2$, $f = 0$, and $g(s) = \lambda |s|^{q-2} s - |s|^{r-2} s - s$, it can be inferred that

$$\frac{n-2}{2} \int_{\mathbb{R}^n_+} |\nabla u|^2 \, \mathrm{d}z = (n-1) \int_{\mathbb{R}^{n-1}} \left[\lambda \frac{|u|^q}{q} - \frac{|u|^r}{r} - \frac{|u|^2}{2}\right] \mathrm{d}x. \tag{21}$$

Incorporating (19) into (21), yields that

$$\lambda \left(\frac{n-2}{2} - \frac{(n-1)}{q}\right) \|u\|^q_{L^q(\mathbb{R}^{n-1})} + \left(\frac{n-1}{r} - \frac{(n-2)}{2}\right) \|u\|^r_{L^r(\mathbb{R}^{n-1})} + \frac{1}{2} \|u\|^2_{L^2(\mathbb{R}^{n-1})} = 0,$$

which implies that

$$\lambda\left(\frac{n-1}{2}-\frac{(n-1)}{q}\right)\|u\|_{L^q(\mathbb{R}^{n-1})}^q \leq \left(\frac{n-2}{2}-\frac{(n-1)}{r}\right)\|u\|_{L^r(\mathbb{R}^{n-1})}^r. \tag{22}$$

Thus, by applying (20) and (22), item 2 follows. To complete the proof of Theorem F, consider $2 < q < r < 2_*$. Define

$$A(u) = \int_{\mathbb{R}^{n-1}} |u|^q \, dx \quad \text{and} \quad B(u) = \int_{\mathbb{R}^{n-1}} |u|^r \, dx.$$

It can be noted that expression (19) can be written as

$$\|u\|_E^2 = \lambda A(u) - B(u). \tag{23}$$

By employing interpolation and Lemma 1.2.1, it follows that for $q = 2(1-\theta) + \theta r$, the following is valid:

$$\begin{aligned} A(u) &= \left(\int_{\mathbb{R}^{n-1}} |u|^2 \, dx\right)^{(1-\theta)} \left(\int_{\mathbb{R}^{n-1}} |u|^r \, dx\right)^{\theta} \\ &\leq \|u\|_E^{2(1-\theta)} (B(u))^{\theta}, \quad \theta \in (0,1). \end{aligned}$$

It is observed that

$$q = 2(1-\theta) + \theta r \quad \Rightarrow \quad \theta = \frac{q-2}{r-2} \in (0,1).$$

This implies that

$$(A(u))^{r-2} \leq \|u\|_E^{2(r-q)} (B(u))^{q-2}. \tag{24}$$

Next, by using (23), it can be deduced that

$$\|u\|_E^2 = \lambda t^{q-2} A(u) - t^{r-2} B(u).$$

Letting $G(t,u) = \lambda t^{q-2} A(u) - t^{r-2} B(u)$, it follows that

$$\lim_{t\to 0^+} G(t,u) = \lim_{t\to 0^+} t^{q-2}\left(\lambda A(u) - t^{r-q} B(u)\right) = 0.$$

Furthermore, since

$$G_t(t,u) = \lambda (q-2) t^{q-3} A(u) - (r-2) t^{r-3} B(u),$$

it can be concluded that

$$G_t(t,u) = 0 \quad \Longleftrightarrow \quad t = \left(\lambda \frac{(q-2) A(u)}{(r-2) B(u)}\right)^{1/(r-q)}.$$

Set $\bar{t}=\left(\lambda\frac{(q-2)A(u)}{(r-2)B(u)}\right)^{1/(r-q)}$. By applying (24), the following holds:

$$\begin{aligned} G\left(\bar{t},u\right) &= \lambda^{\frac{r-2}{r-q}}\frac{\left(A\left(u\right)\right)^{\frac{r-2}{r-q}}}{\left(B\left(u\right)\right)^{\frac{q-2}{r-q}}}\left[\left(\frac{q-2}{r-2}\right)^{\frac{q-2}{r-q}}-\left(\frac{q-2}{r-2}\right)^{\frac{r-2}{r-q}}\right] \\ &\leq \lambda^{\frac{r-2}{r-q}}\left[\left(\frac{q-2}{r-2}\right)^{\frac{q-2}{r-q}}\left(\frac{r-q}{r-2}\right)\right]\|u\|_E^2 . \end{aligned}$$

Thus, by selecting $\lambda^{\frac{r-2}{r-q}}\left[\left(\frac{q-2}{r-2}\right)^{\frac{q-2}{r-q}}\left(\frac{r-q}{r-2}\right)\right]<1$, it can be deduced that

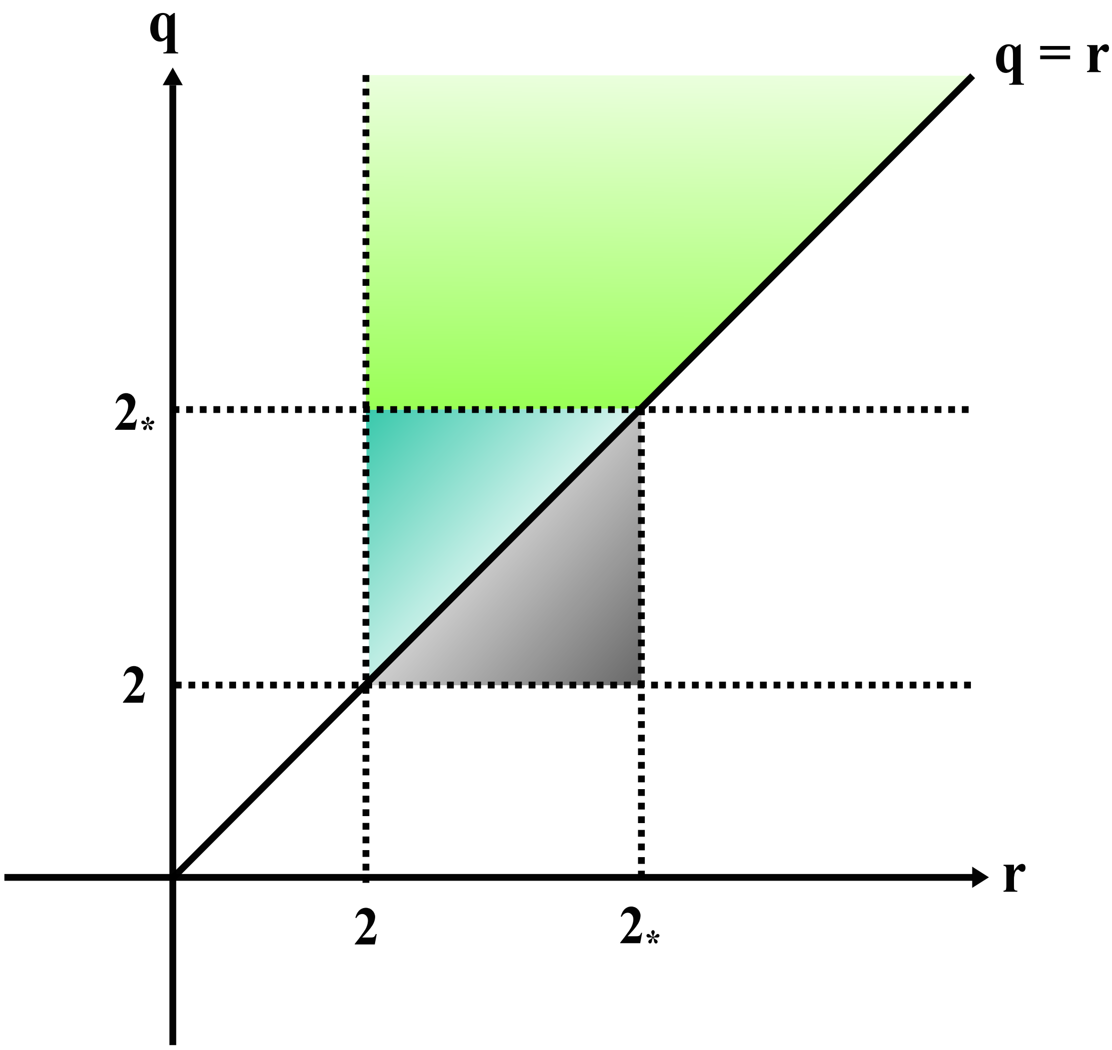
$$G\left(\bar{t},u\right)<\|u\|_E^2=G\left(1,u\right),$$

which is a contradiction, since $G\left(\bar{t},u\right)$ is a maximum value of G. It can be concluded that if

$$\lambda<\frac{1}{\left[\left(\frac{q-2}{r-2}\right)^{\frac{q-2}{r-2}}\left(\frac{r-q}{r-2}\right)\right]^{\frac{r-q}{r-2}}},$$

Problem (P_λ) admits only the trivial solution, thereby completing the proof. ■

Figure 1.1 presents a summary of the results discussed in Chapter 1 concerning the existence and nonexistence of solutions to Problem (P_λ).



1. Only the trivial solution $u_\lambda \equiv 0$ for all $\lambda \in (0, \lambda^*)$, where:

$$\lambda^* = \frac{1}{\left[\left(\frac{q-2}{r-2}\right)^{\frac{q-2}{r-2}}\left(\frac{r-q}{r-2}\right)\right]^{\frac{r-q}{r-2}}}$$

2. Nontrivial weak solution for $\lambda > \Lambda$.

Positive ground state solution for all $\lambda > 0$.

Only the trivial solution $u_\lambda \equiv 0$ for all $\lambda > 0$.

Figure 1.2: Summary - Chapter 1

Chapter 2

Existence and Concentration of Positive Harmonic Functions with Nonlinear Boundary Condition in Expanding Domains

This chapter presents the findings pertinent to the principal problem addressed in this work, namely Problem (P_ϵ). As outlined in the introduction, this problem constitutes a concentration problem as $\epsilon \to 0$. The primary objective of this chapter is to establish the framework that leads to the proof of Theorem J, which is the concentration theorem.

Initially, a result concerning the existence of solutions is established, followed by a comprehensive analysis of the asymptotic behavior of these solutions. It is essential to note that these results are a crucial part of what is required for the proof of Theorem J. The chapter will conclude by exploring the scenario in which the domain contracts, demonstrating that, under these circumstances, the only solution is a constant function.

The chapter commences with an exposition of the fundamental results that establish a rigorous foundation for subsequent analysis.

2.1 Preliminaries

Let the Hilbert space $H^1(\Omega_\epsilon)$ endowed with the inner product

$$\langle u, v\rangle = \int_{\Omega_\epsilon} [\nabla u \nabla v + uv] \, \mathrm{d}z,$$

and the corresponding norm

$$\|u\|_{H^1(\Omega_\epsilon)} = \left(\int_{\Omega_\epsilon} \left[|\nabla u|^2 + |u|^2 \right] \, \mathrm{d}z \right)^{1/2}.$$

Consider the energy functional associated with Problem (P_ϵ), defined on $H^1(\Omega_\epsilon)$ and denoted by I_{Ω_ϵ}, as follows:

$$I_{\Omega_\epsilon}(u) = \frac{1}{2} \int_{\Omega_\epsilon} |\nabla u|^2 \mathrm{d}z + \frac{1}{2} \int_{\partial\Omega_\epsilon} |u|^2 \, \mathrm{d}\sigma + \frac{1}{r} \int_{\partial\Omega_\epsilon} (u^+)^r \mathrm{d}\sigma - \frac{1}{q} \int_{\partial\Omega_\epsilon} (u^+)^q \mathrm{d}\sigma,$$

where $u^+ = \max\{u, 0\}$. As a consequence of the Sobolev trace embedding

$$H^1(\Omega) \hookrightarrow L^s(\partial\Omega), \quad 1 \leq s \leq 2_*, \tag{1}$$

it can be shown that the functional I_{Ω_ϵ} is well-defined in $H^1(\Omega_\epsilon)$. In addition, I_{Ω_ϵ} is continuous Fréchet differentiable, i.e., C^1, and

$$I'_{\Omega_\epsilon}(u)\varphi = \int_{\Omega_\epsilon} \nabla u \nabla \varphi \, \mathrm{d}z + \int_{\partial\Omega_\epsilon} u\varphi \, \mathrm{d}\sigma - \int_{\partial\Omega_\epsilon} \left[(u^+)^{q-1} - (u^+)^{r-1} \right] \varphi \, \mathrm{d}\sigma, \tag{2}$$

for any $\varphi \in H^1(\Omega_\epsilon)$. Through standard arguments, it can be demonstrated that the weak solutions of Problem (P_ϵ) correspond to critical points of I_{Ω_ϵ}, and vice versa (see [34]).

Throughout this chapter, the following equivalent norm on $H^1(\Omega_\epsilon)$ is employed:

$$\|u\|_\tau^2 := \int_\Omega |\nabla u|^2 \mathrm{d}z + \int_{\partial\Omega} u^2 \mathrm{d}\sigma.$$

In accordance with (1), the following result can be established for all $2 \leq s \leq 2_*$.

Lemma 2.1.1 *For some positive constant $S = S(\Omega)$, the inequality*

$$S \left(\int_{\partial\Omega} |u|^s \mathrm{d}\sigma \right)^{2/s} \leq \left(\int_\Omega |\nabla u|^2 \mathrm{d}z + \int_{\partial\Omega} u^2 \mathrm{d}\sigma \right) \tag{3}$$

holds, for all $u \in H^1(\Omega)$.

Proof. By applying the Sobolev trace embedding and the Poincaré inequality, it follows that for every $u \in H^1(\Omega)$,

$$\left(\int_{\partial\Omega} |u|^s \, \mathrm{d}\sigma \right)^{2/s} \leq c_1 \|u\|_{H^1(\Omega)}^2 \leq c_2 \left(\int_\Omega |\nabla u|^2 \, \mathrm{d}z + \int_{\partial\Omega} u^2 \, \mathrm{d}\sigma \right),$$

where c_1 and c_2 are positive constants . ∎

In the subsequent lemma, a geometric property of the energy functional I_{Ω_ϵ} is derived.

Lemma 2.1.2 *For $2 < r < q < 2_*$, the functional I_{Ω_ϵ} admits the geometrical mountain-pass structure on the space $H^1(\Omega_\epsilon)$.*

Proof. As is [16, Lemma 2.1], it is sufficient to demonstrate that I_{Ω_ϵ} satisfies the following conditions:

1. $I_{\Omega_\epsilon}(0) = 0$;
2. There exist $\rho > 0$ and $\alpha > 0$ such that $I_{\partial B_\rho(0)} \geq \alpha$;
3. There exists $e \in H^1(\Omega_\epsilon)$, with $\|e\|_{H^1(\Omega_\epsilon)} > \rho$, such that $I_{\Omega_\epsilon}(e) < 0$.

Item 1 is evident. To demonstrate item 2, it is observed that for $u \in H^1(\Omega_\epsilon)$,

$$I_{\Omega_\epsilon}(u) \geq \|u\|_\tau^2 \left(\frac{1}{2} - \frac{c}{q}\|u\|_\tau^{q-2}\right).$$

Choose $\rho = \|u\|_\tau$ sufficiently small such that the expression

$$\left(\frac{1}{2} - \frac{c}{q}\|u\|_\tau^{q-2}\right)$$

is positive. Thus, it follows that

$$I_{\Omega_\epsilon}(u) \geq \rho^2 c(\rho) > 0,$$

and the item 2. is proved. Finally, to prove item 3, it is noted that for $u \in H^1(\Omega_\epsilon)$ and $t > 0$,

$$I_{\Omega_\epsilon}(tu) = t^q \left[\frac{t^{2-q}}{2}\left(\int_{\Omega_\epsilon} |\nabla u|^2 \,\mathrm{d}x + \int_{\partial\Omega_\epsilon} |u|^2 \,\mathrm{d}\sigma\right) + \frac{t^{r-q}}{r}\int_{\partial\Omega_\epsilon} |u|^r \,\mathrm{d}\sigma - \frac{1}{q}\int_{\partial\Omega_\epsilon} |u|^q \,\mathrm{d}\sigma\right],$$

which implies that

$$\lim_{t\to\infty} I_{\Omega_\epsilon}(tu) = -\infty,$$

since $2 < r < q$. This validates item 3 and concludes the proof of Lemma 2.1.2. ■

The following lemma provides a proof that the functional I_{Ω_ϵ} satisfies the well-known Palais-Smale (PS) condition.

Lemma 2.1.3 *Let $2 < r < q < 2_*$. Then, any sequence $(u_k) \subset H^1(\Omega_\epsilon)$ such that*

$$I_{\Omega_\epsilon}(u_k) \to c \quad \text{and} \quad I'_{\Omega_\epsilon}(u_k) \to 0, \tag{4}$$

has a convergent subsequence.

Proof. First, it can be observed that

$$I_{\Omega_\epsilon}(u_k) - \frac{1}{q} I'_{\Omega_\epsilon}(u_k)\, u_k \;\geq\; \left(\frac{1}{2} - \frac{1}{q}\right) \|u\|_\tau^2 .$$

On the other hand, it is noted that

$$\begin{aligned} I_{\Omega_\epsilon}(u_k) - \frac{1}{q} I'_{\Omega_\epsilon}(u_k)\, u_k &\leq |I_{\Omega_\epsilon}(u_k)| + \frac{1}{q} \left\| I'_{\Omega_\epsilon}(u_k) \right\| \|u_k\|_{H^1(\Omega_\epsilon)} \\ &\leq c + \frac{\epsilon}{q} \|u_k\|_\tau , \quad k \geq k_0. \end{aligned}$$

Hence, the following inequality holds:

$$\left(\frac{1}{2} - \frac{1}{q}\right) \|u_k\|_\tau^2 \leq \frac{\epsilon}{q} \|u_k\|_\tau + c,$$

which implies that (u_k) is bounded in $H^1(\Omega_\epsilon)$. Then, up to a subsequence, $u_k \rightharpoonup u$ weakly in $H^1(\Omega_\epsilon)$ and $u_k \to u$ in $L^s(\partial\Omega_\epsilon)$, $2 \leq s < 2_*$. Now, it can be observed that Hölder's inequality and (4) can be applied to obtain the following:

$$\begin{aligned} o_k(1) &= \left(I'_{\Omega_\epsilon}(u_k) - I'_{\Omega_\epsilon}(u)\right)(u_k - u) = \int_{\Omega_\epsilon} |\nabla u_k - \nabla u|^2 \, dz + \int_{\partial\Omega_\epsilon} |u_k - u|^2 \, d\sigma \\ &- \int_{\partial\Omega_\epsilon} \left(\left(u_k^+\right)^{q-1} - \left(u^+\right)^{q-1}\right)(u_k - u)\, d\sigma \\ &+ \int_{\partial\Omega_\epsilon} \left(\left(u_k^+\right)^{r-1} - (u)^{r-1}\right)(u_k - u)\, d\sigma \\ &= \int_{\Omega_\epsilon} |\nabla u_k - \nabla u|^2 \, dz + \int_{\partial\Omega_\epsilon} |u_k - u|^2 \, d\sigma + o_k(1), \end{aligned}$$

where $o_k(1) \to 0$ as $k \to \infty$. Therefore, it follows that

$$\|u_k - u\|_\tau = o_k(1),$$

so that $u_k \to u$ in $H^1(\Omega_\epsilon)$. This concludes the proof. ∎

Given Lemmas 2.1.2 and 2.1.3, and proceeding as in [5], the following proposition is proved.

Proposition C *For each $\epsilon > 0$, the functional I_{Ω_ϵ} has a positive critical point $u_\epsilon \in H^1(\Omega_\epsilon)$ at the minimax level*

$$c_{q,r}(\Omega_\epsilon) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_{\Omega_\epsilon}(\gamma(t)) > 0,$$

where

$$\Gamma := \left\{\gamma \in C\left([0,1]; H^1(\Omega_\epsilon)\right) : \gamma(0) = 0, \gamma(1) = e\right\},$$

with $e \in H^1(\Omega_\epsilon) \setminus \{0\}$, and $I_{\Omega_\epsilon}(e) \leq 0$.

Proof. By applying Lemmas 2.1.2, 2.1.3, and the mountain pass lemma [8], u_ϵ is obtained. Furthermore, it can be stated that u_ϵ is nonnegative in Ω. In fact, by choosing $\varphi = u_\epsilon^-$ as a test function, the following holds:

$$\int_{\Omega_\epsilon} \left|\nabla u_\epsilon^-\right|^2 \mathrm{d}z + \int_{\partial\Omega_\epsilon} \left(u_\epsilon^-\right)^2 \mathrm{d}\sigma = \int_{\partial\Omega_\epsilon} \left[\left(u_\epsilon^+\right)^{q-1} u_\epsilon^- - \left(u_\epsilon^+\right)^{r-1} u_\epsilon^-\right] \mathrm{d}\sigma = 0.$$

Thus, $u_\epsilon^- \equiv 0$. Finally, by standard elliptic regularity and maximum principle, it can be concluded that $u_\epsilon > 0$ in Ω_ϵ. ∎

Proposition C demonstrates that Problem (P_ϵ) admits a positive mountain-pass solution. Henceforth, the chapter focuses on establishing that this solution is the nonconstant least-energy solution.

Let $\mathcal{N}_\epsilon$ be the Nehari manifold associated with I_{Ω_ϵ}, defined by

$$\mathcal{N}_\epsilon = \left\{v \in H^1\left(\Omega_\epsilon\right) \setminus \{0\} : \; I'_{\Omega_\epsilon}\left(v\right) v = 0\right\}.$$

It is observed that if $v \in \mathcal{N}_\epsilon$, then $v^+ \neq 0$. Furthermore, it is stated that the function $s \mapsto I_{\Omega_\epsilon}\left(sv\right)$ has a maximum $s_\epsilon = s_\epsilon\left(v\right) > 0$, which is its unique critical point. In fact, define the function f as follows:

$$f\left(s\right) = I_{\Omega_\epsilon}\left(sv\right) = \frac{s^2}{2}\left\|v\right\|_\tau^2 + \frac{s^r}{r}\int_{\partial\Omega_\epsilon} \left|v\right|^r \; \mathrm{d}\sigma - \frac{s^q}{q}\int_{\partial\Omega_\epsilon} \left|v\right|^q \; \mathrm{d}\sigma.$$

It is noted that

$$f'\left(s\right) = 0 \quad \Longleftrightarrow \quad As^{q-2} - Bs^{r-2} = C, \tag{5}$$

where

$$A = \int_{\partial\Omega_\epsilon} \left|v\right|^q \; \mathrm{d}\sigma, \quad B = \int_{\partial\Omega_\epsilon} \left|v\right|^r \; \mathrm{d}\sigma, \quad C = \left\|v\right\|_\tau^2.$$

It can be verified that the equation (5) has a unique solution $s_\epsilon = s_\epsilon\left(v\right) > 0$. Moreover,

$$f''\left(\overline{s}\right) = \left(2 - q\right)C + \left(r - q\right)B\left(\overline{s}\right)^{r-2} < 0.$$

Thus, $f\left(s_\epsilon\right) = I_{\Omega_\epsilon}\left(s_\epsilon v\right)$ is a maximum value, which implies that $s_\epsilon v \in \mathcal{N}_\epsilon$, that is, for any $v \in H^1\left(\Omega_\epsilon\right) \setminus \{0\}$, there exists a unique $s_\epsilon = s_\epsilon\left(v\right) > 0$ such that $s_\epsilon v \in \mathcal{N}_\epsilon$, and

$$\max_{s>0} I_{\Omega_\epsilon}\left(sv\right) = I_{\Omega_\epsilon}\left(s_\epsilon v\right).$$

Given that every admissible path in Γ must intersect $\mathcal{N}_\epsilon$, it can be inferred that

$$c_{q,r}(\Omega_\epsilon) = \inf_{v\in N_\epsilon} I_{\Omega_\epsilon}(v).$$

Therefore, it can be concluded that the minimax level $c_{q,r}(\Omega_\epsilon)$ can be characterized as follows:

$$c_{q,r}(\Omega_\epsilon) = \inf_{v \in H^1(\Omega_\epsilon)\setminus\{0\}} \max_{s>0} I_{\Omega_\epsilon}(sv).$$

This characterization aligns more effectively with the objectives of this work. Furthermore, for every nonnegative $v \in H^1(\Omega_\epsilon) \setminus \{0\}$ there exists a unique positive value $s_\epsilon = s_\epsilon(v)$ such that

$$c_{q,r}(\Omega_\epsilon) \leq I_{\Omega_\epsilon}(s_\epsilon v) = \max_{s>0} I_{\Omega_\epsilon}(sv). \tag{6}$$

2.2 The Limit Problem

A fundamental element in the proof that the mountain-pass solution obtained in Proposition C is nonconstant relies on the knowledge of the existence of a positive solution to Problem (P_λ), where $2 < r < q < 2_*$, with $\lambda = 1$. This section addresses this problem, taking into account the developments made in Chapter 1, Section 1.2. The definitions introduced in the aforementioned chapter, along with the findings, will be utilized, adapted to the case $\lambda = 1$, which is directly related to Problem (P_ϵ). In this context, the following proposition is formulated.

Proposition D *Problem (P_λ), with $\lambda = 1$, has a positive solution $w \in C^\infty\left(\mathbb{R}^n_+\right) \cap C^{2,\alpha}\left(\overline{\mathbb{R}^n_+}\right) \cap E$ such that*

1. *$w = w(x,t)$ is radially symmetric with respect to the variable $x \in \mathbb{R}^{n-1}$, that is, $w(x,t) = w(\rho,t)$ if $\rho = |x|$. Moreover, $w_\rho(\rho,t) < 0$ in $(0,+\infty) \times [0,+\infty)$.*

2. *w has exponential decay in the variable x and lower power-type decay in the variable t, that is, there exist $c_1, c_2 > 0$ such that*

$$w(x,t) \leq c_1 \frac{e^{-c_2|x|}}{(1+t^2)^{(n-2)/2}},$$

   for all $(x,t) \in \overline{\mathbb{R}^n_+}$.

3. *The derivatives of w has exponential decay in the variable x and lower power-type decay in the variable t, that is, there exist $c_1, c_2 > 0$ such that*

$$|\nabla w(x,t)| \leq c_1 \frac{e^{-c_2|x|}}{(1+t^2)^{(n-2)/2}},$$

   for all $(x,t) \in \overline{\mathbb{R}^n_+}$.

Proof. For the proof of items 1 and 2, see Chapter 1. At this point, the proof of item 3 will be presented. The idea of the proof is to show that there exists $C > 0$ such that

$$|w_\rho| + |w_t| \leq Cw,$$

and thereby apply item 2. Consider the function $v = w_\rho + Aw$, where A is a positive constant to be chosen later. It is observed that v is a solution to the problem

$$\begin{cases} \Delta v = 0 & \text{in} \quad \mathbb{R}^n_+ \\ \partial_\eta v + v = w^{q-2}\left[(q-1)\, w_\rho + Aw\right] - w^{r-2}\left[(r-1)\, w_\rho + Aw\right] & \text{on} \quad \mathbb{R}^{n-1}. \end{cases} \tag{7}$$

Let $\varphi_1 = (w_\rho + Aw)_- = \max\{0, -(w_\rho + Aw)\}$. Given that w exhibits uniform decay, it is possible to choose $\rho_0 > 0$ such that

$$w^{q-2}(\rho, 0) \leq \frac{1}{2} \qquad \text{if} \qquad \rho \geq \rho_0. \tag{8}$$

By employing the fact that $w_\rho(\rho, t) < 0$ for all $(\rho, t) \in (0, +\infty) \times [0, +\infty)$, it is possible to choose $A > 0$ sufficiently large such that $\varphi_1 \equiv 0$ if $|(\rho, t)| \leq R$. Then, by choosing φ_1 as a test function to the problem (7) and applying the estimate (8), the following holds:

$$\begin{aligned} \int_{|z|\geq R} |\nabla \varphi_1|^2 \, \mathrm{d}z + \int_{|x|\geq R} |\varphi_1|^2 \, \mathrm{d}x &= -\int_{|x|\geq R} \varphi_1 \left[w^{q-2}\left((q-1)\, w_\rho + Aw\right)\right] \mathrm{d}x \\ &+ \int_{|x|\geq R} \varphi_1 \left[w^{r-2}\left((r-1)\, w_\rho + Aw\right)\right] \mathrm{d}x \\ &\leq -\frac{(r-1)}{2} \int_{|x|\geq R} |\varphi_1|^2 \, \mathrm{d}x + \frac{1}{2} \int_{|x|\geq R} |\varphi_1|^2 \, \mathrm{d}x \\ &\leq \frac{1}{2} \int_{|x|\geq R} |\varphi_1|^2 \, \mathrm{d}x, \end{aligned}$$

which implies that $\varphi_1 \equiv 0$ in $\overline{\mathbb{R}^n_+}$. Hence, it follows that

$$0 \leq -w_\rho(\rho, t) \leq Aw(\rho, t). \tag{9}$$

Now, the decay of the derivative of w with respect to the variable t is established. Let $v = w_t - \tilde{A}w$, where $\tilde{A}$ is a positive constant to be chosen later. It is observed that v is a solution to the problem

$$\begin{cases} \Delta v = 0 & \text{in} \quad \mathbb{R}^n_+ \\ v = w^{r-1} - w^{q-1} - w\left(\tilde{A} - 1\right) & \text{on} \quad \mathbb{R}^{n-1}. \end{cases} \tag{10}$$

Since $w(\rho,0)\to 0$ as $\rho\to\infty$, it can be concluded that

$$w_t(\rho,0)-\tilde{A}w(\rho,0)=w(\rho,0)\left[w^{r-2}(\rho,0)-w^{q-2}(\rho,0)-\left(\tilde{A}-1\right)\right]<0,$$

where $\rho\geq R_1$, and $\tilde{A}>0$ is sufficiently large. Thus, $\tilde{A}>0$ can be chosen such that $\varphi_2\equiv 0$, for all $\rho\geq 0$. By choosing φ_2 as a test function for the problem (10), it follows that

$$\int_{\mathbb{R}^n_+}\nabla\left(w_t-\tilde{A}w\right)\nabla\varphi_2\,\mathrm{d}z=\int_{\mathbb{R}^{n-1}}\frac{\partial}{\partial\eta}\left(w_t-\tilde{A}w\right)\varphi_2\,\mathrm{d}\sigma=0,$$

which implies that $\varphi_2\equiv 0$ on $\overline{\mathbb{R}^n_+}$, so that $w_t\leq\tilde{A}w$ on $\overline{\mathbb{R}^n_+}$. In particular, the following inequality holds:

$$(w_t)^+(\rho,t)\leq w(\rho,t),\quad\forall(\rho,t)\in[0,\infty)\times[0,t).\tag{11}$$

Now, let $B>0$ be fixed. Set $v=-w_t-Bw$. It is noted that v satisfies the problem

$$\begin{cases}\Delta v=0 & \text{in}\quad\mathbb{R}^n_+\\ v=w^{q-1}-w^{r-1}-(B+1)\,w & \text{on}\quad\mathbb{R}^{n-1}.\end{cases}\tag{12}$$

Again, since $w(\rho,0)\to 0$ as $\rho\to\infty$, for any $B>0$, it can be deduced that

$$w(\rho,0)\left[w^{q-2}(\rho,0)-w^{r-2}(\rho,0)-(B+1)\right]<0,\quad\rho\geq R_2,$$

which implies that $v\leq 0$ for $\rho\geq R_2$. Thus, it is possible to choose $B>0$ sufficiently large such that

$$\varphi_3(\rho,0)=\left(-w_t(\rho,0)-Bw(\rho,0)\right)_+=0,\quad\rho\geq 0.$$

By choosing φ_3 as a test function for the problem (12), it follows that

$$\int_{\mathbb{R}^n_+}\nabla\left(-w_t-Bw\right)\nabla\varphi_3\,\mathrm{d}z=\int_{\mathbb{R}^{n-1}}\varphi_3\frac{\partial}{\partial\eta}\left(-w_t-Bw\right)\,\mathrm{d}x=0,$$

which implies that $\varphi_3\equiv 0$ on $\overline{\mathbb{R}^n_+}$. Hence, the following inequality holds:

$$(w_t)^-\leq Bw\quad\text{on}\quad\overline{\mathbb{R}^n_+}.\tag{13}$$

Therefore, (9), (11), (13), and item 2 provide evidence. ■

2.3 Upper Bound Estimate to $c_{q,r}(\Omega_\epsilon)$

To demonstrate that the mountain-pass solution u_ϵ is nonconstant for $\epsilon > 0$ sufficiently small (Theorem G), an upper bound estimate to the minimax level $c_{q,r}(\Omega_\epsilon)$ is established. The idea is to consider a positive solution w of Problem (P_λ), where $2 < r < q < 2_*$, with $\lambda = 1$ and, based on this, construct an appropriate function v_ϵ in order to compare the least-energy level $c_{q,r}(\Omega_\epsilon)$ with $\max_{s>0} I_{\Omega_\epsilon}(sv_\epsilon)$ using the characterization given in (6). To simplify the discussion, it is assumed from this point forward that Ω is a strictly convex domain. Let w be a positive solution of Problem (P_λ), where $2 < r < q < 2_*$, with $\lambda = 1$, and let $z_0 \in \partial\Omega$ be fixed. After an appropriate rotation and translation of the coordinate system, it is assumed that z_0 is the origin and the inner normal to Ω at z_0 points in the direction of the positive t-axis. Furthermore, a C^2-function $G : B_{\rho_0} \to \mathbb{R}$ is found, defined on the ball $B_{\rho_0} = \{x = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} : |x| < \rho_0\}$, such that $G(0) = 0$ and $\nabla G(0) = 0$. Given that Ω is strictly convex, the following cylinder in $\mathbb{R}^n$ is considered:

$$\mathcal{U} = \{(x,t) \in \mathbb{R}^n : |x| \le \rho_0 \quad \text{and} \quad 0 \le t \le t_0\},$$

where $t_0 = \min_{|x|=\rho_0} G(x) > 0$. It can be observed that

$$\partial\Omega \cap \mathcal{U} = \{(x,t) : t = G(x)\}, \qquad \text{and} \qquad \Omega \cap \mathcal{U} = \{(x,t) : t > G(x)\},$$

that is, $\Omega \cap \mathcal{U}$ is the epigraph of the smooth function G. Using the minimax characterization of $c_{q,r}(\Omega_\epsilon)$ given in (6) with

$$v_\epsilon(x,t) = w(\epsilon(x,t) - z_0),$$

and denoting by $\mathcal{H}(z)$ the mean curvature of the boundary at the point $z \in \partial\Omega$, the following proposition can be stated.

Proposition E *There exists a positive constant γ, depending on n, q and r, such that*

$$c_{q,r}(\Omega_\epsilon) \le c_{q,r}\left(\mathbb{R}^n_+\right) - \epsilon\gamma \max_{z\in\partial\Omega} \mathcal{H}(z) + o(\epsilon), \quad \text{as} \quad \epsilon \to 0, \tag{14}$$

where $c_{q,r}\left(\mathbb{R}^n_+\right)$ is the least energy level from the associated functional to Problem (P_λ).

An auxiliary result is first established to prove Proposition E. Let

$$g(x) := \langle D^2G(0)\,x, x\rangle, \quad x \in \mathbb{R}^{n-1}, \qquad R_1(\epsilon) := \int_{\Omega_\epsilon} |\nabla w|^2\,\mathrm{d}z - \int_{\mathbb{R}^n_+} |\nabla w|^2\,\mathrm{d}z,$$

and

$$R_2(\epsilon) := \int_{\partial\Omega_\epsilon} \left[\frac{w^2}{2} - \left(\frac{w^q}{q} - \frac{w^r}{r}\right)\right] \mathrm{d}\sigma - \int_{\mathbb{R}^{n-1}} \left[\frac{w^2}{2} - \left(\frac{w^q}{q} - \frac{w^r}{r}\right)\right] \mathrm{d}x,$$

where $2 < r < q < 2_*$, and let $s_\epsilon > 0$ be such that

$$\max_{s>0} I_{\Omega_\epsilon}(sv_\epsilon) = I_{\Omega_\epsilon}(s_\epsilon v_\epsilon).$$

The following lemma may be established.

Lemma 2.3.1 *The following estimates hold as $\epsilon \to 0$,*

$$R_1(\epsilon) = -\epsilon \int_{\mathbb{R}^{n-1}} |\nabla w(x,0)|^2 g(x)\, \mathrm{d}x + o(\epsilon),$$

$$R_2(\epsilon) = \epsilon \int_{\mathbb{R}^{n-1}} w_t^2(x,0)\, g(x)\, \mathrm{d}x + o(\epsilon).$$

Moreover, it follows that

$$s_\epsilon^{q-r} = 1 + O(\epsilon). \tag{15}$$

Proof. Let the following set be considered:

$$\mathcal{U}_\epsilon = \{(x,t) \in \mathbb{R}^n : |\epsilon x| \leq \rho_0 \quad \text{and} \quad 0 \leq \epsilon t \leq t_0\},$$

where $t_0 = \min_{|x|=\rho_0} G(x) > 0$. It is observed that

$$\begin{aligned}
-R_1(\epsilon) &= \int_{\mathbb{R}^n_+\setminus\Omega_\epsilon} |\nabla w|^2\, \mathrm{d}z \\
&= \int_{\mathbb{R}^n_+\setminus\mathcal{U}_\epsilon} |\nabla w|^2\, \mathrm{d}z + \int_{\mathcal{U}_\epsilon\setminus(\Omega_\epsilon\cap\mathcal{U}_\epsilon)} |\nabla w|^2\, \mathrm{d}z - \int_{\Omega_\epsilon\cap\left(\mathbb{R}^n_+\setminus\mathcal{U}_\epsilon\right)} |\nabla w|^2\, \mathrm{d}z \\
&= A_1(\epsilon) + A_2(\epsilon) + A_3(\epsilon).
\end{aligned}$$

By Proposition D, there exists a positive constant $C = C(n)$ such that

$$\begin{aligned}
A_1(\epsilon) &:= \int_{\mathbb{R}^n_+\setminus\mathcal{U}_\epsilon} |\nabla w|^2\, \mathrm{d}z \leq C \int_{\mathbb{R}^n_+\setminus\mathcal{U}_\epsilon} \frac{e^{-2c_2|x|}}{(1+t^2)^{n-1}} \mathrm{d}z \\
&\leq C \left[\int_0^{\epsilon^{-1}\rho_0} e^{-2c_2\rho}\rho^{n-2} \mathrm{d}\rho \int_{\epsilon^{-1}t_0}^{+\infty} \frac{1}{(1+t^2)^{n-2}} \mathrm{d}t\right] \\
&+ C \left[\int_{\epsilon^{-1}\rho_0}^{+\infty} e^{-2c_2\rho}\rho^{n-2} \mathrm{d}\rho \int_0^{+\infty} \frac{1}{(1+t^2)^{n-2}} \mathrm{d}t\right] \\
&\leq C \int_{\epsilon^{-1}t_0}^{+\infty} \frac{1}{t^{2n-4}} \mathrm{d}t + C \int_{\epsilon^{-1}\rho_0}^{+\infty} e^{-2c_2\rho}\rho^{n-2} \mathrm{d}\rho \\
&= o(\epsilon).
\end{aligned}$$

Given that $\left(\Omega_\epsilon \cap \left(\mathbb{R}^n_+ \setminus \mathcal{U}_\epsilon\right)\right) \subset \mathbb{R}^n_+ \setminus \mathcal{U}_\epsilon$, the preceding estimate implies that

$$A_2(\epsilon) := -\int_{\Omega_\epsilon \cap \left(\mathbb{R}^n_+ \setminus \mathcal{U}_\epsilon\right)} |\nabla w|^2 \, \mathrm{d}z = o(\epsilon).$$

Let $D_\epsilon = \{(x,t) : |\epsilon x| \le \rho_0,\ t_0 \le \epsilon t \le G(\epsilon x)\} \subset \mathbb{R}^n_+ \setminus \mathcal{U}_\epsilon$. It is noted that

$$\begin{aligned} A_3(\epsilon) &:= \int_{\mathcal{U}_\epsilon \setminus (\Omega_\epsilon \cap \mathcal{U}_\epsilon)} |\nabla w|^2 \, \mathrm{d}z \\ &= \int_{|\epsilon x| \le \rho_0} \int_0^{\epsilon^{-1} G(\epsilon x)} |\nabla w(x,t)|^2 \, \mathrm{d}t \mathrm{d}x - \int_{D_\epsilon} |\nabla w|^2 \, \mathrm{d}z \\ &= \int_{|\epsilon x| \le \rho_0} \int_0^{\epsilon^{-1} G(\epsilon x)} |\nabla w(x,t)|^2 \, \mathrm{d}t \mathrm{d}x + o(\epsilon). \end{aligned}$$

In accordance with the Mean Value Theorem, there exists $c \in (0,t)$ such that

$$|\nabla w(x,t)|^2 = |\nabla w(x,0)|^2 + 2\langle \nabla w(x,c), \nabla w_t(x,c) \rangle t.$$

Given that G is a C^2-function, the Taylor expansion of G around the point 0 can be written as:

$$G(x) = G(0) + \nabla G(0) \cdot x + \frac{1}{2} \langle D^2 G(0) x, x \rangle + R(x),$$

where $R(x)$ represents the remainder term. Since $G(0) = 0$ and $\nabla G(0) = 0$, and given that the remainder term $R(x)$ is of the order $o\left(|x|^2\right)$ as x approaches 0, it follows that $G(x)$ can be expressed as $G(x) = g(x) + o\left(|x|^2\right)$. Thus, the following holds:

$$\begin{aligned} A_3(\epsilon) &= \int_{|\epsilon x| \le \rho_0} \int_0^{\epsilon^{-1} G(\epsilon x)} |\nabla w(x,0)|^2 + 2\langle \nabla w(x,c), \nabla w_t(x,c) \rangle t \ \mathrm{d}t \mathrm{d}x \\ &= \int_{|\epsilon x| \le \rho_0} \left(|\nabla w(x,0)|^2 + \langle \nabla w(x,c), \nabla w_t(x,c) \rangle\right) \left(\epsilon g(x) + \epsilon o\left(|x|^2\right)\right) \ \mathrm{d}x \\ &= \epsilon \int_{|\epsilon x| \le \rho_0} |\nabla w(x,0)|^2 g(x) \ \mathrm{d}x + o(\epsilon) \\ &= \epsilon \int_{\mathbb{R}^{n-1}} |\nabla w(x,0)|^2 g(x) \ \mathrm{d}x - \epsilon \int_{|\epsilon x| \ge \rho_0} |\nabla w(x,0)|^2 g(x) \ \mathrm{d}x + o(\epsilon). \end{aligned}$$

By following the procedure used in obtaining the estimate for $A_1(\epsilon)$, it can be concluded that

$$\int_{|\epsilon x| \ge \rho_0} |\nabla w(x,0)|^2 g(x) \ \mathrm{d}x = o(\epsilon),$$

which implies that

$$R_1(\epsilon) = -\epsilon \int_{\mathbb{R}^{n-1}} |\nabla w(x,0)|^2 g(x) \ \mathrm{d}x + o(\epsilon). \tag{16}$$

To establish the estimate for $R_2(\epsilon)$, it is first expressed as $R_2(\epsilon) = I_2(\epsilon) + I_r(\epsilon) - I_q(\epsilon)$, where

$$2I_2(\epsilon) = \int_{\partial\Omega_\epsilon} w^2 \, \mathrm{d}\sigma - \int_{\mathbb{R}^{n-1}} w^2 \, \mathrm{d}x,$$

$$rI_r(\epsilon) = \int_{\partial\Omega_\epsilon} w^r \, \mathrm{d}\sigma - \int_{\mathbb{R}^{n-1}} w^r \, \mathrm{d}x, \qquad \text{and} \qquad qI_q(\epsilon) = \int_{\partial\Omega_\epsilon} w^q \, \mathrm{d}\sigma - \int_{\mathbb{R}^{n-1}} w^q \, \mathrm{d}x.$$

Set $\Gamma_\epsilon = \partial\Omega_\epsilon \cap \mathcal{U}_\epsilon$. It is observed that

$$2I_2(\epsilon) = \int_{\Gamma_\epsilon} w^2 \, \mathrm{d}\sigma + \int_{\partial\Omega_\epsilon \setminus \Gamma_\epsilon} w^2 \, \mathrm{d}\sigma - \int_{|\epsilon x| \le \rho_0} w^2(x, 0) \, \mathrm{d}x - \int_{|\epsilon x| \ge \rho_0} w^2(x, 0) \, \mathrm{d}x.$$

The exponential decay of $w(x, t)$ with respect to the variable x leads to the conclusion that

$$\int_{|\epsilon x| \ge \rho_0} w^2(x, 0) \, \mathrm{d}x \le c_1 \int_{|\epsilon x| \ge \rho_0} e^{-2c_2|x|} \mathrm{d}x = c_1 \int_{\epsilon^{-1}\rho_0}^{+\infty} e^{-2c_2\rho} \rho^{n-2} \, \mathrm{d}\rho = o(\epsilon).$$

Consider $\tilde{\Omega_\epsilon} = \Omega_\epsilon \setminus (\Omega_\epsilon \cap \mathcal{U}_\epsilon)$. It can be deduced from the trace embedding theorem that

$$\int_{\partial\Omega_\epsilon \setminus \Gamma_\epsilon} w^2 \, \mathrm{d}\sigma \le \int_{\partial\tilde{\Omega_\epsilon}} w^2 \, \mathrm{d}\sigma \le S\left(\tilde{\Omega_\epsilon}\right) \|w\|^2_{H^1\left(\tilde{\Omega_\epsilon}\right)},$$

where $S\left(\tilde{\Omega_\epsilon}\right)$ is bounded, and independent of ϵ. Since

$$\|w\|^2_{H^1\left(\tilde{\Omega_\epsilon}\right)} = \int_{\Omega_\epsilon \setminus (\Omega_\epsilon \cap \mathcal{U}_\epsilon)} w^2 \, \mathrm{d}z + \int_{\Omega_\epsilon \setminus (\Omega_\epsilon \cap \mathcal{U}_\epsilon)} |\nabla w|^2 \, \mathrm{d}z,$$

the same approach used in the proof of the estimate $R_1(\epsilon)$ can be applied to derive the estimate

$$\int_{\partial\Omega_\epsilon \setminus \Gamma_\epsilon} w^2 \, \mathrm{d}\sigma = o(\epsilon).$$

Consequently, the following conclusion can be drawn:

$$2I_2(\epsilon) = \int_{\Gamma_\epsilon} w^2 \, \mathrm{d}\sigma - \int_{|\epsilon x| \le \rho_0} w^2(x, 0) \, \mathrm{d}x + o(\epsilon).$$

Let $f_\epsilon(s) = w^2(x, s\epsilon^{-1}G(\epsilon x)) \sqrt{1 + s^2 |\nabla G(\epsilon x)|^2}$. By employing the mean value theorem, it follows that

$$2I_2(\epsilon) = \int_{|\epsilon x| \le \rho_0} w^2\left(x, \epsilon^{-1}G(\epsilon x)\right) \sqrt{1 + |\nabla G(\epsilon x)|^2} - w^2(x, 0) \, \mathrm{d}x + o(\epsilon)$$

$$
\begin{aligned}
&= \int_{|\epsilon x|\le\rho_0} \left[f_\epsilon(1) - f_\epsilon(0)\right] \ \mathrm{d}x + o(\epsilon) \\
&= \int_{|\epsilon x|\le\rho_0} f_\epsilon'(s_\epsilon) \ \mathrm{d}x + o(\epsilon) \\
&= 2\int_{|\epsilon x|\le\rho_0} w\left(x, s_\epsilon \epsilon^{-1} G(\epsilon x)\right) w_t\left(x, s_\epsilon\epsilon^{-1}G(\epsilon x)\right) \epsilon^{-1} G(\epsilon x) \sqrt{1 + s_\epsilon^2 \left|\nabla G(\epsilon x)\right|^2} \\
&+ \ o(\epsilon),
\end{aligned}
$$

where $0 < s_\epsilon < 1$. Then, by applying the Dominated Convergence Theorem, the following holds:

$$
\begin{aligned}
I_2(\epsilon) &= \epsilon \int_{\mathbb{R}^{n-1}} w\left(x, s_\epsilon\epsilon^{-1}G(\epsilon x)\right) w_t\left(x, s_\epsilon\epsilon^{-1}G(\epsilon x)\right) \sqrt{1 + s_\epsilon^2 \left|\nabla G(\epsilon x)\right|^2} g(x)\, \chi_{\{|\epsilon x|\le\rho_0\}} \\
&+ \ o(\epsilon) \\
&= \epsilon \int_{\mathbb{R}^{n-1}} w(x,0)\, w_t(x,0)\, g(x) \ \mathrm{d}x + o(\epsilon).
\end{aligned}
$$

By using an approach entirely analogous to that employed in obtaining the estimate for $I_2(\epsilon)$, the following estimates for $I_r(\epsilon)$ and $I_q(\epsilon)$ are derived:

$$
I_r(\epsilon) = \epsilon \int_{\mathbb{R}^{n-1}} w(x,0)\, w_t(x,0)\, g(x) \ \mathrm{d}x + o(\epsilon)
$$

and

$$
I_q(\epsilon) = \epsilon \int_{\mathbb{R}^{n-1}} w^{q-1}(x,0)\, w_t(x,0)\, g(x) \ \mathrm{d}x + o(\epsilon).
$$

Given that w is a solution of Problem (P_λ), it can be concluded that

$$
\begin{aligned}
R_2(\epsilon) &= \epsilon \int_{\mathbb{R}^{n-1}} \left[w(x,0) + w^{r-1}(x,0) - w^{q-1}(x,0)\right] w_t(x,0)\, g(x) \ \mathrm{d}x + o(\epsilon) \\
&= \epsilon \int_{\mathbb{R}^{n-1}} w_t^2(x,0)\, g(x) \ \mathrm{d}x + o(\epsilon).
\end{aligned}
$$

In order to complete the proof, it is necessary to establish the estimate for s_ϵ. First, it is noted that

$$
\begin{aligned}
0 &= I_{\Omega_\epsilon}'(s_\epsilon w)(s_\epsilon w) \\
&= s_\epsilon^2 \int_{\Omega_\epsilon} |\nabla w|^2 \,\mathrm{d}z + s_\epsilon^2 \int_{\partial\Omega_\epsilon} w^2 \mathrm{d}\sigma - s_\epsilon^q \int_{\partial\Omega_\epsilon} w^q \mathrm{d}\sigma + s_\epsilon^r \int_{\partial\Omega_\epsilon} w^r \mathrm{d}\sigma
\end{aligned}
$$

that is,

$$
s_\epsilon^{q-2} \int_{\partial\Omega_\epsilon} w^q d\sigma - s_\epsilon^{r-2} \int_{\partial\Omega_\epsilon} w^r \mathrm{d}\sigma = \int_{\Omega_\epsilon} |\nabla w|^2 \,\mathrm{d}z + \int_{\partial\Omega_\epsilon} w^2 \mathrm{d}\sigma. \qquad (17)
$$

On the other hand, since w is a solution of Problem (P_λ), it follows that

$$\int_{\Omega_\epsilon} |\nabla w|^2 \,\mathrm{d}z + \int_{\partial\Omega_\epsilon} w^2 \mathrm{d}\sigma = R_1(\epsilon) + \int_{\mathbb{R}^{n-1}} w^q \mathrm{d}x - \int_{\mathbb{R}^{n-1}} w^r \mathrm{d}x + 2I_2(\epsilon). \quad (18)$$

Furthermore, the following expressions can be written:

$$s_\epsilon^{q-2} \int_{\partial\Omega_\epsilon} w^q \mathrm{d}\sigma = s_\epsilon^{q-2} q I_q(\epsilon) + s_\epsilon^{q-2} \int_{\mathbb{R}^{n-1}} w^q \mathrm{d}x \quad (19)$$

and

$$s_\epsilon^{r-2} \int_{\partial\Omega_\epsilon} w^r \mathrm{d}\sigma = s_\epsilon^{r-2} r I_r(\epsilon) + s_\epsilon^{r-2} \int_{\mathbb{R}^{n-1}} w^r \mathrm{d}x. \quad (20)$$

By utilizing the identities (18)-(20), the identity (17) can be expressed in the following manner:

$$\left(s_\epsilon^{q-2} - 1\right) \int_{\mathbb{R}^{n-1}} w^q \mathrm{d}x - \left(s_\epsilon^{r-2} - 1\right) \int_{\mathbb{R}^{n-1}} w^r \mathrm{d}x = R_1(\epsilon) + 2I_2(\epsilon) + s_\epsilon^{r-2} r I_r(\epsilon) - s_\epsilon^{q-2} q I_q(\epsilon) \quad (21)$$

Therefore, it can be concluded that

$$s_\epsilon^{q-r} = 1 + O(\epsilon),$$

as was intended to be demonstrated. ■

Proof of Proposition E. Assuming $z_0 = 0$, the estimate provided by (6), with $v_\epsilon(x,t) = w(\epsilon(x,t))$, indicates that

$$\begin{aligned}
c_{q,r}(\Omega_\epsilon) &\leq I_{\Omega_\epsilon}(s_\epsilon w) \\
&= \frac{s_\epsilon^2}{2}\left[\int_{\mathbb{R}^n_+} |\nabla w|^2 \,\mathrm{d}z + \int_{\Omega_\epsilon} |\nabla w|^2 \,\mathrm{d}z - \int_{\mathbb{R}^n_+} |\nabla w|^2 \,\mathrm{d}z\right] \\
&+ \frac{s_\epsilon^2}{2}\left[\int_{\mathbb{R}^{n-1}} w^2 \mathrm{d}x + \int_{\partial\Omega_\epsilon} w^2 \mathrm{d}\sigma - \int_{\mathbb{R}^{n-1}} w^2 \mathrm{d}x\right] \\
&- \frac{s_\epsilon^q}{q}\left[\int_{\mathbb{R}^{n-1}} (w^+)^q \,\mathrm{d}x + \int_{\partial\Omega_\epsilon} (w^+)^q \,\mathrm{d}\sigma - \int_{\mathbb{R}^{n-1}} (w^+)^q \,\mathrm{d}x\right] \\
&+ \frac{s_\epsilon^r}{r}\left[\int_{\mathbb{R}^{n-1}} (w^+)^r \,\mathrm{d}x + \int_{\partial\Omega_\epsilon} (w^+)^r \,\mathrm{d}\sigma - \int_{\mathbb{R}^{n-1}} (w^+)^r \,\mathrm{d}x\right].
\end{aligned}$$

The application of the estimates derived in Lemma 2.3.1, along with the fact that w is a positive solution of Problem (P_λ), where $2 < r < q < 2_*$, with $\lambda = 1$, leads to the conclusion that

$$c_{q,r}(\Omega_\epsilon) \leq c_{q,r}(\mathbb{R}^n_+) + \frac{R_1(\epsilon)}{2} + R_2(\epsilon) + o(\epsilon)$$

$$= \quad c_{q,r}\left(\mathbb{R}^n_+\right) - \epsilon \int_{\mathbb{R}^{n-1}} \left(\frac{|\nabla w(x,0)|^2}{2} - w_t^2(x,0) \right) g(x) \; \mathrm{d}x + o(\epsilon).$$

Now, some ideas presented in the appendix of [3] are employed. First, it is observed that

$$g(x) = \langle D^2 G(0)\, x, x\rangle = \sum_{i=1}^{n-1} \lambda_i x_i^2,$$

where λ_i, $i = 1, \ldots, n-1$, are the eigenvalues of $D^2 G(x)$, the Hessian matrix of G at x. As in [20], the mean curvature of $\partial\Omega$ at $z = (x,t)$ is given by

$$\mathcal{H}(z) = \frac{1}{n-1} \sum_{i=1}^{n-1} \lambda_i = \frac{1}{n-1} \Delta G(x),$$

whenever $\nabla G(x) = 0$. Following the approach in [14], the restricted energy density of w is defined as

$$E(w,x) = \frac{|\nabla w(x,0)|^2}{2} - w_t^2(x,0).$$

Combining these facts, it can be deduced that

$$\int_{\mathbb{R}^{n-1}} E(w,x)\, g(x) = \sum_{i=1}^{n-1} \int_{\mathbb{R}^{n-1}} \lambda_i x_i^2 E(w,x) \; \mathrm{d}x.$$

In accordance with the definition of the mass moment of inertia, the moment of inertia about the x_i-axis, $i = 1, \ldots, n-1$, and the polar moment of inertia are defined, respectively, by

$$I_{x_i} = \int_{\mathbb{R}^{n-1}} x_i^2 E(w,x) \; \mathrm{d}x, \qquad \text{and} \qquad I_0 = \sum_{i=1}^{n-1} I_{x_i} = \sum_{i=1}^{n-1} \int_{\mathbb{R}^{n-1}} x_i^2 E(w,x)\, \mathrm{d}x.$$

Using the fact that $E(w,x)$ is a symmetric function, it can be infered that $I_{x_1} = ... = I_{x_{n-1}}$, which implies that $I_0 = (n-1)\, I_{x_1}$. Thus, it follows that

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} E(w,x)\, g(x) \; \mathrm{d}x &= I_{x_1} \sum_{i=1}^{n-1} \lambda_i \\ &= I_0 \frac{1}{n-1} \sum_{i=1}^{n-1} \lambda_i \\ &= \left(\int_{\mathbb{R}^{n-1}} E(w,x)\, |x|^2 \; \mathrm{d}x \right) \mathcal{H}(z). \end{aligned}$$

Therefore, the following inequality holds:

$$c_{q,r}(\Omega_\epsilon) \le c_{q,r}\left(\mathbb{R}^n_+\right) - \epsilon\gamma\mathcal{H}(z) + o(\epsilon),$$

where

$$\gamma = \int_{\mathbb{R}^{n-1}} \left(\frac{|\nabla w\,(x,0)|^2}{2} - w_t^2\,(x,0) \right) |x|^2 \,\mathrm{d}x.$$

To conclude the proof of Proposition E, it must be shown that γ is positive. The procedure used here follows the one in [5, Lemma 8.2]. It is noted that integration by parts and the fact that w is a solution of Problem (P_λ), where $2 < r < q < 2_*$, with $\lambda = 1$, indicate that

$$0 = \int_{\mathbb{R}^n_+} \nabla w \nabla \varphi \;\mathrm{d}z - \int_{\mathbb{R}^{n-1}} \varphi\,(-w_t)\;\mathrm{d}x.$$

By choosing $\varphi\,(x,t) = |x|^2\, w_t e^{\theta t w_t^+}$, the preceding identity implies that

$$-\int_{\mathbb{R}^{n-1}} |x|^2\, w_t^2 \;\mathrm{d}x = \int_{\mathbb{R}^n_+} \nabla w \nabla \varphi \;\mathrm{d}z,$$

where

$$\begin{aligned} \nabla w \cdot \nabla \varphi \;&=\; \left(|x|^2 \frac{\partial}{\partial t} \left(\frac{|\nabla w|^2}{2} \right) \right) e^{\theta t w_t^+} \\ &+\; \left(2 w_t x \cdot \nabla_x w + \theta\,|x|^2\, w_t \left(t \nabla_x w \nabla_x w_t^+ + w_t \left(w_t^+ + t w_{tt}^+ \right) \right) \right) e^{\theta t w_t^+}, \end{aligned}$$

and the notations $w_t^+ = (w_t)^+$ and $w_{tt}^+ = \left(w_t^+ \right)_t$ are used in the weak sense. Consequently, the following is valid:

$$\begin{aligned} \gamma \;&=\; \int_{\mathbb{R}^{n-1}} \left(\frac{|\nabla w|^2}{2} - w_t^2 \right) |x|^2 \;\mathrm{d}x \\ &\geq\; \int_{\mathbb{R}^n_+} |x|^2 \frac{\partial}{\partial t} \left(\frac{|\nabla w|^2}{2} \right) e^{\theta t w_t^+} \;\mathrm{d}z \\ &+\; \int_{\mathbb{R}^n_+} \left[2 w_t x \cdot \nabla_x w + \theta\,|x|^2\, w_t \left(t \nabla_x w \nabla w_t^+ + w_t \left(w_t^+ + t w_{tt}^+ \right) \right) \right] e^{\theta t w_t^+} \mathrm{d}z. \end{aligned}$$

Furthermore, by using integration by parts, it can be observed that

$$\begin{aligned} \int_{\mathbb{R}^n_+} |x|^2\, e^{\theta t w_t^+} \frac{\partial}{\partial t} \left(\frac{|\nabla w|^2}{2} \right) \mathrm{d}z \;&=\; \int_{\mathbb{R}^{n-1}} |x|^2 \frac{|\nabla w|^2}{2} \;\mathrm{d}x - \int_{\mathbb{R}^n_+} |x|^2 \frac{|\nabla w|^2}{2} \frac{\partial}{\partial t} \left(e^{\theta t w_t^+} \right) \mathrm{d}z \\ &=\; \int_{\mathbb{R}^n_+} \frac{\partial}{\partial t} \left(|x|^2 \frac{|\nabla w|^2}{2} \right) e^{\theta t w_t^+} \;\mathrm{d}z. \end{aligned}$$

Hence, it follows that

$$\gamma \;\geq\; \int_{\mathbb{R}^n_+} |x|^2\, \nabla w \cdot \nabla w_t^+ e^{\theta t w_t^+} \mathrm{d}z + \int_{\mathbb{R}^n_+ \cap \{w_t < 0\}} 2 w_t x \cdot \nabla_x w \;\mathrm{d}z$$

$$+\int_{\mathbb{R}^n_+}\left[2w_t x\cdot\nabla_x w+\theta\left|x\right|^2 w_t\left(t\nabla_x w\nabla w_t^+ + w_t\left(w_t^+ + tw_{tt}^+\right)\right)\right]e^{\theta t w_t^+}\mathrm{d}z$$

$$=\int_{\mathbb{R}^n_+\cap\{w_t<0\}} 2w_t x\cdot\nabla_x w\ \mathrm{d}z+o\left(\theta\right)\quad \text{as}\quad \theta\to-\infty.$$

In conclusion, given that $x\cdot\nabla_x w=\rho w_\rho<0$, it can be deduced that $\gamma>0$. This finding effectively concludes the proof of Proposition E. ■

Proof of Theorem G. By combining Propositions C and E, it can be concluded that Theorem G holds. ■

2.4 Estimates on the Decay of Solutions for Problem (P_ϵ)

This section is dedicated to the formal proof of Theorems H and I. The analysis commences with the application of the Moser iteration method [26] to establish the following result concerning the L^∞ estimate for solutions of Problem (P_ϵ).

Proposition F *There exists a constant $\epsilon_0>0$ and a positive constant $C=C\left(\Omega,q,n\right)$ such that for all nonnegative mountain-pass solutions u_ϵ of Problem (P_ϵ) with $\epsilon\in(0,\epsilon_0)$, it holds that*

$$1<\sup_{\overline{\Omega}} u_\epsilon\left(\epsilon^{-1}z\right)\le C.$$

Proof. Initially, the first inequality is proven. Let z_ϵ be such that $u_\epsilon=\max_{\overline{\Omega}} u_\epsilon\left(z\right)$. By applying Hopf's lemma and using the assumption that u_ϵ is a nonnegative mountain-pass solutions u_ϵ of Problem (P_ϵ), it follows that

$$0<\frac{\partial u_\epsilon}{\partial\eta}\left(z_\epsilon\right)=u_\epsilon^{q-1}\left(z_\epsilon\right)-u_\epsilon^{r-1}\left(z_\epsilon\right)-u_\epsilon\left(z_\epsilon\right).$$

This implies that $u_\epsilon\left(z_\epsilon\right)>1$, thereby validating the first inequality. Now, the second inequality will be established. The procedure applied in the proof is based on the same ideas used in the proof of Proposition A, Chapter 1. The main steps are as follows. For each $k\in\mathbb{N}$ and $\beta>1$, define $\phi_k=uu_k^{2(\beta-1)}$, where $u_k=\min\left\{\left|u\right|,k\right\}$. The choice of ϕ_k as a test function in (2) indicates that

$$\begin{aligned}\int_{\Omega_\epsilon} u_k^{2(\beta-1)}\left|\nabla u\right|^2\mathrm{d}z &= -2\left(\beta-1\right)\int_{\Omega_\epsilon} u_k^{2(\beta-1)-1}u\nabla u\nabla u_k\ \mathrm{d}z-\int_{\partial\Omega_\epsilon} u_k^{2(\beta-1)}u^2\mathrm{d}\sigma\\ &\quad -\int_{\partial\Omega_\epsilon} u_k^{2(\beta-1)}\left(u^+\right)^r\ \mathrm{d}\sigma+\int_{\partial\Omega_\epsilon} u_k^{2(\beta-1)}\left(u^+\right)^q\ \mathrm{d}\sigma.\end{aligned}$$

This implies that

$$\int_{\Omega_\epsilon} |u_k|^{2(\beta-1)} |\nabla u|^2 \ \mathrm{d}z \leq \int_{\partial\Omega_\epsilon} u_k^{2(\beta-1)} u^q \ \mathrm{d}\sigma. \tag{22}$$

By applying the trace embedding theorem, inequality (22), and Holder's inequality with exponents $2_*/(q-2)$ and $2_*/(2_*-q+2)$, the following holds:

$$\begin{aligned}
\left[\int_{\partial\Omega_\epsilon} \left(uu_k^{(\beta-1)}\right)^{2_*} \mathrm{d}\sigma\right]^{2/2_*} &\leq c \int_{\Omega_\epsilon} \left|\nabla(uu_k^{\beta-1})\right|^2 \mathrm{d}z \\
&\leq c\beta^2 \int_{\Omega_\epsilon} u_k^{2(\beta-1)} |\nabla u|^2 \, \mathrm{d}z \\
&\leq c\beta^2 \int_{\partial\Omega_\epsilon} u^q u_k^{2(\beta-1)} \mathrm{d}\sigma \\
&\leq c\beta^2 \int_{\partial\Omega_\epsilon} u^{q-2} u^{2\beta} \mathrm{d}\sigma \\
&\leq c\beta^2 \left(\int_{\partial\Omega_\epsilon} u^{2_*} \ \mathrm{d}\sigma\right)^{\frac{q-2}{2_*}} \left(\int_{\partial\Omega_\epsilon} u^{\frac{22_*\beta}{2_*-q+2}} \ \mathrm{d}\sigma\right)^{\frac{2_*-q+2}{2_*}},
\end{aligned}$$

where $c = c\,(n)$ is a constant. By choosing $\beta = (2_*-q+2)/2$ and $\alpha = (22_*)/(2_*-q+2)$, it can be concluded that

$$\begin{aligned}
\left[\int_{\partial\Omega_\epsilon} \left(uu_k^{\beta-1}\right)^{2_*} \mathrm{d}\sigma\right]^{\frac{2}{2_*}} &\leq c\beta^2 \left[\int_{\partial\Omega_\epsilon} u^{2_*} \ \mathrm{d}\sigma\right]^{\frac{q-2}{2_*}} \left[\int_{\partial\Omega_\epsilon} u^{\alpha\beta} \ \mathrm{d}\sigma\right]^{\frac{2\beta}{\alpha\beta}} \\
&\leq c\beta^2 \left\|u\right\|^{q-2}_{L^{2_*}(\partial\Omega_\epsilon)} \left\|u\right\|^{2\beta}_{L^{\alpha\beta}(\partial\Omega_\epsilon)}.
\end{aligned}$$

It follows from Fatou's Lemma that

$$\left\|u\right\|_{L^{2_*\beta}(\partial\Omega_\epsilon)} \leq c \left(\beta^2 \left\|u\right\|^{q-2}_{L^{2_*}(\partial\Omega_\epsilon)}\right)^{1/(2\beta)} \left\|u\right\|_{L^{\alpha\beta}(\partial\Omega_\epsilon)}. \tag{23}$$

Given the choice of $\beta_0 = \beta$ and $\beta_1\alpha = 2_*\beta_0$ in (23), it follows that

$$\begin{aligned}
\left\|u\right\|_{L^{2_*\beta_1}(\partial\Omega_\epsilon)} &\leq \left(c\beta_1^2 \left\|u\right\|^{q-2}_{L^{2_*}(\partial\Omega_\epsilon)}\right)^{\frac{1}{2\beta_1}} \left\|u\right\|_{L^{\alpha\beta_1}(\partial\Omega_\epsilon)} \\
&\leq \beta_1^{\frac{1}{\beta_1}} \left(c \left\|u\right\|^{q-2}_{L^{2_*}(\partial\Omega_\epsilon)}\right)^{\frac{1}{2\beta_1}} \left(c\beta_0^2 \left\|u\right\|^{q-2}_{L^{2_*}(\partial\Omega_\epsilon)}\right)^{\frac{1}{2\beta_0}} \left\|u\right\|_{L^{(\alpha\beta_0)}(\partial\Omega_\epsilon)} \\
&\leq \left(c \left\|u\right\|^{q-2}_{L^{2_*}(\partial\Omega_\epsilon)}\right)^{(1/(2\beta_1)+1/(2\beta_0))} \beta_1^{\frac{1}{\beta_1}} \beta_0^{\frac{1}{\beta_0}} \left\|u\right\|_{L^{2_*}(\partial\Omega_\epsilon)}.
\end{aligned}$$

Define $\beta_m = (2_*/\alpha)^m \beta_0$, $m = 1, 2, \ldots$. By applying an approach similar to the one employed previously, with the assistance of an iteration process, it can be inferred that

$$\begin{aligned}
&\left\|u\right\|_{L^{(2_*\beta_m)}(\partial\Omega_\epsilon)} \\
\leq\ & \left(c \left\|u\right\|^{q-2}_{L^{2_*}(\partial\Omega_\epsilon)}\right)^{\frac{1}{2\beta_0} \sum_{i=0}^{m} \left(\frac{2_*}{\alpha}\right)^{-i}} \beta_0^{\frac{1}{\beta_0} \sum_{i=0}^{m} \left(\frac{2_*}{\alpha}\right)^{-i}} \left(\frac{2_*}{\alpha}\right)^{\frac{1}{\beta_0} \sum_{i=0}^{m} \left(\frac{2_*}{\alpha}\right)^{-i}} \left\|u\right\|_{L^{2_*}(\partial\Omega_\epsilon)}.
\end{aligned}$$

It is noted that

$$\frac{1}{\beta_0}\sum_{i=0}^{\infty}\left(\frac{2_*}{\alpha}\right)^{-i}=\frac{2}{2_*-q},$$

since

$$\frac{2_*}{\alpha}=\frac{2_*(2_*-q+2)}{22_*}=\beta>1.$$

Consequently, by letting $m\to\infty$, it can be concluded that

$$\|u\|_{L^{\infty}(\partial\Omega_\epsilon)}\leq\left(c\,\|u\|_{L^{2_*}(\partial\Omega_\epsilon)}^{q-2}\right)^{\frac{1}{2_*-q}}\beta_0^{\frac{2}{2_*-q}}\left(\frac{2_*}{\alpha}\right)^{\frac{2}{2_*-q}}\|u\|_{L^{2_*}(\partial\Omega_\epsilon)}.$$

Therefore, $u\in L^{\infty}(\partial\Omega_\epsilon)$. Now, let $M>0$ be such that $\|u\|_{L^{\infty}(\partial\Omega_\epsilon)}\leq M$. For each $k\in\mathbb{N}$, consider the set

$$\Omega_k^{\epsilon}=\left\{z=(x,t)\in\overline{\Omega_\epsilon}:\ |u(z)|>k\right\}.$$

Given that $u\in L^{2_*}(\partial\Omega_\epsilon)$, it is observed that

$$\infty>\int_{\Omega_k^\epsilon}|u|^{2_*}\ \mathrm{d}z\geq\int_{\Omega_k^\epsilon}k^{2_*}dz=k^{2_*}\left|\Omega_k^\epsilon\right|,$$

which implies that Ω_k^ϵ has finite Lebesgue measure. Define the function ψ as follows:

$$\psi(z)=\begin{cases}u(z)-k, & \text{if} \quad z\in\Omega_k^\epsilon\\ 0, & \text{if} \quad z\in\Omega_\epsilon-\Omega_k^\epsilon.\end{cases}$$

It is observed that $\nabla\psi=\nabla u$ in Ω_k^ϵ. Furthermore, given the choice of $k>M$, it can be deduced that $\psi\equiv 0$ in $\partial\Omega_\epsilon$. The choice of ψ as a test function implies that

$$\int_{\Omega_\epsilon}\nabla u\nabla\psi\ \mathrm{d}z+\int_{\partial\Omega_\epsilon}u\psi\ \mathrm{d}\sigma=\int_{\partial\Omega_\epsilon}\left[\left(u^+\right)^{q-1}-\left(u^+\right)^{r-1}\right]\psi\ \mathrm{d}\sigma=0,$$

from which it can be inferred that

$$\int_{\Omega_k^\epsilon}|\nabla u|^2\,\mathrm{d}z=0.$$

Thus, u is constant in Ω_k^ϵ or $|\Omega_k^\epsilon|=0$. Therefore, it can be concluded that $u_\epsilon\in L^{\infty}\left(\overline{\Omega_\epsilon}\right)$.

■

Remark 2.4.1 *An important conclusion can be drawn from Proposition F. Let $u=c$ be an arbitrary constant positive solution to Problem (P_ϵ). It is observed that $c^{q-2}-c^{r-2}=1$ on the boundary $\partial\Omega_\epsilon$. Through the application of fundamental mathematical* *reasoning, it can be inferred that $c > 1$. Furthermore, referencing the initial part of the proof of Proposition F, it can be inferred that*

$$u_\epsilon(z_\epsilon) > 1 \qquad \text{and} \qquad u_\epsilon^{q-2}(z_\epsilon) - u_\epsilon^{r-2} > c^{q-2} - c^{r-2},$$

where $2 < r < q < 2_*$. This leads to the conclusion that $u_\epsilon > c$. In summary, all non-negative mountain-pass solutions u_ϵ of Problem (P_ϵ) with $\epsilon \in (0, \epsilon_0)$ are nonconstant.

The following lemma establishes that u_ϵ exhibits uniform decay at infinity. For instructional purposes, the notation defined next will be introduced:

$$J_A(u) = \frac{1}{2}\int_A |\nabla u|^2 \,\mathrm{d}z + \frac{1}{2}\int_{\partial A} u^2 \,\mathrm{d}\sigma + \frac{1}{r}\int_{\partial A} (u^+)^r \,\mathrm{d}\sigma - \frac{1}{q}\int_{\partial A} (u^+)^q \,\mathrm{d}\sigma,$$

where $A \subset \mathbb{R}^n$ is an open set.

Lemma 2.4.1 *For every $\beta > 0$, there exists a constant $R > 0$ such that*

$$u_\epsilon(z) < \beta \qquad \text{whenever} \qquad |z - z^\epsilon| > R,$$

where z^ϵ denotes any maximum point of u_ϵ in $\overline{\Omega_\epsilon}$.

Proof. Arguing by contradiction, it is assumed that for some $\beta > 0$, there exist sequences $\epsilon_k \to 0$ and $z^k \in \overline{\Omega_{\epsilon_k}}$ such that

$$\left|z^k - z^{\epsilon_k}\right| \to \infty \qquad \text{and} \qquad u_{\epsilon_k}\left(z^k\right) \geq \beta. \tag{24}$$

In this case, it can be asserted that

$$2c_{q,r}\left(\mathbb{R}^n_+\right) \leq \liminf J_{\Omega_{\epsilon_k}}(u_{\epsilon_k}). \tag{25}$$

However, this leads to a contradiction, as Proposition E implies that

$$\limsup J_{\Omega_{\epsilon_k}}(u_{\epsilon_k}) \leq c_{q,r}\left(\mathbb{R}^n_+\right).$$

Thus, to complete the proof, it is sufficient to verify the validity of inequality (25). Given that (u_{ϵ_k}) is uniformly bounded in $C^{1,\alpha}\left(\overline{\Omega_{\epsilon_k}}\right)$, by the application of the Arzelà-Ascoli theorem, it can be assumed, up to a subsequence that

$$u_{\epsilon_k}(z^{\epsilon_k} + z) \to u(z)$$

uniformly over compacts subsets of $\mathbb{R}^n_+$. Furthermore, u satisfies the problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \mathbb{R}^n_+ \\ \partial_\eta u + u = u^{q-1} - u^{r-1} & \text{on} \quad \mathbb{R}^{n-1}. \end{cases} \tag{26}$$

Since $u(0) = \lim_{k\to\infty} u_{\epsilon_k}(z^{\epsilon_k}) \geq \beta$, it follows that $u \geq 0$, and the maximum principle ensures that $u > 0$. Considering that u is a nontrivial solution of (26), it can be concluded that

$$c_{q,r}\left(\mathbb{R}^n_+\right) \leq J_{\mathbb{R}^n_+}(u) = J_{B_R(0)\cap\mathbb{R}^n_+}(u) + J_{B^c_R(0)\cap\mathbb{R}^n_+}(u).$$

By applying the following identities

$$\lim_{R\to\infty} J_{B^c_R(0)\cap\mathbb{R}^n_+}(u) = 0 \quad \text{and} \quad \lim_{k\to\infty} J_{B_R(0)\cap\mathbb{R}^n_+}(u_{\epsilon_k}) = J_{B_R(0)\cap\mathbb{R}^n_+}(u),$$

and given $\delta > 0$, for sufficiently large values of R, the subsequent estimate is valid:

$$\lim_{k\to\infty} J_{B_R(z^{\epsilon_k})\cap\Omega_{\epsilon_k}}(u_{\epsilon_k}) \geq \frac{c_{q,r}\left(\mathbb{R}^n_+\right)}{2} - \delta.$$

Similarly, it can be deduced that

$$\lim_{k\to\infty} J_{B_R\left(z^k\right)\cap\Omega_{\epsilon_k}}(u_{\epsilon_k}) \geq \frac{c_{q,r}\left(\mathbb{R}^n_+\right)}{2} - \delta.$$

Consider $R > 0$ and a smooth cut-off function φ^k_R such that $0 \leq \varphi^k_R \leq 1$ and $\left|\nabla\varphi^k_R\right| \leq C$, where C is independent of R and k, and φ^k_R satisfies the following conditions:

$$\varphi^k_R = 0 \quad \text{on} \quad B_{R-1}(z^{\epsilon_k}) \cap B_{R-1}\left(z^k\right)$$

and

$$\varphi^k_R \equiv 1 \quad \text{on} \quad \mathbb{R}^n_+ \setminus \left(B_R(z^{\epsilon_k}) \cup B_R\left(z^k\right)\right).$$

By choosing $w_{\epsilon_k} = \varphi^k_R u_{\epsilon_k}$ as a test function in the equation $J'_{\Omega_{\epsilon_k}}(u_{\epsilon_k}) = 0$, and defining the set

$$\mathcal{A}^k_R := \left\{z \in \Omega_{\epsilon_k} :\ R-1 < |z - z^{\epsilon_k}| < R \quad \text{or} \quad R-1 < \left|z - z^k\right| < R\right\},$$

it follows that

$$\begin{aligned} 0 &= J'_{\Omega_{\epsilon_k}}(u_{\epsilon_k})\, w_{\epsilon_k} \\ &= J'_{\Omega_{\epsilon_k}\cap\mathcal{A}^k_R}(u_{\epsilon_k})\, w_{\epsilon_k} + J'_{\Omega_{\epsilon_k}\setminus\left(B_R(z^{\epsilon_k})\cup B_R\left(z^k\right)\right)}(u_{\epsilon_k})\, w_{\epsilon_k} \\ &= J'_{\Omega_{\epsilon_k}\cap\mathcal{A}^k_R}(u_{\epsilon_k})\, w_{\epsilon_k} + 2J_{\Omega_{\epsilon_k}\setminus\left(B_R(z^{\epsilon_k})\cup B_R\left(z^k\right)\right)}(u_{\epsilon_k}) - 2J_{\Omega_{\epsilon_k}\setminus\left(B_R(z^{\epsilon_k})\cup B_R\left(z^k\right)\right)}(u_{\epsilon_k}) \\ &+ J'_{\Omega_{\epsilon_k}\setminus\left(B_R(z^{\epsilon_k})\cup B_R\left(z^k\right)\right)}(u_{\epsilon_k})\, w_{\epsilon_k}. \end{aligned}$$

By employing the definitions of J_A and J'_A, the terms outlined previously may be expressed in the following manner:

$$\begin{aligned} J'_{\Omega_{\epsilon_k}\cap\mathcal{A}_R^k}(u_{\epsilon_k})\, w_{\epsilon_k} &= \int_{\Omega_{\epsilon_k}\cap\mathcal{A}_R^k} \nabla u_{\epsilon_k}\nabla\left(\varphi_R^k u_{\epsilon_k}\right) \mathrm{d}z + \int_{\partial\Omega_{\epsilon_k}\cap\mathcal{A}_R^k} \varphi_R^k u_{\epsilon_k}^2 \,\mathrm{d}\sigma \\ &- \int_{\partial\Omega_{\epsilon_k}\cap\mathcal{A}_R^k} \varphi_R^k u_{\epsilon_k}^q \,\mathrm{d}\sigma + \int_{\partial\Omega_{\epsilon_k}\cap\mathcal{A}_R^k} \varphi_R^k u_{\epsilon_k}^r \,\mathrm{d}\sigma; \end{aligned}$$

$$\begin{aligned} -2J_{\Omega_{\epsilon_k}\setminus\left(B_R(z^{\epsilon_k})\cup B_R\left(z^k\right)\right)}(u_{\epsilon_k}) &= -\frac{2}{r}\int_{\partial\Omega_{\epsilon_k}\setminus\left(B_R(z^{\epsilon_k})\cup B_R\left(z^k\right)\right)} u_{\epsilon_k}^r \,\mathrm{d}\sigma \\ &+ \frac{2}{q}\int_{\partial\Omega_{\epsilon_k}\setminus\left(B_R(z^{\epsilon_k})\cup B_R\left(z^k\right)\right)} u_{\epsilon_k}^q \,\mathrm{d}\sigma; \end{aligned}$$

$$\begin{aligned} J'_{\Omega_{\epsilon_k}\setminus\left(B_R(z^{\epsilon_k})\cup B_R\left(z^k\right)\right)}(u_{\epsilon_k})\, w_{\epsilon_k} &= -\int_{\partial\Omega_{\epsilon_k}\setminus\left(B_R(z^{\epsilon_k})\cup B_R\left(z^k\right)\right)} u_{\epsilon_k}^{q-1} w_{\epsilon_k} \,\mathrm{d}\sigma \\ &+ \int_{\partial\Omega_{\epsilon_k}\setminus\left(B_R(z^{\epsilon_k})\cup B_R\left(z^k\right)\right)} u_{\epsilon_k}^{r-1} w_{\epsilon_k} \,\mathrm{d}\sigma. \end{aligned}$$

Consequently, considering these facts, the following is valid:

$$\begin{aligned} 0 &= \tilde{E_{\epsilon_k}} + 2J_{\Omega_{\epsilon_k}\setminus\left(B_R(z^{\epsilon_k})\cup B_R\left(z^k\right)\right)}(u_{\epsilon_k}) \\ &+ \left(\frac{2}{q}-1\right)\int_{\partial\Omega_{\epsilon_k}\setminus\left(B_R(z^{\epsilon_k})\cup B_R\left(z^k\right)\right)} u_{\epsilon_k}^q \,\mathrm{d}\sigma + \left(1-\frac{2}{r}\right)\int_{\partial\Omega_{\epsilon_k}\setminus\left(B_R(z^{\epsilon_k})\cup B_R\left(z^k\right)\right)} u_{\epsilon_k}^r \,\mathrm{d}\sigma \\ &\leq \tilde{E_{\epsilon_k}} - \int_{\partial\Omega_{\epsilon_k}\cap\mathcal{A}_R^k} \varphi_R^k u_{\epsilon_k}^q \,\mathrm{d}\sigma + 2J_{\Omega_{\epsilon_k}\setminus\left(B_R(z^{\epsilon_k})\cup B_R\left(z^k\right)\right)}(u_{\epsilon_k}) \\ &+ 2\frac{(r-q)}{rq}\int_{\partial\Omega_{\epsilon_k}\setminus\left(B_R(z^{\epsilon_k})\cup B_R\left(z^k\right)\right)} u_{\epsilon_k}^q \,\mathrm{d}\sigma, \end{aligned}$$

where

$$\begin{aligned} \tilde{E_{\epsilon_k}} &= \int_{\Omega_{\epsilon_k}\cap\mathcal{A}_R^k} \nabla u_{\epsilon_k}\nabla\left(\varphi_R^k u_{\epsilon_k}\right) \mathrm{d}z + \int_{\partial\Omega_{\epsilon_k}\cap\mathcal{A}_R^k} \varphi_R^k u_{\epsilon_k}^2 \,\mathrm{d}\sigma + \int_{\partial\Omega_{\epsilon_k}\cap\mathcal{A}_R^k} \varphi_R^k u_{\epsilon_k}^r \,\mathrm{d}\sigma \\ &- \int_{\partial\Omega_{\epsilon_k}\cap\mathcal{A}_R^k} \varphi_R^k u_{\epsilon_k}^q \,\mathrm{d}\sigma. \end{aligned}$$

Given that $2 < r < q$, the following statement holds:

$$0 \leq \tilde{E_{\epsilon_k}} + 2J_{\Omega_{\epsilon_k}\setminus\left(B_R(z^{\epsilon_k})\cup B_R\left(z^k\right)\right)}(u_{\epsilon_k}).$$

Furthermore, it can be observed that $\tilde{E_{\epsilon_k}} \to 0$. Thus, it follows that

$$J_{\Omega_{\epsilon_k}\setminus\left(B_R(z^{\epsilon_k})\cup B_R\left(z^k\right)\right)}(u_{\epsilon_k}) \geq -\delta.$$

Conversely,

$$\begin{aligned} J_{\Omega_{\epsilon_k}}\left(u_{\epsilon_k}\right) &= J_{\Omega_{\epsilon_k}\cap B_R(z^{\epsilon_k})}\left(u_{\epsilon_k}\right) + J_{\Omega_{\epsilon_k}\cap B_R\left(z^k\right)}\left(u_{\epsilon_k}\right) + J_{\Omega_{\epsilon_k}\setminus\left(B_R(z^{\epsilon_k})\cup B_R\left(z^k\right)\right)}\left(u_{\epsilon_k}\right) \\ &\geq 2c_{q,r}\left(\mathbb{R}^n_+\right) - \delta, \end{aligned}$$

which implies estimate (25). Therefore, the proof is complete. ■

In order to prove Theorem H, it is necessary to introduce a function U, which will be defined as follows:

$$U(z) = \left(1 + |z|^2\right)^{(2-n)/2}.$$

It can be observed that

$$-\Delta U = n(n-2)\,U^{(n+2)/(n-2)} \qquad \text{in} \qquad \mathbb{R}^n.$$

Proof of Theorem H. Consider the function $W_\epsilon(z) = \frac{u_\epsilon(z)}{U(z)}$ in Ω_ϵ. The aim of the proof is to show that $W_\epsilon(z)$ is uniformly bounded, meaning that for $\epsilon > 0$, there exists a constant $M > 0$ such that

$$|u_\epsilon(z)| \leq MU(z),$$

and this leads to the conclusion that

$$u_\epsilon(z) \leq \frac{M}{\left(1 + |z|^2\right)^{\frac{n-2}{2}}}.$$

Initially, it can be observed that

$$\frac{\partial W_\epsilon}{\partial z_i}(z) = \frac{1}{U(z)}\frac{\partial u_\epsilon}{\partial z_i} - \frac{1}{U(z)}\frac{\partial U}{\partial z_i}W_\epsilon(z),$$

and

$$\begin{aligned} \frac{\partial^2 W_\epsilon}{\partial z_i^2}(z) &= \frac{U(z)\frac{\partial^2 u_\epsilon}{\partial z_i^2} - \frac{\partial u_\epsilon}{\partial z_i}\frac{\partial U}{\partial z_i}}{U^2(z)} \\ &- \left\{\frac{U(z)\left[\frac{\partial^2 U}{\partial z_i}W_\epsilon(z) + \frac{\partial U}{\partial z_i}\frac{\partial W_\epsilon}{\partial z_i}\right] - \frac{\partial U}{\partial z_i}W_\epsilon(z)\frac{\partial U}{\partial z_i}}{U^2(z)}\right\}. \end{aligned}$$

This implies that

$$-\Delta W_\epsilon(z) = \frac{1}{U^2(z)}\sum_{i=1}^n \frac{\partial u_\epsilon}{\partial z_i}\frac{\partial U}{\partial z_i} + \frac{1}{U(z)}W_\epsilon(z)\,\Delta U + \frac{1}{U(z)}\sum_{i=1}^n \frac{\partial U}{\partial z_i}\frac{\partial W_\epsilon}{\partial z_i}$$

$$- \quad \frac{W_\epsilon(z)}{U^2(z)} \sum_{i=1}^{n} \left(\frac{\partial U}{\partial z_i} \right)^2 .$$

Given that

$$\frac{1}{U^2(z)} \sum_{i=1}^{n} \frac{\partial u_\epsilon}{\partial z_i} \frac{\partial U}{\partial z_i} = \frac{1}{U(z)} \sum_{i=1}^{n} \frac{\partial U}{\partial z_i} \frac{\partial W_\epsilon}{\partial z_i} + \frac{W_\epsilon(z)}{U^2(z)} \sum_{i=1}^{n} \left(\frac{\partial U}{\partial z_i} \right)^2,$$

it follows that

$$-\Delta W_\epsilon = \frac{1}{U(z)} \sum_{i=1}^{n} \frac{\partial U}{\partial z_i} \frac{\partial W_\epsilon}{\partial z_i} + \frac{1}{U(z)} \Delta U W_\epsilon(z) + \frac{1}{U(z)} \sum_{i=1}^{n} \frac{\partial U}{\partial z_i} \frac{\partial W_\epsilon}{\partial z_i},$$

which can be rewritten as

$$-\Delta W_\epsilon - \sum_{i=1}^{n} b_i(z) \frac{\partial W_\epsilon}{\partial z_i} + a(z) W_\epsilon(z) = 0,$$

where

$$b_i(z) = \frac{2}{U(z)} \frac{\partial U}{\partial z_i} = -\frac{2(n-2) z_i}{1 + |z|^2}, \quad i = 1, 2, \ldots, n$$

and

$$a(z) = -\frac{\Delta U}{U(z)} = \frac{n(n-2)}{\left(1 + |z|^2\right)^2}, \quad z \in \Omega_\epsilon.$$

Furthermore, it is noted that

$$\begin{aligned} \partial_\eta W_\epsilon &= \frac{1}{U(z)} \partial_\eta u_\epsilon - \frac{1}{U(z)} W_\epsilon(z) \partial_\eta U \\ &= \frac{1}{U(z)} \left(u_\epsilon^{q-1} - u_\epsilon^{r-1} - u_\epsilon \right) - \left(\frac{1}{U(z)} \partial_\eta U \right) W_\epsilon(z), \end{aligned}$$

which can be written as

$$\partial_\eta W_\epsilon + g_1(z) W_\epsilon(z) - g_2(z) W_\epsilon^{q-1} + g_3(z) W_\epsilon^{r-1} = 0,$$

where

$$g_1(z) = 1 + \frac{1}{U(z)} \partial_\eta U, \qquad g_2(z) = U^{q-2}(z) \qquad \text{and} \qquad g_3(z) = U^{r-2}(z).$$

Thus, W_ϵ is a solution of the problem

$$\begin{cases} -\Delta W_\epsilon - \sum_{i=1}^{n} b_i(z) \frac{\partial W_\epsilon}{\partial z_i} + a(z) W_\epsilon = 0 & \text{in} \quad \Omega_\epsilon \\ \partial_\eta W_\epsilon + g_1(z) W_\epsilon - g_2(z) W_\epsilon^{q-1} + g_3(z) W_\epsilon^{r-1} = 0 & \text{on} \quad \partial\Omega_\epsilon, \end{cases} \tag{27}$$

where

$$b_i(z) = -\frac{2(n-2)z_i}{1+|z|^2}\,(i=1,\ldots,n), \quad a(z) = \frac{n(n-2)}{\left(1+|z|^2\right)^2}, \quad z \in \Omega_\epsilon,$$

$$|g_1(z)| = \left|1 + \frac{1}{U(z)}\partial_\eta U\right| \le \left(1 + \frac{n-2}{\left(1+|z|^2\right)}|z|\right), \quad g_2(z) = U^{q-2}(z),$$

and

$$g_3(z) = U^{r-2}(z).$$

It can be observed that there exists a constant $C > 0$, independent of ϵ, such that

$$\|a\|_{L^\infty(\Omega_\epsilon)}, \quad \|b_i\|_{L^\infty(\Omega_\epsilon)}, \quad \|g_j\|_{L^\infty(\Omega_\epsilon)} \le C, \quad j = 1,2,3,$$

for all $i = 1, 2, \ldots, n$. Suppose, for the sake of contradiction, that there exists a sequence $z_\epsilon \in \overline{\Omega_\epsilon}$ such that $W_\epsilon(z_\epsilon) \to \infty$. In accordance with the weak maximum principle, it can be assumed that $z_\epsilon \in \partial\Omega_\epsilon$ for all $\epsilon > 0$. Define $M_\epsilon = W_\epsilon(z_\epsilon)$. Two cases need to be considered:

Case 1 (z_ϵ) is bounded. In this scenario, consider

$$\tilde{W}_\epsilon(z) = \frac{W_\epsilon(z_\epsilon + M_\epsilon^\alpha z)}{M_\epsilon}, \quad z \in \tilde{\Omega}_\epsilon := M_\epsilon^{-\alpha}(\Omega_\epsilon - z_\epsilon), \quad \alpha = \frac{2-q}{2}.$$

Since $\left\|\tilde{W}_\epsilon\right\|_{L^\infty(\Omega_\epsilon)} \le C$, (independent of ϵ), it can be derived from a regularity result established by Lieberman [23] that

$$\left\|\tilde{W}_\epsilon\right\|_{C^{1,\beta}\left(\tilde{\Omega}_\epsilon\right)} \le C, \tag{28}$$

for some $0 < \beta < 1$ and C being a positive constant independent of ϵ. Straightening the boundary in a neighborhood of z_ϵ, it can be shown that $\tilde{\Omega}_\epsilon \to \mathbb{R}^n_+$ as $\epsilon \to 0$. Using (28) and the Arzela-Ascoli theorem, it follows that there exists a nonnegative function $\tilde{W} \in C^{1,\beta/2}\left(\overline{\mathbb{R}^n_+}\right)$ such that

$$\lim_{\epsilon\to 0}\tilde{W}_\epsilon(z) = \tilde{W}(z) \ge 0 \qquad \text{and} \qquad \tilde{W}(0) = 1. \tag{29}$$

Given that the sequence (z_ϵ) is bounded, it can be assumed that $\lim_{\epsilon\to 0} z_\epsilon = 0 \in \partial\Omega$. Thus, it can be deduced that the following statements hold on any compact subset of $\overline{\mathbb{R}^n_+}$:

$$\lim_{\epsilon\to 0} a(z_\epsilon + M_\epsilon^\alpha z) = a(0), \quad \lim_{\epsilon\to 0} b_i(z_\epsilon + M_\epsilon^\alpha) = b_i(0), \quad i = 1, 2, \ldots, n \tag{30}$$

and

$$\lim_{\epsilon\to 0} g_j\left(z_\epsilon + M_\epsilon^\alpha z\right) = g_j\left(0\right), \quad j = 1,2,3. \tag{31}$$

By applying (27)-(31)), it can be verified that the limit function $\tilde{W}$, which is nonnegative, satisfies the following limit problem

$$\begin{cases} \Delta\tilde{W} - n\left(n-2\right)\tilde{W} = 0 & \text{in} \quad \mathbb{R}_+^n \\ \tilde{W}^{q-1} = 0 & \text{on} \quad \mathbb{R}^{n-1}. \end{cases}$$

This leads to a contradiction with (29).

Case 2. There exists a sequence (z_{ϵ_k}) such that $|z_{\epsilon_k}| \to \infty$. In this scenario, consider

$$v_{\epsilon_k}\left(z\right) = \frac{W_{\epsilon_k}\left(z_{\epsilon_k}+z\right)}{M_{\epsilon_k}}, \quad z \in \tilde{\Omega}_{\epsilon_k} = \Omega_{\epsilon_k} - z_{\epsilon_k},$$

where $M_{\epsilon_k} = W_{\epsilon_k}\left(z_{\epsilon_k}\right) = \frac{u_{\epsilon_k}\left(z_{\epsilon_k}\right)}{U\left(z_{\epsilon_k}\right)}$. It can be inferred that $M_{\epsilon_k} \to \infty$ and $\|v_{\epsilon_k}\|_{L^\infty\left(\tilde{\Omega}_{\epsilon_k}\right)} \leq 1$. Furthermore, it is noted that v_{ϵ_k} satisfies the problem

$$\begin{cases} -\Delta v_{\epsilon_k} - \sum_{i=1}^n b_i\left(z_{\epsilon_k}+z\right)\frac{\partial v_{\epsilon_k}}{\partial z_i} + a\left(z_{\epsilon_k}+z\right)v_{\epsilon_k}\left(z\right) = 0, & \text{in} \quad \tilde{\Omega}_{\epsilon_k} \\ \partial_\eta v_{\epsilon_k} + g_1\left(z_{\epsilon_k}+z\right)v_{\epsilon_k}\left(z\right) - \left[u_{\epsilon_k}^{q-2}\left(z_\epsilon+z\right) - u_{\epsilon_k}^{r-2}\left(z_{\epsilon_k}+z\right)\right]v_{\epsilon_k}\left(z\right) = 0, & \text{on} \quad \partial\tilde{\Omega}_{\epsilon_k}. \end{cases}$$

As a result of $u_{\epsilon_k}^{\theta-2}\left(z_{\epsilon_k}+z\right) \to 0$ (where $\theta = r$ or q), an analogous approach to the one previously used leads to the conclusion that

$$v_{\epsilon_k} \to v \in C^{1,\beta/2}\left(\overline{\mathbb{R}_+^n}\right), \qquad \lim_{k\to\infty} v_{\epsilon_k}\left(z\right) = v\left(z\right), \qquad v\left(0\right) = 1,$$

and

$$\begin{cases} -\Delta v = 0 & \text{in} \quad \mathbb{R}_+^n \\ \partial_\eta v = -v & \text{on} \quad \mathbb{R}^{n-1}. \end{cases}$$

Conversely, by applying Hopf's lemma at $z_0 = 0$, it can be concluded that

$$0 < \partial_\eta v = -v < 0,$$

which leads to a contradiction. Therefore, the proof is complete. ∎

To conclude the section, the proof of Theorem I is presented. An auxiliary function v is used, and its definition is given by:

$$v\left(x,t\right) = \varphi_0\left(x\right)\psi_0\left(t\right),$$

where

$$\varphi_0(x) := e^{-\alpha|x|} \qquad \text{and} \qquad \psi_0(t) := \left(\frac{1}{1+t^2}\right)^{(n-2)/2},$$

with α being a positive constant that will be chosen later.

Proof of Theorem I. Initially, it can be verified that

$$\frac{\partial v}{\partial x_i}(x,t) = \left(-\alpha\frac{x_i}{|x|}\right) v(x,t), \qquad \frac{\partial v}{\partial t}(x,t) = \left(\frac{(2-n)t}{1+t^2}\right) v(x,t),$$

$$\frac{\partial^2 v}{\partial x_i^2}(x,t) = \left[\frac{(-\alpha)}{|x|} + \alpha\frac{x_i^2}{|x|^3} + \alpha^2\frac{x_i^2}{|x|^2}\right] v(x,t),$$

and

$$\frac{\partial^2 v}{\partial t^2}(x,t) = \left[\frac{(2-n)}{1+t^2} + n(n-2)\frac{t^2}{(1+t^2)^2}\right] v(x,t),$$

where $i = 1,\ldots,n-1$ and $t \in \mathbb{R}$. Thus, it can be concluded that

$$-\Delta v + c(x,t)\, v = 0, \quad x \in \mathbb{R}^{n-1}\setminus\{0\}, \quad t \in \mathbb{R},$$

where

$$c(x,t) = -\alpha\frac{(n-2)}{|x|} + \alpha^2 + \frac{(n-2)}{(1+t^2)^2}\left[(n-1)t^2 - 1\right].$$

Next, consider $V_\epsilon = u_\epsilon/v$. It is observed that

$$\frac{\partial V_\epsilon}{\partial x_i} = \frac{1}{v}\frac{\partial u_\epsilon}{\partial x_i} - \frac{1}{v}\frac{\partial v}{\partial x_i}V_\epsilon, \qquad \frac{\partial V_\epsilon}{\partial t} = \frac{1}{v}\frac{\partial u_\epsilon}{\partial t} - \frac{1}{v}\frac{\partial v}{\partial t}V_\epsilon,$$

$$\frac{\partial^2 V_\epsilon}{\partial x_i^2} = \frac{1}{v}\frac{\partial^2 u_\epsilon}{\partial x_i^2} - \frac{1}{v^2}\frac{\partial u_\epsilon}{\partial x_i}\frac{\partial v}{\partial x_i} - \frac{1}{v}\frac{\partial^2 v}{\partial x_i^2}V_\epsilon - \frac{1}{v}\frac{\partial v}{\partial x_i}\frac{\partial V_\epsilon}{\partial x_i} + \frac{1}{v^2}\left(\frac{\partial v}{\partial x_i}\right)^2 V_\epsilon,$$

and

$$\frac{\partial^2 V_\epsilon}{\partial t^2} = \frac{1}{v}\frac{\partial^2 u_\epsilon}{\partial t^2} - \frac{1}{v^2}\frac{\partial u_\epsilon}{\partial t}\frac{\partial v}{\partial t} - \frac{1}{v}\frac{\partial^2 v}{\partial t^2}V_\epsilon - \frac{1}{v}\frac{\partial v}{\partial t}\frac{\partial V_\epsilon}{\partial t} + \frac{1}{v^2}\left(\frac{\partial v}{\partial t}\right)^2 V_\epsilon.$$

Consequently, the following statement holds:

$$-\Delta V_\epsilon - \frac{1}{v}\nabla v\cdot\nabla V_\epsilon - c(x,t)\, V_\epsilon = \frac{1}{v^2}\nabla u_\epsilon\cdot\nabla v - \frac{1}{v^2}V_\epsilon\sum_{i=1}^{n}\left(\frac{\partial v}{\partial z_i}\right)^2. \tag{32}$$

Given that

$$\frac{1}{v^2}\nabla u_\epsilon\cdot\nabla v = \frac{\nabla v\cdot\nabla V_\epsilon}{v} + \frac{1}{v^2}V_\epsilon|\nabla v|^2,$$

it can be concluded by (32) that

$$-\Delta V_\epsilon - 2\frac{\nabla v\cdot\nabla V_\epsilon}{v} - c\,(x,t)\,V_\epsilon = 0, \quad \Omega_\epsilon\setminus\{0\}\,,$$

that is,

$$-\Delta V_\epsilon + 2\alpha\sum_{i=1}^{n-1}\frac{x_i}{|x|}\frac{\partial V_\epsilon}{\partial x_i} + \frac{2\,(n-2)\,t}{1+t^2}\frac{\partial V_\epsilon}{\partial t} - c\,(x,t)\,V_\epsilon = 0, \quad \Omega_\epsilon\setminus\{0\}\,.$$

Define the set

$$A_\epsilon = \Big\{(x,t)\in\mathbb{R}^n_+ :\ |x|\geq 1,\ t\geq 0\Big\}\cap\Omega_\epsilon.$$

It is claimed that there exists $C>0$ independent of ϵ such that

$$\|V_\epsilon\|_{L^\infty(A_\epsilon)}\leq C. \tag{33}$$

Assume, for the sake of contradiction, that the inequality (33) is not valid. It means that there exists $y_\epsilon=(x_\epsilon,t_\epsilon)\in A_\epsilon$ such that $V_\epsilon\,(y_\epsilon)\to\infty$. From Theorem H, it can be deduced that $|y_\epsilon|\to+\infty$. Let $\eta_\epsilon=(\eta_1\,(\epsilon)\,,...,\eta_{n-1}\,(\epsilon)\,,\eta_{t_\epsilon})\in\mathbb{R}^n$ be the unit outward normal to $\partial\Omega_\epsilon$ at (x_ϵ,t_ϵ). As a result of applying Hopf's lemma, it is concluded that $\partial_{\eta_\epsilon}V_\epsilon\,(y_\epsilon)>0$. Conversely, for sufficiently small positive ϵ, the following statement holds:

$$\partial_{\eta_\epsilon}V_\epsilon\,(y_\epsilon) = \frac{1}{v}\left[\nabla u_\epsilon\cdot\eta_\epsilon - \frac{u_\epsilon}{v}\,(\nabla v\cdot\eta_\epsilon)\right]\leq 0, \tag{34}$$

which leads to a contradiction. Therefore, to conclude the proof of the theorem, it suffices to verify that the inequality (34) holds. It is noted that

$$\begin{aligned}\partial_{\eta_\epsilon}V_\epsilon\,(y_\epsilon) &= \frac{1}{v}\left[\partial_{\eta_\epsilon}u_\epsilon - \frac{u_\epsilon}{v}\,(\nabla v\cdot\eta_\epsilon)\right]\\ &= \frac{1}{v}\left[\left(u_\epsilon^{q-1}-u_\epsilon^{r-1}-u_\epsilon\right)+u_\epsilon\left(\alpha\sum_{i=1}^{n-1}\frac{x_{i,\epsilon}}{|x_\epsilon|}\eta_i\,(\epsilon)+\frac{(n-2)t_\epsilon}{1+t_\epsilon^2}\eta_{t_\epsilon}\right)\right]\\ &\leq V_\epsilon\left[u_\epsilon^{q-2}-1+\alpha+(n-2)\,\frac{t_\epsilon}{1+t_\epsilon^2}\eta_{t_\epsilon}\right].\end{aligned}$$

Since Ω is strictly convex, it follows that $t_\epsilon\to\infty$ as $\epsilon\to 0$. Thus, it is possible to select a sufficiently small $\alpha>0$ such that

$$u_\epsilon^{q-2}-1+\alpha+(n-2)\,\frac{t_\epsilon}{1+t_\epsilon^2}\eta_{t_\epsilon}\leq-\frac{1}{2} \tag{35}$$

which completes the proof of inequality (34). Therefore, the proof of Theorem I is complete. ■

2.5 Lower Bound Estimate

This section is devoted to completing the proof of Theorem J. For this purpose, a critical estimate concerning the minimax level $c_{q,r}(\Omega_\epsilon)$ will be established in Proposition G, presented below. The discussion commences with the introduction of appropriate notation.

For any sequence $\epsilon_k \to 0$, consider $u_k = u_{\epsilon_k}$ the solution of (P_{ϵ_k}) as given in Proposition C. Let $z_k := z_{\epsilon_k} \in \partial\Omega_{\epsilon_k}$ be chosen such that

$$u_k(z_k) = \max_{z \in \overline{\Omega}_{\epsilon_k}} u_k(z).$$

Given that u_k is harmonic in Ω_{ϵ_k}, the maximum principle implies that the maximum of u_k in $\overline{\Omega_{\epsilon_k}}$ must occur on $\partial\Omega_{\epsilon_k}$. With these notations, the following estimate can be derived.

Proposition G *There exists k_0 such that, for all $k \geq k_0$, the following holds*

$$c_{q,r}(\Omega_{\epsilon_k}) \geq c_{q,r}\left(\mathbb{R}^n_+\right) - \epsilon_k \gamma \mathcal{H}(\epsilon_k z_k) + o(\epsilon_k). \tag{38}$$

Proof. Let the sequence $y_k = \epsilon_k z_k$. Based on the considerations presented at the beginning of the section, it can be assumed, up to a subsequence, that there exists $\overline{z} \in \partial\Omega$ such that $y_k \to \overline{z}$. Define $u_k(y) = u_{\epsilon_k}(y + y_k)$, $y \in \Omega_{\epsilon_k} \setminus \{y_k\}$. Following the approach in [3] (see also [15]), after applying suitable rotation and translation, it can be assumed that $\overline{z} = 0$ and $\Omega \subset \mathbb{R}^n_+$ can be described in a fixed neighborhood $\mathcal{U}$ of $\overline{z}$ as the set $\{(x,t) : \ t > G_k(x)\}$ with G_k smooth, $G_k(0) = 0$, and $\nabla G_k(0) = 0$. Moreover, G_k can be chosen such that it converges in C^2_{loc}-topology to G, the corresponding parametrization of $\partial\Omega$ at $\overline{z}$. Let $\Omega_k = \Omega_{\epsilon_k}$ and $\mathcal{U}_k = \epsilon_k^{-1}\mathcal{U}$, and define

$$\mathcal{V}_k := \{(x,t) \in \mathbb{R}^n \ : \ |\epsilon_k x| \leq \rho_0 \text{ and } 0 \leq \epsilon_k t \leq t_k\} \subset \mathcal{U}_k,$$

where $t_k = \min_{|x|=\rho_0} G_k(x) > 0$. In view of the fact that

$$c_{q,r}(\Omega_k) = I_{\Omega_k}(u_k) \geq I_{\Omega_k}(s u_k),$$

for all $s > 0$, and considering

$$I_{\mathcal{V}_k}(u_k) = \frac{1}{2}\int_{\mathcal{V}_k \cap \Omega_k} |\nabla u_k|^2 \ \mathrm{d}z + \frac{1}{2}\int_{\Gamma_k} |u_k|^2 \ \mathrm{d}\sigma + \frac{1}{r}\int_{\Gamma_k} |u_k|^r \ \mathrm{d}\sigma - \frac{1}{q}\int_{\Gamma_k} |u_k|^q \ \mathrm{d}\sigma,$$

where $\Gamma_k = \mathcal{V}_k \cap \partial\Omega_k$, it can be observed that

$$I_{\mathcal{V}_k\cap\Omega_k}(su_k) = I_{\Omega_k}(su_k) - I_{\Omega_k\setminus(\Omega_k\cap\mathcal{V}_k)}(su_k) \leq c_{q,r}(\Omega_k) - I_{\Omega_k\setminus(\Omega_k\cap\mathcal{V}_k)}(su_k).$$

By employing the decay of u_k, it can be concluded that

$$c_{q,r}(\Omega_k) \geq I_{\mathcal{V}_k\cap\Omega_k}(su_k) + o(\epsilon_k),$$

for all $s > 0$. Next, let $\overline{u}_k(x,t)$ denote the extension of u_k to $\mathcal{V}_k$, defined as follows:

$$\overline{u}_k(x,t) = u_k(x,t) \quad \text{if} \quad \epsilon_k t \geq G_k(\epsilon_k x),$$

and

$$\begin{aligned} \overline{u}_k(x,t) &= u_k\left(x, \epsilon_k^{-1} G_k(\epsilon_k x)\right) + \left(G_k(\epsilon_k x) - \epsilon_k t\right)\left[u_k^q\left(x, G_k(\epsilon_k x)\right) - u_k\left(x, G_k(\epsilon_k x)\right)\right] \\ &+ \left(G_k(\epsilon_k x) - \epsilon_k t\right)\left[u_k^r\left(x, G_k(\epsilon_k x)\right) - u_k\left(x, G_k(\epsilon_k x)\right)\right], \end{aligned}$$

if $\epsilon_k t < G_k(\epsilon_k x)$. By the same reasoning as before, applying the decay of u_k, the following inequality holds:

$$c_{q,r}(\Omega_k) \geq I_{\mathcal{V}_k}(s\overline{u}_k) - I_{\mathcal{V}_k\setminus(\mathcal{V}_k\cap\Omega_k)}(s\overline{u}_k) + o(k).$$

Passing to a subsequence, it can be assumed that $u_k \to w$ in H^1, where w is a least-energy solution to Problem (P_λ), where $2 < r < q < 2_*$, with $\lambda = 1$. Let $s_k > 0$ be such that

$$I_{\mathcal{V}_k}(s_k\overline{u}_k) = \sup_{s>0} I_{\mathcal{V}_k}(s\overline{u}_k).$$

By employing the definition of supremum and the definition of $I_{\mathcal{V}_k}$, it follows that

$$\left(\frac{s_k^2/2 - s_k^q/q + 1/q - 1/2}{s_k^2/2 - s_k^r/r + 1/r - 1/2}\right) \int_{\partial\mathcal{V}_k} |\overline{u_k}|^q \, \mathrm{d}\sigma \geq \int_{\partial\mathcal{V}_k} |\overline{u_k}|^r \, \mathrm{d}\sigma,$$

which leads to the conclusion that $s_k \to 1$. Additionally, it is observed that $I_{\mathcal{V}_k}(s_k\overline{u}_k) \geq c_{q,r}\left(\mathbb{R}^n_+\right) + o(k)$. From these facts, it follows that

$$c_{q,r}(\Omega_k) \geq c_{q,r}\left(\mathbb{R}^n_+\right) - R_1(k) + R_2(k) + o(\epsilon_k), \tag{39}$$

where

$$R_1(k) := \frac{1}{2}\int_{\mathcal{V}_k\setminus(\mathcal{V}_k\cap\Omega_k)} |\nabla\overline{u}_k|^2 \, \mathrm{d}z,$$

and

$$R_2(k) := \frac{1}{q}\int_{\Gamma_k} |s_k\overline{u}_k|^q \,\mathrm{d}\sigma - \frac{1}{r}\int_{\Gamma_k} |s_k\overline{u}_k|^r \,\mathrm{d}\sigma - \frac{1}{2}\int_{\Gamma_k} |s_k\overline{u}_k|^2 \,\mathrm{d}\sigma.$$

Thus, the same approach as in the proof of Lemma 2.3.1 can be followed to obtain the estimates

$$R_1(k) = -\epsilon_k \int_{\mathbb{R}^{n-1}} |\nabla w(x,0)|^2 g(x)\ \mathrm{d}x + o(\epsilon_k),$$

$$R_2(k) = \epsilon_k \int_{\mathbb{R}^{n-1}} w_t^2(x,0)\, g(x)\ \mathrm{d}x + o(\epsilon_k),$$

which, together with (39), implies that estimate (38) holds. Hence, it follows that

$$c_{q,r}(\Omega_k) \geq c_{q,r}\left(\mathbb{R}^n_+\right) - \epsilon_k \int_{\mathbb{R}^{n-1}} \left[\frac{|\nabla w(x,0)|^2}{2} - w_t^2(x,0)\right] g(x)\ \mathrm{d}x + o(\epsilon_k),$$

and by proceeding as in the proof of Proposition E, the following inequality holds:

$$c_{q,r}(\Omega_k) \geq c_{q,r}\left(\mathbb{R}^n_+\right) - \epsilon_k\gamma\mathcal{H}(\epsilon_k z_{\epsilon_k}) + o(\epsilon_k),$$

which completes the proof. ■

To conclude the section, the proof of Theorem J is presented.

Proof of Theorem J. The propositions E and G establish that

$$c_{q,r}(\Omega_\epsilon) \leq c_{q,r}\left(\mathbb{R}^n_+\right) - \epsilon\gamma \max_{z\in\partial\Omega} \mathcal{H}(z) + o(\epsilon), \quad \text{as} \quad \epsilon \to 0$$

and

$$c_{q,r}(\Omega_{\epsilon_k}) \geq c_{q,r}\left(\mathbb{R}^n_+\right) - \epsilon_k\gamma\mathcal{H}(\epsilon_k z_k) + o(\epsilon_k), \quad \text{as} \quad \epsilon_k \to 0.$$

Furthermore, in Proposition G, it was shown that there exists $\overline{z} \in \partial\Omega$ such that $y_k = \epsilon_k z_k \to \overline{z}$. It is stated that $\mathcal{H}(\overline{z}) \geq \mathcal{H}(z)$, for all $z \in \partial\Omega$. In fact, since $\gamma > 0$, the following holds:

$$\begin{aligned} c_{q,r}(\Omega_\epsilon) &\leq c_{q,r}\left(\mathbb{R}^n_+\right) - \epsilon\gamma \max_{z\in\partial\Omega} \mathcal{H}(z) + o(\epsilon) \\ &\leq c_{q,r}\left(\mathbb{R}^n_+\right) - \epsilon\gamma\mathcal{H}(z) + o(\epsilon) \\ &\leq c_{q,r}(\Omega_\epsilon) + \epsilon\gamma\mathcal{H}(\overline{z}) - \epsilon\gamma\mathcal{H}(z) + o(\epsilon), \end{aligned}$$

which implies that $\mathcal{H}(\overline{z}) \geq \mathcal{H}(z)$, for all $z \in \partial\Omega$. Therefore, it can be concluded that

$$\mathcal{H}(\epsilon_k z_k) \to \max_{z\in\partial\Omega} \mathcal{H}(z),$$

which, combined with propositions E and G, completes the proof of Theorem J. ■

2.6 Nonexistence Result

This section is dedicated to proving Theorem K. Let $\overline{u}$ denote the average of u over $\partial\Omega$, which is defined by the expression:

$$\overline{u} = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} u \,\mathrm{d}\sigma.$$

The following Poincaré inequality can be established.

Lemma 2.6.1 *There exists a constant C, depending only on n, such that*

$$\|u-\overline{u}\|_{L^2(\partial\Omega)} \leq C\,\|\nabla u\|_{L^2(\Omega)}\,, \quad \forall u \in H^1(\Omega)\,.$$

Proof. Define the function $v = u - \overline{u}$. It can be noted that

$$\int_{\partial\Omega} v \,\mathrm{d}\sigma = \int_{\partial\Omega} u \,\mathrm{d}\sigma - \overline{u}\,|\partial\Omega| = \overline{u}\,|\partial\Omega| - |\partial\Omega| = 0.$$

By applying the classical Poincaré inequality, the following inequality holds:

$$\|v\|_{L^2(\Omega)} \leq C\,\|\nabla v\|_{L^2(\Omega)}\,.$$

Furthermore, since $v = u - \overline{u}$, it follows that $\nabla v = \nabla u$. Hence, by employing the Sobolev trace inequality, it can be concluded that

$$\|u-\overline{u}\|_{L^2(\partial\Omega)} = \|v\|_{L^2(\partial\Omega)} \leq C\,\|v\|_{H^1(\Omega)} \leq C\,\|\nabla u\|_{L^2(\Omega)}\,,$$

for all $u \in H^1(\Omega)$. This completes the proof. ■

Using the previous lemma, the proof of Theorem K is established.

Proof of Theorem K. The function u is decomposed as $u = \overline{u} + v$, where

$$\overline{u} = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} u \,\mathrm{d}\sigma \quad \text{and} \quad \int_{\partial\Omega} v \,\mathrm{d}\sigma = 0.$$

It can be observed that

$$u^{s-1} - \overline{u}^{s-1} = (s-1)\left[\int_0^1 (\overline{u}+tv)^{s-2}\,\mathrm{d}t\right] v, \tag{40}$$

where $s > 2$. By choosing φ as a test function in (2) and applying (40), the following inequality holds:

$$\epsilon^2 \int_\Omega |\nabla v|^2 \,\mathrm{d}z + \int_{\partial\Omega} v^2 \,\mathrm{d}\sigma \leq \lambda\,(q-1)\int_{\partial\Omega}\left[\int_0^1 (\overline{u}+tv)^{q-2}\,\mathrm{d}t\right] v^2 \,\mathrm{d}\sigma.$$

Furthermore, by Proposition F, it follows that

$$\epsilon^2 \int_\Omega |\nabla v|^2 \, \mathrm{d}z \leq C \int_{\partial\Omega} v^2 \, \mathrm{d}\sigma,$$

which, in combination with Lemma 2.6.1, implies that

$$\epsilon^2 \int_\Omega |\nabla v|^2 \, \mathrm{d}z \leq C \int_\Omega |\nabla v|^2 \, \mathrm{d}z.$$

Thus, for sufficiently large ϵ, v must be constant. Consequently,

$$0 = \int_{\partial\Omega} v \, \mathrm{d}\sigma = |\partial\Omega| \, v.$$

Therefore, $v \equiv 0$, which concludes the proof of Theorem K. ■

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