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Centro de Ciências Exatas e da Natureza
Programa de Pós-Graduação em Matemática
Mestrado em Matemática

Navier-Stokes equations and forward-backward SDEs on the group of diffeomorphisms of a torus

Jean Pereira Soares

João Pessoa – PB
Abril de 2025

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por

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sob a orientação do

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ATA DE DEFESA DE MESTRADO JUNTO AO PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA DA UNIVERSIDADE FEDERAL DA PARAÍBA, REALIZADA NO DIA 30 DE MAIO DE 2025.

Ao trigésimo dia de maio de dois mil e vinte e cinco, às 16:00 horas, na Sala do PIBID/CCEN da Universidade Federal da Paraíba, foi aberta a sessão pública de Defesa de Dissertação intitulada "**Equações de Navier-Stokes e EDEs progressiva-regressiva no grupo de difeomorfismos de um toro**", do aluno **Jean Pereira Soares**, que havia cumprido, anteriormente, todos os requisitos para a obtenção do grau de Mestre em Matemática, na área de **Probabilidade e Estatística**, sob a orientação da Prof.ª Dr.ª Evelina Shamarova. A Banca Examinadora, indicada pelo Colegiado do Programa de Pós-Graduação em Matemática, foi composta pelos professores: Dr. Evelina Shamarova (Orientadora), Dr. João Marcos Bezerra do Ó (membro interno), Dr.ª Maria Fernanda de Almeida Cipriano Salvador Marques (membro externo/UNL) e Dr. Paulo Regis Caron Ruffino (membro externo/UNICAMP).A professora Evelina Shamarova, em virtude da sua condição de orientadora, presidiu os trabalhos e, depois das formalidades de apresentação, convidou o aluno a discorrer sobre o conteúdo da dissertação. Concluída a explanação, o candidato foi arguido pela banca examinadora que, em seguida, sem a presença do aluno, finalizando os trabalhos, reuniu-se para deliberar tendo concedido ao candidato a menção: **Aprovado.** E, para constar, foi lavrada a presente ata que será assinada pelos membros da Banca Examinadora.

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Dedication

I dedicate this work to my parents, Gilvan and Joelsa, my brother, Julian, and my fiancée, Clarisse, who have always supported me on the path I chose to follow.

Acknowledgments

Desde já, não farei esses agradecimentos na língua inglesa, pois as pessoas as quais esse texto é dedicado entendem melhor a língua portuguesa. Sendo assim...

Agradeço especialmente aos meus pais, Gilvan e Joelsa, que sempre me apoiaram no sonho da pós-graduação e que sabem melhor do que ninguém como este caminho do mestrado foi longo e difícil.

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Por fim, guardo também agradecimentos especiais aos meus alunos da Universidade Estadual da Paraíba e colegas de profissão que acompanharam meu caminhar neste mestrado, enquanto fui professor desta instituição de ensino e que uma parte deles se fez presente na minha defesa que ocorreu de forma híbrida.

A todos e todas que participaram disso, desejo os meus mais sinceros votos de gratidão.

"All things are possible for one who believes."

Jesus Christ

Resumo

Ao longo deste estudo, nosso objetivo geral foi o que estabelecer uma relação entre as soluções de um sistema progressivo-regressivo (forward-backward) de equações diferenciais estocásticas no grupo de difeomorfismos do toro plano que são H^α-suaves para $\alpha > 2$ e que preservam volume (denotado como $\mathcal{D}_v^\alpha(\mathbb{T}^2)$) e as soluções das equações de Navier-Stokes no plano, $\mathbb{R}^2$.

Nossa teoria nos capítulos 1 e 2 foi obtida tendo como base principal Shkoller [16], do Carmo [6] e Gliklikh [10] para fornecer os conceitos básicos sobre espaços de Sobolev, Geometria Riemanniana e Análise Estocástica em variedades que seriam necessários para o desenvolvimento dos resultados de Cruzeiro e Shamarova [4].

No capítulo 3, apresentamos o primeiro resultado principal do trabalho. Supomos que existe uma solução para Navier-Stokes equações no plano e, a partir desta solução, encontramos uma tripla de soluções para um sistema de equações diferenciais estocásticas forward-backward em $\mathcal{D}_v^\alpha(\mathbb{T}^2)$.

Por fim, no capítulo 4, apresentamos o resultado recíproco e dada uma solução do sistema supracitado, nós construímos uma solução para as equações de Navier-Stokes.

Palavras-chave: Equações de Navier-Stokes; Sistema FBSDE; Grupo de Difeomorfismos no toro plano.

Abstract

Throughout this study, our general objective was to establish a relationship between the solutions of a forward-backward system of stochastic differential equations in the group of diffeomorphisms of the flat torus that are H^α-smooth for $\alpha \geq 2$ which preserve volume (denoted as $\mathcal{D}_v^\alpha(\mathbb{T}^2)$) and the solutions of the Navier-Stokes equations in the plane, $\mathbb{R}^2$.

The theory in Chapters 1 and 2 was obtained mainly by Shkoller [16], do Carmo [6] and Gliklikh [10] to provide the basic concepts about Sobolev spaces, Riemannian Geometry and Stochastic Analysis on Manifolds that would be necessary for the development of the results of Cruzeiro and Shamarova [4].

In Chapter 3, we present the first main result of the work. We assume that there exists a solution to the Navier-Stokes equations in the plane, and from this solution we find a triple of solutions to a system of forward-backward stochastic differential equations in $\mathcal{D}_v^\alpha(\mathbb{T}^2)$.

Finally, in Chapter 4, we present the converse result and given a solution of the aforementioned system, we construct a solution to Navier-Stokes equations.

Keywords: Navier Stokes equations; FBSDE System; diffeomorphism group on the flat torus

Contents

Notation

Chapter 1

$\exp_q$ Exponential map at a point q on a Riemannian manifold, mapping tangent vectors to the manifold.

$\frac{D\mathbf{V}}{dt}$ Covariant derivative of a vector field $\mathbf{V}$ along a curve, incorporating curvature effects in non-Euclidean spaces.

$\mathbb{T}^n$ The flat n-torus.

$\mathcal{B}$ The σ-algebra representing the information up to time t, often used in stochastic processes.

$\mathcal{B}_t$ Filtration of the σ-algebra $\mathcal{B}$, capturing the evolution of information over time.

$\mathcal{F}_t$ Variation of the filtration notation defined in a σ-algebra $\mathcal{F}$, capturing the evolution of information over time.

∇ Covariant derivative operator, generalizing partial derivatives in curved spaces or manifolds.

H A Hilbert space, a complete inner product space.

Chapter 2

$(-\Delta)^\alpha$ Fractional Laplacian operator, a generalization of Δ, defined as $(-\Delta)^\alpha$ via Fourier transform.

Δ Laplace operator or Laplacian, defined as $\Delta = \nabla \cdot \nabla$, which measures the divergence of the gradient of a function.

$\langle \cdot, \cdot \rangle$ Inner product, typically associated with a Hilbert space, which provides a notion of angle and length.

$\mathcal{D}^\alpha(\mathbb{T}^n)$ Group of diffeomorphisms of the H^α-maps defined on the flat n-torus.

$\mathcal{D}_v^\alpha(\mathbb{T}^n)$ Group of diffeomorphisms of the H^α-maps that preserve volumes defined on the flat n-torus.

e Identity mapping on the flat n-torus $\mathbb{T}^n$, serving as the neutral element in the group of diffeomorphisms $\mathcal{D}^\alpha(\mathbb{T}^n)$ and in $\mathcal{D}_v^\alpha(\mathbb{T}^n)$.

$H^\alpha(\mathbb{T}^n)$ Denote the set of Sobolev H^α-mappings from $\mathbb{T}^n$ to $\mathbb{T}^n$ in which we have $\alpha > \frac{n}{2} + 1$

$T\mathbb{T}^n$ Tangent bundle of the flat n-torus.

$T_e\mathbb{T}^n$ Tangent space at the identity element $e \in \mathbb{T}^n$.

$T_g\mathcal{D}^\alpha(\mathbb{T}^n)$ Tangent space at $g \in \mathcal{D}(\mathbb{T}^n)$, defined by the composition of the vector fields of $T_e\mathcal{D}^\alpha(\mathbb{T}^n)$ by a function $g \in \mathcal{D}^\alpha(\mathbb{T}^n)$.

Chapters 3 e 4

$\bar{A}_k(\theta)$ Vector defined using the Fourier coefficients on $\mathbb{T}^2$, given by the decompositions into cosine series.

$\bar{B}_k(\theta)$ Vector defined using the Fourier coefficients on $\mathbb{T}^2$, given by the decompositions into sine series

$\mathbb{E}[X_t \mid \mathcal{F}_s]$ Conditional expectation of X_t given the σ-algebra $\mathcal{F}_s$, representing the best prediction of X_t based on information available up to time s.

$\mathbb{E}[X_t]$ Expected value of the random variable X_t, representing the average outcome of X_t under the probability measure.

Introduction

The Navier–Stokes equations are among the famous Millennium Prize Problems and have been motivating mathematicians around the world for over two centuries. Researchers have explored numerous theories and performed countless hypothesis tests in an attempt to prove the existence of analytic (or smooth) solutions. However, until this day, the problem remains unsolved for cases beyond two dimensions.

Our goal in this work is not to solve the Navier-Stokes equations in full generality, but rather to outline a possible approach for finding solutions under specific conditions by employing systems of stochastic differential equations.

The main idea of this study is to demonstrate that a solution to a system of forward-backward stochastic differential equations on an infinite-dimensional manifold can be used to formulate a corresponding solution to the Navier-Stokes equations in an appropriate function space.

Conversely, given a solution to the Navier-Stokes equations, it is possible to find the solutions of a given system of stochastic differential equations.

The main idea of this study is to demonstrate that a solution to a system of forward-backward stochastic differential equations on an infinite-dimensional manifold can be used to formulate a corresponding solution to the classical Navier-Stokes equations.

More specifically, the infinite-dimensional manifold under consideration is the set $\mathcal{D}_v^\alpha(\mathbb{T}^2)$ which is also a group with respect to the composition operation and constitutes all volume-preserving H^α-diffeomorphisms of the flat two-dimensional torus $\mathbb{T}^2$ to itself. Similarly, the Navier-Stokes equations are considered on the two-dimensional Euclidean space $\mathbb{R}^2$.

This master's thesis is divided into four chapters. In the first chapter, we are introduced to the most essential and necessary concepts for what we will use in this work on Hilbert manifolds and Stochastic Analysis of manifolds, having as main reference the books by Gliklikh in [9], [10] and [11].

In the second chapter, we address more specific aspects related to Sobolev Spaces and also Sobolev maps, since our diffeomorphisms will be defined under the understanding of these spaces. We also define the metrics and other stronger results related to stochastic analysis on manifolds focusing on diffeomorphism groups, following Schkoller [16], Gliklikh [10] and others.

In the third chapter, we finally begin to deepen the results shown in the article written by Cruzeiro and Shamarova [4], define the notations that we will use next and construct the results that demonstrate that it is possible to obtain a solution for a system of stochastic differential equations from a solution for Navier-Stokes equations.

Finally, in the fourth chapter we have the reciprocal result and present the second main theorem of the work, in which we address that having a solution for the aforementioned stochastic system, we are able to construct the solution for the Navier-Stokes equations.

The appendix presents the author's proof of Itô's formula in Hilbert spaces with a Brownian motion on a finite-dimensional space.

Chapter 1

Riemannian Geometry and Stochastic Analysis

Throughout this chapter, our goal is to discuss the essential concepts related to Riemannian Geometry and Stochastic Analysis that will be covered in the following chapters of this work. However, it is necessary for the reader to be familiar with the most basic notions of Differential Geometry and Measure Theory.

In this way, we will start by addressing the general concepts of Riemannian Geometry throughout this next section.

1.1 Riemannian Geometry

1.1.1 Hilbert Manifolds

Before defining Hilbert manifolds, let us recall the definition of Hilbert space as follows:

Definition 1.1. [Hilbert Space] We say that H is a Hilbert Space if H is a normed vector space which satisfies two properties:

- The norm defined on H is induced by an inner product in this way: $\sqrt{\langle \cdot, \cdot \rangle} = \| \cdot \|$.

- Any Cauchy sequence in H converges with respect to above norm to an element of H.

Definition 1.2. [Hilbert Manifold] A Hilbert manifold is a set M and a family of injective mappings $\phi_\alpha : U_\alpha \subset H \to M$ of open sets U_α of H into M such that:

I) $\bigcup_\alpha \phi_\alpha(U_\alpha) = M$

II) for any pair α, β with $\phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta) = W \neq \emptyset$, the sets $\phi_\alpha^{-1}(W)$ and ϕ_β^{-1} are open sets in H and the mappings $\phi_\beta^{-1} \circ \phi_\alpha$ are differentiable.

III) The family $\{(U_\alpha, \phi_\alpha\})$ is maximal relative to the previous conditions.

Items I) and II) described above define what we call a differentiable atlas; condition III) guarantees what we call a maximal atlas — that is, it cannot be expanded without losing its differentiability conditions. Thus, a maximal atlas is the "best" possible collection of charts from the manifold M.

Let $p \in \phi_\alpha(U_\alpha)$. Then, the pair (U_α, ϕ_α) is called a parametrization or system of coordinates of M at the point p; moreover, $\phi_\alpha(U_\alpha)$ is called a coordinate neighborhood at p.

In what follows, we will discuss properties and main results concerning Riemannian manifolds; our discussion will be restricted to the case of Hilbert manifolds, with appropriate observations.

Example 1.1. *An Euclidean space $\mathbb{R}^n$, and a Hilbert Space H equipped with the differentiable structure generated by the identity map is a trivial example of a Hilbert manifold.*

Example 1.2. *Let M be a Hilbert manifold, any open M' of M is a Hilbert manifold and if $(u_\alpha, M_\alpha)_{\alpha \in A}$ of a differentiable atlas for M, its restriction $(u_\alpha | M_\alpha \cap M', M_\alpha \cap M')_{\alpha \in A}$ to M' gives a differentiable atlas for M'.*

Definition 1.3. [Tangent Vector] Let M be a Hilbert manifold. A differentiable function $\alpha : (-\varepsilon, \varepsilon) \to M$ is called a differentiable curve in M. Suppose that $\alpha(0) = m$, where $m \in M$, and let $\mathcal{D}$ be the set of functions on M that are differentiable at point m. The tangent vector to the curve α at $t = 0$ is a function $\alpha'(0) : \mathcal{D} \to \mathbb{R}$ given by:

$$\alpha'(0)f = \left.\frac{d(f \circ \alpha)}{dt}\right|_{t=0}, \quad f \in \mathcal{D}$$

A tangent vector at m is the tangent vector at $t = 0$ of some curve $\alpha(-\varepsilon, \varepsilon) \to M$ with $\alpha(0) = m$. The set of all tangent vectors to M at m will be denoted by $T_m M$.

Definition 1.4. [Diffeomorphism] Let M_1 and M_2 be Hilbert manifolds. A mapping $\varphi : M_1 \to M_2$ is a diffeomorphism if it is differentiable, bijective, and its inverse φ^{-1} is differentiable. The map φ is said to be a local diffeomorphism at $p \in M$ if there exist neighborhoods U of p and V of $\varphi(p)$ such that $\varphi : U \to V$ is a diffeomorphism.

Definition 1.5. [Tangent Bundle] Let M be a Hilbert manifold and let $TM = \{(m, v);\ m \in M,\ v \in T_m M\}$. That is, we can define TM as the union of all the tangent spaces $T_m M$ of the manifold M at each of its points $m \in M$. Moreover, we will denote by $\pi : TM \to M$ the natural projection of the tangent bundle on the manifold. The set TM will be called the Tangent Bundle of M and this space has manifold structure as will see below.

Let $\{(U_\alpha, \varphi_\alpha)\}$ be a maximal differentiable structure on a manifold M. Denote by $(x_1^\alpha, ..., x_i^\alpha, ...)$ the coordinates of U_α and by $\left\{\frac{\partial}{\partial x_1^\alpha}, ..., \frac{\partial}{\partial x_i^\alpha}, ...\right\}$ the associated bases to the tangent spaces of $\varphi_\alpha(U_\alpha)$. For every α, define:

$$\Phi_\alpha : U_\alpha \times H \to TM,$$

by

$$\Phi_\alpha(x_1^\alpha, u_1, ..., x_i^\alpha, u_i, ...) = \left(\varphi_\alpha(x_1^\alpha), u_1 \frac{\partial}{\partial x_1^\alpha}, ..., \varphi_\alpha(x_i^\alpha), u_i \frac{\partial}{\partial x_i^\alpha}, ...\right)$$

where $(u_1, ..., u_i, ...) \in H$.

In a geometric view, the above equality means that we are taking as coordinates of a point $(m, v) \in TM$ the coordinates $x_1^\alpha, ..., x_i^\alpha, ...$ of m together with the coordinates of v in the basis $\left\{\frac{\partial}{\partial x_1^\alpha}, ..., \frac{\partial}{\partial x_i^\alpha}, ...\right\}$.

Now, we will check that $\{(U_\alpha \times H, \Phi_\alpha)\}$ is a differentiable structure on TM.

Since that $\bigcup_\alpha \varphi_\alpha(U_\alpha) = M$ and $(d\varphi_\alpha)_q(H) = T_{\varphi_\alpha(q)}M$, we obtain:

$$\bigcup_\alpha \Phi_\alpha(U_\alpha \times H) = TM,$$

which verifies condition I) of Definition (1.2). Now, let:

$$(m, v) \in \Phi_\alpha(U_\alpha \times H) \ \cap \ \Phi_\beta(U_\beta \times H).$$

Then,

$$(m, v) = (\varphi_\alpha(q_\alpha), d\varphi_\alpha(v\alpha)) = (\varphi_\beta(q_\beta), d\varphi_\beta(v\beta)),$$

where $q_\alpha \in U_\alpha$, $q_\beta \in U_\beta$, $v_\alpha, v_\beta \in \mathbb{R}^n$. Therefore,

$$\begin{aligned}
&\Phi_\beta^{-1} \circ \Phi_\alpha(q_\alpha, v_\alpha) \\
=&\Phi_\beta^{-1}(\varphi_\alpha(q_\alpha), d\varphi_\alpha(v_\alpha)) \\
=&((\varphi_\beta^{-1} \circ \varphi_\alpha)(q_\alpha), d(\varphi_\beta^{-1} \circ \varphi_\alpha)(v_\alpha)).
\end{aligned}$$

Since $\varphi_\beta^{-1} \circ \varphi_\alpha$ is differentiable, $d(\varphi_\beta^{-1} \circ \varphi_\alpha)$ is as well. So, it follows that $\Phi_\beta^{-1} \circ \Phi_\alpha$ is differentiable, which verifies condition II) of Definition (1.2).

Definition 1.6. [Vector Field] A vector field X of a Hilbert manifold M is a correspondence that associates to each point $m \in M$ a vector $X(m) \in T_mM$. In terms of mappings, X is a mapping of M into the tangent bundle TM. We say that the field is differentiable if the mapping $X : M \to TM$ is differentiable.

Let $\psi : U \subset H \to M$ be a parametrization. Then, any vector field X on M can be locally expressed as

$$X(p) = \sum_{i=1}^{\infty} \psi_i(p) \frac{\partial}{\partial x_i}, \tag{1.1}$$

where each $\psi_i : U \to \mathbb{R}$ is the i-th coordinate function defined by ψ, and $\left\{\frac{\partial}{\partial x_i}\right\}$, where $i \in \mathbb{N}$ is the coordinate basis associated with ψ. Moreover, X is differentiable if and only if the coordinate functions ψ_i are differentiable (with respect to the coordinates given by this parametrization).

It is convenient to use the idea suggested by (1.1) above and consider a vector field as a mapping $X : \mathcal{D} \to \mathcal{F}$ from the set $\mathcal{D}$ of differentiable functions on M to the set $\mathcal{F}$ of functions on M, defined as follows:

$$(Xf)(p) = \sum_i \psi_i(p) \frac{\partial f}{\partial x_i}(p), \tag{1.2}$$

where f denotes, by abuse of notation, the expression of f in the parametrization ψ. The function Xf obtained in equation (1.2) does not depend on the choice of parametrization φ.

Definition 1.7 (Riemannian Metric)**.** *We say that a Riemannian metric is given on a manifold M if in each tangent space T_pM, $p \in M$, a inner product $\langle \cdot, \cdot \rangle_p$ is specified which depends smoothly on p, that is, for any two vector fields X and Y, the function $\langle X(p), Y(p) \rangle_p$ is smooth in $p \in M$. A manifold with a Riemannian metric is called a Riemannian manifold.*

Example 1.3. *Let $M = \mathbb{R}^n$ with $\frac{\partial}{\partial x_i}$ identified with $e_i = (0, ..., 1, ..., 0)$. The metric is given by the relation: $\langle e_i, e_j \rangle = \delta_{ij}$. The manifold $\mathbb{R}^n$ is called Euclidean Space of dimension n and the Riemannian Geometry of this space is called the metric Euclidean Geometry.*

Example 1.4 (Product Metric)**.** *Let M and N be Riemannian Manifolds and consider the cartesian product: $M \times N$ with the product structure. Let $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ be the natural projections. We can define on $M \times N$ a Riemannian Metric as follows:*

$$\langle u, v \rangle_{m,n} = \langle d\pi_1 \cdot u, d\pi_1 \cdot v \rangle_m + \langle d\pi_2 \cdot u, d\pi_2 \cdot v \rangle_n,$$

for all $(m, n) \in M \times N$, $u, v \in T_{(m,n)}(M \times N)$

Example 1.5 (Flat Torus). *As a subcase of previous example, consider the torus* $S^1 \times ... \times S^1 = \mathbb{T}^n$ *equipped with a Riemannian structure obtained by choosing the induced Riemannian Metric of* $\mathbb{R}^2$ *on the circle* $S^1 \subset \mathbb{R}^2$ *and then taking the product metric. The n-dimensional torus, denoted by* $\mathbb{T}^n$*, with this metric will be called the flat torus.*

Definition 1.8. [Lie Group] A Manifold G is called a Lie Group if there is an algebraic operation $*$ on G such that G is a group with respect to $*$ and $g * h$ is jointly smooth in $g, h \in G$ as a map $G \times G \to G$.

Definition 1.9. [Left Action of a Lie Group] A left action of a Lie Group G on a manifold M is defined if a certain C^∞-map

$$G \times M \to M, \quad (g, m) \mapsto gm \tag{1.3}$$

is given such that the following hypotheses hold:

(I) for any $g \in G$, the map (1.3) is a diffeomorphism;

(II) $(g * h)m = g(hm)$ for $g, h \in G, m \in M$.

Analogously, a right action of a Lie Group G on a manifold M is the specification of a certain C^∞-map $M \times G \to G$ for $g \in G$ and $m \in M$ which satisfies (I) and $(\overline{II})$ defined below:

$(\overline{II})$ $(g * h)m = h(gm)$ for $g, h \in G, m \in M$.

When the action is given, the notation mg for $g \in G$ and $m \in M$ is used so that $m(g * h) = (mg)h$. Moreover, in this work, we will denote the unit of the Lie Group by e.

Let $g \in G$, we will define two special maps: the right translation and the left translation (or equivalently right shift and left shift). The right translation is the map $R_g : G \to G$ defined by $R_g h = h * g$ and, analogously, the left translation is the map $L_g : G \to G$ defined by $L_g h = g * h$ for all $h \in G$. From definition of a Lie Group above (1.8), we have that the maps R_g and L_g are smooth.

Definition 1.10. The vector field on G obtained by right translation of a vector $X \in T_eG$ at all points of G is called right-invariant vector field generated by X. In this way, the vector field on G obtained by left translation of a vector $X \in T_eG$ is called left-invariant.

1.1.2 Connection

Definition 1.11. [Affine Connection] Let $\mathcal{X}(M)$ be the set of C^∞-vector fields on M and $\mathcal{D}(M)$ be the set of C^∞-functions on M. A affine connection ∇ on a manifold M is a mapping:

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$

such that:

I) $\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$

II) $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$

III) $\nabla_X(fY) = f\nabla_X Y + X(f)Y$,

in which $X, Y, Z \in \mathcal{X}(M)$ and $f, g \in \mathcal{D}(M)$.

Remark 1.1. *Throughout this work, it will be common for us to use the term connector when referring to the map defined by a connection. In particular, when working in local coordinates, we will call this mapping the local connector.*

Proposition 1.1. *Let M be a differentiable manifold with an affine connection ∇. There exists a unique correspondence which associates to a vector field V along the differentiable curve $c : I \subset \mathbb{R} \to M$ another vector field $\frac{DV}{dt}$ along c, called the covariant derivative of V along c, such that:*

I) $\frac{D}{dt}(V + W) = \frac{DV}{dt} + \frac{DW}{dt}$.

II) $\frac{D}{dt}(fV) = \frac{df}{dt}V + f\frac{DV}{dt}$, *where V is as vector field along c and f is a differentiable function on $I \subset \mathbb{R}$.*

III) If V is induced by a vector field $Y \in \mathcal{X}(M)$, in others words, $V(t) = Y(c(t))$, then $\frac{DV}{dt} = \nabla_{dc/dt} Y$.

Proof. Suppose there is a correspondence satisfying the proposition conditions. Let $\varphi : U \subset H \to M$ be a system of coordinates with $c(I) \cap \varphi(U) \neq \emptyset$ and let $(x_1, x_2, ..., x_i, ...)$ be the local expression of the curve $c(t)$, where $t \in I$. Furthermore, denote by X_i the derivative in the x_i direction, that is, $X_i = \frac{\partial}{\partial x_i}$. In this way, we can express the vector field V locally as $\sum_j v^j X_j$ where $v^j = v^j(t)$ and $X_j = X_j(c(t))$.

By I) and II), we have:

$$\frac{DV}{dt} = \sum_j \frac{dv^j}{dt} + \sum_j \frac{DX_j}{dt}.$$

By III) e I) of the definition 1.11, we have:

$$\begin{aligned} \frac{DX^j}{dt} &= \nabla_{dc/dt} X_j = \nabla_{\sum \frac{dx_i}{dt} X_i} X_j \\ &= \sum_i \frac{dx_i}{dt} \nabla_{X_i} X_j \end{aligned}$$

where $i \in \mathbb{N}$.

Therefore,

$$\frac{DV}{dt} = \sum_j \frac{dv^j}{dt} X_j + \sum_{i,j} \frac{dx_i}{dt} \nabla_{X_i} X_j \tag{1.4}$$

The above expression shows that if there is a correspondence satisfying the conditions of the proposition, then that correspondence is unique.

To prove existence, define $\frac{DV}{dt}$ in $\varphi(U)$ by 1.4. Let $\psi(W)$ be another coordinate neighborhood such that $\psi(W) \cap \varphi(U) \neq \emptyset$ and we define $\frac{DV}{dt}$ in $\psi(W)$ by equation 1.4. Thus, the definitions coincide at the intersection $\psi(W) \cap \varphi(U)$, by the uniqueness of $\frac{DV}{dt}$ in $\varphi(U)$. Finally, it follows from the definition that we can extend it over the entire manifold M and this concludes the proof. □

Definition 1.12 (Parallel Vector Field)**.** *Let M be a differentiable manifold with an affine connection ∇. A vector field V along a curve $c : I \to M$ is called parallel when $\frac{DV}{dt} = 0$, for all $t \in I$.*

Definition 1.13 (Connection Compatible)**.** *We say that a connection ∇ and a Riemannian Metric $\langle \cdot, \cdot \rangle$ are compatible if the metric admits a uniform Riemannian atlas such that, in every chart defined on a ball $V[m, r]$, the local connector $\Gamma_{m'}(X, X)$ at all $m' \in V[m, r]$ is uniformly bounded in the norm generated by the metric, as a quadratic operator of X, by a certain constant $c > 0$ independent of the choice of chart and ball.*

Definition 1.14 (Symmetric Connection)**.** *An affine connection ∇ on a Hilbert manifold M is said to be symmetric when:*

$$\nabla_X Y - \nabla_Y X = [X, Y] \quad \textit{for all} \quad X, Y \in \mathcal{X}(M). \tag{1.5}$$

Theorem 1.1 (Levi-Civita)**.** *Given a Hilbert manifold M, there exists a unique affine connection ∇ on M satisfying the conditions:*

I) ∇ is symetric.

II) ∇ is compatible with the Riemannian Metric.

Proof. Suppose that exists a connection ∇. Then, it holds that:

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \tag{1.6}$$

$$Y\langle Z, X\rangle = \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle, \tag{1.7}$$

$$Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle x, \nabla_Z Y\rangle. \tag{1.8}$$

Adding equations (1.6) and (1.7) and subtracting (1.8), using the symmetry of connection ∇, we obtain:

$$\begin{aligned}&X\langle Y, Z\rangle + Y\langle Z, X\rangle + Z\langle X, Y\rangle \\ =&\langle [X, Z], Y\rangle + \langle [Y, Z], X\rangle + \langle [X, Y], Z\rangle + 2\langle Z, \nabla_Y X\rangle\end{aligned}$$

Therefore,

$$\begin{aligned}\langle Z, \nabla_Y X\rangle = \frac{1}{2}\{&X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ &- \langle [X, Z]Y\rangle - \langle [Y, Z], X\rangle - \langle [X, Y], Z\rangle\}.\end{aligned} \tag{1.9}$$

The expression (1.9) shows that the connection ∇ is uniquely determined from the metric $\langle \cdot, \cdot \rangle$. Hence, if it exists, it will be unique.

To prove existence, define the connection ∇ by (1.9) and note that this connection satisfies the desired conditions. □

Remark 1.2. The connection ∇ as given by above theorem is called as the Levi-Civita (or Riemannian) connection on M.

Now, we will partially rewrite what was shown above using a coordinate system (U, φ). Let be functions Γ_{ij}^k defined on U as follow:

$$\nabla_{X_i} X_j = \sum_k \Gamma_{ij}^k X_k$$

will be called the coefficients of the connection ∇ on U or the Christoffel Symbols of the connection. From equation (1.9), it follows that:

$$\sum_l \Gamma_{ij}^l \langle X_l, X_k\rangle = \frac{1}{2}\left\{\frac{\partial}{\partial x_i}\langle X_j, X_k\rangle + \frac{\partial}{\partial x_j}\langle X_k, X_i\rangle - \frac{\partial}{\partial x_k}\langle X_i, X_j\rangle\right\}.$$

Since the matrix $\langle X_k, X_m\rangle$ has an inverse, denoted by by $\langle X^k, X^m\rangle$, satisfying $\langle X_l, X_k\rangle \langle X^k, X^m\rangle = \delta_l^m$. We obtain that:

$$\Gamma_{ij}^m = \frac{1}{2}\sum_k \left\{\frac{\partial}{\partial x_i}\langle X_j, X_k\rangle + \frac{\partial}{\partial x_j}\langle X_i, X_k\rangle - \frac{\partial}{\partial x_k}\langle X_i, X_j\rangle\right\}\langle X^k, X^m\rangle. \tag{1.10}$$

The above equation given by (1.10) is a classical expression for the Christoffel Symbols of the Riemannian Connection in terms of the $\langle X_i, X_j\rangle$ given by the metric.

As constructed above, it is clear from the Remark 1.1 that the Christofell symbols are an example of a local connector, representing the connection analytically on a given chart.

1.1.3 Geodesics

Definition 1.15 (Geodesic)**.** *A parametrized curve $\gamma : I \to M$ is called a geodesic at $t_0 \in I$ if $\frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = 0$ at the point t_0.*

If γ is a geodesic at t, for all $t \in I$, we say that the curve γ is a geodesic. If $[a, b] \subset I$ and $\gamma : I \to M$ is a geodesic, then the restriction of γ to interval $[a, b]$ is called a geodesic segment joining $\gamma(a)$ to $\gamma(b)$.

If $\gamma : I \to M$ is a geodesic, then:

$$\frac{d}{dt}\left\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \right\rangle = 2\left\langle \frac{D}{dt}\frac{d\gamma}{dt}, \frac{d\gamma}{dt} \right\rangle = 0$$

that is, the length of the tangent vector $\frac{d\gamma}{dt}$ is constant.

In what follow, we will determine the local equations satisfied by a geodesic curve γ in a system of coordinate (U, φ) about $\gamma(t_0)$. In U, a curve γ give by:

$$\gamma(t) = (x_1(t), ..., x_i(t), ...)$$

will be a geodesic if and only if:

$$0 = \frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = \sum_k \left(\frac{d^2 x_k}{dt^2} + \sum_{i,j} \Gamma_{ij}^k \frac{dx_i}{dt}\frac{dx_j}{dt} \right) \frac{\partial}{\partial x^k}.$$

In this way, the second order system:

$$\frac{d^2 x_k}{dt^2} + \sum_{i,j} \Gamma_{ij}^k \frac{dx_i}{dt}\frac{dx_j}{dt} \tag{1.11}$$

in which $k \in \mathbb{N}$ yields the desired equations. To analyze the above system 1.11, it is convenient to consider the tangent bundle TM. This is the set of pairs (q, v), where $q \in M$ and $v \in T_qM$. If (U, φ) is a system of coordinates on the manifold M, then any vector in the tangent space T_qM, where $q \in \varphi(U)$, can written as:

$$\sum_{i=1}^{\infty} y_i \frac{\partial}{\partial x_i}.$$

So, considering $(x_1, y_1, ..., x_i, y_i, ...)$ as coordinates of (q, v) in TU, as we saw previously in the Definition 1.5, TM has differentiable structure. Note that $TU = U \times H$, that is, the tangent bundle is locally a product. Moreover, the canonical projection defined as $\pi : TM \to M$ given by $\pi(q, v) = q$ is differentiable.

This way, any differentiable curve $t \to \gamma(t)$ in the manifold M determines a curve $t \to (\gamma(t), \frac{d\gamma}{dt}(t))$ in TM. If γ is a geodesic, then, on TU, the curve gives by:

$$t \to \left(x_1(t), \frac{dx_1(t)}{dt}, ..., x_i(t), \frac{dx_i(t)}{dt}, ... \right)$$

satisfies the system:

$$\begin{cases} \frac{dx_k}{dt} = y_k \\ \frac{dy_k}{dt} = -\sum_{ij} \Gamma_{ij}^k y_i y_j \end{cases} \tag{1.12}$$

where $k \in \mathbb{N}$ in terms of coordinates $(x_1, y_1, ..., x_i, y_i, ...)$ on TM.

Finally, the second order system described in 1.11 on U is equivalent to the first order system 1.12 on TU above.

Lemma 1.1. *There exists a unique vector field G on the space TM whose trajectories are of the form $t \to (\gamma(t), \gamma'(t))$ where γ is a geodesic curve on M.*

Proof. Initially, we will prove the uniqueness of G, supposing its existence. Consider a system of coordinates (U, φ) on the manifold M. From the hypothesis, the trajectories of the vector field G are of the form: $t \to (\gamma(t), \gamma'(t))$, where γ is a geodesic on M. In this way, it follows that $t \to (\gamma(t), \gamma'(t))$ is a solution of the system shown in (1.12) and results from the uniqueness of the trajectories of a such system that if G exists, then it is unique. Now, to prove the existence, define G locally through the system (1.12) and, by uniqueness, follows that G is well-defined on TM. □

Definition 1.16. *Let X be a vector field on a Hilbert manifold M, and let $p \in M$. Then there exist a neighborhood $U \subset M$ of p, an interval $(-\delta, \delta)$, where $\delta > 0$, and a differentiable mapping $\varphi : (-\delta, \delta) \times U \to M$ such that the curve $t \to \varphi(t, q)$, $t \in (-\delta, \delta)$ and $q \in U$, is the unique curve which satisfies:*

$$\begin{cases} \frac{\partial \varphi}{\partial t} = X(\varphi(t, q)) \\ \varphi(0, q) = q \end{cases}$$

The curve given by $\alpha : (-\delta, \delta) \to M$ which satisfies the conditions $\alpha'(t) = X(\alpha(t))$ and $\alpha(0) = q$ is called a trajectory of the vector field X that passes through point q at time $t = 0$.

Remark 1.3. *Commonly, the notation $\varphi(t, q)$ is used in a simplified form as $\varphi_t(q)$ when we fix a time t. The curve $\varphi_t : U \to M$ is called the local flow of the vector field X.*

Definition 1.17 (Geodesic Flow)**.** *The vector field G defined above is called the geodesic field on TM and its flow is called the geodesic flow on TM.*

In the following results, we will denote a geodesic $\gamma(t, q, v)$ that depends of the triple (t, q, v) defined over the Cartesian product $[-\delta, \delta] \times \mathcal{U}$, with $\delta > 0$, in which $\mathcal{U}$ is a open set in TU, where (U, φ) is a system of coordinates at $p \in M$, such that $\{(q, v); q \in V, v \in T_q M, |v| < \varepsilon\}$ with $\varepsilon > 0$ and V is a neighborhood of $p \in M$.

Lemma 1.2 (Homogeineity of a Geodesic)**.** *If the geodesic $\gamma(t, q, v)$ is defined on the interval $(-\delta, \delta)$, then the geodesic $\gamma(t, q, av)$, with $a \in \mathbb{R}$, $a > 0$, is defined on the interval $\left(-\frac{\delta}{a}, \frac{\delta}{a}\right)$ and*

$$\gamma(t, q, av) = \gamma(at, q, v).$$

Proof. Define the curve:

$$h : \left(-\frac{\delta}{a}, \frac{\delta}{a}\right) \to M$$

$$h(t) = \gamma(at, q, v)$$

Which we have that $h(0) = \gamma(0, q, v) = q$, that is, the curve $h(t)$ passes through the point q at time $t = 0$. Moreover, note that:

$$h'(t) = \frac{d}{dt}\gamma(at, q, v) = a\,\gamma'(at, q, v).$$

Evaluating at $t = 0$, we obtain: $h'(0) = a\,\gamma'(0, q, v) = av$

In addition, since $h'(t) = a\gamma(at, q, v)$, then:

$$\begin{aligned} \frac{D}{dt}\big(h'(t)\big) &= \frac{D}{dt}\big(a\,\gamma'(at, q, v)\big) \\ &= a\,\frac{D}{dt}\big(\gamma'(at, q, v)\big) \\ &= a^2\,\gamma''(at, q, v). \end{aligned}$$

Since the curve γ is a geodesic, then: $\gamma''(at, q, v) = \frac{D}{dt}\big(\gamma'(at, q, v)\big) = 0$. We extend $h'(t)$ to a neighborhood of $h(t)$ in M, therefore, h is a geodesic passing through q with velocity av at the instant $t = 0$. By uniqueness, we obtain:

$$h(t) = \gamma(at, q, v) = \gamma(t, q, av).$$

□

Proposition 1.2. *Given $p \in M$ there exist an open set $V \subset M$, $p \in V$, real numbers $\delta > 0$ and $\epsilon_1 > 0$, and a C^∞ mapping:*

$$\gamma : (-\delta, \delta) \times \mathcal{U}$$

where $\mathcal{U} = \{(q, v); q \in V, v \in T_qM, |v| < \epsilon_1\}$, such that the curve $t \mapsto \gamma(t, q, v)$ with $t \in (-\delta, \delta)$, is the unique geodesic of M which, at the instant $t = 0$, passes through point q with velocity v, for each $q \in V$ and for each $v \in T_qM$ with $|v| < \epsilon_1$.

Proof. Consider $p \in M$, by hypothesis there exists a neighborhood V of p in M and real numbers $\delta > 0$ and $\epsilon_1 > 0$ such that, for each $(q, v) \in \mathcal{U} = \{(q, v) : q \in$

$V,\ v \in T_qM,\ |v| < \epsilon_1\}$, we have:

$$\gamma(0,q,v) = q,$$
$$\frac{D}{dt}\gamma'(t,q,v) = 0,$$
$$\gamma'(0,q,v) = v,$$

where $t \in (-\delta, \delta)$.

We want to prove that the curve $\gamma(t,q,v)$ is the only geodesic under these conditions. By the Homogeneity Lemma 1.2, for any $a > 0$ such that $|av| < \epsilon_1$ we have that $\gamma(t,q,a\,v) = \gamma(a\,t,q,v)$, where $t \in \left(-\frac{\delta}{a}, \frac{\delta}{a}\right)$.

Therefore, the initially defined map $\gamma(\cdot)$ is C^∞-smooth and, by its uniqueness, for each $(q,v) \in \mathcal{U}$ the curve $t \mapsto \gamma(t,q,v)$ is the only geodesic that satisfies the given initial conditions. □

Proposition 1.3. *Let be a Riemannian Manifold M and let $p \in M$, there exist a neighborhood V of p in M, a number $\epsilon > 0$ and a C^∞ mapping $\gamma : (-2,2)\times\mathcal{U} \to M$, where $\mathcal{U} = \{(q,w) \in TM;\ q \in V;\ w \in T_qM,\ |w| < \epsilon\}$ such that $t \mapsto \gamma(t,q,w)$, $t \in (-2,2)$ is the unique geodesic of M which, at the instant $t = 0$, passes through q with velocity w, for every $q \in V$ and for every $w \in T_qM$, with $|w| < \epsilon$.*

Proof. The geodesic $\gamma(t,q,v)$ give by Proposition 1.2 is defined for $|t| < \delta$ and for $|v| < \epsilon_1$. According the Lemma of Homogeneity 1.2, we have that the geodesic $\gamma(t,q,\frac{\delta v}{2})$ is defined for $|t| < 2$.

Taking $\epsilon < \frac{\delta\epsilon_1}{2}$, we obtain that the geodesic $\gamma(t,q,w)$ is defined for $|t| < 2$ and $|w| < \epsilon$. □

The previous proposition permit us to introduce an important concept called exponential map. Let $p \in M$ and $\mathcal{U} \subset TM$ be an open set given by Proposition (1.3), then the map $\exp : \mathcal{U} \to M$ given by:

$$\exp_q(v) = \exp(q,v) = \gamma(1,q,v) = \gamma\left(|v|, q, \frac{v}{|v|}\right)$$

where $(q,v) \in \mathcal{U}$, is called the exponential map on $\mathcal{U}$.

In general, we will restrict of the exponential map to an open subset of the tangent space T_qM, in other words, we will define:

$$\begin{aligned}\exp_q : B[0,\varepsilon] \subset T_qM \to M \\ \exp_q(v) = \exp(q,v)\end{aligned}$$

where $B[0,\varepsilon]$ denotes the open ball with center at the origin 0 of T_qM and of radius ε. Notices that $\exp_q(0) = q$ and map $\exp_q$ is differentiable.

In geometric terms, the $\exp(q,v)$ is a point of the manifold M which is obtained by going out the length equal to $|v|$, starting in the point q, along a geodesic which passes through q with velocity equal to $\frac{v}{|v|}$.

Proposition 1.4. *Let $p \in M$, there exists an $\varepsilon > 0$ such that the exponential map $\exp_q : B[0,\varepsilon] \subset T_qM \to M$ is a diffeomorphism of $B[0,\varepsilon]$ onto an open subset of M.*

Proof. Initially, we will calculate the differential of the exponential map and then we will apply the time $t = 0$. Thus:

$$\begin{aligned} d(\exp_q)_0(v) &= \frac{d}{dt}(\exp_q(tv)) \\ &= \frac{d}{dt}(\gamma(1,q,tv)) \\ &= \frac{d}{dt}(\gamma(t,q,v)) \end{aligned}$$

Evaluating at $t = 0$, we obtain:

$$\frac{d}{dt}(\gamma(0,q,v)) = \gamma'(0,q,v)) = v.$$

Thus, we have that $d(\exp_q)_0$ is the identity of the tangent space T_qM and it follows by the inverse function theorem that the map $\exp_q$ is a local diffeomorphism in a neighborhood of 0. □

If $\exp_p$ is a diffeomorphism of a neighborhood V of the origin in T_pM, the neighborhood gives by $\exp_p(V) = U \subset M$ is called a normal neighborhood or normal chart.

Definition 1.18. [Geodesic Frame] Let M be a Riemannian Manifold, let $p \in M$ and $U \subset M$ be a neighborhood of p, consider the vector fields $E_i \in \mathcal{X}(U)$, where $i \in \mathbb{N}$ and orthonormal at each point of U such that, at point p, $\nabla_{E_i} E_j(0) = 0$. The family of vector fields E_i, in this conditions, is called a local geodesic frame at p.

Definition 1.19. [Divergence] Let M be a Riemannian Manifold, $X \in \mathcal{X}(M)$, $f \in \mathcal{D}(M)$ and E_i, where $i \in \mathbb{N}$ is a local geodesic frame at $p \in M$, we define divergence as:

$$\text{div}X(p) = \sum_i E_i(f_i)(p)$$

where $X = \sum_i f_i E_i$.

Next, we will define the concept of geodesic spray, which will be widely used in the results and theorems of the next chapter.

Definition 1.20. *A vector field $Y(t,(m,X))$ on the tangent bundle TM is called a special vector field on TM or a second order differential equation on M if at every point $(m,X) \in TM$ the equality:*

$$T\pi Y(t,(m,X)) = X_m$$

holds where $\pi : TM \to M$ is the natural projection of TM onto M.

Definition 1.21. Let a connection ∇ be given on a manifold M. Given a point $(m,X) \in TM$, the mapping $T\pi$ is a linear isomorphism of $\nabla_{(m,X)}$ onto $T_m M$. Consequently in $\nabla_{(m,X)}$ there is a unique vector $\mathcal{Z}_{(m,X)}$ such that:

$$T\pi \mathcal{Z}_{(m,X)} = X_m. \tag{1.13}$$

The vector field described above is called the geodesic spray of the connection ∇.

1.2 Stochastic Analysis on Manifolds

In this section, our focus will be the Stochastic Process and their concepts associated, having as a priority the Stochastic Process which occur in Manifolds. Moreover, as mentioned before, we will suppose that reader is familiarized with elementary concepts of Measure and Integration, such as random variables, convergence, and others.

Definition 1.22. [Time] Let $\xi(t)$, for $t \in [0,T]$, be a stochastic process in $\mathbb{R}^n$, defined on a probability space $(\Omega, \mathcal{B}, P)$. Then, for each $t \in [0,T]$, $\xi(t)$ determines three families of σ-subalgebras of $\mathcal{B}_t$:

i) "past" $\mathcal{P}_t^{\xi}$, generated by pre-images of Borel sets in $\mathbb{R}^n$ by all mappings $\xi(s) : \Omega \to \mathbb{R}^n$ for $0 < s < t$;

ii) "future" $\mathcal{F}_t^{\xi}$, generated by pre-images of Borel sets in $\mathbb{R}^n$ by all mappings $\xi(t) : \Omega \to \mathbb{R}^n$ for $t < s < T$;

iii) present ("now") $\mathcal{N}_t^{\xi}$, generated by the mapping $\xi(t)$.

We assume that all of these families of σ-subalgebras are completes, that is, contain all sets of probability $P = 0$.

Definition 1.23. Let $\mathcal{B}_t$ be a filtration of the σ-algebra $\mathcal{B}$, a random process $\mathbb{A}(t)$ is said to be adapted with respect to a filtration $\mathcal{B}_t$ if $\mathbb{A}(t)$ is measurable with respect to $\mathcal{B}_t$ for every t.

1.2.1 Stochastic Integrals

Our objective throughout this section is to review the most essential concepts about stochastic integrals in Itô form and in Stratonovich form, making the necessary connections between these two topics with the aim of clarifying the relationships that exist between stochastic differential equations in Itô and Stratonovich forms, respectively.

First, consider a positive constant T finite. Let $\mathbb{A} : [0,T] \times \Omega \to \mathcal{L}(\mathbb{R}^n, H)$ be a random operator function, in other words, $\mathbb{A}(t)$ is a random linear operator from the n-dimensional space $\mathbb{R}^n$ to a Hilbert space H. Moreover, let us consider a Wiener Process $w(t)$ with respect to filtration $\mathcal{B}_t$ with values in $\mathbb{R}^n$.

To define the Itô Integral of $\mathbb{A}(t)$, choose a partition $\mathcal{P} := (0 = t_0 < t_1 < ... < t_q = T)$ of the time interval $[0, T]$ and consider the integral sum given below:

$$\sum_{i=0}^{q-1} \mathbb{A}(t_i)(w(t_{i+1} - w(t_i)). \tag{1.14}$$

The limit (if it exists) of the above sum, as the length $|\mathcal{P}| \to 0$, is called the H-valued Itô Integral of $\mathbb{A}(t)$ and will be denoted by:

$$\int_0^t \mathbb{A}(t)dw(t)$$

However, under certain boundedness hypothesis, the Itô Integral does exist as the $L^2(\Omega, H)$-limit of the integral sums when we have that $\mathbb{A}(t)$ is adapted with respect to filtration $\mathcal{B}_t$. In particular, it exists if the entries $\mathbb{A}_i^j(t)$ of operator $\mathbb{A}(t)$ satisfy the equality:

$$P\left\{\omega \in \Omega \;\middle|\; \int_0^T \left(\mathbb{A}_i^j\right)^2(t,\omega)dt < \infty\right\} = 1 \tag{1.15}$$

where $i \in \mathbb{N}$ correspond to the lines of the operator $\mathbb{A}(t)$ (interpreted as a matrix) that are indexed by natural numbers (since H has an infinite basis) and j correspond to the columns that vary from 1 to n (since Brownian motion is n-dimensional).

Now, considering the integral sums given by:

$$\sum_{i=0}^{q-1} \frac{\mathbb{A}(t_{i+1}) + \mathbb{A}(t_i)}{2} \cdot (w(t_{i+1}) - w(t_i)), \tag{1.16}$$

we have that the limit of these sums (if it exists) is the called Stratonovich Integral denoted as:

$$\int_0^t \mathbb{A}(s) \circ dw_s \tag{1.17}$$

Remark 1.4. The differentials dw and $\circ dw$ that appear in the previous definitions of Itô and Stratonovich integrals are called the forward and symmetric differentials, which refer to location of t in time interval $[t_i, t_{i+1}]$ where the operator $\mathbb{A}$ is evaluated.

Now, we will show a formula that relates the Itô and Stratonovich integrals. By the definition of the Stratonovich integral, we get:

$$\sum_S^q = \sum_I^q + \frac{1}{2}\sum_{i=0}^{q-1}(\mathbb{A}(t_{i+1}) - \mathbb{A}(t_i))(w(t_{i+1} - w(t_i))) \tag{1.18}$$

where $\sum_I^q$ is the Itô Integral sum presented in (1.14). The limit of the second sum on the right-hand side is a second order integral in $d\mathbb{A}$ and dw, denoted simply by $\int_0^t d\mathbb{A}dw$. In this way, we obtain:

$$\int_0^t \mathbb{A}(\tau) \circ dw(\tau) = \int_0^t \mathbb{A}(\tau)dw(\tau) + \frac{1}{2}\int_0^t d\mathbb{A}(\tau)dw(\tau) \tag{1.19}$$

Definition 1.24. An Itô Process is a process $\xi(t)$ of the form:

$$\xi(t) = \xi(0) + \int_0^t a(s)ds + \int_0^t \mathbb{A}(s)dw(s)$$

where $a(t)$ is a process with sample paths almost surely having bounded variation and $\mathbb{A}$ is defined according to (1.14).

Just for simplicity, we will sometimes denote the differential $dw(s)$ as dw_s for all $s \in [0, T]$. The same goes for X_t which will be used in some situations instead of $X(t)$. Next, we will present one of the most important base results for the theory developed in this master's thesis, the Itô's Formula.

Theorem 1.2 (Itô's Formula)**.** *Let H_1 and H_2 be Hilbert Spaces, W_t be a n-*

dimensional Brownian Motion and $X_t \in H_1$ *a Stochastic Process given by:*

$$X_t = X_0 + \int_0^t a(s)ds + \int_0^t \mathbb{A}(s)dW_s,$$

such that for each direction $i \in \mathbb{N}$*, it is holds:*

$$X_t^i = X_0^i + \int_0^t a_i(s)ds + \sum_{l=1}^{n} \int_0^t \mathbb{A}_i^l(s)dW_s^l$$

in which $a(s)$ *is an* H_1*-valued Stochastic Process and* $\mathbb{A}(s)$ *is a* $L(\mathbb{R}^n, H_1)$*-valued Stochastic Process, and let* $F = (F_1(t,x), ..., F_d(t,x), ...)$ *be a* $C^2([0,t] \times H_1, H_2)$ *application, so we get for each* $d \in \mathbb{N}$*:*

$$\begin{aligned} F_d(t, X_t) = &F_d(0, X_0) + \int_0^t \frac{\partial F_d}{\partial s}(s, X_s)ds + \sum_{i=1}^{\infty} \int_0^t \frac{\partial F_d}{\partial x_i}(s, X_s)a_i(s)ds \\ &+ \sum_{i=1}^{\infty}\sum_{l=1}^{n} \int_0^t \frac{\partial F_d}{\partial x_i}(s, X_s)\mathbb{A}_i^l(s)dW_s^l \\ &+ \frac{1}{2}\sum_{i=1}^{\infty}\sum_{h=1}^{\infty}\sum_{l=1}^{n} \int_0^t \frac{\partial^2 F_d}{\partial x_i \partial x_h}(s, X_s)\mathbb{A}_i^l(s)\mathbb{A}_h^l(s)ds \end{aligned} \tag{1.20}$$

Proof. The proof is extremely extensive and has been added to the appendix. □

To simplify some representations of the integral that contains the second derivative of $F(\cdot)$ (described above), it will be convenient to use the following notations as equivalent when necessary:

$$trF''(\mathbb{A}(s), \mathbb{A}(s)) = \sum_{l=1}^{n} F''(\mathbb{A}(s)e_l, \mathbb{A}(s)e_l) \tag{1.21}$$

where $e_1, ..., e_n$ is an arbitrary orthonormal frame in $\mathbb{R}^n$. And, equivalently,

$$\frac{1}{2}\int_0^t F''(\mathbb{A}(s)dw_s, \mathbb{A}(s)dw_s) = \int_0^t \frac{1}{2}trF''(\mathbb{A}(s), \mathbb{A}(s))ds \tag{1.22}$$

A classic result in linear algebra ensures that the trace introduced above does not depend on the choice of orthonormal frame e_1, ..., e_n in $\mathbb{R}^n$.

Definition 1.25. [Stratonovich Process] A process is called a Stratonovich Process if it is of the form:

$$\xi(t) = \xi(0) + \int_0^t a(s)ds + \int_0^t \mathbb{A}(s) \circ dw_s \tag{1.23}$$

where the integral with the operator $\mathbb{A}(t)$ is constructed as seen in (1.17).

If $F(t, x)$ is a smooth mapping as above, then:

$$F(\xi(t)) = F(\xi(0)) + \int_0^t \left[\frac{\partial F}{\partial t} + F'(a(s))\right] ds + \int_0^t F'(\mathbb{A}(s)) \circ dw_s. \tag{1.24}$$

Itô's formula for the Stratonovich Process will not be demonstrated, as it is similar to the main demonstration of Itô's Formula presented previously and left in the appendix.

Definition 1.26. An Itô Process $\xi(t)$ is called a diffusion-type process if both $a(t)$ and $\mathbb{A}(t)$ are adapted with respect to the past filtration $\mathcal{P}_t^\xi$ of $\xi(\cdot)$ and the Wiener process $w(t)$ is adapted to $\mathcal{P}_t^\xi$.

The diffusion type process exist as solutions of the so-called Itô diffusion type equation as will be presented below.

1.2.2 Stochastic Differential Equations

Let on the Hilbert Manifold M a time dependent vector field $a(t, x)$ and a time dependent field of linear operators $\mathbb{A}(t, x)$. A Stochastic Differential Equation in the Itô form (or Itô SDE) is an integral equation as:

$$\xi(t) = \xi(0) + \int_0^t a(\tau, \xi(\tau))d\tau + \int_0^t \mathbb{A}(\tau, \xi(\tau))dw_\tau \tag{1.25}$$

where the integral with operator $\mathbb{A}(\tau, \xi(\tau))$ on the right-hand side is a Itô Integral. Usually, the equation can be written in the differential form as below:

$$d\xi(t) = a(t, \xi(t))dt + \mathbb{A}(t, \xi(t))dw_t \tag{1.26}$$

On the other hand, a Stochastic Differential Equation in Stratonovich form (or equivalently Stratonovich SDE) is the integral equation presented below which contains the Stratonovich integral:

$$\xi(t) = \xi(0) + \int_0^t a(\tau, \xi(\tau))ds + \int_0^t \mathbb{A}(\tau, \xi(\tau)) \circ dw_\tau \tag{1.27}$$

which can be written in the reduced differential form:

$$d\xi(t) = a(t, \xi(t))dt + \mathbb{A}(t, \xi(t)) \circ dw_t \tag{1.28}$$

Equations of the form present in (1.25) are often called diffusion equations, the reason is clarified in the definition below that describes the diffusion coefficient.

Definition 1.27. The equation of Itô type:

$$d\xi(t) = a(t, \xi(t))dt + \mathbb{A}(t, \xi(t))dw(t) \tag{1.29}$$

is called a diffusion type stochastic differential equation.

The equation shows above in (1.29) is a reduced form to the integral expression:

$$\xi(t) = \xi(0) + \int_0^t a(s, \xi(s))ds + \int_0^t \mathbb{A}(s, \xi(s))dw_s \tag{1.30}$$

Notice that equation present in (1.26) is a particular case of (1.30).

Moreover, we shall often require that the mappings $a(t, x(t))$ and $\mathbb{A}(t, x(t))$ are jointly continuous. Sometimes, it will be necessary to consider equations with random coefficients, that is, coefficients depending on $\omega \in \Omega$.

Definition 1.28. [Strong Solution] The equation (1.29) is said to have a strong solution if for every Wiener Process $w(t)$ on a probability space and a adapted to a filtration $\mathcal{B}_t$, there exists a stochastic process $\xi(t)$ on the same probability space as $w(t)$ and adapted with respect $\mathcal{B}_t$, such that for $\xi(t)$ and $w(t)$ a.s. for every t in some interval, the equality (1.30) is fulfilled.

Definition 1.29. [Weak Solution] The equation (1.29) is said to have a weak solution $\xi(t)$ if there exist a probability space $(\Omega, \mathcal{F}, P)$, a filtration $\mathcal{B}_t$, a process $\xi(t)$ in $\mathbb{R}^n$ adapted with respect to $\mathcal{B}_t$ and a Wiener Process $w(t)$ in $\mathbb{R}^n$ adapted to $\mathcal{B}_t$ such that for $\xi(t)$ and $w(t)$ a.s. for every t in some interval, the equality (1.30) is fulfilled.

Definition 1.30. [Strongly and Weakly Unique] A strong solution is said to be strongly unique if any two strong solutions coincide almost surely. Analogously, a weak solution is called weakly unique if for any two weak solutions, the measures corresponding to them on the path space coincide.

Definition 1.31. [Itô Vector Field] Let M be a Hilbert Manifold, the pair described as $(a(t,m), \mathbb{A}(t,m))$, where $a(t,m)$ is a vector field on M and $\mathbb{A}(t,m)$ is a field of linear operators $\mathbb{A}(t,m) : \mathbb{R}^n \to T_mM$ sending a certain Euclidean Space $\mathbb{R}^n$ to the tangent spaces to M is called an Itô Vector Field.

Let $w(t)$ be a Wiener Process in $\mathbb{R}^n$ and $(a(t,m), \mathbb{A}(t,m))$ be an Itô vector field on the manifold M. The equation below is called a Stochastic Differential Equation in Stratonovich Form on M given by Itô vector field $(a(t,m), \mathbb{A}(t,m))$:

$$d\xi(t) = a(t, \xi(t))dt + \mathbb{A}(t, \xi(t)) \circ dw(t) \tag{1.31}$$

This means that in every chart on the manifold M, the solution $\xi(t)$ satisfies the equation:

$$\xi(t) = \xi(0) + \int_0^t a(s, \xi(s))ds + \int_0^t \mathbb{A}(s, \xi(s)) \circ dw(s) \tag{1.32}$$

previously built and presented in equation (1.27).

Example 1.6. *Let a filtration $\mathcal{B}_t$ of the σ-algebra $\mathcal{B}$, where $t \in [0, T]$ with $T > 0$ to which a Wiener Process $w(t)$ in $\mathbb{R}^n$ is adapted, a stochastic process $a(t)$ in a*

Hilbert Space H and a stochastic process $\mathbb{A}(t)$ with values in the space of linear mappings from $\mathbb{R}^n$ to H that are adapted with respect to $\mathcal{B}_t$ and having almost surely continuous sample trajectories, and the field of linear operators $E_m : \mathbb{R}^n \to T_mM$, smooth in $m \in M$. The pair given by $(E_m a(t), E_m \mathbb{A}(t))$ on M is a random Itô vector field and it generates the stochastic differential equation:

$$d\xi(t) = E_{\xi(t)}a(t)dt + E_{\xi(t)}\mathbb{A}(t) \circ dw_t \tag{1.33}$$

Definition 1.32. *The forward stochastic differential:*

$$a(t,m)dt + \mathbb{A}(t,m)dw(t)$$

at point $m \in M$ given by an Itô vector field $(a, \mathbb{A})$ is the class of stochastic process in the tangent space T_mM that consists of the solution of all stochastic differential equations of the form:

$$X(t+s) = \int_t^{t+s} \tilde{a}(\tau, X(\tau))d\tau + \int_t^{t+s} \tilde{\mathbb{A}}(\tau, X(\tau))dw(\tau)$$

where $\tilde{a}(\tau, X(\tau))$ is a vector field on T_mM; $\tilde{\mathbb{A}}(\tau, X) : \mathbb{R}^n \to T_mM$ is a linear operator depending on the parameters $\tau \in \mathbb{R}$ and $X \in T_mM$ and the following conditions are satisfied: $\tilde{a}(\tau, X)$ and $\bar{\mathbb{A}}(\tau, X)$ are Lipschitz continuous, are equal to zero outside some neighborhood of the origin in T_mM and such that for $\tau \geq t$, the equalities $\tilde{a}(\tau, 0) = a(t,m)$ and $\tilde{\mathbb{A}}(\tau, 0) = \mathbb{A}(t,m)$ hold.

Moreover, since the vector fields $\tilde{a}(\tau, X)$ and $\tilde{\mathbb{A}}(\tau, m)$ are Lipschitz continuous, the process given by $X(t+s)$ is a strong solution of the equation and it is well defined for every Wiener Process in $\mathbb{R}^n$.

Definition 1.33. *A process $\xi(t)$ is said to satisfy the Itô Equation in Belopolskaya-Daletskii Form given by:*

$$d\xi(t) = \exp_{\xi(t)}(a(t,\xi(t))dt + \mathbb{A}(t,\xi(t))dw(t)) \tag{1.34}$$

if for every point $\xi(t)$ there exists a neighborhood of $\xi(t)$ in M such that before

$\xi(t+s)$, $s \geq 0$, leaves this neighborhood, $\xi(t+s)$ almost surely coincides with a process from the class $\exp_{\xi(t)}(a(t,\xi(t))dt + \mathbb{A}(t,\xi(t))dw(t))$

The term "strong" used in the Riemannian metric $G(\cdot,\cdot)$ described in the following theorem means that $G(\cdot,\cdot)$ determines that the metric $\|\cdot\|_H$ of the model space H of the manifold M is defined on the tangent spaces to M.

Theorem 1.3. *Let M be a Hilbert Manifold, ∇ be a connection on M, $a(t,m)$ be a vector field and $\mathbb{A}(t,m)$ be a field of linear operators $\mathbb{A}(t,m) : \mathbb{R}^n \to T_mM$, where $m \in M$, $t \in [0,T]$ and $\mathbb{R}^n$ is the Euclidean space in which a Wiener process $w(t)$ takes values. Assume there exists a strong Riemannian metric $G(\cdot,\cdot)$ on M compatible with ∇, with respect to which $\|a(t,m)\|_H < C$ and $\|\mathbb{A}(t,m)\|_H < C$, where $C > 0$ is a constant, for all t, m. Then, for every $m_0 \in M$ there exists a strong and strongly unique solution $\xi(t)$ of equation*

$$d\xi\left(t\right) = \exp_{\xi(t)}\left(a\left(t,\xi\left(t\right)\right)dt + \mathbb{A}\left(t,\xi\left(t\right)\right)dw(t)\right) \tag{1.35}$$

with initial condition $\xi(0) = m_0$, well-defined for all $t \in [0,T]$.

Proof. Let m_0 be a point belonging to M such that $\xi(0) = m_0$. Since M is a Hilbert manifold, for each point we can take a chart locally given by: $\varphi : U \subset H \to \varphi(U) \subset M$, where H is a Hilbert Space. Consider $B(m_0, r)$ as the open ball with center at m_0 and radius r given according to the metric induced by H , where r is a positive real constant. From the chart, we have that $\varphi(V) \to B(m_0,r) \subset M$, where V is a restriction of U which is the preimage of the ball considered. Furthermore, the chosen radius r is taken independently of the chart and the point.

By hypothesis, in each ball $B(m_0,r)$, we have that there is a positive real constant C such that:

$$\|a(t,m)\|_H \leq C \quad \text{and} \quad \|\mathbb{A}(t,m)\|_H \leq C,$$

for all $t \in [0,T]$ and all m in the ball. This constant C does not depend on the selected point and chart.

Now, applying the classical theorem of existence and uniqueness of strong solutions in Hilbert spaces, according to [1]. By this theorem, we obtain that there exists a strong solution in the chart with center at m_0 in the random interval $[0, \tau_1]$ where τ_1 is the shortest time such that the trajectory of $\xi(t, \omega)$ hits the boundary of the ball $B(m_0, r)$. That is,

$$\tau_1 = \inf\{t > 0 \,:\, \xi(t, \omega) \notin B(m_0, r)\}$$

and if $\xi(t, \omega)$ does not hit the boundary, then $\tau_1 = \infty$.

Similarly, define $\xi(\tau_1, \omega)$ as the initial point at time τ_1 and consider a open ball centered at that point and with radius r as we did before. Similarly, this ball is the image of an open V_2 of H with respect to a chart φ_2 in which the same uniform boundedness conditions hold and, at a certain time τ_2, we have that $\xi(t, \omega)$ hits the boundary of $B(\xi(\tau_1), r)$. Applying the theorem of strong solutions in Hilbert spaces, we are able to find a strong solution locally in the time interval $[\tau_1, \tau_2]$.

We repeat the argument so that we cover the entire time interval $[0, T]$. In this way, we obtain a sequence of stopping times:

$$0 = \tau_0 < \tau_1 < \tau_2 < \cdots \,,$$

on which a unique strong solution is defined locally.

Using estimates, we calculate the probability of the process $\xi(t, \omega)$ hitting the boundary of the respective ball for a small time t, based on the boundedness of the SDE coefficients. It follows, then, that $\sup_n \tau_n = \infty$ which leads us to conclude that when we construct a solution locally and patching these intervals together, the accumulated time almost surely diverges, which guarantees a unique global solution in the complete interval $[0, T]$. $\square$

Definition 1.34. *The stochastic differential $(a(m), \tilde{\mathbb{A}}(m))$ at $m \in M$ given by an Itô vector field $(a, \mathbb{A})$ is the set of stochastic processes in T_mM formed by the solutions of all stochastic differential equations:*

$$X(s) = \int_0^s \tilde{a}(r, X(r))dr + \int_0^s \tilde{\mathbb{A}}(r, X(r))dw_r,$$

where $\tilde{a}(s,X)$ *is a vector field on* T_mM *and* $\tilde{\mathbb{A}}(s,X) : \mathbb{R}^n \to T_mM$ *is a linear operator depending on* $s \in \mathbb{R}$ *and* $X \in T_mM$*. We assume* $\tilde{a}(s,X)$ *and* $\tilde{\mathbb{A}}(s,X)$ *to be Lipschitz, vanish outside a neighborhood of the origin in* T_mM*, and such that* $\tilde{a}(s,0) = a(m)$ *and* $\tilde{\mathbb{A}}(s,0) = \mathbb{A}(m)$*.*

Definition 1.35. *Let* exp *be the exponential map of a fixed connection* ∇ *on a manifold* M*. A process* $\xi(t)$ *is said to satisfy the equation:*

$$d\xi(t) = \exp_{\xi(t)}(a(\xi(t)), \mathbb{A}(\xi(t))) \tag{1.36}$$

if for every t *there exists a neighborhood of* $\xi(t)$ *such that the process* $\xi(t+s)$*,* $s \geq 0$*, a.s. coincides with a process from the set* $\exp_{\xi(t)}(a(\xi(t)), \mathbb{A}(\xi(t)))$*, as long as* $\xi(t+s)$ *belongs to the neighborhood.*

Remark 1.5. If $f : M \to N$ is a C^2-map and $\overline{exp}$ on N is such that $f(\exp X) = \overline{\exp}(Tf \circ X)$ for all $X \in T_mM$, then we have:

$$df(\xi(t)) = \exp_{f(\xi(t))}(Tf \circ a(f(\xi(t))), Tf \circ \mathbb{A}(\xi(t)))$$

for any solution $\xi(t)$ of (1.36).

Next, we will show three important results that will be used in future demonstrations, especially in chapter 4.

Lemma 1.3 (Grönwall's Inequality (Differential))**.** *Let* $\eta(s)$ *be a nonnegative, absolutely continuous function on* $[0,T]$*, which satisfies for almost everywhere the differential inequality:*

$$\eta'(t) \leq \phi(t)\eta(t) + \psi(t) \tag{1.37}$$

where $\phi(t)$ *and* $\psi(t)$ *are nonnegative, integrable functions on* $[0,T]$*. Then,*

$$\eta(t) \leq e^{\int_0^t \phi(s)ds} \left[\eta(0) + \int_0^t \psi(s)ds \right]$$

for all $0 \leq t \leq T$*. In particular, if* $\eta' \leq \phi\eta$ *on* $[0, T]$ *and* $\eta(0) = 0$*, then:* $\eta = 0$ *on* $[0, T]$.

Proof. By inequality (1.37), we obtain:

$$\frac{d}{ds}\left(\eta(s)\eta^{-\int_0^s \phi(r)dr}\right) = e^{-\int_0^s \phi(r)dr}(\eta'(s) - \phi(s)\eta(s))$$
$$\Rightarrow \frac{d}{ds}\left(\eta(s)\eta^{-\int_0^s \phi(r)dr}\right) \leq e^{-\int_0^s \phi(r)dr}\psi(s)$$

for almost everywhere $0 \leq s \leq T$. That way, for each $0 \leq t \leq T$, we get:

$$\eta(t)e^{-\int_0^t \phi(r)dr} \leq \eta(0) + \int_0^s e^{-\int_0^s \phi(r)dr}\psi(s)ds \leq \eta(0) + \int_0^t \psi(s)ds$$

And this inequality implies:

$$\eta(t) \leq e^{\int_0^t \phi(s)ds}\left[\eta(0) + \int_0^t \psi(s)ds\right].$$

□

Lemma 1.4 (Grönwall's Inequality (Integral))**.** *Let* $\xi(t)$ *be a nonnegative, summable function on* $[0, T]$ *which satisfies for almost everywhere (a.e.) time* t *the integral inequality:*

$$\xi(t) \leq C_1 \int_0^t \xi(s)ds + C_2 \tag{1.38}$$

for constants C_1*,* $C_2 \geq 0$*. Then,*

$$\xi(t) \leq C_2\left(1 + C_1 t e^{C_1 t}\right). \tag{1.39}$$

for a.e. $0 \leq t \leq T$*.*

In particular, if:

$$\xi(t) \leq C_1 \int_0^t \xi(t)ds$$

for a.e. $0 \leq t \leq T$*, then* $\xi(t) = 0$ *a.e.*

Proof. Consider a function $\eta(t) = \int_0^t \xi(s)ds$, then $\eta \leq C_1\eta + C_2$ almost everywhere in $[0, T]$. According to the differential version of Grönwall's lemma above, we obtain:

$$\eta(t) \leq e^{C_1 t}(\eta(0) + C_2 t) = C_2 t e^{C_1 t}.$$

In this way, the inequality given by (1.38) results in:

$$\xi(t) \leq C_1\eta(t) + C_2$$
$$\Rightarrow \xi(t) \leq C_2(1 + C_1 t e^{C_1 t})$$

□

Theorem 1.4 (Blumenthal's Zero-One Law)**.** *Let* $(\Omega, \mathcal{B}, (\mathcal{B}_t)_{t\geq 0}, \mathbb{P})$ *be a filtered probability space with a right-continuous filtration, and let* X_t*, with* $t \geq 0$*, be a stochastic process adapted to* $\mathcal{B}_t$ *such that* X_0 *is* $\mathbb{P}$*-almost surely constant. Define the* σ*-subalgebra at zero as*

$$\mathcal{B}_{0+} = \bigcap_{t>0} \mathcal{B}_t.$$

Then, for every event $A \in \mathcal{B}_{0+}$*,*

$$\mathbb{P}(A) \in \{0, 1\}.$$

Proof. The proof can be found in Rogers and Williams [15]. □

Chapter 2

Groups of Diffeomorphisms

In this chapter, we will address the most essential and important concepts about Groups of Diffeomorphisms, in order to prepare for the next section that will focus on stochastic processes that occur in these environments.

But first, we will review Sobolev Spaces and applications involving the n-dimensional Torus, since this will be the focus of our study later on.

2.1 Sobolev Spaces

The study of Sobolev Spaces, in general, begins with the definition of Test Functions, which are functions defined on some specified domain and that are C^∞-differentiable and that are compactly contained in the domain. However, since our study will be restricted to Sobolev applications on the flat $n-$torus, due to the property of the flat n-torus being a closed manifold, we have that all applications defined on it have compact support, that is, it is unnecessary to make this condition explicit, since all diffeomorphisms naturally already satisfy this condition.

Thus, we will consider the set of all maps u defined on the n-dimensional flat torus which are C^∞-differentiable. This set will be denoted as $\mathcal{D}(\mathbb{T}^n)$.

For $u \in C^1(\mathbb{T}^n)$, we can define $\partial_{x_i} u$ for each direction i, where $i \in \{1, ..., n\}$, the integration by parts formula:

$$\int_{\mathbb{T}^n} \frac{\partial u}{\partial x_i}(x)\,\phi\,(x)\,dx = -\int_{\mathbb{T}^n} u\,(x)\,\frac{\partial \phi}{\partial x_i}\,(x)\,dx \quad \forall \phi \in \mathcal{D}(\mathbb{T}^n).$$

The right-hand side of the above equality is well-defined, whenever the function $u \in L^1\,(\mathbb{T}^n)$.

Definition 2.1 (Multi-index)**.** *An element $\alpha \in \mathbb{Z}^n_+$, i.e., α is finite sequence of nonnegative integers, is called a multi-index. For such an $\alpha = (\alpha_1, ..., \alpha_n)$, we write:*

$$D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$$

and $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$.

Definition 2.2 (Weak Derivative)**.** *Let u a function such that $u \in L^1\,(\mathbb{T}^n)$. Then, a function $v^\alpha \in L^1\,(\mathbb{T}^n)$ is called the α-th weak-derivative of u, written as $v^\alpha = D^\alpha u$, if:*

$$\int_{\mathbb{T}^n} u(x) D^\alpha \phi(x) dx = (-1)^{|\alpha|} \cdot \int_{\mathbb{T}^n} v^\alpha(x)\phi(x)dx$$

for all $\phi \in \mathcal{D}(\mathbb{T}^n)$.

Definition 2.3. *Let $p \in [1, \infty)$, define:*

$$W^{1,p}(\mathbb{T}^n) = \{u \in L^p(\mathbb{T}^n); \quad Du \in L^p(\mathbb{T}^n)\}$$

such that the weak derivative exists and Du is the weak derivative of u.

Throughout the work, we will restrict ourselves only to the case $p = 2$, in which we have:

$$W^{1,2}(\mathbb{T}^n) = \{u \in L^2(\mathbb{T}^n); \quad Du \in L^2(\mathbb{T}^n)\}.$$

This set is a Hilbert Space and we will denote simply by: $H^1(\mathbb{T}^n) = W^{1,2}(\mathbb{T}^n)$. The norm for this space will be introduced later.

Remark 2.1. Note that if the weak derivative exists, it is unique. To verify this, suppose that v_1 and v_2 are two weak derivative of u on $\mathbb{T}^n$. Then, it is valid that:

$$\int_{\mathbb{T}^n} (v_1 - v_2)\phi dx = 0$$

for all $\phi \in \mathcal{D}(\mathbb{T}^n)$.

So that $v_1 = v_2$ almost everywhere.

Definition 2.4. *Let $k \geq 0$ an integer and $p = 2$,*

$$W^{k,2}(\mathbb{T}^n) = \{u \in L^1(\mathbb{T}^n);\ D^\alpha u \textit{ exists and is in } L^2(\mathbb{T}^n) \quad \textit{for} \quad |\alpha| \leq k\}$$

Definition 2.5. *Let $k \geq 0$ an integer and $p = 2$, for $u \in W^{k,2}(\mathbb{T}^n)$, define:*

$$||u||_{W^{k,2}(\mathbb{T}^n)} = \left(\sum_{|\alpha| \leq k} ||D^\alpha u||^2_{L^2(\mathbb{T}^n)} \right)^{\frac{1}{2}}$$

The function defined above is a norm, since it is a finite sum of L^2-norms.

Definition 2.6. *Let $k \geq 0$ an integer and $p = 2$, we define: $H^k(\mathbb{T}^n) = W^{k,2}(\mathbb{T}^n)$. This set $H^k(\mathbb{T}^n)$ is a Hilbert Space with inner-product give by:*

$$\langle u, v \rangle_{H^k(\mathbb{T}^n)} = \sum_{|\alpha| \leq k} \langle D^\alpha u, D^\alpha v \rangle_{L^2(\mathbb{T}^n)}.$$

Now, we will dedicate ourselves to reviewing the Fourier coefficients, a tool that will be essential for the development of some proposed calculations in later chapters.

Definition 2.7 (Fourier Coefficient on the n-Torus)**.** *For all $u \in L^2(\mathbb{T}^n)$, the Fourier coefficient $\mathcal{F}$ is defined by:*

$$(\mathcal{F}u)(k) = \hat{u}_k = \frac{1}{(2\pi)^n} \cdot \int_{\mathbb{T}^n} e^{-ik\cdot\theta} u(x) d\theta$$

where $k \cdot \theta$ is a inner product between $k \in \mathbb{Z}^n_+$ and $\theta \in \mathbb{T}^n$.

Equivalently, by Euler's Formula $e^{-ik\cdot\theta} = \cos(k\cdot\theta) - i\sin(k\cdot\theta)$, we obtain:

$$(\mathcal{F}u)(k) = \hat{u}_k = \frac{1}{(2\pi)^n} \cdot \int_{\mathbb{T}^n} (\cos(k\cdot\theta) - i\sin(k\cdot\theta))\, u(\theta)d\theta$$

In the next definition, we will denote $\hat{u} := \{\hat{u}_k\}$, where $k \in \mathbb{Z}^n_+$ as above.

Definition 2.8. *For all $u \in L^1(\mathbb{T}^n)$, we define the inverse operator $\mathcal{F}^*$ by:*

$$(\mathcal{F}^*\hat{u})(\theta) = \sum_{k\in\mathbb{Z}^n} \hat{u}_k e^{ik\cdot\theta}.$$

in other words,in this case, the inverse operator is the Fourier series.

Example 2.1. *Let $u \in C^1(\mathbb{T}^n)$ and $j \in \{1, 2, ..., n\}$, then:*

$$\begin{aligned}\mathcal{F}\left[\frac{\partial u}{\partial\theta_j}\right](k) &= \frac{1}{(2\pi)^n} \cdot \int_{\mathbb{T}^n} \frac{\partial u}{\partial\theta_j} e^{-ik\cdot\theta} d\theta \\ &= -\frac{1}{(2\pi)^n} \cdot \int_{\mathbb{T}^n} u(\theta)(-ik_j)e^{-ik\cdot\theta} d\theta \\ &= ik_j\hat{u}_k\end{aligned}$$

Note that $\mathbb{T}^n$ is a closed manifold without boundary; in a alternative way, we may identify $\mathbb{T}^n$ with the $[0,1]^n$ with periodic boundary conditions, in other words, with opposite faces identified.

Using the Fourier coefficients $\mathcal{F}$ and the Fourier series $\mathcal{F}^*$, over the space $L^2(\mathbb{T}^n)$ we obtain the following applications and equivalences:

$$\begin{aligned} \mathcal{F} : L^2(\mathbb{T}^n) \to \ell^2 \qquad & \mathcal{F}^* : \ell^2 \to L^2(\mathbb{T}^n) \\ \mathcal{F}^*\mathcal{F} = Id \quad \text{on} \quad L^2(\mathbb{T}^n) \qquad & \mathcal{F}\mathcal{F}^* = Id \quad \text{on} \quad \ell^2. \end{aligned}$$

Definition 2.9. *The inner-products on $L^2(\mathbb{T}^n)$ and ℓ^2 are defined as follows. In the $L^2(\mathbb{T}^n)$ space is given by:*

$$\langle u(\theta), v(\theta)\rangle_{L^2(\mathbb{T}^n)} = \frac{1}{\sqrt{(2\pi)^n}} \cdot \int_{\mathbb{T}^n} u(\theta)\overline{v(\theta)}d\theta$$

And in ℓ^2 space is given by:

$$\langle \hat{u}, \hat{v} \rangle_{\ell^2} = \sum_{k \in \mathbb{Z}^n} \hat{u}_k \overline{\hat{v}_k}.$$

Moreover, it is holds that:

$$\|u\|_{L^2(\mathbb{T}^n)} = \|\hat{u}\|_{\ell_2}$$

Definition 2.10 (Sobolev Spaces $H^\alpha(\mathbb{T}^n)$)**.** *For all multi-index $\alpha \in \mathbb{R}^+$, the Sobolev Spaces $H^\alpha(\mathbb{T}^n)$ are defined as follows.*

$$H^\alpha(\mathbb{T}^n) = \{u \in L^2(\mathbb{T}^n);\ \|u\|_{H^\alpha(\mathbb{T}^n)} < \infty\}$$

where the norm on $H^\alpha(\mathbb{T}^n)$ is defined as:

$$\|u\|^2_{H^\alpha(\mathbb{T}^n)} = \sum_{k \in \mathbb{Z}^n} |\hat{u}_k|^2 (1 + |k|^{2\alpha})^{\cdot} \tag{2.1}$$

The space $H^\alpha(\mathbb{T}^n)$ equipped with norm $\|\cdot\|_{H^\alpha(\mathbb{T}^n)}$ mentioned above is a Hilbert Space.

Definition 2.11. *Let $u \in L^2(\mathbb{T}^n)$ and $\alpha > 0$. The fractional Laplacian on the n-torus is given by:*

$$(-\Delta)^\alpha u(\theta) = \sum_{k \in \mathbb{Z}^n} |k|^{2\alpha} \hat{u}_k e^{ik \cdot \theta}$$

2.2 Analysis on Groups of Diffeomorphisms

As seen in the previous section, throughout this master's thesis we will be working only with the flat n-torus $\mathbb{T}^n$. With this in mind, in this section we will present in more detail the structures we will work with that are directly related to the flat n-torus and that will form the base environment for our study in chapters 3 and 4.

Let $\mathbb{T}^n$ be the n-dimensional flat torus equipped with a Riemannian metric $\langle \cdot, \cdot \rangle$ and its Levi-Civita connection ∇. The torus is an example of a compact Riemannian manifold without boundaries, so the concepts presented in this section are just a restriction of the more general results shown in Gliklikh [11] and [9].

2.2.1 The group and manifold structure

Definition 2.12 (Sobolev-Maps between Manifolds)**.** *We will denote by* $H^\alpha = H^\alpha (\mathbb{T}^n, \mathbb{T}^n)$ *the set of Sobolev* H^α*-mappings from* $\mathbb{T}^n$ *to* $\mathbb{T}^n$ *in which we have* $\alpha > \frac{n}{2} + 1$*.*

As shown in Ebin and Marsden [7], for a multi-index $\alpha > \frac{n}{2} + k$, the maps from H^α are C^k-smooth. Furthermore, there is an infinite-dimensional manifold structure on $H^\alpha (\mathbb{T}^n, \mathbb{T}^n)$.

Take the identical mapping $e \in H^\alpha(\mathbb{T}^n, \mathbb{T}^n)$ and consider an open neighborhood $\mathcal{D}^\alpha (\mathbb{T}^n)$ that consists of all H^α-diffeomorphisms. Furthermore, consider also its subset $\mathcal{D}^\alpha_v(\mathbb{T}^n)$ which consists of H^α-diffeomorphisms which preserve the Riemannian volume.

Both these sets, $\mathcal{D}^\alpha(\mathbb{T}^n)$ and $\mathcal{D}^\alpha_v(\mathbb{T}^n)$ have the structures of smooth Hilbert Manifolds as well as multiplicative group structures through composition operation.

To prove the group structure, the verification is quite simple. We will notice that the three properties necessary for $\mathcal{D}^\alpha(\mathbb{T}^n)$ to be a group are satisfied. First, note that the composition operation is closed in the set, that is, given two maps f and g that belong to $\mathcal{D}^\alpha(\mathbb{T}^n)$, the composition $f \circ g$ is also a diffeomorphism of class C^k, therefore, it belongs to $\mathcal{D}^\alpha(\mathbb{T}^n)$.

- Let f, g and h be maps that belong to $\mathcal{D}^\alpha(\mathbb{T}^n)$, note that:

$$(f \circ g) \circ h = f \circ g \circ h = f \circ (g \circ h)$$

  and this composition is a C^k-diffeomorphism, so the first property is satisfied.

- The identity map, denoted by e, is a C^k-diffeomorphism defined as $e : \mathbb{T}^n \mapsto$

$\mathbb{T}^n$, such that $e(m) = m$, for all $m \in M$, so it is clear that it belongs to $\mathcal{D}^\alpha(\mathbb{T}^n)$.

- Finally, consider any map $f : \mathbb{T}^n \mapsto \mathbb{T}^n$ that belongs to $\mathcal{D}^\alpha(\mathbb{T}^n)$, since it is a diffeomorphism, by definition, there exists a $f^{-1} : \mathbb{T}^n \mapsto \mathbb{T}^n$ such that $f \circ f^{-1} = e$. This concludes the proof.

The verification that $\mathcal{D}_v^\alpha(\mathbb{T}^n)$ is a group with respect to the composition operation is analogous, it suffices to observe that the composition of any two applications f and g that preserve the volume results in an application $f \circ g$ that also preserves the volume. Furthermore, all volume-preserving C^k-diffeomorphisms belong to $\mathcal{D}^\alpha(\mathbb{T}^n)$, so the inclusion of $\mathcal{D}_v^\alpha(\mathbb{T}^n)$ in $\mathcal{D}^\alpha(\mathbb{T}^n)$ as a subgroup is clear.

On the other hand, to verify whether these sets are Hilbert manifolds, we need to verify the properties as described in definition 1.2 of this work. First, let us consider again the most general space of functions, that is, let $\alpha > \frac{n}{2}$, consider the H^α-Sobolev maps from $\mathbb{T}^n$ to $\mathbb{T}^n$, which are well defined. This set of maps will be denoted by $H^\alpha(\mathbb{T}^n, \mathbb{T}^n)$.

Consider a map $g \in H^\alpha(\mathbb{T}^n, \mathbb{T}^n)$ and the set given by:

$$T_g H^\alpha(\mathbb{T}^n, \mathbb{T}^n) = \{f \in H^\alpha(\mathbb{T}^n, T\mathbb{T}^n) \,; \pi \circ f = g\}$$

where $\pi : T\mathbb{T}^n \to \mathbb{T}^n$ is the natural projection. Note that the set $T_g H^\alpha(\mathbb{T}^n, \mathbb{T}^n)$ described above, equipped with the standard Sobolev inner product, is a Hilbert Space. The map

$$\omega_{\exp} : T_g H^\alpha(\mathbb{T}^n, \mathbb{T}^n) \to H^\alpha(\mathbb{T}^n, \mathbb{T}^n)$$
$$\omega_{\exp} f = \exp \circ f$$

is defined one-to-one over a sufficiently small neighborhood of the origin in the set $T_g H^\alpha(\mathbb{T}^n, \mathbb{T}^n)$. Thus, this neighborhood of the origin and the map given by $\omega_{\exp}$ can be taken as a chart at the point $g \in H^\alpha(\mathbb{T}^n, \mathbb{T}^n)$. Furthermore, given two charts, it is easy to see that the transition map between these charts is a C^∞-differentiable map. Therefore, we obtain a structure of a C^∞-differentiable Hilbert manifold over $H^\alpha(\mathbb{T}^n, \mathbb{T}^n)$ such that the set $T_g H^\alpha(\mathbb{T}^n, \mathbb{T}^n)$ is the tangent space to $H^\alpha(\mathbb{T}^n, \mathbb{T}^n)$ at the point g.

Furthermore, note that given $\alpha > \frac{n}{2} + 1$, the manifold $H^\alpha(\mathbb{T}^n, \mathbb{T}^n)$ contains the subset $\mathcal{D}^\alpha(\mathbb{T}^n)$ which consists of all H^α-maps that are C^1-diffeomorphisms. Since $\mathcal{D}^\alpha(\mathbb{T}^n)$ is an open in $H^\alpha(\mathbb{T}^n, \mathbb{T}^n)$, then it is a Hilbert submanifold. Looking at e as point of the manifold $\mathcal{D}^\alpha(\mathbb{T}^n)$, we can define the tangent space $T_e\mathcal{D}^\alpha(\mathbb{T}^n)$ which is the space of all vector fields on $\mathbb{T}^n$ belonging to H^α. The tangent space $T_e\mathcal{D}^\alpha(\mathbb{T}^n)$ is the space of all H^α-vector fields on $\mathbb{T}^n$. Moreover, the whole tangent bundle $T\mathcal{D}^\alpha(\mathbb{T}^n)$ can be identified as the subset of $H^\alpha(\mathbb{T}^n, T\mathbb{T}^n)$ which consists of maps that, when composed with the natural projection $\pi : T\mathbb{T}^n \to \mathbb{T}^n$, result in elements of $\mathcal{D}^\alpha(\mathbb{T}^n)$. So, in particular, we have:

$$T_g\mathcal{D}^\alpha(\mathbb{T}^n) = \{f \in H^\alpha(\mathbb{T}^n, T\mathbb{T}^n) \mid \pi \circ f = g\} = \{X \circ g \mid X \in T_e\mathcal{D}^\alpha(\mathbb{T}^n)\}$$

Given the maps g and h in H^α, note that the composition $h \circ g$ may not belong to H^α, however, since $\alpha > \frac{n}{2} + 1$ and g is a local diffeomorphism, then the composition $h \circ g$ belongs to H^α, since that h is a H^α-map. The proof that $\mathcal{D}_v^\alpha(\mathbb{T}^n)$ is a Hilbert manifold is analogous. Moreover, the $T_e\mathcal{D}_v^\alpha(\mathbb{T}^n)$ is the space of all divergent-free vector fields on $\mathbb{T}^n$ belonging to H^α.

Furthermore, it will be important to describe two new spaces that will be of great use in the future which are related to the manifold $\mathcal{D}_v^\alpha(\mathbb{T}^n)$. The tangent space $T_g\mathcal{D}_v^\alpha(\mathbb{T}^n)$, where $g \in \mathcal{D}_v^\alpha(\mathbb{T}^n)$ and that consists of the compositions of the fields from tangent space $T_e\mathcal{D}_v^\alpha(\mathbb{T}^n)$ with g. In symbols, that is:

$$T_g\mathcal{D}_v^\alpha(\mathbb{T}^n) = \{X \circ g \mid X \in T_e\mathcal{D}_v^\alpha(\mathbb{T}^n)\}.$$

This means that given a vector field $Y \in T_g\mathcal{D}_v^\alpha(\mathbb{T}^n)$, $Y : \mathbb{T}^n \to T\mathbb{T}^n$ is a map such that $\pi Y(m) = g(m)$, where $\pi : T\mathbb{T}^n \to \mathbb{T}^n$ is the natural projection. On the other hand, for $X \in T_e\mathcal{D}_v^\alpha(\mathbb{T}^n)$, we have that $\pi X(m) = m$.

Lemma 2.1 (α-lemma)**.** *Let $\mathbb{T}^n$ be the flat n-torus, $g \in \mathcal{D}^\alpha(\mathbb{T}^n)$ and $h \in H^\alpha(\mathbb{T}^n, \mathbb{T}^n)$,* *in which h is a function as described above. Consider the map given by:*

$$\alpha_g : H^{\alpha}(\mathbb{T}^n, \mathbb{T}^n) \to H^{\alpha}(\mathbb{T}^n, \mathbb{T}^n)$$
$$\alpha_g(f) = f \circ g.$$

The map α_g is C^{∞}-smooth and its derivative is also of the form α_g.

Lemma 2.2 (ω-lemma)**.** *Let $\mathbb{T}^n$ be the flat n-torus, let $r \in H^{\alpha}(\mathbb{T}^n, \mathbb{T}^n)$ and $h \in H^{\alpha}(\mathbb{T}^n, \mathbb{T}^n)$, in which h is a function as described above. Consider the map given by:*

$$\omega_h : \mathcal{D}^{\alpha}(\mathbb{T}^n) \to H^{\alpha}(\mathbb{T}^n, \mathbb{T}^n)$$
$$\omega_h(r) = h \circ r.$$

The map ω_h is continuous. If $h \in H^{\alpha+k}$, $\omega_h : \mathcal{D}^{\alpha}(\mathbb{T}^n) \to H^{\alpha}(\mathbb{T}^n, \mathbb{T}^n)$ is a C^k-mapping with derivative of the form ω_{T_h}. In particular, if $h \in C^{\infty}$, ω_h is C^{∞}-smooth.

Proof. The proofs of the ω-lemma and the α-lemma can be found in Ebin and Marsden [7]. □

The right translation defined as:

$$R_g : \mathcal{D}^{\alpha}_v(\mathbb{T}^n) \to \mathcal{D}^{\alpha}_v(\mathbb{T}^n)$$
$$R_g \circ \eta = \eta \circ g$$

where $\eta, f \in \mathcal{D}^{\alpha}_v(\mathbb{T}^n)$ are C^{∞}-smooth and, in this way, can be considered an right-invariant vector fields on $\mathcal{D}^{\alpha}_v(\mathbb{T}^n)$. The tangent map TR_g restricted to the tangent space $T_\eta\mathcal{D}^{\alpha}(\mathbb{T}^n)$ is defined by the formula:

$$TR_g : \; T_\eta\mathcal{D}^{\alpha}(\mathbb{T}^n) \to T_{\eta\circ g}\mathcal{D}^{\alpha}(\mathbb{T}^n),$$
$$X \mapsto X \circ g.$$

for $X \in T\mathcal{D}^{\alpha}_v(\mathbb{T}^n)$. For the Hilbert manifold $\mathcal{D}^{\alpha}(\mathbb{T}^n)$, the properties are analogous.

Remark 2.2. Note that we will consider the map: TR_g given above for all η and $g \in \mathcal{D}_v^\alpha(\mathbb{T}^n)$ as a right action of $\mathcal{D}_v^\alpha(\mathbb{T}^n)$ on $T\mathcal{D}_v^\alpha(\mathbb{T}^n)$.

Theorem 2.1. *Let $X \in T_e\mathcal{D}^\alpha(\mathbb{T}^n)$ be a vector field on $\mathbb{T}^n$ and $\bar{X}$ be the corresponding right-invariant vector field on $\mathcal{D}^\alpha(\mathbb{T}^n)$ denoted by $\bar{X}_g = X \circ g$. The vector field $\bar{X}$ on $\mathcal{D}^\alpha(\mathbb{T}^n)$ is C^k-smooth if and only if the vector field X on $\mathbb{T}^n$ belongs to the class $H^{\alpha+k}$. In particular, $\bar{X}$ is C^∞ smooth if and only if X is C^∞-smooth.*

Proof. Consider the vector field $X \in H^{\alpha+k}(\mathbb{T}^n)$ and its respective right-invariant vector field given by $\bar{X}_g = X \circ g$. Since the composition operation $\circ$ preserves differentiability, let $h = X$ and $g \in H^\alpha(\mathbb{T}^n)$, by the ω-lemma we have that:

$$\omega_X(g) = X \circ g$$

is a function of class C^k if and only if X belongs to $H^{\alpha+k}(\mathbb{T}^n)$.

Furthermore, let X be a C^∞-smooth vector field and $g \in H^\alpha(\mathbb{T}^n)$ as before, again by the ω-lemma, the result follows. □

The same property presented by the previous theorem is satisfied for the case of the Hilbert Manifold $\mathcal{D}_v^\alpha(\mathbb{T}^n)$, it suffices to remember that every diffeomorphism that preserves volume belongs to $\mathcal{D}^\alpha(\mathbb{T}^n)$ and the verification is immediate.

2.2.2 Metrics on Group of Diffeomorphisms

Next, we will define some essential concepts for various theorems and results which will be addressed throughout the work, respectively: the Riemannian Metric (I), the Levi-Civita Connection with connector (II) and the exponential mapping $\overline{\exp}$ (III) are defined as follow:

I) Let $g \in \mathcal{D}^\alpha(\mathbb{T}^n)$ and the tangent space $T_g\mathcal{D}^\alpha(\mathbb{T}^n)$. We will define an inner product $(\cdot,\cdot)$ in $T_g\mathcal{D}^\alpha(\mathbb{T}^n)$ by:

$$(X,Y)_g = \int_{\mathbb{T}^n} \langle X(\theta), Y(\theta)\rangle_{g(\theta)} v(d\theta) \tag{2.2}$$

where $X, Y \in T_g\mathcal{D}^\alpha(\mathbb{T}^n)$ and v is the Riemannian volume form.

II) Let $K : TT\mathbb{T}^n \to T\mathbb{T}^n$ the connector of Levi-Civita Connection ∇ on $\mathbb{T}^n$, we will define the mapping: $\bar{K} : TT\mathcal{D}^\alpha(\mathbb{T}^n) \to T\mathcal{D}^\alpha(\mathbb{T}^n)$ by the equality:

$$\bar{K}(Y) = K \circ Y. \tag{2.3}$$

III) Let $T_e\mathcal{D}^\alpha(\mathbb{T}^n)$ the tangent space of $\mathcal{D}^\alpha(\mathbb{T}^n)$ in the point $e \in \mathcal{D}^\alpha$, by general properties of smooth exponential mapping, we can define the exponential mapping:

$$\overline{exp} : U \subset T_e\mathcal{D}^\alpha(\mathbb{T}^n) \to V \subset \mathcal{D}^\alpha(\mathbb{T}^n)$$

which sends a neighborhood U of zero vector field of $T_e\mathcal{D}^\alpha(\mathbb{T}^n)$ onto a neighborhood V of e in $\mathcal{D}^\alpha(\mathbb{T}^n)$.

The metric described above in equation (2.2) is called a weak metric.

Theorem 2.2. *The connector $\bar{K}$ is invariant with respect to right translations on the manifold $\mathcal{D}^\alpha(\mathbb{T}^n)$.*

Proof. First, $\bar{K}$ is the connector of a connection $\bar{\nabla}$. Moreover, is the Levi-Civita connection of the metric (2.2) (for more details, see Gliklikh [11]).

Consider vector fields X and Y on the manifold $\mathcal{D}^\alpha(\mathbb{T}^n)$ and a vector field $X(t)$ along a certain smooth curve $c(t)$ in $\mathcal{D}^\alpha(\mathbb{T}^n)$. For these vector fields, define the covariant derivatives $\bar{\nabla}_X Y$ and $\frac{\bar{D}}{dt}X(t)$, respectively:

$$\bar{\nabla}_X Y = \bar{K} \circ TY(X) = K \circ TY(X)$$

and

$$\frac{\bar{D}}{dt}X(t) = \bar{K} \circ \frac{d}{dt}X(t) = K \circ \frac{d}{dt}X(t) \tag{2.4}$$

On the space $\mathcal{D}^\alpha(\mathbb{T}^n)$ with connection $\bar{\nabla}$, a vector field X along a curve $c(t)$ is parallel if $\frac{\bar{D}}{dt}X(t) = 0$, according Definition 1.12. The curve $c(t)$ is a geodesic if it satisfies the equation $\frac{\bar{D}}{dt}\left(\frac{d}{dt}c(t)\right) = 0$.

The geodesic spray $\bar{\mathcal{Z}}$ of the connection $\bar{\nabla}$ is give by:

$$\bar{\mathcal{Z}}(X) = \mathcal{Z} \circ X \tag{2.5}$$

for $X \in T\mathcal{D}^\alpha(\mathbb{T}^n)$, where $\mathcal{Z}$ is the geodesic spray of the connection ∇ on $\mathbb{T}^n$.

Since the geodesic spray $\mathcal{Z}$ is C^∞-smooth, according the ω-lemma (2.2), it follows that $\bar{\mathcal{Z}}$ is C^∞-smooth on $T\mathcal{D}^\alpha(\mathbb{T}^n)$. By the equality given by (2.5), it is clear that $\bar{\mathcal{Z}}$ is $\mathcal{D}^\alpha(\mathbb{T}^n)$-right invariant. □

Remark 2.3. In particular, we will consider the exponential map $\overline{\exp} : T\mathcal{D}^\alpha(\mathbb{T}^n) \to \mathcal{D}^\alpha(\mathbb{T}^n)$ as corresponding to $\bar{\mathcal{Z}}$.

Theorem 2.3. *The geodesic spray $\mathcal{S}$ of the Levi-Civita connection $\tilde{\nabla}$ of the metric (2.2) on $\mathcal{D}_v^\alpha(\mathbb{T}^n)$ is a C^∞-smooth right-invariant vector field on $T\mathcal{D}_v^\alpha(\mathbb{T}^n)$ of the form:*

$$\mathcal{S} = T\bar{P}(\bar{\mathcal{Z}})$$

where $\mathcal{Z}$ is the geodesic spray according (2.5) on $T\mathcal{D}^\alpha(\mathbb{T}^n)$.

Proof. We have that $\bar{P}$ and $\bar{\mathcal{Z}}$ are $\mathcal{D}_v^\alpha(\mathbb{T}^n)$-right-invariant and C^∞-smooth on the space $T\mathcal{D}_v^\alpha(\mathbb{T}^n)$, then we have that $\mathcal{S} = T\bar{P}(\bar{\mathcal{Z}})$ is also C^∞-smooth and $\mathcal{D}_v^\alpha(\mathbb{T}^n)$-right-invariant.

Now, we will denote by $\widetilde{\exp}$ the corresponding exponential map of a neighborhood of the 0 in $T\mathcal{D}_v^\alpha(\mathbb{T}^n)$ onto $\mathcal{D}_v^\alpha(\mathbb{T}^n)$. The constructed geodesic spray defines the flow $\phi_t : \mathcal{D}_v^\alpha(\mathbb{T}^n) \to \mathcal{D}_v^\alpha(\mathbb{T}^n)$ is smooth and right-invariant, and since the exponential map is a particular case defined for a fixed time $t = 0$ of this flow, it is also smooth and right-invariant. □

Theorem 2.4 (Neighborhood of unit)**.** *There exists a neighborhood W of the unite e in $\mathcal{D}_v^\alpha(\mathbb{T}^n)$ that is covered by the image of $T_e\mathcal{D}_v^\alpha(\mathbb{T}^n)$ under the exponential mapping of the Levi-Civita connection of $\mathcal{D}_v^\alpha(\mathbb{T}^n)$.*

Proof. According to the previous theorem 2.3, the geodesic spray $\mathcal{S} = T\bar{P}(\bar{\mathcal{Z}})$ is a C^∞-smooth right-invariant vector field on $T\mathcal{D}_v^\alpha(\mathbb{T}^n)$ and furthermore the geodesic spray defines the exponential map given by:

$$\widetilde{\exp}\colon U_0 \subset T\mathcal{D}^{\alpha}_v(\mathbb{T}^n) \to \mathcal{D}^{\alpha}_v(\mathbb{T}^n),$$

which is C^{∞}-smooth and right-invariant.

In particular, consider the exponential map restriction given below:

$$\widetilde{\exp}\colon U \subset T_e\mathcal{D}^{\alpha}_v(\mathbb{T}^n) \to \mathcal{D}^{\alpha}_v(\mathbb{T}^n),$$

where U is a neighborhood of the zero vector in the manifold $T_e\mathcal{D}^{\alpha}_v(\mathbb{T}^n)$.

Since the map $\widetilde{\exp}$ is a smooth local diffeomorphism at 0, by the inverse function theorem, then there exists a neighborhood U of zero in which $\widetilde{\exp}U$ will be a diffeomorphism onto its image, which we will denote by W.

Therefore, since $W = \widetilde{\exp}U$ is an open neighborhood of the identity $e \in \mathcal{D}^{\alpha}_v(\mathbb{T}^n)$, then the theorem is proved. □

Definition 2.13. [Strong Riemannian Metric] Let $\mathbb{T}^n$ be the n-dimensional flat torus, $g \in \mathcal{D}^{\alpha}(\mathbb{T}^n)$, X_g and $Y_g \in T_g\mathcal{D}^{\alpha}(\mathbb{T}^n)$, which $X_g = X \circ g$ and $Y_g = Y \circ g$ such that $X, Y \in T_e\mathcal{D}^{\alpha}(\mathbb{T}^n)$. Introduce on the tangent space $T_g\mathcal{D}^{\alpha}(\mathbb{T}^n)$ a strong inner product $(\cdot,\cdot)_g$ by the formula:

$$(X_g, Y_g)^{(\alpha)}_g = \int_{\mathbb{T}^n} \langle X_g(\theta), Y_g(\theta)\rangle_{g(\theta)} d\theta + \int_{\mathbb{T}^n} \langle (-\Delta)^{\frac{\alpha}{2}} X_g(\theta), (-\Delta)^{\frac{\alpha}{2}} Y_g(\theta)\rangle_{g(\theta)} d\theta \tag{2.6}$$

or

$$(X_g, Y_g)^{(\alpha)}_g = \int_{\mathbb{T}^n} \langle X_g(\theta), (1 + (-\Delta)^{\alpha}))Y_g(\theta)\rangle_{g(\theta)} d\theta$$

where $(-\Delta)^{\alpha}$ is the Laplacian presented in Definition 2.11.

Furthermore, we shall also use another strong right-invariant Riemannian Metric, given by the formula:

$$(X_g, Y_g)^{(\alpha)}_g = (TR_g^{-1}X_g, TR_g^{-1}Y_g)^{(\alpha)}_e. \tag{2.7}$$

2.3 Stochastic Analysis on Groups of Diffeomorphisms

In this section, our goal is to present results that guarantee the existence and uniqueness of SDE solutions specifically associated with the group of diffeomorphisms of the flat $n-$torus. It is worth mentioning that although the theorems presented here are for this case, it is possible to find more general results in the literature, as we can see in Gliklikh [11], in chapter 10. For this, it would be necessary to consider the most general version of the strong metric, introduced in definition 2.13 which would be understood in any finite-dimensional manifold M. However, it would also be necessary to extend the concepts related to Sobolev presented earlier in this work, which can be found in Shkoller [16].

Consider the flat n-dimensional torus $\mathbb{T}^n$, as described before, and note that we can also represent it as $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ and from this we obtain that the metric on the manifold $\mathbb{T}^n$ can be induced through the Euclidean space $\mathbb{R}^n$ as described in Example 1.5. In this case, $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ means that in $\mathbb{R}^n$ the points whose coordinates differ by an integer in each entry are considered equivalent and are therefore identified.

Furthermore, there is a canonical identification of the tangent bundle $T\mathbb{T}^n$ with $\mathbb{T}^n \times \mathbb{R}^n$, which is also inherited from $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$.

Definition 2.14 (Maps on the n-Torus)**.** *Consider the n-dimensional flat-torus as presented previously, define the operators:*

(I) $B : T\mathbb{T}^n \longrightarrow \mathbb{R}^n$*, the projection onto the second factor in* $\mathbb{T}^n \times \mathbb{R}^n$*.*

(II) $\mathbb{A}(m) : \mathbb{R}^n \longrightarrow T_m\mathbb{T}^n$*, the inverse to* B *sending* $\mathbb{R}^n$ *onto the tangent space* $T_m\mathbb{T}^n$ *of* $\mathbb{T}^n$ *at* $m \in \mathbb{T}^n$*.*

(III) $Q_g : \mathbb{A}(g(m)) \circ B$*, the linear isomorphism* $Q_g : T_m\mathbb{T}^n \longrightarrow T_{g(m)}\mathbb{T}^n$*, where* $g \in \mathcal{D}^\alpha$ *and* $m \in \mathbb{T}^n$*.*

Consider a positive constant σ, $a(t, m)$ an H^α-vector field over $\mathbb{T}^n$, where $t \in [0, T]$ and $\alpha > s$ is an integer. Let us denote by $\bar{a}(t, g)$ the right-invariant vector field over $\mathcal{D}^\alpha(\mathbb{T}^n)$ generated by $a(t,m)$ as a vector belonging to the tangent space $T_e\mathcal{D}^\alpha(\mathbb{T}^n)$.

Furthermore, consider a map $\mathbb{A}$ described according to the definition 2.14 above. For any vector $x \in \mathbb{R}^n$, the vector field $\mathbb{A}(x)$ on $\mathbb{T}^n$ is constant, which means that its coordinates with respect to the basis given by $\frac{\partial}{\partial q^1}, ..., \frac{\partial}{\partial q^n}$ are constant. In particular, this implies that $\mathbb{A}(x)$ is C^∞-smooth and divergence-free.

Let $\bar{\mathbb{A}} : \mathcal{D}^\alpha(\mathbb{T}^n) \times \mathbb{R}^n \to T\mathcal{D}^\alpha(\mathbb{T}^n)$ where $\mathbb{A}_e : \mathbb{R}^n \to T_e\mathcal{D}^\alpha_v(\mathbb{T}^n)$ is given by the expression $\mathbb{A}_e(x) = \mathbb{A}(x)$ and for $g \in \mathcal{D}^\alpha(\mathbb{T}^n)$, we have that the mapping $\mathbb{A}_g : \mathbb{R}^n \to T_g\mathcal{D}^\alpha(\mathbb{T}^n)$ is constructed from $\mathbb{A}$ by the right shift $\bar{\mathbb{A}}_g(x) : TR_g\bar{\mathbb{A}}_e(x) = (\mathbb{A} \circ g)(x)$. Since the map $\mathbb{A}$ is C^∞-smooth, it follows by Theorem 2.1 that the map $\bar{\mathbb{A}}$ is also C^∞-smooth on $x \in \mathbb{R}^n$ and $g \in \mathcal{D}^\alpha(\mathbb{T}^n)$. That is, for all $x \in \mathbb{R}^n$, the right-invariant vector field $\mathbb{A}(x)$ over $\mathcal{D}^\alpha(\mathbb{T}^n)$ is C^∞-smooth.

The pair given by $(\bar{a}, \bar{\mathbb{A}})$ is an Itô vector field on $\mathcal{D}^\alpha(\mathbb{T}^n)$. Consider a Wiener process $w(t)$ in $\mathbb{R}^n$ given on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Theorem 2.5. *For every $g \in \mathcal{D}^\alpha(\mathbb{T}^n)$ there exists a unique strong solution $\bar{\xi}^g(t)$ of equation:*

$$d\bar{\xi}(t) = \overline{exp}_{\bar{\xi}(t)}\left(\bar{a}(t,\bar{\xi}(t))dt + \sigma\bar{\mathbb{A}}(t,\bar{\xi}(t))dw(t)\right) \tag{2.8}$$

with initial condition $\bar{\xi}^g(0) = g$ which is well-defined for all $t \in [0,T]$, where $T > 0$ is an arbitrary a priori specified real number.

Proof. Using the exponential map $\overline{exp}$, consider the normal chart in a neighborhood of the point e on the manifold $\mathcal{D}^\alpha(\mathbb{T}^n)$ and note that at each point of this chart, the local connector (2.3) equals zero, this is because the connection is generated by the Euclidean connection on the n-Torus $\mathbb{T}^n$.

Thus, take a strong right-invariant Riemannian Metric on $\mathcal{D}^\alpha(\mathbb{T}^n)$. We say, generated by (2.7).

Since, the normal chart is a open set, there exists a positive real number r such that the ball with radius r (with respect to the strong Riemannian distance on $\mathcal{D}^\alpha(\mathbb{T}^n)$) and with center at e, denoted by $B[e,r]$, is contained in this neighborhood.

Then, in a neighborhood of each point $g \in \mathcal{D}^\alpha(\mathbb{T}^n)$, applying the right shift R_g to the ball $B[e,r]$, we determine an chart. So, in this way, we obtain an atlas on $\mathcal{D}^\alpha(\mathbb{T}^n)$ that is uniformly Riemannian for the strong metric, and in each chart of this atlas considered, the local connectors for connection 2.3 equal zero, this occurs because the connection is right-invariant according to Theorem 2.2.

Therefore, the right-invariant Itô vector field $(\bar{a}, \sigma\bar{\mathbb{A}})$ is uniformly bounded with respect to the strong Riemannian metric, so, the Theorem 1.3 can be applied to equation.

$$d\bar{\xi}(t) = \overline{exp}_{\bar{\xi}(t)}\left(\bar{a}\left(t, \bar{\xi}(t)\right) dt + \sigma\bar{\mathbb{A}}(t, \bar{\xi}(t))dw(t)\right).$$

□

Remark 2.4. Later, in chapters 3 and 4, it will be common to use superscript indices such as: $X^{t,e}(s)$ to highlight that at time $s = t$, the initial condition is $X(t) = e$. This detail will be very useful when we work with different initial conditions caused by the right shift and it will make it common for notations such as $X^{t,\xi}$ and $X^{t,e}$ to be used simultaneously.

Now, denote $\bar{\xi}^e(t)$ by $\xi(t)$ only for simplicity. Since the equation

$$d\bar{\xi}(t) = \overline{exp}_{\bar{\xi}(t)}\left(\bar{a}\left(t, \bar{\xi}(t)\right) dt + \sigma\bar{\mathbb{A}}(t, \bar{\xi}(t))dw(t)\right)$$

is right-invariant, it follows that: $\bar{\xi}^g(t) = \xi(t) \circ g$. So, $\xi(t)$ is the general solution of the stochastic differential equation on $\mathbb{T}^n$ described below:

$$d\xi(t) = \exp_{\xi(t)}\left(a(t, \xi(t))dt + \sigma\mathbb{A}dw(t)\right) \tag{2.9}$$

In other words, for every point m in the n-dimensional torus $\mathbb{T}^n$, the stochastic process $\xi^m(t)$ is a solution of above equation 2.9, with initial condition m at time $t = 0$ on $\mathbb{T}^n$.

Remark 2.5. In the equations previously described in (2.8) and (2.9), we used the general notation of Ito's Equations in Belopolskaya-Daletskii form. But, since that the connection of the manifold $\mathcal{D}^\alpha(\mathbb{T}^n)$ is generated by the flat connection on the torus, the corresponding exponential map is like that on a linear space. In this way, we can employ the same notation that is used for Itô's Equations in linear spaces.

Theorem 2.6. *Let $\sigma > 0$ be a real constant:*

(I) For every $\omega \in \Omega$ and $t \in [0,T]$ the vector field $\mathbb{A}\left(\sigma w(t,\omega)\right)$ on $\mathbb{T}^n$, where $w\left(t\right)$ is a Wiener process in $\mathbb{R}^n$, is divergence-free, i.e. $\mathbb{A}\left(\sigma w\left(t\right)\right)$ is a stochastic process in $T_e\mathcal{D}_v^{\alpha}\left(\mathbb{T}^n\right)$.

(II) For every $\omega \in \Omega$ and $t \in [0,T]$, the mapping:

$$W_{\omega}^{(\sigma)}\left(t\right) = \overline{exp}_e\mathbb{A}\left(\sigma w(t,\omega)\right) : \mathbb{T}^n \to \mathbb{T}^n$$

is a volume-preserving H^{α}-diffeomorphism of $\mathbb{T}^n$, i.e., $W^{\sigma}\left(t\right)$ is a stochastic process in $\mathcal{D}_v^{\alpha}\left(\mathbb{T}^n\right)$.

Proof. Regarding (I), let $\omega \in \Omega$ and $t \in [0,T]$, so the vector field $\mathbb{A}(\sigma w(t,\omega))$ on $\mathbb{T}^n$ has constant coordinates with respect to the basis $\frac{\partial}{\partial q^1}, ..., \frac{\partial}{\partial q^n}$ and so it is C^{∞} and divergence-free.

In respect to (II), the mapping $W_{\omega}^{\sigma}(t)$ sends $m \in \mathbb{T}^n$ to $\exp_m \mathbb{A}(w(t,\omega))$, where $\exp_m : T_m\mathbb{T}^n \to \mathbb{T}^n$ is the exponential mapping of the flat torus n-dimensional $\mathbb{T}^n$. That means that all points of the manifold $\mathbb{T}^n$ under the mapping $W_{\omega}^{(\sigma)}(t)$ perform the same shift as that generated by the shift of the space $\mathbb{R}^n$ by $\sigma w(t,\omega)$. The application $W_{\omega}^{(\sigma)}(t)$ is clearly volume-preserving. □

Theorem 2.7. *For every $g \in \mathcal{D}_v^{\alpha}(\mathbb{T}^n)$ there exists a unique strong solution $\tilde{\xi}^{0,g}(t)$ of*

$$d\tilde{\xi}(t) = \widetilde{exp}_{\tilde{\xi}(t)}\left(\bar{a}(t,\tilde{\xi}(t))dt + \sigma\bar{\mathbb{A}}(t,\tilde{\xi}(t))dw(t)\right) \tag{2.10}$$

with initial condition $\tilde{\xi}^{0,g}(0) = g$ which is well-defined for all $t \in [0,l]$.

Proof. We begin introducing on $\mathcal{D}_v^{\alpha}(\mathbb{T}^n)$ the strong Riemannian Metric, according to formula (2.6), and remember that this metric is right-invariant. On the manifold $\mathcal{D}_v^{\alpha}(\mathbb{T}^n)$, consider a neighborhood U around the point e, according to Theorem 2.4, this neighborhood is covered by the exponential mapping $\exp_{0,e}$, where $\exp_0$ denote the exponential map of a neighborhood of the 0 in $T\mathcal{D}_v^{\alpha}(\mathbb{T}^n)$ onto $\mathcal{D}_v^{\alpha}(\mathbb{T}^n)$.

Consider the normal chart at e in U. The strong norm of the local connector, denoted by $\Gamma_{\eta}(\cdot,\cdot)$, being a quadratic operator, is a continuous function of $\eta \in U$ in this chart.

In this way, at e, we obtain $\Gamma_e(\cdot,\cdot) = 0$. Hence, there exists an open $V \subset U$ such that at each point, the norm (above-mentioned) is less than a priori given positive constant C. Since V is an open, so it contains the ball centered at e and radius $r > 0$ with respect to the strong Riemannian distance on $\mathcal{D}_v^\alpha(\mathbb{T}^n)$ generated by the strong metric described in (2.6), which will be denoted by $B[e, r]$.

Now, we will define a chart at a neighborhood of each point $g \in \mathcal{D}_v^\alpha(\mathbb{T}^n)$ through the right shift of the normal chart U at g. In this way, the atlas constructed is uniformly Riemannian with respect to strong metric (2.6).

Moreover, consider the balls $B[g, r]$, in other words, the balls with center in each point $g \in \mathcal{D}_v^\alpha(\mathbb{T}^n)$ and radius r with respect to the Riemannian metric above-mentioned. On this balls in the charts of the atlas constructed, the norm of the local connector Γ of Levi-Civita Connection of the metric (2.2) on $\mathcal{D}_v^\alpha(\mathbb{T}^n)$ is bounded by constant C, since the connection is right-invariant, according the Theorem 2.2.

So, the right-invariant Itô Vector field $(\bar{a}, \bar{\mathbb{A}})$ on $\mathcal{D}_v^\alpha(\mathbb{T}^n)$ is bounded with respect to the metric (2.6). Thus, the equation 2.10 satisfies the conditions of Theorem 1.3 and therefore it has a strong solution. □

Chapter 3

Navier-Stokes Equations and a system of FBSDE

3.1 Navier-Stokes Equations

We will begin this chapter by presenting the Navier-Stokes equations. Our main results in this work will be developed involving the solutions of these equations and their relations with the solutions of a Forward Backward Stochastic Differential Equations system described later.

The classical Navier-Stokes Equations are described as:

$$\begin{aligned} &\frac{\partial}{\partial t} u(t,x) = -(u,\nabla)u(t,x) + \nu\Delta u(t,x) - \nabla p(t,x), \\ &\operatorname{div} u = 0, \\ &u(0,x) = -u_0(x), \end{aligned} \tag{3.1}$$

where $u_0(x)$ is a divergence-free smooth vector field.

Fixing a time interval $[0,T]$, we can rewrite equations (3.1) with respect to the function

$$\tilde{u}(t,x) = -u(T-t,x).$$

So, the problem (3.1) is equivalent to the following:

$$\begin{aligned} &\frac{\partial}{\partial t}\tilde{u}(t,x) = -(\tilde{u},\nabla)\tilde{u}(t,x) - \nu\Delta\tilde{u}(t,x) - \nabla\tilde{p}(t,x), \\ &\operatorname{div}\tilde{u} = 0, \\ &\tilde{u}(T,x) = u_0(x), \end{aligned} \tag{3.2}$$

where $\tilde{p}(t,x) = p(T-t,x)$.

The above system (3.2) will be referred to as the backward Navier-Stokes Equations.

In what follows, for the development of our calculations, we will use the following notation. Initially, we established the set given by: $\mathbb{Z}^2_+ = \{(k_1,k_2) \in \mathbb{Z}^2 : k_1 > 0 \text{ or } k_1 = 0, k_2 > 0\}$, that is, $\mathbb{Z}^2_+$ will be the set of ordered integers pairs k_1 and k_2 such that one of them is necessarily positive and k_1 is strictly non-negative.

Moreover, let $k = (k_1,k_2) \in \mathbb{Z}^2_+$ an arbitrary element, we will denote its norm as $|k| = \sqrt{k_1^2 + k_2^2}$. And through it, we can define a new vector $\bar{k}$ that will be orthogonal to k denoted by $\bar{k} = (k_2, -k_1)$. Indeed, we see that k and $\bar{k}$ are orthogonal to each other, since: $\langle k, \bar{k}\rangle = 0$.

Furthermore, the element $\theta \in \mathbb{T}^2$ will correspond to the pair (θ_1,θ_2) where $\theta_i \in S_1$, $i \in \{1,2\}$, since the flat 2-torus can be interpreted as $\mathbb{T}^2 = S_1 \times S_1$ as seen before. Thus, using the notation established above, the product $k\cdot\theta$ will correspond to: $k\cdot\theta = k_1\theta_1 + k_2\theta_2$.

Finally, define the following gradient vector:

$$\nabla = \left(\frac{\partial}{\partial\theta_1}, \frac{\partial}{\partial\theta_2}\right)$$

from which we obtain:

$$(\bar{k},\nabla) = k_2\frac{\partial}{\partial\theta_1} - k_1\frac{\partial}{\partial\theta_2}.$$

Using the Fourier coefficients in the $\mathbb{T}^2$, we will define the vectors described below:

$$\bar{A}_k(\theta) = \frac{1}{|k|^{\alpha+1}} \cos(k \cdot \theta) \begin{bmatrix} k_2 \\ -k_1 \end{bmatrix},$$
$$\bar{B}_k(\theta) = \frac{1}{|k|^{\alpha+1}} \sin(k \cdot \theta) \begin{bmatrix} k_2 \\ -k_1 \end{bmatrix},$$
$$\bar{A}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \bar{B}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Let $\{A_k(g), B_k(g)\}$ with $k \in \mathbb{Z}_+^2 \cup \{0\}$, be the right-invariant vector fields on $\mathcal{D}^\alpha(\mathbb{T}^2)$ generated by $\{\bar{A}_k, \bar{B}_k\}_{k \in \mathbb{Z}_+^2 \cup \{0\}}$, i.e.

$$A_k(g) = \bar{A}_k \circ g, \quad B_k(g) = \bar{B}_k \circ g, \quad g \in \mathcal{D}^\alpha(\mathbb{T}^2),$$
$$A_0 = \bar{A}_0, \quad B_0 = \bar{B}_0.$$

According to the ω-lemma 2.2, A_k and B_k are C^∞-smooth vector fields on $\mathcal{D}^\alpha(\mathbb{T}^2)$.

Now, let us define some lemmas that will be important for the discussions presented throughout this chapter. These lemmas describe, in general, the construction of the bases of the tangent spaces $T_e\mathcal{D}_v^\alpha(\mathbb{T}^2)$ and $T_e\mathcal{D}^\alpha(\mathbb{T}^2)$, and also important properties about the right-invariant vector fields.

Lemma 3.1. *The vectors $A_k(g)$, $B_k(g)$, $k \in \mathbb{Z}_+^2 \cup \{0\}$ in which $g \in \mathcal{D}_v^\alpha(\mathbb{T}^2)$, form an orthogonal basis of the tangent space $T_g\mathcal{D}_v^\alpha(\mathbb{T}^2)$ with respect to both the weak and the strong inner products in $T_g\mathcal{D}_v^\alpha(\mathbb{T}^2)$. In particular, the vectors $\bar{A}_k$, $\bar{B}_k$, $k \in \mathbb{Z}_+^2 \cup \{0\}$, form an orthogonal basis of the tangent space $T_e\mathcal{D}_v^\alpha(\mathbb{T}^2)$. Moreover, the weak (2.2) and the strong (2.6) norms of the basis vectors are bounded by the same constant.*

Proof. Proving this lemma for the strong norm given in (2.6) is sufficient, since a strong norm induces the weak norm (2.2). Let us calculate the operator Δ^α applied to the vector $\bar{A}_k$ as follows. First, note that the normalized vectors $\frac{k}{|k|}$ and $\frac{\bar{k}}{|k|}$ form an orthonormal basis of $\mathbb{R}^2$, where $k = (k_1, k_2)$ and $\bar{k} = (k_2, -k_1) \in \mathbb{Z}_+^2$ as described at the beginning of this chapter.

Also, note that:

$$\begin{aligned}
(-\Delta)^{\alpha}\bar{A}_k(\theta) &= (-\Delta)^{\alpha}\left[\frac{1}{|k|^{\alpha}}\cos(k\cdot\theta)\,\frac{\bar{k}}{|k|}\right] \\
&= \frac{1}{|k|^{\alpha}}\,(-\Delta)^{\alpha}\left[\cos(k\cdot\theta)\right]\,\frac{\bar{k}}{|k|} \\
&= \frac{1}{|k|^{\alpha}}\,\left(|k|^{2\alpha}\cos(k\cdot\theta)\right)\,\frac{\bar{k}}{|k|} \\
&= |k|^{\alpha}\,\cos(k\cdot\theta)\,\frac{\bar{k}}{|k|} \\
&= |k|^{2\alpha}\left(\frac{1}{|k|^{\alpha}}\cos(k\cdot\theta)\,\frac{\bar{k}}{|k|}\right) \\
&= |k|^{2\alpha}\,\bar{A}_k(\theta).
\end{aligned}$$

With this last equality and the volume-preserving property of the map $g \in \mathcal{D}_v^{\alpha}(\mathbb{T}^2)$. We have the following results for A_k and B_m, where $k, m \in \mathbb{Z}_+^2$:

$$\begin{aligned}
\langle B_m(g), A_k(g)\rangle_{\alpha} &= \langle \bar{B}_m, \bar{A}_k\rangle_{\alpha} \\
&= (1+|k|^{2\alpha})\,\langle \bar{B}_m, \bar{A}_k\rangle_{L^2} \\
&= (1+|k|^{2\alpha})\cdot\int_{\mathbb{T}^2}\bar{B}_m(\theta)\cdot\bar{A}_k(\theta)\,d\theta \\
&= (1+|k|^{2\alpha})\cdot 0 \\
&= 0.
\end{aligned}$$

Moreover, we obtain:

$$\begin{aligned}
\|A_k(g)\|_{\alpha}^2 &= \|\bar{A}_k\|_{\alpha}^2 \\
&= \left(1+|k|^{2\alpha}\right)\|\bar{A}_k\|_{L_2}^2 \\
&= \left(1+|k|^{2\alpha}\right)\cdot\int_{\mathbb{T}^2}\|\bar{A}_k(\theta)\|_{L^2}^2\,d\theta \\
&= \left(1+|k|^{2\alpha}\right)\cdot\int_{\mathbb{T}^2}\frac{1}{|k|^{2\alpha}}\cos^2(k\cdot\theta)\,d\theta
\end{aligned}$$

$$= \left(1+|k|^{2\alpha}\right)\cdot\frac{1}{|k|^{2\alpha}}\,(2\pi^2).$$
$$= 2\pi^2\left(|k|^{-2\alpha}+1\right)$$

where $\|\cdot\|_\alpha$ is the norm corresponding to the scalar product $\langle\cdot,\cdot\rangle_\alpha$. Therefore, $2\pi^2 \le \|A_k(g)\|_\alpha^2 \le 4\pi^2$.

The same argumentation applies to the directional vectors described as $\bar{B}_k$, from which we obtain the same for $\|B_k(g)\|_\alpha^2$. □

As seen in the first chapter, the weak Riemannian metric has the Levi-Civita connection, geodesics, the exponential map and the geodesic spray. Then, let $\bar{\nabla}$ and $\tilde{\nabla}$ be the covariant derivatives of the Levi-Civita connection with respect to the weak metric (2.2) on the Hilbert manifolds $\mathcal{D}^\alpha(\mathbb{T}^2)$ and $\mathcal{D}^\alpha_v(\mathbb{T}^2)$ respectively. The following equality holds:

$$\tilde{\nabla} = P\circ\bar{\nabla}$$

where the map $P : T\mathcal{D}^\alpha(\mathbb{T}^2)\to T\mathcal{D}^\alpha_v(\mathbb{T}^2)$ is defined in the following way: on each tangent space $T_g\mathcal{D}^\alpha$, we have that $P = P_g$ where $P_g = TR_g\circ P_e\circ TR_{g^{-1}}$, TR_g and $TR_{g^{-1}}$ are tangent maps, and $P_e : T_e\mathcal{D}^\alpha(\mathbb{T}^2)\to\mathcal{D}^\alpha_v(\mathbb{T}^2)$ is the defined projection of the first space into the second.

Next, we will have two similar results, the first addresses the understanding of right invariance in the manifold $\mathcal{D}^\alpha(\mathbb{T}^2)$, while the second brings the same result in $\mathcal{D}^\alpha_v(\mathbb{T}^2)$.

Lemma 3.2. *Let $\hat{U}$ be the right-invariant vector field on $\mathcal{D}^\alpha(\mathbb{T}^2)$ generated by an $H^{\alpha+1}$-vector field U on $\mathbb{T}^2$, and let $\hat{V}$ be the right-invariant vector field on $\mathcal{D}^\alpha(\mathbb{T}^2)$ generated by an H^α-vector field V on $\mathbb{T}^2$. Then $\bar{\nabla}_{\hat{V}}\hat{U}$ is the right-invariant vector field on $\mathcal{D}^\alpha(\mathbb{T}^2)$ generated by the H^α-vector field $\nabla_V U$ on $\mathbb{T}^2$.*

Proof. The proof follows from the right-invariance of covariant derivatives on $\mathcal{D}^\alpha(\mathbb{T}^2)$ and $\mathcal{D}^\alpha_v(\mathbb{T}^2)$, according to [9]. □

Lemma 3.3. *Let $\hat{U}$ be the right-invariant vector field on $\mathcal{D}^\alpha_v(\mathbb{T}^2)$ generated by a divergence-free $H^{\alpha+1}$-vector field U on $\mathbb{T}^2$, and let $\hat{V}$ be the right-invariant vector* *field on $\mathcal{D}^\alpha(\mathbb{T}^2)$ generated by a divergence-free H^α-vector field V on $\mathbb{T}^2$. Then $\tilde{\nabla}_{\hat{V}}\hat{U}$ is the right-invariant vector field on $\mathcal{D}_v^\alpha(\mathbb{T}^2)$ generated by the divergence-free H^α-vector field $P_e\nabla_V U$ on $\mathbb{T}^2$.*

Proof. The proof of the theorem can be done in two ways. The first way consists in observing $\mathcal{D}_v^\alpha(\mathbb{T}^2)$ as a submanifold of $\mathcal{D}^\alpha(\mathbb{T}^2)$, then the result of lemma 3.2 can be simply restricted to divergent-free vectors due to embedding map: $i : \mathcal{D}_v^\alpha(\mathbb{T}^2) \to \mathcal{D}^\alpha(\mathbb{T}^2)$. The second way would be to simply repeat the same argumentation of the proof of 3.2 adapting it to the Hilbert manifold $\mathcal{D}_v^\alpha(\mathbb{T}^2)$. □

Remark 3.1. The basis $\{\bar{A}_k, \bar{B}_k\}_{k\in\mathbb{Z}_2^+\cup\{0\}}$ of $T_e\mathcal{D}_v^\alpha(\mathbb{T}^2)$ can be extended to a basis of the entire tangent space $T_e\mathcal{D}^\alpha(\mathbb{T}^2)$.

Indeed, let us introduce the vectors:

$$\bar{\mathcal{A}}_k(\theta) = \frac{1}{|k|^{\alpha+1}}\cos(k\cdot\theta)\begin{bmatrix}k_1\\ k_2\end{bmatrix},$$

and

$$\bar{\mathcal{B}}_k(\theta) = \frac{1}{|k|^{\alpha+1}}\sin(k\cdot\theta)\begin{bmatrix}k_1\\ k_2\end{bmatrix}$$

where $k \in \mathbb{Z}_2^+$.

The system $\bar{A}_k$, $\bar{B}_k$, in which $k \in \mathbb{Z}_2^+ \cup \{0\}$ and $\bar{\mathcal{A}}_k$ $\bar{\mathcal{B}}_k$ in which $k \in \mathbb{Z}_2^+$, form an orthogonal basis of $T_e\mathcal{D}^\alpha(\mathbb{T}^2)$. Further let $\mathcal{A}_k$ and $\mathcal{B}_k$ denote the right-invariant vector fields on $\mathcal{D}^\alpha(\mathbb{T}^2)$ generated by $\bar{\mathcal{A}}_k$ and $\bar{\mathcal{B}}_k$.

3.2 The FBSDE on group of diffeomorphisms

In this section, we will present the system of stochastic differential equations that we will study in this work, which will be composed by a forward SDE and a backward SDE. First, let us define the vector fields and Brownian motion that we will consider in the system.

Let $h : \mathbb{T}^2 \to \mathbb{R}^2$ be a divergence-free $H^{\alpha+1}$-vector field on the manifold $\mathbb{T}^2$ and let $\hat{h}$ be the right-invariant vector field on $\mathcal{D}^\alpha(\mathbb{T}^2)$ generated by h.

Moreover, consider the function given by $V(s,\cdot)$ such that there exists a function $p : [t,T] \to H^{\alpha+1}(\mathbb{T}^2, \mathbb{R}^2)$ satisfying $V(s,\cdot) = \nabla p(s,\cdot)$ for all s in the time interval $[t,T]$. For each $s \in [t,T]$, $\hat{V}(s,\cdot)$ denotes the right-invariant vector field on $\mathcal{D}^\alpha(\mathbb{T}^2)$ generated by $V(s,\cdot) \in H^\alpha(\mathbb{T}^2, \mathbb{R}^2)$.

Now, consider E as a Euclidean space spanned on an orthonormal, relative to the scalar product in E, system of vectors: $\{e_k^A, e_k^B, e_0^A, e_0^B\}$ which $k \in \mathbb{Z}_+^2, |k| \le N$ and the indices A and B are related to the directions of the vectors A_k and B_k seen in the previous section.

Consider the mapping:

$$\mathbb{A}(g) = \sum_{\substack{k \in \mathbb{Z}_+^2 \cup \{0\}, \\ |k| \le N}} A_k(g) \otimes e_k^A + B_k(g) \otimes e_k^B, \quad g \in \mathcal{D}^\alpha(\mathbb{T}^2),$$

that is, $\mathbb{A}(g)$ is a linear operator $E \to T_g\mathcal{D}^\alpha(\mathbb{T}^2)$ for each $g \in \mathcal{D}^\alpha(\mathbb{T}^2)$. More specifically, if $a = \sum_{\substack{k \in \mathbb{Z}_+^2 \cup \{0\}, \\ |k| \le N}} a_k^A e_k^A + a_k^B e_k^B \in E$, then

$$\mathbb{A}(g)a = \sum_{\substack{k \in \mathbb{Z}_+^2 \cup \{0\}, \\ |k| \le N}} a_k^A A_k(g) + a_k^B B_k(g).$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and W_s, where $s \in [t,T]$, be an E-valued Brownian motion defined as:

$$W_s = \sum_k (\beta_k^A(s) e_k^A + \beta_k^B(s) e_k^B)$$

in which $\{\beta_k^A, \beta_k^B\}$ is a sequence of independent Brownian motions with index $k \in \mathbb{Z}_+^2 \cup \{0\}$ such that $|k| \le N$.

Finally, we shall consider the following system of forward and backward stochastic differential equations (FBSDE):

$$\begin{cases} dZ^{t,e}(s) & = Y^{t,e}(s)ds + \epsilon\,\mathbb{A}(Z^{t,e}(s))dW_s, \\ dY^{t,e}(s) & = -\hat{V}(s, Z^{t,e}(s))ds + X^{t,e}(s)dW_s, \\ Z^{t,e}(t) & = e; \qquad Y^{t,e}(T) = \hat{h}(Z^{t,e}(T)). \end{cases} \tag{3.3}$$

The first line of the above system is the forward SDE (which will be denoted as FSDE throughout the work) and it is an SDE on $\mathcal{D}_v^\alpha(\mathbb{T}^2)$, where $\mathcal{D}_v^\alpha(\mathbb{T}^2)$ is regarded as a Hilbert manifold, as we saw before. Moreover, the stochastic integral in the FSDE can be explicitly written as follows:

$$\int_t^s \mathbb{A}(Z^{t,e}(r))dW_r = \sum_k \int_t^s A_k(Z^{t,e}(r))d\beta_k^A(r) + B_k(Z^{t,e}(r))d\beta_k^B(r). \tag{3.4}$$

which $k \in \mathbb{Z}_+^2 \cup \{0\}$ and we have that $|k| \leq N$.

Let us consider the backward SDE (BSDE) given as follows:

$$Y^{t,e}(s) = \hat{h}(Z^{t,e}(T)) + \int_s^T \hat{V}(r, Z^{t,e}(r))dr - \int_s^T X^{t,e}(r)dW_r. \tag{3.5}$$

Note that the processes

$$\hat{V}(s, Z^{t,e}(s)) = V(s,\cdot) \circ Z^{t,e}(s)$$

and

$$\hat{h}(Z^{t,e}(T)) = h \circ Z^{t,e}(T)$$

are H^α-maps as discussed in the 2.2.1 section, since they are compositions of H^α-maps and H^α-vector fields

In this way, BSDE (3.5) will be interpreted as a BSDE in the Hilbert space $H^\alpha(\mathbb{T}^2, \mathbb{R}^2)$. Let $\mathcal{F}_s = \sigma(W_r, r \in [0,s])$ be a filtration, our objective is to find an $\mathcal{F}_s$-adapted triple of adapted processes $(Z^{t,e}(s), Y^{t,e}(s), X^{t,e}(s))$ solving FBSDEs (3.3) in the following sense: for each $s \in [t,T]$ and $\omega \in \Omega$, we have that $Z^{t,e}(s) \in \mathcal{D}_v^\alpha(\mathbb{T}^2)$ and $Y^{t,e}(s) \in T_{Z^{t,e}(s)}\mathcal{D}_v^\alpha(\mathbb{T}^2)$ and, moreover, $Y^{t,e}(s)$ is an H^α-vector field.

We notice that the forward SDE is well-posed on both Hilbert manifolds $\mathcal{D}^\alpha(\mathbb{T}^2)$ and $\mathcal{D}^\alpha_v(\mathbb{T}^2)$ and can be written in the Dalecky-Belopolskaya form:

$$dZ^{t,e}(s) = \exp_{Z^{t,e}(s)}\{Y^{t,e}(s)ds + \epsilon\mathbb{A}(Z^{t,e}(s))dW_s\} \quad \text{or}$$
$$dZ^{t,e}(s) = \widetilde{\exp}_{Z^{t,e}(s)}\{Y^{t,e}(s)ds + \epsilon\,\mathbb{A}(Z^{t,e}(s))dW_s\}$$

where exp and $\widetilde{\exp}$ are respectively the exponential maps of the Levi-Civita connection of the weak Riemannian metrics (2.2) on $\mathcal{D}^\alpha(\mathbb{T}^2)$ and $\mathcal{D}^\alpha_v(\mathbb{T}^2)$; see Section 1.1.2. In what follows, we will show that using either of these representations leads to the same solution of FBSDEs (3.3).

Finally, the stochastic process $X^{t,e}(s)$ takes values in the space of linear operators $\mathcal{L}(E, H^\alpha(\mathbb{T}^2,\mathbb{R}^2))$, that is,

$$X^{t,e}(s) = \sum_k X^{kA}(s)\otimes e^A_k + X^{kB}(s)\otimes e^B_k \tag{3.6}$$

where the processes $X^{kA}(s)$ and $X^{kB}(s)$ take values in $H^\alpha(\mathbb{T}^2,\mathbb{R}^2)$ and $k\in\mathbb{Z}^2_+\cup\{0\}$ is such that $|k|\leq N$.

3.3 Constructing a solution of the FBSDEs

In this section, our goal will be to study the Forward and the Backward equations separately and then establish results about the solution of the system of Forward Backward Stochastic Differential Equations shown in (3.3)

3.3.1 The forward SDE

First, let us consider the backward Navier-Stokes Equations in $\mathbb{R}^2$ given as:

$$\begin{aligned} &y(s,\theta) = h(\theta) + \int_s^T \big[\nabla p(r,\theta) + \big(y(r,\theta),\nabla\big)y(r,\theta) + \nu\Delta y(r,\theta)\big]dr, \\ &\mathrm{div}\; y(s,\theta) = 0 \end{aligned} \tag{3.7}$$

where $s \in [t,T]$, $\theta \in \mathbb{T}^2$, while Δ and ∇ are the Laplacian and the gradient in $\mathbb{R}^2$, respectively.

Assumption 3.1. *Let us assume that on the interval $[t,T]$ there exists a solution $\big(y(s,\cdot), p(s,\cdot)\big)$ to the backward Navier-Stokes Equations (3.7) such that the functions $p : [t,T] \to H^{\alpha+1}(\mathbb{T}^2,\mathbb{R}^2)$ and $y : [t,T] \to H^{\alpha+1}(\mathbb{T}^2,\mathbb{R}^2)$ are continuous.*

Since $y(s,\cdot)$ is divergence-free, then $y(s,\cdot) \in T_e\mathcal{D}_v^\alpha(\mathbb{T}^2)$. According to Lemma 3.1 and Remark 3.1, we can represent the function $y(s,\cdot)$ with respect to the basis $\{\bar{A}_k, \bar{B}_k\}$ of tangent space $T_e\mathcal{D}_v^\alpha(\mathbb{T}^2)$ using the coordinates: $\{Y^{t;kA}(s),\ Y^{t;kB}(s)\}$, where $k \in \mathbb{Z}_+^2 \cup \{0\}$. In other words,

$$y(s,\theta) = \sum_k Y^{t;kA}(s)\bar{A}_k(\theta) + Y^{t;kB}(s)\bar{B}_k(\theta).$$

in which $k \in \mathbb{Z}_+^2 \cup \{0\}$.

Let $\hat{Y}(s,\cdot)$ denote the right-invariant vector field on $\mathcal{D}_v^\alpha(\mathbb{T}^2)$ generated by the solution $y(s,\cdot)$, that is, $\hat{Y}(s,g) = y(s,\cdot)\circ g$. On each tangent space $T_g\mathcal{D}_v^\alpha(\mathbb{T}^2)$, the vector $\hat{Y}(s,g)$ can be represented by a serie converging with respect to the H^α-norm as follows:

$$\hat{Y}(s,g) = \sum_k Y^{t;kA}(s)A_k(g) + Y^{t;kB}(s)B_k(g). \tag{3.8}$$

with $k \in \mathbb{Z}_+^2 \cup \{0\}$.

First, we will study the SDE

$$dZ^{t,e}(s) = \hat{Y}(s, Z^{t,e}(s))ds + \epsilon\,\mathbb{A}(Z^{t,e}(s))dW_s. \tag{3.9}$$

where $\epsilon > 0$ is a constant.

Later, in Theorem 3.6, we will show that the solution $Z^{t,e}(s)$ to (3.9) and the process $Y^{t,e}(s) = \hat{Y}(s, Z^{t,e}(s))$ are the first two processes in the triple $(Z^{t,e}(s),$ $Y^{t,e}(s)$, $X^{t,e}(s))$ that solves the system of FBSDEs (3.3).

Theorem 3.1. *There exists a unique strong solution $Z^{t,e}(s)$, where $s \in [t,T]$, to (3.9) on the Hilbert Manifold $\mathcal{D}_v^\alpha(\mathbb{T}^2)$, with the initial condition $Z^{t,e}(t) = e$.*

Proof. First, let us check whether the hypotheses of the Theorem 1.3 are satisfied. Analyzing the stochastic integral using the representation given by (3.4), we obtain two cases:

Case 1: If the sum (3.4) representing the stochastic integral $\int_t^s \mathbb{A}(Z^{t,e}(s))dW_s$ contains only the terms $A_0(\beta_0^A(s) - \beta_0^A(t))$ and $B_0(\beta_0^B(s) - \beta_0^B(t))$, in other words, informally speaking, if the Brownian motion runs only along the constant vectors A_0 and B_0, then the result follows from Theorem 2.7.

Case 2: On the other hand, if sum (3.4) contains also terms with A_k and B_k, where $k \in \mathbb{Z}_+^2$, or, informally, when the Brownian motion runs also along non-constant vectors A_k and B_k, then the hypotheses of Theorem 1.3 require the boundedness of vector A_k and B_k with respect to the strong norm. However, this fact is assured by Lemma 3.1.

Therefore, all the hypotheses of Theorem 1.3 are satisfied. Therefore, exists a unique strong solution $Z^{t,e}(s)$ to (3.9) with the initial condition $Z^{t,e}(t) = e$. Indeed, the proof of Theorem 2.7 shows that the Levi-Civita connection of the Weak Riemannian Metric given as (2.2) on $\mathcal{D}_v^\alpha(\mathbb{T}^2)$ is compatible in the sense of Definition 1.13 with the Strong Riemannian metric (2.6). The map:

$$\mathbb{A}(g) = \sum_k A_k(g) \otimes e_k^A + B_k(g) \otimes e_k^B$$

where $k \in \mathbb{Z}_+^2 \cup \{0\}$ such that $|k| \leq N$, is C^∞-smooth since A_k and B_k are C^∞-smooth. Furthermore, according to Lemma 3.1, $\mathbb{A}(g)$ is bounded on $\mathcal{D}_v^\alpha(\mathbb{T}^2)$.

Since $y : [t,T] \to H^{\alpha+1}(\mathbb{T}^2, \mathbb{R}^2)$ is continuous, then $y(s)$ is also bounded with respect to the H^α-norm. In this way, the generated right-invariant vector field $\hat{Y}(s,g)$ is bounded in $s \in [t,T]$ with respect to the strong metric (2.6), and it is at least C^1-smooth in $g \in \mathcal{D}_v^\alpha(\mathbb{T}^n)$. The boundedness of $\hat{Y}(s)$ in g follows from the volume-preserving property of g. □

Now, we will present a similar result in which we verify the existence of a solution to the Forward Stochastic Differential Equation 3.9 on the manifold $\mathcal{D}^\alpha(\mathbb{T}^2)$ and we establish an equivalence between the solutions on $\mathcal{D}^\alpha(\mathbb{T}^2)$ and $\mathcal{D}_v^\alpha(\mathbb{T}^2)$.

Theorem 3.2. *There exists a unique strong solution $Z^{t,e}(s)$, where $s \in [t,T]$, to FSDE (3.9) on the Hilbert manifold $\mathcal{D}^\alpha(\mathbb{T}^2)$, with the initial condition $Z^{t,e}(t) = e$. This solution coincides with the solution to FSDE (3.9) on $\mathcal{D}^\alpha_v(\mathbb{T}^2)$.*

Proof. Consider the identical embedding defined as:

$$i : \mathcal{D}^\alpha_v(\mathbb{T}) \to \mathcal{D}^\alpha(\mathbb{T}^2).$$

According to Remark 1.5, the stochastic process $i(Z^{t,e}(s)) = Z^{t,e}(s)$, where $s \in [t,T]$, is a solution to SDE (3.9) on the manifold $\mathcal{D}^\alpha(\mathbb{T}^2)$, that is, with respect to the exponential map exp. This follows from the fact that $Ti : T\mathcal{D}^\alpha_v(\mathbb{T}^2) \to T\mathcal{D}^\alpha(\mathbb{T}^2)$, where T is the tangent map, is the identical embedding, and that $i\big(\exp(X)\big) = \widetilde{\exp}(Ti \circ X)$.

Therefore, the solution $Z^{t,e}(s)$ to FSDE (3.9) on $\mathcal{D}^\alpha(\mathbb{T}^2)$ is unique. This statement follows from the uniqueness theorem for FSDE (3.9) considered on the manifold $\mathcal{D}^\alpha(\mathbb{T}^2)$ equipped with the weak Riemannian metric.

Indeed, $\mathbb{A}(g)$ and $\hat{Y}(s,g)$ are bounded with respect to the weak metric (2.2) since the functions $\bar{A}_k$, $\bar{B}_k$, where $k \in \mathbb{Z}^2_+ \cup \{0\}$, are bounded on $\mathbb{T}^2$, and $y(\cdot,\cdot)$ is bounded on $[t,T] \times \mathbb{T}^2$. Moreover, the vector field $\mathbb{A}(g)$ is C^∞-smooth and $\hat{Y}(s)$ is at least C^1-smooth on $\mathcal{D}^\alpha(\mathbb{T}^2)$. □

In the following result, we will show that there is a unique strong solution $Z^{t,e}(s)$ for the FSDE if we consider that it takes values in a Hilbert space $H^\alpha(\mathbb{T}^2, \mathbb{R}^2)$ and, furthermore, that this solution coincides with the solutions described in the two previous theorems.

Theorem 3.3. *There exists a unique strong solution $Z^{t,e}(s)$ to the $H^\alpha(\mathbb{T}^2, \mathbb{R}^2)$-valued FSDE (3.9) on time interval $[t,T]$, with the initial condition $Z^{t,e}(t) = e$ where e is the identity of $\mathcal{D}^\alpha_v(\mathbb{T}^2)$. This solution coincides with the solution to FSDE (3.9) on manifolds $\mathcal{D}^\alpha_v(\mathbb{T}^2)$ or $\mathcal{D}^\alpha(\mathbb{T}^2)$.*

Proof. The proof is slightly similar to the proof of the previous theorem. First, according to Theorem 3.1, FSDE (3.9) on the manifold $\mathcal{D}^\alpha_v(\mathbb{T}^2)$ has a unique strong solution $Z^{t,e}(s)$ on $[t,T]$.

Let us prove that the solution $Z^{t,e}(s)$ to (3.9) solves this FSDE considered as an FSDE in Hilbert Space $H^\alpha(\mathbb{T}^2, \mathbb{R}^2)$. Consider the identical embedding given by:

$$f : \mathcal{D}_v^\alpha(\mathbb{T}^2) \to H^\alpha(\mathbb{T}^2, \mathbb{R}^2),$$
$$f(g) \mapsto g.$$

Applying Itô's formula to the embedding $f(\cdot)$ and taking into account that:

$$A_k(g)f(g) = \nabla_{\bar{A}_k}\theta \circ g = A_k(g)$$

and that

$$A_k(g)A_k(g)f(g) = A_k(g)A_k(g) = 0,$$

we obtain that the solution $Z^{t,e}(s)$ to equation (3.9) on the manifold $\mathcal{D}_v^\alpha(\mathbb{T}^2)$ solves the $H^\alpha(\mathbb{T}^2, \mathbb{R}^2)$-valued FSDE (3.9).

Furthermore, by the uniqueness theorem for SDEs in Hilbert spaces, the FSDE (3.9) can have only one solution in the space $L_2(\mathbb{T}^2, \mathbb{R}^2)$.

This proves the uniqueness of its solution in $H^\alpha(\mathbb{T}^2, \mathbb{R}^2)$ as well. Finally, this concludes that the solutions to (3.9) on $\mathcal{D}_v^\alpha(\mathbb{T}^2)$, $\mathcal{D}^\alpha(\mathbb{T}^2)$, and in $H^\alpha(\mathbb{T}^2, \mathbb{R}^2)$ coincide. □

Now, our objective is to find the representations of FSDE (3.9) in normal coordinates on $\mathcal{D}^\alpha(\mathbb{T}^2)$ and $\mathcal{D}_v^\alpha(\mathbb{T}^2)$. But, first, it is necessary to prove the following lemma.

Lemma 3.4. *The following equality holds:*

$$\int_t^s \mathbb{A}(Z^{t,e}(r)) \circ dW_r = \int_t^s \mathbb{A}(Z^{t,e}(r))dW_r.$$

In other words, instead of the Itô stochastic integral in equation (3.9) *we can write the Stratonovich stochastic integral denoted by* $\int_t^s \mathbb{A}(Z^{t,e}(r)) \circ dW_r$.

Proof. We have the following equality:

$$\mathbb{A}(Z^{t,e}(r)) \circ dW_r = \mathbb{A}(Z^{t,e}(r))dW_r + \sum_k dA_k(Z^{t,e}(r))d\beta_k^A(r) + \sum_k dB_k(Z^{t,e}(r))d\beta_k^B(r)$$

in which $k \in \mathbb{Z}_+^2 \cup \{0\}$ is such that $|k| \leq N$.

In this way, our objective is to prove that:

$$dA_k(Z^{t,e}(r))d\beta_k^A(r) = 0 \quad \text{and} \quad dB_k(Z^{t,e}(r))d\beta_k^B(r) = 0.$$

For simplicity of notation, we use the notation C_i for both of the vector fields A_k and B_k and the notation $\bar{C}_i$ for $\bar{A}_k$ and $\bar{B}_k$, $k \in \mathbb{Z}_+^2 \cup \{0\}$.

We will also simplify the notation of Brownian motion and use: $\beta_i(s)$ for the Brownian motions $\{\beta_k^A(s), \beta_k^B(s)\}$ where $k \in \mathbb{Z}_+^2 \cup \{0\}$ is such that $|k| \leq N$.

So, we obtain:

$$d(\bar{C}_i \circ Z^{t,e}(s)) = Y^{t,e}\big(s, (\bar{C}_i \circ Z^{t,e}(s)\big)dt + \sum_j C_j(Z^{t,e}(s))\big(\bar{C}_i \circ Z^{t,e}(s)\big) \circ d\beta_j(s)$$

Multiplying $d\beta_i(s)$ on both sides, we get:

$$\begin{aligned} &d(\bar{C}_i \circ Z^{t,e}(s)) \cdot d\beta_i(s) \\ =&Y^{t,e}\big(s, (\bar{C}_i \circ Z^{t,e}(s)\big)dt \cdot d\beta_i(s) + \sum_j C_j(Z^{t,e}(s))\big(\bar{C}_i \circ Z^{t,e}(s)\big) \circ d\beta_j(s)d \cdot \beta_i(s) \\ =&\sum_j C_j(Z^{t,e}(s))\big(\bar{C}_i \circ Z^{t,e}(s)\big)dt \end{aligned}$$

which vanish by the identity $(\bar{k}, \nabla)\cos(k \cdot \theta) = (\bar{k}, \nabla)\sin(k \cdot \theta) = 0$ or by differentiating of constant vector fields. □

Now, we will bring up a result in which we can represent the Forward Stochastic Differential Equation (3.9) in local coordinates. For this, consider: $\bar{Z}^t(s) = \{Z^{t;kA}(s), Z^{t;kB}(s)\}$ which $k \in \mathbb{Z}_+^2 \cup \{0\}$ be the vector of local coordinates of the solution $Z^{t,e}(s)$ to (3.9) on $\mathcal{D}_v^\alpha(\mathbb{T}^2)$, in other words, the vector of normal coordinates provided by the exponential map $\widetilde{\exp}: T_e\mathcal{D}_v^\alpha(\mathbb{T}^2) \to \mathcal{D}_v^\alpha(\mathbb{T}^2)$. Let U_e be the canonical chart of the map $\widetilde{\exp}$.

Theorem 3.4 (FSDE (3.9) in local coordinates). *Consider the stopping time τ given by:*

$$\tau = \inf\{s \in [t,T] : Z^{t,e}(s) \notin U_e\}. \tag{3.10}$$

On the interval $[t,\tau]$, the FSDE (3.9) has the following representation in local coordinates:

$$\begin{aligned} Z^{t,kA}(s\wedge\tau) &= \int_t^{s\wedge\tau} Y^{t;kA}(r)dr + \delta_k\epsilon\,(\beta_k^A(s\wedge\tau) - \beta_k^A(t)),\\ Z^{t,kB}(s\wedge\tau) &= \int_t^{s\wedge\tau} Y^{t;kB}(r)dr + \delta_k\epsilon\,(\beta_k^B(s\wedge\tau) - \beta_k^B(t)). \end{aligned} \tag{3.11}$$

where $\delta_k = 1$ if $|k| \le N$, and $\delta_k = 0$ if $|k| > N$.

Proof. Let $\bar{g} = \{g^{kA}, g^{kB}\}$ where $k \in \mathbb{Z}_+^2 \cup \{0\}$ be local coordinates in the neighborhood U_e provided by the map $\widetilde{\exp}: T_e\mathcal{D}_v^\alpha(\mathbb{T}^2) \to \mathcal{D}_v^\alpha(\mathbb{T}^2)$.

Let $f \in C^\infty(\mathcal{D}_v^\alpha(\mathbb{T}^2))$, and let $\tilde{f}: T_e\mathcal{D}_v^\alpha(\mathbb{T}^2) \to \mathbb{R}$ be such that $\tilde{f} = f \circ \widetilde{\exp}$. Since $\widetilde{\exp}$ is a C^∞-map (see [7]), then $\tilde{f} \in C^\infty(U_0)$, where $U_0 = \widetilde{\exp}^{-1}U_e$.

Note that $\frac{\partial}{\partial g^{kA}}\tilde{f}(\bar{g}) = A_k(g)f(g)$ and $\frac{\partial}{\partial g^{kB}}\tilde{f}(\bar{g}) = B_k(g)f(g)$. Applying Itô's formula, we obtain:

$$\begin{aligned} & f(Z^{t,e}(s\wedge\tau)) - f(e)\\ =& \tilde{f}(\bar{Z}^{t,0}(s\wedge\tau)) - \tilde{f}(0)\\ =& \sum_k \int_t^{s\wedge\tau} \frac{\partial\tilde{f}}{\partial g^{kA}}(\bar{Z}^t(r))Y^{t;kA}(r)dr + \epsilon\sum_k\int_t^{s\wedge\tau} \delta_k\,\frac{\partial\tilde{f}}{\partial g^{kA}}(\bar{Z}^t(r))Y^{t;kA}(r)\circ d\beta_k^A(r)\\ &+ \sum_k \int_t^{s\wedge\tau} \frac{\partial\tilde{f}}{\partial g^{kB}}(\bar{Z}^t(r))Y^{t;kB}(r)dr + \epsilon\sum_k\int_t^{s\wedge\tau} \delta_k\,\frac{\partial\tilde{f}}{\partial g^{kB}}(\bar{Z}^t(r))Y^{t;kB}(r)\circ d\beta_k^B(r)\\ =& \sum_k\int_t^{s\wedge\tau}\big(Y^{t;kA}(r)A_k(Z^{t,e}(r))f(Z^{t,e}(r)) + Y^{t;kB}(r)B_k(Z^{t,e}(r)\big)f(Z^{t,e}(r))\big)dr \end{aligned}$$

$$+\epsilon \sum_k \int_t^{s\wedge\tau} \delta_k \left(A_k(Z^{t,e}(r))f(Z^{t,e}(r)) \circ d\beta_k^A(r) + B_k(Z^{t,e}(r))f(Z^{t,e}(r)) \circ d\beta_k^B(r)\right).$$

where $k \in \mathbb{Z}_+^2 \cup \{0\}$.

Using the representation of the process $Y^{t,e}(s)$ in the coordinates of the space $\mathcal{D}_v^\alpha(\mathbb{T}^2)$ given by 3.8 and the representation of the stochastic integral as a summation given by 3.4, we obtain:

$$\begin{aligned} & f(Z^{t,e}(s\wedge\tau)) - f(e) \\ = & \int_t^{s\wedge\tau} \hat{Y}(r, Z^{t,e}(r))f(Z^{t,e}(r))dr + \epsilon \int_t^{s\wedge\tau} \mathbb{A}(Z^{t,e}(r))f(Z^{t,e}(r)) \circ dW_r. \end{aligned}$$

This shows that the process given by:

$$\exp\left\{\sum_k Z^{t,kA}(s\wedge\tau)\bar{A}_k + Z^{t,kB}(s\wedge\tau)\bar{B}_k\right\}$$

where $k \in \mathbb{Z}_+^2 \cup \{0\}$ solves the Forward SDE (3.9) on the time interval $[t, \tau]$. $\square$

Now, we will show a result similar to this theorem, however, instead of working with the group of volume-preserving diffeomorphisms $\mathcal{D}_v^\alpha(\mathbb{T}^2)$, we will work with the group of diffeomorphisms $\mathcal{D}^\alpha(\mathbb{T}^2)$. In this way, consider

$$\check{\check{Z}}^t(s) = \{\check{Z}^{t;kA}(s), \check{Z}^{t;kB}(s), \check{Z}^{t;k\mathcal{A}}(s), \check{Z}^{t;k\mathcal{B}}(s), \check{Z}^{t;0A}(s), \check{Z}^{t;0B}(s)\}$$

where $k \in \mathbb{Z}_+^2$ be the vector of local coordinates of the solution $Z^{t,e}(s)$ to Forward SDE (3.9) on $\mathcal{D}^\alpha(\mathbb{T}^2)$. That is, the vector of normal coordinates provided by the exponential map $\exp : T_e\mathcal{D}^\alpha(\mathbb{T}^2) \to \mathcal{D}^\alpha(\mathbb{T}^2)$. Furthermore, let $\check{U}_e$ be the canonical chart of the map exp.

Theorem 3.5. *Let τ be a stopping time given by:*

$$\check{\tau} = \inf\{s \in [t, T] : Z^{t,e}(s) \notin \check{U}_e\}.$$

Then, a.s. $\check{\tau} = \tau$, where the stopping time τ is defined by (3.10), and on $[t, \tau]$, *we have the following equalities:* $\check{Z}^{t;kA}(s) = Z^{t;kA}(s)$, $\check{Z}^{t;kB}(s) = Z^{t;kB}(s)$ *for all* $k \in \mathbb{Z}^2_+ \cup \{0\}$ *and* $\check{Z}^{t;k\mathcal{A}}(s) = \check{Z}^{t;k\mathcal{B}}(s) = 0$ *for all* $k \in \mathbb{Z}^2_+$*, a.s.*

Proof. Let us introduce additional local coordinates $g^{k\mathcal{A}}, g^{k\mathcal{B}}$, $k \in \mathbb{Z}^2_+$. In this way, consider: $\bar{g} = \{g^{kA}, g^{kB}, g^{k\mathcal{A}}, g^{k\mathcal{B}}, g^{0\mathcal{A}}, g^{0\mathcal{B}}\}$ where $k \in \mathbb{Z}^2_+ \cup \{0\}$ be local coordinates in the neighborhood $\check{U}_e$ provided by the map exp mentioned above.

Since the proof follows in a similar way to the previous case proved in 3.4, we will omit it, making only the necessary observations that $Y^{k\mathcal{A}}(s) = Y^{k\mathcal{B}}(s) = 0$ for all $k \in \mathbb{Z}^2_+$, and that the components of the Brownian motion are non-zero only along divergence-free and constant vector fields.

So, we obtain that the coordinate process given by $\check{\check{Z}}^t(s)$ verifies the Forward SDE in local coordinates (3.11) and the equations $\check{Z}^{t;k\mathcal{A}}(s) = \check{Z}^{t;k\mathcal{B}}(s) = 0$, where $k \in \mathbb{Z}^2_+$. □

3.3.2 The Backward SDE and the solution of the FBSDE

Finally, in this subsection, our goal will be to establish a relationship between the solution of Backward Navier-Stokes Equations presented in (3.7) and the solution of the system of forward and backward stochastic differential equations shown in (3.3).

The most important result of this subsection is the theorem described as follow:

Theorem 3.6. *Let* $\hat{Y}(s)$ *be the right-invariant vector field generated by the solution* $y(s, \cdot)$ *to the backward Navier-Stokes equations* (3.7)*. Moreover, let* $Z^{t,e}(s)$ *be the solution to SDE* (3.9) *on* $\mathcal{D}^{\alpha}_{v}(\mathbb{T}^2)$*. Then there exists an* $\epsilon > 0$ *such that the triple of stochastic processes given by:*

$$\begin{aligned} &Z^{t,e}(s), \\ &Y^{t,e}(s) = \hat{Y}(s, Z^{t,e}(s)), \\ &X^{t,e}(s) = \epsilon\, \mathbb{A}(Z^{t,e}(s))\hat{Y}(s, Z^{t,e}(s)). \end{aligned} \tag{3.12}$$

solves the system of FBSDEs (3.3) *on the interval* $[t, T]$*.*

However, before proving this theorem, we need some lemmas and results that we will show below.

Remark 3.2. The expression $\mathbb{A}(Z^{t,e}(s))\hat{Y}(s, Z^{t,e}(s))$ shown above means the following:

$$\mathbb{A}(Z^{t,e}(s))\hat{Y}(s, Z^{t,e}(s)) = \sum_k A_k(Z_s^{t,e})\hat{Y}(s, Z_s^{t,e}) \otimes e_k^A + \sum_k B_k(Z_s^{t,e})\hat{Y}(s, Z_s^{t,e}) \otimes e_k^B$$

where $k \in \mathbb{Z}_+^2 \cup \{0\}$ is such that $|k| \leq N$ and $\hat{Y}(s, \cdot)$ is regarded as a function $\mathcal{D}_v^\alpha(\mathbb{T}^2) \to H^\alpha(\mathbb{T}^2, \mathbb{R}^2)$ in which $s \in [t, T]$ and $A_k(g)\hat{Y}(s, g)$ means differentiation of $\hat{Y}(s, \cdot) : \mathcal{D}_v^\alpha(\mathbb{T}^2) \to H^\alpha(\mathbb{T}^2, \mathbb{R}^2)$ along the vector field A_k at the point $g \in \mathcal{D}_v^\alpha(\mathbb{T}^2)$.

Let $\gamma(\xi)$ be the geodesic on the $\mathcal{D}_v^\alpha(\mathbb{T}^2)$ such that $\gamma(0) = e$ and $\gamma'(0) = \bar{A}_k$.

We obtain:

$$\begin{aligned} A_k(g)\hat{Y}(s, g(\theta)) &= \frac{d}{d\xi}\, \hat{Y}\left(s, (\gamma(\xi) \circ g)(\theta)\right)\Big|_{\xi=0} \\ &= R_g \frac{d}{d\xi}\, \hat{Y}(s, \gamma(\xi(\theta)))\Big|_{\xi=0} \end{aligned}$$

Remember that $\hat{Y}(s, g) = y(s, \cdot) \circ g$, where $g \in \mathcal{D}_v^\alpha(\mathbb{T}^2)$ and, furthermore, $\gamma(0) = e$. Then,

$$\begin{aligned} R_g \frac{d}{d\xi}\, \hat{Y}(s, \gamma(\xi(\theta)))\Big|_{\xi=0} &= R_g \frac{d}{d\xi}\, y(s, \cdot) \circ \gamma(\xi(\theta))\Big|_{\xi=0} \\ &= R_g \frac{d}{d\xi}\, y(s, e(\theta)) \\ &= R_g \nabla_{\bar{A}_k} y(s, \theta) \\ &= \bar{\nabla}_{A_k} \hat{Y}(s, g(\theta)). \end{aligned}$$

Therefore,

$$A_k(g)\hat{Y}(s, g)(\theta) = \bar{\nabla}_{A_k} \hat{Y}(s, g(\theta)) \tag{3.13}$$

Thus, we can represent the process $X^{t,e}(s)$ as:

$$X^{t,e}(s) = \epsilon \sum_k \left[\nabla_{\bar{A}_k} y(s,\cdot) \otimes e_k^A + \nabla_{\bar{B}_k} y(s,\cdot) \otimes e_k^B\right] \circ Z^{t,e}(s), \tag{3.14}$$

where $k \in \mathbb{Z}_+^2 \cup \{0\}$ is such that $|k| \leq N$. Moreover, the stochastic integral presented in BSDE (3.5) can be represented as:

$$\begin{aligned}\int_s^T X^{t,e}(r) dW_r = \epsilon \sum_k \int_s^T \nabla_{\bar{A}_k} y(r,\cdot) \circ Z^{t,e}(r) d\beta_k^A(r) \\ + \epsilon \sum_k \int_s^T \nabla_{\bar{B}_k} y(r,\cdot) \circ Z^{t,e}(r) d\beta_k^B(r).\end{aligned}$$

With k under the conditions described above.

In particular, if $N = 0$, we have:

$$\begin{aligned}\int_s^T X^{t,e}(r) dW_r = \epsilon \int_s^T \frac{\partial}{\partial \theta_1} y(r,\cdot) \circ Z^{t,e}(r) d\beta_0^A(r) \\ + \epsilon \int_s^T \frac{\partial}{\partial \theta_2} y(r,\cdot) \circ Z^{t,e}(r) d\beta_0^B(r).\end{aligned}$$

In next lemma, $\tilde{\alpha}$ is an integer which is not necessary equal to α.

Lemma 3.5 (The Laplacian of a right-invariant vector field)**.** *Let $\hat{V}$ be the right-invariant vector field on $\mathcal{D}^{\tilde{\alpha}}(\mathbb{T}^2)$ generated by an $H^{\tilde{\alpha}+2}$-vector field V on $\mathbb{T}^2$. Furthermore, let $\epsilon > 0$ be a constant such that:*

$$\frac{\epsilon^2}{2}\left(1 + \frac{1}{2}\sum_k \frac{1}{|k|^{2\alpha}}\right) = \nu,$$

where $k \in \mathbb{Z}_+^2, |k| \leq N$. Then for all $g \in \mathcal{D}^{\tilde{\alpha}}(\mathbb{T}^2)$, the following equality holds:

$$\frac{\epsilon^2}{2}\sum_k \left(\bar{\nabla}_{A_k}\bar{\nabla}_{A_k} + \bar{\nabla}_{B_k}\bar{\nabla}_{B_k}\right) \hat{V}(g) = \nu\, \Delta V \circ g. \tag{3.15}$$

where $k \in \mathbb{Z}_+^2 \cup \{0\}, |k| \leq N$.

Proof. According to Lemma 3.2, due to right-invariance of the vector fields $\bar{\nabla}_{A_k}\bar{\nabla}_{A_k}\hat{V}$ and $\bar{\nabla}_{B_k}\bar{\nabla}_{B_k}\hat{V}$, it is sufficient to show that the equation (3.15) is satisfied for the case $g = e$.

We observe that

$$(\bar{k}, \nabla)\cos(k\cdot\theta) = (\bar{k}, \nabla)\sin(k\cdot\theta) = 0.$$

Indeed, both expressions contain the scalar product $(\bar{k}, k) = 0$.

Then, for $k \in \mathbb{Z}^2_+$ and $\theta \in \mathbb{T}^2$, we have:

$$\begin{aligned}\bar{\nabla}_{A_k}\bar{\nabla}_{A_k}\hat{V}(e)(\theta) &= \frac{1}{|k|^{2\alpha+2}}\cos(k\cdot\theta)(\bar{k},\nabla)\left[\cos(k\cdot\theta)(\bar{k},\nabla)V(\theta)\right]\\ &= \frac{1}{|k|^{2\alpha+2}}\cos^2(k\cdot\theta)(\bar{k},\nabla)^2V(\theta).\end{aligned}$$

Analogously,

$$\begin{aligned}\bar{\nabla}_{B_k}\bar{\nabla}_{B_k}\hat{V}(e)(\theta) = &= \frac{1}{|k|^{2\alpha+2}}\sin(k\cdot\theta)(\bar{k},\nabla)\left[\sin(k\cdot\theta)(\bar{k},\nabla)V(\theta)\right]\\ &= \frac{1}{|k|^{2\alpha+2}}\sin^2(k\cdot\theta)(\bar{k},\nabla)^2V(\theta).\end{aligned}$$

Hence, for each $k \in \mathbb{Z}^2_+$,

$$\begin{aligned}&(\bar{\nabla}_{A_k}\bar{\nabla}_{A_k} + \bar{\nabla}_{B_k}\bar{\nabla}_{B_k})\,\hat{V}(e)(\theta) \qquad (3.16)\\ &= \frac{1}{|k|^{2\alpha+2}}\cos^2(k\cdot\theta)(\bar{k},\nabla)^2V(\theta) + \frac{1}{|k|^{2\alpha+2}}\sin^2(k\cdot\theta)(\bar{k},\nabla)^2V(\theta).\\ &= \frac{1}{|k|^{2\alpha+2}}\left(\cos^2(k\cdot\theta)+\sin^2(k\cdot\theta)\right)(\bar{k},\nabla)^2V(\theta).\\ &= \frac{1}{|k|^{2\alpha+2}}(\bar{k},\nabla)^2V(\theta).\end{aligned}$$

Note that for each $k \in \mathbb{Z}^2_+$, either $\bar{k}$ or $-\bar{k}$ is in $\mathbb{Z}^2_+$, and

$$(\bar{k},\nabla)^2 + (k,\nabla)^2 = |k|^2\Delta. \qquad (3.17)$$

Indeed, the equality follows as below:

$$
\begin{aligned}
|k|^2\Delta &= \left(\sqrt{k_1^2+k_2^2}\right)^2 \cdot \left(\frac{\partial^2}{\partial\theta_1^2}+\frac{\partial^2}{\partial\theta_2^2}\right)\\
&= (k_1^2+k_2^2)\cdot\left(\frac{\partial^2}{\partial\theta_1^2}+\frac{\partial^2}{\partial\theta_2^2}\right)\\
&= k_1^2\frac{\partial^2}{\partial\theta_1^2}+k_2^2\frac{\partial^2}{\partial\theta_1^2}+k_1^2\frac{\partial^2}{\partial\theta_2^2}+k_2^2\frac{\partial^2}{\partial\theta_2^2}\\
&= k_1^2\frac{\partial^2}{\partial\theta_1^2}+k_2^2\frac{\partial^2}{\partial\theta_1^2}+k_1^2\frac{\partial^2}{\partial\theta_2^2}+k_2^2\frac{\partial^2}{\partial\theta_2^2}+2\left(k_1k_2\frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2}-k_1k_2\frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2}\right)\\
&= k_2^2\frac{\partial^2}{\partial\theta_1^2}-2k_1k_2\frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2}+k_1^2\frac{\partial^2}{\partial\theta_2^2}+k_1^2\frac{\partial^2}{\partial\theta_1^2}+2k_1k_2\frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2}+k_2^2\frac{\partial^2}{\partial\theta_2^2}\\
&= \left(k_2\frac{\partial}{\partial\theta_1}-k_1\frac{\partial}{\partial\theta_2}\right)^2+\left(k_1\frac{\partial}{\partial\theta_1}+k_2\frac{\partial}{\partial\theta_2}\right)^2\\
&= (\bar{k},\nabla)^2+(k,\nabla)^2.
\end{aligned}
$$

Summation over $k\in\mathbb{Z}_+^2$, such that $|k|\le N$, in (3.16), coupling the terms numbered by k and $\bar{k}$ (or $-\bar{k}$) and using equality (3.17), we get:

$$
\begin{aligned}
\sum_k(\bar{\nabla}_{A_k}\bar{\nabla}_{A_k}+\bar{\nabla}_{B_k}\bar{\nabla}_{B_k})\hat{V}(e)(\theta) &= \frac{1}{|k|^{2\alpha+2}}(\bar{k},\nabla)^2V(\theta)\\
&= \frac{1}{2}\sum_k\frac{1}{|k|^{2\alpha+2}}\cdot|k|^2\,\Delta V(\theta)\\
&= \frac{1}{2}\sum_k\frac{1}{|k|^{2\alpha}}\,\Delta V(\theta).
\end{aligned}
$$

where $k\in\mathbb{Z}_+^2, |k|\le N$.

Let be $k=0$. Note that $(\bar{\nabla}_{A_0}\bar{\nabla}_{A_0}+\bar{\nabla}_{B_0}\bar{\nabla}_{B_0})\hat{V}(e)(\theta)=\Delta V(\theta)$.

Therefore, we obtain:

$$
\sum_{\substack{k\in\mathbb{Z}_+^2\cup\{0\},\\|k|\le N}}\left(\bar{\nabla}_{A_k}\bar{\nabla}_{A_k}+\bar{\nabla}_{B_k}\bar{\nabla}_{B_k}\right)\hat{V}(e)(\theta)=\left(1+\frac{1}{2}\sum_{\substack{k\in\mathbb{Z}_+^2,\\|k|\le N}}\frac{1}{|k|^{2\alpha}}\right)\Delta V(\theta).
$$

The lemma is proved. □

Corollary 3.1. *Let the function $\varphi : \mathbb{T}^2 \to \mathbb{R}^2$ be C^2-smooth. Furthermore, let $A_k(g)[\varphi \circ g]$ and $B_k(g)[\varphi \circ g]$, such that $k \in \mathbb{Z}_2^+$, mean the differentiation of the function $\mathcal{D}^{\tilde{\alpha}}(\mathbb{T}^2) \to L_2(\mathbb{T}^2, \mathbb{R}^2)$, $g \mapsto \varphi \circ g$ along A_k and respectively B_k. Then for all $g \in \mathcal{D}^{\tilde{\alpha}}(\mathbb{T}^2)$,*

$$\frac{\epsilon^2}{2} \sum_k \big(A_k(g)A_k(g) + B_k(g)B_k(g)\big) [\varphi \circ g] = \nu \, \Delta\varphi \circ g. \tag{3.18}$$

where $k \in \mathbb{Z}_+^2 \cup \{0\}$, $|k| \leq N$.

Proof. The proof of this corollary follows two steps performed in previous results. The first is to calculate the derivative of the composite function $\varphi \circ g$ in the directions A_k and B_k.

To calculate $A_k(g)[\varphi \circ g]$ and $B_k(g)[\varphi \circ g]$ we follow as done in 3.13, from where we get:

$$A_k(g)[\varphi \circ g] = \left[\frac{1}{|k|^{\alpha+1}} \cos(k \cdot \theta)(\bar{k}, \nabla)\varphi(\theta) \right] \circ g.$$

and

$$B_k(g)[\varphi \circ g] = \left[\frac{1}{|k|^{\alpha+1}} \sin(k \cdot \theta)(\bar{k}, \nabla)\varphi(\theta) \right] \circ g.$$

Now, the second step is to repeat the proof of the previous Lemma 3.5 to obtain the equation (3.18). In other words, we differentiate A_k and B_k a second time and sum all $A_k(g)A_k(g)[\varphi \circ g]$ and $B_k(g)B_k(g)[\varphi \circ g]$ as we did in (3.16).

With this, it is noted that the terms $\sin^2(k \cdot \theta)$ and $\cos^2(k \cdot \theta)$ are simplified to 1 and we perform the substitution made by (3.17), obtaining the equation:

$$\frac{1}{2} \sum_k \frac{1}{|k|^{2\alpha}} \Delta[\varphi \circ g].$$

where $k \in \mathbb{Z}_+^2$ is such that $|k| \leq N$.

The parcels A_0 and B_0 in which $k = 0$ are obtained in a manner analogous to the previous lemma. Finally, we do the summation over all k and we arrive at the equation (3.18). □

Lemma 3.6. *Let $\psi(r)$, $r \in [t,T]$ in which $t \in [0,T)$, be an $H^\alpha(\mathbb{T}^2,\mathbb{R}^2)$-valued stochastic process whose trajectories are integrable and let $\phi(T)$ be an $H^\alpha(\mathbb{T}^2,\mathbb{R}^2)$-valued random element so that both $\psi(r)$ and $\phi(T)$ possess finite expectations. Then there exists an $\mathcal{F}_s$-adapted $H^\alpha(\mathbb{T}^2,\mathbb{R}^2)\times\mathcal{L}\big(E, H^\alpha(\mathbb{T}^2,\mathbb{R}^2)\big)$-valued pair of stochastic processes $(Y(s), X(s))$ solving the BSDE:*

$$Y(s) = \phi(T) + \int_s^T \psi(r)\,dr - \int_s^T X(r)\,dW_r \tag{3.19}$$

in the time interval $[t,T]$.

The $Y(s)$-part of the solution has the representation

$$Y(s) = \mathbb{E}\left[\phi(T) + \int_s^T \psi(r)\,dr \,\middle|\, \mathcal{F}_s\right], \tag{3.20}$$

and therefore is unique.

The $X(s)$-part of the solution is unique with respect to the norm:

$$\|X\|^2 = \int_t^T \|X(r)\|_{\mathcal{L}}^2\,dr \tag{3.21}$$

where $||\cdot||_{\mathcal{L}}$ is the $\mathcal{L}(E, H^\alpha(\mathbb{T}^2,\mathbb{R}^2))$-norm.

Proof. Extend $Y(s)$, defined by (3.20), to $[0,t]$ as follows:

$$Y(s) = \mathbb{E}\left[\phi(T) + \int_t^T \psi(r)\,dr \,\middle|\, \mathcal{F}_s\right], \tag{3.22}$$

and note that $Y(s)$ is a martingale for $s \in [0,t]$. Also note that

$$M(s) := \mathbb{E}\left[\phi(T) + \int_t^T \psi(r)dr - Y(t) \,\middle|\, \mathcal{F}_s\right] \tag{3.23}$$

is a martingale which is zero for $s \in [0,t]$. Indeed, since for $s \in [0,t]$, $\mathbb{E}[Y(t)|\mathcal{F}_s] =$

$Y(s)$, we obtain

$$\begin{aligned} M(s) =& \mathbb{E}\left[\phi(T) + \int_t^T \psi(r)dr - Y(t) \,\middle|\, \mathcal{F}_s\right] \\ =& \mathbb{E}\left[\phi(T) + \int_t^T \psi(r)\,dr \,\middle|\, \mathcal{F}_s\right] - Y(s) = 0. \end{aligned}$$

By the martingale representation theorem, there exists a process $X(s)$ such that

$$\mathbb{E}\left[\phi(T) + \int_t^T \psi(r)dr - Y(t)\middle|\mathcal{F}_s\right] = \int_0^s X(r)\,dW_r \tag{3.24}$$

Since $M(s) = 0$ on $[0,t]$, we obtain that

$$\int_0^s X(r)\,dW_r = 0 \quad \text{for} \;\; s \in [0,t].$$

Therefore, for $s \in [t,T]$, (3.24) implies that

$$\mathbb{E}\left[\phi(T) + \int_t^T \psi(r)dr - Y(t)\middle|\mathcal{F}_s\right] = \int_t^s X(r)\,dW_r \tag{3.25}$$

$$\mathbb{E}\left[\phi(T) + \int_t^s \psi(r)dr + \int_s^T \psi(r)dr - Y(t)\middle|\mathcal{F}_s\right] = \int_t^s X(r)\,dW_r$$

Since $Y(t)$ is $\mathcal{F}_s$-adapted ($\mathcal{F}_t \subset \mathcal{F}_s$), using the definition of $Y(s)$ by (3.20), we obtain

$$Y(s) + \int_t^s \psi(r)dr - Y(t) = \int_t^s X(r)\,dW_r. \tag{3.26}$$

Evaluating (3.25) at $s = T$, we obtain

$$\phi(T) + \int_t^T \psi(r)dr - Y(t) = \int_t^T X(r)\,dW_r. \tag{3.27}$$

Subtracting (3.26) from (3.27), we obtain that $Y(s)$ and $X(s)$ satisfy (3.19).

To prove the uniqueness, just note that any $\mathcal{F}_s$-adapted solution to BSDE (3.19), this solution takes the forms given by (3.20) and (3.21) . Moreover, consider a process $X'(s)$ such that the processes $X(s)$ and $X'(s)$ satisfy (3.21) , then, again by Itô's isometry, the following equality is valid:

$$\int_t^T \|X(s) - X'(s)\|_{\mathcal{L}}^2 \, dr = \left\| \int_t^T (X(s) - X'(s)) \, dW_r \right\|_{H^\alpha(\mathbb{T}^2,\mathbb{R}^2)}^2 = 0.$$

That is, $X(s)$ and $X'(s)$ are almost surely the same, which concludes the proof.

□

Finally, having these results, we will prove the most important theorem of this section, Theorem 3.6, which establishes a direct connection between the Backward Navier-Stokes (3.7) and FSDE (3.9) solutions for the FBSDE system (3.3).

Proof of Theorem 3.6. Let us consider BSDE (3.5) as an $L_2(\mathbb{T}^2, \mathbb{R}^2)$-valued SDE and $\hat{Y}(s)$ as a map $\mathcal{D}^\alpha(\mathbb{T}^2) \to L_2(\mathbb{T}^2, \mathbb{R}^2)$. Since for each $s \in [t, T]$, the map $y(s, \cdot)$ belongs to $H^{\alpha+1}(\mathbb{T}^2, \mathbb{R}^2)$ and $\alpha > 2$ by assumption, then $\hat{Y}(s) : \mathcal{D}_v^\alpha(\mathbb{T}^2) \to L_2(\mathbb{T}^2, \mathbb{R}^2)$ is at least C^2-smooth.

The Navier-Stokes Equations (3.2) show that the function $\partial_s y(\cdot, \cdot) : [t, T] \to L_2(\mathbb{T}^2, \mathbb{R}^2)$ is continuous since ∇p, Δy, and $(y, \nabla y)$ are continuous functions $[t, T] \to L_2(\mathbb{T}^2, \mathbb{R}^2)$ according to Assumption 3.1. Since the diffeomorphisms of the Hilbert manifold $\mathcal{D}_v^\alpha(\mathbb{T}^2)$ are volume-preserving, we conclude that for each fixed element $g \in \mathcal{D}_v^\alpha(\mathbb{T}^2)$, the map $\partial_s \hat{Y}(s, g) : [t, T] \to L_2(\mathbb{T}^2, \mathbb{R}^2)$ is a continuous function. Consequently, $\hat{Y}(\cdot, \cdot) : [t, T] \times \mathcal{D}_v^\alpha(\mathbb{T}^2) \to L_2(\mathbb{T}^2, \mathbb{R}^2)$ is C^1-smooth in $s \in [t, T]$ and C^2-smooth in $g \in \mathcal{D}_v^\alpha(\mathbb{T}^2)$. In this way, Itô's formula is applicable in the map $\hat{Y}(s, Z^{t,e}(s))$.

In what follows, we will use the fact that $Z^{t,e}(s)$ is a solution to FSDE (3.9) and that the following identity is valid:

$$\frac{\partial \hat{Y}}{\partial s}(s, Z^{t,e}(s)) = \frac{\partial y(s, \cdot)}{\partial s} \circ Z^{t,e}(s).$$

For the latter derivative we substitute the right-hand side of the first equation of (3.2). The notation $\hat{X}(g)[\hat{Y}(s,g)]$ is the same established throughout the Lemma 3.5 and the Corollary 3.18 and means differentiation of the function $\hat{Y}(s,\cdot) : \mathcal{D}_v^\alpha(\mathbb{T}^2) \to L_2(\mathbb{T}^2, \mathbb{R}^2)$ along the right-invariant vector field $\hat{X}$ on $\mathcal{D}_v^\alpha(\mathbb{T}^2)$ at the point $g \in \mathcal{D}_v^\alpha(\mathbb{T}^2)$. The same argument used in Remark 3.2 implies that

$$\hat{X}(g)[\hat{Y}(s,g)] = \bar{\nabla}_{\hat{X}} \hat{Y}(s,g).$$

Considering all these observations and established notations, we get that the Navier-Stokes Equation (3.2) becomes:

$$\begin{aligned} \hat{Y}(s, Z^{t,e}(s)) - \hat{h}(Z^{t,e}(T)) = &- \int_s^T \frac{\partial \hat{Y}}{\partial r}(r, Z^{t,e}(r))\, dr - \int_s^T \hat{Y}(r, Z^{t,e}(r))[\hat{Y}_r(Z_r^{t,e})]dr \\ &- \int_s^T \frac{\epsilon^2}{2} \sum_k A_k(Z^{t,e}(r)) A_k(Z^{t,e}(r)) \hat{Y}_r(Z^{t,e}(r))dr \\ &- \int_s^T \frac{\epsilon^2}{2} \sum_k B_k(Z^{t,e}(r)) B_k(Z^{t,e}(r)) \hat{Y}(r, Z^{t,e}(r))dr \\ &- \int_s^T \epsilon\, \mathbb{A}(Z^{t,e}(r)) \hat{Y}(r, Z^{t,e}(r)) dW_r. \end{aligned} \tag{3.28}$$

where $k \in \mathbb{Z}_+^2 \cup \{0\}$ is such that $|k| \leq N$.

Notice that:

$$\hat{Y}(r, Z^{t,e}(r))[\hat{Y}(r, Z^{t,e}(r))] = [(y(r,\cdot), \nabla) y(r,\cdot)] \circ Z^{t,e}(r).$$

Moreover, let us observe that:

$$\begin{aligned} &\frac{\epsilon^2}{2} \sum_k \big[A_k(Z^{t,e}(r)) A_k(Z^{t,e}(r)) \hat{Y}(r, Z^{t,e}(r)) + B_k(Z^{t,e}(r)) B_k(Z^{t,e}(r)) \hat{Y}(r, Z^{t,e}(r))\big] \\ &= \frac{\epsilon^2}{2} \sum_k \big[\bar{\nabla}_{A_k} \bar{\nabla}_{A_k} \hat{Y}(r, Z^{t,e}(r)) + \bar{\nabla}_{B_k} \bar{\nabla}_{B_k} \hat{Y}(r, Z^{t,e}(r))\big] \\ &= \nu[\Delta y(r,\cdot)] \circ Z^{t,e}(r) \end{aligned}$$

where $k \in \mathbb{Z}_+^2 \cup \{0\}$, $|k| \leq N$ and that the latter equality holds by Lemma 3.5 and

$\epsilon > 0$ is chosen in such a way that equality holds:

$$\frac{\epsilon^2}{2}\left(1+\frac{1}{2}\sum_{k\in\mathbb{Z}^2_+,|k|\le N}\frac{1}{|k|^{2\alpha}}\right)=\nu.$$

Note that the terms $\bar{\nabla}_{A_k}\bar{\nabla}_{A_k}\hat{Y}(r,Z^{t,e}(r))$ and $\bar{\nabla}_{B_k}\bar{\nabla}_{B_k}\hat{Y}(r,Z^{t,e}(r))$ are elements of $T\mathcal{D}^{\alpha-1}(\mathbb{T}^2)$, and therefore are well defined in $L_2(\mathbb{T}^2,\mathbb{R}^2)$.

Replacing the terms with their equivalents presented in the equality above, the new expression of the Navier-Stokes equation (3.28) is given by:

$$\begin{aligned}
&\hat{Y}(s,Z^{t,e}(s))-\hat{h}(Z^{t,e}(T)) \qquad (3.29)\\
&=\int_s^T\Big[\hat{V}(r,Z^{t,e}(r))+[(y(r,\cdot),\nabla)y(r,\cdot)]\circ Z^{t,e}(r)+\nu[\Delta y(r,\cdot)]\circ Z^{t,e}_r\Big]dr\\
&-\int_s^T[(y(r,\cdot),\nabla)y(r,\cdot)]\circ Z^{t,e}(r)\,dr-\int_s^T\nu[\Delta y(r,\cdot)]\circ Z^{t,e}(r)\,dr\\
&-\int_s^T\epsilon\,\mathbb{A}(Z^{t,e}(r))\hat{Y}(r,Z^{t,e}(r))\,dW_r\\
&=\int_s^T\hat{V}(r,Z^{t,e}(r))\,dr-\int_s^T\epsilon\,\mathbb{A}(Z^{t,e}(r))\hat{Y}(r,Z^{t,e}(r))dW_r.
\end{aligned}$$

Then, the pair of stochastic processes $(\hat{Y}(s,Z^{t,e}(s)),\ \epsilon\,\mathbb{A}(Z^{t,e}(s))\hat{Y}(s,Z^{t,e}(s)))$ is a solution to BSDE (3.5) in $L_2(\mathbb{T}^2,\mathbb{R}^2)$. And this pair is $\mathcal{F}_s$ adapted, since the process $Z^{t,e}(s)$ is $\mathcal{F}_s$-adapted.

According to Lemma 3.6, there is a unique $\mathcal{F}_s$-adapted solution $(Y^{t,e}(s),X^{t,e}(s))$ to BSDE (3.5) in the space $H^\alpha(\mathbb{T}^2,\mathbb{R}^2)$. Moreover, $(Y^{t,e}(s),X^{t,e}(s))$ is also a unique $\mathcal{F}_s$-adapted solution to BSDE (3.5) in the space $L_2(\mathbb{T}^2,\mathbb{R}^2)$.

For this reason, $Y^{t,e}(s)=\hat{Y}(s,Z^{t,e}(s))$ and

$$\int_t^T\left\|X^{t,e}(s)-\epsilon\mathbb{A}(Z^{t,e}(s))\hat{Y}(s,Z^{t,e}(s)))\right\|^2_{\mathcal{L}}ds=0, \qquad (3.30)$$

and therefore the pair of stochastic processes given by

$$\big(\hat{Y}(s,Z^{t,e}(s)),\epsilon\,\mathbb{A}(Z^{t,e}(s))\hat{Y}(s,Z^{t,e}(s))\big)$$

is a unique $\mathcal{F}_s$-adapted solution to BSDE (3.5) in $H^\alpha(\mathbb{T}^2, \mathbb{R}^2)$. Finally, the theorem is proved. □

The proof of the above theorem concludes the first major result of this work, which consisted, in simpler terms, that given a solution to the Backward Navier Stokes equations (3.7), we can find a triple of stochastic processes that solve a given FBSDE system (3.3) built on the $\mathcal{D}_v^\alpha(\mathbb{T}^2)$.

3.4 Some Identities involving Navier-Stokes solution

In this section, we will bring an extra result in which we obtain a representation for the solution of the Navier-Stokes equation, in addition to showing how this representation implies the energy identity.

Theorem 3.7. *Let $t \in [0, T]$, and let $Z^{t,e}(s)$ be the solution to FSDE (3.9) on $[t, T]$ with the initial condition $Z^{t,e}(t) = e$. Then, the following representation holds for the solution $y(t, \cdot)$ to (3.7).*

$$y(t, \cdot) = \mathbb{E}\left[\hat{h}(Z^{t,e}(T)) + \int_t^T \nabla p(s, \cdot) \circ Z^{t,e}(s) ds\right]. \tag{3.31}$$

Proof. First, note that due to the initial condition $Z^{t,e}(t) = e$, we obtain that:

$$\hat{Y}(t, Z^{t,e}(t)) = \hat{Y}(t, e) = y(t, \cdot) \circ e = y(t, \cdot),$$

and, moreover, we know that:

$$\mathbb{E}\left[\int_t^T X^{t,e}(r) dW_r\right] = 0.$$

Remember the BSDE presented in (3.5) which is rewritten below:

$$Y^{t,e}(s) = \hat{h}(Z^{t,e}(T)) + \int_s^T \hat{V}(r, Z^{t,e}(r)) dr - \int_s^T X^{t,e}(r) dW_r.$$

Taking the expectation from the both parts of BSDE at time $s = t$ we obtain the representation (3.31). □

Now, we will show a simple derivation of the energy identity. Applying Itô's formula to the squared $L_2(\mathbb{T}^2, \mathbb{R}^2)$-norm of the solution $Y^{t,e}(s)$, we obtain:

$$\begin{aligned} \|Y^{t,e}(s)\|_{L_2}^2 = \|\hat{h}(Z^{t,e}(T))\|_{L_2}^2 + 2\int_s^T \langle Y^{t,e}(r), \hat{V}(Z^{t,e}(r))\rangle_{L_2} dr \\ - 2\int_s^T \langle Y^{t,e}(r), X^{t,e}(r) dW_r\rangle_{L_2} - \int_s^T \|X^{t,e}(s)\|_{L_2}^2 dr. \end{aligned} \tag{3.32}$$

Remember that as shown in equation (3.14), we can decompose the process $X^{t,e}(s)$ with respect to its coordinates, then:

$$\begin{aligned} \|X^{t,e}(s)\|_{L_2}^2 &= \epsilon^2 \left[\sum_{k\in\mathbb{Z}_+^2\cup\{0\},\, |k|\le N} \|\nabla_{\bar{A}_k} y(s,\cdot)\|_{L_2}^2 + \|\nabla_{\bar{B}_k} y(s,\cdot)\|_{L_2}^2 \right] \\ &= \epsilon^2 \left[\sum_{k\in\mathbb{Z}_+^2,\, |k|\le N} \frac{1}{|k|^{2\alpha+2}} \|(\bar{k}, \nabla y(s,\cdot))\|_{L_2}^2 + \|\nabla y(s,\cdot)\|_{L_2}^2 \right] \\ &= \epsilon^2 \left[\frac{1}{2} \sum_{k\in\mathbb{Z}_+^2,\, |k|\le N} \frac{1}{|k|^{2\alpha+2}} \left(\|(\bar{k}, \nabla y(s,\cdot))\|_{L_2}^2 + \|(k, \nabla y(s,\cdot))\|_{L_2}^2 \right) + \|\nabla y(s,\cdot)\|_{L_2}^2 \right] \\ &= \epsilon^2 \left(1 + \frac{1}{2} \sum_{k\in\mathbb{Z}_+^2,\, |k|\le N} \frac{1}{|k|^{2\alpha}} \right) \|\nabla y(s,\cdot)\|_{L_2}^2 \\ &= 2\,\nu\, \|\nabla y(s,\cdot)\|_{L_2}^2. \end{aligned}$$

Taking the expectation in (3.32) and using the volume-preserving property of $Z^{t,e}(s)$, we obtain:

$$\|y(s,\cdot)\|_{L_2}^2 + 2\nu \int_s^T \|\nabla y(r,\cdot)\|_{L_2}^2 dr = \|h(\cdot)\|_{L_2}^2.$$

And that concludes the proof.

Chapter 4

Constructing the solution to the Navier-Stokes from a solution to the FBSDEs

4.1 Looking for a solution for Navier-Stokes

We will present throughout this section the second main result of this work. The theorem shown below is, in a way, a converse to the Theorem 3.6 proved in the previous section. At this point, in simple terms, our goal will be to find a solution to the Navier-Stokes equations starting from the hypothesis that we have a solution to a system of forward and backward equations in $\mathcal{D}_v^\alpha(\mathbb{T}^2)$.

We will consider (3.3) as a system of forward and backward SDEs in the Hilbert space $H^\alpha(\mathbb{T}^2, \mathbb{R}^2)$, where $\alpha \geq 3$. Moreover, as before, let $\hat{V}(s, Z^{t,e}(s))$ be the vector field which denotes $\nabla p(s, \cdot) \circ Z^{t,e}(s)$, and let $\mathcal{F}_s$ denote the filtration $\sigma\{W_r, r \in [0, s]\}$.

Theorem 4.1. *Assume, for an $H^{\alpha+1}$-smooth function $p(s, \cdot)$, with $s \in [0, T]$, and for any $t \in (0, T)$, the existence of an $\mathcal{F}_s$-adapted solution $(Z^{t,e}(s), Y^{t,e}(s), X^{t,e}(s))$*

to FBSDE (3.3) (rewritten below)

$$\begin{cases} dZ^{t,e}(s) &= Y^{t,e}(s)ds + \epsilon\,\mathbb{A}(Z^{t,e}(s))dW_s, \\ dY^{t,e}(s) &= -\hat{V}(s, Z^{t,e}(s))ds + X^{t,e}(s)dW_s, \\ Z^{t,e}(t) &= e; \qquad Y^{t,e}(T) = \hat{h}(Z^{t,e}(T)). \end{cases} \tag{4.1}$$

on $[t,T]$ such that the processes $Z^{t,e}(s)$ and $Y^{t,e}(s)$ have almost surely continuous trajectories and such that $Z^{t,e}(s)$ take values in $\mathcal{D}_v^\alpha(\mathbb{T}^2)$. Then there exists a time $T_0 > 0$ such that for all $T < T_0$ there exists a deterministic function $y(s,\cdot) \in T_e\mathcal{D}_v^\alpha(\mathbb{T}^2)$ on $[0,T]$, such that almost surely on $[t,T]$ the relation $Y^{t,e}(s) = y(s,\cdot)\circ Z^{t,e}(s)$ holds. Moreover, the pair of functions (y,p) solves the backward Navier-Stokes equations (3.7) on $[0,T]$ rewritten below:

$$\begin{aligned} &y(s,\theta) = h(\theta) + \int_s^T \left[\nabla p(r,\theta) + \big(y(r,\theta),\nabla\big)y(r,\theta) + \nu\Delta y(r,\theta)\right]dr, \\ &div\, y(s,\theta) = 0. \end{aligned} \tag{4.2}$$

The proof of this theorem is extensive and for this reason it will be divided into stages, which correspond to the following lemmas.

The first step is to verify that the hypotheses about the triple of solutions $(Z^{t,e}(s), Y^{t,e}(s), X^{t,e}(s))$ are true.

Lemma 4.1. *For all $t \in [0,T)$ and for any $\mathcal{F}_t$-measurable $\mathcal{D}_v^\alpha(\mathbb{T}^2)$-valued random variable ξ, the triple of stochastic processes*

$$(Z^{t,\xi}(s), Y^{t,\xi}(s), X^{t,\xi}(s)) = (Z^{t,e}(s)\circ\xi, Y^{t,e}(s)\circ\xi, X^{t,e}(s)\circ\xi) \tag{4.3}$$

is $\mathcal{F}_s$-adapted and solves the FBSDEs

$$\begin{cases} Z^{t,\xi}(s) = \xi + \int_t^s Y^{t,\xi}(r)\,dr + \int_t^s \mathbb{A}(Z^{t,\xi}(r))\,dW_r \\ Y^{t,\xi}(s) = h(Z^{t,\xi}(T)) + \int_s^T \hat{V}(r, Z^{t,\xi}(r))dr - \int_s^T X^{t,\xi}(r)\,dW_r \end{cases} \tag{4.4}$$

on the interval $[t,T]$ in the space $H^\alpha(\mathbb{T}^2,\mathbb{R}^2)$.

Proof. Let us apply the right translation operator, denoted by R_ξ, on both sides of the system of FBSDEs (3.3). We just need to proof that we can write the operator R_ξ under the signs of both stochastic integrals in (3.3). To verify this, we will show that this statement is true for an arbitrary $\mathcal{F}_t$-measurable stepwise function $\xi = \sum_{i=1}^{\infty} \xi_i \mathbb{I}_{A_i}$, where $\xi_i \in \mathcal{D}_v^\alpha(\mathbb{T}^2)$ and the sets A_i are $\mathcal{F}_t$-measurable.

Indeed, consider the times s and S such that $t \leq s < S \leq T$, and let $\Phi(r)$ be an $\mathcal{F}_r$-adapted stochastically integrable process. We obtain:

$$\begin{aligned}\int_s^S \Phi(r)\, dW_r \circ \sum_{i=1}^{\infty} \xi_i \mathbb{I}_{A_i} &= \sum_{i=1}^{\infty} \mathbb{I}_{A_i} \int_s^S \Phi(r) \circ \xi_i \, dW_r \\ &= \sum_{i=1}^{\infty} \int_s^S \mathbb{I}_{A_i} \Phi(r) \circ \xi_i \, dW_r \\ &= \int_s^S \Phi(r) \circ \sum_{i=1}^{\infty} \xi_i \mathbb{I}_{A_i} \, dW_r.\end{aligned}$$

Now, let us consider a countable number of disjoint Borel sets B_i^n covering the space of continuous functions $C(\mathbb{T}^2, \mathbb{R}^2)$, such that their diameter in the norm of $C(\mathbb{T}^2, \mathbb{R}^2)$ is smaller than $\frac{1}{n}$. And let $A_i^n = \xi^{-1}(B_i^n)$ and $\xi_i^n \in B_i^n \cap \mathcal{D}_v^\alpha(\mathbb{T}^2)$, that is, each set A_i is the pre-image by function ξ_i of a sequence of coverings B_i^n

Define ξ_n as:

$$\xi_n = \sum_{i=1}^{\infty} \xi_i^n \mathbb{I}_{A_i^n}.$$

Then it holds that for all $\omega \in \Omega$, we have that:

$$\|\xi - \xi_n\|_{C(\mathbb{T}^2, \mathbb{R}^2)} < \frac{1}{n}.$$

Our goal will be to prove that almost surely $I(\Phi) \circ \xi = I(\Phi \circ \xi)$. For this, it is enough to check the equalities:

$$\lim_{n\to\infty} \mathbb{E} \left\| \int_s^S \Phi(r) dW_r \circ \xi_n - \int_s^S \Phi(r) dW_r \circ \xi \right\|_{L_2}^2 = 0, \tag{4.5}$$

and

$$\lim_{n\to\infty} \mathbb{E} \left\| \int_s^S \Phi(r) \circ \xi_n \, dW_r - \int_s^S \Phi(r) \circ \xi \, dW_r \right\|_{L_2}^2 = 0, \tag{4.6}$$

where $||\cdot||_{L_2}$ is the $L_2(\mathbb{T}^2, \mathbb{R}^2)$-norm.

Since the functions ξ and ξ_n belong to the space $\mathcal{D}_v^{\alpha}(\mathbb{T}^2)$, they preserve volume, so the following equalities are valid:

$$\left\| \int_s^S \Phi(r) dW_r \circ \xi_n \right\|_{L_2}^2 = \left\| \int_s^S \Phi(r) dW_r \right\|_{L_2}^2$$

and

$$\left\| \int_s^S \Phi(r) dW_r \circ \xi \right\|_{L_2}^2 = \left\| \int_s^S \Phi(r) dW_r \right\|_{L_2}^2 .$$

Applying Lebesgue's theorem to equation (4.5), we can pass to the limit under the expectation sign and the equality presented in (4.5) follows due to the continuity of the stochastic integral of $\phi(r)$ in $\theta \in \mathbb{T}^2$.

On the other hand, to prove equality in (4.6), we will take an additional step which is to apply Itô's Isometry. Note that:

$$\lim_{n\to\infty} \mathbb{E} \left\| \int_s^S \left(\Phi(r) \circ \xi_n - \Phi(r) \circ \xi \right) \, dW_r \right\|_{L_2}^2 = \lim_{n\to\infty} \mathbb{E} \int_s^S \|\Phi(r) \circ \xi_n - \Phi(r) \circ \xi\|_{L_2}^2 \, dr.$$

Finally, we use the same previous argument and by Lebesgue's theorem we can pass the limit under the expectation and the integration sign and the equality presented in (4.6) follows due to continuity follows from the continuity of the function $\Phi(r)$ in $\theta \in \mathbb{T}^2$.

Therefore, the triple $(Z^{t,e}(s) \circ \xi, Y^{t,e}(s) \circ \xi, X^{t,e}(s) \circ \xi)$ is a solution to FBSDE (4.4). This solution is $\mathcal{F}_s$-adapted, since its component processes and functions are $\mathcal{F}_s$-adapted. □

In the following lemma, we will verify that the function considered in the hypotheses of Theorem 4.1 is deterministic, that is, that it does not depend on any stochastic component.

Lemma 4.2. *The map $[0,T]\times\mathbb{T}^2\to\mathbb{R}^2$ defined by $(t,\theta)\mapsto Y^{t,e}(t,\theta)$ is deterministic.*

Proof. Suppose that $(Z^{t,e}(s),Y^{t,e}(s),X^{t,e}(s))$ is a solution to the FBSDE and let us extend this solution to the time interval $[0,t]$ by setting $Z^{t,e}(s)=e$, $Y^{t,e}(s)=Y^{t,e}(t)$, $X^{t,e}(s)=0$ for all $s\in[0,t]$. The extended stochastic process solves the following system:

$$\begin{cases} Z^{t,e}(s)=e+\int_0^s \mathbb{I}_{[t,T]}(r)Y^{t,e}(r)dr+\int_0^s \mathbb{I}_{[t,T]}(r)\mathbb{A}(Z^{t,e}(r))dW_r \\ Y^{t,e}(s)=h(Z^{t,e}(T))+\int_s^T \mathbb{I}_{[t,T]}(r)\hat{V}(r,Z^{t,e}(r))dr-\int_s^T X^{t,e}(r)dW_r. \end{cases} \tag{4.7}$$

In particular, consider $s=0$, then we have that $Y^{t,e}(0)$ is a $\mathcal{F}_0$-measurable function. By Blumenthal's zero-one law (described in 1.4), then $Y^{t,e}(0)$ is deterministic and since $Y^{t,e}(t)=Y^{t,e}(0)$, the result follows. □

Lemma 4.3. *There exists a constant $T_0>0$ such that for $T<T_0$ the function $[0,T]\to H^2(\mathbb{T}^2,\mathbb{R}^2)$, defined by $t\mapsto Y^{t,e}(t)$ is continuous.*

Proof. Consider as solutions to the FBSDE system (4.4) the triples given by $(Z^{t,e}(s),Y^{t,e}(s),X^{t,e}(s))$ and $(Z^{t',e}(s),Y^{t',e}(s),X^{t',e}(s))$ which begin at the point e (identity function) respectively at times t and t', such that $t<t'$. In a procedure analogous to that carried out in the demonstration of the previous lemma, we will extend these solutions to the interval $[0,T]$ by setting the initial conditions: $Z^{t,e}(s)=e$, $Y^{t,e}(s)=Y^{t,e}(t)$, $X^{t,e}(s)=0$ for all $s\in[0,t]$. Therefore, these solutions can be considered as solutions of the extended FBSDE system (4.7).

Using Itô's formula in the $L_2(\mathbb{T}^2,\mathbb{R}^2)$-norm of $Y^{t,e}(s)$ as described in equation (3.32) and the BSDE of system (4.4) ensure that the expectation given by $\mathbb{E}\|Y^{t,e}(s)\|_{L_2}^2$ is bounded.

To verify this, define the function $F(Y)=\|Y\|_{L_2}^2$, so the first-order derivative is given by

$$\frac{\partial F}{\partial Y_i}(Y)=2Y_i,\Rightarrow \langle DF(Y),h\rangle_{L_2}=2\langle Y,h\rangle_{L_2}=2\int_{\mathbb{T}^2}Y(\theta)h(\theta)d\theta.$$

And the second-order derivative is given by:

$$\frac{\partial^2 F}{\partial Y_i \partial Y_j}(Y) = 2\delta_{ij},$$

where $\delta_{ij} = 1$ if $i = j$ and $= 0$ if $i \neq j$.

Using Itô's formula presented in (5.1) we will notice that the derivative of the first term (calculated for time) is zero. This means that Itô's Formula only has in its composition the other derivatives given below:

Using the first derivative, we have:

$$\begin{aligned} -\sum_i \int_r^T 2Y_i(s)\, \hat{V}_i(s)\, ds &= -2 \int_r^T \langle Y(s), \hat{V}(s) \rangle_{L_2}\, ds \\ &= -2 \int_r^T \langle Y(s), \hat{V}(s, Z^{t,\xi}(s)) \rangle_{L_2}\, ds. \end{aligned}$$

The stochastic term is given by:

$$\sum_i \sum_l \int_r^T 2Y_i(s)\, X_i^l(s)\, dW_s^l = 2 \int_r^T \langle Y(s),\ X^{t,\xi}(s)\, dW_s \rangle_{L_2}.$$

and the second-order term becomes:

$$\begin{aligned} \frac{1}{2} \sum_{i,j} \sum_l \int_r^T 2\, \delta_{ij}\, X_i^l(s)\, X_j^l(s)\, ds &= \int_r^T \sum_i \sum_l \left(X_i^{t,\xi}(s) \right)^2 ds \\ &= \int_r^T \| X^{t,\xi}(s) \|_{L_2}^2\, ds. \end{aligned}$$

where i and j correspond to the infinite directions of the basis of the Hilbert space and l are the finite directions of the basis of the space in which the Wiener process takes values.

Applying Itô's formula between times s and T and remembering that the final condition is given by $Y^{t,\xi}(T) = \hat{h}(Z^{t,\xi}(T))$, we obtain:

$$\| Y^{t,\xi}(T) \|_{L_2}^2 - \| Y^{t,\xi}(s) \|_{L_2}^2 = -2 \int_s^T \langle Y^{t,\xi}(r), \hat{V}(r, Z^{t,\xi}(r)) \rangle_{L_2}\, dr$$

$$+ 2\int_s^T \langle Y^{t,\xi}(r), X^{t,\xi}(r)\, dW_r\rangle_{L_2} + \int_s^T \|X^{t,\xi}(r)\|_{L_2}^2\, dr.$$

Rearranging the terms, we have:

$$\|Y^{t,\xi}(s)\|_{L_2}^2 = \|\hat{h}(Z^{t,\xi}(T))\|_{L_2}^2 + 2\int_s^T \langle Y^{t,\xi}(r), \hat{V}(Z^{t,\xi}(r))\rangle_{L_2} dr \qquad (4.8)$$
$$- 2\int_s^T \langle Y^{t,\xi}(r), X^{t,\xi}(r) dW_r\rangle_{L_2} - \int_s^T \|X^{t,\xi}(r)\|_{L_2}^2 dr.$$

where $\|\cdot\|_{L_2}$ is the $L_2(\mathbb{T}^2, \mathbb{R}^2)$-norm.

Applying the expectation, we have to make some observations: the expectation of the initial condition $\hat{h}(Z^{t,\xi}(T))$ will be bounded, since it is a constant, the expectation of the stochastic term will be zero due to the martingale property and the expectation of the term $X^{t,\xi}(s)$ can be estimated by any positive value, since it is negative. Thus, it is necessary to verify that the expectation of the norm of $2\int_s^T \langle Y^{t,\xi}(r), \hat{V}(Z^{t,\xi}(r))\rangle_{L_2} dr$ is bounded.

Note that by the Cauchy-Schwarz inequality, we have: $\langle Y^{t,\xi}(r), \hat{V}(Z^{t,\xi}(r))\rangle_{L_2} \leq \|Y^{t,\xi}(r)\|_{L_2} \cdot \|\hat{V}(Z^{t,\xi}(r))\|_{L_2}$, which implies:

$$2\int_s^T \langle Y^{t,\xi}(r), \hat{V}(Z^{t,\xi}(r))\rangle_{L_2} dr \leq 2\int_s^T \|Y^{t,\xi}(r)\|_{L_2} \cdot \|\hat{V}(Z^{t,\xi}(r))\|_{L_2} dr \qquad (4.9)$$

For simplicity, the constant 2 previously presented will be omitted, since it will naturally be implied in the estimate below. By hypothesis, the function $\hat{V}(\cdot)$ is Lipchitz, so there is a positive constant C_1 which is worth the equation:

$$\|\hat{V}(Z^{t,\xi}(r))\|_{L_2} \leq C_1\big(1 + \|Z^{t,\xi}(r)\|_{L_2}\big).$$

Applying the control condition described above to the equation (4.9), we obtain:

$$\int_s^T \langle Y^{t,\xi}(r), \hat{V}(Z^{t,\xi}(r))\rangle_{L_2} dr \leq \int_s^T \|Y^{t,\xi}(r)\|_{L_2} \cdot C_1\big(1 + \|Z^{t,\xi}(r)\|_{L^2}\big) dr$$
$$\leq \int_s^T C_1 \|Y^{t,\xi}(r)\|_{L_2} dr + \int_s^T C_1 \|Y^{t,\xi}(r)\|_{L_2} \cdot \|Z^{t,\xi}(r)\|_{L_2} dr$$

Finally, recall that $Z^{t,\xi}(s)$ is a solution of the FSDE and, therefore, is bounded with respect to the L_2-norm by hypothesis. To estimate the product of its norm with the norm of $\|Y^{t,\xi}(s)\|$ we can apply Grönwall's Lemma and obtain a new constant C_3 which will guarantee the estimate.

Now, considering the FSDE of system (4.7), we can calculate the difference of $Z^{t,e}(s) - Z^{t',e}(s)$ as:

$$Z^{t,e}(s) - Z^{t',e}(s) = \int_0^s \mathbb{I}_{[t,T]}(r)\big(Y^{t,e}(r) - Y^{t',e}(r)\big)dr + \int_0^s \mathbb{I}_{[t,T]}(r)\big(\mathbb{A}(Z^{t,e}(r)) - \mathbb{A}(Z^{t',e}(r))\big)dW_r.$$

Applying the L_2-norm to both sides we obtain the following equality:

$$\|Z^{t,e}(s) - Z^{t',e}(s)\|_{L_2}^2 \leq 2\left\|\int_0^s \mathbb{I}_{[t,T]}(r)\big(Y^{t,e}(r) - Y^{t',e}(r)\big)dr\right\|_{L_2}^2 \tag{4.10}$$

$$+ 2\left\|\int_0^s \mathbb{I}_{[t,T]}(r)\big(\mathbb{A}(Z^{t,e}(r)) - \mathbb{A}(Z^{t',e}(r))\big)dW_r\right\|_{L_2}^2. \tag{4.11}$$

Then, we apply the expectation to both sides of the previous equation and using Itô's Isometry in the stochastic term, we obtain:

$$\mathbb{E}\|Z^{t,e}(s) - Z^{t',e}(s)\|_{L_2}^2 \leq 2\,\mathbb{E}\left\|\int_0^s \mathbb{I}_{[t,T]}(r)\big(Y^{t,e}(r) - Y^{t',e}(r)\big)dr\right\|_{L_2}^2 + 2\int_0^s \mathbb{E}\left\|\mathbb{I}_{[t,T]}(r)\big(\mathbb{A}(Z^{t,e}(r)) - \mathbb{A}(Z^{t',e}(r))\big)\right\|_{L_2}^2 dr. \tag{4.12}$$

By hypothesis, the operator $\mathbb{A}(\cdot)$ is bounded with respect to the norm (see conditions of Theorem 1.3) then there exists a positive constant C_2 such that:

$$\int_0^s \mathbb{I}_{[t,T]}(r)\cdot\mathbb{E}\left\|\mathbb{A}(Z^{t,e}(r)) - \mathbb{A}(Z^{t',e}(r))\right\|_{L_2}^2 dr$$
$$< C_2\int_0^s \mathbb{I}_{[t,T]}(r)\cdot\mathbb{E}\left\|(Z^{t,e}(r)) - (Z^{t',e}(r))\right\|_{L_2}^2 dr$$

Using the above estimate in the previous equality given by (4.12), we obtain:

$$\mathbb{E}\|Z^{t,e}(s) - Z^{t',e}(s)\|_{L_2}^2 < 2\,\mathbb{E}\left\|\int_0^s \mathbb{I}_{[t,T]}(r)\big(Y^{t,e}(r) - Y^{t',e}(r)\big)dr\right\|_{L_2}^2 \\ + C_2 \int_0^s \mathbb{I}_{[t,T]}(r)\cdot\mathbb{E}\left\|(Z^{t,e}(r)) - (Z^{t',e}(r))\right\|_{L_2}^2 dr$$

Applying Grönwall's Lemma, we can obtain a new positive constant K_1 such that the following inequality holds:

$$\mathbb{E}\left\|Z^{t,e}(s) - Z^{t',e}(s)\right\|_{L_2}^2 < K_1\left[\int_0^s \mathbb{I}_{[t,T]}\mathbb{E}\left\|Y^{t,e}(r) - Y^{t',e}(r)\right\|_{L_2}^2 dr + (t'-t)\right]. \tag{4.13}$$

Similarly, we will apply Itô's Formula to: $\|Y^{t,e}(s) - Y^{t',e}(s)\|_{L_2}^2$ using the BSDE of (4.7), where $\|\cdot\|_{L_2}$ denotes the $L_2(\mathbb{T}^2,\mathbb{R}^2)$-norm as before. From which we can conclude that $\|Y^{t,e}(s) - Y^{t',e}(s)\|_{L_2}^2$ is bounded, since it is a sum of bounded norms. Again, applying the Grönwall's lemma, usual stochastic integral estimates and the above estimate for expectation $\mathbb{E}\|Z^{t,e}(s) - Z^{t',e}(s)\|_{L_2}^2$, we obtain that there is a positive constant K_2 such that the following inequality holds:

$$\mathbb{E}\left\|Y^{t,e}(s) - Y^{t',e}(s)\right\|_{L_2}^2 < K_2\left[\int_0^T \mathbb{E}\left\|Y^{t,e}(r) - Y^{t',e}(r)\right\|_{L_2}^2 dr + (t'-t)\right].$$

Take T_0 smaller than $\frac{1}{K_2}$. Then there exists a positive constant K for which the following estimate holds:

$$\sup_{s\in[0,T]} \mathbb{E}\left\|Y^{t,e}(s) - Y^{t',e}(s)\right\|_{L_2}^2 < K(t'-t). \tag{4.14}$$

Evaluating the previous inequality at time $s = t$ and taking into account that initial condition we have: $Y^{t,e}(t) = e = Y^{t',e}(t')$, then we obtain that:

$$\left\|Y^{t,e}(t) - Y^{t',e}(t')\right\|_{L_2}^2 \leq K(t'-t). \tag{4.15}$$

Differentiating the extended FBSDE (4.7) with respect to θ, we obtain the following system of forward and backward SDEs:

$$\begin{cases} \nabla Z^{t,e}(s) = 1 + \int_0^s \mathbb{I}_{[t,T]}(r)\nabla Y_r^{t,e}\,dr + \int_0^s \mathbb{I}_{[t,T]}(r)\nabla\mathbb{A}(Z^{t,e}(r))\nabla Z^{t,e}(r)dW_r \\ \nabla Y^{t,g}(s) = \nabla h(Z^{t,e}(T))\nabla Z^{t,e}(T) + \int_s^T \mathbb{I}_{[t,T]}(r)\nabla\hat{V}(r, Z^{t,e}(r))\nabla Z^{t,e}(r)dr \\ \qquad\qquad - \int_s^T \nabla X^{t,e}(r)\,dW_r. \end{cases} \tag{4.16}$$

Through the usual stochastic integral estimates, we obtain that the expectations $\mathbb{E}\|\nabla Z^{t,e}(s)\|_{L_2}^2$ and $\mathbb{E}\|\nabla Y^{t,e}(s)\|_{L_2}^2$ are bounded.

Note that when we apply the control condition of inequality (4.14) to the estimate given for $\|Z^{t,e}(s) - Z^{t',e}(s)\|$ in (4.13), we also obtain an estimate for $\sup_{s\in[0,T]}\mathbb{E}\|Z^{t,e}(s) - Z^{t',e}(s)\|_{L_2}^2$ Using this estimate and considering the FSDE of system (4.16), when evaluating $s = t$ and remembering that $Y^{t,e}(t) = Y^{t',e}(t')$, then there is a positive constant L such that for all t and t' from the time interval $[0,T]$ holds:

$$\|\nabla Y^{t,e}(t) - \nabla Y^{t',e}(t')\|_{L_2}^2 < L|t' - t|. \tag{4.17}$$

Differentiating the extended system of FBSDE (4.7) with respect to θ a second time and using the same argument, we will obtain a new positive constant M such that for all times t and t' belonging to $[0,T]$ the following inequality holds:

$$\|\nabla\nabla Y^{t,e}(t) - \nabla\nabla Y^{t',e}(t')\|_{L_2}^2 < M|t' - t|. \tag{4.18}$$

Finally, note that the estimates given for the first and second derivatives at (4.15), (4.17), and (4.18) show that they are bounded and therefore continuous. This leads us to conclude that map $t \mapsto Y^{t,e}(t)$ is continuous with respect to the $H^2(\mathbb{T}^2, \mathbb{R}^2)$-norm. □

For the following lemmas, we will assume that $T < T_0$, where T_0 is a positive definite constant, as described in Lemma 4.3.

Lemma 4.4. *For every $t \in [0,T)$ and for every $\mathcal{F}_t$-measurable random variable ξ, the solution $(Z^{t,\xi}(s), Y^{t,\xi}(s), X^{t,\xi}(s))$ to FBSDE (4.4) is unique in the time interval $[t,T]$.*

Proof. Let $(Z^{t,\xi}(s), Y^{t,\xi}(s), X^{t,\xi}(s))$ be a solution to the FBSDE (4.4), to prove uniqueness, consider another solution $(\tilde{Z}^{t,\xi}(s), \tilde{Y}^{t,\xi}(s), \tilde{X}^{t,\xi}(s))$ to (4.4) in time $[t,T]$. We will extend these solutions to the interval $[0,T]$ by setting the initial conditions:

$$\begin{aligned} Z^{t,\xi}(s) &= \xi = \tilde{Z}^{t,\xi}(s), \\ Y^{t,\xi}(s) = Y^{t,\xi}(t) &= \tilde{Y}^{t,\xi}(t) = \tilde{Y}^{t,\xi}(s), \\ X^{t,\xi}(s) &= 0 = \tilde{X}^{t,\xi}(s). \end{aligned}$$

for all $s \in [0,t]$. In this way, these triples of solutions can be considered as solutions of the extended FBSDE system presented below:

$$\begin{cases} Z^{t,\xi}(s) = e + \displaystyle\int_0^s \mathbb{I}_{[t,T]}(r) Y^{t,\xi}(r) dr + \int_0^s \mathbb{I}_{[t,T]}(r) \mathbb{A}(Z^{t,\xi}(r)) dW_r \\ Y^{t,\xi}(s) = h(Z^{t,\xi}(T)) + \displaystyle\int_s^T \mathbb{I}_{[t,T]}(r) \hat{V}(r, Z^{t,\xi}(r)) dr - \int_s^T X^{t,\xi}(r) dW_r. \end{cases} \tag{4.19}$$

The rest of the proof follows in a similar manner to what we did in the previous lemma 4.3 and will be omitted.

In short, our goal would be to verify the norms of differences $\|Y^{t,\xi}(s) - \tilde{Y}^{t,\xi}(s)\|^2_{L_2}$ and $\|Z^{t,\xi}(s) - \tilde{Z}^{t,\xi}(s)\|^2_{L_2}$ are 0, which means that these processes are almost surely equal. The calculations are developed in a similar way to what we did using the inequality (4.10). □

Lemma 4.5. *Let the function $y : [0,T] \times \mathbb{T}^2 \to \mathbb{R}^2$ be defined by the formula:*

$$y(t,\theta) = Y^{t,e}(t,\theta). \tag{4.20}$$

Then, for every $t \in [0,T]$, $y(t,\cdot)$ is H^α-smooth, and a.s.

$$Y^{t,e}(u) = y(u,\cdot) \circ Z^{t,e}(u). \tag{4.21}$$

Proof. Recall that in lemma 4.2, we proved that the equality presented by the triple of solutions in (4.3) is true, which implies that the function ξ is $\mathcal{F}_t$-measurable, in this way:

$$Y^{t,\xi}(t) = y(t,\cdot) \circ \xi$$

Furthermore, for each fixed time $u \in [t,T]$, the triple $(Z^{t,e}(s), Y^{t,e}(s), X^{t,e}(s))$ is a solution of the following FBSDE system on the time interval $[u,T]$:

$$\begin{cases} Z^{t,e}(s) = Z^{t,e}(u) + \int_u^s Y^{t,e}(r)dr + \int_u^s \mathbb{A}(Z^{t,e}(r))dW_r \\ Y^{t,e}(s) = h(Z^{t,e}(T)) + \int_s^T \hat{V}(r, Z^{t,e}(r)) - \int_s^T X^{t,e}(r)dW_r. \end{cases}$$

Note that by the uniqueness of the solution, it is valid that if $u = t$, then $Z^{t,e}(u) = e$ which implies $Y^{t,e}(s) = Y^{u,Z^{t,e}(u)}(s)$ almost surely in the interval $[u,T]$. By the equality shown in 4.21, we obtain:

$$Y^{u,Z^{t,e}(u)}(s) = y(u,\cdot) \circ Z^{t,e}(u).$$

This implies that there exists a set that depends on time $u \in [t,T]$, which we will denote as Ω_u, of full $\mathbb{P}$-measure such that the equality 4.21 holds everywhere on Ω_u. Thus, we can find a set $\Omega_{\mathbb{Q}}$ such that the probability of this set is $\mathbb{P}(\Omega_{\mathbb{Q}}) = 1$ and that the equality given by 4.21 holds over all $\Omega_{\mathbb{Q}}$ for all rational $u \in [t,T]$.

However, the processes given by $Z^{t,e}(s)$ and $Y^{t,e}(s)$ have almost surely continuous trajectories. Furthermore, according to lemma 4.3, the function $y(t,\cdot)$ is continuous at time t with respect to $L_2(\mathbb{T}^2, \mathbb{R}^2)$-norm. Therefore, since the equality (4.21) is a composition of functions of this space, we conclude that almost surely the equality (4.21) holds with respect to $L_2(\mathbb{T}^2, \mathbb{R}^2)$-norm.

Finally, since both sides of the equality (4.21) are continuous in $\theta \in \mathbb{T}^2$, then it also holds for almost surely every $\theta \in \mathbb{T}^2$. □

The following lemma presents the same function introduced in lemma 4.5, however, it addresses the differentiability of this function with respect to the time variable.

Lemma 4.6. *Consider the function* $y : [0, T] \times \mathbb{T}^2 \to \mathbb{R}^2$ *defined as:*

$$y(t, \theta) = Y^{t,e}(t, \theta). \tag{4.22}$$

The function $y(t, \cdot)$ *defined above is* C^1*-smooth in* $t \in [0, T]$.

Proof. Consider a positive increment δ, applying this increment to the time variable t in the function $y(t, \cdot)$, we obtain:

$$\begin{aligned} y(t+\delta, \cdot) - y(t, \cdot) &= Y^{t+\delta,e}(t+\delta) - Y^{t,e}(t) \\ &= Y^{t+\delta,e}(t+\delta) - Y^{t,e}(t+\delta) + Y^{t,e}(t+\delta) - Y^{t,e}(t). \end{aligned}$$

Recall that by $\hat{Y}(s)$, we denote the right-invariant vector field over the manifold $\mathcal{D}^\alpha(\mathbb{T}^2)$ generated by the vector field $y(s, \cdot)$. By Lemma 4.5, we have that the following equality holds almost surely:

$$\begin{aligned} Y^{t,e}(t+\delta) &= y(t+\delta, \cdot) \circ Z^{t,e}(t+\delta) \\ &= \hat{Y}(t+\delta,\ Z^{t,e}(t+\delta)). \end{aligned}$$

Thus, the equation we initially considered becomes almost surely:

$$y(t+\delta, \cdot) - y(t, \cdot) = \hat{Y}(t+\delta, e) - \hat{Y}(t+\delta, Z^{t,e}(t+\delta)) + Y^{t,e}(t+\delta) - Y^{t,e}(t). \tag{4.23}$$

For the parcel $\hat{Y}(t+\delta, e) - \hat{Y}(t+\delta, Z^{t,e}(t+\delta))$, we apply Itô's formula when considering $\hat{Y}(t+\delta)$ as a C^2-smooth function $\mathcal{D}^\alpha_v(\mathbb{T}^2) \to L_2(\mathbb{T}^2, \mathbb{R}^2)$. So, we obtain:

$$\begin{aligned} \hat{Y}(t+\delta, Z^{t,e}(t+\delta)) - \hat{Y}(t+\delta, e) = &\int_t^{t+\delta} \hat{Y}(r, Z^{t,e}(r))[\hat{Y}(t+\delta, Z^{t,e}(r))]dr \\ &+ \sum_k \int_t^{t+\delta} A_k(Z^{t,e}(r)) A_k(Z^{t,e}(r)) \hat{Y}(t+\delta, Z^{t,e}(r))dr \\ &+ \sum_k \int_t^{t+\delta} B_k(Z^{t,e}(r)) B_k(Z^{t,e}(r))\, \hat{Y}(t+\delta, Z^{t,e}(r))dr \end{aligned}$$

$$+\int_t^{t+\delta} \epsilon\, \mathbb{A}(Z^{t,e}(r))\, \hat{Y}(t+\delta, Z^{t,e}(r))dW_r$$

where $k \in \mathbb{Z}_2^+ \cup \{0\}$.

Note that this equation is similar to the one presented in Theorem 3.6, specifically in equation (3.28), an argument similar to that used in this theorem implies:

$$\begin{aligned}\hat{Y}(t+\delta, Z^{t,e}(t+\delta)) - \hat{Y}(t+\delta, e) = &\int_t^{t+\delta} \nabla_{y(r,\cdot)}\, y(t+\delta,\cdot)\circ Z^{t,e}(r)dr \\ &+ \int_t^{t+\delta} \nu\, \Delta\, y(t+\delta,\cdot)\circ Z^{t,e}(r)dr \\ &+ \int_t^{t+\delta} \epsilon\, \mathbb{A}(Z^{t,e}(r))\, \hat{Y}(t+\delta, Z^{t,e}(r))\, dW_r.\end{aligned} \tag{4.24}$$

where ν is defined according to the lemma 3.5 and is given as:

$$\nu = \frac{\epsilon^2}{2}\left(1 + \frac{1}{2}\sum_k \frac{1}{|k|^{2\alpha}}\right)$$

in which $k \in \mathbb{Z}_+^2$, $|k| \le N$.

On other hand, for the parcel $Y^{t,e}(t+\delta) - Y^{t,e}(t)$ of equation (4.23) we use the equality given by BSDE, obtaining:

$$Y^{t,e}(t) - Y^{t,e}(t+\delta) = \int_t^{t+\delta} \nabla p(r,\cdot)\circ Z^{t,e}(r)dr - \int_t^{t+\delta} X^{t,e}(r)\, dW_r. \tag{4.25}$$

Finally, adding both developments made in (4.24) and (4.25), we arrive at the equation:

$$\begin{aligned}&\hat{Y}(t+\delta, Z^{t,e}(t+\delta)) - \hat{Y}(t+\delta, e) + Y^{t,e}(t) - Y^{t,e}(t+\delta) \\ &= \int_t^{t+\delta} \nabla_{y(r,\cdot)}\, y(t+\delta,\cdot)\circ Z^{t,e}(r)dr + \int_t^{t+\delta} \nu\, \Delta\, y(t+\delta,\cdot)\circ Z^{t,e}(r)dr \\ &+ \int_t^{t+\delta} \epsilon\, \mathbb{A}(Z^{t,e}(r))\, \hat{Y}(t+\delta, Z^{t,e}(r))\, dW_r + \int_t^{t+\delta} \nabla p(r,\cdot)\circ Z^{t,e}(r)dr - \int_t^{t+\delta} X^{t,e}(r)\, dW_r\end{aligned}$$

As seen previously in (3.30), the stochastic terms are equal and cancel each other out, thus we obtain.

$$
\begin{aligned}
&\hat{Y}(t+\delta, Z^{t,e}(t+\delta)) - \hat{Y}(t+\delta, e) + Y^{t,e}(t) - Y^{t,e}(t+\delta) \\
&= \int_t^{t+\delta} \nabla_{y(r,\cdot)}\, y(t+\delta,\cdot) \circ Z^{t,e}(r)dr + \int_t^{t+\delta} \nu\, \Delta\, y(t+\delta,\cdot) \circ Z^{t,e}(r)dr \\
&+ \int_t^{t+\delta} \nabla p(r,\cdot) \circ Z^{t,e}(r)dr
\end{aligned}
$$

Now, note that $\nabla_{y(r,\cdot)} y(t+\delta,\cdot) = (y(r,\cdot),\nabla) y(t+\delta,\cdot)$. With this equality, applying the expectation to both sides of the equation and multiplying by $\frac{1}{\delta}$, we obtain:

$$
\begin{aligned}
&\frac{1}{\delta}\big(y(t+\delta,\cdot) - y(t,\cdot)\big) && (4.26)\\
&= -\frac{1}{\delta}\,\mathbb{E}\left[\int_t^{t+\delta} [\,(y(r,\cdot),\nabla)\, y(t+\delta,\cdot) + \nu\,\Delta\, y(t+\delta,\cdot) + \nabla p(r,\cdot)] \circ Z^{t,e}(r)\; dr\right].
\end{aligned}
$$

Note that functions $Z^{t,e}(r)$, $\nabla p(r,\cdot)$, and $(y(r,\cdot),\nabla)\, y(t+\delta,\cdot) \circ Z^{t,e}(r)$ are continuous in time r almost surely with respect to the $L_2(\mathbb{T}^2,\mathbb{R}^2)$-norm. In this way, by Lemma 4.3, the functions $\nabla y(t,\cdot)$ and $\Delta y(t,\cdot)$ are continuous in time t with respect to the $L_2(\mathbb{T}^2,\mathbb{R}^2)$-norm. Formula (4.26) and the initial condition $Z^{t,e}(t) = e$ imply that in the $L_2(\mathbb{T}^2,\mathbb{R}^2)$-norm, we have:

$$
\partial_t y(t,\cdot) = -[\nabla_{y(t,\cdot)}\, y(t,\cdot) + \nu\, \Delta\, y(t,\cdot) + \nabla p(t,\cdot)]. \tag{4.27}
$$

Since the right-hand side of equation (4.27) is an $H^{\alpha-2}$-map, then this implies that the left side is also an $H^{\alpha-2}$-map. Therefore, the function $\partial_t y(t,\cdot)$ is continuous in $\theta \in \mathbb{T}^2$.

The relation given by equality (4.27) is obtained at this point for the right derivative of $y(t,\theta)$ with respect to t. However, notice that the right-hand side of equation (4.27) is continuous in t which clearly implies that the right derivative $\partial_t y(t,\theta)$ is continuous in each t of the time interval $[0,T)$. Finally, this function is uniformly continuous on every compact sub-interval of $[0,T)$. For this reason, there is a left derivative of $y(t,\theta)$ in t, and therefore, this proves the existence of the continuous derivative $\partial_t y(t,\theta)$ everywhere of the time interval $[0,T]$. □

Finally, we come to the last lemma that will conclude the proof of Theorem 4.1.

Lemma 4.7. *For every time $t \in [0,T]$, the function $y(t,\cdot) : \mathbb{T}^2 \to \mathbb{R}^2$ is divergence-free. Moreover, the pair of functions $(y(t,\cdot), p(t,\cdot))$ verifies the backward Navier-Stokes equations* (4.2).

Proof. Initially, fix a time $t > 0$ and consider the $T_e\mathcal{D}^\alpha_v(\mathbb{T}^2)$-valued curve $\gamma(u) = \mathbb{E}\left[\exp^{-1} Z^{t,e}(u)\right]$, where $u \geq t$, in a neighborhood of the origin of $T_e\mathcal{D}^\alpha_v(\mathbb{T}^2)$.

The FSDE of the system (4.4) can be represented as an SDE on the manifold $\mathcal{D}^\alpha(\mathbb{T}^2)$ as follows:

$$\begin{cases} dZ^{t,e}(s) = \exp\left\{\hat{Y}(s, Z^{t,e}(s))\, ds + \mathbb{A}(Z^{t,e}(s))\, dW_s\right\}, \\ Z^{t,e}(t) = e, \end{cases}$$

where $\hat{Y}(s)$ is the right-invariant vector field on $\mathcal{D}^\alpha(\mathbb{T}^2)$ generated by function $y(s,\cdot)$.

This implies that the derivative of the curve $\gamma(u)$ with respect to time u and evaluated at time t is given by

$$\frac{\partial}{\partial u}\gamma(u)\Big|_{u=t} = y(t,\cdot),$$

and therefore $y(t,\cdot) \in T_e\mathcal{D}^\alpha_v(\mathbb{T}^2)$.

Next, the BSDE of the FBSDE system (4.4) implies that the final conditions $Y^{t,e}(T) = h(Z^{t,e}(T))$. This observation and the equality given by Lemma 4.5 in (4.21) imply that $y(T,\cdot) = h$, just note that:

$$h(Z^{t,e}(T)) = Y^{t,e}(T) = y(T,\cdot) \circ e = y(T,\cdot)$$

Since we already obtained $\partial_t y(t,\cdot)$ to which it associates the solution $p(t\cdot)$ using the equation (4.27) in Lemma 4.6, then the proof of the lemma is now complete. □

4.2 Conclusion

In the previous section, we conclude our study of the relationship between the solutions of the system of classical Navier-Stokes equations and the solutions of a forward-backward system of stochastic differential equations on the $\mathcal{D}_v^\alpha(\mathbb{T}^n)$. However, new questions arise and remain open within this issue, the most general being: is it possible to generalize or extend this concept?

Indeed, other approaches are possible to test, and I will list some hypotheses that can be studied below:

- I) By extending the torus to higher dimensions.
- II) By testing other free-divergence maps.
- III) By studying other cases of diffeomorphism groups.
- IV) By testing homeomorphisms instead of diffeomorphisms.

Finally, we emphasize how stochastic analysis is both interesting and challenging as a research area, in addition to clearly being a field of study with diverse applications that connect well with other areas, as seen in this work that permeated geometry, algebra, and analysis content.

Chapter 5

Appendix

Theorem 5.1 (Itô's Formula). *Let H_1 and H_2 be Hilbert Spaces, W_t be a n-dimensional Brownian Motion and $X_t \in H_1$ a Stochastic Process given by:*

$$X_t = X_0 + \int_0^t a(s)ds + \int_0^t \mathbb{A}(s)dW_s,$$

such that for each direction $i \in \mathbb{N}$, it is holds:

$$X_t^i = X_0^i + \int_0^t a_i(s)ds + \sum_{l=1}^{n} \int_0^t \mathbb{A}_i^l(s)dW_s^l$$

in which $a(s)$ is an H_1-valued Stochastic Process and $\mathbb{A}(s)$ is a $L(\mathbb{R}^n, H_1)$-valued Stochastic Process, and let $F = (F_1(t,x), ..., F_d(t,x), ...)$ be a $C^2([0,t] \times H_1, H_2)$ application, so we get for each $d \in \mathbb{N}$:

$$\begin{aligned} F_d(t, X_t) =& F_d(0, X_0) + \int_0^t \frac{\partial F_d}{\partial s}(s, X_s)ds + \sum_{i=1}^{\infty} \int_0^t \frac{\partial F_d}{\partial x_i}(s, X_s)a_i(s)ds \\ &+ \sum_{i=1}^{\infty} \sum_{l=1}^{n} \int_0^t \frac{\partial F_d}{\partial x_i}(s, X_s)\mathbb{A}_i^l(s)dW_s^l \\ &+ \frac{1}{2} \sum_{i=1}^{\infty} \sum_{h=1}^{\infty} \sum_{l=1}^{n} \int_0^t \frac{\partial F_d}{\partial x_i \partial x_h}(s, X_s)\mathbb{A}_i^l(s)\mathbb{A}_h^l(s)ds \end{aligned} \quad (5.1)$$

Proof. Throughout this demonstration, the subscripted d is used to indicate the

d-th component function of F, while the subscripts i and h indicate, respectively, the i-th and h-th components of the process X_t or the corresponding i-th and h-th derivatives of each function F_d. Moreover, we will use the index l to indicate the dimensions of the Brownian motion dW_s and m when there are two distinct Brownian motions.

First, assume that F_d, X_t, $\int_0^t \mathbb{A}(s)dW_s$, $\int_0^t |a(s)|ds$ and $\int_0^t \langle \mathbb{A}_i(s), \mathbb{A}_h(s)\rangle ds$, for all $l \in \{1, \ldots, n\}$ and $i, h \in \mathbb{N}$ are bounded in (t, X_t).

In which,

$$\int_0^t \langle \mathbb{A}_i(s), \mathbb{A}_h(s)\rangle ds = \sum_{l=1}^{n} \int_0^t \mathbb{A}_i^l(s)\mathbb{A}_h^l(s)ds$$

By simplicity, we will denote $a_i(s)$ and $\mathbb{A}_i(s)$ by a_i and $\mathbb{A}_i$ for all $s \in [0, t]$ and $i \in \mathbb{N}$.

Define the stopping time:

$$\tau_c = \begin{cases} 0, \quad \text{if} \quad |X_0| > c \\ \inf\left\{t : \max\left(\left|\int_0^t \mathbb{A}dW_s\right|, \int_0^t |a|ds, \int_0^t \langle \mathbb{A}_i, \mathbb{A}_h\rangle ds\right) > c\right\}, \quad \text{if} \quad |X_0| \leq c. \end{cases}$$

We have that τ_c tends to ∞ when c tends to ∞. So, if we prove the equation (5.1) for $X_{\tau_c \wedge t}$ on the set $\{\tau_c > 0\}$, when $c \to \infty$, then the equation (5.1) is proved for the general case.

Assume that a and $\mathbb{A}$ are bounded elementary functions and define the partition:

$$t_0 = 0 < t_1 < ... < t_n < t$$

for the time interval $[0, t]$. Using Taylor's Expansion in each direction $d \in \mathbb{N}$ of function F, we obtain:

$$\begin{aligned} F_d(t, X_t) =& F_d(0, X_0) + \sum_j \Delta F_d(t_j, X_j) \\ =& F_d(0, X_0) + \sum_j \frac{\partial F_d}{\partial t}\Delta t_j + \sum_{i=1}^{\infty}\sum_j \frac{\partial F_d}{\partial x_i}\Delta X_j^i \end{aligned}$$

$$+\frac{1}{2}\sum_j \frac{\partial^2 F_d}{\partial t^2}(\Delta t_j)^2 + \sum_{i=1}^{\infty}\sum_j \frac{\partial^2 F_d}{\partial t \partial x_i}\Delta t_j \Delta X_j^i$$
$$+\frac{1}{2}\sum_{i=1}^{\infty}\sum_{h=1}^{\infty}\sum_j \frac{\partial^2 F_d}{\partial x_i \partial x_h}\Delta X_j^i \Delta X_j^h + \sum_j R_j$$

where $\frac{\partial F_d}{\partial t}$, $\frac{\partial F_d}{\partial x}$, and others, are evaluated at the points (t_j, X_{t_j}), $\Delta t_j = t_{j+1} - t_j$, $\Delta X_j^i = X_{t_{j+1}}^i - X_{t_j}^i$, $i = 1, 2, \ldots$, $\Delta F_d = F_d(t_{j+1}, X_{t_{j+1}}) - F_d(t_j, X_{t_j})$ and $R_j = o(|\Delta t_j|^2 + |\Delta X_j|^2)$ for all j.

If $\Delta t_j \to 0$, by the definition of the Riemann integral, we obtain:

$$\sum_j \frac{\partial F_d}{\partial t}\Delta t_j = \sum_j \frac{\partial F_d}{\partial t}(t_j, X_{t_j})\Delta t_j \to \int_0^t \frac{\partial F_d}{\partial t}(s, X_s)ds$$

Now, our goal will be to prove that:

$$\sum_{i=1}^{\infty}\sum_j \frac{\partial F_d}{\partial x_i}(t_j, X_{t_j})\Delta X_j^i \to \sum_{i=1}^{\infty}\int_0^t \frac{\partial F_d}{\partial x_i}(s, X_s)dX_s^i$$

To demonstrate this convergence, consider the decomposition of ΔX_j^i into elementary functions a_i and $\mathbb{A}_i$ as follow:

$$\Delta X_j^i = a_i(t_j)\Delta t_j + \sum_{l=1}^{n}\mathbb{A}_i^l(t_j)\Delta W_j^l$$

In this way, we obtain:

$$\sum_{i=1}^{\infty}\sum_j \frac{\partial F_d}{\partial x_i}(t_j, X_{t_j})\Delta X_j^i$$
$$=\sum_{i=1}^{\infty}\sum_j \frac{\partial F_d}{\partial x_i}(t_j, X_{t_j})\left(a_i(t_j)\Delta t_j + \sum_{l=1}^{n}\mathbb{A}_i^l(t_j)\Delta W_j^l\right)$$

$$=\sum_{i=1}^{\infty}\sum_{j}\frac{\partial F_d}{\partial x_i}(t_j,X_{t_j})a_i(t_j)\Delta t_j+\sum_{l=1}^{n}\sum_{i=1}^{\infty}\sum_{j}\frac{\partial F_d}{\partial x_i}(t_j,X_{t_j})\mathbb{A}_i^l(t_j)\Delta W_j^l$$

Which, by the definitions of Riemann and stochastic integrals, converge respectively to:

$$\begin{aligned}&\sum_{i=1}^{\infty}\int_0^t\frac{\partial F_d}{\partial x_i}(s,X_s)a_i(s)ds+\sum_{l=1}^{n}\sum_{i=1}^{\infty}\int_0^t\frac{\partial F_d}{\partial x_i}(s,X_s)\mathbb{A}_i^l(s)dW_s^l\\&=\sum_{l=1}^{n}\sum_{i=1}^{\infty}\int_0^t\frac{\partial F_d}{\partial x_i}(s,X_s)a_i(s)ds+\frac{\partial F_d}{\partial x_i}(s,X_s)\mathbb{A}_i^l(s)dW_s^l\\&=\sum_{i=1}^{\infty}\int_0^t\frac{\partial F_d}{\partial x_i}(s,X_s)dX_s^i\end{aligned}$$

Now, for the component with $(\Delta t_j)^2$ in Taylor's Expansion, we have that:

$$\left|\sum_j\frac{\partial^2 F_d}{\partial t^2}(\Delta t_j)^2\right|\le\max_j|\Delta t_j|\cdot\sum_j\left|\frac{\partial^2 F_d}{\partial t^2}(t_j,X_{t_j})\right|\Delta t_j$$

and this vanish as $|\mathcal{P}|\to 0$. Indeed,

$$\sum_j\left|\frac{\partial^2 F_d}{\partial t^2}(t_j,X_{t_j})\right|\Delta t_j\to\int_0^t\left|\frac{\partial^2 F_d}{\partial t^2}(t,X_t)\right|ds,\quad|\mathcal{P}|\to 0.$$

For the component of Taylor's Expansion with $\Delta t_j\Delta X_j^i$. We will verify that:

$$\sum_{i=1}^{\infty}\sum_j\frac{\partial^2 F_d}{\partial t\partial x_i}(t_j,X_{t_j})\Delta t_j\Delta X_j^i\to 0\quad\text{in}\quad L_2(\Omega).$$

Let $\beta_i(t_j):=\frac{\partial F_d}{\partial t\partial x_i}(t_j,X_{t_j})$ and let $\Delta X_j^i=a_i(t_j)(\Delta t_j)+\sum_{l=1}^n\mathbb{A}_i^l(t_j)\Delta W_j^l$, since this converges to zero, we will use the L_2-norm to prove it as follows:

$$\begin{aligned}
&\sum_{i=1}^{\infty}\sum_{j}\beta_i(t_j)\Delta t_j\Delta X_j^i \\
=&\sum_{i=1}^{\infty}\sum_{j}\beta_i(t_j)\Delta t_j\left(a_i(t_j)\Delta t_j+\sum_{l=1}^{n}\mathbb{A}_i^l(t_j)\Delta W_j^l\right) \\
=&\sum_{i=1}^{\infty}\left(\sum_{j}\beta_i(t_j)a_i(t_j)(\Delta t_j)^2+\sum_{l=1}^{n}\sum_{j}\beta_i(t_j)\mathbb{A}_i^l(t_j)\Delta W_j^l\Delta t_j\right).
\end{aligned}$$

Let us prove the first term goes to zero:

$$\begin{aligned}
\left|\sum_{i=1}^{\infty}\sum_{j}\beta_i(t_j)a_i(t_j)(\Delta t_j)^2\right| &\leq \max_j(\Delta t_j)\sum_{i=1}^{\infty}\sum_{j}|\beta_i(t_j)a_i(t_j)|\Delta t_j \\
&\leq \max_j(\Delta t_j)\sum_{j}|\beta(t_j)|\cdot|a(t_j)|\Delta t_j \to 0.
\end{aligned}$$

Indeed, let us observe that:

$$\sum_{j}\|\beta(t_j)\|\,\|a(t_j)\|\Delta t_j \to \int_0^t \|\beta(s)\|\,\|a(s)\|ds$$

Now, let us prove that the second term vanishes. For this, we will use the $L^2(H_1)$-norm:

$$\begin{aligned}
&\mathbb{E}\left|\sum_{l=1}^{n}\sum_{i=1}^{\infty}\sum_{j}\beta_i(t_j)\mathbb{A}_i^l(t_j)\Delta W_j^l\Delta t_j\right|^2 \\
=&\mathbb{E}\left[\left(\sum_{l=1}^{n}\sum_{i=1}^{\infty}\sum_{j}\beta_i(t_j)\mathbb{A}_i^l(t_j)\Delta t_j\Delta W_j^l\right)\cdot\left(\sum_{m=1}^{n}\sum_{h=1}^{\infty}\sum_{k}\beta_h(t_k)\mathbb{A}_h^m(t_k)\Delta t_k\Delta W_k^m\right)\right] \\
=&\mathbb{E}\left[\sum_{l,m=1}^{n}\sum_{i,h=1}^{\infty}\sum_{j,k}\beta_i(t_j)\beta_h(t_k)\mathbb{A}_i^l(t_j)\mathbb{A}_h^m(t_k)\Delta t_j\Delta t_k\Delta W_j^l\Delta W_k^m\right]
\end{aligned}$$

$$=\mathbb{E}\left[\mathbb{E}\left[\sum_{l,m=1}^{n}\sum_{i,h=1}^{\infty}\sum_{j,k}\beta_i(t_j)\beta_h(t_k)\mathbb{A}_i^l(t_j)\mathbb{A}_h^m(t_k)\Delta t_j\Delta t_k\Delta W_j^l\Delta W_k^m\middle|\mathcal{F}_{t_k}\right]\right]$$

Let $j < k$, since Brownian Motion has independents trajectories, so ΔW_j^l and ΔW_k^m are independents for every case $l \neq m$, thus, the expectation of their product is 0, remaining the case $l = m$. In this way, we obtain:

$$\begin{aligned}&\sum_{l=1}^{n}\sum_{i,h=1}^{\infty}\sum_{j,k}\mathbb{E}\left[\beta_i(t_j)\beta_h(t_k)\mathbb{A}_i^l(t_j)\mathbb{A}_h^l(t_k)\Delta t_j\Delta t_k\right]\cdot\mathbb{E}\left[\mathbb{E}\left[\Delta W_j^l\Delta W_k^l\,|\mathcal{F}_{t_k}\right]\right]\\ =&\sum_{l=1}^{n}\sum_{i,h=1}^{\infty}\sum_{j,k}\mathbb{E}\left[\beta_i(t_j)\beta_h(t_k)\mathbb{A}_i^l(t_j)\mathbb{A}_h^l(t_k)\Delta t_j\Delta t_k\right]\cdot 0\\ =&\,0.\end{aligned}$$

The case $k < j$ is analogous, so we will analyze the case where $j = k$:

$$\begin{aligned}&\mathbb{E}\left[\mathbb{E}\left[\sum_{l=1}^{n}\sum_{i,h=1}^{\infty}\sum_{j,k}\beta_i(t_j)\beta_h(t_k)\mathbb{A}_i^l(t_j)\mathbb{A}_h^l(t_k)\Delta t_j\Delta t_k\Delta W_j^l\Delta W_k^l\middle|\mathcal{F}_{t_j}\right]\right]\\ =&\mathbb{E}\left[\mathbb{E}\left[\sum_{l=1}^{n}\sum_{i,h=1}^{\infty}\sum_{j}\beta_i(t_j)\beta_h(t_j)\mathbb{A}_i^l(t_j)\mathbb{A}_h^l(t_j)(\Delta t_j)^2(\Delta W_j^l)^2\middle|\mathcal{F}_{t_j}\right]\right]\\ =&\sum_{l=1}^{n}\sum_{i,h=1}^{\infty}\sum_{j}\mathbb{E}\left[\beta_i(t_j)\beta_h(t_j)\mathbb{A}_i^l(t_j)\mathbb{A}_h^l(t_j)(\Delta t_j)^2\right]\cdot\mathbb{E}\left[\mathbb{E}\left[(\Delta W_j^l)^2\middle|\mathcal{F}_{t_j}\right]\right]\\ =&\sum_{l=1}^{n}\sum_{i,h=1}^{\infty}\sum_{j}\mathbb{E}\left[\beta_i(t_j)\beta_h(t_j)\mathbb{A}_i^l(t_j)\mathbb{A}_h^l(t_j)(\Delta t_j)^2\right]\cdot(\Delta t_j)\end{aligned}$$

This expression tends to 0 as Δt_j tends to 0.

Now, let us analyze the last component of Taylor's Expansion, which has variation: $\Delta X_j^i \Delta X_j^h$. Let

$$\Delta X_j^i = a_i \Delta t_j + \sum_{l=1}^{n}\mathbb{A}_i^l \Delta W_j^l,$$

and for ΔX_j^h is analogous with functions a_h and $\mathbb{A}_h$, then:

$$\begin{aligned}
&\sum_{i=1}^{\infty}\sum_{h=1}^{\infty}\sum_{j}\frac{\partial^2 F_d}{\partial x_i \partial x_h}\Delta X_j^i \Delta X_j^h \\
&= \sum_{i=1}^{\infty}\sum_{h=1}^{\infty}\sum_{j}\frac{\partial^2 F_d}{\partial x_i \partial x_h}\left(a_i \Delta t_j + \sum_{l=1}^{n}\mathbb{A}_i^l \Delta W_j^l\right)\cdot\left(a_h \Delta t_j + \sum_{m=1}^{n}\mathbb{A}_h^m \Delta W_j^m\right) \\
&= \sum_{i=1}^{\infty}\sum_{h=1}^{\infty}\sum_{j}\frac{\partial^2 F_d}{\partial x_i \partial x_h} a_i a_h (\Delta t_j)^2 \\
&+ \sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{i=1}^{\infty}\sum_{h=1}^{\infty}\sum_{j}\frac{\partial^2 F_d}{\partial x_i \partial x_h}\left(a_i \mathbb{A}_h^m \Delta t_j \Delta W_j^m + a_h \mathbb{A}_i^l \Delta t_j \Delta W_j^l\right) \\
&+ \sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{i=1}^{\infty}\sum_{h=1}^{\infty}\sum_{j}\frac{\partial^2 F_d}{\partial x_i \partial x_h}\mathbb{A}_i^l \mathbb{A}_h^m \Delta W_j^l \Delta W_j^m \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

Where, each a_i and $\mathbb{A}_i$ is evaluated in t_j and $\frac{\partial^2 F_d}{\partial x_i \partial x_h}$ is evaluated in (t_j, X_{t_j}). In this equality, the terms I_1 and I_2, respectively, the terms with $(\Delta t_j)^2$, $\Delta t_j \Delta W_j^m$ and $t_j \Delta W_j^l$ tends to 0 as Δt_j tends to 0, as we saw in the previous demonstrations.

However, it remains to see what happening with the term I_3 with $\Delta W_j^l \Delta W_j^m$.

We will assert that:

$$\begin{aligned}
&\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{i=1}^{\infty}\sum_{h=1}^{\infty}\sum_{j}\frac{\partial^2 F_d}{\partial x_i \partial x_h}\mathbb{A}_i^l(t_j)\mathbb{A}_h^m(t_j)\Delta W_j^l \Delta W_j^m \\
&\qquad \to \sum_{l=1}^{n}\sum_{m=1}^{n}\sum_{i=1}^{\infty}\sum_{h=1}^{\infty}\int_0^t \frac{\partial^2 F_d}{\partial x_i \partial x_h}\mathbb{A}_i^l(s)\mathbb{A}_h^m(s)ds
\end{aligned}$$

Since Brownian Motions W_j^l and W_j^m have independents trajectories, so the cases where $l \neq m$ vanishes in the product. So, the equality is reduced to the case:

$$\sum_{l=1}^{n}\sum_{h=1}^{\infty}\sum_{i=1}^{\infty}\sum_{j}\frac{\partial^2 F_d}{\partial x_i \partial x_h}(t_j, X_{t_j})\mathbb{A}_i^l(t_j)\mathbb{A}_h^l(t_j)(\Delta W_j^l)^2$$
$$\to \sum_{l=1}^{n}\sum_{h=1}^{\infty}\sum_{i=1}^{\infty}\int_0^t \frac{\partial^2 F_d}{\partial x_i \partial x_h}\mathbb{A}_i(s)\mathbb{A}_h(s)ds$$

To prove the assert above, define:

$$\gamma^l(t) = \sum_{h=1}^{\infty}\sum_{i=1}^{\infty}\frac{\partial^2 F_d}{\partial x_i \partial x_h}(t, X_t)\mathbb{A}_i^l(t)\mathbb{A}_h^l(t)$$

and $\gamma_j^l = \gamma^l(t_j)$, $\gamma_j = \sum_{l=1}^n \gamma_j^l$. So,

$$\mathbb{E}\left[\sum_{l=1}^{n}\sum_{j}\gamma_j^l(\Delta W_j^l)^2 - \sum_{l=1}^{n}\sum_{j}\gamma_j^l \Delta t_j\right]^2$$
$$= \mathbb{E}\left[\sum_{l=1}^{n}\sum_{j}\gamma_j^l\big((\Delta W_j^l)^2 - \Delta t_j\big)\right]^2$$
$$= \mathbb{E}\left[\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{j,k}\gamma_j^l\big((\Delta W_j^l)^2 - \Delta t_j\big)\cdot\gamma_k^m\big((\Delta W_k^m)^2 - \Delta t_k\big)\right]$$
$$= \sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{j,k}\mathbb{E}\left[\gamma_j^l\gamma_k^m\big((\Delta W_j^l)^2 - \Delta t_j)(\Delta W_k^m)^2 - \Delta t_k)\big)\right]$$
$$= \sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{j,k}\mathbb{E}\left[\mathbb{E}[\gamma_j^l\gamma_k^m\big((\Delta W_j^l)^2 - \Delta t_j)(\Delta W_k^m)^2 - \Delta t_k)\big)|\mathcal{F}_{t_k}]\right]$$

If $j < k$, then

$$\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{j,k}\mathbb{E}\left[\mathbb{E}[\gamma_j^l\gamma_k^m\big((\Delta W_j^l)^2 - \Delta t_j)(\Delta W_k^m)^2 - \Delta t_k)\big)|\mathcal{F}_{t_k}]\right]$$
$$= \sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{j,k}\mathbb{E}[\gamma_j^l\gamma_k^m((\Delta W_j^l)^2 - \Delta t_j)]\cdot\mathbb{E}[(\Delta W_k^m)^2 - \Delta t_k|\mathcal{F}_{t_k}]$$

$$= \sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{j,k}\mathbb{E}[\gamma_j^l\gamma_k^m((\Delta W_j^l)^2 - \Delta t_j)]\cdot 0$$
$$=0.$$

For the case $k < j$, the proof is analogous. Finally, consider $j = k$ e we get:

$$\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{j}\mathbb{E}\left[\gamma_j^l\gamma_j^m\big((\Delta W_j^l)^2 - \Delta t_j)(\Delta W_j^m)^2 - \Delta t_j)\big)\right]$$

Again, since Brownian Motion have trajectories independents, the cases $l \neq m$ vanish, so:

$$\sum_{l=1}^{n}\sum_{j}\mathbb{E}\left[(\gamma_j^l)^2(\Delta W_j^l)^2 - \Delta t_j)^2\right]$$
$$\sum_{l=1}^{n}\sum_{j}\mathbb{E}\left[\mathbb{E}\left[(\gamma_j^l)^2(\Delta W_j^l)^2 - \Delta t_j)^2|\mathcal{F}_{t_j}\right]\right]$$
$$\sum_{l=1}^{n}\sum_{j}\mathbb{E}\left[\mathbb{E}\left[(\gamma_j^l)^2|\mathcal{F}_{t_j}\right]\cdot\mathbb{E}\left[(\Delta W_j^l)^2 - \Delta t_j)^2|\mathcal{F}_{t_j}\right]\right]$$
$$\sum_{l=1}^{n}\sum_{j}\mathbb{E}[(\gamma_j^l)^2]\cdot\mathbb{E}\left[(\Delta W_j^l)^2 - \Delta t_j)^2\right]$$

This occurs because $(\gamma_j^l)^2$ is $\mathcal{F}_{t_j}$-measurable and $(\Delta W_j^l)^2$ is independent of $\mathcal{F}_{t_j}$. So,

$$\sum_{l=1}^{n}\sum_{j}\mathbb{E}[(\Delta W_j^l)^2 - \Delta t_j]^2 = \sum_{l=1}^{n}\sum_{j}\mathbb{E}[(\Delta W_j^l)^4 - 2(\Delta W_j^l)^2(\Delta t_j) + (\Delta t_j)^2]$$
$$= 3n(\Delta t_j)^2 - 2n(\Delta t_j)^2 + (\Delta t_j)^2$$

$$= (n+1)(\Delta t_j)^2$$

In this way,

$$\sum_{l=1}^{n} \sum_{j} \mathbb{E}[(\gamma_j^l)^2] \cdot \mathbb{E}[(\Delta W_j^l)^2 - \Delta t_j)^2] = (n+1) \sum_{j} \mathbb{E}[\gamma_j^2](\Delta t_j)^2$$

And this expression tends to 0 as Δt_j tends to 0.

Then, in the end,

$$\sum_{l=1}^{n} \sum_{j} \gamma_j^l (\Delta W_j^l)^2 \to \sum_{l=1}^{n} \int_0^t \gamma^l(s)ds \tag{5.2}$$

By equation 5.2, we obtain:

$$\sum_{l=1}^{n} \sum_{i=1}^{\infty} \sum_{j} \frac{\partial^2 F_d}{\partial x_i^2}(t_j, X_{t_j}) \mathbb{A}_i(t_j) \mathbb{A}_i(t_j)(\Delta W_j^l)^2$$
$$\to \sum_{i=1}^{\infty} \int_0^t \frac{\partial^2 F_d}{\partial x_i^2}(s, X_s) \mathbb{A}_i(s) \mathbb{A}_i(s) ds$$

Moreover, it is clear that:

$$\sum_{j} R_j = \sum_{j} o(|\Delta t_j|^2 + |\Delta X_j|^2) \to 0$$

as $\Delta t_j \to 0$.

That completes the demonstration of Itô Formula in Hilbert Spaces. □

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