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Doutorado em Matemática

A Search for Linearity in the Universe of Topological Vector Spaces

By

Geivison dos Santos Ribeiro

João Pessoa - PB
July 24, 2024

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under the supervision of

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ATA DE DEFESA DE DOUTORADO JUNTO AO PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA DA UNIVERSIDADE FEDERAL DA PARAÍBA, REALIZADA NO DIA 24 DE JULHO DE 2024.

Ao vigésimo quarto dia de julho de dois mil e vinte e quatro, às 15:00 horas, em formato híbrido, presencialmente na sala 02 da Pós Graduação em Matemática, no Departamento de Matemática do Centro de Ciências Exatas e da Natureza na Universidade Federal da Paraíba, e remotamente por meio da plataforma virtual Google Meet, através do link: https://meet.google.com/caz-gbmc-zsc, em conformidade com o parágrafo único do Art. 80 da Resolução CONSEPE nº 79/2013, que regulamenta a defesa de trabalho final por videoconferência, foi aberta a sessão pública de Defesa de Tese intitulada "**A Search for Linearity in the Universe of Topological Vector Spaces**", do aluno **Geivison dos Santos Ribeiro** que havia cumprido, anteriormente, todos os requisitos para a obtenção do grau de Doutor em Matemática, sob a orientação do Prof. Dr. Daniel Marinho Pellegrino e sob a coorientação do Prof. Dr. Anselmo Baganha Raposo Júnior. A Banca Examinadora, indicada pelo Colegiado do Programa de Pós-Graduação em Matemática, foi composta pelos professores: Daniel Marinho Pellegrino (Orientador), Anselmo Baganha Raposo Júnior (Coorientador/UFMA), Fernando Vieira Costa Júnior (UFPB), Nacib André Gurgel e Albuquerque (UFPB), Diana Marcela Serrano Rodríguez (Universidad Nacional de Colombia) e Daniel Núñez Alarcón (Universidad Nacional de Colombia). O professor Daniel Marinho Pellegrino, em virtude da sua condição de orientador, presidiu os trabalhos e, depois das formalidades de apresentação, convidou o aluno a discorrer sobre o conteúdo da tese. Concluída a explanação, o candidato foi arguido pela banca examinadora que, em seguida, sem a presença do aluno, finalizando os trabalhos, reuniu-se para deliberar tendo concedido ao discente a menção: **APROVADO.** E, para constar, foi lavrada a presente ata que será assinada pelos membros da Banca Examinadora.

João Pessoa, 24 de julho de 2024.

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Cada tentativa, cada erro, importa!
Tente de novo, falhe de novo, falhe melhor!

Resumo

Esta tese fornece critérios (tanto negativos quanto positivos) que contribuem para a literatura existente e aborda problemas em aberto dentro das noções de (α, β)-lineabilidade/espaçabilidade, (α, β)-lineabilidade densa, lineabilidade pontual e $[\mathcal{S}]$-lineabilidade. Em nossa exploração, iniciamos investigando o comportamento de estruturas algébricas e topológicas presentes no conjunto de funções não limitadas, contínuas e integráveis no intervalo $[0, \infty)$. Essa investigação foi iniciada por Calderón-Moreno, Gerlach-Mena e Prado-Bassas, onde eles demonstraram, entre outros resultados, que o conjunto

$$\mathcal{A} := \left\{ f \in \mathcal{C}\,[0, \infty) \cap L_1\,[0, \infty) : \limsup_{x \to \infty} |f\,(x)| = \infty \right\}$$

é lineável. Para compreender melhor as relações dimensionais nesse ambiente, empregamos novas técnicas e obtivemos insights adicionais tanto na estrutura topológica quanto na algébrica desse conjunto. Especificamente, provamos sua espaçabilidade pontual (e, portanto, espaçabilidade).

Além disso, demonstramos que o conjunto $L_p[0,1] \setminus \bigcup_{q \in (p,\infty)} L_q[0,1]$, para $p \in (0, \infty)$, embora $(1, \mathfrak{c})$-espaçável (see [21]), não é $(\aleph_0, \mathfrak{c})$-espaçável. Estabelecemos um critério geral para resultados negativos referentes à (α, β)-espaçabilidade e verificamos que o conjunto $\mathcal{ND}[0,1]$ das funções que não possuem derivada, não pode ser (α, β)-espaçável para qualquer cardinal infinito α. Também fornecemos critérios para resultados positivos, mostrando em particular que os conjuntos $\ell_\infty \setminus F$, onde $F \in \{c, c_0\}$, e $L_p[0,1] \setminus \bigcup_{q \in (p,\infty)} L_q[0,1]$, para $p \in (0, \infty)$, são $(\alpha, \mathfrak{c})$-espaçáveis se, e somente se, α for finito.

Introduzimos a noção de (α, β)-lineabilidade densa e fornecemos um critério para demonstrar em particular que o conjunto $L_p[0,1] \setminus \bigcup_{q \in (p,\infty)} L_q[0,1]$, para $p \in (0, \infty)$, é também (α, β)-denso lineável para cada $0 \leq \alpha \leq \beta$ e $\max\{\alpha, \aleph_0\} \leq \beta \leq \mathfrak{c}$. Nossos achados destacam que a geometria dos conjuntos estudados sozinha é insuficiente e que o tipo de topologia considerada em cada ambiente também desempenha um papel crucial.

Palavras-chave: Lineabilidade, Espaçabilidade, Espaço de Sequências, Sequência básica, Séries, Convergência, Topologia.

Abstract

This thesis provides criteria (both negative and positive) that contribute to the existing literature and address open problems within the notions of (α, β)-lineability/spaceability, (α, β)-dense lineability, pointwise lineability, and $[\mathcal{S}]$-lineability. In our exploration, we began by investigating the behavior of algebraic and topological structures present in the set of unbounded, continuous, and integrable functions on the interval $[0, \infty)$. This investigation was initiated by Calderón-Moreno, Gerlach-Mena, and Prado-Bassas, where they demonstrated, among other results, that the set

$$\mathcal{A} := \left\{ f \in \mathcal{C}\left[0, \infty\right) \cap L_1\left[0, \infty\right) : \limsup_{x \to \infty} \left| f\left(x\right) \right| = \infty \right\}$$

is lineable. To better understand the dimensional relationships in this environment, we employed new techniques and gained additional insights into both the topological and algebraic structure of this set. Specifically, we proved its pointwise spaceability (and thus, spaceability).

Additionally, we demonstrated that the set $L_p[0, 1] \setminus \bigcup_{q \in (p, \infty)} L_q[0, 1]$, para $p \in (0, \infty)$, although $(1, \mathfrak{c})$-spaceable (see [21]), is not $(\aleph_0, \mathfrak{c})$-spaceable. We established a general criterion for negative results concerning (α, β)-spaceability and verified that the set $\mathcal{ND}[0, 1]$ of nowhere differentiable functions cannot be (α, β)-spaceable for any infinite cardinal α. We also provided criteria for positive results, showing in particular that the sets $\ell_\infty \setminus F$, where $F \in \{c, c_0\}$, and $L_p[0, 1] \setminus \bigcup_{q \in (p, \infty)} L_q[0, 1]$, for $p \in (0, \infty)$, are $(\alpha, \mathfrak{c})$-espaceable if and only if α is finite.

We introduced the notion of (α, β)-dense lineability and provided a criterion to demonstrate in particular that the set $L_p[0, 1] \setminus \bigcup_{q \in (p, \infty)} L_q[0, 1]$, for $p \in (0, \infty)$ is (α, β)-dense lineable for every $0 \leq \alpha \leq \beta$ and $\max\{\alpha, \aleph_0\} \leq \beta \leq \mathfrak{c}$. Our findings highlight that the geometry of the studied sets alone is insufficient and that the type of topology considered in each environment also plays a crucial role.

Keywords: Lineability, Spaceability, Sequence space, Basic Sequence, Series, Convergence, Topology.

Contents

Introduction

Modern mathematics often seeks to structure and understand seemingly disordered and chaotic phenomena. A line of research exemplifying this quest is lineability, which aims to find vector spaces (closed or not) in environments initially devoid of linearity.

Historically, the first hints of the term lineability appear in the classic work of V. Gurariy (1966, [25]). He proved that the family of continuous functions defined on the compact interval $[0, 1]$, which are not differentiable at any point (in short nowhere differentiable), contains, except for the identically zero function, an infinite dimensional vector subspace. This result surprised the academic community, given the difficulty of explicitly exemplifying functions with such chaotic behavior.

Inspired by the work of Levine and Milman (1940, [28]), who proved that the subset of $C[0, 1]$ of all functions with bounded variation does not contain an infinite dimensional closed subspace, Gurariy and other researchers began exploring linearity in various mathematical contexts. The research progressed, and in recent years, the search for vector structures in exotic environments has gained prominence. Several works have been developed in areas such as Algebraic Geometry [9], Real and Complex Analysis [5], Topology and Measure Theory [6], Linear Chaos [8], Linear and Multilinear Algebra, or even Operator Theory [12], among others.

With the development of the theory, positive results of lineability have become increasingly common, while techniques and general criteria for both positive and negative results have not appeared as frequently. From this perspective, some more restrictive notions have emerged, which add not only geometric contours but also qualitative information to the problems, such as the notion of *maximal lineability*, introduced by L. Bernal-González [7], which revolves around optimality in terms of dimension.

The terms *lineability* and *spaceability* were formally introduced by Gurariy in the early 2000s. These notions are established as follows:

For a vector space X and a cardinal number α, we say that a set $A \subseteq X$ is:

- α-lineable if it contains an α-dimensional vector subspace of X, excluding the zero vector.

Moreover, if X has a topology, the subset A is:

- α-spaceable in X whenever it contains a closed α-dimensional vector subspace of X, excluding the zero vector.

For the case where α is infinite, for simplicity, we simply say that A is lineable (respectively spaceable).

One of the motivations in the area of lineability is to construct vector spaces that are the best possible (in terms of dimension). This is precisely where the notion of maximal lineability, introduced by L. Bernal-González, as mentioned earlier, comes into play. Formally, we say that a set A in a vector space X is *maximal-lineable* if it contains a vector subspace whose dimension is the same as that of X.

It is important to mention that the "order" of lineability may not be achieved, i.e., there may not exist a maximum cardinal α such that a set A is α-lineable [3].

The book [4], entitled *Lineability: The Search for Linearity in Mathematics*, published in 2015, provides a comprehensive overview of what has been developed in this direction and the number of works done by various authors over the past 15 years.

To name just a few, there are results of lineability and spaceability in the following topics: subsets of real functions, functions that are continuous and differentiable at no point, norm-attaining functionals, hypercyclicity, series and summability, polynomials and zero sets in Banach spaces, non-absolutely summing operators, operator ideals, complex and holomorphic analysis, measurable and non-measurable functions, Peano curves in topological vector spaces.

It is worth mentioning that, at a first sight, negative results concerning lineability might be more likely common since many of these non-linear sets are referred as "monsters". However, most of the results present in the literature are in fact positive.

Concerning this thesis, one of its objectives is to provide criteria that extend those already existing in the literature, as well as to address some open problems mentioned throughout this introduction.

We explore some preliminary notions that are essential for understanding the subsequent chapters. This initial part is dedicated to introducing key concepts related to the notion of lineability as well as some preliminary results. Here, we explore the concepts of (α, β)-lineability/spaceability, pointwise lineability, $[\mathcal{S}]$-lineability, cardinality, among others. These foundational ideas provide the necessary background and framework for the more advanced topics discussed in later chapters.

In the first chapter, we delve into the concepts of pointwise lineability and (α, β)-lineability, presenting results pertaining to the set of unbounded, continuous, and integrable functions on $[0, \infty)$. The central paper on this topic was constructed in collaboration with V.V. Fávaro, D. Pellegrino, and A. Raposo Jr :

[22] V.V. Fávaro, D. Pellegrino, A. Raposo Jr, and G. Ribeiro, *Lineability and unbounded, continuous and integrable functions.* Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **117** (2023), no. 3, Paper No. 104, 21pp.

The investigation of the algebraic structure of the set of unbounded, continuous, and integrable functions on $[0, \infty)$ was initiated by Calderón-Moreno, Gerlach-Mena and Prado-Bassas in [34]. In their foundational work, the authors proved, among other results, that the set

$$\mathcal{A} := \left\{ f \in \mathcal{C}\,[0, \infty) \cap L_1\,[0, \infty) : \limsup_{x \to \infty} |f\,(x)| = \infty \right\}$$

is lineable. Building upon this foundation, our study in [22], extends the investigation further, employing different techniques to gain additional insights into both the topological and algebraic structure of this set. Notably, we enhance the understanding of this set by proving its pointwise spaceability (and thus, spaceability) and achieving $\mathfrak{c}$-lineability. These advancements represent a significant step forward in the field.

In the second chapter, our first objective was to establish a criterion for obtaining some negative results concerning (α, β)-spaceability. This allowed us to verify, for example, that the set $\mathcal{ND}[0,1]$ of nowhere differentiable functions cannot be (α, β)-spaceable for any infinite cardinal α regardless of the cardinal β. Following this, we provided criteria for positive results related to the notion of (α, β)-lineability. Based on these criteria, we verified, in particular, that the sets $\ell_\infty \setminus F$, where $F \in \{c, c_0\}$, and $L_p[0,1] \setminus \bigcup_{q\in(p,\infty)} L_q[0,1]$, with $p \in (0, \infty)$, are $(\alpha, \mathfrak{c})$-spaceable if and only if α is finite.

Our exploration in this chapter also included the introduction of the notion of (α, β)-dense lineability. We provided a criterion for obtaining some results, and in particular demonstrated that the set $L_p[0,1] \setminus \bigcup_{q\in(p,\infty)} L_q[0,1]$, with $p \in (0, \infty)$, is (α, β)-dense lineable for every $0 \leq \alpha \leq \beta$ and $\max\{\alpha, \aleph_0\} \leq \beta \leq \mathfrak{c}$.

Throughout these results, we observed that the geometry of the studied sets alone was not sufficient; the type of topology present in each set also played a crucial role.

The central papers for this stage of our research were constructed in collaboration with G. Araújo, A. Barbosa, V.V. Fávaro, D.M. Pellegrino, and A. Raposo Jr.:

[2] G. Araújo, A. Barbosa, A. Raposo Jr., and G. Ribeiro, *On the spaceability of the set of functions in the Lebesgue space L_p which are not in L_q*, Bull Braz Math Soc, New Series 54, **44** (2023).

[23] V.V. Fávaro, D. Pellegrino, A. Raposo Jr, and G. Ribeiro, *General criteria for a stronger notion of lineability.* Proc. Amer. Math. Soc. 152 (2024), 941-954

The third chapter is dedicated to extending the concept of $[\mathcal{S}]$-lineability, originally developed by L. Bernal-González, J.A. Conejero, M. Murillo-Arcila, and J.B. Seoane-Sepúlveda in [10]. Firstly, we characterize lineability in the context of complements of unions of closed subspaces in F-spaces, introducing $\mathcal{S}$-topologically linearly independent sequences. Additionally, we present a negative result in normed spaces and in p-Banach spaces, highlighting the importance of the properties of sequences and their distances from the origin in infinite dimensional vector spaces. This chapter culminates with a characterization of the spaceability of the complements of non-enumerable unions of subspaces in terms of lineability, offering a comprehensive view of linear structures in different mathematical contexts.

The central paper for this stage was:

[31] ______________, *A Quest for Convergence: Exploring Series in Non-Linear Environments*, Arch. Math. (2024) to appear.

The exploration of the results presented in this thesis was conducted in collaboration with G. Araújo, A. Barbosa, V.V. Fávaro, D.M. Pellegrino, and A. Raposo Jr.

Preliminaries

In this section, we present the fundamental concepts that will be used throughout this thesis. These preliminaries are essential for understanding the more advanced topics discussed in the subsequent chapters.

Basic Notions, Classical Spaces and Some Notations

In this thesis, we denote $\mathbb{N}$ as the set of positive integers, $\mathbb{R}$ as the real scalar field, and $\mathbb{C}$ as the complex scalar field. Unless explicity stated otherwise, all linear spaces are over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The term *subspace* will be used instead of vector subspace. Additionally, α and β represent cardinal numbers, $\operatorname{card}(A)$ denotes the cardinality of set A, $\aleph_0 := \operatorname{card}(\mathbb{N})$ and $\mathfrak{c} := \operatorname{card}(\mathbb{R})$. For $p \in (0, \infty)$, as usual, we denote ℓ_p as the classical set of absolutely p-summable sequences, defined as:

$$\ell_p := \left\{ (t_n)_{n=1}^{\infty} \in \mathbb{K}^{\mathbb{N}} : \sum_{n=1}^{\infty} |t_n|^p < \infty \right\}.$$

Let X be a vector space and $p \in (0, 1)$. A p-norm over X is a function denoted by $\|\cdot\|_p$, which satisfies the following conditions:

1. $\|x\|_p = 0$ if and only if $x = 0$.

2. $\|tx\|_p = |t|^p \|x\|_p$ for all $x \in X$ and $t \in \mathbb{K}$.

3. $\|x + y\|_p \leq \|x\|_p + \|y\|_p$ for all $x, y \in X$.

When X is complete with the metric endowed by the p-norm $\|\cdot\|_p$, we say that the pair $\left(X, \|\cdot\|_p\right)$ is a p-Banach space.

It is known that for $p \in (0, 1]$, ℓ_p is a p-Banach space with the p-norm given by $\|(t_n)_{n=1}^{\infty}\|_p := \sum_{n=1}^{\infty} |t_n|^p$. These spaces are usually non-locally convex and still play an important role in the geometry of F-spaces (metrizable topological vector spaces). For more information on the theory of p-Banach spaces, we refer to [27, 32].

Still in terms of notations and terminologies, as usual, we denote ℓ_∞ as the classical space of bounded sequences defined as:

$$\ell_\infty := \left\{ (t_n)_{n=1}^{\infty} \in \mathbb{K}^{\mathbb{N}} : \sup_{n \in \mathbb{N}} |t_n| < \infty \right\}.$$

Furthermore, if $(\Omega, \mathcal{M}, \mu)$ is a measure space, with μ a positive measure, and $p \in (0, \infty]$, we denote $L_p(\Omega)$ as the classical vector space of all (Lebesgue classes of) measurable functions $f : \Omega \to \mathbb{K}$ such that

$$\begin{cases} |f|^p \text{ is } \mu\text{-integrable on } \Omega, & \text{for } 0 < p < \infty, \\ f \text{ is } \mu\text{-essentially bounded in } \Omega, & \text{for } p = \infty. \end{cases}$$

Some Concepts Related to the Notion of Lineability

In what follows, interconnected concepts related to the notion of lineability, which are central to this thesis, are introduced.

- (α, β)-**lineability/spaceability:** For α and β with $\alpha \leq \beta$, a set A in a vector space X is said be (α, β)-*lineable* if it is α-lineable and for every subspace $F_\alpha \subseteq X$ with $F_\alpha \subset A \cup \{0\}$ and $\dim(F_\alpha) = \alpha$, there is a subspace $F_\beta \subset X$ with $\dim(F_\beta) = \beta$ and

$$F_\alpha \subset F_\beta \subset A \cup \{0\}. \tag{1}$$

  (hence $(0, \beta)$-lineability $\Leftrightarrow$ β-lineability).

If in addition, X has a topology, then A is called (α, β)-*spaceable* if the subspace F_β satisfying (1) can always be chosen closed, (see, for instance, [21]).

- **Pointwise lineability/spaceability**: A set A in a vector space X is *pointwise lineable* if, for every x in A, there is an infinite dimensional subspace $Y \subseteq X$ such that

$$x \in Y \subseteq A \cup \{0\}\,. \tag{2}$$

  If in addition, X has a topology, then A is called *pointwise spaceable* if the subspace Y satisfying (2) can always be chosen closed, (see, for instance, [29]).

  In both the cases, if $\dim Y = \dim X$ then A is called *pointwise maximal lineable* (respectively, *pointwise maximal spaceable*).

- $[\mathcal{S}]$-**lineability**: For a subspace $\mathcal{S}$ of $\mathbb{K}^{\mathbb{N}}$, we say that a subset A of a Hausdorff topological vector space X is $[(u_n)_{n=1}^\infty, \mathcal{S}]$-*lineable* in X if, for each sequence $(c_n)_{n=1}^\infty \in \mathcal{S}$, the series $\sum_{n=1}^\infty c_n u_n$ converges in X to a vector in $A \cup \{0\}$. Moreover, A is $[\mathcal{S}]$-*lineable* in X if it is $[(u_n)_{n=1}^\infty, \mathcal{S}]$-lineable for some sequence $(u_n)_{n=1}^\infty$ of linearly independent elements in X.

Other concepts will be introduced throughout the subsequent chapter.

Chapter 1

Linearity in Unbounded, Continuous, and Integrable Functions

In this chapter, we explore the notions of (α, β)-lineability/spaceability, pointwise lineability, and spaceability within the context of unbounded, continuous, and integrable functions, seeking to understand more about this environment in terms of algebraic and topological structure.

This line of investigation was initiated by Calderón-Moreno, Gerlach-Mena and Prado-Bassas in [34], where the authors prove, among other results, that the set

$$\mathcal{A} := \left\{ f \in \mathcal{C}\,[0,\infty) \cap L_1\,[0,\infty) : \limsup_{x\to\infty} |f\,(x)| = \infty \right\}$$

is lineable.

In,

[22] V.V. Fávaro, D. Pellegrino, A. Raposo Jr, and G. Ribeiro, *Lineability and unbounded, continuous and integrable functions.* Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **117** (2023), no. 3, Paper No. 104, 21pp.

we extend this investigation and conclude by proving, with different techniques, further results on the topological and algebraic structure of this set. More precisely, we improve by obtaining pointwise spaceability (and so spaceability) and obtaining pointwise $\mathfrak{c}$-lineability. Utilizing a specific technique, we also ascertain that, contrary to the $(1, \mathfrak{c})$-spaceability demonstrated in [21] for the set $L_p[0,1] \setminus \bigcup_{q\in(p,\infty)} L_q[0,1]$, for $p > 0$, this set does not possess $(\aleph_0, \mathfrak{c})$-spaceability in $L_p[0,1]$.

In our study, we denote by $\mathcal{C}\,[0,\infty)$ the space of all continuous functions $f\colon\ [0,\infty) \to \mathbb{R}$, endowed with the topology of uniform convergence on compact sets. According to [34], this set becomes a **complete metrizable topological vector space** (F-space). We consider, as in [34], the set

$$X := \mathcal{C}\,[0,\infty) \cap L_1\,[0,\infty)$$

equipped with the metric $d_X\colon X \times X \to \mathbb{R}$ given by

$$d_X(f,g) := \|f-g\|_1 + \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\|f-g\|_{\infty,n}}{1+\|f-g\|_{\infty,n}},$$

where $\|f\|_{\infty,n} = \max\{|f(x)| : x \in [0,n]\}$. In [34, Theorem 2.4] it was shown that

$$\mathcal{A} := \left\{ f \in X : \limsup_{x\to\infty} |f(x)| = \infty \right\}$$

is maximal dense-lineable.

A prototype of a function $\omega\colon [0,\infty) \to \mathbb{R}$ in $\mathcal{A}$ is the following:

$$\omega(x) = \begin{cases} 2^n s_n^2 (x-n), & \text{if } n \le x \le n + \dfrac{1}{2^n s_n}, \\ -2^n s_n^2 \left(x - n - \dfrac{1}{2^{n-1} s_n} \right), & \text{if } n + \dfrac{1}{2^n s_n} \le x \le n + \dfrac{1}{2^{n-1} s_n}, \\ 0, & \text{if } x \notin \bigcup_{n=1}^{\infty} \left[n, n + \dfrac{1}{2^{n-1} s_n} \right], \end{cases}$$

where $s_n = 1 + 1/2 + \cdots + 1/n$.

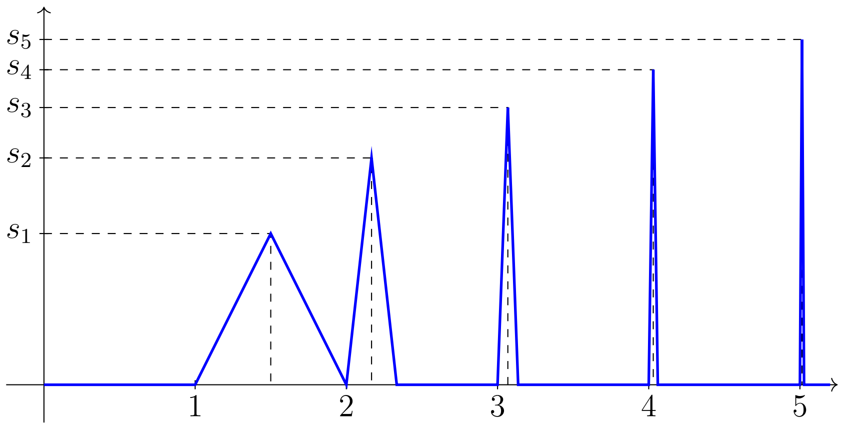

Figure 1.1: The graph of ω.

1.1 Pointwise Spaceability

In this section, we will present a positive result related to the pointwise spaceability of set $\mathcal{A}$. Our finding demonstrates that $\mathcal{A}$ is pointwise $\mathfrak{c}$-spaceable. This discovery reveals more about the geometry of this set, providing new perspectives on its structure.

Theorem 1.1. *The set $\mathcal{A}$ is pointwise $\mathfrak{c}$-spaceable in (X, d_X).*

Proof. Let $f \in \mathcal{A}$ and $(x_n)_{n=1}^{\infty}$ be an increasing sequence in $[0,\infty)$ such that

$$\begin{cases} x_n \overset{n\to\infty}{\longrightarrow} \infty, \\ f(x_n) \neq 0 \text{ for each } n = 1, 2, \ldots, \\ |f(x_n)| \overset{n\to\infty}{\longrightarrow} \infty. \end{cases} \tag{1.1}$$

Given $n, k \in \mathbb{N}$, we define $t_n^{(k)}$ as follows:

$$t_n^{(1)} = \frac{x_n + x_{n+1}}{2} \qquad \text{and} \qquad t_n^{(k+1)} = \frac{x_n + t_n^{(k)}}{2}.$$

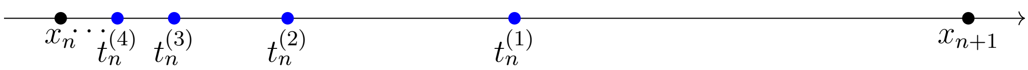

Figure 1.2: A geometric view of $t_n^{(k)}$.

Now, given $k, n \in \mathbb{N}$, let

$$0 < \varepsilon_n^{(k)} \leq \frac{t_n^{(k)} - t_n^{(k+1)}}{2}$$

be such that

$$|f(x_n)| \leq \frac{1}{2^n \varepsilon_n^{(k)}}.$$

For each positive integer k, let $f_k\colon [0, \infty) \to \mathbb{R}$ be the function

$$f_k(x) = \begin{cases} \dfrac{f(x_n)}{\varepsilon_n^{(k)}} \left(x - t_n^{(k)} + \varepsilon_n^{(k)} \right), & \text{if } t_n^{(k)} - \varepsilon_n^{(k)} \leq x \leq t_n^{(k)}, \\ -\dfrac{f(x_n)}{\varepsilon_n^{(k)}} \left(x - t_n^{(k)} - \varepsilon_n^{(k)} \right), & \text{if } t_n^{(k)} \leq x \leq t_n^{(k)} + \varepsilon_n^{(k)}, \\ 0, & \text{if } x \notin \bigcup\limits_{n=1}^{\infty} \left[t_n^{(k)} - \varepsilon_n^{(k)}, t_n^{(k)} + \varepsilon_n^{(k)} \right]. \end{cases} \tag{1.2}$$

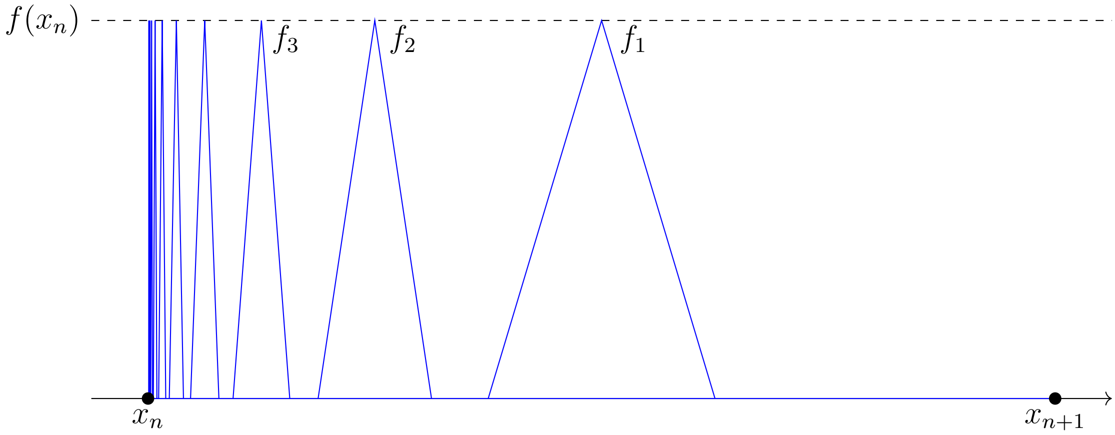

Figure 1.3: Overlaid graphs of the functions f_k on $[x_n, x_{n+1}]$.

Roughly speaking, f_k takes the k-th triangle on each interval $[x_n, x_{n+1}]$. Observe that:

(*i*) f_k is continuous, for each $k \in \mathbb{N}$;

(*ii*) $f_k \in L_1[0,\infty)$, for each $k \in \mathbb{N}$, because

$$\int_0^\infty |f_k|\, dx = \sum_{n=1}^{\infty} \varepsilon_n^{(k)} |f(x_n)| \le \sum_{n=1}^{\infty} \frac{1}{2^n} = 1;$$

(*iii*) For each $k \in \mathbb{N}$,

$$t_n^{(k)} \overset{n\to\infty}{\longrightarrow} \infty \text{ and } \left|f_k\left(t_n^{(k)}\right)\right| = |f(x_n)| \overset{n\to\infty}{\longrightarrow} \infty.$$

Therefore, $\limsup_{x\to\infty} |f_k(x)| = \infty$;

(*iv*) If $x \in \left(t_n^{(k)} - \varepsilon_n^{(k)}, t_n^{(k)} + \varepsilon_n^{(k)}\right)$ for some positive integers k and n, then, for each $j \neq k$, we have $f_k(x) \neq 0 = f_j(x)$.

Thus, if $x \notin \bigcup_{k,n\in\mathbb{N}} \left(t_n^{(k)} - \varepsilon_n^{(k)}, t_n^{(k)} + \varepsilon_n^{(k)}\right)$, then $f_k(x) = 0$, for each $k \in \mathbb{N}$. In particular, $f_k(x_n) = 0$ for all positive integers n, k. Fixing a non zero sequence $(a_n)_{n=0}^{\infty} \in \ell_1$, let $g\colon [0,\infty) \to \mathbb{R}$ be the function

$$g(x) = \sum_{k=0}^{\infty} a_k f_k(x),$$

where $f_0 = f$. Let us verify that g is well-defined. If $x \in \left(t_{n_0}^{(k_0)} - \varepsilon_{n_0}^{(k_0)}, t_{n_0}^{(k_0)} + \varepsilon_{n_0}^{(k_0)}\right)$ for some pair k_0, n_0 of positive integers, then

$$g(x) = a_0 f(x) + a_{k_0} f_{k_0}(x)$$

and, if $x \notin \bigcup_{k,n\in\mathbb{N}} \left(t_n^{(k)} - \varepsilon_n^{(k)}, t_n^{(k)} + \varepsilon_n^{(k)}\right)$, then

$$g(x) = a_0 f(x).$$

Now we will show that g is continuous. If x_0 lies in some interval $\left(t_n^{(k)} - \varepsilon_n^{(k)}, t_n^{(k)} + \varepsilon_n^{(k)}\right)$, then

$$\lim_{x\to x_0} g(x) = \lim_{x\to x_0} [a_0 f(x) + a_k f_k(x)] = a_0 f(x_0) + a_k f_k(x_0) = g(x_0).$$

If x_0 is an interior point of $\mathbb{R} \setminus \bigcup_{k,n\in\mathbb{N}} \left(t_n^{(k)} - \varepsilon_n^{(k)}, t_n^{(k)} + \varepsilon_n^{(k)}\right)$, then

$$\lim_{x\to x_0} g(x) = \lim_{x\to x_0} a_0 f(x) = a_0 f(x_0) = g(x_0).$$

If $x_0 = t_n^{(k)} - \varepsilon_n^{(k)}$ for some pair k, n of positive integers, then

$$\lim_{x\to x_0^-} g(x) = \lim_{x\to x_0^-} a_0 f(x) = a_0 f(x_0) = g(x_0)$$

and

$$\lim_{x\to x_0^+} g(x) = \lim_{x\to x_0^+} [a_0 f(x) + a_k f_k(x)] = a_0 f(x_0) + a_k f_k(x_0) = a_0 f(x_0) = g(x_0).$$

Analogously, we verify that

$$\lim_{x\to x_0^-} g(x) = g(x_0) = \lim_{x\to x_0^+} g(x)$$

if $x_0 = t_n^{(k)} + \varepsilon_n^{(k)}$ for some positive integers k and n. This assures that g is continuous.

Now, since

$$\sum_{k=0}^{\infty} \|a_k f_k\|_1 \le |a_0| \, \|f\|_1 + \sum_{k=1}^{\infty} |a_k| < \infty,$$

then $g = \sum_{k=0}^{\infty} a_k f_k$ belongs to $L_1[0,\infty)$.

Finally, notice that, if $a_0 \neq 0$, then

$$|g(x_n)| = |a_0 f(x_n)| = |a_0| \, |f(x_n)| \overset{n\to\infty}{\longrightarrow} \infty$$

and, therefore, $\limsup_{x\to\infty} |g(x)| = \infty$. If $a_0 = 0$, let j be such that $a_j \neq 0$. Then,

$$\left|g\left(t_n^{(j)}\right)\right| = \left|a_j f_j\left(t_n^{(j)}\right)\right| = |a_j| \, |f(x_n)| \overset{n\to\infty}{\longrightarrow} \infty$$

and, consequently, $\limsup_{x\to\infty} |g(x)| = \infty$. Hence, $g \in \mathcal{A}$; in particular, we have proved that the operator

$$\begin{aligned} T\colon \ell_1 &\to \mathcal{C}[0,\infty) \cap L_1[0,\infty) \\ (a_k)_{k=0}^{\infty} &\mapsto \sum_{k=0}^{\infty} a_k f_k \end{aligned}$$

is well-defined and $T(\ell_1) \subset \mathcal{A} \cup \{0\}$. It is easy to see that T is linear and injective. Since

$$\mathfrak{c} = \dim(\ell_1) = \dim(T(\ell_1))$$

and $f = T(1,0,0,\ldots) \in T(\ell_1)$, we just need to show that $\overline{T(\ell_1)}^{(X,d_X)} \subset \mathcal{A} \cup \{0\}$, where $\overline{T(\ell_1)}^{(X,d_X)}$, denotes the closure of $T(\ell_1)$ in (X, d_X).

Let $h \in \overline{T(\ell_1)}^{(X,d_X)} \setminus \{0\}$. Then, there exists a sequence $\left(c^{(j)}\right)_{j=1}^{\infty}$ in ℓ_1, with $c^{(j)} = \left(c_k^{(j)}\right)_{k=0}^{\infty}$ such that

$$\lim_{j\to\infty} \sum_{k=0}^{\infty} c_k^{(j)} f_k = h$$

in (X, d_X). In particular, $\sum_{k=0}^{\infty} c_k^{(j)} f_k$ converges uniformly to h over each compact interval $[a,b] \subset [0,\infty)$. Therefore, fixing $n \in \mathbb{N}$ and recalling that $f_k(x_n) = 0$ for all positive integers n, k, we have

$$\lim_{j\to\infty} c_0^{(j)} f(x_n) = \lim_{j\to\infty} \sum_{k=0}^{\infty} c_k^{(j)} f_k(x_n) = h(x_n),$$

which implies

$$\lim_{j\to\infty} c_0^{(j)} = \frac{h(x_n)}{f(x_n)} := c_0.$$

The uniqueness of the limit of a sequence and the arbitrariness of $n \in \mathbb{N}$ assure that $h(x_n) = c_0 f(x_n)$, for each $n \in \mathbb{N}$.

If $c_0 \neq 0$, then

$$\lim_{n\to\infty} |h(x_n)| = \lim_{n\to\infty} |c_0| |f(x_n)| = \infty$$

and we have $h \in \mathcal{A}$.

If $c_0 = 0$, it follows from (iv) that, if $x \notin \bigcup_{k,n\in\mathbb{N}} \left(t_n^{(k)} - \varepsilon_n^{(k)}, t_n^{(k)} + \varepsilon_n^{(k)}\right)$, then $f_k(x) = 0$ for each $k \in \mathbb{N}$ and, hence,

$$h(x) = \lim_{j\to\infty} \sum_{k=1}^{\infty} c_k^{(j)} f_k(x) = 0.$$

If $x \in \left(t_n^{(r)} - \varepsilon_n^{(r)}, t_n^{(r)} + \varepsilon_n^{(r)}\right)$ for some pair of positive integers (n, r), it follows from (iv) that $f_r(x) \neq 0 = f_k(x)$ whenever $k \neq r$ and, therefore,

$$h(x) = \lim_{j\to\infty} \sum_{k=1}^{\infty} c_k^{(j)} f_k(x) = \lim_{j\to\infty} c_r^{(j)} f_r(x).$$

Thus

$$\lim_{j\to\infty} c_r^{(j)} = \frac{h(x)}{f_r(x)} := c_r$$

and

$$h(x) = c_r f_r(x). \tag{1.3}$$

Since $h \neq 0$, there is a pair (n, k) of positive integers such that $h(x_0) \neq 0$ for a certain $x_0 \in \left(t_n^{(k)} - \varepsilon_n^{(k)}, t_n^{(k)} + \varepsilon_n^{(k)}\right)$. Thus $c_k \neq 0$ and, since (1.3) is valid for all $x \in \left(t_n^{(k)} - \varepsilon_n^{(k)}, t_n^{(k)} + \varepsilon_n^{(k)}\right)$, we conclude that

$$\lim_{n\to\infty} \left|h\left(t_n^{(k)}\right)\right| = \lim_{n\to\infty} |c_k| \left|f_k\left(t_n^{(k)}\right)\right| = \infty.$$

Thus, $h \in \mathcal{A}$ and the proof is done. □

As an immediate consequence we get:

Corollary 1.2. *The set $\mathcal{A}$ is $(1, \mathfrak{c})$-spaceable in (X, d_X).*

Remark 1.3. Let $\mathcal{C}^n[0,\infty)$, $n = 1, 2, \ldots$, be the linear space of the functions $f\colon [0,\infty) \to \mathbb{R}$ whose derivative of order n is continuous and let $\mathcal{C}^\infty[0,\infty)$ be the linear space of the functions $f\colon [0,\infty) \to \mathbb{R}$ having derivatives of any order. Using bump functions, a slight adaptation of the above proof assures that the sets

$$\mathcal{A}^n = \left\{ f \in \mathcal{C}^n[0,\infty) \cap L_1[0,\infty) : \limsup_{x\to\infty} |f(x)| = \infty \right\}, \ n = 1, 2, \ldots$$

and

$$\mathcal{A}^\infty = \left\{ f \in \mathcal{C}^\infty[0,\infty) \cap L_1[0,\infty) : \limsup_{x\to\infty} |f(x)| = \infty \right\}$$

are pointwise $\mathfrak{c}$-lineable.

1.2 (α, β)-Spaceability: A Negative Result

In this section, we present a negative result concerning the $(\aleph_0, \mathfrak{c})$-spaceability of set $\mathcal{A}$. Our finding demonstrates that $\mathcal{A}$ does not possess $(\aleph_0, \mathfrak{c})$-spaceability within the space (X, d_X).

Theorem 1.4. *$\mathcal{A}$ is not $(\aleph_0, \mathfrak{c})$-spaceable in (X, d_X).*

Proof. Let $f \in \mathcal{A}$ and let $\left(x_n^{(1)}\right)_{n=1}^{\infty}$ be an increasing sequence in $[0, \infty)$, with $x_1^{(1)} > 0$, such that

$$x_n^{(1)} \stackrel{n\to\infty}{\longrightarrow} \infty, \qquad f\left(x_n^{(1)}\right) \neq 0 \text{ for each } n = 1, 2, \ldots \qquad \text{and} \qquad \left|f\left(x_n^{(1)}\right)\right| \stackrel{n\to\infty}{\longrightarrow} \infty.$$

For each $n \in \mathbb{N}$, let $t_n^{(1)}$ be the midpoint of the interval $\left[x_n^{(1)}, x_{n+1}^{(1)}\right]$ and let $\varepsilon_n^{(1)} > 0$ be such that

$$\varepsilon_n^{(1)} \leq \frac{x_{n+1}^{(1)} - x_n^{(1)}}{2} \qquad \text{and} \qquad \varepsilon_n^{(1)} \left|f\left(x_n^{(1)}\right)\right| \leq \frac{1}{2^n}.$$

Let us define $x_n^{(1)} = x_n$ for each $n \in \mathbb{N}$ and $f_1\colon [0, \infty) \to \mathbb{R}$ by

$$f_1(x) = \begin{cases} \dfrac{f(x_n)}{\varepsilon_n^{(1)}}\left(x - t_n^{(1)} + \varepsilon_n^{(1)}\right), & \text{if } t_n^{(1)} - \varepsilon_n^{(1)} \leq x \leq t_n^{(1)}, \\ -\dfrac{f(x_n)}{\varepsilon_n^{(1)}}\left(x - t_n^{(1)} - \varepsilon_n^{(1)}\right), & \text{if } t_n^{(1)} \leq x \leq t_n^{(1)} + \varepsilon_n^{(1)}, \\ 0, & \text{if } x \notin \bigcup_{n=1}^{\infty}\left[t_n^{(1)} - \varepsilon_n^{(1)}, t_n^{(1)} + \varepsilon_n^{(1)}\right]. \end{cases}$$

Let $\left(x_n^{(2)}\right)_{n=1}^{\infty}$ be the unique increasing ordering of $\{x_n : n \in \mathbb{N}\} \cup \left\{t_n^{(1)} : n \in \mathbb{N}\right\}$. For each $n \in \mathbb{N}$, let $t_n^{(2)}$ be the midpoint of the interval $\left[x_n^{(2)}, x_{n+1}^{(2)}\right]$ and let $\varepsilon_n^{(2)} > 0$ be such that

$$\left[t_n^{(2)} - \varepsilon_n^{(2)}, t_n^{(2)} + \varepsilon_n^{(2)}\right] \subset \left[x_n^{(2)}, x_{n+1}^{(2)}\right] \qquad \text{and} \qquad \varepsilon_n^{(2)} \left|f(x_n)\right| \leq \frac{1}{2^n}.$$

Let $f_2\colon [0, \infty) \to \mathbb{R}$ be the function defined by

$$f_2(x) = \begin{cases} f_1(x), & \text{if } x_1 \leq x \leq x_2, \\ \dfrac{f(x_n)}{\varepsilon_n^{(2)}}\left(x - t_n^{(2)} + \varepsilon_n^{(2)}\right), & \text{if } t_n^{(2)} - \varepsilon_n^{(2)} \leq x \leq t_n^{(2)} \text{ and } x \notin [x_1, x_2], \\ -\dfrac{f(x_n)}{\varepsilon_n^{(2)}}\left(x - t_n^{(2)} - \varepsilon_n^{(2)}\right), & \text{if } t_n^{(2)} \leq x \leq t_n^{(2)} + \varepsilon_n^{(2)} \text{ and } x \notin [x_1, x_2], \\ 0, & \text{if } x \notin \left(\bigcup_{n=1}^{\infty}\left[t_n^{(2)} - \varepsilon_n^{(2)}, t_n^{(2)} + \varepsilon_n^{(2)}\right]\right) \cup [x_1, x_2]. \end{cases}$$

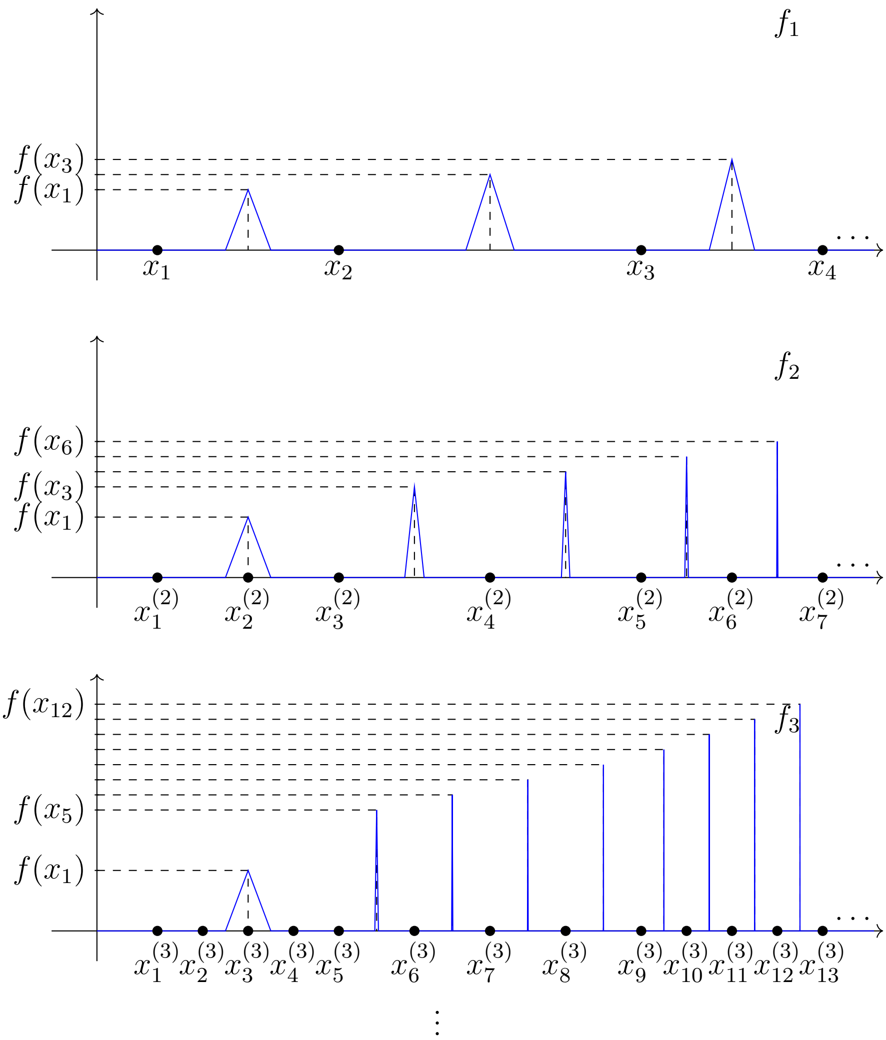

Figure 1.4: The sequence $(f_k)_{k=1}^{\infty}$.

In general, for each integer $k > 1$, let $\left(x_n^{(k)}\right)_{n=1}^{\infty}$ be the unique increasing ordering of $\left\{x_n^{(k-1)} : n \in \mathbb{N}\right\} \cup \left\{t_n^{(k-1)} : n \in \mathbb{N}\right\}$, $t_n^{(k)}$ be the midpoint of the interval $\left[x_n^{(k)}, x_{n+1}^{(k)}\right]$ and $\varepsilon_n^{(k)} > 0$ be such that

$$\varepsilon_n^{(k)} \leq \frac{x_{n+1}^{(k)} - x_n^{(k)}}{2} \qquad \text{and} \qquad \varepsilon_n^{(k)} \left|f\left(x_n\right)\right| \leq \frac{1}{2^n}.$$

Then $f_k \colon [0, \infty) \to \mathbb{R}$ is the function defined by

$$f_k\left(x\right) = \begin{cases} f_1(x), & \text{if } x_1 \leq x \leq x_2, \\ \dfrac{f\left(x_n\right)}{\varepsilon_n^{(k)}} \left(x - t_n^{(k)} + \varepsilon_n^{(k)}\right), & \text{if } t_n^{(k)} - \varepsilon_n^{(k)} \leq x \leq t_n^{(k)} \text{ and } x \notin \left[x_1, x_2\right], \\ -\dfrac{f\left(x_n\right)}{\varepsilon_n^{(k)}} \left(x - t_n^{(k)} - \varepsilon_n^{(k)}\right), & \text{if } t_n^{(k)} \leq x \leq t_n^{(k)} + \varepsilon_n^{(k)} \text{ and } x \notin \left[x_1, x_2\right], \\ 0, & \text{if } x \notin \left(\bigcup_{n=1}^{\infty} \left[t_n^{(k)} - \varepsilon_n^{(k)}, t_n^{(k)} + \varepsilon_n^{(k)}\right]\right) \cup [x_1, x_2]. \end{cases}$$

Given $k \in \mathbb{N}$, let n_k be the smallest index such that $x_{n_k} > \max\{x_2, k\}$. Let $g_k\colon [0,\infty) \to \mathbb{R}$ be defined by

$$g_k(x) = \begin{cases} f_k(x), & x \in [0, x_2] \cup [x_{n_k}, \infty), \\ 0, & \text{if } x \in [x_2, x_{n_k}]. \end{cases}$$

Let

$$g(x) = \begin{cases} f_1(x), & \text{if } 0 \leq x \leq x_2, \\ 0, & \text{if } x \geq x_2. \end{cases}$$

It is simple to check that $\{g_1, g_2, \ldots\}$ is linearly independent and that

$$\operatorname{span}\{g_n : n \in \mathbb{N}\} \setminus \{0\} \subset \mathcal{A}.$$

We will prove that $g_k \overset{k\to\infty}{\longrightarrow} g$ in (X, d_X) and, to do this, we just need to prove that

$$\|g_k - g\|_1 \overset{k\to\infty}{\longrightarrow} 0$$

and

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|g_k - g\|_{\infty,n}}{1 + \|g_k - g\|_{\infty,n}} \overset{k\to\infty}{\longrightarrow} 0.$$

It is easy to see that

$$(g_k - g)(x) = \begin{cases} 0, & \text{if } x \leq x_{n_k}, \\ g_k(x) = f_k(x), & \text{if } x \geq x_{n_k}. \end{cases}$$

Since $x_{n_k} \to \infty$, we obtain

$$\|g_k - g\|_1 = \int_{x_{n_k}}^{\infty} |g_k| \overset{k\to\infty}{\longrightarrow} 0.$$

Since $x_{n_k} > k$, we get $\|g_k - g\|_{\infty,n} = 0$, for every $n = 1, \ldots, k$, and then

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|g_k - g\|_{\infty,n}}{1 + \|g_k - g\|_{\infty,n}} = \sum_{n=k+1}^{\infty} \frac{1}{2^n} \frac{\|g_k - g\|_{\infty,n}}{1 + \|g_k - g\|_{\infty,n}} \leq \sum_{n=k+1}^{\infty} \frac{1}{2^n} = \frac{1}{2^k}$$

for each $k \in \mathbb{N}$. Therefore

$$d_X(g_k, g) \overset{k\to\infty}{\longrightarrow} 0.$$

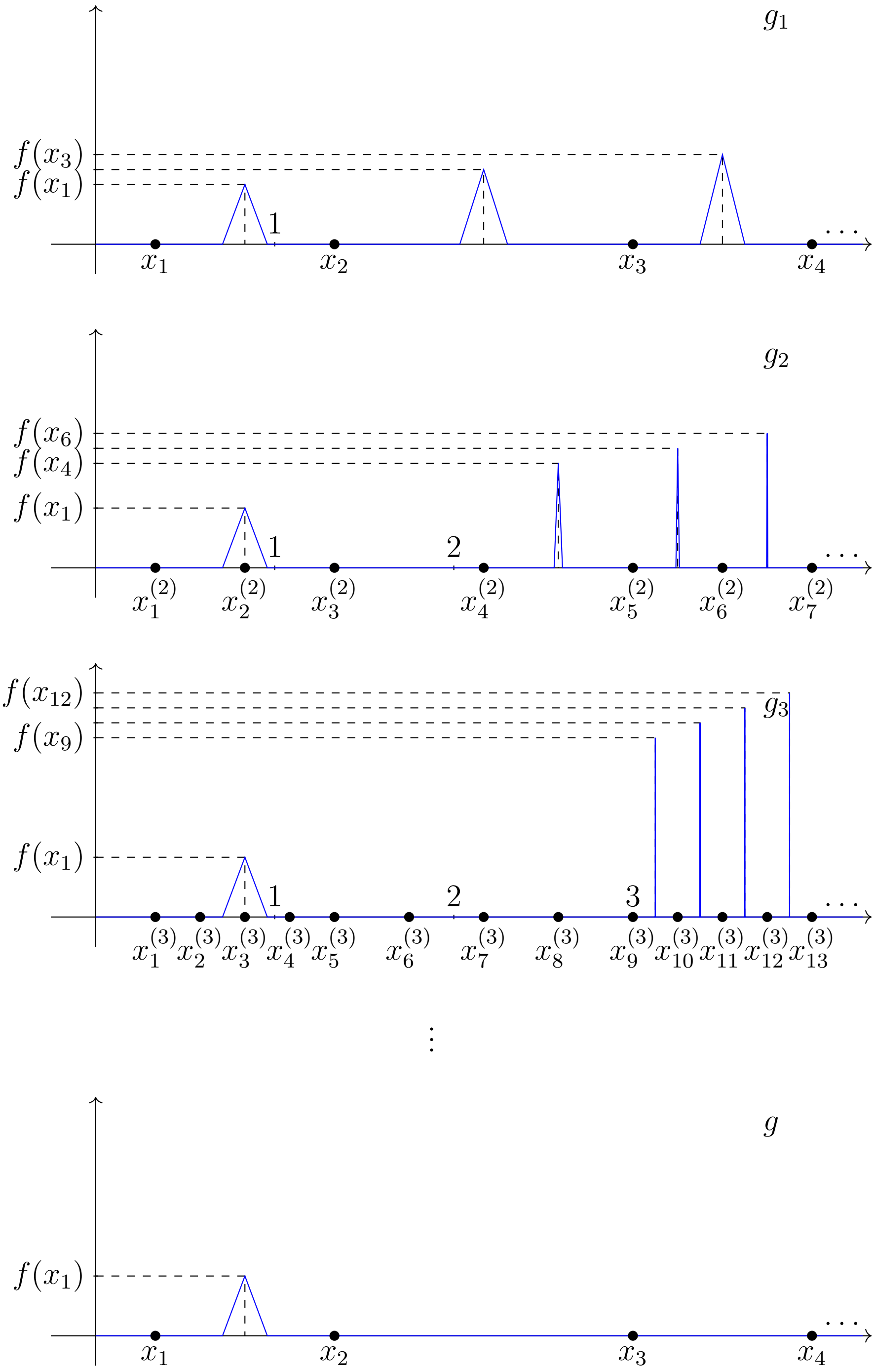

Figure 1.5: The sequence $(g_k)_{k=1}^{\infty}$ and its limit g.

Since $g \in X \setminus \mathcal{A}$, it follows that

$$\overline{\operatorname{span}}\,\{g_n : n \in \mathbb{N}\} \setminus \{0\} \nsubseteq \mathcal{A}.$$

Thus, there is no infinite dimensional closed subspace containing span $\{g_n : n \in \mathbb{N}\}$ and contained in $\mathcal{A} \cup \{0\}$. Hence $\mathcal{A}$ is not $(\aleph_0, \mathfrak{c})$-spaceable in (X, d_X). □

Corollary 1.5. *$\mathcal{A}$ is not $(\aleph_0, \mathfrak{c})$-spaceable in $L_1\,[0, \infty)$.*

Corollary 1.6. *$\mathcal{A}$ is not $(\aleph_0, \mathfrak{c})$-spaceable in $\mathcal{C}[0,\infty)$ endowed with the topology of the uniform convergence on compacta.*

1.3 Sequences of Unbounded, Continuous, and Integrable Functions

The following remark will be useful in this section:

Remark 1.7. Let $f \in \mathcal{A}$ and let $(x_n)_{n=1}^{\infty}$ be an increasing sequence in $[0,\infty)$ satisfying *(1.1)*. For each $k \in \mathbb{N}$, let f_k as in *(1.2)*. Fixing $k \in \mathbb{N}$, let j_k be the smallest index such that $x_{j_k} \geq k$. Let us define $g_0 = f$ and $g_k\colon [0,\infty) \to \mathbb{R}$ by

$$g_k(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq x_{j_k}, \\ f_k(x), & \text{if } x \geq x_{j_k}. \end{cases}$$

It is clear that the proof of *Theorem 1.1* holds if we replace the sequence $(f_k)_{k=0}^{\infty}$ by the sequence $(g_k)_{k=0}^{\infty}$.

Let $c_0(X)$ be the sequence space defined by

$$c_0(X) := \left\{(f_r)_{r=1}^{\infty} : f_r \in X \text{ for each } r \text{ and } d_X(f_r, 0) \overset{r\to\infty}{\longrightarrow} 0\right\}.$$

In this space we shall consider the distance $d_{c_0(X)}\colon c_0(X) \times c_0(X) \to \mathbb{R}$ given by

$$d_{c_0(X)}\left((f_r)_{r=1}^{\infty}, (g_r)_{r=1}^{\infty}\right) = \sup_{r\in\mathbb{N}} d_X(f_r, g_r).$$

Now, let $\mathcal{A}_0 \subset c_0(X)$ be the set defined by

$$\mathcal{A}_0 := \left\{(f_r)_{r=1}^{\infty} : f_r \in \mathcal{A} \text{ for each } r \text{ and } d_X(f_r, 0) \overset{r\to\infty}{\longrightarrow} 0\right\}.$$

In [34, Theorem 3.2] it was proved that $\mathcal{A}_0$ is *maximal dense-lineable* in $\left(c_0(X), d_{c_0(X)}\right)$, that is, $\mathcal{A}_0$ contains, except by the null vector, a dense vector space with the same dimension of $c_0(X)$. Here we establish the following result:

Theorem 1.8. *$\mathcal{A}_0$ is pointwise $\mathfrak{c}$-spaceable in $\left(c_0(X), d_{c_0(X)}\right)$.*

Proof. Let $f = (f_r)_{r=1}^{\infty} \in \mathcal{A}_0$. For each $r \in \mathbb{N}$, let $\left(x_j^{(r)}\right)_{j=1}^{\infty}$ be an increasing sequence in $[0,\infty)$ such that

$$x_j^{(r)} \overset{j\to\infty}{\longrightarrow} \infty, \qquad f_r\left(x_j^{(r)}\right) \neq 0 \text{ for each } j = 1, 2, \ldots \qquad \text{and} \qquad \left|f_r\left(x_j^{(r)}\right)\right| \overset{j\to\infty}{\longrightarrow} \infty.$$

Let us consider the function $f_{k,r}$ defined from f_r and $\left(x_j^{(r)}\right)_{j=1}^{\infty}$ as in (1.2), i.e.

$$f_{k,r}(x) = \begin{cases} \dfrac{f_r\left(x_n^{(r)}\right)}{\varepsilon_n^{(k,r)}}\left(x - t_n^{(k,r)} + \varepsilon_n^{(k,r)}\right), & \text{if } t_n^{(k,r)} - \varepsilon_n^{(k,r)} \leq x \leq t_n^{(k,r)}, \\ -\dfrac{f_r\left(x_n^{(r)}\right)}{\varepsilon_n^{(k,r)}}\left(x - t_n^{(k,r)} - \varepsilon_n^{(k,r)}\right), & \text{if } t_n^{(k,r)} \leq x \leq t_n^{(k,r)} + \varepsilon_n^{(k,r)}, \\ 0, & \text{if } x \notin \bigcup_{n=1}^{\infty}\left[t_n^{(k,r)} - \varepsilon_n^{(k,r)}, t_n^{(k,r)} + \varepsilon_n^{(k,r)}\right], \end{cases}$$

where $t_n^{(k,r)}$ is recursively defined by

$$t_n^{(1,r)} = \frac{x_n^{(r)} + x_{n+1}^{(r)}}{2} \qquad \text{and} \qquad t_n^{(k+1,r)} = \frac{x_n^{(r)} + t_n^{(k,r)}}{2},$$

$\varepsilon_n^{(k,r)}$ is such that

$$0 < \varepsilon_n^{(k,r)} < \frac{t_n^{(k,r)} - t_n^{(k+1,r)}}{2}$$

and

$$\varepsilon_n^{(k,r)} \left| f_r \left(x_n^{(r)} \right) \right| \leq \frac{1}{2^n}.$$

For each $r \in \mathbb{N}$ let j_r be the smallest index such that $x_{j_r}^{(r)} \geq r$. For each $k \in \mathbb{N}$, let $g_{k,r} \colon [0, \infty) \to \mathbb{R}$ be defined by

$$g_{k,r}(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq x_{j_r}^{(r)}, \\ f_{k,r}(x), & \text{if } x \geq x_{j_r}^{(r)}. \end{cases}$$

Denoting $g_j = (2^{-r} g_{j,r})_{r=1}^{\infty}$, we need to show that $g_j \in \mathcal{A}_0$, for each j. (In fact, the factor 2^{-r} in the definition of g_j is not really necessary. However, as we will see, its use simplifies the arguments.) Fixing $j \in \mathbb{N}$, since $g_{j,r} \in \mathcal{A}$ for each $r \in \mathbb{N}$, we only need to prove that $d_X(2^{-r} g_{j,r}, 0) \overset{r\to\infty}{\longrightarrow} 0$. If $j \geq 1$, then

$$\left\| 2^{-r} g_{j,r} \right\|_1 = 2^{-r} \int_0^{\infty} |g_{j,r}| \, dx \leq 2^{-r} \overset{r\to\infty}{\longrightarrow} 0.$$

Given $n \in \mathbb{N}$, it is clear that

$$\left\| 2^{-r} g_{j,r} \right\|_{\infty,n} = \max \left\{ \left| 2^{-r} g_{j,r}(x) \right| : x \in [0, n] \right\} = 0$$

whenever $r \geq n$. Consequently,

$$\left\| 2^{-r} g_{j,r} \right\|_{\infty,n} \overset{r\to\infty}{\longrightarrow} 0$$

for each $n \in \mathbb{N}$. Hence, for $r \geq 1$,

$$\begin{aligned} d_X\left(2^{-r} g_{j,r}, 0\right) &= \|2^{-r} g_{j,r}\|_1 + \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\|2^{-r} g_{j,r}\|_{\infty,n}}{1 + \|2^{-r} g_{j,r}\|_{\infty,n}} \\ &\leq 2^{-r} + \sum_{n=r+1}^{\infty} \frac{1}{2^n} \cdot \frac{\|2^{-r} g_{j,r}\|_{\infty,n}}{1 + \|2^{-r} g_{j,r}\|_{\infty,n}} \\ &\leq 2^{-r} + \sum_{n=r+1}^{\infty} \frac{1}{2^n} = 2^{-r+1} \overset{r\to\infty}{\longrightarrow} 0. \end{aligned}$$

Now, let $(a_n)_{r=0}^{\infty} \in \ell_1$ be non null and define

$$h = a_0 f + \sum_{n=1}^{\infty} a_n g_n = \left(a_0 f_1 + 2^{-1} \sum_{n=1}^{\infty} a_n g_{n,1}, a_0 f_2 + 2^{-2} \sum_{r=1}^{\infty} a_n g_{n,2}, \ldots \right).$$

Following the same lines of the proof of Theorem 1.1, we get $h_r = a_0 f_r + 2^{-r} \sum_{n=1}^{\infty} a_n g_{n,r} \in \mathcal{A}$ for each r. We also notice that, if $r \geq n$, then

$$\|h_r\|_1 = \left\| a_0 f_r + 2^{-r} \sum_{n=1}^{\infty} a_n g_{n,r} \right\|_1 \leq |a_0| \, \|f_r\|_1 + 2^{-r} \sum_{n=1}^{\infty} |a_n| \, \|g_{n,r}\|_1 \leq |a_0| \, \|f_r\|_1 + 2^{-r} \sum_{n=1}^{\infty} |a_n| .$$

Since

$$\|h_r\|_{\infty,n} = \max \{ |a_0 f_r(x)| : x \in [0,n] \} = |a_0| \, \|f_r\|_{\infty,n}$$

whenever $r \geq n$, then it follows that

$$\|h_r\|_{\infty,n} \overset{r\to\infty}{\longrightarrow} 0$$

for each $n \in \mathbb{N}$. Since $d_X(f_r, 0) \to 0$, we conclude that

$$\|f_r\|_1 \overset{r\to\infty}{\longrightarrow} 0 \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\|f_r\|_{\infty,n}}{1 + \|f_r\|_{\infty,n}} \overset{r\to\infty}{\longrightarrow} 0.$$

If $a_0 = 0$, then

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|a_0| \, \|f_r\|_{\infty,n}}{1 + |a_0| \, \|f_r\|_{\infty,n}} = 0.$$

Notice that, if $0 < |a_0| \leq 1$, then

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|a_0| \, \|f_r\|_{\infty,n}}{1 + |a_0| \, \|f_r\|_{\infty,n}} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|a_0| \, \|f_r\|_{\infty,n}}{|a_0| + |a_0| \, \|f_r\|_{\infty,n}} = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\|f_r\|_{\infty,n}}{1 + \|f_r\|_{\infty,n}} \overset{r\to\infty}{\longrightarrow} 0$$

and, if $|a_0| > 1$, then

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|a_0| \, \|f_r\|_{\infty,n}}{1 + |a_0| \, \|f_r\|_{\infty,n}} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|a_0| \, \|f_r\|_{\infty,n}}{1 + \|f_r\|_{\infty,n}} = |a_0| \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\|f_r\|_{\infty,n}}{1 + \|f_r\|_{\infty,n}} \overset{r\to\infty}{\longrightarrow} 0.$$

Therefore,

$$\begin{aligned} d_X(h_r, 0) &= \|h_r\|_1 + \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\|h_r\|_{\infty,n}}{1 + \|h_r\|_{\infty,n}} \\ &= \|h_r\|_1 + \sum_{n=1}^{r} \frac{1}{2^n} \cdot \frac{|a_0| \, \|f_r\|_{\infty,n}}{1 + |a_0| \, \|f_r\|_{\infty,n}} + \sum_{n=r+1}^{\infty} \frac{1}{2^n} \cdot \frac{\|h_r\|_{\infty,n}}{1 + \|h_r\|_{\infty,n}} \\ &\leq |a_0| \, \|f_r\|_1 + 2^{-r} \sum_{n=1}^{\infty} |a_n| + \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|a_0| \, \|f_r\|_{\infty,n}}{1 + |a_0| \, \|f_r\|_{\infty,n}} + \sum_{n=r+1}^{\infty} \frac{1}{2^n} \\ &= |a_0| \, \|f_r\|_1 + 2^{-r} \left(\sum_{n=1}^{\infty} |a_n| + 1 \right) + \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|a_0| \, \|f_r\|_{\infty,n}}{1 + |a_0| \, \|f_r\|_{\infty,n}} \overset{r\to\infty}{\longrightarrow} 0. \end{aligned}$$

Hence $h \in \mathcal{A}_0$. In particular, denoting $g_0 = f$, we have that the operator

$$\begin{aligned} T \colon \ell_1 &\to c_0(X) \\ (a_n)_{n=0}^{\infty} &\mapsto \sum_{n=0}^{\infty} a_n g_n, \end{aligned}$$

is well-defined, linear, injective and satisfies

$$T(\ell_1) \subset \mathcal{A}_0 \cup \{0\}.$$

Since

$$\mathfrak{c} = \dim(\ell_1) = \dim(T(\ell_1))$$

and $f = T(1,0,0,\ldots) \in T(\ell_1)$, at this moment, we shown that $\mathcal{A}_0$ is pointwise $\mathfrak{c}$-lineable. The result will be fully proved if we show that $\overline{T(\ell_1)}^{\left(c_0(X),d_{c_0}(X)\right)} \subset \mathcal{A}_0 \cup \{0\}$.

Let $w = (w_r)_{r=1}^{\infty} \in \overline{T(\ell_1)}^{\left(c_0(X),d_{c_0(X)}\right)} \setminus \{0\}$. Notice that all we have to do to complete this proof is to show that $w_r \in \mathcal{A}$ for each $r \in \mathbb{N}$.

Let $\left(a^{(j)}\right)_{j=1}^{\infty}$ be a sequence in ℓ_1, with $a^{(j)} = \left(a_n^{(j)}\right)_{n=0}^{\infty}$, such that

$$v^{(j)} = \sum_{n=0}^{\infty} a_n^{(j)} g_n \overset{j\to\infty}{\longrightarrow} w$$

in $\left(c_0(X), d_{c_0(X)}\right)$. Hence, denoting

$$v_r^{(j)} = a_0^{(j)} f_r + 2^{-r} \sum_{n=1}^{\infty} a_n^{(j)} g_{n,r},$$

we have

$$d_{c_0(X)}\left(v^{(j)}, w\right) = \sup_{r\in\mathbb{N}} d_X\left(v_r^{(j)}, w_r\right) \overset{j\to\infty}{\longrightarrow} 0.$$

In particular,

$$d_X\left(v_r^{(j)}, w_r\right) \overset{j\to\infty}{\longrightarrow} 0$$

for each $r \in \mathbb{N}$. Therefore, for each $r \in \mathbb{N}$, $v_r^{(j)} \overset{j\to\infty}{\longrightarrow} w_r$ uniformly over each non-degenerate compact interval $[a,b] \subset [0,\infty)$. Proceeding as in Theorem 1.1, we conclude that $w_r \in \mathcal{A}$. ☐

Corollary 1.9. *$\mathcal{A}_0$ is $(1,\mathfrak{c})$-spaceable in $\left(c_0(X), d_{c_0(X)}\right)$.*

Theorem 1.10. *$\mathcal{A}_0$ is not $(\aleph_0,\mathfrak{c})$-spaceable in $\left(c_0(X), d_{c_0(X)}\right)$.*

Proof. Let us take a sequence $f = (f_r)_{r=1}^{\infty} \in \mathcal{A}_0$ and consider the functions $f_{k,r}$ defined as in (1.2) from f_r and a suitable sequence $\left(x_k^{(r)}\right)_{k=1}^{\infty}$ in $[0,\infty)$ as in (1.1). For each pair (n,r) of positive integers, let n_r be the smallest index such that $x_{n_r}^{(r)} \geq rn$. Let $g_{n,r}\colon [0,\infty) \to \mathbb{R}$ defined by

$$g_{n,r}(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq x_{n_r}^{(r)}, \\ f_{n,r}(x), & \text{if } x \geq x_{n_r}^{(r)}. \end{cases}$$

Defining $\phi_n\colon [0,\infty) \to \mathbb{R}$ by

$$\phi_n(x) = \begin{cases} 1-x, & \text{if } 0 \leq x \leq 1, \\ 0 & \text{if } 1 \leq x \leq x_{n_1}^{(1)}, \\ n^{-1} g_{n,1}(x), & \text{if } x \geq x_{n_1}^{(1)}, \end{cases}$$

denote $g_n = (\phi_n, 2^{-2n} g_{n,2}, 2^{-3n} g_{n,3}, \ldots, 2^{-rn} g_{n,r}, \ldots)$.

We need to show that $g_n \in \mathcal{A}_0$, for each n. Since $g_{n,r} \in \mathcal{A}$ for each $n, r \in \mathbb{N}$, we only need to prove that $d_X\left(2^{-rn} g_{n,r}, 0\right) \overset{r\to\infty}{\longrightarrow} 0$. If $r > 1$, then

$$\left\|2^{-rn} g_{n,r}\right\|_1 = 2^{-rn} \int_0^\infty |g_{n,r}| \le 2^{-rn} \overset{r\to\infty}{\longrightarrow} 0.$$

Since $x_{n_r}^{(r)} \ge rn$, we have

$$\left\|2^{-rn} g_{n,r}\right\|_{\infty,m} = \max\left\{\left|2^{-rn} g_{n,r}(x)\right| : x \in [0, m]\right\} = 0,$$

whenever $m \le rn$. Consequently,

$$\left\|2^{-rn} g_{n,r}\right\|_{\infty,m} \overset{r\to\infty}{\longrightarrow} 0$$

for each $m \in \mathbb{N}$. Hence, given $r > 1$,

$$\begin{aligned} d_X\left(2^{-rn} g_{n,r}, 0\right) &= \|2^{-rn} g_{n,r}\|_1 + \sum_{m=1}^{\infty} \frac{1}{2^m} \cdot \frac{\|2^{-rn} g_{n,r}\|_{\infty,m}}{1 + \|2^{-rn} g_{n,r}\|_{\infty,m}} \\ &\le 2^{-rn} + \sum_{m=rn+1}^{\infty} \frac{1}{2^m} \cdot \frac{\|2^{-rn} g_{n,r}\|_{\infty,m}}{1 + \|2^{-rn} g_{n,r}\|_{\infty,m}} \\ &\le 2^{-rn} + \sum_{m=rn+1}^{\infty} \frac{1}{2^m} = 2^{-rn+1} \overset{r\to\infty}{\longrightarrow} 0. \end{aligned} \tag{1.4}$$

Now, let us prove that $\{g_n : n \in \mathbb{N}\}$ is linearly independent. Suppose that

$$\sum_{n=1}^{N} a_n g_n = 0.$$

Since

$$\sum_{n=1}^{N} a_n g_n = \left(\sum_{n=1}^{N} a_n \phi_n, \sum_{n=1}^{N} 2^{-2n} a_n g_{n,2}, \sum_{n=1}^{N} 2^{-3n} a_n g_{n,3}, \sum_{n=1}^{N} 2^{-4n} a_n g_{n,4}, \ldots\right),$$

we have

$$\sum_{n=1}^{N} 2^{-2n} a_n g_{n,2} = 0.$$

Since $\{g_{n,2} : n \in \mathbb{N}\}$ is linearly independent, we obtain $a_n = 0$, for each $n = 1, \ldots, N$. Consider $W := \operatorname{span}\{g_n : n \in \mathbb{N}\}$. Let us prove that $W \setminus \{0\} \subset \mathcal{A}_0$. Let

$$\sum_{n=1}^{N} a_n g_n = \left(\sum_{n=1}^{N} a_n \phi_n, \sum_{n=1}^{N} 2^{-2n} a_n g_{n,2}, \sum_{n=1}^{N} 2^{-3n} a_n g_{n,3}, \sum_{n=1}^{N} 2^{-4n} a_n g_{n,4}, \ldots\right) \in W \setminus \{0\}.$$

Following the same lines of the proof of Theorem 1.1, we get $\sum_{n=1}^{N} a_n \phi_n \in \mathcal{A}$ and $\sum_{n=1}^{N} 2^{-rn} a_n g_{n,r} \in \mathcal{A}$ for each $r > 1$. We also notice that

$$\left\|\sum_{n=1}^{N} 2^{-rn} a_n g_{n,r}\right\|_1 \le \sum_{n=1}^{N} \left|2^{-rn} a_n\right| \|g_{n,r}\|_1 \le \sum_{n=1}^{N} \left|2^{-rn} a_n\right| \le 2^{-r} \sum_{n=1}^{N} |a_n|.$$

It is obvious that, given $m \in \mathbb{N}$,

$$\left\| \sum_{n=1}^{N} 2^{-rn} a_n g_{n,r} \right\|_{\infty,m} = \max \left\{ \left| \sum_{n=1}^{N} 2^{-rn} a_n g_{n,r}(x) \right| : x \in [0,m] \right\} = 0$$

whenever $r > m$. Consequently,

$$\left\| \sum_{n=1}^{N} 2^{-rn} a_n g_{n,r} \right\|_{\infty,m} \overset{r\to\infty}{\longrightarrow} 0$$

for each $m \in \mathbb{N}$. Therefore,

$$\begin{aligned} d_X\left(\sum_{n=1}^{N} 2^{-rn} a_n g_{n,r}, 0 \right) &= \left\| \sum_{n=1}^{N} 2^{-rn} a_n g_{n,r} \right\|_1 + \sum_{m=1}^{\infty} \frac{1}{2^m} \cdot \frac{\left\| \sum_{n=1}^{N} 2^{-rn} a_n g_{n,r} \right\|_{\infty,m}}{1 + \left\| \sum_{n=1}^{N} 2^{-rn} a_n g_{n,r} \right\|_{\infty,m}} \\ &= \left\| \sum_{n=1}^{N} 2^{-rn} a_n g_{n,r} \right\|_1 + \sum_{m=r+1}^{\infty} \frac{1}{2^m} \cdot \frac{\left\| \sum_{n=1}^{N} 2^{-rn} a_n g_{n,r} \right\|_{\infty,m}}{1 + \left\| \sum_{n=1}^{N} 2^{-rn} a_n g_{n,r} \right\|_{\infty,m}} \\ &\leq 2^{-r} \sum_{n=1}^{N} |a_n| + \sum_{m=r+1}^{\infty} \frac{1}{2^m} \overset{r\to\infty}{\longrightarrow} 0. \end{aligned}$$

Hence, $\sum_{n=1}^{N} a_n g_n \in \mathcal{A}_0$ and

$$W \setminus \{0\} \subset \mathcal{A}_0.$$

Let us define $v := (v_r)_{r=1}^{\infty}$, where

$$v_1(x) := \begin{cases} 1 - x, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{if } x \geq 1. \end{cases}$$

and $v_r \equiv 0$ for each $r \geq 2$. We have:

(*i*) $d_X(v_r, 0) \overset{r\to\infty}{\longrightarrow} 0$;

(*ii*) $v_r \in X$ for each $r \in \mathbb{N}$;

(*iii*) $v_r \notin \mathcal{A}$ for each $r \in \mathbb{N}$.

It follows from (*i*), (*ii*) and (*iii*) that $v \in c_0(X) \setminus \mathcal{A}_0$.

Since, for each $n \in \mathbb{N}$,

$$\begin{aligned} d_X(\phi_n, v_1) &= \|\phi_n - v_1\|_1 + \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{\|\phi_n - v_1\|_{\infty,m}}{1 + \|\phi_n - v_1\|_{\infty,m}} \\ &= n^{-1} \int_{x_{n_1}^{(1)}}^{\infty} |g_{n,1}| + \sum_{m=n+1}^{\infty} \frac{1}{2^m} \frac{\|g_{n,1}\|_{\infty,m}}{1 + \|g_{n,1}\|_{\infty,m}} \end{aligned}$$

$$\begin{aligned}&\leq n^{-1}+\sum_{m=n+1}^{\infty}2^{-m}\\&=n^{-1}+2^{-n}\\&\leq 2n^{-1}\end{aligned}$$

it follows from (2.1) that

$$\begin{aligned}d_{c_0(X)}\left(g_n,v\right)&=\sup\left\{d_X\left(\phi_n,v_1\right),d_X\left(2^{-2n}g_{n,2},v_2\right),d_X\left(2^{-3n}g_{n,3},v_3\right),\ldots\right\}\\&\leq\sup\left\{2n^{-1},d_X\left(2^{-2n}g_{n,2},v_2\right),d_X\left(2^{-3n}g_{n,3},v_3\right),\ldots\right\}\\&=\sup\left\{2n^{-1},d_X\left(2^{-2n}g_{n,2},0\right),d_X\left(2^{-3n}g_{n,3},0\right),\ldots\right\}\\&\leq\sup\left\{2n^{-1},2^{-2n+1},2^{-3n+1},\ldots\right\}=2n^{-1}\overset{n\to\infty}{\longrightarrow}0.\end{aligned}$$

This means that $v\in\overline{W}$. Since $v\notin\mathcal{A}_0\cup\{0\}$, it follows that $\overline{W}\setminus\{0\}\not\subset\mathcal{A}_0$ and, hence, $\mathcal{A}_0$ is not $(\aleph_0,\mathfrak{c})$-spaceable in $\left(c_0\left(X\right),d_{c_0(X)}\right)$. □

For $p>0$, consider the sequence space

$$\ell_p\left(X\right):=\{(f_r)_{r=1}^{\infty}:f_r\in X\text{ for each }r\text{ and }(d_X(f_r,0))_{r=1}^{\infty}\in\ell_p\}$$

endowed with the following distance:

$$d_{\ell_p(X)}\left((f_r)_{r=1}^{\infty},(g_r)_{r=1}^{\infty}\right)=\begin{cases}\sum\limits_{r=1}^{\infty}\left[d_X\left(f_r,g_r\right)\right]^p, & \text{if }0<p<1\\ \left(\sum\limits_{r=1}^{\infty}\left[d_X\left(f_r,g_r\right)\right]^p\right)^{1/p}, & \text{if }1\leq p<\infty.\end{cases}$$

Observe that $\left(\ell_p\left(X\right),d_{\ell_p(X)}\right)$ is a metric vector space. Let $\mathcal{A}_p\subset\ell_p\left(X\right)$ be the set defined by

$$\mathcal{A}_p:=\{(f_r)_{r=1}^{\infty}:f_r\in\mathcal{A}\text{ for each }r\text{ and }(d_X(f_r,0))_{r=1}^{\infty}\in\ell_p\}.$$

Let us prove that $\mathcal{A}_p$ is non-void. In fact, for each $r\in\mathbb{N}$, let $f_r\colon[0,\infty)\to\mathbb{R}$ given by

$$f_r\left(x\right)=\begin{cases}n^2 2^n\left(x-n\right), & \text{if }n\leq x\leq n+\dfrac{1}{n2^n}\text{ for some integer }n\geq r,\\ -n^2 2^n\left(x-n-\dfrac{1}{n2^n}\right), & \text{if }n+\dfrac{1}{n2^n}\leq x\leq n+\dfrac{1}{n2^{n-1}}\text{ for some integer }n\geq r,\\ 0, & \text{if }x\notin\bigcup\limits_{n=r}^{\infty}\left[n,n+\dfrac{1}{n2^{n-1}}\right].\end{cases}$$

Obviously, each f_r is continuous and it is simple to check that $\left\|f_r\right\|_1=2^{-r+1}$ and $f_r\left(n+\dfrac{1}{n2^n}\right)=n$ if $n\geq r$. Hence $f_r\in\mathcal{A}$ for each $r\in\mathbb{N}$. Fixing $p>0$, let $(a_r)_{r=1}^{\infty}\in\ell_p$ such that $a_r\neq 0$ for each $r\in\mathbb{N}$. If we take

$$(g_r)_{r=1}^{\infty}=\left(2^{r-1}a_r f_r\right)_{r=1}^{\infty},$$

then

$$d_X(g_r, 0) = \|g_r\|_1 + \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|g_r\|_{\infty,n}}{1 + \|g_r\|_{\infty,n}} = |a_r| + \sum_{n=r+1}^{\infty} \frac{1}{2^n} \frac{\|g_r\|_{\infty,n}}{1 + \|g_r\|_{\infty,n}}.$$

Note that

$$\left(\sum_{n=r+1}^{\infty} \frac{1}{2^n} \frac{\|g_r\|_{\infty,n}}{1 + \|g_r\|_{\infty,n}} \right)^p \leq \left(\sum_{n=r+1}^{\infty} \frac{1}{2^n} \right)^p = \frac{1}{2^{rp}}.$$

Hence,

$$\sum_{r=1}^{\infty} \left(\sum_{n=r+1}^{\infty} \frac{1}{2^n} \frac{\|g_r\|_{\infty,n}}{1 + \|g_r\|_{\infty,n}} \right)^p \leq \sum_{r=1}^{\infty} \frac{1}{2^{rp}} = \frac{1}{2^p - 1}$$

and so

$$\left(\sum_{n=r+1}^{\infty} \frac{1}{2^n} \frac{\|g_r\|_{\infty,n}}{1 + \|g_r\|_{\infty,n}} \right)_{r=1}^{\infty} \in \ell_p.$$

Consequently,

$$(d_X(g_r, 0))_{r=1}^{\infty} = (|a_r|)_{r=1}^{\infty} + \left(\sum_{n=r+1}^{\infty} \frac{1}{2^n} \frac{\|g_r\|_{\infty,n}}{1 + \|g_r\|_{\infty,n}} \right)_{r=1}^{\infty} \in \ell_p$$

and then $(g_r)_{r=1}^{\infty} \in \mathcal{A}_p$.

The proofs of Theorems 1.8 and 1.10 can be adapted, *mutatis mutandis*, to prove the following:

Theorem 1.11. *For each $p > 0$, the set $\mathcal{A}_p$*

(a) *is pointwise $\mathfrak{c}$-spaceable in $\left(\ell_p(X), d_{\ell_p(X)}\right)$, for each $p > 0$;*

(b) *is $(1, \mathfrak{c})$-spaceable in $\left(\ell_p(X), d_{\ell_p(X)}\right)$;*

(c) *is not $(\aleph_0, \mathfrak{c})$-spaceable in $\left(\ell_p(X), d_{\ell_p(X)}\right)$.*

Chapter 2

Criteria for Stronger Notions of Lineability

This chapter explores (α, β)-lineability/spaceability, enhancing classical lineability and spaceability by adding constraints to subspaces. We identify conditions where sets are not (α, β)-spaceable and develop methods to prove (α, β)-spaceability in various contexts. Additionally, we introduce (α, β)-dense lineability, examining the relationships between subspaces, particularly when one is infinite-dimensional and dense. The refined framework aids in understanding the properties of function spaces and complex sets, opening new research opportunities and applications in mathematical analysis.

2.1 (α, β)-Spaceability: Negative Results

We start off by recalling that (see [33, Definition 2.1]) if A, B are subsets of a vector space X, then A is called stronger than B whenever $A + B \subset A$.

Before presenting the main result, we introduce the concept of characteristic density. The **characteristic density** $\mathrm{dens}\,(X)$ of a topological space X is the smallest cardinality of a dense subset of X. Formally, $\mathrm{dens}\,(X)$ is given by:

$$\mathrm{dens}\,(X) = \min\{\mathrm{card}\,(D) : D \subseteq X \text{ and } D \text{ is dense in } X\}.$$

The ideas of this stage are part of paper:

[23] V.V. Fávaro, D. Pellegrino, A. Raposo Jr, and G. Ribeiro, *General criteria for a stronger notion of lineability.* Proc. Amer. Math. Soc. 152 (2024), 941-954

The main result of this section is a criterion for non (α, β)-spaceability, which is independent of the characteristic density of the space, as we shall see next.

Theorem 2.1. *Let $\alpha \geq \aleph_0$ and X be an F-space. Let A, B be subsets of X such that A is α-lineable and B is 1-lineable. If $A \cap B = \varnothing$ and A is stronger than B, then A is not (α, β)-spaceable, regardless of the cardinal number β.*

Proof. Consider a complete translation-invariant metric d on X whose topology is the one of X. Since B is 1-lineable, let $v \in X \setminus \{0\}$ be such that

$$\mathbb{K}v \setminus \{0\} \subset B.$$

Since A is α-lineable, let Γ be a set with cardinality α and $\{v_a : a \in \Gamma\} \subset X$ be a set of linearly independent vectors such that

$$E \setminus \{0\} \subset A,$$

where $E = \text{span}\,\{v_a : a \in \Gamma\}$. Let $\{a_m : m \in \mathbb{N}\} \subset \Gamma$ be an infinite countable set. Given $n \in \mathbb{N}$, the continuity of scalar multiplication on $s \mapsto sv_{a_n}$ yields that there exists $\delta_n > 0$ such that

$$d\left(sv_{a_n}, 0v_{a_n}\right) < n^{-1}, \text{ for all } s \in (0, \delta_n).$$

In particular, for $s = 2^{-1}\delta_n$, we have

$$d\left(2^{-1}\delta_n v_{a_n}, 0\right) < n^{-1}.$$

Given $b \in \Gamma$, define $u_b := t_b v_b + v$, where

$$t_b = \begin{cases} 2^{-1}\delta_n, & \text{if } b = a_n \in \{a_m\}_{m=1}^{\infty} \text{ for some } n, \\ 1, & \text{otherwise.} \end{cases}$$

Since $A + B \subset A$, it follows that $u_b \in A$, for each $b \in \Gamma$. Since $A \cap B = \varnothing$, it is clear that $\{u_b : b \in \Gamma\}$ is linearly independent. Defining

$$M := \text{span}\,\{u_b : b \in \Gamma\},$$

let us prove that $M \setminus \{0\} \subset A$. For $f \in M \setminus \{0\}$, there are $N \in \mathbb{N}$ and $(c_1, \ldots, c_N) \neq (0, \ldots, 0)$ in $\mathbb{K}^N$ such that

$$f = \sum_{j=1}^{N} c_j t_{b_j} v_{_{bj}} + \sum_{j=1}^{N} c_j v.$$

Since $\{v_b : b \in \Gamma\}$ is linearly independent and $(c_1, \ldots, c_N) \neq (0, \ldots, 0)$, we have

$$\sum_{j=1}^{N} c_j t_{b_j} v_{_{bj}} \in E \setminus \{0\} \subset A,$$

and since

$$\sum_{j=1}^{N} c_j v \in \mathbb{K}v \subset B \cup \{0\},$$

we get

$$f \in A + (B \cup \{0\}) \subset A.$$

Hence $M \setminus \{0\} \subset A$. Given $\varepsilon > 0$, let $n \in \mathbb{N}$ be such that $n^{-1} < \varepsilon$. Since d is a translation-invariant metric, we conclude that

$$d\left(u_{a_n}, v\right) = d\left(2^{-1}\delta_n v_{a_n}, 0\right) < n^{-1} < \varepsilon.$$

This implies that $v \in \overline{M}$. Since $v \notin A$, it follows that

$$\overline{M} \not\subset A \cup \{0\}$$

and this means that A is not (α, β)-spaceable, regardless of the $\beta \geq \alpha$. $\square$

From [24] we know that the set $\mathcal{ND}[0,1]$ of continuous nowhere differentiable functions $f\colon [0,1] \to \mathbb{R}$ is $\mathfrak{c}$-spaceable in $C[0,1]$. The following consequence of the previous result shows that it is not $(\alpha, \mathfrak{c})$-spaceable regardless of the $\alpha \geq \aleph_0$:

Corollary 2.2. *Let $\alpha \geq \aleph_0$ and β be a cardinal number. The set $\mathcal{ND}[0,1]$ is not (α, β)-spaceable.*

Proof. Consider $A = \mathcal{ND}[0,1]$ and $B = \{f \in C[0,1] : f \text{ is differentiable}\}$. Note that

$$A \cap B = \varnothing \text{ and } A + B \subset A.$$

If $\aleph_0 \leq \alpha \leq \mathfrak{c}$, since A is α-lineable, the result follows from Theorem 1.1. If $\alpha > \mathfrak{c}$ the result is immediate. □

It is well known (see [13]) that $L_p[0,1] \setminus \bigcup_{q\in(p,\infty)} L_q[0,1]$, for $p > 0$, is $\mathfrak{c}$-spaceable in $L_p[0,1]$. Considering $X = L_p[0,1]$ and $Y = \bigcup_{q\in(p,\infty)} L_q[0,1]$ in Corollary 2.5, since $X \setminus Y$ is α-lineable for all $\aleph_0 \leq \alpha \leq c$ and $\dim(X) = \mathfrak{c}$, we have:

Corollary 2.3. *Let $\alpha \geq \aleph_0$ and β be a cardinal number. For $p > 0$, the set $L_p[0,1] \setminus \bigcup_{q\in(p,\infty)} L_q[0,1]$ is not (α, β)-spaceable.*

This result complements Theorem 2.16 from the previous chapter. In Theorem 2.16, we established the non $(\aleph_0, \mathfrak{c})$-spaceability of this same set. It is important to note that the techniques used in both results are different, highlighting the diversity of approaches that can be employed to study spaceability properties in different contexts.

Remark 2.4. (*a*) The *Corollary* 2.2 is just to illustrate the applicability of *Theorem* 1.1. Notice that if X is a topological vector space and A is a proper subset of X which is α-dense lineable and $\alpha = \operatorname{dens}(X)$ then A cannot be (α, β)-spaceable for any $\beta \geq \alpha$.

(*b*) Observing the proof of *Theorem* 1.1, we are led to believe that a (natural) variant of the definition of (α, β)-spaceability, demanding W_α to be closed, would provide a completely different kind of results. We think that this is worth of further investigation in the future.

Recall that a subspace F of a Banach space X is *quasicomplemented* in X if it is closed, and there exists a closed linear subspace G of X such that $F \cap G = \{0\}$ and $F + G$ is dense in X (see [35, 39, 42]). In [39], Lindenstrauss asks whether or not c_0 is quasicomplemented in ℓ_∞, and, in [42, Theorem 1.7], Rosenthal shows that this is so. Furthermore, every separable subspace of ℓ_∞ is quasicomplemented in ℓ_∞. In particular, if $F = c_0$ or c, the set $\ell_\infty \setminus F$ is $\mathfrak{c}$-spaceable. However, in [41, Proposition 1.5], Papathanasiou shows that the dimension of every dense linear subspace of ℓ_∞ is $\mathfrak{c}$. This ensures that $\ell_\infty \setminus F$ is not α-dense lineable whenever $\aleph_0 \leq \alpha < \mathfrak{c}$.

The above comment ensures that the our next corollary cannot follow from Remark 2.4. Specifically, the next corollary is independent of the characteristic density of the space.

Corollary 2.5. *Let $\alpha \geq \aleph_0$ and β be a cardinal number. Let X be a Banach space or p-Banach space ($p > 0$) and Y be a non-trivial subspace of X. If $X \setminus Y$ is α-lineable then $X \setminus Y$ is not (α, β)-spaceable.*

Proof. Considering $A = X \setminus Y$ and $B = Y$, we have $A + B \subset A$ and $A \cap B = \varnothing$ and the result follows by Theorem 1.1. □

As an immediate consequence of Corollary 2.5 we have the following result:

Corollary 2.6. *If $F = c_0$ or $F = c$, then $\ell_\infty \setminus F$ is not (α, β)-spaceable if $\alpha \geq \aleph_0$.*

2.2 (α, β)-Spaceability: Positive Results

A classical result due to Wilansky and Kalton says that if F is a closed subspace of a Fréchet space X, then $X \setminus F$ is spaceable if, and only if, $\dim(X/F) = \infty$. In fact, Wilansky [44, p.12] proved this result for Banach spaces and Kalton noticed that the same proof works for Fréchet spaces. This result appears in [37] in the following way:

Theorem 2.7. ([37, Theorem 2.2]) *If F is a closed subspace of a Fréchet space X, then $X \setminus F$ is spaceable if, and only if, F has infinite codimension.*

The main result of this section is a criterion for (α, β)-spaceability. In some sense, we complement the result above by showing that, if F is a closed subspace of a Banach space X such that F has a regular basic sequence and $X \setminus F$ is $\aleph_0$-lineable, then

$$X \setminus F \text{ is } (\alpha, \mathfrak{c})\text{-spaceable if, and only if, } \alpha < \aleph_0.$$

In particular, this assures that $\ell_\infty \setminus c_0$ and $\ell_\infty \setminus c$ are $(\alpha, \mathfrak{c})$-spaceable if, and only if, $\alpha < \aleph_0$.

The ideas of this section are also part of paper:

[23] V.V. Fávaro, D. Pellegrino, A. Raposo Jr, and G. Ribeiro, *General criteria for a stronger notion of lineability.* Proc. Amer. Math. Soc. 152 (2024), 941-954

We start off recalling some results concerning Schauder basis and basic sequences.

Definition 2.8. A sequence $(e_n)_{n=1}^\infty$ in a Banach space X is a basic sequence if it is a Schauder basis for $\overline{\text{span}\{e_n : n \in \mathbb{N}\}}$. Furthermore, as in [36] and [43], a basic sequence is called regular if it is bounded away from zero, that is, if it lies entirely outside some neighbourhood of zero.

The following lemmas are probably folklore, but we present their proofs for the sake of completeness.

Lemma 2.9. *Let X be an infinite dimensional Banach space and let W be a finite-dimensional subspace of X. Then, given $\varepsilon \in (0, 1)$, there exists $v \in X$ such that $\|v\| = 1$ and*

$$\|w + \lambda v\| \geq (1 - \varepsilon)\|w\|$$

for all $(\lambda, w) \in \mathbb{K} \times W$.

Proof. Let $\mathbb{S}_W$ be the unit sphere of W. Since $\mathbb{S}_W$ is compact, we can find a finite set $\{w_1,\ldots,w_r\}$ in $\mathbb{S}_W$ such that $\{B_\varepsilon(w_k) : k=1,\ldots,r\}$, where $B_\varepsilon(w_k)$ is the open ball of center w_k and radius ε, covers $\mathbb{S}_W$. Let X^* be the topological dual of V and let us consider $w_1^*,\ldots,w_r^* \in V^*$ such that, for every $k=1,\ldots,r$, $\|w_k^*\|=1$ and $w_k^*(w_k)=1$. Since X is infinite dimensional, $\bigcap_{k=1}^{r} \ker w_k^*$ is non-trivial subspace of X.

Let us fix a unit vector $v \in \bigcap_{k=1}^{r} \ker w_k^*$. Given $w \in W \setminus \{0\}$, there is $k \in \{1,\ldots,r\}$ such that

$$\left\| \frac{w}{\|w\|} - w_k \right\| < \varepsilon.$$

For any $\lambda \in \mathbb{K}$, we have

$$\begin{aligned}
\frac{\|w+\lambda v\|}{\|w\|} &\geq \left\| w_k + \frac{\lambda v}{\|w\|} \right\| - \left\| \frac{w}{\|w\|} - w_k \right\| \\
&> \left\| w_k + \frac{\lambda v}{\|w\|} \right\| - \varepsilon \\
&\geq \left| w_k^* \left(w_k + \frac{\lambda v}{\|w\|} \right) \right| - \varepsilon \\
&= |w_k^*(w_k)| - \varepsilon \\
&= 1 - \varepsilon.
\end{aligned}$$

Hence

$$\|w+\lambda v\| \geq (1-\varepsilon)\|w\|,$$

for all $w \in W \setminus \{0\}$ and all scalars λ. Since the case $w=0$ is immediate, the proof is done. □

Lemma 2.10. *Let X be an infinite dimensional Banach space. If $v_1,\ldots,v_n \in X$ are linearly independent with $\|v_i\|=1$, for each $i=1,\ldots,n$, then X contains a basic sequence $(u_k)_{k=1}^{\infty}$ where $u_k = v_k$ for each $k=1,\ldots,n$.*

Proof. Let $W_1 = \operatorname{span}\{v_1,\ldots,v_n\}$. Considering the sequence $(\varepsilon_k)_{k=1}^{\infty}$ defined by

$$\varepsilon_k = \left(\frac{10^k}{9} + \sum_{j=0}^{k-1} 10^j \right)^{-1},$$

observe that

$$\left(\prod_{k=1}^{m} (1-\varepsilon_k) \right)^{-1} \leq \prod_{k=1}^{\infty} (1-\varepsilon_k)^{-1} = 2$$

for all m.

Define $u_k = v_k$ for each $k=1,\ldots,n$. By Lemma 2.9 there is $u_{n+1} \in X$ such that $\|u_{n+1}\|=1$ and

$$\|w+\lambda u_{n+1}\| \geq (1-\varepsilon_1)\|w\|$$

for all $w \in W_1$ and $\lambda \in \mathbb{K}$, that is

$$\left\| \sum_{k=1}^{n} \lambda_k u_k + \lambda u_{n+1} \right\| \geq (1-\varepsilon_1) \left\| \sum_{k=1}^{n} \lambda_k u_k \right\|$$

whenever $\lambda, \lambda_1, \ldots, \lambda_n \in \mathbb{K}$. Defining $W_2 = \operatorname{span}\{u_1, \ldots, u_n, u_{n+1}\}$, it follows from Lemma 2.9 that there is $u_{n+2} \in V$ such that $\|u_{n+2}\| = 1$ and

$$\|w + \lambda u_{n+2}\| \geq (1 - \varepsilon_2)\|w\|$$

for all $w \in W_2$ and $\lambda \in \mathbb{K}$, that is,

$$\left\|\sum_{k=1}^{n+2} \lambda_k u_k\right\| \geq (1 - \varepsilon_2)\left\|\sum_{k=1}^{n+1} \lambda_k u_k\right\|$$

whenever $\lambda_1, \ldots, \lambda_{n+2} \in \mathbb{K}$. Repeating this process, we obtain a sequence $(u_k)_{k=1}^{\infty}$ in V with $\|u_k\| = 1$, for each $k \in \mathbb{N}$, such that for each $N \geq 1$ and any scalars $(\lambda_k)_{k=1}^{n+N}$,

$$\left\|\sum_{k=1}^{n+N} \lambda_k u_k\right\| \geq (1 - \varepsilon_N)\left\|\sum_{k=1}^{n+N-1} \lambda_k u_k\right\|.$$

Therefore,

$$\begin{aligned}\left\|\sum_{k=1}^{n+N-1} \lambda_k u_k\right\| &\leq \prod_{k=1}^{m+1} (1 - \varepsilon_{N+k-1})^{-1}\left\|\sum_{k=1}^{n+N+m} \lambda_k u_k\right\| \\ &\leq 2\left\|\sum_{k=1}^{n+N+m} \lambda_k u_k\right\|\end{aligned}$$

for each $m \in \mathbb{N}$. In particular,

$$\left\|\sum_{k=1}^{s} \lambda_k u_k\right\| \leq 2\left\|\sum_{k=1}^{t} \lambda_k u_k\right\| \tag{2.1}$$

for each $t \geq s \geq n$.

Now let us suppose $s \leq r \leq n$. Since the correspondence

$$\sum_{k=1}^{n} \alpha_k u_k \mapsto \sum_{k=1}^{n} |\alpha_k|$$

defines a norm on W_1 and two norms are always equivalent in finite dimensional spaces, there are positive constants L and M, such that

$$\left\|\sum_{k=1}^{n} \alpha_k u_k\right\| \leq L\sum_{k=1}^{n} |\alpha_k| \leq M\left\|\sum_{k=1}^{n} \alpha_k u_k\right\|.$$

Hence,

$$\left\|\sum_{k=1}^{s} \lambda_k u_k\right\| \leq L\sum_{k=1}^{s} |\lambda_k| \leq L\sum_{k=1}^{r} |\lambda_k| \leq M\left\|\sum_{k=1}^{r} \lambda_k u_k\right\|. \tag{2.2}$$

Combining the previous inequality with (2.1) we have

$$\left\|\sum_{k=1}^{s} \lambda_k u_k\right\| \leq 2M\left\|\sum_{k=1}^{t} \lambda_k u_k\right\|, \tag{2.3}$$

for each $t \geq n$.

Finally, by (2.1), (2.2) and (2.3), we conclude that, in general, if $r, s \in \mathbb{N}$ are such that $s \leq r$, we have

$$\left\|\sum_{k=1}^{s} \lambda_k u_k\right\| \leq C\left\|\sum_{k=1}^{r} \lambda_k u_k\right\|$$

for a certain constant C. This shows that $(u_k)_{k=1}^{\infty}$ is a basic sequence. □

Lemma 2.11. ([36, Lemma 4.3]) *Let X be a Banach space and $(w_k)_{k=1}^{\infty}$ be a regular basic sequence. Let $(u_k)_{n=1}^{\infty}$ be a sequence in X such that $\sum_{n=1}^{\infty} \|u_k\| < \infty$. If*

$$\sum_{n=1}^{\infty} a_k (w_k + u_k) = 0 \Rightarrow a_k = 0,$$

then $(w_k + u_k)_{n=1}^{\infty}$ is a basic sequence.

Now we are able to state the main result of this section.

Theorem 2.12. *Let X be an infinite dimensional Banach space and F be a closed subspace of X. If F has infinite codimension, then $X \setminus F$ is $(n, \mathfrak{c})$-spaceable, for every $n \in \mathbb{N}$.*

Proof. Let Z be an n-dimensional subspace of X such that

$$Z \setminus \{0\} \subset X \setminus F.$$

Let $\{v_1, \ldots, v_n\}$ be a basis of Z. Since F is closed, the quotient space X/F is Banach when endowed with the norm $\|\cdot\|_{X/F}$ given by $\|\overline{v}\|_{X/F} = \inf \{\|v - w\| : w \in F\}$, where $\overline{v}$ denotes the equivalence class of $v \in X$. Let $(y_k)_{k=1}^{\infty}$ be a regular basic sequence in F. For each $i = 1, \ldots, n$, let

$$x_i = \frac{(v_i - y_i)}{\|\overline{v}_i\|_{X/F}}. \tag{2.4}$$

Thus $\overline{x}_1, \ldots, \overline{x}_n$ are linearly independent. In fact, if

$$\sum_{i=1}^{n} \lambda_i \overline{x}_i = \overline{0},$$

then

$$\overline{0} = \sum_{i=1}^{n} \frac{\lambda_i}{\|\overline{v}_i\|_{X/F}} \overline{v}_i = \overline{\sum_{i=1}^{n} \frac{\lambda_i}{\|\overline{v}_i\|_{X/F}} v_i}.$$

This implies that

$$\sum_{i=1}^{n} \frac{\lambda_i}{\|\overline{v}_i\|_{X/F}} v_i \in W \cap Z = \{0\}.$$

Since $\{v_1, \ldots, v_n\}$ is a basis of Z, we conclude that

$$\lambda_i = 0, \text{ for each } i = 1, \ldots, n.$$

By Lemma 2.10, we obtain a normalized basic sequence $(\overline{x}_k)_{k=1}^{\infty}$ on X/F, with $x_1, \ldots, x_n$ as in (2.4). Since $\|\overline{x}_k\|_{X/F} := \inf \{\|x_k - w\| : w \in F\}$, for every $k > n$, there is $w_k \in F$ such that

$$\|x_k - w_k\| \leq \|\overline{x}_k\|_{X/F} + 2^{-k} = 1 + 2^{-k}.$$

Let

$$u_k := \begin{cases} \|\overline{v}_k\|_{X/F} \, x_k, & \text{if } k \leq n, \\ 2^{-k} (x_k - w_k), & \text{if } k > n, \end{cases}$$

and let $(a_k)_{k=1}^{\infty}$ be a sequence in $\mathbb{K}$ such that

$$\sum_{k=1}^{\infty} a_k (y_k + u_k) = 0. \tag{2.5}$$

Then, in particular,

$$\lim_{k\to\infty} \|a_k\| \, \|y_k + u_k\| = 0. \tag{2.6}$$

Since $(y_k)_{k=1}^{\infty}$ is a regular basic sequence, there is $L > 0$ such that $\|y_k\| \geq L$ for each $k \in \mathbb{N}$. Thus, if $k > n$, then

$$\|y_k + u_k\| \geq \|y_k\| - \|u_k\| \geq L - 2^{-k}\left(1 + 2^{-k}\right) \overset{k\to\infty}{\longrightarrow} L > 0. \tag{2.7}$$

From (2.6) and (2.7) we conclude that

$$\lim_{k\to\infty} a_k = 0. \tag{2.8}$$

The inequality

$$\sum_{k=n+1}^{\infty} \|u_k\| = \sum_{k=n+1}^{\infty} \left\|2^{-k}(x_k - w_k)\right\| \leq \sum_{k=n+1}^{\infty} 2^{-k}\left(1 + 2^{-k}\right) < \infty, \tag{2.9}$$

combined with (2.8), allow us to conclude that $\sum_{k=1}^{\infty} a_k u_k$ converges absolutely and, hence, converges. Therefore, by (2.5), we have

$$\sum_{k=1}^{\infty} a_k y_k = -\sum_{k=1}^{\infty} a_k u_k.$$

Consequently, $\sum_{k=1}^{\infty} a_k u_k \in F$. Thus,

$$\overline{0} = \sum_{k=1}^{\infty} a_k \overline{u}_k = \sum_{k=1}^{n} a_k \overline{u}_k + \sum_{k=n+1}^{\infty} a_k \overline{u}_k = \sum_{k=1}^{n} a_k \|\overline{v}_k\|_{X/F}\, \overline{x}_k + \sum_{k=n+1}^{\infty} a_k 2^{-k} \overline{x}_k$$

and, since $\{\overline{x}_k : k \in \mathbb{N}\}$ is a basic sequence in X/F, it follows that

$$a_k = 0 \tag{2.10}$$

for each $k \in \mathbb{N}$. By (2.9) and (2.10) we can invoke Lemma 2.11 to conclude that the sequence $(y_k + u_k)_{k=1}^{\infty}$ is a basic sequence in X. Defining the (norm) closure of span $\{y_k + u_k : k \in \mathbb{N}\}$ by E, let us to prove that $E \setminus \{0\} \subset X \setminus F$. If $v \in E \cap F$, then there are scalars c_k such that

$$v = \sum_{k=1}^{\infty} c_k (y_k + u_k)$$

and

$$\overline{0} = \overline{v} = \sum_{k=1}^{\infty} \overline{c_k (y_k + u_k)} = \sum_{k=1}^{\infty} c_k \overline{u}_k = \sum_{k=1}^{\infty} d_k \overline{x}_k,$$

where

$$d_k = \begin{cases} c_k \|\overline{v}_k\|_{X/F}, & \text{if } k \leq n, \\ c_k 2^{-k}, & \text{if } k > n. \end{cases}$$

Since $(\overline{x}_k)_{k=1}^{\infty}$ is a basic sequence in X/F, it follows that $d_k = c_k = 0$ for all $k \in \mathbb{N}$ and, hence $v = 0$. Thus $E \cap F = \{0\}$, that is,

$$E \setminus \{0\} \subset X \setminus F.$$

Since

$$y_k + u_k = y_k + \|\overline{v}_k\|_{X/F}\, x_k = y_k + \|\overline{v}_k\|_{X/F} \frac{v_k - y_k}{\|\overline{v}_k\|_{X/F}} = v_k,$$

for all $k = 1, \ldots, n$, we have $Z \subset F$ and the result is done, since $\dim(F) = \mathfrak{c}$. □

The next result is a consequence of Corollary 2.5 and Theorem 2.12.

Corollary 2.13. *If F is a closed subspace of a Banach space X and* $\dim(X/F) \geq \aleph_0$*, then*

$$X \setminus F \text{ is } (\alpha, \mathfrak{c})\text{-spaceable if, and only if, } \alpha < \aleph_0.$$

Proof. Theorem 2.12 assures that $X \setminus F$ is $(\alpha, \mathfrak{c})$-spaceable, for every $\alpha < \aleph_0$. Conversely, Corollary 2.5 guarantees that, if $\aleph_0 \leq \alpha \leq \mathfrak{c}$, then $X \setminus F$ is not $(\alpha, \mathfrak{c})$-spaceable. □

Corollary 2.14. *Let $F = c$ or c_0. Then $\ell_\infty \setminus F$ is $(\alpha, \mathfrak{c})$-spaceable if, and only if, $\alpha < \aleph_0$.*

Proof. The subspaces c_0 and c are both closed and have infinite codimension in ℓ_∞. Hence, if we take $X = \ell_\infty$ and $F = c_0$ or c in Corollary 2.13, we conclude that both $\ell_\infty \setminus c$ and $\ell_\infty \setminus c_0$ are $(\alpha, \mathfrak{c})$-spaceable if, and only if, $\alpha < \aleph_0$. □

Alternatively, Corollary 2.14 can be obtained replacing Corollary 2.5 by Corollary 2.6 in the proof of Corollary 2.13.

The following result is an immediate consequence of Theorem 2.12 and the fact that, for $X \setminus F$, pointwise $\mathfrak{c}$-spaceability coincides with $(1, \mathfrak{c})$-spaceability.

Corollary 2.15. *Let X be an infinite dimensional Banach space and F be a closed vector subspace of X. If* $\dim(X/F) \geq \aleph_0$*, then $X \setminus F$ is pointwise $\mathfrak{c}$-spaceable.*

2.3 $(\alpha, \mathfrak{c})$-Spaceability of Set $L_p[0,1] \setminus \bigcup_{q\in(p,\infty)} L_q[0,1]$ for $p \in (0,\infty)$

The aforementioned results, while satisfactory with respect to the complement of closed subspaces in Banach spaces, do not guarantee, for instance, that the set $L_p[0,1] \setminus \bigcup_{q\in(p,\infty)} L_q[0,1]$ with $p \in (0,\infty)$, is $(\alpha, \mathfrak{c})$-spaceable for some cardinal $\alpha > 0$. A result by Fávaro et al. [21] shows that this is true for $\alpha = 1$ and in the same article they inquire about the $(\alpha, \mathfrak{c})$-spaceability of this same set for a cardinal $1 < \alpha < \mathfrak{c}$ (this same issue is again highlighted in [20]).

The first result of this section demonstrates, using a different technique from the one used in Corollary 2.3 that thes set $L_p[0,1] \setminus \bigcup_{q\in(p,\infty)} L_q[0,1]$ is not $(\aleph_0, \mathfrak{c})$-spaceable. We believe that this new result further deepens our understanding of the geometric structure of these function spaces.

Theorem 2.16. *Let $p > 0$. The set $L_p[0,1] \setminus \bigcup_{q\in(p,\infty)} L_q[0,1]$ is not $(\aleph_0, \mathfrak{c})$-spaceable.*

Proof. Let

$$[1/2, 1) = \bigcup_{n=1}^{\infty} I_n,$$

where $I_n = [a_n, b_n)$, $a_n = 1 - 1/2^n$ and $b_n = 1 - 1/2^{n+1}$. For each $x \in I_n$, there is a unique $t_{x,n} \in [0,1)$ such that

$$x = (1 - t_{x,n})\, a_n + t_{x,n} b_n.$$

Let $f \in L_p[0,1] \setminus \bigcup_{q\in(p,\infty)} L_q[0,1]$. For each $n \in \mathbb{N}$, let $f_n\colon\ [0,1] \to \mathbb{R}$ be the function given by

$$f_n(x) = \begin{cases} 1, & \text{if } x \in [0,1/2), \\ f(t_{x,n}), & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n \cup [0,1/2). \end{cases}$$

It is easy to see that $f_n \in L_p[0,1]$, for every $n \in \mathbb{N}$. Furthermore, if $q > p$, denoting the length of the interval I_n by $|I_n|$, $n \in \mathbb{N}$, and making a change of variables, we conclude that

$$\begin{aligned}\int_0^1 |f_n|^q = \int_{[0,1/2)} |f_n|^q + \int_{[1/2,1]} |f_n|^q = \frac{1}{2} + \int_{I_n} |f_n|^q \\ = \frac{1}{2} + |I_n| \cdot \int_0^1 |f|^q = \infty.\end{aligned}$$

Thus, $f_n \in L_p[0,1] \setminus \bigcup_{q\in(p,\infty)} L_q[0,1]$ for each $n \in \mathbb{N}$.

Note that $\{f_1, f_2, \ldots\}$ is linearly independent. In fact, let $a_1, a_2, \ldots, a_m$ be scalars such that

$$a_1 f_1 + a_2 f_2 + \cdots + a_m f_m = 0.$$

For each $k = 1, \ldots, m$, let $x_k \in I_k$ be such that $f_k(x_k) \neq 0$. Hence

$$a_k f_k(x_k) = (a_1 f_1 + a_2 f_2 + \cdots + a_m f_m)(x_k) = 0$$

and thus $a_k = 0$. It is also simple to check that

$$W \setminus \{0\} \subset L_p[0,1] \setminus \bigcup_{q\in(p,\infty)} L_q[0,1],$$

where $W = \operatorname{span}\{f_n : n \in \mathbb{N}\}$.

Finally, denoting by $\chi_{[0,1/2)}$ the characteristic function at $[0,1/2)$, we have

$$\left\| f_n - \chi_{[0,1/2)} \right\|_p^p = \int_{[0,1]} \left| f_n \chi_{[1/2,1)} \right|^p = \int_{I_n} |f_n|^p = |I_n| \cdot \int_0^1 |f|^p \overset{n\to\infty}{\longrightarrow} 0,$$

and $\chi_{[0,1/2)} \in \overline{W}$. Since $\chi_{[0,1/2)} \notin L_p[0,1] \setminus \bigcup_{q\in(p,\infty)} L_q[0,1]$, the proof is done. ☐

Summarizing all the information above, together with Corollary 2.3, we have the following question:

For $0 < p < \infty$ and $2 \leq \alpha < \aleph_0$, is the set $L_p[0,1] \setminus \bigcup_{q\in(p,\infty)} L_q[0,1]$ $(\alpha, \mathfrak{c})$-spaceable?

In view of [21, Theorem 2.2], many authors conjectured the veracity of this question. In this thesis, using a slightly different technique than the one usually used in this type of problem, namely the *mother vector technique*, we address and answer the above question. The result was published in:

[2] G. Araújo, A. Barbosa, A. Raposo Jr., and G. Ribeiro, *On the spaceability of the set of functions in the Lebesgue space L_p which are not in L_q*, Bull Braz Math Soc, New Series 54, **44** (2023).

Theorem 2.17. *For all $0 < p < \infty$ the set*

$$L_p[0,1] \setminus \bigcup_{q\in(p,\infty)} L_q[0,1]$$

is $(\alpha, \mathfrak{c})$-spaceable in $L_p[0,1]$ if, and only if, $\alpha < \aleph_0$.

Proof. From the previous discussion the question remains open only for $2 \leq \alpha < \aleph_0$.

Let $g_1, \ldots, g_n \in L_p[0,1]$ be linearly independent normalized vectors so that

$$\operatorname{span}\{g_1, \ldots, g_n\} \setminus \{0\} \subset L_p[0,1] \setminus \bigcup_{q\in(p,\infty)} L_q[0,1].$$

Let us consider the representation of the semi-open interval $(0,1]$ as the following disjoint union

$$(0,1] = \bigcup_{k=1}^{\infty} I_k,$$

where $I_k := \left(\frac{1}{k+1}, \frac{1}{k}\right]$. Let us fix $k \in \mathbb{N}$. Since $\bigcup_{q\in(p,\infty)} L_q(I_k)$ is a vector subspace of $L_p(I_k)$ and $\bigcup_{q\in(p,\infty)} L_q(I_k)$ has infinite codimension (see [6, Theorem 4.4]), we can take an infinite dimensional subspace V_k of $L_p(I_k)$ so that

$$L_p(I_k) = V_k \oplus \bigcup_{q\in(p,\infty)} L_q(I_k).$$

Now, consider the canonical projection $P_k \colon L_p(I_k) \to V_k$ of $L_p(I_k)$ onto V_k and let

$$\tilde{f}_k \in V_k \setminus P_k\left(\operatorname{span}\{g_1|_{I_k}, \ldots, g_n|_{I_k}\}\right)$$

with $\|\tilde{f}_k\|_p = 1$. Let us prove that, for all $a_1, \ldots, a_n \in \mathbb{K}$,

$$\tilde{f}_k + \sum_{i=1}^{n} a_i g_i|_{I_k} \notin \bigcup_{q\in(p,\infty)} L_q(I_k). \tag{2.11}$$

In fact, if there exists $a_1, \ldots, a_n \in \mathbb{K}$ such that $\tilde{f}_k + \sum_{i=1}^{n} a_i g_i|_{I_k} \in \bigcup_{q\in(p,\infty)} L_q(I_k)$, since

$$\tilde{f}_k + \sum_{i=1}^{n} a_i g_i|_{I_k} = \tilde{f}_k + P_k\left(\sum_{i=1}^{n} a_i g_i|_{I_k}\right) + \left(-P_k\left(\sum_{i=1}^{n} a_i g_i|_{I_k}\right) + \sum_{i=1}^{n} a_i g_i|_{I_k}\right),$$

we would conclude that $\tilde{f}_k + P_k\left(\sum_{i=1}^n a_i g_i|_{I_k}\right) = 0$ and, hence, $\tilde{f}_k \in P_k\left(\mathrm{span}\{g_1|_{I_k}, \ldots, g_n|_{I_k}\}\right)$, which we know doesn't happen.

Define $\tilde{p} = 1$ if $p \geq 1$ and $\tilde{p} = p$ if $0 < p < 1$. Furthermore, consider $f_k \in L_p[0,1] \setminus \bigcup_{q \in (p,\infty)} L_q[0,1]$, where

$$f_k = \begin{cases} 0 & \text{in } [0,1] \setminus I_k \\ \tilde{f}_k & \text{in } I_k. \end{cases}$$

For $(a_i)_{i=1}^\infty \in \ell_{\tilde{p}}$,

$$\|a_1 g_1\|_p^{\tilde{p}} + \cdots + \|a_n g_n\|_p^{\tilde{p}} + \sum_{i=n+1}^\infty \|a_i f_{i-n}\|_p^{\tilde{p}} = \sum_{i=1}^\infty |a_i|^{\tilde{p}} < \infty.$$

Since $L_p[0,1]$ is a Banach space for $p \geq 1$ and a quasi Banach space for $0 < p < 1$, it follows that $a_1 g_1 + \cdots + a_n g_n + \sum_{i=n+1}^\infty a_i f_{i-n} \in L_p[0,1]$. Therefore we can define the operator

$$T \colon \ell_{\tilde{p}} \to L_p[0,1], \quad T\left((a_i)_{i=1}^\infty\right) = a_1 g_1 + \cdots + a_n g_n + \sum_{i=n+1}^\infty a_i f_{i-n}.$$

For an arbitrary function $f \colon X \to \mathbb{K}$ whose domain is an arbitrary set X, let $\mathrm{supp}\,(f) = \{x \in X : f(x) \neq 0\}$. Since $\mathrm{supp}(f_i) \cap \mathrm{supp}(f_j) = \varnothing$ for $i \neq j$, we can conclude that $T(\ell_{\tilde{p}})$ has infinite dimension.

Below we will show that there exists a positive integer m_0 such that

$$\{g_1|_{\cup_{i=1}^{m_0} I_i}, \ldots, g_n|_{\cup_{i=1}^{m_0} I_i}, f_1|_{\cup_{i=1}^{m_0} I_i}, \ldots, f_{m_0}|_{\cup_{i=1}^{m_0} I_i}\}$$

is a linearly independent set in $L_p(\bigcup_{i=1}^{m_0} I_i)$. We first need to prove the following lemma:

Lemma 2.18. *There exist a positive integer m_1 such that*

$$\{g_1|_{\cup_{i=1}^{m_1} I_i}, \ldots, g_n|_{\cup_{i=1}^{m_1} I_i}\}$$

is a linearly independent set in $L_p(\bigcup_{i=1}^{m_1} I_i)$.

Proof of Lemma 2.18. Fix $j \in \{1, \ldots, n\}$. Since $g_j|_{\cup_{i=1}^m I_i} \overset{m\to\infty}{\longrightarrow} g_j$ in $L_p[0,1]$, we have $g_j|_{\cup_{i=1}^m I_i} \neq 0$ for all large enough m. By contradiction, suppose there is not a positive integer m_1 such that $\{g_1|_{\cup_{i=1}^{m_1} I_i}, \ldots, g_n|_{\cup_{i=1}^{m_1} I_i}\}$ is linearly independent in $L_p(\bigcup_{i=1}^{m_1} I_i)$. Thus, the set $\{g_1|_{\cup_{i=1}^m I_i}, \ldots, g_n|_{\cup_{i=1}^m I_i}\}$ is linearly dependent on $L_p(\bigcup_{i=1}^m I_i)$ for all $m \in \mathbb{N}$. For each $m \in \mathbb{N}$, let $\{g_{1(m)}|_{\cup_{i=1}^m I_i}, \ldots, g_{r(m)}|_{\cup_{i=1}^m I_i}\}$ be a smaller linearly dependent subset of $\{g_1|_{\cup_{i=1}^m I_i}, \ldots, g_n|_{\cup_{i=1}^m I_i}\}$ and define $\varphi \colon \mathbb{N} \to \mathcal{P}(\{1, \ldots, n\})$ by $\varphi(m) = \{1(m), \ldots, r(m)\}$, where $\mathcal{P}(\{1, \ldots, n\})$ is the set of all subsets of $\{1, \ldots, n\}$. Since $\mathrm{card}(\mathcal{P}(\{1, \ldots, n\})) < \mathrm{card}(\mathbb{N}) = \aleph_0$, there is $\{j_1, \ldots, j_r\} \in \varphi(\mathbb{N})$ such that $\mathrm{card}(\varphi^{-1}(\{j_1, \ldots, j_r\})) = \aleph_0$. Define $\mathbb{N}' := \varphi^{-1}(\{j_1, \ldots, j_r\}) \subset \mathbb{N}$ and note that

$$\{g_{1(m)}|_{\cup_{i=1}^m I_i}, \ldots, g_{r(m)}|_{\cup_{i=1}^m I_i}\} = \{g_{j_1}|_{\cup_{i=1}^m I_i}, \ldots, g_{j_r}|_{\cup_{i=1}^m I_i}\}.$$

Thus, if $m, \tilde{m} \in \mathbb{N}'$ are such that $m < \tilde{m}$, then there are $b_1, \dots, b_{r-1}, \tilde{b}_1, \dots, \tilde{b}_{r-1} \in \mathbb{K}$ so that

$$g_{j_r}|_{\cup_{i=1}^{m} I_i} = b_1 g_{j_1}|_{\cup_{i=1}^{m} I_i} + \cdots + b_{r-1} g_{j_{r-1}}|_{\cup_{i=1}^{m} I_i}$$

and

$$g_{j_r}|_{\cup_{i=1}^{\tilde{m}} I_i} = \tilde{b}_1 g_{j_1}|_{\cup_{i=1}^{\tilde{m}} I_i} + \cdots + \tilde{b}_{r-1} g_{j_{r-1}}|_{\cup_{i=1}^{\tilde{m}} I_i}. \tag{2.12}$$

Restricting (2.12) to $\bigcup_{i=1}^{m} I_i$ we get

$$\begin{aligned} \tilde{b}_1 g_{j_1}|_{\cup_{i=1}^{m} I_i} + \cdots + \tilde{b}_{r-1} g_{j_{r-1}}|_{\cup_{i=1}^{m} I_i} &= g_{j_r}|_{\cup_{i=1}^{m} I_i} \\ &= b_1 g_{j_1}|_{\cup_{i=1}^{m} I_i} + \cdots + b_{r-1} g_{j_{r-1}}|_{\cup_{i=1}^{m} I_i} \end{aligned}$$

and consequently

$$(\tilde{b}_1 - b_1) g_{j_1}|_{\cup_{i=1}^{m} I_i} + \cdots + (\tilde{b}_{r-1} - b_{r-1}) g_{j_{r-1}}|_{\cup_{i=1}^{m} I_i} = 0.$$

Since $\{g_{j_1}|_{\cup_{i=1}^{m} I_i}, \dots, g_{j_r}|_{\cup_{i=1}^{m} I_i}\}$ is a smaller linearly dependent subset of $\{g_1|_{\cup_{i=1}^{m} I_i}, \dots, g_n|_{\cup_{i=1}^{m} I_i}\}$ we can conclude that $\tilde{b}_k = b_k$, $k = 1, \dots, r-1$. Since $m \in \mathbb{N}'$ is arbitrary, we obtain

$$g_{j_r}|_{\cup_{i=1}^{m} I_i} = b_1 g_{j_1}|_{\cup_{i=1}^{m} I_i} + \cdots + b_{r-1} g_{j_{r-1}}|_{\cup_{i=1}^{m} I_i}$$

for all $m \in \mathbb{N}'$.

Therefore

$$\begin{aligned} g_{j_r} &= \lim_{m\in\mathbb{N}'} g_{j_r}|_{\cup_{i=1}^{m} I_i} \\ &= \lim_{m\in\mathbb{N}'} \left(b_1 g_{j_1}|_{\cup_{i=1}^{m} I_i} + \cdots + b_{r-1} g_{j_{r-1}}|_{\cup_{i=1}^{m} I_i} \right) \\ &= b_1 g_{j_1} + \cdots + b_{r-1} g_{j_{r-1}}, \end{aligned}$$

which is contrary to the fact that $\{g_1, \dots, g_n\}$ is linearly independent. □

Let us return to the proof of Theorem 2.17. Let us prove that the set

$$\{g_1|_{\cup_{i=1}^{m} I_i}, \dots, g_n|_{\cup_{i=1}^{m} I_i}, f_1|_{\cup_{i=1}^{m} I_i}, \dots, f_m|_{\cup_{i=1}^{m} I_i}\}$$

is linearly independet in $L_p(\bigcup_{i=1}^{m} I_i)$ for all $m \geq m_0$, where

$$m_0 = \min\left\{ m_1 : \{g_1|_{\cup_{i=1}^{m} I_i}, \dots, g_n|_{\cup_{i=1}^{m} I_i}\} \text{ is linearly independent in } L_p\left(\bigcup_{i=1}^{m} I_i\right) \right\}.$$

Given $m \geq m_0$, let $b_1, \dots, b_n, b_{n+1}, \dots, b_{n+m} \in \mathbb{K}$ such that

$$b_1 g_1|_{\cup_{i=1}^{m} I_i} + \cdots + b_n g_n|_{\cup_{i=1}^{m} I_i} + b_{n+1} f_1|_{\cup_{i=1}^{m} I_i} + \cdots + b_{n+m} f_m|_{\cup_{i=1}^{m} I_i} = 0,$$

i.e.,

$$b_1 g_1|_{\cup_{i=1}^{m} I_i} + \cdots + b_n g_n|_{\cup_{i=1}^{m} I_i} = -b_{n+1} f_1|_{\cup_{i=1}^{m} I_i} - \cdots - b_{n+m} f_m|_{\cup_{i=1}^{m} I_i}. \tag{2.13}$$

Restricting the equality in (2.13) to I_j, $j = 1, \ldots, m$, we have

$$b_1 g_1|_{I_j} + \cdots + b_n g_n|_{I_j} = -b_{n+j}\tilde{f}_j,$$

i.e., $-b_{n+j}\tilde{f}_j = P_j\left(b_1 g_1|_{I_j} + \cdots + b_n g_n|_{I_j}\right) \in P_j\left(\operatorname{span}\{g_1|_{I_j}, \ldots, g_n|_{I_j}\}\right)$, and we can conclude that $b_{n+j} = 0$. From (2.13) we have

$$b_1 g_1|_{\cup_{i=1}^m I_i} + \cdots + b_n g_n|_{\cup_{i=1}^m I_i} = 0,$$

and from the Lemma 2.18 we obtain $b_1 = \ldots = b_n = 0$.

Now let us see that

$$\overline{T(\ell_{\tilde{p}})} \setminus \{0\} \subset L_p[0,1] \setminus \bigcup_{q \in (p,\infty)} L_q[0,1].$$

Indeed, given $h \in \overline{T(\ell_{\tilde{p}})} \setminus \{0\}$, let $(a_i^{(k)})_{i=1}^\infty \in \ell_{\tilde{p}}$ $(k \in \mathbb{N})$ such that

$$T\left((a_i^{(k)})_{i=1}^\infty\right) \overset{k\to\infty}{\longrightarrow} h \text{ in } L_p[0,1].$$

Observe that $T\left((a_i^{(k)})_{i=1}^\infty\right)|_I \overset{k\to\infty}{\longrightarrow} h|_I$ in $L_p(I)$ for any subinterval I of $[0,1]$. In order to go further, the strategy shall be to prove that there is a sequence of scalars $(a_i)_{i\in\mathbb{N}}$ such that

$$a_1 g_1 + \cdots + a_n g_n + \sum_{i=1}^\infty a_{n+i} f_i = h.$$

In fact, for a fixed $m \geq m_0$, note that

$$\begin{aligned} a_1^{(k)} g_1|_{\cup_{i=1}^m I_i} + \cdots + a_n^{(k)} g_n|_{\cup_{i=1}^m I_i} + a_{n+1}^{(k)} f_1|_{\cup_{i=1}^m I_i} + \cdots + a_{n+m}^{(k)} f_m|_{\cup_{i=1}^m I_i} \\ = T\left(\left(a_i^{(k)}\right)_{i=1}^\infty\right)|_{\cup_{i=1}^m I_i} \overset{k\to\infty}{\longrightarrow} h|_{\cup_{i=1}^m I_i}, \end{aligned}$$

and that $\operatorname{span}\left\{g_1|_{\cup_{i=1}^m I_i}, \ldots, g_n|_{\cup_{i=1}^m I_i}, f_1|_{\cup_{i=1}^m I_i}, \ldots, f_m|_{\cup_{i=1}^m I_i}\right\}$ is finite dimensional on $L_p(\bigcup_{i=1}^m I_i)$. Since every finite-dimensional subspace of a topological vector space is closed, there are scalars $a_1(m), \ldots, a_{n+m}(m)$ such that

$$h|_{\cup_{i=1}^m I_i} = a_1(m) g_1|_{\cup_{i=1}^m I_i} + \cdots + a_n(m) g_n|_{\cup_{i=1}^m I_i} + a_{n+1}(m) f_1|_{\cup_{i=1}^m I_i} + \cdots + a_{n+m}(m) f_m|_{\cup_{i=1}^m I_i}, \tag{2.14}$$

Obviously the same reasoning can be applied to $\tilde{m} > m$ and therefore

$$h|_{\cup_{i=1}^{\tilde{m}} I_i} = a_1(\tilde{m}) g_1|_{\cup_{i=1}^{\tilde{m}} I_i} + \cdots + a_n(\tilde{m}) g_n|_{\cup_{i=1}^{\tilde{m}} I_i} + a_{n+1}(\tilde{m}) f_1|_{\cup_{i=1}^{\tilde{m}} I_i} + \cdots + a_{n+\tilde{m}}(\tilde{m}) f_{\tilde{m}}|_{\cup_{i=1}^{\tilde{m}} I_i}. \tag{2.15}$$

Restricting (2.15) to $\bigcup_{i=1}^m I_i$ and comparing with (2.14) we get

$$\begin{aligned} & a_1(\tilde{m}) g_1|_{\cup_{i=1}^m I_i} + \cdots + a_n(\tilde{m}) g_n|_{\cup_{i=1}^m I_i} + a_{n+1}(\tilde{m}) f_1|_{\cup_{i=1}^m I_i} + \cdots + a_{n+m}(\tilde{m}) f_m|_{\cup_{i=1}^m I_i} \\ & = h|_{\cup_{i=1}^m I_i} \\ & = a_1(m) g_1|_{\cup_{i=1}^m I_i} + \cdots + a_n(m) g_n|_{\cup_{i=1}^m I_i} + a_{n+1}(m) f_1|_{\cup_{i=1}^m I_i} + \cdots + a_{n+m}(m) f_m|_{\cup_{i=1}^m I_i}. \end{aligned}$$

Since the set $\{g_1|_{\cup_{i=1}^m I_i},\ldots,g_n|_{\cup_{i=1}^m I_i},f_1|_{\cup_{i=1}^m I_i},\ldots,f_m|_{\cup_{i=1}^m I_i}\}$ is linearly independent, we obtain $a_j(m)=a_j(\tilde{m})$ for every $j=1,\ldots,n+m$. Thus we conclude that there is a sequence of scalars $(a_i)_{i=1}^\infty$ such that

$$\left(a_1g_1+\cdots+a_ng_n+\sum_{i=1}^\infty a_{n+i}f_i\right)|_{\cup_{i=1}^m I_i}=(a_1g_1+\cdots+a_ng_n)|_{\cup_{i=1}^m I_i}+\left(\sum_{i=1}^m a_{n+i}f_i\right)|_{\cup_{i=1}^m I_i}$$
$$=h|_{\cup_{i=1}^m I_i}$$

and so we finally have

$$a_1g_1+\cdots+a_ng_n+\sum_{i=1}^\infty a_{n+i}f_i=h.$$

Since $h\neq 0$, it follows that $(a_i)_{i=1}^\infty\neq 0$. Therefore, if $a_{n+i}=0$ for all $i\in\mathbb{N}$, we have

$$h=a_1g_1+\cdots+a_ng_n\in\operatorname{span}\{g_1,\ldots,g_n\}\smallsetminus\{0\}\subset L_p[0,1]\smallsetminus\bigcup_{q\in(p,\infty)}L_q[0,1].$$

On the other hand, if $a_{n+i}\neq 0$ for some $i\in\mathbb{N}$, from (2.11) we obtain

$$\frac{1}{a_{n+i}}h|_{I_i}=\tilde{f}_i+\frac{1}{a_{n+i}}\left(a_1g_1+\cdots+a_ng_n\right)|_{I_i}\notin\bigcup_{q\in(p,\infty)}L_q(I_i).$$

Consequently, $h\notin\bigcup_{q\in(p,\infty)}L_q[0,1]$ and the result is done. □

2.4 $(\alpha,\mathfrak{c})$-Dense Lineability of Set $L_p[0,1]\setminus\bigcup_{q\in(p,\infty)}L_q[0,1]$ for $p\in(0,\infty)$

The criteria we will examine next were published in:

[23] V.V. Fávaro, D. Pellegrino, A. Raposo Jr, and G. Ribeiro, *General criteria for a stronger notion of lineability.* Proc. Amer. Math. Soc. 152 (2024), 941-954

and will allow us to further investigate the topological structure of the set $L_p[0,1]\setminus\bigcup_{q\in(p,\infty)}L_q[0,1]$ in terms of density.

We will now return to discussing additional results that were published in the aforementioned paper. These criteria will provide a deeper understanding of the concept of dense lineability and its implications within the broader context of our study. The insights gained from these findings are crucial for the comprehensive analysis and conclusions presented in this thesis.

The Concept of (α,β)-Dense Lineability

This section delves into a refined version of lineability, known as (α,β)-dense lineability, which considers not only the relationship between the dimensions of the subspaces within the investigated environment, but also the density of these algebraic structures.

This notion extends the idea of dense lineability, thus providing a more comprehensive understanding of the distribution of linear structures within a given space. The foundation for this study is laid by Bernal et al. [6], who established a fundamental result regarding $\aleph_0$-dense lineability in metrizable separable topological vector spaces.

The concept of (α, β)-dense lineability was first introduced in the aforementioned paper. Essentially, this notion is defined as follows: Let X be a topological vector space and let A be a non-void subset of X.

- We shall say that A is (α, β)-*dense lineable* if it is α-lineable and for each α-dimensional subspace $W_\alpha \subset A \cup \{0\}$, there is a β-dimensional dense subspace W_β such that
$$W_\alpha \subset W_\beta \subset A \cup \{0\}.$$

An interesting example related to these concepts is the set of continuous nowhere differentiable functions on $[0,1]$, denoted as $\mathcal{ND}[0,1]$. It is known that $\mathcal{ND}[0,1]$ is $\mathfrak{c}$-dense lineable. However, it is still an open question whether $\mathcal{ND}[0,1]$ is $(\mathfrak{c},\mathfrak{c})$-dense lineable. This means that while we know a $\mathfrak{c}$-dimensional dense subspace exists, it remains unknown whether for every $\mathfrak{c}$-dimensional subspace, there is a corresponding $\mathfrak{c}$-dimensional dense extension within $\mathcal{ND}[0,1]$. In the same direction, in [29], the notions of pointwise β-lineability were introduced. Now we introduce the concept of pointwise β-dense lineability:

- A subset A of a topological vector space X is called *pointwise* β*-dense lineable* if, for each $x \in A$, there is a β-dimensional dense subspace W_x such that
$$x \in W_x \subset A \cup \{0\}.$$
- If $\beta = \dim(X)$, we say that A is *maximally pointwise dense lineable.*

It is plain that the notions of pointwise β-dense lineability imply $(1,\beta)$-dense lineability. In general, the converse is not true (see [29], Example 2.2). However, whenever $A \cup \{0\}$ is closed under scalar products, the pointwise notions coincide with the $(1,\beta)$ corresponding notions.

Theorem 2.19. ([6, Theorem 2.5]) *Let X be a metrizable separable topological vector space and W be a vector subspace of X. If $\dim(X/W) \geq \aleph_0$, then $X \setminus W$ is $\aleph_0$-dense lineable.*

To extend these ideas to non-separable spaces, we used the concept of the weight of a topological vector space. Given a topological vector space X, let B_X be the set of all bases for the topology of X. The cardinality of a basis with minimal cardinality, denoted by $w(X)$, is called the *weight* of X.

Inspired by [38, Lemma 3.1], we present a main result that offers a non-separable and stronger variant of Theorem 2.19. This theorem leverages the powerful technique of transfinite induction, a method that extends the principle of mathematical induction to well-ordered sets, allowing for the construction and verification of properties across larger cardinalities.

Transfinite induction will be the key technique used to prove the following theorem, providing a robust framework for demonstrating (α,β)-dense lineability in a wide range of topological vector spaces.

This result not only broadens the scope of dense lineability but also underscores the significance of cardinality constraints in the context of topological vector spaces. By utilizing transfinite induction, we are able to systematically construct and verify the existence of densely lineable structures, providing deeper insights into the algebraic and topological properties of these spaces.

Theorem 2.20. *Let $X\neq\{0\}$ be a topological vector space and $W\subset X$ be a linear subspace such that $w(X)\leq\dim(X/W)$. Then $X\setminus W$ is (α,β)-dense lineable for each $\alpha<\dim(X/W)$ and*

$$\max\{\alpha,w(X)\}\leq\beta\leq\dim(X/W).$$

Proof. Fix $\alpha<\dim(X/F)$ and let $\mathcal{C}=\{u_\lambda\}_{\lambda\in\alpha}$ be a linearly independent subset of X such that

$$\operatorname{span}(\mathcal{C})\subset(X\setminus W)\cup\{0\}.$$

Consider the subspace $M_\alpha:=W\oplus\operatorname{span}(\mathcal{C})$ and let $B_0=\{U_\mu\}_{\mu\in w(X)}\in B_X$ be a basis of minimal cardinality for the topology of X. Since $\alpha=\dim(\operatorname{span}(\mathcal{C}))<\dim(X/F)$, we have $M_\alpha\subsetneq X$. Thus,

$$X\setminus M_\alpha \text{ is dense in } X. \tag{2.16}$$

This implies that there is a vector $v_{\mu_0}\in U_{\mu_0}\cap(V\setminus M_\alpha)$. Given $\mu<w(X)$, let us suppose by *transfinite induction* that

$$v_\eta\in U_\eta\cap\Big(X\setminus\Big(M_\alpha\oplus\operatorname{span}\{v_\gamma\}_{\gamma\in\eta}\Big)\Big)$$

has been defined for each $\eta<\mu$. Since

$$\dim\Big(\operatorname{span}(\mathcal{C})\oplus\operatorname{span}\{v_\gamma\}_{\gamma\in\mu}\Big)=\alpha+\mu<\max\{\alpha,w(V)\}\leq\dim(X/F),$$

the subspace $M_\alpha\oplus\operatorname{span}\{v_\gamma\}_{\gamma\in\mu}\subsetneq V$. That is,

$$X\setminus\Big(M_\alpha\oplus\operatorname{span}\{v_\gamma\}_{\gamma\in\mu}\Big)\text{ is dense in } X.$$

Therefore, there is a vector $v_\mu\in U_\mu\cap\Big(X\setminus\Big(M_\alpha\oplus\operatorname{span}\{v_\gamma\}_{\gamma\in\mu}\Big)\Big)$. This assures the existence of vectors $\{v_\mu\}_{\mu\in w(X)}$ satisfying

$$v_\mu\in U_\mu\cap(X\setminus G_\mu),\text{ for each }\mu<w(X),$$

where $G_\mu:=W\oplus H_\mu$ and $H_\mu:=\operatorname{span}(\mathcal{C})\oplus\operatorname{span}\{v_\gamma\}_{\gamma\in\mu}$. It is plain that the set $\{v_\mu\}_{\mu\in w(X)}$ is dense and linearly independent in X, consequently the subspace

$$Y:=\operatorname{span}(\mathcal{C})\oplus\operatorname{span}\{v_\mu\}_{\mu\in w(X)}$$

is dense in X. Furthermore, if $v\in Y\setminus\{0\}$, then $v=\sum_{j=1}^{n}a_ju_{\lambda j}+\sum_{k=n+1}^{m}a_kv_{\mu_k}$. Hence $v\in H_{\mu_m}$ and, since $H_{\mu_m}\cap W=\{0\}$, we conclude that $v\notin W$. Therefore,

$$\operatorname{span}(\mathcal{C})\subset\operatorname{span}(\mathcal{C})\oplus\operatorname{span}\{v_\mu\}_{\mu\in w(X)}=:Y\subset(X\setminus W)\cup\{0\}.$$

Since

$$\dim(Y) = \max\{\alpha, w(X)\} \text{ and } \max\{\alpha, w(X)\} \geq \alpha,$$

the set $X \setminus W$ is $(\alpha, \max\{\alpha, w(X)\})$-dense lineable. Since we can extend Y to $\tilde{Y}$ such that $\tilde{Y} \subset (X \setminus W) \cup \{0\}$ and $\dim\left(\tilde{Y}\right) = \beta$, the result follows. □

For a topological vector space $X \neq \{0\}$, recall that the $\operatorname{dens}(X)$ denotes the *density character* of X, that is, the minimal infinite cardinal number of a dense subset of X.

Corollary 2.21. *Let X be a metrizable, infinite dimensional topological vector space with $\kappa = \operatorname{dens}(X)$ and W be a linear subspace. The following conditions are equivalent:*

(i) *$X \setminus W$ is (α, β)-dense lineable for each $\alpha < \dim(X/W)$ and*

$$\max\{\kappa, \alpha\} \leq \beta \leq \dim(X/W).$$

(ii) *$X \setminus W$ is κ-lineable.*

(iii) *$\kappa \leq \dim(X/W)$.*

Proof. $(i) \Rightarrow (ii)$. It is obvious.

$(ii) \Rightarrow (iii)$. Let W_κ be a κ-dimensional subspace of X such that $W \cap W_\kappa = \{0\}$ and let $\pi_W \colon X \to X/W$ be the canonical projection. Since $\pi_W|_{W_\kappa}$ is injective, we have

$$\kappa = \dim(W_\kappa) = \dim(\pi_W(W_\kappa)) \leq \dim(X/W).$$

$(iii) \Rightarrow (i)$. Since X is a metric space, we have

$$\operatorname{dens}(X) = w(X)$$

and, hence, Theorem 2.20 assures the result. □

In [6, Theorem 2.5] the authors proved that if X is a separable metrizable topological vector space and W is a subspace of X with infinite codimension, then $X \setminus W$ is $\aleph_0$-dense lineable. The next result improves the aforementioned result:

Corollary 2.22. *Let X be a metrizable, infinite dimensional topological vector space with $\kappa = \operatorname{dens}(X)$ and W be a linear subspace such that $\kappa \leq \dim(X/W)$, then $X \setminus W$ is $\dim(X/W)$-pointwise dense lineable. If furthermore, $\dim(X/W) = \dim(X)$, then $X \setminus W$ is pointwise maximal dense lineable.*

Proof. In fact, Corollary 2.21 assures the $(1, \dim(X/W))$-dense lineability of $X \setminus W$ and, since $(X \setminus W) \cup \{0\}$ is closed under scalar multiplication, as pointed before, pointwise notions and $(1, \beta)$ notions coincide. This shows that the set $X \setminus W$ is pointwise $\dim(X/W)$-dense lineable. □

In [21, Theorem 3.2] the authors proved that the set $L_p\,[0,1] \setminus \bigcup_{q\in(p,\infty)} L_q\,[0,1]$ is $(1,\mathfrak{c})$-spaceable. Considering $X = L_p[0,1]$ and $W = \bigcup_{q\in(p,\infty)} L_q\,[0,1]$ in previous corollary, we have:

Corollary 2.23. *$L_p\,[0,1] \setminus \bigcup_{q\in(p,\infty)} L_q\,[0,1]$ is pointwise maximal dense-lineable, for every $p > 0$.*

Corollary 2.24. *Let X be a metrizable separable topological vector space and let W be a subspace of X. If $\dim(X/W) \geq \aleph_0$, then $X \setminus W$ is (k,β)-dense lineable, for each $k \in \mathbb{N}$ and each $\aleph_0 \leq \beta \leq \dim(X/W)$.*

Proof. It is an immediate consequence of Corollary 2.21 since, in this case,

$$\operatorname{dens}(X) = \aleph_0 \leq \dim(X/W)\,.$$

□

Considering $X = L_p[0,1]$ and $W = \bigcup_{q\in(p,\infty)} L_q[0,1]$ in Corollary 2.21, we have:

Corollary 2.25. *Let $0 < p < \infty$. The set $L_p[0,1]\setminus\bigcup_{q\in(p,\infty)} L_q[0,1]$ is (α,β)-dense lineable, for each $\alpha < \mathfrak{c}$ and*

$$\max\{\alpha, \aleph_0\} \leq \beta \leq \mathfrak{c}.$$

In [40, Section 2], Nestoridis posed the following question:

Does the set $\ell_\infty \setminus c_0$ contain a dense linear subspace?

In [41, Theorem 1.2] Papathanasiou gave a positive answer to this question. More precisely, he proved that the set $\ell_\infty \setminus c_0$ is maximal dense-lineable. Now we improve the result of Papathanasiou.

Corollary 2.26. *Let $F = c_0$ or c.*

(i) *$\ell_\infty \setminus F$ is $(\alpha,\mathfrak{c})$-dense lineable for each $\alpha < \mathfrak{c}$.*

(ii) *$\ell_\infty \setminus F$ is not $(n,\aleph_0)$-dense lineable for each $n \in \mathbb{N}$.*

Proof. *(i)* It follows from Corollary 2.21 that $\ell_\infty \setminus F$ is $(\alpha,\mathfrak{c})$-dense lineable for each $\alpha < \mathfrak{c} = \dim(\ell_\infty/F)$, since

$$\operatorname{dens}(\ell_\infty) = \mathfrak{c}.$$

(ii) It is a straightforward consequence of that fact that the dimension of every dense linear subspace of ℓ_∞ is $\mathfrak{c}$. □

Remark 2.27. Corollaries 2.14 and 2.26 allow us to conclude, in particular, that $\ell_\infty \setminus F$ is $(\aleph_0,\mathfrak{c})$-dense lineable, but it is not $(\aleph_0,\mathfrak{c})$-spaceable.

Chapter 3

A Quest of Convergence

This chapter presents an extension of a result within the concept of $[\mathcal{S}]$-lineability, originally developed in 2019 by L. Bernal-González, J.A. Conejero, M. Murillo-Arcila, and J.B. Seoane-Sepúlveda in [10]. Additionally, we provide a characterization in terms of lineability in the context of complements of unions of closed subspaces in F-spaces, and finally, we present a negative result in both normed spaces and p-Banach spaces.

$[\mathcal{S}]$-Lineability in the Context of Complements of Unions

In this section, we aim to characterize the family of complements of unions of closed subspaces through the notion of $[\mathcal{S}]$-lineability. To this end, let us begin with the following notion as appeared in [18].

Definition 3.1. We say that a sequence $(u_n)_{n=1}^{\infty}$ of elements of a topological vector space X is topologically linearly independent if, for each sequence $(c_n)_{n=1}^{\infty} \in \mathbb{K}^{\mathbb{N}}$ with $\sum_{n=1}^{\infty} c_n u_n = 0$, we have $(c_n)_{n=1}^{\infty} = 0$.

Based on this definition, if $\mathcal{S} \neq \{0\}$ is a subspace of $\mathbb{K}^{\mathbb{N}}$ then we will say that a sequence $(u_n)_{n=1}^{\infty}$ of elements of a topological vector space X is $\mathcal{S}$-topologically linearly independent in X (or $\mathcal{S}$-independent) if for each sequence $(c_n)_{n=1}^{\infty} \in \mathcal{S}$ with $\sum_{n=1}^{\infty} c_n u_n = 0$, we have $(c_n)_{n=1}^{\infty} = 0$.

Within this perspective, still in [18], Drewnowski et al. demonstrated the following result, which will be a crucial ingredient for the proof of the Theorem 3.4. It is worth mentioning that this result establishes a connection between linear ℓ_∞-independence and linear independence.

Proposition 3.2. *Assume that $(x_n)_{n=1}^{\infty}$ is a linearly independent sequence in a Hausdorff topological vector space X. Then there is $\lambda_n > 0$ such that $(\lambda_n x_n)_{n=1}^{\infty}$ is ℓ_∞-independent.*

Em [37], the authors made a notable contribution with the following result:

Theorem 3.3. [37, Theorem 7.4.2] *Let Z_r $(r \in \mathbb{N})$ be Banach spaces and let X be a Fréchet space. Let $T_r \colon Z_r \longrightarrow X$ be continuous linear mappings and $Y = \operatorname{span}\left(\bigcup_{r=1}^{\infty} T_r(Z_r)\right)$. If Y is not closed in X then the complement $X \setminus Y$ is spaceable.*

At this point, a characterization of spaceability for the complements of non-enumerable unions of subspaces in the context of F-spaces in terms of lineability has yet to be developed. It is not even known if these complements are $[\mathcal{S}]$-lineable, and it is precisely for this purpose that we present the following result:

Theorem 3.4. *Let $X \neq \{0\}$ be an F-space and $\{Y_i\}_{i\in I}$ be a family of nontrivial closed subspaces of X. The set $X \setminus \bigcup_{i\in I} Y_i$ is lineable if and only if it is $[\mathcal{S}]$-lineable for every subspace $\mathcal{S} \neq \{0\}$ of ℓ_∞.*

Proof. Assume that $X \setminus \bigcup_{i\in I} Y_i$ is lineable and consider the quotient map $Q_i \colon X \longrightarrow X/Y_i$ $(i \in I)$. Let E be an infinite dimensional subspace of X such that

$$E \cap \left(\bigcup_{i\in I} Y_i \right) = \{0\}.$$

Let $(x_n)_{n=1}^{\infty}$ be a sequence of elements linearly independent in E such that

$$\sum_{n=1}^{\infty} \|x_n\|_X < \infty. \tag{3.1}$$

Fix $i \in I$. Due to the fact that $X \setminus \bigcup_{i\in I} Y_i$ is lineable, we can infer that the sequence $(Q_i(x_n))_{n=1}^{\infty}$ is also linearly independent in X/Y_i. Indeed, if for some $N \in \mathbb{N}$, we have

$$\sum_{n=1}^{N} a_n Q_i(x_n) = 0,$$

then

$$Q_i \left(\sum_{n=1}^{N} a_n x_n \right) = 0.$$

Hence

$$\sum_{n=1}^{N} a_n x_n \in Y_i \cap E = \{0\}.$$

That is,

$$a_n = 0,$$

which proves the desired linear independence in X/Y_i. Since X/Y_i is a Hausdorff topological vector space, we can invoke Proposition 3.2 to obtain a sequence of positive real numbers $(\lambda_n)_{n=1}^{\infty}$ such that the sequence $(Q_i(\lambda_n x_n))_{n=1}^{\infty}$ is ℓ_∞-independent in X/Y_i. We also claim that the sequence $(\alpha_n)_{n=1}^{\infty}$, where $\alpha_n := \frac{\lambda_n}{1+\lambda_n}$ is such that $(Q_i(\alpha_n x_n))_{n=1}^{\infty}$ is ℓ_∞-independent in X/Y_i. Indeed, let $(t_n)_{n=1}^{\infty} \in \ell_\infty$ be such that

$$\sum_{n=1}^{\infty} t_n Q_i(\alpha_n x_n) = 0.$$

Hence,

$$\sum_{n=1}^{\infty} \frac{t_n}{1+\lambda_n} Q(\lambda_n x_n) = 0. \tag{3.2}$$

Since

$$\left(\frac{t_n}{1+\lambda_n}\right)_{n=1}^{\infty} \in \ell_\infty,$$

and $(Q_i(\lambda_n x_n))_{n=1}^{\infty}$is ℓ_∞-independent in X/Y_i, we get

$$\frac{t_n}{1+\lambda_n} = 0 \text{ for each } n \in \mathbb{N},$$

and this entails that

$$t_n = 0 \text{ for each } n \in \mathbb{N}.$$

Therefore, $(Q_i(\alpha_n x_n))_{n=1}^{\infty}$ is really ℓ_∞-independent in X/Y_i. Furthermore, since $|\alpha_n| \leq 1$ for all $n \in N$, we have

$$\|\alpha_n x_n\|_X \leq \|x_n\|_X \text{ for all } n \in \mathbb{N}.$$

Since

$$\sum_{n=1}^{\infty} \|x_n\|_X < \infty,$$

we obtain

$$\sum_{n=1}^{\infty} \|\alpha_n x_n\|_X \leq \sum_{n=1}^{\infty} \|x_n\|_X < \infty.$$

Now, let $c = (c_n)_{n=1}^{\infty} \in \ell_\infty \setminus \{0\}$. We claim that the series $\sum\limits_{n=1}^{\infty} c_n \alpha_n x_n$ converges in X. Indeed, due to the fact that

$$\left\| \|c\|_{\ell_\infty}^{-1} c_n \alpha_n x_n \right\|_X \leq \|\alpha_n x_n\|_X < \infty,$$

if we take $\varepsilon > 0$, then there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0+1}^{\infty} \left\| \|c\|_{\ell_\infty}^{-1} c_n \alpha_n x_n \right\|_X < \varepsilon.$$

Thus, for $r, s \in \mathbb{N}$ with $r, s \geq n_0$, we have

$$\begin{aligned}
\left\| \sum_{n=1}^{r} \|c\|_{\ell_\infty}^{-1} c_n \alpha_n x_n - \sum_{n=1}^{s} \|c\|_{\ell_\infty}^{-1} c_n \alpha_n x_n \right\|_X &= \left\| \sum_{n=s+1}^{r} \|c\|_{\ell_\infty}^{-1} c_n \alpha_n x_n \right\|_X \\
&\overset{s<r}{\leq} \sum_{n=s+1}^{r} \left\| \|c\|_{\ell_\infty}^{-1} c_n \alpha_n x_n \right\|_X \\
&\leq \sum_{n=n_0+1}^{\infty} \left\| \|c\|_{\ell_\infty}^{-1} c_n \alpha_n x_n \right\|_X \\
&< \varepsilon.
\end{aligned}$$

Since X is complete and Hausdorff, we can conclude that sequence $\left(\sum_{n=1}^{r} \|c\|_{\ell_\infty}^{-1} c_n \alpha_n x_n\right)_{r=1}^{\infty}$ converges in X for the series

$$\sum_{n=1}^{\infty} \|c\|_{\ell_\infty}^{-1} c_n \alpha_n x_n.$$

Hence,

$$\sum_{n=1}^{\infty} \|c\|_{\ell_\infty}^{-1} c_n \alpha_n x_n \in X,$$

and consequently,

$$\sum_{n=1}^{\infty} c_n \alpha_n x_n = \|c\|_{\ell_\infty} \sum_{n=1}^{\infty} \|c\|_{\ell_\infty}^{-1} c_n \alpha_n x_n \in X.$$

Our aim from this point forward will be to demonstrate that the series $\sum_{n=1}^{\infty} c_n \alpha_n x_n$ converges to a vector in $\left(X \setminus \bigcup_{i \in I} Y_i\right) \cup \{0\}$. To achieve this, we will assume, by way of contradiction, the existence of $y \in \left(\bigcup_{i \in I} Y_i\right) \setminus \{0\}$ such that

$$\sum_{n=1}^{\infty} c_n \alpha_n x_n = y. \tag{3.3}$$

Since $y \in Y_i$ for some $i \in I$, we have

$$Q_i\left(\sum_{n=1}^{\infty} c_n \alpha_n x_n\right) = Q_i(y) = 0.$$

That is,

$$\sum_{n=1}^{\infty} c_n Q_i(\alpha_n x_n) = 0.$$

Since $(Q_i(\alpha_n x_n))_{n=1}^{\infty}$ is ℓ_∞-independent in X/Y_i and $(c_n)_{n=1}^{\infty} \in \ell_\infty$, we get

$$(c_n)_{n=1}^{\infty} = 0.$$

From (3.3), we obtain

$$y = 0,$$

which leads to a contradiction. Therefore, the series $\sum_{n=1}^{\infty} c_n \alpha_n x_n$ converges to a vector in $\left(X \setminus \bigcup_{i \in I} Y_i\right) \cup \{0\}$. This shows that $X \setminus \bigcup_{i \in I} Y_i$ is $[\ell_\infty]$-lineable and by [10], we conclude that $X \setminus \bigcup_{i \in I} Y_i$ is $[\mathcal{S}]$-lineable for each subspace $\mathcal{S}$ of ℓ_∞. For the proof of the converse, it suffices to take $\mathcal{S} = c_{00}$ and utilize the fact that $X \setminus Y$ is $[c_{00}]$-lineable if and only if it is lineable. □

A Kind of Extension

A result due Bernal et al. in [10] states that if A is a subset $[(x_n)_{n=1}^{\infty}, \mathcal{S}]$-lineable of a metrizable vector space X, where $\mathcal{S}$ is an infinite dimensional subspace of $\mathbb{K}^{\mathbb{N}}$ and $(x_n)_{n=1}^{\infty}$ is a basic sequence, (recall that a sequence $(x_n)_{n=1}^{\infty}$of a metrizable topological vector space X is said to be basic whenever every $x \in \overline{\text{span}}\{x_n : n \in \mathbb{N}\}$, the closed linear span of the x_n's, can be uniquely represented as a convergent series $x = \sum_{n=1}^{\infty} c_n x_n$, then A is lineable. More precisely, they showed the following:

Proposition 3.5. *Assume that $(x_n)_{n=1}^{\infty}$ is a basic sequence in X, where X is metrizable, and that A is a $[(x_n)_{n=1}^{\infty}, \mathcal{S}]$-lineable subset of X, where $\mathcal{S}$ is an infinite dimensional subspace of $\mathbb{K}^{\mathbb{N}}$. Then A is lineable.*

In this section, we complement this result by removing the assumption of metrizability and the requirement for the sequence to be basic.

Theorem 3.6. *Let $X \neq \{0\}$ be a Hausdorff topological vector space and A be a nonvoid subset of X. If $(x_n)_{n=1}^{\infty}$ is an ℓ_∞-independent sequence of elements of X and A is $[(x_n)_{n=1}^{\infty}, \mathcal{S}]$-lineable, for some infinite dimensional subspace $\mathcal{S}$ of ℓ_∞, then A is lineable.*

Proof. Let $\mathcal{S}$ be an infinite dimensional subspace of ℓ_∞ and assume that A is $[(x_n)_{n=1}^{\infty}, \mathcal{S}]$-lineable. Since $(x_n)_{n=1}^{\infty}$ is ℓ_∞-independent in X, we can conclude that the operator $T\colon \mathcal{S} \longrightarrow X$ given by $T\left((c_n)_{n=1}^{\infty}\right) = \sum_{n=1}^{\infty} c_n x_n$ is not only well-defined and linear, but also injective. Thus,

$$\dim T(\mathcal{S}) = \dim \mathcal{S} \geq \aleph_0.$$

Furthermore, due to the fact that X is Hausdorff, we have

$$T(\mathcal{S}) \subset A \cup \{0\}.$$

Hence, A is lineable and the proof is complete. □

Negative Results

In this section, we consider results that provide an enlightening insight into the $[\mathcal{S}]$-lineability of subsets in some classes of infinite dimensional vector spaces. The next proposition highlights the importance of sequence properties concerning their distance from the origin in normed vector spaces. Roughly speaking, the next statement shows that sequences away from zero do not generate $[\mathcal{S}]$-lineability.

Proposition 3.7. *Let X be an infinite dimensional normed space. If $(x_n)_{n=1}^{\infty}$ is a linearly independent sequence in X such that $\inf_{n\in\mathbb{N}} \|x_n\|_X > 0$, then for any infinite dimensional subspace $\mathcal{S}$ of ℓ_∞ properly containing c_0, there is no subset of X that is $[(x_n)_{n=1}^{\infty}, \mathcal{S}]$-lineable.*

Proof. Assume that A is a subset of X which is $[(x_n)_{n=1}^{\infty}, \mathcal{S}]$-lineable in X for some infinite dimensional subspace $\mathcal{S}$ of ℓ_∞ properly containing c_0. Given $(c_n)_{n=1}^{\infty} \in \mathcal{S}$, we have

$$\|c_n x_n\|_X = |c_n| \, \|x_n\|_X \geq |c_n| \inf_{n\in\mathbb{N}} \|x_n\|_X \quad \text{for all } n \in \mathbb{N}.$$

That is,

$$\left(\inf_{n\in\mathbb{N}} \|x_n\|_X\right)^{-1} \cdot \|c_n x_n\|_X \geq |c_n| \quad \text{for all } n \in \mathbb{N}. \tag{3.4}$$

Due to (3.4), if we take $(c_n)_{n=1}^{\infty} \in \mathcal{S} \setminus c_0$, we can infer that the sequence $(\|c_n x_n\|_X)_{n=1}^{\infty}$ does not converges to zero. However, this leads to a contradiction, since the series $\sum_{n=1}^{\infty} c_n x_n$ converges in X for some vector of $A \cup \{0\}$. Therefore, the desired result follows. □

The result above remains valid in the context of p-Banach spaces as well. If we consider the sequence $(e_n)_{n=1}^{\infty}$ in ℓ_∞, where $e_n = (0, 0, \ldots, 0, 1, 0, 0, \ldots)$ (with the 1 at the nth place), we obtain the following result:

Corollary 3.8. *For $p \in (0, 1)$ and any infinite dimensional subspace $\mathcal{S}$ of ℓ_∞ properly containing c_0, there is no subset of ℓ_p that is $[(e_n)_{n=1}^{\infty}, \mathcal{S}]$-lineable.*

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