

Universidade Federal da Paraíba
Programa de Pós-Graduação em Matemática
Doutorado em Matemática

Controllability and Observability of Coupled PDE Systems: Boussinesq and Thermoelasticity

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February 14, 2026

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Thesis submitted to the Post-graduate Program of the
Department of Mathematics of the Federal University of
Paraíba in partial fulfillment of the necessary requirements
for the degree of PhD in Mathematics.

João Pessoa - PB

February 14, 2026

Catálogo na publicação
Seção de Catalogação e Classificação

S586c Silva, Hudson Cavalcante da.

Controllability and observability of coupled PDE systems : Boussinesq and thermoelasticity / Hudson Cavalcante da Silva. - João Pessoa, 2026.
140 f.

Orientação: Maurício Cardoso Santos.

Coorientação: Diego Araujo de Souza.

Tese (Doutorado) - UFPB/CCEN.

1. Matemática. 2. Sistema de Boussinesq. 3. Sistema termoelástico linear. 4. Controlabilidade nula local. 5. Posicionamento ótimo sensores. I. Santos, Maurício Cardoso. II. Souza, Diego Araujo de. III. Título.

UFPB/BC

CDU 51(043)



UNIVERSIDADE FEDERAL DA PARAÍBA
CENTRO DE CIÊNCIAS EXATAS E DA NATUREZA
CAMPUS I – DEPARTAMENTO DE MATEMÁTICA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

ATA DE DEFESA DE DOUTORADO JUNTO AO PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA DA UNIVERSIDADE FEDERAL DA PARAÍBA, REALIZADA NO DIA 20 DE JANEIRO DE 2026.

Ao vigésimo dia de janeiro de dois mil e vinte e seis, às 14:00 horas, no Auditório do Departamento de Matemática da Universidade Federal da Paraíba, foi aberta a sessão pública de Defesa de Tese intitulada “**Controlabilidade e observabilidade de sistemas acopladores de EDPs: Boussinesq e Termoelasticidade**”, do aluno **Hudson Cavalcante da Silva**, que havia cumprido, anteriormente, todos os requisitos para a obtenção do grau de Doutor em Matemática, sob a orientação do Prof. Dr. Maurício Cardoso Santos. A Banca Examinadora, indicada pelo Colegiado do Programa de Pós-Graduação em Matemática, foi composta pelos professores: Maurício Cardoso Santos (Orientador), Everaldo Souto de Medeiros (UFPB), Alberto Carlos Mercado Saucedo (UTFSM/Chile), Alessandro Alla (Sapienza/Itália), Felipe Wergete Cruz (UFPE), Lucas Catão de Freitas Ferreira (UNICAMP) e Marko Antonio Rojas Medar (UTA/Chile). O professor Maurício Cardoso Santos, em virtude da sua condição de orientador, presidiu os trabalhos e, depois das formalidades de apresentação, convidou o aluno a discorrer sobre o conteúdo da tese. Concluída a explanação, o candidato foi arguido pela banca examinadora que, em seguida, sem a presença do aluno, finalizando os trabalhos, reuniu-se para deliberar tendo concedido a menção: APROVADO. E, para constar, foi lavrada a presente ata que será assinada pelos membros da Banca Examinadora.

João Pessoa, 20 de Janeiro de 2026.

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Everaldo Souto de Medeiros

Alberto Carlos Mercado Saucedo

Alessandro Alla

Felipe Wergete Cruz

Lucas Catão de Freitas Ferreira

Marko Antonio Rojas Medar

Acknowledgments

Agradeço, primeiramente, a Deus, por me conceder força, saúde e perseverança ao longo de toda essa caminhada, sustentando-me nos momentos de maior dificuldade.

Agradeço ao meu orientador, Dr. Mauricio Cardoso Santos, pela orientação atenta, pela disponibilidade, paciência, pelas discussões produtivas e pelos valiosos ensinamentos ao longo do doutorado. Agradeço também ao meu coorientador, Dr. Diego Araujo de Souza, pelo apoio, pelas contribuições acadêmicas e pela parceria ao longo deste trabalho.

Agradeço aos membros da banca examinadora, pela disponibilidade, pelas valiosas contribuições, sugestões e observações, que certamente enriqueceram este trabalho e contribuíram para o seu aprimoramento.

Agradeço à Universidade Federal do Agreste de Pernambuco (UFAPE), que, por meio da concessão de afastamento institucional, possibilitou a realização do curso de doutorado. Agradeço, de forma especial, aos meus colegas professores de Matemática da UFAPE, pelo suporte, compreensão e incentivo, fundamentais para que essa etapa pudesse ser concluída com êxito. Agradeço também à Universidade Federal da Paraíba (UFPB), em especial ao Programa de Pós-Graduação em Matemática (PPGMat), pelo acolhimento, pela infraestrutura e pelo ambiente acadêmico que tornaram possível o desenvolvimento desta pesquisa.

À minha família, expresso minha mais profunda gratidão. Agradeço aos meus pais, em especial à minha mãe, *in memoriam*, cuja presença, valores e ensinamentos permanecem vivos e foram fundamentais para que eu chegasse até aqui. Aos meus irmãos Alex, Denison e Alessandra, e às minhas cunhadas Iady e Jasiane, pelo apoio, incentivo e compreensão ao longo dessa trajetória. Em especial, agradeço à minha esposa Valéria, pelo amor, paciência e apoio incondicional, e à minha filha Laura Sophia, que é minha maior motivação.

Agradeço aos amigos de doutorado Luiz Felipe, Dennys, Lucas, Gabriel, Milena, Jaislayne, Shirley e Manoel, pela convivência e pelo companheirismo durante essa etapa tão importante da minha formação. Agradeço também aos professores Fagner e Flank, pelos incentivos e pelo apoio ao longo da minha trajetória acadêmica.

Aos meus amigos Everson, Anderson e Anthony, agradeço pelas palavras de apoio, incentivo e amizade, que foram essenciais em diversos momentos. Aos meus amigos de fé Rafaela, Larissa e Jollysson, agradeço pelas orações, pela torcida e pelo encorajamento constante.

Por fim, agradeço à família que ganhei por meio da minha esposa, pelo carinho, apoio e torcida, em especial ao meu cunhado Vitor, à minha sogra Vera e ao meu sogro Walter e a tia dela (minha esposa), Neide. Estendo meus agradecimentos a todos que, de forma direta ou indireta, contribuíram para que esta conquista se tornasse possível.

Resumo

Nesta tese, examinamos dois problemas distintos no âmbito de sistemas de parâmetros distribuídos: um diz respeito ao controle de sistemas, mais precisamente, à questão da controlabilidade nula local, e o outro refere-se à observabilidade e problemas inversos.

Primeiro, estabelecemos a controlabilidade nula local do sistema de Boussinesq 3D utilizando um único controle escalar distribuído atuando exclusivamente na equação da temperatura. Para isso, primeiro linearizamos o sistema em torno de uma trajetória adequada. Inspirados no trabalho clássico de M. Gromov e nas contribuições recentes de J. M. Coron e P. Lissy sobre a equação de Navier-Stokes, empregamos um método algébrico para reduzir sistematicamente o número de controles necessários, aproveitando resultados de controlabilidade conhecidos para sistemas com um número maior de controles. Finalmente, utilizando um teorema da função inversa, provamos a controlabilidade nula local do sistema não linear. Esses resultados demonstram que o sistema de Boussinesq é localmente controlável sem ação direta sobre o campo de velocidade, destacando a força do acoplamento entre as equações térmicas e de dinâmica de fluidos.

Segundo, investigamos o problema da colocação ótima de sensores para um sistema de termoelasticidade linear. Adotando uma abordagem randomizada para os dados iniciais, reduzimos o problema à maximização de um funcional espectral definido pelas autofunções do operador de Laplace e por uma sequência de pesos correspondentes ao inverso dos maiores autovalores de uma certa sequência de matrizes.

Do ponto de vista teórico, analisamos a existência de conjuntos ótimos estudando o hiato de relaxação entre funções características e funções densidade. Demonstramos que, se os pesos espectrais convergem para seu limite por baixo, a existência de um conjunto ótimo clássico é garantida sob a hipótese de Ergodicidade Quântica (QUE) ou sob condições assintóticas de separação mais fracas. Também tratamos casos em que a QUE falha, como no ortotope n -dimensional, provando resultados de existência sob restrições geométricas específicas.

Aplicando esse arcabouço ao sistema termoelástico, realizamos uma análise assintótica

detalhada de altas frequências dos pesos espectrais. Identificamos dois regimes físicos distintos dependendo da velocidade de propagação das ondas e dos parâmetros de acoplamento. Derivamos condições algébricas suficientes explícitas sobre os parâmetros físicos (força de acoplamento, viscosidade, velocidade de onda e horizonte de tempo) que garantem a convergência dos pesos espectrais por baixo, assegurando assim a existência de um domínio ótimo para sensores.

Palavras-chave: Sistema de Boussinesq; Controlabilidade nula local; Controle distribuído; Sistema termoelástico linear; Posicionamento ótimo de sensores; Hipótese QUE.

Abstract

In this thesis, we examine two distinct problems in the context of distributed parameter systems: one concerns system control, more precisely, the issue of local null controllability, and the other concerns observability and inverse problems.

First, we establish the local null controllability of the 3D Boussinesq system using a single distributed scalar control acting solely on the temperature equation. To achieve this, we first linearize the system around a suitable trajectory. Inspired by the classical work of M. Gromov and the recent contributions of J. M. Coron and P. Lissy on the Navier-Stokes equation, we employ an algebraic method to systematically reduce the number of required controls, leveraging known controllability results for systems with more controls. Finally, using an inverse function theorem, we prove the local null controllability of the nonlinear system. These results demonstrate that the Boussinesq system is locally null-controllable without direct action on the velocity field, highlighting the strength of the coupling between the thermal and fluid dynamic equations.

Second, we investigate the optimal sensor placement problem for a linear thermoelasticity system on a bounded domain. Adopting a randomized approach to the initial data, we reduce the problem to the maximization of a spectral functional defined by the eigenfunctions of the Laplacian and a sequence of weights corresponding to the inverse of the maximum eigenvalues of a particular sequence of matrices.

From a theoretical perspective, we analyze the existence of optimal sets by studying the relaxation gap between characteristic functions and density functions. We prove that if the spectral weights converge to their limit from below, the existence of a classical optimal set is guaranteed under Quantum Ergodicity (QUE) assumptions or weaker asymptotic separation conditions. We also address cases where QUE fails, such as the n -dimensional orthotope, proving existence results under specific geometric constraints.

Applying this framework to the thermoelasticity system, we perform a detailed high-frequency asymptotic analysis of the spectral weights. We identify two distinct physical regimes depending on the wave speed and coupling parameters. We derive explicit

algebraic sufficient conditions on the physical parameters (coupling strength, viscosity, wave speed, and time horizon) that ensure the spectral weights converge from below, thereby guaranteeing the existence of an optimal sensor domain.

Keywords: Boussinesq system; Local null controllability;
Distributed control; Thermoelastic system; Optimal sensor placement;
QUE assumption.

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Introduction

Observability, Controllability and Inverse Problems

The interest in understanding the behavior of natural phenomena has led humanity to develop models capable of describing and predicting their dynamics. From this motivation arises a fundamental question: beyond observing these phenomena, is it possible to influence or control them in order to achieve a desired outcome?

With the advancement of mathematics and physics, it became clear that many of these phenomena can be represented by mathematical models, particularly differential equations, which describe the temporal evolution of dynamical systems. This formalization has enabled the use of analytical and numerical tools to study and modify the behavior of such systems, giving rise to the field known as Modern Control Theory.

A central question in this area concerns controllability, the ability to drive a time-evolving system to a predetermined target state via appropriate external intervention, a concept crucial across disciplines ranging from engineering to biology. Intricately linked to this is the dual concept of observability, which ensures that the system's complete internal state can be reconstructed from partial measurements taken over a restricted region or boundary. Beyond its fundamental role in effective control, observability lies at the heart of inverse problems, where the objective shifts from predicting future evolution to determining unknown parameters, initial conditions, or internal sources based solely on observed effects. Unlike direct problems, these inverse formulations require identifying causes from consequences, a frequently ill-posed task that necessitates delicate mathematical techniques, such as Carleman estimates and regularization methods, to ensure rigorous analysis.

Thus, observability, controllability, and inverse problems form a set of complementary approaches that allow not only the understanding but also the influence and reconstruction of the behavior of complex systems described by differential equations. They constitute the theoretical foundation for modeling, intervention, and identification of real

phenomena, reinforcing the central role of mathematics at the interface between theory and applications.

The main goal of this thesis is to study the notion of controllability and observability for certain problems arising in the theory of partial differential equations (PDEs). In broad terms, we investigate under which conditions it is possible, through appropriate control actions, to influence the evolution of systems governed by PDEs in order to reach specific configurations or trajectories.

From a mathematical point of view, **observability** is often expressed by inequalities of the type

$$\|u(0)\|_H^2 \leq C \int_0^T \int_{\omega} |u(x, t)|^2 dx dt,$$

where H is the natural energy space of the system and $\omega \subset \Omega$ represents the observation region. This inequality guarantees that the total initial energy of the system can be estimated from partial measurements, ensuring the possibility of reconstructing the complete state from limited observations.

Controllability refers to the ability to drive the system, through distributed or boundary controls, from an initial state u_0 to a desired final state u_T . For linear systems of the form

$$\partial_t u = Au + Bf,$$

exact controllability is achieved when there exists a control $f \in L^2(0, T; U)$ such that

$$u(T) = u_T.$$

Mathematically, exact controllability is equivalent to the observability of the adjoint equation, a relation formalized by the *Hilbert Uniqueness Method* (HUM).

Inverse problems aim to determine unknown parameters, sources, or initial conditions from observed data, reversing the usual direction of the modeling process. In general, these problems are ill-posed, meaning that small perturbations in the observed data can cause large variations in the solution. Analytical tools such as *Carleman estimates* provide stability inequalities of the form

$$\|u_0\| \leq Ce^{C/\varepsilon} \|\text{observed data}\|,$$

and ensure uniqueness conditions for parameter reconstruction. Regularization techniques, variational formulations, and optimization methods are typically employed to stabilize numerical solutions.

In general, the control problem consists of determining a control function capable of making the system evolve according to a desired behavior, driving the state toward a pre-established configuration. Depending on the objective and the imposed constraints, different notions of controllability can be defined. The optimal control problem, for instance, seeks a control that minimizes a cost functional, taking into account both the proximity to the desired final state and the effort associated with the control.

Exact controllability refers to the ability to steer the system from an initial state to a final state within a finite time interval. A weaker notion, called approximate controllability, requires only that the final state can be made arbitrarily close to the target. Null controllability aims to drive the system to the zero state, while controllability along trajectories seeks to follow a pre-established reference trajectory. These notions form the analytical foundation of modern control theory, providing the necessary tools to study active intervention in dynamical systems, particularly in partial differential equations, whose analysis is naturally more complex due to the infinite-dimensional nature of the state spaces and the presence of dissipative or dispersive effects.

As examples of applications involving controllability, observability, and inverse problems, we may refer, for instance, to the following works: [5], [13],[22], [41], [42] and [53].

To provide a concrete application of these general concepts, this thesis investigates two specific problems situated at the interface of control theory and inverse problems. The first part is dedicated to the controllability of fluid dynamics, specifically addressing the Boussinesq system. The primary objective here is to establish local null controllability using a reduced number of controls, solely acting on the temperature equation, thereby exploiting the physical coupling between the thermal and velocity fields. The second part shifts focus to the optimal design of sensors for a system of linear thermoelasticity. In this context, we adopt a probabilistic framework to determine the optimal observation domain that maximizes the observability constant, relying on a detailed spectral analysis of the coupled hyperbolic-parabolic dynamics.

Controllability for Boussinesq system with a single thermal control

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary $\partial\Omega$, and let ω be a nonempty open subset of Ω . For a given final time $T > 0$, and using the following notations $Q_T := (0, T) \times \Omega$ and $\Sigma_T := (0, T) \times \partial\Omega$.

We will consider the following controlled Boussinesq system

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla)y + \nabla p = \theta e_3 & \text{in } Q_T, \\ \theta_t - \Delta \theta + y \cdot \nabla \theta = \chi_\omega v & \text{in } Q_T, \\ \nabla \cdot y = 0 & \text{in } Q_T, \\ y = 0, \theta = 0 & \text{on } \Sigma_T, \\ y(x, 0) = y_0(x), \theta(x, 0) = \theta_0(x) & \text{in } \Omega, \end{cases} \quad (1)$$

where (y, θ) represents the state of the fluid, with y being the velocity and θ the temperature, while p is the associated pressure. The pair (y_0, θ_0) denotes the initial state. The control action is given by the term $\chi_\omega v$, where χ_ω is a fixed, non-negative smooth function with compact support contained in ω (that is, $\chi_\omega \in C_0^\infty(\omega)$). The function $v = v(x, t)$ is the control to be determined, and $e_3 = (0, 0, 1)$ is the third vector of the canonical basis of $\mathbb{R}^3$.

In this thesis, one of our objectives is to study the null controllability of the Boussinesq system under a single thermal control.

The Boussinesq system is a fundamental model in fluid dynamics used to describe fluid motion caused by temperature differences. Essentially, the model combines the Navier-Stokes equations, which describe the fluid's velocity and pressure, with a heat equation for the temperature. The link between them is the buoyancy effect: as the temperature changes, so does the fluid's density, causing warmer, lighter fluid to rise and cooler, denser fluid to sink. The model is designed to capture this gravity-driven motion.

Because it describes this process well, the Boussinesq system is widely applied in many fields. For example, it is used in oceanography to understand sea currents, in meteorology to predict weather patterns, and in geophysics to study the Earth's mantle convection. For a more detailed mathematical study of the model, the reader can consult standard reference books in the field, such as [2, 28].

Beyond its theoretical significance, establishing controllability results for the Boussinesq system has profound practical implications. The ability to steer or stabilize heat-driven flows is a central challenge in many areas of engineering and environmental science. In advanced manufacturing, for example, controlling the Boussinesq dynamics is essential for processes like semiconductor crystal growth, where suppressing natural convection is necessary to avoid imperfections and ensure material homogeneity [27, 21]. In energy and thermal management, the goal is often the opposite: to enhance convective flows to maximize the efficiency of heat exchangers or to ensure the safe cooling of critical components, a topic extensively detailed in foundational texts on heat transfer [23]. Furthermore, understanding the control mechanisms of such systems offers insights into managing en-

vironmental phenomena, such as the dispersion of pollutants in lakes or coastal waters. Thus, developing a robust control theory for the Boussinesq system is a crucial step towards creating effective strategies for these diverse applications.

The controllability of the Boussinesq system has been the subject of extensive research, primarily focusing on local results under various boundary conditions. Key early contributions include the work of Fursikov and Imanuvilov, who established local exact boundary controllability with controls on the entire boundary in [14] and analyzed exact controllability with distributed controls under periodic conditions in [15]. For the widely studied Dirichlet boundary conditions, Guerrero [19] proved local exact controllability to trajectories. This line of research was further advanced by reducing the number of necessary controls, as demonstrated in seminal works such as Carreño [3] and in [11, 16]. The inviscid case has also been explored by Fernández-Cara, Santos, and Souza in [12].

Regarding the more specific case of Navier-slip boundary conditions, the literature is more recent. For the Navier-Stokes system, a significant breakthrough was achieved in [6], providing a positive answer to Lions' well-known open question on small-time global exact controllability; this result was recently extended to smooth solutions in [24]. Earlier works include the small-time global result for the 2D case by Coron [4] and the local controllability to trajectories with general nonlinear Navier conditions by Guerrero [18]. For the Boussinesq system with these boundary conditions, we highlight the local results obtained in [22, 26].

A central theme in the control of coupled systems is the possibility of steering the entire state by acting on fewer equations than the system possesses. This 'control reduction' problem is fundamental, as minimizing the number of actuators has significant practical implications.

A pivotal contribution in this direction for fluid dynamics was made by Coron and Lissy in [5]. They established the local null controllability of the 3D Navier-Stokes system using a distributed control with only one non-vanishing component. To do so, they introduced a powerful algebraic method, inspired by the work of Gromov, to demonstrate how a single control could indirectly influence all components of the system.

Recent progress has been made in reducing the number of controls for the Boussinesq system, as demonstrated in the seminal works of Carreño [3] and in [11, 16]. However, a notable feature of these results, and indeed of all previous works on this topic, is that the control mechanism always retains at least one component acting directly on the fluid's velocity field. In this paper, we demonstrate for the first time that such direct intervention is not necessary. We establish the local null controllability of the system using a single scalar control that acts exclusively on the temperature equation. This finding is particularly significant from a practical standpoint, as it demonstrates that the entire

fluid state can be driven to rest by only manipulating its temperature. This highlights a powerful indirect control mechanism inherent in the system's physical coupling.

Optimal Shape and Location of Sensors for Linear Thermoelasticity

The optimal design of sensors for linear partial differential equation systems (PDEs) is an important research topic. The goal is to determine the optimal location of sensors to monitor and control systems governed by linear PDEs. The appropriate choice of location and number of sensors can significantly impact the accuracy and efficiency of the monitoring and control system.

In this sense, the optimal design of sensors problem can be formulated as follows:

- Determine the optimal location: find the optimal location of sensors to minimize or maximize an objective function, such as the accuracy of measurements or the efficiency of the control.
- Satisfy constraints: Satisfy physical and practical constraints, such as limiting the number of sensors or accessibility to certain regions of the system.

The optimal design of sensors for linear PDE systems has applications in several areas, for example in process control, such as material production and energy generation, in environmental monitoring, including air and water quality assessment, and in aerospace engineering, involving the monitoring and control of aeronautical and space systems, among others.

In the context of observability inequalities, the optimal sensor design problem can be formulated as follows: Find the best observation region ω such that the observability constants are minimized.

Observability inequalities are fundamental tools for establishing the observability of a system, meaning that its state can be reconstructed from a set of measurements. The associated observability constants characterize the quality of this observability; consequently, minimizing them is crucial to ensure an accurate reconstruction of the system's state. For more details, see [46], [47], [48], [49] and [50].

In this sense, we consider a linear system of thermoelasticity defined on a bounded open subset $\Omega \subset \mathbb{R}^N$. The system consists of coupled wave and heat equations given by:

$$\begin{cases} u_{tt} - c^2 \Delta u + \alpha \Delta \theta = 0 & \text{in } \Omega \times (0, T), \\ \theta_t - \nu \Delta \theta + \beta u_t = 0 & \text{in } \Omega \times (0, T), \\ u = \theta = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u^0, u_t(0) = u^1, \theta(0) = \theta^0 & \text{in } \Omega, \end{cases} \quad (2)$$

where $T > 0$ is the final time, $\partial\Omega$ denotes the boundary of Ω , and α, β, c, ν are positive real numbers representing the coupling parameters, wave speed, and thermal diffusivity, respectively.

For system (2), we address the problem of determining the optimal shape and location of sensors to measure the signals propagating in Ω over a given time horizon. The objective is to place these sensors in a configuration that allows for the reconstruction of the solutions in the most efficient way possible (i.e., optimizing the well-posedness of the inverse problem).

To address the inverse problem of reconstructing solutions from partial measurements, a standard mathematical strategy involves establishing a suitable observability inequality. For the system (2), such an inequality generally takes the form:

$$\|(u, \theta)(T)\|_X^2 \leq A_0 \|u\|_{X_{0,\omega}}^2 + A_1 \|\theta\|_{X_{1,\omega}}^2, \quad (3)$$

where X and $X_{i,\omega}$ ($i = 0, 1$) are appropriate Banach spaces. Typically, X corresponds to the energy space (e.g., a Sobolev space $H^s(\Omega)$), while the observation spaces are often $X_{i,\omega} = L^2(\omega \times (0, T))$. The coefficients A_0 and A_1 are non-negative constants depending on the physical parameters α, β, c, ν, T and the geometry of ω , satisfying the condition $A_0 + A_1 \neq 0$.

In the context of observability inequalities, the problem of optimizing the sensor shape and location reduces to finding the "best" domain ω such that the constants A_0 and A_1 are minimized. This is because these constants effectively quantify the well-posedness of the corresponding inverse problem. Specifically, the smaller the observability constants in (3), the more stable and better-conditioned the reconstruction of the solution becomes.

Generally, determining the optimal shape and location of sensors is a challenging task, particularly when considering fixed initial data. Indeed, expanding the solution of (2) in an orthonormal basis introduces cross-terms on the right-hand side of (3), rendering the corresponding optimization problem highly complex and difficult to solve (see [42]). Moreover, in many practical scenarios, the exact initial data is unknown, and reconstruction is typically performed using a large set of measurements. Consequently, to robustly model the observation problem, it is natural to search for optimal sets that solve the problem on average.

In the literature, various approaches address the problem of optimal sensor placement. Applied studies in physics and engineering often focus on finite-dimensional optimization derived from the discretization of continuous models, for instance [30, 32]. However, this approach can suffer from numerical instabilities such as the *spillover* phenomenon. Alternatively, some authors circumvent these issues by restricting the admissible sets

to those satisfying strict geometric constraints, such as a finite number of sensors with prescribed shapes [31, 34, 37, 45]. From a mathematical perspective, relaxed formulations have been extensively studied, particularly for the optimal design of actuators for the heat equation [47, 29], as well as numerical investigations [38, 39]. However, significant challenges arise when considering fixed initial data: for the wave equation, the optimal shape problem can yield highly complex solutions [43] or exhibit spillover [44, 33].

In this work, we will also focus on the problem of optimal sensor placement for the system (2), assuming that the initial data to be reconstructed are not fully known. To address this, we model the data as random, meaning that their coefficients in a given basis are treated as random variables. This probabilistic framework is motivated by several key results in the literature.

Existing literature reveals a striking contrast between hyperbolic and parabolic equations regarding shape optimization. For the wave equation, as shown in [41], the optimal sensor placement problem is generally ill-posed in the class of characteristic functions. A relaxation gap typically exists, meaning the optimal solution is a density function taking values in $[0, 1]$ rather than a classical set. Exceptions exist only in specific scenarios, such as the damped wave equation [40]. Conversely, for the heat equation, [42] proves that the problem is well-posed: an optimal classical set always exists.

This dichotomy makes the analysis of the thermoelasticity system (2) particularly intriguing, as it couples hyperbolic and parabolic dynamics. Determining whether the optimization problem admits a classical solution is therefore non-trivial and depends heavily on which dynamic dominates the observation.

As we will demonstrate, under the randomized framework, the problem of maximizing the observability constant reduces to analyzing the spectral functional:

$$J(\chi_\omega) = \inf_{i \in \mathbb{N}^*} \gamma_i \int_{\omega} |\phi_i(x)|^2 dx, \quad (4)$$

where the weights γ_i are derived from the system's spectral properties. The existence of an optimal set is governed by the asymptotic monotonicity of this sequence. Specifically, convergence from below typically favors attainability. In contrast, the case of convergence from above is more intricate; while no general existence or non-existence result is available, we demonstrate that attainability still holds in certain specific scenarios.

Consequently, before addressing the specific problem for the thermoelasticity system, we dedicate Section 3.2 to a rigorous study of the abstract functional J . We investigate sufficient conditions on the sequence (γ_i) and the eigenfunctions (ϕ_i) that guarantee the existence of optimal sets, thereby providing the theoretical foundation for our subsequent physical analysis.

Preliminaries and Fundamental Results

We present some fundamental results that will be used throughout this thesis. These theorems and propositions establish the analytical and functional foundation necessary for the theoretical development and for the study of the problems investigated in the following chapters.

A central step in our analysis relies on a Carleman estimate adapted to the linearized Navier–Stokes system. This type of inequality provides a weighted control of the solutions and plays a relevant role in establishing observability properties for fluid systems. In the context of the present work, this result is particularly important, as it serves as the foundation for deriving a new Carleman estimate for the Boussinesq system. For reference, we recall below the estimate originally proved in [20, Proposition 3.1], which will be essential for the development of one of the main results of this thesis.

Proposition 1.1 (Carleman Estimate for the Navier–Stokes System) *Let Ω be a bounded domain of $\mathbb{R}^n$. Define $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$. For any open subset ω_0 of Ω , there exists a positive constant $C > 0$, depending only on Ω and ω_0 , such that*

$$I(s, \lambda; \nabla \times \psi) \leq Cs^3\lambda^4 \int_{\omega_0 \times (0, T)} e^{-2s\alpha} e^{-2s\alpha^*} \xi^3 |\nabla \times \psi|^2 dx dt + C \int_Q e^{-2s\alpha} |g_1|^2 dx dt,$$

for any $\lambda \geq C$, $s \geq C(T^5 + T^{10})$, and for every ψ satisfying the system

$$\begin{cases} \psi_t - \Delta \psi + \nabla \kappa = g_1, & \text{in } Q, \\ \nabla \cdot \psi = 0, & \text{in } Q, \\ \psi = 0, & \text{on } \Sigma, \\ \psi|_{t=0} = \psi_0, & \text{in } \Omega. \end{cases}$$

Here, the functional $I(s, \lambda; u)$ is defined by

$$I(s, \lambda; u) = s^3 \lambda^4 \int_Q e^{-2s\alpha} e^{-2s\alpha^*} \xi^3 |u|^2 dx dt + s \lambda^2 \int_Q e^{-2s\alpha} e^{-2s\alpha^*} \xi |\nabla u|^2 dx dt.$$

The next result allows us to uniquely represent certain continuous linear functionals defined on the underlying Hilbert space, thus playing a punctual yet essential role in our analysis. In particular, it ensures the precise identification of the elements required for the formulation of the control terms of the problem. For completeness and to clarify its specific use within our framework, we recall its classical statement below.

Theorem 1.2 (Riesz Representation Theorem) *Let H be a Hilbert space over $\mathbb{R}$ or $\mathbb{C}$, with inner product $\langle \cdot, \cdot \rangle$. Then for every continuous linear functional $f : H \rightarrow \mathbb{K}$ (with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), there exists a unique element $y \in H$ such that*

$$f(x) = \langle x, y \rangle, \quad \text{for all } x \in H.$$

Moreover, the norm of f satisfies

$$\|f\| = \|y\|.$$

The proof can be found in [51].

Another important ingredient in our analysis concerns the regularity of the control terms associated with the Boussinesq system. In order to ensure that the region where the control acts can be treated within functional spaces compatible with both the Carleman estimates and the algebraic resolvability established previously, we rely on a classical interpolation result in Hilbert spaces. This result provides a rigorous framework for combining different levels of regularity in a coherent manner and guarantees that the control functions attain the functional precision required for the development of our approach.

To introduce the functional setting used in what follows, and for the sake of completeness, we recall the definition of the interpolation spaces involved. More precisely, for $\theta \in (0, 1)$, the space $[D(A_0) \cap D(A_1), X]_\theta$ is defined, in the sense of Lions–Magenes, as the domain of the fractional power $A^{1-\theta}$, where A is a positive self-adjoint operator in X whose domain coincides with $D(A_0) \cap D(A_1)$, endowed with the corresponding graph norm.

Theorem 1.3 *Let X be a separable Hilbert space, and let A_0 and A_1 be two self-adjoint, strictly positive, and commuting operators on X . Denote by $D(A_i)$, $i = 0, 1$, the domain of A_i , equipped with the corresponding graph norm.*

Then, for any $\theta \in (0, 1)$, we have

$$[D(A_0) \cap D(A_1), X]_\theta = [D(A_0), X]_\theta \cap [D(A_1), X]_\theta,$$

with equivalence of norms.

The proof of this theorem can be found in [25, Theorem 13.1].

Remark 1.4 As proved in Section 13.2 of [25], the following application of Theorem 1.3 proves particularly useful when dealing with functional spaces defined over product domains, such as $\mathbb{R}_x^n \times \mathbb{R}_y^m$ (or $Q_x \times Q_y$, where $Q_x \subset \mathbb{R}_x^n$ and $Q_y \subset \mathbb{R}_y^m$ are open sets). In these cases, the variables may exhibit distinct properties.

Such situations commonly arise in the study of evolution equations, quasi-elliptic equations, and more broadly in problems involving non-elliptic differential operators. To illustrate this, the authors provide a simple example.

On the space $\mathbb{R}_x^n \times \mathbb{R}_y^{n'}$, define

$$H_{x,y}^{s,\sigma} = \left\{ u \mid [(1 + |\xi|^2)^{s/2} + (1 + |\eta|^2)^{\sigma/2}] \hat{u} \in L^2(\mathbb{R}_{\xi,\eta}^{n+n'}) \right\},$$

where the Fourier transform $\hat{u}$ is given by

$$\hat{u}(\xi, \eta) = \frac{1}{(2\pi)^{(n+n')/2}} \int_{\mathbb{R}_x^n \times \mathbb{R}_y^{n'}} e^{-i(x \cdot \xi + y \cdot \eta)} u(x, y) dx dy.$$

Note that

$$H_{x,y}^{0,0} = H_{x,y}^0 = L_{x,y}^2(\mathbb{R}_{x,y}^{n+n'}).$$

It follows directly from Theorem 1.3 that, for non-negative indices s_i and σ_i , the following interpolation identity holds:

$$[H_{x,y}^{s_1,\sigma_1}, H_{x,y}^{s_2,\sigma_2}]_\theta = H_{x,y}^{(1-\theta)s_1+\theta s_2, (1-\theta)\sigma_1+\theta\sigma_2}.$$

The Inverse Mapping Theorem plays a decisive role in our analysis. After establishing null controllability for the linearized system, we invoke this classical result from functional analysis to extend such a property to the fully nonlinear Boussinesq system. In broad terms, the theorem ensures that whenever the associated linearized operator is surjective, the nonlinear mapping becomes locally invertible, allowing us to solve the controllability problem for sufficiently small perturbations.

We now state precisely the theorem that will be used.

Theorem 1.5 (Inverse Mapping Theorem) *Let E and F be two Banach spaces, and let $\mathcal{A} : E \rightarrow F$ be a C^1 map in a neighborhood of $e_0 \in E$. Assume that the Fréchet derivative $\mathcal{A}'(e_0) \in \mathcal{L}(E; F)$ is a surjective (onto) operator.*

Then, there exist positive constants $\eta > 0$ and $C > 0$ such that for every $f \in F$ verifying $\|f - f_0\|_F < \eta$, where $f_0 = \mathcal{A}(e_0)$, there exists $e \in E$ satisfying:

- a) $\mathcal{A}(e) = f$;
- b) $\|e - e_0\|_E \leq C\|f - f_0\|_F$.

For more details, see, e.g., [1, Chapter 2, p. 107].

An additional compactness result that plays a structural role in our analysis is the Banach–Alaoglu Theorem. It ensures the *weak** compactness of bounded sets in dual spaces, which is essential whenever we need to extract convergent subsequences from families of controls or uniformly bounded functions. This compactness property provides the stability required to pass to the limit in several functional arguments used throughout the work.

We recall its statement below for completeness.

Theorem 1.6 (Banach–Alaoglu Theorem) *Let X be a normed vector space and denote by X' its dual space. Then the closed unit ball*

$$B_{X'} := \{f \in X' : \|f\| \leq 1\}$$

is compact in the weak- topology $\sigma(X', X)$.*

Moreover, if X is separable, then $B_{X'}$ is metrizable in the weak- topology. The proof can be found in [55].*

In addition to the previously mentioned results, two further theorems play a significant role in the analysis developed in this work. In particular, they will be used in the study of the truncated problem associated with (4), allowing us to properly characterize the structure of optimal solutions. These results are stated as follows:

Theorem 1.7 (Sion’s Minimax Theorem) *Let X be a convex subset of a linear topological space, and let Y be a compact convex subset of a linear topological space. Let $f : X \times Y \rightarrow \mathbb{R}$ be a real-valued function such that:*

- a) *for every fixed $y \in Y$, the function $f(\cdot, y)$ is upper semicontinuous and quasi-concave on X ;*

b) for every fixed $x \in X$, the function $f(x, \cdot)$ is lower semicontinuous and quasi-convex on Y .

Then,

$$\sup_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \sup_{x \in X} f(x, y).$$

For further details and proofs, see, e.g., [56, 57].

Theorem 1.8 (Bathtub Principle) Let (Ω, Σ, μ) be a measure space and let $f : \Omega \rightarrow \mathbb{R}$ be a measurable function such that

$$\mu(\{x \in \Omega : f(x) < t\}) < \infty, \quad \forall t \in \mathbb{R}.$$

Let $G > 0$ be a given number and define the class of measurable functions

$$\mathcal{C} = \left\{ g : \Omega \rightarrow [0, 1] \text{ measurable} \mid \int_{\Omega} g(x) \mu(dx) = G \right\}.$$

Consider the minimization problem

$$I = \inf_{g \in \mathcal{C}} \int_{\Omega} f(x) g(x) \mu(dx).$$

Then this problem is solved by the function

$$g(x) = \chi_{\{f < s\}}(x) + c \chi_{\{f = s\}}(x),$$

where

$$s = \sup\{t \in \mathbb{R} : \mu(\{x \in \Omega : f(x) < t\}) \leq G\},$$

and the constant $c \in [0, 1]$ is determined by

$$c \mu(\{x \in \Omega : f(x) = s\}) = G - \mu(\{x \in \Omega : f(x) < s\}).$$

The proof of this theorem can be found in (see, e.g., [36, Theorem 1.14]).

Finally, another result that plays an important analytical role in this work is the Riemann–Lebesgue Lemma. It will be used to guarantee the decay of certain Fourier coefficients associated with the functions under consideration, which is relevant for asymptotic estimates and approximation arguments throughout the analysis. Its statement is as follows:

Lemma 1.9 (Riemann–Lebesgue Lemma) Let $f \in L^1(\mathbb{R})$. Then its Fourier

transform

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} dx$$

satisfies

$$\lim_{|\xi| \rightarrow \infty} \widehat{f}(\xi) = 0.$$

Equivalently, if $f \in L^1([0, \pi])$, then the Fourier coefficients

$$a_n = \int_0^\pi f(x) \cos(nx) dx, \quad b_n = \int_0^\pi f(x) \sin(nx) dx$$

satisfy

$$\lim_{n \rightarrow \infty} a_n = 0, \quad \lim_{n \rightarrow \infty} b_n = 0.$$

For a proof, see, e.g., [52].

In the following, we present a summary of each chapter, detailing the control and observability problems addressed in this thesis. To this end, we outline the two works that constitute this study in the following order:

Chapter 2

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary $\partial\Omega$, and let ω be a nonempty open subset of Ω . For a given final time $T > 0$, we use the notations $Q_T := (0, T) \times \Omega$ and $\Sigma_T := (0, T) \times \partial\Omega$.

We are concerned with the controlled Boussinesq system:

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla)y + \nabla p = \theta e_3 & \text{in } Q_T, \\ \theta_t - \Delta \theta + y \cdot \nabla \theta = \chi_\omega v & \text{in } Q_T, \\ \nabla \cdot y = 0 & \text{in } Q_T, \\ y = 0, \theta = 0 & \text{on } \Sigma_T, \\ y(x, 0) = y_0(x), \theta(x, 0) = \theta_0(x) & \text{in } \Omega, \end{cases} \quad (1.1)$$

where (y, θ) represents the state of the fluid, with y being the velocity and θ the temperature, while p is the associated pressure. The pair (y_0, θ_0) denotes the initial state. The control action is given by the term $\chi_\omega v$, where χ_ω is a fixed, non-negative smooth function with compact support contained in ω (that is, $\chi_\omega \in C_0^\infty(\omega)$). The function $v = v(x, t)$ is the control to be determined, and $e_3 = (0, 0, 1)$ is the third vector of the canonical basis of $\mathbb{R}^3$.

We first introduce the function space for divergence-free fields, which is standard in fluid mechanics:

$$V := \{w \in H_0^1(\Omega)^3 \mid \nabla \cdot w = 0 \text{ in } \Omega\}.$$

The main result of this paper concerns the local null controllability for the system (1.1). More precisely, we have the following result:

Theorem 1.10 *For every $T > 0$, there exists $\eta > 0$ such that for any initial data $y_0 \in V$ and $\theta_0 \in H_0^1(\Omega)$ verifying*

$$\|y_0\|_V^2 + \|\theta_0\|_{H_0^1(\Omega)}^2 \leq \eta,$$

there exists a scalar control $v \in L^2(Q_T)$ and a corresponding solution (y, θ) with regularity

$$\begin{aligned} y &\in L^2(0, T; H^2(\Omega)^3 \cap V) \cap L^\infty(0, T; V), \\ \theta &\in L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap L^\infty(0, T; H_0^1(\Omega)), \end{aligned}$$

that, together with some pressure $p \in L^2(Q_T)$, solves system (1.1) and satisfies the null controllability condition:

$$y(T, \cdot) = 0 \quad \text{and} \quad \theta(T, \cdot) = 0 \quad \text{in } \Omega. \quad (1.2)$$

The proof of Theorem 1.10 is obtained by following the strategies below:

Linearization and the Return Method. Since system (1.1) is nonlinear, our strategy relies on an inverse mapping theorem. This requires first proving the null controllability of a linearized system. We employ the "Return Method," whose goal is to find a specific trajectory $(\bar{y}, \bar{\theta}, \bar{p}, \bar{v})$ of (1.1) such that the system linearized around it is null-controllable. This linearized system, with state (y, θ, p) , is given by:

$$\begin{cases} y_t - \Delta y + (\bar{y} \cdot \nabla)y + (y \cdot \nabla)\bar{y} + \nabla p = \theta e_3 + F & \text{in } Q_T, \\ \theta_t - \Delta \theta + \bar{y} \cdot \nabla \theta + y \cdot \nabla \bar{\theta} = f + \chi_\omega v & \text{in } Q_T, \\ \nabla \cdot y = 0 & \text{in } Q_T, \\ y = 0, \theta = 0 & \text{on } \Sigma_T, \\ y(0) = y_0, \theta(0) = \theta_0 & \text{in } \Omega, \end{cases} \quad (1.3)$$

where the source terms F and f are assumed to have a suitable exponential decay as $t \rightarrow T^-$.

Algebraic Solvability. To prove the null controllability of the linearized system (1.3) with a single control v , we use the concept of "algebraic solvability". This strategy splits the problem into two parts:

1. An **analytic part**, where we prove the existence of a control u that solves the

null controllability problem for a system with full actuation:

$$\begin{cases} y_t^1 - \Delta y^1 + (\bar{y} \cdot \nabla) y^1 + (y \cdot \nabla) \bar{y}^1 + \partial_{x_1} p = F_1 + \mathfrak{B}^1 u, & \text{in } Q_T, \\ y_t^2 - \Delta y^2 + (\bar{y} \cdot \nabla) y^2 + (y \cdot \nabla) \bar{y}^2 + \partial_{x_2} p = F_2 + \mathfrak{B}^2 u, & \text{in } Q_T, \\ y_t^3 - \Delta y^3 + (\bar{y} \cdot \nabla) y^3 + (y \cdot \nabla) \bar{y}^3 + \partial_{x_3} p = F_3 + \theta + \mathfrak{B}^3 u, & \text{in } Q_T, \\ \theta_t - \Delta \theta + \bar{y} \cdot \nabla \theta + y \cdot \nabla \bar{\theta} = f + \mathfrak{B}^4 u, & \text{in } Q_T, \\ \nabla \cdot y = 0, & \text{in } Q_T, \\ y = 0, \theta = 0 & \text{on } \Sigma_T, \\ y(0) = y_0, \theta(0) = \theta_0 & \text{in } \Omega, \end{cases} \quad (1.4)$$

Here, $\mathfrak{B} = (\mathfrak{B}^1, \mathfrak{B}^2, \mathfrak{B}^3, \mathfrak{B}^4)$ is a carefully chosen control operator.

2. An **algebraic part**, where we show that for the control u found above, we can find a new control $\tilde{v}$ such that the following system, with zero initial data, has a solution:

$$\begin{cases} y_t^1 - \Delta y^1 + (\bar{y} \cdot \nabla) y^1 + (y \cdot \nabla) \bar{y}^1 + \partial_{x_1} p = -\mathfrak{B}^1 u, & \text{in } Q_T, \\ y_t^2 - \Delta y^2 + (\bar{y} \cdot \nabla) y^2 + (y \cdot \nabla) \bar{y}^2 + \partial_{x_2} p = -\mathfrak{B}^2 u, & \text{in } Q_T, \\ y_t^3 - \Delta y^3 + (\bar{y} \cdot \nabla) y^3 + (y \cdot \nabla) \bar{y}^3 + \partial_{x_3} p = \theta - \mathfrak{B}^3 u, & \text{in } Q_T, \\ \theta_t - \Delta \theta + \bar{y} \cdot \nabla \theta + y \cdot \nabla \bar{\theta} = -\mathfrak{B}^4 u + \tilde{v}, & \text{in } Q_T, \\ \nabla \cdot y = 0, & \text{in } Q_T, \\ y = 0, \theta = 0 & \text{on } \Sigma_T, \\ y(0) = 0, \theta(0) = 0 & \text{in } \Omega, \end{cases} \quad (1.5)$$

The null controllability for (1.3) then follows by linearity, summing the solutions of (1.4) and (1.5).

Observability for the Adjoint System. To address the analytic part (system (1.4)), we use duality. It is well known that its null controllability is equivalent to an observability inequality for the corresponding adjoint system:

$$\begin{cases} -\varphi_t - \Delta \varphi - D\varphi \bar{y} + \nabla \pi = G + \bar{\theta} \nabla \psi & \text{in } Q_T, \\ -\psi_t - \Delta \psi - \bar{y} \nabla \psi = g + \varphi \cdot e_3 & \text{in } Q_T, \\ \nabla \cdot \varphi = 0 & \text{in } Q_T, \\ \varphi = 0, \psi = 0 & \text{on } \Sigma_T, \\ \varphi(T) = \varphi^T, \quad \psi(T) = \psi^T & \text{in } \Omega, \end{cases} \quad (1.6)$$

where $D\varphi = \nabla \varphi + \nabla \varphi^t$. The desired observability inequality will be a consequence

of a global Carleman inequality for (1.6) (see Lemma 2.10).

Conclusion via Inverse Mapping Theorem. Finally, once the null controllability for the nonhomogeneous linearized system (1.3) is established, we apply an inverse mapping theorem to deduce the local null controllability for the original nonlinear system (1.1).

Chapter 3

Consider a linear thermoelastic system defined on a bounded open subset $\Omega \subset \mathbb{R}^N$. The system consists of a wave equation coupled with a heat equation, given by

$$\begin{cases} u_{tt} - c^2 \Delta u + \alpha \Delta \theta = 0 & \text{in } \Omega \times (0, T), \\ \theta_t - \nu \Delta \theta + \beta u_t = 0 & \text{in } \Omega \times (0, T), \\ u = \theta = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u^0, u_t(0) = u^1, \theta(0) = \theta^0 & \text{in } \Omega, \end{cases} \quad (1.7)$$

where $T > 0$ is the final time, $\partial\Omega$ denotes the boundary of Ω , and α, β, c, ν are positive constants representing the coupling parameters, the wave speed, and the thermal diffusivity.

We aim to study the optimal sensor placement problem for this system; that is, to determine which regions of Ω provide the best capacity for observing the thermoelastic dynamics. As will be explained throughout the text, this problem can be reduced to the following optimization problem:

$$\sup_{\chi_\omega \in \mathcal{U}_L} J(\chi_\omega), \quad (1.8)$$

where

$$J(\chi_\omega) = \inf_{i \in \mathbb{N}^*} \gamma_i \int_{\omega} |\phi_i(x)|^2 dx, \quad (1.9)$$

with $(\phi_j)_{j \in \mathbb{N}^*}$ being a Hilbert basis of $L^2(\Omega, \mathbb{C})$, consisting of eigenfunctions of the Laplace operator Δ with homogeneous Dirichlet boundary conditions.

The maximization is performed over the set of admissible characteristic functions

$$\mathcal{U}_L = \{ \chi_\omega \mid \omega \subset \Omega \text{ is measurable and satisfies } |\omega| = L|\Omega| \}, \quad (1.10)$$

for a fixed parameter $L \in (0, 1)$, representing the fraction of the domain where the sensors may be placed.

Through a detailed asymptotic expansion in the high-frequency regime, we character-

ized the behavior of the thermoelastic weights as

$$\gamma_j = \gamma^* \left(1 - \frac{R}{\omega_j^k} + o\left(\frac{1}{\omega_j^k}\right) \right),$$

where the exponent k and the coefficient R depend on the physical parameters α, β, c, ν and on the time horizon T .

Moreover, under assumptions of Quantum Ergodicity on the eigenfunctions, we investigate the existence of an optimal sensor domain for the thermoelastic problem. Even in geometries where this hypothesis fails (for example, n -dimensional orthotopes), we show that in the case of the orthotope, optimal sets may still exist in this regime.

Local null controllability for Boussinesq system with a single thermal control

In this chapter, we show that the 3D Boussinesq system is locally controllable using only a single control in the temperature equation, exploiting the coupling between temperature and velocity through linearization, algebraic techniques, and an inverse function theorem.

2.1 Main Result

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary $\partial\Omega$, and let ω be a nonempty open subset of Ω . For a given final time $T > 0$, we use the notations $Q_T := (0, T) \times \Omega$ and $\Sigma_T := (0, T) \times \partial\Omega$.

We are concerned with the controlled Boussinesq system:

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla)y + \nabla p = \theta e_3 & \text{in } Q_T, \\ \theta_t - \Delta \theta + y \cdot \nabla \theta = \chi_\omega v & \text{in } Q_T, \\ \nabla \cdot y = 0 & \text{in } Q_T, \\ y = 0, \theta = 0 & \text{on } \Sigma_T, \\ y(x, 0) = y_0(x), \theta(x, 0) = \theta_0(x) & \text{in } \Omega, \end{cases} \quad (2.1)$$

where (y, θ) represents the state of the fluid, with y being the velocity and θ the temperature, while p is the associated pressure. The pair (y_0, θ_0) denotes the initial state. The control action is given by the term $\chi_\omega v$, where χ_ω is a fixed, non-negative smooth function with compact support contained in ω (that is, $\chi_\omega \in C_0^\infty(\omega)$). The function $v = v(x, t)$ is the control to be determined, and $e_3 = (0, 0, 1)$ is the third vector of the

canonical basis of $\mathbb{R}^3$.

We first introduce the function space for divergence-free fields, which is standard in fluid mechanics:

$$V := \{w \in H_0^1(\Omega)^3 \mid \nabla \cdot w = 0 \text{ in } \Omega\}.$$

The main result of this paper concerns the local null controllability for the system (2.1). More precisely, we have the following result:

Theorem 2.1 *For every $T > 0$, there exists $\eta > 0$ such that for any initial data $y_0 \in V$ and $\theta_0 \in H_0^1(\Omega)$ verifying*

$$\|y_0\|_V^2 + \|\theta_0\|_{H_0^1(\Omega)}^2 \leq \eta,$$

there exists a scalar control $v \in L^2(Q_T)$ and a corresponding solution (y, θ) with regularity

$$\begin{aligned} y &\in L^2(0, T; H^2(\Omega)^3 \cap V) \cap L^\infty(0, T; V), \\ \theta &\in L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap L^\infty(0, T; H_0^1(\Omega)), \end{aligned}$$

that, together with some pressure $p \in L^2(Q_T)$, solves system (2.1) and satisfies the null controllability condition:

$$y(T, \cdot) = 0 \quad \text{and} \quad \theta(T, \cdot) = 0 \quad \text{in } \Omega. \quad (2.2)$$

2.2 Preliminary results

In this section, we introduce the fundamental ideas developed in [17], which have been further examined and extended in [5], particularly in the context of controllability problems.

2.2.1 Gromov's Algebraic Solvability

This section introduces the main concepts of algebraic solvability, following [17, 5]. Let Q_0 be an open subset of Q_T . We begin with the formal definition of a linear partial differential operator.

Definition 2.2 (Linear Partial Differential Operator) *Let $k, l, n \in \mathbb{N}$. A linear map $\mathfrak{L} : C^\infty(Q_0)^k \rightarrow C^\infty(Q_0)^l$ is a linear partial differential operator of order n if there exist matrix-valued functions $A_\alpha \in C^\infty(Q_0; \mathcal{L}(\mathbb{R}^k, \mathbb{R}^l))$ for all multi-indices $\alpha \in \mathbb{N}^4$ with $|\alpha| \leq n$, such that for any $\varphi \in C^\infty(Q_0)^k$:*

$$\mathfrak{L}(\varphi)(\xi) = \sum_{|\alpha| \leq n} A_\alpha(\xi) \partial^\alpha \varphi(\xi), \quad \forall \xi \in Q_0.$$

An operator $\mathfrak{L}$ is called a linear partial differential operator if it is of order n for some $n \in \mathbb{N}$.

For every linear partial differential operator $\mathfrak{M} = \sum_{|\alpha| \leq m} A_\alpha \partial^\alpha$, its formal adjoint is the operator $\mathfrak{M}^* : C^\infty(Q_0)^l \rightarrow C^\infty(Q_0)^k$ defined by

$$\mathfrak{M}^*(\psi) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \partial^\alpha (A_\alpha^T \psi), \quad (2.3)$$

where A_α^T is the transpose of the matrix-valued function A_α .

Remark 2.3 (Properties of the Adjoint) *The formal adjoint satisfies the following fundamental properties:*

1. $(\mathfrak{M}^*)^* = \mathfrak{M}$.
2. If $\mathfrak{N} : C^\infty(Q_0)^l \rightarrow C^\infty(Q_0)^m$ is another linear partial differential operator, then $(\mathfrak{N} \circ \mathfrak{M})^* = \mathfrak{M}^* \circ \mathfrak{N}^*$.

Now, let k, l, m be positive integers, and let $\mathfrak{L}_0 : C^\infty(Q_0)^l \rightarrow C^\infty(Q_0)^m$ and $\mathfrak{B}_0 : C^\infty(Q_0)^k \rightarrow C^\infty(Q_0)^m$ be linear partial differential operators. We consider the problem of finding a solution $z \in C^\infty(Q_0)^l$ to the linear system

$$\mathfrak{L}_0 z = \mathfrak{B}_0 u, \quad (2.4)$$

for a given datum $u \in C^\infty(Q_0)^k$.

Following [17, p.148], we introduce the key concept of algebraic solvability for this system.

Definition 2.4 (Algebraic Solvability) *The linear equation (2.4) is said to be algebraically solvable if there exists a linear partial differential operator $\mathfrak{M}_0 : C^\infty(Q_0)^k \rightarrow C^\infty(Q_0)^l$ such that, for every datum $u \in C^\infty(Q_0)^k$, the function $z = \mathfrak{M}_0 u$ is a solution. This condition is equivalent to the operator identity:*

$$\mathfrak{L}_0 \circ \mathfrak{M}_0 = \mathfrak{B}_0. \quad (2.5)$$

Following the approach in [17, 5], we analyze the condition (2.5) by considering the adjoint operators. Using the properties from Remark 2.3, taking the adjoint of both sides of the identity $\mathfrak{L}_0 \circ \mathfrak{M}_0 = \mathfrak{B}_0$ yields the equivalent condition:

$$\mathfrak{M}_0^* \circ \mathfrak{L}_0^* = \mathfrak{B}_0^*. \quad (2.6)$$

Therefore, establishing the algebraic solvability for the original problem (2.4) is equivalent to establishing it for the corresponding adjoint problem.

2.2.2 Constructing a suitable trajectory

In this section, we construct an explicit trajectory, denoted by $(\bar{y}, \bar{p}, \bar{\theta}, \bar{v})$, for the Boussinesq system (2.1). This trajectory is designed to start and end at the zero state and to be such that the linearized control system around it is null-controllable. Our strategy is to first adopt the specific trajectory $(\bar{y}, \bar{p})$ constructed for the Navier-Stokes equations in [5]. Then, leveraging the cascade structure of the Boussinesq system, we construct the corresponding temperature component $\bar{\theta}$ and the required control $\bar{v}$.

For completeness, we now recall the main ideas from [5] for the construction of the fluid trajectory $(\bar{y}, \bar{p})$.

Without loss of generality, we assume that the origin belongs to the control domain, i.e., $0 \in \omega$. Since ω is an open set, we can fix a radius $r_0 > 0$ small enough such that the closed cylinder centered at the origin, defined by

$$\mathfrak{C}_1 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 \leq r_0^2, |x_3| \leq r_0\}, \quad (2.7)$$

is compactly contained in ω .

We now introduce two auxiliary smooth functions, g and h , which are essential for building the trajectory. Let $g = g(t, w, x_3)$ and $h = h(t, w, x_3)$ be functions in $C^\infty(\mathbb{R}^3; \mathbb{R})$ satisfying the support condition:

$$\text{supp}(g) \cup \text{supp}(h) \subset [T/4, T] \times [-\infty, r_0^2] \times [-r_0, r_0]. \quad (2.8)$$

Using the auxiliary functions g and h , we define the velocity trajectory $\bar{y} \in C^\infty(\mathbb{R}^4; \mathbb{R}^3)$ as

$$\bar{y}(t, x_1, x_2, x_3) := \begin{pmatrix} g(t, r^2, x_3)x_1 \\ g(t, r^2, x_3)x_2 \\ h(t, r^2, x_3) \end{pmatrix}, \quad (2.9)$$

where $r := \sqrt{x_1^2 + x_2^2}$ denotes the radial coordinate.

From equations (2.7), (2.8), and (2.9), it follows directly that the support of the trajectory $\bar{y}$ is contained within the space-time cylinder:

$$\text{supp}(\bar{y}) \subset [T/4, T] \times \mathfrak{C}_1. \quad (2.10)$$

Since $\mathfrak{C}_1$ is compactly contained in Ω , this implies, in particular, that $\bar{y}$ has a null trace

on the lateral boundary Σ_T .

For the constructed velocity field $\bar{y}$ to be divergence-free, the functions g and h must satisfy a compatibility condition. A direct computation using the definition (2.9) shows that the divergence of $\bar{y}$ is

$$\nabla \cdot \bar{y} = 2g + 2r^2 \partial_w g + \partial_{x_3} h.$$

Therefore, if

$$\partial_{x_3} h = -2(g + w \partial_w g), \quad (2.11)$$

then the incompressibility condition $\nabla \cdot \bar{y} = 0$ holds.

The corresponding pressure trajectory $\bar{p} \in C^\infty(\mathbb{R}^4)$ is defined via an auxiliary function $\hat{p}$ as

$$\bar{p}(t, x_1, x_2, x_3) := \hat{p}(t, x_1^2 + x_2^2, x_3), \quad (2.12)$$

where $\hat{p} = \hat{p}(t, w, x_3)$ is constructed to satisfy the momentum equation and is given by the integral formula:

$$\hat{p}(t, w, x_3) := -\frac{1}{2} \int_w^{r_0^2} \left[\partial_t g - (4w' \partial_{ww} g + 8 \partial_w g + \partial_{x_3 x_3} g) + 2w' g \partial_w g + g^2 + h \partial_{x_3} g \right] dw'. \quad (2.13)$$

The definitions above directly imply the properties of the constructed trajectory. From the support condition (2.8) and the definitions of $\bar{y}$ and $\bar{p}$, it follows that both have support strictly inside the space-time domain:

$$\text{supp}(\bar{y}) \cup \text{supp}(\bar{p}) \subset [T/4, T] \times \mathfrak{C}_1. \quad (2.14)$$

Furthermore, the pressure $\bar{p}$ is defined by (2.13) precisely so that the first two components of the momentum equation are satisfied:

$$\begin{cases} \bar{y}_t^1 - \Delta \bar{y}^1 + (\bar{y} \cdot \nabla) \bar{y}^1 + \partial_{x_1} \bar{p} = 0 & \text{in } Q_T, \\ \bar{y}_t^2 - \Delta \bar{y}^2 + (\bar{y} \cdot \nabla) \bar{y}^2 + \partial_{x_2} \bar{p} = 0 & \text{in } Q_T. \end{cases} \quad (2.15)$$

Now we proceed with the construction of the temperature and control trajectories. First, we define the temperature $\bar{\theta} \in C^\infty(\mathbb{R}^4)$ by rearranging the third component of the momentum equation:

$$\bar{\theta} := \bar{y}_t^3 - \Delta \bar{y}^3 + (\bar{y} \cdot \nabla) \bar{y}^3 + \partial_{x_3} \bar{p}. \quad (2.16)$$

Then, the corresponding control trajectory $\bar{v} \in C^\infty(\mathbb{R}^4)$ is defined as the source term that

makes $(\bar{y}, \bar{\theta})$ a solution to the heat equation:

$$\bar{v} := \bar{\theta}_t - \Delta \bar{\theta} + \bar{y} \cdot \nabla \bar{\theta}. \quad (2.17)$$

By construction, the tuple $(\bar{y}, \bar{p}, \bar{\theta}, \bar{v})$ is a smooth solution to the full Boussinesq system (2.1) (with control term $\chi_\omega v$ identified as $\bar{v}$).

Since the trajectory components $\bar{\theta}$ and $\bar{v}$ are defined in (2.16) and (2.17) using derivatives of $\bar{y}$ and $\bar{p}$, they inherit the same support properties. Specifically,

$$\text{supp}(\bar{\theta}) \cup \text{supp}(\bar{v}) \subset [T/4, T] \times \mathfrak{C}_1. \quad (2.18)$$

This property is crucial, as it implies that the constructed trajectory $(\bar{y}, \bar{\theta})$ is zero at the initial and final times ($t = 0$ and $t = T$) and satisfies the required null boundary conditions on Σ_T .

Combining the previous steps, specifically, the incompressibility condition (2.11) and the definitions of $\bar{p}$ in (2.12), $\bar{\theta}$ in (2.16), and $\bar{v}$ in (2.17), we conclude that the constructed trajectory satisfies the Boussinesq system. That is, the tuple $(\bar{y}, \bar{p}, \bar{\theta}, \bar{v})$ solves:

$$\begin{cases} \bar{y}_t - \Delta \bar{y} + (\bar{y} \cdot \nabla) \bar{y} + \nabla \bar{p} = \bar{\theta} e_3, & \text{in } Q_T, \\ \nabla \cdot \bar{y} = 0, & \text{in } Q_T, \\ \bar{\theta}_t - \Delta \bar{\theta} + \bar{y} \cdot \nabla \bar{\theta} = \bar{v}, & \text{in } Q_T. \end{cases} \quad (2.19)$$

The construction of the trajectory depends on the functions g and h , which must satisfy the support condition (2.8) and the compatibility condition (2.11). We now provide an explicit construction of such functions. These are crucial in proving that the linearized system around the trajectory $(\bar{y}, \bar{\theta}, \bar{p}, \bar{v})$ is null-controllable.

To this end, let $\nu > 0$ be a large constant whose value will be chosen later. We define three auxiliary smooth functions $a, b, c \in C^\infty(\mathbb{R})$ with the following properties:

$$\text{supp}(a) \subset [T/4, T] \quad \text{and} \quad a(t) = e^{-\frac{\nu}{(T-t)^5}} \quad \text{for } t \in [T/2, T], \quad (2.20)$$

$$\text{supp}(b) \subset (-\infty, r_0^2) \quad \text{and} \quad b(w) = w \quad \text{for } w \in (-\infty, r_0^2/4], \quad (2.21)$$

$$\text{supp}(c) \subset (-r_0, r_0) \quad \text{and} \quad c(x_3) = x_3^2 \quad \text{for } x_3 \in [-r_0/2, r_0/2]. \quad (2.22)$$

Using these functions, we set

$$g(t, w, x_3) := \varepsilon a(t) b(w) c'(x_3) \quad (2.23)$$

and

$$h(t, w, x_3) := -2\varepsilon a(t)(b(w) + wb'(w))c(x_3), \quad (2.24)$$

where $\varepsilon > 0$ is a small parameter. A direct computation confirms that these definitions of g and h satisfy the divergence-free condition (2.11).

2.3 Null Controllability of the Linearized System

As outlined in Chapter 1, the proof of controllability for the linearized system is divided into two main parts: an analytic part and an algebraic part. In this section, we will combine these strategies to establish the null controllability of the linearized Boussinesq system (1.3).

Writing the Boussinesq system component-wise, linearized around the trajectory $(\bar{y}, \bar{\theta}, \bar{p})$, we obtain the following system for the state (y, θ, p) :

$$\begin{cases} y_t^1 - \Delta y^1 + (\bar{y} \cdot \nabla)y^1 + (y \cdot \nabla)\bar{y}^1 + \partial_{x_1}p = F^1, & \text{in } Q_T, \\ y_t^2 - \Delta y^2 + (\bar{y} \cdot \nabla)y^2 + (y \cdot \nabla)\bar{y}^2 + \partial_{x_2}p = F^2, & \text{in } Q_T, \\ y_t^3 - \Delta y^3 + (\bar{y} \cdot \nabla)y^3 + (y \cdot \nabla)\bar{y}^3 + \partial_{x_3}p = F^3 + \theta, & \text{in } Q_T, \\ \theta_t - \Delta \theta + \bar{y} \cdot \nabla \theta + y \cdot \nabla \bar{\theta} = f + \chi_\omega v, & \text{in } Q_T, \\ \nabla \cdot y = 0, & \text{in } Q_T, \\ y = 0, \theta = 0, & \text{on } \Sigma_T, \\ y(0) = y_0, \theta(0) = \theta_0, & \text{in } \Omega, \end{cases} \quad (2.25)$$

where $F = (F^1, F^2, F^3)$ and f are source terms and v is the control function. Then, the main result of this section is the following:

Proposition 2.5 *For any time $T > 0$, the linear system (2.25) is null-controllable.*

The proof of Proposition 2.5 is based on the concept of algebraic solvability. This strategy reduces the problem of controlling system (2.25) with a single input to a related problem with a larger set of controls, for which controllability can be established by analytic methods.

2.3.1 Algebraic Solvability for the Linearized Boussinesq System

In this section, we apply the algebraic solvability framework to our problem. First, we define a smaller subdomain where the algebraic construction will take place. Let $\mathfrak{C}_2$

be the closed cylinder strictly contained within $\mathfrak{C}_1$, defined by

$$\mathfrak{C}_2 := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 \leq \frac{r_0^2}{4}, |x_3| \leq \frac{r_0}{2} \right\}. \quad (2.26)$$

Now, let ω_0 be any nonempty open subset of $\mathfrak{C}_2$. We define the space-time subdomain for our algebraic problem as

$$Q_0 := \left(\frac{T}{2}, T \right) \times \omega_0. \quad (2.27)$$

We define the linear partial differential operator $\mathfrak{L} : C^\infty(Q_0)^6 \rightarrow C^\infty(Q_0)^5$ associated with the linearized system. Its action on the tuple (y, p, θ, v) is given by:

$$\mathfrak{L} \begin{pmatrix} y \\ p \\ \theta \\ v \end{pmatrix} := \begin{pmatrix} y_t^1 - \Delta y^1 + (\bar{y} \cdot \nabla) y^1 + (y \cdot \nabla) \bar{y}^1 + \partial_{x_1} p \\ y_t^2 - \Delta y^2 + (\bar{y} \cdot \nabla) y^2 + (y \cdot \nabla) \bar{y}^2 + \partial_{x_2} p \\ y_t^3 - \Delta y^3 + (\bar{y} \cdot \nabla) y^3 + (y \cdot \nabla) \bar{y}^3 + \partial_{x_3} p - \theta \\ \theta_t - \Delta \theta + \bar{y} \cdot \nabla \theta + y \cdot \nabla \bar{\theta} - v \\ \nabla \cdot y \end{pmatrix}. \quad (2.28)$$

Let k be a positive integer and let $\mathfrak{B} := (\mathfrak{B}^1, \mathfrak{B}^2, \mathfrak{B}^3, \mathfrak{B}^4) : C^\infty(Q_0)^k \rightarrow C^\infty(Q_0)^4$ be a linear partial differential operator. We now consider the following fully actuated linear system:

$$\mathfrak{L} \begin{pmatrix} y \\ p \\ \theta \\ v \end{pmatrix} = \begin{pmatrix} \mathfrak{B}^1 u \\ \mathfrak{B}^2 u \\ \mathfrak{B}^3 u \\ \mathfrak{B}^4 u \\ 0 \end{pmatrix}, \quad (2.29)$$

where $u \in C^\infty(Q_0)^k$ is the given control (datum) and the tuple $(y, p, \theta, v) \in C^\infty(Q_0)^6$ is the unknown solution.

Based on Definition 2.4, we will adapt the definition of algebraic solvability to the system 2.29.

Definition 2.6 (Algebraic Solvability) *The linear system (2.29) is said to be algebraically solvable if there exists a linear partial differential operator $\mathfrak{M} : C^\infty(Q_0)^k \rightarrow C^\infty(Q_0)^6$ such that for any datum $u \in C^\infty(Q_0)^k$, the tuple $(y, p, \theta, v) = \mathfrak{M}u$ is a solution.*

This is equivalent to finding an operator $\mathfrak{M}$ that satisfies the identity:

$$\mathfrak{L} \circ \mathfrak{M} = \begin{pmatrix} \mathfrak{B} \\ 0 \end{pmatrix}. \quad (2.30)$$

Here, the operator on the right-hand side maps $u \in C^\infty(Q_0)^k$ to the 5-component vector

$(\mathfrak{B}^1 u, \mathfrak{B}^2 u, \mathfrak{B}^3 u, \mathfrak{B}^4 u, 0) \in C^\infty(Q_0)^5$, consistent with the output of $\mathfrak{L}$.

Following [5], we choose the dimension of the auxiliary control space to be $k = 8$. We define the linear partial differential operator $\mathfrak{B} : C^\infty(Q_0)^8 \rightarrow C^\infty(Q_0)^4$ by its action on a control $u = (u^1, \dots, u^8)$ as follows:

$$\mathfrak{B}(u) := \begin{pmatrix} \partial_{x_1} u^1 + \partial_{x_2} u^2 + \partial_{x_3} u^3 \\ \partial_{x_1} u^4 + \partial_{x_2} u^5 + \partial_{x_3} u^6 \\ u^7 \\ u^8 \end{pmatrix}. \quad (2.31)$$

With the operators $\mathfrak{L}$ and $\mathfrak{B}$ now explicitly defined in (2.28) and (2.31), we can state the main algebraic solvability result of this section.

Proposition 2.7 *There exist constants $\epsilon^* > 0$ and $T^* > 0$ such that for every $\epsilon \in (0, \epsilon^*)$ and every $T \in (0, T^*)$, one can find a nonempty open subset $\omega_0 \subset \mathfrak{C}_2$ and a linear partial differential operator*

$$\mathfrak{M} : C^\infty(Q_0)^8 \rightarrow C^\infty(Q_0)^6$$

for which the algebraic solvability condition (2.30) holds.

Proof: Following the arguments of [5], we first reduce the problem to a smaller set of operators. Let the linear partial differential operator $\mathfrak{L}_0 : C^\infty(Q_0)^4 \rightarrow C^\infty(Q_0)^3$ be defined by its action on a state $(y, p) \in C^\infty(Q_0)^4$ as:

$$\mathfrak{L}_0 \begin{pmatrix} y \\ p \end{pmatrix} := \begin{pmatrix} y_t^1 - \Delta y^1 + (\bar{y} \cdot \nabla) y^1 + (y \cdot \nabla) \bar{y}^1 + \partial_{x_1} p \\ y_t^2 - \Delta y^2 + (\bar{y} \cdot \nabla) y^2 + (y \cdot \nabla) \bar{y}^2 + \partial_{x_2} p \\ \nabla \cdot y \end{pmatrix}. \quad (2.32)$$

Correspondingly, let $\mathfrak{B}_0 : C^\infty(Q_0)^6 \rightarrow C^\infty(Q_0)^3$ be the operator acting on the first six control components $u_0 := (u^1, \dots, u^6)$ as follows:

$$\mathfrak{B}_0(u_0) := \begin{pmatrix} \partial_{x_1} u^1 + \partial_{x_2} u^2 + \partial_{x_3} u^3 \\ \partial_{x_1} u^4 + \partial_{x_2} u^5 + \partial_{x_3} u^6 \\ 0 \end{pmatrix}. \quad (2.33)$$

With the operators $\mathfrak{L}_0$ and $\mathfrak{B}_0$ defined, we can decompose the fully actuated system (2.29). The first two momentum equations and the divergence constraint can be written compactly as the reduced system:

$$\mathfrak{L}_0(y, p) = \mathfrak{B}_0(u_0), \quad (2.34)$$

where $u_0 := (u^1, \dots, u^6)$ denotes the first six components of the control.

Recalling that we set $k = 8$ and the specific form of $\mathfrak{B}$ from (2.31), the choices $\mathfrak{B}^3 u = u^7$ and $\mathfrak{B}^4 u = u^8$ mean that the third and fourth equations of the full system (2.29) can be read as:

$$y_t^3 - \Delta y^3 + (\bar{y} \cdot \nabla) y^3 + (y \cdot \nabla) \bar{y}^3 + \partial_{x_3} p - \theta = u^7, \quad (2.35)$$

and

$$\theta_t - \Delta \theta + \bar{y} \cdot \nabla \theta + y \cdot \nabla \bar{\theta} - v = u^8. \quad (2.36)$$

This means that the structure of the fully actuated system allows for a crucial simplification. Since the variables θ and v can be determined algebraically from equations (2.35) and (2.36) once (y, p) and the controls (u^7, u^8) are known, the algebraic solvability of the full system (2.29) reduces to that of the subsystem (2.34).

Therefore, the task is to prove the existence of a linear partial differential operator $\mathfrak{M}_0 : C^\infty(Q_0)^6 \rightarrow C^\infty(Q_0)^4$ that satisfies the reduced solvability condition $\mathfrak{L}_0 \circ \mathfrak{M}_0 = \mathfrak{B}_0$. As established in Section 2.2.1, this is equivalent to solving the adjoint formulation:

$$\mathfrak{M}_0^* \circ \mathfrak{L}_0^* = \mathfrak{B}_0^*. \quad (2.37)$$

This means we need to find a linear partial differential operator $\mathfrak{N}$ (which will define $\mathfrak{M}_0^*$) such that for any $\varphi \in C^\infty(Q_0)^4$, if $(z, \pi) \in C^\infty(Q_0)^3$ is a solution to the system $\mathfrak{L}_0^*(z, \pi) = \varphi$, i.e.,

$$\begin{cases} -z_t^1 - \Delta z^1 - (\bar{y} \cdot \nabla) z^1 + (\partial_{x_1} \bar{y}^1) z^1 + (\partial_{x_1} \bar{y}^2) z^2 - \partial_{x_1} \pi = \varphi^1, \\ -z_t^2 - \Delta z^2 - (\bar{y} \cdot \nabla) z^2 + (\partial_{x_2} \bar{y}^1) z^1 + (\partial_{x_2} \bar{y}^2) z^2 - \partial_{x_2} \pi = \varphi^2, \\ (\partial_{x_3} \bar{y}^1) z^1 + (\partial_{x_3} \bar{y}^2) z^2 - \partial_{x_3} \pi = \varphi^3, \\ -\nabla \cdot z = \varphi^4, \end{cases} \quad (2.38)$$

then $\mathfrak{B}_0^*(z, \pi) = \mathfrak{N}(\varphi)$.

Following the strategy in [5], we address this by treating system (2.38) as a system of algebraic equations. While it is "analytically" overdetermined (4 equations for 3 unknown functions), it is "algebraically" underdetermined if we treat the functions and all their partial derivatives as independent algebraic unknowns. The core idea is to differentiate the equations in (2.38) a sufficient number of times to generate a new, larger system of linear algebraic equations that is overdetermined (i.e., has more equations than unknowns).

This algebraic procedure was carried out in [5]. The procedure involves the following steps:

- **Differentiation:** The equations of the system are differentiated multiple times. For instance, in the reference, the first two equations were differentiated 19 times and the last 21 times, creating a massive system of 30,360 equations for 29,900 algebraic unknowns (the functions' derivatives up to a certain order).
- **Matrix Formulation:** The large system is written in matrix form, $L_0 Z = \Phi$, where Z is the vector of all algebraic unknowns (derivatives of z) and Φ is the vector of source terms (derivatives of φ).
- **Inversion:** A suitable square submatrix P of the large matrix L_0 is found and proven to be invertible. This is done with the help of a computer performing symbolic computations, and it is sufficient to prove this invertibility at a single, well-chosen point. The Dulmage-Mendelsohn decomposition can be used to find such a submatrix.
- **Construction of $\mathfrak{N}$:** The desired operator $\mathfrak{N}$ (and thus $\mathfrak{M}_0^*$) is then constructed based on the inverse of this submatrix, P^{-1} .

By applying this established procedure, the existence of the operator $\mathfrak{M}_0$ is guaranteed, which concludes the proof. ■

Remark 2.8 (On the choice of the operator $\mathfrak{B}$) *The most natural choice for the operator $\mathfrak{B}$ would be the identity, a strategy that is successful in other contexts (see, e.g., [8, 9, 7]). However, for the Boussinesq and Navier-Stokes systems, this approach fails. As explained in [5, Remark 4], the presence of the pressure term and the incompressibility constraint prevents the algebraic solvability condition from holding with such a simple choice of $\mathfrak{B}$.*

Remark 2.9 (Extension to Sobolev Spaces) *For simplicity, we have formulated the algebraic solvability proposition in a C^∞ setting. This framework, however, must be compatible with the Sobolev spaces used in the analytic part of the proof. The key idea is that the solution operator $\mathfrak{M}$, as a linear partial differential operator, extends to a continuous linear map between appropriate weighted Sobolev spaces, say $\mathfrak{M} : H_1 \rightarrow H_2$, where H_2 has lower regularity than H_1 . The analytic part of our proof consists of establishing a null controllability result for the fully actuated system (1.4) with a control $u \in H_1$. The algebraic solvability result then guarantees the existence of our single control $v \in H_2$ for the original linearized system (1.3).*

2.3.2 Controllability of the Linear System with Fictitious Control

In this section, we establish a null controllability result for the fully actuated system (1.4), which constitutes the analytic part of our proof. The argument is based on the Hilbert Uniqueness Method (HUM), which relies on proving a suitable observability inequality for the adjoint system (1.6). This inequality is, in turn, a consequence of a global Carleman estimate. Following the approach in [5], this method allows us to construct a control u in an appropriate function space such that the control term $\mathfrak{B}u$ drives the solution to zero at the final time.

Carleman Estimate for the Adjoint System

We establish a suitable Carleman inequality for the adjoint system (1.6), which is the key tool to prove the null controllability of the fully actuated system (1.4). Our approach is adapted from the proofs by Gueye for the Navier-Stokes system in [20] and by Guerrero for the Boussinesq system in [19].

Before stating the Carleman inequality, we introduce the classical weight functions used in the analysis of parabolic equations [13]. Let ω_0 be the nonempty open set previously defined in (2.27). It is a well-known result that there exists a function $\eta^0 \in C^2(\overline{\Omega})$ satisfying the following properties:

- $\eta^0(x) > 0$ for all $x \in \Omega$;
- $\eta^0(x) = 0$ for all $x \in \partial\Omega$;
- $|\nabla\eta^0(x)| > 0$ for all $x \in \overline{\Omega} \setminus \omega_0$.

For a proof of the existence of such a function η^0 , see, for instance, [13].

We now define the weight functions required for the Carleman estimate, which depend on a large parameter $\lambda > 0$. Let the functions α and ξ be defined on $(T/2, T) \times \Omega$ by

$$\alpha(t, x) := \frac{e^{12\lambda\|\eta^0\|_{L^\infty(\Omega)}} - e^{\lambda(10\|\eta^0\|_{L^\infty(\Omega)} + \eta^0(x))}}{(t - T/2)^5(T - t)^5}, \quad (2.39)$$

$$\xi(t, x) := \frac{e^{\lambda(10\|\eta^0\|_{L^\infty(\Omega)} + \eta^0(x))}}{(t - T/2)^5(T - t)^5}. \quad (2.40)$$

Furthermore, we define their time-dependent spatial extrema:

$$\alpha^*(t) := \max_{x \in \overline{\Omega}} \alpha(t, x),$$

$$\xi^*(t) := \max_{x \in \overline{\Omega}} \xi(t, x),$$

$$\widehat{\alpha}(t) := \min_{x \in \Omega} \alpha(t, x).$$

We state the Carleman inequality for the adjoint system (1.6), which is the key to proving the null controllability of the fully actuated system. Our approach is adapted from the methods developed in [20] and [19].

Lemma 2.10 (Carleman Estimate) *Let $\chi : \Omega \rightarrow [0, +\infty)$ be a lower semicontinuous function that is not identically zero. There exist constants $C_1 > 0, C_3 > 0, \varepsilon_0 > 0$, and $\nu_0 > 0$ with the following property.*

For any choice of parameters $K_1 > C_1, K_3 > C_3, r, \tilde{r} \in (0, 1), \nu \geq \nu_0$, and $\varepsilon \in (0, \varepsilon_0)$, there exists a constant $C > 0$ such that for any source terms $G \in (L^2(Q_T))^3$ and $g \in L^2(Q_T)$, and any final data (φ_T, ψ_T) , the solution (φ, ψ) of the adjoint system (1.6) satisfies:

$$\begin{aligned} & \|\varphi(T/2, \cdot)\|_{L^2(\Omega)^3}^2 + \|\psi(T/2, \cdot)\|_{L^2(\Omega)}^2 + \iint_{Q_{T/2}} \exp\left(\frac{-K_1}{r(t - \frac{T}{2})^5(T-t)^5}\right) |\varphi|^2 dx dt \\ & \quad + \iint_{Q_{T/2}} \exp\left(\frac{-K_3}{\tilde{r}(t - \frac{T}{2})^5(T-t)^5}\right) |\psi|^2 dx dt \\ & \leq C \left(\iint_{Q_{T/2}} \exp\left(\frac{-K_1}{(T-t)^5}\right) |G|^2 dx dt + \iint_{Q_{T/2}} \chi \exp\left(\frac{-K_1}{(t - \frac{T}{2})^5(T-t)^5}\right) |\nabla \times \varphi|^2 dx dt \right. \\ & \quad \left. + \iint_{Q_{T/2}} \exp\left(\frac{-K_3}{(T-t)^5}\right) |g|^2 dx dt + \iint_{Q_{T/2}} \chi \exp\left(\frac{-K_3}{(t - \frac{T}{2})^5(T-t)^5}\right) |\psi|^2 dx dt \right). \end{aligned} \quad (2.41)$$

Here, the domain of integration is $Q_{T/2} := (T/2, T) \times \Omega$.

Proof: Without loss of generality, we may assume that the observation function χ is the characteristic function of a nonempty open subset, i.e., $\chi = \mathbf{1}_{\omega^*}$ for some open set $\omega^* \subset \Omega$. For the trajectory constructed in Section 2.2.2, we now specify that the control domain ω corresponds to this observation set, so $\omega = \omega^*$. The Carleman inequality will be derived for the adjoint system (1.6), considered on the restricted domain $Q_{T/2}$ and the boundary $\Sigma_{T/2} := (T/2, T) \times \partial\Omega$.

By the explicit construction of the trajectory (see Section 2.2.2), the following estimates hold for some constant $C > 0$ that does not depend on ε or ν . The velocity trajectory satisfies:

$$|\bar{y}(t, x)| + |\nabla \bar{y}(t, x)| \leq C\varepsilon \exp\left(\frac{-\nu}{(T-t)^5}\right) \quad \text{for all } (t, x) \in \left(\frac{T}{2}, T\right) \times \Omega. \quad (2.42)$$

Similarly, using the definitions of the pressure and temperature trajectories, we obtain

the estimate for $\bar{\theta}$:

$$|\bar{\theta}(t, x)| \leq \frac{C\varepsilon}{(T-t)^4} \exp\left(\frac{-\nu}{(T-t)^5}\right) \quad \text{for all } (t, x) \in \left(\frac{T}{2}, T\right) \times \Omega. \quad (2.43)$$

Applying the Carleman estimate from Proposition 1.1, there exists a constant $C > 0$ such that the following inequality holds:

$$\begin{aligned} & s^3 \lambda^4 \int_{Q_{T/2}} e^{-2s\alpha - 2s\mu\alpha^*} \xi^3 |\nabla \times \varphi|^2 dx dt + s\lambda^2 \int_{Q_{T/2}} e^{-2s\alpha - 2s\mu\alpha^*} \xi |\nabla(\nabla \times \varphi)|^2 dx dt \\ & \leq C \left(s^3 \lambda^4 \int_{(T/2, T) \times \omega^*} e^{-2s\alpha - 2s\mu\alpha^*} \xi^3 |\nabla \times \varphi|^2 dx dt + \int_{Q_{T/2}} e^{-2s\mu\alpha^*} |G|^2 dx dt \right. \\ & \quad \left. + \varepsilon^2 \int_{Q_{T/2}} \exp\left(\frac{-2\nu}{(T-t)^5}\right) e^{-2s\mu\alpha^*} (|\varphi|^2 + |\nabla \varphi|^2) dx dt \right. \\ & \quad \left. + \varepsilon^2 \int_{Q_{T/2}} e^{-2s\mu\alpha^*} |\bar{\theta}|^2 |\nabla \psi|^2 dx dt \right). \quad (2.44) \end{aligned}$$

This estimate holds for all $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$, where $\mu > 0$ is a parameter to be specified later. The domain of integration is $Q_{T/2} := (T/2, T) \times \Omega$.

Remark 2.11 *The Carleman estimate in [20] is stated for the case $\mu = 1$. A simple adaptation of the proof allows for the inclusion of the additional parameter μ , which provides more flexibility.*

Next, we apply a standard Carleman estimate for the heat equation (see, e.g., [13]) to the equation for ψ in the adjoint system (1.6). For this, we introduce a positive constant $m > 0$ to adjust the weight. For sufficiently large parameters $C, \lambda, s > 0$, we obtain the inequality:

$$\begin{aligned} & s^3 \lambda^4 \int_{Q_{T/2}} e^{-2ms\alpha} \xi^3 |\psi|^2 dx dt + s\lambda^2 \int_{Q_{T/2}} e^{-2ms\alpha} \xi |\nabla \psi|^2 dx dt \\ & \quad + s^{-1} \int_{Q_{T/2}} e^{-2ms\alpha} \xi^{-1} |\Delta \psi|^2 dx dt \\ & \leq C \left(s^3 \lambda^4 \int_{(T/2, T) \times \omega^*} e^{-2ms\alpha} \xi^3 |\psi|^2 dx dt + \int_{Q_{T/2}} e^{-2ms\alpha} |g|^2 dx dt \right. \\ & \quad \left. + \varepsilon^2 \int_{Q_{T/2}} \exp\left(\frac{-2\nu}{(T-t)^5}\right) e^{-2ms\alpha} |\nabla \psi|^2 dx dt + \int_{Q_{T/2}} e^{-2ms\alpha} |\varphi \cdot e_3|^2 dx dt \right). \quad (2.45) \end{aligned}$$

A simple computation shows that the last term on the right-hand side of (2.45) can

be estimated. First, since $\alpha(t, x) \geq \hat{\alpha}(t)$, we have:

$$\begin{aligned} \int_{Q_{T/2}} e^{-2ms\alpha} |\varphi \cdot e_3|^2 dx dt &\leq \int_{Q_{T/2}} e^{-2ms\hat{\alpha}} |\varphi \cdot e_3|^2 dx dt \\ &\leq C \int_{Q_{T/2}} e^{-2ms\hat{\alpha}} \xi^3 |\nabla \times \varphi|^2 dx dt. \end{aligned} \quad (2.46)$$

The second inequality is a consequence of a weighted Poincaré-type estimate for divergence-free vector fields, which bounds the L^2 norm of the field by the L^2 norm of its curl.

Moreover, the term involving $|\nabla \psi|^2$ on the right-hand side of (2.45) can be absorbed by the corresponding, larger term on the left-hand side. This is achieved by choosing the parameter ε in the trajectory construction to be sufficiently small relative to the large Carleman parameters s and λ .

Combining this absorption argument with the estimate for the coupling term (2.46), the inequality (2.45) simplifies to:

$$\begin{aligned} &s^3 \lambda^4 \int_{Q_{T/2}} e^{-2ms\alpha} \xi^3 |\psi|^2 dx dt + s \lambda^2 \int_{Q_{T/2}} e^{-2ms\alpha} \xi |\nabla \psi|^2 dx dt \\ &\quad + s^{-1} \int_{Q_{T/2}} e^{-2ms\alpha} \xi^{-1} |\Delta \psi|^2 dx dt \\ &\leq C \left(s^3 \lambda^4 \int_{(T/2, T) \times \omega^*} e^{-2ms\alpha} \xi^3 |\psi|^2 dx dt + \int_{Q_{T/2}} e^{-2ms\alpha} |g|^2 dx dt \right. \\ &\quad \left. + \int_{Q_{T/2}} e^{-2ms\hat{\alpha}} \xi^3 |\nabla \times \varphi|^2 dx dt \right). \end{aligned} \quad (2.47)$$

Combining the Carleman estimate for the curl of φ , (2.44), and the estimate for ψ ,

(2.47), we obtain the following combined inequality:

$$\begin{aligned}
& s^3 \lambda^4 \int_{Q_{T/2}} e^{-2s\alpha - 2s\mu\alpha^*} \xi^3 |\nabla \times \varphi|^2 dx dt + s \lambda^2 \int_{Q_{T/2}} e^{-2s\alpha - 2s\mu\alpha^*} \xi |\nabla(\nabla \times \varphi)|^2 dx dt \\
& + s^3 \lambda^4 \int_{Q_{T/2}} e^{-2ms\alpha} \xi^3 |\psi|^2 dx dt + s \lambda^2 \int_{Q_{T/2}} e^{-2ms\alpha} \xi |\nabla \psi|^2 dx dt \\
& + s^{-1} \int_{Q_{T/2}} e^{-2ms\alpha} \xi^{-1} |\Delta \psi|^2 dx dt \\
& \leq C \left(s^3 \lambda^4 \int_{(T/2, T) \times \omega^*} e^{-2s\alpha - 2s\mu\alpha^*} \xi^3 |\nabla \times \varphi|^2 dx dt \right. \\
& + s^3 \lambda^4 \int_{(T/2, T) \times \omega^*} e^{-2ms\alpha} \xi^3 |\psi|^2 dx dt + \int_{Q_{T/2}} e^{-2s\mu\alpha^*} |G|^2 dx dt \\
& + \int_{Q_{T/2}} e^{-2ms\alpha} |g|^2 dx dt + \varepsilon^2 \int_{Q_{T/2}} \exp\left(\frac{-2\nu}{(T-t)^5}\right) e^{-2s\mu\alpha^*} (|\varphi|^2 + |\nabla \varphi|^2) dx dt \\
& \left. + \varepsilon^2 \int_{Q_{T/2}} e^{-2s\mu\alpha^*} |\bar{\theta}|^2 |\nabla \psi|^2 dx dt + \int_{Q_{T/2}} e^{-2ms\hat{\alpha}} \xi^3 |\nabla \times \varphi|^2 dx dt \right). \tag{2.48}
\end{aligned}$$

Finally, the last term on the right-hand side of (2.48) can be absorbed by the corresponding term for the curl of φ on the left-hand side. This absorption is possible for large s and λ if the exponential weights satisfy the condition $m\hat{\alpha}(t) \geq (1 + \mu)\alpha^*(t)$ for $t \in (T/2, T)$. As demonstrated in [5], this inequality can be satisfied for a sufficiently large λ by choosing the parameters appropriately, for instance by taking $m = \frac{3}{2}(1 + \mu)$.

To handle the penultimate term on the right-hand side of (2.48), we apply integration by parts to the term involving $|\nabla \psi|^2$:

$$\begin{aligned}
\int_{Q_{T/2}} e^{-2s\mu\alpha^*} |\bar{\theta}|^2 |\nabla \psi|^2 dx dt &= \int_{Q_{T/2}} e^{-2s\mu\alpha^*} \operatorname{div}(|\bar{\theta}|^2 \psi \nabla \psi) dx dt \\
&- \int_{Q_{T/2}} e^{-2s\mu\alpha^*} (\nabla(|\bar{\theta}|^2) \cdot \nabla \psi) \psi dx dt \\
&- \int_{Q_{T/2}} e^{-2s\mu\alpha^*} |\bar{\theta}|^2 (\Delta \psi) \psi dx dt. \tag{2.49}
\end{aligned}$$

By the divergence theorem, the first term on the right-hand side of (2.49) vanishes, since $\psi = 0$ on the boundary of the spatial domain. The identity thus simplifies to:

$$\int_{Q_{T/2}} e^{-2s\mu\alpha^*} |\bar{\theta}|^2 |\nabla \psi|^2 dx dt = -A - B,$$

where we have defined the terms A and B as

$$A := \int_{Q_{T/2}} e^{-2s\mu\alpha^*} (\nabla(|\bar{\theta}|^2) \cdot \nabla\psi) \psi \, dx \, dt, \quad (2.50)$$

$$B := \int_{Q_{T/2}} e^{-2s\mu\alpha^*} |\bar{\theta}|^2 (\Delta\psi) \psi \, dx \, dt. \quad (2.51)$$

We now estimate the terms A and B . For term A , defined in (2.50), we apply the Cauchy-Schwarz and Young inequalities. Since the support of $\bar{\theta}$ is contained in $(T/2, T) \times \omega^*$, the integrals are non-zero only in this region. For any $\delta > 0$:

$$\begin{aligned} |A| &\leq C \int_{Q_{T/2}} e^{-2s\mu\alpha^*} |\bar{\theta}| |\nabla\bar{\theta}| |\nabla\psi| |\psi| \, dx \, dt \\ &= C \int_{Q_{T/2}} (s^{1/2} \lambda \xi^{1/2} e^{-sm\alpha} |\nabla\psi|) \cdot (s^{-1/2} \lambda^{-1} \xi^{-1/2} e^{-2s\mu\alpha^* + sm\alpha} |\bar{\theta}| |\nabla\bar{\theta}| |\psi|) \, dx \, dt \\ &\leq \delta s \lambda^2 \int_{Q_{T/2}} e^{-2ms\alpha} \xi |\nabla\psi|^2 \, dx \, dt \\ &\quad + C_\delta s^{-1} \lambda^{-2} \int_{(T/2, T) \times \omega^*} e^{-4s\mu\alpha^* + 2sm\alpha} \xi^{-1} |\bar{\theta}|^2 |\nabla\bar{\theta}|^2 |\psi|^2 \, dx \, dt. \end{aligned} \quad (2.52)$$

To bound the final integral in the estimate (2.52), we require the exponent in its weight function to be non-positive. This gives the condition:

$$-4s\mu\alpha^* + 2sm\alpha \leq 0 \quad \Longleftrightarrow \quad m\alpha \leq 2\mu\alpha^*.$$

Recalling that $\alpha(t, x) \leq \alpha^*(t)$ and using the specific choice $m = \frac{3}{2}(1 + \mu)$, a sufficient condition for this to hold is

$$\frac{3}{2}(1 + \mu) \leq 2\mu,$$

which is satisfied for any $\mu \geq 3$.

Similarly, we estimate term B defined in (2.51). For any $\delta > 0$, applying the Cauchy-Schwarz and Young inequalities yields:

$$\begin{aligned} |B| &\leq C \int_{Q_{T/2}} e^{-2s\mu\alpha^*} |\bar{\theta}|^2 |\Delta\psi| |\psi| \, dx \, dt \\ &\leq \delta s^{-1} \int_{Q_{T/2}} e^{-2ms\alpha} \xi^{-1} |\Delta\psi|^2 \, dx \, dt \\ &\quad + C_\delta s \lambda^2 \int_{(T/2, T) \times \omega^*} e^{-4s\mu\alpha^* + 2sm\alpha} \xi |\bar{\theta}|^4 |\psi|^2 \, dx \, dt. \end{aligned} \quad (2.53)$$

As before, the exponential weight in the final integral is bounded by choosing $\mu \geq 3$.

By substituting the estimates (2.52) and (2.53) into the combined inequality (2.48) and performing the absorption of the remainder terms (by choosing the parameters $\varepsilon, m, \mu, s, \lambda$ appropriately), we arrive at the final Carleman estimate:

$$\begin{aligned}
& s^3 \lambda^4 \int_{Q_{T/2}} e^{-2s(1+\mu)\alpha^*} \xi^3 |\nabla \times \varphi|^2 dx dt + s \lambda^2 \int_{Q_{T/2}} e^{-2s(1+\mu)\alpha^*} \xi |\nabla(\nabla \times \varphi)|^2 dx dt \\
& + s^3 \lambda^4 \int_{Q_{T/2}} e^{-2ms\alpha} \xi^3 |\psi|^2 dx dt + s \lambda^2 \int_{Q_{T/2}} e^{-2ms\alpha} \xi |\nabla \psi|^2 dx dt \\
& + s^{-1} \int_{Q_{T/2}} e^{-2ms\alpha} \xi^{-1} |\Delta \psi|^2 dx dt \\
& \leq C \left(s^3 \lambda^4 \int_{(T/2, T) \times \omega^*} e^{-2s\alpha - 2s\mu\alpha^*} \xi^3 |\nabla \wedge \varphi|^2 dx dt \right. \\
& + s^3 \lambda^4 \int_{(T/2, T) \times \omega^*} e^{-2ms\alpha} \xi^3 |\psi|^2 dx dt + \int_{Q_{T/2}} e^{-2s\mu\alpha^*} |G|^2 dx dt \\
& + \int_{Q_{T/2}} e^{-2ms\alpha} |g|^2 dx dt \\
& \left. + \varepsilon^2 \int_{Q_{T/2}} \exp\left(\frac{-2\nu}{(T-t)^5}\right) e^{-2s\mu\alpha^*} (|\varphi|^2 + |\nabla \varphi|^2) dx dt \right). \tag{2.54}
\end{aligned}$$

By setting $m = \frac{3}{2}(1 + \mu)$ and simplifying the exponential weights using $\hat{\alpha} \leq \alpha \leq \alpha^*$, the estimate (2.54) can be expressed as:

$$\begin{aligned}
& s^3 \lambda^4 \int_{Q_{T/2}} e^{-2s(1+\mu)\alpha^*} \xi^3 |\nabla \times \varphi|^2 dx dt + s \lambda^2 \int_{Q_{T/2}} e^{-2s(1+\mu)\alpha^*} \xi |\nabla(\nabla \times \varphi)|^2 dx dt \\
& + s^3 \lambda^4 \int_{Q_{T/2}} e^{-3s(1+\mu)\alpha^*} \xi^3 |\psi|^2 dx dt + s \lambda^2 \int_{Q_{T/2}} e^{-3s(1+\mu)\alpha^*} \xi |\nabla \psi|^2 dx dt \\
& \leq C \left(s^3 \lambda^4 \int_{(T/2, T) \times \omega^*} e^{-2s\mu\alpha^*} \xi^3 |\nabla \times \varphi|^2 dx dt \right. \\
& + s^3 \lambda^4 \int_{(T/2, T) \times \omega^*} e^{-3s(1+\mu)\hat{\alpha}} \xi^3 |\psi|^2 dx dt + \int_{Q_{T/2}} e^{-2s\mu\alpha^*} |G|^2 dx dt \\
& + \int_{Q_{T/2}} e^{-3s(1+\mu)\hat{\alpha}} |g|^2 dx dt \\
& \left. + \varepsilon^2 \int_{Q_{T/2}} \exp\left(\frac{-2\nu}{(T-t)^5}\right) e^{-2s\mu\alpha^*} (|\varphi|^2 + |\nabla \varphi|^2) dx dt \right). \tag{2.55}
\end{aligned}$$

We now simplify the left-hand side of the Carleman estimate using two classical inequalities for vector fields. First, since $\nabla \cdot \varphi = 0$ in Q_T and $\varphi = 0$ on Σ_T , the full

gradient of φ is controlled by its curl:

$$\int_{Q_{T/2}} e^{-2s(1+\mu)\alpha^*} |\nabla \varphi|^2 dx dt \leq C \int_{Q_{T/2}} e^{-2s(1+\mu)\alpha^*} |\nabla \times \varphi|^2 dx dt. \quad (2.56)$$

Second, the Poincaré inequality provides control of the function itself by its gradient:

$$\int_{Q_{T/2}} e^{-2s(1+\mu)\alpha^*} |\varphi|^2 dx dt \leq C \int_{Q_{T/2}} e^{-2s(1+\mu)\alpha^*} |\nabla \varphi|^2 dx dt. \quad (2.57)$$

Combining the estimates (2.56) and (2.57) with the main inequality (2.55), we obtain the Carleman estimate for the adjoint solution. For sufficiently large parameters s and λ , we have:

$$\begin{aligned} & s^3 \lambda^4 \int_{Q_{T/2}} e^{-2s(1+\mu)\alpha^*} (|\varphi|^2 + |\psi|^2) dx dt + s \lambda^2 \int_{Q_{T/2}} e^{-2s(1+\mu)\alpha^*} (|\nabla \varphi|^2 + |\nabla \psi|^2) dx dt \\ & \leq C \left(s^3 \lambda^4 \int_{(T/2, T) \times \omega^*} e^{-2s\mu\alpha^*} \gamma^3 |\nabla \times \varphi|^2 dx dt + s^3 \lambda^4 \int_{(T/2, T) \times \omega^*} e^{-3s(1+\mu)\hat{\alpha}} \gamma^3 |\psi|^2 dx dt \right. \\ & \quad + \int_{Q_{T/2}} e^{-2s\mu\alpha^*} |G|^2 dx dt + \int_{Q_{T/2}} e^{-3s(1+\mu)\hat{\alpha}} |g|^2 dx dt \\ & \quad \left. + \varepsilon^2 \int_{Q_{T/2}} e^{\frac{-2\nu}{(T-t)^5}} e^{-2s\mu\alpha^*} (|\varphi|^2 + |\nabla \varphi|^2) dx dt \right). \end{aligned} \quad (2.58)$$

The next step is dedicated to absorbing the remainder term involving ε^2 on the right-hand side of the main estimate (2.58). This requires a careful comparison of the weight functions. First of all, we split the time interval $[T/2, T]$ into three parts:

$$\begin{aligned} \varepsilon^2 \int_{Q_{T/2}} e^{\frac{-2\nu}{(T-t)^5}} e^{-2s\mu\alpha^*} (|\varphi|^2 + |\nabla \varphi|^2) dx dt = \\ \varepsilon^2 \int_{(T/2, 5T/8) \times \Omega} e^{\frac{-2\nu}{(T-t)^5}} e^{-2s\mu\alpha^*} (|\varphi|^2 + |\nabla \varphi|^2) dx dt \\ + \varepsilon^2 \int_{(5T/8, 3T/4) \times \Omega} e^{\frac{-2\nu}{(T-t)^5}} e^{-2s\mu\alpha^*} (|\varphi|^2 + |\nabla \varphi|^2) dx dt \\ + \varepsilon^2 \int_{(3T/4, T) \times \Omega} e^{\frac{-2\nu}{(T-t)^5}} e^{-2s\mu\alpha^*} (|\varphi|^2 + |\nabla \varphi|^2) dx dt = \Lambda_1 + \Lambda_2 + \Lambda_3. \end{aligned} \quad (2.59)$$

To deal with Λ_1 , we apply the following energy estimate:

$$\begin{aligned} & \int_{T/2}^{5T/8} \left(\|\varphi(t)\|_{H^1(\Omega)}^2 + \|\psi(t)\|_{H^1(\Omega)}^2 \right) dt \\ & \leq C \left(\int_{5T/8}^{3T/4} \left(\|\varphi(t)\|_{L^2(\Omega)}^2 + \|\psi(t)\|_{L^2(\Omega)}^2 \right) dt + \int_{(T/2, 3T/4) \times \Omega} (|G|^2 + |g|^2) dx dt \right). \end{aligned} \quad (2.60)$$

Therefore, using the boundness of the weight function combined with (2.60):

$$\Lambda_1 \leq \varepsilon^2 \int_{Q_{T/2}} e^{\frac{-2\nu}{(t-\frac{T}{2})^5(T-t)^5}} e^{-2s\mu\alpha^*} (|\varphi|^2 + |\nabla\varphi|^2) dx dt + C \int_{(T/2, 3T/4) \times \Omega} (|G|^2 + |g|^2) dx dt. \quad (2.61)$$

analogously

$$\Lambda_2 \leq \varepsilon^2 \int_{Q_{T/2}} e^{\frac{-2\nu}{(t-\frac{T}{2})^5(T-t)^5}} e^{-2s\mu\alpha^*} (|\varphi|^2 + |\nabla\varphi|^2) dx dt. \quad (2.62)$$

Finally, using the bound

$$\begin{aligned} & \frac{1}{(t - \frac{T}{2})(T - t)} \leq \frac{4}{T(T - t)} \quad \text{for } t \in (3T/4, T) \\ & \Lambda_3 \leq \varepsilon^2 \int_{(3T/4, T) \times \Omega} e^{-(\frac{T}{4})^5 \frac{2\nu}{(t-\frac{T}{2})^5(T-t)^5}} e^{-2s\mu\alpha^*} (|\varphi|^2 + |\nabla\varphi|^2) dx dt. \end{aligned} \quad (2.63)$$

By defining the constants

$$\begin{aligned} R_0 &:= e^{12\lambda\|\eta^0\|_{L^\infty(\Omega)}} - e^{10\lambda\|\eta^0\|_{L^\infty(\Omega)}}, \\ R_1 &:= e^{12\lambda\|\eta^0\|_{L^\infty(\Omega)}} - e^{11\lambda\|\eta^0\|_{L^\infty(\Omega)}}, \end{aligned}$$

we have that

$$\alpha^*(t) = \frac{R_0}{(t - \frac{T}{2})^5(T - t)^5} \quad \text{and} \quad \hat{\alpha}(t) = \frac{R_1}{(t - \frac{T}{2})^5(T - t)^5}. \quad (2.64)$$

Substituting the bounds for the weight functions (2.64) into the estimate (2.58), and after choosing ε small enough to absorb the last remainder term, we obtain the Carleman

inequality:

$$\begin{aligned}
& \int_{Q_{T/2}} e^{\frac{-2s(1+\mu)R_0}{(t-\frac{T}{2})^5(T-t)^5}} (|\varphi|^2 + |\nabla\varphi|^2) dx dt + \int_{Q_{T/2}} e^{\frac{-3s(1+\mu)R_0}{(t-\frac{T}{2})^5(T-t)^5}} (|\psi|^2 + |\nabla\psi|^2) dx dt \\
& \leq C \left(\int_{(T/2, T) \times \omega^*} e^{\frac{-2s\mu R_1}{(t-\frac{T}{2})^5(T-t)^5}} |\nabla \times \varphi|^2 dx dt + \int_{(T/2, T) \times \omega^*} e^{\frac{-3s(1+\mu)R_1}{(t-\frac{T}{2})^5(T-t)^5}} |\psi|^2 dx dt \right. \\
& + \varepsilon^2 \int_{Q_{T/2}} e^{\frac{-2c^*\nu + 2s\mu R_1}{(t-\frac{T}{2})^5(T-t)^5}} (|\varphi|^2 + |\nabla\varphi|^2) dx dt \\
& \left. + \int_{Q_{T/2}} e^{\frac{-2s\mu R_1}{(T-t)^5}} |G|^2 dx dt + \int_{Q_{T/2}} e^{\frac{-3s(1+\mu)R_1}{(T-t)^5}} |g|^2 dx dt \right). \tag{2.65}
\end{aligned}$$

where $c^* = \min\{1, (T/4)^5\}$. Note that the weight functions appearing on the integral of G and g have an exponential decay only at $t = T$. This is due to the fact that in (2.61) the time integrals of these terms are localized at the time interval $(\frac{T}{2}, \frac{3T}{4})$, zone where the corresponding weight can be bounded from below.

Finally, the last term on the right-hand side of (2.58) involving ε^2 can be absorbed into the left-hand side. For a sufficiently small ε , this is possible if its exponential weight is smaller than the corresponding weight on the left-hand side. That is, using that $R_1 < R_0$, we require:

$$e^{\frac{-2c^*\nu + 2s\mu R_1}{(t-\frac{T}{2})^5(T-t)^5}} \leq e^{\frac{-2s(1+\mu)R_0}{(t-\frac{T}{2})^5(T-t)^5}}.$$

This is equivalent to the condition $2s(1+\mu)R_0 \leq 2c^*\nu + 2s\mu R_1$, which we can write as a requirement for the parameter ν :

$$\nu \geq \frac{s(1+\mu)R_0 - s\mu R_1}{c^*}. \tag{2.66}$$

Since all other parameters (s, μ, R_0, R_1, c^*) are fixed at this stage, this inequality can always be satisfied by choosing the parameter ν in our trajectory construction to be sufficiently large.

Therefore, after choosing the parameter ν large enough to satisfy (2.66), the final remainder term is absorbed, and the inequality (2.65) becomes:

$$\begin{aligned}
& \int_{Q_{T/2}} e^{\frac{-2s(1+\mu)R_0}{(t-\frac{T}{2})^5(T-t)^5}} (|\varphi|^2 + |\nabla\varphi|^2) dx dt + \int_{Q_{T/2}} e^{\frac{-3s(1+\mu)R_0}{(t-\frac{T}{2})^5(T-t)^5}} (|\psi|^2 + |\nabla\psi|^2) dx dt \\
& \leq C \left(\int_{(T/2, T) \times \omega^*} e^{\frac{-2s\mu R_1}{(t-\frac{T}{2})^5(T-t)^5}} |\nabla \times \varphi|^2 dx dt + \int_{(T/2, T) \times \omega^*} e^{\frac{-3s(1+\mu)R_1}{(t-\frac{T}{2})^5(T-t)^5}} |\psi|^2 dx dt \right. \\
& \left. + \int_{Q_{T/2}} e^{\frac{-2s\mu R_1}{(T-t)^5}} |G|^2 dx dt + \int_{Q_{T/2}} e^{\frac{-3s(1+\mu)R_1}{(T-t)^5}} |g|^2 dx dt \right). \tag{2.67}
\end{aligned}$$

To simplify the weight expressions, we define the constants

$$\begin{aligned} K_0 &:= 2s(1 + \mu)R_0, \\ K_1 &:= 2s\mu R_1. \end{aligned}$$

With this notation, for the condition (2.66) required for the absorption of the final remainder term to hold, it is sufficient to have:

$$\nu \geq \frac{1}{c^*} \left(\frac{K_0 - K_1}{2} \right). \quad (2.68)$$

As noted before, since all parameters on the right-hand side are fixed, this inequality can always be satisfied by choosing the trajectory parameter ν to be sufficiently large.

Observe that if ν satisfies

$$\nu \geq \frac{1}{c^*} (K_0 - K_1). \quad (2.69)$$

then (2.68) holds.

We now analyze the asymptotic behavior of the constants K_0 and K_1 as the parameters λ and μ become large. We first note the following limits:

$$\lim_{\lambda \rightarrow +\infty} \frac{e^{12\lambda\|\eta^0\|_\infty} - e^{10\lambda\|\eta^0\|_\infty}}{e^{12\lambda\|\eta^0\|_\infty} - e^{11\lambda\|\eta^0\|_\infty}} = 1 \quad \text{and} \quad \lim_{\mu \rightarrow +\infty} \frac{1 + \mu}{\mu} = 1.$$

Using the definitions of R_0 and R_1 , the ratio of K_0 and K_1 is

$$\begin{aligned} \frac{K_0}{K_1} &= \frac{2s(1 + \mu)R_0}{2s\mu R_1} = \frac{(1 + \mu)(e^{12\lambda A} - e^{10\lambda A})}{\mu(e^{12\lambda A} - e^{11\lambda A})} \\ &= \left(\frac{1 + \mu}{\mu} \right) \left(\frac{e^{12\lambda A} - e^{10\lambda A}}{e^{12\lambda A} - e^{11\lambda A}} \right), \end{aligned}$$

where $A := \|\eta^0\|_{L^\infty(\Omega)}$. Taking the limit as $\lambda, \mu \rightarrow +\infty$, we find that:

$$\lim_{\lambda, \mu \rightarrow +\infty} \frac{K_0}{K_1} = 1.$$

Consequently, from the asymptotic behavior of the ratio K_0/K_1 , we can conclude the following. For any $r \in (0, 1)$, there exist constants $\mu_0 > 0$ and $\lambda_0 > 0$ such that for all $\mu \geq \mu_0$ and $\lambda \geq \lambda_0$:

$$K_0 < \frac{K_1}{r}. \quad (2.70)$$

By making a similar analysis with

$$K_2 := 3s(1 + \mu)R_0,$$

$$K_3 := 3s(1 + \mu)R_1,$$

For any $\tilde{r} \in (0, 1)$, there exist constants $\mu_0 > 0$ and $\lambda_0 > 0$ such that for all $\mu \geq \mu_0$ and $\lambda \geq \lambda_0$:

$$K_2 < \frac{K_3}{\tilde{r}}. \quad (2.71)$$

Therefore, (2.67) becomes

$$\begin{aligned} & \int_{Q_{T/2}} e^{\left(\frac{-K_1}{r(t-\frac{T}{2})^5(T-t)^5}\right)} (|\varphi|^2 + |\nabla \varphi|^2) dx dt + \int_{Q_{T/2}} e^{\left(\frac{-K_3}{\tilde{r}(t-\frac{T}{2})^5(T-t)^5}\right)} (|\psi|^2 + |\nabla \psi|^2) dx dt \\ & \leq C \left(\int_{(T/2, T) \times \omega^*} e^{\left(\frac{-K_1}{(t-\frac{T}{2})^5(T-t)^5}\right)} |\nabla \times \varphi|^2 dx dt + \int_{Q_{T/2}} e^{\left(\frac{-K_1}{(T-t)^5}\right)} |G|^2 dx dt \right. \\ & \quad \left. + \int_{(T/2, T) \times \omega^*} e^{\left(\frac{-K_3}{(t-\frac{T}{2})^5(T-t)^5}\right)} |\psi|^2 dx dt + \int_{Q_{T/2}} e^{\left(\frac{-K_3}{(T-t)^5}\right)} |g|^2 dx dt \right). \end{aligned} \quad (2.72)$$

for all $\mu \geq \mu_0$, $\lambda \geq \lambda_0$, and $r, \tilde{r} \in (0, 1)$.

Finally, by combining the inequality (2.72) with classical energy estimates for the adjoint system, one can deduce the full estimate stated in the lemma. This concludes the proof. ■

Remark 2.12 (Choice of the Trajectory Parameter ν) *We can now choose the trajectory parameter ν explicitly to satisfy the absorption condition (2.69). Let us define*

$$\nu := \frac{1}{c^*} \left(\frac{1}{r} - 1 \right) K_1. \quad (2.73)$$

For this choice of ν to be positive, we must have $r < 1$. Using that $K_1 > rK_0$, it is not difficult to see that this parameter satisfies (2.69).

2.3.3 Null Controllability with Fictitious Controls

We begin by defining the various time-dependent weight functions used in this section, which depend on the constant $K_1, K_3 > 0$ from the Carleman estimate.

First, we define the one-sided weights, which are singular only at the final time $t = T$:

$$\rho_1(t) := e^{\left(-\frac{K_1}{(T-t)^5}\right)}, \quad (2.74)$$

$$\rho_r(t) := e^{\left(-\frac{K_3}{r(T-t)^5}\right)}. \quad (2.75)$$

Next, we define the symmetric weights, which are singular at both $t = T/2$ and $t = T$:

$$\rho_1^*(t) := e^{\left(-\frac{K_1}{(t-T/2)^5(T-t)^5}\right)}, \quad (2.76)$$

$$\rho_r^*(t) := e^{\left(-\frac{K_1}{r(t-T/2)^5(T-t)^5}\right)}. \quad (2.77)$$

$$\rho_1^{**}(t) := e^{\left(-\frac{K_3}{(t-T/2)^5(T-t)^5}\right)}, \quad (2.78)$$

$$\rho_r^{**}(t) := e^{\left(-\frac{K_3}{\tilde{r}(t-T/2)^5(T-t)^5}\right)}. \quad (2.79)$$

Using the Carleman inequality established above, we now prove a null-controllability result for a system with controls that have a specific derivative structure. Let $\chi_{\omega_0} \in C_0^\infty(\omega_0)$ be a non-trivial cutoff function. We have the following result:

Proposition 2.13 *Under the assumptions of Lemma 2.10, and for any parameter satisfying $1/2 < r < 1$ and $1/2 < \tilde{r} < 1$, consider the linearized Boussinesq control system:*

$$\begin{cases} y_t - \Delta y + (\bar{y} \cdot \nabla)y + (y \cdot \nabla)\bar{y} + \nabla p = f_1 + \theta e_3 + \nabla \times ((\nabla \times v)\chi_{\omega_0}) & \text{in } Q_T, \\ \theta_t - \Delta \theta + \bar{y} \cdot \nabla \theta + y \cdot \nabla \bar{\theta} = f_2 + \chi_{\omega_0} u & \text{in } Q_T, \\ \nabla \cdot y = 0 & \text{in } Q_T, \\ y = 0, \theta = 0 & \text{on } \Sigma_T, \end{cases} \quad (2.80)$$

with controls (v, u) and with source terms satisfying $\rho_r^{-1}(f_1, f_2) \in L^2(Q_T)^4$.

Then for every initial data $(y_0, \theta_0) \in V \times H_0^1(\Omega)$, there exists a solution (y, θ, p) and controls (u, v) to (2.80) satisfying the initial condition $(y(0), \theta(0)) = (y_0, \theta_0)$ and the null controllability condition $(y(T), \theta(T)) = (0, 0)$.

Moreover, for every $\tilde{K}_1 \in (0, K_1)$, $\tilde{K}_3 \in (0, K_3)$, the solution and controls satisfy the following regularity properties:

$$e^{\frac{\tilde{K}_1(2-1/r)}{2(t-\frac{T}{2})^5(T-t)^5}} (\nabla \times v)\chi_{\omega_0} \in L^2(0, T; H^{53}(\Omega)^3) \cap H^{27}(0, T; H^{-1}(\Omega)^3), \quad (2.81)$$

$$e^{\frac{\tilde{K}_3(2-1/\tilde{r})}{2(t-\frac{T}{2})^5(T-t)^5}} u\chi_{\omega_0} \in L^2(0, T; H^{54}(\Omega)) \cap H^{27}(0, T; L^2(\Omega)), \quad (2.82)$$

$$e^{\left(\frac{\tilde{K}_1}{2(T-t)^5}\right)} y \in L^2(0, T; H^2(\Omega)^3) \cap L^\infty(0, T; H^1(\Omega)^3), \quad (2.83)$$

$$e^{\left(\frac{\tilde{K}_1}{2(T-t)^5}\right)} \theta \in L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega)). \quad (2.84)$$

Proof: The proof is divided into two time intervals. For a given initial state $(y_0, \theta_0) \in$

$V \times H_0^1(\Omega)$, we first let the solution of system (2.80) evolve naturally with zero controls ($u = v = 0$) on the interval $(0, T/2)$. We denote the state reached at time $T/2$ as:

$$(y^{T/2}, \theta^{T/2}) := (y(T/2, \cdot), \theta(T/2, \cdot)).$$

The core of the proof is then to construct controls (u, v) on the interval $(T/2, T)$ that steer the system from this intermediate state $(y^{T/2}, \theta^{T/2})$ to the zero state at time T . A key property of these constructed controls is that they decay exponentially as $t \rightarrow T/2^+$ and $t \rightarrow T^-$, allowing for a smooth transition to zero control outside this interval.

To reduce the problem to one with zero initial data at $t = T/2$, we introduce a smooth cutoff function in time. Let $\epsilon \in (0, T/4)$ be a small constant, and let $\gamma \in C^\infty([T/2, T])$ be a function such that

$$\gamma(t) = \begin{cases} 1 & \text{if } t \in [T/2, T/2 + \epsilon], \\ 0 & \text{if } t \in [T - \epsilon, T]. \end{cases} \quad (2.85)$$

We then perform the change of variables $(z, q, \pi) := (y, \theta, p) - \gamma(y^*, \theta^*, p^*)$, where (y^*, θ^*, p^*) is a reference solution designed to satisfy the initial data at $t = T/2$. This reference solution solves the following system on $Q_{T/2} := (T/2, T) \times \Omega$:

$$\begin{cases} y_t^* - \Delta y^* + (\bar{y} \cdot \nabla) y^* + (y^* \cdot \nabla) \bar{y} + \nabla p^* = f_1 + \theta^* e_3, & \text{in } Q_{T/2}, \\ \theta_t^* - \Delta \theta^* + \bar{y} \cdot \nabla \theta^* + y^* \cdot \nabla \bar{\theta} = f_2, & \text{in } Q_{T/2}, \\ \nabla \cdot y^* = 0, & \text{in } Q_{T/2}, \\ y^* = 0, \theta^* = 0, & \text{on } \Sigma_{T/2}, \\ y^*(T/2, \cdot) = y^{T/2}, \theta^*(T/2, \cdot) = \theta^{T/2}, & \text{in } \Omega. \end{cases} \quad (2.86)$$

A direct computation shows that the new variables (z, q, π) solve the following system, which now has the desired zero initial data:

$$\begin{cases} z_t - \Delta z + (\bar{y} \cdot \nabla) z + (z \cdot \nabla) \bar{y} + \nabla \pi = F_1 + q e_3 + \nabla \times ((\nabla \times v) \chi_{\omega_0}), & \text{in } Q_{T/2}, \\ q_t - \Delta q + \bar{y} \cdot \nabla q + z \cdot \nabla \bar{\theta} = F_2 + \chi_{\omega_0} u, & \text{in } Q_{T/2}, \\ \nabla \cdot z = 0, & \text{in } Q_{T/2}, \\ z = 0, q = 0, & \text{on } \Sigma_{T/2}, \\ z(T/2, \cdot) = 0, q(T/2, \cdot) = 0, & \text{in } \Omega, \end{cases} \quad (2.87)$$

where the new source terms F_1 and F_2 are defined as:

$$F_1 := (1 - \gamma)f_1 - \gamma' y^*, \quad (2.88)$$

$$F_2 := (1 - \gamma)f_2 - \gamma'\theta^*.$$

In summary, this change of variables converts the problem of controlling system (2.80) from a non-zero initial state into an equivalent problem for system (2.87) with a zero initial state. This transformation is advantageous because the new source terms (F_1, F_2) preserve the required weighted regularity,

$$(\rho_r^{-1/2}F_1, \rho_r^{-1/2}F_2) \in (L^2(Q_T))^3 \times L^2(Q_T), \quad (2.89)$$

and also have the additional property of vanishing in a neighborhood of the initial time $t = T/2$.

Let P denote the Leray projector, which is the orthogonal projector from $(L^2(\Omega))^3$ onto the space of divergence-free vector fields. For any field $\varphi \in (L^2(\Omega))^3$, its projection is given by the Helmholtz-Hodge decomposition:

$$P\varphi := \varphi - \nabla p, \quad (2.90)$$

where the scalar potential p is the unique (up to an additive constant) solution to the Neumann problem:

$$\begin{cases} \Delta p = \nabla \cdot \varphi & \text{in } \Omega, \\ \frac{\partial p}{\partial \eta} = \varphi \cdot \eta & \text{on } \partial\Omega. \end{cases} \quad (2.91)$$

Here, η is the unit outward normal vector to the boundary $\partial\Omega$.

The operator P has two fundamental properties. First, it exhibits a commutation property with the Laplacian for smooth vector fields with compact support, meaning $P(\Delta\varphi) = \Delta(P\varphi)$ for all $\varphi \in C_0^\infty(\Omega)^3$. Furthermore, by density, P extends uniquely to a continuous linear operator on Sobolev spaces. As a zero-order operator, it preserves the regularity space, yielding the bounded map:

$$P : H^{-1}(\Omega)^3 \rightarrow H^{-1}(\Omega)^3. \quad (2.92)$$

For simplicity, we continue to denote this extended operator by P .

We now define the main differential operators for the linearized and adjoint systems. Let P be the Leray projector. These operators map from a space of functions to a space of distributions as follows:

$$S, S^* : \mathcal{D}'((T/2, T); H_0^1(\Omega)^3 \times H_0^1(\Omega)) \rightarrow \mathcal{D}'((T/2, T); H^{-1}(\Omega)^3 \times H^{-1}(\Omega)).$$

The operator for the adjoint system, S , is given by:

$$S(\varphi, \psi) := \begin{pmatrix} -\varphi_t - P(\Delta\varphi + (\nabla\varphi)\bar{y} + (\nabla\bar{y})^T\varphi - \psi\nabla\bar{\theta}) \\ -\psi_t - \Delta\psi - \bar{y} \cdot \nabla\psi - \varphi \cdot e_3 \end{pmatrix}. \quad (2.93)$$

Its formal adjoint, the operator for the linearized system, S^* , is:

$$S^*(y, \theta) := \begin{pmatrix} y_t - P(\Delta y - (\bar{y} \cdot \nabla)y - (y \cdot \nabla)\bar{y} + \theta e_3) \\ \theta_t - \Delta\theta + \bar{y} \cdot \nabla\theta + y \cdot \nabla\bar{\theta} \end{pmatrix}. \quad (2.94)$$

Recall that $Q_{T/2} := (T/2, T) \times \Omega$. We define a closed unbounded differential operator

$$\mathcal{S} : (L^2(Q_{T/2}))^4 \rightarrow (L^2(Q_{T/2}))^4$$

on the domain

$$\begin{aligned} \mathcal{D}(\mathcal{S}) := \Big\{ (\varphi, \psi) \mid & \varphi \in L^2((T/2, T); (H^2 \cap H_0^1)(\Omega)^3) \cap H^1((T/2, T); (L^2(\Omega))^3), \\ & \psi \in L^2((T/2, T); (H^2 \cap H_0^1)(\Omega)) \cap H^1((T/2, T); L^2(\Omega)), \\ & \varphi(T, \cdot) = 0, \psi(T, \cdot) = 0 \text{ in } \Omega \Big\}. \end{aligned} \quad (2.95)$$

The action of the operator is given by

$$\mathcal{S}(\varphi, \psi) := \begin{pmatrix} -\varphi_t - P(\Delta\varphi + (\nabla\varphi)\bar{y} + (\nabla\bar{y})^T\varphi - \psi\nabla\bar{\theta}) \\ -\psi_t - \Delta\psi - \bar{y} \cdot \nabla\psi - \varphi \cdot e_3 \end{pmatrix}. \quad (2.96)$$

Based on the operator $\mathcal{S}$, we define a scale of Hilbert spaces. For any non-negative integer m , we define the space X_m as the domain of the m -th power of $\mathcal{S}$:

$$X_m := \mathcal{D}(\mathcal{S}^m). \quad (2.97)$$

The spaces with negative indices are defined by duality:

$$X_{-m} := (X_m)', \quad (2.98)$$

with $(L^2(Q_{T/2}))^4$ serving as the pivot space. Consequently, for any integers $k \leq l$, we have the continuous embedding:

$$X_l \hookrightarrow X_k. \quad (2.99)$$

The space X_m is a Hilbert space when endowed with the inner product

$$\langle (\varphi_1, \psi_1), (\varphi_2, \psi_2) \rangle_{X_m} := \langle \mathcal{S}^m(\varphi_1, \psi_1), \mathcal{S}^m(\varphi_2, \psi_2) \rangle_{(L^2(Q_{T/2}))^4}, \quad (2.100)$$

for any $(\varphi_1, \psi_1), (\varphi_2, \psi_2) \in X_m$. The norm induced by this inner product is denoted by $\|\cdot\|_{X_m}$.

By duality, the adjoint operator $\mathcal{S}^*$ can be extended to a bounded linear operator on the scale of spaces $\{X_m\}_{m \in \mathbb{Z}}$. For any non-negative integer m , this extension maps X_{-m} to X_{-m-1} and is defined by the relation:

$$\langle \mathcal{S}^* z_1, z_2 \rangle_{X_{-m-1}, X_{m+1}} = \langle z_1, \mathcal{S} z_2 \rangle_{X_{-m}, X_m}, \quad (2.101)$$

which holds for all $z_1 \in X_{-m}$ and $z_2 \in X_{m+1}$.

We define the space $\mathcal{H}_0$ as the set of pairs (φ, ψ) belonging to the regularity class

$$\begin{aligned} & (L^2((T/2, T); (V \cap H^2(\Omega))^3) \cap H^1((T/2, T); (L^2(\Omega))^3)) \\ & \times (L^2((T/2, T); (H_0^1 \cap H^2)(\Omega)) \cap H^1((T/2, T); L^2(\Omega))) \end{aligned}$$

that additionally satisfy the following two conditions:

$$\sqrt{\rho_1} S(\varphi, \psi) \in X_{26}, \quad (2.102)$$

$$(\sqrt{\chi_{\omega_0} \rho_1^*} \nabla \wedge \varphi, \sqrt{\chi_{\omega_0} \rho_1^{**}} \psi) \in (L^2(Q_{T/2}))^4. \quad (2.103)$$

Note that condition (2.102) involves the operator S , not the unbounded operator $\mathcal{S}$. This is a crucial distinction. The domain of $\mathcal{S}$, $\mathcal{D}(\mathcal{S})$, requires functions to be zero at the final time $t = T$. Since the space $\mathcal{H}_0$ is being defined without this a priori condition, we use the formal operator S to represent the action of the differential expression in the sense of distributions.

We endow the space $\mathcal{H}_0$ with the inner product $a(\cdot, \cdot)$ defined by

$$\begin{aligned} a((\varphi_1, \psi_1), (\varphi_2, \psi_2)) &:= \langle \sqrt{\rho_1} S(\varphi_1, \psi_1), \sqrt{\rho_1} S(\varphi_2, \psi_2) \rangle_{X_{26}} \\ &+ \int_{Q_{T/2}} \chi_{\omega_0} \rho_1^* (\nabla \wedge \varphi_1) \cdot (\nabla \wedge \varphi_2) dx dt \\ &+ \int_{Q_{T/2}} \chi_{\omega_0} \rho_1^{**} \psi_1 \psi_2 dx dt. \end{aligned} \quad (2.104)$$

where (φ_1, ψ_1) and (φ_2, ψ_2) are two elements of $\mathcal{H}_0$. The Carleman estimate (2.72) ensures that the induced norm $\|(\varphi, \psi)\|_{\mathcal{H}_0} := \sqrt{a((\varphi, \psi), (\varphi, \psi))}$ is indeed a norm on $\mathcal{H}_0$.

Let $\mathcal{H}$ denote the completion of $\mathcal{H}_0$ with respect to the inner product $a(\cdot, \cdot)$. By construction, any element $(\varphi, \psi) \in \mathcal{H}$ satisfies the regularity conditions (2.102) and (2.103).

Furthermore, as a consequence of the full Carleman proof, we have the final weighted

estimate that links the norm of the space $\mathcal{H}$ to the standard L^2 norm. There exists a constant $C > 0$ such that for all $(\varphi, \psi) \in \mathcal{H}$:

$$\|\sqrt{\rho_r^*}\varphi\|_{L^2(Q_{T/2})^3} + \|\sqrt{\rho_r^{**}}\psi\|_{L^2(Q_{T/2})} \leq C\|(\varphi, \psi)\|_{\mathcal{H}}. \quad (2.105)$$

With the source terms (F_1, F_2) defined as in (2.88), we now define the linear functional $l : \mathcal{H} \rightarrow \mathbb{R}$ by

$$l(\varphi, \psi) := \int_{Q_{T/2}} F_1 \cdot \varphi \, dx \, dt + \int_{Q_{T/2}} F_2 \psi \, dx \, dt. \quad (2.106)$$

This linear form is continuous on $\mathcal{H}$. More precisely, using the Cauchy-Schwarz inequality and the estimate (2.105), we have:

$$\begin{aligned} |l(\varphi, \psi)| &\leq \int_{Q_{T/2}} |F_1 \cdot \varphi| \, dx \, dt + \int_{Q_{T/2}} |F_2 \psi| \, dx \, dt \\ &\leq \|(\rho_r^*)^{-1/2} F_1\|_{L^2(Q_{T/2})^3} \|(\rho_r^*)^{1/2} \varphi\|_{L^2(Q_{T/2})^3} + \|(\rho_r^{**})^{-1/2} F_2\|_{L^2(Q_{T/2})} \|(\rho_r^{**})^{1/2} \psi\|_{L^2(Q_{T/2})} \\ &\leq C \left(\|(y^*, \theta^*)\|_{L^2(Q_{T/2})} + \|\rho_r^{-1/2} f_1\|_{L^2(Q_{T/2})^3} + \|\rho_r^{-1/2} f_2\|_{L^2(Q_{T/2})} \right) \|(\varphi, \psi)\|_{\mathcal{H}}, \end{aligned} \quad (2.107)$$

where $C > 0$ is a constant independent of (φ, ψ) , and the original source terms (f_1, f_2) belong to the appropriate weighted L^2 space. Note that we changed the weight from ρ_r^* and ρ_r^{**} to ρ_r due to the fact that (F_1, F_2) vanishes near $t = \frac{T}{2}$.

Since l is a continuous linear functional on the Hilbert space $\mathcal{H}$, the Riesz representation theorem guarantees the existence of a unique pair

$$(\hat{\varphi}, \hat{\psi}) \in \mathcal{H} \quad (2.108)$$

such that for every $(\varphi, \psi) \in \mathcal{H}$, the following variational identity holds:

$$a((\hat{\varphi}, \hat{\psi}), (\varphi, \psi)) = l(\varphi, \psi). \quad (2.109)$$

By identifying the terms in the variational identity (2.109), we can now define the control terms (w_u, w_v) that solve the problem. They are constructed from the Riesz representer $(\hat{\varphi}, \hat{\psi})$ by first defining the intermediate controls

$$(u, v) := (\rho_1^{**} \hat{\psi}, \rho_1^* \hat{\varphi}) \quad (2.110)$$

and then setting the control terms as follows:

$$w_v := \nabla \times (\chi_{\omega_0} \nabla \times v), \quad (2.111)$$

$$w_u := \chi_{\omega_0} u. \quad (2.112)$$

The remainder of the proof is dedicated to showing that these controls, and the corresponding solution they generate, indeed satisfy the high-order regularity properties stated in Proposition 2.13.

First, we use the Riesz identity to get an estimate on the adjoint solution $(\hat{\varphi}, \hat{\psi})$. We start with the identity (2.109) and choose the test function to be the solution itself, i.e., $(\varphi, \psi) = (\hat{\varphi}, \hat{\psi})$. This gives:

$$\|(\hat{\varphi}, \hat{\psi})\|_{\mathcal{H}}^2 = l(\hat{\varphi}, \hat{\psi}).$$

Using the continuity estimate for l from (2.107), we have:

$$\|(\hat{\varphi}, \hat{\psi})\|_{\mathcal{H}} \leq C \left(\|(y^*, \theta^*)\|_{L^2(Q_{T/2})} + \|\rho_r^{-1/2} f_1\|_{L^2(Q_{T/2})^3} + \|\rho_r^{-1/2} f_2\|_{L^2(Q_{T/2})} \right). \quad (2.113)$$

This is the fundamental estimate for the adjoint solution.

Now, with the estimate (2.113) in hand, we can estimate the controls (w_u, w_v) . For the control w_v , we use a duality argument to estimate its norm in $L^2(T/2, T; H^{-1}(\Omega))$, for any test function $\xi \in L^2(T/2, T; H_0^1(\Omega))$ with unit norm:

$$\begin{aligned} \langle (\rho_1^*)^{-1/2} w_v, \xi \rangle_{L^2(H^{-1}), L^2(H_0^1)} &= \langle (\rho_1^*)^{-1/2} (-\rho_1^* \nabla \times (\chi_{\omega_0} \nabla \times \hat{\varphi})), \xi \rangle \\ &= \int_{Q_{T/2}} (\rho_1^*)^{1/2} \chi_{\omega_0} (\nabla \wedge \hat{\varphi}) \cdot (\nabla \wedge \xi) \, dx dt \\ &\leq \|\sqrt{\chi_{\omega_0} \rho_1^*} (\nabla \wedge \hat{\varphi})\|_{L^2(Q_{T/2})} \|\nabla \wedge \xi\|_{L^2(Q_{T/2})} \\ &\leq C \|\sqrt{\chi_{\omega_0} \rho_1^*} (\nabla \wedge \hat{\varphi})\|_{L^2(Q_{T/2})}. \end{aligned}$$

By the definition of the dual norm and the inner product $a(\cdot, \cdot)$, this implies:

$$\|(\rho_1^*)^{-1/2} w_v\|_{L^2(T/2, T; H^{-1}(\Omega))} \leq C \|\sqrt{\chi_{\omega_0} \rho_1^*} (\nabla \wedge \hat{\varphi})\|_{L^2(Q_{T/2})} \leq C \|(\hat{\varphi}, \hat{\psi})\|_{\mathcal{H}}.$$

For the control w_u , the computation is simpler as it requires no duality:

$$\begin{aligned} \|(\rho_1^{**})^{-1/2} w_u\|_{L^2(Q_{T/2})} &= \|(\rho_1^{**})^{-1/2} (-\rho_1^{**} \chi_{\omega_0} \hat{\psi})\|_{L^2(Q_{T/2})} = \|\sqrt{\rho_1^{**}} \chi_{\omega_0} \hat{\psi}\|_{L^2(Q_{T/2})} \\ &\leq C \|(\hat{\varphi}, \hat{\psi})\|_{\mathcal{H}}. \end{aligned}$$

The crucial consequence of these weighted estimates is that they formally imply that the controls (w_u, w_v) have exponential decay as the time t approaches the endpoints of the control interval, $T/2$ and T .

The final step is to construct the solution to the linearized problem using the element $(\hat{\varphi}, \hat{\psi}) \in \mathcal{H}$ found via the Riesz representation theorem. We first define an auxiliary

variable:

$$(\tilde{y}, \tilde{\theta}) := (\mathcal{S}^*)^{26} \mathcal{S}^{26} \left(\sqrt{\rho_1} S(\hat{\varphi}, \hat{\psi}) \right), \quad (2.114)$$

which, by construction, belongs to the space X_{-26} . The solution to our control problem is then given by:

$$(\hat{y}, \hat{\theta}) := \sqrt{\rho_1}(\tilde{y}, \tilde{\theta}). \quad (2.115)$$

By unwinding the variational identity (2.109), one can show that the pair $(\hat{y}, \hat{\theta})$ satisfies the following equation in the distributional sense:

$$\mathcal{S}^*(\hat{y}, \hat{\theta}) = (F_1, F_2) + (w_v, w_u) \in X_{-27}, \quad (2.116)$$

where (F_1, F_2) are the source terms defined in (2.88), and (w_v, w_u) are the control terms constructed from (v, u) as they appear in system (2.80).

Let us show that the controls are indeed more regular. For that, we need the following lemma.

Lemma 2.14 (Bootstrap Regularity) *For any integer $m \geq 0$, if a distribution y satisfies $y \in X_{-m}$ and $\mathcal{S}^*y \in X_{-m}$, then $y \in X_{-m+1}$.*

Proof: The proof is by induction on m . The base case $m = 0$ is a direct consequence of standard parabolic regularity theory for the operator $\mathcal{S}^*$.

Now, assume $m \geq 1$. Suppose $y \in X_{-m}$ and $\mathcal{S}^*y \in X_{-m}$. Our goal is to show that $y \in X_{-m+1}$. By definition, this is equivalent to showing that the linear functional $L_y(h) := \langle y, h \rangle$ is continuous on the space X_{m-1} .

Let $h \in X_{m-1}$. Since the operator $\mathcal{S} : X_m \rightarrow X_{m-1}$ is surjective, there exists a state $\alpha \in X_m$ such that $\mathcal{S}\alpha = h$. Using the definition of the adjoint operator, we can write the pairing as:

$$\langle y, h \rangle = \langle y, \mathcal{S}\alpha \rangle = \langle \mathcal{S}^*y, \alpha \rangle.$$

We now apply the Cauchy-Schwarz inequality. Since $\mathcal{S}^*y \in X_{-m}$ by hypothesis and $\alpha \in X_m$, we have:

$$|\langle y, h \rangle| = |\langle \mathcal{S}^*y, \alpha \rangle| \leq \|\mathcal{S}^*y\|_{X_{-m}} \|\alpha\|_{X_m}.$$

A key identity for the norms on this scale of spaces is that $\|\alpha\|_{X_m} = \|\mathcal{S}\alpha\|_{X_{m-1}}$. Since $h = \mathcal{S}\alpha$, this implies $\|\alpha\|_{X_m} = \|h\|_{X_{m-1}}$. Substituting this back into our estimate, we get:

$$|\langle y, h \rangle| \leq \|\mathcal{S}^*y\|_{X_{-m}} \|h\|_{X_{m-1}}.$$

Since $\|\mathcal{S}^*y\|_{X_{-m}}$ is a fixed finite constant, this inequality shows that the functional $L_y(h)$ is bounded on X_{m-1} . Therefore, by definition of the dual space, $y \in (X_{m-1})' = X_{-m+1}$.

■

Our starting point is the final observability estimate (2.105), which clearly implies that:

$$(\rho_1^*(\rho_1^*)^{-1}(\rho_r^*)^{\frac{1}{2}}\hat{\varphi}, \rho_1^{**}(\rho_1^{**})^{-1}(\rho_r^{**})^{\frac{1}{2}}\hat{\psi}) \in L^2(Q_{T/2})^4.$$

This baseline regularity of $(\hat{\varphi}, \hat{\psi})$ translates, via the control definitions (2.110), (2.111), and (2.112) into an initial, low-order regularity for the controls (u, v) . These controls then act as source terms in the direct system (2.80) for (y, θ) . By applying standard parabolic regularity theory, this gives a slightly improved regularity for the solution (y, θ) .

The Bootstrap Regularity Lemma 2.14 is the technical tool that formally justifies the gain of regularity at each step of this iteration.

To precisely measure the regularity at each step, we introduce slightly weaker weight functions by choosing parameters $\tilde{K}_1 < K_1$ and $\tilde{K}_3 < K_3$, for instance:

$$\tilde{\rho}_1^*(t) := e^{\left(-\frac{\tilde{K}_1}{(t-T/2)^5(T-t)^5}\right)}, \quad (2.117)$$

$$\tilde{\rho}_1^{**}(t) := e^{\left(-\frac{\tilde{K}_3}{(t-T/2)^5(T-t)^5}\right)}. \quad (2.118)$$

Applying $\mathcal{S}$, we get that:

$$\begin{aligned} \mathcal{S}(\rho_1^*(\tilde{\rho}_1^*)^{-1}(\rho_r^*)^{\frac{1}{2}}\hat{\varphi}, \rho_1^{**}(\tilde{\rho}_1^{**})^{-1}(\rho_r^{**})^{\frac{1}{2}}\hat{\psi}) &= \left(\left(\frac{1}{\tilde{\rho}_1^*} \right)' (\rho_r^*)^{\frac{1}{2}}\rho_1^*\hat{\varphi} + (\rho_1^*(\rho_r^*)^{\frac{1}{2}})' \frac{1}{\tilde{\rho}_1^*}\hat{\varphi} \right. \\ &+ \rho_1^*(\tilde{\rho}_1^*)^{-1}(\rho_r^*)^{\frac{1}{2}}L_1\hat{\varphi} + P(\rho_1^{**}(\tilde{\rho}_1^{**})^{-1}(\rho_r^{**})^{\frac{1}{2}}\hat{\psi}\nabla\bar{\theta}), \left(\frac{1}{\tilde{\rho}_1^{**}} \right)' (\rho_r^{**})^{\frac{1}{2}}\rho_1^{**}\hat{\psi} + (\rho_1^{**}(\rho_r^{**})^{\frac{1}{2}})' \frac{1}{\tilde{\rho}_1^{**}}\hat{\psi} \\ &\left. + \rho_1^{**}(\tilde{\rho}_1^{**})^{-1}(\rho_r^{**})^{\frac{1}{2}}L_2(\hat{\psi}) - \rho_1^*(\tilde{\rho}_1^*)^{-1}(\rho_r^*)^{\frac{1}{2}}\hat{\varphi} \cdot e_3 \right) \\ &= (D_1 + D_2 + D_3 - D_4, D_5 + D_6 + D_7 - D_8), \end{aligned} \quad (2.119)$$

where $L_1(\hat{\varphi}) = \hat{\varphi}_t - P(\Delta\hat{\varphi} + (\nabla\hat{\varphi})\bar{y} + (\nabla\bar{y})^T\hat{\varphi})$ and $L_2(\hat{\psi}) = \hat{\psi}_t - \Delta\hat{\psi} + \bar{y} \cdot \nabla\hat{\psi}$.

Since

$$\left(\frac{1}{\tilde{\rho}_1^*} \right)' \rho_1^* = \frac{-5\tilde{K}_1}{(t - \frac{T}{2})^6(T-t)^6} e^{\frac{\tilde{K}_1 - K_1}{(t - \frac{T}{2})^5(T-t)^5}} \leq C,$$

we get that $D_1 \in L^2(Q_{T/2})$. A similar analysis can be made to prove that D_2, D_5 and D_6 are in $L^2(Q_{T/2})$, by switching conveniently K_1 for K_3 .

For D_3 , we write it in the form:

$$D_3 = \rho_1^*(\tilde{\rho}_1^*)^{-1}(\rho_r^*)^{\frac{1}{2}}(\rho_1)^{-\frac{1}{2}}(\rho_1)^{\frac{1}{2}}L_1(\hat{\varphi}).$$

From (2.102) and

$$\rho_1^*(\tilde{\rho}_1^*)^{-1}(\rho_r^*)^{\frac{1}{2}}(\rho_1)^{-\frac{1}{2}} = e^{\frac{\tilde{K}_1 - K_1 + \frac{K_1(t-T/2)^5}{2} - \frac{K_1}{2r}}{(t-\frac{T}{2})^5(T-t)^5}}.$$

Since, $\tilde{K}_1 < K_1$ in order for the above exponential to be bounded, it suffices to require that

$$\frac{K_1(t - T/2)^5}{2} - \frac{K_1}{2r} < 0,$$

for every $t \in (T/2, T)$. Which hold if

$$\left(\frac{T}{2}\right)^5 < \frac{1}{r},$$

which is equivalent to $r < \frac{2^5}{T^5}$. Since r is a parameter belonging to $(0, 1)$, we take T sufficiently small so that $\frac{2^5}{T^5} \geq 1$.

A similar analysis can be made for D_7 by switching K_1 to K_3 . Indeed

$$\rho_1^{**}(\tilde{\rho}_1^{**})^{-1}(\rho_r^{**})^{\frac{1}{2}}(\rho_1)^{-\frac{1}{2}} = e^{\frac{\tilde{K}_3 - K_3 + \frac{K_1(t-T/2)^5}{2} - \frac{K_3}{2r}}{(t-\frac{T}{2})^5(T-t)^5}}.$$

Analogously to what was done previously, to obtain the boundedness of the function, it suffices to assume that

$$\frac{K_1(t - T/2)^5}{2} - \frac{K_3}{2r} < 0.$$

Since that $K_1 < K_3$, alternatively,

$$\frac{K_3(t - T/2)^5}{2} - \frac{K_3}{2r} < 0,$$

for every $t \in (T/2, T)$. Which hold again by asking $r \leq \frac{2^5}{T^5}$ and T small.

Therefore, we obtain that $D_7 \in L^2(Q_{T/2})$. The coupling terms D_4 and D_8 can be estimated in a straightforward manner using the Cauchy-Schwarz inequality and the properties of the weight functions. These estimates readily show that D_4 and D_8 also belong to $(L^2(Q_{T/2}))^4$.

We have established that all terms D_i belong to $X_0 = (L^2(Q_{T/2}))^4$. This is precisely the setup for the Bootstrap Regularity Lemma 2.14. Applying the lemma with $m = 0$ allows us to deduce a gain in regularity:

$$(\Phi, \Psi) = (\rho_1^*(\tilde{\rho}_1^*)^{-1}(\rho_r^*)^{1/2}\hat{\varphi}, \rho_1^{**}(\tilde{\rho}_1^{**})^{-1}(\rho_r^{**})^{1/2}\hat{\psi}) \in X_1. \quad (2.120)$$

This means the weighted state belongs to the domain of the operator $\mathcal{S}$.

By repeatedly applying the Bootstrap Regularity Lemma 2.14, and adjusting the parameters $\tilde{K}_1 < K_1$ and $\tilde{K}_3 < K_3$ as needed at each step, we can raise the regularity of the weighted state iteratively, getting $(\Phi, \Psi) \in X_2$.

After a sufficient number of iterations, we reach the final desired regularity:

$$(\Phi, \Psi) = (\rho_1^*(\tilde{\rho}_1^*)^{-1}(\rho_r^*)^{1/2}\hat{\varphi}, \rho_1^{**}(\tilde{\rho}_1^{**})^{-1}(\rho_{\tilde{r}}^{**})^{1/2}\hat{\psi}) \in X_{27}. \quad (2.121)$$

This final regularity for the weighted adjoint state is sufficient to prove the regularity of the controls and solution stated in Proposition 2.13.

In this way, we conclude the final regularity property for the controls. By substituting the definitions of u and v , the result that $(\Phi, \Psi) \in X_{27}$ from the iterative argument implies that this pair of weighted controls possesses high regularity:

$$\left(\frac{\sqrt{\rho_r^*}}{\tilde{\rho}_1^*} w_v, \frac{\sqrt{\rho_{\tilde{r}}^{**}}}{\tilde{\rho}_1^{**}} w_u \right) \in X_{25}. \quad (2.122)$$

Now, we return to the equation (2.116) to establish the final regularity for the solution $(\hat{y}, \hat{\theta})$. For any $\tilde{K}_1 \in (0, K_1)$, let's define the one-sided weight function:

$$\tilde{\rho}_1(t) := e^{\frac{-\tilde{K}_1}{(T-t)^5}}. \quad (2.123)$$

Our goal is to analyze the regularity of the weighted solution $(\tilde{\rho}_1)^{-1/2}(\hat{y}, \hat{\theta})$. First of all, it is not difficult to see that

$$(\tilde{\rho}_1)^{-1/2}(\hat{y}, \hat{\theta}) = (\tilde{\rho}_1)^{-1/2} \rho_1^{1/2}(\tilde{y}, \tilde{\theta}) \in X_{-26}, \quad (2.124)$$

since $\tilde{K}_1 < K_1$ and, therefore, $(\tilde{\rho}_1)^{-1/2} \rho_1^{1/2}$ is bounded.

From (2.116), we can see that:

$$\mathcal{S} \left(\frac{1}{\sqrt{\tilde{\rho}_1}}(\hat{y}, \hat{\theta}) \right) = \left(\frac{1}{\sqrt{\tilde{\rho}_1}} \right)' (\hat{y}, \hat{\theta}) + \frac{1}{\sqrt{\tilde{\rho}_1}}((F_1, F_2) + (w_v, w_u)). \quad (2.125)$$

Observe that

$$\frac{1}{\sqrt{\tilde{\rho}_1}}((F_1, F_2)) = \frac{1}{\sqrt{\tilde{\rho}_1}}(\rho_r^*)^{1/2}(\rho_r^*)^{-1/2}(F_1, F_2)$$

and

$$\frac{1}{\sqrt{\tilde{\rho}_1}}(\rho_r^*)^{1/2} = e^{\frac{\tilde{K}_1}{2(T-t)^5}} e^{-\frac{K_1}{2r(t-T/2)^5(T-t)^5}}$$

The above function is bounded for t close $T/2$ and for t close to T it is bounded if

$$\tilde{K}_1(t - T/2)^5 - \frac{K_1}{r} < 0$$

for every $t \in (T/2, T)$. Which hold if

$$\tilde{K}_1 \left(\frac{T}{2} \right)^5 < \frac{K_1}{r}$$

since $\tilde{K}_1 < K_1$, then $\frac{1}{r}\tilde{K}_1 < \frac{1}{r}K_1$, and the estimate follows by asking $\frac{1}{r} \geq \left(\frac{T}{2}\right)^5$. Which is equivalent to $r \leq \frac{2^5}{T^5}$. Since r is a parameter belonging to $(0, 1)$, we take T sufficiently small so that $\frac{2^5}{T^5} \geq 1$.

We concluded that

$$\frac{1}{\sqrt{\tilde{\rho}_1}}(F_1, F_2) \in L^2(Q_{T/2}). \quad (2.126)$$

Now, let us work with the controls. First, we writte

$$\frac{1}{\sqrt{\tilde{\rho}_1}}(w_v, w_u) = \left(\frac{\tilde{\rho}_1^*}{\sqrt{\tilde{\rho}_r^*}} \frac{1}{\sqrt{\tilde{\rho}_1}} \frac{\sqrt{\tilde{\rho}_r^*}}{\tilde{\rho}_1^*} w_v, \frac{\tilde{\rho}_1^{**}}{\sqrt{\tilde{\rho}_r^{**}}} \frac{1}{\sqrt{\tilde{\rho}_1}} \frac{\sqrt{\tilde{\rho}_r^{**}}}{\tilde{\rho}_1^{**}} w_u \right).$$

Analyzing the term involving w_v , using (2.122), we have that this term will lies in X_{25} if $\frac{\tilde{\rho}_1^*}{\sqrt{\tilde{\rho}_r^*}} \frac{1}{\sqrt{\tilde{\rho}_1}}$ is bounded. Indeed

$$\frac{\tilde{\rho}_1^*}{\sqrt{\tilde{\rho}_r^*}} \frac{1}{\sqrt{\tilde{\rho}_1}} = e^{-\frac{\tilde{K}_1}{(t-\frac{T}{2})^5(T-t)^5}} e^{\frac{K_1}{2r(t-\frac{T}{2})^5(T-t)^5}} e^{\frac{\tilde{K}_1}{2(T-t)^5}}.$$

This weight will be bounded if

$$\tilde{K}_1(t - T/2)^5 + \frac{K_1}{r} - 2\tilde{K}_1 < 0,$$

which is true if

$$\tilde{K}_1(T/2)^5 + \frac{K_1}{r} - 2\tilde{K}_1 < 0$$

which is equivalent to asking

$$\frac{1}{2 - (T/2)^5} < r.$$

Similarly, we have that for the term involving w_u that $\frac{\tilde{\rho}_1^{**}}{\sqrt{\tilde{\rho}_r^{**}}} \frac{1}{\sqrt{\tilde{\rho}_1}}$ is bounded.

Indeed

$$\frac{\tilde{\rho}_1^{**}}{\sqrt{\tilde{\rho}_r^{**}}} \frac{1}{\sqrt{\tilde{\rho}_1}} = e^{-\frac{\tilde{K}_3}{(t-\frac{T}{2})^5(T-t)^5}} e^{\frac{K_3}{2r(t-\frac{T}{2})^5(T-t)^5}} e^{\frac{\tilde{K}_1}{2(T-t)^5}}.$$

This weight will be bounded if

$$\frac{1}{(2 - (T/2)^5)} < \tilde{r}.$$

Again, for T sufficiently small $\frac{1}{2 - (T/2)^5} \in (0, 1)$, which turns feasible this choice for r and $\tilde{r}$.

In this way $\frac{1}{\sqrt{\rho_1}}(w_v, w_u) \in X_{25}$, which combined with (2.124), (2.125) and (2.126), implies that $\frac{1}{\sqrt{\rho_1}}(\hat{y}, \hat{\theta}) \in X_{-25}$. We can start the argument all over again in (2.124), by taking $\tilde{K}_1$ even smaller if necessary, but still sufficiently close to K_1 , replacing X_{-26} by X_{-25} . After using again (2.124), (2.125) and (2.126) a that $\frac{1}{\sqrt{\rho_1}}(w_v, w_u) \in X_{25}$ we conclude that $\frac{1}{\sqrt{\rho_1}}(\hat{y}, \hat{\theta}) \in X_{-24}$. We can proceed with this bootstrap argument until we reach

$$\frac{1}{\sqrt{\tilde{\rho}_1}}(\hat{y}, \hat{\theta}) \in X_1. \quad (2.127)$$

It is important to remark that this is the maximum regularity we can reach since we are limited to de regularity (2.126) for the source terms. ■

2.3.4 Null Controllability of the Linear System: Proof of Proposition 2.5

We are now in a position to prove Proposition 2.5. The proof synthesizes the key results from the preceding sections: the null controllability with fictitious controls established in Subsection 2.3.3 (the analytic part) and the algebraic solvability from Subsection 2.3.1 (the algebraic part).

The main technical step is to ensure these two frameworks are compatible. Specifically, while the algebraic solvability (Proposition 2.7) was established in a setting of smooth functions (C^∞), the analytic results rely on weighted Sobolev spaces. As we will show, the linear partial differential operator $\mathfrak{M}$, whose existence is guaranteed by the algebraic theory, can be extended to operate on these function spaces, thereby bridging the two results.

Proposition 2.15 *For $T > 0$ sufficiently small, there exist constants $r_0, r_1 \in (0, 1)$ and $C_1, C_2 > 0$ with the following property:*

For any initial data $(y_0, \theta_0) \in V \times H_0^1(\Omega)$, any source terms (f_1, f_2) such that $(\rho_r^{-1/2} f_1, \rho_{\tilde{r}}^{-1/2} f_2) \in L^2(Q_T)^3 \times L^2(Q_T)$, and any parameters $r \in (r_0, 1)$, $r^ \in (r_1, 1)$, $K_1 > C_1$, $K_3 > C_2$, with the trajectory parameter ν defined by*

$$\nu = \frac{1}{c^*} \left(\frac{1}{r} - 1 \right) K_1, \quad (2.128)$$

there exists a scalar control v and a corresponding solution (y, θ, p) to the linearized Boussinesq system

$$\begin{cases} y_t - \Delta y + (\bar{y} \cdot \nabla)y + (y \cdot \nabla)\bar{y} + \nabla p = f_1 + \theta e_3 & \text{in } Q_T, \\ \theta_t - \Delta \theta + \bar{y} \cdot \nabla \theta + y \cdot \nabla \bar{\theta} = f_2 + \chi_{\omega_0} v & \text{in } Q_T, \\ \nabla \cdot y = 0 & \text{in } Q_T, \\ y = 0, \theta = 0 & \text{on } \Sigma_T, \\ y(0, \cdot) = y_0, \theta(0, \cdot) = \theta_0 & \text{in } \Omega, \end{cases} \quad (2.129)$$

such that $(y(T, \cdot), \theta(T, \cdot)) = (0, 0)$.

Moreover, for some constant $\alpha > 0$ independent of the data, the solution satisfies the regularity estimates:

$$e^{\frac{\alpha K_1}{2(T-t)^5}} y \in L^2(0, T; H^2(\Omega)^3 \cap V) \cap L^\infty(0, T; V), \quad (2.130)$$

$$e^{\frac{\alpha K_1}{2(T-t)^5}} \theta \in L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap L^\infty(0, T; H_0^1(\Omega)), \quad (2.131)$$

$$e^{\frac{\alpha K_1}{2(T-t)^5}} v \in L^2(Q_T), \quad (2.132)$$

$$p \in L^2(0, T; H^1(\Omega)). \quad (2.133)$$

Proof:[Proof of Proposition 2.15] The proof is constructive and follows the two-step strategy outlined in the introduction. First, we solve a null controllability problem for an auxiliary system with a larger set of "fictitious" controls. Second, we use the algebraic solvability theory to show that these fictitious controls can be generated by a new state driven by a single thermal control. The sum of these two solutions will then solve the original problem.

We begin by applying Proposition 2.13. For the given initial data (y_0, θ_0) and source terms (f_1, f_2) , there exists a solution (y^*, θ^*, p^*) and a pair of fictitious controls (u^*, v^*) that solve the system (2.80), satisfying the null controllability condition $(y^*(T, \cdot), \theta^*(T, \cdot)) = (0, 0)$.

Furthermore, for any constants $\tilde{K}_1 \in (0, K_1)$ and $\tilde{K}_3 \in (0, K_3)$, this solution and its corresponding controls possess the following high-order regularity:

$$e^{\frac{\tilde{K}_1(2-1/r)}{2(t-T/2)^5(T-t)^5}} (\nabla \times v^*) \chi_{\omega_0} \in L^2(0, T; H^{53}(\Omega)^3) \cap H^{27}(0, T; H^{-1}(\Omega)^3), \quad (2.134)$$

$$e^{\frac{\tilde{K}_3(2-1/r^*)}{2(t-T/2)^5(T-t)^5}} u^* \chi_{\omega_0} \in L^2(0, T; H^{54}(\Omega)) \cap H^{27}(0, T; L^2(\Omega)), \quad (2.135)$$

$$e^{\frac{\tilde{K}_1}{2(T-t)^5}} y^* \in L^2(0, T; H^2(\Omega)^3 \cap V) \cap L^\infty(0, T; V), \quad (2.136)$$

$$e^{\frac{\tilde{K}_1}{2(T-t)^5}} \theta^* \in L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap L^\infty(0, T; H_0^1(\Omega)). \quad (2.137)$$

Now, we must connect this analytic result to the algebraic solvability framework from Subsection 2.3.1. The algebraic operator $\mathfrak{M}$ is defined on the space-time cylinder $Q_0 = (T/2, T) \times \omega_0$. Therefore, we first need to establish the regularity of the fictitious controls on this subdomain.

By applying classical interpolation theory for such spaces (see, e.g., [25, Section 13.2, p.96]), we can trade spatial regularity for temporal regularity to obtain a more standard Sobolev regularity on Q_0 . This yields:

$$e^{\frac{\tilde{K}_1(2-1/r)}{2(t-T/2)^5(T-t)^5}} (\nabla \times v^*) \chi_{\omega_0} \in H^{17}(Q_0)^3, \quad (2.138)$$

and

$$e^{\frac{\tilde{K}_3(2-1/r^*)}{2(t-T/2)^5(T-t)^5}} u^* \chi_{\omega_0} \in H^{17}(Q_0). \quad (2.139)$$

The solution (y^*, θ^*) solves the desired linearized system (2.129) but with additional "fictitious" source terms on the right-hand side. Let us define these terms, localized to the subdomain Q_0 :

$$w := (\nabla \times v^*) \chi_{\omega_0}, \quad (2.140)$$

$$h := u^* \chi_{\omega_0}. \quad (2.141)$$

With these definitions, the system solved by (y^*, θ^*, p^*) can be written as:

$$\begin{cases} y_t^* - \Delta y^* + (\bar{y} \cdot \nabla) y^* + (y^* \cdot \nabla) \bar{y} + \nabla p^* = f_1 + \theta^* e_3 + \nabla \times w, \\ \theta_t^* - \Delta \theta^* + \bar{y} \cdot \nabla \theta^* + y^* \cdot \nabla \bar{\theta} = f_2 + h. \end{cases} \quad (2.142)$$

Our goal is to find a new state $(\tilde{y}, \tilde{\theta})$, generated by a single control $\tilde{v}$, that solves a system with source terms $-\nabla \times w$ and $-h$. By linearity, the sum $(y, \theta) = (y^* + \tilde{y}, \theta^* + \tilde{\theta})$ will then be the desired solution to (2.129).

To apply the algebraic solvability result (Proposition 2.7), we must first construct an appropriate input datum $u \in C^\infty(Q_0)^8$ for the operator $\mathfrak{B}$ defined in (2.31). We need to find u such that $\mathfrak{B}u$ matches the target source terms. Specifically, for $\mathfrak{B}$ given by (2.31) we take u as:

$$u^1 = 0, u^2 = -w^3, u^3 = w^2, u^4 = -w^3, u^5 = 0, u^6 = w^1, u^7 = \partial_{x_2} w^1 - \partial_{x_1} w^2, u^8 = h. \quad (2.143)$$

In this way:

$$\mathfrak{B}u = \begin{pmatrix} \partial_{x_3} w^2 - \partial_{x_2} w^3 \\ \partial_{x_3} w^1 - \partial_{x_1} w^3 \\ \partial_{x_2} w^1 - \partial_{x_1} w^2 \\ h \end{pmatrix}. \quad (2.144)$$

Therefore, one has that $(\nabla \wedge w, h) \in \text{Im}(\mathfrak{B})$.

Now, we apply the algebraic solvability result from Proposition 2.7. using the datum $u \in H^{17}(Q_0)^8$ constructed in (2.143). A crucial technical point is the transition from the C^∞ framework of the proposition to the Sobolev spaces required here. This extension is valid because the solution operator $\mathfrak{M}$ is a linear partial differential operator (of order 17, as shown in [5]). Its action is well-defined on u , and it produces a new solution.

This guarantees the existence of $(\tilde{y}, \tilde{\theta}, \tilde{p}, \tilde{v}) = \mathfrak{M}u$ that solve the following system from zero initial data:

$$\begin{cases} \tilde{y}_t - \Delta \tilde{y} + (\bar{y} \cdot \nabla) \tilde{y} + (\tilde{y} \cdot \nabla) \bar{y} + \nabla \tilde{p} = -(\nabla \times w) + \tilde{\theta} e_3, & \text{in } Q_T, \\ \tilde{\theta}_t - \Delta \tilde{\theta} + \bar{y} \cdot \nabla \tilde{\theta} + \tilde{y} \cdot \nabla \bar{\theta} = -h + \chi_{\omega_0} \tilde{v}, & \text{in } Q_T, \\ \nabla \cdot \tilde{y} = 0, & \text{in } Q_T, \\ \tilde{y}(0, \cdot) = 0, \tilde{\theta}(0, \cdot) = 0, & \text{in } \Omega. \end{cases} \quad (2.145)$$

However, we must analyze the behavior of this new solution near the final time $t = T$. The coefficients of the operator $\mathfrak{M}$ depend on the trajectory $(\bar{y}, \bar{\theta})$, which has a strong temporal behavior governed by the parameter ν . As established in the proof of [[5], Proposition 2], the solution $(\tilde{y}, \tilde{\theta})$ generated by $\mathfrak{M}$ can therefore grow as $t \rightarrow T^-$, with a rate at most $e^{\frac{7321\nu}{(T-t)^5}}$.

We must now verify that this potential growth from the operator $\mathfrak{M}$ is overcome by the rapid decay inherent in our specific datum u . This requires a direct comparison of the competing exponential weights.

The decay of u is governed by the two weights from (2.138) and (2.139), which involve the constants $\tilde{K}_1$ and $\tilde{K}_3$, respectively. By choosing the parameters appropriately, we can ensure that $\tilde{K}_3 > \tilde{K}_1$. Consequently, the decay associated with the $\tilde{K}_1$ term is slower and thus represents the limiting factor.

Therefore, the problem reduces to showing that the decay from the weight $e^{-\frac{\tilde{K}_1(2-1/r)}{2(t-T/2)^5(T-t)^5}}$ is strong enough to dominate the growth from the operator, which is of the order $e^{\frac{7321\nu}{(T-t)^5}}$.

To formalize this comparison, the combined exponent must remain negative for all

$t \in (T/2, T)$. This leads to the condition:

$$\frac{7321\nu}{(T-t)^5} - \frac{\tilde{K}_1(2-1/r)}{2(t-T/2)^5(T-t)^5} < 0. \quad (2.146)$$

Multiplying by the positive term $2(t-T/2)^5(T-t)^5$ gives the equivalent inequality:

$$2 \cdot 7321 \cdot \nu \cdot (t-T/2)^5 - \tilde{K}_1(2-1/r) < 0. \quad (2.147)$$

The left-hand side of (2.147) is an increasing function of t . Therefore, the inequality holds for all $t \in (T/2, T)$ if it holds at the endpoint of the interval, i.e., in the limit as $t \rightarrow T^-$. Taking this limit yields the sufficient condition:

$$2 \cdot 7321 \cdot \nu \cdot (T/2)^5 - \tilde{K}_1(2-1/r) < 0. \quad (2.148)$$

Since we can choose $\tilde{K}_1$ arbitrarily close to K_1 , the condition (2.148) is satisfied if the stricter inequality holds:

$$2 \cdot 7321 \cdot \nu \cdot (T/2)^5 - K_1(2-1/r) < 0. \quad (2.149)$$

To satisfy this, we substitute the expression for ν from (2.128) into the inequality, which leads to:

$$2 \cdot 7321 \cdot (T/2)^5 \cdot \frac{1}{c^*} \left(\frac{1}{r} - 1 \right) K_1 - \left(2 - \frac{1}{r} \right) K_1 < 0. \quad (2.150)$$

To obtain a sufficient, time-independent condition, we use the bound $(t-T/2)^5 \leq (T/2)^5$. For $T > 0$ sufficiently small, the definition of c^* gives $c^* = (T/4)^5$. Applying these bounds to the previous inequality reduces the problem to analyzing:

$$2 \cdot 7321 \cdot (T/2)^5 \cdot \frac{1}{(T/4)^5} \left(\frac{1}{r} - 1 \right) - \left(2 - \frac{1}{r} \right) < 0. \quad (2.151)$$

It is straightforward to show that the inequality (2.151) is equivalent to:

$$\frac{1}{r} (2^6 \cdot 7321 + 1) < 2^6 \cdot 7321 + 2, \quad (2.152)$$

which, in turn, yields the explicit condition on the parameter r :

$$r > \frac{2^6 \cdot 7321 + 1}{2^6 \cdot 7321 + 2}. \quad (2.153)$$

This condition can be ensured, since the right-hand side of (2.153) is a constant less than 1, allowing for a choice of r arbitrarily close to 1. Let $\alpha \in (0, 1)$ be a fixed parameter. We define

$$r_0 := \frac{2^6 \cdot 7321 + 1}{2^6 \cdot 7321 + 2 - \alpha}. \quad (2.154)$$

Then, for any $r \in (r_0, 1)$, the inequality $r > r_0$ holds, which is algebraically equivalent to, with $T > 0$ sufficiently small:

$$\begin{aligned} \alpha K_1 &< \left(2 - \frac{1}{r}\right) K_1 - 2 \cdot 7321 \cdot \frac{(T/2)^5}{(T/4)^5} \left(\frac{1}{r} - 1\right) K_1 \\ &< \left(2 - \frac{1}{r}\right) \frac{1}{(T/2)^5} K_1 - 2 \cdot 7321 \cdot \frac{1}{c^*} \left(\frac{1}{r} - 1\right) K_1 \\ &= \left(2 - \frac{1}{r}\right) \frac{1}{(T/2)^5} K_1 - 2 \cdot 7321 \nu \end{aligned} \quad (2.155)$$

Let $\alpha \in (0, 1)$, $r \in (r_0, 1)$ and let $T > 0$ be sufficiently small. Assume that K_2 is chosen such that

$$\alpha K_1 < K_2 < \left(2 - \frac{1}{r}\right) \frac{K_1}{(T/2)^5} - 2 \cdot 7321 \nu. \quad (2.156)$$

Such a choice is possible thanks to the positive margin ensured by the previous conditions.

Observe now that, for every $t \in (T/2, T)$, one has

$$(t - T/2)^5 < (T/2)^5,$$

and consequently,

$$\frac{1}{(T/2)^5} < \frac{1}{(t - T/2)^5}.$$

Therefore, inequality (2.156) immediately implies

$$\alpha K_1 < K_2 < \left(2 - \frac{1}{r}\right) \frac{K_1}{(t - T/2)^5} - 2 \cdot 7321 \nu, \quad \forall t \in (T/2, T). \quad (2.157)$$

The inequality (2.157) is constructed precisely to guarantee that the decay induced by the exponential weight dominates the growth generated by the operator. Indeed, rearranging the terms, we obtain that for all $t \in (T/2, T)$,

$$2 \cdot 7321 \nu - \frac{K_1 \left(2 - \frac{1}{r}\right)}{(t - T/2)^5} \leq -K_2. \quad (2.158)$$

Dividing (2.158) by $2(T - t)^5 > 0$, we conclude that the corresponding exponents

satisfy the strict chain of inequalities

$$\frac{7321\nu}{(T-t)^5} - \frac{K_1(2 - \frac{1}{r})}{2(t - T/2)^5(T-t)^5} < -\frac{K_2}{2(T-t)^5} < -\frac{\alpha K_1}{2(T-t)^5}, \quad \forall t \in (T/2, T). \quad (2.159)$$

Because this inequality is strict, we can choose the parameter $\tilde{K}_1 < K_1$ from our fictitious control estimate to be sufficiently close to K_1 such that the full estimate holds. This leads to the final chain of inequalities:

$$e^{\frac{7321\nu}{(T-t)^5}} e^{-\frac{\tilde{K}_1(2-1/r)}{2(t-T/2)^5(T-t)^5}} \leq e^{\frac{-\alpha K_1}{2(T-t)^5}}. \quad (2.160)$$

Finally, we set

$$(y, \theta, p, v) = (y^* + \tilde{y}, \theta^* + \tilde{\theta}, p^* + \tilde{p}, \tilde{v}),$$

Using usual regularity result for the linearized Navier-Stokes and Heat operators (now considered on the intire time interval $(0, T)$), we obtain

$$e^{\frac{\alpha K_1}{2(T-t)^5}} y \in L^2((0, T), H^2(\Omega)^3) \cap L^\infty((0, T), H^1(\Omega)^3), \quad (2.161)$$

$$e^{\frac{\alpha K_1}{2(T-t)^5}} \theta \in L^2((0, T), H^2(\Omega)) \cap L^\infty((0, T), H^1(\Omega)), \quad (2.162)$$

as soon $(y_0, \theta_0) \in V \times H^1(\Omega)$ the proof of Proposition 2.15 is completed. ■

2.4 Null-controllability of nonlinear system

2.4.1 Proof of the Nonlinear Controllability Result

In this final section, we prove our main result, Theorem 2.1. The strategy is to leverage the null controllability of the linearized system, established in Proposition 2.15, to deduce a local result for the original nonlinear system.

The core tool for this transition from a linear to a local nonlinear result is a generalized inverse mapping theorem. We rely on the version Theorem 1.5.

We are going to use similar ideas as those in [5, 10, 13, 14, 15]. Let α , r_0 , and r_1 be the parameters given in Proposition 2.15, and let $r \in (\tilde{r}, 1)$ with $\tilde{r} := \max\{r_0, r_1\}$.

We will now apply Theorem 1.5 with the Banach spaces E and F defined as follows. First, the space E is defined as the set of tuples (y, p, θ, v) such that:

$$E := \left\{ (y, p, \theta, v) \in L^2(Q_T)^3 \times L^2(Q_T) \times L^2(Q_T) \times L^2(Q_T) \mid \right.$$

$$\begin{aligned}
& e^{\frac{\alpha K_1}{2(T-t)^5}} y \in L^2(0, T; H^2(\Omega)^3 \cap V) \cap L^\infty(0, T; V), \quad \nabla p \in L^2(Q_T) \\
& e^{\frac{\alpha K_1}{2(T-t)^5}} \theta \in L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap L^\infty(0, T; H_0^1(\Omega)), \\
& e^{\frac{\alpha K_1}{2(T-t)^5}} v \in L^2(Q_T), \\
& e^{\frac{K_3}{2r(T-t)^5}} [y_t - \Delta y + (\bar{y} \cdot \nabla)y + (y \cdot \nabla)\bar{y} + \nabla p - \theta e_3] \in L^2(Q_T)^3, \\
& e^{\frac{K_3}{2r(T-t)^5}} [\theta_t - \Delta \theta + \bar{y} \cdot \nabla \theta + y \cdot \nabla \bar{\theta}] \in L^2(Q_T), \\
& y(0) \in V, \theta(0) \in H_0^1(\Omega) \}.
\end{aligned} \tag{2.163}$$

And the target space F is defined as:

$$F := \left\{ (G, h, y_0, \theta_0) \mid e^{\frac{\alpha K_1}{2(T-t)^5}} G \in L^2(Q_T)^3, e^{\frac{\alpha K_1}{2(T-t)^5}} h \in L^2(Q_T), y_0 \in V, \theta_0 \in H_0^1(\Omega) \right\}. \tag{2.164}$$

In these space we consider the usual norms.

We endow these spaces with their natural product norms.

Now, we introduce the mapping $\mathcal{A} : E \rightarrow F$:

$$\begin{aligned}
\mathcal{A}(y, p, \theta, v) := & \left(y_t - \Delta y + (y \cdot \nabla)y + (\bar{y} \cdot \nabla)y + (y \cdot \nabla)\bar{y} + \nabla p - \theta e_3, \right. \\
& \theta_t - \Delta \theta + y \cdot \nabla \theta + \bar{y} \cdot \nabla \theta + y \cdot \nabla \bar{\theta} - \chi_{\omega_0} v, \\
& \left. y(0), \theta(0) \right)
\end{aligned}$$

Let us now verify that the mapping $\mathcal{A}$ satisfies the hypotheses of the Inverse Mapping Theorem. To do this, we begin with the following lemma.

Lemma 2.16 *The mapping $\mathcal{A}$ has its range included in F and is of class C^1 on E .*

Proof:[Proof] We decompose the mapping $\mathcal{A}$ as the sum of a linear operator $\mathcal{A}_1$ and a nonlinear operator $\mathcal{A}_2$, i.e., $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$.

The linear operator $\mathcal{A}_1 : E \rightarrow F$ is defined as:

$$\begin{aligned}
\mathcal{A}_1(y, p, \theta, v) := & \left(y_t - \Delta y + (\bar{y} \cdot \nabla)y + (y \cdot \nabla)\bar{y} + \nabla p - \theta e_3, \right. \\
& \theta_t - \Delta \theta + \bar{y} \cdot \nabla \theta + y \cdot \nabla \bar{\theta} - \chi_{\omega_0} v, \\
& \left. y(0), \theta(0) \right)
\end{aligned}$$

The nonlinear operator $\mathcal{A}_2 : E \rightarrow F$ is defined as:

$$\mathcal{A}_2(y, p, \theta, v) := \left((y \cdot \nabla)y, y \cdot \nabla \theta, 0, 0 \right).$$

By the construction of the spaces E and F , the linear operator $\mathcal{A}_1 : E \rightarrow F$ is continuous by definition, and therefore is of class C^1 .

The operator $\mathcal{A}_2$ is quadratic. For such a mapping between Banach spaces, being of class C^1 is equivalent to being continuous. Thus, to prove that $\mathcal{A}_2$ maps E into F and is C^1 , it suffices to show its continuity. This requires proving the following estimate for some constant $C > 0$:

$$\left\| e^{\frac{K_3}{2r(T-t)^5}} (y \cdot \nabla) y \right\|_{L^2(Q_T)^3}^2 + \left\| e^{\frac{K_3}{2r(T-t)^5}} (y \cdot \nabla) \theta \right\|_{L^2(Q_T)}^2 \leq C \| (y, p, \theta, v) \|_E^4. \quad (2.165)$$

Let us begin by introducing the weighted functions

$$\tilde{y}(t, x) := e^{\frac{K_3}{4r(T-t)^5}} y(t, x) \quad \text{and} \quad \tilde{\theta}(t, x) := e^{\frac{K_3}{4r(T-t)^5}} \theta(t, x).$$

We recall that $K_1 := 2s\mu R_1$ and $K_3 := 3s(1 + \mu)R_1$, which implies $K_1 < K_3$ for $\mu > 0$. Since r and α can be taken arbitrarily close to 1, we can choose them such that $\frac{K_3}{2r} < \alpha K_1$. This ensures that the weight defining $\tilde{y}$ and $\tilde{\theta}$ is weaker than the primary weight in the norm of E :

$$\frac{K_3}{4r} < \frac{\alpha K_1}{2}.$$

From the definition of the norm $\| (y, p, \theta, v) \|_E$, we deduce:

$$\| \tilde{y} \|_{L^\infty(0, T; V)} \leq \| (y, p, \theta, v) \|_E, \quad (2.166)$$

$$\| \tilde{y} \|_{L^2(0, T; H^2(\Omega)^3 \cap V)} \leq \| (y, p, \theta, v) \|_E, \quad (2.167)$$

$$\| \tilde{\theta} \|_{L^\infty(0, T; H_0^1(\Omega))} \leq \| (y, p, \theta, v) \|_E. \quad (2.168)$$

Now, from the classical Sobolev embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ in dimension 3, and applying the $L^2(0, T; H^2)$ estimate from (2.167), we deduce that there exists a constant $C > 0$ such that:

$$\| \tilde{y} \|_{L^2(0, T; L^\infty(\Omega)^3)} \leq C \| (y, \theta, p, v) \|_E. \quad (2.169)$$

We now estimate the nonlinear term $(\tilde{y} \cdot \nabla) \tilde{y}$. By applying a standard product inequality, we have:

$$\| (\tilde{y} \cdot \nabla) \tilde{y} \|_{L^2(0, T; L^2(\Omega)^3)} \leq C \| \nabla \tilde{y} \|_{L^\infty(0, T; L^2(\Omega)^3)} \| \tilde{y} \|_{L^2(0, T; L^\infty(\Omega)^3)}. \quad (2.170)$$

By substituting the bounds from (2.166) (for the $L^\infty L^2$ norm of $\nabla \tilde{y}$) and (2.169) (for the $L^2 L^\infty$ norm of $\tilde{y}$) into (2.170), we obtain the desired estimate for the quadratic term:

$$\| (\tilde{y} \cdot \nabla) \tilde{y} \|_{L^2(0, T; L^2(\Omega)^3)} \leq C \| (y, p, \theta, v) \|_E^2. \quad (2.171)$$

Similarly, we estimate the second nonlinear term $(\tilde{y} \cdot \nabla)\tilde{\theta}$:

$$\|\tilde{y} \cdot \nabla \tilde{\theta}\|_{L^2(0,T;L^2(\Omega)^3)} \leq C \|\tilde{y}\|_{L^2(0,T;L^\infty(\Omega)^3)} \|\nabla \tilde{\theta}\|_{L^\infty(0,T;L^2(\Omega)^3)}. \quad (2.172)$$

By substituting the bounds from (2.170) (for the $L^2 L^\infty$ norm of $\tilde{y}$) and (2.168) (for the $L^\infty L^2$ norm of $\nabla \tilde{\theta}$) into the product inequality (2.172), we obtain:

$$\|\tilde{y} \cdot \nabla \tilde{\theta}\|_{L^2(0,T;L^2(\Omega)^3)} \leq C \|(y, p, \theta, v)\|_E^2. \quad (2.173)$$

By substituting the weighted functions $\tilde{y}$ and $\tilde{\theta}$ back in terms of y and θ , the estimates (2.171) and (2.173) become:

$$\begin{aligned} \left\| e^{\frac{K_3}{2r(T-t)^5}} (y \cdot \nabla) y \right\|_{L^2(Q_T)^3}^2 &\leq C \|(y, p, \theta, v)\|_E^4, \\ \left\| e^{\frac{K_3}{2r(T-t)^5}} (y \cdot \nabla) \theta \right\|_{L^2(Q_T)}^2 &\leq C \|(y, p, \theta, v)\|_E^4. \end{aligned}$$

Summing these two inequalities directly yields the target estimate (2.165). This confirms the continuity of the quadratic map $\mathcal{A}_2$.

Since $\mathcal{A}_1$ is continuous and linear, and $\mathcal{A}_2$ has been shown to be continuous, their sum $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$ is a C^1 map from E to F . ■

We now verify the final hypothesis of Theorem 1.5: the surjectivity of the Fréchet derivative $\mathcal{A}'(e_0)$.

To do this, we consider the derivative at the origin, $e_0 = (0, 0, 0, 0) \in E$. The derivative $\mathcal{A}'(e_0)$ is the linear operator $\mathcal{A}_1$. A direct computation of the derivative of $\mathcal{A}$ at e_0 shows that for any $(y, p, \theta, v) \in E$:

$$\begin{aligned} [\mathcal{A}'(e_0)](y, p, \theta, v) = & \left(y_t - \Delta y + (\bar{y} \cdot \nabla) y + (y \cdot \nabla) \bar{y} + \nabla p - \theta e_3, \right. \\ & \theta_t - \Delta \theta + \nabla \cdot (\theta \bar{y}) - \chi_{\omega_0} v, \\ & \left. y(0), \theta(0) \right) \end{aligned}$$

We must show that this operator $\mathcal{A}'(e_0) : E \rightarrow F$ is onto (surjective). This is precisely the result established in Proposition 2.15.

Consequently, all hypotheses of the Inverse Mapping Theorem (Theorem 1.5) are satisfied. We apply the theorem at the point $e_0 = (0, 0, 0, 0) \in E$. The image of this point is $f_0 = \mathcal{A}(e_0) = (0, 0, 0, 0) \in F$.

The theorem guarantees that there exists a $\delta > 0$ such that for any target element $f = (G, h, y_0, \theta_0) \in F$ satisfying $\|f - f_0\|_F < \delta$, there exists $e = (y, p, \theta, v) \in E$ with

$$\mathcal{A}(e) = f.$$

We are interested in the specific target $f = (0, 0, y_0, \theta_0)$, which corresponds to solving the nonlinear system with zero source terms. The condition on the norm becomes:

$$\|(0, 0, y_0, \theta_0)\|_F = \|(y_0, \theta_0)\|_{V \times H_0^1(\Omega)} \leq \delta.$$

For any initial data (y_0, θ_0) small enough to satisfy this condition, the theorem provides a solution $e = (y, p, \theta, v) \in E$. The condition $\mathcal{A}(e) = f$ means that (y, p, θ, v) is a solution to the full nonlinear Boussinesq system (2.1) with initial data (y_0, θ_0) .

The condition $e \in E$ is crucial. The definition of the space E requires that the solution possesses the specified weighted regularity, such as $e^{\frac{\alpha K_1}{2(T-t)^5}} y \in L^\infty(0, T; V)$. Since the weight $e^{\frac{\alpha K_1}{2(T-t)^5}}$ explodes as $t \rightarrow T^-$, this regularity is only possible if the solution itself vanishes. Thus, the property $e \in E$ inherently implies the null controllability condition:

$$y(T, \cdot) = 0 \quad \text{and} \quad \theta(T, \cdot) = 0 \quad \text{in } \Omega.$$

This concludes the proof of Theorem 2.1.

2.5 Future Problems

As a proposal for future work, we intend to develop studies involving Boussinesq systems, which couple the Navier–Stokes equations with the temperature transport equation and provide a natural framework for the investigation of insensitizing controls. In these problems, the objective is to design control strategies capable of keeping certain observables practically invariant under small perturbations in the initial data, even when the control acts only on a subset of the system equations. A central issue in this context is the possibility of achieving such controls with a reduced number of actuators, which represents both a theoretical and practical challenge. Similarly, we intend to investigate micropolar fluid systems, which incorporate particle micro-rotation effects and angular momentum tensors, and which present additional challenges both in the definition of insensitizing controls and in the characterization of controllability in the presence of limited controls.

Optimal design of sensors for a system of linear thermoelasticity

In this chapter, we will study the problem of optimal sensor placement for the system thermoelasticity, assuming that the initial data to be reconstructed are not fully known.

3.1 Linear thermoelastic system

Let a linear system of thermoelasticity, in a bounded and open subset Ω of $\mathbb{R}^N$. The system consists of coupled wave and heat equations and reads as follows:

$$\begin{cases} u_{tt} - c^2 \Delta u + \alpha \Delta \theta = 0 & \text{in } \Omega \times (0, T), \\ \theta_t - \nu \Delta \theta + \beta u_t = 0 & \text{in } \Omega \times (0, T), \\ u = \theta = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u^0, u_t(0) = u^1, \theta(0) = \theta^0 & \text{in } \Omega, \end{cases} \quad (3.1)$$

where $T > 0$ is a final time, $\partial\Omega$ is the boundary of Ω , and α, β, c, ν are positive real numbers.

For system (3.1), we want to solve the problem of deciding the best place and shape for sensors that will make measurements of signals propagating in Ω over a given horizon of time. It is desirable to place these sensors so that we can reconstruct the solutions in the best possible way.

To solve a prescribed inverse problem for partial differential equations, one strategy which is usually applied is to prove that a suitable observability inequality holds. For

This chapter was developed in collaboration with Yannick Privat (Institut Élie Cartan de Lorraine, Université de Lorraine) and Emmanuel Trélat (Sorbonne Université).

system (3.1), we will see that this inequality has the general form

$$\|(u, \theta)(T)\|_X^2 \leq A_0 \|u\|_{X_{0,\omega}}^2 + A_1 \|\theta\|_{X_{1,\omega}}^2, \quad (3.2)$$

where X and $X_{i,\omega}$, $i = 0, 1$, are suitable Banach spaces, and A_0 or A_1 are positive constants depending only on c, α, ν, β, T and ω , and so that $A_0 + A_1 \neq 0$. In most of problems we find that X is a Sobolev space like $W^{s,2}(\Omega)$, for $s \in \mathbb{R}$, and $X_{i,\omega} = L^2(\omega \times (0, T))$.

3.2 Existence of Optimal Solutions

Let $(\phi_j)_{j \in \mathbb{N}^*}$ be a Hilbert basis of $L^2(\Omega, \mathbb{C})$, consisting of eigenfunctions of the Laplace operator Δ with null Dirichlet boundary conditions. In particular, these functions are analytic in Ω .

We will see in the following subsections that finding the optimal sets that minimize the observability constant for the thermoelasticity system reduces to studying the maximizers of functionals of the form

$$J(\chi_\omega) = \inf_{i \in \mathbb{N}^*} \gamma_i \int_{\omega} |\phi_i(x)|^2 dx, \quad (3.3)$$

where the maximization is performed over the set of admissible characteristic functions

$$\mathcal{U}_L = \{\chi_\omega \mid \omega \text{ is a measurable subset of } \Omega \text{ with Lebesgue measure } |\omega| = L|\Omega|\}, \quad (3.4)$$

for a given $L \in (0, 1)$.

The optimization problem under consideration is therefore

$$\sup_{\chi_\omega \in \mathcal{U}_L} J(\chi_\omega). \quad (3.5)$$

In what follows, we assume that the sequence $(\gamma_i)_{i \in \mathbb{N}^*}$ in (3.3) converges to a limit $\gamma^* \in \mathbb{R}$ as $i \rightarrow +\infty$.

As is common in shape optimization problems, we first study a relaxed version of the problem (3.5). This approach consists of replacing the set of characteristic functions $\mathcal{U}_L$ with its convex closure in the weak-* topology of $L^\infty(\Omega)$. This relaxed set of admissible functions is given by

$$\overline{\mathcal{U}}_L = \left\{ a \in L^\infty(\Omega; [0, 1]) \mid \int_{\Omega} a(x) dx = L|\Omega| \right\}. \quad (3.6)$$

The functional J , defined in (3.3), is then extended to functions $a \in \overline{\mathcal{U}}_L$ as

$$J(a) = \inf_{i \in \mathbb{N}^*} \gamma_i \int_{\Omega} a(x) |\phi_i(x)|^2 dx. \quad (3.7)$$

The corresponding relaxed optimization problem is

$$\sup_{a \in \overline{\mathcal{U}}_L} J(a). \quad (3.8)$$

Since $\mathcal{U}_L \subset \overline{\mathcal{U}}_L$, the relaxed problem provides an upper bound for the original one, that is,

$$\sup_{\chi_{\omega} \in \mathcal{U}_L} J(\chi_{\omega}) \leq \sup_{a \in \overline{\mathcal{U}}_L} J(a). \quad (3.9)$$

The existence of a solution to the relaxed problem is guaranteed by the following lemma.

Lemma 3.1 *For any $L \in (0, 1)$, the relaxed problem (3.8) admits at least one solution $a^* \in \overline{\mathcal{U}}_L$.*

Proof: For each $i \in \mathbb{N}^*$, the functional $J_i(a) := \gamma_i \int_{\Omega} a(x) |\phi_i(x)|^2 dx$ is linear in a . As the integral of a function in $L^{\infty}(\Omega)$ against a function in $L^1(\Omega)$, it is also continuous on $L^{\infty}(\Omega)$ with respect to the weak-* topology.

The functional $J(a) = \inf_{i \in \mathbb{N}^*} J_i(a)$, being the infimum of a family of weak-* continuous functionals, is weak-* upper semicontinuous. Furthermore, the set of admissible functions $\overline{\mathcal{U}}_L$ is compact in the weak-* topology of $L^{\infty}(\Omega)$. This is because $\overline{\mathcal{U}}_L$ is a weak-* closed set contained within the closed unit ball of $L^{\infty}(\Omega)$, which is compact by the Banach-Alaoglu theorem. Therefore, by the generalized Weierstrass extreme value theorem, the upper semicontinuous functional J attains its maximum value on the compact set $\overline{\mathcal{U}}_L$. ■

A standard approach to this type of optimization problem is to first analyze a truncated version of the functional. This leads to the following existence and regularity result for the non-relaxed problem.

Proposition 3.2 *For any $N \in \mathbb{N}^*$, let the truncated functional $J_N : \overline{\mathcal{U}}_L \rightarrow \mathbb{R}$ be defined by*

$$J_N(a) = \inf_{1 \leq i \leq N} \gamma_i \int_{\Omega} a(x) |\phi_i(x)|^2 dx,$$

where $(\gamma_i)_{i=1}^N$ are positive real numbers.

Then, the optimization problem

$$\sup_{\chi_{\omega} \in \mathcal{U}_L} J_N(\chi_{\omega})$$

possesses a unique solution $\chi_{\omega^N} \in \mathcal{U}_L$. Furthermore, the optimal set ω^N is semi-analytic and, as a consequence, has a finite number of connected components.

Proof: The proof is analogous to the arguments presented in [41, Theorem 8] and [42, Proposition 2]. For completeness, we give a detailed account here. First, we note that the truncated functional J_N is concave and continuous on the weak-* compact set $\overline{\mathcal{U}_L}$. By the Weierstrass theorem, the problem $\sup_{a \in \overline{\mathcal{U}_L}} J_N(a)$ admits at least one solution $a^N \in \overline{\mathcal{U}_L}$. We now show that any such solution must be a characteristic function.

The problem can be rewritten using the simplex set

$$\mathcal{S}_N = \left\{ \beta = (\beta_j)_{1 \leq j \leq N} \in \mathbb{R}_+^N \mid \sum_{j=1}^N \beta_j = 1 \right\}$$

. By applying the Sion minimax theorem, we can interchange the supremum and the infimum:

$$\begin{aligned} \sup_{a \in \overline{\mathcal{U}_L}} \inf_{1 \leq j \leq N} \gamma_j \int_{\Omega} a(x) |\phi_j(x)|^2 dx &= \sup_{a \in \overline{\mathcal{U}_L}} \inf_{\beta \in \mathcal{S}_N} \sum_{j=1}^N \beta_j \gamma_j \int_{\Omega} a(x) |\phi_j(x)|^2 dx \\ &= \inf_{\beta \in \mathcal{S}_N} \sup_{a \in \overline{\mathcal{U}_L}} \sum_{j=1}^N \beta_j \gamma_j \int_{\Omega} a(x) |\phi_j(x)|^2 dx. \end{aligned}$$

This ensures the existence of a saddle point $(a^N, \beta^N) \in \overline{\mathcal{U}_L} \times \mathcal{S}_N$. The saddle point property implies that a^N solves the linear optimization problem

$$\max_{a \in \overline{\mathcal{U}_L}} \int_{\Omega} a(x) G_N(x) dx, \quad (3.10)$$

where the kernel $G_N(x)$ is defined as $G_N(x) := \sum_{j=1}^N \beta_j^N \gamma_j |\phi_j(x)|^2$.

To characterize the solution a^N , we apply the bathtub principle (see, Theorem 1.8). The principle asserts that there exists a level $\lambda^N \in \mathbb{R}$ such that any solution a^N to (3.10) must satisfy

$$a^N(x) = 1 \text{ a.e. on } \{G_N > \lambda^N\} \quad \text{and} \quad a^N(x) = 0 \text{ a.e. on } \{G_N < \lambda^N\}.$$

The key is to show that the level-set $E_{\lambda^N} := \{x \in \Omega \mid G_N(x) = \lambda^N\}$ has zero Lebesgue measure. We argue by contradiction. Assume $|E_{\lambda^N}| > 0$. Since $G_N(x)$ is real analytic, this implies that $G_N(x) = \lambda^N$ for all $x \in \Omega$. However, since the eigenfunctions satisfy the zero Dirichlet boundary condition ($\phi_j|_{\partial\Omega} = 0$), we must have $G_N|_{\partial\Omega} = 0$, which forces the

constant to be zero, i.e., $\lambda^N = 0$. The equation thus becomes

$$\sum_{j=1}^N \beta_j^N \gamma_j |\phi_j(x)|^2 = 0, \quad \text{for all } x \in \Omega.$$

Since this is a sum of non-negative terms, each term must be individually zero. For each j , we have $\beta_j^N \gamma_j |\phi_j(x)|^2 = 0$ for all x . As eigenfunctions are non-trivial by definition ($\phi_j \not\equiv 0$) and $\gamma_j > 0$, we must conclude that $\beta_j^N = 0$. This holds for all j , which contradicts the fact that $\beta^N \in \mathcal{S}_N$ (since its components must sum to 1). The initial assumption is therefore false, and any level-set of G_N must have zero Lebesgue measure.

Since $|E_{\lambda^N}| = 0$, the solution to (3.10) is unique (up to a set of measure zero) and is given by the characteristic function

$$a^N(x) = \chi_{\{x \in \Omega \mid G_N(x) > \lambda^N\}}(x).$$

This proves that any solution to the relaxed problem is, in fact, a classical solution in $\mathcal{U}_L$.

The uniqueness of the maximizer for J_N follows from this characterization and the concavity of J_N . If there were two distinct maximizers, their convex combination would also be a maximizer, but it would not be a characteristic function, leading to a contradiction.

Finally, since the eigenfunctions ϕ_j are analytic, $G_N(x)$ is also analytic. Consequently, the optimal set $\omega^N = \{x \in \Omega \mid G_N(x) > \lambda^N\}$, being a superlevel-set of an analytic function, is semi-analytic and thus has a finite number of connected components. ■

The relationship between the truncated functionals and the relaxed functional $J(a)$ is precisely established in the following lemma. While the proof follows arguments similar to those in [41, Theorem 8], we include it here for completeness.

Lemma 3.3 *The truncated problems approximate the full relaxed problem in the following sense:*

- (i) **(Convergence of maximal values)** *The sequence of maximal values of the truncated problems converges to the maximal value of the full problem, i.e.,*

$$\lim_{N \rightarrow \infty} \left(\max_{a \in \overline{\mathcal{U}}_L} J_N(a) \right) = \max_{a \in \overline{\mathcal{U}}_L} J(a).$$

- (ii) **(Convergence of maximizers)** *Let $(a^N)_{N \in \mathbb{N}^*}$ be a sequence such that, for each N , a^N is a maximizer of J_N over $\overline{\mathcal{U}}_L$. Then this sequence admits a subsequence that converges in the weak- $*$ topology of $L^\infty(\Omega)$ to an element a^* in $\overline{\mathcal{U}}_L$, and any such limit point a^* is a maximizer for the full functional J .*

Proof: Let $(a^N)_{N \in \mathbb{N}^*}$ be a sequence where, for each N , a^N is a maximizer of J_N . Since the set $\overline{\mathcal{U}}_L$ is weak-* compact, this sequence admits a subsequence (which we still denote by a^N) that converges to a limit point $\bar{a} \in \overline{\mathcal{U}}_L$.

For any fixed integer $p \in \mathbb{N}^*$, and for all $N \geq p$, we have $J_N(a) \leq J_p(a)$ for any a , because the infimum for J_N is taken over a larger set of indices. This leads to the following chain of inequalities for any $N \geq p$:

$$\max_{a \in \overline{\mathcal{U}}_L} J(a) \leq \max_{a \in \overline{\mathcal{U}}_L} J_N(a) = J_N(a^N) \leq J_p(a^N).$$

Taking the limit superior as $N \rightarrow \infty$ across this chain yields:

$$\max_{a \in \overline{\mathcal{U}}_L} J(a) \leq \limsup_{N \rightarrow \infty} J_N(a^N) \leq \limsup_{N \rightarrow \infty} J_p(a^N).$$

The functional J_p is the infimum of a finite number of continuous linear functionals, which makes it weak-* upper semicontinuous. By the definition of upper semicontinuity, we have $\limsup_{N \rightarrow \infty} J_p(a^N) \leq J_p(\bar{a})$.

Combining these results, we find that for any fixed p :

$$\max_{a \in \overline{\mathcal{U}}_L} J(a) \leq J_p(\bar{a}).$$

This inequality holds for all $p \in \mathbb{N}^*$. We can now take the limit as $p \rightarrow \infty$. By definition of J , we have $\lim_{p \rightarrow \infty} J_p(\bar{a}) = J(\bar{a})$. This gives:

$$\max_{a \in \overline{\mathcal{U}}_L} J(a) \leq J(\bar{a}).$$

Since $\bar{a} \in \overline{\mathcal{U}}_L$, the reverse inequality $J(\bar{a}) \leq \max_{a \in \overline{\mathcal{U}}_L} J(a)$ is immediate. We must therefore have equality:

$$J(\bar{a}) = \max_{a \in \overline{\mathcal{U}}_L} J(a).$$

This shows that any limit point $\bar{a}$ of the sequence of maximizers (a^N) is a maximizer for the full functional J . This proves part **(ii)** of the lemma.

Part **(i)** follows directly. Our chain of inequalities shows that

$$\max J(a) \leq \liminf J_N(a^N) \leq \limsup J_N(a^N) \leq J(\bar{a})$$

. Since we just proved that $J(\bar{a}) = \max J(a)$, all inequalities must be equalities. This

implies that the limit exists and

$$\lim_{N \rightarrow \infty} \left(\max_{a \in \overline{\mathcal{U}}_L} J_N(a) \right) = \max_{a \in \overline{\mathcal{U}}_L} J(a).$$

This completes the proof of part (i). ■

Thus far, our analysis has guaranteed two key results: the existence of an optimal solution a^* in the relaxed set $\overline{\mathcal{U}}_L$, and the existence of a unique classical solution χ_{ω^N} for the truncated problem (Proposition 3.2). However, our original problem (3.5) seeks a classical solution, that is, a characteristic function $\chi_{\omega} \in \mathcal{U}_L$, for the full, non-relaxed functional J . In the following section, we present a first positive result in this direction, guaranteeing the existence of such a classical solution under certain ergodicity assumptions.

3.2.1 Existence of optimal solutions under ergodicity assumptions

In this subsection, we investigate the existence of a classical maximizer for the original problem (3.5). This question is equivalent to determining whether the optimal value of the relaxed problem (3.8) is attained by a characteristic function. In other words, we check if there is a "gap" in the inequality (3.9):

$$\sup_{\chi_{\omega} \in \mathcal{U}_L} J(\chi_{\omega}) = \sup_{a \in \overline{\mathcal{U}}_L} J(a).$$

To do this, we now work under the following ergodicity and regularity assumptions on the eigenfunction basis.

(H¹) Quantum Ergodicity (QUE): The sequence of probability measures $\mu_j := \phi_j^2 dx$ converges vaguely to the uniform measure on Ω . That is, for every continuous function $f \in C_c^\infty(\overline{\Omega})$,

$$\int_{\Omega} f(x) \phi_j(x)^2 dx \xrightarrow{j \rightarrow \infty} \frac{1}{|\Omega|} \int_{\Omega} f(x) dx.$$

(H²) Uniform L^∞ -Boundedness: The eigenfunctions are uniformly bounded. That is, there exists a constant $A > 0$ such that

$$\|\phi_j\|_{L^\infty(\Omega)} \leq A, \quad \text{for all } j \in \mathbb{N}^*.$$

Remark 3.4 *The assumptions (H¹) and (H²) together imply that $(|\phi_j|^2)$ converges in the weak- $*$ topology of $L^\infty(\Omega)$ to the constant function $\frac{1}{|\Omega|}$. In fact, let $g \in L^1(\Omega)$. Since*

$C_c^\infty(\bar{\Omega})$ is dense in $L^1(\Omega)$, we can take a sequence $(f_n)_{n \in \mathbb{N}} \subset C_c^\infty(\bar{\Omega})$ such that $f_n \rightarrow g$ strongly in $L^1(\Omega)$. Then, for every $n \in \mathbb{N}$ and $j \in \mathbb{N}^*$,

$$\int_{\Omega} g(x) |\phi_j(x)|^2 dx = \int_{\Omega} (g(x) - f_n(x)) |\phi_j(x)|^2 dx + \int_{\Omega} f_n(x) |\phi_j(x)|^2 dx.$$

Using hypothesis **(H²)**, we can bound the first term on the right-hand side. By the Hölder inequality, we have:

$$\left| \int_{\Omega} (g(x) - f_n(x)) |\phi_j(x)|^2 dx \right| \leq \| |\phi_j|^2 \|_{L^\infty(\Omega)} \|g - f_n\|_{L^1(\Omega)} \leq A^2 \|g - f_n\|_{L^1(\Omega)}.$$

Since $f_n \rightarrow g$ strongly in $L^1(\Omega)$, the right-hand side converges to zero as $n \rightarrow \infty$. This convergence is uniform with respect to j because the bound A^2 is independent of j .

For any given $\epsilon > 0$, we can first fix an $n_0 \in \mathbb{N}$ large enough so that:

$$\left| \int_{\Omega} (g(x) - f_{n_0}(x)) |\phi_j(x)|^2 dx \right| \leq \epsilon, \quad \text{for all } j \in \mathbb{N}^*.$$

Now, with n_0 held constant, we apply hypothesis **(H¹)** to the fixed continuous function $f_{n_0}(x)$. This implies that there exists an index j_0 (which depends on our choice of n_0) such that for all $j \geq j_0$:

$$\left| \int_{\Omega} f_{n_0}(x) |\phi_j(x)|^2 dx - \frac{1}{|\Omega|} \int_{\Omega} f_{n_0}(x) dx \right| \leq \epsilon.$$

Furthermore, the term involving g and f_{n_0} is also small by taking n_0 even bigger if necessary:

$$\left| \frac{1}{|\Omega|} \int_{\Omega} f_{n_0}(x) dx - \frac{1}{|\Omega|} \int_{\Omega} g(x) dx \right| \leq \frac{1}{|\Omega|} \|f_{n_0} - g\|_{L^1(\Omega)} \leq \frac{\epsilon}{|\Omega|}.$$

By combining these estimates using the triangle inequality (from the decomposition in the previous step), we find that for all $j \geq j_0$:

$$\left| \int_{\Omega} g(x) |\phi_j(x)|^2 dx - \frac{1}{|\Omega|} \int_{\Omega} g(x) dx \right| \leq \epsilon + \epsilon + \frac{\epsilon}{|\Omega|}.$$

Since ϵ was arbitrary, the left-hand side must converge to zero.

Proposition 3.5 Assume hypotheses **(H¹)** and **(H²)** hold. Suppose the sequence (γ_i) has the following asymptotic expansion for some $k \in \mathbb{N}$, $\gamma^* > 0$, $R \neq 0$, and for all i large

enough:

$$\gamma_i = \gamma^* \left(1 - \frac{R}{\omega_i^k} \right) + o \left(\frac{1}{\omega_i^k} \right). \quad (3.11)$$

Furthermore, if the sequence (γ_i) has a global minimum, let i_0 be a minimizer and assume that $\gamma_{i_0} < \gamma^*$ and the volume fraction L satisfies $\frac{\gamma_{i_0}}{\gamma^*} < L < 1$.

Then, there exists an integer $N_0 \in \mathbb{N}^*$ such that for all $N \geq N_0$, the value of the full problem is equal to the value of the N_0 -truncated problem:

$$\max_{\chi_\omega \in \mathcal{U}_L} J(\chi_\omega) = \max_{\chi_\omega \in \mathcal{U}_L} J_{N_0}(\chi_\omega).$$

In particular, the full optimization problem for J possesses a unique maximizer. This maximizer is the set ω^{N_0} from Proposition 3.2, which is semi-analytic and has a finite number of connected components.

The proof of this proposition follows from analogous arguments as the ones of [41], Theorem 9. However, in [41] they use the fact that the corresponding sequence (γ_j) is increasing to γ^* . By investigating their proof, we have seen that it is sufficient to ask $\gamma_{i_0} < \gamma^*$. To understand better the role of $(\gamma_i)_{i \in \mathbb{N}^*}$, we will provide a proof.

Proof:

Let $b^* \in \overline{\mathcal{U}}_L$ be a maximizer for the relaxed functional $J(a)$ in (3.7), whose existence is guaranteed by Lemma 3.1. We first prove that there exists an $N_0 \in \mathbb{N}^*$ such that b^* is also a maximizer for the truncated functional $J_{N_0}(a)$.

Our analysis relies on the asymptotic behavior of the terms defining J . By hypothesis $(\mathbf{H}^1)$ and $(\mathbf{H}^2)$, and since $\int_\Omega b^*(x) dx = L|\Omega|$, we have the convergence

$$\lim_{i \rightarrow \infty} \int_\Omega b^*(x) |\phi_i(x)|^2 dx = L.$$

From the asymptotic expansion (3.11), we also have $\lim_{i \rightarrow \infty} \gamma_i = \gamma^*$.

Combining these two limits, it follows that for any $\epsilon > 0$, there exists an integer N_0 such that for all $i \geq N_0$:

$$\gamma_i \int_\Omega b^*(x) |\phi_i(x)|^2 dx \geq \gamma^*(L - \epsilon). \quad (3.12)$$

On the one hand, from (3.12) and the hypothesis that $\frac{\gamma_{i_0}}{\gamma^*} < L$, we can choose ϵ small enough such that $\gamma^*(L - \epsilon) > \gamma_{i_0}$. This implies the “tail” of the infimum is bounded below:

$$\inf_{i > N_0} \gamma_i \int_\Omega b^*(x) |\phi_i(x)|^2 dx \geq \gamma^*(L - \epsilon) > \gamma_{i_0}.$$

On the other hand, we analyze the “head” of the infimum, $J_{N_0}(b^*)$. Since N_0 was chosen large enough so that the global minimizer i_0 is in the set $\{1, \dots, N_0\}$, and using that $b^*(x) \leq 1$, we get:

$$\inf_{1 \leq i \leq N_0} \gamma_i \int_{\Omega} b^*(x) |\phi_i(x)|^2 dx \leq \inf_{1 \leq i \leq N_0} \gamma_i \int_{\Omega} |\phi_i(x)|^2 dx = \inf_{1 \leq i \leq N_0} \gamma_i = \gamma_{i_0}.$$

We now recall that

$$J(b^*) = \min \left(J_{N_0}(b^*), \inf_{i > N_0} \gamma_i \int_{\Omega} b^*(x) |\phi_i(x)|^2 dx \right).$$

Substituting our findings, we have

$$J(b^*) = J_{N_0}(b^*).$$

We now show that $J(b^*) = J_{N_0}(b^{N_0})$, where b^{N_0} is the unique maximizer of the truncated functional J_{N_0} (see Proposition 3.2).

We have already shown that $J(b^*) = J_{N_0}(b^*)$. Since b^{N_0} is the unique maximizer of J_{N_0} (by definition), it follows that

$$J(b^*) = J_{N_0}(b^*) \leq J_{N_0}(b^{N_0}).$$

We argue by contradiction. Assume that $J_{N_0}(b^*) < J_{N_0}(b^{N_0})$ and consider the perturbation $b_t = b^* + t(b^{N_0} - b^*)$. Since the functional J_{N_0} is concave,

$$J_{N_0}(b_t) \geq tJ_{N_0}(b^{N_0}) + (1-t)J_{N_0}(b^*) > J_{N_0}(b^*), \quad (3.13)$$

for every $t \in (0, 1]$. The first inequality follows from concavity, and the second (strict) inequality holds because $J_{N_0}(b^{N_0}) > J_{N_0}(b^*)$ by our assumption.

We must also analyze the “tail” of the infimum for b_t (i.e., for $i \geq N_0$). Since $b^{N_0}, b^* \in \overline{\mathcal{U}}_L$, their values lie in $[0, 1]$, which implies $b^{N_0}(x) - b^*(x) \in [-1, 1]$.

Furthermore, since the sequence (γ_i) converges, it is bounded. Let $M = \sup_{i \in \mathbb{N}^*} \gamma_i < \infty$. For every $i \geq N_0$, we can establish the following lower bound (assuming the basis is normalized, $\int_{\Omega} |\phi_i|^2 dx = 1$):

$$\begin{aligned} \gamma_i \int_{\Omega} b_t(x) |\phi_i(x)|^2 dx &= \gamma_i \int_{\Omega} b^*(x) |\phi_i(x)|^2 dx + t\gamma_i \int_{\Omega} (b^{N_0}(x) - b^*(x)) |\phi_i(x)|^2 dx \\ &\geq (\gamma^*(L - \epsilon)) + t\gamma_i \int_{\Omega} (-1) |\phi_i(x)|^2 dx \\ &= \gamma^*(L - \epsilon) - t\gamma_i \geq \gamma^*(L - \epsilon) - tM, \end{aligned} \quad (3.14)$$

for every $i \geq N_0$.

We now select t to be small enough to ensure the "tail" of the infimum (i.e., for $i \geq N_0$) remains strictly greater than γ_{i_0} . Recall from our assumption that $\gamma^*(L - \epsilon) > \gamma_{i_0}$. Let us take t such that

$$0 < t < \frac{\gamma^*}{M} \left(L - \epsilon - \frac{\gamma_{i_0}}{\gamma^*} \right),$$

which is equivalent to $0 < t < (\gamma^*(L - \epsilon) - \gamma_{i_0})/M$.

Applying this to our previously derived lower bound, $\gamma_i \int_{\Omega} b_t(x) |\phi_i(x)|^2 dx \geq \gamma^*(L - \epsilon) - tM$, we obtain:

$$\gamma_i \int_{\Omega} b_t(x) |\phi_i(x)|^2 dx > \gamma^*(L - \epsilon) - M \left(\frac{\gamma^*(L - \epsilon) - \gamma_{i_0}}{M} \right) = \gamma_{i_0}.$$

This gives the following complete inequality chain:

$$\gamma_i \int_{\Omega} b_t(x) |\phi_i(x)|^2 dx > \gamma_{i_0} \geq \gamma_{i_0} \int_{\Omega} b^*(x) |\phi_{i_0}(x)|^2 dx \geq J(b^*), \quad (3.15)$$

for every $i \geq N_0$. ■

Remark 3.6 *If we take $R > 0$ in the asymptotic expansion (3.11) of Proposition 3.5, the sequence converges to γ^* from below (at least for i large enough). This implies that the global minimum $\gamma_{i_0} = \inf_j \gamma_j$ must be strictly less than the limit γ^* . Therefore, the assumption $\gamma_{i_0} < \gamma^*$ is automatically satisfied and is not needed as an independent hypothesis.*

Remark 3.7 *The assumptions $(\mathbf{H}^1)$ and $(\mathbf{H}^2)$ hold true in dimension one. For $\Omega = (0, \pi)$, the eigenfunctions $\phi_j(x) = \sqrt{2/\pi} \sin(jx)$ satisfy: $(\mathbf{H}^2)$ $\|\phi_j\|_{L^\infty(\Omega)} = \sqrt{2/\pi}$ for all $j \in \mathbb{N}^*$, which is uniform. $(\mathbf{H}^1)$ The sequence $(\phi_j^2)_{j \in \mathbb{N}^*}$ converges to $1/\pi$ in the weak-* topology of $L^\infty(0, \pi)$.*

Remark 3.8 *The QUE assumption $(\mathbf{H}^1)$ is not general. It fails for domains like the square $\Omega = (0, \pi)^2$ (with sine-product eigenfunctions) and the unit disk (with Bessel-function eigenfunctions), as shown in [41, Proposition 1].*

As highlighted in Remarks 3.7 and 3.8, the QUE assumption $(\mathbf{H}^1)$ is strong and geometry-dependent. We now investigate cases where $(\mathbf{H}^1)$ might fail.

3.2.2 Existence of optimal solutions under asymptotic lower bound assumption

Since the QUE assumption does not hold in all scenarios, we now consider, in some sense, a weaker set of conditions.

Let Ω be a bounded, connected subset of $\mathbb{R}^n$, and let $L \in (0, 1)$ and $T > 0$ be arbitrary. We assume there exists a Hilbert basis $(\phi_j)_{j \in \mathbb{N}^*}$ of $L^2(\Omega, \mathbb{C})$ consisting of (complex-valued) eigenfunctions of $-\Delta$ with Dirichlet boundary condition, associated with (complex) eigenvalues $(\lambda_j)_{j \in \mathbb{N}^*}$ such that $\operatorname{Re}(\lambda_1) \leq \dots \leq \operatorname{Re}(\lambda_j) \leq \dots$. We further assume that the following property is satisfied:

(H_{*}¹) (Asymptotic Lower Bound) For every $a \in L^\infty(\Omega; [0, 1])$ such that $\int_\Omega a(x) \, dx = L|\Omega|$, there exists an integer $N_0 \in \mathbb{N}^*$ (which may depend on a) such that

$$\liminf_{j \rightarrow +\infty} \gamma_j(T) \int_\Omega a(x) |\phi_j(x)|^2 \, dx > \gamma_{N_0}(T).$$

We now prove that under assumptions **(H_{*}¹)**, a result similar to Proposition 3.5 holds.

Proposition 3.9 *Under assumptions **(H_{*}¹)**, the optimal shape design problem (3.5) admits a unique solution $\chi_{\omega^*} \in \mathcal{U}_L$.*

Moreover, there exists an integer $N_1(T)$ such that for every $N \geq N_1(T)$, this solution χ_{ω^} is also the unique maximizer of the N -truncated problem, and their maximal values are equal:*

$$\max_{\chi_\omega \in \mathcal{U}_L} J(\chi_\omega) = \max_{\chi_\omega \in \mathcal{U}_L} J_N(\chi_\omega).$$

Proof: According to Lemma 3.1, the relaxed optimal design problem (3.8) has at least one solution $a^* \in \overline{\mathcal{U}}_L$. The assumption **(H_{*}¹)** applied to a^* implies that there exists $N_1 \in \mathbb{N}^*$, with $N_1 \geq N_0$, such that

$$\inf_{j > N_1} \gamma_j(T) \int_\Omega a^*(x) |\phi_j|^2 \, dx > \gamma_{N_0}(T). \quad (3.16)$$

Since $N_1 \geq N_0$, the truncated functional $J_{N_1}(a^*)$ satisfies

$$J_{N_1}(a^*) = \inf_{1 \leq j \leq N_1} \gamma_j(T) \int_\Omega a^*(x) |\phi_j|^2 \, dx \leq \gamma_{N_0}(T) \int_\Omega a^*(x) |\phi_{N_0}(x)|^2 \, dx.$$

Assuming $a^*(x) \leq 1$ and normalized eigenfunctions ($\int_\Omega |\phi_{N_0}|^2 \, dx = 1$), this gives $J_{N_1}(a^*) \leq \gamma_{N_0}(T)$.

By definition, the full functional $J(a^*)$ can be split at N_1 :

$$J(a^*) = \min \left(J_{N_1}(a^*), \inf_{j > N_1} \gamma_j(T) \int_\Omega a^*(x) |\phi_j|^2 \, dx \right).$$

From (3.16), the asymptotic part (the infimum for $j > N_1$) is strictly greater than $\gamma_{N_0}(T)$.

Since we just showed $J_{N_1}(a^*) \leq \gamma_{N_0}(T)$, the minimum must be the truncated functional:

$$J(a^*) = J_{N_1}(a^*).$$

Using hypotheses the analiticity of the eigenfunctions, and Proposition 3.2, let $a^{N_1} \in \mathcal{U}_L$ be the unique maximizer of J_{N_1} over $\overline{\mathcal{U}}_L$. We now prove that $J(a^*) = J_{N_1}(a^{N_1})$.

Since a^{N_1} is the maximizer, we have $J_{N_1}(a^*) \leq J_{N_1}(a^{N_1})$. We proceed by contradiction, assuming the inequality is strict:

$$J(a^*) < J_{N_1}(a^{N_1}).$$

From the previous step, $J(a^*) = J_{N_1}(a^*)$, so our assumption is $J_{N_1}(a^*) < J_{N_1}(a^{N_1})$.

Consider the perturbation $a_t = (1-t)a^* + ta^{N_1}$ for $t \in (0, 1]$. By the concavity of J_{N_1} and our assumption,

$$J_{N_1}(a_t) \geq (1-t)J_{N_1}(a^*) + tJ_{N_1}(a^{N_1}) > (1-t)J_{N_1}(a^*) + tJ_{N_1}(a^*) = J_{N_1}(a^*). \quad (3.17)$$

Since $J(a^*) = J_{N_1}(a^*)$, equation (3.17) directly implies:

$$J_{N_1}(a_t) > J(a^*), \quad \text{for every } t \in (0, 1]. \quad (3.18)$$

Next, we analyze the asymptotic part of the functional $J(a_t)$. From (3.16), we know that $\inf_{j>N_1} \gamma_j(T) \int_{\Omega} a^*(x)|\phi_j|^2 dx > \gamma_{N_0}(T)$. Let $\delta = \inf_{j>N_1} \gamma_j(T) \int_{\Omega} a^*(x)|\phi_j|^2 dx - \gamma_{N_0}(T) > 0$.

For any $j > N_1$, we have

$$\gamma_j(T) \int_{\Omega} a_t(x)|\phi_j|^2 dx = (1-t)\gamma_j(T) \int_{\Omega} a^*(x)|\phi_j|^2 dx + t\gamma_j(T) \int_{\Omega} a^{N_1}(x)|\phi_j|^2 dx.$$

Since $a^{N_1} \geq 0$, this implies $\gamma_j(T) \int_{\Omega} a_t(x)|\phi_j|^2 dx \geq (1-t)\gamma_j(T) \int_{\Omega} a^*(x)|\phi_j|^2 dx$. Thus, for all $j > N_1$:

$$\gamma_j(T) \int_{\Omega} a_t(x)|\phi_j|^2 dx \geq (1-t)(\gamma_{N_0}(T) + \delta).$$

By choosing $t > 0$ sufficiently small (e.g., $t < \delta/(2(\gamma_{N_0}(T) + \delta))$), we can ensure $(1-t)(\gamma_{N_0}(T) + \delta) > \gamma_{N_0}(T)$. Therefore, for such t ,

$$\inf_{j>N_1} \gamma_j(T) \int_{\Omega} a_t(x)|\phi_j|^2 dx > \gamma_{N_0}(T). \quad (3.19)$$

Furthermore, since $N_0 \leq N_1$, we have $J_{N_1}(a_t) \leq \gamma_{N_0}(T)$. Combining that with (3.19), the

minimum must be:

$$J(a_t) = J_{N_1}(a_t).$$

This result, combined with (3.18), gives $J(a_t) > J(a^*)$, which contradicts the optimality of a^* . $\blacksquare$

Remark 3.10 *The assumptions of Proposition 3.5 (specifically $(\mathbf{H}^1)$, $(\mathbf{H}^2)$, and the condition $L > \gamma_{i_0}/\gamma^*$) are strong enough to imply hypothesis $(\mathbf{H}_*^1)$.*

Indeed, from Remark 3.4, $(\mathbf{H}^1)$ and $(\mathbf{H}^2)$ give

$$\lim_{j \rightarrow +\infty} \gamma_j(T) \int_{\Omega} a(x) |\phi_j(x)|^2 dx = L\gamma^*.$$

The condition $L > \gamma_{i_0}/\gamma^$ is equivalent to $L\gamma^* > \gamma_{i_0}$. By choosing $N_0 = i_0$, we have*

$$\liminf_{j \rightarrow +\infty} \gamma_j(T) \int_{\Omega} a(x) |\phi_j(x)|^2 dx = L\gamma^* > \gamma_{i_0} = \gamma_{N_0}(T).$$

Thus, hypothesis $(\mathbf{H}_^1)$ is satisfied.*

Remark 3.11 *Hypothesis $(\mathbf{H}_*^1)$ fails if $\gamma^* \leq \gamma_k$ for all k , i.e., if γ^* is the global minimum.*

Indeed, since $\int_{\Omega} a |\phi_j|^2 \leq 1$ (assuming normalized ϕ_j), we have:

$$\liminf_{j \rightarrow \infty} \gamma_j(T) \int_{\Omega} a(x) |\phi_j(x)|^2 dx \leq \liminf_{j \rightarrow \infty} \gamma_j(T) = \gamma^*.$$

So $(\mathbf{H}_^1)$ cannot hold.*

Although ergodicity $((\mathbf{H}^1), (\mathbf{H}^2))$ implies $(\mathbf{H}_*^1)$ (see Remark 3.10), this hypothesis is not always necessary for attainability. We now demonstrate this by analyzing the n -dimensional orthotope case.

Case: The n -Dimensional Orthotope

Assume $\Omega = (0, \pi)^n$ for $n \in \mathbb{N}^*$. We consider the usual Hilbert basis of eigenfunctions, given by

$$\phi_j(x) = \left(\frac{2}{\pi}\right)^{n/2} \prod_{k=1}^n \sin(j_k x_k), \quad (3.20)$$

for any multi-index $j = (j_1, \dots, j_n) \in (\mathbb{N}^*)^n$. The corresponding eigenvalues are $\lambda_j = \sum_{k=1}^n j_k^2$.

Remark 3.12 *The eigenfunctions (3.20) for $n > 1$ generally fail the QUE assumption $(\mathbf{H}^1)$, as noted for $n = 2$ in Remark 3.8.*

We first recall the following lemma, whose proof can be found in [41, Lemma 5].

Lemma 3.13 *Let $M > 0$ and $a \in L^\infty((0, \pi)^n; [0, M])$. Then*

$$\inf_{j \in (\mathbb{N}^*)^n} \int_{(0, \pi)^n} a(x) \phi_j(x)^2 dx \geq \frac{M}{\pi} F^{[n]} \left(\frac{\int_{(0, \pi)^n} a(x) dx}{M \pi^{n-1}} \right),$$

where $j = (j_1, \dots, j_n)$, ϕ_j is defined in (3.20), F is the function defined on $[0, \pi]$ by $F(s) = (s - \sin s)$, and $F^{[n]}$ is the n -fold composition of F with itself.

Proposition 3.14 *Assume $\Omega = (0, \pi)^n$, with eigenfunctions (3.20). Let $\gamma^* = \lim_{j \rightarrow \infty} \gamma_j$ and $j_0 = \operatorname{argmin}_{j \in \mathbb{N}^*} \gamma_j$. If the sequence satisfies $\gamma_{j_0} < \gamma^*$, then there exist $L_0 \in (0, 1)$ and $N_0 \in \mathbb{N}^*$ such that*

$$\max_{\chi_\omega \in \mathcal{U}_L} J(\chi_\omega) = \max_{\chi_\omega \in \mathcal{U}_L} J_N(\chi_\omega), \quad (3.21)$$

for every $L \in [L_0, 1)$ and every $N \geq N_0$.

In particular, for $L \in [L_0, 1)$, the problem (3.5) has a unique solution $\chi_{\omega^{N_0}}$, and this optimal set ω^{N_0} is semi-analytic with a finite number of connected components.

Proof: Let $j_0 = \operatorname{argmin}_{j \in \mathbb{N}^*} \gamma_j$ and $\gamma^* = \lim_{j \rightarrow \infty} \gamma_j$. We assume $L \in [L_0, 1)$ is chosen sufficiently close to 1. Since $\gamma_{j_0} < \gamma^*$ (by hypothesis of Theorem 3.14), we can select $\epsilon > 0$ small enough such that $L > \frac{\gamma_{j_0}}{\gamma^* - \epsilon} + \epsilon$.

Applying Lemma 3.13 with $M = 1$ and for the functions a satisfying $\int_\Omega a(x) dx = L|\Omega| = L\pi^n$, the lemma gives a uniform lower bound for all $j \in (\mathbb{N}^*)^n$:

$$\int_{(0, \pi)^n} a(x) \phi_j(x)^2 dx \geq \frac{1}{\pi} F^{[n]} \left(\frac{L\pi^n}{1 \cdot \pi^{n-1}} \right) = \frac{1}{\pi} F^{[n]}(L\pi).$$

Since $F(\pi) = (\pi - \sin \pi) = \pi$, it follows by induction that $F^{[n]}(\pi) = \pi$. As $L \rightarrow 1$, we have $L\pi \rightarrow \pi$, and by continuity, $F^{[n]}(L\pi) \rightarrow \pi$. Thus, the term $\frac{1}{\pi} F^{[n]}(L\pi)$ converges to 1 as $L \rightarrow 1$. We can therefore choose $L_0 \in (0, 1)$ such that for any $L \in [L_0, 1)$, $\frac{1}{\pi} F^{[n]}(L\pi) \geq L - \epsilon$.

Since $\gamma_j \rightarrow \gamma^*$ as $j \rightarrow \infty$, there exists $N_0 \in \mathbb{N}^*$ such that $\gamma_j \geq \gamma^* - \epsilon$ for all $j \geq N_0$. Combining these, for $L \in [L_0, 1)$ and $j \geq N_0$:

$$\gamma_j \int_{(0, \pi)^n} a(x) \phi_j(x)^2 dx \geq (\gamma^* - \epsilon)(L - \epsilon). \quad (3.22)$$

By our initial choice of L , we have $(\gamma^* - \epsilon)(L - \epsilon) > \gamma_{j_0}$, which in particular implies on $(\mathbf{H}_*^1)$. The conclusion follows from Proposition 3.9. ■

Remark 3.15 *Another key example, studied in [42], is the unit disk, where eigenfunctions are defined using Bessel functions. A uniform lower bound analogous to Lemma 3.13 is not known for this framework and remains an open question.*

In [42], a property similar to $(\mathbf{H}_^1)$ was proven, but that proof relied on spectral inequalities and the fact that their sequence $\gamma_j \rightarrow \infty$. Since the sequence in our paper converges to a finite limit ($\gamma_j \rightarrow \gamma^*$), that argument does not apply. Thus, the validity of $(\mathbf{H}_*^1)$ for the disk in our framework is also an open question.*

3.2.3 On the Condition $\inf \gamma_j < \gamma^*$

In our previous attainability proofs (e.g., Proposition 3.5 and Proposition 3.14), the condition $\inf \gamma_j < \gamma^*$ was essential. This assumption allowed us to show that the relaxed solution a^* is also a maximizer for a truncated functional J_{N_0} , which, by Proposition 3.2, has a unique classical solution.

Nevertheless, this condition is sufficient but may not be necessary. We now explore cases where attainability holds even when this assumption is not met.

We now analyze the specific case of Laplacian eigenfunctions with Dirichlet boundary conditions on $\Omega = (0, \pi)$, assuming that $\gamma^* = \inf \gamma_j$. The eigenfunctions are $\phi_j(x) = \sqrt{2/\pi} \sin(jx)$, so the functional J defined in (3.5) becomes:

$$J(\chi_\omega) = \inf_{j \in \mathbb{N}^*} \gamma_j \int_{\omega} \frac{2}{\pi} \sin^2(jx) dx. \quad (3.23)$$

The corresponding relaxed (or "convex") version, defined for $a \in \overline{\mathcal{U}}_L$, is given by:

$$J(a) = \inf_{j \in \mathbb{N}} \gamma_j \int_0^\pi \frac{2}{\pi} a(x) \sin^2(jx) dx. \quad (3.24)$$

Our focus remains on the optimization problems (3.5) and (3.8), which we now analyze in the specific 1D setting defined by (3.23) and (3.24).

Lemma 3.16 *The value of the supremum for the relaxed problem (3.8) in the 1D case (3.24) is $L\gamma^*$.*

$$\sup_{a \in \overline{\mathcal{U}}_L} J(a) = L\gamma^*. \quad (3.25)$$

Moreover, this supremum is attained by any function $a \in \overline{\mathcal{U}}_L$ whose Fourier series is of the form

$$a(x) = L + \sum_{j=1}^{+\infty} (a_j \cos(2jx) + b_j \sin(2jx)),$$

provided the cosine coefficients satisfy $a_j \leq 0$ for all $j \in \mathbb{N}^$.*

Proof:

The proof is inspired by [44, Proposition 1], which addresses the specific case where $\gamma_j \equiv 1$.

For any $a \in \overline{\mathcal{U}}_L$, the infimum is bounded by the limit of its terms:

$$J(a) = \inf_{j \in \mathbb{N}^*} \gamma_j \int_0^\pi \frac{2}{\pi} a(x) \sin^2(jx) dx \leq \lim_{j \rightarrow \infty} \left(\gamma_j \cdot \frac{2}{\pi} \int_0^\pi a(x) \sin^2(jx) dx \right).$$

We have two limits: $\lim_{j \rightarrow \infty} \gamma_j = \gamma^*$. For the integral, using $\sin^2(jx) = (1 - \cos(2jx))/2$, we get

$$\int_0^\pi a(x) \sin^2(jx) dx = \frac{1}{2} \int_0^\pi a(x) dx - \frac{1}{2} \int_0^\pi a(x) \cos(2jx) dx.$$

Since $\int_0^\pi a(x) dx = L\pi$ and the second term tends to 0 by the Riemann-Lebesgue Lemma, $\lim_{j \rightarrow \infty} \int_0^\pi a(x) \sin^2(jx) dx = \frac{L\pi}{2}$. This shows $J(a) \leq L\gamma^*$ for all $a \in \overline{\mathcal{U}}_L$.

To show the supremum is attained, assume $a \in \overline{\mathcal{U}}_L$ satisfies the hypothesis $a_j \leq 0$ (its Fourier coefficients for $\cos(2jx)$). Using $\sin^2(jx) = \frac{1}{2}(1 - \cos(2jx))$:

$$\begin{aligned} \gamma_j \int_0^\pi \frac{2}{\pi} a(x) \sin^2(jx) dx &= \gamma_j \left(\frac{1}{\pi} \int_0^\pi a(x) dx - \frac{1}{\pi} \int_0^\pi a(x) \cos(2jx) dx \right) \\ &\geq \gamma_j L. \end{aligned}$$

Taking the infimum over j :

$$J(a) \geq L \inf_{j \in \mathbb{N}^*} \gamma_j = L\gamma^*.$$

■

Remark 3.17 The result in [44, Proposition 1] establishes a necessary and sufficient condition: the supremum is $L\gamma^*$ if and only if the Fourier coefficients satisfy $a_j \leq 0$. However, this equivalence relies on the fact that $\gamma_j \equiv \gamma^* = 1$ in their context.

By repeating their proof in our framework, we cannot derive the necessary condition $a_j \leq 0$. This is evident from the following estimate:

$$\begin{aligned} L\gamma^* &= \inf_{j \in \mathbb{N}^*} \gamma_j \int_0^\pi \frac{2}{\pi} a(x) \sin^2(jx) dx \\ &= \inf_{j \in \mathbb{N}^*} \gamma_j \int_0^\pi \frac{2}{\pi} a(x) \left(\frac{1}{2} - \frac{1}{2} \cos(2jx) \right) dx \\ &\geq \inf_{j \in \mathbb{N}^*} \left(\gamma_j \int_0^\pi \frac{1}{\pi} a(x) dx \right) + \inf_{j \in \mathbb{N}^*} \left(-\gamma_j \int_0^\pi \frac{1}{\pi} a(x) \cos(2jx) dx \right) \\ &= L\gamma^* - \sup_{j \in \mathbb{N}^*} \left(\gamma_j \int_0^\pi \frac{1}{\pi} a(x) \cos(2jx) dx \right). \quad (3.26) \end{aligned}$$

This inequality implies

$$\sup_{j \in \mathbb{N}^*} \left(\gamma_j \int_0^\pi \frac{1}{\pi} a(x) \cos(2jx) dx \right) \geq 0.$$

Consequently, we cannot conclude that the cosine coefficients must be non-positive for all j .

Therefore, following the approach in [44], we will show that in the specific case $L = \frac{1}{2}$, an optimal set exists and attains the supremum value $L\gamma^*$

Proposition 3.18 *Consider the admissible class*

$$\overline{\mathcal{U}}_{1/2} = \left\{ a \in L^\infty((0, \pi); [0, 1]) : \int_0^\pi a(x) dx = \frac{\pi}{2} \right\}$$

and the functional

$$J(a) = \inf_{j \in \mathbb{N}^*} \gamma_j \int_0^\pi \frac{2}{\pi} a(x) \sin^2(jx) dx,$$

where $(\gamma_j)_{j \geq 1}$ is a sequence of positive numbers satisfying

$$\lim_{j \rightarrow \infty} \gamma_j = \gamma^* \quad \text{with} \quad \gamma^* = \inf_{j \in \mathbb{N}^*} \gamma_j.$$

Then, the supremum of J on $\mathcal{U}_{1/2}$ is attained and is equal to $\gamma^*/2$.

Proof: Let $\omega \subset [0, \pi]$ be a measurable subset with measure $|\omega| = \pi/2$, satisfying the symmetry condition:

$$x \in \omega \iff \pi - x \notin \omega.$$

Define the characteristic function $\chi_\omega(x) = \mathbf{1}_\omega(x)$. By construction, $\chi_\omega \in \mathcal{U}_{1/2}$. We first claim that the Fourier cosine coefficients vanish:

$$\int_0^\pi \chi_\omega(x) \cos(2jx) dx = 0 \quad \text{for every } j \geq 1.$$

To see this, first observe that the integral over the full domain is zero:

$$\int_0^\pi \cos(2jx) dx = \left[\frac{\sin(2jx)}{2j} \right]_0^\pi = 0.$$

We decompose the integral over $[0, \pi]$ into the sum over ω and its complement $\omega' = [0, \pi] \setminus \omega$:

$$0 = \int_0^\pi \cos(2jx) dx = \int_\omega \cos(2jx) dx + \int_{\omega'} \cos(2jx) dx.$$

To analyze the term over ω' , we apply the change of variable $y = \pi - x$. Given the symmetry of the set construction ($x \in \omega' \iff \pi - x \in \omega$), the domain of integration transforms from ω' to ω . Consequently,

$$\int_{\omega'} \cos(2jx) dx = \int_{\omega} \cos(2j(\pi - y)) dy.$$

Using the periodicity and evenness of the cosine function, we have $\cos(2j(\pi - y)) = \cos(2jy)$ for any integer j . Thus, the integral over the complement becomes:

$$\int_{\omega'} \cos(2jx) dx = \int_{\omega} \cos(2jy) dy.$$

Substituting this back into the decomposition of the total integral, we obtain:

$$0 = \int_{\omega} \cos(2jx) dx + \int_{\omega'} \cos(2jx) dx = 2 \int_{\omega} \cos(2jx) dx.$$

This implies $\int_{\omega} \cos(2jx) dx = 0$. Consequently, the Fourier cosine coefficients of the characteristic function vanish:

$$\int_0^{\pi} \chi_{\omega}(x) \cos(2jx) dx = 0 \quad \text{for every } j \geq 1.$$

Finally, we evaluate the functional for this characteristic function χ_{ω} . Using the identity $\sin^2(jx) = \frac{1}{2}(1 - \cos(2jx))$, we compute the integral term for any $j \in \mathbb{N}^*$:

$$\begin{aligned} \frac{2}{\pi} \int_0^{\pi} \chi_{\omega}(x) \sin^2(jx) dx &= \frac{2}{\pi} \int_0^{\pi} \chi_{\omega}(x) \left(\frac{1}{2} - \frac{1}{2} \cos(2jx) \right) dx \\ &= \frac{1}{\pi} \int_0^{\pi} \chi_{\omega}(x) dx - \frac{1}{\pi} \int_0^{\pi} \chi_{\omega}(x) \cos(2jx) dx. \end{aligned}$$

Substituting the known values $\int_0^{\pi} \chi_{\omega}(x) dx = \frac{\pi}{2}$ and $\int_0^{\pi} \chi_{\omega}(x) \cos(2jx) dx = 0$, we obtain:

$$\frac{2}{\pi} \int_0^{\pi} \chi_{\omega}(x) \sin^2(jx) dx = \frac{1}{\pi} \cdot \frac{\pi}{2} - 0 = \frac{1}{2}.$$

This value is independent of j . Therefore, the functional becomes:

$$J(\chi_{\omega}) = \inf_{j \in \mathbb{N}^*} \left(\gamma_j \cdot \frac{1}{2} \right) = \frac{1}{2} \inf_{j \in \mathbb{N}^*} \gamma_j = \frac{\gamma^*}{2}.$$

■

A similar strategy allows us to extend the previous existence results to the orthotope $\Omega = (0, \pi)^n$. For the sake of simplicity, we focus on the two-dimensional case $n = 2$.

We investigate the behavior of the functional J_2 restricted to Cartesian product sets. Specifically, we consider characteristic functions $a = \chi_\omega$ where $\omega = \omega_1 \times \omega_2 \subset (0, \pi)^2$. The functional is given by:

$$J_2(\chi_\omega) = \inf_{(j,k) \in (\mathbb{N}^*)^2} \gamma_{j,k} \iint_{\omega} \frac{4}{\pi^2} \sin^2(jx) \sin^2(ky) dx dy.$$

Our goal is to explicitly construct product sets $\omega = \omega_1 \times \omega_2$ satisfying the area constraint $|\omega| = |\omega_1||\omega_2| = L\pi^2$ for the specific values $L \in \{1/4, 1/2\}$. We will demonstrate that for these sets, the functional attains the optimal value:

$$J_2(\chi_\omega) = L\gamma^*, \quad \text{where } \gamma^* := \inf_{(j,k) \in (\mathbb{N}^*)^2} \gamma_{j,k}.$$

Throughout this analysis, we assume that $\gamma_{j,k} > 0$ and that the sequence converges to this limit, i.e., $\lim_{j,k \rightarrow \infty} \gamma_{j,k} = \gamma^*$.

Considering a product set $\omega = \omega_1 \times \omega_2$, the separability of the integrand allows us to factorize the double integral:

$$\frac{4}{\pi^2} \iint_{\omega} \sin^2(jx) \sin^2(ky) dx dy = \left(\frac{2}{\pi} \int_{\omega_1} \sin^2(jx) dx \right) \left(\frac{2}{\pi} \int_{\omega_2} \sin^2(ky) dy \right).$$

We introduce, for $i \in \{1, 2\}$, the quantities

$$\alpha_i(m) := \frac{2}{\pi} \int_{\omega_i} \sin^2(mt) dt, \quad \text{for } m \in \mathbb{N}^*.$$

With this notation, the functional takes the compact form:

$$J_2(\chi_\omega) = \inf_{(j,k) \in (\mathbb{N}^*)^2} \gamma_{j,k} \alpha_1(j) \alpha_2(k).$$

To analyze the functions $\alpha_i(j)$, we apply the identity $\sin^2(nt) = \frac{1}{2}(1 - \cos(2nt))$, yielding:

$$\alpha_i(j) = \frac{|\omega_i|}{\pi} - \frac{1}{\pi} \int_{\omega_i} \cos(2jx) dx.$$

To ensure that $\alpha_i(j)$ is constant (independent of j), we require the oscillatory integrals to vanish. This is achieved by choosing ω_i to satisfy the anti-reflective symmetry condition:

$$x \in \omega_i \iff \pi - x \notin \omega_i,$$

which implies $|\omega_i| = \pi/2$, or by choosing the full domain $\omega_i = (0, \pi)$, which implies $|\omega_i| = \pi$.

For such product sets $\omega = \omega_1 \times \omega_2$, the functional becomes:

$$J_2(\chi_\omega) = \gamma^* \frac{|\omega_1||\omega_2|}{\pi^2} = \gamma^* \frac{L\pi^2}{\pi^2} = L\gamma^*.$$

We remark that there are infinitely many sets (ω_1, ω_2) satisfying these properties. However, since the measure of each component ω_i is constrained to be either $\pi/2$ or π , this construction yields optimal solutions specifically for the cases $L = \frac{1}{4}$ (where $|\omega_1| = |\omega_2| = \pi/2$) and $L = \frac{1}{2}$ (where one set has measure $\pi/2$ and the other π).

For the general case $\Omega = (0, \pi)^n$, a similar strategy allows us to deduce the existence of optimal sets specifically when the volume fraction takes the form $L = \frac{1}{2^m}$ for any integer $m \in \{1, \dots, n\}$.

By constructing product sets where m components are symmetric subsets of measure $\pi/2$ (eliminating oscillatory terms) and the remaining $n - m$ components are the full interval $(0, \pi)$, we obtain a set with volume fraction $L = (1/2)^m \cdot 1^{n-m} = 1/2^m$ that attains the optimal value $L\gamma^*$. In this way we have the following result for the n -dimensional orthotope:

Proposition 3.19 *Consider the domain $\Omega = (0, \pi)^n$ and the functional J defined by*

$$J(\chi_\omega) = \inf_{j \in (\mathbb{N}^*)^n} \gamma_j \int_{\omega} |\phi_j(x)|^2 dx,$$

where ϕ_j are the sine-product eigenfunctions defined in (3.20). Assume the sequence (γ_j) satisfies $\lim_{|j| \rightarrow \infty} \gamma_j = \gamma^*$ and $\gamma_j \geq \gamma^*$ for all j .

If the volume fraction satisfies $L = \frac{1}{2^m}$ for some integer $m \in \{1, \dots, n\}$, then the supremum of J over $\mathcal{U}_L$ is attained and is equal to $L\gamma^*$.

Specifically, the maximum is reached by any product set $\omega = \omega_1 \times \dots \times \omega_n$ constructed as follows:

- m components are measurable subsets of $(0, \pi)$ with measure $\pi/2$ satisfying the symmetry condition $x \in \omega_k \iff \pi - x \notin \omega_k$;
- The remaining $n - m$ components are the full interval $(0, \pi)$.

Remark 3.20 Proposition 3.19 establishes the existence of optimal sets in cases where the sequence converges from above ($\gamma_j \geq \gamma^*$). However, as shown in Remark 3.11, the condition $(\mathbf{H}_*^1)$ fails in this exact scenario. This demonstrates that $(\mathbf{H}_*^1)$ is not a necessary condition for the existence of classical solutions in general.

A similar reasoning applies to the case where $\Omega = \{x \in \mathbb{R}^2; \|x\| < 1\}$. In this case, the Hilbert basis of eigenfunctions is defined by the triply indexed sequence of functions

$$\phi_{j,k,m}(r, \theta) = \begin{cases} R_{0,k}(r)/\sqrt{2\pi} & \text{if } j = 0, \\ R_{j,k}Y_{j,m}(\theta) & \text{if } j \geq 1, \end{cases} \quad (3.27)$$

for $j \in \mathbb{N}, k \in \mathbb{N}^*$ and $m = 1, 2$, where (r, θ) are the usual polar coordinates. The functions $Y_{j,m}(\theta)$ are defined by $Y_{j,1}(\theta) = \frac{1}{\sqrt{\pi}}\cos(j\theta)$ and $Y_{j,2}(\theta) = \frac{1}{\sqrt{\pi}}\sin(j\theta)$ and the functions $R_{j,k}$ are defined by

$$R_{j,k}(r) = \frac{\sqrt{2}J_j(z_{j,k}r)}{|J'_j(z_{j,k})|} = \frac{\sqrt{2}J_j(z_{j,k}r)}{\sqrt{\int_0^1 J_j(z_{j,k})^2 r dr}}, \quad (3.28)$$

where J_j is the Bessel function of the first kind of order j , and $z_{j,k} > 0$ is the k^{th} positive zero of J_j . The corresponding eigenvalues consist of a doubly indexed sequence $(-\lambda_{j,k})_{j \in \mathbb{N}, k \in \mathbb{N}^*}$, where $\lambda_{j,k} = z_{j,k}^2$ is of multiplicity 1, if $j = 0$ and 2, if $j \geq 1$.

The similarity arises from the separation of variables in the eigenfunctions. Specifically, using the identity $\phi_{j,k,m}(r, \theta) = R_{j,k}(r)Y_{j,m}(\theta)$ and the normalization of the radial part ($\int_0^1 R_{j,k}^2(r)r dr = 1$), we obtain:

$$\begin{aligned} L\gamma_{j,k} \int_0^{2\pi} \int_0^1 R_{j,k}(r)^2 r Y_{j,m}(\theta)^2 dr d\theta &= L\gamma_{j,k} \left(\int_0^1 R_{j,k}(r)^2 r dr \right) \left(\int_0^{2\pi} Y_{j,m}(\theta)^2 d\theta \right) \\ &= L\gamma_{j,k} \int_0^{2\pi} Y_{j,m}(\theta)^2 d\theta, \end{aligned}$$

This effectively reduces the problem to an angular integral, analogous to the 1D interval case.

In this framework, we establish the following result for the unit disk.

Proposition 3.21 *Assume $\Omega = \{x \in \mathbb{R}^2 : \|x\| < 1\}$ is the unit disk, equipped with the basis of eigenfunctions defined in terms of Bessel functions. Let $\mathcal{A}$ be the class of measurable subsets $\omega \subset \Omega$ expressible in polar coordinates as product sets:*

$$\omega = \{(r \cos \theta, r \sin \theta) : r \in \omega_r, \theta \in \omega_\theta\},$$

where $\omega_r \subset [0, 1]$ and $\omega_\theta \subset [0, 2\pi]$ are measurable sets such that $|\omega| = L|\Omega| = L\pi$.

If $L = \frac{1}{2}$, then the supremum of J over the class $\mathcal{A}$ is equal to $L\gamma^*$ and is attained.

Specifically, the maximum is reached by any set $\omega \in \mathcal{A}$ of the form $\omega = \{(r, \theta) : r \in [0, 1], \theta \in \omega_\theta\}$, where the angular set ω_θ has measure π and satisfies the symmetry

condition:

$$\omega_\theta \cap \omega'_\theta = \emptyset \quad \text{and} \quad \omega_\theta \cup \omega'_\theta = [0, 2\pi] \quad (\text{up to measure zero}),$$

where $\omega'_\theta = \{2\pi - \theta : \theta \in \omega_\theta\}$ is the symmetric image of ω_θ .

3.3 System of Linear Thermoelasticity

We consider a linear system of thermoelasticity defined on a bounded open subset $\Omega \subset \mathbb{R}^N$. The system consists of coupled wave and heat equations given by :

$$\begin{cases} u_{tt} - c^2 \Delta u + \alpha \Delta \theta = 0 & \text{in } \Omega \times (0, T), \\ \theta_t - \nu \Delta \theta + \beta u_t = 0 & \text{in } \Omega \times (0, T), \\ u = \theta = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u^0, u_t(0) = u^1, \theta(0) = \theta^0 & \text{in } \Omega, \end{cases} \quad (3.29)$$

where $T > 0$ is the final time and $\partial\Omega$ denotes the boundary of Ω . The coupling parameters α, β and the viscosity ν are assumed to be positive constants. The System (3.29) models the coupling between a wave equation for the displacement u and a parabolic equation for the temperature θ .

We introduce the following functional spaces:

$$H = H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega), \quad (3.30)$$

and

$$H' = L^2(\Omega) \times H^{-1}(\Omega) \times H^{-1}(\Omega). \quad (3.31)$$

The system (3.29) is well-posed in the energy space H . The associated energy is given by:

$$E(t) = \frac{1}{2} \int_{\Omega} \left(|u_t(x, t)|^2 + c^2 |\nabla u(x, t)|^2 + \frac{\alpha}{\beta} |\nabla \theta(x, t)|^2 \right) dx. \quad (3.32)$$

3.3.1 Spectral Decomposition

In [35], the authors established the null controllability of system (3.29) in the energy space H . To achieve this, they performed a spectral decomposition of the system using the basis generated by the eigenfunctions of the Laplacian operator in $H_0^1(\Omega)$. They observed that, for high frequencies, the spectrum splits into distinct parabolic and hyperbolic parts. By projecting the system onto the corresponding subspaces, they obtained two weakly

coupled systems: one behaving like a heat equation and the other like a wave equation. They then applied arguments from [53] and [54], respectively, to control each component.

Although this paper does not focus on controllability, the spectral decomposition analysis provided in [35] is fundamental to our work.

We consider the operator $A : D(A) \subset H \rightarrow H$ associated with the system (3.29). Its domain is defined by

$$D(A) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times \{\theta \in H^3(\Omega) : \theta = \Delta\theta = 0 \text{ on } \partial\Omega\},$$

and its matrix representation is given by

$$A = \begin{pmatrix} 0 & I & 0 \\ c^2\Delta & 0 & -\alpha\Delta \\ 0 & -\beta I & \nu\Delta \end{pmatrix}.$$

Consequently, the system (3.29) can be rewritten as the abstract first-order evolution equation:

$$\begin{cases} U_t = AU, \\ U(0) = U_0, \end{cases} \quad (3.33)$$

where $U(t) = (u(t), u_t(t), \theta(t))^\top$ and initial data $U_0 = (u_0, u_1, \theta_0)^\top$.

Let $(\phi_j)_{j \in \mathbb{N}^*}$ be a Hilbert basis of $L^2(\Omega)$ consisting of the eigenfunctions of the Dirichlet Laplacian:

$$\begin{cases} -\Delta\phi_j = \omega_j^2\phi_j & \text{in } \Omega, \\ \phi_j = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.34)$$

The associated eigenfrequencies form a non-decreasing sequence of positive real numbers:

$$0 < \omega_1 \leq \omega_2 \leq \dots \leq \omega_n \leq \dots \rightarrow +\infty.$$

We decompose the solution $U(t)$ of (3.33) into a Fourier series

$$U(t) = \sum_{j=1}^{\infty} U_j(t) \phi_j(x), \quad (3.35)$$

where, for each j , the coefficient vector $U_j(t) \in \mathbb{C}^3$ satisfies the ordinary differential equation:

$$\frac{dU_j(t)}{dt} = A_j U_j(t), \quad \text{with} \quad A_j = \begin{pmatrix} 0 & 1 & 0 \\ -c^2\omega_j^2 & 0 & \alpha\omega_j^2 \\ 0 & -\beta & -\nu\omega_j^2 \end{pmatrix}. \quad (3.36)$$

Eigenvalues and Eigenfunctions

The eigenvalues of A_j are the roots of the characteristic polynomial:

$$(\lambda^2 + c^2\omega_j^2)(\lambda + \nu\omega_j^2) + \alpha\beta\lambda\omega_j^2 = 0. \quad (3.37)$$

We seek the eigenvalues as perturbations of the uncoupled spectrum. Specifically, we define the parabolic branch $\lambda_{j,1} = -\nu\omega_j^2 + \epsilon_j$ and the hyperbolic branch $\lambda_{j,2} = ic\omega_j + \eta_j$. Substituting these expressions into (3.37), we find that the perturbation terms ϵ_j and η_j must satisfy the following implicit equations:

$$\epsilon_j = \frac{\alpha\beta\left(\nu - \frac{\epsilon_j}{\omega_j^2}\right)}{\left(\nu - \frac{\epsilon_j}{\omega_j^2}\right)^2 + \frac{c^2}{\omega_j^2}} \quad \text{and} \quad \eta_j = \frac{-\alpha\beta\left(ic + \frac{\eta_j}{\omega_j}\right)}{\left(2ic + \frac{\eta_j}{\omega_j}\right)\left(\nu + \frac{ic}{\omega_j} + \frac{\eta_j}{\omega_j^2}\right)}. \quad (3.38)$$

We define the auxiliary functions $F^1, F^2 : \mathbb{C} \times \mathbb{R} \rightarrow \mathbb{C}$ by:

$$F^1(\epsilon, z) = \epsilon - \frac{\alpha\beta(\nu - z^2\epsilon)}{(\nu - z^2\epsilon)^2 + z^2c^2} \quad \text{and} \quad F^2(\eta, z) = \eta + \frac{\alpha\beta(ic + z\eta)}{(2ic + z\eta)(\nu + zic + z^2\eta)}. \quad (3.39)$$

Evaluating these functions at $z = 0$, we find the following roots:

$$F^1\left(\frac{\alpha\beta}{\nu}, 0\right) = 0 \quad \text{and} \quad F^2\left(-\frac{\alpha\beta}{2\nu}, 0\right) = 0.$$

Moreover, we compute the partial derivatives with respect to the first arguments (ϵ and η). For F^1 , we obtain:

$$\frac{\partial F^1}{\partial \epsilon}(\epsilon, z) = 1 - \frac{-\alpha\beta z^2((\nu - z^2\epsilon)^2 + z^2c^2) + 2\alpha\beta z^2(\nu - z^2\epsilon)^2}{((\nu - z^2\epsilon)^2 + z^2c^2)^2}.$$

Similarly, for F^2 :

$$\begin{aligned} & \frac{\partial F^2}{\partial \eta}(\eta, z) \\ &= 1 + \frac{\alpha\beta z(2ic + z\eta)(\nu + zic + z^2\eta) - \alpha\beta(ic + z\eta)(z(\nu + zic + z^2\eta) + z^2(2ic + z\eta))}{(2ic + z\eta)^2(\nu + zic + z^2\eta)^2}. \end{aligned}$$

Evaluating these expressions at $z = 0$, the rational terms vanish due to the factors of z in the numerators. Thus, we find:

$$\frac{\partial F^1}{\partial \epsilon}\left(\frac{\alpha\beta}{\nu}, 0\right) = 1 \quad \text{and} \quad \frac{\partial F^2}{\partial \eta}\left(-\frac{\alpha\beta}{2\nu}, 0\right) = 1.$$

By the Implicit Function Theorem, there exists a neighborhood $U \subset \mathbb{C}$ of $z = 0$ and unique analytic functions $\epsilon, \eta : U \rightarrow \mathbb{C}$, satisfying the initial conditions $\epsilon(0) = \frac{\alpha\beta}{\nu}$ and $\eta(0) = -\frac{\alpha\beta}{2\nu}$, such that

$$F^1(\epsilon(z), z) = 0 \quad \text{and} \quad F^2(\eta(z), z) = 0 \quad \text{for all } z \in U.$$

Furthermore, these functions satisfy the differential relations:

$$\frac{d\epsilon}{dz}(z) = -\frac{\frac{\partial F^1}{\partial z}(\epsilon(z), z)}{\frac{\partial F^1}{\partial \epsilon}(\epsilon(z), z)} \quad \text{and} \quad \frac{d\eta}{dz}(z) = -\frac{\frac{\partial F^2}{\partial z}(\eta(z), z)}{\frac{\partial F^2}{\partial \eta}(\eta(z), z)}. \quad (3.40)$$

We explicitly compute the partial derivatives with respect to z at the origin:

$$\frac{\partial F^1}{\partial z} \left(\frac{\alpha\beta}{\nu}, 0 \right) = 0 \quad \text{and} \quad \frac{\partial F^2}{\partial z} \left(-\frac{\alpha\beta}{2\nu}, 0 \right) = -\frac{i\alpha\beta(4c^2 - \alpha\beta)}{8c\nu^2}.$$

Recall from the previous step that the partial derivatives with respect to the first variables (ϵ and η) are equal to 1 at these points. Substituting these values into (3.40), we obtain the derivatives of the implicit functions at $z = 0$:

$$\frac{d\epsilon}{dz}(0) = 0 \quad \text{and} \quad \frac{d\eta}{dz}(0) = \frac{i\alpha\beta(4c^2 - \alpha\beta)}{8c\nu^2}.$$

For subsequent computations, we require a higher-order asymptotic expansion for the eigenvalues λ than the one provided in [35]. To this end, we compute the second derivative of $\epsilon(z)$. Differentiating (3.40), we obtain:

$$\frac{d^2\epsilon}{dz^2}(z) = -\frac{\left(\frac{\partial^2 F^1}{\partial \epsilon \partial z} \frac{d\epsilon}{dz} + \frac{\partial^2 F^1}{\partial z^2} \right) \frac{\partial F^1}{\partial \epsilon} - \frac{\partial F^1}{\partial z} \left(\frac{\partial^2 F^1}{\partial \epsilon^2} \frac{d\epsilon}{dz} + \frac{\partial^2 F^1}{\partial \epsilon \partial z} \right)}{\left(\frac{\partial F^1}{\partial \epsilon} \right)^2}, \quad (3.41)$$

where all partial derivatives of F^1 are evaluated at $(\epsilon(z), z)$.

Evaluating this expression at $z = 0$, and recalling that $\frac{d\epsilon}{dz}(0) = 0$ and $\frac{\partial F^1}{\partial z}(\frac{\alpha\beta}{\nu}, 0) = 0$, the expression simplifies significantly. We obtain:

$$\frac{d^2\epsilon}{dz^2}(0) = \frac{2\alpha\beta}{\nu^3}(\alpha\beta - c^2).$$

By a completely analogous computation, we obtain the second derivative for η at the origin:

$$\frac{d^2\eta}{dz^2}(0) = \frac{\alpha\beta c^2}{\nu^3} - \frac{\alpha^3\beta^3}{16c^2\nu^3} - \frac{3\alpha^2\beta^2}{4\nu^3} + \frac{\alpha\beta}{2c\nu} \left(\frac{\alpha\beta ci}{2\nu^2} - \frac{\alpha^2\beta^2 i}{8c\nu^2} \right) i.$$

Simplifying the last term (using $i^2 = -1$):

$$\frac{d^2\eta}{dz^2}(0) = \frac{\alpha\beta(c^2 - \alpha\beta)}{\nu^3}.$$

Thus, for j sufficiently large, we substitute $z = \omega_j^{-1}$ into the Taylor expansions. Recalling that $\lambda_{j,1} = -\nu\omega_j^2 + \epsilon(\omega_j^{-1})$ and $\lambda_{j,2} = ic\omega_j + \eta(\omega_j^{-1})$, we obtain the following asymptotic expansions:

For the parabolic branch:

$$\lambda_j^p = -\nu\omega_j^2 + \frac{\alpha\beta}{\nu} + \frac{\alpha\beta(\alpha\beta - c^2)}{\nu^3} \frac{1}{\omega_j^2} + \mathcal{O}\left(\frac{1}{\omega_j^3}\right). \quad (3.42)$$

For the hyperbolic branch, utilizing the simplified form of the second derivative derived previously:

$$\lambda_j^{h+} = ic\omega_j - \frac{\alpha\beta}{2\nu} + \frac{i\alpha\beta(4c^2 - \alpha\beta)}{8c\nu^2} \frac{1}{\omega_j} + \frac{\alpha\beta(c^2 - \alpha\beta)}{2\nu^3} \frac{1}{\omega_j^2} + \mathcal{O}\left(\frac{1}{\omega_j^3}\right). \quad (3.43)$$

The eigenvalues derived in (3.42) and (3.43) correspond to the roots of the characteristic polynomial (3.37) of the matrix A_j . The spectrum is thus partitioned into a "parabolic" branch, represented by the single real eigenvalue λ_j^p , and a "hyperbolic" branch, consisting of the complex conjugate pair λ_j^{h+} and $\lambda_j^{h-} = \overline{\lambda_j^{h+}}$.

Computation of the Eigenvectors

The eigenvectors U corresponding to an eigenvalue $\lambda \in \sigma_p(A)$ satisfy the equation

$$AU = \lambda U.$$

By expressing the eigenvector in the Fourier series form $U = \sum_{j=1}^{\infty} (U_j^1, U_j^2, U_j^3)^\top \phi_j$, we find that for each mode j , the coefficients must satisfy the following system:

$$\begin{cases} U_j^2 = \lambda U_j^1, \\ -c^2\omega_j^2 U_j^1 + \alpha\omega_j^2 U_j^3 = \lambda U_j^2, \\ -\beta U_j^2 - \nu\omega_j^2 U_j^3 = \lambda U_j^3. \end{cases}$$

Substituting $U_j^2 = \lambda U_j^1$ into the system, we eliminate U_j^2 and obtain the following

coupled equations for U_j^1 and U_j^3 :

$$\begin{cases} -c^2\omega_j^2 U_j^1 + \alpha\omega_j^2 U_j^3 = \lambda^2 U_j^1, \\ -\beta\lambda U_j^1 - \nu\omega_j^2 U_j^3 = \lambda U_j^3. \end{cases}$$

Thus, for a given eigenvalue λ , the eigenvectors have the form:

$$U_\lambda = U_j^3 \begin{pmatrix} \frac{\alpha\omega_j^2}{\lambda^2 + c^2\omega_j^2} \\ \frac{\alpha\lambda\omega_j^2}{\lambda^2 + c^2\omega_j^2} \\ 1 \end{pmatrix} \phi_j \quad \text{or equivalently} \quad U_\lambda = U_j^1 \begin{pmatrix} 1 \\ \lambda \\ -\frac{\beta\lambda}{\lambda + \nu\omega_j^2} \end{pmatrix} \phi_j. \quad (3.44)$$

Therefore, a natural strategy is to associate the first structural form in (3.44) with the parabolic eigenvalue λ_j^p , and the second form with the hyperbolic eigenvalues $\lambda_j^{h\pm}$. To ensure the eigenvectors remain bounded in the energy space H , we select the normalizations $U_j^3 = \frac{1}{\omega_j}$ for the parabolic mode and $U_j^1 = (\lambda_j^{h\pm})^{-1}$ for the hyperbolic modes. In summary, the eigenvectors associated with high frequencies are given by:

$$U_j^p = \begin{pmatrix} \frac{\alpha\omega_j}{(\lambda_j^p)^2 + c^2\omega_j^2} \\ \frac{\alpha\lambda_j^p\omega_j}{(\lambda_j^p)^2 + c^2\omega_j^2} \\ \frac{1}{\omega_j} \end{pmatrix} \phi_j \quad \text{and} \quad U_j^{h\pm} = \begin{pmatrix} (\lambda_j^{h\pm})^{-1} \\ 1 \\ \frac{-\beta}{\nu\omega_j^2 + \lambda_j^{h\pm}} \end{pmatrix} \phi_j. \quad (3.45)$$

As established in [35], by incorporating the generalized eigenvectors associated with low frequencies into the set (3.45), we obtain a Riesz basis $\{U_{j,1}, U_{j,2}, U_{j,3}\}$ for the energy space H . For j sufficiently large, these correspond to $U_{j,1} = U_j^p$, $U_{j,2} = U_j^{h+}$, and $U_{j,3} = U_j^{h-}$.

For the subsequent analysis, we introduce the space

$$\widehat{H} = H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega). \quad (3.46)$$

To ensure suitable properties in this space, we renormalize the parabolic eigenvectors (specifically, multiplying the previous U_j^p by ω_j). The high-frequency eigenvectors for $\widehat{H}$ take the form:

$$U_j^p = \begin{pmatrix} \frac{\alpha\omega_j^2}{(\lambda_j^p)^2 + c^2\omega_j^2} \\ \frac{\alpha\lambda_j^p\omega_j^2}{(\lambda_j^p)^2 + c^2\omega_j^2} \\ 1 \end{pmatrix} \phi_j \quad \text{and} \quad U_j^{h\pm} = \begin{pmatrix} (\lambda_j^{h\pm})^{-1} \\ 1 \\ \frac{-\beta}{\nu\omega_j^2 + \lambda_j^{h\pm}} \end{pmatrix} \phi_j. \quad (3.47)$$

These vectors, together with the corresponding generalized eigenvectors for the low frequencies, form a Riesz basis for $\widehat{H}$.

3.3.2 Observability Inequalities

This section is primarily devoted to establishing the following observability estimates.

Lemma 3.22 *There exist positive constants C_0 and C_1 , depending on the coefficients c, α, ν, β , the time T , and the observation domain ω , such that:*

$$\beta^2 \|u(T)\|_{L^2(\Omega)}^2 + \beta^2 \|u_t(T)\|_{H^{-1}(\Omega)}^2 + \frac{\alpha^2}{\nu^2} \|(\theta + \beta u)(T)\|_{H^{-1}(\Omega)}^2 \leq C_0 \int_0^T \int_{\omega} |u|^2 dx dt, \quad (3.48)$$

and

$$\beta^2 \|u_t(T)\|_{H^{-2}(\Omega)}^2 + \beta^2 c^4 \|u(T)\|_{H^{-1}(\Omega)}^2 + \alpha^2 \|\theta(T)\|_{H^{-1}(\Omega)}^2 \leq C_1 \int_0^T \int_{\omega} |\theta|^2 dx dt, \quad (3.49)$$

for any solution (u, θ) of system (3.29) with initial data $(u^0, u^1, \theta^0) \in H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega)$.

Proof:

for any initial data $(u^0, u^1, \theta^0) \in H$, there exists a unique solution satisfying:

$$(u, u_t, \theta) \in C([0, T]; H). \quad (3.50)$$

The adjoint system associated with (3.29) is given by:

$$\begin{cases} v_{tt} - (c^2 + \alpha\beta)\Delta v + \nu\beta\Delta\psi = 0 & \text{in } \Omega \times (0, T), \\ -\psi_t - \nu\Delta\psi + \alpha\Delta v = 0 & \text{in } \Omega \times (0, T), \\ v = \psi = 0 & \text{on } \partial\Omega \times (0, T), \\ v(T) = v^0, v_t(T) = v^1, \psi(T) = \psi^0 & \text{in } \Omega. \end{cases} \quad (3.51)$$

Based on the results established in [35], we know that there exist positive constants K_0 and K_1 such that the following observability inequalities hold :

$$\|v(0)\|_{L^2(\Omega)}^2 + \|v_t(0) - \beta\psi(0)\|_{H^{-1}(\Omega)}^2 + \|\psi(0)\|_{H^{-1}(\Omega)}^2 \leq K_0 \int_0^T \int_{\omega} |v|^2 dx dt, \quad (3.52)$$

and

$$\|v(0)\|_{L^2(\Omega)}^2 + \|v_t(0) - \beta\psi(0)\|_{H^{-1}(\Omega)}^2 + \|\psi(0)\|_{H^{-1}(\Omega)}^2 \leq K_1 \int_0^T \int_{\omega} |\psi|^2 dx dt, \quad (3.53)$$

for every solution of (3.51) with initial data $(v^0, v^1, \psi^0) \in H'$.

The strategy to prove the desired inequalities (3.48) and (3.49) relies on a duality argument. By applying a suitable change of variables, we can transform the established adjoint observability estimates (3.52) and (3.53) into the corresponding estimates for the direct system.

Observation of u : We assume the initial data satisfies $(u^0, u^1, \theta^0) \in H$. We introduce the transformed variables:

$$\tilde{\theta} = \alpha \Delta \theta, \quad \tilde{u} = \beta u_t.$$

From system (3.29), substituting the transformed variables $\tilde{u} = \beta u_t$ and $\tilde{\theta} = \alpha \Delta \theta$, it is straightforward to verify that $(\tilde{u}, \tilde{\theta})$ satisfies the following system:

$$\left\{ \begin{array}{ll} \tilde{u}_{tt} - (c^2 + \alpha\beta)\Delta\tilde{u} + \nu\beta\Delta\tilde{\theta} = 0 & \text{in } \Omega \times (0, T), \\ \tilde{\theta}_t - \nu\Delta\tilde{\theta} + \alpha\Delta\tilde{u} = 0 & \text{in } \Omega \times (0, T), \\ \tilde{u} = \tilde{\theta} = 0 & \text{on } \partial\Omega \times (0, T), \\ \tilde{u}(0) = \beta u^1, \quad \tilde{u}_t(0) = \beta(c^2\Delta u^0 - \alpha\Delta\theta^0), \quad \tilde{\theta}(0) = \alpha\Delta\theta^0 & \text{in } \Omega. \end{array} \right. \quad (3.54)$$

Since the transformed system (3.54) has the same structure as the adjoint system (3.51), we can apply the observability estimate (3.52) to the pair $(\tilde{u}, \tilde{\theta})$. Substituting the definitions $\tilde{u} = \beta u_t$ and $\tilde{\theta} = \alpha \Delta \theta$ back into the inequality, we obtain the following estimate in terms of (u, θ) :

$$\beta^2 \|u_t(T)\|_{L^2(\Omega)}^2 + \beta^2 c^4 \|\Delta u(T)\|_{H^{-1}(\Omega)}^2 + \alpha^2 \|\Delta \theta(T)\|_{H^{-1}(\Omega)}^2 \leq \beta^2 K_0 \int_0^T \int_{\omega} |u_t|^2 dx dt, \quad (3.55)$$

which holds for every initial data $(u^0, u^1, \theta^0) \in H$.

Now, for any initial data $(u^0, u^1, \theta^0) \in H$, consider the auxiliary initial data $(\hat{u}^0, \hat{u}^1, \hat{\theta}^0) \in D(A)$ determined by the following system:

$$\left\{ \begin{array}{l} \hat{u}^1 = u^0, \\ c^2 \Delta \hat{u}^0 - \alpha \Delta \hat{\theta}^0 = u^1, \\ \nu \Delta \hat{\theta}^0 - \beta \hat{u}^1 = \theta^0. \end{array} \right. \quad (3.56)$$

Let $(\hat{u}, \hat{\theta})$ denote the unique solution of system (3.29) associated with this initial data $(\hat{u}^0, \hat{u}^1, \hat{\theta}^0)$.

It is straightforward to verify that the functions $u = \hat{u}_t$ and $\theta = \hat{\theta}_t$ constitute the unique solution of system (3.29) with initial data $(u^0, u^1, \theta^0) \in H$. Consequently, we can apply the observability estimate (3.55) to the solution $(\hat{u}, \hat{\theta})$. Substituting $u = \hat{u}_t$ and $\theta = \hat{\theta}_t$, and using the system equations to express the Laplacian terms $\Delta \hat{u}$ and $\Delta \hat{\theta}$ in

terms of u and θ , we obtain:

$$\begin{aligned} \beta^2 \|u(T)\|_{L^2(\Omega)}^2 + \beta^2 \left\| u_t(T) + \frac{\alpha}{\nu}(\theta(T) + \beta u(T)) \right\|_{H^{-1}(\Omega)}^2 \\ + \frac{\alpha^2}{\nu^2} \|(\theta + \beta u)(T)\|_{H^{-1}(\Omega)}^2 \leq \beta^2 K_0 \int_0^T \int_{\omega} |u|^2 dx dt. \end{aligned} \quad (3.57)$$

Using the inequality $\|x + y\|^2 \geq \frac{1}{2}\|x\|^2 - \|y\|^2$, we establish the following lower bound for the mixed term:

$$\beta^2 \left\| u_t(T) + \frac{\alpha}{\nu}(\theta(T) + \beta u(T)) \right\|_{H^{-1}(\Omega)}^2 \geq \frac{\beta^2}{2} \|u_t(T)\|_{H^{-1}(\Omega)}^2 - \frac{\beta^2 \alpha^2}{\nu^2} \|(\theta + \beta u)(T)\|_{H^{-1}(\Omega)}^2. \quad (3.58)$$

Combining this estimate with (3.57), we obtain:

$$\begin{aligned} \beta^2 \|u(T)\|_{L^2(\Omega)}^2 + \frac{\beta^2}{2} \|u_t(T)\|_{H^{-1}(\Omega)}^2 + \frac{\alpha^2}{\nu^2} \|(\theta + \beta u)(T)\|_{H^{-1}(\Omega)}^2 \\ \leq \beta^2 K_0 \int_0^T \int_{\omega} |u|^2 dx dt + \frac{\alpha^2 \beta^2}{\nu^2} \|(\theta + \beta u)(T)\|_{H^{-1}(\Omega)}^2. \end{aligned} \quad (3.59)$$

Finally, we use the estimate (3.57) to bound the second term on the right-hand side of (3.59). This yields the final inequality (3.48).

Observation of θ : We proceed analogously. Applying the observability inequality (3.53) to the transformed pair $(\tilde{u}, \tilde{\theta})$, we obtain:

$$\beta^2 \|u_t(T)\|_{L^2(\Omega)}^2 + \beta^2 c^4 \|\Delta u(T)\|_{H^{-1}(\Omega)}^2 + \alpha^2 \|\Delta \theta(T)\|_{H^{-1}(\Omega)}^2 \leq \alpha^2 K_1 \int_0^T \int_{\omega} |\Delta \theta|^2 dx dt, \quad (3.60)$$

which holds for any initial data $(u^0, u^1, \theta^0) \in H$.

Now, for arbitrary data $(u^0, u^1, \theta^0) \in H$, we introduce auxiliary initial data $(\hat{u}^0, \hat{u}^1, \hat{\theta}^0)$ defined as the solutions to the elliptic problems:

$$\Delta \hat{u}^0 = u^0, \quad \Delta \hat{u}^1 = u^1, \quad \Delta \hat{\theta}^0 = \theta^0.$$

Let $(\hat{u}, \hat{\theta})$ be the solution of (3.29) corresponding to this auxiliary data. Linearity implies that $\Delta \hat{u} = u$ and $\Delta \hat{\theta} = \theta$. Applying estimate (3.60) to $(\hat{u}, \hat{\theta})$, we observe that terms such as $\|\Delta \hat{\theta}(T)\|_{H^{-1}(\Omega)}$ become $\|\theta(T)\|_{H^{-1}(\Omega)}$. This yields the final inequality (3.49). $\blacksquare$

In fact, it is possible to establish improved observability estimates with a stronger energy norm on the left-hand side. Due to the regularizing effect of the heat equation, the thermal component can be estimated in the $L^2(\Omega)$ norm rather than $H^{-1}(\Omega)$. Accordingly, we have the following result for the adjoint system:

Lemma 3.23 *There exist positive constants $\tilde{K}_0$ and $\tilde{K}_1$ such that*

$$\|v(0)\|_{L^2(\Omega)}^2 + \|v_t(0) - \beta\psi(0)\|_{H^{-1}(\Omega)}^2 + \|\psi(0)\|_{L^2(\Omega)}^2 \leq \tilde{K}_0 \int_0^T \int_{\omega} |v|^2 dx dt, \quad (3.61)$$

and

$$\|v(0)\|_{L^2(\Omega)}^2 + \|v_t(0) - \beta\psi(0)\|_{H^{-1}(\Omega)}^2 + \|\psi(0)\|_{L^2(\Omega)}^2 \leq \tilde{K}_1 \int_0^T \int_{\omega} |\psi|^2 dx dt, \quad (3.62)$$

for every solution of (3.51) with initial data $(v^0, v^1, \psi^0) \in H'$.

Proof: The proof follows by establishing that system (3.29) is null controllable for initial thermal data $\theta^0 \in L^2(\Omega)$. This constitutes a regularity improvement over the results in [35], which require $\theta^0 \in H_0^1(\Omega)$.

Given any initial data $U^0 = (u^0, u^1, \theta^0) \in \hat{H}$, we have the spectral expansion:

$$U^0 = \sum_{j=1}^{\infty} (a_j^+ U_j^{h+} + a_j^- U_j^{h-} + b_j U_j^p),$$

where the coefficients satisfy the square-summability condition required by the Riesz basis structure:

$$\sum_{j=1}^{\infty} (|a_j^+|^2 + |a_j^-|^2 + |b_j|^2) < \infty.$$

Consequently, the solution $U(t) = (u, u_t, \theta)^\top$ of system (3.29) admits the following spectral representation:

$$U(t) = \sum_{j=1}^{\infty} \left(a_j^+ U_j^{h+} e^{\lambda_j^{h+} t} + a_j^- U_j^{h-} e^{\lambda_j^{h-} t} + b_j U_j^p e^{\lambda_j^p t} \right).$$

In particular, by extracting the third component of the vector solution, the temperature variable can be written as:

$$\theta(t) = \sum_{j=1}^{\infty} \left(a_j^+ \frac{-\beta}{\nu\omega_j^2 + \lambda_j^{h+}} e^{\lambda_j^{h+} t} + a_j^- \frac{-\beta}{\nu\omega_j^2 + \lambda_j^{h-}} e^{\lambda_j^{h-} t} + b_j e^{\lambda_j^p t} \right) \phi_j(x). \quad (3.63)$$

Using the asymptotic expansions derived in (3.42) and (3.43), and recalling that $|e^{zt}| = e^{\operatorname{Re}(z)t}$, we deduce the following decay estimates.

For the hyperbolic branch, observing that the ω_j^{-1} term is purely imaginary, the real

part decay is governed by:

$$\left| e^{2\lambda_j^{h\pm}t} \right| = e^{2\operatorname{Re}(\lambda_j^{h\pm})t} \leq \exp \left(2 \left[-\frac{\alpha\beta}{2\nu} + \mathcal{O} \left(\frac{1}{\omega_j^2} \right) \right] t \right) \leq e^{-\frac{\alpha\beta}{\nu}t} \left(1 + \mathcal{O} \left(\frac{1}{\omega_j^2} \right) \right). \quad (3.64)$$

Similarly, for the parabolic branch:

$$\left| e^{2\lambda_j^p t} \right| \leq \exp \left(2 \left[-\nu\omega_j^2 + \frac{\alpha\beta}{\nu} + \mathcal{O} \left(\frac{1}{\omega_j^2} \right) \right] t \right) \leq e^{2(-\nu\omega_j^2 + \frac{\alpha\beta}{\nu})t} \left(1 + \mathcal{O} \left(\frac{1}{\omega_j^2} \right) \right). \quad (3.65)$$

Furthermore, we estimate the magnitude of the coefficients appearing in the series expansion:

$$\left| \frac{\beta\omega_j^2}{\nu\omega_j^2 + \lambda_j^{h\pm}} \right|^2 = \frac{\beta^2}{\nu^2 + (c^2 - \alpha\beta)\frac{1}{\omega_j^2} + \mathcal{O} \left(\frac{1}{\omega_j^3} \right)}. \quad (3.66)$$

Consequently, summing over the high frequencies $j > N_0$, we obtain the following estimate:

$$\begin{aligned} & \sum_{j>N_0} \omega_j^4 \left| a_j^+ \frac{-\beta}{\nu\omega_j^2 + \lambda_j^{h+}} e^{\lambda_j^{h+}t} + a_j^- \frac{-\beta}{\nu\omega_j^2 + \lambda_j^{h-}} e^{\lambda_j^{h-}t} + b_j e^{\lambda_j^p t} \right|^2 \\ & \leq 3 \sum_{j>N_0} \left(|a_j^+|^2 \left| \frac{\beta\omega_j^2}{\nu\omega_j^2 + \lambda_j^{h+}} \right|^2 e^{2\operatorname{Re}(\lambda_j^{h+})t} \right. \\ & \quad \left. + |a_j^-|^2 \left| \frac{\beta\omega_j^2}{\nu\omega_j^2 + \lambda_j^{h-}} \right|^2 e^{2\operatorname{Re}(\lambda_j^{h-})t} + |b_j|^2 \omega_j^4 e^{2\lambda_j^p t} \right) \\ & \leq C(\alpha, \beta, c, \nu) \sum_{j>N_0} (|a_j^+|^2 + |a_j^-|^2 + |b_j|^2). \end{aligned} \quad (3.67)$$

This demonstrates that the temperature variable exhibits a regularizing effect, ensuring that $\theta(t) \in H^2(\Omega) \cap H_0^1(\Omega)$ for all $t > 0$. Consequently, the conclusion of Lemma 3.23 is established through the following argument: Consider the system (3.29) with initial data $(u^0, u^1, \theta^0) \in \widehat{H}$. From the estimates derived above, we deduce that the thermal component satisfies $\theta(t) \in H^2(\Omega) \cap H_0^1(\Omega)$ for $t > 0$. This improved regularity implies that the full state vector satisfies $(u, u_t, \theta)(t) \in H$ for any $t > 0$. Since the system enters the standard energy space H instantaneously, we can apply the null controllability results from [35] to drive the system to rest using distributed controls. By duality, this controllability result implies the validity of the observability estimates stated in Lemma 3.23. ■

3.3.3 The Observability Constant

To address the problem of optimizing the shape and location of sensors for system (3.29), we adopt the approach developed in [41, 42], which is based on randomized initial data. We utilize the spectral expansion of the solution in terms of the Riesz basis $\{U_{j,k}\}_{j \in \mathbb{N}^*, k \in \{1,2,3\}}$ derived in Section 3.3.1. Thus, we write the solution as:

$$U(t) = \sum_{j=1}^{\infty} \left(x_j^1 U_{j,1} e^{\lambda_j^1 t} + x_j^2 U_{j,2} e^{\lambda_j^2 t} + x_j^3 U_{j,3} e^{\lambda_j^3 t} \right), \quad (3.68)$$

where, for each $j \in \mathbb{N}^*$, the scalar coefficients x_j^1, x_j^2, x_j^3 satisfy the linear system determined by the initial data:

$$\begin{cases} x_j^1 U_{j,1}^1 + x_j^2 U_{j,2}^1 + x_j^3 U_{j,3}^1 = \int_{\Omega} u^0(x) \phi_j(x) dx, \\ x_j^1 U_{j,1}^2 + x_j^2 U_{j,2}^2 + x_j^3 U_{j,3}^2 = \int_{\Omega} u^1(x) \phi_j(x) dx, \\ x_j^1 U_{j,1}^3 + x_j^2 U_{j,2}^3 + x_j^3 U_{j,3}^3 = \int_{\Omega} \theta^0(x) \phi_j(x) dx. \end{cases}$$

Here, the vector components are denoted by

$$U_{j,k} = \begin{pmatrix} U_{j,k}^1 \\ U_{j,k}^2 \\ U_{j,k}^3 \end{pmatrix}, \quad \text{for } j \in \mathbb{N}^*, k \in \{1, 2, 3\}.$$

The sequence $\{\lambda_j^i\}_{i=1,2,3}$ consists of the generalized eigenvalues of the matrix A_j . In particular, there exists a threshold integer $R_0 > 0$ such that for all high frequencies $j > R_0$, the spectral elements coincide with the parabolic and hyperbolic branches derived earlier:

$$\lambda_j^1 = \lambda_j^p, \quad \lambda_j^2 = \lambda_j^{h+}, \quad \lambda_j^3 = \lambda_j^{h-},$$

and the corresponding eigenvectors satisfy:

$$U_{j,1} = U_j^p, \quad U_{j,2} = U_j^{h+}, \quad U_{j,3} = U_j^{h-}.$$

Observability Constant for Wave Observation

In this section, we compute the observability constant for the case where the observation is performed on the wave equation component u .

We begin by defining the scaled coefficients:

$$\tilde{a}_j = x_j^1 U_{j,1}^1, \quad \tilde{b}_j = x_j^2 U_{j,2}^1, \quad \tilde{c}_j = x_j^3 U_{j,3}^1, \quad \text{for } j \in \mathbb{N}^*.$$

Next, we perform a change of variables to shift the reference time to T :

$$a_j = \tilde{a}_j e^{\lambda_j^1 T}, \quad b_j = \tilde{b}_j e^{\lambda_j^2 T}, \quad c_j = \tilde{c}_j e^{\lambda_j^3 T}.$$

Substituting these into (3.68), we obtain the following expansions for the state variables:

$$\begin{cases} u(t) = \sum_{j=1}^{\infty} \left(a_j e^{-\lambda_j^1(T-t)} + b_j e^{-\lambda_j^2(T-t)} + c_j e^{-\lambda_j^3(T-t)} \right) \phi_j, \\ u_t(t) = \sum_{j=1}^{\infty} \left(a_j \lambda_j^1 e^{-\lambda_j^1(T-t)} + b_j \lambda_j^2 e^{-\lambda_j^2(T-t)} + c_j \lambda_j^3 e^{-\lambda_j^3(T-t)} \right) \phi_j, \\ \theta(t) = \sum_{j=1}^{\infty} \left(a_j \frac{U_{j,1}^3}{U_{j,1}^1} e^{-\lambda_j^1(T-t)} + b_j \frac{U_{j,2}^3}{U_{j,2}^1} e^{-\lambda_j^2(T-t)} + c_j \frac{U_{j,3}^3}{U_{j,3}^1} e^{-\lambda_j^3(T-t)} \right) \phi_j. \end{cases} \quad (3.69)$$

For any non-zero initial data $(u^0, u^1, \theta^0) \in H'$, the *observability constant with wave observation* is defined by the following variational formula:

$$C_T(\chi_\omega) = \inf_{\substack{(u^0, u^1, \theta^0) \in H' \\ (u^0, u^1, \theta^0) \neq 0}} \frac{\int_0^T \int_\Omega \chi_\omega |u|^2 dx dt}{\beta^2 \|u(T)\|_{L^2(\Omega)}^2 + \beta^2 \|u_t(T)\|_{H^{-1}(\Omega)}^2 + \frac{\alpha^2}{\nu^2} \|(\theta + \beta u)(T)\|_{L^2(\Omega)}^2}. \quad (3.70)$$

We define the denominator of the observability ratio as the functional:

$$D(a, b, c) = \beta^2 \|u(T)\|_{L^2(\Omega)}^2 + \beta^2 \|u_t(T)\|_{H^{-1}(\Omega)}^2 + \frac{\alpha^2}{\nu^2} \|(\theta + \beta u)(T)\|_{L^2(\Omega)}^2. \quad (3.71)$$

Substituting the spectral expansions from (3.69) and utilizing the orthonormality of the basis $(\phi_j)_{j \in \mathbb{N}^*}$ (along with the isometric isomorphism $\|\cdot\|_{H^{-1}(\Omega)} \cong \|\omega_j^{-1} \cdot\|_{L^2}$), we obtain the following series representation:

$$\begin{aligned} D(a, b, c) = \sum_{j=1}^{\infty} & \left(\beta^2 |a_j + b_j + c_j|^2 + \frac{\beta^2}{\omega_j^2} |\lambda_j^1 a_j + \lambda_j^2 b_j + \lambda_j^3 c_j|^2 \right. \\ & \left. + \frac{\alpha^2}{\nu^2} \left| a_j \left(\frac{U_{j,1}^3}{U_{j,1}^1} + \beta \right) + b_j \left(\frac{U_{j,2}^3}{U_{j,2}^1} + \beta \right) + c_j \left(\frac{U_{j,3}^3}{U_{j,3}^1} + \beta \right) \right|^2 \right). \end{aligned} \quad (3.72)$$

Similarly, using the spectral expansion (3.69), we express the observation integral (the

numerator of the observability ratio) as:

$$\int_0^T \int_{\omega} |u|^2 dx dt = \int_0^T \int_{\omega} \left| \sum_{j=1}^{\infty} \left(a_j e^{-\lambda_j^1(T-t)} + b_j e^{-\lambda_j^2(T-t)} + c_j e^{-\lambda_j^3(T-t)} \right) \phi_j(x) \right|^2 dx dt. \quad (3.73)$$

Thus, combining the expressions (3.73) and (3.71), we can rewrite the observability constant as the solution to the following constrained minimization problem:

$$C_T(\chi_{\omega}) = \inf_{\substack{\mathbf{a}, \mathbf{b}, \mathbf{c} \in \ell^2(\mathbb{C}) \\ D(\mathbf{a}, \mathbf{b}, \mathbf{c})=1}} \int_0^T \int_{\omega} \left| \sum_{j=1}^{\infty} \left(a_j e^{-\lambda_j^1(T-t)} + b_j e^{-\lambda_j^2(T-t)} + c_j e^{-\lambda_j^3(T-t)} \right) \phi_j(x) \right|^2 dx dt, \quad (3.74)$$

where $\mathbf{a} = (a_j)$, $\mathbf{b} = (b_j)$, and $\mathbf{c} = (c_j)$ denote the coefficient sequences.

We define the randomized solution $u^{\nu}(t)$ corresponding to an event $\nu \in \mathcal{X}$ by the series expansion:

$$u^{\nu}(t) = \sum_{j=1}^{\infty} \left(\beta_{1,j}^{\nu} a_j e^{-\lambda_j^1(T-t)} + \beta_{2,j}^{\nu} b_j e^{-\lambda_j^2(T-t)} + \beta_{3,j}^{\nu} c_j e^{-\lambda_j^3(T-t)} \right) \phi_j,$$

defined on a probability space $(\mathcal{X}, \mathcal{A}, \mathbb{P})$. Here, for each $k \in \{1, 2, 3\}$, the sequence $(\beta_{k,j})_{j \in \mathbb{N}^*}$ consists of independent random variables of Bernoulli type, having mean equal to 0, variance equal to 1, and a super-exponential decay.

Given the stochastic nature of ν , it is natural to analyze the *randomized observability constant*, which is based on the expected value of the measurements.

Since the solution u^{ν} depends on the random event $\nu \in \mathcal{X}$, it is natural to study the *randomized observability constant*, which considers the expected value of the measurements. This is defined by:

$$\begin{aligned} C_{T,\text{Rand}}(\chi_{\omega}) &= \inf \left\{ \mathbb{E} \left(\frac{\int_0^T \int_{\Omega} \chi_{\omega} \left| \sum_{j=1}^{\infty} (\beta_{1,j}^{\nu} a_j e^{-\lambda_j^1(T-t)} + \beta_{2,j}^{\nu} b_j e^{-\lambda_j^2(T-t)} + \beta_{3,j}^{\nu} c_j e^{-\lambda_j^3(T-t)}) \phi_j \right|^2 dx dt}{\beta^2 \|u(T)\|_{L^2(\Omega)}^2 + \beta^2 \|u_t(T)\|_{H^{-1}(\Omega)}^2 + \frac{\alpha^2}{\nu^2} \|(\theta + \beta u)(T)\|_{L^2(\Omega)}^2} \right) \right. \\ &\quad \left. \text{such that } (u^0, u^1, \theta^0) \in H' \right\} \end{aligned}$$

Equivalently, using the functional D defined in (3.71), we can write:

$$\begin{aligned} C_{T,\text{Rand}}(\chi_{\omega}) &= \inf \left\{ \mathbb{E} \left(\int_0^T \int_{\omega} \left| \sum_{j=1}^{\infty} (\beta_{j,i}^{\nu} a_j e^{-\lambda_j^1(T-t)} + \beta_{2,j}^{\nu} b_j e^{-\lambda_j^2(T-t)} + \beta_{3,j}^{\nu} c_j e^{-\lambda_j^3(T-t)}) \phi_j \right|^2 dx dt \right) \right\} \end{aligned}$$

$$\text{such that } \mathbf{a}, \mathbf{b}, \mathbf{c} \in \ell^2(\mathbb{C}), D(\mathbf{a}, \mathbf{b}, \mathbf{c}) = 1 \}. \quad (3.75)$$

Using the statistical properties of the expectation operator $\mathbb{E}$ —specifically, that the random variables have mean zero, unit variance, and are independent—the cross-terms in the squared modulus vanish. Consequently, the randomized observability constant simplifies to the ratio of diagonal quadratic forms:

$$C_{T,\text{Rand}}(\chi_\omega) = \inf_{\mathbf{a}, \mathbf{b}, \mathbf{c} \in \ell^2(\mathbb{C})} \frac{N(\mathbf{a}, \mathbf{b}, \mathbf{c})}{D(\mathbf{a}, \mathbf{b}, \mathbf{c})}, \quad (3.76)$$

where the numerator N is given by:

$$\begin{aligned} N(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \sum_{j=1}^{\infty} & \left(|a_j|^2 \int_0^T e^{-2\operatorname{Re}(\lambda_j^1)(T-t)} dt + |b_j|^2 \int_0^T e^{-2\operatorname{Re}(\lambda_j^2)(T-t)} dt \right. \\ & \left. + |c_j|^2 \int_0^T e^{-2\operatorname{Re}(\lambda_j^3)(T-t)} dt \right) \int_{\omega} |\phi_j(x)|^2 dx, \end{aligned} \quad (3.77)$$

and the denominator $D(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is given by (3.72). We introduce the change of variables to normalize the time-integral weights:

$$d_j = a_j \sqrt{\int_0^T e^{-2\operatorname{Re}(\lambda_j^1)(T-t)} dt}, \quad e_j = b_j \sqrt{\int_0^T e^{-2\operatorname{Re}(\lambda_j^2)(T-t)} dt},$$

and

$$f_j = c_j \sqrt{\int_0^T e^{-2\operatorname{Re}(\lambda_j^3)(T-t)} dt}.$$

Substituting these into the expression for the randomized observability constant, we find that the inverse of the constant is given by:

$$\begin{aligned} \frac{1}{C_{T,\text{Rand}}(\chi_\omega)} = & \sup_{\mathbf{d}, \mathbf{e}, \mathbf{f} \in \ell^2} \frac{\sum_{j=1}^{\infty} \left(A_j^{11} |d_j|^2 + A_j^{22} |e_j|^2 + A_j^{33} |f_j|^2 + 2\Re(A_j^{12} d_j \bar{e}_j) + 2\Re(A_j^{31} d_j \bar{f}_j) + 2\Re(A_j^{32} e_j \bar{f}_j) \right)}{\sum_{j=1}^{\infty} (|d_j|^2 + |e_j|^2 + |f_j|^2) \int_{\omega} |\phi_j|^2 dx}, \end{aligned} \quad (3.78)$$

where the coefficients are defined by:

$$\begin{aligned}
A_j^{11} &= \frac{1}{\int_0^T e^{-2\operatorname{Re}\lambda_j^1 t} dt} \left(\beta^2 + \beta^2 \frac{|\lambda_j^1|^2}{\omega_j^2} + \frac{\alpha^2}{\nu^2} \left| \frac{U_{j,1}^3}{U_{j,1}^1} + \beta \right|^2 \right), \\
A_j^{22} &= \frac{1}{\int_0^T e^{-2\operatorname{Re}\lambda_j^2 t} dt} \left(\beta^2 + \beta^2 \frac{|\lambda_j^2|^2}{\omega_j^2} + \frac{\alpha^2}{\nu^2} \left| \frac{U_{j,2}^3}{U_{j,2}^1} + \beta \right|^2 \right), \quad A_j^{33} = A_j^{22}, \\
A_j^{12} &= \frac{1}{\sqrt{\int_0^T e^{-2\operatorname{Re}\lambda_j^1 t} dt \int_0^T e^{-2\operatorname{Re}\lambda_j^2 t} dt}} \left(\beta^2 + \beta^2 \frac{\lambda_j^1 \bar{\lambda}_j^2}{\omega_j^2} + \frac{\alpha^2}{\nu^2} \left(\frac{U_{j,1}^3}{U_{j,1}^1} + \beta \right) \overline{\left(\frac{U_{j,2}^3}{U_{j,2}^1} + \beta \right)} \right), \quad A_j^{31} = \overline{A_j^{12}}, \\
A_j^{32} &= \frac{1}{\sqrt{\int_0^T e^{-2\operatorname{Re}\lambda_j^2 t} dt \int_0^T e^{-2\operatorname{Re}\lambda_j^3 t} dt}} \left(\beta^2 + \beta^2 \frac{\lambda_j^2 \bar{\lambda}_j^3}{\omega_j^2} + \frac{\alpha^2}{\nu^2} \left(\frac{U_{j,2}^3}{U_{j,2}^1} + \beta \right) \overline{\left(\frac{U_{j,3}^3}{U_{j,3}^1} + \beta \right)} \right).
\end{aligned}$$

Thus, we can express the inverse of the randomized observability constant in a compact form:

$$\frac{1}{C_{T,\text{Rand}}(\chi_\omega)} = \sup_{\mathbf{d}, \mathbf{e}, \mathbf{f} \in \ell^2} \frac{\sum_{j=1}^{\infty} \langle M_j \mathbf{v}_j, \mathbf{v}_j \rangle_{\mathbb{C}^3}}{\sum_{j=1}^{\infty} |\mathbf{v}_j|^2 \int_\omega |\phi_j(x)|^2 dx}, \quad (3.79)$$

where $\mathbf{v}_j = (d_j, e_j, f_j)^\top$ and $M_j = (A_j^{k,l})_{k,l=1}^3$ is a Hermitian matrix for each j .

The next result characterizes the randomized observability constant:

Proposition 3.24 *The inverse of the randomized observability constant is given by the supremum of the spectral modes:*

$$\frac{1}{C_{T,\text{Rand}}(\chi_\omega)} = \sup_{j \in \mathbb{N}^*} \frac{\lambda_{\max}(M_j)}{\int_\omega |\phi_j(x)|^2 dx}. \quad (3.80)$$

Consequently, the observability constant itself is characterized by:

$$C_{T,\text{Rand}}(\chi_\omega) = \inf_{j \in \mathbb{N}^*} \frac{\int_\omega |\phi_j(x)|^2 dx}{\lambda_{\max}(M_j)}, \quad (3.81)$$

where $\lambda_{\max}(M_j)$ denotes the largest eigenvalue of the Hermitian matrix M_j .

Proof: Let $Q(\mathbf{v})$ denote the quotient in (3.79) and let $l_j = \int_\omega |\phi_j(x)|^2 dx$. We define the supremum of the spectral modes as $S = \sup_{j \in \mathbb{N}^*} \frac{\lambda_{\max}(M_j)}{l_j}$.

Upper Bound: For any sequence $\mathbf{v} \in \ell^2$, we have the bound $\langle M_j \mathbf{v}_j, \mathbf{v}_j \rangle \leq \lambda_{\max}(M_j) |\mathbf{v}_j|^2$. By definition of S , we have $\lambda_{\max}(M_j) \leq S l_j$ for all j . Summing over j , we obtain:

$$\sum_{j=1}^{\infty} \langle M_j \mathbf{v}_j, \mathbf{v}_j \rangle \leq \sum_{j=1}^{\infty} S l_j |\mathbf{v}_j|^2 = S \sum_{j=1}^{\infty} |\mathbf{v}_j|^2 l_j.$$

This implies $Q(\mathbf{v}) \leq S$ for all $\mathbf{v} \neq 0$, and consequently $\sup_{\mathbf{v}} Q(\mathbf{v}) \leq S$.

Lower Bound: By definition of the supremum, for any given $\epsilon > 0$, there exists an index $k \in \mathbb{N}^*$ such that

$$S - \epsilon < \frac{\lambda_{\max}(M_k)}{l_k}.$$

Let $\mathbf{e}_{\max}^{(k)} \in \mathbb{C}^3$ be a normalized eigenvector of M_k corresponding to $\lambda_{\max}(M_k)$. We construct the test sequence $\boldsymbol{\xi}^{(k)} = (\mathbf{v}_j)_{j \in \mathbb{N}^*}$ defined by:

$$\mathbf{v}_j = \begin{cases} \mathbf{e}_{\max}^{(k)} & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

Substituting this sequence into the quotient yields:

$$Q(\boldsymbol{\xi}^{(k)}) = \frac{\langle M_k \mathbf{e}_{\max}^{(k)}, \mathbf{e}_{\max}^{(k)} \rangle}{l_k |\mathbf{e}_{\max}^{(k)}|^2} = \frac{\lambda_{\max}(M_k)}{l_k} > S - \epsilon.$$

Since ϵ is arbitrary, we conclude that $\sup_{\mathbf{v}} Q(\mathbf{v}) \geq S$.

Combining the upper and lower bounds, we have $\sup_{\mathbf{v}} Q(\mathbf{v}) = S$. The result follows by taking the inverse. ■

Next, we analyze the asymptotic behavior of the spectral quantity $\lambda_{\max}(M_j)$ in terms of the frequency ω_j . The following technical lemma provides the necessary expansions.

Lemma 3.25 *Consider functions defined for $z \in \mathbb{C}$ near the origin by the following local behaviors:*

$$\begin{aligned} H(z) &= \frac{A + Bz + Cz^2 + \mathcal{O}(z^3)}{1 + Dz^2 + \mathcal{O}(z^3)}, \\ I(z) &= \sqrt{A + Bz^2 + \mathcal{O}(z^3)}, \\ J(z) &= \frac{1}{A + Bz + Cz^2 + Dz^3 + \mathcal{O}(z^4)}, \\ K(z) &= \frac{1}{A + Bz + \mathcal{O}(z^2)}. \end{aligned}$$

Then, for z sufficiently small, we have the following asymptotic expansions:

$$\begin{aligned} H(z) &= A + Bz + (C - AD)z^2 + \mathcal{O}(z^3), \\ I(z) &= \begin{cases} \sqrt{A} + \frac{B}{2\sqrt{A}}z^2 + \mathcal{O}(z^3) & \text{if } A \neq 0, \\ \sqrt{B}z + \mathcal{O}(z^2) & \text{if } A = 0, \end{cases} \\ J(z) &= \frac{1}{A} - \frac{B}{A^2}z + \frac{B^2 - AC}{A^3}z^2 + \frac{2ABC - A^2D - B^3}{A^4}z^3 + \mathcal{O}(z^4), \\ K(z) &= \frac{1}{A} - \frac{B}{A^2}z + \mathcal{O}(z^2). \end{aligned}$$

To simplify the notation for the matrix coefficients, we define the inverse time-integral

factors:

$$\mathcal{E}_j^1 = \left(\int_0^T e^{-2\operatorname{Re}(\lambda_j^1)t} dt \right)^{-1} \quad \text{and} \quad \mathcal{E}_j^2 = \left(\int_0^T e^{-2\operatorname{Re}(\lambda_j^2)t} dt \right)^{-1}.$$

Using these definitions and substituting the expansions above, the coefficients of the matrix M_j for $j > R_0$ are given by:

$$\begin{aligned} A_j^{1,1} &= \mathcal{E}_j^1 \left(\beta^2 + \beta^2 \frac{|\lambda_j^p|^2}{|\omega_j|^2} + \frac{\alpha^2}{\nu^2} \left| \frac{(c^2 + \alpha\beta)\omega_j^2 + |\lambda_j^p|^2}{\alpha\omega_j^2} \right|^2 \right), \\ A_j^{2,2} &= \mathcal{E}_j^2 \left(\beta^2 + \beta^2 \frac{|\lambda_j^{h+}|^2}{|\omega_j|^2} + \frac{\alpha^2}{\nu^2} \frac{(\nu\beta\omega_j^2)^2}{|\nu\omega_j^2 + \lambda_j^{h+}|^2} \right), \\ A_j^{1,2} &= \sqrt{\mathcal{E}_j^1 \mathcal{E}_j^2} \left(\beta^2 + \beta^2 \frac{\lambda_j^p \overline{\lambda_j^{h+}}}{\omega_j^2} + \frac{\alpha^2}{\nu^2} \frac{\nu\beta}{\alpha} \frac{(c^2 + \alpha\beta)\omega_j^2 + |\lambda_j^p|^2}{\nu\omega_j^2 + \lambda_j^{h-}} \right), \\ A_j^{3,2} &= \mathcal{E}_j^2 \left(\beta^2 + \beta^2 \frac{(\lambda_j^{h+})^2}{\omega_j^2} + \frac{\alpha^2}{\nu^2} \frac{(\nu\beta\omega_j^2)^2}{(\nu\omega_j^2 + \lambda_j^{h+})^2} \right). \end{aligned} \tag{3.82}$$

We now compute the asymptotic behavior of each matrix coefficient in (3.82). For high frequencies $j \geq R_0$, the following expansions are required:

$$\left| \frac{\lambda_j^p}{\omega_j} \right|^2 = \nu^2 \omega_j^2 - 2\alpha\beta + \frac{\alpha\beta}{\nu^2} (2c^2 - \alpha\beta) \frac{1}{\omega_j^2} + \mathcal{O}\left(\frac{1}{\omega_j^3}\right), \tag{3.83}$$

$$\frac{\lambda_j^{h+}}{\omega_j^2} = \frac{ic}{\omega_j} - \frac{\alpha\beta}{2\nu} \frac{1}{\omega_j^2} + \frac{i\alpha\beta(4c^2 - \alpha\beta)}{8c\nu^2} \frac{1}{\omega_j^3} + \mathcal{O}\left(\frac{1}{\omega_j^4}\right), \tag{3.84}$$

$$\frac{|\lambda_j^{h+}|^2}{\omega_j^2} = c^2 + \frac{c^2\alpha\beta}{\nu^2} \frac{1}{\omega_j^2} + \mathcal{O}\left(\frac{1}{\omega_j^4}\right), \tag{3.85}$$

$$\frac{U_{j,1}^3}{U_{j,1}^1} + \beta = \frac{\nu^2}{\alpha} \omega_j^2 + \left(\frac{c^2}{\alpha} - \beta \right) + \frac{\beta(2c^2 - \alpha\beta)}{\nu^2} \frac{1}{\omega_j^2} + \mathcal{O}\left(\frac{1}{\omega_j^3}\right), \tag{3.86}$$

$$\frac{U_{j,2}^3}{U_{j,2}^1} + \beta = \beta - i \frac{c\beta}{\nu\omega_j} + \frac{\beta}{2\nu^2} (\alpha\beta - 2c^2) \frac{1}{\omega_j^2} + \mathcal{O}\left(\frac{1}{\omega_j^3}\right), \tag{3.87}$$

$$\frac{U_{j,3}^3}{U_{j,3}^1} + \beta = \beta + i \frac{c\beta}{\nu\omega_j} + \frac{\beta}{2\nu^2} (\alpha\beta - 2c^2) \frac{1}{\omega_j^2} + \mathcal{O}\left(\frac{1}{\omega_j^3}\right). \tag{3.88}$$

Using the expansion established in (3.86), we compute the square of the term appearing

in $A_j^{1,1}$:

$$\frac{\alpha^2}{\nu^2} \left(\frac{(c^2 + \alpha\beta)\omega_j^2 + |\lambda_j^p|^2}{\alpha\omega_j^2} \right)^2 = \nu^2\omega_j^4 + 2(c^2 - \alpha\beta)\omega_j^2 + \frac{c^4 + 2c^2\alpha\beta - \alpha^2\beta^2}{\nu^2} + \mathcal{O}\left(\frac{1}{\omega_j}\right). \quad (3.89)$$

Combining the first expansion in (3.82) with the results from (3.83) and (3.89), and using the notation $\mathcal{E}_j^1$, we obtain the asymptotic behavior for $A_j^{1,1}$:

$$A_j^{1,1} = \mathcal{E}_j^1 \left(\nu^2\omega_j^4 + (2c^2 - 2\alpha\beta + \beta^2\nu^2)\omega_j^2 + \left(\beta^2 - 2\alpha\beta^3 + \frac{c^4 + 2c^2\alpha\beta - \alpha^2\beta^2}{\nu^2} \right) + \mathcal{O}\left(\frac{1}{\omega_j}\right) \right). \quad (3.90)$$

Similarly, utilizing the expansion (3.87), we derive the asymptotic behavior of the term appearing in $A_j^{2,2}$:

$$\frac{\alpha^2}{\nu^2} \left| \frac{\nu\beta\omega_j^2}{\nu\omega_j^2 + \lambda_j^{h+}} \right|^2 = \frac{\alpha^2\beta^2}{\nu^2} \left(1 + \frac{\alpha\beta - c^2}{\nu^2} \frac{1}{\omega_j^2} + \mathcal{O}\left(\frac{1}{\omega_j^3}\right) \right). \quad (3.91)$$

Substituting this result and the expansion from (3.85) into the expression for $A_j^{2,2}$, and recalling the definition of $\mathcal{E}_j^2$, we obtain:

$$A_j^{2,2} = \mathcal{E}_j^2 \left(\frac{\beta^2}{\nu^2} (\nu^2(1 + c^2) + \alpha^2) + \frac{\beta^2\alpha}{\nu^4} (-\alpha c^2 + \beta c^2\nu^2 + \beta\alpha^2) \frac{1}{\omega_j^2} + \mathcal{O}\left(\frac{1}{\omega_j^3}\right) \right). \quad (3.92)$$

Proceeding in a similar manner, we derive the asymptotic expansion for $A_j^{3,2}$. Using the notation $\mathcal{E}_j^2$, we obtain:

$$A_j^{3,2} = \mathcal{E}_j^2 \left(\frac{\beta^2}{\nu^2} (\nu^2(1 - c^2) + \alpha^2) - \frac{i\alpha\beta^2 c}{\nu^3} (2\alpha + \beta\nu^2) \frac{1}{\omega_j} + \frac{\alpha\beta^2}{\nu^4} \left(\alpha^2\beta + \frac{\alpha\beta^2\nu^2}{2} - 3\alpha c^2 - \beta c^2\nu^2 \right) \frac{1}{\omega_j^2} + \mathcal{O}\left(\frac{1}{\omega_j^3}\right) \right). \quad (3.93)$$

We investigate the asymptotic behavior of the integral term $\int_0^T e^{-2\operatorname{Re}(\lambda_j^{h+})t} dt$ for large

j . Let $z = \omega_j^{-1}$. Considering the expansion for sufficiently small z , we have:

$$\begin{aligned} \int_0^T e^{-2 \operatorname{Re}(\lambda_j^{h+})t} dt &= \int_0^T e^{\left(\frac{\alpha\beta}{\nu} - \frac{\alpha\beta}{\nu^3}(c^2 - \alpha\beta)z^2 + \mathcal{O}(z^3)\right)t} dt \\ &= \left[\frac{e^{\left(\frac{\alpha\beta}{\nu} - \frac{\alpha\beta}{\nu^3}(c^2 - \alpha\beta)z^2 + \mathcal{O}(z^3)\right)t}}{\frac{\alpha\beta}{\nu} - \frac{\alpha\beta}{\nu^3}(c^2 - \alpha\beta)z^2 + \mathcal{O}(z^3)} \right]_0^T \\ &= \frac{e^{\left(\frac{\alpha\beta}{\nu} - \frac{\alpha\beta}{\nu^3}(c^2 - \alpha\beta)z^2 + \mathcal{O}(z^3)\right)T} - 1}{\frac{\alpha\beta}{\nu} - \frac{\alpha\beta}{\nu^3}(c^2 - \alpha\beta)z^2 + \mathcal{O}(z^3)}. \end{aligned} \quad (3.94)$$

Thus, evaluating the expression at the limit $z = 0$, we obtain:

$$\int_0^T e^{-2 \operatorname{Re}(\lambda_j^{h+})t} dt \Big|_{z=0} = \int_0^T e^{\frac{\alpha\beta}{\nu}t} dt.$$

We compute the first derivative with respect to z . Applying the quotient rule to the expression derived in (3.94), we obtain:

$$\begin{aligned} \frac{d}{dz} \left(\int_0^T e^{-2 \operatorname{Re}(\lambda_j^{h+})t} dt \right) &= \frac{1}{\left(\frac{\alpha\beta}{\nu} + \mathcal{O}(z^2)\right)^2} \left[\left(-2z \frac{\alpha\beta}{\nu^3}(c^2 - \alpha\beta) + \mathcal{O}(z^2) \right) T e^{\left(\frac{\alpha\beta}{\nu} + \mathcal{O}(z^2)\right)T} \left(\frac{\alpha\beta}{\nu} + \mathcal{O}(z^2) \right) \right. \\ &\quad \left. - \left(e^{\left(\frac{\alpha\beta}{\nu} + \mathcal{O}(z^2)\right)T} - 1 \right) \left(-2z \frac{\alpha\beta}{\nu^3}(c^2 - \alpha\beta) + \mathcal{O}(z^2) \right) \right] \\ &= \frac{1}{\frac{\alpha^2\beta^2}{\nu^2} + \mathcal{O}(z^2)} \left[-2zT \frac{\alpha^2\beta^2}{\nu^4}(c^2 - \alpha\beta) e^{\frac{\alpha\beta}{\nu}T} \right. \\ &\quad \left. + 2z \frac{\alpha\beta}{\nu^3}(c^2 - \alpha\beta) \left(e^{\frac{\alpha\beta}{\nu}T} - 1 \right) + \mathcal{O}(z^2) \right] \\ &= \frac{-2z \frac{\alpha\beta}{\nu^3}(c^2 - \alpha\beta) \left[e^{\frac{\alpha\beta}{\nu}T} \left(\frac{\alpha\beta}{\nu}T - 1 \right) + 1 \right] + \mathcal{O}(z^2)}{\frac{\alpha^2\beta^2}{\nu^2} + \mathcal{O}(z^2)}. \end{aligned}$$

Consequently, evaluating the derivative at $z = 0$, we find:

$$\frac{d}{dz} \left(\int_0^T e^{-2 \operatorname{Re}(\lambda_j^{h+})t} dt \right) \Big|_{z=0} = 0.$$

We compute the second derivative with respect to z . Differentiating the result from the previous step, we focus on the leading order terms. Applying the quotient rule and noting that the term proportional to z in the numerator simplifies the calculation at

$z \rightarrow 0$, we obtain:

$$\begin{aligned}
& \frac{d^2}{dz^2} \left(\int_0^T e^{-2\operatorname{Re}(\lambda_j^{h+})t} dt \right) \\
&= \frac{1}{\left(\frac{\alpha^2 \beta^2}{\nu^2} + \mathcal{O}(z^2) \right)^2} \left[\left(-2(c^2 - \alpha\beta) \frac{\alpha\beta}{\nu^3} \left(e^{\frac{\alpha\beta}{\nu}T} \left(\frac{\alpha\beta}{\nu}T - 1 \right) + 1 \right) + \mathcal{O}(z) \right) \right. \\
&\quad \left. \times \left(\frac{\alpha^2 \beta^2}{\nu^2} + \mathcal{O}(z^2) \right) - \mathcal{O}(z^2) \right] \\
&= \frac{-2(c^2 - \alpha\beta) \frac{\alpha^3 \beta^3}{\nu^5} \left[e^{\left(\frac{\alpha\beta}{\nu} - \frac{\alpha\beta}{\nu^3}(c^2 - \alpha\beta)z^2 + \mathcal{O}(z^3) \right)T} \left(\frac{\alpha\beta}{\nu}T - 1 \right) + 1 \right] + \mathcal{O}(z)}{\frac{\alpha^4 \beta^4}{\nu^4} - \frac{\alpha^4 \beta^4}{\nu^6}(c^2 - \alpha\beta)z^2 + \mathcal{O}(z^3)}.
\end{aligned}$$

Consequently, evaluating the second derivative at $z = 0$, the expression simplifies to:

$$\left. \frac{d^2}{dz^2} \left(\int_0^T e^{-2\operatorname{Re}(\lambda_j^{h+})t} dt \right) \right|_{z=0} = \frac{-2(c^2 - \alpha\beta)}{\alpha\beta\nu} \left[e^{\frac{\alpha\beta}{\nu}T} \left(\frac{\alpha\beta}{\nu}T - 1 \right) + 1 \right].$$

Consequently, applying the Taylor expansion up to second order (noting that the first derivative vanishes) and substituting $z = 1/\omega_j$, we obtain:

$$\int_0^T e^{-2\operatorname{Re}(\lambda_j^{h+})t} dt = \int_0^T e^{\frac{\alpha\beta t}{\nu}} dt - \frac{(c^2 - \alpha\beta) \left[e^{\frac{\alpha\beta}{\nu}T} \left(\frac{\alpha\beta}{\nu}T - 1 \right) + 1 \right]}{\alpha\beta\nu} \frac{1}{\omega_j^2} + \mathcal{O}\left(\frac{1}{\omega_j^3}\right). \quad (3.95)$$

Combining the asymptotic expansion for the numerator from (3.92) and the denominator from (3.95), we express $A_j^{2,2}$ as:

$$A_j^{2,2} = \frac{\beta^2 (\nu^2(1 + c^2) + \alpha^2) + \frac{\alpha(-\alpha c^2 + \beta c^2 \nu^2 + \beta \alpha^2)}{\nu^2} \frac{1}{\omega_j^2} + \mathcal{O}\left(\frac{1}{\omega_j^3}\right)}{\int_0^T e^{\frac{\alpha\beta t}{\nu}} dt - \frac{(c^2 - \alpha\beta) \left[e^{\frac{\alpha\beta}{\nu}T} \left(\frac{\alpha\beta}{\nu}T - 1 \right) + 1 \right]}{\alpha\beta\nu} \frac{1}{\omega_j^2} + \mathcal{O}\left(\frac{1}{\omega_j^3}\right)}. \quad (3.96)$$

Similarly, by combining the asymptotic expansion from (3.93) with the denominator expansion (3.95), we obtain the expression for $A_j^{3,2}$:

$$A_j^{3,2} = \frac{\beta^2 (\nu^2(1 - c^2) + \alpha^2) - \frac{i c \alpha (2\alpha + \beta \nu^2)}{\nu} \frac{1}{\omega_j} + \frac{\alpha}{\nu^2} \left(\alpha^2 \beta + \frac{\alpha \beta^2 \nu^2}{2} - 3\alpha c^2 - \beta c^2 \nu^2 \right) \frac{1}{\omega_j^2} + \mathcal{O}\left(\frac{1}{\omega_j^3}\right)}{\int_0^T e^{\frac{\alpha\beta t}{\nu}} dt - \frac{(c^2 - \alpha\beta) \left[e^{\frac{\alpha\beta}{\nu}T} \left(\frac{\alpha\beta}{\nu}T - 1 \right) + 1 \right]}{\alpha\beta\nu} \frac{1}{\omega_j^2} + \mathcal{O}\left(\frac{1}{\omega_j^3}\right)}. \quad (3.97)$$

At this stage, we reintroduce the notation $z = 1/\omega_j$ to perform precise asymptotic

simplifications. Applying the expansion rules from Lemma 3.25 to (3.96), we obtain:

$$A_j^{2,2} = \frac{\beta^2}{\nu^2 \int_0^T e^{\frac{\alpha\beta t}{\nu}} dt} \left(\nu^2(1+c^2) + \alpha^2 + \mathcal{C}_{2,2}z^2 + \mathcal{O}(z^3) \right), \quad (3.98)$$

where the second-order coefficient is given by:

$$\begin{aligned} \mathcal{C}_{2,2} &= \frac{\alpha(\alpha^2\beta + c^2(-\alpha + \beta\nu^2))}{\nu^2} \\ &\quad + \left(\int_0^T e^{\frac{\alpha\beta t}{\nu}} dt \right)^{-1} \frac{(\nu^2(1+c^2) + \alpha^2)(c^2 - \alpha\beta) \left[e^{\frac{\alpha\beta}{\nu}T} \left(\frac{\alpha\beta}{\nu}T - 1 \right) + 1 \right]}{\alpha\beta\nu} \\ &= \frac{\alpha(\alpha^2\beta + c^2(-\alpha + \beta\nu^2))}{\nu^2} + \frac{(\nu^2(1+c^2) + \alpha^2)(c^2 - \alpha\beta) \left[e^{\frac{\alpha\beta}{\nu}T} \left(\frac{\alpha\beta}{\nu}T - 1 \right) + 1 \right]}{\nu^2(e^{\frac{\alpha\beta T}{\nu}} - 1)}. \end{aligned}$$

We also have that:

$$A_j^{3,2} = \frac{\beta^2}{\nu^2 \int_0^T e^{\frac{\alpha\beta t}{\nu}} dt} \left(\nu^2(1-c^2) + \alpha^2 - \frac{ic\alpha(2\alpha + \beta\nu^2)}{\nu}z + \mathcal{C}_{3,2}z^2 + \mathcal{O}(z^3) \right), \quad (3.99)$$

where the second-order coefficient is given by:

$$\begin{aligned} \mathcal{C}_{3,2} &= \frac{\alpha(\alpha\beta(2\alpha + \beta\nu^2) - 2c^2(3\alpha + \beta\nu^2))}{2\nu^2} \\ &\quad + \frac{(\nu^2(1-c^2) + \alpha^2)(c^2 - \alpha\beta) \left[e^{\frac{\alpha\beta}{\nu}T} \left(\frac{\alpha\beta}{\nu}T - 1 \right) + 1 \right]}{\nu^2(e^{\frac{\alpha\beta T}{\nu}} - 1)}. \end{aligned}$$

In particular, squaring the moduli of these expansions, we obtain:

$$\begin{aligned} |A_j^{2,2}|^2 &= \frac{\beta^4}{\nu^4 \left(\int_0^T e^{\frac{\alpha\beta t}{\nu}} dt \right)^2} \left((\nu^2(1+c^2) + \alpha^2)^2 \right. \\ &\quad \left. + 2(\nu^2(1+c^2) + \alpha^2) \mathcal{C}_{2,2}z^2 + \mathcal{O}(z^3) \right), \end{aligned} \quad (3.100)$$

$$\begin{aligned} |A_j^{3,2}|^2 &= \frac{\beta^4}{\nu^4 \left(\int_0^T e^{\frac{\alpha\beta t}{\nu}} dt \right)^2} \left((\nu^2(1-c^2) + \alpha^2)^2 \right. \\ &\quad \left. + \left[\frac{c^2\alpha^2(2\alpha + \beta\nu^2)^2}{\nu^2} + 2(\nu^2(1-c^2) + \alpha^2) \mathcal{C}_{3,2} \right] z^2 + \mathcal{O}(z^3) \right). \end{aligned} \quad (3.101)$$

Given the specific structure of the matrix M_j , its eigenvalues λ satisfy the characteristic

equation:

$$(A_j^{1,1} - \lambda) ((A_j^{2,2} - \lambda)^2 - |A_j^{3,2}|^2) + 2|A_j^{1,2}|^2 (\operatorname{Re}(A_j^{3,2}) - (A_j^{2,2} - \lambda)) = 0. \quad (3.102)$$

Expanding this expression, we obtain the cubic polynomial:

$$\begin{aligned} & -\lambda^3 + (A_j^{1,1} + 2A_j^{2,2})\lambda^2 \\ & + (-2A_j^{1,1}A_j^{2,2} - (A_j^{2,2})^2 + |A_j^{3,2}|^2 + 2|A_j^{1,2}|^2)\lambda \\ & + A_j^{1,1}((A_j^{2,2})^2 - |A_j^{3,2}|^2) + 2|A_j^{1,2}|^2(\operatorname{Re}(A_j^{3,2}) - A_j^{2,2}) = 0. \end{aligned} \quad (3.103)$$

We regard the coefficients $A_j^{k,l}$ as functions of the continuous variable $z = 1/\omega_j$. Let $P(\lambda, z)$ denote the characteristic polynomial defined by the left-hand side of (3.103).

We observe that any coefficient involving the parabolic index 1 (namely $A^{1,1}$ and $A^{1,2}$) decays exponentially to zero as $z \rightarrow 0$ (i.e., $\omega_j \rightarrow \infty$). This is due to the exponential growth of the parabolic eigenvalue real part in the time integral denominators. Consequently, evaluating the polynomial at the limit $z = 0$, the constant term (which depends entirely on these vanishing coefficients) becomes zero. Thus, we have:

$$P(0, 0) = 0.$$

Computing the partial derivative of P with respect to λ , we obtain:

$$\frac{\partial P}{\partial \lambda}(\lambda, z) = -3\lambda^2 + 2(A_j^{1,1} + 2A_j^{2,2})\lambda + (-2A_j^{1,1}A_j^{2,2} - |A_j^{2,2}|^2 + |A_j^{3,2}|^2 + 2|A_j^{1,2}|^2). \quad (3.104)$$

Evaluating this at the limit $(\lambda, z) = (0, 0)$, and recalling that the coefficients associated with the parabolic mode ($A_j^{1,1}$ and $A_j^{1,2}$) vanish at $z = 0$, we have:

$$\frac{\partial P}{\partial \lambda}(0, 0) = -|A_j^{2,2}|^2 \Big|_{z=0} + |A_j^{3,2}|^2 \Big|_{z=0}.$$

Substituting the zero-order terms from the expansions (3.100) and (3.101), a straightforward algebraic calculation yields:

$$\frac{\partial P}{\partial \lambda}(0, 0) = -\frac{4c^2\beta^4}{\nu^2 \left(\int_0^T e^{\frac{\alpha\beta t}{\nu}} dt \right)^2} (\nu^2 + \alpha^2).$$

By the Implicit Function Theorem, since we established that $\frac{\partial P}{\partial \lambda}(0, 0) \neq 0$, there exists a unique solution $\lambda_1 = \lambda_1(z)$ to the characteristic equation (3.103), defined and continuously differentiable in a neighborhood of $z = 0$, satisfying $\lambda_1(0) = 0$.

Furthermore, observe that $P(0, z)$ depends exclusively on the coefficients $A^{1,1}$ and $A^{1,2}$.

As shown previously, these coefficients exhibit exponential decay as $z \rightarrow 0$. Consequently, the function $z \mapsto P(0, z)$ is flat at the origin, implying that all its derivatives with respect to z vanish at $z = 0$. This property transfers to the root $\lambda_1(z)$, meaning it decays faster than any polynomial order:

$$\lambda_1(z) = \mathcal{O}(z^k) \quad \text{for any } k \in \mathbb{N}.$$

We factorize the polynomial $P(\lambda, z)$ by isolating the root $\lambda_1(z)$:

$$P(\lambda, z) = (\lambda - \lambda_1(z))(-\lambda^2 + a(z)\lambda + b(z)).$$

By comparing coefficients with the original polynomial (3.103), and recalling that $\lambda_1(z)$, $A_j^{1,1}$, and $A_j^{1,2}$ decay faster than any polynomial order, we identify the asymptotic behavior of the coefficients $a(z)$ and $b(z)$:

$$a(z) = 2A_j^{2,2}(z) + \mathcal{O}(z^k) \quad \text{and} \quad b(z) = -|A_j^{2,2}(z)|^2 + |A_j^{3,2}(z)|^2 + \mathcal{O}(z^k),$$

for any integer $k > 0$.

Solving the quadratic equation $-\lambda^2 + a(z)\lambda + b(z) = 0$, we obtain the two remaining roots:

$$\lambda_k(z) = \frac{1}{2} \left(a(z) + (-1)^k \sqrt{a(z)^2 + 4b(z)} \right), \quad \text{for } k = 2, 3.$$

Substituting the asymptotic expressions for $a(z)$ and $b(z)$ derived previously, the discriminant simplifies as follows:

$$a(z)^2 + 4b(z) \approx (2A_j^{2,2})^2 + 4(-|A_j^{2,2}|^2 + |A_j^{3,2}|^2) = 4|A_j^{3,2}|^2 + \mathcal{O}(z^k).$$

Among the three eigenvalues ($\lambda_1 \approx 0$, λ_2 , and λ_3), the largest one for sufficiently small z is clearly $\lambda_2(z)$ (corresponding to the positive sign), as $A_j^{2,2}$ is positive. Thus, evaluating at $z = 1/\omega_j$, we obtain the expansion:

$$\lambda_2(1/\omega_j) = A_j^{2,2} + \sqrt{|A_j^{3,2}|^2 + \mathcal{O}\left(\frac{1}{\omega_j^k}\right)} + \mathcal{O}\left(\frac{1}{\omega_j^k}\right). \quad (3.105)$$

Applying Lemma 3.25 to the expansion (3.101), we distinguish two cases based on the behavior of the leading term. To simplify the notation, we introduce the auxiliary constants:

$$\Delta^+ = \nu^2(1 + c^2) + \alpha^2 \quad \text{and} \quad \Delta^- = \nu^2(1 - c^2) + \alpha^2. \quad (3.106)$$

Case 1: $\Delta^- \neq 0$

In this general case, the leading term of the discriminant is non-zero. Using the notation Δ^- defined in (3.106), and applying Lemma 3.25 to the expansion (3.101), we obtain:

$$\sqrt{|A_j^{3,2}|^2 + \mathcal{O}\left(\frac{1}{\omega_j^k}\right)} = \frac{\beta^2}{\nu^2 \int_0^T e^{\frac{\alpha\beta t}{\nu}} dt} \left(|\Delta^-| + \frac{\frac{c^2\alpha^2(2\alpha+\beta\nu^2)^2}{\nu^2} + 2\Delta^- \mathcal{C}_{3,2}}{2|\Delta^-|} \frac{1}{\omega_j^2} + \mathcal{O}\left(\frac{1}{\omega_j^3}\right) \right). \quad (3.107)$$

Substituting this result and the expansion for $A_j^{2,2}$ into the expression for λ_2 , and utilizing the notation $\Delta^\pm$, we find:

$$\begin{aligned} \lambda_2\left(\frac{1}{\omega_j}\right) &= \frac{\beta^2}{\nu^2 \int_0^T e^{\frac{\alpha\beta t}{\nu}} dt} \left(\Delta^+ + |\Delta^-| \right. \\ &\quad \left. + \left[\mathcal{C}_{2,2} + \frac{\frac{c^2\alpha^2(2\alpha+\beta\nu^2)^2}{\nu^2} + 2\Delta^- \mathcal{C}_{3,2}}{2|\Delta^-|} \right] \frac{1}{\omega_j^2} + \mathcal{O}\left(\frac{1}{\omega_j^3}\right) \right). \end{aligned} \quad (3.108)$$

Finally, applying the inversion expansion from Lemma 3.25 yields the asymptotic behavior of the inverse eigenvalue:

$$\frac{1}{\lambda_2(\omega_j^{-1})} = \frac{\nu^2 \int_0^T e^{\frac{\alpha\beta t}{\nu}} dt}{\beta^2(\Delta^+ + |\Delta^-|)} \left(1 - R(\alpha, \beta, c, \nu, T) \frac{1}{\omega_j^2} + \mathcal{O}\left(\frac{1}{\omega_j^3}\right) \right), \quad (3.109)$$

where the coefficient $R := R(\alpha, \beta, c, \nu, T)$ groups the second-order terms and is defined by:

$$R = \frac{1}{\Delta^+ + |\Delta^-|} \left(\mathcal{C}_{2,2} + \text{sgn}(\Delta^-) \mathcal{C}_{3,2} + \frac{c^2\alpha^2(2\alpha + \beta\nu^2)^2}{2\nu^2|\Delta^-|} \right). \quad (3.110)$$

Case 2: $\Delta^- = 0$

This case proceeds analogously, with the key difference being that the leading term of the discriminant vanishes. Consequently, we apply the expansion for $I(z)$ from Lemma 3.25 in the case $A = 0$. This yields a first-order expansion for the square root term:

$$\sqrt{|A_j^{3,2}|^2 + \mathcal{O}\left(\frac{1}{\omega_j^k}\right)} = \frac{\beta^2}{\nu^2 \int_0^T e^{\frac{\alpha\beta t}{\nu}} dt} \left(\frac{c\alpha(2\alpha + \beta\nu^2)}{\nu} \frac{1}{\omega_j} + \mathcal{O}\left(\frac{1}{\omega_j^2}\right) \right). \quad (3.111)$$

Substituting this first-order expansion for the square root term into the expression for λ_2 , and recalling that in this regime the leading term of $A_j^{2,2}$ is proportional to Δ^+ , we obtain:

$$\lambda_2\left(\frac{1}{\omega_j}\right) = \frac{\beta^2}{\nu^2 \int_0^T e^{\frac{\alpha\beta t}{\nu}} dt} \left(\Delta^+ + \frac{c\alpha(2\alpha + \beta\nu^2)}{\nu} \frac{1}{\omega_j} + \mathcal{O}\left(\frac{1}{\omega_j^2}\right) \right). \quad (3.112)$$

Finally, applying the inversion expansion from Lemma 3.25 yields the asymptotic behavior of the inverse eigenvalue:

$$\frac{1}{\lambda_2(\omega_j^{-1})} = \frac{\nu^2 \int_0^T e^{\frac{\alpha\beta t}{\nu}} dt}{\beta^2 \Delta^+} \left(1 - R(\alpha, \beta, c, \nu, T) \frac{1}{\omega_j} + \mathcal{O}\left(\frac{1}{\omega_j^2}\right) \right), \quad (3.113)$$

where, in this case, the first-order coefficient is defined by:

$$R(\alpha, \beta, c, \nu, T) = \frac{c\alpha(2\alpha + \beta\nu^2)}{\nu\Delta^+}. \quad (3.114)$$

Summarizing the analysis, we arrive at the main characterization result.

Theorem 3.26 *The randomized observability constant defined in (3.75) takes the form*

$$C_{T,Rand}(\chi_\omega) = \inf_{j \in \mathbb{N}^*} \gamma_j \int_\omega |\phi_j(x)|^2 dx, \quad (3.115)$$

where the sequence of weights is given by $\gamma_j = \frac{1}{\lambda_{\max}(M_j)}$.

For sufficiently large j , these weights satisfy the asymptotic expansions derived in (3.109) if the parameter $\Delta^- \neq 0$, and (3.113) if $\Delta^- = 0$.

Motivated by Propositions 3.5 and 3.14, it is crucial to determine the sign of the coefficients R and $\tilde{R}$. Specifically, if these coefficients are positive, the asymptotic behavior $\gamma_j \approx \gamma^*(1 - R\omega_j^{-k})$ implies that the sequence converges from below for sufficiently high frequencies. This guarantees that the condition $\inf_{j \in \mathbb{N}^*} \gamma_j < \gamma^*$ is satisfied.

Corollary 3.27 *We have the following simplified expressions for the coefficient R :*

Subcase 1: $\Delta^- > 0$

$$R(\alpha, \beta, c, \nu, T) = \frac{1}{\Delta^+ + \Delta^-} \left(\frac{\alpha^2}{\nu^2} \left(2\alpha\beta + \frac{\beta^2\nu^2}{2} - 4c^2 \right) + \frac{c^2\alpha^2(2\alpha + \beta\nu^2)^2}{\nu^2\Delta^-} + \frac{2(\nu^2 + \alpha^2)(c^2 - \alpha\beta)(e^{\frac{\alpha\beta}{\nu}T}(\frac{\alpha\beta}{\nu}T - 1) + 1)}{\nu^2(e^{\frac{\alpha\beta T}{\nu}} - 1)} \right). \quad (3.116)$$

Subcase 2: $\Delta^- < 0$

$$R(\alpha, \beta, c, \nu, T) = \frac{1}{\Delta^+ - \Delta^-} \left(\alpha \left(2\beta c^2 + \frac{2\alpha c^2}{\nu^2} - \frac{\alpha\beta^2}{2} \right) + \frac{c^2\alpha^2(2\alpha + \beta\nu^2)^2}{\nu^2|\Delta^-|} + \frac{2c^2(c^2 - \alpha\beta)(e^{\frac{\alpha\beta}{\nu}T}(\frac{\alpha\beta}{\nu}T - 1) + 1)}{(e^{\frac{\alpha\beta T}{\nu}} - 1)} \right). \quad (3.117)$$

where $\Delta^\pm$ are defined in (3.106).

Then, we deduce the following sufficient conditions for the positivity of the coefficients, which guarantees convergence from below.

Corollary 3.28 *The following conditions are sufficient to ensure that R is strictly positive:*

i. If $\Delta^- = 0$;

ii. If $\Delta^- < 0$, it is sufficient that:

$$\alpha\beta \leq c^2 \quad \text{and} \quad 2\beta c^2 + \frac{2\alpha c^2}{\nu^2} - \frac{\alpha\beta^2}{2} \geq 0.$$

iii. If $\Delta^- > 0$, it is sufficient that:

$$\alpha\beta \leq c^2 \quad \text{and} \quad 2\alpha\beta + \frac{\beta^2\nu^2}{2} - 4c^2 \geq 0.$$

Remark 3.29 *It is important to remark that the sufficient conditions (ii) and (iii) in Corollary 3.28 are not sharp; rather, they serve as simplified criteria to ensure R is positive.*

To illustrate this, consider the case where $c = 1$. In this scenario, we are in the scenario $\Delta^- = \alpha^2 > 0$ (Case 1). Examining the constant polynomial terms in the expression for R , we observe that they simplify to a strictly positive quantity:

$$\begin{aligned} \frac{\alpha^2}{\nu^2} \left(2\alpha\beta + \frac{\beta^2\nu^2}{2} - 4 \right) + \frac{(2\alpha + \beta\nu^2)^2}{\nu^2} \\ = \frac{\beta(\alpha^2 + 2\nu^2)(\beta\nu^2 + 4\alpha)}{2\nu^2} > 0. \end{aligned} \tag{3.118}$$

Since this polynomial part is always positive, the algebraic condition " $2\alpha\beta + \dots \geq 0$ " from item (iii) is not needed.

Consequently, the only remaining condition from item (iii) is the one required to ensure the non-negativity of the time-dependent term, which is $\alpha\beta \leq c^2$.

By continuity, since the polynomial component of R is strictly positive at $c = 1$ (as shown in (3.118)), it remains positive for values of c sufficiently close to 1. Consequently, the condition $\alpha\beta \leq c^2$ is also sufficient to ensure the positivity of R in a neighborhood of $c = 1$.

Remark 3.30 *A similar conclusion holds for sufficiently large values of the wave speed c . Specifically, consider the regime where c is large enough such that $\Delta^- < 0$ (which occurs when $c^2 > 1 + \alpha^2/\nu^2$).*

Examining the sufficient condition derived in Corollary 3.28 (ii), we require:

$$2\beta c^2 + \frac{2\alpha c^2}{\nu^2} - \frac{\alpha\beta^2}{2} \geq 0.$$

Rearranging this expression to isolate c^2 , we obtain:

$$c^2 \left(2\beta + \frac{2\alpha}{\nu^2} \right) \geq \frac{\alpha\beta^2}{2}.$$

Since the coefficients α, β, ν are positive, the term multiplying c^2 is strictly positive. Therefore, for sufficiently large c , this inequality is automatically satisfied.

Consequently, in the high-velocity regime (where $\Delta^- < 0$), the only remaining condition to ensure the positivity of R is $\alpha\beta \leq c^2$. Note that the condition $\Delta^- < 0$ itself implies $c^2 > 1$, so if the coupling is weak ($\alpha\beta \leq 1$), the positivity of R is guaranteed unconditionally for all such large c .

Remark 3.31 *Motivated by Proposition 3.19, it is of significant interest to identify conditions ensuring that the sequence converges from above globally, i.e., $\inf_{j \in \mathbb{N}^*} \gamma_j = \gamma^*$.*

Our asymptotic analysis provides a partial answer: if we establish conditions such that the coefficient R is negative, we guarantee that $\gamma_j \geq \gamma^$ for all sufficiently large j . However, since these computations are valid only asymptotically (for $\omega_j \rightarrow \infty$), they do not control the behavior of the sequence at low frequencies. Even if $R < 0$, it is possible that $\gamma_j < \gamma^*$ for some small j . Consequently, determining whether the global infimum coincides with the limit for the thermoelasticity system remains an open question.*

Remark 3.32 *Consider the case $c = 1$. Using Corollary 3.27 and the expression derived in (3.118), we find that the limit of the coefficient R as the coupling parameter α vanishes is:*

$$\lim_{\alpha \rightarrow 0} R(\alpha, \beta, 1, \nu, T) = \frac{\beta^2}{2} > 0.$$

Since a positive R implies that the weights γ_j converge from below, this result might suggest that the limiting problem (observing a single wave equation, as seen in (3.29)) always possesses optimal sets. This would contradict the results of [41], which show that optimal sets for the wave equation do not generally exist.

However, this contradiction is resolved by observing that the limits do not commute. The randomized observability constant for the coupled system does not converge to the

constant for the wave equation as $\alpha \rightarrow 0$ in this regime. Specifically, when $c = 1$, the structural parameter $\Delta^- = \nu^2(1 - c^2) + \alpha^2$ reduces to α^2 . As $\alpha \rightarrow 0$, we have $\Delta^- \rightarrow 0$.

The expansion used to derive R (see (3.107)) relies on the first case of Lemma 3.25 ($I(z)$ with $A \neq 0$), valid only when $\Delta^- \neq 0$. As $\alpha \rightarrow 0$, the constant term A vanishes, shifting the system into the case where $A = 0$. Since the asymptotic expansion for $\sqrt{A + Bz^2}$ is not continuous with respect to A at $A = 0$, the limit of the coefficient R does not capture the true behavior of the uncoupled wave equation.

Observability Constant for Heat Observation

In the case where the observation is performed on the thermal component θ , we utilize the improved regularity estimate derived in Lemma 3.23. The observability constant is defined by:

$$C_T(\chi_\omega) = \inf_{\substack{(u^0, u^1, \theta^0) \in H' \\ (u^0, u^1, \theta^0) \neq 0}} \frac{\int_0^T \int_\Omega \chi_\omega |\theta|^2 dx dt}{\beta^2 \|u_t(T)\|_{H^{-2}(\Omega)}^2 + \beta^2 c^4 \|u(T)\|_{H^{-1}(\Omega)}^2 + \alpha^2 \|\theta(T)\|_{L^2(\Omega)}^2}. \quad (3.119)$$

We proceed analogously to the previous section by rewriting the observability constant (3.119) in terms of the orthonormal basis. Starting from the general spectral expansion (3.68), we introduce a change of variables normalized with respect to the third component (the thermal component) of the eigenvectors:

$$\tilde{a}_j = x_j^1 U_{j,1}^3, \quad \tilde{b}_j = x_j^2 U_{j,2}^3, \quad \tilde{c}_j = x_j^3 U_{j,3}^3, \quad \text{for } j \in \mathbb{N}^*.$$

Applying the time-shift transformation as before:

$$a_j = \tilde{a}_j e^{\lambda_j^1 T}, \quad b_j = \tilde{b}_j e^{\lambda_j^2 T}, \quad c_j = \tilde{c}_j e^{\lambda_j^3 T}.$$

Substituting these into the spectral expansion, we obtain the following expressions for the state variables:

$$\begin{cases} u(t) = \sum_{j=1}^{\infty} \left(a_j \frac{U_{j,1}^1}{U_{j,1}^3} e^{-\lambda_j^1(T-t)} + b_j \frac{U_{j,2}^1}{U_{j,2}^3} e^{-\lambda_j^2(T-t)} + c_j \frac{U_{j,3}^1}{U_{j,3}^3} e^{-\lambda_j^3(T-t)} \right) \phi_j, \\ u_t(t) = \sum_{j=1}^{\infty} \left(a_j \lambda_j^1 \frac{U_{j,1}^1}{U_{j,1}^3} e^{-\lambda_j^1(T-t)} + b_j \lambda_j^2 \frac{U_{j,2}^1}{U_{j,2}^3} e^{-\lambda_j^2(T-t)} + c_j \lambda_j^3 \frac{U_{j,3}^1}{U_{j,3}^3} e^{-\lambda_j^3(T-t)} \right) \phi_j, \\ \theta(t) = \sum_{j=1}^{\infty} \left(\tilde{a}_j e^{-\lambda_j^1(T-t)} + \tilde{b}_j e^{-\lambda_j^2(T-t)} + \tilde{c}_j e^{-\lambda_j^3(T-t)} \right) \phi_j. \end{cases} \quad (3.120)$$

We define the energy functional $\hat{D}$ associated with the heat observation, using the L^2

norm for the temperature component as discussed:

$$\hat{D}(a, b, c) = \beta^2 \|u_t(T)\|_{H^{-2}(\Omega)}^2 + \beta^2 c^4 \|u(T)\|_{H^{-1}(\Omega)}^2 + \alpha^2 \|\theta(T)\|_{L^2(\Omega)}^2. \quad (3.121)$$

Substituting the spectral expansions into this definition, we obtain:

$$\begin{aligned} \hat{D}(a, b, c) = & \sum_{j=1}^{\infty} \left(\frac{\beta^2 c^4}{\omega_j^2} \left| a_j \frac{U_{j,1}^1}{U_{j,1}^3} + b_j \frac{U_{j,2}^1}{U_{j,2}^3} + c_j \frac{U_{j,3}^1}{U_{j,3}^3} \right|^2 \right. \\ & + \frac{\beta^2}{\omega_j^4} \left| \lambda_j^1 a_j \frac{U_{j,1}^1}{U_{j,1}^3} + \lambda_j^2 b_j \frac{U_{j,2}^1}{U_{j,2}^3} + \lambda_j^3 c_j \frac{U_{j,3}^1}{U_{j,3}^3} \right|^2 \\ & \left. + \alpha^2 |a_j + b_j + c_j|^2 \right). \end{aligned} \quad (3.122)$$

Similarly, using the spectral expansion (3.120), the observation integral for the heat component is given by:

$$\int_0^T \int_{\omega} |\theta|^2 dx dt = \int_0^T \int_{\omega} \left| \sum_{j=1}^{\infty} \left(a_j e^{-\lambda_j^1(T-t)} + b_j e^{-\lambda_j^2(T-t)} + c_j e^{-\lambda_j^3(T-t)} \right) \phi_j(x) \right|^2 dx dt. \quad (3.123)$$

Thus, combining the observation integral (3.123) and the energy functional (3.121), we rewrite the observability constant as the solution to the following minimization problem:

$$C_T(\chi_{\omega}) = \inf_{\substack{\mathbf{a}, \mathbf{b}, \mathbf{c} \in \ell^2(\mathbb{C}) \\ \hat{D}(\mathbf{a}, \mathbf{b}, \mathbf{c})=1}} \int_0^T \int_{\omega} \left| \sum_{j=1}^{\infty} \left(a_j e^{-\lambda_j^1(T-t)} + b_j e^{-\lambda_j^2(T-t)} + c_j e^{-\lambda_j^3(T-t)} \right) \phi_j(x) \right|^2 dx dt. \quad (3.124)$$

The randomized solution for the temperature variable θ , corresponding to an event ν , is given by:

$$\theta^{\nu}(t) = \sum_{j=1}^{\infty} \left(\beta_{1,j}^{\nu} a_j e^{-\lambda_j^1(T-t)} + \beta_{2,j}^{\nu} b_j e^{-\lambda_j^2(T-t)} + \beta_{3,j}^{\nu} c_j e^{-\lambda_j^3(T-t)} \right) \phi_j.$$

Consequently, the randomized observability constant is defined by:

$$\begin{aligned} C_{T,\text{Rand}}(\chi_{\omega}) = & \inf_{\substack{(u^0, u^1, \theta^0) \in H' \\ (u^0, u^1, \theta^0) \neq 0}} \left[\left(\beta^2 \|u_t(T)\|_{H^{-2}(\Omega)}^2 + \beta^2 c^4 \|u(T)\|_{H^{-1}(\Omega)}^2 + \alpha^2 \|\theta(T)\|_{L^2(\Omega)}^2 \right)^{-1} \right. \\ & \times \mathbb{E} \left(\int_0^T \int_{\omega} \chi_{\omega} \left| \sum_{j=1}^{\infty} \left(\beta_{1,j}^{\nu} a_j e^{-\lambda_j^1(T-t)} + \beta_{2,j}^{\nu} b_j e^{-\lambda_j^2(T-t)} + \beta_{3,j}^{\nu} c_j e^{-\lambda_j^3(T-t)} \right) \phi_j \right|^2 dx dt \right) \left. \right]. \end{aligned}$$

Equivalently, in terms of the spectral coefficients and the functional $\hat{D}$ defined in (3.121):

$$C_{T,\text{Rand}}(\chi_\omega) = \inf_{\substack{\mathbf{a}, \mathbf{b}, \mathbf{c} \in \ell^2(\mathbb{C}) \\ \hat{D}(\mathbf{a}, \mathbf{b}, \mathbf{c})=1}} \mathbb{E} \left(\int_0^T \int_\omega \left| \sum_{j=1}^{\infty} \left(\beta_{1,j}^\nu a_j e^{-\lambda_j^1(T-t)} + \beta_{2,j}^\nu b_j e^{-\lambda_j^2(T-t)} + \beta_{3,j}^\nu c_j e^{-\lambda_j^3(T-t)} \right) \phi_j(x) \right|^2 dx dt \right). \quad (3.125)$$

Using the properties of the expectation $\mathbb{E}$ (specifically, the independence and unit variance of the random variables), the cross-terms vanish. Consequently, the randomized observability constant simplifies to a ratio of diagonal quadratic forms:

$$C_{T,\text{Rand}}(\chi_\omega) = \inf_{\mathbf{a}, \mathbf{b}, \mathbf{c} \in \ell^2(\mathbb{C})} \frac{\hat{N}(\mathbf{a}, \mathbf{b}, \mathbf{c})}{\hat{D}(\mathbf{a}, \mathbf{b}, \mathbf{c})}. \quad (3.126)$$

Here, the numerator $\hat{N}$ is defined by:

$$\begin{aligned} \hat{N}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = & \sum_{j=1}^{\infty} \left(|a_j|^2 \int_0^T e^{-2\operatorname{Re}(\lambda_j^1)t} dt + |b_j|^2 \int_0^T e^{-2\operatorname{Re}(\lambda_j^2)t} dt \right. \\ & \left. + |c_j|^2 \int_0^T e^{-2\operatorname{Re}(\lambda_j^3)t} dt \right) \int_\omega |\phi_j(x)|^2 dx, \end{aligned} \quad (3.127)$$

and the denominator $\hat{D}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ corresponds to the energy expansion given in (3.122).

We introduce the change of variables to normalize the time integrals:

$$d_j = a_j \sqrt{\int_0^T e^{-2\operatorname{Re}(\lambda_j^1)t} dt}, \quad e_j = b_j \sqrt{\int_0^T e^{-2\operatorname{Re}(\lambda_j^2)t} dt}, \quad f_j = c_j \sqrt{\int_0^T e^{-2\operatorname{Re}(\lambda_j^3)t} dt}.$$

Substituting these into the ratio, we find that the inverse of the randomized observability constant is given by:

$$\frac{1}{C_{T,\text{Rand}}(\chi_\omega)} = \sup_{\mathbf{d}, \mathbf{e}, \mathbf{f} \in \ell^2} \frac{\sum_{j=1}^{\infty} \left(A_j^{1,1} |d_j|^2 + A_j^{2,2} |e_j|^2 + A_j^{3,3} |f_j|^2 + 2\operatorname{Re}(A_j^{1,2} d_j \bar{e}_j) + 2\operatorname{Re}(A_j^{3,1} d_j \bar{f}_j) + 2\operatorname{Re}(A_j^{3,2} e_j \bar{f}_j) \right)}{\sum_{j=1}^{\infty} (|d_j|^2 + |e_j|^2 + |f_j|^2) \int_\omega |\phi_j(x)|^2 dx}, \quad (3.128)$$

To simplify the notation, we reuse the inverse time-integral factors defined previously:

$$\mathcal{E}_j^1 = \left(\int_0^T e^{-2\operatorname{Re}(\lambda_j^1)t} dt \right)^{-1} \quad \text{and} \quad \mathcal{E}_j^2 = \left(\int_0^T e^{-2\operatorname{Re}(\lambda_j^2)t} dt \right)^{-1}.$$

Recalling that $\operatorname{Re}(\lambda_j^3) = \operatorname{Re}(\lambda_j^2)$, the factor for the third eigenvalue is identical to $\mathcal{E}_j^2$. The matrix coefficients for the heat observation case can thus be written compactly as:

$$\begin{aligned} A_j^{1,1} &= \mathcal{E}_j^1 \left(\frac{\beta^2}{\omega_j^2} \left| \frac{U_{j,1}^1}{U_{j,1}^3} \right|^2 \left(c^4 + \frac{|\lambda_j^1|^2}{\omega_j^2} \right) + \alpha^2 \right), \\ A_j^{2,2} &= \mathcal{E}_j^2 \left(\frac{\beta^2}{\omega_j^2} \left| \frac{U_{j,2}^1}{U_{j,2}^3} \right|^2 \left(c^4 + \frac{|\lambda_j^2|^2}{\omega_j^2} \right) + \alpha^2 \right), \quad A_j^{3,3} = A_j^{2,2}, \\ A_j^{1,2} &= \sqrt{\mathcal{E}_j^1 \mathcal{E}_j^2} \left(\frac{\beta^2}{\omega_j^2} \frac{U_{j,1}^1}{U_{j,1}^3} \frac{\overline{U_{j,2}^1}}{U_{j,2}^3} \left(c^4 + \frac{\lambda_j^1 \overline{\lambda_j^2}}{\omega_j^2} \right) + \alpha^2 \right), \quad A_j^{3,1} = \overline{A_j^{1,2}} \\ A_j^{3,2} &= \mathcal{E}_j^2 \left(\frac{\beta^2}{\omega_j^2} \frac{U_{j,2}^1}{U_{j,2}^3} \frac{\overline{U_{j,3}^1}}{U_{j,3}^3} \left(c^4 + \frac{\lambda_j^2 \overline{\lambda_j^3}}{\omega_j^2} \right) + \alpha^2 \right). \end{aligned}$$

Thus, the inverse of the randomized observability constant takes the compact matrix form:

$$\frac{1}{C_{T,\text{Rand}}(\chi_\omega)} = \sup_{\mathbf{d}, \mathbf{e}, \mathbf{f} \in \ell^2} \frac{\sum_{j=1}^{\infty} \langle M_j \mathbf{v}_j, \mathbf{v}_j \rangle_{\mathbb{C}^3}}{\sum_{j=1}^{\infty} |\mathbf{v}_j|^2 \int_{\omega} |\phi_j(x)|^2 dx}, \quad (3.129)$$

where $M_j = (A_j^{k,l})_{k,l=1}^3$ is the Hermitian matrix defined above.

Proceeding analogously to the wave equation case (Corollary 3.24), we characterize the constant by the spectral modes:

$$C_{T,\text{Rand}}(\chi_\omega) = \left(\sup_{j \in \mathbb{N}^*} \frac{\lambda_{\max}(M_j)}{\int_{\omega} |\phi_j(x)|^2 dx} \right)^{-1} = \inf_{j \in \mathbb{N}^*} \frac{\int_{\omega} |\phi_j(x)|^2 dx}{\lambda_{\max}(M_j)}, \quad (3.130)$$

where $\lambda_{\max}(M_j)$ denotes the largest eigenvalue of M_j . In what follows, we analyze the asymptotic behavior of $\lambda_{\max}(M_j)$ as $\omega_j \rightarrow \infty$.

In what follows, we derive the asymptotic expansions for the coefficients $A_j^{k,l}$. As the computational method closely parallels the wave observation case, we present the results concisely. We again express the coefficients as functions of the continuous variable $z = 1/\omega_j$.

We start by analyzing the coefficient $A_j^{1,1}$. Based on the spectral definitions, we have:

$$A_j^{1,1} = \frac{1}{\int_0^T e^{-2\operatorname{Re}(\lambda_j^1)t} dt} (\alpha^2 + \mathcal{O}(z^2)).$$

Since $\operatorname{Re}(\lambda_j^1) \approx -\nu\omega_j^2 \rightarrow -\infty$, the integral in the denominator grows exponentially. Consequently, $A_j^{1,1}$ decays exponentially to zero as $j \rightarrow \infty$ (or equivalently, as $z \rightarrow 0$).

Next, we consider the coefficient $A_j^{2,2}$. Substituting the asymptotic expansions for the

eigenvalues and eigenvectors into its definition, we obtain:

$$A_j^{2,2} = \frac{\alpha^2 + \nu^2(1 + c^2) + [c^4 - (2\alpha\beta - 1)c^2 - \alpha\beta] z^2 + \mathcal{O}(z^3)}{\int_0^T e^{\frac{\alpha\beta t}{\nu}} dt - \frac{(c^2 - \alpha\beta) \left[e^{\frac{\alpha\beta}{\nu} T} \left(\frac{\alpha\beta}{\nu} T - 1 \right) + 1 \right]}{\alpha\beta\nu}} z^2 + \mathcal{O}(z^3). \quad (3.131)$$

Applying the rational expansion from Lemma 3.25 to (3.131), we obtain:

$$A_j^{2,2} = \frac{1}{\int_0^T e^{\frac{\alpha\beta t}{\nu}} dt} \left(\alpha^2 + (1 + c^2)\nu^2 + \tilde{\mathcal{C}}_{2,2} z^2 + \mathcal{O}(z^3) \right), \quad (3.132)$$

where the second-order coefficient is given by:

$$\begin{aligned} \tilde{\mathcal{C}}_{2,2} = & (c^4 - (2\alpha\beta - 1)c^2 - \alpha\beta) \\ & + \frac{(\alpha^2 + (1 + c^2)\nu^2)(c^2 - \alpha\beta) \left[e^{\frac{\alpha\beta}{\nu} T} \left(\frac{\alpha\beta}{\nu} T - 1 \right) + 1 \right]}{\nu^2(e^{\frac{\alpha\beta T}{\nu}} - 1)}. \end{aligned}$$

In particular, squaring the expansion derived above, we obtain:

$$|A_j^{2,2}|^2 = \frac{1}{\left(\int_0^T e^{\frac{\alpha\beta t}{\nu}} dt \right)^2} \left((\alpha^2 + (1 + c^2)\nu^2)^2 + 2(\alpha^2 + (1 + c^2)\nu^2) \tilde{\mathcal{C}}_{2,2} z^2 + \mathcal{O}(z^3) \right). \quad (3.133)$$

Next, we compute the asymptotic expansion for the coefficient $A_j^{3,2}$. Substituting the relevant spectral expansions, we obtain:

$$A_j^{3,2} = \frac{\alpha^2 + \nu^2(1 - c^2) + i c \nu (\alpha\beta - 2c^2 + 2)z + \left(\frac{1}{2} \alpha^2 \beta^2 - \alpha\beta + c^4 - c^2 \right) z^2 + \mathcal{O}(z^3)}{\int_0^T e^{\frac{\alpha\beta t}{\nu}} dt - \frac{(c^2 - \alpha\beta) \left[e^{\frac{\alpha\beta}{\nu} T} \left(\frac{\alpha\beta}{\nu} T - 1 \right) + 1 \right]}{\alpha\beta\nu}} z^2 + \mathcal{O}(z^3). \quad (3.134)$$

Applying Lemma 3.25 to (3.134), we derive the final asymptotic form for $A_j^{3,2}$:

$$A_j^{3,2} = \frac{1}{\int_0^T e^{\frac{\alpha\beta t}{\nu}} dt} \left((\alpha^2 + (1 - c^2)\nu^2) + i c \nu (\alpha\beta - 2c^2 + 2)z + \tilde{\mathcal{C}}_{3,2} z^2 + \mathcal{O}(z^3) \right), \quad (3.135)$$

where the second-order coefficient is given by:

$$\begin{aligned} \tilde{\mathcal{C}}_{3,2} = & \left(\frac{1}{2} \alpha^2 \beta^2 - \alpha\beta + c^4 - c^2 \right) \\ & + \frac{(\alpha^2 + (1 - c^2)\nu^2)(c^2 - \alpha\beta) \left[e^{\frac{\alpha\beta}{\nu} T} \left(\frac{\alpha\beta}{\nu} T - 1 \right) + 1 \right]}{\nu^2(e^{\frac{\alpha\beta T}{\nu}} - 1)}. \end{aligned}$$

Squaring the modulus of the expansion, we obtain:

$$|A_j^{3,2}|^2 = \frac{1}{\left(\int_0^T e^{\frac{\alpha\beta t}{\nu}} dt\right)^2} \left((\alpha^2 + (1 - c^2)\nu^2)^2 + \left[c^2\nu^2(\alpha\beta - 2c^2 + 2)^2 + 2(\alpha^2 + (1 - c^2)\nu^2) \tilde{\mathcal{C}}_{3,2} \right] z^2 + \mathcal{O}(z^3) \right). \quad (3.136)$$

Proceeding as in the wave observation analysis, we distinguish between the generic and critical regimes based on whether the parameter Δ^- , defined in (3.106), vanishes or not.

Case 1: $\Delta^- \neq 0$

Proceeding analogously to the previous section, the largest eigenvalue behaves as $\lambda_2(z) \approx A_j^{2,2} + \sqrt{|A_j^{3,2}|^2}$. Applying Lemma 3.25 to the expansion derived in (3.136), we obtain:

$$\sqrt{|A_j^{3,2}|^2 + \mathcal{O}(z^k)} = \frac{1}{\int_0^T e^{\frac{\alpha\beta t}{\nu}} dt} \left(|\Delta^-| + \frac{c^2\nu^2(\alpha\beta - 2c^2 + 2)^2 + 2\Delta^- \tilde{\mathcal{C}}_{3,2}}{2|\Delta^-|} z^2 + \mathcal{O}(z^3) \right). \quad (3.137)$$

Consequently, substituting the expansions into the expression for the largest eigenvalue, we obtain:

$$\lambda_2(z) = \frac{1}{\int_0^T e^{\frac{\alpha\beta t}{\nu}} dt} \left(\Delta^+ + |\Delta^-| + \left(\tilde{\mathcal{C}}_{2,2} + \operatorname{sgn}(\Delta^-) \tilde{\mathcal{C}}_{3,2} + \frac{c^2\nu^2(\alpha\beta - 2c^2 + 2)^2}{2|\Delta^-|} \right) z^2 + \mathcal{O}(z^3) \right), \quad (3.138)$$

provided that $\Delta^- \neq 0$.

Therefore, applying the inversion expansion from Lemma 3.25, we derive the asymptotic behavior of the inverse:

$$\frac{1}{\lambda_2(z)} = \int_0^T e^{\frac{\alpha\beta t}{\nu}} dt \left(\frac{1}{\Delta^+ + |\Delta^-|} - R_1(\alpha, \beta, c, \nu, T) z^2 + \mathcal{O}(z^3) \right), \quad (3.139)$$

where the coefficient $R_1 := R_1(\alpha, \beta, c, \nu, T)$ is defined by:

$$R_1 = \frac{\tilde{\mathcal{C}}_{2,2} + \operatorname{sgn}(\Delta^-) \tilde{\mathcal{C}}_{3,2} + \frac{c^2\nu^2(\alpha\beta - 2c^2 + 2)^2}{2|\Delta^-|}}{(\Delta^+ + |\Delta^-|)^2}. \quad (3.140)$$

Case 2: $\Delta^- = 0$

In this specific case, the constant term inside the square root vanishes. Consequently, the leading behavior is determined by the linear term in z . Applying the expansion for $I(z)$ from Lemma 3.25 (case $A = 0$), we obtain a first-order expansion:

$$\sqrt{|A_j^{3,2}|^2 + \mathcal{O}(z^k)} = \frac{1}{\int_0^T e^{\frac{\alpha\beta t}{\nu}} dt} (c\nu| - 2c^2 + \alpha\beta + 2|z + \mathcal{O}(z^2)) . \quad (3.141)$$

Consequently, substituting this expansion into the expression for λ_2 and applying the inversion formula , we obtain the asymptotic behavior of the inverse eigenvalue.

$$\frac{1}{\lambda_2(z)} = \int_0^T e^{\frac{\alpha\beta t}{\nu}} dt \left(\frac{1}{\Delta^+} - R_1(\alpha, \beta, c, \nu, T)z + \mathcal{O}(z^2) \right) , \quad (3.142)$$

where the first-order coefficient is defined by:

$$R_1(\alpha, \beta, c, \nu, T) = \frac{c\nu| - 2c^2 + \alpha\beta + 2|}{(\Delta^+)^2} . \quad (3.143)$$

Summarizing the analysis for the heat observation, we arrive at the main characterization result.

Theorem 3.33 *The randomized observability constant defined in (3.125) takes the form*

$$C_{T,Rand}(\chi_\omega) = \inf_{j \in \mathbb{N}^*} \gamma_j \int_\omega |\phi_j(x)|^2 dx, \quad (3.144)$$

where the sequence of weights is given by $\gamma_j = \frac{1}{\lambda_{\max}(M_j)}$.

For sufficiently large j , these weights satisfy the asymptotic expansions derived in (3.139) if $\Delta^- \neq 0$, and (3.142) if $\Delta^- = 0$.

Analogous to the wave equation case, if the coefficient R_1 is positive, the sequence of weights converges from below. As discussed in Section 3.2, this condition is sufficient to guarantee the existence of an optimal set, provided that the eigenfunctions satisfy specific properties (such as the Quantum Ergodicity assumption or the alternative hypotheses $(\mathbf{H}_*^1)$).

We now derive simplified expressions for R_1 to facilitate this sign analysis.

Corollary 3.34 *Recalling the notation $\Delta^- = \nu^2(1 - c^2) + \alpha^2$, we have the following simplifications for R_1 :*

i. Subcase 1: $\Delta^- > 0$ *The global denominator simplifies to $2(\nu^2 + \alpha^2)$.*

$$R_1(\alpha, \beta, c, \nu, T) = \frac{1}{2(\nu^2 + \alpha^2)} \left(\frac{1}{2} \alpha^2 \beta^2 - 2\alpha\beta(1 + c^2) + 2c^4 \right. \\ \left. + \frac{c^2 \nu^2 (\alpha\beta - 2c^2 + 2)^2}{2\Delta^-} \right) + \frac{c^2 - \alpha\beta}{\nu^2} \left(\frac{e^{\frac{\alpha\beta}{\nu} T} \left(\frac{\alpha\beta}{\nu} T - 1 \right) + 1}{e^{\frac{\alpha\beta}{\nu} T} - 1} \right). \quad (3.145)$$

ii. **Subcase 2:** $\Delta^- < 0$ The global denominator simplifies to $2\nu^2 c^2$.

$$R_1(\alpha, \beta, c, \nu, T) = \frac{1}{2\nu^2 c^2} \left(2c^2(1 - \alpha\beta) - \frac{1}{2} \alpha^2 \beta^2 \right. \\ \left. + \frac{c^2 \nu^2 (\alpha\beta - 2c^2 + 2)^2}{2|\Delta^-|} \right) + \frac{c^2 - \alpha\beta}{\nu^2} \left(\frac{e^{\frac{\alpha\beta}{\nu} T} \left(\frac{\alpha\beta}{\nu} T - 1 \right) + 1}{e^{\frac{\alpha\beta}{\nu} T} - 1} \right). \quad (3.146)$$

Corollary 3.35 *The following conditions are sufficient to ensure that the coefficient R_1 is strictly positive:*

i. If $\Delta^- = 0$, then $R_1 \geq 0$ holds automatically for all parameters (and is strictly positive provided $\alpha\beta \neq 2c^2 - 2$).

ii. If $\Delta^- < 0$, it is sufficient that:

$$\alpha\beta \leq c^2 \quad \text{and} \quad 2c^2(1 - \alpha\beta) - \frac{1}{2} \alpha^2 \beta^2 > 0.$$

iii. If $\Delta^- > 0$, it is sufficient that:

$$\alpha\beta \leq c^2 \quad \text{and} \quad \frac{1}{2} \alpha^2 \beta^2 - 2\alpha\beta(1 + c^2) + 2c^4 > 0.$$

In [42], the authors established the existence of optimal sets under suitable conditions. A crucial feature of their analysis is that the sequence γ_j grows exponentially, which facilitates the proof of attainability. In our framework, however, the sequence is bounded. A natural question is whether a similar existence result holds if we decouple the system by letting the coupling parameter vanish, i.e., $\beta \rightarrow 0$.

Corollary 3.36 *In the limit as $\beta \rightarrow 0$, the coefficient R_1 converges to a strictly positive value. Consequently, the sequence γ_j converges from below.*

Proof: We analyze the limit of the coefficients derived in Corollary 3.34 and Corollary 3.35 as $\beta \rightarrow 0$. Note that Δ^- is independent of β .

Case 1: $\Delta^- = 0$

Using the definition of R_1 in (3.143), and noting that $\alpha\beta \rightarrow 0$:

$$\lim_{\beta \rightarrow 0} R_1(\alpha, \beta, c, \nu, T) = \frac{c\nu| - 2c^2 + 2|}{(\Delta^+)^2} = \frac{2c\nu|1 - c^2|}{(\Delta^+)^2} > 0.$$

Case 2: $\Delta^- \neq 0$

We use the simplified expressions from Corollary 3.34. In the limit $\beta \rightarrow 0$, the singular term simplifies as:

$$\frac{c^2\nu^2(\alpha\beta - 2c^2 + 2)^2}{2|\Delta^-|} \rightarrow \frac{c^2\nu^2(2 - 2c^2)^2}{2|\Delta^-|} = \frac{2c^2\nu^2(1 - c^2)^2}{|\Delta^-|}.$$

Subcase $\Delta^- > 0$: The global denominator is $2(\nu^2 + \alpha^2) = \Delta^+ + \Delta^-$. The polynomial part simplifies to $2c^4$. Thus:

$$\lim_{\beta \rightarrow 0} R_1 = \frac{1}{\Delta^+ + \Delta^-} \left(2c^4 + \frac{2c^2\nu^2(1 - c^2)^2}{\Delta^-} \right).$$

Subcase $\Delta^- < 0$: The global denominator is $2\nu^2c^2 = \Delta^+ - \Delta^-$. The polynomial part simplifies to $2c^2$. Thus:

$$\lim_{\beta \rightarrow 0} R_1 = \frac{1}{\Delta^+ - \Delta^-} \left(2c^2 + \frac{2c^2\nu^2(1 - c^2)^2}{|\Delta^-|} \right).$$

In all cases, the limits are sums of strictly positive terms, confirming that $R_1 > 0$. ■

3.4 Conclusion

In this work, we addressed the optimal sensor placement problem for a system of linear thermoelasticity on a bounded domain Ω . By adopting a randomized approach to the initial data, we reduced the problem to maximizing a spectral functional of the form

$$J(\chi_\omega) = \inf_{j \in \mathbb{N}^*} \gamma_j \int_{\omega} |\phi_j(x)|^2 dx,$$

where the weights γ_j are determined in terms of the eigenvalues of a matrix determined by the observability inequality.

The core of our analysis focused on bridging the gap between the relaxed optimization problem (where solutions are density functions) and the original problem (where solutions must be characteristic functions of sets). As discussed in Section 3.2, the existence of a classical optimal set ω^* is closely linked to the asymptotic behavior of the sequence

$(\gamma_j)_{j \in \mathbb{N}^*}$. Specifically, if the sequence converges to its limit γ^* *from below* (i.e., $\gamma_j \leq \gamma^*$ for large j), the relaxation gap vanishes, and an optimal set exists under mild ergodicity assumptions on the eigenfunctions.

Through a detailed asymptotic expansion in the high-frequency regime, we characterized the behavior of the thermoelastic weights as

$$\gamma_j = \gamma^* \left(1 - \frac{R}{\omega_j^k} + o\left(\frac{1}{\omega_j^k}\right) \right),$$

where the exponent k and the coefficient R depend on the physical parameters α, β, c, ν and the time horizon T . We identified two distinct regimes:

1. $\Delta^- \neq 0$: Here, the convergence is of second order ($k = 2$). We derived explicit algebraic conditions on the parameters (Corollary 3.28) sufficient to ensure $R > 0$. Notably, we showed that for sufficiently weak coupling ($\beta \rightarrow 0$), sufficiently large time horizons, or wave speeds satisfying specific inequalities, the condition $R > 0$ holds. In these scenarios, the existence of an optimal sensor shape is guaranteed.
2. $\Delta^- = 0$: Here, the convergence is of first order ($k = 1$). We established that the corresponding coefficient R is strictly positive, similarly ensuring the existence of optimal sets.

The sign of R has significant theoretical implications:

- **When $R > 0$ (Convergence from Below):** The sequence approaches the limit from below for high frequencies. If the eigenfunctions satisfy the Quantum Ergodicity assumptions $(\mathbf{H}^1)$ and $(\mathbf{H}^2)$, the existence of an optimal sensor domain is guaranteed. Even in geometries where QUE fails (e.g., n -dimensional orthotopes), we demonstrated in Section 3.2.2 that optimal sets can still exist in this regime.
- **When $R < 0$ (Convergence from Above):** The sufficient condition $(\mathbf{H}_*^1)$ fails. As discussed in Section 3.2.2, while optimal sets may exist for specific volume fractions, the general existence problem in this regime remains challenging.

Finally, we highlight an important open question. Our asymptotic analysis guarantees the behavior of γ_j only for sufficiently large j . Establishing the global condition $\inf_{j \in \mathbb{N}^*} \gamma_j = \gamma^*$ requires control over the low frequencies, which the asymptotic coefficient R does not provide. Therefore, determining conditions under which the global infimum coincides with the limit for the fully coupled thermoelasticity system remains a subject for future investigation.

3.5 Future Problems

From the perspective of the present work, some directions for future research involve conducting more detailed analyses or developing alternative lines of investigation (including from a numerical point of view) regarding the case in which superior convergence is observed, specifically when $R < 0$.

Another relevant issue is the consideration of other random variables or stochastic perturbations. Extending the current framework to include different probabilistic scenarios could provide a deeper understanding of the robustness and sensitivity of optimal sensor configurations.

Additionally, the challenge of producing results on the Disk remains, as we were unable to obtain satisfactory outcomes due to the lack of exponential growth of our γ_j . These represent possible directions for future work stemming from the present study.

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