Universidade Federal da Paraíba
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Uma desigualdade do tipo Trudinger-Moser em espaços de Sobolev com peso e aplicações

por

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João Pessoa - PB
Abril/2014

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sob orientação do

Prof. Dr. Everaldo Souto de Medeiros

Tese apresentada ao Corpo Docente do Programa Associado de Pós-Graduação em Matemática - UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

Co-orientador: Prof. Dr. Claudianor Oliveira Alves

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"Sua sabedoria determina sua força

Sua força determina sua resistência

Sua resistência determina seu sucesso."

Mike Murdock

Notações

- Em todas as integrais, com excessão daquelas de outros trabalhos, omitiremos o símbolo dx e usaremos $C, \widetilde{C}, C_0, C_1, C_2, \ldots$ para denotar constantes positivas (possivelmente diferentes de uma linha para outra);

- $C = C(\alpha, \beta, a, b, c, d, \ldots)$ denota uma constante positiva dependente dos valores $\alpha, \beta, a, b, c, d, \ldots$;

- B_r denota a bola aberta de $\mathbb{R}^2$ centrada na origem e raio r;

- $B_R \setminus B_r$ denota o anel de raio interior r e raio exterior R;

- Para qualquer subconjunto $A \subset \mathbb{R}^2$, A^c denota o complemento de A;

- $|A|$ denota a medida de Lebesgue do conjunto A;

- $o_n(1)$ denota uma sequência de números reais convergindo para 0 quando $n \to +\infty$;

- $p \gg q$ significa que o número p é muito maior do que o número q;

- $f(s) = o(g(s))$ quando $s \to 0$ significa que $\lim_{s \to 0} \dfrac{f(s)}{g(s)} = 0$;

- a.e.: Abreviação em inglês de *almost everywhere* para designar em quase todo ponto, ou seja, a menos de um conjunto de medida nula;

- $\blacksquare$ indica o fim de uma demonstração;

- $(PS)_c$: Sequência Palais-Smale no nível c;

- $\nabla u = \left(\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}\right)$ denota o gradiente da função $u : \mathbb{R}^2 \to \mathbb{R}$;

- $\Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$ denota o Laplaciano de u;

- X' é o dual topológico do espaço de Banach X;

- $\langle \cdot, \cdot \rangle$ denota o par de dualidade entre X' e X;

- $\text{supp}(u)$ denota o suporte de u;

- $C_0^\infty(\mathbb{R}^2)$ denota o conjunto de funções suaves com suporte compacto;

- $C_{0,\text{rad}}^\infty(\mathbb{R}^2) = \{u \in C_0^\infty(\mathbb{R}^2) : u \text{ é radial}\}$;

- $D_{\text{rad}}^{1,2}(\mathbb{R}^2)$ denota o fecho de $C_{0,\text{rad}}^\infty(\mathbb{R}^2)$ sob a norma $\|\nabla u\|_2 = \left(\int_{\mathbb{R}^2} |\nabla u|^2\right)^{1/2}$;

- $C^{k,\gamma}(\overline{\Omega}) = \{u \in C^k(\overline{\Omega}) : D^k u \text{ é } \gamma\text{-Hölder contínua}\}$;

- $\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p\right)^{1/p}$;

- $L^p(\Omega) = \{u : \Omega \to \mathbb{R} : u \text{ é mensurável e } \|u\|_{L^p(\Omega)} < \infty\}$;

- $\|u\|_{L^\infty(\Omega)} = \inf\{C \geq 0 : |\{x \in \Omega : |u(x)| > C\}| = 0\}$;

- $L^\infty(\Omega) = \{u : \Omega \to \mathbb{R} : u \text{ é mensurável e } \|u\|_{L^\infty(\Omega)} < \infty\}$;

- $W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^\gamma u \in L^p(\Omega),\ \forall |\gamma| \leq k\}$, onde γ é um multi-índice;

- $\|u\|_{W^{k,p}(\Omega)} = \|u\|_{k,p} = \left(\sum_{i=0}^k \|D^i u\|_{L^p(\Omega)}\right)^{1/p}$;

- $W_0^{k,p}(\Omega)$ denota o fecho de $C_0^\infty(\Omega)$ sob a norma $\|\cdot\|_{W^{k,p}(\Omega)}$;

- $H^k(\Omega) = W^{k,2}(\Omega)$;

- $H_0^k(\Omega)$ denota o fecho de $C_0^\infty(\Omega)$ sob a norma $\|\cdot\|_{H^k(\Omega)}$.

Resumo

Este trabalho aborda uma classe de desigualdades do tipo Trudinger-Moser em espaços de Sobolev com peso em $\mathbb{R}^2$. Como aplicação destas desigualdades e usando métodos variacionais, estabeleceremos condições suficientes para a existência, multiplicidade e não-existência de soluções para algumas classes de equações (e sistemas de equações) de Schrödinger elípticas não-lineares com potenciais radiais ilimitados, singulares na origem ou decaindo a zero no infinito e envolvendo não-linearidades com crescimento crítico exponencial do tipo Trudinger-Moser.

Palavras-chave: Desigualdade de Trudinger-Moser; Espaços de Sobolev com peso; Equação de Schrödinger não-linear; Potenciais radiais ilimitados ou decaindo a zero; Crescimento crítico exponencial.

Abstract

This work addresses a class of Trudinger-Moser type inequalities in weighted Sobolev spaces in $\mathbb{R}^2$. As an application of these inequalities and by using variational methods, we establish sufficient conditions for the existence, multiplicity and nonexistence of solutions for some classes of nonlinear Schrödinger elliptic equations (and systems of equations) with unbounded, singular or decaying radial potentials and involving nonlinearities with exponential critical growth of Trudinger-Moser type.

Keywords: Trudinger-Moser inequality; Weighted Sobolev spaces; Nonlinear Schrödinger equation; Unbounded or decaying radial potentials; Exponential critical growth.

Contents

Introdução

Seja $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ é um domínio limitado e suave. Sabemos do Teorema de Imersão de Sobolev (veja [2, Teorema 5.4]) que o espaço $H_0^1(\Omega)$ está imerso continuamente nos espaços de Lebesgue $L^p(\Omega)$ para todo $1 \leq p \leq 2^* \doteq 2N/(N-2)$. Equivalentemente,

$$\sup_{\|u\|_D \leq 1} \int_\Omega |u|^p \mathrm{dx} < +\infty, \quad \text{para} \quad 1 \leq p \leq 2^*, \tag{1}$$

onde

$$\|u\|_D = \left(\int_\Omega |\nabla u|^2 \mathrm{dx} \right)^{1/2}$$

é a norma de Dirichlet ou usual do espaço $H_0^1(\Omega)$. Ademais, o supremo (1) é infinito para $p > 2^*$. Diante disso, depreendemos o seguinte: o expoente 2^* é a "linha de divisão" entre a finitude e a infinitude de (1), sendo portanto chamado de *expoente crítico*, e sua potência correspondente, $|u|^{2^*}$, é o tão conhecido *crescimento crítico de Sobolev*. Um fato curioso ocorre quando $N = 2$. Neste caso, temos formalmente que $2^* = \infty$, porém a imersão de $H_0^1(\Omega)$ em $L^\infty(\Omega)$ não é válida. Para justificar tal fato, basta considerarmos o clássico contra-exemplo em que $\Omega = B_1$ e $u(x) = \log(1 - \log |x|)$, que por uma integração direta verifica-se que $u \in H_0^1(B_1)$, mas no entanto $u \notin L^\infty(B_1)$. Diante deste fenômeno, surge a seguinte questão: qual a função $h(s) : \mathbb{R} \to \mathbb{R}^+$ de maior crescimento possível de tal sorte que

$$\sup_{\|u\|_D \leq 1} \int_\Omega h(u) \mathrm{dx} < +\infty?$$

Os primeiros trabalhos no sentido de responderem a essa questão são devidos a S.I. Pohozhaev [49], J. Moser [46] e N.S. Trudinger [64], que mostraram que tal crescimento maximal é do tipo exponencial. Mais precisamente, usando um argumento de simetrização, Moser e Trudinger mostraram que se $\Omega \subset \mathbb{R}^2$ é um domínio limitado e suave, então existe $C > 0$ tal que

$$S_\alpha \doteq \sup_{\|u\|_D \leq 1} \int_\Omega e^{\alpha u^2} \mathrm{dx} \leq C|\Omega|, \quad \text{para} \quad \alpha \leq 4\pi. \tag{2}$$

Ademais, em analogia com o caso Sobolev, eles mostraram que a constante 4π é a "linha de divisão" entre a finitude e a infinitude de S_α e sua potência correspondente, $e^{4\pi u^2}$, é o crescimento maximal ou crítico, porém agora no intitulado *sentido de Trudinger-Moser*. Em símbolos,

$$S_\alpha \leq C|\Omega|, \qquad \text{para} \quad \alpha \leq 4\pi,$$

$$S_\alpha = +\infty, \qquad \text{para} \quad \alpha > 4\pi.$$

Nos últimos anos, generalizações das mais diversas da desigualdade (2) tem surgido motivadas principalmente pelo estudo de problemas elípticos envolvendo não-linearidades comportando-se no infinito como a função exponencial. Para citar alguns trabalhos, destacamos [1, 3, 4, 5, 21, 24, 32, 34, 35, 44]. Enunciados precisos das generalizações tratadas em alguns desses trabalhos, bem como suas respectivas aplicações, serão apresentados no primeiro capítulo da tese. Seguindo a mesma linha desses artigos, o presente trabalho de tese se propõe a estabelecer uma desigualdade do tipo obtida por Trudinger e Moser em (2) em todo o espaço Euclidiano $\mathbb{R}^2$ e aplicá-la ao estudo de problemas elípticos não-lineares no que concerne a obtenção de resultados de existência, não-existência, multiplicidade e comportamento assintótico de soluções para tais problemas.

Este trabalho está organizado em cinco capítulos.

No **Capítulo 1** estabeleceremos alguns resultados de imersão envolvendo espaços de Sobolev com peso, bem como uma desigualdade do tipo Trudinger-Moser em tais espaços que será uma das principais ferramentas nas aplicações que se seguem nos demais capítulos da tese. Para isso, algumas definições iniciais se fazem necessárias. Sejam $V, Q : \mathbb{R}^2 \to \mathbb{R}$ funções contínuas e $1 \leq p < \infty$. O conjunto

$$L^p(\mathbb{R}^2; Q) \doteq \left\{ u : \mathbb{R}^2 \to \mathbb{R} : u \text{ é mensurável e } \int_{\mathbb{R}^2} Q(x)|u|^p < \infty \right\}$$

denota o espaço de Lebesgue com peso Q. Similarmente, definimos $L^2(\mathbb{R}^2; V)$. Definimos o espaço vetorial

$$H^1_{\mathrm{rad}}(\mathbb{R}^2; V) \doteq D^{1,2}_{\mathrm{rad}}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2; V),$$

o qual mostraremos ser um espaço de Hilbert quando munido do produto interno

$$\langle u, v\rangle_{H^1_{\rm rad}(\mathbb{R}^2;V)} \doteq \int_{\mathbb{R}^2} (\nabla u \nabla v + V(|x|)uv)\,, \quad u, v \in H^1_{\rm rad}(\mathbb{R}^2;V). \tag{3}$$

Associado ao produto interno (3) temos a norma

$$\|u\|_{H^1_{\rm rad}(\mathbb{R}^2;V)} \doteq \left(\int_{\mathbb{R}^2} |\nabla u|^2 + V(|x|)|u|^2\right)^{1/2}, \quad u \in H^1_{\rm rad}(\mathbb{R}^2;V), \tag{4}$$

e, em todo o trabalho, $H^1_{\rm rad}(\mathbb{R}^2;V)$ será denotado por E e sua norma (4) por $\|\cdot\|$. Voltando às funções V e Q, iremos supor que as mesmas são radialmente simétricas e satisfazendo as seguintes hipóteses na origem e no infinito:

(V) $V \in C(0,\infty)$, $V(r) > 0$ e existe $a > -2$ tal que

$$\liminf_{r\to+\infty} \frac{V(r)}{r^a} > 0.$$

(Q) $Q \in C(0,\infty)$, $Q(r) > 0$ e existem $b < (a-2)/2$ e $b_0 > -2$ tais que

$$\limsup_{r\to 0} \frac{Q(r)}{r^{b_0}} < \infty \quad \text{e} \quad \limsup_{r\to+\infty} \frac{Q(r)}{r^b} < \infty.$$

Os principais resultados deste capítulo são:

Lema 1 *Suponhamos que* $(V)-(Q)$ *valem. Então as imersões* $E \hookrightarrow L^p(\mathbb{R}^2;Q)$ *são compactas para todo* $2 \le p < \infty$.

Teorema 1 *Suponhamos que* $(V)-(Q)$ *valem. Então, para quaisquer* $u \in E$ *e* $\alpha > 0$, *temos que* $(e^{\alpha u^2}-1) \in L^1(\mathbb{R}^2;Q)$. *Ademais, se* $\alpha < \alpha' \doteq \min\{4\pi, 4\pi(1+b_0/2)\}$, *então existe* $C = C(\alpha, a, b, b_0) > 0$ *tal que*

$$\sup_{u\in E;\ \|u\|\le 1} \int_{\mathbb{R}^2} Q(|x|)(e^{\alpha u^2}-1) \le C. \tag{5}$$

Em seguida, motivados pelos trabalhos [34, 35, 44], obteremos o seguinte refinamento da desigualdade (5):

Corolário 1 *Suponhamos que* $(V)-(Q)$ *valem. Seja* (v_n) *uma sequência em* E *com* $\|v_n\| = 1$ *e suponhamos que* $v_n \rightharpoonup v$ *fracamente em* E *com* $\|v\| < 1$. *Então, para cada* $0 < \beta < \alpha'(1-\|v\|^2)^{-1}$, *a menos de subsequência, vale*

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^2} Q(|x|)(e^{\beta v_n^2}-1) < \infty.$$

Por fim, no intuito de explorarmos um pouco mais a desigualdade (5) no que tange aos estudos da optimalidade da constante α' e da existência de função extremal, necessitaremos das seguintes condições adicionais sobre $V(|x|)$ e $Q(|x|)$ na origem:

$(\widetilde{V})$ existe $a_0 > -2$ tal que $\displaystyle\limsup_{r\to 0} \frac{V(r)}{r^{a_0}} < \infty$;

$(\widetilde{Q})$ $Q \in C(0,\infty)$, $Q(r) > 0$ e existem $b < (a-2)/2$ e $-2 < b_0 \leq 0$ tais que

$$0 < \liminf_{r\to 0} \frac{Q(r)}{r^{b_0}} \leq \limsup_{r\to 0} \frac{Q(r)}{r^{b_0}} < \infty \quad \text{e} \quad \limsup_{r\to+\infty} \frac{Q(r)}{r^{b}} < \infty.$$

Desta forma, o último resultado deste capítulo pode ser sumarizado como segue:

Teorema 2 *Suponhamos que* $(V), (\widetilde{V})$ *e* $(\widetilde{Q})$ *valem. Então,*

$$S_\alpha = \sup_{u\in E;\ \|u\|\leq 1} \int_{\mathbb{R}^2} Q(|x|)(e^{\alpha u^2} - 1) < +\infty \tag{6}$$

se, e somente se, $0 < \alpha \leq \alpha'$. *Ademais, o supremo* (6) *é atingido desde que* $0 < \alpha < \alpha'$.

As demonstrações dos Teoremas 1 e 2 seguem basicamente os mesmos argumentos desenvolvidos por [21, 55] e contam com a ajuda da desigualdade clássica de Trudinger-Moser (2) e de uma versão singular da mesma devida a Adimurthi-Sandeep [4].

No **Capítulo 2** estudaremos a existência, comportamento assintótico e multiplicidade de soluções fracas, bem como a não-existência de solução clássica para uma classe de problemas elípticos não-lineares da forma

$$-\Delta u + V(x)u = Q(x)f(u) \quad \text{em} \quad \mathbb{R}^2, \tag{7}$$

onde os pesos $V, Q : \mathbb{R}^2 \to \mathbb{R}$ são funções radiais satisfazendo as condições (V) e (Q) do **Capítulo 1** e a não-linearidade $f(s)$ tem crescimento crítico do tipo Trudinger-Moser (ou do tipo exponencial) a ser definido como segue: dizemos que $f(s)$ tem *crescimento crítico do tipo exponencial* se existe $\alpha_0 > 0$ tal que

$$(f_{\alpha_0}) \quad \lim_{|s|\to+\infty} \frac{|f(s)|}{e^{\alpha s^2}} = \begin{cases} 0, & \forall \alpha > \alpha_0, \\ +\infty, & \forall \alpha < \alpha_0. \end{cases}$$

Assumiremos também que $f(s)$ é contínua e satisfaz:

(f_1) $f(s) = o(s)$ quando $s \to 0$;

(f_2) existe $\theta > 2$ tal que

$$0 < \theta F(s) \doteq \theta \int_0^s f(t)\mathrm{dt} \leq sf(s), \quad \forall s \neq 0;$$

(f_3) existem constantes $R_0, M_0 > 0$ tais que

$$0 < F(s) \leq M_0|f(s)|, \quad \forall |s| \geq R_0;$$

(f_4) existem $\nu > 2$ e $\mu > 0$ tais que

$$F(s) \geq \frac{\mu}{\nu}|s|^\nu, \quad \forall s \in \mathbb{R}.$$

Observação 1 *O estudo do problema* (7) *é motivado por trabalhos recentes focados na busca de soluções do tipo ondas estacionárias para EQUAÇÕES DE SCHRÖDINGER NÃO-LINEARES da forma*

$$i\frac{\partial \psi}{\partial t} = -\Delta\psi + W(x)\psi - Q(x)\xi(|\psi|)\psi, \quad (x,t) \in \mathbb{R}^2 \times \mathbb{R},$$

ou EQUAÇÕES DE KLEIN-GORDON NÃO-LINEARES do tipo

$$i\frac{\partial^2 \psi}{\partial t^2} = \Delta\psi + (W(x) - m^2)\psi - Q(x)\xi(|\psi|)\psi, \quad (x,t) \in \mathbb{R}^2 \times \mathbb{R},$$

i.e., soluções da forma $\psi(x,t) = \exp(-i\mathcal{E}t)u(x)$, *onde* $\mathcal{E} \in \mathbb{R}$, $i = \sqrt{-1}$, m *é um número positivo,* $W(x), Q(x)$ *são potenciais de valor real e* $\xi : \mathbb{R}^+ \to \mathbb{R}$ *é um termo não-linear. Tais equações surgem em vários ramos da Física-Matemática e Biologia-Matemática e tem sido extensivamete estudadas nos últimos anos. Entre outros trabalhos, citamos por exemplo [11, 15, 16, 34, 54, 59, 60, 61, 62] e suas referências.*

Os principais resultados deste capítulo são enunciados a seguir:

Teorema 3 *Suponhamos que* $(V) - (Q)$ *valem. Se* f *satisfaz* $(f_{\alpha_0}) - (f_4)$, *com*

$$\mu > \left[\frac{\alpha_0(\nu-2)}{\alpha'\nu}\right]^{(\nu-2)/2} S_\nu^{\nu/2},$$

onde S_ν *denota a melhor constante da imersão* $E \hookrightarrow L^\nu(\mathbb{R}^2; Q)$ *estabelecida no Lema 1, então o problema* (7) *possui uma solução fraca positiva* u *em* E. *Ademais, se vale a hipótese adicional* $(\widetilde{V})$, *então existem constantes* $c_0, c_1 > 0$ *tais que*

$$u(x) \leq c_0 \exp\left(-c_1|x|^{(a+2)/4}\right), \quad \forall x \in \mathbb{R}^2.$$

Nosso resultado de multiplicidade é referente ao problema

$$-\Delta u + V(x)u = \lambda Q(x)f(u) \quad \text{em} \quad \mathbb{R}^2, \tag{8}$$

onde λ é um parâmetro positivo, e está enunciado como segue:

Teorema 4 *Suponhamos que $(V)-(Q)$ valem. Se f é ímpar e satisfaz $(f_{\alpha_0})-(f_4)$, então existe uma sequência crescente $(\lambda_k)\subset\mathbb{R}_+$ com $\lambda_k\to\infty$ quando $k\to\infty$ tal que, para $\lambda>\lambda_k$, o problema (8) possui pelo menos k pares de soluções fracas em E.*

As principais ferramentas utilizadas para se demonstrar os Teoremas 3 e 4 são a desigualdade do tipo Trudinger-Moser estabelecida no Teorema 1 (bem como seu refinamento; Corolário 1) e o Teorema do Passo da Montanha em suas versões clássica sem a condição de Palais-Smale [54] e simétrica [28].

Com intuito de obtermos um resultado de não-existência de soluções para o problema (8), assumiremos a seguinte hipótese simultânea sobre V e Q:

$$(VQ) \quad \lim_{|x|\to+\infty}\frac{V(x)}{|x|^a}<\infty \quad \text{e} \quad \lim_{|x|\to+\infty}\frac{Q(x)}{|x|^b}>0, \quad \text{com} \quad a<-2<b.$$

Ademais, como consequência das hipóteses (f_{α_0}) com $\alpha<\alpha_0$, (f_2) e (f_4), segue-se que existe $C_0>0$ tal que, para qualquer $p\geq\nu-1$,

$$f(s)\geq C_0 s^p, \quad \text{para todo} \quad s\geq 0. \tag{9}$$

Com isso, obteremos o seguinte resultado de não-existência para o problema (8):

Teorema 5 *Suponhamos que (VQ) vale. Se f satisfaz (9), então o problema (8) não possui solução positiva de classe C^2 para λ grande.*

Observação 2 *Observamos que na hipótese (VQ) não houve a necessidade de supor que V e Q fossem radialmente simétricas. Isso ficará claro na demonstração do Teorema 5.*

A demonstração do Teorema 5 está baseada na análise de uma inequação diferencial ordinária advinda do problema (8) via propriedades envolvendo a média esférica.

No **Capítulo 3** estudaremos a existência e multiplicidade de soluções fracas para a seguinte classe de problemas elípticos não-lineares e não-homogêneos da forma

$$-\Delta u + V(x)u = Q(x)f(u) + h(x) \quad \text{em} \quad \mathbb{R}^2, \tag{10}$$

onde o potencial $V(|x|)$ satisfaz a hipótese (V) do **Capítulo 1**, a função peso $Q(|x|)$ satisfaz a hipótese (Q) num primeiro momento, $f(s)$ ainda apresenta crescimento crítico do tipo Trudinger-Moser e $h \in E' = E^{-1}$ é uma pequena pertubação não identicamente nula.

Observação 3 *Nesse caso, o estudo do problema* (10) *é motivado pela busca de soluções do tipo ondas estacionárias da seguinte classe de equações de Schrödinger não-lineares:*

$$i\frac{\partial \psi}{\partial t} = -\Delta\psi - Q(x)\xi(|\psi|)\psi - e^{i\mathcal{E}t}h(x), \quad (x,t) \in \mathbb{R}^2 \times \mathbb{R},$$

ou equações de Klein-Gordon não-lineares do tipo

$$i\frac{\partial^2 \psi}{\partial t^2} = \Delta\psi + (W(x) - m^2)\psi - Q(x)\xi(|\psi|)\psi - e^{i\mathcal{E}t}h(x), \quad (x,t) \in \mathbb{R}^2 \times \mathbb{R}.$$

Os principais resultados deste capítulo são enunciados como seguem:

Teorema 6 *Suponhamos que* $(V)-(Q)$ *valem. Se* f *satisfaz* $(f_{\alpha_0})-(f_2)$*, então existe* $\delta_1 > 0$ *tal que se* $0 < \|h\|_{E^{-1}} < \delta_1$*, o problema* (10) *possui uma solução fraca* $u_h \not\equiv 0$ *em* E.

No intuito de estabelecermos um resultado de multiplicidade e observando que a hipótese $(\widetilde{V})$ implica que existem $r_0 > 0$ e $C_0 > 0$ tais que

$$V(|x|) \leq C_0|x|^{a_0} \quad \text{para todo} \quad 0 < |x| \leq r_0, \tag{11}$$

necessitaremos das seguintes condições adicionais sobre $f(s)$:

(f_3) existem constantes $R_0, M_0 > 0$ tais que

$$0 < F(s) \leq M_0|f(s)|, \quad \forall |s| \geq R_0;$$

(f_4) existe $\beta_0 > 0$ tal que

$$\liminf_{|s|\to\infty} \frac{sf(s)}{e^{\alpha_0 s^2}} \geq \beta_0 > \begin{cases} \dfrac{4}{C_0\alpha_0}\dfrac{e^{2m(r_0)}}{r_0^2}, & \text{se} \quad b_0 = 0; \\ \dfrac{b_0+2}{C_0\alpha_0}\dfrac{1}{r_0^{b_0+2}}, & \text{se} \quad -2 < b_0 < 0, \end{cases}$$

onde

$$m(r) = \frac{2C_0 r^{a_0+2}}{(a_0+2)^3},$$

com $0 < r \leq r_0$ e r_0 dado em (11).

Desta forma, o resultado de multiplicidade pode ser enunciado como segue:

Teorema 7 *Suponhamos que* $(V)-(\widetilde{Q})$ *e* $(\widetilde{V})$ *valem. Se* f *satisfaz* $(f_{\alpha_0})-(f_4)$*, então existe* $\delta_2>0$ *tal que se* $0<\|h\|_{E^{-1}}<\delta_2$*, o problema* (10) *possui pelo menos duas soluções fracas não-triviais em* E*.*

As demonstrações dos Teoremas 6 e 7, assim como todo o capítulo, seguem as mesmas ideias utilizadas no recente trabalho de Furtado-Medeiros-Severo [38], valendo-se da desigualdade do tipo Trudinger-Moser estabelecida no Teorema 1 e seu refinamento em conjunto com o Teorema do Passo da Montanha [12] e o Princípio Variacional de Ekeland [36, 68].

Nos capítulos subsequentes, nosso objeto de estudo serão sistemas do tipo variacional, ou seja, sistemas de equações de Euler-Lagrange de algum funcional. No **Capítulo 4** estudaremos a existência e multiplicidade de soluções fracas para a seguinte classe de sistemas elípticos do tipo gradiente (ou Lagrangeano)

$$\begin{cases} -\Delta u+V(x)u=Q(x)f(u,v) \quad \text{em} \quad \mathbb{R}^2, \\ -\Delta v+V(x)v=Q(x)g(u,v) \quad \text{em} \quad \mathbb{R}^2, \end{cases} \tag{12}$$

onde os pesos $V,Q:\mathbb{R}^2\to\mathbb{R}$ são funções radialmente simétricas satisfazendo as condições (V) e (Q) do **Capítulo 1** e consideraremos a situação variacional que caracteriza o sistema (12) como sendo do tipo gradiente, ou seja, iremos supor que

$$(f(u,v),g(u,v))=\nabla F(u,v),$$

para alguma função $F:\mathbb{R}^2\to\mathbb{R}$ de classe C^1. Denotando $w=(u,v)\in\mathbb{R}^2$ e visando uma analogia com o caso escalar, podemos reescrever o sistema (12) na forma matricial como segue

$$-\Delta w+V(x)w=Q(x)\nabla F(w) \quad \text{em} \quad \mathbb{R}^2,$$

onde $\Delta=(\Delta,\Delta)$ e $Q(x)\nabla F(w)=(Q(x)f(w),Q(x)g(w))$. Consideraremos novamente o caso em que as não-linearidades f e g apresentam crescimento crítico do tipo exponencial no sentido da desigualdade de Trudinger-Moser. Mais precisamente:

(F_{α_0}) existe $\alpha_0>0$ tal que

$$\lim_{|w|\to+\infty}\frac{|f(w)|}{e^{\alpha|w|^2}}=\lim_{|w|\to+\infty}\frac{|g(w)|}{e^{\alpha|w|^2}}=\begin{cases} 0, & \forall\alpha>\alpha_0, \\ +\infty, & \forall\alpha<\alpha_0. \end{cases}$$

Além disso, assumiremos as seguintes condições:

(F_1) $f(w) = o(|w|)$ e $g(w) = o(|w|)$ quando $|w| \to 0$;

(F_2) existe $\theta > 2$ tal que

$$0 < \theta F(w) \leq w \cdot \nabla F(w), \quad \forall w \in \mathbb{R}^2 \backslash \{0\};$$

(F_3) existem constantes $R_0, M_0 > 0$ tais que

$$0 < F(w) \leq M_0 |\nabla F(w)|, \quad \forall |w| \geq R_0;$$

(F_4) existem $\nu > 2$ e $\mu > 0$ tais que

$$F(w) \geq \frac{\mu}{\nu}|w|^\nu, \quad \forall w \in \mathbb{R}^2.$$

Os principais resultados deste capítulo são enunciados a seguir:

Teorema 8 *Suponhamos que* $(V) - (Q)$ *valem. Se* $(F_{\alpha_0}) - (F_4)$ *são satisfeitas, então o sistema* (12) *possui uma solução fraca não-trivial* w_0 *em* $E \times E$ *desde que*

$$\mu > \left[\frac{2\alpha_0(\nu - 2)}{\alpha' \nu}\right]^{(\nu-2)/2} S_\nu^{\nu/2}.$$

Nosso resultado de multiplicidade é referente ao problema

$$-\Delta w + V(x)w = \lambda Q(x) \nabla F(w) \quad \text{em} \quad \mathbb{R}^2, \tag{13}$$

onde λ é um parâmetro positivo, e está enunciado como segue:

Teorema 9 *Suponhamos que* $(V) - (Q)$ *valem. Se* F *é ímpar e* $(F_{\alpha_0}) - (F_4)$ *são satisfeitas, então para qualquer* $k \in \mathbb{N}$ *dado existe* $\Lambda_k > 0$ *tal que o sistema* (13) *possui pelo menos* $2k$ *pares de soluções fracas não-triviais em* $E \times E$ *desde que* $\lambda > \Lambda_k$.

As demonstrações dos Teoremas 8 e 9 seguem as mesmas ideias de seus análogos escalares no **Capítulo 2**, com uma pequena ressalva que a ferramenta principal para se provar o Teorema 9, a saber, o Teorema do Passo da Montanha Simétrico, foi utilizada em uma forma mais geral, a qual pode ser encontrada em [12, 13, 57].

Finalmente, no **Capítulo 5** estudaremos a existência de solução fraca para a seguinte classe de sistemas elípticos do tipo Hamiltoniano:

$$\begin{cases} -\Delta u + V(x)u = Q(x)g(v) & \text{em} \quad \mathbb{R}^2, \\ -\Delta v + V(x)v = Q(x)f(u) & \text{em} \quad \mathbb{R}^2, \end{cases} \tag{14}$$

onde as funções peso $V, Q : \mathbb{R}^2 \to \mathbb{R}$ são radialmente simétricas, satisfazem as mesmas hipóteses do **Capítulo 1** e as não-linearidades f e g ainda apresentam crescimento crítico do tipo exponencial, porém não necessariamente com a mesma constante α_0 para ambas, ou seja, existem $\alpha_0 \geq \beta_0 > 0$ tais que

$$\lim_{s\to+\infty} \frac{f(s)}{e^{\alpha s^2}} = \begin{cases} 0, & \forall \alpha > \alpha_0, \\ +\infty, & \forall \alpha < \alpha_0 \end{cases} \quad \text{e} \quad \lim_{s\to+\infty} \frac{g(s)}{e^{\alpha s^2}} = \begin{cases} 0, & \forall \alpha > \beta_0, \\ +\infty, & \forall \alpha < \beta_0. \end{cases} \tag{15}$$

Além disso, assumiremos que $f, g : \mathbb{R} \to [0, +\infty)$ são funções contínuas satisfazendo:

(H_1) $f(s) = o(s)$ e $g(s) = o(s)$ quando $s \to 0$;

(H_2) existe $\theta > 2$ tal que para todo $s > 0$

$$0 < \theta F(s) \doteq \theta \int_0^s f(t)\mathrm{dt} \leq sf(s) \quad \text{e} \quad 0 < \theta G(s) \doteq \theta \int_0^s g(t)\mathrm{dt} \leq sg(s);$$

(H_3) existem constantes $s_0, M_0 > 0$ tais que para todo $s \geq s_0$

$$0 < F(s) \leq M_0 f(s) \quad \text{e} \quad 0 < G(s) \leq M_0 g(s);$$

(H_4) existem constantes $\nu > 2$ e $\mu > 0$ tais que

$$F(s), G(s) \geq \frac{\mu}{\nu} s^\nu, \quad \forall s \geq 0.$$

Observação 4 *(i) Sistemas do tipo Hamiltoniano possuem inúmeras aplicações em ciências e, em especial, na Biologia. Por exemplo, a QUIMIOTAXIA, movimento dirigido que desenvolvem alguns seres vivos em resposta aos gradientes químicos presentes no seu ambiente, foi estudada por Keller-Segel [41] na década de 70 usando um sistema de equações parabólicas cujos estados estacionários devem satisfazer, sob certas hipóteses, a um sistema do tipo Hamiltoniano. Mais tarde, Gierer-Meinhardt [39] estudaram o processo de Ativação-Inibição de dois componentes químicos como um modelo de formação de padrão e também recaíram num sistema do tipo Hamiltoniano quando vistos em seus estados estacionários. Para maiores detalhes sobre estes e outros fenômenos naturais em que suas modelagens se dão por meio de sistemas do tipo Hamiltoniano, indicamos os livros de Murray [47, 48].*

(ii) Torna-se natural pensarmos em considerar o funcional

$$I(u, v) = \int_{\mathbb{R}^2} (\nabla u \nabla v + V(x)uv) - \int_{\mathbb{R}^2} Q(x)[F(u) + G(v)],$$

de modo que, formalmente, (14) *é o sistema de equações de Euler-Lagrange associado ao funcional I. Nessa direção, a primeira grande dificuldade que surge no estudo do sistema* (14) *e, em geral, no estudo de sistemas do tipo Hamiltoniano, é que o mesmo tem a característica de ser FORTEMENTE INDEFINIDO, ou seja, se o espaço onde o funcional I estiver definido for decomposto em soma direta de dois subespaços de dimensão infinita, então sua parte quadrática,*

$$\int_{\mathbb{R}^2} (\nabla u \nabla v + V(x)uv),$$

será coerciva num deles e anti-coerciva no outro. Para maiores detalhes, recomendamos [20]. Outras dificuldades que naturalmente surgem no estudo dos problemas (7)*,* (10) *e dos sistemas* (12)*,* (14) *são as já esperadas, a saber, uma possível perda de compacidade por estarmos trabalhando em domínio ilimitado e o crescimento crítico das não-linearidades envolvidas.*

O principal resultado deste capítulo é o seguinte:

Teorema 10 *Suponhamos que* $(V)-(Q)$ *valem. Se f e g satisfazem* (15) *e* $(H_1)-(H_4)$*, com*

$$\mu > \left[\frac{(\alpha_0+\beta_0)(\nu-2)}{\nu\alpha'}\right]^{(\nu-2)/2} 2^{\nu/2} S_\nu^\nu,$$

então o sistema (14) *possui uma solução fraca não-trivial em $E \times E$.*

Visto que o funcional associado ao sistema (14) é fortemente indefinido, não podemos utilizar as versões clássicas dos teoremas do Passo da Montanha e do Ponto de Sela. Desta forma, a abordagem ao sistema (14) se dará por meio de um procedimento de aproximação devido a Galerkin [37], seguindo as mesmas ideias utilizadas por de Figueiredo-Felmer [26], de Figueiredo-Miyagaki-Ruf [27] e de Figueiredo-do Ó-Ruf [25]. Por fim, destacamos que uma nova dificuldade surgirá no nosso problema em resposta à escolha de tal método de aproximação, a saber, a impossibilidade de usarmos um *teorema de interseção* (vide Proposição 5.9 em [54]).

Com o intuito de não ficarmos recorrendo à **Introdução** e de tornar os capítulos independentes, enunciaremos novamente, em cada capítulo, os resultados acima, bem como as hipóteses sobre as funções em geral com mais detalhes.

Chapter 1

A Trudinger-Moser type inequality in weighted Sobolev spaces

This chapter is devoted to establish some embedding results and a Trudinger-Moser type inequality in weighted Sobolev spaces. We point out that part of this chapter is contained in the published paper [7].

1.1 Introduction and main results

We recall that if Ω is a bounded domain in $\mathbb{R}^2$, the classical Trudinger-Moser inequality (cf. [46, 64]) asserts that $e^{\alpha u^2} \in L^1(\Omega)$ for all $u \in H_0^1(\Omega)$ and $\alpha > 0$. Moreover, there exists a constant $C = C(\Omega) > 0$ such that

$$\sup_{\|u\|_{H_0^1(\Omega)} \leq 1} \int_\Omega e^{\alpha u^2} \mathrm{dx} \leq C, \quad \text{if} \quad \alpha \leq 4\pi, \tag{1.1}$$

where

$$\|u\|_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla u|^2 \mathrm{dx} \right)^{1/2}.$$

Furthermore, (1.1) is sharp in the sense that, if $\alpha > 4\pi$, then the supremum in (1.1) is $+\infty$. Related inequalities for unbounded domains have been proposed by Cao [21] and Ruf [55] (and by Tanaka [1], do Ó [32] and Li-Ruf [43] in general dimension). However in [1], [21] and [32] they assumed the growth $e^{\alpha u^2}$ with $\alpha < 4\pi$, i.e. with subcritical growth (see also Adams [2]). In [55], the author proved that there exists a constant

$d > 0$ such that for any domain $\Omega \subset \mathbb{R}^2$,

$$\sup_{\|u\|_{1,2}\leq 1} \int_\Omega (e^{4\pi u^2} - 1)\mathrm{dx} \leq d, \tag{1.2}$$

where

$$\|u\|_{1,2} = \left(\int_\Omega (|\nabla u|^2 + |u|^2)\mathrm{dx} \right)^{1/2}.$$

Moreover, the inequality (1.2) is sharp in the sense that for any growth $e^{\alpha u^2}$ with $\alpha > 4\pi$ the supremum in (1.2) is $+\infty$. Furthermore, he proved that the supremum in (1.2) is attained whenever it is finite. On the other hand, Adimurthi-Sandeep [4] extended the Trudinger-Moser inequality (1.1) for singular weights. More precisely, they proved that if Ω is a bounded domain in $\mathbb{R}^2$ containing the origin, $u \in H_0^1(\Omega)$ and $\beta \in [0, 2)$, then there exists a positive constant $C = C(\alpha, \beta)$ such that

$$\sup_{\|u\|_{H_0^1(\Omega)}\leq 1} \int_\Omega \frac{e^{\alpha u^2} - 1}{|x|^\beta}\mathrm{dx} \leq C|\Omega| \tag{1.3}$$

if, and only if, $0 < \alpha \leq 4\pi(1 - \beta/2)$. We remark that in the same work, the authors used the inequality (1.3) in order to study the corresponding critical exponent problem.

Throughout, we consider weight functions $V(|x|)$ and $Q(|x|)$ satisfying the following assumptions:

(V) $V \in C(0, \infty)$, $V(r) > 0$ and there exists $a > -2$ such that

$$\liminf_{r\to+\infty} \frac{V(r)}{r^a} > 0.$$

(Q) $Q \in C(0, \infty)$, $Q(r) > 0$ and there exist $b < (a - 2)/2$ and $b_0 > -2$ such that

$$\limsup_{r\to 0} \frac{Q(r)}{r^{b_0}} < \infty \quad \text{and} \quad \limsup_{r\to+\infty} \frac{Q(r)}{r^b} < \infty.$$

Example 1.1.1 1) *In [11], Ambrosetti, Felli and Malchiodi considered the potentials $V(|x|)$ and $Q(|x|)$ satisfying*

$$\frac{A_1}{1 + |x|^\alpha} \leq V(|x|) \leq A_2 \quad \text{and} \quad 0 < Q(|x|) \leq \frac{A_3}{1 + |x|^\beta}$$

for positive constants A_1, A_2, A_3, with $\alpha \in (0, 2)$ and $\beta \geq 0$, which verify (V) and (Q) for $\beta > 1$. Indeed, it just takes $a = -\alpha \in (-2, 0)$, $b_0 = 0$ and $b = -\beta$.

2) *Singular potentials of the form*

$$V(x) = |x|^\alpha \quad \text{and} \quad Q(x) = |x|^\beta$$

with $2(\beta + 1) < \alpha < 0$. Indeed, it just takes $a = \alpha$ and $b = b_0 = \beta$.

With the aid of inequalities (1.1), (1.3) and inspired by similar arguments developed in [21, 55], we establish in this work the following Trudinger-Moser type inequality in the functional space E.

Theorem 1.1.2 *Assume that $(V)-(Q)$ hold. Then, for any $u \in E$ and $\alpha > 0$, we have that $(e^{\alpha u^2}-1) \in L^1(\mathbb{R}^2; Q)$. Furthermore, if $\alpha < \alpha' \doteq \min\{4\pi, 4\pi(1+b_0/2)\}$, then there exists $C = C(\alpha, a, b, b_0) > 0$ such that*

$$\sup_{u\in E;\ \|u\|\le 1} \int_{\mathbb{R}^2} Q(|x|)(e^{\alpha u^2}-1) \le C. \tag{1.4}$$

Remark 1.1.3 *Since the weight $Q(|x|)$ can assume a singular behavior (see Example 1.1.1) we refer the reader to [35] where the authors investigated the Trudinger-Moser type inequality with a singular weight for any domain $\Omega \subset \mathbb{R}^2$ containing the origin as well as some applications. More precisely, they proved that if $\alpha > 0$, $\beta \in [0,2)$ is such that $\alpha/4\pi + \beta/2 < 1$ and $\|u\|_{L^2(\Omega)} \le M$, then there exists a constant $C = C(\alpha, M) > 0$ (independent of Ω) such that*

$$\sup_{\|\nabla u\|_{L^2(\Omega)}\le 1} \int_\Omega \frac{e^{\alpha u^2}-1}{|x|^\beta} dx \le C \tag{1.5}$$

and the above inequality does not holds if $\alpha/4\pi + \beta/2 > 1$. As in the paper [4], the authors also used the inequality (1.5) to study the corresponding subcritical and critical exponent nonhomogeneous problem. We also refer the reader to [5] for a Trudinger-Moser type inequality with a singular weight in high dimensions.

The inequality (1.1) was improved by Lions in [44, Theorem I.6 pp.196-199]. More precisely, he proved that if (u_n) is a sequence of functions in $H_0^1(\Omega)$ with $\|\nabla u_n\|_{L^2(\Omega)} = 1$ such that $u_n \rightharpoonup u \neq 0$ weakly in $H_0^1(\Omega)$, then for any $0 < p < 4\pi\left(1-\|\nabla u\|^2_{L^2(\Omega)}\right)^{-1}$ we have

$$\sup_{n\in\mathbb{N}} \int_\Omega e^{pu_n^2} \mathrm{dx} < \infty. \tag{1.6}$$

Recently, do Ó-Medeiros-Severo in [34, Lemma 2.6] established a version of the inequality (1.6) for the whole $\mathbb{R}^2$. They proved that if (u_n) is a sequence of functions in $H^1(\mathbb{R}^2)$ with $\|u_n\|_{1,2} = 1$ such that $u_n \rightharpoonup u \neq 0$ weakly in $H^1(\mathbb{R}^2)$ with $\|u\|_{1,2} < 1$, then for any $0 < p < 4\pi\left(1-\|u\|^2_{1,2}\right)^{-1}$ we have

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^2} \left(e^{pu_n^2}-1\right) \mathrm{dx} < \infty. \tag{1.7}$$

Three years later, do Ó-de Souza in [35, Lemma 2.1] established a singular version of the inequality (1.7). Under the same conditions above on the sequence (u_n), they proved that for all $0 < p < 2\pi(2-\beta)\left(1 - \|u\|_{1,2}^2\right)^{-1}$ and $\beta \in [0,2)$, we have

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^2} \frac{e^{pu_n^2}-1}{|x|^\beta} \mathrm{dx} < \infty.$$

With the purpose to control the Palais-Smale sequences in our applications we establish the following *improvement* of the Trudinger-Moser inequality considering our variational setting.

Corollary 1.1.4 *Assume that $(V)-(Q)$ hold. Let (v_n) be in E with $\|v_n\| = 1$ and suppose that $v_n \rightharpoonup v$ weakly in E with $\|v\| < 1$. Then, for each $0 < \beta < \alpha' \left(1 - \|v\|^2\right)^{-1}$, up to a subsequence, it holds*

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^2} Q(|x|)(e^{\beta v_n^2} - 1) < \infty.$$

1.2 Preliminary results

In this section, we establish some embeddings from E into the weighted Lebesgue space $L^p(\mathbb{R}^2; Q)$. We start by recalling a version of the Radial Lemma (see [62]) due to Strauss [58]. Before let us to check that $(E, \|\cdot\|)$ is a Banach space.

Proposition 1.2.1 *E is a Banach space with respect to the norm given in* (4).

Proof. First, it is standard to check that $(E, \|\cdot\|)$ is a linear space. Let (u_n) be a Cauchy sequence in E. Since the embedding $E \hookrightarrow L^2(\mathbb{R}^2; V)$ is continuous, (u_n) is a Cauchy sequence in $L^2(\mathbb{R}^2; V)$. Hence, there exists $u \in L^2(\mathbb{R}^2; V)$ such that $u_n \to u$ in $L^2(\mathbb{R}^2; V)$ and so, up to a subsequence, $u_n(x) \to u(x)$ a.e. $x \in \mathbb{R}^2$. Analogously, since $E \hookrightarrow D^{1,2}_{\mathrm{rad}}(\mathbb{R}^2)$, (u_n) is a Cauchy sequence in $D^{1,2}_{\mathrm{rad}}(\mathbb{R}^2)$. Thus, there exists $v \in D^{1,2}_{\mathrm{rad}}(\mathbb{R}^2)$ such that $u_n \to v$ in $D^{1,2}_{\mathrm{rad}}(\mathbb{R}^2)$ and so, up to a subsequence, $u_n(x) \to v(x)$ a.e. $x \in \mathbb{R}^2$. Consequently, $u(x) = v(x)$ a.e. $x \in \mathbb{R}^2$. Therefore, $u_n \to u \in E$ in E. ∎

Lemma 1.2.2 *Assume that (V) holds. Then, there exists $C > 0$ such that for all $u \in E$,*

$$|u(x)| \le C\|u\||x|^{-(a+2)/4}, \quad |x| \gg 1. \tag{1.8}$$

Proof. By a standard density argument, it suffices to prove (1.8) for $u \in C^\infty_{0,\mathrm{rad}}(\mathbb{R}^2)$. Let $\rho = |x|$ and $\varphi : [0, +\infty) \to \mathbb{R}$ be such that $\varphi(\rho) = u(|x|)$. Since $a > -2$, one has

$$\frac{d}{d\rho}\left[\rho^{(a+2)/2}\varphi^2(\rho)\right] = \frac{a+2}{2}\rho^{a/2}\varphi^2(\rho) + 2\rho^{(a+2)/2}\varphi(\rho)\varphi'(\rho) \ge 2\rho^{(a+2)/2}\varphi(\rho)\varphi'(\rho).$$

It follows from (V) that there exist $R_0 > 0$ and $C_0 > 0$ such that

$$V(|x|) \geq C_0|x|^a \quad \text{for} \quad |x| \geq R_0.$$

Then for $\rho > R_0$, the Hölder's inequality implies that

$$\begin{aligned}
\rho^{(a+2)/2}\varphi^2(\rho) &\leq 2\int_\rho^\infty s^{(a+2)/2}|\varphi(s)||\varphi'(s)|\mathrm{ds} \\
&= 2\int_\rho^\infty \left(|\varphi'(s)|\sqrt{s}\right)\left(s^{a/2}|\varphi(s)|\sqrt{s}\right)\mathrm{ds} \\
&\leq 2\left(\int_\rho^\infty |\varphi'(s)|^2 s\mathrm{ds}\right)^{1/2}\left(\int_\rho^\infty s^a|\varphi(s)|^2 s\mathrm{ds}\right)^{1/2} \\
&\leq \frac{1}{\sqrt{C_0}\pi}\left(\int_{B_\rho^c}|\nabla u|^2\right)^{1/2}\left(\int_{B_\rho^c}V(|x|)|u|^2\right)^{1/2} \\
&\leq \frac{1}{\sqrt{C_0}\pi}\int_{\mathbb{R}^2}\left(|\nabla u|^2 + V(|x|)|u|^2\right).
\end{aligned}$$

Thus, we conclude that

$$|u(x)| \leq C\|u\||x|^{-(a+2)/4}, \quad \forall |x| > R_0,$$

which completes the proof. ■

Next, we recall some basic embeddings (see Su-Wang-Willem [62, Lemmas 3 and 4]). Let $A \subset \mathbb{R}^2$ and define

$$H^1_{\mathrm{rad}}(A;V) = \left\{u_{|_A} : u \in H^1_{\mathrm{rad}}(\mathbb{R}^2;V)\right\}.$$

Lemma 1.2.3 *Assume that* $(V)-(Q)$ *hold and let* $1 \leq p < \infty$. *For any* $0 < r < R < \infty$, *with* $R \geq 1$,

i) the embeddings $H^1_{rad}(B_R \setminus B_r;V) \hookrightarrow L^p(B_R \setminus B_r;Q)$ *are compacts;*

ii) the embedding $H^1_{rad}(B_R;V) \hookrightarrow H^1(B_R)$ *is continuous.*

In particular, as a consequence of *ii*) we have that $H^1_{\mathrm{rad}}(B_R;V)$ is compactly immersed in $L^q(B_R)$ for all $1 \leq q < \infty$. We also need the following Hardy type inequality with remainder terms (see Wang-Willem [67, Theorem 2]).

Lemma 1.2.4 *For all* $u \in H^1_0(B_1)$

$$\int_{B_1}|\nabla u|^2 \geq \frac{1}{4}\int_{B_1}\left[|x|^{-2}\left(\log\frac{1}{|x|}\right)^{-2}|u|^2\right].$$

From the previous lemmas we have:

Lemma 1.2.5 *Assume that* $(V)-(Q)$ *hold. Then the embeddings* $E \hookrightarrow L^p(\mathbb{R}^2;Q)$ *are compacts for all* $2 \le p < \infty$.

Proof. For the continuity of the embedding, it suffices to show that

$$S_p \doteq \inf_{u\in E} \frac{\int_{\mathbb{R}^2}(|\nabla u|^2 + V|u|^2)}{\left(\int_{\mathbb{R}^2} Q|u|^p\right)^{2/p}} > 0.$$

Otherwise, there exists (u_n) in E such that

$$\int_{\mathbb{R}^2} Q|u_n|^p = 1 \quad \text{and} \quad \lim_{n\to\infty}\int_{\mathbb{R}^2}\left(|\nabla u_n|^2 + V|u_n|^2\right) = 0. \tag{1.9}$$

By the hypotheses $(V)-(Q)$, there exist $R_0 > 0$ and $C_0 > 0$ such that

$$Q(|x|) \le C_0|x|^b \quad \text{for} \quad |x| \ge R_0,$$
$$V(|x|) \ge C_0|x|^a \quad \text{for} \quad |x| \ge R_0.$$

Now for $R > R_0$, by Lemma 1.2.2, we have

$$\begin{aligned}\int_{B_R^c} Q|u_n|^p &\le C_0\int_{B_R^c}|x|^b|u_n|^p \\ &= C_0\int_{B_R^c}|x|^{b-a}|u_n|^{p-2}|x|^a|u_n|^2 \\ &\le C_1\|u_n\|^{p-2}\int_{B_R^c}|x|^{b-a-(p-2)\frac{a+2}{4}}V|u_n|^2.\end{aligned}$$

Since $a > -2$, $b < (a-2)/2 < a$ and $p \ge 2$, we have that $b-a-(p-2)(a+2)/4 < 0$. Thus, we obtain

$$\begin{aligned}\int_{B_R^c} Q|u_n|^p &\le C_1 R^{b-a-(p-2)\frac{a+2}{4}}\|u_n\|^p \\ &= C_1 R^{b-a-(p-2)\frac{a+2}{4}} o_n(1).\end{aligned} \tag{1.10}$$

On the other hand, again by (Q) there exist $0 < r_0 < R_0$ and $C_0 > 0$ such that

$$Q(|x|) \le C_0|x|^{b_0} \quad \text{for} \quad 0 < |x| < r_0. \tag{1.11}$$

In what follows for $0 < r < \min\{r_0, 1/2\}$ and $p \ge 2$, we will estimate the integral

$$\int_{B_r} Q|u_n|^p.$$

We distinguish two cases.

Case 1. $b_0 > 0$. It follows from (1.11) and Lemma 1.2.3 that

$$\begin{aligned}\int_{B_r} Q|u_n|^p &\leq C_0 r^{b_0}\int_{B_r}|u_n|^p \leq C_0 r^{b_0}\int_{B_1}|u_n|^p\\ &\leq C_2 r^{b_0}\|u_n\|^{p/2}\\ &= C_2 r^{b_0} o_n(1).\end{aligned}$$

Case 2. $-2 < b_0 \leq 0$. We choose $\delta > 0$ such that $b_0 - \delta > -2$ and take a cut-off function $\phi \in C_0^\infty(B_1), 0 \leq \phi \leq 1$ in B_1 and $\phi \equiv 1$ in $B_{1/2}$. Invoking (1.11) and the Hölder's inequality we get

$$\begin{aligned}&\int_{B_r} Q|u_n|^p\\ &\leq C_0\int_{B_r}|x|^{b_0}|u_n|^p\\ &= C_0\int_{B_r}|x|^{b_0-\delta}\left(\log\frac{1}{|x|}\right)^{b_0-\delta}|u_n\phi|^{\delta-b_0}|x|^{\delta}\left(\log\frac{1}{|x|}\right)^{\delta-b_0}|u_n|^{p+b_0-\delta}\\ &\leq C_3 r^{\delta}\left(\log\frac{1}{r}\right)^{\delta-b_0}\left[\int_{B_1}|x|^{-2}\left(\log\frac{1}{|x|}\right)^{-2}|u_n\phi|^2\right]^{\frac{\delta-b_0}{2}}\left(\int_{B_1}|u_n|^{\frac{2(p+b_0-\delta)}{2+b_0-\delta}}\right)^{\frac{2+b_0-\delta}{2}}.\end{aligned}$$

Since $u_n\phi \in H_0^1(B_1)$, Lemmas 1.2.3 and 1.2.4 imply that

$$\begin{aligned}\int_{B_r} Q|u_n|^p &\leq C_4 r^{\delta}\left(\log\frac{1}{r}\right)^{\delta-b_0}\|u_n\|^{\delta-b_0}\|u_n\|^{p+b_0-\delta}\\ &= C_4 r^{\delta}\left(\log\frac{1}{r}\right)^{\delta-b_0} o_n(1).\end{aligned}$$

Hence, in any case

$$\lim_{n\to\infty}\int_{B_r} Q|u_n|^p = 0. \tag{1.12}$$

Now writing

$$\int_{\mathbb{R}^2} Q|u_n|^p = \int_{B_r} Q|u_n|^p + \int_{B_R\setminus B_r} Q|u_n|^p + \int_{B_R^c} Q|u_n|^p,$$

using (1.10), (1.12) and $i)$ from Lemma 1.2.3 we get

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} Q|u_n|^p = 0,$$

which is a contradiction with (1.9). This proves the continuity of the embedding. For the compactness, let (u_n) be a sequence in E be such that $\|u_n\| \leq C$. Without loss of generality, we may assume that $u_n \rightharpoonup 0$ weakly in E. We need to prove that $u_n \to 0$ strongly in $L^p(\mathbb{R}^2; Q)$ for all $2 \le p < \infty$. As in (1.10), we get

$$\int_{B_R^c} Q|u_n|^p \le CR^{b-a-(p-2)\frac{a+2}{4}}\|u_n\|^p \le CR^{b-a-(p-2)\frac{a+2}{4}}.$$

Since $b - a - (p-2)(a+2)/4 < 0$, given $\varepsilon > 0$, for $R > 0$ sufficiently large we have that

$$\int_{B_R^c} Q|u_n|^p \le CR^{b-a-(p-2)\frac{a+2}{4}} < \frac{\varepsilon}{3}. \tag{1.13}$$

On the other hand, if $b_0 > 0$ then as in **Case 1** we have

$$\int_{B_r} Q|u_n|^p \le Cr^{b_0}\|u_n\|^{p/2} \le Cr^{b_0}.$$

If $-2 < b_0 \le 0$, similarly to the **Case 2** we have for $0 < r < \min\{r_0, 1/2, \delta\}$ that

$$\int_{B_r} Q|u_n|^p \le Cr^{\delta}\left(\log\frac{1}{\delta}\right)^{\delta-b_0}\|u_n\|^p \le Cr^{\delta}\left(\log\frac{1}{\delta}\right)^{\delta-b_0}.$$

In any case,

$$\int_{B_r} Q|u_n|^p < \frac{\varepsilon}{3}, \tag{1.14}$$

for $r > 0$ small enough. Now, by $i)$ from Lemma 1.2.3, $u_n \to 0$ strongly in $L^p(B_R \backslash B_r; Q)$ for all $1 \le p < \infty$. Thus, for $n \in \mathbb{N}$ large enough

$$\int_{B_R \backslash B_r} Q|u_n|^p < \frac{\varepsilon}{3}. \tag{1.15}$$

From (1.13), (1.14) and (1.15), we get

$$\lim_{n\to\infty}\|u_n\|_{L^p(\mathbb{R}^2;Q)} = \lim_{n\to\infty}\int_{\mathbb{R}^2} Q|u_n|^p = 0,$$

and this finish the proof of lemma. ■

Remark 1.2.6 *Notice that lemmas 1.2.2 and 1.2.5 hold in fact for $a \ge -2$ and $b < a$.*

1.3 Proof of the main results

Now we are ready to present the proof of the main results of this chapter.

Proof of Theorem 1.1.2. Recall that by hypothesis (Q), there exist $0 < r_0 < R_0$ and $C_0 > 0$ such that

$$\begin{aligned} Q(|x|) &\le C_0|x|^b \quad &&\text{for} \quad |x| \ge R_0,\\ Q(|x|) &\le C_0|x|^{b_0} \quad &&\text{for} \quad 0 < |x| \le r_0. \end{aligned} \tag{1.16}$$

Let $R > 0$ to be chosen later. We write

$$\int_{\mathbb{R}^2} Q(e^{\alpha u^2} - 1) = \int_{B_R} Q(e^{\alpha u^2} - 1) + \int_{B_R^c} Q(e^{\alpha u^2} - 1). \tag{1.17}$$

We are going to estimate each integral in (1.17). For the integral on B_R, we have two cases to consider:

Case 1. $b_0 \geq 0$. From the second inequality in (1.16) and the continuity of $Q(r)$, there exists $C > 0$ such that

$$\int_{B_R} Q(e^{\alpha u^2} - 1) \leq C \int_{B_R} e^{\alpha u^2}.$$

Let $v \in H_0^1(B_R)$ defined by

$$v(x) = u(x) - u(R),$$

for $x \in B_R$. Then by the Young's inequality, for each $\varepsilon > 0$ given, there exists a constant $C_\varepsilon > 0$ such that

$$u^2(x) \leq (1+\varepsilon)v^2(x) + (1 + C_\varepsilon)u^2(R).$$

Thus, fixing

$$R \gg \max\left\{1, R_0, \left[(1+C_\varepsilon)C^2\right]^{2/(a+2)}\right\},$$

it follows from Lemma 1.2.2 that

$$\begin{aligned} u^2(x) &\leq (1+\varepsilon)v^2(x) + (1+C_\varepsilon)C^2 R^{-(a+2)/2}\|u\|^2 \\ &\leq (1+\varepsilon)v^2(x) + \|u\|^2. \end{aligned}$$

Hence,

$$\begin{aligned} \int_{B_R} Q(e^{\alpha u^2} - 1) &\leq C \int_{B_R} e^{\alpha[(1+\varepsilon)v^2 + \|u\|^2]} \\ &= C e^{\alpha\|u\|^2} \int_{B_R} e^{\alpha(1+\varepsilon)v^2}. \end{aligned} \tag{1.18}$$

Since $v \in H_0^1(B_R)$, $\|v\|_{H_0^1(B_R)} = \|\nabla v\|_{L^2(B_R)} \leq \|u\| \leq 1$ and in this case $\alpha < 4\pi$, we can take $\varepsilon > 0$ such that $\alpha(1+\varepsilon) \leq 4\pi$. Then, from the classical Trudinger-Moser inequality (1.1) we get

$$\sup_{v \in H_0^1(B_R);\ \|v\|_{H_0^1(B_R)} \leq 1} \int_{B_R} e^{\alpha(1+\varepsilon)v^2} \leq C.$$

Thus, from (1.18), we obtain

$$\sup_{u\in E;\ \|u\|\leq 1}\int_{B_R} Q(e^{\alpha u^2}-1)\leq C(\alpha).$$

Case 2. $-2<b_0<0$. Since $0<r_0<R_0<R$, we write

$$\begin{aligned}\int_{B_R} Q(e^{\alpha u^2}-1) &= \int_{B_{r_0}} Q(e^{\alpha u^2}-1)+\int_{B_R\setminus B_{r_0}} Q(e^{\alpha u^2}-1)\\ &\leq C_0\int_{B_{r_0}}|x|^{b_0}e^{\alpha u^2}+C\int_{B_R\setminus B_{r_0}} e^{\alpha u^2}\\ &\leq C_0\int_{B_R}|x|^{b_0}e^{\alpha u^2}+C\int_{B_R} e^{\alpha u^2},\end{aligned}$$

where we have used again the continuity of $Q(r)$ and the second inequality in (1.16). By similar computations done above, we obtain

$$\int_{B_R}|x|^{b_0}e^{\alpha u^2}=\int_{B_R}\frac{e^{\alpha u^2}}{|x|^{-b_0}}\leq e^{\alpha\|u\|^2}\int_{B_R}\frac{e^{\alpha(1+\varepsilon)v^2}}{|x|^{-b_0}}. \tag{1.19}$$

Since in this case $\alpha<4\pi(1+b_0/2)$, we can take $\varepsilon>0$ such that $\alpha(1+\varepsilon)\leq 4\pi(1+b_0/2)$. Thus, since $v\in H_0^1(B_R)$, $\|v\|_{H_0^1(B_R)}=\|\nabla v\|_{L^2(B_R)}\leq\|u\|\leq 1$ and $-b_0\in(0,2)$, thanks to inequality (1.3)

$$\sup_{v\in H_0^1(B_R);\ \|v\|_{H_0^1(B_R)}\leq 1}\int_{B_R}\frac{e^{\alpha(1+\varepsilon)v^2}}{|x|^{-b_0}}\leq C(\alpha,b_0).$$

Using this in (1.19) we obtain

$$\sup_{u\in E;\ \|u\|\leq 1}\int_{B_R}|x|^{b_0}e^{\alpha u^2}\leq C(\alpha,b_0).$$

Therefore, in both cases we have

$$\sup_{u\in E;\ \|u\|\leq 1}\int_{B_R} Q(e^{\alpha u^2}-1)\leq C(\alpha,b_0). \tag{1.20}$$

Next, we estimate the second integral in (1.17). It follows from the first inequality in (1.16) and Monotone Convergence Theorem that for any $u\in E$

$$\begin{aligned}\int_{B_R^c} Q(e^{\alpha u^2}-1)\leq C_0\int_{B_R^c}|x|^b(e^{\alpha u^2}-1) &= C_0\int_{B_R^c}|x|^b\sum_{j=1}^{\infty}\frac{\alpha^j u^{2j}}{j!}\\ &\leq C_0\sum_{j=1}^{\infty}\frac{\alpha^j}{j!}\int_{B_R^c}|x|^b u^{2j}.\end{aligned}$$

By using Lemma 1.2.2 we can estimate the last integral above as follows

$$\begin{aligned}\int_{B_R^c} |x|^b u^{2j} &\leq (C\|u\|)^{2j} \int_{B_R^c} |x|^{b-j\frac{a+2}{2}} \\ &= 2\pi \left(C\|u\|\right)^{2j} \int_R^\infty s^{1+b-j\frac{a+2}{2}} ds \\ &\leq \frac{2\pi}{\left(\frac{a-2}{2}-b\right) R^{\frac{a-2}{2}-b}} \left(C\|u\|\right)^{2j},\end{aligned}$$

where we have used that $b < (a-2)/2$, $j \geq 1$ and $R > 1$. Thus,

$$\begin{aligned}\int_{B_R^c} Q(e^{\alpha u^2}-1) &\leq \frac{2\pi C_0}{\left(\frac{a-2}{2}-b\right) R^{\frac{a-2}{2}-b}} \sum_{j=1}^{\infty} \frac{(\alpha C^2\|u\|^2)^j}{j!} \\ &= \frac{2\pi C_0}{\left(\frac{a-2}{2}-b\right) R^{\frac{a-2}{2}-b}} \left(e^{\alpha C^2\|u\|^2}-1\right) < \infty,\end{aligned}$$

for all $u \in E$. Furthermore,

$$\sup_{u\in E;\ \|u\|\leq 1} \int_{B_R^c} Q(e^{\alpha u^2}-1) \leq C(\alpha, a, b). \tag{1.21}$$

Therefore, from (1.20) and (1.21) we have that

$$\sup_{u\in E;\ \|u\|\leq 1} \int_{\mathbb{R}^2} Q(e^{\alpha u^2}-1) \leq C(\alpha, a, b, b_0)$$

and the proof of theorem is finished. ■

Using Theorem 1.1.2 and following the same steps as in the proof of [34, Lemma 2.6] we present the

Proof of Corollary 1.1.4. Recall that if y, z and ε are positive numbers, the Young's inequality implies that

$$y^2 = (y-z)^2 + z^2 + 2\varepsilon(y-z)\frac{z}{\varepsilon} \leq (1+\varepsilon^2)(y-z)^2 + \left(1+\frac{1}{\varepsilon^2}\right) z^2.$$

Hence, we can use the Young's inequality again to get

$$\begin{aligned}\int_{\mathbb{R}^2} Q(e^{\beta v_n^2}-1) &\leq \int_{\mathbb{R}^2} \left(Q^{1/r_1} e^{\beta(1+\varepsilon^2)(v_n-v)^2} \cdot Q^{1/r_2} e^{\beta(1+1/\varepsilon^2)v^2} - \frac{Q}{r_1} - \frac{Q}{r_2}\right) \\ &\leq \frac{1}{r_1} \int_{\mathbb{R}^2} Q\left(e^{r_1\beta(1+\varepsilon^2)(v_n-v)^2}-1\right) + \frac{1}{r_2} \int_{\mathbb{R}^2} Q\left(e^{r_2\beta(1+1/\varepsilon^2)v^2}-1\right),\end{aligned}$$

where $r_1, r_2 > 1$ and $1/r_1 + 1/r_2 = 1$. It follows from Theorem 1.1.2 that the last integral above is finite and therefore it suffices to prove that

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^2} Q\left(e^{r_1\beta(1+\varepsilon^2)(v_n-v)^2}-1\right) < \infty.$$

Since $v_n \rightharpoonup v$ weakly in E and $\|v_n\| = 1$, we conclude that

$$\lim_{n\to\infty} \|v_n - v\|^2 = 1 - \|v\|^2 < \frac{\alpha'}{\beta} \Rightarrow \lim_{n\to\infty} \beta\|v_n - v\|^2 < \alpha'.$$

Consequently, for $n \in \mathbb{N}$ large, there exist $r_1 > 1$ sufficiently close to 1, $\varepsilon > 0$ small enough and $0 < \alpha < \alpha'$ such that

$$r_1\beta(1+\varepsilon^2)\|v_n - v\|^2 \le \alpha < \alpha'.$$

Hence, invoking the Theorem 1.1.2, we obtain $C > 0$ independent of n such that

$$\int_{\mathbb{R}^2} Q\left(e^{r_1\beta(1+\varepsilon^2)(v_n-v)^2} - 1\right) = \int_{\mathbb{R}^2} Q\left(e^{r_1\beta(1+\varepsilon^2)\|v_n-v\|^2((v_n-v)/\|v_n-v\|)^2} - 1\right) \le C,$$

and the corollary is proved. ■

1.4 Sharp constant and existence of extremal function

In this section we are going to explore further properties of the Trudinger-Moser inequality (1.4) concerned with the sharpness and the existence of extremal function. Throughout the section, we need the following additional hypotheses on $V(|x|)$ and $Q(|x|)$ at the origin:

($\widetilde{V}$) there exists $a_0 > -2$ such that $\limsup_{r\to 0} \dfrac{V(r)}{r^{a_0}} < \infty$;

($\widetilde{Q}$) $Q \in C(0,\infty)$, $Q(r) > 0$ and there exist $b < (a-2)/2$ and $-2 < b_0 \le 0$ such that

$$0 < \liminf_{r\to 0} \frac{Q(r)}{r^{b_0}} \le \limsup_{r\to 0} \frac{Q(r)}{r^{b_0}} < \infty \quad \text{and} \quad \limsup_{r\to+\infty} \frac{Q(r)}{r^{b}} < \infty.$$

Remark 1.4.1 1) *Observe that the singular potentials considered in the Example 1.1.1 satisfy* ($\widetilde{V}$) *and* ($\widetilde{Q}$)*;*

2) *Notice that* ($\widetilde{V}$) *implies that there exist* $r_0 > 0$ *and* $C_0 > 0$ *such that*

$$V(|x|) \le C_0|x|^{a_0}, \quad \text{for all} \quad 0 < |x| \le r_0. \tag{1.22}$$

The main result of this section is the following:

Theorem 1.4.2 *Assume that* $(V), (\widetilde{V})$ *and* $(\widetilde{Q})$ *hold. Then there holds*

$$S_\alpha = \sup_{u \in E;\ \|u\| \leq 1} \int_{\mathbb{R}^2} Q(|x|)(e^{\alpha u^2} - 1) < +\infty \tag{1.23}$$

if and only if $0 < \alpha \leq \alpha'$. *Moreover, the supremum* (1.23) *is attained provided* $0 < \alpha < \alpha'$.

In order to use similar arguments developed in [55] we need the following version of the Radial Lemma for functions in $L^2(\mathbb{R}^2; V)$.

Lemma 1.4.3 *Assume that* (V) *holds. If* $u \in L^2(\mathbb{R}^2; V)$ *is a radial non-increasing function (i.e.* $0 \leq u(x) \leq u(y)$ *if* $|x| \geq |y|$*), then one has*

$$|u(x)| \leq C\|u\|_{L^2(\mathbb{R}^2;V)}|x|^{-(a+2)/2}, \quad |x| \gg 1.$$

Proof. Recall that from the hypothesis (V) that there exists $R_0 > 0$ such that for some $C_0 > 0$,

$$V(|x|) \geq C_0|x|^a \quad \text{for} \quad |x| \geq R_0.$$

Then for $\rho > 0$ such that $\rho/2 > R_0$, we have (setting $\rho = |x|$)

$$\begin{aligned}
\|u\|^2_{L^2(\mathbb{R}^2;V)} &\geq 2\pi \int_{\rho/2}^{\rho} V(s)u^2(s)s\mathrm{ds} \\
&\geq 2\pi C_0 u^2(\rho) \int_{\rho/2}^{\rho} s^{a+1}\mathrm{ds} \\
&= C\rho^{a+2}u^2(\rho).
\end{aligned}$$

Thus we conclude that

$$|u(x)| \leq C\|u\|_{L^2(\mathbb{R}^2;V)}|x|^{-(a+2)/2}, \quad \forall |x| > R_0.$$

Hence, the lemma is proved. ■

In order to prove the sharpness of (1.23), we recall the Moser's function sequence (see [46]):

$$\widetilde{M}_n(x,r) = \frac{1}{(2\pi)^{1/2}} \begin{cases} (\log n)^{1/2}, & |x| \leq r/n, \\ \dfrac{\log \frac{r}{|x|}}{(\log n)^{1/2}}, & r/n < |x| \leq r, \\ 0, & |x| > r, \end{cases}$$

with $0 < r \leq r_0$ fixed and r_0 given in (1.22). We have the following estimate for $\|\widetilde{M}_n\|$:

Lemma 1.4.4 *Under the hypothesis* $(\widetilde{V})$,

$$\|\widetilde{M}_n\|^2 \leq 1 + \frac{m(r)}{\log n}\left(1 + o_n(1)\right),$$

where $m(r) = 2C_0 r^{a_0+2}/(a_0+2)^3$.

Proof. It is easy to compute

$$\int_{\mathbb{R}^2} |\nabla \widetilde{M}_n|^2 = \frac{1}{2\pi}\int_{r/n\leq|x|\leq r} \frac{1}{|x|^2 \log n} = 1.$$

On the other hand, (1.22) and integration by parts give

$$\begin{aligned}
&\int_{\mathbb{R}^2} V(|x|)|\widetilde{M}_n|^2 \\
&\leq \frac{C_0}{2\pi}\int_{|x|\leq r/n} |x|^{a_0}\log n + \frac{C_0}{2\pi}\int_{r/n\leq|x|\leq r} |x|^{a_0}\frac{\left(\log\frac{r}{|x|}\right)^2}{\log n} \\
&= -\frac{2C_0 r^{a_0+2}}{(a_0+2)^2}\left(\frac{1}{n}\right)^{a_0+2} + \frac{2C_0 r^{a_0+2}}{(a_0+2)^3}\frac{1}{\log n} - \frac{2C_0 r^{a_0+2}}{(a_0+2)^3}\frac{1}{\log n}\left(\frac{1}{n}\right)^{a_0+2} \\
&= \frac{2C_0 r^{a_0+2}/(a_0+2)^3}{\log n}(1+o_n(1)) \\
&= \frac{m(r)}{\log n}(1+o_n(1)),
\end{aligned}$$

and thus

$$\begin{aligned}
\|\widetilde{M}_n\|^2 &= \int_{\mathbb{R}^2} |\nabla \widetilde{M}_n|^2 + \int_{\mathbb{R}^2} V(|x|)|\widetilde{M}_n|^2 \\
&\leq 1 + \frac{m(r)}{\log n}(1+o_n(1)).
\end{aligned}$$

Hence, the lemma is proved. ■

Proof of Theorem 1.4.2. 1. *Necessity.* By hypothesis $(\widetilde{Q})$, there exist $0 < r_0 < R_0$ and $C_0 > 0$ such that

$$\begin{aligned}
Q(|x|) &\leq C_0|x|^b \qquad \text{for} \quad |x| \geq R_0, \\
Q(|x|) &\leq C_0|x|^{b_0} \qquad \text{for} \quad 0 < |x| \leq r_0.
\end{aligned} \tag{1.24}$$

Let $R > 0$ be large enough. We write

$$\int_{\mathbb{R}^2} Q(e^{\alpha u^2} - 1) = \int_{B_R} Q(e^{\alpha u^2} - 1) + \int_{B_R^c} Q(e^{\alpha u^2} - 1). \tag{1.25}$$

We are going to estimate each integral in (1.25). For the integral on B_R, we have two cases to consider:

Case 1. $b_0 = 0$. From the second inequality in (1.24) and the continuity of $Q(r)$, there exists $C > 0$ such that

$$\int_{B_R} Q(e^{\alpha u^2} - 1) \leq C \int_{B_R} e^{\alpha u^2}. \tag{1.26}$$

As in [46, 55], we use Schwarz symmetrization theory (see [42]) by defining the radially symmetric function u^* as follows: for all $s > 0$

$$|\{x \in B_R : u^*(x) > s\}| = |\{x \in B_R : u(x) > s\}|.$$

It follows from the properties of this construction that:

- u^* is a non-increasing function in $|x|$;
- $u^* \in H_0^1(B_R)$ and $\int_{B_R} |\nabla u^*|^2 \leq \int_{B_R} |\nabla u|^2$;
- $\int_{B_R} e^{\alpha|u^*|^2} = \int_{B_R} e^{\alpha|u|^2}$.

Thus, we may assume that u in the second integral from (1.26) is non-increasing. Let

$$v(r) = \begin{cases} u(r) - u(R), & \text{if} \quad 0 \leq r \leq R; \\ 0, & \text{if} \quad r \geq R. \end{cases}$$

By Lemma 1.4.3,

$$\begin{aligned} u^2(r) &= v^2(r) + 2v(r)u(R) + u^2(R) \\ &\leq v^2(r) + Cv^2(r)R^{-(a+2)/2}\|u\|^2_{L^2(\mathbb{R}^2;V)} + 1 + CR^{-(a+2)/2}\|u\|^2_{L^2(\mathbb{R}^2;V)} \\ &= v^2(r)\left[1 + CR^{-(a+2)/2}\|u\|^2_{L^2(\mathbb{R}^2;V)}\right] + d(R). \end{aligned}$$

Hence

$$u(r) \leq v(r)\left[1 + CR^{-(a+2)/2}\|u\|^2_{L^2(\mathbb{R}^2;V)}\right]^{1/2} + d^{1/2}(R) \doteq w(r) + d^{1/2}(R).$$

By assumption

$$\int_{B_R} |\nabla v|^2 = \int_{B_R} |\nabla u|^2 \leq 1 - \|u\|^2_{L^2(\mathbb{R}^2;V)}$$

and so

$$\begin{aligned}\int_{B_R}|\nabla w|^2 &= \int_{B_R}\left|\nabla v\left[1+CR^{-(a+2)/2}\|u\|^2_{L^2(\mathbb{R}^2;V)}\right]^{1/2}\right|^2\\ &= \left[1+CR^{-(a+2)/2}\|u\|^2_{L^2(\mathbb{R}^2;V)}\right]\int_{B_R}|\nabla v|^2\\ &\leq \left[1+CR^{-(a+2)/2}\|u\|^2_{L^2(\mathbb{R}^2;V)}\right]\left[1-\|u\|^2_{L^2(\mathbb{R}^2;V)}\right]\\ &= 1+CR^{-(a+2)/2}\|u\|^2_{L^2(\mathbb{R}^2;V)}-\|u\|^2_{L^2(\mathbb{R}^2;V)}-CR^{-(a+2)/2}\|u\|^4_{L^2(\mathbb{R}^2;V)}\\ &\leq 1.\end{aligned}$$

Since

$$u^2(r)\leq w^2(r)+d(R),$$

we get

$$\int_{B_R}Q(e^{\alpha u^2}-1)\leq C\int_{B_R}e^{\alpha u^2}\leq Ce^{\alpha d}\int_{B_R}e^{\alpha w^2}.$$

Taking into account that $w\in H_0^1(B_R)$ and

$$\|w\|_{H_0^1(B_R)}=\|\nabla w\|_{L^2(B_R)}\leq 1,$$

we conclude that

$$\sup_{u\in E;\ \|u\|\leq 1}\int_{B_R}Q(e^{\alpha u^2}-1)<+\infty,$$

by the classical Trudinger-Moser inequality (1.1).

Case 2. $-2<b_0<0$. It was done in the proof of Theorem 1.1.2 as well as the estimative of the second integral in (1.25).

2. *Sufficiency.* Next we will show that (1.23) does not hold if $\alpha>\alpha'$. Setting

$$M_n(x,r)=\frac{1}{\|\widetilde{M}_n\|}\widetilde{M}_n(x,r),$$

then M_n belongs to E with its support in $\overline{B}_r$ and $\|M_n\|=1$. From Lemma 1.4.4, when $|x|\leq r/n$, we have

$$M_n^2(x)\geq\frac{1}{2\pi}\frac{\log n}{1+\frac{m(r)}{\log n}(1+o_n(1))}=(2\pi)^{-1}\log n-(2\pi)^{-1}m(r)+o_n(1).$$

By hypothesis $(\widetilde{Q})$,

$$Q(|x|)\geq C_0|x|^{b_0}\quad\text{for}\quad 0<|x|\leq r_0.$$

Thus, for $0 < r \leq r_0$ we have

$$\begin{aligned}\int_{\mathbb{R}^2} Q(e^{\alpha M_n^2}-1) &\geq \int_{B_{r/n}} Q(e^{\alpha M_n^2}-1)\\ &\geq C_0 \int_{B_{r/n}} |x|^{b_0}\left(e^{\alpha[(2\pi)^{-1}\log n-(2\pi)^{-1}m(r)+o_n(1)]}-1\right)\\ &= C_0\left(e^{\alpha[(2\pi)^{-1}\log n-(2\pi)^{-1}m(r)+o_n(1)]}-1\right)\int_{B_{r/n}}|x|^{b_0}\\ &= \pi C_0 r^{2+b_0}\frac{1}{n^{2+b_0}}\left(e^{\alpha[(2\pi)^{-1}\log n-(2\pi)^{-1}m(r)+o_n(1)]}-1\right)\\ &= Cn^{\alpha(2\pi)^{-1}-(2+b_0)}e^{o_n(1)}+o_n(1).\end{aligned}$$

Thus if $b_0 = 0$ ($\Rightarrow \alpha' = 4\pi$), then

$$\alpha > 4\pi \therefore \alpha(2\pi)^{-1} - 2 > 0$$

and we conclude that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} Q(e^{\alpha M_n^2}-1) = +\infty.$$

Now, if $-2 < b_0 < 0$ ($\Leftrightarrow \alpha' = 4\pi(1+b_0/2)$), then

$$\alpha > \alpha' = 4\pi(1+b_0/2) = 2\pi(2+b_0) \therefore \alpha(2\pi)^{-1} > 2+b_0 \therefore \alpha(2\pi)^{-1}-(2+b_0) > 0$$

and consequently we also obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} Q(e^{\alpha M_n^2}-1) = +\infty,$$

concluding the first part of theorem.

For the last part of theorem, we consider $0 < \alpha < \alpha'$. Let $(u_n) \subset E$ be a maximizing sequence, with $\|u_n\| \leq 1$. Then, up to subsequences, we can assume that $u_n \rightharpoonup u_0$ weakly in E and, by Lemma 1.2.5, $u_n \to u_0$ strongly in $L^p(\mathbb{R}^2; Q)$ for $2 \leq p < \infty$. Using the inequality

$$|e^x - e^y| \leq |x-y|(e^x+e^y), \quad \forall x,y \in \mathbb{R}, \tag{1.27}$$

we estimate

$$\left|\int_{\mathbb{R}^2} Q(e^{\alpha u_n^2}-e^{\alpha u_0^2})\right| \leq \alpha\int_{\mathbb{R}^2} Qe^{\alpha u_n^2}|u_n^2-u_0^2| + \alpha\int_{\mathbb{R}^2} Qe^{\alpha u_0^2}|u_n^2-u_0^2|. \tag{1.28}$$

Writing

$$\int_{\mathbb{R}^2} Qe^{\alpha u_n^2}|u_n^2-u_0^2| = \int_{\mathbb{R}^2} Q(e^{\alpha u_n^2}-1)|u_n^2-u_0^2| + \int_{\mathbb{R}^2} Q|u_n^2-u_0^2|$$

and taking $r_1 > 1$ sufficiently close to 1 such that $r_1\alpha \leq \alpha'$ (it is possible because we are assuming $\alpha < \alpha'$) and $r_2 \geq 2$ such that $1/r_1 + 1/r_2 = 1$, the Hölder's inequality implies that

$$\int_{\mathbb{R}^2} Qe^{\alpha u_n^2}|u_n^2 - u_0^2| \leq \left(\int_{\mathbb{R}^2} Q(e^{r_1\alpha u_n^2} - 1)\right)^{1/r_1} \left(\int_{\mathbb{R}^2} Q|u_n^2 - u_0^2|^{r_2}\right)^{1/r_2} + \left(\int_{\mathbb{R}^2} Q|u_n - u_0|^2\right)^{1/2} \left(\int_{\mathbb{R}^2} Q|u_n + u_0|^2\right)^{1/2}$$

and likewise for the integral in (1.28) containing $e^{\alpha u_0^2}$. Thus, it follows from the first part of theorem and Lemma 1.2.5 that

$$S_\alpha + o_n(1) = \int_{\mathbb{R}^2} Q(e^{\alpha u_n^2} - 1) = \int_{\mathbb{R}^2} Q(e^{\alpha u_0^2} - 1) + o_n(1).$$

Finally, since $\|u_0\| \leq 1$, we see that u_0 is the required extremal function. This completes the proof of the result. ■

Remark 1.4.5 *The maximizer u_0 can be chosen unitary, i.e, $\|u_0\| = 1$. Indeed, since for instance if $\|u_0\| < 1$, then setting $v_0 = u_0/\|u_0\|$, we would have*

$$\int_{\mathbb{R}^2} Q(e^{\alpha v_0^2} - 1) > \int_{\mathbb{R}^2} Q(e^{\alpha u_0^2} - 1) = S_\alpha,$$

which is a contradiction.

Chapter 2

On a class of nonlinear Schrödinger equations involving exponential critical growth in $\mathbb{R}^2$

This chapter is concerned with the existence, multiplicity and nonexistence of solutions for nonlinear elliptic equations of the form

$$-\Delta u + V(|x|)u = Q(|x|)f(u) \quad \text{in} \quad \mathbb{R}^2, \tag{2.1}$$

when the nonlinear term $f(s)$ is allowed to enjoy the exponential critical growth by means of the Trudinger-Moser inequality and the radial potentials V and Q may be unbounded, singular or decaying to zero. We point out that part of this chapter is contained in the published paper [6].

2.1 Introduction and main results

In the papers [62, 63], Su-Wang-Willem studied the existence of solutions for the problem

$$\begin{cases} -\Delta u + V(|x|)u = Q(|x|)|u|^{p-2}u & \text{in} \quad \mathbb{R}^N \\ |u(x)| \to 0 & \text{as} \quad |x| \to \infty \end{cases}$$

with $2 < p < 2^* = 2N/(N-2)$ for $N \geq 3$, $2 < p < \infty$ for $N = 2$ and $V, Q \in C(0, \infty)$ are radial potentials which are singular at the origin or vanish super-quadratically at infinity. It is natural to ask if this result is true, under a similar local condition on

$V(|x|)$ and $Q(|x|)$, when we consider nonlinearities with exponential critical growth in dimension two. Explicitly, we assume the following hypotheses on $V(|x|)$ and $Q(|x|)$:

(V) $V \in C(0,\infty)$, $V(r) > 0$ and there exists $a > -2$ such that

$$\liminf_{r \to +\infty} \frac{V(r)}{r^a} > 0.$$

(Q) $Q \in C(0,\infty)$, $Q(r) > 0$ and there exist $b < (a-2)/2$ and $b_0 > -2$ such that

$$\limsup_{r \to 0} \frac{Q(r)}{r^{b_0}} < \infty \quad \text{and} \quad \limsup_{r \to +\infty} \frac{Q(r)}{r^b} < \infty.$$

Here, we are interested in the case where the nonlinear term $f(s)$ has maximal growth on s which allows us to treat problem (2.1) variationally. Explicitly, in view of the classical Trudinger-Moser inequality, we recall that $f(s)$ has *exponential critical growth* if there exists $\alpha_0 > 0$ such that

$$(f_{\alpha_0}) \quad \lim_{|s| \to +\infty} \frac{|f(s)|}{e^{\alpha s^2}} = \begin{cases} 0, & \forall \alpha > \alpha_0, \\ +\infty, & \forall \alpha < \alpha_0. \end{cases}$$

We will assume that the nonlinearity $f(s)$ is continuous and satisfies:

(f_1) $f(s) = o(s)$ as $s \to 0$;

(f_2) there exists $\theta > 2$ such that

$$0 < \theta F(s) \doteq \theta \int_0^s f(t)\mathrm{d}t \le s f(s), \quad \forall s \neq 0;$$

(f_3) there exist constants $R_0, M_0 > 0$ such that

$$0 < F(s) \le M_0 |f(s)|, \quad \forall |s| \ge R_0;$$

(f_4) there exist $\nu > 2$ and $\mu > 0$ such that

$$F(s) \ge \frac{\mu}{\nu}|s|^\nu, \quad \forall s \in \mathbb{R}.$$

We point out that the hypotheses $(f_{\alpha_0}) - (f_4)$ has been used in many papers, see for instance [21, 24, 27].

Example 2.1.1 *Let $\nu > 2$ and $\mu > 0$ be constants. A simple model of a function that verifies our assumptions is*

$$f(s) = \mu|s|^{\nu-2}s + 2se^{s^2} - 2s,$$

for $s \in \mathbb{R}$.

Clearly (f_{α_0}) is satisfied with $\alpha_0 = 1$ and (f_1) holds provided $\nu > 2$. In order to prove that (f_2) is satisfied, notice that

$$F(s) = \frac{\mu}{\nu}|s|^\nu + e^{s^2} - s^2 - 1.$$

Thus

$$\begin{aligned} sf(s) - \theta F(s) &= \mu\left(1 - \frac{\theta}{\nu}\right)|s|^\nu + e^{s^2}(2s^2 - \theta) - s^2(2-\theta) + \theta \\ &\geq e^{s^2}(2s^2 - \theta) - s^2(2-\theta) + \theta \\ &\geq 2s^2 - \theta - 2s^2 + \theta s^2 + \theta \\ &= \theta s^2 \geq 0, \end{aligned}$$

for $|s| \geq \sqrt{\theta/2}$ and $2 < \theta < \nu$. If $|s| < \sqrt{\theta/2}$, then

$$sf(s) - \theta F(s) \geq -\theta - 2s^2 + \theta s^2 + \theta = s^2(\theta - 2) > 0,$$

provided $s \neq 0$ and $\theta > 2$. For (f_3), it is enough to notice that

$$\lim_{|s|\to\infty} \frac{F(s)}{f(s)} = \lim_{|s|\to\infty} \frac{\frac{\mu}{\nu}|s|^\nu + e^{s^2} - s^2 - 1}{\mu|s|^{\nu-2}s + 2se^{s^2} - 2s} = 0.$$

Finally, since $e^{s^2} \geq s^2 + 1$ for all $s \in \mathbb{R}$, it is easy to see that (f_4) is satisfied.

Denoting by $S_\nu > 0$ the best constant of the Sobolev embedding $E \hookrightarrow L^\nu(\mathbb{R}^2; Q)$ (see Lemma 1.2.5), as an application of the Theorem 1.1.2 and using a minimax procedure, we have the following existence result for problem (2.1).

Theorem 2.1.2 (Existence) *Assume that $(V) - (Q)$ hold. If f satisfies $(f_{\alpha_0}) - (f_4)$, with*

$$\mu > \left[\frac{\alpha_0(\nu-2)}{\alpha'\nu}\right]^{(\nu-2)/2} S_\nu^{\nu/2},$$

then the problem (2.1) has a nontrivial positive weak solution u in E. Moreover, if in addition $(\widetilde{V})$ holds, then there exist constants $c_0, c_1 > 0$ such that

$$u(x) \leq c_0 \exp\left(-c_1|x|^{(a+2)/4}\right), \quad \forall x \in \mathbb{R}^2. \tag{2.2}$$

Remark 2.1.3 *The existence result above completes those of [62] in the sense that we treat the exponential critical case.*

The first difficulty in treating this class of Schrödinger equations (2.1) is the possible lack of compactness due to the unboundedness of the domain. The second difficulty is the critical growth. In both cases, it is not standard to verify that the associated functional to the problem (2.1) satisfies the Palais-Smale condition at some level $c \in \mathbb{R}$.

Our multiplicity result is concerned with the problem

$$-\Delta u + V(|x|)u = \lambda Q(|x|)f(u) \quad \text{in} \quad \mathbb{R}^2, \tag{2.3}$$

where λ is a positive parameter. In this result we introduce more symmetry in the problem (2.3) and shows that the value of the parameter $\lambda > 0$ affects the number of solutions. It can be stated as follows.

Theorem 2.1.4 (Multiplicity) *Assume that* $(V)-(Q)$ *hold. If* f *is odd and satisfies* $(f_{\alpha_0})-(f_4)$*, then there exists an increasing sequence* $(\lambda_k) \subset \mathbb{R}_+$ *with* $\lambda_k \to \infty$ *as* $k \to \infty$ *such that, for* $\lambda > \lambda_k$*, the problem* (2.3) *has at least* k *pairs of weak solutions in* E*.*

2.2 Variational setting

We establish the necessary functional framework where solutions are naturally studied by variational method. Let $\alpha > \alpha_0$ given by (f_{α_0}) and $q \geq 1$. We claim that it follows from (f_{α_0}) and (f_1), for any given $\varepsilon > 0$, there exist constants $b_1, b_2 > 0$ such that

$$|f(s)| \leq \varepsilon|s| + b_1|s|^{q-1}(e^{\alpha s^2} - 1), \quad \forall s \in \mathbb{R} \tag{2.4}$$

and

$$|F(s)| \leq \frac{\varepsilon}{2}s^2 + b_2|s|^q(e^{\alpha s^2} - 1), \quad \forall s \in \mathbb{R}. \tag{2.5}$$

Indeed, from (f_1), given $\varepsilon > 0$ there exists $\delta_1 > 0$ such that

$$|f(s)| < \varepsilon|s|, \quad \forall\, 0 < |s| < \delta_1. \tag{2.6}$$

For $\alpha > \alpha_0$, the condition (f_{α_0}) ensures that

$$\lim_{s \to +\infty} \frac{f(s)}{|s|^{q-1}(e^{\alpha s^2} - 1)} = 0,$$

which implies that there exists $\delta_2 > \delta_1 > 0$ such that

$$|f(s)| < \varepsilon |s|^{q-1}(e^{\alpha s^2} - 1), \quad \forall |s| > \delta_2. \tag{2.7}$$

In the case that $s \in [\delta_1, \delta_2]$, we have

$$(e^{\alpha\delta_1^2} - 1) \leq (e^{\alpha s^2} - 1) \therefore 1 \leq (e^{\alpha\delta_1^2} - 1)^{-1}(e^{\alpha s^2} - 1),$$

for all $s \in [\delta_1, \delta_2]$. On the other hand, since $f(s)$ is bounded in $[\delta_1, \delta_2]$, we have

$$\begin{aligned}
|f(s)| &\leq C, \quad \forall s \in [\delta_1, \delta_2] \\
&= C|s|^{q-1}|s|^{1-q}, \quad \forall s \in [\delta_1, \delta_2] \\
&\leq C|s|^{q-1}, \quad \forall s \in [\delta_1, \delta_2] \\
&\leq C|s|^{q-1}(e^{\alpha\delta_1^2} - 1)^{-1}(e^{\alpha s^2} - 1), \quad \forall s \in [\delta_1, \delta_2] \\
&= C|s|^{q-1}(e^{\alpha s^2} - 1), \quad \forall s \in [\delta_1, \delta_2].
\end{aligned} \tag{2.8}$$

Hence, from (2.6), (2.7) and (2.8) we get (2.4). To verify (2.5) we use (2.4) and the fact that $e^{\alpha s^2}$ is increasing. So, we have:

$$\begin{aligned}
|F(s)| &\leq \int_0^s |f(t)|\mathrm{dt} \\
&\leq \varepsilon \int_0^s |t|\mathrm{dt} + b_1 \int_0^s |t|^{q-1}(e^{\alpha t^2} - 1)\mathrm{dt} \\
&\leq \frac{\varepsilon}{2}s^2 + b_2|s|^q(e^{\alpha s^2} - 1), \quad \forall s \in \mathbb{R}.
\end{aligned}$$

Therefore, the claim follows.

Given $u \in E$ we can use (2.5) with $q = 2$ to obtain

$$\int_{\mathbb{R}^2} QF(u) \leq \frac{\varepsilon}{2}\int_{\mathbb{R}^2} Q|u|^2 + b_2 \int_{\mathbb{R}^2} Q|u|^2(e^{\alpha u^2} - 1).$$

By Lemma 1.2.5,

$$\int_{\mathbb{R}^2} Q|u|^2 < \infty.$$

If we apply the inequality $(1+t)^r \geq 1 + t^r$ with $t = e^s - 1 \geq 0$, we get

$$(e^s - 1)^r \leq (e^{rs} - 1), \tag{2.9}$$

for all $r \geq 1, s \geq 0$. Now, let $r_1, r_2 > 1$ be such that $1/r_1 + 1/r_2 = 1$. The Hölder's inequality and (2.9) imply that

$$\int_{\mathbb{R}^2} Q|u|^2(e^{\alpha u^2} - 1) \leq \left(\int_{\mathbb{R}^2} Q|u|^{2r_1}\right)^{1/r_1}\left(\int_{\mathbb{R}^2} Q(e^{r_2\alpha u^2} - 1)\right)^{1/r_2} < \infty,$$

where we have used Lemma 1.2.5 and Theorem 1.1.2 to conclude that the latter two terms are finites. Therefore, the energy functional associated to problem (2.1) $I : E \to \mathbb{R}$ defined by

$$I(u) \doteq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^2} QF(u)$$

is well defined and $I \in C^1(E, \mathbb{R})$ with derivative given by

$$I'(u)v = \int_{\mathbb{R}^2} (\nabla u \nabla v + Vuv) - \int_{\mathbb{R}^2} Qf(u)v, \quad \forall u, v \in E.$$

Thus, since we are searching for weak solutions for problem (2.1), that is, functions $u \in E$ such that

$$\int_{\mathbb{R}^2} (\nabla u \nabla v + Vuv) - \int_{\mathbb{R}^2} Qf(u)v = 0,$$

for all $v \in E$, we conclude that a weak solution of (2.1) is exactly a critical point of I and reciprocally.

Next lemma describe the geometric structure of the functional I required by the Mountain-Pass Theorem.

Lemma 2.2.1 *Assume that* $(V) - (Q)$ *hold. If* $(f_{\alpha_0}) - (f_2)$ *are satisfied, then:*

i) there exist $\tau, \rho > 0$ *such that* $I(u) \geq \tau$ *if* $\|u\| = \rho$;

ii) there exists $e_* \in E$, *with* $\|e_*\| > \rho$, *such that* $I(e_*) < 0$.

Proof. By using (2.5) with $q > 2$, the Hölder's inequality, Lemma 1.2.5 and Theorem 1.1.2 we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} QF(u) &\leq \frac{\varepsilon}{2}\int_{\mathbb{R}^2} Q|u|^2 + b_2 \int_{\mathbb{R}^2} Q|u|^q(e^{\alpha u^2} - 1) \\ &\leq \frac{C\varepsilon}{2}\|u\|^2 + b_2 \left(\int_{\mathbb{R}^2} Q|u|^{qr_1}\right)^{1/r_1} \left(\int_{\mathbb{R}^2} Q(e^{r_2\alpha u^2} - 1)\right)^{1/r_2} \\ &\leq \frac{C\varepsilon}{2}\|u\|^2 + \widetilde{C}\|u\|^q,\end{aligned}$$

whenever $\|u\| \leq M < (\alpha'/\alpha)^{1/2}$ and $r_2 > 1$ is sufficiently close to 1. Consequently,

$$I(u) \geq \left(\frac{1}{2} - \frac{C\varepsilon}{2}\right)\|u\|^2 - \widetilde{C}\|u\|^q,$$

which implies $i)$. In order to verify $ii)$, if we take a function $\phi \in C^\infty_{0,\mathrm{rad}}(\mathbb{R}^2) \setminus \{0\}$, then it follows from (f_4) that for $t \geq 0$

$$I(t\phi) \leq \frac{t^2}{2}\|\phi\|^2 - Ct^\nu \int_{\mathrm{supp}(\phi)} Q|\phi|^\nu,$$

which implies *ii*) with $e_* = t_*\phi$ and $t_* > 0$ large, because $\nu > 2$. ■

In view of Lemma 3.2.1 the minimax level

$$c = \inf_{g\in\Gamma} \max_{0\le t\le 1} I(g(t))$$

is positive, where

$$\Gamma = \{g \in C([0,1], E) : g(0) = 0 \quad \text{and} \quad I(g(1)) < 0\}\,.$$

Hence, by the Mountain-Pass Theorem without the Palais-Smale condition (see [12]) there exists a $(PS)_c$ sequence (u_n) in E, that is,

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0. \tag{2.10}$$

Lemma 2.2.2 *The sequence (u_n) above is bounded.*

Proof. Notice that by (f_2)

$$\begin{aligned} I(u_n) - \frac{1}{\theta} I'(u_n)u_n &= \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2 + \int_{\mathbb{R}^2} Q\left[\frac{1}{\theta} f(u_n)u_n - F(u_n)\right] \\ &\ge \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2. \end{aligned}$$

Combining the last inequality with

$$I(u_n) - \frac{1}{\theta} I'(u_n)u_n \le c + 1 + \|u_n\|$$

for large $n \in \mathbb{N}$, we conclude the proof of lemma. ■

Lemma 2.2.3 *For each $p > 2$, S_p is attained for a non-negative function $u_p \in E\setminus\{0\}$.*

Proof. The proof is based on the direct method of the calculus of variations. Given any $p > 2$ choose a sequence of functions $(u_n) \in E$ such that

$$\int_{\mathbb{R}^2} Q|u_n|^p = 1 \quad \text{and} \quad \lim_{n\to\infty} \int_{\mathbb{R}^2} \left(|\nabla u_n|^2 + V|u_n|^2\right) = S_p.$$

Thus, (u_n) is bounded in E. Hence, up to subsequences, we can assume

$$\begin{aligned} &u_n \rightharpoonup u_p && \text{weakly in } E, \\ &u_n \to u_p && \text{strongly in } L^q(\mathbb{R}^2; Q) \text{ for all } 2 \le q < \infty, \\ &u_n(x) \to u_p(x) && \text{for almost everywhere } x \in \mathbb{R}^2. \end{aligned}$$

In particular,

$$1 = \lim_{n\to\infty} \int_{\mathbb{R}^2} Q|u_n|^p = \int_{\mathbb{R}^2} Q|u_p|^p.$$

On the other hand,

$$\|u_p\|^2 \le \liminf_{n\to\infty} \|u_n\|^2 = S_p.$$

Therefore,

$$S_p = \|u_p\|^2,$$

which completes the proof of lemma. ■

We obtain the following estimate for the minimax level c.

Lemma 2.2.4 *If*

$$\mu > \left[\frac{\alpha_0(\nu-2)}{\alpha'\nu}\right]^{(\nu-2)/2} S_\nu^{\nu/2},$$

then $c < \dfrac{\alpha'}{2\alpha_0}$.

Proof. Since the embeddings $E \hookrightarrow L^p(\mathbb{R}^2;Q)$ are compacts for $2 \le p < \infty$, there exists a function $\bar{u} \in E$ such that S_ν is attained, that is,

$$S_\nu = \int_{\mathbb{R}^2} (|\nabla \bar{u}|^2 + V\bar{u}^2) \quad \text{and} \quad \int_{\mathbb{R}^2} Q|\bar{u}|^\nu = 1.$$

By the definition of c, one has

$$c \le \max_{t\ge 0} \left[\frac{t^2}{2} S_\nu - \int_{\mathbb{R}^2} QF(t\bar{u})\right]$$

and thus in view of (f_4) we conclude that

$$c \le \max_{t\ge 0} \left[\frac{t^2}{2} S_\nu - t^\nu \frac{\mu}{\nu}\right] = \frac{\nu-2}{2\nu} \frac{S_\nu^{\nu/(\nu-2)}}{\mu^{2/(\nu-2)}} < \frac{\alpha'}{2\alpha_0}.$$

Hence, the lemma is proved. ■

In order to prove that the weak limit of a sequence is a weak solution of (2.1) we will need the following convergence results.

Lemma 2.2.5 *[27, Lemma 2.1] Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $h : \mathbb{R} \to \mathbb{R}$ a continuous function. Then, for any sequence (u_n) in $L^1(\Omega)$ such that*

$$u_n \to u \quad \text{in} \quad L^1(\Omega), \quad h(u_n) \in L^1(\Omega) \quad \text{and} \quad \int_\Omega Q|h(u_n)u_n| \le C,$$

up to a subsequence we have

$$h(u_n) \to h(u) \quad \text{in} \quad L^1(\Omega).$$

Lemma 2.2.6 *There are a subsequence of (u_n), still denoted by (u_n), and $u \in E$ such that $u_n \rightharpoonup u$ in E, $u_n \to u$ in $L^p(\mathbb{R}^2; Q)$, for all $2 \le p < \infty$, and*

$$\begin{aligned} f(u_n) \to f(u) \qquad & \textit{in} \quad L^1_{loc}(\mathbb{R}^2; Q), \\ F(u_n) \to F(u) \qquad & \textit{in} \quad L^1(\mathbb{R}^2; Q). \end{aligned}$$

Proof. From Lemma 2.2.2, up to a subsequence, we assume that there exists $u \in E$ such that $u_n \rightharpoonup u$ weakly in E. Therefore, by Lemma 1.2.5, it follows that $u_n \to u$ strongly in $L^p(\mathbb{R}^2; Q)$, for all $2 \le p < \infty$, and $u_n(x) \to u(x)$ a.e. $x \in \mathbb{R}^2$. By (2.10),

$$\frac{1}{2}\|u_n\|^2 - \int_{\mathbb{R}^2} QF(u_n) = c + o_n(1)$$

and

$$\|u_n\|^2 - \int_{\mathbb{R}^2} Qf(u_n)u_n = o_n(1)$$

as $n \to \infty$. Since $(u_n) \subset E$ is bounded, there exists $C > 0$ such that

$$\int_{\mathbb{R}^2} Q|F(u_n)| \le C \quad \text{and} \quad \int_{\mathbb{R}^2} Q|f(u_n)u_n| \le C.$$

For the convergence in $L^1_{\text{loc}}(\mathbb{R}^2; Q)$, recalling that $H^1_{\text{rad}}(B_R; V)$ is compactly embedding in $L^q(B_R)$ for all $1 \le q < \infty$, up to a subsequence, we can assume that $u_n \to u$ strongly in $L^1(B_R)$. Moreover,

$$Qf(u_n) \in L^1(B_R), \quad Qf(u) \in L^1(B_R) \quad \text{and} \quad \int_{B_R} Q|f(u_n)u_n| \le C,$$

$n \in \mathbb{N}$. Therefore, thanks to Lemma 2.2.5, the convergence follows. Next, we prove the second convergence. From the first convergence, there exists $g \in L^1(B_R)$ such that $Q|f(u_n)| \le g$ a.e. in B_R. By (f_3),

$$Q|F(u_n)| \le Q \sup_{[-R_0, R_0]} |F(u_n)| + M_0 Q g$$

a.e. in B_R. Thus, by Lebesgue's Dominated Convergence Theorem

$$QF(u_n) \to QF(u) \quad \text{in} \quad L^1(B_R),$$

for all $R > 0$. Now, we are going to estimate

$$\int_{B_R^c} QF(u_n) \quad \text{and} \quad \int_{B_R^c} QF(u).$$

Using (2.5) with $q = 2$ we have

$$\int_{B_R^c} QF(u_n) \le \frac{\varepsilon}{2} \int_{B_R^c} Q|u_n|^2 + b_2 \int_{B_R^c} Q|u_n|^2 (e^{\alpha u_n^2} - 1), \tag{2.11}$$

for $\alpha > \alpha_0$. By the Hölder's inequality, Lemmas 1.2.5, 2.2.2, and similar calculations to estimate the second integral in (1.17), we get

$$\int_{B_R^c} Q|u_n|^2(e^{\alpha u_n^2} - 1) \leq \frac{C}{R^\xi},$$

$\xi > 0$. Hence, given $\delta > 0$, there exists $R > 0$ sufficiently large such that

$$\int_{B_R^c} Q|u_n|^2(e^{\alpha u_n^2} - 1) < \delta \quad \text{and} \quad \int_{B_R^c} Q|u_n|^2 < \delta.$$

Thus, from (2.11)

$$\int_{B_R^c} QF(u_n) \leq C\delta \quad \text{and} \quad \int_{B_R^c} QF(u) \leq C\delta.$$

Finally, since

$$\begin{aligned} &\left|\int_{\mathbb{R}^2} QF(u_n) - \int_{\mathbb{R}^2} QF(u)\right| \\ &\leq \left|\int_{B_R} QF(u_n) - \int_{B_R} QF(u)\right| + \left|\int_{B_R^c} QF(u_n) + \int_{B_R^c} QF(u)\right|, \end{aligned}$$

we get

$$\lim_{n\to\infty} \left|\int_{\mathbb{R}^2} QF(u_n) - \int_{\mathbb{R}^2} QF(u)\right| \leq C\delta.$$

Since δ is arbitrary, the result follows and the lemma is proved. ■

2.2.1 Proof of the existence theorem

Once we intend to find positive solutions, without loss of generality, we will assume that $f(s) = 0$ for $s < 0$. Thus, (f_2) holds for $s > 0$, (f_3) holds for $s \geq R_0$ and (f_4) holds for $s \geq 0$. Let us check the veracity of the reduction argument mentioned. Set

$$\tilde{f}(s) = \begin{cases} 0, & f(s) < 0, \\ f(s), & f(s) \geq 0. \end{cases}$$

Assume that $u \in E$ is a weak solution of

$$-\Delta u + V(|x|)u = Q(|x|)\tilde{f}(u). \tag{2.12}$$

Then the negative part of u, namely

$$u_-(x) = \begin{cases} 0, & u(x) > 0, \\ u(x), & u(x) \leq 0 \end{cases}$$

belongs to the functional space E and satisfies

$$\int_{\mathbb{R}^2} (|\nabla u_-|^2 + V|u_-|^2) = \int_{\mathbb{R}^2} Q\tilde{f}(u)u_- = 0.$$

Hence, $u_-(x) = 0$ for a.e. $x \in \mathbb{R}^2$ and thus u is a positive weak solution of (2.12). This together with the condition (f_2) imply that $f(u) \geq 0$. It follows that

$$\tilde{f}(u) = f(u).$$

Therefore, u is also a positive weak solution of problem (2.1).

Noticing that the above lemmas are valid also for this modified nonlinearity, we are ready to prove our existence result.

Proof of Theorem 2.1.2. By (2.10),

$$I'(u_n)\phi = \int_{\mathbb{R}^2} (\nabla u_n \nabla \phi + V u_n \phi) - \int_{\mathbb{R}^2} Qf(u_n)\phi = o_n(1), \tag{2.13}$$

for $\phi \in C^\infty_{0,\mathrm{rad}}(\mathbb{R}^2)$ fixed. Passing to the limit in (2.13), using that $u_n \rightharpoonup u$ weakly in E and Lemma 2.2.6 we obtain

$$\int_{\mathbb{R}^2} (\nabla u \nabla \phi + V u \phi) - \int_{\mathbb{R}^2} Qf(u)\phi = 0,$$

for all $\phi \in C^\infty_{0,\mathrm{rad}}(\mathbb{R}^2)$. Thus, we conclude that u is a weak solution of (2.1). Next, we prove that u is nontrivial. Arguing by contradiction, if $u \equiv 0$, Lemma 2.2.6 implies that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} QF(u_n) = 0.$$

Since

$$I(u_n) = \frac{1}{2}\|u_n\|^2 - \int_{\mathbb{R}^2} QF(u_n) = c + o_n(1),$$

we get

$$\lim_{n\to\infty} \|u_n\|^2 = 2c > 0. \tag{2.14}$$

From this and Lemma 2.2.4, given $\varepsilon > 0$, we have that

$$\|u_n\|^2 < \frac{\alpha'}{\alpha_0} + \varepsilon,$$

for $n \in \mathbb{N}$ large. Thus, it is possible to choice $r_2 > 1$ sufficiently close to 1 and $\alpha > \alpha_0$ close to α_0 such that $r_2\alpha\|u_n\|^2 \leq \beta' < \alpha'$. Hence, from Theorem 1.1.2,

$$\int_{\mathbb{R}^2} Q(e^{\alpha u_n^2} - 1)^{r_2} \leq \int_{\mathbb{R}^2} Q\left(e^{r_2\alpha\|u_n\|^2(u_n/\|u_n\|)^2} - 1\right) \leq C.$$

Thus, using this, (2.4) in combination with the Hölder's inequality and the compactness of the embedding $E \hookrightarrow L^p(\mathbb{R}^2; Q)$ for all $2 \le p < \infty$, up to a subsequence, we conclude that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} Qf(u_n)u_n = 0.$$

Hence, since $I'(u_n)u_n = o_n(1)$, we obtain that

$$\lim_{n\to\infty} \|u_n\|^2 = 0,$$

which is a contradiction with (2.14). Therefore, u is a nontrivial positive weak solution of problem (2.1). To finish, we are going to prove the exponential decay of u. Using (V) and that $a > (a-2)/2$, we have

$$V(|x|) \ge C_0|x|^{(a-2)/2} \quad \text{for} \quad |x| \ge R_0.$$

Consider

$$\phi(r) = \exp\left(-c_1 r^{(a+2)/4}\right),$$

with $c_1 = \sqrt{8C_0}/(a+2) > 0$. Then, since $a > -2$, a straightforward computation shows that for $|x| \ge R_0$,

$$-\Delta\phi + V(|x|)\phi \ge \frac{C_0}{2}|x|^{(a-2)/2}\phi. \tag{2.15}$$

On the other hand, by (Q), (f_1) and the decay to zero of u at infinity, there exists $\widetilde{R}_0 > R_0 > 0$ sufficiently large such that for $|x| \ge \widetilde{R}_0$

$$Q(|x|)f(u) \le \frac{C_0}{2}|x|^b u(x) \le \frac{C_0}{2}|x|^{(a-2)/2}u(x), \tag{2.16}$$

where in the last inequality we used that $b < (a-2)/2$. Then, combining (2.15) (with $c_0\phi$ instead of ϕ, where c_0 is a positive constant such that $u \le c_0\phi$ on $|x| = \widetilde{R}_0$) and (2.16), we get

$$-\Delta(u - c_0\phi) + \left(V(|x|) - \frac{C_0}{2}|x|^{(a-2)/2}\right)(u - c_0\phi) \le 0 \quad \text{in} \quad |x| \ge \widetilde{R}_0,$$

$$u - c_0\phi \le 0 \quad \text{on} \quad |x| = \widetilde{R}_0,$$

$$\lim_{|x|\to\infty}(u - c_0\phi) = 0.$$

Therefore, by the maximum principle,

$$u(x) \le c_0\phi(x) \quad \text{for} \quad |x| \ge \widetilde{R}_0.$$

To complete the proof of estimate (2.2) for all $x \in \mathbb{R}^2$ it is sufficient to prove that $u \in L^\infty(B_{\widetilde{R}_0})$. Initially, we will show that $u \in L^\infty(B_r)$ for all $0 < r \leq r_0$, where r_0 is given in (1.22). For this notice that

$$-\Delta u = w \quad \text{in} \quad \mathbb{R}^2$$

in the weak sense, where

$$w(x) = Q(|x|)f(u(x)) - V(|x|)u(x), \quad x \in \mathbb{R}^2.$$

According to (2.4) with $q = 1$, we obtain

$$|w| \leq \varepsilon Q|u| + b_1 Q(e^{\alpha u^2} - 1) + V|u|. \tag{2.17}$$

By using the Trudinger-Moser inequality (1.4), Lemma 1.2.5 and the conditions on V and Q at the origin, it follows that $Qu, Q(e^{\alpha u^2} - 1), Vu \in L^p(B_r)$ for some $p \gg 1$. Hence, $w \in L^p(B_r)$ for some $p > 1$. Thus, by elliptic regularity theory $u \in W^{2,p}(B_r)$ and so from Sobolev embedding Theorem (see [2, Theorem 5.4]) $u \in C^\gamma(\overline{B}_r)$ for some $\gamma \in (0,1)$. In the annulus $B_{\widetilde{R}_0} \setminus B_{r_0}$, it follows from the continuity of the potentials V, Q and the consequence of $ii)$ from Lemma 1.2.3 that $Qu, Vu \in L^p(B_{\widetilde{R}_0} \setminus B_{r_0})$ for all $1 \leq p < \infty$. To estimate the second term on the right hand side of (2.17), it is enough to use similar computations as in the **Case 1** of the proof of Theorem 1.1.2 to conclude that $Q(e^{\alpha u^2} - 1) \in L^p(B_{\widetilde{R}_0} \setminus B_{r_0})$ for all $1 \leq p < \infty$. Hence, $w \in L^p(B_{\widetilde{R}_0} \setminus B_{r_0})$ for all $1 \leq p < \infty$. Thus, by elliptic regularity theory, $u \in C^{1,\gamma}(\overline{B_{\widetilde{R}_0} \setminus B_{r_0}})$ for some $\gamma \in (0,1)$ and this completes the proof of theorem. ■

2.3 Proof of the multiplicity theorem

In a general context, let $E = E_1 \oplus E_2$ be a real Banach space with $\dim(E_1) = k < +\infty$. Suppose that I is a $C^1(E, \mathbb{R})$ functional satisfying the following conditions:

(I_1) $I(0) = 0$ and I is even;

(I_2) there exist $\tau, \rho > 0$ such that $I(u) \geq \tau$ if $\|u\| = \rho$;

(I_3) there exists $\mathcal{S} > 0$ such that I satisfies the $(PS)_c$ condition for all $c \in (0, \mathcal{S})$;

(I_4) for any finite-dimensional subspace $\widetilde{F} \subset E$, there exists $R = R(\widetilde{F}) > 0$ such that

$$I(u) \leq 0, \quad \forall u \in \widetilde{F} \setminus B_R.$$

Let $\{e_1, e_2, ..., e_k\}$ be a base of E_1. For each $l \geq k$, take $e_{l+1} \notin E_l \doteq \text{span}\{e_1, e_2, ..., e_l\}$. Consider $R_l = R_l(E_l)$ given by (I_4) and define the sets

$$D_l \doteq B_{R_l} \cap E_l,$$

$$G_l \doteq \{g \in C(D_l, E) : g \text{ is odd and } g(u) = u, \forall u \in \partial B_{R_l} \cap E_l\}$$

and

$$\Gamma_j \doteq \left\{ g\left(\overline{D_l \setminus Y}\right) : g \in G_l, l \geq j, Y \in \Sigma \text{ and } \gamma(Y) \leq l - j \right\},$$

where $\gamma(Y)$ is the genus of the set $Y \in \Sigma$, with

$$\Sigma \doteq \{Y \subset E \setminus \{0\} : Y \text{ is closed in } E \text{ and } Y = -Y\}.$$

Defining now for each $j \in \mathbb{N}$ the following minimax levels

$$c_j \doteq \inf_{K \in \Gamma_j} \sup_{u \in K} I(u)$$

and the set $K_c \doteq \{u \in E : I'(u) = 0 \quad \text{and} \quad I(u) = c\}$, we employ the following abstract result to prove the multiplicity in Theorem 2.1.4. See [28, Theorem 3.1 p. 74] or [53, p. 55].

Proposition 2.3.1 *Under the conditions on I above, the following claims are true:*

i) *for each $j > k$, we have $0 < \tau \leq c_j \leq c_{j+1}$;*

ii) *if $j > k$ and $c_j < \mathcal{S}$, then c_j is a critical value of I;*

iii) *if $j > k$ and $c_j = c_{j+1} = c_{j+2} = ... = c_{j+l} = c < \mathcal{S}$, then $\gamma(K_c) \geq l + 1$.*

In our case, we will consider E our original space in which we are working and E_1 the trivial subspace of E. We see that the energy functional associated to problem (2.3)

$$I_\lambda(u) \doteq \frac{1}{2}\|u\|^2 - \lambda \int_{\mathbb{R}^2} QF(u), \quad u \in E,$$

is well defined and $I_\lambda \in C^1(E, \mathbb{R})$ with derivative given by

$$I'_\lambda(u)v = \int_{\mathbb{R}^2} (\nabla u \nabla v + Vuv) - \lambda \int_{\mathbb{R}^2} Qf(u)v, \quad \forall u, v \in E.$$

Hence, a weak solution $u \in E$ of (2.3) is exactly a critical point of I_λ and reciprocally. Furthermore, since $f(0) = 0$ and f is odd, I_λ satisfies (I_1) and with similar computations to prove $i)$ in Lemma 3.2.1 we conclude that I_λ also verifies (I_2).

To verify (I_3) and (I_4) we consider the following lemma.

Lemma 2.3.2 *Assume that $(V)-(Q)$ hold. If f satisfies $(f_{\alpha_0})-(f_4)$, we have:*

i) the functional I_λ satisfies the $(PS)_c$ condition for all $c \in (0, \alpha'/2\alpha_0)$, that is, any sequence (u_n) in E such that

$$I_\lambda(u_n) \to c \quad \textit{and} \quad I'_\lambda(u_n) \to 0 \tag{2.18}$$

admits a convergent subsequence in E;

ii) for any finite-dimensional subspace $\widetilde{F} \subset E$, there exists $R = R(\widetilde{F}) > 0$ such that

$$I_\lambda(u) \leq 0, \quad \forall u \in \widetilde{F} \setminus B_R.$$

Proof. Using (f_2), a standard computation gives that (u_n) is bounded in E and so, up to a subsequence, $u_n \rightharpoonup u$ weakly in E. Now, as in the Lemma 2.2.6, the convergences

$$\begin{aligned} f(u_n) \to f(u) \qquad & \text{in} \quad L^1_{\text{loc}}(\mathbb{R}^2; Q), \\ F(u_n) \to F(u) \qquad & \text{in} \quad L^1(\mathbb{R}^2; Q). \end{aligned} \tag{2.19}$$

hold. We claim that

$$\int_{\mathbb{R}^2} Qf(u_n)u \to \int_{\mathbb{R}^2} Qf(u)u \quad \text{as} \quad n \to \infty. \tag{2.20}$$

Indeed, since $C^\infty_{0,\text{rad}}(\mathbb{R}^2)$ is dense in E, for all $\delta > 0$, there exists $v \in C^\infty_{0,\text{rad}}(\mathbb{R}^2)$ such that $\|u - v\| < \delta$. Observe that

$$\begin{aligned} &\left| \int_{\mathbb{R}^2} Q[f(u_n) - f(u)]u \right| \\ &\leq \left| \int_{\mathbb{R}^2} Qf(u_n)(u - v) \right| + \left| \int_{\mathbb{R}^2} Qf(u)(u - v) \right| + \|v\|_\infty \int_{\text{supp}(v)} Q|f(u_n) - f(u)|. \end{aligned}$$

To estimate the first integral we use that $|I'_\lambda(u_n)(u - v)| \leq \varepsilon_n \|u - v\|$ with $\varepsilon_n \to 0$ and we conclude that

$$\left| \int_{\mathbb{R}^2} Qf(u_n)(u - v) \right| \leq \varepsilon_n \|u - v\| + \|u_n\| \|u - v\| \leq C\|u - v\| < C\delta,$$

where we have used that (u_n) is bounded in E. Similarly, since the second limit in (2.18) implies that $I'_\lambda(u)(u-v)=0$, we have

$$\left|\int_{\mathbb{R}^2} Qf(u_n)(u-v)\right| < C\delta.$$

To estimate the last integral we use the first limit in 2.19 and conclude by the previous inequalities that

$$\lim_{n\to\infty}\left|\int_{\mathbb{R}^2} Q[f(u_n)-f(u)]u\right| < 2C\delta.$$

Since $\delta>0$ is arbitrary, the claim follows. Hence, passing to the limit when $n\to\infty$ in

$$o_n(1) = I'_\lambda(u_n)u = \int_{\mathbb{R}^2}(\nabla u_n\nabla u + Vu_nu) - \lambda\int_{\mathbb{R}^2} Qf(u_n)u$$

and using that $u_n \rightharpoonup u$ weakly in E, (2.20) and (f_2) we get

$$\|u\|^2 = \lambda\int_{\mathbb{R}^2} Qf(u)u \geq 2\lambda\int_{\mathbb{R}^2} QF(u),$$

which implies that

$$I_\lambda(u)\geq 0. \tag{2.21}$$

We have two cases to consider:

Case 1. $u=0$. This case is similar to the checking that the solution u obtained in the Theorem 2.1.2 is nontrivial.

Case 2. $u\neq 0$. In this case, we define

$$v_n = \frac{u_n}{\|u_n\|} \quad \text{and} \quad v = \frac{u}{\lim\|u_n\|}.$$

It follows that $v_n \rightharpoonup v$ weakly in E, $\|v_n\|=1$ and $\|v\|\leq 1$. If $\|v\|=1$, we conclude the proof. If $\|v\|<1$, we claim that there exist $r_1>1$ sufficiently close to 1, $\alpha>\alpha_0$ close to α_0 and $\beta>0$ such that $r_1\alpha\|u_n\|^2\leq\beta<\alpha'(1-\|v\|^2)^{-1}$ for $n\in\mathbb{N}$ large. Indeed, since

$$I_\lambda(u_n) = c + o_n(1)$$

and

$$F(u_n)\to F(u) \quad \text{in} \quad L^1(\mathbb{R}^2;Q)$$

we have that

$$\frac{1}{2}\lim_{n\to\infty}\|u_n\|^2 = c + \lambda\int_{\mathbb{R}^2} QF(u). \tag{2.22}$$

Setting

$$A \doteq \left(c + \lambda \int_{\mathbb{R}^2} QF(u)\right)\left(1 - \|v\|^2\right),$$

then by (2.22) and the definition of v, we obtain

$$A = c - I_\lambda(u).$$

Hence, coming back to (2.22) and using (2.21), we conclude that

$$\frac{1}{2}\lim_{n\to\infty}\|u_n\|^2 = \frac{A}{1-\|v\|^2} = \frac{c - I_\lambda(u)}{1-\|v\|^2} \leq \frac{c}{1-\|v\|^2} < \frac{\alpha'}{2\alpha_0\left(1-\|v\|^2\right)}.$$

Consequently, for $n \in \mathbb{N}$ large there exist $r_1 > 1$ sufficiently close to 1, $\alpha > \alpha_0$ close to α_0 and $\beta > 0$ such that $r_1\alpha\|u_n\|^2 \leq \beta < \alpha'(1-\|v\|^2)^{-1}$, and the claim is proved. Therefore, from Corollary 1.1.4,

$$\int_{\mathbb{R}^2} Q(e^{\alpha u_n^2} - 1)^{r_1} \leq C. \tag{2.23}$$

Next, we claim that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} Qf(u_n)(u_n - u) = 0.$$

Indeed, let $r_1, r_2, r_3 > 1$ be such that $1/r_1 + 1/r_2 + 1/r_3 = 1$ and $(q-1)r_2 \geq 2$. Thus, by (2.4) and the Hölder's inequality we conclude that

$$\begin{aligned}\left|\int_{\mathbb{R}^2} Qf(u_n)(u_n - u)\right| &\leq \varepsilon\left(\int_{\mathbb{R}^2} Q|u_n|^2\right)^{1/2}\left(\int_{\mathbb{R}^2} Q|u_n - u|^2\right)^{1/2} \\ &+ b_1\left(\int_{\mathbb{R}^2} Q(e^{\alpha u_n^2} - 1)^{r_1}\right)^{1/r_1}\left(\int_{\mathbb{R}^2} Q|u_n|^{(q-1)r_2}\right)^{1/r_2}\left(\int_{\mathbb{R}^2} Q|u_n - u|^{r_3}\right)^{1/r_3}.\end{aligned}$$

Then, by Lemma 1.2.5 and (2.23) the claim follows. This convergence together with the fact that $I'_\lambda(u_n)(u_n - u) = o_n(1)$ imply that

$$\|u_n\|^2 = \int_{\mathbb{R}^2}(\nabla u_n \nabla u + V u_n u) + o_n(1).$$

Since $u_n \rightharpoonup u$ weakly in E, we obtain $u_n \to u$ strongly in E. Therefore, $i)$ is proved. Given $u \in E \hookrightarrow L^\nu(\mathbb{R}^2; Q)$, from (f_4) we get

$$\begin{aligned}I_\lambda(u) &\leq \frac{1}{2}\|u\|^2 - \frac{\mu\lambda}{\nu}\int_{\mathbb{R}^2} Q|u|^\nu \\ &= \frac{1}{2}\|u\|^2 - \frac{\mu\lambda}{\nu}\|u\|^\nu_{L^\nu(\mathbb{R}^2;Q)}.\end{aligned}$$

For any finite-dimensional subspace $\widetilde{F} \subset E$, since all norms on $\widetilde{F}$ are equivalent, it follows that there exists $C > 0$ such that

$$\begin{aligned} I_\lambda(u) &\leq \frac{1}{2}\|u\|_{\widetilde{F}}^2 - \frac{\mu\lambda}{C\nu}\|u\|_{\widetilde{F}}^\nu \\ &= \|u\|_{\widetilde{F}}^2\left(\frac{1}{2} - \frac{\mu\lambda}{C\nu}\|u\|_{\widetilde{F}}^{\nu-2}\right), \end{aligned}$$

for $u \in \widetilde{F}$. Choosing $R > 0$ such that

$$\frac{1}{2} - \frac{\mu\lambda}{C\nu}R^{\nu-2} < 0,$$

we obtain

$$I_\lambda(u) \leq 0, \quad \forall u \in \widetilde{F} \setminus B_R.$$

The proof of the lemma is concluded. ■

For each $k \in \mathbb{N}$, consider E_k a finite-dimensional subspace with $\dim(E_k) = k$. By $ii)$ from Lemma 2.3.2, there exists $R_k > 0$ such that

$$I_\lambda(u) \leq 0, \quad \forall u \in E_k \setminus B_{R_k}.$$

Thus, considering D_k, G_k and Γ_k above we define

$$c_k^\lambda \doteq \inf_{K\in\Gamma_k} \sup_{u\in K} I_\lambda(u).$$

Lemma 2.3.3 *For each $k \in \mathbb{N}$, there exists $0 < M_k < \infty$ such that*

$$c_k^\lambda \leq M_k \lambda^{2/(2-\nu)},$$

where $\nu > 2$ is given in (f_4).

Proof. Since the identity map is in G_k, we will consider $K = \overline{D}_k \in \Gamma_k$. By the definition of the minimax level, we have

$$c_k^\lambda = \inf_{K\in\Gamma_k} \max_{u\in K} I_\lambda(u) \leq \max_{u\in K}\left[\frac{1}{2}\|u\|^2 - \lambda\int_{\mathbb{R}^2} QF(u)\right].$$

Thus, by (f_4)

$$c_k^\lambda \leq \max_{u\in K}\left[\frac{1}{2}\|u\|^2 - \frac{\mu\lambda}{\nu}\int_{\mathbb{R}^2} Q|u|^\nu\right] = \max_{u\in K}\left[\frac{1}{2}\|u\|^2 - \frac{\mu\lambda}{\nu}\|u\|_{L^\nu(\mathbb{R}^2;Q)}^\nu\right].$$

Now, since $\dim(K) < \infty$, there exists $C > 0$ such that

$$c_k^\lambda \leq \max_{u\in K}\left[\frac{1}{2}\|u\|_K^2 - \frac{\mu\lambda}{C\nu}\|u\|_K^\nu\right] = \left(\frac{1}{2} - \frac{1}{\nu}\right)\left(\frac{C}{\mu}\right)^{2/(\nu-2)}\lambda^{2/(2-\nu)}.$$

Therefore, setting

$$M_k \doteq \left(\frac{1}{2}-\frac{1}{\nu}\right)\left(\frac{C}{\mu}\right)^{2/(\nu-2)}$$

and observing that $\nu > 2$ we obtain the desired result. ■

Proof of Theorem 2.1.4. For each $k \in \mathbb{N}$, let M_k from the Lemma 2.3.3. Choosing λ_k such that

$$M_k < \frac{\alpha'}{2\alpha_0}\lambda_k^{2/(\nu-2)} \tag{2.24}$$

and combining $i)$ from Proposition 2.3.1, Lemma 2.3.3 and (2.24) we conclude that, for all $\lambda > \lambda_k$,

$$0 < c_1^\lambda \leq c_2^\lambda \leq ... \leq c_k^\lambda \leq M_k\lambda^{2/(2-\nu)} < M_k\lambda_k^{2/(2-\nu)} < \frac{\alpha'}{2\alpha_0} \doteq \mathcal{S}.$$

Furthermore, by $ii)$, the levels $c_1^\lambda \leq c_2^\lambda \leq \ldots \leq c_k^\lambda$ are critical values of I_λ. Since I_λ is even in E, we can associate at least k pairs of critical points. Finally, we observe that if $c_j^\lambda = c_{j+1}^\lambda$ for some $j = 1, 2, \ldots, k-1$, $iii)$ implies that $\gamma(K_{c_j^\lambda}) \geq 2$ and consequently $K_{c_j^\lambda}$ is an infinite set. Therefore, in this case, the problem (2.3) has infinitely many solutions in E. This completes the proof of Theorem 2.1.4. ■

2.4 A nonexistence result

With the purpose to investigate new rages of a and b for the nonexistence of solutions of problem (2.3), we shall need to assume the following simultaneous condition on V and Q:

$$(VQ)\quad \lim_{|x|\to+\infty}\frac{V(x)}{|x|^a} < \infty \quad \text{and} \quad \lim_{|x|\to+\infty}\frac{Q(x)}{|x|^b} > 0, \quad \text{with} \quad a < -2 < b.$$

We quote that in this hypothesis we are not supposing that V and Q are radial.

We remark that from (f_{α_0}) with $\alpha < \alpha_0$, conditions (f_2) and (f_4), for any $p \geq \nu-1$ there exists $C_0 > 0$ such that

$$f(s) \geq C_0 s^p, \quad \text{for all} \quad s \geq 0. \tag{2.25}$$

Indeed, from (f_{α_0}) with $\alpha < \alpha_0$ there exists $s_0 \gg 1$ such that, for any given $p > 1$,

$$f(s) \geq C_1 s^p, \quad \text{for all} \quad s \geq s_0.$$

On the other hand, (f_2) and (f_4) imply that, for any given $p \geq \nu - 1$,

$$f(s) \geq C_2 s^{\nu-1} \geq C_2 s^p, \quad \text{for all} \quad s \in [0, s_0].$$

Hence, for any $p \geq \nu - 1$,

$$f(s) \geq \min\{C_1, C_2\} s^p = C_0 s^p, \quad \text{for all} \quad s \geq 0.$$

Our main result of this section is summarized in the following:

Theorem 2.4.1 (Nonexistence) *Assume that* (VQ) *holds. If* f *satisfies* (2.25), *then the problem* (2.3) *has no* C^2 *positive solutions for* λ *large.*

Remark 2.4.2 *In this way, the above ranges of* a *and* b *of existence and nonexistence of solutions of* (2.3) *can be summarized in the figure 2.1.*

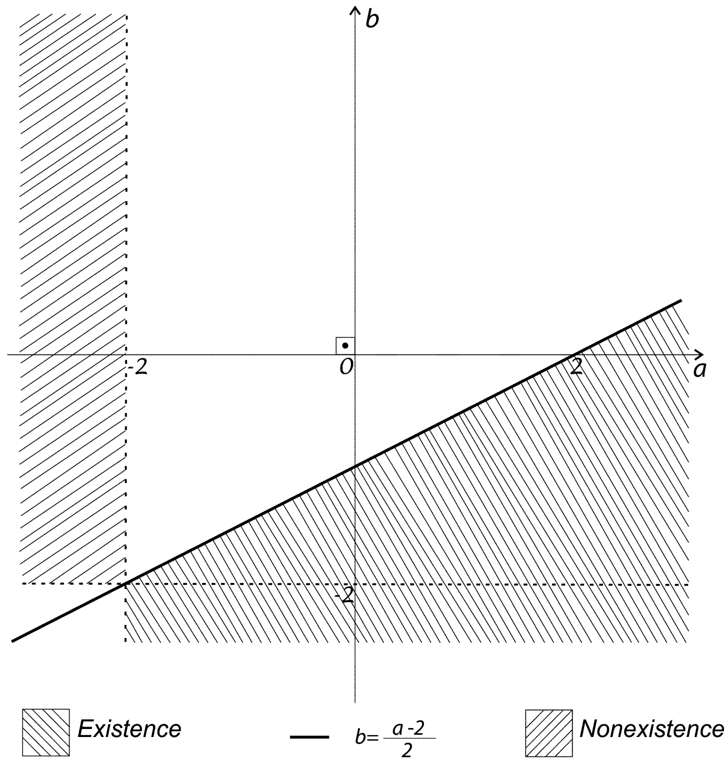

Figure 2.1: Regions of existence and nonexistence of solutions

The proof of Theorem 2.4.1 is based on an averaging process in which we reduce the problem to an ordinary differential inequality in order to get a contradiction via some elementary arguments. We quote that this method was used in the papers [14, 19, 69]. Before we need some technical lemmas. We denote the spherical average $\bar{u}$ of a function $u \in C(\mathbb{R}^2)$ by

$$\bar{u}(r) \doteq \frac{1}{|\partial B_r|} \int_{\partial B_r} u(x) d\sigma,$$

where $d\sigma$ is the standard volume element on ∂B_r. It is standard to verify that if $u \in C^2(\mathbb{R}^2)$ then the following holds:

Lemma 2.4.3 *(i)* $\overline{u+v} = \bar{u} + \bar{v}$;

(ii) $\dfrac{d}{dr}(r\bar{u}'(r)) = r\Delta\bar{u}(r)$;

(iii) $\bar{u}''(r) + \dfrac{1}{r}\bar{u}'(r) = \Delta\bar{u}(r)$ *(Darboux's equation);*

(iv) $\overline{\Delta u}(r) = \Delta\bar{u}(r)$;

(v) $\bar{u}^p \leq \overline{u^p}$, *for all* $p > 1$ *(Jensen's inequality).*

Proof. See [37]. ■

Lemma 2.4.4 *Assume that* (VQ) *holds. If* u *is a* C^2 *positive solution of problem* (2.3)*, then setting* $w(t) = r^m\bar{u}(r)$ *with* $m = (b+2)/(p-1)$*,* $p \geq \nu - 1$ *and* $t = \log r$*, there exist real numbers* l_1 *and* l_2 *such that for* t *large* w *satisfies*

$$w'' + l_1 w' + (l_2 - V(r)r^2)w + w^p \leq 0, \tag{2.26}$$

where $V(r) = \max_{|x|=r} V(x)$.

Proof. If u is C^2 nontrivial positive solution of problem (2.3), we have

$$-\Delta u - V(x)u = \lambda Q(x)f(u) \quad \text{in} \quad \mathbb{R}^2. \tag{2.27}$$

Now, from the second limit in hypothesis (VQ) there exist $C_0, R_0 > 0$ such that

$$Q(x) \geq C_0|x|^b, \quad \forall |x| \geq R_0.$$

From this and (2.25), it follows for any $p \geq \nu - 1$ that

$$\lambda Q(x)f(u) \geq \lambda C_0 Q(x)u^p \geq \lambda C_1|x|^b u^p,$$

for all $|x| \geq R_0$. Thus for $\lambda C_1 \geq 1$, we get

$$\lambda Q(x)f(u) \geq |x|^b u^p, \quad \forall |x| \geq R_0. \tag{2.28}$$

Hence, from (2.27) and (2.28) u satisfies

$$\Delta u - V(x)u + |x|^b u^p \leq 0 \quad \text{in} \quad \mathbb{R}^2 \setminus B_{R_0}. \tag{2.29}$$

By taking the spherical average in (2.29) and using Lemma 2.4.3, we obtain

$$\bar{u}''(r) + \frac{1}{r}\bar{u}'(r) - V(r)\bar{u}(r) + r^b\bar{u}^p(r) \leq 0, \quad \text{for} \quad r > R_0, \tag{2.30}$$

where $V(r) = \max_{|x|=r} V(x)$. Setting $w(t) = r^m\bar{u}(r)$ with $m = (b+2)/(p-1)$ and $t = \log r$, we see that

$$\begin{aligned} w'(t) &= mr^m\bar{u}(r) + r^{m+1}\bar{u}'(r), \\ w''(t) &= m^2r^m\bar{u}(r) + mr^{m+1}\bar{u}'(r) + (m+1)r^{m+1}\bar{u}'(r) + r^{m+2}\bar{u}''(r), \\ l_1w'(t) &= -2m^2r^m\bar{u}(r) - 2mr^{m+1}\bar{u}'(r), \\ (l_2 - V(r)r^2)w(t) &= m^2r^m\bar{u}(r) - V(r)r^{m+2}\bar{u}(r), \end{aligned}$$

where $l_1 = -2m$ and $l_2 = m^2$. Thus, by using (2.30), we get (2.26) for t large. The proof of lemma is finished. ■

Proof of Theorem 2.4.1. We shall use similar arguments developed in [19]. Suppose by contradiction that u is a C^2 nontrivial positive solution of problem (2.3). We have three cases to consider:

Case 1. $w'(T) < 0$ for some T sufficiently large. We set $B(r) \doteq l_2 - V(r)r^2$. We claim that from the first limit in hypothesis (VQ) with the condition $a < -2$, we have $B + w^{p-1} \geq 0$ at infinity. In fact, it just observes that

$$V(x)|x|^2 = \frac{V(x)}{|x|^a}|x|^{a+2} \to 0 \quad \text{as} \quad |x| \to +\infty.$$

Thus,

$$V(x)|x|^2 \leq l_2, \quad \forall |x| \gg 1,$$

which implies

$$V(r)r^2 \leq l_2,$$

for $r \gg 1$. Consequently, $B(r) \geq 0$ at infinity and the claim follows. Integrating (2.26) over $[T, t]$ for T large, we have

$$w'(t) \leq e^{-l_1(t-T)}w'(T) - e^{-l_1t}\int_T^t (B + w^{p-1})we^{l_1s}\mathrm{ds} \leq e^{-l_1(t-T)}w'(T).$$

Since $l_1 < 0$, integrating the above inequality over $[T, t]$, we obtain

$$0 < w(t) \leq w(T) + \frac{1}{l_1}w'(T)\left(1 - e^{-l_1(t-T)}\right) \to -\infty \quad \text{as} \quad t \to +\infty,$$

which is a contradiction.

Case 2. w is non-decreasing and bounded at infinity. Then there exists $w_\infty > 0$ such that $w(t) \to w_\infty$ as $t \to \infty$. Thus, there exists a real sequence (t_n) with $\lim_{n\to\infty} t_n = \infty$ such that $w'(t_n), w''(t_n) \to 0$ as $n \to \infty$, which implies by passing to the inferior limit in (2.26)

$$0 < w_\infty^p \leq m^2 + \liminf_{n\to\infty} w(t_n)^p \leq \liminf_{n\to\infty} \left(\left(B(e^{t_n}) + w(t_n)^{p-1} \right) w(t_n) \right) \leq 0,$$

which is a contradiction.

Case 3. w is non-decreasing and unbounded at infinity. Setting $v(t) = e^{\frac{l_1}{2}t} w(t)$, we have

$$v''(t) + D(t)v \leq 0, \tag{2.31}$$

where $D(t) \doteq B(e^t) - l_1^2/4 + w(t)^{p-1}$. Multiplying both sides of (2.31) by $\sin t$ and integrating by parts twice over $[2k\pi, (2k+1)\pi]$ with integer $k > 0$, we obtain

$$\int_{2k\pi}^{(2k+1)\pi} (D(t) - 1)v(t) \sin t \mathrm{d}t \leq -v(2k\pi) - v((2k+1)\pi) \leq 0. \tag{2.32}$$

Since $D(t) \to \infty$ as $t \to \infty$, we have in particular that $D > 1$ on $[2k\pi, (2k+1)\pi]$ for $k > 0$ sufficiently large, which contradicts inequality (2.32) and this completes the proof of theorem. ∎

Chapter 3

On a class of nonhomogeneous Schrödinger equations involving exponential critical growth in $\mathbb{R}^2$

This chapter is concerned with the existence and multiplicity of solutions for nonlinear elliptic equations of the form

$$-\Delta u + V(|x|)u = Q(|x|)f(u) + h(x) \quad \text{in} \quad \mathbb{R}^2, \tag{3.1}$$

where V and Q are unbounded, singular or decaying radial potentials, the nonlinearity $f(s)$ has exponential critical growth and the nonhomogeneous term h belongs to the dual of an appropriate functional space. By combining minimax methods and the Trudinger-Moser inequality (1.4), we establish the existence and multiplicity of weak solutions for this class of equations by controlling the size of h. We point out that part of this chapter is contained in the preprint [9].

3.1 Introduction and main results

Initially, we assume the following hypotheses on $V(|x|)$ and $Q(|x|)$:

(V) $V \in C(0,\infty)$, $V(r) > 0$ and there exists $a > -2$ such that

$$\liminf_{r\to+\infty} \frac{V(r)}{r^a} > 0.$$

(Q) $Q \in C(0,\infty)$, $Q(r) > 0$ and there exist $b < (a-2)/2$ and $b_0 > -2$ such that

$$\limsup_{r\to 0} \frac{Q(r)}{r^{b_0}} < \infty \quad \text{and} \quad \limsup_{r\to +\infty} \frac{Q(r)}{r^{b}} < \infty.$$

We will assume that the nonlinearity $f(s)$ is continuous and satisfies:

(f_1) $f(s) = o(s)$ as $s \to 0$;

(f_2) there exists $\theta > 2$ such that

$$0 < \theta F(s) \doteq \theta \int_0^s f(t)\mathrm{d}t \leq s f(s), \quad \forall s \neq 0.$$

Now, we are ready to state our first existence result.

Theorem 3.1.1 (Existence) *Suppose that* $(V)-(Q)$ *hold. If* f *satisfies* $(f_{\alpha_0})-(f_2)$, *then there exists* $\delta_1 > 0$ *such that if* $0 < \|h\|_{E^{-1}} < \delta_1$, *the problem* (3.1) *has a nontrivial weak solution* u_h *in* E.

In order to establish our next result, instead of (Q) we will need of the more restrict hypothesis $(\widetilde{Q})$ introduced in the Chapter 1:

$(\widetilde{Q})$ $Q \in C(0,\infty)$, $Q(r) > 0$ and there exist $b < (a-2)/2$ and $-2 < b_0 \leq 0$ such that

$$0 < \liminf_{r\to 0} \frac{Q(r)}{r^{b_0}} \leq \limsup_{r\to 0} \frac{Q(r)}{r^{b_0}} < \infty \quad \text{and} \quad \limsup_{r\to +\infty} \frac{Q(r)}{r^{b}} < \infty.$$

In relation to the nonlinearity $f(s)$, make the following additional hypothesis:

(f_3) there exist constants $R_0, M_0 > 0$ such that

$$0 < F(s) \leq M_0 |f(s)|, \quad \forall |s| \geq R_0;$$

(f_4) there exists $\beta_0 > 0$ such that

$$\liminf_{|s|\to\infty} \frac{s f(s)}{e^{\alpha_0 s^2}} \geq \beta_0 > \begin{cases} \dfrac{4}{C_0\alpha_0} \dfrac{e^{2m(r_0)}}{r_0^2}, & \text{if} \quad b_0 = 0; \\ \dfrac{b_0+2}{C_0\alpha_0} \dfrac{1}{r_0^{b_0+2}}, & \text{if} \quad -2 < b_0 < 0, \end{cases}$$

where

$$m(r) = \frac{2C_0 r^{a_0+2}}{(a_0+2)^3},$$

with $0 < r \leq r_0$ and r_0 given in (1.22).

Remark 3.1.2 *A simple model of a function that verifies our assumptions is*

$$f(s) = s|s| + 2se^{s^2} - 2s,$$

for $s \in \mathbb{R}$. *Clearly* (f_{α_0}) *is satisfied with* $\alpha_0 = 1$ *and* (f_1) *holds since*

$$\lim_{|s|\to 0} \frac{f(s)}{s} = \lim_{|s|\to 0} \left(|s| + 2e^{s^2} - 2\right) = 0.$$

In order to prove that (f_2) *is satisfied, notice that*

$$F(s) = \frac{1}{3}|s|^3 + e^{s^2} - s^2 - 1.$$

Thus

$$\begin{aligned} sf(s) - \theta F(s) &= \left(1 - \frac{\theta}{3}\right)|s|^3 + e^{s^2}(2s^2 - \theta) - s^2(2-\theta) + \theta \\ &\geq e^{s^2}(2s^2 - \theta) - s^2(2-\theta) + \theta \\ &\geq 2s^2 - \theta - 2s^2 + \theta s^2 + \theta \\ &= \theta s^2 \geq 0, \end{aligned}$$

for $|s| \geq \sqrt{\theta/2}$ *and* $2 < \theta < 3$. *If* $|s| < \sqrt{\theta/2}$, *then*

$$sf(s) - \theta F(s) \geq -\theta - 2s^2 + \theta s^2 + \theta = s^2(\theta - 2) > 0,$$

provided $s \neq 0$ *and* $\theta > 2$. *For* (f_3), *it is enough to notice that*

$$\lim_{|s|\to\infty} \frac{F(s)}{f(s)} = \lim_{|s|\to\infty} \frac{\frac{1}{3}|s|^3 + e^{s^2} - s^2 - 1}{s|s| + 2se^{s^2} - 2s} = 0.$$

Finally, it is easy to see that

$$\lim_{|s|\to\infty} sf(s)e^{-s^2} = +\infty,$$

showing that (f_4) *holds.*

Our multiplicity result can be stated as follows.

Theorem 3.1.3 (Multiplicity) *Assume that* $(V) - (\widetilde{Q})$ *and* $(\widetilde{V})$ *hold. If* f *satisfies* $(f_{\alpha_0}) - (f_4)$, *then there exists* $\delta_2 > 0$ *such that if* $0 < \|h\|_{E^{-1}} < \delta_2$, *the problem* (3.1) *has at least two nontrivial weak solutions in* E.

3.2 Proof of the existence theorem

We establish the necessary functional framework where solutions are naturally studied by variational methods. Given $u \in E$ we can use (2.5) with $q = 2$ to obtain

$$\int_{\mathbb{R}^2} QF(u) \leq \frac{\varepsilon}{2}\int_{\mathbb{R}^2} Q|u|^2 + b_2\int_{\mathbb{R}^2} Q|u|^2(e^{\alpha u^2}-1).$$

By Lemma 1.2.5,

$$\int_{\mathbb{R}^2} Q|u|^2 < \infty.$$

Now, let $r_1, r_2 > 1$ be such that $1/r_1 + 1/r_2 = 1$. The Hölder's inequality, Lemma 1.2.5 and Theorem 1.1.2 imply that

$$\int_{\mathbb{R}^2} Q|u|^2(e^{\alpha u^2}-1) \leq \left(\int_{\mathbb{R}^2} Q|u|^{2r_1}\right)^{1/r_1}\left(\int_{\mathbb{R}^2} Q(e^{r_2\alpha u^2}-1)\right)^{1/r_2} < \infty,$$

where we have used the elementary inequality $(e^s-1)^r \leq (e^{rs}-1)$, for all $r \geq 1, s \geq 0$. Therefore, the energy functional associated to problem (3.1) $I : E \to \mathbb{R}$ defined by

$$I(u) \doteq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^2} QF(u) - \int_{\mathbb{R}^2} hu$$

is well defined and $I \in C^1(E, \mathbb{R})$ with derivative given by

$$I'(u)v = \int_{\mathbb{R}^2}(\nabla u\nabla v + Vuv) - \int_{\mathbb{R}^2} Qf(u)v - \int_{\mathbb{R}^2} hv, \quad \forall u, v \in E.$$

Thus, since we are searching for weak solutions for problem (3.1), that is, functions $u \in E$ such that

$$\int_{\mathbb{R}^2}(\nabla u\nabla v + Vuv) - \int_{\mathbb{R}^2} Qf(u)v - \int_{\mathbb{R}^2} hv = 0,$$

for all $v \in E$, we conclude that a weak solution of (3.1) is exactly a critical point of I and reciprocally.

Next lemma describe the geometric structure of the functional I required by the Mountain-Pass Theorem. We will denote $\int_{\mathbb{R}^2} hu$ by the dual pairing $\langle h, u\rangle$, for any $u \in E$.

Lemma 3.2.1 *Suppose that $(V) - (Q)$ hold and f satisfies $(f_{\alpha_0}) - (f_2)$. There exists $\delta_1 > 0$ such that, for each $h \in E^{-1}$ with $0 < \|h\|_{E^{-1}} \leq \delta_1$, there hold:*

i) there exist $\tau_h, \rho_h > 0$ such that $I(u) \geq \tau_h$ if $\|u\| = \rho_h$. Furthermore, ρ_h can be chosen such that $\rho_h \to 0$ as $\|h\|_{E^{-1}} \to 0$;

ii) there exists $e_h \in E$, with $\|e_h\| > \rho_h$, such that

$$I(e_h) < \inf_{B_{\rho_h}} I < 0.$$

Proof. By using (2.5) with $q > 2$, the Hölder's inequality, Lemma 1.2.5 and Theorem 1.1.2 we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} QF(u) &\leq \frac{\varepsilon}{2}\int_{\mathbb{R}^2} Q|u|^2 + b_2\int_{\mathbb{R}^2} Q|u|^q(e^{\alpha u^2}-1)\\ &\leq \frac{C\varepsilon}{2}\|u\|^2 + b_2\left(\int_{\mathbb{R}^2} Q|u|^{qr_1}\right)^{1/r_1}\left(\int_{\mathbb{R}^2} Q(e^{r_2\alpha u^2}-1)\right)^{1/r_2}\\ &\leq \frac{C\varepsilon}{2}\|u\|^2 + \widetilde{C}\|u\|^q,\end{aligned}$$

whenever $\|u\| \leq M < (\alpha'/\alpha)^{1/2}$ and $r_2 > 1$ is sufficiently close to 1. Consequently,

$$I(u) \geq \left(\frac{1}{2} - \frac{C\varepsilon}{2}\right)\|u\|^2 - \widetilde{C}\|u\|^q - \|h\|_{E^{-1}}\|u\|.$$

Choosing $\varepsilon = 1/(2C)$, we get

$$I(u) \geq \|u\|\left(\frac{1}{4}\|u\| - \widetilde{C}\|u\|^{q-1} - \|h\|_{E^{-1}}\right).$$

Since $q > 2$, we may choose $\rho > 0$ such that $\frac{1}{4}\rho - \widetilde{C}\rho^{q-1} > 0$. Thus, for $\|h\|_{E^{-1}}$ sufficiently small there exists $0 < \rho_h < (\alpha'/\alpha)^{1/2}$ such that

$$I(u) > 0 \quad \text{if} \quad \|u\| = \rho_h$$

and

$$\rho_h \to 0 \quad \text{as} \quad \|h\|_{E^{-1}} \to 0.$$

In order to verify *ii*), we note that from (f_2) there exists constants $A, B > 0$ such that

$$F(s) \geq A|s|^\theta - B,$$

for all $s \in \mathbb{R}$. If we take a function $\phi \in C^\infty_{0,\mathrm{rad}}(\mathbb{R}^2) \setminus \{0\}$, then for $t \geq 0$ we have

$$I(t\phi) \leq \frac{t^2}{2}\|\phi\|^2 - At^\theta\int_G Q|\phi|^\theta + B\int_{\mathrm{supp}(\phi)} Q + t\|h\|_{E^{-1}}\|\phi\|.$$

Hence, since $\theta > 2$ we conclude that $I(t\phi) \to -\infty$ as $t \to \infty$. Thus, if we set $e_h = t_*\phi$ for $t_* > 0$ large, then we conclude that

$$I(e_h) < \inf_{u \in B_{\rho_h}} I(u).$$

Finally, it remains to show that

$$\inf_{u\in B_{\rho_h}} I(u) < 0.$$

Since $h \in E'$, by the Riesz Representation Theorem, there exists an unique function in E, denoted by v_h, such that

$$\langle h, u\rangle = \int_{\mathbb{R}^2} (\nabla u \nabla v_h + V u v_h),$$

for $u \in E$. Thus, we have that $\langle h, v_h\rangle = \|v_h\|^2 > 0$, whenever $h \neq 0$. Since $f(0) = 0$, it follows by continuity that there exists $\eta_h > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{dt}} I(tv_h) = t\|v_h\|^2 - \int_{\mathbb{R}^2} Qf(tv_h)v_h - \langle h, v_h\rangle < 0,$$

for all $0 < t < \eta_h$. Hence the function $t \mapsto I(tv_h)$ is decreasing in $(0, \eta_h)$. Since $I(0) = 0$, we must have $I(tv_h) < 0$ for all $0 < t < \eta_h$ and the result follows. ∎

Proof of Theorem 3.1.1. Let ρ_h be given by Lemma 3.2.1. Since $\rho_h \to 0$ as $\|h\|_{E^{-1}} \to 0$, we can choose $\|h\|_{E^{-1}}$ small enough in such way $\rho_h < (\alpha'/\alpha_0)^{1/2}$. Let $I_\infty \doteq \inf_{B_{\rho_h}} I < 0$. Consider the complete metric space $\overline{B}_{\rho_h} \doteq \{u \in E : \|u\| \leq \rho_h\}$ with metric given by $d(u, v) = \|u - v\|$. The functional I is bounded from below in B_{ρ_h}, for $0 < \|h\|_{E^{-1}} \leq \delta_1$ (see Lemma 3.2.1). Applying the Ekeland's Variational Principle [36, 68], we obtain a minimizing sequence (u_n) in $\overline{B}_{\rho_h}$ such that

$$I(u_n) \to I_\infty \quad \text{and} \quad I'(u_n) \to 0.$$

Observing that

$$\liminf_{n\to\infty} \|u_n\|^2 \leq \rho_h^2 < \frac{\alpha'}{\alpha_0},$$

we infer that for $n \in \mathbb{N}$ large, there exist $r_1 > 1$ sufficiently close to 1 and $\alpha > \alpha_0$ close to α_0 such that $r_1\alpha\|u_n\|^2 \leq \rho_h^2 < \alpha'$. Thus, by Theorem 1.1.2, there exists $C > 0$ such that

$$\int_{\mathbb{R}^2} Q(e^{\alpha u_n^2} - 1)^{r_1} \leq \int_{\mathbb{R}^2} Q\left(e^{r_1\alpha\|u_n\|^2(u_n/\|u_n\|)^2} - 1\right) \leq C. \tag{3.2}$$

Since $(u_n) \subset E$ is bounded we may suppose that there exists $u_h \in E$ such that $u_n \rightharpoonup u_h$ weakly in E. Next, we claim that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} Qf(u_n)(u_n - u_h) = 0.$$

Indeed, let $r_1, r_2, r_3 > 1$ be such that $1/r_1 + 1/r_2 + 1/r_3 = 1$ and $(q-1)r_2 \geq 2$. By (2.4) and the Hölder's inequality we conclude that

$$\left|\int_{\mathbb{R}^2} Qf(u_n)(u_n - u_h)\right| \leq \varepsilon \left(\int_{\mathbb{R}^2} Q|u_n|^2\right)^{1/2} \left(\int_{\mathbb{R}^2} Q|u_n - u_h|^2\right)^{1/2}$$
$$+ b_1 \left(\int_{\mathbb{R}^2} Q(e^{\alpha u_n^2} - 1)^{r_1}\right)^{1/r_1} \left(\int_{\mathbb{R}^2} Q|u_n|^{(q-1)r_2}\right)^{1/r_2} \left(\int_{\mathbb{R}^2} Q|u_n - u_h|^{r_3}\right)^{1/r_3}.$$

Thus, by Lemma 1.2.5 and (3.2), the claim follows. This convergence together with the fact that $I'(u_n)(u_n - u_h) = o_n(1)$ imply that

$$\|u_n\|^2 = \int_{\mathbb{R}^2} (\nabla u_n \nabla u_h + V u_n u_h) + o_n(1).$$

Hence, since $u_n \rightharpoonup u_h$ weakly in E, we obtain by passing the limit that

$$\lim_{n\to\infty} \|u_n\|^2 = \|u_h\|^2$$

and then $u_n \to u_h$ strongly in E. Therefore, $I(u_h) = I_\infty < 0$ and consequently u_h is a nontrivial weak solution of problem (3.1). ■

3.3 Proof of the multiplicity theorem

The proof that we are going to present is based on the Mountain-Pass Theorem [12]. Before we need to obtain a local compactness result and make a careful estimate of the minimax level of the functional I. We state below these results.

Proposition 3.3.1 *Suppose that $(V) - (Q)$ hold. If f satisfies $(f_{\alpha_0}) - (f_3)$, then the functional I satisfies the $(PS)_d$ condition for any*

$$d < I(u_h) + \frac{\alpha'}{2\alpha_0},$$

provided 0 *and u_h are the only critical points of I.*

Proposition 3.3.2 *Assume that $(V) - (\widetilde{Q})$ and $(\widetilde{V})$ hold. Suppose f satisfies $(f_{\alpha_0}) - (f_2)$ and (f_4) and let $\delta_1 > 0$ and $u_h \in E$ be given by Theorem 3.1.1. Then there exists $0 < \delta_2 \leq \delta_1$ such that, for all $h \in E^{-1}$ such that $0 < \|h\|_{E^{-1}} < \delta_2$, there exists $v \in E$ with compact support such that*

$$\max_{t\geq 0} I(tv) < I(u_h) + \frac{\alpha'}{2\alpha_0}. \tag{3.3}$$

Assuming the propositions above, which will be proved later, we show how they can be applied to prove our multiplicity result.

Proof of Theorem 3.1.3. Let δ_2 be obtained in the Proposition 3.3.2. Arguing by contradiction, we suppose that $0 < \|h\|_{E^{-1}} < \delta_2$ but the function I has no critical points different from 0 and u_h. Now let v be obtained in the Proposition 3.3.1. Recalling that from (f_2) we get $F(s) \geq A|s|^\theta - B$ for all $s \in \mathbb{R}$, we have for $t > 0$

$$I(tv) \leq \frac{t^2}{2}\|v\|^2 - At^\theta \int_{\text{supp}(v)} Q|v|^\theta + B \int_{\text{supp}(v)} Q + t\|h\|_{E^{-1}}\|v\|.$$

Since $\theta > 2$, we conclude that $I(tv) \to -\infty$ as $t \to \infty$. Hence, there exists $t_0 > 0$ large enough such that $I(t_0 v) < 0$. This and $i)$ from Lemma 3.2.1 show that I has the mountain-pass geometry, and therefore we can define the minimax level

$$c_M = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$

where

$$\Gamma = \{\gamma : [0,1] \to E : \gamma(0) = 0 \quad \text{and} \quad \gamma(1) = t_0 v\}.$$

The definition of c_M and (3.3) imply that

$$c_M \leq \max_{t \geq 0} I(tv) < I(u_h) + \frac{\alpha'}{2\alpha_0}.$$

By Proposition 3.3.1, the functional I satisfies the Palais-Smale condition at the level c_M. Thus, it follows from the Mountain-Pass Theorem [12] that I possesses a critical point $u_M \in E$ with $I(u_M) = c_M > 0$. Hence, since $I(0) = 0$ and $I(u_h) < 0$, we conclude that

$$u_M \notin \{0, u_h\},$$

which is a contradiction, since we are supposing that the only critical points of I are 0 and u_h. Therefore, the proof of theorem is finished. ∎

Now we are ready to prove our compactness result.

Proof of Proposition 3.3.1. Let $(u_n) \subset E$ be a sequence such that

$$I'(u_n) \to 0 \quad \text{and} \quad I(u_n) \to d < I(u_h) + \frac{\alpha'}{2\alpha_0}.$$

Notice that by (f_2)

$$\begin{aligned} I(u_n) - \frac{1}{\theta} I'(u_n)u_n &= \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2 + \int_{\mathbb{R}^2} Q\left[\frac{1}{\theta} f(u_n)u_n - F(u_n)\right] \\ &\geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2. \end{aligned}$$

Combining the last inequality with

$$I(u_n) - \frac{1}{\theta} I'(u_n)u_n \leq d + 1 + \|u_n\|,$$

for large $n \in \mathbb{N}$, we conclude that (u_n) is bounded in E. Thus, from Lemma 2.2.6 there exists $u \in E$ such that, up to a subsequence, $u_n \rightharpoonup u$ weakly in E, $u_n \to u$ strongly in $L^p(\mathbb{R}^2; Q)$, for all $2 \leq p < \infty$, and

$$\begin{aligned} f(u_n) \to f(u) \qquad & \text{in} \quad L^1_{\mathrm{loc}}(\mathbb{R}^2; Q), \\ F(u_n) \to F(u) \qquad & \text{in} \quad L^1(\mathbb{R}^2; Q). \end{aligned}$$

Moreover, the weak convergence of (u_n) also implies that $\langle h, u_n \rangle \to \langle h, u \rangle$. We have two possible cases to consider:

Case 1. $u = 0$. In this case,

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} QF(u_n) = 0.$$

Since

$$I(u_n) = \frac{1}{2}\|u_n\|^2 - \int_{\mathbb{R}^2} QF(u_n) - \langle h, u_n \rangle = d + o_n(1),$$

we get

$$\lim_{n\to\infty} \|u_n\|^2 = 2d < 2I(u_h) + \frac{\alpha'}{\alpha_0} < \frac{\alpha'}{\alpha_0}.$$

Hence, we can arguing as in the proof of Theorem 3.1.1 to conclude that, up to a subsequence, $u_n \to 0 = u$ strongly in E.

Case 2. $u = u_h \neq 0$. In this case, we define

$$v_n = \frac{u_n}{\|u_n\|} \quad \text{and} \quad v = \frac{u_h}{\lim \|u_n\|}.$$

It follows that $v_n \rightharpoonup v$ weakly in E, $\|v_n\| = 1$ and $\|v\| \leq 1$. If $\|v\| = 1$, we conclude the proof. If $\|v\| < 1$, we claim that there exist $r_1 > 1$ sufficiently close to 1, $\alpha > \alpha_0$ close to α_0 and $\beta > 0$ such that $r_1 \alpha \|u_n\|^2 \leq \beta < \alpha'(1 - \|v\|^2)^{-1}$ for $n \in \mathbb{N}$ large. Indeed, since $I(u_n) = d + o_n(1)$,

$$\frac{1}{2} \lim_{n\to\infty} \|u_n\|^2 = d + \int_{\mathbb{R}^2} QF(u_h) + \langle h, u_h \rangle. \tag{3.4}$$

Setting

$$A \doteq \left(d + \int_{\mathbb{R}^2} QF(u_h) + \langle h, u_h \rangle \right) \left(1 - \|v\|^2\right),$$

then by (3.4) and the definition of v, we obtain that

$$A = d - I(u_h).$$

Hence, coming back to (3.4), we conclude that

$$\frac{1}{2}\lim_{n\to\infty}\|u_n\|^2 = \frac{A}{1-\|v\|^2} = \frac{d-I(u_h)}{1-\|v\|^2} < \frac{\alpha'}{2\alpha_0\,(1-\|v\|^2)}.$$

Consequently, for $n \in \mathbb{N}$ large, there exist $r_1 > 1$ sufficiently close to 1, $\alpha > \alpha_0$ close to α_0 and $\beta > 0$ such that $r_1\alpha\|u_n\|^2 \le \beta < \alpha'(1-\|v\|^2)^{-1}$, and the claim is proved. Therefore, from Corollary 1.1.4,

$$\int_{\mathbb{R}^2} Q(e^{\alpha u_n^2}-1)^{r_1} \le C.$$

Next, by similar computations done above we have that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} Qf(u_n)(u_n-u_h) = 0.$$

This convergence together with the fact that $I'(u_n)(u_n-u_h) = o_n(1)$ imply that

$$\|u_n\|^2 = \int_{\mathbb{R}^2}(\nabla u_n\nabla u_h + Vu_nu_h) + o_n(1).$$

Since $u_n \rightharpoonup u_h$ weakly in E, we obtain $u_n \to u_h = u$ strongly in E and the proof of proposition is finished. ■

3.3.1 Proof of Proposition 3.3.2

In order to prove Proposition 3.3.2, we recall the Moser's function sequence introduced in Chapter 1:

$$\widetilde{M}_n(x,r) = \frac{1}{(2\pi)^{1/2}}\begin{cases}(\log n)^{1/2}, & |x| \le r/n,\\ \dfrac{\log\frac{r}{|x|}}{(\log n)^{1/2}}, & r/n < |x| \le r,\\ 0, & |x| > r,\end{cases}$$

with $0 < r \le r_0$ fixed and r_0 given in (1.22). Let

$$M_n(x,r) = \frac{1}{\|\widetilde{M}_n\|}\widetilde{M}_n(x,r).$$

Then M_n belongs to E with its support in $\overline{B}_r$ and $\|M_n\| = 1$. From Lemma 1.4.4 we have

Lemma 3.3.3 *Suppose that* $(\widetilde{V}) - (\widetilde{Q})$ *hold. If* f *satisfies* (f_2) *and* (f_4)*, then there exists* $n \in \mathbb{N}$ *such that*

$$\max_{t \geq 0} \left\{ \frac{t^2}{2} - \int_{\mathbb{R}^2} QF(tM_n) \right\} < \frac{\alpha'}{2\alpha_0}. \tag{3.5}$$

Proof. Suppose by contradiction that for each $n \in \mathbb{N}$ there exists a real sequence (t_n) such that

$$\frac{t_n^2}{2} - \int_{\mathbb{R}^2} QF(t_n M_n) \geq \frac{\alpha'}{2\alpha_0}.$$

Since $F(s) \geq 0$, for all $s \in \mathbb{R}$, we have

$$t_n^2 \geq \frac{\alpha'}{\alpha_0}. \tag{3.6}$$

It is easy to check that at $t = t_n$,

$$\frac{\mathrm{d}}{\mathrm{dt}} \left(\frac{t^2}{2} - \int_{\mathbb{R}^2} QF(tM_n) \right) = 0$$

or equivalently

$$t_n^2 = \int_{\mathbb{R}^2} Q t_n M_n f(t_n M_n). \tag{3.7}$$

By (f_4), for all $0 < \varepsilon < \beta_0$, there exists $R = R(\varepsilon) > 0$ such that for all $|s| \geq R$

$$sf(s) \geq (\beta_0 - \varepsilon) e^{\alpha_0 s^2}. \tag{3.8}$$

By Lemma 1.4.4, when $|x| \leq r/n$, we have

$$\begin{aligned} M_n^2(x) &\geq \frac{1}{2\pi} \frac{\log n}{1 + \frac{m(r)}{\log n}(1 + o_n(1))} \\ &= (2\pi)^{-1} \log n - (2\pi)^{-1} m(r) + o_n(1). \end{aligned} \tag{3.9}$$

Recall that by hypothesis $(\widetilde{Q})$,

$$Q(|x|) \geq C_0 |x|^{b_0} \quad \text{for} \quad 0 < |x| \leq r_0. \tag{3.10}$$

Thus, combining (3.8), (3.9) and (3.10) we get

$$\begin{aligned} t_n^2 &\geq (\beta_0 - \varepsilon) \int_{|x| \leq r/n} Q e^{\alpha_0 t_n^2 M_n^2} \\ &\geq (\beta_0 - \varepsilon) C_0 \int_{|x| \leq r/n} |x|^{b_0} e^{\alpha_0 t_n^2 (2\pi)^{-1} (\log n - m(r) + o_n(1))} \\ &\geq (\beta_0 - \varepsilon) 2\pi C_0 \left(\frac{r}{n} \right)^{b_0 + 2} e^{\alpha_0 t_n^2 (2\pi)^{-1} (\log n - m(r) + o_n(1))}. \end{aligned} \tag{3.11}$$

This yields that (t_n) is bounded. Indeed, since

$$\log n - m(r) + o_n(1) \geq \frac{1}{2}\log n,$$

for $n \in \mathbb{N}$ sufficiently large, it follows from (3.11) that

$$\begin{aligned} t_n^2 &\geq (\beta_0-\varepsilon)2\pi C_0 r^{b_0+2}\frac{1}{n^{b_0+2}}e^{\frac{\alpha_0 t_n^2}{2}\log n} \\ &= (\beta_0-\varepsilon)2\pi C_0 r^{b_0+2} n^{\frac{\alpha_0 t_n^2}{2}-(b_0+2)}, \end{aligned}$$

for $n \in \mathbb{N}$ sufficiently large. Then,

$$\begin{aligned} t_n^2 &\geq \log t_n^2 \\ &\geq \log\left[(\beta_0-\varepsilon)2\pi C_0 r^{b_0+2}\right] + \left[\frac{\alpha_0 t_n^2}{2} - (b_0+2)\right]\log n \end{aligned}$$

and consequently

$$1 \geq \frac{1}{t_n^2}\log\left[(\beta_0-\varepsilon)2\pi C_0 r^{b_0+2}\right] + \left[\frac{\alpha_0}{2} - \frac{b_0+2}{t_n^2}\right]\log n. \tag{3.12}$$

Thus, if $t_n \to \infty$ as $n \to \infty$, the right hand side of (3.12) goes to infinity when $n \to \infty$, which leads a contradiction. Therefore, (t_n) is bounded. Hence, passing to a subsequence if necessary, we may use (3.6), (3.11) and the condition $-2 < b_0 \leq 0$ to conclude that

$$\lim_{n\to\infty} t_n^2 = \frac{\alpha'}{\alpha_0}. \tag{3.13}$$

Indeed, otherwise there exists some $\delta > 0$ such that for $n \in \mathbb{N}$ sufficiently large

$$t_n^2 \geq \frac{\alpha'}{\alpha_0} + \delta.$$

Thus

$$\alpha_0 t_n^2 (2\pi)^{-1} \geq \alpha'(2\pi)^{-1} + \alpha_0(2\pi)^{-1}\delta.$$

From this and (3.11),

$$\begin{aligned} t_n^2 &\geq (\beta_0-\varepsilon)2\pi C_0 r^{b_0+2}\frac{1}{n^{b_0+2}}e^{\left(\alpha'(2\pi)^{-1}+\alpha_0(2\pi)^{-1}\delta\right)(\log n - m(r) + o_n(1))} \\ &\geq C n^{\alpha'(2\pi)^{-1}-(b_0+2)} n^{\alpha_0(2\pi)^{-1}\delta}. \end{aligned} \tag{3.14}$$

Hence, it is easy to check that in any case, $\alpha' = 4\pi$ or $\alpha' = 4\pi(1+b_0/2)$, the right hand side of (3.14) tends to infinity when $n \to \infty$, which contradicts the boundedness of (t_n). Now we estimate β_0 to get a contradiction. We have two cases to consider:

Case 1. $b_0 = 0$ ($\Rightarrow \alpha' = 4\pi$). It follows from (3.7) and (3.8) that

$$t_n^2 \geq (\beta_0 - \varepsilon) \int_{|x|\leq r} Q e^{\alpha_0 t_n^2 M_n^2} + \int_{t_n M_n < R} Q t_n M_n f(t_n M_n) - (\beta_0 - \varepsilon) 2\pi C_0 r^2. \quad (3.15)$$

Since $M_n \to 0$ a.e. in $\mathbb{R}^2$, it follows from Lebesgue's Dominated Convergence Theorem that

$$\lim_{n\to\infty} \int_{t_n M_n < R} Q t_n M_n f(t_n M_n) = 0. \quad (3.16)$$

Using (3.6), we obtain

$$\int_{|x|\leq r} Q e^{\alpha_0 t_n^2 M_n^2} \geq \int_{|x|\leq r/n} Q e^{4\pi M_n^2} + \int_{r/n \leq |x| \leq r} Q e^{4\pi M_n^2}. \quad (3.17)$$

Now we are going to estimate each integral in (3.17). From (3.9) and (3.10),

$$\begin{aligned} \int_{|x|\leq r/n} Q e^{4\pi M_n^2} &\geq C_0 \int_{|x|\leq r/n} e^{4\pi\left[(2\pi)^{-1}\log n - (2\pi)^{-1} m(r) + o_n(1)\right]} \\ &= C_0 e^{[2\log n - 2m(r) + o_n(1)]} \int_{|x|\leq r/n} \mathrm{dx} \\ &= \pi C_0 r^2 e^{-2m(r) + o_n(1)}, \end{aligned}$$

and by definition of M_n,

$$\begin{aligned} \int_{r/n\leq|x|\leq r} Q e^{4\pi M_n^2} &\geq C_0 \int_{r/n\leq|x|\leq r} e^{2\pi\left[(\log n)^{-1/2}\|\widetilde{M}_n\|^{-1}\log(r/|x|)\right]^2} \\ &= 2\pi C_0 \int_{r/n}^{r} t e^{2\pi\left[(\log n)^{-1/2}\|\widetilde{M}_n\|^{-1}\log(r/t)\right]^2} \mathrm{dt} \\ &= 2\pi C_0 r^2 \int_0^{\|\widetilde{M}_n\|^{-1}(\log n)^{1/2}} (\log n)^{1/2}\|\widetilde{M}_n\| e^{2\pi s^2 - \|\widetilde{M}_n\|(\log n)^{1/2} s} \mathrm{ds} \\ &\geq 2\pi C_0 r^2 \int_0^{\|\widetilde{M}_n\|^{-1}(\log n)^{1/2}} (\log n)^{1/2}\|\widetilde{M}_n\| e^{-\|\widetilde{M}_n\|(\log n)^{1/2} s} \mathrm{ds} \\ &= 2\pi C_0 r^2 (1 - 1/n), \end{aligned}$$

where we have used the change of variable $t = r e^{-\|\widetilde{M}_n\|(\log n)^{1/2} s}$ in the second equality. Thus

$$\liminf_{n\to\infty} \int_{|x|\leq r} Q e^{\alpha_0 t_n^2 M_n^2} \geq \pi C_0 r^2 (2 + e^{-2m(r)}). \quad (3.18)$$

Hence passing to the limit in (3.15) and using (3.13), (3.16) and (3.18), we get

$$\frac{4\pi}{\alpha_0} \geq (\beta_0 - \varepsilon)\pi C_0 r^2 e^{-2m(r)}.$$

Letting $\varepsilon \to 0$ we conclude that

$$\beta_0 \leq \frac{4}{C_0\alpha_0}\frac{1}{r^2}e^{2m(r)}.$$

Since $r > 0$ is arbitrary, the last expression contradicts (f_4).

Case 2. $-2 < b_0 < 0$ $(\Leftrightarrow \alpha' = 4\pi(1 + b_0/2))$. By similar computations done in the last case, we have

$$\int_{|x|\leq r/n} Qe^{\alpha' M_n^2} \geq \pi C_0 r^{b_0+2}\frac{1}{n^{-b_0}}e^{-(2+b_0)m(r)+o_n(1)}$$

and

$$\int_{r/n\leq|x|\leq r} Qe^{\alpha' M_n^2} \geq 2\pi C_0 r^{b_0+2}(1 - 1/n).$$

Thus

$$\liminf_{n\to\infty}\int_{|x|\leq r} Qe^{\alpha_0 t_n^2 M_n^2} \geq 2\pi C_0 r^{b_0+2}. \tag{3.19}$$

Hence passing to the limit in

$$t_n^2 \geq (\beta_0 - \varepsilon)\int_{|x|\leq r} Qe^{\alpha_0 t_n^2 M_n^2} + \int_{t_nM_n<R} Qt_nM_nf(t_nM_n)$$

and using (3.13), (3.16) and (3.19), we get

$$\frac{\alpha'}{\alpha_0} \geq (\beta_0 - \varepsilon)2\pi C_0 r^{b_0+2}.$$

Letting $\varepsilon \to 0$ we conclude that

$$\beta_0 \leq \frac{\alpha'}{2\pi C_0\alpha_0}\frac{1}{r^{b_0+2}} = \frac{b_0+2}{C_0\alpha_0}\frac{1}{r^{b_0+2}}.$$

Since $r > 0$ is arbitrary, the last expression also contradicts (f_4). Therefore, the proof of lemma is finished. ■

Now we are ready to finish this chapter by presenting the

Proof of Proposition 3.3.2. Let $n \in \mathbb{N}$ be obtained in Lemma 3.3.3 and set $v \doteq M_n$. Since $\langle h, v\rangle \leq \|h\|_{E^{-1}}$, we can use (3.5) from Lemma 3.3.3 to obtain $0 < \delta_2 \leq \delta_1$ such that

$$\max_{t\geq 0} I(tv) < \frac{\alpha'}{2\alpha_0},$$

whenever $0 < \|h\|_{E^{-1}} < \delta_2$. By *i)* from Lemma 3.2.1,

$$u_h \to 0 \quad \text{as} \quad \rho_h \to 0$$

and

$$\rho_h \to 0 \quad \text{as} \quad \|h\|_{E^{-1}} \to 0.$$

Thus, taking δ_2 small enough we obtain the desired result. ■

Chapter 4

On a class of gradient elliptic systems

This chapter is concerned with the existence and multiplicity of solutions for the following class of elliptic systems

$$\begin{cases} -\Delta u + V(|x|)u = Q(|x|)f(u,v) \quad \text{in} \quad \mathbb{R}^2, \\ -\Delta v + V(|x|)v = Q(|x|)g(u,v) \quad \text{in} \quad \mathbb{R}^2, \end{cases} \tag{4.1}$$

when the nonlinearities f and g are allowed to enjoy the exponential critical growth by means of the Trudinger-Moser inequality and the radial potentials V and Q may be unbounded, singular or decaying to zero. The approaches used here are based on the Trudinger-Moser type inequality established in Theorem 1.1.2 and a minimax theorem. We point out that part of this chapter is contained in the published paper [6].

4.1 Introduction and main results

We shall consider the variational situation in which

$$(f(u,v), g(u,v)) = \nabla F(u,v)$$

for some function $F : \mathbb{R}^2 \to \mathbb{R}$ of class C^1, where ∇F stands for the gradient of F in the variables $w = (u,v) \in \mathbb{R}^2$. Aiming an analogy with the scalar case, we rewrite (4.1) in the matrix form as

$$-\Delta w + V(|x|)w = Q(|x|)\nabla F(w) \quad \text{in} \quad \mathbb{R}^2,$$

where we denote $\Delta = (\Delta, \Delta)$ and $Q(|x|)\nabla F(w) = (Q(|x|)f(w), Q(|x|)g(w))$.

Since the Schrödinger equation plays the roles in many areas of mathematical-physic, in recent years, much attention has been paid to the nonlinear the Schrödinger system

$$\begin{cases} i\dfrac{\partial \phi}{\partial t} = -\Delta\phi + W(x)\phi - Q(x)\xi(|\phi|)\phi, & x \in \mathbb{R}^2, \\ i\dfrac{\partial \psi}{\partial t} = -\Delta\psi + W(x)\psi - Q(x)\xi(|\psi|)\psi, & x \in \mathbb{R}^2, \end{cases} \tag{4.2}$$

where $\phi, \psi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{C}$ are Schrödinger wave functions, $W, Q : \mathbb{R}^2 \to \mathbb{R}$ are given potentials and $\xi : \mathbb{R}^+ \to \mathbb{R}$ is a suitable function. In particular, solutions of system (4.1) provide *standing waves solutions* of system (4.2). Systems of this type under various hypotheses on the potentials and the nonlinearities have been investigated extensively, see for example [17, 23, 29, 45, 50, 51, 52, 66] and references therein.

We make the following assumptions on the potentials $V(|x|)$ and $Q(|x|)$:

(V) $V \in C(0, \infty)$, $V(r) > 0$ and there exists $a > -2$ such that

$$\liminf_{r\to+\infty} \frac{V(r)}{r^a} > 0.$$

(Q) $Q \in C(0, \infty)$, $Q(r) > 0$ and there exist $b < (a-2)/2$ and $b_0 > -2$ such that

$$\limsup_{r\to 0} \frac{Q(r)}{r^{b_0}} < \infty \quad \text{and} \quad \limsup_{r\to+\infty} \frac{Q(r)}{r^{b}} < \infty.$$

Let us introduce the precise assumptions under which our problem is studied.

(F_{α_0}) f and g have α_0-*exponential critical growth*, i.e., there exists $\alpha_0 > 0$ such that

$$\lim_{|w|\to+\infty} \frac{|f(w)|}{e^{\alpha|w|^2}} = \lim_{|w|\to+\infty} \frac{|g(w)|}{e^{\alpha|w|^2}} = \begin{cases} 0, & \forall\alpha > \alpha_0, \\ +\infty, & \forall\alpha < \alpha_0. \end{cases}$$

(F_1) $f(w) = o(|w|)$ and $g(w) = o(|w|)$ as $|w| \to 0$;

(F_2) there exists $\theta > 2$ such that

$$0 < \theta F(w) \le w \cdot \nabla F(w), \quad \forall w \in \mathbb{R}^2\backslash\{0\};$$

(F_3) there exist constants $R_0, M_0 > 0$ such that

$$0 < F(w) \le M_0|\nabla F(w)|, \quad \forall |w| \ge R_0;$$

(F_4) there exist $\nu > 2$ and $\mu > 0$ such that

$$F(w) \geq \frac{\mu}{\nu}|w|^{\nu}, \quad \forall w \in \mathbb{R}^2.$$

We denote the product space $Z = E \times E$ endowed with the inner product

$$\langle w_1, w_2 \rangle_Z \doteq \int_{\mathbb{R}^2} (\nabla u_1 \nabla u_2 + V u_1 u_2) + \int_{\mathbb{R}^2} (\nabla v_1 \nabla v_2 + V v_1 v_2),$$

where $w_1 = (u_1, v_1)$ and $w_2 = (u_2, v_2)$, to which corresponds the norm

$$\|w\|_Z = \langle w, w \rangle_Z^{1/2}.$$

Recalling that $S_\nu > 0$ is the best constant of the Sobolev embedding

$$E \hookrightarrow L^{\nu}(\mathbb{R}^2; Q)$$

(see Lemma 1.2.5), we have the following existence result for system (4.1).

Theorem 4.1.1 (Existence) *Suppose that $(V) - (Q)$ hold. If $(F_{\alpha_0}) - (F_4)$ are satisfied, then the system* (4.1) *has a nontrivial weak solution w_0 in Z provided*

$$\mu > \left[\frac{2\alpha_0(\nu - 2)}{\alpha' \nu}\right]^{(\nu-2)/2} S_\nu^{\nu/2},$$

where $\alpha' \doteq \min\{4\pi, 4\pi(1 + b_0/2)\}$.

Our multiplicity result is concerned with the problem

$$-\Delta w + V(|x|)w = \lambda Q(|x|)\nabla F(w) \quad \text{in} \quad \mathbb{R}^2, \tag{4.3}$$

where λ is a positive parameter. As in the scalar case, in this result we introduce more symmetry in the problem (4.3) and we show again that the value of the parameter $\lambda > 0$ affects the number of solutions. It can be stated as follows.

Theorem 4.1.2 (Multiplicity) *Suppose that $(V) - (Q)$ hold. If F is odd and $(F_{\alpha_0}) - (F_4)$ are satisfied, then for any given $k \in \mathbb{N}$ there exists $\Lambda_k > 0$ such that the system* (4.3) *has at least $2k$ pairs of nontrivial weak solutions in Z provided $\lambda > \Lambda_k$.*

We finish this section by remarking that the main tool to prove Theorem 4.1.2, the Symmetric Mountain-Pass Theorem due to Ambrosetti-Rabinowitz [12], it will be used in a more common version in comparison to the one used to prove the analogous theorem in the scalar case (Theorem 2.1.4), which leads us to a more direct conclusion of the result.

4.2 A version of the improvement in the product space

In line with [34, 35, 44] and in order to prove our multiplicity result; Theorem 4.1.2, we establish a version of the Corollary 1.1.4 on the space Z. Using Theorem 1.1.2 and following the same steps as in the proof of Corollary 1.1.4 we have

Corollary 4.2.1 *Suppose that* $(V)-(Q)$ *hold. Let* (w_n) *be in* Z *with* $\|w_n\|_Z = 1$ *and suppose that* $w_n \rightharpoonup w$ *weakly in* Z *with* $\|w\|_Z < 1$. *Then, for each* $0 < \beta < \frac{\alpha'}{2}\left(1-\|w\|_Z^2\right)^{-1}$, *up to a subsequence, it holds*

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^2} Q(|x|)(e^{\beta|w_n|^2}-1) < +\infty.$$

Proof. Since $w_n \rightharpoonup w$ weakly in Z and $\|w_n\|_Z = 1$, we conclude that

$$\lim_{n\to\infty} \|w_n - w\|_Z^2 = 1 - \|w\|_Z^2 < \frac{\alpha'}{2\beta}.$$

Thus, for large $n \in \mathbb{N}$ we have

$$2\beta\|w_n - w\|_Z^2 < \alpha'.$$

Now choosing $r_1 > 1$ close to 1 and $\varepsilon > 0$ satisfying

$$2r_1\beta(1+\varepsilon^2)\|w_n - w\|_Z^2 < \alpha',$$

the Young's inequality and Theorem 1.1.2 imply that

$$\begin{aligned}\int_{\mathbb{R}^2} Q\left(e^{r_1\beta(1+\varepsilon^2)|w_n-w|^2}-1\right) &\leq \frac{1}{2}\int_{\mathbb{R}^2} Q\left(e^{2r_1\beta(1+\varepsilon^2)\|w_n-w\|_Z^2\left(\frac{|u_n-u|}{\|w_n-w\|_Z}\right)^2}-1\right)\\ &+\frac{1}{2}\int_{\mathbb{R}^2} Q\left(e^{2r_1\beta(1+\varepsilon^2)\|w_n-w\|_Z^2\left(\frac{|v_n-v|}{\|w_n-w\|_Z}\right)^2}-1\right) \leq C.\end{aligned}$$

Moreover, since

$$\beta|w_n|^2 \leq \beta(1+\varepsilon^2)|w_n-w|^2 + \beta(1+1/\varepsilon^2)|w|^2,$$

it follows again by the Young's inequality that

$$\begin{aligned}&\int_{\mathbb{R}^2} Q\left(e^{\beta|w_n|^2}-1\right)\\ &\leq \frac{1}{r_1}\int_{\mathbb{R}^2} Q\left(e^{r_1\beta(1+\varepsilon^2)|w_n-w|^2}-1\right) + \frac{1}{r_2}\int_{\mathbb{R}^2} Q\left(e^{r_2\beta(1+1/\varepsilon^2)|w|^2}-1\right) \leq C,\end{aligned}$$

for $n \in \mathbb{N}$ large and $r_2 = r_1/(r_1-1)$. Therefore, the result is proved. ■

4.3 Variational setting

The natural functional associated with system (4.1) is

$$I(w) \doteq \frac{1}{2}\|w\|_Z^2 - \int_{\mathbb{R}^2} QF(w),$$

for $w \in Z$. Under our assumptions we have that I is well defined and it is C^1 on Z. Indeed, by (F_1), for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\nabla F(w)| \leq \varepsilon |w|$$

always that $|w| < \delta$. On the other hand, for $\alpha > \alpha_0$, there exist $C_0, C_1 > 0$ such that

$$f(w) \leq C_0(e^{\alpha|w|^2} - 1) \quad \text{and} \quad g(w) \leq C_1(e^{\alpha|w|^2} - 1),$$

for all $|w| \geq \delta$. Thus, for all $w \in \mathbb{R}^2$ we have

$$\begin{aligned} |\nabla F(w)| &\leq \varepsilon|w| + |f(w)| + |g(w)| \\ &\leq \varepsilon|w| + C(e^{\alpha|w|^2} - 1). \end{aligned} \tag{4.4}$$

Hence, using (F_2), (4.4) and the Hölder's inequality, we have

$$\begin{aligned} &\int_{\mathbb{R}^2} Q|F(w)| \\ &\leq \varepsilon \int_{\mathbb{R}^2} Q|w|^2 + C\int_{\mathbb{R}^2} Q|w|(e^{\alpha|w|^2} - 1) \\ &\leq \varepsilon \left(\int_{\mathbb{R}^2} Q|u|^2 + \int_{\mathbb{R}^2} Q|v|^2\right) + C\left(\int_{\mathbb{R}^2} Q|w|^r\right)^{1/r} \left(\int_{\mathbb{R}^2} Q(e^{s\alpha|w|^2} - 1)\right)^{1/s}, \end{aligned}$$

with $r, s \geq 1$ such that $1/r + 1/s = 1$. Considering Lemma 1.2.5 we have for $r \geq 4$ that

$$\left(\int_{\mathbb{R}^2} Q|w|^r\right)^{1/r} = \|u^2 + v^2\|_{L^{r/2}(\mathbb{R}^2;Q)}^{1/2} \leq C\|u^2 + v^2\|^{1/2} \leq C\|w\|_Z < \infty.$$

On the other hand, by the Young's inequality and Theorem 1.1.2,

$$\int_{\mathbb{R}^2} Q(e^{s\alpha|w|^2} - 1) \leq \frac{1}{2}\int_{\mathbb{R}^2} Q(e^{2s\alpha u^2} - 1) + \frac{1}{2}\int_{\mathbb{R}^2} Q(e^{2s\alpha v^2} - 1) < \infty. \tag{4.5}$$

Hence, $QF(w) \in L^1(\mathbb{R}^2)$, which implies that I is well defined, for $\alpha > \alpha_0$. In the following, we will show that $I \in C^1(Z, \mathbb{R})$ with

$$I'(w)z = \langle w, z\rangle_Z - \int_{\mathbb{R}^2} Qz \cdot \nabla F(w),$$

for all $z \in Z$. Thus, since we are searching for weak solutions for system (4.1), that is, functions $w \in Z$ such that

$$\langle w, z\rangle_Z - \int_{\mathbb{R}^2} Qz \cdot \nabla F(w) = 0,$$

for all $z \in Z$, we conclude that critical points of the functional I are precisely weak solutions of system (4.1) and reciprocally. Setting

$$\Phi(w) = \int_{\mathbb{R}^2} Q(|x|)F(w),$$

by Gâteaux's derivative definition

$$\frac{\partial \Phi}{\partial z}(w) = \lim_{t\to 0} \int_{\mathbb{R}^2} Q\frac{F(w+tz) - F(w)}{t}.$$

Defining

$$h_t(x) = Q(|x|)\frac{F(w(x) + tz(x)) - F(w(x))}{t}, \quad x \in \mathbb{R}^2,$$

by Mean Value Theorem, there exists $\theta_t(x) \in [w(x), w(x)+tz(x)]$ (or $[w(x)+tz(x), w(x)]$), with $x \in \mathbb{R}^2$, such that

$$h_t(x) = Q(|x|)z \cdot \nabla F(\theta_t(x)), \quad x \in \mathbb{R}^2.$$

Thus, by (4.4) and the Hölder's inequality

$$\int_{\mathbb{R}^2} |h_t| \le \varepsilon \left(\int_{\mathbb{R}^2} Q|z|^2 + \int_{\mathbb{R}^2} Q|\theta_t|^2 \right) + C \left(\int_{\mathbb{R}^2} Q|z|^r \right)^{1/r} \left(\int_{\mathbb{R}^2} Q(e^{s\alpha|\theta_t|^2} - 1) \right)^{1/s}.$$

Considering Lemma 1.2.5 and Theorem 1.1.2, it follows that $h_t \in L^1(\mathbb{R}^2)$. On the other hand,

$$\lim_{t\to 0} h_t = Qz \cdot \nabla F(w).$$

Hence, by Lebesgue's Dominated Convergence Theorem

$$\frac{\partial \Phi}{\partial z}(w) = \lim_{t\to 0} \int_{\mathbb{R}^2} h_t = \int_{\mathbb{R}^2} Qz \cdot \nabla F(w).$$

Now, since $\frac{\partial \Phi}{\partial (\cdot)}(w) \in Z'$ for each $w \in Z$, it remains to verify that

$$w_n \to w \quad \text{in} \quad Z \Rightarrow \frac{\partial \Phi}{\partial (\cdot)}(w_n) \to \frac{\partial \Phi}{\partial (\cdot)}(w) \quad \text{in} \quad Z'$$

to conclude the differentiability of Φ. Since $w_n = (u_n, v_n) \to w = (u, v)$ in Z we have $u_n \to u$ and $v_n \to v$ in E. Hence, by Lemma 1.2.5

$$u_n \to u \quad \text{and} \quad v_n \to v \quad \text{in} \quad L^p(\mathbb{R}^2; Q),$$

for all $2 \leq p < \infty$. Thus, up to a subsequence, $w_n(x) = (u_n(x), v_n(x)) \to w(x) = (u(x), v(x))$ a.e. $x \in \mathbb{R}^2$. Moreover, there exist $h_1, h_2 \in L^1(\mathbb{R}^2)$ such that $|u_n(x)| \leq h_1(x)$ and $|v_n(x)| \leq h_2(x)$ a.e. $x \in \mathbb{R}^2$. Thus, defining

$$G_n(x) \doteq Q(|x|)z(x) \cdot \nabla F(w_n(x)), \quad x \in \mathbb{R}^2,$$

we conclude that $G_n(x) \to G(x) \doteq Q(|x|)z(x) \cdot \nabla F(w(x))$, a.e. $x \in \mathbb{R}^2$. In addition, by similar computations done to verify that I was well-defined, we obtain that $G_n(x) \in L^1(\mathbb{R}^2)$. Hence, again by Lebesgue's Dominated Convergence Theorem

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} G_n(x) = \int_{\mathbb{R}^2} G(x).$$

Therefore, for each $z \in Z$, we get

$$\frac{\partial \Phi}{\partial z}(w_n) \to \frac{\partial \Phi}{\partial z}(w) \quad \text{in} \quad Z'.$$

In the next lemma we check that the functional I satisfies the geometric conditions of the Mountain-Pass Theorem.

Lemma 4.3.1 *Suppose that* $(V)-(Q)$ *hold. If* $(F_{\alpha_0})-(F_2)$ *are satisfied, then:*

i) there exist $\tau, \rho > 0$ *such that* $I(w) \geq \tau$ *whenever* $\|w\|_Z = \rho$;

ii) there exists $e_* \in Z$, *with* $\|e_*\|_Z > \rho$, *such that* $I(e_*) < 0$.

Proof. As in the proof of (4.4), we have

$$|\nabla F(w)| \leq \varepsilon|w| + C|w|^{q-1}(e^{\alpha|w|^2} - 1) \tag{4.6}$$

for all $w \in \mathbb{R}^2$ and $q \geq 1$. Thus, using (F_2), the Hölder's inequality and Lemma 1.2.5, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} Q|F(w)| \\
&\leq \varepsilon \int_{\mathbb{R}^2} Q|w|^2 + C \int_{\mathbb{R}^2} Q|w|^q(e^{\alpha|w|^2} - 1) \\
&\leq \varepsilon \left(\int_{\mathbb{R}^2} Q|u|^2 + \int_{\mathbb{R}^2} Q|v|^2 \right) + C \left(\int_{\mathbb{R}^2} Q|w|^{qr} \right)^{1/r} \left(\int_{\mathbb{R}^2} Q(e^{s\alpha|w|^2} - 1) \right)^{1/s} \\
&\leq C\varepsilon\|w\|_Z^2 + C_0\|w\|_Z^q \left(\int_{\mathbb{R}^2} Q(e^{s\alpha|w|^2} - 1) \right)^{1/s},
\end{aligned}$$

provided $r \geq 2, s > 1$ such that $1/r + 1/s = 1$. Now for $\|w\|_Z \leq M < (\alpha'/2\alpha)^{1/2}$, which implies that

$$2\alpha\|u\|^2 \leq 2\alpha M^2 < \alpha' \quad \text{and} \quad 2\alpha\|v\|^2 \leq 2\alpha M^2 < \alpha',$$

and s sufficiently close to 1, it follows from (4.5) that

$$\int_{\mathbb{R}^2} Q(e^{s\alpha|w|^2} - 1) \leq C.$$

Thus,

$$\int_{\mathbb{R}^2} Q|F(w)| \leq C\varepsilon\|w\|_Z^2 + C_1\|w\|_Z^q.$$

Hence,

$$I(w) \geq \left(\frac{1}{2} - C\varepsilon\right)\|w\|_Z^2 - C_1\|w\|_Z^q,$$

which implies *i*), if $q > 2$. In order to verify *ii*), let $w \in Z$ be a function with compact support. Thus, using (F_4) we get

$$I(tw) \leq \frac{t^2}{2}\|w\|^2 - Ct^\nu \int_{\text{supp}(w)} Q|w|^\nu,$$

for all $t > 0$, which yields $I(tw) \to -\infty$ as $t \to +\infty$, provided $\nu > 2$. Setting $e_* = t_*w$ with $t_* > 0$ large enough, the proof of lemma is finished. ∎

To prove that a Palais-Smale sequence converges to the weak solution of system (4.1) we need to establish the following lemmas:

Lemma 4.3.2 *Suppose that (F_2) holds. Let (w_n) be a sequence in Z such that*

$$I(w_n) \to c \quad \textit{and} \quad I'(w_n) \to 0.$$

Then

$$\|w_n\|_Z \leq C, \quad \int_{\mathbb{R}^2} QF(w_n) \leq C \quad \textit{and} \quad \int_{\mathbb{R}^2} Qw_n \cdot \nabla F(w_n) \leq C.$$

Proof. Let (w_n) be a sequence in Z such that $I(w_n) \to c$ and $I'(w_n) \to 0$. Thus, for any $z \in Z$,

$$I(w_n) = \frac{1}{2}\|w_n\|_Z^2 - \int_{\mathbb{R}^2} QF(w_n) = c + o_n(1) \tag{4.7}$$

and

$$I'(w_n)z = \langle w_n, z\rangle_Z - \int_{\mathbb{R}^2} Qz \cdot \nabla F(w_n) = o_n(1). \tag{4.8}$$

Taking $z = w_n$ in (4.8) and using (F_2) we have

$$\begin{aligned} c + \|w_n\|_Z + o_n(1) &\geq I(w_n) - \frac{1}{\theta} I'(w_n) w_n \\ &= \left(\frac{1}{2} - \frac{1}{\theta}\right) \|w_n\|_Z^2 + \int_{\mathbb{R}^2} Q \left[\frac{1}{\theta} w_n \cdot \nabla F(w_n) - F(w_n)\right] \\ &\geq \left(\frac{1}{2} - \frac{1}{\theta}\right) \|w_n\|_Z^2. \end{aligned}$$

Consequently, $\|w_n\|_Z \leq C$. From (4.7) and (4.8) we get

$$\int_{\mathbb{R}^2} QF(w_n) \leq C \quad \text{and} \quad \int_{\mathbb{R}^2} Qw_n \cdot \nabla F(w_n) \leq C.$$

Therefore, the lemma is proved. ■

We will also use the following convergence result:

Lemma 4.3.3 *Suppose that $(F_2) - (F_3)$ hold. If $(w_n) \subset Z$ is a Palais-Smale sequence for I and w_0 is its weak limit then, up to a subsequence,*

$$\nabla F(w_n) \to \nabla F(w_0) \quad \textit{in} \quad L^1_{loc}(\mathbb{R}^2, \mathbb{R}^2)$$

and

$$F(w_n) \to F(w_0) \quad \textit{in} \quad L^1(\mathbb{R}^2; Q).$$

Proof. Suppose that (w_n) is a Palais-Smale sequence. According to Lemma 4.3.2, $w_n = (u_n, v_n) \rightharpoonup w_0 = (u_0, v_0)$ weakly in Z, that is, $u_n \rightharpoonup u_0$ and $v_n \rightharpoonup v_0$ weakly in E. Thus, recalling that $H^1_{\text{rad}}(B_R; V) \hookrightarrow L^q(B_R)$ compactly for all $1 \leq q < \infty$ and $R > 0$ (see the consequence of *ii*) from Lemma 1.2.3), up to a subsequence, we can assume that $u_n \to u_0$ and $v_n \to v_0$ in $L^1(B_R)$. Hence, $w_n \to w_0$ in $L^1(B_R, \mathbb{R}^2)$ and $w_n(x) \to w_0(x)$ a.e. $x \in \mathbb{R}^2$. Since $\nabla F(w_n) \in L^1(B_R, \mathbb{R}^2)$, the first convergence follows from Lemma 2.2.5. Hence,

$$f(w_n) \to f(w_0) \quad \text{and} \quad g(w_n) \to g(w_0) \quad \text{in} \quad L^1_{loc}(\mathbb{R}^2).$$

Thus, there exist $h_1, h_2 \in L^1(B_R)$ such that $Q|f(w_n)| \leq h_1$ and $Q|g(w_n)| \leq h_2$ a.e. in B_R. From (F_3) we can conclude that

$$|F(w_n)| \leq \sup_{[-R_0, R_0]} |F(w_n)| + M_0 |\nabla F(w_n)|,$$

a.e. in B_R. Thus, by Lebesgue's Dominated Convergence Theorem

$$F(w_n) \to F(w_0) \quad \text{in} \quad L^1(B_R; Q).$$

On the other hand, by (4.6) with $q = 2$

$$\int_{B_R^c} QF(w_n) \le \varepsilon \int_{B_R^c} Q|w_n|^2 + C \int_{B_R^c} Q|w_n|(e^{\alpha|w_n|^2} - 1), \tag{4.9}$$

for $\alpha > \alpha_0$. From Lemma 1.2.5, the Hölder's inequality, $\|w_n\|_Z \le C$ and similar computations to estimate the second integral in (1.17), we get

$$\varepsilon \int_{B_R^c} Q|w_n|^2 \le C\varepsilon \quad \text{and} \quad \int_{B_R^c} Q|w_n|(e^{\alpha|w_n|^2} - 1) \le \frac{C}{R^\xi},$$

for some $\xi > 0$. Hence, given $\delta > 0$, there exists $R > 0$ sufficiently large such that

$$\int_{B_R^c} Q|w_n|^2 < \delta \quad \text{and} \quad \int_{B_R^c} Q|w_n|(e^{\alpha|w_n|^2} - 1) < \delta.$$

Thus, from (4.9)

$$\int_{B_R^c} QF(w_n) \le C\delta \quad \text{and} \quad \int_{B_R^c} QF(w_0) \le C\delta.$$

Finally, since

$$\begin{aligned} &\left| \int_{\mathbb{R}^2} QF(w_n) - \int_{\mathbb{R}^2} QF(w_0) \right| \\ &\le \left| \int_{B_R} QF(w_n) - \int_{B_R} QF(w_0) \right| + \int_{B_R^c} QF(w_n) + \int_{B_R^c} QF(w_0), \end{aligned}$$

we get

$$\lim_{n\to\infty} \left| \int_{\mathbb{R}^2} QF(w_n) - \int_{\mathbb{R}^2} QF(w_0) \right| \le C\delta.$$

Since δ is arbitrary, the result follows and the lemma is proved. ∎

In view of Lemma 4.3.1 the minimax level

$$c = \inf_{g\in\Gamma} \max_{0\le t\le 1} I(g(t))$$

is positive, where $\Gamma = \{g \in C([0,1], Z) : g(0) = 0 \quad \text{and} \quad I(g(1)) < 0\}$. Hence, by the Mountain-Pass Theorem without the Palais-Smale condition (see [12]) there exists a $(PS)_c$ sequence (w_n) in Z, that is,

$$I(w_n) \to c \quad \text{and} \quad I'(w_n) \to 0. \tag{4.10}$$

Lemma 4.3.4 *If*

$$\mu > \left[\frac{2\alpha_0(\nu-2)}{\alpha'\nu}\right]^{(\nu-2)/2} S_\nu^{\nu/2},$$

then $c < \dfrac{\alpha'}{4\alpha_0}$.

Proof. Since the embeddings $E \hookrightarrow L^p(\mathbb{R}^2;Q)$ are compacts for all $2 \le p < \infty$, there exists a function $\bar{u} \in E$ such that

$$S_\nu = \|\bar{u}\|^2 \quad \text{and} \quad \|\bar{u}\|_{L^\nu(\mathbb{R}^2;Q)} = 1.$$

Thus, considering $\bar{w} = (\bar{u}, \bar{u})$, by the definition of c and (F_4), one has

$$c \le \max_{t\ge 0}\left[S_\nu t^2 - \int_{\mathbb{R}^2} QF(t\bar{w})\right] \le \max_{t\ge 0}\left[S_\nu t^2 - \frac{2^{\nu/2}\mu}{\nu}t^\nu\right] = \frac{\nu-2}{2\nu}\frac{S_\nu^{\nu/(\nu-2)}}{\mu^{2/(\nu-2)}} < \frac{\alpha'}{4\alpha_0}.$$

Hence, the lemma is proved. ■

4.4 Proof of the existence theorem

Proof of Theorem 4.1.1. It follows from Lemmas 4.3.2 and 4.3.3 that the Palais-Smale sequence (w_n) is bounded and converges weakly to a weak solution of system (4.1) denoted by w_0. To prove that w_0 is nontrivial we argue by contradiction. If $w_0 \equiv 0$, Lemma 4.3.3 implies that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} QF(w_n) = 0.$$

Thus, by (4.7)

$$\lim_{n\to\infty}\|w_n\|_Z^2 = 2c > 0. \tag{4.11}$$

From this and Lemma 4.3.4, given $\varepsilon > 0$, we have that

$$\|w_n\|_Z^2 < \frac{\alpha'}{2\alpha_0} + \varepsilon,$$

for $n \in \mathbb{N}$ large. Thus, it is possible to choice $s > 1$ sufficiently close to 1 and $\alpha > \alpha_0$ close to α_0 such that $s\alpha\|w_n\|_Z^2 \le \beta' < \alpha'/2$, which implies that

$$2s\alpha\|u_n\|^2 \le 2\beta' < \alpha' \quad \text{and} \quad 2s\alpha\|v_n\|^2 \le 2\beta' < \alpha'.$$

Thus, using (4.5), (4.4) in combination with the Hölder's inequality and Lemma 1.2.5, up to a subsequence, we conclude that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} Qw_n \cdot \nabla F(w_n) = 0.$$

Hence, by (4.8), we obtain that

$$\lim_{n\to\infty}\|w_n\|_Z^2 = 0,$$

which is a contradiction with (4.11). Therefore, w_0 is a nontrivial weak solution of system (4.1). ■

4.5 Proof of the multiplicity theorem

To prove our multiplicity result we shall use the following version of the Symmetric Mountain-Pass Theorem (see [12, 13, 57]).

Theorem 4.5.1 *Let $X = X_1 \oplus X_2$, where X is a real Banach space and X_1 is finite-dimensional. Suppose that J is a $C^1(X, \mathbb{R})$ functional satisfying the following conditions:*

- (J_1) *$J(0) = 0$ and J is even;*
- (J_2) *there exist $\tau, \rho > 0$ such that $J(u) \geq \tau$ if $\|u\|_X = \rho,\ u \in X_2$;*
- (J_3) *there exists a finite-dimensional subspace $W \subset X$ with $\dim(X_1) < \dim(W)$ and there exists $\mathcal{S} > 0$ such that $\max_{u \in W} J(u) \leq \mathcal{S}$;*
- (J_4) *J satisfies the $(PS)_c$ condition for all $c \in (0, \mathcal{S})$.*

Then J possesses at least $\dim(W) - \dim(X_1)$ pairs of nontrivial critical points.

Given $k \in \mathbb{N}$, we are going to apply this abstract result with $X = Z$, $X_1 = \{0\}$, $J = I_\lambda$ and $W = \widetilde{W} \times \widetilde{W}$ with $\widetilde{W} \doteq \text{span}\{\psi_1, \ldots, \psi_k\}$, where $\{\psi_i\}_{i=1}^k \subset C_{0,\text{rad}}^\infty(\mathbb{R}^2)$ is a collection of smooth function with disjoint supports. We see that the energy functional associated to system (4.3)

$$I_\lambda(w) \doteq \frac{1}{2}\|w\|_Z^2 - \lambda \int_{\mathbb{R}^2} QF(w), \quad w \in Z,$$

is well defined and $I_\lambda \in C^1(Z, \mathbb{R})$ with derivative given by

$$I'_\lambda(w)z = \langle w, z \rangle_Z - \lambda \int_{\mathbb{R}^2} Qz \cdot \nabla F(w), \quad \forall w, z \in Z.$$

Hence, a weak solution $w \in Z$ of system (4.3) is exactly a critical point of I_λ. Furthermore, since $I_\lambda(0) = 0$ and F is odd, I_λ satisfies (J_1) and with similar computations to prove $i)$ in Lemma 4.3.1 we conclude that I_λ also verifies (J_2). In order to verify (J_3) and (J_4) we consider the following lemma.

Lemma 4.5.2 *Suppose that $(V) - (Q)$ hold. If F satisfies $(F_{\alpha_0}) - (F_4)$, we have:*

- *$i)$ there exists $\mathcal{S} > 0$ such that $\max_{w \in W} I_\lambda(w) \leq \mathcal{S}$;*
- *$ii)$ the functional I_λ satisfies the $(PS)_c$ condition for all $c \in (0, \mathcal{S})$, that is, any sequence (w_n) in Z such that*

$$I_\lambda(w_n) \to c \quad \text{and} \quad I'_\lambda(w_n) \to 0 \tag{4.12}$$

admits a convergent subsequence in Z.

Proof. By (F_4),

$$\begin{aligned}\max_{w\in W} I_\lambda(w) &= \max_{w\in W}\left[\frac{1}{2}\|w\|_Z^2 - \lambda\int_{\mathbb{R}^2} QF(w)\right]\\ &\le \max_{w\in W}\left[\frac{1}{2}\|u\|_{\widetilde{W}}^2 + \frac{1}{2}\|v\|_{\widetilde{W}}^2 - \frac{\mu\lambda}{\nu}\|u\|^\nu_{L^\nu(\mathbb{R}^2;Q)} - \frac{\mu\lambda}{\nu}\|v\|^\nu_{L^\nu(\mathbb{R}^2;Q)}\right]\\ &\le \max_{u\in\widetilde{W}}\left[\frac{1}{2}\|u\|_{\widetilde{W}}^2 - \frac{\mu\lambda}{\nu}\|u\|^\nu_{L^\nu(\mathbb{R}^2;Q)}\right] + \max_{v\in\widetilde{W}}\left[\frac{1}{2}\|v\|_{\widetilde{W}}^2 - \frac{\mu\lambda}{\nu}\|v\|^\nu_{L^\nu(\mathbb{R}^2;Q)}\right].\end{aligned}$$

Now, once $\dim(\widetilde{W}) < \infty$, the equivalence of the norms in this space gives a constant $C > 0$ such that

$$\max_{u\in\widetilde{W}}\left[\frac{1}{2}\|u\|_{\widetilde{W}}^2 - \frac{\mu\lambda}{C\nu}\|u\|^\nu_{\widetilde{W}}\right] + \max_{v\in\widetilde{W}}\left[\frac{1}{2}\|v\|_{\widetilde{W}}^2 - \frac{\mu\lambda}{C\nu}\|v\|^\nu_{\widetilde{W}}\right] = M_k(\lambda),$$

where

$$M_k(\lambda) \doteq \frac{\nu-2}{\nu}\left(\frac{C}{\mu}\right)^{2/(\nu-2)}\lambda^{2/(2-\nu)}.$$

Since $2/(2-\nu) < 0$ we have that

$$\lim_{\lambda\to+\infty} M_k(\lambda) = 0,$$

which implies that there exists $\Lambda_k > 0$ such that

$$M_k(\lambda) < \frac{\alpha'}{4\alpha_0} \doteq \mathcal{S},$$

for any $\lambda > \Lambda_k$. Therefore, *i*) is proved. For *ii*), by Lemma 4.3.2, (w_n) is bounded in Z and so, up to a subsequence, $w_n \rightharpoonup w$ weakly in Z. We claim that

$$\int_{\mathbb{R}^2} Qw\cdot\nabla F(w_n) \to \int_{\mathbb{R}^2} Qw\cdot\nabla F(w) \quad \text{as} \quad n\to\infty. \tag{4.13}$$

Indeed, since $C^\infty_{0,\mathrm{rad}}(\mathbb{R}^2)$ is dense in E for all $\delta > 0$, there exists $v \in C^\infty_{0,\mathrm{rad}}(\mathbb{R}^2,\mathbb{R}^2)$ such that $\|w-v\|_Z < \delta$. Observing that

$$\left|\int_{\mathbb{R}^2} Qw\cdot[\nabla F(w_n) - \nabla F(w)]\right| \le \left|\int_{\mathbb{R}^2} Q(w-v)\cdot\nabla F(w_n)\right|$$
$$+ \|v\|_\infty \int_{\mathrm{supp}(v)} Q|\nabla F(w_n) - \nabla F(w)| + \left|\int_{\mathbb{R}^2} Q(w-v)\cdot\nabla F(w)\right|$$

and using Cauchy-Schwarz and the fact that $|I'_\lambda(w_n)(w-v)| \le \varepsilon_n\|w-v\|$ with $\varepsilon_n \to 0$, we get

$$\left|\int_{\mathbb{R}^2} Q(w-v)\cdot\nabla F(w_n)\right| \le \varepsilon_n\|w-v\| + \|w_n\|\|w-v\| \le C\|w-v\| < C\delta,$$

where we have used that (w_n) is bounded in Z. Similarly, since the second limit in (4.12) implies that $I'_\lambda(w)(w-v)=0$, we have

$$\left|\int_{\mathbb{R}^2} Q(w-v)\cdot\nabla F(w_n)\right| < C\delta.$$

From Lemma 4.3.3

$$\lim_{n\to\infty}\int_{\mathrm{supp}(v)} Q|\nabla F(w_n)-\nabla F(w)| = 0.$$

Thus,

$$\lim_{n\to\infty}\left|\int_{\mathbb{R}^2} Qw\cdot[\nabla F(w_n)-\nabla F(w)]\right| < 2C\delta.$$

Since $\delta>0$ is arbitrary, the claim follows. Hence, passing to the limit when $n\to\infty$ in

$$o_n(1) = I'_\lambda(w_n)w = \langle w_n, w\rangle_Z - \lambda\int_{\mathbb{R}^2} Qw\cdot\nabla F(w_n)$$

and using that $w_n \rightharpoonup w$ weakly in Z, (4.13) and (F_2) we get

$$\|w\|_Z^2 = \lambda\int_{\mathbb{R}^2} Qw\cdot\nabla F(w) \geq 2\lambda\int_{\mathbb{R}^2} QF(w).$$

Hence

$$I_\lambda(w)\geq 0. \tag{4.14}$$

Now, we have two cases to consider:

Case 1. $w=0$. This case is similar to the checking that the solution w_0 obtained in the Theorem 4.1.1 is nontrivial.

Case 2. $w\neq 0$. In this case, we define

$$z_n = \frac{w_n}{\|w_n\|_Z} \quad \text{and} \quad z = \frac{w}{\lim\|w_n\|_Z}.$$

It follows that $z_n \rightharpoonup z$ weakly in Z, $\|z_n\|_Z = 1$ and $\|z\|_Z\leq 1$. If $\|z\|_Z=1$, we conclude the proof. If $\|z\|_Z<1$, it follows from Lemma 4.3.3 and (4.12) that

$$\frac{1}{2}\lim_{n\to\infty}\|w_n\|_Z^2 = c+\lambda\int_{\mathbb{R}^2} QF(w). \tag{4.15}$$

Setting

$$A \doteq \left(c+\lambda\int_{\mathbb{R}^2} QF(w)\right)\left(1-\|z\|_Z^2\right),$$

then by (4.15) and the definition of z, we obtain

$$A = c - I_\lambda(w).$$

Hence, coming back to (4.15) and using (4.14), we conclude that

$$\frac{1}{2}\lim_{n\to\infty}\|w_n\|_Z^2 = \frac{A}{1-\|z\|_Z^2} = \frac{c-I_\lambda(w)}{1-\|z\|_Z^2} \le \frac{c}{1-\|z\|_Z^2} < \frac{\alpha'}{4\alpha_0\,(1-\|z\|_Z^2)}.$$

Consequently, for $n \in \mathbb{N}$ large, there exist $r > 1$ sufficiently close to 1, $\alpha > \alpha_0$ close to α_0 and $\beta > 0$ such that

$$r\alpha\|w_n\|_Z^2 \le \beta < \frac{\alpha'}{2}(1-\|z\|_Z^2)^{-1}.$$

Therefore, from Corollary 4.2.1,

$$\int_{\mathbb{R}^2} Q(e^{\alpha|w_n|^2}-1)^r < +\infty. \tag{4.16}$$

Next, we claim that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} Q(w_n-w)\cdot\nabla F(w_n) = 0.$$

Indeed, let $r, s > 1$ be such that $1/r + 1/s = 1$. Invoking (4.4) and the Hölder's inequality we conclude that

$$\left|\int_{\mathbb{R}^2} Q(w_n-w)\cdot\nabla F(w_n)\right| \le \varepsilon\left(\int_{\mathbb{R}^2} Q|w_n|^2\right)^{1/2}\left(\int_{\mathbb{R}^2} Q|w_n-w|^2\right)^{1/2} + C\left(\int_{\mathbb{R}^2} Q(e^{\alpha|w_n|^2}-1)^r\right)^{1/r}\left(\int_{\mathbb{R}^2} Q|w_n-w|^s\right)^{1/s}.$$

Then, from Lemma 1.2.5 and (4.16), the claim follows. This convergence together with the fact that $I'_\lambda(w_n)(w_n-w) = o_n(1)$ imply that

$$\lim_{n\to\infty}\|w_n\|_Z^2 = \|w\|_Z^2$$

and so $w_n \to w$ strongly in Z. The proof of the lemma is concluded. ■

Proof of Theorem 4.1.2. Since I_λ verifies $(J_1)-(J_4)$, the result follows directly from Theorem 4.5.1. ■

Chapter 5

On a class of Hamiltonian elliptic systems

This chapter is concerned with the existence of solution for the following class of Hamiltonian elliptic systems

$$\begin{cases} -\Delta u + V(|x|)u = Q(|x|)g(v) & \text{in} \quad \mathbb{R}^2, \\ -\Delta v + V(|x|)v = Q(|x|)f(u) & \text{in} \quad \mathbb{R}^2, \end{cases} \tag{5.1}$$

when the nonlinearities f and g are allowed to enjoy the exponential critical growth by means of the Trudinger-Moser inequality and the radial potentials V and Q may be unbounded, singular or decaying to zero. The approach relies on an approximation procedure and the Trudinger-Moser type inequality established in Theorem 1.1.2. We point out that part of this chapter is contained in the preprint [8].

5.1 Introduction and main results

We make the following assumptions on the potentials $V(|x|)$ and $Q(|x|)$:

(V) $V \in C(0,\infty)$, $V(r) > 0$ and there exists $a > -2$ such that

$$\liminf_{r\to+\infty} \frac{V(r)}{r^a} > 0.$$

(Q) $Q \in C(0,\infty)$, $Q(r) > 0$ and there exist $b < (a-2)/2$ and $b_0 > -2$ such that

$$\limsup_{r\to 0} \frac{Q(r)}{r^{b_0}} < \infty \quad \text{and} \quad \limsup_{r\to+\infty} \frac{Q(r)}{r^{b}} < \infty.$$

We suppose that the nonlinearities $f(s)$ and $g(s)$ have maximal growth on s which allows us to treat system (5.1) variationally. Explicitly, in view of the classical Trudinger-Moser inequality, we say that f and g have *exponential critical growth* at $+\infty$ if there exist $\alpha_0 \geq \beta_0 > 0$ such that

$$\lim_{s\to+\infty} \frac{f(s)}{e^{\alpha s^2}} = \begin{cases} 0, & \forall \alpha > \alpha_0, \\ +\infty, & \forall \alpha < \alpha_0 \end{cases} \quad \text{and} \quad \lim_{s\to+\infty} \frac{g(s)}{e^{\alpha s^2}} = \begin{cases} 0, & \forall \alpha > \beta_0, \\ +\infty, & \forall \alpha < \beta_0. \end{cases} \tag{5.2}$$

Throughout we will assume that $f, g : \mathbb{R} \to [0, +\infty)$ are continuous functions satisfying:

(H_1) $f(s) = o(s)$ and $g(s) = o(s)$ as $s \to 0$;

(H_2) there exists $\theta > 2$ such that for all $s > 0$

$$0 < \theta F(s) \doteq \theta \int_0^s f(t)\mathrm{dt} \leq sf(s) \quad \text{and} \quad 0 < \theta G(s) \doteq \theta \int_0^s g(t)\mathrm{dt} \leq sg(s);$$

(H_3) there exist constants $s_0, M_0 > 0$ such that for all $s \geq s_0$

$$0 < F(s) \leq M_0 f(s) \quad \text{and} \quad 0 < G(s) \leq M_0 g(s);$$

(H_4) there exist constants $\nu > 2$ and $\mu > 0$ such that

$$F(s), G(s) \geq \frac{\mu}{\nu} s^\nu, \quad \forall s \geq 0.$$

We recall that in $Z \doteq E \times E$ we defined the norm of an element $z = (u, v) \in Z$ by

$$\|z\|_Z \doteq \left(\|u\|^2 + \|v\|^2\right)^{1/2}.$$

Denoting by $S_\nu > 0$ the best constant of the Sobolev embedding

$$E \hookrightarrow L^\nu(\mathbb{R}^2; Q),$$

see Lemma 1.2.5, we state our main result as follows

Theorem 5.1.1 (Existence) *Suppose that* $(V) - (Q)$ *hold. If* f *and* g *have exponential critical growth and* $(H_1) - (H_4)$ *are satisfied, with*

$$\mu > \left[\frac{(\alpha_0 + \beta_0)(\nu - 2)}{\nu \alpha'}\right]^{(\nu-2)/2} 2^{\nu/2} S_\nu^\nu,$$

then system (5.1) *has a nontrivial positive weak solution in* Z.

Remark 5.1.2 (Subcritical case) *We point out that if $g(s)$ has exponential subcritical growth, that is, if $\beta_0 = 0$ in (5.2), $f(s)$ has exponential subcritical ($\alpha_0 = 0$ in (5.2)) or critical growth and $(V)-(Q)$, $(H_1)-(H_2)$ are satisfied, then using similar arguments developed in [25, Theorem 1.1] we can prove that system (5.1) possesses a nontrivial weak solution in Z. This remark will be verified at the end of the chapter.*

Remark 5.1.3 *Our result complements the study made in [62] in the sense that, in this chapter, we study a class of Hamiltonian systems involving exponential critical growth and in [62] only the Sobolev subcritical growth and scalar problem was considered. We refer the reader to [63] for a related result involving p-Laplace equation.*

As it is well known in dimensions $N \geq 3$, the nonlinearities are required to have polynomial growth at infinity, so that one can define associated functionals in Sobolev spaces. Coming to dimension $N = 2$, much faster growth is allowed for the nonlinearity. In fact, the Trudinger-Moser inequalities in dimension two replace the Sobolev embedding theorem used in $N \geq 3$. After the seminal work of Brezis-Nirenberg [18] on elliptic problems involving critical growth, many advances have been done on this class of problems. Recently, elliptic systems in dimensions $N \geq 3$ was treated for instance in [26, 31, 40] and references therein by using a variational approach. When $N = 2$, Hamiltonian systems in bounded domain $\Omega \subset \mathbb{R}^2$ was studied by de Figueiredo et al. [25] and in the whole space $\mathbb{R}^2$ by de Souza [30], do Ó et al. [33] and Zhang-Liu [70].

One very interesting characteristic of the Hamiltonian elliptic systems of the form

$$\begin{cases} -\Delta u + u = |v|^{q-1}v \quad \text{in} \quad \mathbb{R}^N, \\ -\Delta v + v = |u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N, \\ u, v > 0 \quad \text{in} \quad \mathbb{R}^N, \quad N \geq 3, \end{cases} \tag{5.3}$$

is presented by the *critical hyperbola* (see [22, 65])

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N},$$

which plays a very important role for to be the dividing line between the existence and nonexistence of solutions for (5.3). Indeed, it is well known that (5.3) has a radial solutions (u, v) whenever

$$p, q > 0 \quad \text{and} \quad 1 > \frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}.$$

In analogy, as pointed out in [56], the author considered the following Hamiltonian elliptic systems

$$\begin{cases} -\Delta u + u = ve^{v^q} \quad \text{in} \quad \mathbb{R}^2, \\ -\Delta v + v = ue^{u^p} \quad \text{in} \quad \mathbb{R}^2, \\ u, v > 0 \quad \text{in} \quad \mathbb{R}^2, \end{cases}$$

with

$$p, q > 1 \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} \geq 1,$$

which means a criticality notion similar that one established for high dimensions and is usually known by *exponential critical hyperbola*.

To finish this section, we remark that there are at least three main difficulties in our problem; the possible lack of the compactness of the Sobolev embedding since the domain $\mathbb{R}^2$ is unbounded, the critical growth of the nonlinearities and the fact that the energy functional associated with system (5.1) is strongly indefinite, as explained in Introduction, which has more complex geometry structure than functionals with mountain-pass geometry. Besides these difficulties described, which are very common for the study of Hamiltonian elliptic systems in unbounded domains, there is one arisen when we are in infinite dimensional spaces, that is, the impossibility to use an *intersection theorem* (see Proposition 5.9 in [54]) which is crucial to get lower bound of minimax levels as well as allows us to show that they are indeed critical values.

5.2 Variational setting

Since we are interested in find positive solutions and $f(0) = g(0) = 0$, without loss of generality, we will assume $f(s) = g(s) = 0$ for all $s \leq 0$. The natural functional associated with system (5.1) is given by

$$I(u, v) = \int_{\mathbb{R}^2} (\nabla u \nabla v + Vuv) - \int_{\mathbb{R}^2} Q[F(u) + G(v)],$$

for $(u, v) \in Z$. To ensures that I is well-defined on Z, observes that for $\alpha > \alpha_0 \geq \beta_0$ given by (5.2) and $q \geq 1$, it follows from (5.2) and (H_1), for any given $\varepsilon > 0$, there exist constants $b_1, b_2 > 0$ such that

$$f(s), g(s) \leq \varepsilon|s| + b_1|s|^{q-1}(e^{\alpha s^2} - 1), \quad \forall s \in \mathbb{R} \tag{5.4}$$

and

$$F(s), G(s) \leq \frac{\varepsilon}{2}s^2 + b_2|s|^q(e^{\alpha s^2} - 1), \quad \forall s \in \mathbb{R}. \tag{5.5}$$

Given $u \in E$ we can use (5.5) with $q = 2$ to obtain

$$\int_{\mathbb{R}^2} QF(u) \leq \frac{\varepsilon}{2}\int_{\mathbb{R}^2} Q|u|^2 + b_2\int_{\mathbb{R}^2} Q|u|^2(e^{\alpha u^2} - 1).$$

By Lemma 1.2.5,

$$\int_{\mathbb{R}^2} Q|u|^2 < \infty.$$

Now, let $r_1, r_2 > 1$ be such that $1/r_1 + 1/r_2 = 1$. The Hölder's inequality, Lemma 1.2.5 and Theorem 1.1.2 imply that

$$\int_{\mathbb{R}^2} Q|u|^2(e^{\alpha u^2} - 1) \leq \left(\int_{\mathbb{R}^2} Q|u|^{2r_1}\right)^{1/r_1}\left(\int_{\mathbb{R}^2} Q(e^{r_2\alpha u^2} - 1)\right)^{1/r_2} < \infty,$$

where we have used the elementary inequality $(e^s - 1)^r \leq (e^{rs} - 1)$, for all $r \geq 1, s \geq 0$. Hence

$$\int_{\mathbb{R}^2} QF(u) < \infty$$

and analogously given $v \in E$ we have $\int_{\mathbb{R}^2} QG(v) < \infty$. Therefore, I is well-defined and $I \in C^1(Z, \mathbb{R})$ with

$$I'(u,v)(\phi,\psi) = \int_{\mathbb{R}^2}[\nabla u\nabla\psi + \nabla v\nabla\phi + V(u\psi + v\phi)] - \int_{\mathbb{R}^2} Q[f(u)\phi + g(v)\psi],$$

for all $(\phi, \psi) \in Z$. Thus, since we are searching for a weak solution for system (5.1), that is, a function $(u, v) \in Z$ such that

$$\int_{\mathbb{R}^2}[\nabla u\nabla\psi + \nabla v\nabla\phi + V(u\psi + v\phi)] - \int_{\mathbb{R}^2} Q[f(u)\phi + g(v)\psi] = 0,$$

for all $(\phi, \psi) \in Z$, a critical point of I turns out to be a weak solution of system (5.1) and reciprocally.

Remark 5.2.1 *In the proof of Theorem 5.1.1, we shall need a more precise estimate for $f(s)$ and $g(s)$, namely, given $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that*

$$f(s), g(s) \leq C_\varepsilon e^{(\alpha_0 + \beta_0 + \varepsilon)s^2}, \quad \forall s \in \mathbb{R}. \tag{5.6}$$

5.2.1 Linking geometry

If we define:

$$Z^+ = \{(u,u) : u \in E\} \quad \text{and} \quad Z^- = \{(u,-u) : u \in E\},$$

it is easy to check that $Z = Z^+ \oplus Z^-$, since

$$(u,v) = \frac{1}{2}(u+v,u+v) + \frac{1}{2}(u-v,v-u).$$

The next lemmas are essential to establish the geometry of the Linking Theorem of the functional I.

Lemma 5.2.2 *Assume that $(V)-(Q)$ hold. If f and g satisfy (5.2) and $(H_1)-(H_2)$, then there exist $\rho, \sigma > 0$ such that $I(z) \geq \sigma$, for all $z \in S \doteq \partial B_\rho \cap Z^+$.*

Proof. Invoking (5.5) with $q > 2$, it follows from the Hölder's inequality, Lemma 1.2.5 and Theorem 1.1.2 that

$$\begin{aligned}\int_{\mathbb{R}^2} QF(u), \int_{\mathbb{R}^2} QG(u) &\leq \frac{\varepsilon}{2}\int_{\mathbb{R}^2} Q|u|^2 + b_2 \int_{\mathbb{R}^2} Q|u|^q(e^{\alpha u^2}-1) \\ &\leq \frac{C\varepsilon}{2}\|u\|^2 + b_2\left(\int_{\mathbb{R}^2} Q|u|^{qr_1}\right)^{1/r_1}\left(\int_{\mathbb{R}^2} Q(e^{r_2\alpha u^2}-1)\right)^{1/r_2} \\ &\leq \frac{C\varepsilon}{2}\|u\|^2 + C_1\|u\|^q,\end{aligned}$$

whenever $\|u\| \leq M < (\alpha'/\alpha)^{1/2}$ and $r_2 > 1$ is sufficiently close to 1. Consequently,

$$I(u,u) \geq \left(\frac{1}{2} - \frac{C\varepsilon}{2}\right)\|(u,u)\|_Z^2 - C_2\|(u,u)\|_Z^q.$$

Since $q > 2$, we can find $\rho, \sigma > 0$, ρ sufficiently small, such that $I(u,u) \geq \sigma$ for $\|(u,u)\|_Z = \rho$. Therefore, the proof of lemma is finished. ■

Let $y \in E$ be a fixed nonnegative function with $\|y\| = 1$ and

$$Q_y \doteq \{r(y,y) + w : w \in Z^-, \quad \|w\|_Z \leq R_0 \quad \text{and} \quad 0 \leq r \leq R_1\},$$

where R_0, R_1 are positive constants to be chosen later.

Lemma 5.2.3 *Assume that $(V)-(Q)$ hold. If (H_2) is satisfied, then there exist positive constants R_0, R_1, which depend on y, such that $I(z) \leq 0$ for all $z \in \partial Q_y$.*

Proof. Since the boundary ∂Q_y of Q_y lives in the space $\mathbb{R}(y,y) \oplus Z^-$, it consists of three parts (see Figure 5.1 at the end of the proof). For this reason, we have to estimate the functional I on these parts as follows:

(*i*) If $z \in \partial Q_y \cap Z^-$, then $z = (u, -u) \in Z^-$ and thus

$$I(z) = -\|u\|^2 - \int_{\mathbb{R}^2} Q[F(u) + G(-u)] \leq 0.$$

(*ii*) If $z = R_1(y, y) + (u, -u) \in \partial Q_y$ with $\|(u, -u)\|_Z \leq R_0$,

$$I(z) = R_1^2\|y\|^2 - \|u\|^2 - \int_{\mathbb{R}^2} Q[F(R_1 y + u) + G(R_1 y - u)].$$

It follows from assumption (H_4) that

$$I(z) \leq R_1^2 - C \int_{\mathbb{R}^2} Q[\xi(R_1 y + u) + \xi(R_1 y - u)],$$

where we have used Lemma 1.2.5 and we introduced the real function

$$\xi(s) = \begin{cases} s^\nu, & \forall s \geq 0, \\ 0, & \forall s < 0. \end{cases}$$

Now, using the convexity of ξ, it follows that

$$\begin{aligned} I(z) &\leq R_1^2 - C \int_{\mathbb{R}^2} Q\xi(R_1 y) \\ &= R_1^2 - C R_1^\nu \int_{\mathbb{R}^2} Q|y|^\nu \\ &= R_1^2 - C R_1^\nu. \end{aligned}$$

Finally taking $R_1 = R_1(y)$ sufficiently large, we get $I(z) \leq 0$.

(*iii*) If $z = r(y, y) + (u, -u) \in \partial Q_y$ with $\|(u, -u)\|_Z = R_0$ and $0 \leq r \leq R_1$,

$$\begin{aligned} I(z) &= r^2\|y\|^2 - \|u\|^2 - \int_{\mathbb{R}^2} Q[F(ry + u) + G(ry - u)] \\ &\leq R_1^2\|y\|^2 - \|u\|^2 \\ &= R_1^2 - \frac{1}{2}R_0^2. \end{aligned}$$

Thus, $I(z) \leq 0$ if $\sqrt{2}R_1 \leq R_0$. Therefore, the proof of lemma is finished. ∎

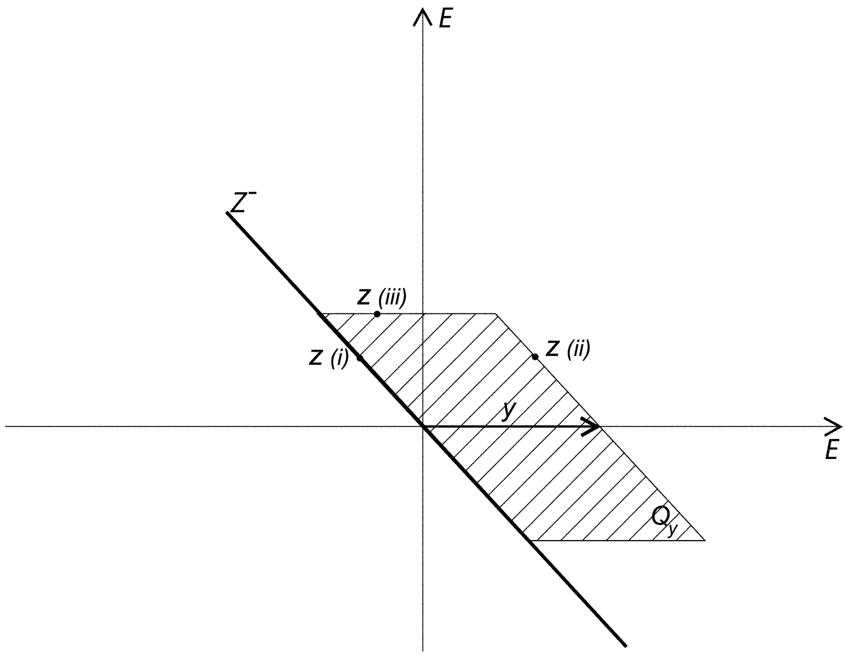

Figure 5.1: Q_y and its boundary ∂Q_y

5.2.2 Palais-Smale condition

In order to obtain the Palais-Smale condition we need the following technical lemma due to [25] which we include the proof for completes.

Lemma 5.2.4 *The following inequality holds*

$$st \leq \begin{cases} (e^{t^2}-1)+s(\log s)^{1/2}, & \text{for all } t \geq 0 \text{ and } s \geq e^{1/4}, \\ (e^{t^2}-1)+\frac{1}{2}s^2, & \text{for all } t \geq 0 \text{ and } 0 \leq s \leq e^{1/4}. \end{cases} \tag{5.7}$$

Proof. Clearly, for $s=0$ the inequality is obviously satisfied. For $s>0$ given, let us consider the function $\varphi : \mathbb{R}^+ \to \mathbb{R}$ given by

$$\varphi(t) = st - (e^{t^2}-1).$$

If t_s denote the maximum point of φ, then $s = 2t_s e^{t_s^2}$. Now, we have three cases to consider:

Case 1. $t_s \geq 1/2$. In this case, $s = 2t_s e^{t_s^2} \geq e^{t_s^2}$, which implies $(\log s)^{1/2} \geq t_s$. Thus

$$\varphi(t) \leq st_s - (e^{t_s^2}-1) \leq st_s \leq s(\log s)^{1/2}.$$

Case 2. $0 \leq t_s \leq 1/2$ and $s \geq e^{1/4}$. In this case,

$$st_s \leq s/2 \quad \text{and} \quad s/2 \leq s(\log s)^{1/2} \Leftrightarrow s \geq e^{1/4},$$

which imply that

$$st_s \leq s(\log s)^{1/2} \leq (e^{t_s^2}-1)+s(\log s)^{1/2} \therefore \varphi(t) \leq st_s-(e^{t_s^2}-1) \leq s(\log s)^{1/2}, \quad \forall t \geq 0.$$

Case 3. $0 \leq t_s \leq 1/2$ and $s \leq e^{1/4}$. In fact, the second part of the inequality holds always, since

$$ts \leq \frac{1}{2}t^2+\frac{1}{2}s^2 \leq \frac{1}{2}(e^{t^2}-1)+\frac{1}{2}s^2 \leq (e^{t^2}-1)+\frac{1}{2}s^2.$$

Therefore, the lemma is proved. ■

Under the same conditions assumed in Lemma 5.2.2, we have the following

Proposition 5.2.5 *Let $((u_n, v_n))$ be a $(PS)_c$ sequence in Z, that is,*

i) $I(u_n, v_n) = c+\delta_n$, *where* $\delta_n \to 0$ *as* $n \to \infty$*;*

ii) $|I'(u_n, v_n)(\phi, \psi)| \leq \varepsilon_n \|(\phi, \psi)\|_Z$, *for* $(\phi, \psi) \in Z$, *where* $\varepsilon_n \to 0$ *as* $n \to \infty$*.*

Then $((u_n, v_n))$ is bounded in Z and

$$\int_{\mathbb{R}^2} Qf(u_n)u_n \leq C, \quad \int_{\mathbb{R}^2} Qg(v_n)v_n \leq C,$$
$$\int_{\mathbb{R}^2} QF(u_n) \leq C, \quad \int_{\mathbb{R}^2} QG(v_n) \leq C.$$

Proof. From $i)$ and $ii)$ (taking $(\phi, \psi) = (u_n, v_n)$), we have

$$\int_{\mathbb{R}^2} Q[f(u_n)u_n-2F(u_n)]+\int_{\mathbb{R}^2} Q[g(v_n)v_n-2G(v_n)] \leq 2c+2\delta_n+\varepsilon_n\|(u_n, v_n)\|_Z. \quad (5.8)$$

This together with the hypothesis (H_2) imply that

$$(\theta-2)\int_{\mathbb{R}^2} Q[F(u_n)+G(v_n)] \leq 2c+2\delta_n+\varepsilon_n\|(u_n, v_n)\|_Z. \quad (5.9)$$

Thus, using (5.9) in (5.8) we get

$$\int_{\mathbb{R}^2} Q[f(u_n)u_n+g(v_n)v_n] \leq \frac{\theta}{\theta-2}(2c+2\delta_n+\varepsilon_n\|(u_n, v_n)\|_Z). \quad (5.10)$$

Next taking $(\phi, \psi) = (v_n, 0)$ and $(\phi, \psi) = (0, u_n)$ in $ii)$ we have

$$\|v_n\|^2-\varepsilon_n\|v_n\| \leq \int_{\mathbb{R}^2} Qf(u_n)v_n,$$
$$\|u_n\|^2-\varepsilon_n\|u_n\| \leq \int_{\mathbb{R}^2} Qg(v_n)u_n.$$

Setting

$$U_n = \frac{u_n}{\|u_n\|} \quad \text{and} \quad V_n = \frac{v_n}{\|v_n\|},$$

we infer that

$$\|v_n\| \leq \int_{\mathbb{R}^2} Qf(u_n)V_n + \varepsilon_n \tag{5.11}$$

and

$$\|u_n\| \leq \int_{\mathbb{R}^2} Qg(v_n)U_n + \varepsilon_n. \tag{5.12}$$

Observing that from (5.2) and (H_1) we have for $\alpha > \alpha_0$

$$f(s) \leq C_1 e^{\alpha s^2}, \quad \forall s \geq 0 \tag{5.13}$$

and

$$[f(s)]^2 \leq C_2 f(s)s \tag{5.14}$$

in $\{s \in \mathbb{R} : s \geq 0 \text{ and } |f(s)|/C_1 \leq e^{1/4}\}$, we shall use Lemma 5.2.4 to estimate the integrals in (5.11) and (5.12). Choosing $\alpha < \alpha'$ and using the inequality (5.7), with

$$t = \sqrt{\alpha}V_n \quad \text{and} \quad s = f(u_n)/C_1,$$

and Theorem 1.1.2 we obtain

$$\begin{aligned} C_1 \int_{\mathbb{R}^2} Q\frac{f(u_n)}{C_1}V_n &\leq 2C_1 \int_{\mathbb{R}^2} Q(e^{\alpha V_n^2} - 1) + C_1 \int_{\{f(u_n)/C_1 \geq e^{1/4}\}} Q\frac{f(u_n)}{C_1}\left(\log \frac{f(u_n)}{C_1}\right)^{1/2} \\ &+ \frac{1}{2}\int_{\{f(u_n)/C_1 \leq e^{1/4}\}} Q\frac{1}{C_1^2}[f(u_n)]^2 \\ &\leq C_3 + C_4 \int_{\mathbb{R}^2} Qf(u_n)u_n. \end{aligned}$$

This estimate together with (5.11) imply that

$$\|v_n\| \leq C_3 + C_4 \int_{\mathbb{R}^2} Qf(u_n)u_n. \tag{5.15}$$

Repeating the same argument above we get

$$\|u_n\| \leq C_5 + C_6 \int_{\mathbb{R}^2} Qg(v_n)v_n. \tag{5.16}$$

Now joining the estimates (5.15) and (5.16) and using (5.10) we achieved

$$\|(u_n, v_n)\|_Z \leq \frac{\theta}{\theta - 2}(2c + 2\delta_n + \varepsilon_n\|(u_n, v_n)\|_Z + \varepsilon_n),$$

which implies that $\|(u_n, v_n)\|_Z \leq C$. From this estimate, inequalities (5.9) and (5.10) we obtain the other estimates in the statement of the proposition. ■

To prove that a Palais-Smale sequence converges to a weak solution of system (5.1) we will use the following convergence result:

Lemma 5.2.6 *Let $((u_n, v_n)) \subset Z$ be a Palais-Smale sequence for I and (u, v) its weak limit. Then, up to a subsequence,*

$$f(u_n) \to f(u), \quad g(v_n) \to g(v) \quad in \quad L^1_{loc}(\mathbb{R}^2; Q)$$

and

$$F(u_n) \to F(u), \quad G(v_n) \to G(v) \quad in \quad L^1(\mathbb{R}^2; Q).$$

Proof. Using Proposition 5.2.5 and recalling that $H^1_{\text{rad}}(B_R; V)$ is compactly immersed in $L^q(B_R)$ for all $1 \leq q < \infty$ (see consequence of *ii*) from Lemma 1.2.3), up to a subsequence, we may assume that $u_n \to u$ strongly in $L^1(B_R)$. Moreover,

$$Qf(u_n) \in L^1(B_R), \quad Qf(u) \in L^1(B_R) \quad \text{and} \quad \int_{B_R} Q|f(u_n)u_n| \leq C,$$

$n \in \mathbb{N}$. Therefore, from Lemma 2.2.5,

$$Qf(u_n) \to Qf(u) \quad \text{in} \quad L^1(B_R),$$

for all $R > 0$. Analogously,

$$Qg(v_n) \to Qg(v) \quad \text{in} \quad L^1(B_R),$$

for all $R > 0$. Finally, by using (H_3), it just follows the same steps used in Lemma 2.2.6 to prove the convergences in $L^1(\mathbb{R}^2)$. Therefore, the lemma is proved. ■

5.3 Estimate of the minimax level

Lemma 5.3.1 *Suppose that $(V) - (Q)$ hold. If (H_4) is satisfied with*

$$\mu > \left[\frac{(\alpha_0 + \beta_0)(\nu - 2)}{\nu \alpha'} \right]^{(\nu-2)/2} 2^{\nu/2} S_\nu^\nu,$$

then

$$\sup_{\mathbb{R}_+(u_\nu, u_\nu) \oplus Z^-} I < \frac{\alpha'}{2(\alpha_0 + \beta_0)},$$

where $u_\nu \in E \setminus \{0\}$ is a nonnegative function such that S_ν is attained.

Proof. Since the embeddings $E \hookrightarrow L^p(\mathbb{R}^2; Q)$ are compacts for $2 \leq p < \infty$, there exists a function $u_\nu \in E$ such that S_ν is attained, that is, there exists a nonnegative function $u_\nu \in E \setminus \{0\}$ satisfying

$$S_\nu = \int_{\mathbb{R}^2} (|\nabla u_\nu|^2 + V u_\nu^2) \quad \text{and} \quad \int_{\mathbb{R}^2} Q u_\nu^\nu = 1.$$

For each $z = t(u_\nu, u_\nu) + (v, -v)$ with $t \geq 0$ and $v \in E$, we have

$$
\begin{aligned}
I(z) &\leq t^2\|u_\nu\|^2 - \|v\|^2 - \int_{\mathbb{R}^2} QF(tu_\nu + v) - \int_{\mathbb{R}^2} QG(tu_\nu - v) \\
&\leq t^2 S_\nu^2 - \int_{\mathbb{R}^2} Q[F(tu_\nu + v) + G(tu_\nu - v)].
\end{aligned}
$$

By using condition (H_4) and the elementary inequality

$$
|s|^\nu \leq |s+t|^\nu + |s-t|^\nu, \quad \forall s, t \in \mathbb{R},
$$

we obtain

$$
I(z) \leq t^2 S_\nu^2 - t^\nu \frac{\mu}{\nu} \int_{\mathbb{R}^2} Qu_\nu^\nu, \quad t \geq 0.
$$

Consequently,

$$
I(z) \leq \max_{t \geq 0} \left[t^2 S_\nu^2 - t^\nu \frac{\mu}{\nu} \right] = \frac{2^{2/(\nu-2)}\nu - 2^{\nu/(\nu-2)}}{\nu} \frac{S_\nu^{2\nu/(\nu-2)}}{\mu^{2/(\nu-2)}} < \frac{\alpha'}{2(\alpha_0 + \beta_0)},
$$

which completes the proof of lemma. ■

5.4 Approximation procedure

Since the functional I is strongly indefinite and defined in an infinite dimensional space, we can not to apply an *intersection theorem* (see Proposition 5.9 in [54]) which is crucial to get lower bound of minimax levels as mentioned in Section 5.1. Moreover, linking theorems are not suitable. To overcome these difficulties, we shall approximate our problem (5.1) with a sequence of finite dimensional problems by using a approximation procedure due to Galerkin. To do this, let us start with a basic result of spectral theory.

Proposition 5.4.1 *If $(V) - (Q)$ hold, then the eigenvalue problem*

$$
-\Delta w + V(|x|)w = \lambda w \quad in \quad \mathbb{R}^2,
$$

possesses a sequence of eigenvalues $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_j \to +\infty$ and an orthonormal basis $\{\phi_1, \phi_2, \ldots\}$ of corresponding eigenfunctions in E.

Proof. Indeed, for every $h \in L^2(\mathbb{R}^2; Q)$, by Riesz Representation Theorem, there exists an unique $w \in E$ such that

$$
-\Delta w + V(|x|)w = h \quad \text{in} \quad \mathbb{R}^2,
$$

in the weak sense. Denote $L = -\Delta + V$. Then the operator L has an inverse L^{-1}. Next, we check that L^{-1} is continuous:

$$\begin{aligned}\|L^{-1}(h)\|^2 = \|w\|^2 &= \int_{\mathbb{R}^2} (|\nabla w|^2 + Vw^2) \\ &= \langle h, w\rangle \\ &\le \|h\|_{L^2(\mathbb{R}^2;Q)}\|w\|_{L^2(\mathbb{R}^2;Q)} \\ &\le C\|h\|_{L^2(\mathbb{R}^2;Q)}\|w\|.\end{aligned}$$

Thus

$$\|L^{-1}(h)\| \le C\|h\|_{L^2(\mathbb{R}^2;Q)}.$$

Moreover, using the fact that the embedding

$$E \hookrightarrow L^2(\mathbb{R}^2; Q)$$

is compact (see Lemma 1.2.5), we conclude that the operator $L : E \to E$ is compact. Therefore, from spectral theory of symmetric compact operators on Hilbert spaces, we obtain the result. ■

In the following, we define some liner spaces and sets in order to treat of finite dimensional problem. Firstly, we set

$$\begin{aligned}&Z_n^+ = \operatorname{span}\{(\phi_1, \phi_1), \ldots, (\phi_n, \phi_n)\}, \\ &Z_n^- = \operatorname{span}\{(\phi_1, -\phi_1), \ldots, (\phi_n, -\phi_n)\}, \\ &Z_n = Z_n^+ \oplus Z_n^-.\end{aligned}$$

Now, let $y \in E$ be a fixed nonnegative function with $\|y\| = 1$ and

$$Q_{n,y} \doteq \{r(y, y) + w : w \in Z_n^-, \quad \|w\|_Z \le R_0 \quad \text{and} \quad 0 \le r \le R_1\},$$

where R_0 and R_1 are given in Lemma 5.2.3. We recall that these constants depend on y only. We use the following notation:

$$H_{n,y} = \mathbb{R}(y, y) \oplus Z_n, \quad H_{n,y}^+ = \mathbb{R}(y, y) \oplus Z_n^+, \quad H_{n,y}^- = \mathbb{R}(y, y) \oplus Z_n^-.$$

Furthermore, define the class of mappings

$$\Gamma_{n,y} = \{h \in C(Q_{n,y}, H_{n,y}) : h(z) = z \quad \text{on} \quad \partial Q_{n,y}\}$$

and set

$$c_{n,y} = \inf_{\Gamma_{n,y}} \max_{z \in Q_{n,y}} I(h(z)).$$

Using an *intersection theorem* (see Proposition 5.9 in [53]), we have

$$h(Q_{n,y}) \cap S \neq \emptyset, \quad \forall h \in \Gamma_{n,y},$$

which in combination with Lemma 5.2.2 implies that

$$c_{n,y} \geq \sigma > 0.$$

On the other hand, an upper bound for the minimax level $c_{n,y}$ can be obtained as follows. Since the identity mapping $Id : Q_{n,y} \to H_{n,y}$ belongs to $\Gamma_{n,y}$, we have for $z = r(y, y) + (u, -u) \in Q_{n,y}$ that

$$I(z) = r^2\|y\|^2 - \|u\|^2 - \int_{\mathbb{R}^2} Q[F(ry + u) + G(ry - u)] \leq r^2\|y\|^2 \leq R_1^2.$$

Therefore we have

$$0 < \sigma \leq c_{n,y} \leq R_1^2.$$

We remark that the upper bound does not depend on n, but it depends on y.

Let us denote by $I_{n,y}$ the functional I restricted to the finite dimensional subspace $H_{n,y}$. Thus, in view of Lemmas 5.2.2 and 5.2.3 we see that the geometry of a linking theorem holds for the functional $I_{n,y}$. Therefore, applying the linking theorem for $I_{n,y}$ (see Theorem 5.3 in [53]), we obtain a Palais-Smale sequence, which is bounded in view of Proposition 5.2.5. Finally, using the fact that $H_{n,y}$ is a finite dimensional space, we get the main result of this section (see [37]).

Proposition 5.4.2 *For each $n \in \mathbb{N}$ and for each $y \in E$, a fixed nonnegative function, the functional $I_{n,y}$ has a critical point at level $c_{n,y}$. More precisely, there exists $z_{n,y} \in H_{n,y}$ such that*

$$\begin{aligned} I_{n,y}(z_{n,y}) &= c_{n,y} \in [\sigma, R_1^2] \\ I'_{n,y}(z_{n,y}) &= 0. \end{aligned} \tag{5.17}$$

Furthermore, $\|z_{n,y}\|_Z \leq C$, where C does not depend on n.

5.5 Proof of the existence theorem

In the proof of Theorem 5.1.1, we shall need of the following technical lemma:

Lemma 5.5.1 *Assume that* $(V)-(Q)$ *hold and* $(5.2)-(H_1)$ *is satisfied. Let* $y \in E$ *be a fixed nonnegative function and* $z_{n,y} = (u_n, v_n) \in H_{n,y}$ *such that*

$$\begin{aligned} I_{n,y}(z_{n,y}) &= c_{n,y} \in [\sigma, \alpha'/2(\alpha_0+\beta_0)-\delta) \\ I'_{n,y}(z_{n,y}) &= 0, \end{aligned} \tag{5.18}$$

for some $\delta > 0$. *In addition, suppose that*

$$\|u_n\|, \|v_n\| \geq C > 0, \ \textit{for all } n \in \mathbb{N}$$

and

$$(u_n, v_n) \rightharpoonup (0,0) \ \textit{weakly in } Z.$$

Then for any given $\varepsilon > 0$ *we have the following estimate*

$$\|u_n\| + \|v_n\| \leq o_n(1) + 2\left(1 + \frac{\varepsilon}{\alpha_0+\beta_0}\right)^{1/2}\left(\frac{\alpha'}{2(\alpha_0+\beta_0)} - \delta\right)^{1/2}. \tag{5.19}$$

Proof. Taking $(0, u_n)$ as a test function in (5.18) we obtain

$$\|u_n\|^2 = \int_{\mathbb{R}^2} Qg(v_n)u_n. \tag{5.20}$$

Setting

$$\overline{u}_n \doteq \left(\frac{\alpha'}{2(\alpha_0+\beta_0)} - \delta\right)^{1/2} \frac{u_n}{\|u_n\|}$$

and using the inequality (5.7) with

$$s = g(v_n)/\sqrt{\alpha_0+\beta_0} \quad \text{and} \quad t = \sqrt{\alpha_0+\beta_0}\,\overline{u}_n,$$

we obtain

$$\begin{aligned} \left(\frac{\alpha'}{2(\alpha_0+\beta_0)} - \delta\right)^{1/2} \|u_n\| &= \int_{\mathbb{R}^2} Qg(v_n)\overline{u}_n \leq \int_{\mathbb{R}^2} Q(e^{(\alpha_0+\beta_0)\overline{u}_n^2} - 1) \\ &+ \int_{\{g(v_n)/\sqrt{\alpha_0+\beta_0} \geq e^{1/4}\}} Q\frac{g(v_n)}{\sqrt{\alpha_0+\beta_0}}\left[\log\left(\frac{g(v_n)}{\sqrt{\alpha_0+\beta_0}}\right)\right]^{1/2} \\ &+ \int_{\{g(v_n)/\sqrt{\alpha_0+\beta_0} \leq e^{1/4}\}} Q\frac{[g(v_n)]^2}{\alpha_0+\beta_0}. \end{aligned} \tag{5.21}$$

To estimate the third integral in (5.21) we observe that by (H_1), we have

$$[g(v_n)]^2 \leq Cv_n^2$$

in $\{s \in \mathbb{R} : s \geq 0 \text{ and } g(s)/\sqrt{\alpha_0+\beta_0} \leq e^{1/4}\}$ and so by Lemma 1.2.5 we have that the third term tends to zero. In relation to the first integral, by using the elementary inequality (1.27) with

$$x = (\alpha_0+\beta_0)\overline{u}_n^2 \quad \text{and} \quad y = 0,$$

we estimate

$$\begin{aligned}\int_{\mathbb{R}^2} Q(e^{(\alpha_0+\beta_0)\overline{u}_n^2}-1) &\le \int_{\mathbb{R}^2} Q(\alpha_0+\beta_0)\overline{u}_n^2(e^{(\alpha_0+\beta_0)\overline{u}_n^2}+1)\\ &= \int_{\mathbb{R}^2} Q(\alpha_0+\beta_0)\overline{u}_n^2(e^{(\alpha_0+\beta_0)\overline{u}_n^2}-1+2)\\ &= (\alpha_0+\beta_0)\int_{\mathbb{R}^2} Q(e^{(\alpha_0+\beta_0)\overline{u}_n^2}-1)\overline{u}_n^2 + 2(\alpha_0+\beta_0)\int_{\mathbb{R}^2} Q\overline{u}_n^2.\end{aligned}$$

By the Hölder's inequality

$$\int_{\mathbb{R}^2} Q(e^{(\alpha_0+\beta_0)\overline{u}_n^2}-1)\overline{u}_n^2 \le \left(\int_{\mathbb{R}^2} Q(e^{r_1(\alpha_0+\beta_0)\overline{u}_n^2}-1)\right)^{1/r_1}\left(\int_{\mathbb{R}^2} Q\overline{u}_n^{2r_2}\right)^{1/r_2},$$

where $1/r_1+1/r_2=1$. Since

$$\|\overline{u}_n\|^2 = \frac{\alpha'}{2(\alpha_0+\beta_0)}-\delta,$$

it is possible to take $r_1>1$ sufficiently close to 1 such that $r_1(\alpha_0+\beta_0)\|\overline{u}_n\|^2<\alpha'$. Hence, from Theorem 1.1.2 and Lemma 1.2.5 we conclude that the first integral in (5.21) also tends to zero. Now using (5.6) and Lemma 5.2.6 we can estimate the second integral in (5.21) as follows

$$\begin{aligned}&\int_{\mathbb{R}^2} Q\frac{g(v_n)}{\sqrt{\alpha_0+\beta_0}}\left[\log\left(\frac{C_\varepsilon e^{(\alpha_0+\beta_0+\varepsilon)v_n^2}}{\sqrt{\alpha_0+\beta_0}}\right)\right]^{1/2}\\ &\le \int_{\mathbb{R}^2} Q\frac{g(v_n)}{\sqrt{\alpha_0+\beta_0}}\left\{\left[\log\left(\frac{C_\varepsilon}{\sqrt{\alpha_0+\beta_0}}\right)\right]^{1/2}+(\alpha_0+\beta_0+\varepsilon)^{1/2}v_n\right\}\\ &= o_n(1)+\left(1+\frac{\varepsilon}{\alpha_0+\beta_0}\right)^{1/2}\int_{\mathbb{R}^2} Qg(v_n)v_n,\end{aligned}$$

and hence by (5.21), we get

$$\left(\frac{\alpha'}{2(\alpha_0+\beta_0)}-\delta\right)^{1/2}\|u_n\| \le o_n(1)+\left(1+\frac{\varepsilon}{\alpha_0+\beta_0}\right)^{1/2}\int_{\mathbb{R}^2} Qg(v_n)v_n. \tag{5.22}$$

Next taking $(v_n,0)$ as a test function in (5.18) we obtain

$$\|v_n\|^2 = \int_{\mathbb{R}^2} Qf(u_n)v_n.$$

Then, setting

$$\overline{v}_n \doteq \left(\frac{\alpha'}{2(\alpha_0+\beta_0)}-\delta\right)^{1/2}\frac{v_n}{\|v_n\|}$$

and repeating the same argument above we also obtain

$$\left(\frac{\alpha'}{2(\alpha_0+\beta_0)}-\delta\right)^{1/2}\|v_n\| \le o_n(1)+\left(1+\frac{\varepsilon}{\alpha_0+\beta_0}\right)^{1/2}\int_{\mathbb{R}^2} Qf(u_n)u_n. \tag{5.23}$$

From Lemma 5.2.6

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} QF(u_n) = \lim_{n\to\infty}\int_{\mathbb{R}^2} QG(v_n) = 0. \tag{5.24}$$

Thus, we conclude from Lemma 5.3.1 that

$$\int_{\mathbb{R}^2}(\nabla u_n \nabla v_n + V u_n v_n) \le o_n(1) + \frac{\alpha'}{2(\alpha_0+\beta_0)} - \delta,$$

which together with (5.25) implies that

$$\int_{\mathbb{R}^2} Qf(u_n)u_n + \int_{\mathbb{R}^2} Qg(v_n)v_n \le o_n(1) + 2\left(\frac{\alpha'}{2(\alpha_0+\beta_0)} - \delta\right).$$

Hence, from (5.22) and (5.23) we obtain the desired estimate (5.19). Therefore, the proof is finished. ■

Proof of Theorem 5.1.1. By Lemma 5.3.1 there exists $\delta > 0$ such that

$$c_n \doteq c_{n,u_\nu} \le \frac{\alpha'}{2(\alpha_0+\beta_0)} - \delta,$$

where c_{n,u_ν} is defined in Section 5.4. Next, applying the Proposition 5.4.2 we find a bounded sequence $z_n \doteq z_{n,u_\nu} = (u_n, v_n) \in H_{n,u_\nu}$ such that

$$\begin{aligned} &I_{n,u_\nu}(u_n,v_n) = c_n \in [\sigma, \alpha'/2(\alpha_0+\beta_0) - \delta),\\ &I'_{n,u_\nu}(u_n,v_n) = 0,\\ &(u_n,v_n) \rightharpoonup (u_0,v_0) \text{ weakly in } Z. \end{aligned} \tag{5.25}$$

Taking as test functions $(0,\psi)$ and $(\varphi,0)$ in (5.25), where φ and ψ are arbitrary functions in $\operatorname{span}\{\phi_1,\ldots,\phi_n\}$, we get

$$\int_{\mathbb{R}^2}(\nabla u_n\nabla\psi + Vu_n\psi) = \int_{\mathbb{R}^2} Qg(v_n)\psi, \quad \forall\psi\in\operatorname{span}\{\phi_1,\ldots,\phi_n\} \tag{5.26}$$

and

$$\int_{\mathbb{R}^2}(\nabla v_n\nabla\varphi + Vv_n\varphi) = \int_{\mathbb{R}^2} Qf(u_n)\varphi, \quad \forall\varphi\in\operatorname{span}\{\phi_1,\ldots,\phi_n\}. \tag{5.27}$$

Thus, taking the limit in (5.26) and (5.27), using Lemma 5.2.6 and the fact that $\bigcup_{n\in\mathbb{N}}\operatorname{span}\{\phi_1,\ldots,\phi_n\}$ is dense in E, it follows that

$$\int_{\mathbb{R}^2}(\nabla u_0\nabla\psi + Vu_0\psi) = \int_{\mathbb{R}^2} Qg(v_0)\psi, \quad \forall\psi\in E \tag{5.28}$$

and

$$\int_{\mathbb{R}^2}(\nabla v_0\nabla\varphi + Vv_0\varphi) = \int_{\mathbb{R}^2} Qf(u_0)\varphi, \quad \forall\varphi\in E. \tag{5.29}$$

Therefore, from (5.28) and (5.29) we conclude that (u_0, v_0) is a weak solution of system (5.1). Finally, it only remains to prove that u_0 and v_0 are nontrivial. Assume by contradiction that $u_0 \equiv 0$. This and (5.29) imply that $v_0 \equiv 0$. Now, if $\|u_n\| \to 0$ or $\|v_n\| \to 0$, then since $(u_n), (v_n) \subset E$ are bounded it follows from the Cauchy-Schwarz inequality that

$$|\langle u_n, v_n\rangle_E| \leq \|u_n\|\|v_n\| \to 0,$$

that is,

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} (\nabla u_n \nabla v_n + V u_n v_n) = 0.$$

Hence, using this convergence together that ones in (5.24) in the first equation of (5.25) we obtain that $c_n = 0$, which is a contradiction. Then assume that

$$\|u_n\|, \|v_n\| \geq C > 0,$$

for all $n \in \mathbb{N}$. Thus, from (5.19) we have in particular that

$$\begin{aligned} \|v_n\|^2 &\leq \frac{2\alpha'}{\alpha_0 + \beta_0} - 4\delta + o_n(1) + o(\varepsilon) \\ &\leq \frac{2\alpha'}{\alpha_0 + \beta_0} - \frac{\delta}{2}, \end{aligned}$$

for $\varepsilon > 0$ sufficiently small and $n \in \mathbb{N}$ sufficiently large, where $o(\varepsilon) \to 0$ as $\varepsilon \to 0$. It follows that there exists a subsequence of (v_n) (also denoted by (v_n)) such that

$$\|v_n\|^2 \leq \frac{2\alpha'}{\alpha_0 + \beta_0} - \frac{\delta}{2}.$$

Thus using the Hölder's inequality, Lemma 1.2.5 and Theorem 1.1.2, we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} Q g(v_n) v_n = 0.$$

This together with (5.22) provide

$$\lim_{n\to\infty} \|u_n\| = 0,$$

which is a contradiction. Consequently, we have a nontrivial critical point of I, and thereby we conclude the proof of the Theorem 5.1.1. ∎

5.6 On the Remark 5.1.2 (Subcritical case)

Let $y \in E$ be a nonnegative function fixed. Applying Proposition 5.4.2 there exists a sequence $z_{n,y} = (u_{n,y}, v_{n,y}) \in H_{n,y}$ such that $\|(u_{n,y}, v_{n,y})\|_Z \leq C$ and

$$\begin{aligned} &I_{n,y}(u_{n,y}, v_{n,y}) = c_{n,y} \in [\sigma, R_1^2], \\ &I'_{n,y}(u_{n,y}, v_{n,y}) = 0, \\ &(u_{n,y}, v_{n,y}) \rightharpoonup (u_0, v_0) \text{ weakly in } Z. \end{aligned} \tag{5.30}$$

Arguing as in the critical case by (5.30), we obtain

$$\int_{\mathbb{R}^2} (\nabla u_0 \nabla \psi + V u_0 \psi) = \int_{\mathbb{R}^2} Q g(v_0) \psi, \quad \forall \psi \in E \tag{5.31}$$

and

$$\int_{\mathbb{R}^2} (\nabla v_0 \nabla \varphi + V v_0 \varphi) = \int_{\mathbb{R}^2} Q f(u_0) \varphi, \quad \forall \varphi \in E. \tag{5.32}$$

Therefore, from (5.31) and (5.32) we conclude that (u_0, v_0) is a weak solution of system (5.1). Finally, it only remains to prove that u_0 and v_0 are nontrivial. Assume by contradiction that $u_0 \equiv 0$. This and (5.32) imply that $v_0 \equiv 0$. Since g has subcritical growth, we see that for all $\alpha > 0$

$$g(s) \leq C_1 |s| + C_2 (e^{\alpha s^2} - 1), \quad \forall s \in \mathbb{R}.$$

For $r, s > 1$ such that $1/r + 1/s = 1$, the Hölder's inequality implies that

$$\begin{aligned} \int_{\mathbb{R}^2} Q g(v_{n,y}) u_{n,y} \leq\ & C_1 \left(\int_{\mathbb{R}^2} Q |u_{n,y}|^2 \right)^{1/2} \left(\int_{\mathbb{R}^2} Q |v_{n,y}|^2 \right)^{1/2} \\ & + C_2 \left(\int_{\mathbb{R}^2} Q |u_{n,y}|^r \right)^{1/r} \left(\int_{\mathbb{R}^2} Q (e^{s\alpha v_{n,y}^2} - 1) \right)^{1/s}. \end{aligned}$$

Choosing $\alpha > 0$ and $s > 1$ sufficiently close to 1 such that $s\alpha \|v_{n,y}\|^2 < \alpha'$ and $1/r + 1/s = 1$, we conclude that

$$\int_{\mathbb{R}^2} (|\nabla u_{n,y}|^2 + V |u_{n,y}|^2) = \int_{\mathbb{R}^2} Q g(v_{n,y}) u_{n,y} \to 0,$$

since $u_{n,y}, v_{n,y} \to 0$ in $L^p(\mathbb{R}^2; Q)$ for all $2 \leq p < \infty$. Consequently, $u_{n,y} \to 0$ in E. This implies that

$$\int_{\mathbb{R}^2} (\nabla u_{n,y} \nabla v_{n,y} + V u_{n,y} v_{n,y}) \to 0. \tag{5.33}$$

Then by (5.30) and (5.33)

$$\int_{\mathbb{R}^2} Qf(u_{n,y})u_{n,y} \to 0 \quad \text{and} \quad \int_{\mathbb{R}^2} Qg(v_{n,y})v_{n,y} \to 0.$$

Using this limits and (H_2) it follows that

$$\int_{\mathbb{R}^2} QF(u_{n,y}) \to 0 \quad \text{and} \quad \int_{\mathbb{R}^2} QG(v_{n,y}) \to 0. \tag{5.34}$$

Finally, using (5.33) and (5.34) we get $c_{n,y} = 0$, which is a contradiction. Consequently, we have a nontrivial critical point of I, and thereby the claim in the Remark 5.1.2 is verified.

Final remarks

We finish this thesis with some discussions on further results which are contained in the preprint [10].

(i) Considering now radial functions $V, Q : \mathbb{R}^N \to \mathbb{R}$ $(N \geq 2)$ satisfying:

(V) $V \in C(0,\infty)$, $V(r) > 0$ and there exists $a > -N$ such that

$$\liminf_{r\to+\infty} \frac{V(r)}{r^a} > 0.$$

(Q) $Q \in C(0,\infty)$, $Q(r) > 0$ and there exist $b < [a(N-1) - N]/N$, $b_0 > -N$ such that

$$\limsup_{r\to 0} \frac{Q(r)}{r^{b_0}} < \infty \quad \text{and} \quad \limsup_{r\to+\infty} \frac{Q(r)}{r^b} < \infty,$$

it is possible to establish a natural generalization for the Trudinger-Moser inequality (1.4) in high dimensions by using similar arguments developed in [32]. It can be stated as follows.

Theorem 5.6.1 *Assume that* $(V) - (Q)$ *hold. Then, for any* $u \in W^{1,N}_{rad}(\mathbb{R}^N; V)$ *and* $\alpha > 0$, *we have that* $\Phi_\alpha(u) \in L^1(\mathbb{R}^N; Q)$, *where*

$$\Phi_\alpha(s) \doteq e^{\alpha|s|^{N/(N-1)}} - \sum_{j=0}^{N-2} \frac{\alpha^j |s|^{jN/(N-1)}}{j!}.$$

Furthermore, if $\alpha < \alpha' \doteq \min\{\alpha_N, \alpha_N(1 + b_0/N)\}$ *there exists* $C > 0$ *such that*

$$\sup_{u\in W^{1,N}_{rad}(\mathbb{R}^N;V);\ \|u\|_{W^{1,N}_{rad}(\mathbb{R}^N;V)}\leq 1} \int_{\mathbb{R}^N} Q(|x|)\Phi_\alpha(u)dx \leq C,$$

where $\alpha_N = N\omega_{N-1}^{1/(N-1)}$ *and* ω_{N-1} *is the* $(N-1)$*-dimensional measure of the* $(N-1)$*-sphere.*

(ii) As an application of the previous theorem and using a minimax procedure we can prove the existence of nontrivial solution for the following quasilinear elliptic problem:

$$-\text{div}(|\nabla u|^{N-2}\nabla u) + V(|x|)|u|^{N-2}u = Q(|x|)f(u) \quad \text{in} \quad \mathbb{R}^N \quad (N \geq 2), \quad (5.35)$$

when the nonlinear term $f(s)$ is allowed to enjoy the exponential critical growth and satisfies similar hypotheses assumed in the bi-dimensional problem. More precisely, the nonlinearity $f(s)$ is a continuous function with *exponential critical growth* at $+\infty$, i.e, there exists $\alpha_0 > 0$ such that

$$(f_{\alpha_0}) \quad \lim_{s\to+\infty} f(s)e^{-\alpha|s|^{N/(N-1)}} = \begin{cases} 0, & \forall \alpha > \alpha_0, \\ +\infty, & \forall \alpha < \alpha_0 \end{cases}$$

and satisfies the following conditions:

(f_1) $f(s) = o(|s|^{N-1})$ as $s \to 0$;

(f_2) there exists $\theta > N$ such that

$$0 < \theta F(s) \doteq \theta \int_0^s f(t)\text{dt} \leq sf(s), \quad \forall s > 0;$$

(f_3) there exist constants $R_0, M_0 > 0$ such that

$$0 < F(s) \leq M_0 f(s), \quad \forall s \geq R_0;$$

(f_4) there exist $\nu > N$ and $\mu > 0$ such that

$$F(s) \geq \frac{\mu}{\nu}|s|^\nu, \quad \forall s \geq 0.$$

Under these conditions, the existence result for problem (5.35) can be stated as follows.

Theorem 5.6.2 *Suppose that $(V)-(Q)$ and $(f_{\alpha_0})-(f_4)$ hold. Then there exists $\mu_0 > 0$ such that problem (5.35) has a nontrivial positive weak solution $u \in W^{1,N}_{rad}(\mathbb{R}^N; V)$ for all $\mu > \mu_0$.*

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