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Crepant resolutions of Gorenstein abelian quotient singularities

By

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João Pessoa - PB
February/ 2025

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under the supervision of

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Dedicatory

To my family

Resumo

Seja G um subgrupo finito e abeliano de $\mathrm{SL}(n,\mathbb{C})$ e suponha que existe uma resolução crepante $\phi: X \longrightarrow \mathbb{C}^n/G$ da variedade quociente (Gorenstein) $\mathbb{C}^n/G$. Nesta situação, o conjunto excepcional de ϕ é puro de codimensão 1, isto é, quando decompomos tal conjunto em componentes irredutíveis, $\mathrm{Exc}(\phi) = E_1 \cup \cdots \cup E_s$, podemos verificar que $\dim(E_i) = n - 1$, para todo i. Além disso, é conhecido que $\mathbb{C}^n/G$ é uma variedade tórica e que qualquer resolução crepante de tal variedade também é tórica. Usando as ferramentas oferecidas pela geometria tórica, neste trabalho provamos que, para cada componente irredutível E_i de $\mathrm{Exc}(\phi)$, existe um subconjunto aberto U_i de X tal que $E_i \subset U_i$ e U_i é o espaço total do feixe canônico ω_{E_i} de E_i. Além disso, $X = U_1 \cup \cdots \cup U_s$. Em particular, quando $s = 1$ obtemos que a resolução X em si é um fibrado de linhas sobre $\mathrm{Exc}(\phi)$.

Palavras-chave: Resolução crepante, variedade quociente, conjunto excepcional, geometria tórica, feixe canônico.

Abstract

Let G be a finite abelian subgroup of $\mathrm{SL}(n,\mathbb{C})$, and suppose there exists a crepant resolution $\phi : X \longrightarrow \mathbb{C}^n/G$ of the (Gorenstein) quotient variety $\mathbb{C}^n/G$. In this situation, the exceptional set of ϕ is pure of codimension 1. That is, when we decompose such a set into irreducible components, $\mathrm{Exc}(\phi) = E_1 \cup \cdots \cup E_s$, we can verify that $\dim(E_i) = n - 1$, for all i. Furthermore, it is known that $\mathbb{C}^n/G$ is a toric variety, and any crepant resolution of such a variety is also toric. Using the tools provided by toric geometry, in this work we prove that, for each irreducible component E_i of $\mathrm{Exc}(\phi)$, there exists an open subset U_i of X such that $E_i \subset U_i$, and U_i is the total space of the canonical bundle ω_{E_i} of E_i. Furthermore, $X = U_1 \cup \cdots \cup U_s$. In particular, when $s = 1$, we obtain that the resolution X itself is a line bundle over $\mathrm{Exc}(\phi)$.

Keywords: Crepant Resolution, quotient variety, exceptional set, toric geometry, canonical bundle.

Contents

Introduction

Quotient singularities of the form $\mathbb{C}^n/G$, where G is a finite subgroup of $\mathrm{SL}(n,\mathbb{C})$, are central objects of study in algebraic geometry due to their rich structure and numerous applications in both mathematics and physics [7]. As G is a finite subgroup of $\mathrm{SL}(n,\mathbb{C})$, the quotient variety $\mathbb{C}^n/G$ is a Gorenstein singularity [52], a desirable property which means that the canonical sheaf of $\mathbb{C}^n/G$ is locally trivial. These singularities arise naturally in invariant theory, representation theory, toric geometry, and the study of moduli spaces [18, 13, 4].

A classical example occurs in dimension $n = 2$, where G is a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$. The resulting quotient singularities, known as Kleinian or Du Val singularities, are classified by the A_n, D_n, and E_n series and exhibit a deep connection with the representation theory of simple Lie algebras [40].

Given a resolution of singularities, $f : X \to \mathbb{C}^n/G$, one has the ramification formula

$$K_X = f^*(K_{\mathbb{C}^n/G}) + \sum_{i=1}^{s} a_i E_i,$$

where K_X is a canonical divisor of X, $K_{\mathbb{C}^n/G}$ is a canonical divisor of $\mathbb{C}^n/G$, the divisors $E_1, \ldots, E_s$ are those that appear in the exceptional locus $\mathrm{Exc}(f)$ of f (called of exceptional divisors), and $a_1, \ldots, a_s$ are integer numbers. f is said to be a crepant resolution of singularities if $a_i = 0$ for every $i = 1, \ldots, s$. The existence of crepant resolutions is a fundamental problem for quotient singularities. For $n = 2$, it is well-understood that $\mathbb{C}^2/G$ admits a unique (up to isomorphisms) crepant resolution of singularities and in this case all the exceptional divisors are isomorphic to $\mathbb{P}^1$ [16]. For $n = 3$, the problem of the existence of a crepant resolution was solved first case by case [29, 30, 34, 42, 41], since the conjugacy classes of finite subgroups of $\mathrm{SL}(3, \mathbb{C})$ were listed [9]. Although in the 3-dimensional case, a crepant resolution is not necessarily unique, there is a canonical resolution, as shown in [6], which is the G-orbit Hilbert scheme $\mathrm{Hilb}^G(\mathbb{C}^3)$. The G-orbit Hilbert scheme is, by definition, the scheme parametrizing all G-invariant zero-dimensional subschemes of $\mathbb{A}^3$ of length $|G|$ (the order of G), with their structure sheaf isomorphic to the regular representation of the group G as G-modules. This canonical crepant resolution had already been found in the case where G is an abelian group [36], by using the tools from algebraic geometry.

When $G \subset \mathrm{SL}(n, \mathbb{C})$ is abelian, $\mathbb{C}^n/G$ is a toric variety, and toric geometry provides powerful tools to describe both the singularity and its resolutions [17]. The McKay correspondence establishes a striking relationship between the geometry of a crepant resolution and the representation theory of G. This correspondence has been extensively studied in dimensions $n = 2$ [51] and $n = 3$ [31] and continues to motivate research in higher dimensions [6].

Crepant resolutions of quotient singularities also appear in theoretical physics, where they are used as "compactification" spaces in quantum field theories or string theories in spaces possessing extra dimensions. The structure of the singularity encodes indeed specific properties of the physical models. [2].

In this thesis, we focus on the structure and properties of $\mathbb{C}^n/G$ singularities, with particular emphasis on the case where G is abelian. We explore the toric nature of these singularities and their crepant resolutions, proving results about the geometry of the exceptional set and its decomposition into irreducible components. Using tools from toric geometry, we find an original result claiming that crepant resolutions of $\mathbb{C}^n/G$ admit a decomposition into line bundles over their exceptional sets. Our results provide new insights into the interplay between geometry, topology, and the junior elements of G, as has been done, for example, in [31, 43].

Toric geometry is a powerful framework in algebraic geometry that provides a combinatorial approach to studying varieties. These varieties, called toric varieties, are defined as varieties containing a torus $(\mathbb{C}^*)^n$ as a dense open subset, where the action of the torus extends to the entire variety. In chapter 1, we introduce the basics of toric geometry, including cones, fans, and their connection to affine and projective toric varieties, where the main references for this chapter are [11], [17], and [38].

In chapter 2, we discuss the properties of the quotient variety $\mathbb{C}^n/G$ in the case where G is a finite abelian subgroup of $\mathrm{SL}(n, \mathbb{C})$ and how to realize this variety as a toric variety.

Applying techniques from toric geometry, we also determine when $\mathbb{C}^n/G$ has terminal or canonical singularities [39] and give a recipe for constructing minimal models of $\mathbb{C}^n/G$. Hilbert basis resolutions are also discussed. The main references in this chapter are [53], [54], [14], and [33].

In chapter 3, we prove our original result, which is well understood in the following context: Let $\{g_1, \ldots, g_s\}$ be the set of junior classes of G. Suppose that there is a Hilbert basis resolution $\phi : X_\Xi \to \mathbb{C}^n/G = U_{\sigma,N}$ of the quotient variety $\mathbb{C}^n/G$. The fractional expressions $\hat{g}_1, \ldots, \hat{g}_s$ are elements of $\mathbf{Hlb}_N(\sigma)$. Thus, $E_{g_i}; = V(\mathrm{Cone}(\hat{g}_i))$ is a exceptional prime divisor of the resolution, for every $i = 1, \ldots, s$. We prove that there is an open toric set U_i of X_Σ that contains E_{g_i} together with a torus invariant divisor D_i of E_{g_i} such that an isomorphism $\varphi : \mathrm{tot}(\mathcal{O}_{E_{g_i}}(D_i)) \to U_i$ is found and it is the identity in the zero section. In particular, when $\mathbb{C}^n/G$ admits a crepant resolution we get our main theorem:

Theorem 3.4 Let G be an abelian finite subgroup of $\mathrm{SL}(n, \mathbb{C})$ and suppose that there exists a crepant resolution $\phi : X_\Xi \to \mathbb{C}^n/G = U_{\sigma,N}$ of the quotient variety. If g is a junior element of G, then E_g is normally embedded in X_Ξ. In particular, the total space of the canonical bundle of E_g, $\mathrm{tot}(\omega_{E_g})$, is isomorphic to the toric variety X_{Ξ_g}, and

$$X_\Xi = \bigcup_{\hat{g} \in \hat{G} \cap \triangle_1} X_{\Xi_g}$$

where Ξ_g is the fan consisting of all the faces of the cones that appear in the set

$$\Xi_g(n) := \{\eta \in \Xi(n) | \rho_g \preceq \eta\}.$$

In particular, X_{Ξ_g} is open in X_Ξ.

The term "normally embedded" in the above theorem means that X_{Ξ_g} is a tubular neighborhood of E_g (i.e. it is isomorphic to the total space of the normal bundle $\mathcal{N}_{E_g}/X_\Xi$) and then this result contributes to the collection of results aimed at solving the very classical problem: determine which subvarieties of algebraic varieties have a neighborhood, isomorphic to the neighborhood of the zero section of their normal bundle. This problem is trivial in the category of real manifolds (see [26], Chapter 4), but it becomes highly nontrivial in the holomorphic and algebraic geometry [35]. Many mathematicians studied it [1], starting with the works of Grauert [19] and Van de Ven [50]. The main result of the thesis gives a partial solution to a global version of this problem, providing a class of examples in which the subvariety has a neighborhood isomorphic to the whole of its normal bundle, and not just to a neighborhood of its zero section. This is the second main result of this thesis.

CHAPTER 1

A glimpse of Toric geometry

This chapter discusses the main concepts and theorems about normal toric varieties. A toric variety may be formally described as an algebraic variety X that contains an algebraic torus $(\mathbb{C}^*)^n$ as a Zariski-dense open subset, together with a $(\mathbb{C}^*)^n$-action on X extending the standard action of the torus on itself. Such varieties are the subject of intensive study precisely because they unify methods from algebraic geometry, combinatorics, and convex geometry. Normal toric varieties are closely linked to the combinatorics of rational polyhedral fans. Each cone in a fan corresponds to an affine piece of the toric variety, and the whole variety is obtained by gluing these affine charts along common faces. Singularity types and their resolutions can thus be characterized purely in combinatorial terms. For this chapter, the main references are [11], [17], and [38].

1.1 Convex polyhedral cones and fans

We start with the notion of convex cone. These objects provide the main tools for studying toric varieties. In this work, $N \cong \mathbb{Z}^n$ will always denote a lattice (a free $\mathbb{Z}$-module of rank n) with dual $\mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ denoted by M. The vector spaces $N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M \otimes_{\mathbb{Z}} \mathbb{R}$ are denoted by $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$, respectively. Notice that the canonical map $N \to N_{\mathbb{R}}$ is injective. Thus, we can consider the points of N as points in $N_{\mathbb{R}}$. Moreover, the canonical pairing

$$\langle , \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$$

is such that $\langle u, v\rangle \in \mathbb{Z}$ whenever $u \in M$ and $v \in N$. Notice that the same considerations are true when $\mathbb{R}$ is replaced by $\mathbb{Q}$.

Definition 1.1. *(a) A subset $\sigma \subset N_{\mathbb{R}}$ is a* ***rational convex polyhedral cone*** *(rcp cone for short) if there are $v_1, \ldots, v_s \in N$ such that*

$$\sigma = \{a_1 v_1 + \cdots + a_s v_s | a_s \geq 0\}.$$

In this situation, σ is said to be generated by $v_1, \ldots, v_s$ as a cone, and the notation $\sigma = \mathrm{Cone}(v_1, \ldots, v_s)$ is used to mean that;

(b) The dimension of σ, denoted by dim(σ), is the dimension of the vector space $\mathbb{R}\sigma = \sigma + (-\sigma)$, where $-\sigma = \{-v | v \in \sigma\}$. A rational polyhedral convex cone σ is said to be ***strongly convex*** *if $\sigma \cap (-\sigma) = 0$; The* ***relative interior*** *of σ is the interior of σ in $\sigma + (-\sigma)$, and it is denoted by $\mathrm{Relint}(\sigma)$;*

(c) The dual of any subset $C \subset N_{\mathbb{R}}$ is the set

$$C^{\vee} := \{u \in M_{\mathbb{R}} | \langle u, v\rangle \geq 0 \textit{ for every } v \in C\};$$

(d) Given $u \in M_{\mathbb{R}}$, consider $H_u := \{v \in N_{\mathbb{R}} | \langle u, v\rangle = 0\}$. A subset τ of a rational polyhedral convex cone σ in $N_{\mathbb{R}}$ is a ***face*** *if there is $u \in \sigma^{\vee}$ such that $\tau = H_u \cap \sigma$.*

Before giving an example of a strongly convex rational polyhedral cone, we list the main properties of such objects below.

Proposition 1.1. Let σ be a rcp cone in $N_{\mathbb{R}}$. One has the following properties:

(a) (Farkas' theorem) $(\sigma^{\vee})^{\vee} = \sigma$ and $\sigma^{\vee}$ is a rcp cone in $M_{\mathbb{R}}$;

(b) Every face of σ is also a rcp cone and every proper face is contained in a facet (a facet is a face of dimension equal $\dim(\sigma) - 1$). In particular, any proper face is the intersection of all the facets containing it;

(c) Let τ be a face of σ and let $v \in \mathrm{Relint}(\tau)$. Define $\tau^* := H_v \cap \sigma^{\vee}$. Then

$$\tau^* = \{u \in \sigma^{\vee} | \langle u, w\rangle = 0 \text{ for every } w \in \tau\};$$

this means that the definition of the face τ^* of $\sigma^{\vee}$ does not depend on the choice of $u \in \mathrm{Relint}(\tau)$. Moreover, the map $\tau \to \tau^*$ is a bijection between the faces of σ and the faces of $\sigma^{\vee}$ such that $\dim(\tau) + \dim(\tau^*) = n (= \dim N_{\mathbb{R}})$;

(d) If σ is strongly convex then every edge ρ of σ (an edge is a 1-dimensional face) is a ray, i.e, a half line in $N_{\mathbb{R}}$. Moreover, there is a unique primitive element $u_\rho \in N$, called the **ray generator** of ρ, such that $\rho = \text{Cone}(u_\rho)$.

(e) If σ is strongly convex, then $\sigma = \text{Cone}(u_\rho | \rho \text{ is an edge of } \sigma)$.

Proof. for (a) and (b), see [17] p.10 and p.11. For (c), see [38] p.173. For (d) and (e), see [11] p.29. ■

In the case that $\sigma \subset N_{\mathbb{R}}$ is a rcp cone of maximal dimension, its dual $\sigma^\vee$ is a strongly convex rcp cone. Moreover, in this situation, there is a recipe for finding a set of generators for $\sigma^\vee$ if a set of generators for σ is known: Suppose that $\sigma = \text{Cone}(u_1, \ldots, u_s)$ and pick a subset $\{u_{k_1}, \ldots u_{k_{n-1}}\}$ of $n-1$ linearly independent vectors in $\{u_1, \ldots, u_s\}$ and find $w \in M_{\mathbb{R}}$ such that w annihilates the chosen subset. If w or $-w$ is an element of $\sigma^\vee$, say w, it means that $\text{Cone}(u_{k_1}, \ldots, u_{k_{n-1}})$ is a facet of σ and by parts (c) and (d) of the above proposition, $\text{Cone}(w)$ is a ray of σ. If neither w nor $-w$ is in $\sigma^\vee$ then both are discarded and another subset should be chosen. Since each facet of σ comes from a subset of $n-1$ linearly independent vectors among the generators of σ, by the bijection in part (c) and the fact that $\sigma^\vee$ is the sum of its rays, the set of all elements w obtained in that process is a set of generators of $\sigma^\vee$.

Example 1.1. Consider $N = \mathbb{Z}^3$. Thus, $N_{\mathbb{R}} = \mathbb{R}^3$ and M can also be identified as $\mathbb{Z}^3$. Take $\sigma = \text{Cone}(u_1, u_2, u_3) \subseteq N_{\mathbb{R}}$, where $u_1 = (-1, 2, 0), u_2 = (3, -1, -1), u_3 = (0, 0, 1)$. Notice that σ is a strongly convex cone and u_1, u_2, u_3 are primitive elements of N. Moreover, those are the ray generators of σ and $\dim(\sigma) = 3$. The set $\{u_1, u_2\}$ is annihilated by $w_1 = (2, 1, 5)$ and $\langle w_1, u_3 \rangle = 5$, hence $w_1 \in \sigma^\vee$, $\text{Cone}(u_1, u_2)$ is a facet of σ, and $\text{Cone}(w_1)$ is a ray of $\sigma^\vee$. In the same way, $w_2 = (2, 1, 0)$, $w_3 = (1, 3, 0)$ annihilate $\{u_1, u_3\}$, $\{u_2, u_3\}$, respectively, and both are in $\sigma^\vee$. Therefore, $\sigma^\vee = \text{Cone}(w_1, w_2, w_3) \subset M_{\mathbb{R}}$. △

Definition 1.2. *Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone.*

*(a) σ is **smooth** or **regular** if the set of its ray generators form part of a $\mathbb{Z}$-basis of N.*

*(b) σ is **simplicial** if the set of its ray generators is linearly independent over $\mathbb{R}$.*

Note that the cone σ of Example 1.1 is not smooth, because $\{u_1, u_2, u_3\}$ is not a $\mathbb{Z}$-basis for $\mathbb{Z}^3$. However, since u_1, u_2, u_3 are linearly independent over $\mathbb{R}$, σ is a simplicial cone.

Since $\sigma^\vee$ is also a rcp cone, the set $S_{\sigma,N} := \sigma^\vee \cap M$ is a subsemigroup of M. When there is no confusion, we denote $S_{\sigma,N}$ only by S_σ. This object is very important for studying affine toric varieties. The main properties of such a semigroup are listed below.

Proposition 1.2. (a) (Gordan's Lemma). $S_\sigma = \sigma^\vee \cap M$ is finitely generated as a semigroup by the set $D \cup (K \cap M)$, where $\sigma^\vee = \mathrm{Cone}(D)$ and

$$K = \Big\{ \sum_{m \in D} \delta_m m \;\Big|\; 0 \leq \delta_m < 1 \text{ for all } m \in D \Big\}.$$

(b) If σ is strongly convex then S_σ is saturated (i.e, for all $k \in \mathbb{N} \setminus \{0\}$ and $m \in M$, $km \in S_\sigma$ implies $m \in S_\sigma$);

(c) All the faces of σ are of the type $\sigma \cap H_u = \tau$ for some $u \in \sigma^\vee \cap M$. Moreover,

$$S_\tau = S_\sigma + \mathbb{Z}_{\geq 0} \cdot (-u).$$

Proof. See [11] section 1.2. ■

Now, we define the objects that will be related to normal toric varieties that are not necessarily affine.

Definition 1.3. *A **fan** Σ in $N_\mathbb{R}$ is a finite collection of cones $\sigma \subseteq N_\mathbb{R}$ such that:*

1. *Every $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone.*

2. *For all $\sigma \in \Sigma$, each face of σ is also in Σ.*

3. *For all $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of each (hence it is a cone in Σ).*

Furthermore, if Σ is a fan, then:

- *The **support** of Σ is*

$$|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma \subseteq N_\mathbb{R}.$$

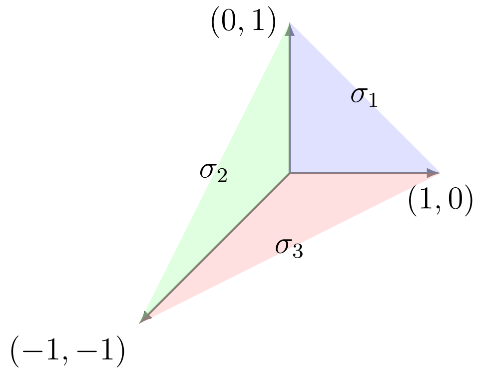
- *$\Sigma(r)$ is the set of r-dimensional cones of Σ.*

- *Σ is **smooth** (resp. **simplicial**) if all of its cones are smooth (resp. simplicial).*

Example 1.2. These are important examples of fans:

(a) Let σ be a strongly convex rpc cone and let Σ be the collection of cones that consists of all the faces of σ. Then Proposition 1.1 implies that Σ is a fan.

(b) For $N = \mathbb{Z}^2$ and the standard basis $\{e_1, e_2\}$ consider the cones $\sigma_1 = \mathrm{Cone}(e_1, e_2)$, $\sigma_2 = \mathrm{Cone}(e_2, -e_1 - e_2)$, and $\sigma_3 = \mathrm{Cone}(-e_1 - e_2, e_1)$. Let Σ be the collection of cones that consists of all of the faces of σ_i, for $i = 1, 2, 3$, as in the following picture



this is,

$$\Sigma = \{\sigma_1, \sigma_2, \sigma_3, \text{Cone}(e_1), \text{Cone}(e_2), \text{Cone}(-e_1 - e_2), \{0\}\}.$$

One can check that Σ is in fact a fan. Moreover, since $\{-e_1-e_2, e_1\}$ and $\{-e_1-e_2, e_1\}$ are also bases for $\mathbb{Z}^2$ it follows that Σ is a smooth fan (this is the fan of the projective space $\mathbb{P}^2$).

$\triangle$

1.2 Toric varieties

Given an abelian algebraic group T over $\mathbb{C}$, a homomorphism of algebraic groups $\chi : T \to \mathbb{C}^*$ is said to be a **character** of T, and a homomorphism of algebraic groups $\lambda : \mathbb{C}^* \to T$ is said to be a **cocharacter** of T. Denote by $X(T)$ (resp. $Y(T)$) the set of all characters (resp. cocharacters) of T. One can check that both $X(T)$ and $Y(T)$ are groups with binary operations coming from the group operations of T and $\mathbb{C}^*$. T is said to be a **algebraic torus** if it is isomorphic to $(\mathbb{C}^*)^n$ as algebraic group, for some integer $n \geq 1$.

Example 1.3. In this example, our aim is to describe the groups of characters of $(\mathbb{C}^*)^n$. Note that the coordinate ring of $(\mathbb{C}^*)^n$ is $\mathbb{C}[t_1, \ldots, t_n]_{t_1 \ldots, t_n} = \mathbb{C}[t_1^{\pm}, \ldots, t_n^{\pm}]$, that is, it is the localization of the polynomial ring $\mathbb{C}[t_1, \ldots, t_n]$ with respect to the element $t_1 \ldots t_n$. Consider the case $n = 1$: In this situation, a character

$$\chi : \mathbb{C}^* \to \mathbb{C}^*$$

is an element of $\mathbb{C}[t_1, t_1^{-1}]$. Thus, one can write $\chi = \frac{f}{t_1^s}$ for some $f \in \mathbb{C}[t_1]$ and for some integer $s \geq 0$. If f has a nonzero root, say $a \in \mathbb{C}^*$, then $\chi(a) = \frac{f(a)}{a^s} \in \mathbb{C}^*$, which is a contradition because $f(a) = 0$. Therefore, by the fundamental theorem of algebra,

$f = t_1^m$ for some integer $m \geq 0$. This way, $\chi = t_1^{m-s}$. Since every Laurent monomial t_i^r is a character of $\mathbb{C}^*$, it follows

$$X(\mathbb{C}^*) = \{t_1^s \in \mathbb{C}[t_1, t_1^{-1}] | s \in \mathbb{Z}\} \cong \mathbb{Z}.$$

For the case $n > 1$, note that

$$G_i := \{(t_1, \dots, t_i, \dots, t_n) \in (\mathbb{C}^*)^n | t_j = 1, \text{if } j \neq i\}$$

is a closed subgroup of $(\mathbb{C}^*)^n$ such $G_i \cong \mathbb{C}^*$. This implies that for any character $\xi \in X(G_i)$, $\xi(1, \dots, 1, t_i, 1 \dots, 1) = t_i^r$ for some integer r. Given a character $\chi \in X((\mathbb{C}^*)^n$, it follows that $\chi|_{G_i} \in X(G_i)$, therefore there is an integer r_i such that

$$\chi(1, \dots, 1, t_i, 1, \dots, 1) = t_i^i.$$

For each element $(t_1, \dots, t_n) \in (\mathbb{C}^*)$, one has

$$\begin{aligned} \chi(t_1, \dots, t_n) &= \chi(\prod_{i=1}^n (1, \dots, 1, t_i, 1, \dots, 1)) \\ &= \prod_{i=1}^n \chi((1, \dots, 1, t_i, 1, \dots, 1)) \\ &= t_1^{r_1} \dots t_i^{r_i} \dots t_n^{r_n}. \end{aligned}$$

Since any Laurent monomial in n variables also corresponds to a character of $(\mathbb{C}^*)^n$, one can conclude that

$$X((\mathbb{C}^*)^n) = \{t_1^{r_1} \dots t_n^{r_n} \in \mathbb{C}[t_1^{\pm}, \dots, t_n^{\pm}] | (r_1, \dots, r_n) \in \mathbb{Z}^n\} \cong \mathbb{Z}^n.$$

$\triangle$

From the above example, we can denote an element of $X((\mathbb{C}^*)^n)$ by $\chi^{(r_1, \dots, r_n)}$, which means that $\chi^{(r_1, \dots, r_n)}(t_1, \dots, t_n) = t_1^{r_1} \dots t_n^{r_n}$. One can also check that given a cocharacter $\lambda \in Y((\mathbb{C}^*)^n)$, there is a unique element $(a_1, \dots, a_n) \in \mathbb{Z}^n$ such that $\lambda(t) = (t^{a_1}, \dots, t^{a_n})$ for every $t \in \mathbb{C}^*$, in this case we use the notation $\lambda = \lambda^{(a_1, \dots, a_n)}$. In particular, $Y((\mathbb{C}^*)^n) \cong \mathbb{Z}^n$. With this notation, we have the perfect pairing

$$\begin{gathered} \langle \, , \, \rangle : X((\mathbb{C}^*)^n) \times Y((\mathbb{C}^*)^n) \to X(\mathbb{C}^*) \cong \mathbb{Z}, \\ \langle \chi^{(r_1, \dots, r_n)}, \lambda^{(a_1, \dots, a_n)} \rangle \mapsto \chi^{(r_1, \dots, r_n)} \circ \lambda^{(a_1, \dots, a_n)} = \chi^{r_1 a_1 + \dots + r_n a_n} \end{gathered}$$

which makes $X((\mathbb{C}^*)^n)$ and $Y((\mathbb{C}^*)^n)$ become dual $\mathbb{Z}$-modules.

Let $T \cong (\mathbb{C}^*)^n$ be an algebraic torus. One can check that $X(T) \cong X((\mathbb{C}^*)^n)$ and $Y(T) \cong Y((\mathbb{C}^*)^n)$, thus both $X(T)$ and $Y(T)$ are lattices of rank n, which are dual to each other. Normally, a lattice has a binary operation given by an addition, that is why the following construction is commonly used. Let M be a set that has the same cardinality as $X(T)$. By axiom of choice, for each element $u \in M$, we can associate it to a unique element χ^u of $X(T)$, so that the map

$$\begin{aligned} \eta : M &\to X(T) \\ u &\mapsto \chi^u \end{aligned}$$

is a bijection. Given $u_1, u_2 \in M$, we define $u_1 + u_2 = \eta^{-1}(\chi^{u_1}\chi^{u_2})$. Thus, $(M, +)$ becomes a $\mathbb{Z}$-module such that η is an isomorphism. In this case M is said to be the **lattice of the characters** of T. Moreover, following the same steps, we can construct the lattice $(N, +)$ of cocharacters of T. By construction, one has $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. Usually, T is written as T_N to mean that it has lattice of cocharacters given by N. For example, $\mathbb{C}^* = T_{\mathbb{Z}}$. For more details about algebraic groups and their groups of characters, see [27].

Definition 1.4. *A* ***toric variety*** *is an irreducible variety W containing an algebraic torus $T_N \cong (\mathbb{C}^*)^n$ as an open subset, along with a algebraic action $T_N \times W \to W$ that extends the multiplication of T_N.*

Example 1.4. Here are some examples of toric varieties:

(a) The affine space $\mathbb{C}^n$, whose torus is $(\mathbb{C}^*)^n$;

(b) The projective space $\mathbb{P}^n$, whose torus is

$$\{(x_0 : \cdots : x_n) | x_i \neq 0, i = 0, \ldots, n\} \cong (\mathbb{C}^*)^n;$$

(c) The cuspidal curve $C = V(x^3 - y^2) \subset \mathbb{C}^2$ which is a non-normal affine toric variety with torus

$$C - 0 = \{(t^2, t^3); t \in \mathbb{C}^*\} \cong \mathbb{C}^*;$$

(d) The variety $V = V(xy - zw) \subset \mathbb{C}^4$ with torus given by

$$V \cap (\mathbb{C}^*)^4 = \{(t_1, t_2, t_3, t_1 t_2 t_3^{-1}); t_i \in (\mathbb{C}^*)\} \cong (\mathbb{C}^*)^3.$$

△

Any strongly convex rcp cone σ provides a normal toric variety the following way: Denote by $\mathbb{C}[S_\sigma]$ the vector space over $\mathbb{C}$ with a basis given by the set of symbols $\{\chi^u | u \in S_\sigma\}$. This vector space has a structure of $\mathbb{C}$-algebra defined as follows:

- Multiplication is given by addition in S_σ:

$$\chi^u \cdot \chi^{u'} = \chi^{u+u'}.$$

- The identity $0 \in S_\sigma$ corresponds to the unit in $\mathbb{C}[S_\sigma]$, i.e., $\chi^0 = 1$.

If S_σ has generators $\{u_i\}$, then $\{\chi^{u_i}\}$ are generators of $C[S_\sigma]$ as a $\mathbb{C}$-algebra. Notice that $\mathbb{C}[S_\sigma] \subset \mathbb{C}[M] = \mathbb{C}[S_{\{0\}}]$, where $\{0\} = \text{Cone}(0)$ is a face of any strongly convex cone in $N_\mathbb{R}$. Moreover, since $\mathbb{C}[M]$ is the coordinate ring of the algebraic torus T_N, $\mathbb{C}[M] \cong \mathbb{C}[t_1, \ldots, t_n]_{t_1 \ldots t_n}$, i.e, it is isomorphic to a localization of the polynomial ring in $n = \text{rank}(N)$ variables. In particular, by Gordan's lemma, $\mathbb{C}[S_\sigma]$ is a finitely generated $\mathbb{C}$-algebra which is integral. Thus, its set of maximal ideals $U_\sigma := \text{Specm}(\mathbb{C}[S_\sigma])$ has a structure of irreducible complex algebraic variety.

Given a face $\tau = H_u \cap \sigma$ of σ, for some $u \in \sigma^\vee \cap M$, the equality $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-u)$ implies that $\mathbb{C}[S_\tau] = \mathbb{C}[S_\sigma]_{\chi^u}$. Thus, U_τ can be seen as a principal open set of U_σ. In particular, T_N is an open subset of U_σ.

Proposition 1.3. Let $\sigma \subset N_\mathbb{R} \cong \mathbb{R}^n$ be a strongly convex rational polyhedral cone.

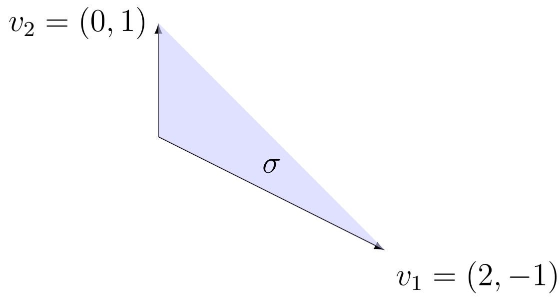
(a) U_σ is a normal toric variety of dimension n;

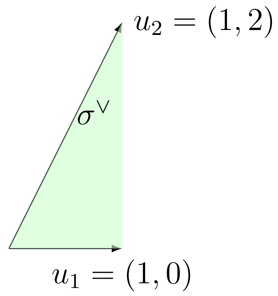
(b) σ is smooth if and only if U_σ is a smooth variety;

(c) σ is simplicial if and only if U_σ is a $\mathbb{Q}$-factorial variety.

Proof. See [10], p. 16. ■

Example 1.5. Let $N = \mathbb{Z}^2$ and consider the cone σ generated by $v_1 = (2, -1)$ and $v_2 = (0, 1)$. This cone is not smooth. In fact, the point $(1, 1)$ is not in the integer span of v_1, v_2. This way, U_σ is singular. The dual cone $\sigma^\vee$ is generated by $u_1 = (1, 0)$ and $u_2 = (1, 2)$.





One can check that the semigroup $S_\sigma = \sigma^\vee \cap M$ is generated by u_1, u_2, and $u_3 := (1,1)$. Thus, $\mathbb{C}[S_\sigma] = \mathbb{C}[\chi^{u_1}, \chi^{u_2}, \chi^{u_3}]$. Consider the homomorphism of $\mathbb{C}$-algebras

$$\phi : C[t_1, t_2, t_3] \longrightarrow \mathbb{C}[S_\sigma]$$

from the polynomial ring in three variable to $\mathbb{C}[S_\sigma]$, such that $\phi(t_i) = \chi^{u_i}$. Since $u_1 + u_2 = 2u_3$, one can check that $ker(\phi) = (t_1t_2 - t_3^2)$. Therefore

$$U_\sigma \cong \text{Specm}(\mathbb{C}[t_1, t_2, t_3]/(t_1t_2 - t_3^2).$$

Thus, this toric variety is the affine normal cone in affine 3-space (corresponding to a projective variety in $\mathbb{P}^2$), which is singular at the vertex. This variety is actually the quotient of $\mathbb{C}^2$ by the group $\mathbb{Z}_2$ acting as $(x, y) \to (-x, -y)$. In fact, one knows from the theory of affine quotients (see, e.g., [37]) that

$$\mathbb{C}^2/\mathbb{Z}_2 = \text{Spec}\,\mathbb{C}[x, y]^{\mathbb{Z}_2}.$$

It is not difficult to show that the ring of $\mathbb{Z}_2$-invariant polynomials in x, y is generated by x^2, xy, and y^2, this is, $\mathbb{C}[x, y]^{\mathbb{Z}_2} = \mathbb{C}[x^2, y^2, xy]$ (for some general facts about invariant polynomials under group actions, see [15]). The homomorphism

$$\psi : \mathbb{C}[t_1, t_2, t_3] \longrightarrow \mathbb{C}[x, y]^{\mathbb{Z}_2}$$

such that $\psi(t_1) = x^2, \psi(t_2) = y^2, \psi(t_3) = xy$ also has kernel given by $(t_1t_2 - t_3^2)$. Therefore,

$$\mathbb{C}^2/\mathbb{Z}_2 = \text{Specm}\,\mathbb{C}[t_1, t_2, t_3]/(t_1t_2 - t_3^2),$$

which shows that $\mathbb{C}^2/\mathbb{Z}_2$ is indeed U_σ. △

Given a strongly convex cone $\sigma \subset N_{\mathbb{R}}$, we have shown how to construct an affine normal toric variety $U_{\sigma,N} = \text{Specm}(\mathbb{C}[S_{\sigma,N}])$. The next result says that this correspondence from the "category" of strongly convex rpc cones with rational points in some lattice N to the category of affine normal toric varieties is actually surjective.

Proposition 1.4. Let W be an affine toric variety with torus T_N. Then the following are equivalent:

(a) W is normal.

(b) $W = \text{Specm}(\mathbb{C}[S])$, where $S \subseteq M$ is a saturated affine semigroup.

(c) $W = \text{Specm}(\mathbb{C}[S_\sigma]) = U_{\sigma,N}$, where $S_\sigma = \sigma^\vee \cap M$ and $\sigma \subseteq N_{\mathbb{R}}$ is a strongly convex rational polyhedral cone.

Proof. See [11], p.37. ■

In addition to the properties of a normal toric variety $U_{\sigma,N}$, there is a very nice bijection between its points and the set $\text{Hom}_{sg}(S_\sigma, \mathbb{C})$ of semigroup morphisms $S_\sigma \longrightarrow \mathbb{C}$. Such correspondence is defined as follows: given a point $p \in U_{\sigma,N}$, define

$$\begin{aligned} \gamma_p : S_\sigma &\to \mathbb{C} \\ u &\mapsto \chi^u(p). \end{aligned}$$

This makes sense since $\chi^u \in \mathbb{C}[S_\sigma]$. One can check that γ_p is a semigroup homomorphism. On the other way, let $\eta : S_\sigma \to \mathbb{C}$ be a semigroup homomorphism. Since $\{\chi^m\}_{m \in S}$ is a basis of $\mathbb{C}[S_\sigma]$, η induces a surjective linear map $\hat{\eta} : \mathbb{C}[S_\sigma] \to \mathbb{C}$ which is a $\mathbb{C}$-algebra homomorphism. The kernel of $\hat{\eta}$ is a maximal ideal and thus gives a point $p_\eta \in U_{S_\sigma}$. Let $\mathfrak{m}_p \subset \mathbb{C}[S_\sigma]$ be the maximal ideal corresponding to $p \in U_\sigma$. If $f \in ker(\hat{\gamma_p})$, then

$$f = \sum_{i=1}^{r} a_i \chi^{u_i}$$

for some $a_i \in \mathbb{C}$ and $u_i \in S_\sigma$, and

$$0 = \hat{\gamma_p}(f) = \sum_{i=1}^{r} a_i \chi^{u_i}(p) = f(p).$$

Thus, $ker(\hat{\gamma_p}) = \mathfrak{m}_p$ and therefore $p_{\gamma_p} = p$. Conversely, if $\eta : S_\sigma \to \mathbb{C}$ is a semigroup homomorphism, $\hat{\eta}(\chi^u) = \eta(u)$ by definition. However, $ker(\hat{\gamma}) = \mathfrak{m}_{p_\eta}$, which means that

$$\eta(u) = \underbrace{\chi^u + \mathfrak{m}_{p_\eta}}_{=\chi^u(p_\eta)} \in \mathbb{C}[S_\sigma] \cong \mathbb{C}.$$

Hence, $\gamma_{p_\eta} = \eta$. Therefore, the correspondence $p \to \gamma_p$ is indeed a bijection.

Now we shall construct normal toric varieties associated to a fan Σ. In this situation, such varieties are not necessarily affine, but Σ encodes the information needed to glue together the affine toric varieties U_σ, for $\sigma \in \Sigma$, to create an abstract variety X_Σ.

Given a fan Σ, let $\sigma \in \Sigma$ and let τ be a face of σ. In this case, there is $u \in S_\sigma$ such that $\mathbb{C}[S_\tau] = \mathbb{C}[S_\sigma]_{\chi^u}$, which gives an inclusion $U_\tau \to U_\sigma$ so that U_τ is a principal open set of U_σ. Given $\sigma, \sigma' \in \Sigma$, their intersection $\sigma \cap \sigma'$ is a common face of both cones. Hence, we get open immersions:

$$U_{\sigma\cap\sigma'} \to U_\sigma,$$

$$U_{\sigma\cap\sigma'} \to U_{\sigma'}.$$

If we denote the images of these maps by $V_{\sigma\cap\sigma'}$ and $V_{\sigma'\cap\sigma}$, respectively, then we have an isomorphism:

$$g_{\sigma\sigma'} : U_{\sigma'\cap\sigma} \cong U_{\sigma\cap\sigma'}.$$

This provides gluing data $\{U_\sigma, U_{\sigma'}, g_{\sigma\sigma'}\}$ (a cocycle condition is satisfied on triple intersections, if any), so we get the following variety.

Definition 1.5. *Given a fan Σ in $N_\mathbb{R}$, X_Σ is the abstract variety constructed using the above gluing data.*

To be more precise, X_Σ is constructed from affine varieties $U_\sigma, \sigma \in \Sigma$, by gluing U_σ and $U_{\sigma'}$ along their common open subset $U_{\sigma\cap\sigma'}$ for all $\sigma, \sigma' \in \Sigma$. The next result summarizes the main properties of such varieties.

Proposition 1.5. Let Σ be a fan of cones in $N_\mathbb{R} \cong \mathbb{R}^n$.

(a) The variety X_Σ is a separated normal toric variety;

(b) Σ is smooth if and only if X_Σ is a smooth variety;

(c) Σ is simplicial if and only if X_Σ is $\mathbb{Q}$-factorial variety;

(d) $|\Sigma| = N_\mathbb{R}$ if and only if X_Σ is a complete variety (which also means that it is compact in the analytic topology by [46]);

(e) X_Σ has no torus factors (i.e, X_Σ cannot be equivariantly isomorphic to the product of a nontrivial torus and a toric variety of smaller dimension) if and only if the u_ρ, $\rho \in \Sigma(1)$, span $N_\mathbb{R}$.

Proof. See [11], chapter 3. ■

Example 1.6. Consider the fan Σ in $\mathbb{Z}^2_\mathbb{R} \cong \mathbb{R}^2$ of Example 1.2, where

- The maximal cones are:

$$\sigma_1 = \text{Cone}(e_1, e_2), \quad \sigma_2 = \text{Cone}(-e_1 - e_2, e_1), \quad \sigma_3 = \text{Cone}(-e_1 - e_2, e_1),$$

- The 1-dimensional cones are $\rho_1 = \text{Cone}(e_1)$, $\rho_2 = \text{Cone}(e_2)$, $\rho_3 = \text{Cone}(-e_1 - e_3)$.

- The 0-dimensional cone is the origin $\{0\}$.

In this situation, Σ is a smooth cone, such that $|\Sigma| = \mathbb{R}^2$ and its ray generators e_1, e_2 span $\mathbb{R}^2$, hence the toric variety X_Σ is a complete and smooth toric variety without toric factors. Now we shall give a description of X_Σ. Such variety is covered by the affine open sets:

$$U_{\sigma_1} = \text{Spec}(\mathbb{C}[S_{\sigma_0}]) \cong \text{Spec}(\mathbb{C}[x, y]),$$

$$U_{\sigma_2} = \text{Spec}(\mathbb{C}[S_{\sigma_1}]) \cong \text{Spec}(\mathbb{C}[x^{-1}, x^{-1}y]),$$

$$U_{\sigma_3} = \text{Spec}(\mathbb{C}[S_{\sigma_2}]) \cong \text{Spec}(\mathbb{C}[xy^{-1}, y^{-1}]).$$

Thus one can check that the gluing data on the coordinate rings is given by:

$$g^*_{\sigma_1\sigma_2} : \mathbb{C}[x, y]_x \cong \mathbb{C}[x^{-1}, x^{-1}y]_{x^{-1}},$$

$$g^*_{\sigma_1\sigma_2} : \mathbb{C}[x, y]_y \cong \mathbb{C}[xy^{-1}, y^{-1}]_{y^{-1}},$$

$$g^*_{\sigma_2\sigma_3} : \mathbb{C}[x^{-1}, x^{-1}y]_{x^{-1}y} \cong \mathbb{C}[xy^{-1}, y^{-1}]_{xy^{-1}}.$$

Let $(x_1 : x_2 : x_3)$ be the usual homogeneous coordinates on $\mathbb{P}^2$. Then the mappings

$$x \mapsto \frac{x_2}{x_1} \quad \text{and} \quad y \mapsto \frac{x_3}{x_1}$$

identify the standard affine open $U_i \subseteq \mathbb{P}^2$ with $U_{\sigma_i} \subseteq X_\Sigma$. Hence, we can check that $\mathbb{P}^2 \cong X_\Sigma$.

△

Remark 1.1. *One of the main theorems of Sumihiro states that any point x of normal toric variety X has an affine open neighborhood that is invariant under the torus action (see [48], corollary 2). Bearing this result in mind, one can check that any normal toric variety can be obtained from a fan. Therefore, as in the affine case, the correspondence of the "category" of fans of strongly convex rpc cones to the category of normal toric varieties is surjective.*

1.3 The orbit-cone correspondence

Let Σ be a fan of cones in $N_{\mathbb{R}}$. There is a correspondence between the cones in Σ and the orbits of the variety X_Σ, which is given in the following way: an orbit O corresponds to a cone σ if and only if $\lim_{t\to 0} \lambda^u(t)$ exists and lies in O for all u in the relative interior of σ and the cocharacter $\lambda^u \in T_N$. The following results ensure that this correspondence is well-defined.

Proposition 1.6. With the notation as above:

(a) If τ is a proper face of σ then $\tau \cap \mathrm{Relint}(\sigma) = \emptyset$;

(b) $u \in \sigma$ is and only if $\lim_{t\to 0} \lambda^u(t)$ exists in U_σ. Moreover, there is a unique point $\gamma_\sigma \in U_\sigma$, such that for every $u \in \mathrm{Relint}(\sigma)$, one has

$$\lim_{t\to 0} \lambda^u(t) = \gamma_\sigma.$$

In this case γ_σ is said to be the distinguished point of U_σ.

Proof. For (a), see [38], appendix A. For (b), see [11], p.116 ■

The above proposition says that for every $\sigma, \sigma' \in \Sigma$, $\gamma_\sigma \neq \gamma_{\sigma'}$, thus the correspondence $\sigma \to O(\sigma) := T_N.\gamma_\sigma$ between the cones of Σ and the orbits of X_Σ injective, and it coincides with the correspondence that it has been defined at the beginning of this section. The next theorem shows that this correspondence is also surjective and provides much information about the orbits of X_Σ coming from the cones in Σ.

Proposition 1.7. (Orbit-Cone Correspondence) Let X_Σ be the toric variety of the fan Σ in $N_{\mathbb{R}}$. Then:

1. The correspondence:

$$\{\text{cones } \sigma \text{ in } \Sigma\} \longrightarrow \{T_N\text{-orbits in } X_\Sigma\}$$

$$\sigma \to O(\sigma)$$

is bijective;

2. Let $n = \dim N_{\mathbb{R}}$. For each cone $\sigma \in \Sigma$, $\dim O(\sigma) = n - \dim \sigma$;

3. The affine open subset U_σ is the union of orbits:

$$U_\sigma = \bigcup_{\tau \preceq \sigma} O(\tau);$$

4. $\tau \preceq \sigma$ if and only if $O(\sigma) \subseteq \overline{O(\tau)}$, and

$$\overline{O(\tau)} = \bigcup_{\tau \preceq \sigma} O(\sigma),$$

where $\overline{O(\tau)}$ denotes the closure in both the analytic and Zariski topologies.

Proof. See [17], section 3.1. ■

Example 1.7. In Example 1.6, the fan of $\mathbb{P}^2$ is described. For $i \in \{1, 2, 3\}$, the chart U_{σ_i} of $\mathbb{P}^2$ has coordinates of the form (t_1, t_2, t_2), such that $t_i = 1$, $t_j \in \mathbb{C}$, for $j \neq i$. The torus $(\mathbb{C}^*)^2 = T_{\mathbb{Z}^2} \subset \mathbb{P}^2$, in this chart, is the set of points (t_1, t_2, t_3), such that $t_j \neq 0$. A similar situation occurs with the charts U_{ρ_i}, but two of the coordinates should be equal 1. Given a point $v = (a, b) \in \mathbb{Z}^2$, we shall see how are the limits of the cocharacter $\lambda^v(t)$ with respect these charts. The limits are as in the following table:

v is in	The limit is
σ_1	$(1, 0, 0)$
σ_2	$(0, 1, 0)$
σ_3	$(0, 0, 1)$
ρ_1	$(1, 0, 1)$
ρ_2	$(1, 1, 0)$
ρ_3	$(0, 1, 1)$
the origin	$(1, 1, 1)$

The right side of the table describes the distinguished points. In particular, given $(1, t_2, t_3) \in T_{\mathbb{Z}^2} \subset U_{\sigma_1}$, one has $(1, t_2, t_3).(1, 0, 0) = (1, 0, 0)$. Therefore,

$$O(\sigma_1) = T_{\mathbb{Z}^2}.(0, 1, 0) = \{(1, 0, 0)\}$$

is an orbit of dimension 0, as expected. △

Given a strongly convex rcp cone σ in $N_{\mathbb{R}} \cong \mathbb{R}^n$, let N_σ be the $\mathbb{Z}$-submodule of N generated by $\sigma \cap N$. By Gordan's lemma, $\sigma \cap N$ is a finitely generated semigroup. Hence, N_σ is a finitely generated $\mathbb{Z}$ -submodule of N. Therefore, N_σ is also free (see [32], p.146), that is, N_σ is a sublattice of N. Furthermore, since σ is saturated, so is N_σ as a submodule of N, so that the quotient $N(\sigma) := N/N_\sigma$ is torsion-free and is therefore also a lattice.

The exact sequence

$$0 \to N_\sigma \to N \to N(\sigma) \to 0$$

splits, as $N(\sigma)$ is free over $\mathbb{Z}$. Tensoring the above exact sequence by $\mathbb{C}^*$ and letting

$$T_{N(\sigma)} = N(\sigma) \otimes_{\mathbb{Z}} \mathbb{C}^*$$

we obtain a surjective group homomorphism $T_N \to T_{N(\sigma)}$, so that T_N acts transitively on $T_{N(\sigma)}$.

Proposition 1.8. Let σ be a strongly convex rational polyhedral cone in $N_{\mathbb{R}}$.

1. The pairing $M \times N \to \mathbb{Z}$ induces a non-degenerated pairing

$$(\sigma^\perp \cap M) \times N(\sigma) \to \mathbb{Z};$$

2. This pairing induces isomorphisms

$$O(\sigma) = \{\gamma : S_\sigma \to \mathbb{C} \mid \gamma(m) \neq 0 \iff m \in \sigma^\perp \cap M\}$$

$$\cong \mathrm{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*) \cong T_{N(\sigma)}.$$

Proof. See [17], section 3.1. ■

Let Σ be a fan of cones in $N_{\mathbb{R}}$. Given $\tau \in \Sigma$, we denote

$$V(\tau) = \overline{O(\tau)}.$$

One has that $\tau \preceq \sigma$ if and only if $O(\sigma) \subseteq V(\tau)$, and

$$V(\tau) = \bigcup_{\tau \preceq \sigma} O(\sigma).$$

The torus $O(\tau) = T_{N(\tau)}$ is an open subset of $V(\tau)$. We shall check that $V(\tau)$ is a normal toric variety by constructing its fan. For each cone $\sigma \in \Sigma$ containing τ, let $\overline{\sigma}$ be the image cone in $N(\tau)_{\mathbb{R}}$ under the quotient map

$$N_{\mathbb{R}} \to N(\tau)_{\mathbb{R}}.$$

Then

$$\mathrm{Star}(\tau) = \{\overline{\sigma} \subseteq N(\tau)_{\mathbb{R}} \mid \tau \preceq \sigma \in \Sigma\}$$

is a fan in $N(\tau)_{\mathbb{R}}$.

Proposition 1.9. For any $\tau \in \Sigma$, the orbit closure $V(\tau) = \overline{O(\tau)}$ is isomorphic to the toric variety $X_{\mathrm{Star}(\tau),N(\tau)}$.

Proof. See [11], p.121. ∎

Let X be an algebraic variety. The set of singular points of X (the singular locus of X) is denoted by X_{sing}. In pages 92 and 126 of [47], one can check that X_{sing} is a closed subvariety of X, and if X is a normal variety, then $\dim X_{\mathrm{sing}} \leq \dim X - 2$. In the case where X is a normal toric variety, there is a relation between its singular locus and the singular cones of the corresponding fan.

Proposition 1.10. Let X_Σ be the toric variety of the fan Σ. Then:

$$(X_\Sigma)_{\mathrm{sing}} = \bigcup_{\sigma \text{ not smooth}} V(\sigma),$$

$$X_\Sigma \setminus (X_\Sigma)_{\mathrm{sing}} = \bigcup_{\sigma \text{ smooth}} U_\sigma.$$

Proof. See [11] p. 514. ∎

1.4 Morphisms of normal toric varieties

We define **FAN** as the category whose class of its objects Ob(**FAN**) consists of pairs (Σ, N), where N is a lattice and Σ is a fan of strongly convex rpc cones in $N_{\mathbb{R}}$, and given two pairs $(\Sigma, N), (\Sigma', N') \in$ **FAN**, the set of arrows between these objects, denoted by

$$\mathrm{Hom}_{\mathbf{FAN}}((\Sigma, N), (\Sigma', N')),$$

consists of lattice morphisms $\phi : N \longrightarrow N'$ with the condition that the induced morphism $\phi_{\mathbb{R}} : N_{\mathbb{R}} \longrightarrow N'_{\mathbb{R}}$ satisfies the following property: given a cone $\sigma \in \Sigma$, there is a cone

$\sigma' \in \Sigma'$ such that $\phi_{\mathbb{R}}(\sigma) \subset \sigma'$. In this case, ϕ is said to be **compatible** with the fans Σ and Σ'. One can check that **FAN** is a locally small category with the usual map composition (for an introduction to the language of categories, see [8], chapter 1).

Now we define **NTORIC** as the category such that Ob(**NTORIC**) consists of normal toric varieties. Given $X, X' \in$ Ob(**NTORIC**) with tori T, T', respectively; the set of arrows between these objects, denoted by

$$\mathrm{Hom}_{\mathbf{NTORIC}}(X, X'),$$

consists of equivariant algebraic variety morphisms $\eta : X \longrightarrow X'$, such that $\eta(T) \subset T'$ and $\eta|_T$ is a homomorphism of algebraic groups. In this case, η is said to be a **toric morphism**. This is also a locally small category.

From remark 1.1, it is expected that the above two categories are isomorphic. We shall check it. Define the functor

$$\mathcal{F} : \mathbf{FAN} \longrightarrow \mathbf{NTORIC}$$

such that $\mathcal{F}(\Sigma, N) = X_{\Sigma,N}$, and for $\phi \in \mathrm{Hom}_{\mathbf{FAN}}((\Sigma, N), (\Sigma', N'))$, $\mathcal{F}(\phi)$ is defined as follows:

- If Σ consists only of the faces of a cone σ, then $\mathcal{F}(\Sigma, N) = U_{\sigma,N}$. In this case, choose $\sigma' \in \Sigma'$ such that $\phi_{\mathbb{R}}(\sigma) \subset \sigma'$. Hence, its dual $\phi^{\vee}_{\mathbb{R}} : M'_{\mathbb{R}} \longrightarrow M_{\mathbb{R}}$ is such that, for $u \in \sigma'$,

  $$\phi^{\vee}_{\mathbb{R}}(u) = u \circ \phi_{\mathbb{R}} \in \sigma^{\vee}.$$

  Thus $\phi^{\vee}_{\mathbb{R}}(S_{\sigma'}) \subset (S_{\sigma})$. This inclusion induces an algebra morphism

  $$\mathbb{C}[S_{\sigma'}] \longrightarrow \mathbb{C}[S_{\sigma}]$$

  which, in its turn, induces a toric morphism $\phi_{\sigma} : U_{\sigma,N} \longrightarrow U_{\sigma',N'}$ (it is not straightforward, but with some effort it can be checked by using the fact that ϕ also induces a group morphism between T and T'). $\mathcal{F}(\phi)$ is defined as the composition of ϕ_{σ} followed by the inclusion $U_{\sigma',N'} \hookrightarrow X_{\Sigma',N'}$.

- For the general case, let σ be a cone in Σ. Since ϕ is compatible with Σ and Σ', there is a cone $\sigma' \in \Sigma'$ with $\phi_{\mathbb{R}}(\sigma) \subseteq \sigma'$. The previous step shows that ϕ induces a toric morphism

  $$\phi_{\sigma} : U_{\sigma,N} \to U_{\sigma',N'}.$$

Using the general criterion for gluing morphisms (see [22], chapter 2), one can check that the ϕ_σ, for $\sigma \in \Sigma$, glue together to give a toric morphism

$$\mathcal{F}(\phi) : X_{\Sigma,N} \to X_{\Sigma',N'}.$$

Proposition 1.11. With the above notation, $\mathcal{F}$: **FAN** $\longrightarrow$ **NTORIC** is an isomorphism of categories.

Proof. From Remark 1.1, $\mathcal{F}$ is essentially surjective. In [10], p. 126, one can check that $\mathcal{F}$ is fully faithful. ∎

The main properties of toric morphisms are summarized in the following proposition.

Proposition 1.12. Let $\mathcal{F}(\phi) : X_\Sigma \to X_{\Sigma'}$ be the toric morphism corresponding to a homomorphism $\phi : N \to N'$ that is compatible with fans Σ in $N_\mathbb{R}$ and Σ' in $N'_\mathbb{R}$.

(a) $\mathcal{F}(\phi) : X_\Sigma \to X_{\Sigma'}$ is a proper morphism if and only if $\phi_\mathbb{R}^{-1}(|\Sigma'|) = |\Sigma|$;

(b) If $X_{\Sigma',N'}$ is an affine toric variety and $\mathcal{F}$ is proper, then $X_{\Sigma,N}$ is quasi projective if and only if $\mathcal{F}(\phi)$ is a projective morphism.

Proof. For part (a), see [17], section 2.4. For part (b), see [11], p.330. ∎

1.5 Divisors on toric varieties

Let Σ be a fan in $N_\mathbb{R}$. From the orbit-cone theorem, the rays of Σ correspond to the codimension 1 orbit closures in the normal toric variety X_Σ. We let

$$D_\rho := V(\rho) = \overline{O(\rho)}$$

denote the irreducible torus-invariant divisor corresponding to $\rho \in \Sigma(1)$. Note that the torus of X_Σ is $T_N = X_\Sigma \setminus \bigcup_{\rho\in\Sigma(1)} D_\rho$. Let $\mathbb{C}(X_\Sigma)$ denote the field of rational funtions on X_Σ. Since X_Σ is normal and D_ρ is irreducible, one has that D_ρ is a prime divisor and the ring

$$\mathcal{O}_{X_\Sigma,D_\rho} = \{\phi \in \mathbb{C}(X_\Sigma) \mid \phi \text{ is defined on } U \subseteq X \text{ open with } U \cap D_\rho \neq \emptyset\}$$

is a discrete valuation ring. Its discrete valuation is denoted by

$$\nu_\rho : \mathbb{C}(X_\Sigma)^* \longrightarrow \mathbb{Z}.$$

For $f \in \mathbb{C}(X_\Sigma)^*$, $\nu_\rho(f)$ is said to be the **order of vanishing** of f along D_ρ.

There is a good relationship between the divisors D_ρ and the characters χ^m coming from $m \in M$. We can regard χ^m as a rational function on X_Σ which is non-vanishing on T_N. Hence, the divisor of χ^m is supported on $\bigcup_{\rho\in\Sigma(1)} D_\rho$. Thus the order of vanishing $\nu_\rho(\chi^m)$ is defined. By [17], Section 3.3, we have the remarkable formula

$$\nu_\rho(\chi^m) = \langle m, u_\rho \rangle,$$

where u_ρ is the ray generator of ρ, for $\rho \in \Sigma(1)$. It follows that the divisor of χ^u is given by

$$\mathrm{div}(\chi^m) = \sum_\rho \langle m, u_\rho \rangle D_\rho.$$

We denote by $\mathrm{Div}_{T_N}(X_\Sigma)$ the subgroup of $\mathrm{Div}(X_\Sigma)$ generated by the T_N-invariant prime divisors. We also define $\mathrm{CDiv}_{T_N}(X_\Sigma) := \mathrm{Div}_{T_N}(X_\Sigma) \cap \mathrm{CDiv}(X_\Sigma)$, where $\mathrm{CDiv}(X_\Sigma)$ is the group of Cartier divisors on X_Σ. Given $D \in \mathrm{Div}_{T_N}(X_\Sigma)$, let $\mathrm{Supp}(D)$ be its support. Note that $T_N \cap \mathrm{Supp}(D) = \emptyset$, otherwise $T_N \subset \mathrm{Supp}(D)$ since D in T_N-invariant. Thus

$$\mathrm{Supp}(D) \subset X_\Sigma \setminus T_N = \bigcup_{\rho\in\Sigma(1)} D_\rho,$$

which means that $D \in \bigoplus_{\rho\in\Sigma(1)} \mathbb{Z}D_\rho$. Therefore, $\mathrm{Div}_{T_N}(X_\Sigma) = \bigoplus_{\rho\in\Sigma(1)} \mathbb{Z}D_\rho$ and $\mathrm{rank}(\mathrm{Div}_{T_N}(X_\Sigma)) = \#(\Sigma(1))$. Furthermore, the main feature of this group is that any Weil divisor in X_Σ is linearly equivalent to one of its elements, as we can see in the following proposition, which lists the main properties of the Class group $\mathrm{Cl}(X_\Sigma)$ and the Picard group $\mathrm{Pic}(X_\Sigma)$ of X_Σ.

Proposition 1.13. With the above notation, one has:

(a) There is an exact sequence:

$$M \longrightarrow \mathrm{Div}_{T_N}(X_\Sigma) \longrightarrow \mathrm{Cl}(X_\Sigma) \longrightarrow 0,$$

where the first map is $m \mapsto \mathrm{div}(\chi^m)$ and the second sends a T_N-invariant divisor D to its divisor class $[D]$ in $\mathrm{Cl}(X_\Sigma)$. Furthermore, we have a short exact sequence:

$$0 \longrightarrow M \longrightarrow \mathrm{Div}_{T_N}(X_\Sigma) \longrightarrow \mathrm{Cl}(X_\Sigma) \longrightarrow 0$$

if and only if $\{u_\rho \mid \rho \in \Sigma(1)\}$ spans $N_\mathbb{R}$, i.e, X_Σ has no torus factors.

(b) Replacing $\mathrm{Div}_{T_N}(X_\Sigma)$ by $\mathrm{CDiv}_{T_N}(X_\Sigma)$ and $\mathrm{Cl}(X_\Sigma)$ by $\mathrm{Pic}(X_\Sigma)$ in part (a), the same result remains true;

(c) If X_Σ is affine, then $\mathrm{Pic}(X_\Sigma) = 0$;

(d) If $N_\mathbb{R} \cong \mathbb{R}^n$ and $\Sigma(n) \neq \emptyset$, then $\mathrm{Pic}(X_\Sigma)$ is a free abelian group;

(e) $\mathrm{Pic}(X_\Sigma) = \mathrm{Cl}(X_\Sigma)$ if and only if X_Σ is smooth.

Proof. See [11], Chapter 4. ■

When a smooth toric variety X_Σ is such that Σ contains at least a cone of maximal dimension, then X_Σ has no torus factors and the above proposition provides a method to compute $\mathrm{Pic}(X_\Sigma)(= \mathrm{Cl}(X_\Sigma))$. In this situation, $\mathrm{Pic}(X_\Sigma)$ is a free abelian group, and one has the exact sequence

$$0 \longrightarrow M \longrightarrow \mathrm{Div}_{T_N}(X_\Sigma) \longrightarrow \mathrm{Pic}(X_\Sigma) \longrightarrow 0.$$

Thus, since the groups appearing in the sequence are free, one has

$$\mathrm{rank}(\mathrm{Pic}(X_\Sigma)) = \mathrm{rank}(\mathrm{Div}_{\mathrm{T_N}}(X_\Sigma) - \mathrm{rank}(M) = \#(\Sigma(1)) - \dim X_\Sigma. \tag{1.1}$$

Example 1.8. Consider the fan $(\Sigma, \mathbb{Z}^2)$ of $\mathbb{P}^2$, described in Example 1.2. Since $\mathbb{P}^2$ is smooth and its fan contains cones of maximal dimension, it follows $\mathrm{rank}(\mathrm{Pic}(\mathbb{P}^2) = \#(\Sigma(1)) - \dim \mathbb{P}^2 = 1$, because Σ has 3 rays $\rho_1, \rho_2,$ and ρ_3 defined in Example 1.6. Note that

$$\begin{aligned} \mathrm{div}(\chi^{(1,0)}) = &\quad D_{\rho_1} - D_{\rho_3} \\ \mathrm{div}(\chi^{(0,1)}) = &\quad D_{\rho_2} - D_{\rho_3}. \end{aligned}$$

Hence, $[D_{\rho_1}] = [D_{\rho_2}] = [D_{\rho_3}]$, and $\mathrm{Pic}(\mathbb{P}^2)$ is generated by $[D_{\rho_3}]$. △

Remark 1.2. *Note that if X_Σ is smooth and projective, then its Picard group* $\mathrm{Pic}(X_\Sigma)$ *is not trivial. In fact, if X_Σ is projective, then $\Sigma(1) \neq \emptyset$, otherwise X_Σ would be equal to T_N. In this case, consider $\rho \in \Sigma(1)$ and suppose that* $\mathrm{Pic}(X_\Sigma) = 0$. *It follows that there is $f \in \mathbb{C}(X_\Sigma)^*$ such that* $\mathrm{div}(f) = D_\rho$, *which implies that $f \in \Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}) = \mathbb{C}$. Since f is constant, one has $\nu_\rho(f) = 0$, what is a contradiction. Therefore,* $\mathrm{Pic}(X_\Sigma) \neq 0$.

Definition 1.6. *Let Σ be a fan in $N_\mathbb{R}$.*

*(a) A **support function** is a function $\varphi : |\Sigma| \to \mathbb{R}$ that is linear on each cone of Σ. The set of all support functions is denoted* $\mathrm{SF}(\Sigma)$.

(b) *A support function* φ *is* ***integral*** *with respect to the lattice* N *if*

$$\varphi(|\Sigma| \cap N) \subseteq \mathbb{Z}.$$

The set of all such support functions is denoted $\mathrm{SF}(\Sigma, N)$.

There is a nice correspondence between T_N-invariant Cartier divisors of X_Σ and the support functions which are integral with respect to N, described as follows. Given

$$D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho \in \mathrm{CDiv}_{\mathrm{T_N}}(X_\Sigma),$$

one can check that for every cone $\sigma \in \Sigma$, there exists $m_\sigma \in M$ with the following properties:

- $\langle m_\sigma, u_\rho \rangle = -a_\rho \quad$ for all $\rho \in \sigma(1)$;
- m_σ is unique modulo $M(\sigma) = \sigma^\perp \cap M$;
- If τ is a face of σ, then $m_\sigma \equiv m_\tau \pmod{M(\tau)}$.

In this situation, D is principal in U_σ and $\{U_\sigma, \chi^{m_\sigma}\}_{\sigma \in \Sigma}$ is the local data for D (see [17], p. 62). We call $\{m_\sigma\}_{\sigma \in \Sigma}$ of the Cartier data for D. Bearing this in mind, the following proposition provides the description of Cartier divisors in terms of support functions.

Proposition 1.14. Let Σ be a fan in $N_\mathbb{R}$. Then:

(a) Given $D = \sum_\rho a_\rho D_\rho$ with Cartier data $\{m_\sigma\}_{\sigma \in \Sigma}$, the function

$$\varphi_D : |\Sigma| \to \mathbb{R}, \quad u \mapsto \varphi_D(u) = \langle m_\sigma, u \rangle \text{ when } u \in \sigma,$$

is a well-defined support function that is integral with respect to N;

(b) $\varphi_D(u_\rho) = -a_\rho$ for all $\rho \in \Sigma(1)$, so that

$$D = -\sum_\rho \varphi_D(u_\rho) D_\rho;$$

(c) The map $D \mapsto \varphi_D$ induces an isomorphism

$$\mathrm{CDiv}_{T_N}(X_\Sigma) \cong \mathrm{SF}(\Sigma, N).$$

Proof. See [11] p.184. ■

Remark 1.3. *The identification of* $\mathrm{CDiv}_T(X_\Sigma)$ *with* $\mathrm{SF}(\Sigma)$ *allows us to write the exact sequence of Proposition 1.13 (b) in the form:*

$$M \to \mathrm{SF}(\Sigma) \to \mathrm{Pic}(X_\Sigma) \to 0,$$

where the morphism $M \to \mathrm{SF}(\Sigma)$ *maps* $m \in M$ *to the support function* $\varphi(v) = -m(v)$*, which is linear on the entire fan* Σ*. On the other hand, if a support function* φ *is linear, it defines an element of* M*, that is,* $m(v) = -\varphi(v)$ *for all* $v \in N_\mathbb{R}$*. Thus, the image of* M *in* $\mathrm{SF}(\Sigma)$ *is the subspace of linear integral support functions. It follows that if all integral support functions* $\mathrm{SF}(\Sigma)$ *are linear, then* $\mathrm{Pic}(X_\Sigma) = 0$ *(and* X_Σ *is not projective).*

1.6 Homogeneous coordinates

From our basic example of $\mathbb{P}^2$, the usual homogeneous coordinates give not only the graded ring $\mathbb{C}[x_0, x_1, x_2]$ but also the quotient construction $\mathbb{P}^2 \cong (\mathbb{C}^3 \setminus \{0\})/\mathbb{C}^*$. Given a toric variety X_Σ with no torus factors, we can generalize this as follows. For each $\rho \in \Sigma(1)$, introduce a variable x_ρ, which gives the polynomial ring

$$S = \mathbb{C}[x_\rho \,:\, \rho \in \Sigma(1)].$$

We call S the total coordinate ring of X_Σ. In particular, $\mathrm{Specm}(S) = \mathbb{C}^{\Sigma(1)}$. Given a cone $\sigma \in \Sigma$, define the monomial

$$x_{\widehat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho.$$

Thus, $x_{\widehat{\sigma}}$ is the product of the variables corresponding to rays not in σ. Then define the ideal

$$B(\Sigma) = \langle x_{\widehat{\sigma}} \mid \sigma \in \Sigma \rangle \subseteq S.$$

$B(\Sigma)$ is called the **irrelevant ideal** of X_Σ. Note that $x_{\widehat{\tau}}$ is a multiple of $x_{\widehat{\sigma}}$ whenever $\tau \preceq \sigma$. Hence, if $\Sigma_{\max}$ is the set of maximal cones of Σ with respect the inclusion, then

$$B(\Sigma) = \langle x_{\widehat{\sigma}} \mid \sigma \in \Sigma_{\max} \rangle.$$

Finally, define $Z(\Sigma)$ as the zero locus of $B(\Sigma)$ in $\mathbb{C}^{\Sigma(1)}$. Note that a monomial $\prod_\rho x_\rho^{a_\rho}$

determines a divisor

$$D = \sum_\rho a_\rho D_\rho,$$

and in this case, we will write such monomial as x^D. We will grade S as follows:

The degree of a monomial $x^D \in S$ is $\deg(x^D) = [D] \in \mathrm{Cl}(X_\Sigma)$.

Since X_Σ has no torus factors, one has the exact sequence

$$0 \longrightarrow M \longrightarrow \mathrm{Div}_{T_N}(X_\Sigma) \longrightarrow \mathrm{Cl}(X_\Sigma) \longrightarrow 0 \tag{1.2}$$

from Proposition 1.13. It follows that two monomials $\prod_\rho x_\rho^{a_\rho}$ and $\prod_\rho x_\rho^{b_\rho}$ in S have the same degree if and only if there is some $m \in M$ such that $a_\rho = \langle m, n_\rho \rangle + b_\rho$ for all ρ. Then, let

$$S_\alpha = \bigoplus_{\deg(x^D)=\alpha} \mathbb{C} \cdot x^D,$$

so that the ring S can be written as the direct sum

$$S = \bigoplus_{\alpha \in \mathrm{Cl}(X_\Sigma)} S_\alpha.$$

Note also that $S_\alpha \cdot S_\beta \subseteq S_{\alpha+\beta}$. The plan is that X_Σ should be a quotient of $\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)$, with homogeneous coordinate ring given by S. The quotient is by the group G, which is defined to be

$$G = \mathrm{Hom}_\mathbb{Z}(\mathrm{Cl}(X_\Sigma), \mathbb{C}^*).$$

Note that applying $\mathrm{Hom}_\mathbb{Z}(-, \mathbb{C}^*)$ to (1.2) gives the exact sequence

$$1 \to G \to (\mathbb{C}^*)^{\Sigma(1)} \to T_N \to 1$$

since $T_N = N \otimes_\mathbb{Z} \mathbb{C}^* = \mathrm{Hom}_\mathbb{Z}(M, \mathbb{C}^*)$. This shows that G acts naturally on $\mathbb{C}^{\Sigma(1)}$ and leaves $Z(\Sigma)$ invariant since this subvariety consists of coordinate subspaces.

Proposition 1.15. Assume that X_Σ is a toric variety with no torus factors. Then:

(a) Given a basis $m_1, \ldots, m_n$ of M, we have

$$G = \{(t_\rho) \in (\mathbb{C}^*)^{\Sigma(1)} \mid \prod_\rho t_\rho^{\langle m_i, u_\rho \rangle} = 1 \text{ for } 1 \leq i \leq n\}.$$

(b) X_Σ is the universal categorical quotient $(\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma))/G$.

(c) X_Σ is a geometric quotient $(\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma))/G$ if and only if Σ is simplicial.

(d) If X_Σ is simplicial, then for every homogeneous ideal $I \subseteq S$ one has

$$V(I) = \{[x] \in X_\Sigma \mid f(x) = 0 \text{ for all } f \in I\}$$

is a closed subvariety of X_Σ, where $[x]$ denotes the orbit of $x \in \mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)$ in X_Σ.

Proof. See [12]. ■

Example 1.9. For $N = \mathbb{Z}^2$ and $\sigma = \text{Cone}(v_1, v_2)$, where $v_1 = (2, -1)$ and $v_2 = (0, 1)$, we have shown in Example 1.5 that $U_\sigma \cong \mathbb{C}^2/\mathbb{Z}_2$ by computing their coordinate rings. It is also possible to check this using Proposition 1.15, as follows. Let Σ be the fan that consists of σ and its faces. Thus, $X_\Sigma = U_\sigma$ is a simplicial toric variety with no torus factors. In this case, we write the homogeneous coordinate ring of X_Σ as $S = \mathbb{C}[x_1, x_2]$ where x_i corresponds to the ray ρ_i generated by v_i. The unique maximal cone of Σ with respect the inclusion is σ, hence $B(\Sigma) = \langle x_{\hat{\sigma}} \rangle$. However, since $\Sigma(1)$ consists only of rays of σ, $x_{\hat{\sigma}} = 1$. This way, $Z(\Sigma) = \emptyset$ and then $X_\Sigma = \mathbb{C}^2/G$, where $G = \text{Hom}_\mathbb{Z}(\text{Cl}(X_\Sigma), \mathbb{C}^*) \subset (\mathbb{C}^*)^2$. By part (a) of Proposition 1.15, G consists of the points (x_1, x_2) of $(\mathbb{C}^*)^2$ such that

$$\begin{cases} x_1^{\langle (1,0),(2,-1)\rangle} x_2^{\langle (1,0),(0,1)\rangle} = 1, \\ x_1^{\langle (0,1),(2,-1)\rangle} x_2^{\langle (0,1),(0,1)\rangle} = 1. \end{cases}$$

The above equations imply that $G = \{(1,1), (-1,-1)\}$. Furthermore, G can be identified with $\mathbb{Z}_2$ acting on $\mathbb{C}^2$ as $(x, y) \to (-x, -y)$. Therefore, $X_\Sigma = \mathbb{C}^2/\mathbb{Z}^2$. △

1.7 The canonical sheaf of a toric variety

Before describing the canonical sheaf of a toric variety, we recall some definitions and constructions. For an affine variety $\text{Specm}(R)$, and a R-module M, we denote the associated sheaf associated with M by $\widetilde{M}$. For any sheaf $\mathcal{F}$ of abelian groups on topological space X, $\Gamma(X, \mathcal{F})$ denotes $\mathcal{F}(X)$. For any abelian group G and connected topological space X, we denote the constant sheaf on X with values in G by $\underline{G}_X$.

Let X be a irreducible variety over $\mathbb{C}$ and let $\mathcal{G}$ be a coherent sheaf of $\mathcal{O}_X$-modules on X. Denote by $\mathbb{C}(X)$ the the field of rational functions on X. We shall check that $\mathcal{G} \otimes_{\mathcal{O}_X} \underline{\mathbb{C}(X)}_X$ is a constant sheaf. Given an affine open set $U = \text{Specm}(R)$ of X, there is a finitely generated R-module M such that $\mathcal{G}|_U \cong \widetilde{M}$. For $p \in U$, denote by $\mathfrak{m}_p$ the corresponding maximal ideal in R. Since $\mathbb{C}(X)$ equals the fraction field of R, one has

$$(\mathcal{G} \otimes_{\mathcal{O}_X} \underline{\mathbb{C}(X)}_X)_p = (\mathcal{G}|_U \otimes_{\mathcal{O}_X|_U} \underline{\mathbb{C}(X)}_X)_p = M_{\mathfrak{m}_p} \otimes_R \mathbb{C}(X) = M \otimes_R \mathbb{C}(X). \tag{1.3}$$

Denote $M \otimes_R \mathbb{C}(X)$ by G. Note that

$$\Gamma(X, \mathcal{G}|_U \otimes_{\mathcal{O}_X|_U} \underline{\mathbb{C}(X)}_X) = G$$

and (1.3) provides that

$$\mathcal{G}|_U \otimes_{\mathcal{O}_X|_U} \underline{\mathbb{C}(X)}_X) = \underline{G}_U.$$

Given another affine open set W of X, the fact that there is $p \in W \cap U$ along with (1.3) imply that

$$\mathcal{G}|_W \otimes_{\mathcal{O}_X|_W} \underline{\mathbb{C}(X)}_X) = \underline{G}_W.$$

Hence, $\mathcal{G} \otimes_{\mathcal{O}_X} \underline{\mathbb{C}(X)}_X$ is a locally constant sheaf and therefore it is a constant sheaf (see [20], p. 33) with module of global sections G, which is a finitely generated vector space over $\mathbb{C}(X)$.

Definition 1.7. *The rank of a coherent sheaf $\mathcal{G}$ on a irreducible variety X (over $\mathbb{C}$) is the dimension over $\mathbb{C}(X)$ of $\Gamma(X, \mathcal{G} \otimes_{\mathcal{O}_X} \underline{\mathbb{C}(X)}_X)$.*

Remark 1.4. *Note that the rank of a coherent sheaf $\mathcal{G}$ on a irreducible variety X is equal to the rank of $\mathcal{G}|_U$ on U, for every open set U of X.*

We recall that a coherent sheaf $\mathcal{G}$ on a variety X is **reflexive** if the canonical map $\mathcal{G} \to \mathcal{G}^{\vee\vee}$ is an isomorphism. In the context below, an open subset U in X is said to be **big** if $\operatorname{codim}(X - U) \geq 2$.

Proposition 1.16. Let X be a normal variety, $\mathcal{G}$ a coherent sheaf on X, and U a big subset of X. Denote by $j : U \hookrightarrow X$ the inclusion and j_* the functor direct image. Then

(a) $\mathcal{G}^\vee$ is reflexive;

(b) If $\mathcal{G}$ is reflexive then $\mathcal{G} \simeq j_*(\mathcal{G}|_U)$;

(c) If $\mathcal{G}|_U$ is locally free then $\mathcal{G}^{\vee\vee} \simeq j_*(\mathcal{G}|_U)$.

Proof. For (a) and (b), see [23]. For (c), note that

$$(\mathcal{G}^\vee)|_{U_0} = (\mathcal{G}|_{U_0})^\vee$$

for any coherent sheaf $\mathcal{G}$ on X. This way,

$$\mathcal{F}^{\vee\vee} \simeq j_*((\mathcal{F}^{\vee\vee})|_U) = j_*((\mathcal{F}|_U)^{\vee\vee}) \simeq j_*(\mathcal{F}|_U),$$

where the first isomorphism follows from (a) and (b), and the last comes from the fact that $\mathcal{F}|_U$ is locally free and therefore reflexive. ■

Reflexive sheaves of rank 1 on a normal variety X have a deep connection with the Weil divisors of such varieties, described as follows. Given a Weil divisor D in X, we associate with it a sheaf $\mathcal{O}_X(D)$ by letting, for every open subset $U \subset X$,

$$\mathcal{O}_X(D)(U) = \{f \in \mathbb{C}(X)^* \mid (\mathrm{div}(f) + D)|_U \geq 0\} \cup \{0\}.$$

One can check that $\mathcal{O}_X(D)$ is a reflexive $\mathcal{O}_X$-module (see [45]). Since $\mathcal{O}_X(D)$ is isomorphic to $\mathcal{O}_X$ away from the support of D, it has rank one, by Remark 1.4.

Proposition 1.17. Let X be a normal variety, and $\mathcal{F}$ a coherent sheaf on X.

(a) $\mathcal{F}$ is reflexive and has rank one if and only if $\mathcal{F} \cong \mathcal{O}_X(D)$ for some Weil divisor D if and only if there is an big open subset $j : U \hookrightarrow X$ such that $\mathcal{L}|_U$ is a line bundle on U, and $\mathcal{L} \cong j_*(\mathcal{L}|_U)$;

(b) If D and E are Weil divisors on X, then

$$(\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(E))^{\vee\vee} \cong \mathcal{O}_X(D + E);$$

(c) $\mathcal{O}_X(D) \cong \mathcal{O}_X(E)$ if and only if D linearly equivalent to E;

(d) D is Cartier if and only if $\mathcal{O}_X(D)$ is a line bundle.

Proof. See [45]. ■

Let X be a normal variety. In general, the cotangent sheaf Ω^1_X is not locally free (it fails to be so at the singular points of X), and in particular, the sheaf

$$\Omega^n_X = \wedge^n \Omega^1_X$$

may fail to be a line bundle. However, we can at least define reflexive sheaves of "differential forms" by using the fact that the smooth locus U of X is a big subset. We set

$$\widehat{\Omega}^p_X = (\Omega^p_X)^{\vee\vee} \cong j_*(\Omega^p_X|_U),$$

where $\Omega_X^p = \wedge^p \Omega_X^1$. The sections of $\widehat{\Omega}_X^p$ are called **Zariski p-forms**. In particular, the rank one reflexive sheaf

$$\omega_X = \widehat{\Omega}_X^n$$

is the **canonical sheaf** of X. From the previous proposition, there is a class $\alpha \in \mathrm{Cl}(X)$, such that $D \in \alpha$ implies $\omega_X \cong \mathcal{O}_X(D)$. Every element of α, which is generally written as K_X, is said to be a **canonical divisor** on X. Furthermore, X is said to be **Gorenstein** if one of its canonical divisors (and hence each of them) is Cartier. In the case that X is a toric variety, there is a very nice description of a canonical divisor on X, given as follows.

Proposition 1.18. For a normal toric variety X_Σ, the canonical sheaf ω_{X_Σ} is given by

$$\omega_{X_\Sigma} \cong \mathcal{O}_{X_\Sigma}(-\sum_{\rho \in \Sigma(1)} D_\rho).$$

Thus, $K_{X_\Sigma} = -\sum_\rho D_\rho$ is a torus-invariant canonical divisor on X_Σ.

Proof. See [11], p. 366. ■

Example 1.10. For any smooth variety, every Weil divisor is Cartier, and hence such type of variety is in particular Gorenstein. However, many Gorenstein varieties are not smooth. For instance, we have seen in Examples 1.9 and 1.5 that $\mathbb{C}^2/\mathbb{Z}_2$ is the affine toric variety corresponding to the pair $(\Sigma, \mathbb{Z}^2)$, where Σ consists of all the faces of the cone $\sigma = \mathrm{Cone}((2,-1),(0,1))$; which is singular because $\{(2,-1),(0,1)\}$ is not a basis of $\mathbb{Z}^2$. Following the notation of Example 1.9, note that

$$\mathrm{div}(\chi^{(-1,-1)}) = \langle(-1,-1),(2,-1)\rangle D_{\rho_1} + \langle(-1,-1),(0,1)\rangle D_{\rho_2} = -D_{\rho_1} - D_{\rho_2} = K_{\mathbb{C}^2/\mathbb{Z}_2}.$$

Thus, $K_{\mathbb{C}^2/\mathbb{Z}_2}$ is a principal divisor and therefore $\mathbb{C}^2/\mathbb{Z}_2$ is a Gorenstein variety. △

CHAPTER 2

Gorenstein abelian Quotient singularities

Let G be a finite abelian subgroup of $\mathrm{SL}(n, \mathbb{C})$. This chapter discusses the properties of the geometric quotient $\mathbb{C}^n/G$ and the situations in which it admits a crepant resolution. In addition, Hilbert resolutions are also discussed, and under which conditions such type of resolution is crepant.

2.1 Gorenstein abelian quotient singularities $\mathbb{C}^n/G$ as toric varieties

From now on, G denotes abelian finite subgroup of $\mathrm{SL}(n, \mathbb{C})$. Since G consists of automorphisms of $\mathbb{C}^n$, it acts algebraically on that affine space. Because G is finite, it is a reductive group. In this way, the orbit space $\mathbb{C}^n/G$ is an algebraic variety with a coordinate ring $\mathbb{C}[x_1, \dots, x_n]^G$ and the canonical map $\pi : \mathbb{C}^n \to \mathbb{C}^n/G$ is a morphism of affine varieties corresponding to the inclusion $\mathbb{C}[x_1, \dots, x_n]^G \subset \mathbb{C}[x_1, \dots, x_n]$. As noted below, $\mathbb{C}^n/G$ is a singular Gorenstein affine variety. In this section, we discuss how to realize that kind of object as a toric variety, which enables us to discuss the resolution of singularities for $\mathbb{C}^n/G$ by using tools from toric geometry.

Proposition 2.1. Let G be as above. Let $S := \{x \in \mathbb{C}^n | g.x \neq x \text{ for some } g \in G\}$ be the set of points whose isotropy group is nontrivial. Then the quotient $\pi : \mathbb{C}^n \to \mathbb{C}^n/G$ satisfies the following properties

(a) $\pi(S)$ is the set of singular points of $\mathbb{C}^n/G$;

(b) $\mathbb{C}^n/G$ is a normal variety whose canonical sheaf is a line bundle (i.e $\mathbb{C}^n/G$ is a Gorenstein variety).

Proof. See [52]. ■

Part (a) of the previous proposition says that at least the image of the origin in $\mathbb{C}^n$ is a singular point of the quotient. As shown in the following, part (b) can be recovered by the toric description of $\mathbb{C}^n/G$ with a stronger property: its canonical sheaf is actually trivial.

Since $G \subset \mathrm{SL}(n,\mathbb{C})$ is abelian and finite, there is a matrix $h \in GL(n,\mathbb{C})$ such that the subgroup hGh^{-1} of $\mathrm{SL}(n,\mathbb{C})$ is made of diagonal matrices. Notice $\mathbb{C}^n/G \simeq \mathbb{C}^n/hGh^{-1}$ and for this reason, throughout the present work, G is supposed to be also a group of diagonal matrices. Let be r the order of G. If $g \in G$, then $g = \mathrm{diag}(\lambda_1,\ldots,\lambda_n)$ and $g^r = Id_{n\times n}$ ($Id_{n\times n}$ means the $n\times n$ identity matrix) so that λ_i is a r-th root of 1. Let ϵ_r be a fixed primitive r-th root of 1. There are $a_1,\ldots,a_n \in \mathbb{Z}_{\geq 0}$ such that $\lambda_i = \epsilon_r^{a_i}$. Since $\det(g) = 1$, it follows that $a_1+\cdots+a_n \equiv 0 \mod r$, and hence $\frac{1}{r}(a_1+\cdots+a_n) = m$, for some integer $m \in \{0,1,\ldots,n-1\}$.

Definition 2.1. *Let $g = \mathrm{diag}(\epsilon_r^{a_1},\ldots,\epsilon_r^{a_n}) \in G$ as above.*

(a) $\hat{g} := \frac{1}{r}(a_1,\ldots,a_n) \in \mathbb{Q}^n$ is called the ***fractional expression*** *of g. Define $\hat{G} := \{\hat{g} | g \in G\}$.*

(b) The ***age*** *of g is given by the number $\mathrm{age}(g) := \frac{1}{r}(a_1+\cdots+a_n)$. If $\mathrm{age}(g) = 0$ then $g = Id_{n\times n}$. If $\mathrm{age}(g) = 1$, then g is called a* ***junior element*** *of G. If $\mathrm{age}(g) > 1$ then g is called a* ***senior element*** *of G.*

Since G is finite, it can be considered an algebraic or topological group. The groups of characters of G in both cases are the same as a group, which is $\mathrm{Hom}_{\mathbb{Z}}(G,\mathbb{C}^*)$. For $i \in \{1,\ldots,n\}$, let $\eta_i : G \longrightarrow \mu_r$ be the projections, i.e,

$$\eta_i(\mathrm{diag}(\epsilon_r^{a_1},\ldots,\epsilon_r^{a_i},\ldots\epsilon_r^{a_n})) = \epsilon_r^{a_i}.$$

The characters $\eta_i' s$ provide the monomorphism $\eta : G \longrightarrow \mu_r^n$, defined by $\eta(g) = (\eta_1(g),\ldots,\eta_n(g))$. Roughly speaking, G is a closed subgroup of $(\mathbb{C}^*)^n$ by the the inclusions

$$G \xrightarrow{\eta} \mu_r^n \subset (\mathbb{C}^*)^n;$$

Applying the functor $X(\_)$, which takes an algebraic group to the group of its characters, one gets the surjective group homomorphism

$$X((\mathbb{C}^*)^n) \longrightarrow X(G)$$

that is obtained by the properties of closed subgroups of algebraic groups (see [49] pg.342). One can identify $\mathbb{Z}^n$ with $X((\mathbb{C}^*)^n)$ by the isomorphism that takes a n-tuple of integers $(b_1, \ldots, b_n)$ to the Laurent monomial $x_1^{b_1} \ldots x_n^{b_n}$, as usual in toric geometry. In this way, from the arrow above we get the homomorphism $\psi : \mathbb{Z}^n \twoheadrightarrow X(G)$ which takes a n-tuple $(b_1, \ldots, b_n)$ to the character $b_1\eta_1 + \cdots + b_n\eta_n$. Letting M be the kernel of this morphism we have the exact sequence

$$0 \longrightarrow M \stackrel{i}{\longrightarrow} \mathbb{Z}^n \stackrel{\psi}{\longrightarrow} X(G) \longrightarrow 0. \tag{2.1}$$

Applying the functor $\mathrm{Hom}(-, \mathbb{Z})$ to the above sequence and denoting $N := \mathrm{Hom}(M, \mathbb{Z})$, we see that N is a lattice that contains $\mathbb{Z}^n$. Note that M is a sublattice of finite index in $\mathbb{Z}^n$, and it is also the set of invariant monomials for the action of G on $\mathbb{C}^n$.

Proposition 2.2. With the notation above, one has

$$N = \mathbb{Z}^n + \sum_{g \in G} \mathbb{Z}\hat{g} \text{ and } N/\mathbb{Z}^n \cong G.$$

Proof. By definition, $(b_1, \ldots, b_n) \in M$ if and only if $b_1\eta_1 + \cdots + b_n\eta_n = 1$, and this is true if and only if for every element $g = \mathrm{diag}((\epsilon_r^{a_1}, \ldots, \epsilon_r^{a_n}) \in G$,

$$\sum_{i=1}^{n} b_i a_i \equiv 0 \mod r.$$

Thus

$$M = \{m \in \mathbb{Z}^n | \langle m, \hat{g} \rangle \in \mathbb{Z} \text{ for every } \hat{g} \in \hat{G}\}. \tag{2.2}$$

Let N' denote $\mathbb{Z}^n + \sum_{g \in G} \mathbb{Z}\hat{g}$ and let M' be the dual of N'. By the above description of M, we have $\mathbb{Z}^n \subset N' \subset N$ and $M \subset M' \subset \mathbb{Z}^n$. Note that every element of M' is in the kernel M of ψ. Therefore, $M = M'$ and $N = N'$. In particular, $N/\mathbb{Z}^n \cong G$ where the isomorphism assigns an element $g = \mathrm{diag}(\epsilon^{a_1}, \ldots, \epsilon^{a_n}) \in G$ to the class of its fractional expression, that is, to the class of $\hat{g} = \frac{1}{r}(a_1, \ldots, a_n)$. ■

Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$ and $\sigma := \mathrm{Cone}(e_1, \ldots, e_n)$. It is straightforward to check that $\mathbb{C}^n$ is the toric variety $U_{\sigma,\mathbb{Z}^n} := \text{Specm}(\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n])$. The fact that $N_\mathbb{R} = \mathbb{R}^n$ implies the existence of a toric variety $U_{\sigma,N}$ with torus T_N. In the next proposition, we show that $\mathbb{C}^n/G$ is, in fact, $U_{\sigma,N}$.

Proposition 2.3. With the above notation, we have $\mathbb{C}^n/G = U_{\sigma,N} = \text{Specm}(\mathbb{C}[\sigma^\vee \cap M])$.

Proof. Using the theory presented in Section 1.6, we will express $U_{\sigma,N}$ in homogeneous coordinates. Let (Σ, N) be the fan that consists of σ and its faces. Thus, $X_{\Sigma,N} = U_{\sigma,N}$ is a simplicial toric variety with no torus factors, and in this case we can apply Proposition 1.15. Since Σ has n rays, $\rho_1 = \text{Cone}(e_1), \rho_2 = \text{Cone}(e_2), \ldots, \rho_n = \text{Cone}(e_n)$, one can denote the total coordinate ring of $X_{\Sigma,N}$ by the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$, where x_i is the variable corresponding to the ray ρ_i. The unique maximal cone of Σ with respect the inclusion is σ, hence its irrelevant ideal is given by $B(\Sigma) = \langle x_{\hat{\sigma}} \rangle$. However, since $\Sigma(1)$ consists only of rays of σ, $x_{\hat{\sigma}} = 1$. In this way, $Z(\Sigma) = \emptyset$ and then $X_\Sigma = \mathbb{C}^n/G'$, where $G' = \text{Hom}_\mathbb{Z}(\text{Cl}(X_\Sigma), \mathbb{C}^*) \subset (\mathbb{C}^*)^n$. By part (a) of Proposition 1.15, G' consists of the points $(x_1, \ldots, x_n)$ of $(\mathbb{C}^*)^n$ such that, for every $m = (b_1, \ldots b_n) \in M$,

$$1 = x_1^{\langle e_1, m \rangle} \ldots x_1^{\langle e_1, m \rangle} = x_1^{b_1} \ldots x_n^{b_n}.$$

all of the elements of G satisfy this equation, hence $G \subset G'$. Since $X_{\Sigma,N}$ has not torus factors, the sequence

$$0 \longrightarrow M \longrightarrow \text{Div}_{T_N}(X_{\Sigma,N}) \longrightarrow \text{Cl}(X_{\Sigma,N}) \longrightarrow 0$$

is exact. Note that $\text{Div}_{T_N}(X_{\Sigma,N}) = \bigoplus_{i=1}^n \mathbb{Z}D_{\rho_i} \cong \mathbb{Z}^n$, and the induced morphism $M \to \mathbb{Z}^n$ is the inclusion of M in $\mathbb{Z}^n$. Thus, by (2.1), $X(G) \cong \text{Cl}(X_{\Sigma,N})$. Since a finite group has the same order of its group of characters, we have

$$|G| = |X(G)| = |\,\text{Cl}(X_{\Sigma,N})| = |G'|.$$

Hence $G = G'$ and therefore $U_{\sigma,N} = \mathbb{C}^n/G$. ■

From the above proposition, $\mathbb{C}[x_1, \ldots, x_n]^G = \mathbb{C}[\sigma^\vee \cap M]$. In particular, the canonical morphism $\pi : \mathbb{C}^n \to \mathbb{C}^n/G = U_{\sigma,N}$ is the toric morphism coming from the inclusion $\mathbb{Z}^n \subset N$. Moreover $\pi((\mathbb{C}^*)^n) = (\mathbb{C}^*)^n/G = T_N$, and if τ is a face of σ then $\pi(O_{\mathbb{Z}^n}(\tau)) = O_{\mathbb{Z}^n}(\tau)/G = O_N(\tau)$ is the orbit in $U_{\sigma,N}$ corresponding to τ. Another way to see the latter reasoning is to check that the action of G on $\mathbb{C}^n$ commutes with the action of the torus $(\mathbb{C}^*)^n$.

Remark 2.1. *Note that the pair* $(\sigma, \mathbb{Z}^n)$ *is a smooth cone, but the pair* (σ, N) *is no longer smooth, since it corresponds to the quotient variety* $\mathbb{C}^n/G$*. However, the ray generators* $e_1, \ldots, e_n$ *of* σ *remain linearly independent in* $N_\mathbb{R} = \mathbb{R}^n$*, and hence* (σ, N) *is simplicial. Thus, by Proposition 1.3,* $\mathbb{C}^n/G$ *is a* $\mathbb{Q}$*-factorial variety.*

Remark 2.2. *From Proposition 1.18, one has that a canonical divisor of* $\mathbb{C}^n/G = U_{\sigma,N}$ *is given as* $K_{U_{\sigma,N}} = -(D_{\rho_{e_1}} + \cdots + D_{\rho_{e_n}})$*, where* $\rho_{e_i} = \mathrm{Cone}(e_i)$*. Since the monomial* $x_1 \ldots x_n$ *is* G*-invariant,* $m = (1, \ldots, 1) \in \sigma^\vee \cap M$*. The support function of* $K_{U_{\sigma,N}}$ *is given by* $\varphi_{K_{U_{\sigma,N}}}(u) = \langle m, u\rangle$ *for every* $u \in \sigma$*. In particular,* $K_{U_{\sigma,N}}$ *is a principal divisor, and this also shows that* $\mathbb{C}^n/G$ *is, in fact, a Gorenstein variety.*

The set

$$\square := \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n | 0 \leq \alpha_i < 1\}$$

is a fundamental domain for the action of $\mathbb{Z}^n$ on $\mathbb{R}^n$, thus every element of G has a unique representative in $\square$, i.e, $N \cap \square = \{\hat{g} | g \in G\} := \hat{G}$. In particular, the junior classes of G have their unique representations contained in the slice $\{(\alpha_1, \ldots, \alpha_n) \in \square | \alpha_1 + \cdots + \alpha_n = 1\}$.

Let $H_{m,1}$ be affine hyperplane defined by $\langle m, x\rangle = 1$. Note that

$$H_{m,1} \cap \sigma = \mathrm{Conv}(e_1, \ldots, e_n) := \triangle$$

and the junior elements of G have their fractional expressions contained in $\triangle$. Moreover,

$$N \cap \triangle = \{\hat{g} \in \hat{G} | \operatorname{age}(g) = 1\} \cup \{e_1, \ldots, e_n\} := \nu_G.$$

The set ν_G is called the **junior simplex** of G and the picture of the points of ν_G in $\triangle$ is called **graph** of G.

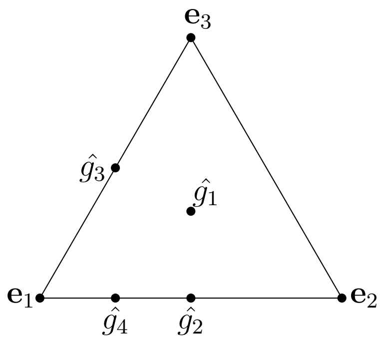
Before giving an example that illustrates the discussion above, it is important to define a very interesting type of Gorenstein quotient singularity, which is the type that admits the canonical Fujiki-Oka resolutions (see [43]) and singles out a class of quotient singularities that are easier to work with.

Definition 2.2. *An element* $\frac{1}{r}(a_1, \ldots, a_n) \in \hat{G}$ *is called* ***semi-unimodular*** *if at least one of the* a_i'*s is* 1*. Suppose* G *is a cyclic group generated by an element* g *corresponding to a semi-unimodular element of* $\hat{G}$*. Up to a change of coordinates, that element can be written as* $\hat{g} = \frac{1}{r}(1, a_1, \ldots, a_{n-1})$*. In this case,* G *is denoted by* $\mathbb{Z}_{r,(1,a_1,\ldots,a_{n-1})}$ *and* $\mathbb{C}^n/G$ *is said to be a* ***cyclic quotient singularity*** *of* $\frac{1}{r}(1, a_1, \ldots, a_{n-1})-$*type.*

Example 2.1. Denote by $\mathbb{Z}_{6,(1,2,3)}$ the cyclic subgroup of $\mathrm{SL}(3,\mathbb{C})$ generated by the diagonal matrix

$$g_1 = \begin{bmatrix} \epsilon_6 & 0 & 0 \\ 0 & \epsilon_6^2 & 0 \\ 0 & 0 & \epsilon_6^3 \end{bmatrix}.$$

In this case, $\mathbb{C}^3/\mathbb{Z}_{6,(1,2,3)}$ is the toric variety $U_{\sigma,N}$ where $N = \mathbb{Z}^3 + \mathbb{Z}\frac{1}{6}(1,2,3)$ and $\sigma = \mathrm{Cone}(e_1, e_2, e_3)$. The junior elements of this group are $g_i := g_1^i$ for $i \in \{1,2,3,4\}$ and its unique senior element is $g_5 := g_1^5$. Their representations $\hat{g}_1 = \frac{1}{6}(1,2,3), \hat{g}_2 = \frac{1}{6}(2,4,0), \hat{g}_3 = \frac{1}{6}(3,0,3), \hat{g}_4 = \frac{1}{6}(4,2,0)$ in $\triangle$ provide the following picture



which is the so called graph of $\mathbb{Z}_{6,(1,2,3)}$. The singular faces of σ with respect to N are $\tau_1 = \mathrm{Cone}(e_1, e_2), \tau_2 = \mathrm{Cone}(e_1, e_3)$ and σ itself. By Proposition 1.10,

$$(\mathbb{C}^3/\mathbb{Z}_{6,(1,2,3)})_{\mathrm{sing}} = V_N(\tau_1) \cup V_N(\tau_2) \cup V_N(\sigma) = \pi(C_1) \cup \pi(C_2) \cup \pi(\{0\})$$

where $C_1 = \{(x,y,z) \in \mathbb{C}^3 | x = z = 0\}$, $C_2 = \{(x,y,z) \in \mathbb{C}^3 | x = y = 0\}$, and $\pi : \mathbb{C}^3 \to \mathbb{C}^3/\mathbb{Z}_{6,(1,2,3)}$ is the canonical morphism. $\triangle$

Note that in the previous example, the singular faces of σ are those that contain at least a fractional expression of a junior element of $\mathbb{Z}_{6,(1,2,3)}$ in their relative interior. In the next section, it is shown that this is not a coincidence.

2.2 Crepant resolutions of $\mathbb{C}^n/G$

Definition 2.3. *Given an irreducible variety Y, a **resolution of singularities** of X is a morphism $f : X \to Y$ such that:*

(a) X is smooth and irreducible.

(b) f *is proper.*

(c) f *induces an isomorphism of varieties* $\mathrm{Exc(f)} \cong Y \setminus Y_{\mathrm{sing}}$*, where* $\mathrm{Exc}(f) = f^{-1}(Y \setminus Y_{\mathrm{sing}})$ *is said to be the* ***exceptional locus*** *of* f*.*

Note that if Y is normal, then the fibers of f are connected by Zariski's main theorem (see [21], section 4.3.)

Remark 2.3. *If* Y *is normal and locally* $\mathbb{Q}$*-factorial (for example,* $Y = \mathbb{C}^n/G$*, by Remark 2.1), then every irreducible component of* $\mathrm{Exc}(f)$ *has codimension* 1 *in* X*. In fact, let* $x \in \mathrm{Exc}(f)$ *and set* $y = f(x)$*. Identify the quotient fields* $\mathbb{C}(X)$ *and* $\mathbb{C}(Y)$ *via the isomorphism induced by* f*, so that* $\mathcal{O}_{Y,y}$ *is a proper subring of* $\mathcal{O}_{X,x}$*. Let* t *be an element of* $\mathfrak{m}_{X,x}$ *not in* $\mathcal{O}_{Y,y}$*, and write its divisor as the difference of two effective divisors* D' *and* D'' *in* Y *without common components. There exists a positive integer* m *such that* mD' *and* mD'' *are Cartier divisors, hence there are rational functions* u' *and* v' *defined in a neighborhood* U *of* y*, which are invertible in a smaller open set of* Y *not necessarily containing* y*, such that* $\mathrm{div}(t^m) = \mathrm{div}(u') - \mathrm{div}(v')$ *in* U*. This way, we can choose* u *and* $v \in \mathcal{O}_{Y,y}$ *such that* $t^m = \frac{u}{v}$ *in* $\mathbb{C}(Y)$*. Both* u *and* v *are actually in* $\mathfrak{m}_{Y,y}$ *because* t^m *is not in* $\mathcal{O}_{Y,y}$ *(otherwise* t *would be, since* $\mathcal{O}_{Y,y}$ *is integrally closed), and* $u = t^m v$ *because* u *is in* $\mathfrak{m}_{X,x} \cap \mathcal{O}_{Y,y} = \mathfrak{m}_{Y,y}$*. But the equations* $u = v = 0$ *define a subscheme* Z *containing* y*, which has codimension* 2 *in some neighborhood of* y*. It is the intersection of the codimension-*1 *subschemes* $\mathrm{Supp}(mD')$ *and* $\mathrm{Supp}(mD'')$*, whereas* $f^{-1}(Z)$ *is defined by* $t^m v = v = 0$*. Hence by the sole equation* $v = 0$*,* $\pi^{-1}(Z)$ *has codimension* 1 *in* Y*, and is contained in* $\mathrm{Exc}(f)$*. It follows that there is a codimension-*1 *component of* $\mathrm{Exc}(f)$ *through every point of* $\mathrm{Exc(f)}$*.*

Hironaka proved that any irreducible variety over a algebraically closed field of characteristic 0 has a resolution of singularities, obtained by a sequence of blow-ups (see [25]). In the case of $\mathbb{C}^3/G$, we will see that toric geometry provides tools to find toric resolutions of singularities for such a variety, by means of "toric blow-ups". We are interested in resolutions with an extra property, described in the following discussion. From now on, varieties are always assumed to be normal, irreducible, and over $\mathbb{C}$.

Given a Gorenstein variety Y, a birational morphism $f : X \to Y$, where X is smooth, is said to be **crepant** if $K_X = f^*K_Y$, where $f^*(K_Y)$ is the pullback of a canonical divisor of Y. Notice that this makes sense because K_Y is Cartier. In general, the existence of a crepant resolution of singularities is uncommon, so for most varieties, we have $K_X \neq f^*K_Y$ regardless of the resolution we use. However, we can check that the support of the difference of such divisors is contained in the exceptional locus of f. In fact, note that

$$X \setminus \mathrm{Exc}(f) \xrightarrow{\sim} Y \setminus Y_{\mathrm{sing}}$$

and

$$\omega_{X\setminus \mathrm{Exc}(f)} = \omega_X|_{X\setminus \mathrm{Exc}(f)} \text{ and } \omega_{Y\setminus Y_{\mathrm{sing}}} = \omega_Y|_{Y\setminus Y_{\mathrm{sing}}}.$$

Thus, given a canonical divisor K_Y of Y, there exists a canonical divisor K_X of X such that

$$K_X|_{X\setminus \mathrm{Exc}(f)} = (f|_{X\setminus \mathrm{Exc}(f)})^*(K_Y|_{Y\setminus Y_{\mathrm{sing}}}) = f^*(K_Y)|_{X\setminus \mathrm{Exc(f)}}.$$

Therefore, we get the so called ramification formula

$$K_X = f^*(K_Y) + \sum_{i=1}^{s} a_i E_i \tag{2.3}$$

where $a_i \in \mathbb{Z}$ and the E_i are the irreducible divisors lying in the exceptional locus $\mathrm{Exc}(f)$. The divisor $\sum_{i=1}^{s} a_i E_i$ is said to be the **discrepancy** of the difference $K_X - f^*(K_Y)$. This difference of divisors leads to some interesting classes of singularities.

Definition 2.4. *Let Y be Gorenstein variety.*

*(a) Y has **terminal singularities** if there is a resolution of singularities $f : X \to Y$, such that the coefficients a_i in* (2.3) *satisfy $a_i > 0$ for all i.*

*(b) Y has **canonical singularities** if there is resolution of singularities $f : X \to Y$, such that the coefficients a_i in* (2.3) *satisfy $a_i \geq 0$ for all i.*

If Y has terminal (resp. canonical) singularities, then the inequalities $a_i > 0$ (resp. $a_i \geq 0$) hold for all resolutions of singularities $f : X \to Y$ (see [28], p.108).

Definition 2.5. *A **minimal model** of $\mathbb{C}^n/G$ is a $\mathbb{Q}$-factorial normal variety X which has only terminal singularities together with a crepant proper birational morphism $X \longrightarrow \mathbb{C}^n/G$.*

In [54] p.11, it is proven that any minimal model of $\mathbb{C}^n/G$ is toric. In particular, if $\mathbb{C}^n/G$ admits a crepant resolution $X \to \mathbb{C}^n/G$, then X should be toric, because any smooth variety has canonical singularities (see [28], p.102). In this way, following the steps provided in [14] and in the Appendix of [33], we will give the recipe to get (toric) minimal models for $\mathbb{C}^n/G$. Especially in dimensions 2 and 3 any minimal model for such quotient variety is a crepant resolution, as we will see.

To begin with, we give the definition of multiplicity of a cone. Let L be a lattice. Given a simplicial strongly convex rational polyhedral cone $\xi \subset L_{\mathbb{R}}$ with ray generators $u_1, \dots u_d$, let L_ξ be $\operatorname{span}(\xi) \cap L$. The multiplicity of ξ in L is defined by

$$\operatorname{mult}(\xi)_L := [L_\xi : u_1\mathbb{Z} + \cdots + u_d\mathbb{Z}]$$

i.e, the index of $u_1\mathbb{Z} + \cdots + u_d\mathbb{Z}$ in L_ξ. The following result provides some important relations to that number:

Proposition 2.4. With the notation above:

(a) ξ is smooth if and only if $\operatorname{mult}(\xi) = 1$.

(b) $\operatorname{mult}(\xi)$ is the number of points in $P_\xi \cap L$, where

$$P_\xi := \{\sum_{i=1}^{d} \alpha_i u_i | 0 \leq \alpha_i < 1\}.$$

(c) Let $v_1, ..., v_d$ be a basis of L_ξ and write $u_i = \sum_{j=1}^{d} a_{ij} v_j$. Then

$$\operatorname{mult}(\xi) = |\det[a_{ij}]|.$$

Proof. See [11] pg. 519. ■

Now, following the notation of the previous section, since $\sigma = \operatorname{Cone}(e_1, \dots, e_n)$, it follows $N = N_\sigma$ and

$$\operatorname{mult}_N(\sigma) = [N : \mathbb{Z}^n] = |G| = r.$$

Moreover, $P_\sigma = \square$, thus $P_\sigma \cap N = \hat{G}$.

We say that a fan (Σ', L) refines (or subdivides) a fan (Σ, L) if $|\Sigma'| = |\Sigma|$ and every cone of Σ' is contained in a cone of Σ. In this situation, the identity map $\operatorname{id}\colon L \longrightarrow L$ induces a toric morphism $\mathcal{F}(\operatorname{id})\colon X_{\Sigma'} \longrightarrow X_\Sigma$ (following the notation of Section 1.4) which is proper by Proposition 1.12 and birational since it is the identity on T_L. There is a way to construct very nice refinements of a fan, defined as follows.

Definition 2.6. *Let Ξ be a fan in $L_{\mathbb{R}}$. Given a primitive element $\mu \in |\Xi| \cap L$, the* ***star subdivision*** *of Ξ at μ is the fan denoted by $\Xi^*(\mu)$ and defined as the set containing the following cones:*

(a) $\xi \in \Xi$, where $\mu \notin \xi$;

(*b*) $\mathrm{Cone}(\tau, \mu)$, *where* $\mu \notin \tau \in \Xi$ *and* $\{\mu\} \cup \tau \subset \xi \in \Xi$.

The main properties of the star subdivision are the following:

- The 1-dimensional cones of $\Xi^*(\mu)$ are the 1-dimensional cones of Ξ plus the cone generated by μ;
- $\Xi^*(\mu)$ is a refinement of Ξ;
- The correspondent toric morphism $X_{\Xi^*(\mu)} \to X_\Xi$ is projective.
- If Ξ is simplicial then $\Xi^*(\mu)$ is simplicial.

We will apply this method to find minimal models for $\mathbb{C}^n/G$ after some considerations.

Lemma 2.5. Suppose that X_Σ is a Gorenstein toric variety. Let φ be the support function of the Cartier divisor K_{X_Σ}, and let $\phi\colon X_{\Sigma'} \longrightarrow X_\Sigma$ be the toric morphism coming from a refinement Σ' of Σ. Then the ramification formula becomes

$$K_{X_{\Sigma'}} \;=\; \phi^*(K_{X_\Sigma}) \;+ \sum_{\rho \in \Sigma'(1) \setminus \Sigma(1)} \big(\varphi(u_\rho) - 1\big)\, D_\rho.$$

Proof. Note that that K_{X_Σ} and its pullback $\phi^* K_{X_\Sigma}$ have the same support function. Thus

$$\phi^*(K_{X_\Sigma}) \;=\; -\sum_{\rho \in \Sigma'(1)} \varphi(u_\rho)\, D_\rho.$$

Hence, the required formula now follows at once, relying on the fact that $\varphi(u_\rho) = 1$ for each $\rho \in \Sigma(1)$. ■

Proposition 2.6. Let $X_{\Sigma,L}$ be a normal Gorenstein toric variety. Then X_Σ has canonical singularities.

Proof. Observe first that the support function φ associated with K_{X_Σ} is integral along L by virtue of X_Σ being Gorenstein. Next, choose a smooth refinement Σ' of Σ (its existence is guaranteed by [17], p. 48). Then for each $u_\rho \in \Sigma'(1)$, the following properties hold:

1. $\varphi(u_\rho) \in \mathbb{Z}$, since u_ρ lies in $|\Sigma| \cap N$.
2. $\varphi(u_\rho) > 0$, because u_ρ lies in some cone $\sigma \in \Sigma$, and φ takes the value 1 on each minimal generator of σ. It follows that φ remains strictly positive on σ.

Hence $\varphi(u_\rho) \geq 1$. By the ramification formula for toric morphisms, we conclude that X_Σ must have canonical singularities. ■

Bearing this result in mind, it follows that $\mathbb{C}^n/G$ has canonical singularities since it is a Gorenstein toric variety. The next proposition give us the ideia of which points we should choose to refine the fan of $\mathbb{C}^n/G$ in order to get a minimal model for it.

Proposition 2.7. Let L be a lattice and let $\xi \subset L_\mathbb{R}$ be a simplicial strongly convex rational polyhedral cone with ray generators $u_1, \ldots u_d$. Define

$$\Psi_\xi = \text{Conv}(0, u_i | i = 1, \ldots d).$$

If U_ξ is $\mathbb{Q}$-Gorenstein, then

(i) Ψ_ξ has a unique facet not containing the origin.

(ii) $U_{\xi,L}$ has terminal singularities if and only if the only lattice points of Ψ_ξ are given by its vertices.

(iii) $U_{\xi,L}$ has canonical singularities if and only if the only nonzero lattice points of Ψ_ξ lie in the facet not containing the origin.

Proof. See [11], p.551. ■

In particular, if $U_{\xi,L}$ is Goreinstein and n-dimensional, and its fan has exactly rays $u_1, \ldots, u_n$, with $rank(L) = n$ then the face of Ψ_ξ that does not contain the origin is

$$T_\xi := \text{Conv}(u_1, \ldots, u_n), \tag{2.4}$$

which is a triangle that lies in the hyperplane defined by $m \in M$ such that $K_{U_\xi} = Div(\chi^m)$. Applying this proposition to our case, we get $\Psi_\sigma = \text{Conv}(0, e_1, \ldots, e_n)$ and the facet not containing the origin is simply $\triangle = \text{Conv}(e_1, \ldots, e_n)$. Since $\mathbb{C}^n/G = U_{\sigma,N}$ is Gorenstein and has canonical singularities, it follows the lattice points of Ψ_σ are given by the set

$$\nu_G = \{\hat{g} \in \hat{G} |\, \text{age}(g) = 1\} \cup \{e_1, \ldots, e_n\}.$$

Now we can understand how to find the toric minimal models of $\mathbb{C}^n/G = U_{\sigma,N}$. For every junior class g of G, $\varphi_{K_{U_{\sigma,N}}}(\hat{g}) = 1$, and along with Propositions 2.4 and 2.7, it follows that any simplicial refinement of σ such that new rays added are those ones generated by each element of the set $\nu_G \cap \hat{G}$ provides a toric crepant morphism $\phi : X_\Sigma \to \mathbb{C}^n/G$, because of the toric ramification formula

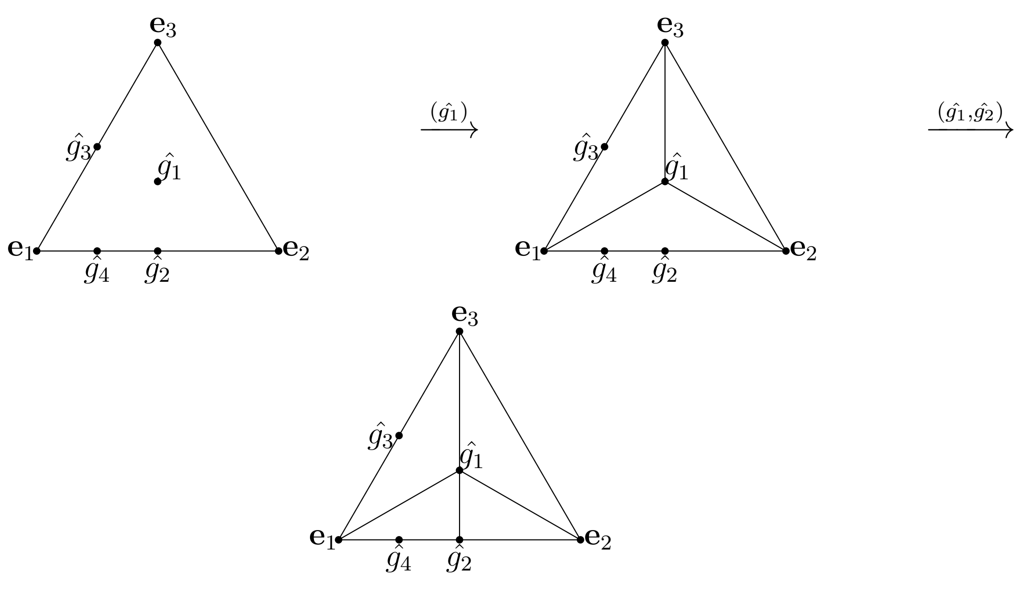
$$K_{X_\Sigma} \;=\; \phi^*(K_{U_{\sigma,N}}) \;+\; \sum_{\hat{g}\in\nu_G\cap\hat{G}} \big(\varphi_{K_{U_{\sigma,N}}}(\hat{g}) - 1\big)\, D_{Cone(\hat{g})}.$$

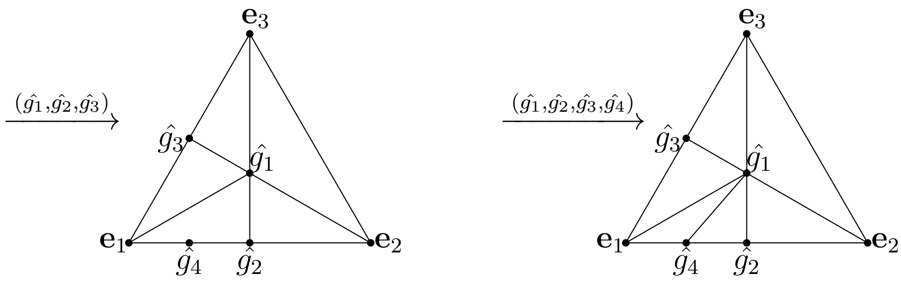
In this situation, X_Σ has terminal singularities.

Remark 2.4. *This type of morphism is a crepant resolution of singularities when $n = 2, 3$, because every Gorenstein toric surface and every Gorenstein simplicial 3-dimensional toric variety with terminal singularities are smooth (see [11], p.555) .*

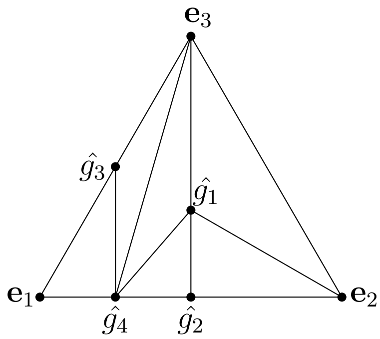
The easiest way to obtain this type of morphism along with the property of being projective (i.e. a minimal model) is by doing a sequence of star subdivisions in σ at each element of $\nu_G \cap \hat{G}$. Performing a star subdivision of the fan that consists of σ and its faces provides us with an n-dimensional picture. However, when such star subdivision is done at a point $\hat{g} \in \nu_G$, the picture can be simplified into the graph of G. It is reduced to a triangulation of $\triangle$ with new vertex $\hat{g}$, which we will denote by $\triangle_{\hat{g}}$ and call of (roughly) star subdivision of $\triangle$ at $\hat{g}$. The fan associated to $\triangle_{\hat{g}}$ is denoted by $\Sigma^*(\hat{g})$. Another star subdivision of $\triangle_{\hat{g}}$ at a point $\hat{h} \in \nu\backslash\{\hat{g}\}$ will be denoted by $\triangle_{(\hat{g},\hat{h})}$ (and its associated fan by $\Sigma^*(\hat{g},\hat{h})$) and so on. Note that it can happen $\triangle_{(\hat{g},\hat{h})} \neq \triangle_{(\hat{h},\hat{g})}$, as it is shown in the next example.

Example 2.2. Following the notation of Example 2.1, we will find two projective crepant resolutions of $\mathbb{C}^3/\mathbb{Z}_{6,(1,2,3)}$ by performing different sequences of star subdivisions of $\triangle$. Firstly, the sequence of star subdivisions that we will take is given simply by $(\hat{g}_1, \hat{g}_2, \hat{g}_3, \hat{g}_4)$, and the sequence of pictures correspondent is displayed in the following way:





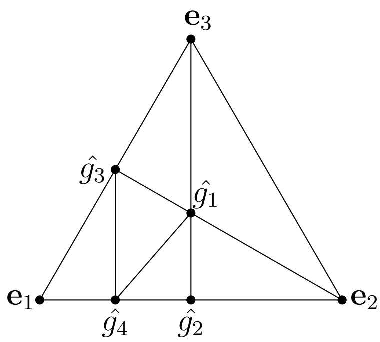
The last graph $\triangle_{(\hat{g_1},\hat{g_2},\hat{g_3},\hat{g_4})}$ has a corresponding fan $\Sigma^*(\hat{g}_1, \hat{g}_2, \hat{g}_3, \hat{g}_4)$. This is a refinement of σ and the corresponding toric morphism $\phi_{(\hat{g_1},\hat{g_2},\hat{g_3},\hat{g_4})} : X_{\Sigma^*(\hat{g_1},\hat{g_2},\hat{g_3},\hat{g_4})} \longrightarrow \mathbb{C}^3/\mathbb{Z}_{6,(1,2,3)}$ is a minimal model and, therefore, by Remark 2.4, a crepant resolution for the quotient singularity. On the other hand, after performing the sequence $(\hat{g}_4, \hat{g}_3, \hat{g}_1, \hat{g}_2)$ of star subdivisions, we get $\triangle_{(\hat{g_4},\hat{g_3},\hat{g_1},\hat{g_2})}$, whose graph is



and the corresponding fan is $\Sigma^*(\hat{g}_4, \hat{g}_3, \hat{g}_1, \hat{g}_1)$. This fan provides another toric minimal model $X_{\Sigma^*(\hat{g_4},\hat{g_3},\hat{g_1},\hat{g_2})} \longrightarrow \mathbb{C}^3/\mathbb{Z}_{6,(1,2,3)}$ for the singularity.

$\triangle$

Now, two things should be noted. First, not all the toric minimal models of the singularity $\mathbb{C}^n/G$ have a fan which is a sequence of star subdivisions of the points of $\hat{G} \cap \nu_G$. For instance, the fan related to the graph



provides another minimal model for $\mathbb{C}^3/\mathbb{Z}_{6,(1,2,3)}$. However, both fan and graph can not be obtained by a sequence of star subdivisions of the set $\{\hat{g}_1, \hat{g}_2, \hat{g}_3, \hat{g}_4\}$. Second, suppose

Ξ is the fan of a crepant morphism of $\mathbb{C}^n/G$. In that case, the number of elements of $\Xi(m)$ equals the number of m-dimensional faces corresponding to the triangulation of $\triangle$, for $m > 0$.

Remark 2.5. *If $\mathbb{C}^n/G = U_{\sigma,N}$ admits a toric crepant resolution $\phi : X_\Sigma \to \mathbb{C}^n/G$ coming from a simplicial refinement of σ such that new rays added are those generated by each element of the set $\nu_G \cap \hat{G}$, then all the irreducible components of $Exc(\phi)$ have codimension* 1 *in X_Σ, by Remark 2.3. Moreover, one can check that*

$$\mathrm{Exc}(\phi) = \bigcup_{\hat{g} \in \nu_G \cap \hat{G}} E_g,$$

where $E_g = V(Cone(\hat{g})) \subset X_\Sigma$ (following the notation of Section 1.3*), that is, there is a bijection between the irreducible components of* $\mathrm{Ext}(\phi)$*, which are called of* ***exceptional divisors*** *of ϕ, and the junior elements of G (see [31]). In particular, each irreducible component of* $\mathrm{Ext}(\phi)$ *is a smooth toric variety by Proposition 1.9. Moreover, the complete (=compact in the analytic topology) exceptional divisors of ϕ are those that are located in the variety $\phi^{-1}(V_N(\sigma))$, where $V(\sigma) = \overline{O_N(\sigma)} = O_N(\sigma)$. Therefore, E_g is a complete exceptional divisor of ϕ if and only if $\hat{g}$ lies in the interior of σ (see [11], p.* 521*).*

Example 2.3. The projective crepant resolution

$$\phi_{(\hat{g_1},\hat{g_2},\hat{g_3},\hat{g_4})} : X_{\Sigma^*(\hat{g_1},\hat{g_2},\hat{g_3},\hat{g_4})} \longrightarrow \mathbb{C}^3/\mathbb{Z}_{6,(1,2,3)},$$

described in Example 2.2, has a unique complete exceptional divisor E_{g_1}, since $\hat{g}_1$ is the unique element that lies in the interior of σ. Using Proposition 1.9, we will compute E_{g_1}. Since $\{\hat{g}_1, e_2, e_3\}$ is a basis of N, we can identify N with $\mathbb{Z}^3$ by changing the coordinates in the following way

$$\hat{g_1} \mapsto (1,0,0), \quad e_2 \mapsto (0,1,0), \quad e_3 \mapsto (0,0,1).$$

In these new coordinates, we have

$$e_1 = (6,-2,-3), \quad \hat{g_2} = (2,0,-1), \quad \hat{g_3} = (3,-1,-1), \quad \hat{g_4} = (4,-1,-2).$$

Let ρ_{g_1} be the cone generated by $\hat{g}_1$. In this case, we have

$$N(\rho_{g_1}) = \mathbb{Z}^2,$$

$$\operatorname{Star}(\rho_{g_1}) = \left\{ \overline{\xi} \subseteq N(\rho_{g_1})_{\mathbb{R}} \mid \rho_{g_1} \preceq \xi \in \Sigma^*\big(\hat{g}_1, \hat{g}_2, \hat{g}_3, \hat{g}_4\big) \right\},$$

$$E_g \cong X_{\operatorname{Star}(\rho_{g_1}), N(\rho_{g_1})}.$$

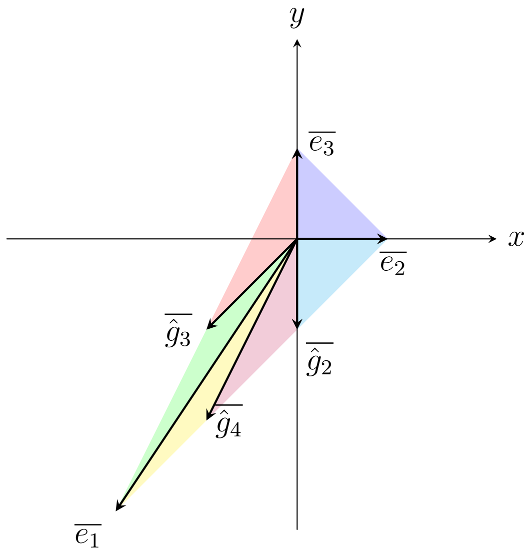
Since all the maximal cones of $\Sigma^*\big(\hat{g}_1, \hat{g}_2, \hat{g}_3, \hat{g}_4\big)$ contain the ray ρ_{g_1} and $\overline{e_1} = (-2,-3)$, $\overline{\hat{g}_2} = (0,-1)$, $\overline{\hat{g}_3} = (-1,-1)$, $\overline{\hat{g}_4} = (-1,-2)$, $\overline{e_2} = (1,0)$ and $\overline{e_3} = (0,1)$, it follows that the fan $\operatorname{Star}(\rho_{g_1})$ also has 6 maximal cones, which are

$$\xi_1 := \operatorname{Cone}\big(\overline{e_2}, \overline{e_3}\big), \qquad \xi_2 := \operatorname{Cone}\big(\overline{e_3}, \overline{\hat{g}_3}\big),$$

$$\xi_3 := \operatorname{Cone}\big(\overline{e_1}, \overline{\hat{g}_3}\big), \qquad \xi_4 := \operatorname{Cone}\big(\overline{e_1}, \overline{\hat{g}_4}\big),$$

$$\xi_5 := \operatorname{Cone}\big(\overline{\hat{g}_2}, \overline{\hat{g}_4}\big), \qquad \xi_6 := \operatorname{Cone}\big(\overline{e_2}, \overline{\hat{g}_2}\big).$$

Thus, the representation of $\operatorname{Star}(\rho_{g_1})$ in $\mathbb{R}^2$ is given by the following picture



This is a sequence of 3 star subdivisions in the fan of $\mathbb{P}^2$ defined in Example 1.6. It can also be seen as a sequence of 3 blowups starting in $\mathbb{P}^2$, with centers at 3 points. (see [11], p.130). In chapter 3, we show in our main result that $X_{\Sigma^*(\hat{g}_1,\hat{g}_2,\hat{g}_3,\hat{g}_4)}$ is actually a geometric line bundle over E_g, and that explains why their fan has the same number of maximal cones.

△

The method used to find crepant resolutions of $\mathbb{C}^3/G$ (if they exist) also helps us to find crepant resolutions for singularities that are not affine, as shown in the next example.

Example 2.4. From [5], Remark 4.4, a weighted projective space $\mathbb{P}(w_0, \ldots, w_s)$ is Gorenstein if and only if w_i divides $\sum_{j=0}^{s} w_j$ for any i. It follows that the $n-1$-dimensional weighted projective space $\mathbb{P}(1, 1, \ldots, n-1)$ is Gorenstein. Our goal is to find a crepant resoltuion for this variety. Let $e_1, \ldots, e_{n-1}$ be the canonical basis of $\mathbb{R}^{n-1} = \mathbb{Z}^{n-1} \otimes_{\mathbb{Z}} \mathbb{R}$ and

$$u := (\sum_{i=1}^{n-2}(-1)e_i) + (-1)(n-1)e_{n-1}.$$

Consider the fan Ξ, with ray generators in $\mathbb{Z}^{n-1}$, that consists of all the strongly convex rational polyhedral cones generated by the proper subsets of $\{e_1, \ldots, e_{n-1}, u\}$. It is known that $\mathbb{P}(1, 1, \ldots, n-1)$ is the toric variety $X_{\Xi,\mathbb{Z}^{n-1}}$. Notice that

$$\xi_0 := \text{Cone}(e_1, \ldots, e_{n-1}),$$

$$\xi_i := \text{Cone}\big(e_1, \ldots, e_i,\, u,\, e_{i+1}, \ldots, e_{n-1}\big), \quad 1 \le i \le n-1$$

are the maximal cones of Ξ. Moreover, for $1 \le i \le n-3$,

$$\begin{aligned} \big|\det\big([\,e_1, \ldots, e_i,\, u,\, e_{i+1}, \ldots, e_{n-1}\,]\big)\big| &= \big|\det\big([\,(-1,-1),\,(0,1)\,]\big)\big| \\ &= 1. \end{aligned}$$

where the equality is obtained by the Laplace expansion. For $i = n-2$, it can be reduced to

$$|\det([(-1, n-1), (0, 1)])| = 1.$$

Finally, for $i = n-1$, it can be reduced to

$$|\det([(1, 0), (-1, n-1)])| = n-1.$$

This way, the unique singular maximal cone of Ξ is ξ_{n-1}, by Proposition 2.4 . Since X_Ξ is Gorenstein, $U_{\xi_{n-1}} \subset X_\Xi$ is also a Gorenstein toric variety. From Proposition 2.7, the lattice points of $\Psi_{\xi_{n-1}}$ lies in $\text{Conv}(e_1, \ldots, e_{n-2}, u)$. If μ is such a point that is not a vertex, then

$$\begin{aligned} \mu &= \Big(\sum_{i=1}^{n-2} \alpha_i\, e_i\Big) \;+\; \alpha\, u \\ &= \big(\alpha_1 - \alpha,\ \alpha_2 - \alpha,\ \ldots,\ \alpha_{n-1} - \alpha,\ -(n-1)\alpha\big), \end{aligned}$$

with $\alpha_i, \alpha \in \mathbb{R}_{\geq 0}$ and $(\sum_{i=1}^{n-2} \alpha_i) + \alpha = 1$. Since $\mu \in \mathbb{Z}^{n-1}$, it implies that $\alpha_i = \alpha = \frac{1}{n-1}$ for any i. Therefore

$$\mu = (0, \ldots, 0, -1)$$

is the unique point of $\text{Conv}(e_1, \ldots, e_{n-2}, u)$ that it is not a vertex. Consider the star-subdivision $\Xi^*(\mu)$ of Ξ with the induced projective morphism $f : X_{\Xi^*(\mu)} \to X_\Xi$. From the toric ramification formula, one gets

$$K_{\Xi^*(\mu)} = f^*(K_{X_\Xi}) + (\varphi_{K_{X_\Xi}}(\mu) - 1))D_\rho$$

where $\rho = \text{Cone}(\mu)$ and $\varphi_{K_{X_\Xi}}$ is the support function of K_{X_Ξ}. Since $\mu \in \xi_{n-1}$, it follows that $\varphi_{K_{X_\Xi}}(\mu) = m_{\xi_{n-1}}(\mu)$, with $K_{X_\Xi}|_{U_{\xi_{n-1}}} = \text{Div}(\chi^{m_{\xi_{n-1}}})$. By construction, $m_{\xi_{n-1}}(\mu) = 1$. Thus, f is crepant. Notice that the maximal cones of $\Xi^*(\mu)$ are $\xi_0, \xi_1, \ldots \xi_{n-2}$, and $\xi_{n-1,j} := \text{Cone}(e_1, \ldots, e_{j-1}, \mu, e_{j+1}, \ldots, e_{n-2}, u)$ for $1 \leq j \leq n-2$, and $\xi_{n-1,n-1} := \text{Cone}(e_1, \ldots, e_{n-2}, \mu)$. All of those maximal cones of $\Xi^*(\mu)$ are smooth, and therefore, f is a projective crepant resolution of $\mathbb{P}(1, 1, \ldots, n-1)$. △

In Example 2.2 , we have seen that all the exhibited resolutions of $\mathbb{C}^3/\mathbb{Z}_{6,(1,2,3)}$ have a corresponding graph with six triangles and this number is simply the order $\mathbb{Z}_{6,(1,2,3)}$. The next results show that this is not a coincidence.

Proposition 2.8. Let X_Ξ be a n-dimensional toric variety and let

$$e(X_\Xi) := \sum_{i=0}^{2n} (-1)^k \text{rank} H_c^i(X_\Xi, \mathbb{Z})$$

be its topological Euler number characteristic. Then $e(X_\Xi) = |\Xi(n)|$.

Proof. See [17], p.59. ■

Proposition 2.9. Let G be a finite subgroup of $\text{SL}(n, \mathbb{C})$. If a crepant resolution $X \to \mathbb{C}^n/G$ exists, then the Euler number of X equals the number of conjugacy classes of G.

Proof. See [3]. ■

It follows, from the previous propositions, that when G is an abelian finite subgroup of $\text{SL}(n, \mathbb{C})$ and if there exists a toric crepant resolution $X_\Sigma \to \mathbb{C}^n/G$ then $|\Sigma(n)| = |G|$.

2.3 Hilbert basis resolutions

There are many examples of quotientes $\mathbb{C}^n/G$ that does not admit a crepant resolution. The next result provides a big extent of such examples.

Proposition 2.10. If $\mathbb{C}^n/G$ admits a (not necessarily projective) crepant resolution, then G is generated by junior elements.

Proof. See [53], p.6. ∎

Thus, for example, the quotient variety $\mathbb{C}^4/\mathbb{Z}_{2,(1,1,1,1)}$ does not admit crepant resolutions because $\mathbb{Z}_{2,(1,1,1,1)}$ contains only senior elements. The reciprocal of the above proposition is not true, as we will see further. But for the cases where G is generated by junior elements, or more generally, it is possible to find another type of resolution that has very important properties and gives another criterion for $\mathbb{C}^n/G$ to have a crepant resolution or not, described below.

Given a rational strongly convex polyhedral cone of maximal dimension ξ in $L_{\mathbb{R}}$ it is proven in [44] p.233 that $\xi \cap L$ has a unique minimal system of generators $\mathbf{Hlb}_L(\xi)$, which is given by

$$\mathbf{Hlb}_L(\xi) = \{n \in \xi \cap L | \text{n is irreducible}\}$$

where $n \in \xi \cap L$ is an irreducible element if $n = n_1 + n_2$, with $n_i \in \xi \cap L$, then $n_i = n$ and $n_j = 0$ for $i \neq j$. In particular, for the our case $\mathbb{C}^n/G = U_{\sigma,N}$, we have that $\nu_G \subset \mathbf{Hlb}_N(\sigma)$. Furthermore, since $N = \mathbb{Z}^n + \sum_{g \in G} \hat{g}\mathbb{Z}$, one has

$$\mathbf{Hlb}_N(\sigma) \subset \hat{G} \cup \{e_1, \ldots, c_n\}. \tag{2.5}$$

In fact, by part (a) of Proposition 1.2, $\sigma \cap N$ is generated by union of the sets $\nu_G \setminus \hat{G} = \{e_1, \ldots, e_n\}$ and $K \cap N$, where

$$K = \Big\{ \sum_{m \in \nu_G \setminus \hat{G}} \delta_m m \;\Big|\; 0 \leq \delta_m < 1 \text{ for all } m \in \nu_G \setminus \hat{G} \Big\}.$$

Note that $K = \mathrm{Conv}(0, e_1, \ldots, e_n)$. Hence, by Proposition 2.7,

$$K \cap N = \{\hat{g} \in \hat{G} | \operatorname{age}(g) = 1\}.$$

Therefore, $\nu_G \setminus \hat{G} \cup (K \cap N) = \nu_G$ and the desired inclusion follows from the fact that $\mathbf{Hlb}_N(\sigma)$ is the minimal system of generators of $\sigma \cap N$.

Definition 2.7. *A subdivision Σ of σ is called a* ***Hilbert basis resolution*** *of σ if Σ satisfies the following conditions:*

- *Σ is smooth.*
- *$\mathbf{Hlb}_N(\sigma)$ is the set of ray generators of Σ.*

In particular, such subdivision provides a toric morphism $X_\Sigma \to U_{\sigma,N}$ which is also called Hilbert basis resolution of $U_{\sigma,N}$.

Example 2.5. Consider the group

$$G := \mathbb{Z}_{7,((1,1,2,3)} \subset \mathrm{SL}(4,\mathbb{C}).$$

In this case,

$$\mathbb{C}^4/G \;=\; U_{\sigma,\,N},$$

where

$$N \;=\; \mathbb{Z}^3 \;+\; \mathbb{Z}\,\frac{1}{7}(1,1,2,3) \quad \text{and} \quad \sigma \;=\; \mathrm{Cone}\big(e_1,\, e_2,\, e_3,\, e_4\big).$$

Following the established notation, one has

$$\hat{G} = \{(0,0,0,0), \frac{1}{7}(1,1,2,3), \frac{1}{7}(2,2,4,6), \frac{1}{7}(3,3,6,2), \frac{1}{7}(4,4,1,5), \frac{1}{7}(5,5,3,1), \frac{1}{7}(6,6,5,4)\}.$$

Defining

$$\begin{aligned}\hat{g}_1 &:= \tfrac{1}{7}(1,1,2,3),\\ \hat{g}_2 &:= \tfrac{1}{7}(3,3,6,2),\\ \hat{g}_3 &:= \tfrac{1}{7}(4,4,1,5),\\ \hat{g}_4 &:= \tfrac{1}{7}(5,5,3,1),\end{aligned}$$

we can check that

$$\mathbf{Hlb}_N(\sigma) = \{e_1, e_1, e_3, e_4, \hat{g}_1, \hat{g}_2, \hat{g}_3, \hat{g}_4\}.$$

The unique junior element G corresponds to $\hat{g}_1$. Performing a star subdivision of σ at $\hat{g}_1$, one gets a fan $\Sigma_1 := \Sigma^*(\hat{g}_1)$, such that

$$\Sigma_1(4) = \{\mathrm{Cone}(\hat{g}_1, e_2, e_3, e_4), \mathrm{Cone}(\hat{g}_1, e_1, e_3, e_4), \mathrm{Cone}(\hat{g}_1, e_1, e_2, e_3), \mathrm{Cone}(\hat{g}_1, e_1, e_2, e_3)\}.$$

Such refinement provides a minimal model $X_{\Sigma_1} \to \mathbb{C}^n/G$, but this is not a resolution since $\mathrm{Cone}(\hat{g}_1, e_1, e_2, e_3)$ is not a smooth cone. Furthermore, it is not possible to get a crepant resolution of $\mathbb{C}^4/G$ by a sequence of star subdivisions because $\nu_G = \{e_1, e_2, e_3, e_4, \hat{g}_1\}$.

Nevertheless, it is possible to get a Hilbert basis resolution by performing the star subdivision of σ at the sequence $(\hat{g}_1, \hat{g}_2, \hat{g}_3, \hat{g}_4)$. In this case, the resulting fan Σ is such that

$$\begin{cases} \text{Cone}(\hat{g}_1, e_2, e_3, e_4), & \text{Cone}(\hat{g}_1, e_1, e_3, e_4), & \text{Cone}(\hat{g}_1, e_1, e_2, \hat{g}_4), & \text{Cone}(\hat{g}_1, e_1, e_4, \hat{g}_4), \\ \text{Cone}(\hat{g}_1, e_2, e_4, \hat{g}_4), & \text{Cone}(e_1, e_2, e_4, \hat{g}_4), & \text{Cone}(\hat{g}_1, e_1, e_3, \hat{g}_3), & \text{Cone}(\hat{g}_1, e_2, e_3, \hat{g}_3), \\ \text{Cone}(\hat{g}_1, \hat{g}_3, e_2, \hat{g}_5), & \text{Cone}(\hat{g}_1, \hat{g}_3, e_1, \hat{g}_5), & \text{Cone}(\hat{g}_1, e_1, e_2, \hat{g}_5), & \\ \text{Cone}(\hat{g}_5, e_1, e_2, e_3), & \text{Cone}(\hat{g}_3, e_2, e_3, \hat{g}_5), & \text{Cone}(\hat{g}_3, e_2, e_3, \hat{g}_5) & \end{cases}$$

is the set of maximal cones of Σ, this is, $|\Sigma(4)| = 14$.

△

The next result provides a necessary condition for the existence of crepant resolutions for an arbitrary Gorenstein quotient singularity $\mathbb{C}^n/G$ via Hilbert basis.

Proposition 2.11. Let $\mathbb{C}^n/G = U_{\sigma,N}$ be an abelian Gorenstein Quotient singularity. If there exists a toric crepant resolution $X_\Sigma \to \mathbb{C}^n/G$ such that the set of ray generators of Σ is ν_G then $\mathbf{Hlb}_N(\sigma) = \nu_G$.

Proof. See [14], p.30. ■

The above proposition says that whenever $\mathbf{Hlb}_N(\sigma)$ contains senior elements, the quotient $\mathbb{C}^n/G = U_{\sigma,N}$ does not admit a crepant resolution. In particular, $\mathbb{C}^4/\mathbb{Z}_{7,((1,1,2,3)}$ in fact do not have a crepant resolution, although $\mathbb{Z}_{7,((1,1,2,3)}$ is generated by a junior element.

CHAPTER 3

Line bundles and exceptional divisors

In this chapter, we explore the structure of Hilbert basis resolutions and toric crepant resolutions of quotient singularities $\mathbb{C}^n/G$, where G is an abelian finite subgroup of $\mathrm{SL}(n, \mathbb{C})$. The results presented here extend the understanding of the exceptional sets of these resolutions and their relationship to junior elements of G. We begin by proving a connection between the deformation retracts of the exceptional divisors corresponding to junior elements of G and specific toric varieties associated with the fan of the resolution. More precisely, we show that if a junior element of G corresponds to a ray in the fan of a Hilbert basis resolution of $\mathbb{C}^n/G$, the associated irreducible component of the exceptional set is a deformation retract of an open toric subvariety (Theorem 3.3). In the case where the resolution is crepant, this result also establishes that each exceptional divisor is normally embedded and the total space of its normal bundle is isomorphic to an open toric subvariety of the resolution.

Furthermore, by combining these results, we describe the global structure of the resolution. For $n = 3$, we identify compact junior elements of G, and show that the minimal toric model of $\mathbb{C}^3/G$ is a toric gluing of the total spaces of the canonical bundles over the exceptional divisors associated with these compact junior elements (Corollary 3.5). A special case arises when there is only one compact junior element, where the resolution itself is isomorphic to the total space of the canonical bundle over the corresponding exceptional divisor (Corollary 3.6).

The results in this chapter highlight the interplay between the combinatorial data of the fan, the geometry of the exceptional divisors, and the structure of the crepant resolution. This chapter serves as a bridge between the foundational toric tools discussed earlier and their application to the study of quotient singularities in dimensions $n = 3$ and higher. The proofs of these results rely heavily on the rich combinatorial and geometric properties of toric varieties, as well as the decomposition of the fan into subsets corresponding to junior elements of G.

3.1 Exceptional divisors and their algebraic tubular neighborhoods

Definition 3.1. *Let X be a complex manifold and let E be a submanifold of X.*

*(a) E is a **holomorphic neighborhood retract** of X if there is a neighborhood U of E and a holomorphic map $h : U \to E$ such that $h|_E =$ identity;*

*(b) E is **normally embedded** in X if there is a neighborhood U_0 of the zero section $Z_{E,X}$ of the normal bundle of E with respect to X, $N_{E,X}$, and a biholomorphic map of U_0 onto a neighborhood U of E in X which is the identity on $Z_{E,X}$. In this case U is called of **tubular neighborhood** of E.*

For more details about the above definitions see [35]. In the category of real smooth manifolds, tubular neighborhoods always exist in the above context, with the morphisms in this category. In the category of complex manifolds, tubular neighborhoods do not exist in general because the normal bundle sequence

$$0 \to T_E \to T_X|_E \to N_{E,X} \to 0$$

does not split in general (see [1]), where T_X is the tangent bundle of X and $N_{E,X}$ is the normal bundle of E with respect to X.

In this section, we will prove the existence of "algebraic neighborhood retract" in the following context. Let G be a finite abelian subgroup of $\mathrm{SL}(n, \mathbb{C})$ and let $\{g_1, \ldots, g_s\}$ be the set of junior classes of G. Suppose that there is a Hilbert basis resolution $f : X_\Sigma \to \mathbb{C}^n/G = U_{\sigma,N}$ of the quotient variety $\mathbb{C}^n/G$. By Proposition 2.7, $\hat{g}_1, \ldots, \hat{g}_s$ are elements of $\mathbf{Hlb}_N(\sigma)$. Thus, $E_{g_i} ;= V(\mathrm{Cone}(\hat{g}_i))$ is a exceptional prime divisor of the resolution, for every $i = 1, \ldots, s$. We will see that there is an open toric set U_i of X_Σ containing E_{g_i} together with a torus invariant divisor D_i of E_{g_i} such that one finds an isomorphism $\varphi : \mathrm{tot}(\mathcal{O}_{E_{g_i}}(D_i)) \to U_i$ that is the identity in the zero section. In particular, when f is a crepant resolution of $\mathbb{C}^n/G$, we will get that $X_\Sigma = \bigcup_{i=1}^{s} U_i$, $\mathcal{O}_{E_{g_i}}(D_i)$ is actually the normal bundle $\mathcal{N}_{E_{g_i}/X_\Sigma}$ of E_{g_i} in X_Σ, and U_i is an "algebraic" tubular neighborhood of E_{g_i}.

As a warm-up, we will start with an example that somewhat describes what has been said.

Example 3.1. Let G be the cyclic group $\mathbb{Z}_{5,(1,2,2)} \subset \mathrm{SL}(3,\mathbb{C})$. In this case

$$\hat{G} = \{(0,0,0), \frac{1}{5}(1,2,2), \frac{1}{5}(2,4,4), \frac{1}{5}(3,1,1), \frac{1}{5}(4,3,3)\}$$

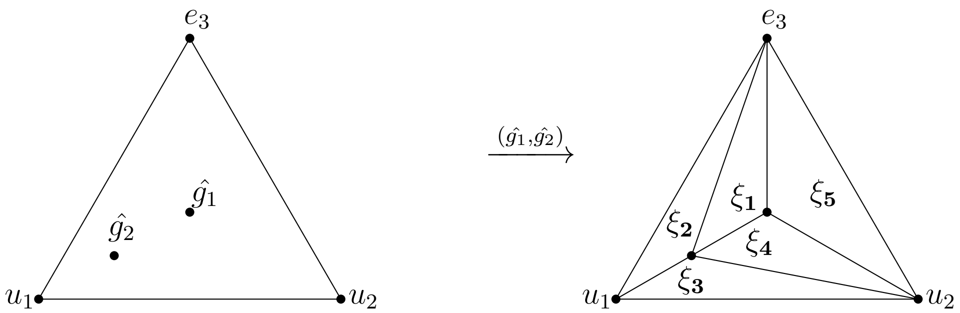
and $\mathbb{C}^3/G = U_{\sigma,N}$, where $N = \mathbb{Z}^3 + \mathbb{Z}\frac{1}{5}(1,2,2)$. Note that $\{\frac{1}{5}(1,2,2), \frac{1}{5}(3,1,1), (0,0,1)\}$ is a basis of N and

$$\begin{aligned} (1,0,0) \quad &= -(\tfrac{1}{5}(1,2,2)) + 2(\tfrac{1}{5}(3,1,1)) + 0(0,0,1) \\ (0,1,0) \quad &= 3(\tfrac{1}{5}(1,2,2)) + (-1)(\tfrac{1}{5}(3,1,1)) + (-1)(0,0,1). \end{aligned}$$

Thus, after changing the coordinates, one can assume $\mathbb{C}^4/G = U_{\xi,\mathbb{Z}^3}$, where

$$\xi = \mathrm{Cone}(u_1 := (-1,2,0), u_2 := (3,-1,-1), e_3).$$

Moreover, the junior elements of G are given as $\hat{g}_1 = e_1$ and $\hat{g}_2 = e_2$. This way, after performing a star subdivision of ξ at the sequence $(\hat{g}_1, \hat{g}_2)$ one has the following picture



Let $\phi : X_\Xi \longrightarrow \mathbb{C}^3/G$ the corresponding crepant resolution and let $\rho_1 = \mathrm{Cone}(\hat{g}_1), \rho_2 = \mathrm{Cone}(\hat{g}_2), \rho_3 = \mathrm{Cone}(e_3), \rho_4 = \mathrm{Cone}(u_1), \rho_5 = \mathrm{Cone}(u_2)$ be the rays of Ξ. Note that $Exc(\phi) = E_{g_1} \cup E_{g_2}$, where $E_{g_i} := D_{\rho_i} = \overline{O(\rho_i)}$ for $i = 1,2$. Furthermore, one can check that $E_{g_1} \cong \mathbb{P}^2$ and $E_{g_2} \cong \mathcal{H}_3$, where $\mathcal{H}_3$ is the third Hirzebruch surface. Now let us describe X_Ξ in homogeneous coordinates. Since X_Ξ is smooth and has no torus factors, it follows that $\mathrm{Cl}(X_\Xi) = \mathrm{Pic}(X_\Xi)$ is a free group and the following sequence is exact

$$0 \longrightarrow \mathbb{Z}^3 \xrightarrow{\varphi} \bigoplus_{i=1}^5 D_{\rho_i} \xrightarrow{\psi} \mathrm{Pic}(X_\Xi) \longrightarrow 0 .$$

Moreover,

$$\begin{aligned}\operatorname{div}(\chi^{(1,0,0)}) &= D_{\rho_1} - D_{\rho_4} + 3D_{\rho_5}\\ \operatorname{div}(\chi^{(0,1,0)}) &= D_{\rho_2} + 2D_{\rho_4} - D_{\rho_5}\\ \operatorname{div}(\chi^{(0,0,1)}) &= D_{\rho_3} - D_{\rho_5}.\end{aligned}$$

Thus $[D_{\rho_1}] = [D_{\rho_4}] - 3[D_{\rho_5}], [D_{\rho_2}] = -2[D_{\rho_4}] + [D_{\rho_5}], [D_{\rho_3}] = [D_{\rho_5}]$ in $\operatorname{Pic}(X_\Xi)$. In particular, $\{[D_{\rho_4}], [D_{\rho_5}]\}$ is basis of $\operatorname{Pic}(X_\Xi)$, and hence one can identify $\operatorname{Pic}(X_\Xi) \cong \mathbb{Z}^2$. This way, the previous exact sequence can be rewritten as

$$0 \longrightarrow \mathbb{Z}^3 \xrightarrow{\varphi} \mathbb{Z}^5 \xrightarrow{\psi} \mathbb{Z}^2 \longrightarrow 0$$

where $\varphi(m_1, m_2, m_3) = (m_1, m_2, m_3, -m_1+2m_2, 3m_1-m_2-m_3)$ and $\psi(n_1, n_2, n_3 n_4, n_5) = (n_1-2n_2+n_4, -3n_1+n_2+n_3+n_5)$. Applying the functor $\operatorname{Hom}(\_, \mathbb{C}^*)$ to the latter sequence, one gets

$$0 \longrightarrow (\mathbb{C}^*)^2 \xrightarrow{\psi^*} (\mathbb{C}^*)^5 \xrightarrow{\varphi^*} (\mathbb{C}^*)^3 \longrightarrow 0$$

where $\psi^*(t_1, t_2) = (t_1 t_2^{-3}, t_1^{-2} t_2, t_2, t_1, t_2)$ and $\varphi^*(q_1, q_2, q_3, q_4, q_5) = (q_1 q_4^{-1} q_5^3, q_2 q_4^2 q_5^{-1}, q_3 q_5^{-1})$. Notice that ψ^* defines a action of $(\mathbb{C}^*)^2$ on $\mathbb{C}^5$. Let $S = \mathbb{C}[x_1, x_2, x_3, x_4, x_5]$ be total coordinate ring of X_Ξ, where each variable x_i corresponds to the ray ρ_i of Ξ. S is a graded ring such that $deg(x_i) = [D_{\rho_i}] \in \operatorname{Pic}(X_\Xi)$. Consider the monomial ideal

$$B(\Xi) := (x_4x_5, x_1x_5, x_1x_3, x_3x_4, x_2x_4)$$

and its zero locus $Z(\Xi) := V(B(\Xi)) \subset \mathbb{C}^5$. Given polynomial $g \in S$, denote by $D(g)$ the principal open set of $\mathbb{C}^5$ where $g \neq 0$. Note that

$$\mathbb{C}^5 \backslash Z(\Xi) = D(x_4x_5) \cup D(x_1x_5) \cup D(x_1x_3) \cup D(x_3x_4) \cup D(x_2x_4)$$

and each principal open set that appears in this union is invariant under the action of $(\mathbb{C}^*)^2$. By part (c) of Proposition 1.15 one can check that

$$(\mathbb{C}^5 \backslash Z(\Xi)/(\mathbb{C}^*)^2) \cong X_\Xi$$

and each homogeneous ideal of S which is contained in $B(\Xi)$ provides a closed subset of

X_Ξ. In this situation, one has

$$\begin{aligned}
X_\Xi &= \{[x_1 : x_2 : x_3 : x_4 : x_5] | (x_1, x_2, x_3, x_4, x_5) \in \mathbb{C}^5 \backslash Z(\Xi)\} \\
E_{g_1} &= \{[0 : x_2 : x_3 : x_4 : x_5] | (0, x_2, x_3, x_4, x_5) \in D(x_4x_5) \cup D(x_3x_4) \cup D(x_2x_4)\} \\
E_{g_2} &= \{[x_1 : 0 : x_3 : x_4 : x_5] | (x_1, 0, x_3, x_4, x_5) \in D(x_4x_5) \cup D(x_1x_5) \cup D(x_1x_3) \cup D(x_3x_4)\} \\
X_{\Xi_{g_1}} &:= \{[x_1 : x_2 : x_3 : x_4 : x_5] | (x_1, x_2, x_3, x_4, x_5) \in D(x_4x_5) \cup D(x_3x_4) \cup D(x_2x_4)\} \\
X_{\Xi_{g_2}} &:= \{[x_1 : x_2 : x_3 : x_4 : x_5] | (x_1, x_2, x_3, x_4, x_5) \in D(x_4x_5) \cup D(x_1x_5) \cup D(x_1x_3) \cup D(x_3x_4)\}.
\end{aligned}$$

Note that each $X_{\Xi_{g_i}}$ is an open set that contains E_{g_i}. Furthermore, the projections $\pi_i : X_{\Xi_{g_i}} \longrightarrow E_{\hat{g_1}}$, given by $\pi_1([x_1 : x_2 : x_3 : x_4 : x_5]) = [0 : x_2 : x_3 : x_4 : x_5], \pi_2([x_1 : x_2 : x_3 : x_4 : x_5]) = [x_1 : 0 : x_3 : x_4 : x_5]$, are well defined morphisms that are the identity in E_{g_i}. Therefore, E_{g_i} is a algebraic neighborhood retract of X_Ξ. △

Given Cartier divisor $D = \sum_{\rho \in \Xi(1)} a_\rho u_\rho$ of a toric variety $X_{L,\Xi} = X_\Xi$, the following construction will be important for our goals: For each $\tau \in \Xi$ consider the cone in $L_\mathbb{R} \times \mathbb{R}$ defined by

$$\tilde{\tau} := \text{Cone}((0, 1), (u_\rho, -a_\rho) | u_\rho \in \tau).$$

Note that $\tilde{\tau}$ is a strongly convex rational polyhedral cone in $L_\mathbb{R} \times \mathbb{R}$. Let Ξ_D be the set consisting of the cones $\tilde{\tau}$ and their faces, for $\tau \in \Xi$. This is a fan of cones in $L_\mathbb{R} \times \mathbb{R}$ and the projection $L_\mathbb{R} \times \mathbb{R} \longrightarrow L$ is a compatible morphism for the fans Ξ_D and Ξ. Hence, we get the toric morphism

$$\pi : X_{\Xi_D} \longrightarrow X_\Xi.$$

From this construction we have the following proposition, whose proof can be found in [11] pg 335.

Proposition 3.1. $\pi : X_{\Xi_D} \longrightarrow X_\Xi$ is a rank 1 vector bundle whose sheaf of sections is $\mathcal{O}_{X_\Xi}(D)$.

In particular, this proposition implies that the total space of the line bundle corresponding to a Cartier divisor of a toric variety is also a toric variety.

Lemma 3.2. Let $\{u_1, \dots, u_n\}$ be a basis of $\mathbb{R}^n$. If $b_i \in \text{Conv}(k_i u_1, \dots, k_i u_n)$, for $i = 1, \dots, n$ and $k_i \in \mathbb{Z}_{>0}$, are such that $\{b_1, \dots, b_n\}$ is another basis of $\mathbb{R}^n$ and $b \in \text{Conv}(ku_1, \dots, ku_n)$, for some $k \in \mathbb{Z}_{>0}$, and $b = a_1 b_1 + \dots + a_n b_n$, then $k_1 a_1 + \dots + k_n a_n = k$.

Proof. One can write $b = \sum_{i=1}^n \alpha_i k u_i$, where $\sum_{i=1}^n \alpha_i = 1$, and $b_j = \sum_{i=1}^n a_{ij} k_j u_i$, where $\sum_{i=1}^{n} a_{ij} = 1$. The latter implies that $b = \sum_{i=1}^{n}(\sum_{j=1}^{n} a_j k_j a_{ij})u_i$. Thus $\alpha_i k = \sum_{j=1}^{n} k_j a_j a_{ij}$. It follows that

$$k = \sum_{i=1}^{n} \alpha_i k = (\sum_{i=1}^{n} a_{i1})k_1 a_1 + \cdots + (\sum_{i=1}^{n} a_{in})k_n a_n = k_1 a_1 + \cdots + k_n a_n$$

■

Now we can prove our first important result.

Theorem 3.3. *Let G be an abelian finite subgroup of $\mathrm{SL}(n, \mathbb{C})$. Suppose that there is a Hilbert desingularization $\phi : X_\Xi \to \mathbb{C}^n/G = U_{\sigma,N}$ of the quotient variety, and for each $\hat{g} \in \hat{G} \cap \mathbf{Hlb}_N(\sigma)$ denote by E_g the irreducible component of the exceptional set of ϕ, corresponding to the ray $\rho_g := \mathrm{Cone}(\hat{g})$. If $\mathrm{age}(\hat{g}) = 1$ then E_g is a deformation retract of the toric variety X_{Ξ_g} , where Ξ_g is the fan that consists of all the faces of the cones that appear in the set*

$$\Xi_g(n) := \{\eta \in \Xi(n) | \rho_g \preceq \eta\}.$$

In particular X_{Ξ_g} is open in X_Ξ.

Proof. Consider the new lattice $N(\hat{g}) = N/\hat{g}\mathbb{Z} \cong \mathbb{Z}^{n-1}$ and the set

$$\mathrm{Star}(\hat{g}) := \{\overline{\eta} \subseteq N(\hat{g}) | \rho_g \preceq \eta \in \Xi\}.$$

Note that $\mathrm{Star}(g)$ is a fan in $N(\hat{g})_\mathbb{R}$. By Proposition 1.9, $\mathrm{Star}(\hat{g})$ is the fan of E_g. Furthermore, all of the elements of $\mathrm{Star}(g)(1)$ are of the form $\overline{\mathrm{Cone}(\hat{g}, u)} = \mathrm{Cone}(\overline{u})$, for some $u \in \mathbf{Hlb}_N(\sigma)$ and $\mathrm{Cone}(\hat{g}, u) \in \Xi_g(2)$. Since E_g is a smooth variety any of its Weil divisors are Cartier. Consider the Cartier divisor

$$D := \sum_{\mathrm{Cone}(\overline{u}) \in \mathrm{Star}(g)(1)} -(\mathrm{age}(u))D_{\mathrm{Cone}(\overline{u})},$$

where $D_{\mathrm{Cone}(\overline{u})}$ is the toric prime divisor corresponding to the ray $\mathrm{Cone}(\overline{u})$. Let $\xi := Cone(\hat{g}, u_1, \ldots, u_{n-1})$ be a cone in $\Xi(n)$. This way, $\{\overline{u_1} \ldots, \overline{u_{n-1}}\}$ is a basis of $N(g)$ and hence $\{(\overline{0}, 1), (\overline{u_1}, \mathrm{age}(u_1)), \ldots, (\overline{u_{n-1}}, \mathrm{age}(u_{n-1}))\}$ is a basis of $N(g) \times \mathbb{Z}$. Consider the lattice isomorphism

$$\overline{f} : N(g) \times \mathbb{Z} \longrightarrow N$$

defined by:

$$\overline{f}(\overline{0}, 1) = \hat{g}, \quad \overline{f}(\overline{u_1}, \mathrm{age}(u_1)) = u_1, \ldots, \quad \overline{f}(\overline{u_{n-1}}, \mathrm{age}(u_{n-1})) = u_{n-1}.$$

Let ρ be an element of $\Xi_g(1)$ and denote by u_ρ its ray generator and k_ρ the age of u_ρ. Note that $u_\rho \in \mathrm{Conv}(k_\rho e_1, \ldots, k_\rho e_n)$. One can write $u_\rho = a\hat{g} + a_1 u_1 + \cdots + a_{n-1}u_{n-1}$, hence, by Lemma 3.2,

$$k_\rho = a(\mathrm{age}(\hat{g})) + a_1(\mathrm{age}(u_1)) + \cdots + a_{n-1}(\mathrm{age}(u_{n-1})).$$

Thus, one gets

$$\begin{aligned}
\overline{f}(\overline{u_\rho}, k_\rho) = & \quad \overline{f}(\overline{a\hat{g} + a_1u_1 + \cdots + a_{n-1}u_n}, a(\mathrm{age}) + a_1(\mathrm{age}(u_1)) \cdots + a_{n-1}(\mathrm{age}(u_{n-1})) \\
= & \quad a\overline{f}(\overline{0}, 1) + a_1 f(\overline{u_1}, \mathrm{age}(u_1)) + \cdots + a_{n-1}\overline{f}(\overline{u_{n-1}}, \mathrm{age}(u_{n-1})) \\
= & \quad u_\rho.
\end{aligned}$$

Let $\eta = \mathrm{Cone}(v_1, \ldots, v_s)$ be a cone in Ξ_g, where $v_i \in \mathbf{Hlb}_N(\sigma)$. If $\hat{g} \in \eta$, then we can suppose $u_1 = \hat{g}$, and we have

$$\overline{n} \in \mathrm{Star}(g), \quad \hat{\overline{\eta}} \in \mathrm{Star}(g)_D,$$

and hence,

$$\overline{f}_{\mathbb{R}}(\hat{\overline{\eta}}) = \eta.$$

If $\hat{g} \notin \eta$, then $\xi := \rho_g + \eta$ is a cone in Ξ_g such that η is one of its faces. In this case,

$$\hat{\overline{\xi}} = \mathrm{Cone}((\overline{0}, 1), (u_1, \mathrm{age}(u_1)), \ldots, (u_n, \mathrm{age}(u_n)))$$

is a cone in $\mathrm{Star}(g)_D$. Thus, its face

$$\eta' := \mathrm{Cone}((u_1, \mathrm{age}(u_1)), \ldots, (u_n, \mathrm{age}(u_n)))$$

is also in $\mathrm{Star}(g)_D$. Moreover,

$$\overline{f}_{\mathbb{R}}(\eta') = \eta.$$

The two cases analyzed above show that $\overline{f}$ is compatible with the fans $\mathrm{Star}(g)_D$ and Ξ_g, and hence, one obtains an isomorphism of toric varieties:

$$f : X_{\mathrm{Star}(\hat{g})_{K_{E_g}}} \longrightarrow X_{\Xi_g}.$$

The last statement implies that X_{Ξ_g} is a line bundle over its subset E_g, and therefore, E_g is a (strong) deformation retract of X_{Ξ_g}.

■

Theorem 3.4. *Let G be an abelian finite subgroup of* $\mathrm{SL}(n,\mathbb{C})$ *and suppose that there is a crepant resolution* $\phi : X_\Xi \to \mathbb{C}^n/G = U_{\sigma,N}$ *of the quotient variety. If g is a junior element of G, then E_g is normally embedded in X_Ξ. In particular, the total space of the canonical bundle of E_g,* $\mathrm{tot}(\omega_{E_g})$, *is isomorphic to the toric variety X_{Ξ_g}, and*

$$X_\Xi = \bigcup_{\hat{g}\in\hat{G}\cap\triangle_1} X_{\Xi_g}$$

where X_{Ξ_g} is defined in the same way as the previous theorem.

Proof. The proof is very similar to the proof of the previous theorem, but with some adjustment. Again, one considers the lattice $N(g)$ and the fan $\mathrm{Star}(g)$ of E_g. Since E_g is a smooth variety, its canonical divisor K_{E_g} must be a Cartier divisor. Since ϕ is a crepant morphism, the age of all the ray generators of the rays in Ξ should equal 1. This way, one has

$$\sum_{Cone(\overline{u})\in\mathrm{Star}(g)(1)} -(\mathrm{age}(u))D_{Cone(\overline{u})} = \sum_{Cone(\overline{u})\in Star(g)(1)} -D_{Cone(\overline{u})} = K_{E_g}.$$

Now consider the fan $Star(g)_{K_{E_g}}$ in $(N(g)\times\mathbb{Z})_\mathbb{R}$. By Proposition 3.1, $X_{\mathrm{Star}(g)_{K_{E_g}}}$ is the total space of $\mathcal{O}_{E_g}(K_{E_g}) = \omega_{E_g}$. Let $\xi := Cone(\hat{g}, u_1, \ldots, u_{n-1})$ be a cone in $\Xi(n)$. This way, $\{\overline{u_1}\ldots, \overline{u_{n-1}}\}$ is a basis of $N(g)$ and hence $\{(\overline{0},1), (\overline{u_1}, \mathrm{age}(u_1)), \ldots, (\overline{u_{n-1}}, \mathrm{age}(u_{n-1}))\}$ is a basis of $N(g)\times\mathbb{Z}$. Consider the isomorphism $\overline{f} : N(g)\times\mathbb{Z} \longrightarrow N$ of lattices defined from $\overline{f}(\overline{0},1) = g$, $\overline{f}(\overline{u_1},1) = u_1, \ldots, \overline{f}(\overline{u_{n-1}},1) = u_{n-1}$. Write $u_\rho = a_1 g + a_2 v_1 + \cdots + a_n v_{n-1}$. Since $u_\rho, g, v_1, \ldots v_{n-1} \in T_\sigma$, Lemma 3.2 guarantees $a_1 + \cdots + a_n = 1$. With the same arguments of the proof of the last theorem, one gets that $\overline{f}_\mathbb{R}$ induces an isomorphism of toric varieties $f : X_{\mathrm{Star}(\hat{g})_{K_{E_g}}} \longrightarrow X_{\Xi_g}$.

Since ϕ is a crepant resolution of $\mathbb{C}^n/G$ then $\omega_{X_\Xi} \cong \mathcal{O}_{X_\Xi}$. By the adjunction formula

$$\omega_{E_g} \simeq \omega_{X_\Xi} \otimes_{\mathcal{O}_{X_\Xi}} \mathcal{N}_{E_g/X_\Xi},$$

one has that $\omega_{E_g} \cong \mathcal{N}_{E_g/X_\Xi}$. Therefore E_g is normally embedded in X_Ξ.

■

Corollary 3.5. *Let G be an abelian finite subgroup of* $\mathrm{SL}(3,\mathbb{C})$. *Suppose G has compact junior elements $g_{c_1}, \ldots, g_{c_s}$ (those elements whose fractional expressions lie in* $\mathrm{Relint}(\sigma)$*). Then there is a toric minimal model X_Ξ of $\mathbb{C}^3/G = U_{\sigma,N}$ such that X_Ξ is a toric gluing of* $\mathrm{tot}(\omega_{E_{g_{c_1}}}), \ldots, \mathrm{tot}(\omega_{E_{g_{c_s}}})$.

Proof. Since, in dimension 3, every Gorenstein toric variety with terminal singularities is smooth, every sequence of star subdivisions at all the points of ν_G provides a crepant resolution of $\mathbb{C}^3/G = U_{\sigma,N}$. This way, if one starts a sequence of star subdivisions at the points $\hat{g_{c_1}}, \ldots, \hat{g_{c_s}}$, and finish with the other remaining points of ν_G, one gets a fan Ξ such that any of its maximal fan contains the ray $\text{Cone}(g_{c_i})$ for some $i = 1, \ldots, s$. From the previous Corollary, it follows that $X_\Xi = \bigcup_{i=1}^{s} X_{\Xi_{g_{c_i}}}$ ■

Corollary 3.6. *Let G be an abelian finite subgroup of* $\mathrm{SL}(3, \mathbb{C})$*. Then G has only one compact junior element g if and only if $\mathbb{C}^3/G = U_{N,\sigma}$ has a toric crepant resolution X_Ξ that is isomorphic to* $\text{tot}(\omega_{E_g})$*.*

Proof. One way comes from the previous corollary. For the other way around, suppose that such resolution X_Ξ exists. Since X_Ξ is a line bundle of E_g, by Poincaré duality, one gets $H_c^4(X_\Xi, \mathbb{Q}) \cong H_2(X_\Xi, \mathbb{Q}) \cong H_2(E_g, \mathbb{Q}) \cong H^0(E_g, \mathbb{Q})$ (see [24], for more details). Because $E_{\hat{g}}$ is a complete toric surface, one must have $\dim H^0(E_g, \mathbb{Q}) = 1$. Thus $\dim H_c^4(X_\Xi, \mathbb{Q}) = 1$. By [31], $\dim H_c^4(X_\Xi, \mathbb{Q})$ is the number of compact junior elements of G. ■

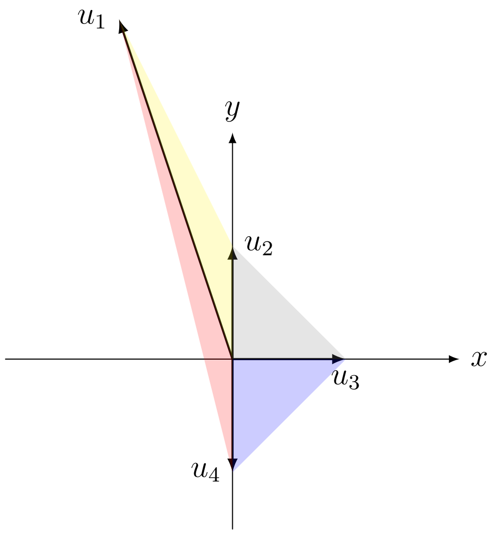
Example 3.2. In Example 3.1, we have seen that $G = \mathbb{Z}_{5,(1,2,2)}$ has two junior elements g_1 and g_2 such that $\hat{g_1} = \frac{1}{5}(1, 2, 2)$ and $\hat{g_2} = \frac{1}{5}(3, 1, 1)$. In particular, these elements are compact and $E_{g_1} \cong \mathbb{P}^2$ and $E_{g_2} \cong \mathcal{H}_3$. For $0 \leq i \leq 6$, we will compute the dimension of the singular cohomology groups $H^i(X_\Xi, \mathbb{Q})$ of the variety X_Ξ obtained in the resolution $\phi : X_\Xi \longrightarrow \mathbb{C}^3/G$ presented in Example 3.1, according to the Mckay correspondence. By Theorem 3.4, $X_\Xi = X_{\Xi_{g_1}} \cup X_{\Xi_{g_2}}$ and $X_{\Xi_{g_1}} \cong \text{tot}(\omega_{\mathbb{P}^2})$ and $X_{\Xi_{g_2}} \cong \text{tot}(\omega_{\mathcal{H}_3})$. In this case, $H^i(X_{\Xi_{g_1}}, \mathbb{Q}) = H^i(\mathbb{P}^2, \mathbb{Q})$ and $H^i(X_{\Xi_{g_2}}, \mathbb{Q}) = H^i(\mathcal{H}_3, \mathbb{Q})$ for every i. For any complete simplicial toric variety X_Σ of dimension n, one has:

- $H^{2k+1}(X_\Sigma, \mathbb{Q}) = 0$ for every k;
- $\dim H^{2k}(X_\Sigma, \mathbb{Q}) = \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} |\Sigma(n-i)|$;
- $\dim H^{2k}(X_\Sigma, \mathbb{Q}) = \dim H^{2n-2k}(X_\Sigma, \mathbb{Q})$.

The above identities can be seen in chapter 12 of [11]. Let Σ_1 be the fan of $\mathbb{P}^2$, presented in example 1.2. One has $|\Sigma_1(2)| = 3, |\Sigma_1(1)| = 3$ and $|\Sigma_1(0)| = 1$. Thus,

$$\begin{aligned}
\dim H^0(X_{\Xi_{g_1}}, \mathbb{Q}) &= \dim H^0(\mathbb{P}^2, \mathbb{Q}) = \sum_{i=0}^{2}(-1)^{i-0}\binom{i}{0}|\Sigma_1(2-i)| = 3-3+1 = 1;\\
\dim H^2(X_{\Xi_{g_1}}, \mathbb{Q}) &= \dim H^2(\mathbb{P}^2, \mathbb{Q}) = \sum_{i=1}^{2}(-1)^{i-1}\binom{i}{1}|\Sigma_1(2-i)| = 3-2\times 1 = 1;\\
\dim H^4(X_{\Xi_{g_1}}, \mathbb{Q}) &= \dim H^4(\mathbb{P}^2, \mathbb{Q}) = \dim H^0(\mathbb{P}^2, \mathbb{Q}) = 1;\\
\dim H^6(X_{\Xi_{g_1}}, \mathbb{Q}) &= \dim H^6(\mathbb{P}^2, \mathbb{Q}) = 0;
\end{aligned}$$

where for any manifold M of real dimension n, $\dim H^k(M, \mathbb{Q}) = 0$, for every $k > n$ (see [24], Chapter 3). The fan Σ_2 of $\mathcal{H}_3$ is described by the following picture:



where $u_1 = (-1, 3), u_2 = (0, 1), u_3 = (1, 0), u_4 = (0, -1)$ are the ray generators of Σ_2 and $\text{Cone}(u_1, u_2), \text{Cone}(u_2, u_3), \text{Cone}(u_3, u_4), \text{Cone}(u_4, u_1)$ are its maximal cones. In this situation, $|\Sigma_2(2)| = 4, |\Sigma_2(1)| = 4$ and $|\Sigma_2(0)| = 1$. Thus,

$$\begin{aligned}
\dim H^0(X_{\Xi_{g_2}}, \mathbb{Q}) &= \dim H^0(\mathcal{H}_3, \mathbb{Q}) = \sum_{i=0}^{2}(-1)^{i-0}\binom{i}{0}|\Sigma_2(2-i)| = 4-4+1 = 1;\\
\dim H^2(X_{\Xi_{g_2}}, \mathbb{Q}) &= \dim H^2(\mathcal{H}_3, \mathbb{Q}) = \sum_{i=1}^{2}(-1)^{i-1}\binom{i}{1}|\Sigma_2(2-i)| = 4-2\times 1 = 2;\\
\dim H^4(X_{\Xi_{g_2}}, \mathbb{Q}) &= \dim H^4(\mathcal{H}_3, \mathbb{Q}) = \dim H^0(\mathcal{H}_3, \mathbb{Q}) = 1;\\
\dim H^6(X_{\Xi_{g_2}}, \mathbb{Q}) &= \dim H^6(\mathcal{H}_3, \mathbb{Q}) = 0.
\end{aligned}$$

Note that $U := X_{\Xi_{g_1}} \cap X_{\Xi_{g_2}} = U_{\xi_1} \cup U_{\xi_2}$ and $U_{\xi_1} \cap U_{\xi_2} = U_\tau$, where ξ_1, ξ_2 are defined in example 3.1 and $\tau := \text{Cone}(\hat{g}_1, \hat{g}_2)$. In general an affine toric variety $U_{\sigma,N}$ is a deformation retract of the orbit $O_N(\sigma)$ (see [11], chapter 12). Since $\dim \xi_i = 3$, it follows that U_{ξ_i} is a deformation retract of its distinguished point γ_{ξ_i}. Thus

$$H^i(U_{\xi_i}, \mathbb{Q}) = H^i(\{\gamma_{\xi_i}\}, \mathbb{Q}) = \begin{cases} \mathbb{Q}, & i = 0, \\ 0, & i \geq 1. \end{cases}$$

Since τ is a smooth cone of dimension 2, one has $N(\tau) \cong \mathbb{Z}$. By Proposition 1.8, $O_N(\tau) \cong T_{N(\tau)} \cong \mathbb{C}^*$ which is a deformation retraction of the real sphere S^1. Hence

$$H^i(U_\tau, \mathbb{Q}) = H^i(S^1, \mathbb{Q}) = \begin{cases} \mathbb{Q}, & i = 0, 1, \\ 0, & n \geq 2. \end{cases}$$

Applying the Mayer–Vietoris sequence to the cover $U = U_{\xi_1} \cup U_{\xi_2}$, we get

$$\begin{aligned}
0 &\to H^0(U, \mathbb{Q}) \to H^0(U_{\xi_1}, \mathbb{Q}) \oplus H^0(U_{\xi_2}, \mathbb{Q}) \to H^0(U_\tau, \mathbb{Q}) \\
&\to H^1(U, \mathbb{Q}) \to H^1(U_{\xi_1}, \mathbb{Q}) \oplus H^1(U_{\xi_2}, \mathbb{Q}) \to H^1(U_\tau, \mathbb{Q}) \\
&\to H^2(U, \mathbb{Q}) \to H^2(U_{\xi_1}, \mathbb{Q}) \oplus H^2(U_{\xi_2}, \mathbb{Q}) \to H^2(U_\tau, \mathbb{Q}) \\
&\to H^3(U, \mathbb{Q}) \to H^3(U_{\xi_1}, \mathbb{Q}) \oplus H^3(U_{\xi_2}, \mathbb{Q}) \to H^3(U_\tau, \mathbb{Q}) \\
&\to H^4(U, \mathbb{Q}) \to H^4(U_{\xi_1}, \mathbb{Q}) \oplus H^4(U_{\xi_2}, \mathbb{Q}) \to H^4(U_\tau, \mathbb{Q}) \\
&\to H^5(U, \mathbb{Q}) \to H^5(U_{\xi_1}, \mathbb{Q}) \oplus H^5(U_{\xi_2}, \mathbb{Q}) \to H^5(U_\tau, \mathbb{Q}) \\
&\to H^6(U, \mathbb{Q}) \to H^6(U_{\xi_1}, \mathbb{Q}) \oplus H^6(U_{\xi_2}, \mathbb{Q}) \to H^6(U_\tau, \mathbb{Q}) \to \cdots,
\end{aligned}$$

thus $\dim H^i(U, \mathbb{Q}) = 0$ for $i > 2$ and $\dim H^2(U, \mathbb{Q}) = 1$. Since U is path-connected, one has $\dim H^0(U, \mathbb{Q}) = 1$ (see [24], p.109). Hence $\dim H^1(U, \mathbb{Q}) = 0$.

Applying the Mayer-Vietoris sequence to the cover $X_\Xi = X_{\Xi_{g_1}} \cup X_{\Xi_{g_2}}$, we get

$$\begin{aligned}
0 &\to H^0(X_\Xi, \mathbb{Q}) \to H^0(X_{\Xi_{g_1}}, \mathbb{Q}) \oplus H^0(X_{\Xi_{g_2}}, \mathbb{Q}) \to H^0(U, \mathbb{Q}) \\
&\to H^1(X_\Xi, \mathbb{Q}) \to H^1(X_{\Xi_{g_1}}, \mathbb{Q}) \oplus H^1(X_{\Xi_{g_2}}, \mathbb{Q}) \to H^1(U, \mathbb{Q}) \\
&\to H^2(X_\Xi, \mathbb{Q}) \to H^2(X_{\Xi_{g_1}}, \mathbb{Q}) \oplus H^2(X_{\Xi_{g_2}}, \mathbb{Q}) \to H^2(U, \mathbb{Q}) \\
&\to H^3(X_\Xi, \mathbb{Q}) \to H^3(X_{\Xi_{g_1}}, \mathbb{Q}) \oplus H^3(X_{\Xi_{g_2}}, \mathbb{Q}) \to H^3(U, \mathbb{Q}) \\
&\to H^4(X_\Xi, \mathbb{Q}) \to H^4(X_{\Xi_{g_1}}, \mathbb{Q}) \oplus H^4(X_{\Xi_{g_2}}, \mathbb{Q}) \to H^4(U, \mathbb{Q}) \\
&\to H^5(X_\Xi, \mathbb{Q}) \to H^5(X_{\Xi_{g_1}}, \mathbb{Q}) \oplus H^5(X_{\Xi_{g_2}}, \mathbb{Q}) \to H^5(U, \mathbb{Q}) \\
&\to H^6(X_\Xi, \mathbb{Q}) \to H^6(X_{\Xi_{g_1}}, \mathbb{Q}) \oplus H^6(X_{\Xi_{g_2}}, \mathbb{Q}) \to H^6(U, \mathbb{Q}) \to \cdots,
\end{aligned}$$

thus

$$\dim H^0(X_\Xi,\mathbb{Q})=1,\quad \dim H^1(X_\Xi,\mathbb{Q})=0,\quad \dim H^2(X_\Xi,\mathbb{Q})=2,$$
$$\dim H^3(X_\Xi,\mathbb{Q})=0,\quad \dim H^4(X_\Xi,\mathbb{Q})=2,\quad \dim H^5(X_\Xi,\mathbb{Q})=0,$$
$$\dim H^6(X_\Xi,\mathbb{Q})=0.$$

Note that $\dim H^2(X_\Xi,\mathbb{Q})$ is the number of junior elements of G and $\dim H^4(X_\Xi,\mathbb{Q})$ is the number of senior elements of G. These numbers were actually expected due to the main theorem of [31]. △

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