Universidade Federal da Paraíba
Universidade Federal de Campina Grande
Programa Associado de Pós-Graduação em Matemática
Doutorado em Matemática

Controlabilidade exata de sistemas parabólicos, hiperbólicos e dispersivos

por

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Julho/2014

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sob orientação do

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e sob co-orientação do

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Tese apresentada ao Corpo Docente do Programa Associado de Pós-Graduação em Matemática - UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

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Resumo

Nesta tese, estudaremos resultados de controle para alguns problemas da teoria das equações diferenciais parciais (EDPs):

- Problema de controle multi objetivo para um problema parabólico, seguindo estratégias do tipo Stackelberg-Nash: para cada controle líder, que impõe a controlabilidade nula para o estado, encontramos seguidores, em equilíbrio de Nash, associados a funcionais custo. Em seguida, determinamos o líder de menor custo.

- Controlabilidade nula para a equação de Schrödinger linear: com uma discretização *espaço-tempo* adequada, construímos numericamente controles-fronteira que conduzem a solução de Schrödinger a zero; utilizando técnicas de Fursikov-Imanuvilov (veja [Lecture Notes Series, Vol 34, 1996]) contruímos controles que decaem exponencialmente no tempo final.

- Controlabilidade nula para um sistema acoplado Schrödinger-KdV: neste trabalho, combinando estimativas globais de Carleman com estimativas de energia, obtemos uma *desigualdade de observabilidade.* O resultado de controlabilidade segue pelo método de unicicade Hilbert (HUM).

- Controlabilidade para um sistema do tipo Euler, incompressível, invíscido, sob influência de uma temperatura: Utilizamos os métodos de extensão seguido do método do retorno para provar resultados de controlabilidade para este sistema.

Palavras-chave: Controlabilidade, Estratégias do tipo Stackelberg-Nash, Desigualdade de Carleman, Equação de Schrödinger-1D, Equação do Calor, Equação KdV, Elementos finitos, Sistema de Boussinesq-Invíscido.

Abstract

In this thesis, we study controllability results of some phenomena modeled by Partial Differential Equations (PDEs):

- Multi objective control problem, for parabolic equations, following the Stackelber-Nash strategy is considered: for each leader control which impose the null controllability for the state variable, we find a Nash equilibrium associated to some costs. The leader control is chosen to be the one of minimal cost.

- Null controllability for the linear Schrödinger equation: with a convenient *space-time* discretization, we numerically construct boundary controls which lead the solution of the Schrödinger equation to zero; using some arguments of Fursikov-Imanuvilov (see [Lecture Notes Series, Vol 34, 1996]) we construct controls with exponential decay at final time.

- Null controllability for a Schrödinger-KdV system: in this work, we combine global Carleman estimates with energy estimates to obtain an *observability inequality.* The controllability result holds by the Hilbert Uniqueness Method (HUM).

- Controllability results for a Euler type system, incompressible, inviscid, under the influence of a temperature are obtained: we mainly use the extension and return methods.

Keywords: Controllability, Stackelberg-Nash strategies, Carleman inequalities, 1D Schrödinger equation, Heat Equation, KdV equation, Finite element methods, Carleman inequalities, Inviscid Boussinesq system.

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Sumário

Introdução

Sempre foi de interesse da humanidade investigar o comportamento de determinados fenômenos da natureza. Uma pergunta natural que surge é a possibilidade de atuar ou influenciar tal fenômeno de maneira a obter um comportamento desejado. Uma vez que estes fenômenos são compreendidos e representados matematicamente, uma grande quantidade de ferramentas e métodos estão disponíveis para serem aplicados e é neste ponto que se baseia a teoria moderna de Controle.

O objetivo desta tese é mostrar o estudo da controlabilidade (em um sentido que será explicado posteriormente) de alguns problemas da teoria das equações diferenciais parciais. Ao longo desta introdução, descreveremos um pouco a evolução histórica da teoria do controle e, em seguida, motivaremos brevemente cada trabalho que será estudado nesta tese.

Relembrando um pouco a história, encontramos que os romanos utilizaram alguns elementos de controle para a construção de seus arquedutos. Mais precisamente, sistemas engenhosos regulavam válvulas de modo a manter um nível de água constante. Muitos estudiosos afirmam que na antiga mesopotâmia, mais de 2000 anos antes de Cristo, o sistema de controle de irrigação também era uma arte conhecida. O trabalho de Ch. Huygens e R. Hooke ao final do século XVII sobre oscilação do pêndulo é um relevante trabalho no desenvolvimento da teoria do controle. Estes trabalhos futuramente foram adaptados para regular a velocidade de moinhos de vento. J. Watt adaptou este modelo em seu motor a vapor que constituiu um mecanismo importantíssimo na revolução industrial. Neste mecanismo, quando a velocidade das esferas aumentava, uma ou várias esferas destapavam algumas válvulas diminuindo a pressão e reduzindo a velocidade, consequentemente as esferas voltavam a tapar as válvulas novamente de modo a velocidade aumentar. Este mecanismo tinha o objetivo de controlar a velocidade de forma a ficar aproximadamente constante. O astrônomo britânico G. Airy foi o primeiro a analisar matematicamente o sistema regulador apresentado por Watt. Porém, a primeira descrição matemática definitiva foi dada apenas no trabalho de J. C. Maxwell, em 1868, em que alguns comportamentos indesejados encontrados no motor a vapor foram descritos e alguns mecanismos de controle foram propostos. Com o passar dos anos, as ideias centrais da teoria de controle sofreram um impacto notável. Em meados de 1920, os engenheiros já utilizavam processamento contínuo usando técnicas de controle automático ou semi automático. Assim, a engenharia de controle germinou e ganhou o reconhecimento de uma área independente. Durante a segunda guerra mundial e os anos seguintes, engenheiros e cientistas melhoraram suas experiências com mecanismos de controle de rastreamento , mísseis balísticos e modelagem de esquadrões aéreos. Depois dos anos 60, os métodos e teorias mencionados acima passaram a ser considerados como parte da teoria "clássica"do controle.

A segunda guerra mundial serviu para perceber que os modelos considerados até o momento não eram suficientes para descrever a complexidade do mundo real. Na verdade, já se sabia que os verdadeiros sistemas eram não lineares ou indetermináveis, desde que estes são afetados por alguma "pertubação". As contribuições de R. Bellman no contexto de programação dinâmica, R. Kalman nas técnicas de filtragem e aproximações algébricas a sistemas lineares e L. Pontryagin com o princípio do máximo para problemas de controle óptimo não linear, estabeleceram a base para a teoria do controle moderna. Tal teoria ganhou um formalismo ou uma representação matemática de modo a conseguirmos usar as ferramentas matemáticas que temos para solucionarmos tais problemas de controle.

Um sistema de controle é uma equação de evolução (EDO ou EDP) que depende de um parâmetro u, que escreveremos da seguinte forma:

$$\dot{y} = f(t, y, u),$$

onde $t \in [0, T]$ é o tempo, $y : [0, T] \to \mathfrak{Y}$ é a função estado e $u : [0, T] \to \mathfrak{U}$ é o controle. Temos que $\mathfrak{Y}$ e $\mathfrak{U}$ são espaços de funções adequados. Na equação acima, $\dot{y}$ representa a derivada de y em relação ao tempo t.

O problema de controle consiste em encontrar um controle u tal que a função estado se comporta de uma forma desejada. Exemplificaremos alguns, dentre os vários, problemas de controlabilidade presentes na literatura.

Controle óptimo: Encontrar um controle que minimiza algum funcional custo, por exemplo,

$$J(u) = \|y(T; u) - \bar{y}\|^2_{\mathfrak{Y}} + \|u\|^2_{\mathfrak{U}},$$

em que $\bar{y}$ é um alvo desejado e $y(T; u)$ é o estado alcançado pelo sistema no tempo final T.

Controlabilidade exata: Dado dois tempos $T_0 < T_1$ e y_0, y_1 dois possíveis estados do sistema, encontrar $u : [T_0, T_1] \to \mathfrak{U}$ tal que

$$\begin{cases} \dot{y} = f(y, u) & \text{em} \quad [T_0, T_1] \\ y(T_0) = y_0, & y(T_1) = y_1. \end{cases}$$

Em outras palavras, partindo de qualquer configuração inicial y_0, podemos conduzir a solução y para o estado y_1 sob a ação do controle u.

Controlabilidade aproximada: Dados $T_0 < T_1$, dois possíveis estados y_0, y_1 e $\epsilon > 0$, encontrar $u : [T_0, T_1] \to \mathfrak{U}$ tal que

$$\begin{cases} \dot{y} = f(y, u) & \text{em} \quad [T_0, T_1] \\ y(T_0) = y_0, & \|y(T_1) - y_1\|_{\mathfrak{Y}} < \epsilon. \end{cases}$$

A controlabilidade aproximada é uma versão mais fraca se comparada a controlabilidade exata. De fato, em vez de pedirmos que a função estado seja exatamente y_1 em T_1, pedimos apenas que o estado esteja arbitrariamente perto de y_1.

Controlabilidade Nula: Dados dois tempos $T_0 < T_1$ e y_0 um estado do sistema, encontrar $u : [T_0, T_1] \to \mathfrak{U}$ tal que

$$\begin{cases} \dot{y} = f(y, u) & \text{em} \quad [T_0, T_1] \\ y(T_0) = y_0, & y(T_1) = 0. \end{cases}$$

Para finalizar temos

Controlabilidade exata para as trajetórias: Dados $T_0 < T_1$, $y_0 \in \mathfrak{Y}$ e $\bar{y}$ uma trajetória (uma solução com controle $\bar{u} : [T_0, T_1] \to \mathfrak{U}$). encontrar um controle $u : [T_0, T_1] \to \mathfrak{U}$ tal que

$$\begin{cases} \dot{y} = f(y,u) \quad \text{em} \quad [T_0, T_1] \\ y(T_0) = y_0, \quad y(T_1) = \bar{y}(T_1). \end{cases}$$

Os conceitos de controlabilidade nula e controlabilidade exata para as trajetórias são de especial importância em sistemas não reversíveis e sistemas com efeito regularizante. Nestes casos, a controlabilidade exata não é esperada.

Sejamos mais específico sobre os problemas de controle que serão abordados nesta tese. Introduziremos os quatros trabalhos que serão mostrados na ordem seguinte:

Capítulo 1
Stackelberg-Nash exact controllability for linear and semilinear parabolic equations

Seja N um número inteiro e positivo, $\Omega \subset \mathbb{R}^N$ e T um número real. Consideremos em $Q = \Omega \times (0,T)$ um sistema distribuído, governado por uma equação parabólica com um controle v de suporte ω.

Abordaremos o seguinte método: Associado a este sistema, temos três (ou mais) objetivos, um do tipo "controlabilidade" e outros dois, possivelmente conflitivos, do tipo "controle óptimo" com a tarefa de fazer com que o estado não seja "muito distante" de um determinado valor desejado. Dividimos v em três partes, digamos f, v^1 e v^2 correspondendo, respectivamente, à divisão de ω em três regiões $\mathcal{O}$, $\mathcal{O}_1$ e $\mathcal{O}_2$. Utilizamos a noção de optimização de Stackelberg (muito utilizado em economia) em que v^1 e v^2 são os seguidores e f é o líder. Fixado f, resolvemos um problema de controle óptimo para v^1 e v^2; o par óptimo é escolhido por meio de um critério de optimização, não cooperativo de J. Nash a ser detalhado posteriormente. Desta forma, escrevemos o par em função de f de uma forma $(v^1, v^2) = \mathcal{F}(f)$, obtendo um sitema de optimalidade associado, dependendo apenas de f, onde estudamos um problema de controlabilidade com controle f.

Consideremos, por simplicidade, a equação do calor com seus respectivos controles definidos segundo o método de Stackelberg:

$$\begin{cases} y_t - \Delta y + a(x,t)y = F(y) + f1_{\mathcal{O}} + v^1 1_{\mathcal{O}_1} + v^2 1_{\mathcal{O}_2} & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(x,0) = y^0(x) & \text{in} \quad \Omega, \end{cases} \tag{1}$$

onde $F : \mathbb{R} \to \mathbb{R}$ é semilinear. Para $i = 1,2$, sejam $\mathcal{O}_{i,d}$ subconjuntos abertos de Ω e consideremos os seguintes funcionais (secundários):

$$J_i(f; v^1, v^2) := \frac{\alpha_i}{2} \iint_{\mathcal{O}_{i,d} \times (0,T)} |y - y_{i,d}|^2 \, dx\, dt + \frac{\mu_i}{2} \iint_{\mathcal{O}_i \times (0,T)} |v^i|^2 \, dx\, dt. \tag{2}$$

Seja também o funcional principal

$$J(f) := \frac{1}{2} \iint_{\mathcal{O} \times (0,T)} |f|^2 \, dx\, dt,$$

onde $\alpha_i > 0$, $\mu_i > 0$ são constantes e $y_{i,d} = y_{i,d}(x,t)$ são funções dadas. A estrutura do processo é da forma: Os seguidores v^1 e v^2 assumem que o líder f fez uma escolha e posteriormente serão um Equilíbrio de Nash para os custos J_i $(i = 1, 2)$. Fixado f, procuramos por controles $v^i \in L^2(\mathcal{O}_i \times (0,T))$ que satisfazem

$$J_1(f; v^1, v^2) = \min_{\hat{v}^1} J_1(f; \hat{v}^1, v^2), \quad J_2\left(f; v^1, v^2\right) = \min_{\hat{v}^2} J_2(f; v^1, \hat{v}^2), \tag{3}$$

e o par (v^1, v^2) será chamado equilíbrio de Nash para J_1 e J_2. Observemos que, se os funcionais J_i $(i = 1, 2)$ são convexos, então (v^1, v^2) é um equilíbrio de Nash se, e somente se,

$$J_1'(f; v^1, v^2)(\hat{v}^1, 0) = 0, \quad \forall \hat{v}^1 \in L^2\left(\mathcal{O}_1 \times (0,T)\right); \ v^1 \in L^2(\mathcal{O}_1 \times (0,T)) \tag{4}$$

e

$$J_2'(f; v^1, v^2)(0, \hat{v}^2) = 0, \quad \forall \hat{v}^2 \in L^2\left(\mathcal{O}_2 \times (0,T)\right); \ v^2 \in L^2(\mathcal{O}_2 \times (0,T)). \tag{5}$$

Fixemos uma trajetória suficientemente regular, isto é, solução do problema:

$$\begin{cases} \overline{y}_t - \Delta \overline{y} + a(x,t)\overline{y} = F(\overline{y}) & \text{in} \quad Q, \\ \overline{y} = 0 & \text{on} \quad \Sigma, \\ \overline{y}(x,0) = \overline{y}^0(x) & \text{in} \quad \Omega. \end{cases} \tag{6}$$

Uma vez que o equilíbrio de Nash foi determinado para cada f, procuramos um controle óptimo $\hat{f} \in L^2(\mathcal{O} \times (0,T))$ tal que

$$J(\hat{f}) = \min_{f} J(f), \tag{7}$$

sujeito à restrição

$$y(x,T) = \overline{y}(x,T) \quad \text{em} \quad \Omega. \tag{8}$$

Existem vários trabalhos relacionados a este tópico:

- Os trabalhos de J.-L. Lions [61] e [62], onde o autor apresenta alguns resultados relacionados às estratégias de Pareto e Stackelberg, respectivamente.

- O trabalho de Díaz e Lions [32], onde a controlabilidade aproximada de um sistema é estabelecido seguindo a estratégia de Stackelberg-Nash e a extensão em Díaz [31], que fornece uma caracterização da solução por meio do teorema de dualidade de Fenchel-Rockafellar.

- Os trabalhos [74] e [75], onde Ramos, Glowinski e Periaux estudam o equilíbrio de Nash do ponto de vista teórico e numérico para EDPs parabólicas e para a equação de Burgers, respectivamente.

- Finalmente, mencionamos o equilíbrio de Stackelberg-Nash para o sistema de Stokes que foi estudado por Guillén-González et al. em [48].

Os resultados de controlabilidade presentes nos trabalhos citados acima dão respostas apenas no nível de controle aproximado; a grande novidade neste trabalho é estender os resultados para um nível de controle exato.

Suponhamos que $\mathcal{O}_{1,d} = \mathcal{O}_{2,d} = \mathcal{O}_d$. Os resultados pincipais são os seguintes:

Teorema (Caso linear): *Suponhamos que* $F \equiv 0$, $\mathcal{O}_d \cap \mathcal{O} \neq \emptyset$ *e que* μ_i *são suficientemente grandes*

$$\mu_i \geq C(\Omega, T, \mathcal{O}_i, \mathcal{O}_d, \alpha_i, \|a\|_{L^\infty(Q)}), \quad i = 1, 2.$$

Assumimos que as funções $y_{i,d}$ *satisfazem a seguinte propriedade de compatibilidade: existe uma função positiva* $\hat{\rho} = \hat{\rho}(x,t)$ *que explode em* $t = T$ *tal que se* $\overline{y}$ *é a única solução de* (6) *com* $F \equiv 0$ *associada ao dado inicial* $\overline{y}^0 \in L^2(\Omega)$ *então*

$$\iint_{\mathcal{O}_d \times (0,T)} \hat{\rho}^2 |\overline{y} - y_{i,d}|^2 \, dx \, dt < +\infty, \quad i = 1, 2. \tag{9}$$

Para qualquer $y^0 \in L^2(\Omega)$ *existem controles* $f \in L^2(\mathcal{O} \times (0,T))$ *associado ao equilíbrio de Nash* (v^1, v^2) *tais que a correspondente solução* (1) *satisfaz* (8).

Os próximos resultados estão relacionados ao caso em que F é semilinear e não identicamente nula. A dificuldade neste caso se encontra no fato que os funcionais (2) perdem a convexidade e, portanto, o conceito de equilíbrio (3) não pode mais ser associado ao conceito diferencial (4)-(5). Desta forma, é necessário definir um conceito mais fraco de equilíbrio de Nash. Dizemos que o par (v^1, v^2) é um quase equlíbrio de Nash se satisfaz (4)-(5). Assim temos o segundo resultado:

Teorema (Caso semilinear, $F \in W^{1,\infty}$**):** *Suponhamos que* $F \in W^{1,\infty}(\mathbb{R})$ *e que* $\mu_i > 0$ *são suficientemente grandes. Seja* $\bar{y}$ *é a única solução de* (6) *com dado inicial* $\overline{y}^0 \in L^2(\Omega)$ *e suponhamos que* (9) *é verdadeiro. Então, para cada* $y_0 \in L^2(\Omega)$*, existem controles* $f \in L^2(\mathcal{O} \times (0,T))$ *e quase equilíbrio de Nash* (v^1, v^2) *tal que a solução de* (1) *satisfaz* (8).

Uma questão natural é sob que condição um quase equilíbrio de Nash é equivalente ao equilíbrio de Nash. A resposta está no terceiro resultado:

Teorema (Caso semilinear, $F \in W^{2,\infty}$**):** *Suponha que* $F \in W^{2,\infty}(\mathbb{R})$, $y_{i,d} \in L^\infty(\mathcal{O}_{i,d} \times (0,T))$ $(i = 1, 2)$*. Suponha também que* $y_0 \in H_0^1(\Omega)$ *(resp.* $y_0 \in L^2(\Omega)$*) e* $N \leq 14$ *(resp.* $N \leq 12$*). Então, existe* $C > 0$ *tal que, se* $f \in L^2(\mathcal{O} \times (0,T))$ *e ainda se* μ_i *satisfaz*

$$\mu_i \geq C(1 + \|f\|_{L^2(\mathcal{O} \times (0,T))}),$$

então as condições (3) *e* (4)-(5) *são equivalentes.*

O quarto e último resultado consiste em analisar a situação se seguidores estão restritos a um subconjunto convexo e fechado $\mathcal{U}_i \subset L^2(\mathcal{O}_i \times (0,T))$. Seja I_1 e I_2 dois intervalos convexos e fechados com $0 \in I_1 \cap I_2$, consideremos

$$\mathcal{U}_i = \{v \in L^2(\mathcal{O}_i \times (0,T)) : \ v(x,t) \in I_i\}, \quad i = 1, 2, \tag{10}$$

e suponhamos que a minimização de J_1 e J_2 em (3) é sujeita à restrição $\hat{v}^1 \in \mathcal{U}_1$ e $\hat{v}^2 \in \mathcal{U}_2$. Temos o seguinte resultado:

Teorema (Caso com restrições): *Suponhamos que* $F \equiv 0$ *e que* $\mu_i > 0$ *são suficientemente grandes. Seja* $\overline{y}$ *a única solução de* (6) *com dado inicial* $\bar{y}^0 \in L^2(\Omega)$*. Então, para cada* $y_0 \in$

$L^2(\Omega)$, *existem controles* $f \in L^2(\mathcal{O} \times (0,T))$ *e equilíbrio de Nash associado* $(v^1, v^2) \in \mathcal{U}_1 \times \mathcal{U}_2$ *tal que a solução de* (1) *satisfaz* (8).

Os resultados deste trabalho são encontrado em [6].

Capítulo 2
Numerical null controllability of the 1D linear Schrödinger equation

Este capítulo lida com a controlabilidade exata, com controle atuando na fronteira, para a equação de Schrödinger unidimensional. A equação do estado é dada por:

$$\begin{cases} iy_t - y_{xx} + V(x,t)y = 0, & (x,t) \in Q = (0,1) \times (0,T), \\ y(0,t) = u(t), \quad y(1,t) = 0, & t \in (0,T), \\ y(x,0) = y_0(x), & x \in (0,1). \end{cases} \tag{11}$$

Estamos supondo que $T > 0$, $y_0 \in H_0^1((0,1);\mathbb{C})$ e V, $V_x \in L^\infty(Q;\mathbb{R})$. Em (11), $u \in L^2((0,T);\mathbb{C})$ é o controle e $y = y(x,t)$ é o estado associado.

O principal resultado deste capítulo, é calcular aproximações numéricas de controles que conduzem a solução de (11) a zero (controlabilidade nula). Devido à reversibilidade em tempo da equação de Schrödinger, as propriedade de controlabilidade nula e exata são equivalentes.

É bem conhecido que, para qualquer $T > 0$, (11) possui a propriedade de controlabilidade nula, veja [65]. Isto significa que, para qualquer $y_0 \in L^2((0,1);\mathbb{C})$, existem controles $u \in L^2((0,T);\mathbb{C})$ tal que o estado associado satisfaz $y(\cdot,T) = 0$; mais ainda, o controle de norma mínima em $L^2((0,T);\mathbb{C})$ é dado por $u = \phi_x(0,\cdot)$, onde ϕ resolve o problema adjunto

$$\begin{cases} i\phi_t - \phi_{xx} + V\phi = 0, & (x,t) \in Q, \\ \phi(x,t) = 0, & (x,t) \in \{0,1\} \times (0,T), \\ \phi(x,T) = \phi^T(x), & x \in (0,1), \end{cases} \tag{12}$$

com ϕ^T em um espaço apropriado. Citamos os trabalhos [60, 83, 84, 85] onde a controlabilidade para a equação de Schrödinger foi investigada.

Neste trabalho, usaremos as ideias inspiradas nos trabalhos de Fursikov e Imanuvilov em [38], para problemas parabólicos similares. Mais precisamente, consideremos o seguinte problema de minimização:

$$\begin{cases} \text{Minimizar } J(y,u) = \dfrac{1}{2} \displaystyle\iint_Q \rho^2 |y|^2 \, dx\, dt + \dfrac{1}{2} \int_0^T \rho_1(0,t)^2 |u|^2 \, dt \\ \text{Sujeito a } (y,v) \in \mathcal{C}(y_0,T). \end{cases} \tag{13}$$

Em (13)

$$\mathcal{C}(y_0,T) = \{\, (y,u) \in X : y \text{ é solução de (11) e satisfaz } y(\cdot,T) = 0 \,\},$$

onde

$$X = L^2(Q;\mathbb{C}) \times L^2((0,T);\mathbb{C}).$$

Assumimos também que

$$\begin{cases} \rho = \rho(x,t), \rho_1 = \rho_1(x,t) \text{ são contínuas, reais e } \geq \rho_* > 0, \\ \rho, \rho_1 \in L^\infty((0,1) \times (0,T-\delta);\mathbb{R}) \ \ \forall \delta > 0, \end{cases}$$

são funções peso que, em princípio, podem explodir em $t = T$.

Nosso objetivo consistirá em resolver numericamente o problema de minização (13). O fato de buscarmos um controle e um estado que são soluções de (13) pode ser justificado como segue: Primeiramente, com esse método obtemos um "bom" par estado-controle que satisfaz (11) com a propriedade de controlabilidade nula. Segundo que, naturalmente, os controles obtidos terão uma propriedade de decaimento exponencial, evitando oscilação indesejadas do controle quando $t \to T$; este comportamento já foi observado em problemas parabólicos similares quando se calcula numericamente os controles de norma mínima. Para este propósito, veremos que $\mathcal{C}(y_0, T)$ em (2.3) é não-vazio e que (2.3) possui uma única solução.

O par (y, u) solução para (13) será aproximado de duas formas distintas: primeiramente em termos de uma nova variável p, pertencente a um espaço adequado P, solução do seguinte problema variacional:

$$\begin{cases} \displaystyle\iint_Q \rho^{-2} Lp\, \overline{Lq}\, dx\, dt + \int_0^T \rho_1^{-2} p_x(0,t)\, \overline{q_x(0,t)}\, dt = i\langle y_0\,, \overline{q(\cdot\,,0)}\rangle \\ \forall q \in P; \quad p \in P. \end{cases} \tag{14}$$

Neste caso teremos

$$y = \rho^{-2} Lp, \quad u = -\rho_1^{-2} p_x|_{x=0}. \tag{15}$$

A segunda forma consiste em aplicar uma mudança de variável, escrevendo o par (y, u) em termos de uma variável w (que dependerá de p), pertencente a um espaço adequado W, solução do problema

$$\begin{cases} \displaystyle\iint_Q (A_1 w + A_2 w_t + A_3 w_x + A_4 w_{xx})\, \overline{(A_1 m + A_2 m_t + A_3 m_x + A_4 m_{xx})}\, dx\, dt \\ \displaystyle\qquad + \int_0^T (T-t)^{2\gamma} w_x(0,t)\, \overline{m_x(0,t)}\, dt = iT^\gamma \langle y_0\,, \rho_1(\cdot\,,0)\, \overline{m(\cdot\,,0)}\rangle \\ \forall m \in W; \; w \in W, \end{cases} \tag{16}$$

onde os coeficientes A_i serão funções em $L^\infty(Q; \mathbb{C})$. Neste caso teremos

$$y = \rho^{-1} \left(A_1 w + A_2 w_t + A_3 w_x + A_4 w_{xx}\right), \quad u = -(T-t)^\gamma \rho_1(0,\cdot)^{-1} w_x(0,\cdot). \tag{17}$$

Para calcularmos as aproximações para a solução (y, u) de (13), em ambos os casos, faremos uso do método de elementos finitos para determinar aproximações numéricas para p e w. Em seguida, utilizaremos as expressões (15) e (17).

Por (14) e (16), vemos que p e w são soluções fracas de um problema de ordem quatro no espaço e ordem dois no tempo. Por esta razão é natural a utilização da discretização em termos de polinômios que pertencem a $(\mathbb{P}_{3,x} \otimes \mathbb{P}_{1,t})(R)$. Os resultados obtidos serão apresentados em termos de gráficos e tabelas.

Os resultados deste trabalho são encontrado em [36].

Capítulo 3
Internal null controllability of a linear Schrödinger-KdV system on a bounded interval

Nos últimos anos, muitos artigos foram voltados ao estudo de propriedades de controlabilidade para sistemas acoplados de equações diferenciais parciais onde novos fenômenos surgem. De fato, em alguns sistemas parabólicos foi provado controlabilidade para tempo grande, ao contrário do que ocorre quando se estuda a equação individualmente. Grande parte desses resultados lidam com a controlabilidade para sistemas parabólicos (veja [3]) ou hiperbólicos (veja [1, 2, 7]). Abordagens por estimativas de Carleman, problema de momentos e métodos de energia foram aplicados para obter controlabilidade interna e fronteira.

Existem poucos resultados relacionados com a controlabilidade de sistemas dispersivos. Vários sistemas do tipo Boussinesq foram considerados em [68] onde resultados de controlabilidade exata são provados. Outros sistemas acoplados com equações do tipo Korteweg-de Vries foram estudados em [22, 67] onde resultados de controlabilidade exata na fronteira foram estabelecidos.

Neste trabalho, estamos interessados em um sistema linear dispersivo definido no intervalo $[0,1]$ e formado por duas EDPs: uma equação de Schrödinger e uma equação Korteweg-de Vries (KdV). Consideramos um controle interno com suporte em um subconjunto aberto $\omega \subset (0,1)$ e condição de fronteira homogênea.

Dado $T>0$, denotamos $Q=(0,1)\times(0,T)$. Mais ainda, $\mathbf{1}_\omega$ é a função característica de ω e M, a_1, a_2, a_3, a_4 são funções dadas. Neste tabalho, para um número complexo z, denotamos por $Re(z)$ e $Im(z)$ a parte real e a parte imaginária de z, respectivamente.

O sistema é dado conforme segue:

$$\begin{cases} iw_t + w_{xx} = a_1 w + a_2 y + \ell \mathbf{1}_\omega & \text{in} \quad Q, \\ y_t + y_{xxx} + (My)_x = Re(a_3 w) + a_4 y + h\mathbf{1}_\omega & \text{in} \quad Q, \\ w(0,t) = w(1,t) = 0 & \text{in} \quad (0,T), \\ y(0,t) = y(1,t) = y_x(1,t) = 0 & \text{in} \quad (0,T), \\ w(x,0) = w_0(x), \quad y(x,0) = y_0(x) & \text{in} \quad (0,1), \end{cases} \tag{18}$$

onde o estado é formado pela função complexa w e a função real y. Os controles são a função complexa ℓ e a função real h. Este sistema é uma versão linearizada de um sistema Schrödinger-Korteweg-de Vries, não linear, presente na mecânica dos fluidos assim como na física de plasma para modelar interações entre *ondas curtas* $w = w(x,t)$ e *ondas longas* $y = y(x,t)$ (veja [58] onde ondas capilar-gravidade são consideradas). Resultados de boa colocação foram obtidos quando o sistema é estudado em toda reta real [14, 25] ou no toro [8]. Este sistema pode ser visto como a acoplamento de três equações reais considerando a parte real e a parte imaginária da equação de Schrödinger. Neste trabalho queremos provar um resultado de controlabilidade com menos controles que equações. De fato, provaremos que este sistema é nulamente controlável utilizando o controle h e também um controle puramente real ou puramente imaginário ℓ. Então, necessitaremos dois controles reais para controlar todo o sistema. É importante mencionar que a equação de Schrödinger é controlável com um controle complexo. Graças ao acoplamento com a equação KdV, podemos remover ou a parte real ou a parte imaginária do controle.

O principal resultado do trabalho segue:

Teorema (Controlabilidade nula): *Seja* $T>0$. *Suponhamos que* $M \in L^2(0,T;H^1(0,1)) \cap L^\infty(0,T;L^2(0,1))$, a_1, $a_4 \in L^\infty(0,T;W^{1,\infty}(0,1))$, *and* $a_2, a_3 \in L^\infty(0,T;\boldsymbol{W}^{1,\infty}(0,1))$. *Supo-*

nhamos também que

$$Im(a_2) \in C((0,T);W^{1,\infty}(0,1)) \ \text{ com } \ |Im(a_2)| \geq \delta > 0 \ \text{ em } \omega. \tag{19}$$

Para qualquer $(w_0, y_0) \in \boldsymbol{H}^{-1}(0,1) \times L^2(0,1)$*, existem controles* $(\ell, h) \in L^2(0,T;H^{-1}(\omega)) \times L^2(0,T;L^2(\omega))$*, tal que a única solução* $(w,y) \in C([0,T], \boldsymbol{H}^{-1}(0,1) \times L^2(0,1))$ *de (18) satisfaz*

$$w(T,\cdot) = 0, \quad y(T,\cdot) = 0.$$

Acima, espaços em negrito dentotam espaços de funções complexas, do contrário, denotam espaços de funções reais.

De forma a provar o teorema anterior, seguimos o processo padrão de observabilidade-controlabilidade, que reduz a propriedade de controlabilidade nula à seguinte desigualdade de observabilidade:

Teorema (Desigualdade de observabilidade): *Seja* $Q_\omega = \omega \times (0,T)$*. Existe* $C > 0$ *tal que*

$$\|\phi(\cdot,0)\|^2_{H^1_0(0,1)} + \|\psi(\cdot,0)\|^2_{L^2(0,1)} \leq C \left(\iint_{Q_\omega} (|Re(\phi)|^2 + |Re(\phi_x)|^2 + |\psi|^2) dxdt \right),$$

para qualquer $(\phi^T, \psi^T) \in \boldsymbol{H}^1_0(0,1) \times L^2(0,1)$*, onde* (ϕ, ψ) *satisfaz o sistema adjunto*

$$\begin{cases} i\phi_t + \phi_{xx} = a_1\phi + \bar{a}_3\psi & \text{in} \quad Q, \\ -\psi_t - \psi_{xxx} - M\psi_x = Re(\bar{a}_2\phi) + a_4\psi & \text{in} \quad Q, \\ \phi(0,t) = \phi(1,t) = 0 & \text{in} \quad (0,T), \\ \psi(0,t) = \psi(1,t) = \psi_x(0,t) = 0 & \text{in} \quad (0,T), \\ \phi(x,T) = \phi^T(x), \quad \psi(x,T) = \psi^T & \text{in} \quad (0,1). \end{cases}$$

Estes resultados podem ser vistos em [5].

Capítulo 4
Boundary controllability of incompressible Euler fluids with Boussinesq heat effects

Seja Ω um subconjunto aberto, limitado e não-vazio de $\mathbb{R}^N$ de classe $\mathcal{C}^\infty$ ($N = 2$ ou $N = 3$). Suponhamos que Ω é conexo e, por simplicidade, simplesmente conexo. Seja Γ_0 um subconjunto aberto e não vazio da fronteira Γ de Ω.

Neste capítulo estamos interessados na controlabilidade fronteira do sistema:

$$\begin{cases} \mathbf{y}_t + (\mathbf{y} \cdot \nabla)\mathbf{y} = -\nabla p + \vec{\mathbf{k}}\,\theta & \text{in} \quad \Omega \times (0,T), \\ \nabla \cdot \mathbf{y} = 0 & \text{in} \quad \Omega \times (0,T), \\ \theta_t + \mathbf{y} \cdot \nabla\theta = \kappa\,\Delta\theta & \text{in} \quad \Omega \times (0,T), \\ \mathbf{y} \cdot \mathbf{n} = 0 & \text{on} \quad (\Gamma \backslash \Gamma_0) \times (0,T), \\ \mathbf{y}(\mathbf{x},0) = \mathbf{y}_0(\mathbf{x}), \quad \theta(\mathbf{x},0) = \theta_0(\mathbf{x}) & \text{in} \quad \Omega, \end{cases} \tag{20}$$

onde

- O campo $\mathbf{y}$ e a função escalar p representam, respectivamente, a velocidade e pressão do fluido em $\Omega \times (0,T)$, respectivamente.
- A função θ representa a distribuição de temperatura do fluido.
- O lado direito $\vec{\mathbf{k}}\,\theta$ pode ser visto como a densidade da *força de flutuação* ($\vec{\mathbf{k}} \in \mathbb{R}^N$ é um vetor não nulo).
- A constante não negativa $\kappa \geq 0$ é o coeficiente de difusão de calor.

Na mecânica dos fluidos, o sistema (20) descreve o movimento de um fluido invíscido e incompressível sujeito a uma transferência de calor convectiva sob influência de um campo gravitacional, veja [64].

Abordaremos os casos $\kappa = 0$ e $\kappa > 0$. No caso $\kappa = 0$ denominamos (20) de sistema de Boussinesq invíscido e incompressível.

Seja $\alpha \in (0,1)$ e definimos

$$\begin{aligned}
\mathbf{C}_0^{m,\alpha}(\overline{\Omega};\mathbb{R}^N) &:= \{\, \mathbf{u} \in \mathbf{C}^{m,\alpha}(\overline{\Omega};\mathbb{R}^N) \,:\, \nabla \cdot \mathbf{u} = 0 \ \text{ in } \ \overline{\Omega}, \quad \mathbf{u} \cdot \mathbf{n} = 0 \ \text{ on } \ \Gamma \,\}, \\
\mathbf{C}(m,\alpha,\Gamma_0) &:= \{\, \mathbf{u} \in \mathbf{C}^{m,\alpha}(\overline{\Omega};\mathbb{R}^N) \,:\, \nabla \cdot \mathbf{u} = 0 \ \text{ in } \ \overline{\Omega}, \quad \mathbf{u} \cdot \mathbf{n} = 0 \ \text{ on } \ \Gamma \backslash \Gamma_0 \,\},
\end{aligned}$$

onde $\mathbf{C}^{m,\alpha}(\overline{\Omega};\mathbb{R}^N)$ denota o espaço das funções que assumem valores em $\mathbb{R}^N$ e que suas derivadas até a ordem m são Hölder-contínuas em $\overline{\Omega}$ com expoente α.

Quando $\kappa = 0$, é apropriado considerar a controlabilidade exata fronteira para (20). Em termos gerais, pode ser formulada como segue

Dados $\mathbf{y}_0$, $\mathbf{y}_1$, θ_0 e θ_1 em espaços apropriados com $\mathbf{y}_0 \cdot \mathbf{n} = \mathbf{y}_1 \cdot \mathbf{n} = 0$ sobre $\Gamma \backslash \Gamma_0$, encontrar $(\mathbf{y}, p, \theta)$ tal que (20) é satisfeito, mais ainda,

$$\mathbf{y}(\mathbf{x},T) = \mathbf{y}_1(\mathbf{x}), \quad \theta(\mathbf{x},T) = \theta_1(\mathbf{x}) \quad \text{em} \ \ \Omega. \tag{21}$$

Observe que, quando $\kappa = 0$, de modo a determinar, sem ambiguidade, uma única solução regular local em tempo para (20), é suficiente fornecer a componente normal da velocidade na fronteira, todo o vetor $\mathbf{y}$ e a temperatura θ, na região da fronteira por onde o fluido entra, i.e. apenas onde $\mathbf{y} \cdot \mathbf{n} < 0$, veja por exemplo [63]. Assim, podemos assumir que os controles tem a forma

$$\begin{cases}
\mathbf{y} \cdot \mathbf{n} \text{ sobre } \Gamma_0 \times (0,T), \text{ com } \displaystyle\int_{\Gamma_0} \mathbf{y} \cdot \mathbf{n}\, d\Gamma = 0; \\
\mathbf{y} \text{ em qualquer ponto de } \Gamma_0 \times (0,T) \text{ satisfazendo } \mathbf{y} \cdot \mathbf{n} < 0, \\
\theta \text{ em qualquer ponto de } \Gamma_0 \times (0,T) \text{ satisfazendo } \mathbf{y} \cdot \mathbf{n} < 0.
\end{cases}$$

O significado da propriedade de controlabilidade exata é que, quando vale, poderemos conduzir s solução de qualquer estado $(\mathbf{y}_0, \theta_0)$ exatamente a qualquer estado final $(\mathbf{y}_1, \theta_1)$, atuando apenas sobre uma pequena parte da fronteira durante um intervalo de tempo arbitrariamente pequeno.

No caso $\kappa > 0$, a situação é diferente. Devido ao efeito regularizante da equação da temperatura, não podemos esperar um resultado de controlabilidade exata, pelo menos para a temperatura.

De forma a apresentar um problema de controlabilidade na fronteira adequado, definimos $\gamma \subset \Gamma$. Então o problema de controlabilidade segue

> *Dados* $\mathbf{y}_0$*,* $\mathbf{y}_1$ *e* θ_0 *em espaços apropriados com* $\mathbf{y}_0 \cdot \mathbf{n} = \mathbf{y}_1 \cdot \mathbf{n} = 0$ *sobre* $\Gamma\backslash\Gamma_0$ *e* $\theta_0 = 0$ *sobre* $\Gamma\backslash\gamma$*, encontrar* $(\mathbf{y}, p, \theta)$ *com* $\theta = 0$ *sobre* $(\Gamma\backslash\gamma) \times (0, T)$ *tal que* (20) *é válido e, mais ainda,*
>
> $$\mathbf{y}(\mathbf{x}, T) = \mathbf{y}_1(\mathbf{x}), \quad \theta(\mathbf{x}, T) = 0 \quad in \quad \Omega. \tag{22}$$

Se é sempre possível encontrar $\mathbf{y}$, p e θ, dizemos que o sistema de Boussinesq, invíscido, incompressível com temperatura difusiva (20) possui a propriedade de controle exato-nulo para $(\Omega, \Gamma_0, \Gamma)$ no tempo T.

Observe que, se $\kappa > 0$ e fixamos as mesmas condições de fronteira de antes e (por exemplo) condição de Dirichlet para θ da forma

$$\theta = \theta * 1_\gamma \text{ sobre } \Gamma \times (0, T),$$

existe uma única solução para (20). Podemos assumir, neste caso, que os controles tem a forma

$$\begin{cases} \mathbf{y} \cdot \mathbf{n} \text{ sobre } \Gamma_0 \times (0, T), \text{ com } \displaystyle\int_{\Gamma_0} \mathbf{y} \cdot \mathbf{n}\, d\Gamma = 0, \\ \mathbf{y} \text{ em qualquer ponto de } \Gamma_0 \times (0, T) \text{ satisfazendo } \mathbf{y} \cdot \mathbf{n} < 0, \\ \theta \text{ em qualquer ponto de } \gamma \times (0, T). \end{cases}$$

O significado da propriedade de controle exato-nulo é que, quando vale, podemos conduzir o par $(\mathbf{y}, \theta)$ de qualquer estado inicial $(\mathbf{y}_0, \theta_0)$ exatamente a qualquer estado da forma $(\mathbf{y}_1, 0)$, atuando apenas em uma pequena parte Γ_0 e γ da fronteira, durante um intervalo de tempo arbitrariamente pequeno $(0, T)$.

Nas últimas décadas, muita investigação no contexto de fluidos incompressíveis perfeitos foi realizada. Temos, principalmente, os trabalhos de Coron [28, 29] e Glass [41, 42, 43]. Neste trabalho, adaptaremos as idéias de [28] e [43] para o problema modelado por (20).

Para finalizar, apresentaremos dois dos principais resultados obtidos neste trabalho.

Teorema ($\kappa = 0$): *Se* $\kappa = 0$*, então então o sistema de Boussinesq, invíscido* (20) *é exatamente controlável para* (Ω, Γ_0) *em qualquer tempo* $T > 0$*. Mais precisamente, para qualquer* $\mathbf{y}_0, \mathbf{y}_1 \in \mathbf{C}(2, \alpha, \Gamma_0)$ *e qualquer* $\theta_0, \theta_1 \in C^{2,\alpha}(\overline{\Omega})$*, existem* $\mathbf{y} \in C^0([0, T]; \mathbf{C}(1, \alpha, \Gamma_0))$*,* $\theta \in C^0([0, T]; C^{1,\alpha}(\overline{\Omega}))$ *e* $p \in \mathcal{D}'(\Omega \times (0, T))$ *tal que temos* (20) *and* (21)*.*

Nos argumentos que provam o teorema anterior, vemos a necessidade da regularidade $C^{2,\alpha}$ para o dado inicial e final. Entretanto, provamos a existência da solução controlada apenas no espaço $C^{1,\alpha}$. Seria interessante melhorar este resultado mas, até o momento, é um problema em aberto.

O segundo resultado do trabalho segue

Teorema ($\kappa > 0$): *Seja* Ω*,* Γ_0 *e* γ *dados, e suponhamos que* $\kappa > 0$*. Então* (20) *é localmente exato-nulo controlável. Mais precisamente, para qualquer* $T > 0$ *e qualquer* $\mathbf{y}_0, \mathbf{y}_1 \in \mathbf{C}(2, \alpha, \emptyset)$*, existe* $\eta > 0$*, dependendo de* $\mathbf{y}_0$*, tal que, para cada* $\theta_0 \in C^{2,\alpha}(\overline{\Omega})$ *com*

$$\theta_0 = 0 \quad on \quad \Gamma\backslash\gamma, \quad \|\theta_0\|_{2,\alpha} \leq \eta,$$

podemos encontrar $\mathbf{y} \in C^0([0,T];\mathbf{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^N))$, $\theta \in C^0([0,T];C^{1,\alpha}(\overline{\Omega}))$ *com* $\theta = 0$ *sobre* $(\Gamma\backslash\gamma) \times (0,T)$, *e* $p \in \mathcal{D}'(\Omega \times (0,T))$ *satisfazendo* (20) *e* (22).

Estes resultados podem ser vistos em [82].

Problemas em aberto e trabalhos futuros

Comentaremos brevemente uma série de perguntas e problemas em aberto que os resultados contidos nesta tese produziram.

- *Outros tipos de equilíbrio:* No Capítulo 1, o par óptimo é determinado segundo o critério não cooperativo de Nash. Um problema natural a seguir é utilizar outros tipos de estratégias para determinar o par. Um exemplo clássico é o de equilíbrio de Pareto; seguindo a notação do Capítulo 1 temos:

  Definição (Equilíbrio de Pareto): *Para cada* $f \in L^2(\mathcal{O} \times (0,T))$ *dizemos que o par* $(u^1(f), u^2(f)) \in \mathcal{H}$ *é um equilíbrio de Pareto se não existe* $(\hat{u}^1, \hat{u}^2) \in \mathcal{H}$ *satisfazendo*

  $$J_i(\hat{u}^1, \hat{u}^2) \leq J_i(u^1(f), u^2(f)) \quad \textit{for} \quad i = 1, 2,$$

  com alguma das desigualdades sendo estrita.

  Uma vez que o par $(u^1(f), u^2(f))$ está fixado, queremos determinar f tal que o estado y associado a f e a $(u^1(f), u^2(f))$ satisfaz (8).

  Este tema é o alvo de um trabalho em andamento.

- *Estratégias do tipo Stackelberg-Nash para o problema de Stokes:*

  Problemas similares aos do Capítulo 1 podem ser postulados para sistemas do tipo Stokes

  $$\begin{cases} y_t - \Delta y + (w \cdot \nabla)y + \nabla p = f 1_{\mathcal{O}} + v^1 1_{\mathcal{O}_1} + v^2 1_{\mathcal{O}_2} & \text{in} \quad Q, \\ \nabla \cdot y = 0 & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(x,0) = y^0(x) & \text{in} \quad \Omega. \end{cases} \tag{23}$$

  O dado inicial y^0 pertence ao espaço de Hilbert

  $$H := \{\, z \in L^2(\Omega)^N : \nabla \cdot z = 0 \ \text{ in } \ \Omega, \ z \cdot n = 0 \ \text{ on } \ \Gamma \,\},$$

  o campo w pertence a $L^\infty(0,T;H)$ e os controles f e v^i satisfazem

  $$f \in L^2(\mathcal{O} \times (0,T))^N, \quad v^i \in L^2(\mathcal{O}_i \times (0,T))^N.$$

  Com os funcionais J e J_i poderemos formular as estratégias do tipo Stackelberg-Nash associada a uma propriedade de controllabilidade nula para (23).

  A situação se torna mais difícil quando analisamos o sistema de Navier-Stokes

  $$\begin{cases} y_t - \Delta y + (y \cdot \nabla)y + \nabla p = f 1_{\mathcal{O}} + v^1 1_{\mathcal{O}_1} + v^2 1_{\mathcal{O}_2} & \text{in} \quad Q, \\ \nabla \cdot y = 0 & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(x,0) = y^0(x) & \text{in} \quad \Omega. \end{cases}$$

A existência de um equilíbrio ou quase-equilíbrio para cada f e, é claro, quando vale a propriedade de controlabilidade nula associada a este equilíbrio é um problema em aberto.

Para outros resultados de controlabilidade para sistemas de Stokes e Navier-Stokes, veja [39, 53, 34, 46, 47].

- *Estudo numérico da controlabilidade nula para a equação de Schrödinger* $1D$ *semilinear*

  Este problema consiste em encontrar aproximações numéricas para controles u que conduzem a solução de

$$\begin{cases} iy_t - y_{xx} + V(x,t)y = F(y)y, & (x,t) \in (0,1) \times (0,T), \\ y(0,t) = u(t), \quad y(1,t) = 0, & t \in (0,T), \\ y(x,0) = y_0(x), & x \in (0,1), \end{cases} \tag{24}$$

  a zero. Devido às propriedades de regularidade da equação de Schrödinger, este problema possui um certo nível de dificuldade, entretanto, resultados preliminares já foram obtidos

- *Estudo numérico da controlabilidade bilinear para a equação de Schrödinger* $1D$

  Este problema consiste em encontrar aproximações numéricas para controles u que conduzem a solução de

$$\begin{cases} iy_t - y_{xx} + u(x,t)y = 0, & (x,t) \in (0,1) \times (0,T), \\ y(0,t) = 0, \quad y(1,t) = 0, & t \in (0,T), \\ y(x,0) = y_0(x), & x \in (0,1). \end{cases} \tag{25}$$

  a um estado final desejado. Note que o controle é bilinear e isso gera grandes dificuldades do ponto de vista técnico.

Capítulo 1

Stackelberg-Nash exact controllability for linear and semilinear parabolic equations

Stackelberg-Nash exact controllability for linear and semilinear parabolic equations

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Abstract. This paper is concerned with Stackelberg-Nash strategies to control parabolic equations. We assume that we can act on the system through a hierarchy of controls. A first control (the leader) is assumed to choose the policy. Then, a Nash equilibrium pair (corresponding to a noncooperative multiple-objective optimization strategy) is identified; this governs the action of the other controls (the followers). The main novelty in this paper is that we can impose and obtain exact controllability to a prescribed (but arbitrary) trajectory. We study linear and semilinear problems and also problems with constraints on the followers.

1.1 Introduction

In classical control theory, we usually find a state equation or system and one control with the mission of achieving a predetermined goal. Usually (but not always), the goal is to minimize a cost functional in a prescribed family of admissible controls.

A more interesting situation arises when several (in general, conflictive or contradictory) objectives are considered. This may happen, for example, if the cost function is the sum of several terms and it is not clear how to average. It can also be expectable to have more than one control acting on the equation. In these case, we are led to consider *multi-objective* control problems.

In contrast with the mono-objective case, various strategies for the choice of good controls can appear, depending of the characteristics of the problem. Moreover, these strategies can be *cooperative* (when the controls mutually cooperate in order to achieve some goals) or *noncooperative.*

There exist various equilibrium concepts for multi-objective problems, with origin in *game theory,* mainly motivated by economics. Each of them determines a strategy. Thus, let us mention the noncooperative optimization strategy proposed by Nash [69], the Pareto cooperative strategy [70] and the Stackelberg hierarchical-cooperative strategy [87].

In the context of the control of PDEs, a relevant question is whether one is able to lead the system to a desired state (exactly or approximately) by applying controls that fulfill one of these equilibrium conditions. There have been up to date several works on this subject that intended to provide an answer to this question:

- The papers by J.-L. Lions [61] and [62], where the author gives some results concerning Pareto and Stackelberg strategies, respectively.

- The paper by Díaz and Lions [32], where the approximate controllability of a system is established following a Stackelberg-Nash strategy and the extension in Díaz [31], that provides a characterization of the solution by means of Fenchel-Rockafellar duality theory.
- The papers [74] and [75], where Ramos, Glowinski and Periaux study Nash equilibrium from the theoretical and numerical viewpoints for linear parabolic PDEs and for the Burgers equation, respectively.
- Finally, let us mention that the Stackelberg-Nash strategy for the Stokes systems has been studied by Guillén-González et al in [48].

The controllability issues considered in these works only provide answers at the approximate level. This means that the main results assert that one can lead the system to a state that is arbitrarily close to a desired target.

The main novelty of the present paper is to extend the analysis and the results to the *exact controllability* framework.

1.1.1 The problems and their motivations

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain whose boundary Γ is regular enough. Let $T > 0$ be given and let us consider the cylinder $Q = \Omega \times (0, T)$, with lateral boundary $\Sigma = \Gamma \times (0, T)$. In the sequel, we will denote by C a generic positive constant. Sometimes, we will indicate the data on which C depends by writing $C(\Omega)$, $C(\Omega, T)$, etc. The usual norm and scalar product in $L^2(\Omega)$ will be respectively denoted by $\|\cdot\|$ and $(\cdot\,,\cdot)$.

We are interested in the proof of the exact controllability to the trajectories of a multi-objective parabolic PDE problem in Q, where we apply a Stackelberg-Nash strategy. For simplicity, we will assume that only three controls are applied (one leader and two followers), but very similar considerations hold for systems with a higher number of controls.

Moe precisely, we will consider systems of the form

$$\begin{cases} y_t - \Delta y + a(x,t)y = F(y) + f 1_{\mathcal{O}} + v^1 1_{\mathcal{O}_1} + v^2 1_{\mathcal{O}_2} & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(x,0) = y^0(x) & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $y = y(x,t)$ is the state, $a \in L^\infty(Q)$, F is a locally Lipschitz-continuous function and y^0 is prescribed.

In (1.1), the set $\mathcal{O} \subset \Omega$ is the *main control domain* and $\mathcal{O}_1, \mathcal{O}_2 \subset \Omega$ are the *secondary control domains* (all them are supposed to be small); $1_{\mathcal{O}}$, $1_{\mathcal{O}_1}$ and $1_{\mathcal{O}_2}$ are the characteristic functions of $\mathcal{O}$, $\mathcal{O}_1$ and $\mathcal{O}_2$, respectively; the controls are f, v^1 and v^2, f is the *leader* and v^1 and v^2 are the *followers*.

Let $\mathcal{O}_{1,d}$, $\mathcal{O}_{2,d} \subset \Omega$ be open sets, representing observation domains for the followers. We will consider the (secondary) functionals

$$J_i(f; v^1, v^2) := \frac{\alpha_i}{2} \iint_{\mathcal{O}_{i,d} \times (0,T)} |y - y_{i,d}|^2 \, dx\, dt + \frac{\mu_i}{2} \iint_{\mathcal{O}_i \times (0,T)} |v^i|^2 \, dx\, dt, \quad i = 1, 2 \tag{1.2}$$

and the main functional

$$J(f) := \frac{1}{2} \iint_{\mathcal{O}\times(0,T)} |f|^2 \, dx \, dt, \tag{1.3}$$

where the $\alpha_i > 0$, $\mu_i > 0$ are constants and the $y_{i,d} = y_{i,d}(x,t)$ are given functions.

The structure of the control process can be described as follows:

1. The followers v^1 and v^2 assume that the leader f has made a choice and intend to be a *Nash equilibrium* for the costs J_i $(i = 1, 2)$.

   Thus, once f has been fixed, we look for controls $v^i \in L^2(\mathcal{O}_i \times (0,T))$ that satisfy

   $$J_1(f; v^1, v^2) = \min_{\hat{v}^1} J_1(f; \hat{v}^1, v^2), \quad J_2\left(f; v^1, v^2\right) = \min_{\hat{v}^2} J_2(f; v^1, \hat{v}^2). \tag{1.4}$$

   Any pair (v^1, v^2) satisfying (1.4) is called a Nash equilibrium for J_1 and J_2.

   Note that, if the functionals J_i $(i = 1, 2)$ are convex, then (v^1, v^2) is a Nash equilibrium if and only if

   $$J_1'(f; v^1, v^2)(\hat{v}^1, 0) = 0, \quad \forall \hat{v}^1 \in L^2\left(\mathcal{O}_1 \times (0,T)\right); \ v^1 \in L^2(\mathcal{O}_1 \times (0,T)) \tag{1.5}$$

   and

   $$J_2'(f; v^1, v^2)(0, \hat{v}^2) = 0, \quad \forall \hat{v}^2 \in L^2\left(\mathcal{O}_2 \times (0,T)\right); \ v^2 \in L^2(\mathcal{O}_2 \times (0,T)). \tag{1.6}$$

2. Let us fix an uncontrolled trajectory of (1.1), that is, a sufficiently regular solution to the system:

   $$\begin{cases} \overline{y}_t - \Delta \overline{y} + a(x,t)\overline{y} = F(\overline{y}) & \text{in} \quad Q, \\ \overline{y} = 0 & \text{on} \quad \Sigma, \\ \overline{y}(x,0) = \overline{y}^0(x) & \text{in} \quad \Omega. \end{cases} \tag{1.7}$$

   Once the Nash equilibrium has been identified and fixed for each f, we look for an optimal control $\hat{f} \in L^2(\mathcal{O} \times (0,T))$ such that

   $$J(\hat{f}) = \min_{f} J(f), \tag{1.8}$$

   subject to the restriction of exact controllability

   $$y(x,T) = \overline{y}(x,T) \quad \text{in} \quad \Omega. \tag{1.9}$$

Several motivations can be found for control problems of this kind:

- If $y = y(x,t)$ is viewed as a temperature distribution in a body, we interpret that our intention is to drive y to a desired $\overline{y}$ at time T by heating and cooling (acting only on the small subdomains $\mathcal{O}$, $\mathcal{O}_1$ and $\mathcal{O}_2$), trying at the same time to keep reasonable temperatures in $\mathcal{O}_{1,d}$ and $\mathcal{O}_{2,d}$ during the whole time interval $(0,T)$.

- A similar control process makes sense in the context of fluid mechanics. Thus, we can replace (1.1) and (1.7) by similar Stokes and/or Navier-Stokes systems and we can look for controls $\bar{f}$ and associated Nash equilibria (v_1, v_2) satisfying (1.8)-(1.9). In this case, it is assumed that we act on the system through mechanical forces applied on $\mathcal{O}, \mathcal{O}_1$ and $\mathcal{O}_2$ and the goal is to reach $\overline{y}$ at time T keeping the velocity field y not too far from $y_{i,d}$ in $\mathcal{O}_{i,d} \times (0,T)$ for $i = 1, 2$.

- In the framework of mathematical finance, this can also be an interesting question. For instance, it is well known that the price of an European call option is governed by a backward PDE similar to (1.1). Now, the independent variable x must be interpreted as the stock price and t is in fact the reverse of time (we fix a situation at $t = T$ and we want to know what to do in order to arrive at this situation from a well chosen state). In this regard, it can be interesting to control the solution of the system with the composed action of several agents, each of them corresponding to a different range of values of x. For further information on the modeling and control of phenomena of this kind, see for instance [30, 79, 88].

1.1.2 The main results

We will have to impose the following assumption:

$$\mathcal{O}_{1,d} = \mathcal{O}_{2,d}. \tag{1.10}$$

Accordingly, we will denote these sets by $\mathcal{O}_d$; see below, in Section 1.5, some comments on the necessity of the hypothesis (1.10).

In the linear case ($F \equiv 0$), the exact controllability to the trajectories is equivalent to the null controllability property. The following result holds:

Theorem 1. *Let us assume that $F \equiv 0$, $\mathcal{O}_d \cap \mathcal{O} \neq \emptyset$ and the μ_i are sufficiently large:*

$$\mu_i \geq C(\Omega, T, \mathcal{O}_i, \mathcal{O}_d, \alpha_i, \|a\|_{L^\infty(Q)}) \quad \text{for} \quad i = 1, 2. \tag{1.11}$$

There exists a positive function $\hat{\rho} = \hat{\rho}(x,t)$ blowing up at $t = T$ with the following property: if $\bar{y}$ is the unique solution to (1.7) with $F \equiv 0$ associated to the initial state $\overline{y}^0 \in L^2(\Omega)$ and the $y_{i,d}$ are such that

$$\iint_{\mathcal{O}_d \times (0,T)} \hat{\rho}^2 |\overline{y} - y_{i,d}|^2 \, dx \, dt < +\infty, \quad i = 1, 2, \tag{1.12}$$

for any $y^0 \in L^2(\Omega)$ there exist controls $f \in L^2(\mathcal{O} \times (0,T))$ and associated Nash equilibria (v^1, v^2) such that the corresponding solutions to (1.1) satisfy (1.9).

Roughly speaking, the assumption (1.11) means that it is important for us to get followers with moderate L^2 norms. On the other hand, the assumption (1.12) means that both $y_{1,d}$ and $y_{2,d}$ approach $\overline{y}$ as $t \to T$.

In the semilinear case, with F being a locally Lipschitz-continuous function, we can consider the same controllability questions. However, it is important to note that, in this case, we lose the convexity of the functionals J_i and the Nash equilibrium condition (1.4) is not necessarily equivalent to (1.5) and (1.6). For this reason, it is convenient to weaken the definition of equilibrium as follows:

Definition 1. *Let $f \in L^2(\mathcal{O} \times (0,T))$ be given. The pair (v^1, v^2) is a Nash quasi-equilibrium when the conditions* (1.5) *and* (1.6) *are satisfied.*

For the semilinear case, we have the following result:

Theorem 2. *Let us assume that $F \in W^{1,\infty}(\mathbb{R})$ and the $\mu_i > 0$ are sufficiently large. Let $\overline{y}$ be the unique solution to* (1.7) *associated to the initial state $\overline{y}^0 \in L^2(\Omega)$ and let us assume that* (1.12) *holds, where $\hat{\rho}$ is the weight furnished by Theorem 1. Then, for each $y_0 \in L^2(\Omega)$, there exist controls $f \in L^2(\mathcal{O} \times (0,T))$ and associated Nash quasi-equilibria (v^1, v^2) such that the corresponding solutions to* (1.1) *satisfy* (1.9).

A natural question is whether there are semilinear systems for which the concepts of Nash equilibrium and Nash quasi-equilibrium are equivalent. The answer is furnished by the following result:

Proposition 1. *Let us assume that $F \in W^{2,\infty}(\mathbb{R})$ and $y_{i,d} \in L^\infty(\mathcal{O}_{i,d} \times (0,T))$ for $i = 1,2$. Suppose that $y_0 \in H_0^1(\Omega)$ (resp. $y_0 \in L^2(\Omega)$) and $N \leq 14$ (resp. $N \leq 12$). Then, there exists $C > 0$ such that, if $f \in L^2(\mathcal{O} \times (0,T))$ and the μ_i satisfy*

$$\mu_i \geq C(1 + \|f\|_{L^2(\mathcal{O}\times(0,T))}),$$

the conditions (1.4) *and* (1.5)*-*(1.6) *are equivalent.*

In this paper, we also analyze if a result like Theorem 1 holds true when the followers are constrained to belong to appropriate convex sets $\mathcal{U}_i \subset L^2(\mathcal{O}_i \times (0,T))$. Thus, let I_1 and I_2 be two nonempty closed intervals with $0 \in I_1 \cap I_2$, let us take

$$\mathcal{U}_i = \{\, v \in L^2(\mathcal{O}_i \times (0,T)) : \ v(x,t) \in I_i \ \text{ a.e.} \,\}, \quad i = 1,2, \tag{1.13}$$

and let us suppose that the minimization of J_1 and J_2 in (1.4) is subject to the restrictions $\hat{v}^1 \in \mathcal{U}_1$ and $\hat{v}^2 \in \mathcal{U}_2$.

The controllability result is the following:

Theorem 3. *Let us assume that $F \equiv 0$ and the $\mu_i > 0$ are sufficiently large. Let $\overline{y}$ be the unique solution to* (1.7) *associated to the initial state $\overline{y}^0 \in L^2(\Omega)$. Then, for each $y_0 \in L^2(\Omega)$, there exist controls $f \in L^2(\mathcal{O} \times (0,T))$ and associated Nash equilibria $(v^1, v^2) \in \mathcal{U}_1 \times \mathcal{U}_2$ such that the corresponding solutions to* (1.1) *satisfy* (1.9).

As mentioned above, the main novelty of this paper is that we deal with exact and not approximate controllability. There are other points that distinguish our contribution as well. Thus, contrarily to what was imposed in other previous papers (see for instance [48]), we do not make any assumption on the open sets $\mathcal{O}_i$. In particular, the $\mathcal{O}_i$ can be disjoint of $\mathcal{O}$, which is obviously the most interesting situation. On the other hand, the analysis and results also hold, after appropriate modifications, for m followers with $m \geq 2$.

The rest of the paper is organized as follows.

In Section 1.2, we prove Theorem 1, which concerns the linear case. This result will be strongly used in the other sections. In Section 1.3, we prove Theorem 2 and Proposition 1. As a consequence, we see that the Stackelberg-Nash strategy can be applied to nonlinear problems and, also, that under adequate hypotheses on F, we still obtain a Nash equilibrium. Section 1.4 deals with the proof of Theorem 3. Finally, we present some additional comments and questions in Section 1.5.

1.2 The linear case

In this section we prove Theorem 1. The proof is long and, for clarity, has been decomposed in two parts. In Section 2.4 we recall the existence, uniqueness and characterization of a Nash equilibrium (for fixed but arbitrary f); then, in Section 1.2.2, we prove the desired controllability result.

By the linearity of the problem, we may reduce the exact controllability to the trajectories to a null controllability property. In fact, after the change of variable $y = z + \overline{y}$, it is immediate to see from (1.1) and (1.7) for $F \equiv 0$ that z is the solution to the problem

$$\begin{cases} z_t - \Delta z + a(x,t)z = f 1_{\mathcal{O}} + v^1 1_{\mathcal{O}_1} + v^2 1_{\mathcal{O}_2} & \text{in} \quad Q, \\ z = 0 & \text{on} \quad \Sigma, \\ z(\cdot, 0) = z^0 & \text{in} \quad \Omega, \end{cases} \tag{1.14}$$

where $z^0 = y^0 - \overline{y}^0$.

It is clear that $y(x,T) \equiv \overline{y}(x,T)$ if and only if $z(x,T) \equiv 0$. Also, we can write the functionals J_i in (1.2) in terms of z, which gives

$$J_i(f; v^1, v^2) = \frac{\alpha_i}{2} \iint_{\mathcal{O}_{i,d} \times (0,T)} |z - z_{i,d}|^2 \, dx \, dt + \frac{\mu_i}{2} \iint_{\mathcal{O}_i \times (0,T)} |v^i|^2 \, dx \, dt, \quad i = 1, 2,$$

where $z_{i,d} := y_{i,d} - \overline{y}$ for $i = 1, 2$.

1.2.1 Nash equilibrium

In this section, we recall an existence/uniqueness result concerning a Nash equilibrium, in the sense of (1.4), for any $f \in L^2(\mathcal{O} \times (0,T))$. Then, we recall a result which characterizes this Nash equilibrium in terms of the solution to an adjoint system. These results are due to Díaz and Lions; see [31, 32, 62].

For the moment, we do not have to impose the assumption (1.10). This requirement only appears later, in Section 1.2.2, when the choice of f has to be made.

Existence and uniqueness

Let us introduce the spaces $\mathcal{H}_i := L^2(\mathcal{O}_i \times (0,T))$ and $\mathcal{H} := \mathcal{H}_1 \times \mathcal{H}_2$ and let us consider the operators $L_i \in \mathcal{L}(\mathcal{H}_i; L^2(Q))$ with $L_i v^i = z^i$, where z^i is the solution to the system

$$\begin{cases} z^i_t - \Delta z^i + a(x,t)z = v^i 1_{\mathcal{O}_i} & \text{in} \quad Q, \\ z^i = 0 & \text{on} \quad \Sigma, \\ z^i(x,0) = 0 & \text{in} \quad \Omega. \end{cases}$$

By definition, for any control f, the pair (v^1, v^2) is a Nash equilibrium if and only if it satisfies (1.5) and (1.6), that is to say,

$$\alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} (z - z_{i,d}) w^i \, dx \, dt + \mu_i \iint_{\mathcal{O}_i \times (0,T)} v^i \hat{v}^i \, dx \, dt = 0 \quad \forall \hat{v}^i \in \mathcal{H}_i, \tag{1.15}$$

where w^i is the derivative of z with respect to v^i in the direction $\hat{v}^i$.

Note that

$$\begin{cases} w^i_t - \Delta w^i + a(x,t)w^i = \hat{v}^i 1_{\mathcal{O}_i} & \text{in} \quad Q, \\ w^i = 0 & \text{on} \quad \Sigma, \\ w^i(x,0) = 0 & \text{in} \quad \Omega. \end{cases}$$

Consequently, $L_i \hat{v}^i = w^i$. We also have $z = L_1 v^1 + L_2 v^2 + u$, where

$$\begin{cases} u_t - \Delta u + a(x,t)u = f 1_{\mathcal{O}} & \text{in} \quad Q, \\ u = 0 & \text{on} \quad \Sigma, \\ u(x,0) = z^0 & \text{in} \quad \Omega. \end{cases}$$

Therefore, we may rewrite (1.15) in the form

$$\alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} (L_1 v^1 + L_2 v^2 - (z_{i,d} - u)) L_i \hat{v}^i \, dx \, dt \\ + \mu_i \iint_{\mathcal{O}_i \times (0,T)} v^i \hat{v}^i \, dx \, dt = 0, \quad \forall \hat{v}^i \in \mathcal{H}_i$$

or

$$\iint_{\mathcal{O}_i \times (0,T)} \left(\alpha_i L_i^*((L_1 v^1 + L_2 v^2 - (z_{i,d} - u)) 1_{\mathcal{O}_{i,d}}) + \mu_i v^i\right) \hat{v}^i \, dx \, dt = 0, \quad \forall \hat{v}^i \in \mathcal{H}_i,$$

where $L_i^* : \mathcal{L}(L^2(Q); \mathcal{H}_i)$ is the adjoint of L_i.

In other words, (v^1, v^2) is a Nash equilibrium if and only if

$$\alpha_i L_i^*((L_1 v^1 + L_2 v^2) 1_{\mathcal{O}_{i,d}}) + \mu_i v^i = \alpha_i L_i^*((z_{i,d} - u) 1_{\mathcal{O}_{i,d}}) \ \text{ in } \ \mathcal{H}_i, \quad i = 1, 2.$$

Let us introduce the operator $\mathbb{L} \in \mathcal{L}(\mathcal{H}; \mathcal{H})$, given by

$$\mathbb{L}(v^1, v^2) = (\alpha_1 L_1^*((L_1 v^1 + L_2 v^2) 1_{\mathcal{O}_{1,d}}) + \mu_1 v^1, \alpha_2 L_2^*((L_1 v^1 + L_2 v^2) 1_{\mathcal{O}_{2,d}}) + \mu_2 v^2), \tag{1.16}$$

for all $(v^1, v^2) \in \mathcal{H}$. Then, the task is to prove the existence and uniqueness of a solution for the equation

$$\mathbb{L}(v^1, v^2) = \Psi, \quad (v^1, v^2) \in \mathcal{H}, \tag{1.17}$$

where

$$\Psi = (\alpha_1 L_1^*((z_{1,d} - u) 1_{\mathcal{O}_{1,d}}), \alpha_2 L_2^*((z_{2,d} - u) 1_{\mathcal{O}_{2,d}})). \tag{1.18}$$

In this direction, the following holds:

Proposition 2. *Let us assume that*

$$\alpha_1\|1_{\mathcal{O}_{1,d}}L_2\|_{(1)} < 4\mu_2 \quad \text{and} \quad \alpha_2\|1_{\mathcal{O}_{2,d}}L_1\|_{(2)} < 4\mu_1, \tag{1.19}$$

where $\|\cdot\|_{(i)}$ *denotes the norm in the space* $\mathcal{L}(\mathcal{H}_{3-i}; L^2(\mathcal{O}_{i,d}\times(0,T)))$. *Then* $\mathbb{L}$ *is an isomorphism. In particular, for each* $f \in L^2(\mathcal{O}\times(0,T))$, *there exists exactly one Nash equilibrium* $(v^1(f), v^2(f))$ *in the sense of* (1.4).

Proof: From (1.16) and Young's inequality, we observe that

$$\begin{aligned}
(\mathbb{L}(v^1,v^2),(v^1,v^2))_{\mathcal{H}} &= \sum_{i=1}^{2}\mu_i\|v^i\|^2_{\mathcal{H}_i} + \sum_{i,j=1}^{2}\alpha_i(L_jv^{(j)},L_iv^i)_{L^2(\mathcal{O}_{i,d}\times(0,T))}\\
&\geq \sum_{i=1}^{2}\Big(\mu_i\|v^i\|^2_{\mathcal{H}_i} + \alpha_i\|L_iv^i\|^2_{L^2(\mathcal{O}_{i,d}\times(0,T))}\Big)\\
&\quad -\sum_{i=1}^{2}\alpha_i\left(\|L_iv^i\|^2_{L^2(\mathcal{O}_{i,d}\times(0,T))} + \frac{1}{4}\|L_{3-i}v^{3-i}\|^2_{L^2(\mathcal{O}_{i,d}\times(0,T))}\right)\\
&\geq \sum_{i=1}^{2}\Big(\mu_i - \frac{\alpha_{3-i}}{4}\|1_{\mathcal{O}_{3-i,d}}L_i\|^2_{(3-i)}\Big)\|v^i\|^2_{\mathcal{H}_i}.
\end{aligned}$$

Therefore,

$$(\mathbb{L}((v^1,v^2),(v^1,v^2))_{\mathcal{H}} \geq \gamma\|(v^1,v^2)\|^2_{\mathcal{H}} \;\; \forall (v^1,v^2)\in\mathcal{H}, \tag{1.20}$$

where $\gamma = \min_i\{\mu_i - \alpha_{3-i}\|1_{\mathcal{O}_{3-i,d}}L_i\|^2_{(3-i)}\} > 0$, see (1.19).

Now, let us introduce the bilinear form $a : \mathcal{H}\times\mathcal{H}\mapsto\mathbb{R}$, with

$$a((v^1,v^2),(\hat{v}^1,\hat{v}^2)) := (\mathbb{L}((v^1,v^2),(\hat{v}^1,\hat{v}^2)))_{\mathcal{H}}.$$

From the definition of the operator $\mathbb{L}$ and the inequality (1.20), we readily see that $a(\cdot\,,\cdot)$ is continuous and coercive on $\mathcal{H}$. Consequently, the Lax-Milgram Theorem implies that, for any $\Phi\in\mathcal{H}'$, there exists exactly one $(v^1,v^2)\in\mathcal{H}$ satisfying

$$a((v^1,v^2),(\hat{v}^1,\hat{v}^2)) = \langle\Phi,(\hat{v}^1,\hat{v}^2)\rangle_{\mathcal{H}'\times\mathcal{H}} \quad \forall\,(\hat{v}^1,\hat{v}^2)\in\mathcal{H};\;\; (v^1,v^2)\in\mathcal{H}.$$

In particular, we get (1.17) and the proof is done. ■

From the proof, it becomes clear that, under the assumptions of Proposition 2, for any $f\in L^2(\mathcal{O}\times(0,T))$ the associated Nash equilibrium $(v^1(f),v^2(f))$ satisfies

$$\|(v^1(f),v^2(f))\|_{\mathcal{H}} \leq C\left(1+\|f\|_{L^2(\mathcal{O}\times(0,T))}\right), \tag{1.21}$$

where C depends on $\Omega,\mathcal{O},T,\mathcal{O}_i,\mathcal{O}_{i,d},\alpha_i,\mu_i,\|z_0\|$ and $\|a\|_{L^\infty(Q)}$. These estimates will be used below. Notice that, in view of (1.21), the state z associated to f and $(v^1(f),v^2(f))$ satisfies

$$\|z\|_{L^2(0,T;H^1_0(\Omega))} + \|z_t\|_{L^2(0,T;H^{-1}(\Omega))} \leq C(1+\|f\|_{L^2(\mathcal{O}\times(0,T))}), \tag{1.22}$$

where C is as above.

Characterization of the Nash equilibrium

In this section, we express the followers $v^1(f)$ and $v^2(f)$ in terms of a (new) adjoint variable.

Let $f \in L^2(\mathcal{O} \times (0,T))$ be given. For any $(v^1, v^2) \in \mathcal{H}$, let us consider the associated state z (the solution for (1.14)). In view of (1.15), it is very natural to introduce the adjoint states ϕ^i $(i = 1, 2)$, with

$$\begin{cases} -\phi^i_t - \Delta\phi^i + a(x,t)\phi^i = \alpha_i(z - z_{i,d})1_{\mathcal{O}_{i,d}} & \text{in} \quad Q, \\ \phi^i = 0 & \text{on} \quad \Sigma, \\ \phi^i(\cdot, T) = 0 & \text{in} \quad \Omega. \end{cases}$$

Using integration by parts, we see that (v^1, v^2) is a Nash equilibrium if and only if

$$\iint_{\mathcal{O}_i \times (0,T)} \left(\phi^i + \mu_i v^i\right) \hat{v}^i \, dx \, dt = 0 \quad \forall \hat{v}^i \in \mathcal{H}_i; \quad v^i \in \mathcal{H}_i.$$

This directly implies that

$$v^i = -\frac{1}{\mu_i}\phi^i\Big|_{\mathcal{O}_i \times (0,T)} \quad \text{for} \quad i = 1, 2.$$

Let us gather all these informations in the same system. We obtain the following:

$$\begin{cases} z_t - \Delta z + a(x,t)z = f\,1_{\mathcal{O}} - \displaystyle\sum_{i=1}^{2} \frac{1}{\mu_i}\phi^i 1_{\mathcal{O}_i} & \text{in} \quad Q, \\ -\phi^i_t - \Delta\phi^i + a(x,t)\phi^i = \alpha_i(z - z_{i,d})1_{\mathcal{O}_{i,d}} & \text{in} \quad Q, \\ z = 0, \quad \phi^i = 0 & \text{on} \quad \Sigma, \\ z(x,0) = z^0(x), \quad \phi^i(x,T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.23}$$

Recall that our main objective is to prove the null controllability of z at time $t = T$. Therefore, the task is to find a distributed control $f \in L^2(\mathcal{O} \times (0,T))$ such that the solution to (1.23) satisfies

$$z(x,T) = 0 \quad \text{in} \quad \Omega. \tag{1.24}$$

1.2.2 Null controllability

In this section, we achieve the proof of Theorem 1.

We will establish an *observability inequality* for the system

$$\begin{cases} -\psi_t - \Delta\psi + a(x,t)\psi = \displaystyle\sum_{i=1}^{2} \alpha_i \gamma^i 1_{\mathcal{O}_{i,d}} & \text{in} \quad Q, \\ \gamma^i_t - \Delta\gamma^i + a(x,t)\gamma^i = -\dfrac{1}{\mu_i}\psi 1_{\mathcal{O}_i} & \text{in} \quad Q, \\ \psi = 0, \quad \gamma^i = 0 & \text{on} \quad \Sigma, \\ \psi(x,T) = \psi^T(x), \quad \gamma^i(x,0) = 0 & \text{in} \quad \Omega, \end{cases} \tag{1.25}$$

which can be viewed as the adjoint of (1.23). This will suffice.

The observability estimate is given in the following result:

Proposition 3. *Assume that* (1.10) *holds and the* μ_i *are sufficiently large. There exist* $C > 0$, *only depending on* $\Omega, \mathcal{O}, T, \mathcal{O}_i, \mathcal{O}_d, \alpha_i, \mu_i$ *and* $\|a\|_{L^\infty(Q)}$ *and a weight function* $\hat{\rho} = \hat{\rho}(x,t)$, *only depending on* Ω, $\mathcal{O}$, T *and* $\|a\|_{L^\infty(Q)}$, *such that for any* $\psi^T \in L^2(\Omega)$ *the following inequality holds true for the solution* (ψ, γ^i) *of* (1.25)*:*

$$\int_\Omega |\psi(x,0)|^2\,dx + \sum_{i=1}^{2} \iint_Q \hat{\rho}^{-2}|\gamma^i|^2\,dx\,dt \le C \iint_{\mathcal{O}\times(0,T)} |\psi|^2\,dx\,dt. \tag{1.26}$$

Let us assume for a moment that Proposition 3 holds and let us prove the controllability result in Theorem 1. From a well known duality argument, we have that, for any $z^0 \in L^2(\Omega)$ and any $\psi^T \in L^2(\Omega)$,

$$\begin{aligned}\int_\Omega \left[z(x,T)\psi^T(x) - z^0(x)\psi(x,0)\right]\,dx &= \iint_{\mathcal{O}\times(0,T)} f\psi\,dx\,dt \\ &\quad - \sum_{i=1}^{2} \alpha_i \iint_{\mathcal{O}_d\times(0,T)} z_{i,d}\gamma^i\,dx\,dt,\end{aligned} \tag{1.27}$$

where (z, ϕ^1, ϕ^2) and $(\psi, \gamma^1, \gamma^2)$ are the solutions to (1.23) and (1.25), respectively associated to z^0 and ψ^T.

Thus, to prove the null controllability property is equivalent to find, for each $z^0 \in L^2(\Omega)$, a control f such that, for any $\psi^T \in L^2(\Omega)$, one has

$$\iint_{\mathcal{O}\times(0,T)} f\psi\,dx\,dt = -\int_\Omega z^0(x)\psi(x,0)\,dx + \sum_{i=1}^{2} \alpha_i \iint_{\mathcal{O}_d\times(0,T)} z_{i,d}\gamma^i\,dx\,dt.$$

There are several ways to show that (3.4) implies the existence of such a control. They rely on well known arguments. For completeness, let us sketch one of them.

For each $\varepsilon > 0$, let us consider the following functional:

$$\begin{aligned}F_\epsilon(\psi^T) := \frac{1}{2} \iint_{\mathcal{O}\times(0,T)} |\psi|^2\,dx\,dt + \epsilon\|\psi^T\| + \int_\Omega z^0(x)\psi(x,0)\,dx \\ - \sum_{i=1}^{2} \alpha_i \iint_{\mathcal{O}\times(0,T)} z_{i,d}\gamma^i\,dx\,dt \quad \forall \psi^T \in L^2(\Omega).\end{aligned}$$

It is then clear that $F_\varepsilon : L^2(\Omega) \mapsto \mathbb{R}$ is continuous and strictly convex. Moreover,

$$\begin{aligned}F_\epsilon(\psi^T) \ge \frac{1}{4} &\iint_{\mathcal{O}\times(0,T)} |\psi|^2\,dx\,dt \\ &- C\left(\int_\Omega |z^0|^2\,dx + \sum_{i=1}^{2} \alpha_i^2 \iint_{\mathcal{O}_d\times(0,T)} \hat{\rho}^2|z_{i,d}|^2\,dx\,dt\right) \\ &+ \varepsilon\|\psi^T\|,\end{aligned}$$

where C and $\hat{\rho}$ are furnished by Proposition 1. Consequently, F_ϵ is also coercive in $L^2(\Omega)$. Note that, here, we have used the assumption (1.12) on $z_{i,d} = y_{i,d} - \bar{y}$.

Let ψ_ϵ^T be the unique minimizer of F_ϵ. Then, either $\psi_\epsilon^T = 0$ or

$$\left\langle F_\epsilon'(\psi_\epsilon^T), \psi^T \right\rangle = 0 \quad \forall \psi^T \in L^2(\Omega).$$

Suppose that $\psi_\epsilon^T \neq 0$. In this case, we have

$$\begin{aligned} \iint_{\mathcal{O}\times(0,T)} \psi_\varepsilon \psi \, dx\, dt + \epsilon\Big(\frac{\psi_\varepsilon^T}{\|\psi_\varepsilon^T\|}, \psi^T\Big) + \int_\Omega z^0(x)\psi_\varepsilon(x,0)\, dx \\ - \sum_{i=1}^{2} \alpha_i \iint_{\mathcal{O}\times(0,T)} z_{i,d}\gamma^i \, dx\, dt = 0 \quad \forall \psi^T \in L^2(\Omega), \end{aligned} \tag{1.28}$$

where we have denoted by $(\psi_\epsilon, \gamma_\epsilon^1, \gamma_\epsilon^2)$ the solution to (1.25) corresponding to $\psi^T = \psi_\epsilon^T$. Taking $f = f_\epsilon := \psi_\epsilon 1_{\mathcal{O}\times(0,T)}$ in (1.27), denoting by z_ϵ the associated state and comparing to (1.28), we see that

$$\int_\Omega \left(z_\epsilon(x,T) - \frac{\epsilon}{\|\psi_\epsilon^T\|}\psi_\epsilon^T \right) \psi^T(x)\, dx = 0 \quad \forall \psi^T \in L^2(\Omega),$$

which implies

$$\|z_\epsilon(\cdot\,,T)\| = \varepsilon. \tag{1.29}$$

On the other hand, from (1.28) and (3.4) we also have

$$\|f_\epsilon\|_{L^2(\mathcal{O}\times(0,T))} \leq C \left(\int_\Omega |z^0|^2 \, dx + \sum_{i=1}^{2} \iint_{\mathcal{O}_{i,d}\times(0,T)} \hat\rho^2 |z_{i,d}|^2 \, dx\, dt \right)^{1/2}, \tag{1.30}$$

that is, f_ϵ is uniformly bounded in $L^2(\mathcal{O}\times(0,T))$. Obviously, we also have (1.29) and (1.30) when $\psi_\epsilon^T = 0$ and we take $f_\epsilon = 0$.

Consequently, we can easily deduce a uniform estimate for z_ϵ. Then, taking limits as $\epsilon \to 0$, we conclude that null controllability holds.

This ends the proof of Theorem 1.

Remark 1. The leader control we have constructed is the unique solution to the extremal problem (1.8)–(1.9). This claim can be justified as follows:

1. For each $\varepsilon > 0$, there exists exactly one minimal L^2 norm control f_ϵ such that the associated state, i.e. the corresponding solution to (1.23), satisfies

$$\|z_\epsilon(\cdot\,,T)\| \leq \varepsilon.$$

2. From the weak lower semicontinuity of the terms in J_ε, it is clear that any weak limit of a subsequence of $\{f_\varepsilon\}$ minimizes the L^2 norm in the family of the null controls for z. Consequently, this is the case for f. □

Proof of Proposition 3. The assumption (1.10) will be used here.

Let ω be a non-empty open set satisfying $\omega \subset\subset \mathcal{O}_d \cap \mathcal{O}$. Let $\eta_0 = \eta_0(x)$ be a function satisfying

$$\begin{cases} \eta_0 \in C^2(\overline{\Omega}),\ \eta_0 > 0 \text{ in } \Omega,\ \eta_0 = 0 \text{ on } \partial\Omega, \\ |\nabla \eta_0| > 0 \text{ in } \overline{\Omega} \setminus \omega. \end{cases}$$

Such a function η_0 always exists; see [38].

Let us introduce the weight functions

$$\alpha(x,t) = \frac{e^{2\lambda\|\eta^0\|_\infty} - e^{\lambda(\|\eta^0\|_\infty+\eta^0(x))}}{t(T-t)}, \quad \xi(x,t) = \frac{e^{\lambda(\|\eta^0\|_\infty+\eta^0(x))}}{t(T-t)} \tag{1.31}$$

and the notation

$$\begin{aligned} I_m(\psi) := {} & s^{m-4}\lambda^{m-3}\iint_Q e^{-2s\alpha}\xi^{m-4}(|\psi_t|^2 + |\Delta\psi|^2)\,dx\,dt \\ & + s^{m-2}\lambda^{m-1}\iint_Q e^{-2s\alpha}\xi^{m-2}|\nabla\psi|^2\,dx\,dt \\ & + s^m\lambda^{m+1}\iint_Q e^{-2s\alpha}\xi^m|\psi|^2\,dx\,dt. \end{aligned}$$

From the usual Carleman inequalities (see [38, 54, 33]), we have:

$$\begin{aligned} I_3(\psi) \le C \Bigg(& \iint_{\Omega\times(0,T)} e^{-2s\alpha}|\alpha_1\gamma^1 1_{\mathcal{O}_{1,d}} + \alpha_2\gamma^2 1_{\mathcal{O}_{2,d}}|^2\,dx\,dt \\ & + s^3\lambda^4\iint_{\omega\times(0,T)} e^{-2s\alpha}\xi^3|\psi|^2\,dx\,dt\Bigg). \end{aligned} \tag{1.32}$$

Since (1.10) holds, we introduce $h := \alpha_1\gamma^1 + \alpha_2\gamma^2$. One has

$$\begin{aligned} & \iint_Q e^{-2s\alpha}|h|^2\,dx\,dt \le I_0(h) \\ & \quad \le C\left(s^{-3}\lambda^{-2}\iint_Q e^{-2s\alpha}\xi^{-3}|\psi|^2\,dx\,dt + \lambda\iint_{\omega\times(0,T)} e^{-2s\alpha}|h|^2\,dx\,dt\right) \end{aligned} \tag{1.33}$$

for all large s and λ and some C only depending on Ω, ω and T.

But, in $\omega\times(0,T)$, one has $h = -\psi_t - \Delta\psi + a\psi$. Consequently, by introducing an appropriate cut-off function ζ and integrating by parts, we get

$$\begin{aligned} \iint_{\omega\times(0,T)} e^{-2s\alpha}|h|^2\,dx\,dt & \le \iint_{\omega'\times(0,T)} \zeta\, e^{-2s\alpha}h\,(-\psi_t - \Delta\psi + a\psi)\,dx\,dt \\ & \le \varepsilon I_0(h) + C_\varepsilon s^4\lambda^5\iint_{\omega'\times(0,T)} \xi^4 e^{-2s\alpha}|\psi|^2\,dx\,dt, \end{aligned} \tag{1.34}$$

where ω' is a new open set satisfying $\omega \subset \omega' \subset \mathcal{O}_d \cap \mathcal{O}$. From (1.32), (1.33) and (1.34), we find that, for some $C > 0$,

$$I_3(\psi) + I_0(h) \le C\iint_{\omega'\times(0,T)} \xi^4 e^{-2s\alpha}|\psi|^2\,dx\,dt. \tag{1.35}$$

Let $\hat\rho = \hat\rho(x,t)$ be a positive nondecreasing C^1 function which blows up at $t = T$. From the PDE satisfied by γ^i in (1.25), we readily see that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega \hat\rho^{-2}|\gamma^i|^2\,dx + \int_\Omega \hat\rho^{-2}|\nabla\gamma^i|^2\,dx & = -\frac{1}{\mu_i}\int_{\mathcal{O}_i} \hat\rho^{-2}\psi\gamma^i\,dx - \int_\Omega \hat\rho^{-3}\hat\rho_t|\gamma^i|^2\,dx \\ & \le \frac{1}{\mu_i^2}\int_{\mathcal{O}_i} \hat\rho^{-2}|\psi|^2\,dx + \int_\Omega \hat\rho^{-2}|\gamma^i|^2\,dx \end{aligned}$$

and, using Gronwall Lemma and the fact that $\gamma^i(x,0) \equiv 0$, it follows that

$$\left(\int_\Omega \hat\rho^{-2}|\gamma^i|^2\,dx\right)(\tau) \le C \iint_{\mathcal{O}_i\times(0,T)} \hat\rho^{-2}|\psi|^2\,dx\,dt \tag{1.36}$$

for all $\tau \in [0,T]$.

Let us choose $\hat\rho$ satisfying $\hat\rho > \xi^{-3/2}e^{s\alpha}$ in Q; then, the right hand side of (1.36) is bounded, up to a multiplicative constant, by $I_3(\psi)$. Therefore, in view of (1.35) and (1.36), we see that

$$I_3(\psi) + \sum_{i=1}^{2} \iint_Q \hat\rho^{-2}|\gamma^i|^2\,dx\,dt \le C \iint_{\mathcal{O}\times(0,T)} \xi^4 e^{-2s\alpha}|\psi|^2\,dx\,dt. \tag{1.37}$$

Observe that the choice of $\hat\rho$ is determined by the Carleman weight $\xi^{-\frac{3}{2}}e^{s\alpha}$, that depends on Ω, $\mathcal{O}$, T and $\|a\|_{L^\infty(Q)}$; but $\hat\rho$ can be chosen independent of the $\mathcal{O}$, $\mathcal{O}_d$, α_i and μ_i.

To end the proof, we need an energy estimate for ψ.

Multiplying the first PDE in (1.25) by ψ and integrating in $\Omega\times(\tau,t)$, we have

$$\begin{aligned}\|\psi(\cdot\,,\tau)\|^2 - \|\psi(\cdot\,,t)\|^2 + \iint_{\Omega\times(\tau,t)} |\nabla\psi|^2\,dx\,ds \\ \le C\int_\tau^t \|\psi(\cdot,s)\|^2\,ds + C\int_\tau^t \|(\alpha_1\gamma^1+\alpha_2\gamma^2)1_{\mathcal{O}_d}\|^2\,ds\end{aligned}$$

for all $\tau, t \in [0,T]$, with $\tau \le t$. For the γ^i, in view of the second and third PDE in (1.25), we get:

$$\|\gamma^i(\cdot\,,s)\|^2 - \|\gamma^i(\cdot\,,\tau)\|^2 \le C\int_\tau^s \|\gamma^i(\cdot\,,\sigma)\|^2\,d\sigma + \frac{C}{\mu_i^2}\iint_{\mathcal{O}_i\times(\tau,s)} |\psi(x,\sigma)|^2\,dx\,d\sigma \tag{1.38}$$

for all $s \in [\tau,t]$. Using again Gronwall Lemma, the following is found:

$$\|\gamma^i(\cdot\,,s)\|^2 \le C\left(\|\gamma^i(\cdot\,,\tau)\|^2 + \int_\tau^s \|\psi(\cdot\,,\sigma)\|^2\,d\sigma\right).$$

Consequently,

$$\|\psi(\cdot\,,\tau)\|^2 \le \|\psi(\cdot\,,t)\|^2 + C\left[\int_\tau^t \|\psi(\cdot\,,s)\|^2\,ds + \sum_{i=1}^{2}\|\gamma^i(\cdot\,,\tau)\|^2\right]$$

for all $\tau, t \in [0,T]$, with $\tau \le t$, whence

$$\|\psi(\cdot\,,\tau)\|^2 \le C\left(\|\psi(\cdot\,,t)\|^2 + \sum_{i=1}^{2}\|\gamma^i(\cdot\,,\tau)\|^2\right).$$

In particular, we find that

$$\|\psi(\cdot\,,0)\|^2 \le C\|\psi(\cdot\,,t)\|^2, \quad \forall t \in [0,T].$$

This yields

$$\|\psi(\cdot\,,0)\|^2 \le \frac{C}{T}\iint_{\Omega\times(T/4,3T/4)} |\psi|^2\,dx\,dt \le CI_3(\psi).$$

Combining the last inequality and (1.37), we deduce (3.4). □

Remark 2. If, instead of (1.10), we suppose that

$$\mathcal{O}_i \subset \mathcal{O} \text{ for } i = 1, 2,$$

the same result holds. Indeed, we have from (1.38) that

$$\|\gamma^i(\cdot\,, s)\|^2 \leq C \iint_{\mathcal{O}_i \times (0,s)} |\psi(x,\sigma)|^2 \, dx \, d\sigma \leq C \iint_{\mathcal{O} \times (0,s)} |\psi(x,\sigma)|^2 \, dx \, d\sigma.$$

By replacing this inequality in the first term on the right hand side of (1.32) and taking into account (1.36), we get easily (1.37). □

1.3 The semilinear case

In this section, we analyze the controllability of a more general model, with a not necessarily vanishing function F. Our goals are to prove Theorem 2 and Proposition 1.

1.3.1 Characterization of Nash quasi-equilibria

As already mentioned in Section 1.1, in the semilinear case, the convexity of the functionals J_i is lost. Consequently, it is not clear whether the definition of Nash equilibria used in the linear case is the good one. For this reason, we must re-define the concept of Nash optimality (recall Definition 1).

Notice that (1.5)–(1.6) is equivalent to

$$\left\{ \begin{array}{l} \alpha_i \displaystyle\iint_{\mathcal{O}_d \times (0,T)} (y - y_{i,d}) \, p^i \, dx \, dt + \mu_i \iint_{\mathcal{O}_i \times (0,T)} v^i \hat{v}^i dx dt = 0 \\ \forall \hat{v}^i \in \mathcal{H}_i; \quad v^i \in \mathcal{H}_i, \quad i = 1, 2, \end{array} \right. \tag{1.39}$$

where we have denoted by p^i the derivative of the state y with respect to v^i in the direction $\hat{v}^i$. Obviously, one has

$$\left\{ \begin{array}{lll} p^i_t - \Delta p^i + a(x,t) p^i = F'(y) p^i + \hat{v}^i 1_{\mathcal{O}_i} & \text{in} & Q, \\ p^i = 0 & \text{on} & \Sigma, \\ p^i(x,0) = 0 & \text{in} & \Omega. \end{array} \right.$$

Let us introduce the adjoint systems

$$\left\{ \begin{array}{lll} -\phi^i_t - \Delta \phi^i + a(x,t) \phi^i = F'(y) \phi^i + \alpha_i (y - y_{i,d}) 1_{\mathcal{O}_d} & \text{in} & Q, \\ \phi^i = 0 & \text{on} & \Sigma, \\ \phi^i(x,T) = 0 & \text{in} & \Omega. \end{array} \right.$$

Then, a short computation shows that (1.39) can be written equivalently as

$$\iint_{\mathcal{O}_i \times (0,T)} (\phi^i + \mu_i v^i) \, \hat{v}^i \, dx \, dt = 0 \;\; \forall \hat{v}^i \in \mathcal{H}_i; \quad v^i \in \mathcal{H}_i, \quad i = 1, 2.$$

As a consequence, we get the following characterization of any Nash quasi-equilibrium:

$$v^i = -\frac{1}{\mu_i}\phi^i\Big|_{\mathcal{O}_i\times(0,T)}, \quad i = 1,2, \tag{1.40}$$

with

$$\begin{cases} y_t - \Delta y + a(x,t)y = F(y) + f1_{\mathcal{O}} - \dfrac{1}{\mu_1}\phi^1 1_{\mathcal{O}_1} - \dfrac{1}{\mu_2}\phi^2 1_{\mathcal{O}_2} & \text{in} \quad Q, \\ -\phi^i_t - \Delta\phi^i + a(x,t)\phi^i = F'(y)\phi^i + \alpha_i(y - y_{i,d})1_{\mathcal{O}_{i,d}} & \text{in} \quad Q, \\ y = 0, \quad \phi^i = 0 & \text{on} \quad \Sigma, \\ y(x,0) = y^0(x), \quad \phi^i(x,T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.41}$$

1.3.2 Proof of Theorem 2

The proof of Theorem 2 follows some arguments that are nowadays standard and well known; see [38, 89]. It is divided in three steps: first, we perform a change of variable that reduces the task to solve a null controllability problem; then, this is rewritten as a fixed-point equation in $L^2(Q)$; in particular, we use again Carleman inequalities and energy estimates to deduce an observability inequality for the adjoint of a linearized system; finally, in a third step, we use some compactness properties of the system and we prove the existence of a fixed-point.

Step 1: We must find a leader control $f \in L^2(\mathcal{O}\times(0,T))$ such that the solution (y, ϕ^1, ϕ^2) to (1.41) satisfies (1.9). In fact, by introducing the change of variable $z = y - \overline{y}$, we can rewrite (1.41) in the form

$$\begin{cases} z_t - \Delta z + a(x,t)z = G(x,t;z)z + f1_{\mathcal{O}} - \dfrac{1}{\mu_1}\phi^1 1_{\mathcal{O}_1} - \dfrac{1}{\mu_2}\phi^2 1_{\mathcal{O}_2} & \text{in} \quad Q, \\ -\phi^i_t - \Delta\phi^i + a(x,t)\phi^i = F'(z+\overline{y})\phi^i + \alpha_i(z - z_{i,d})1_{\mathcal{O}_d} & \text{in} \quad Q, \\ z = 0, \quad \phi^i = 0 & \text{on} \quad \Sigma, \\ z(x,0) = z^0(x), \quad \phi^i(x,T) = 0 & \text{in} \quad \Omega, \end{cases} \tag{1.42}$$

where $z_{i,d} := y_{i,d} - \overline{y}$, $z^0 = y^0 - \overline{y}(\cdot\,,0)$ and

$$G(x,t;z) = \int_0^1 F'(\overline{y}(x,t) + \sigma z)\,d\sigma.$$

Obviously, what we have to prove is the null controllability for z in (1.42).

Step 2: For each $z \in L^2(Q)$ and each $f \in L^2(\mathcal{O}\times(0,T))$, let us introduce the linear system

$$\begin{cases} w_t - \Delta w + a(x,t)w = G(x,t;z)w + f1_{\mathcal{O}} - \dfrac{1}{\mu_1}\phi^1 1_{\mathcal{O}_1} - \dfrac{1}{\mu_2}\phi^2 1_{\mathcal{O}_2} & \text{in} \quad Q, \\ -\phi^i_t - \Delta\phi^i + a(x,t)\phi^i = F'(z+\overline{y})\phi^i + \alpha_i(w - z_{i,d})1_{\mathcal{O}_d} & \text{in} \quad Q, \\ w = 0, \quad \phi^i = 0 & \text{on} \quad \Sigma, \\ w(x,0) = z^0, \quad \phi^i(x,T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.43}$$

By assumption, there exists $K > 0$ such that

$$|G(x,t;s)| + |F'(s)| \le K \quad \forall (x,t,s) \in Q \times \mathbb{R}.$$

Note that, arguing as in Section 2.4, it can be proved that, if μ_1 and μ_2 are sufficiently large, (1.43) possesses exactly one solution for each $f \in L^2(\mathcal{O} \times (0,T))$. Furthermore, one has

$$\|w\|_{L^2(0,T;H_0^1(\Omega))} + \|w_t\|_{L^2(0,T;H^{-1}(\Omega))} \le C\left(1 + \|f\|_{L^2(\mathcal{O}\times(0,T))}\right), \tag{1.44}$$

where C depends on Ω, $\mathcal{O}$, T, $\mathcal{O}_d$, α_i, μ_i, K, $\|a\|_{L^\infty(Q)}$ and $\|z_0\|$.

Let us introduce the mapping $\Lambda : L^2(Q) \mapsto L^2(Q)$, with $\Lambda(z) = w_z$ for all $z \in L^2(Q)$, where w_z is the state associated to the minimal L^2 norm null control f_z for the linear system (1.43). In other words, w_z is, together with ϕ_z^1, ϕ_z^2 and f_z, the unique solution to (1.43) and f_z minimizes (1.3) subject to the constraint

$$w(x,T) = 0 \ \text{ in } \ \Omega.$$

The existence and uniqueness of a solution to (1.43) proves that Λ is well defined.

The goal is now to prove the null controllability for w in (1.43). To this purpose, we will make use of a suitable global Carleman inequality for the solutions to the adjoint system, that is,

$$\begin{cases} -\psi_{z,t} - \Delta\psi_z + a(x,t)\psi_z = G(x,t;z)\psi_z + (\alpha_1\gamma_z^1 + \alpha_2\gamma_z^2)1_{\mathcal{O}_d} & \text{in} \quad Q, \\ \gamma_{z,t}^i - \Delta\gamma_z^i = F'(z+\overline{y})\gamma_z^i - \dfrac{1}{\mu_i}\psi_z 1_{\mathcal{O}_i} & \text{in} \quad Q, \\ \psi_z = 0, \quad \gamma_z^i = 0 & \text{on} \quad \Sigma, \\ \psi_z(x,T) = \psi^T, \quad \gamma_z^i(x,0) = 0 & \text{in} \quad \Omega. \end{cases}$$

In this context, we have the following:

Proposition 4. *There exist a constant $C > 0$, only depending on Ω, $\mathcal{O}$, T, $\mathcal{O}_i$, $\mathcal{O}_d$, α_i, μ_i, K and $\|a\|_{L^\infty(Q)}$ and a weight function $\hat\rho = \hat\rho(x,t)$, only depending on Ω, $\mathcal{O}$, T, K and $\|a\|_{L^\infty(Q)}$, such that the following observability inequality holds true for any $\psi^T \in L^2(\Omega)$ and any $z \in L^2(Q)$:*

$$\int_\Omega |\psi_z(x,0)|^2\,dx + \sum_{i=1}^{2} \iint_Q \hat\rho^{-2}|\gamma_z^i|^2\,dx\,dt \le C \iint_{\mathcal{O}\times(0,T)} |\psi_z|^2\,dx\,dt.$$

The proof is almost identical to the proof of Proposition 3 and, for brevity, is omitted.

This result leads, as in Section 1.2.2, to the existence of a minimal norm null control $f_z \in L^2(\mathcal{O}\times(0,T))$ for (1.43). Furthermore it is clear that there exists a positive constant C, only depending on Ω, $\mathcal{O}$, T, $\mathcal{O}_i$, $\mathcal{O}_d$, α_i, μ_i, K, $\|a\|_{L^\infty(Q)}$ and $\|z_0\|$, such that

$$\|f_z\|^2_{L^2(\mathcal{O}\times(0,T))} \le C, \quad \forall z \in L^2(Q). \tag{1.45}$$

Step 3: Taking into account (1.44) and (1.45) , we see that w_z is uniformly bounded in $L^2(0,T;H_0^1(\Omega))$ and $w_{z,t}$ is uniformly bounded in $L^2(0,T,H^{-1}(\Omega))$. In view of the classical Aubin-Lions Compactness Theorem, this means that Λ maps the whole space $L^2(Q)$ into a compact set.

On the other hand, the mapping $z \mapsto \Lambda(z)$ is obviously continuous. Therefore, we can use Schauder Fixed-Point Theorem to ensure the semilinear controllability result.

This ends the proof of Theorem 2.

1.3.3 Equilibria and quasi-equilibria

The aim of this subsection is to prove Proposition 1, that is, to investigate whether, in the semilinear case, we may have a Nash equilibrium. Let us show that the answer is positive when $F \in W^{2,\infty}(\mathbb{R})$.

Let $f \in L^2(\mathcal{O} \times (0,T))$ be given and let (v^1, v^2) be the associated Nash quasi-equilibrium. Note that, for any $s \in \mathbb{R}$ and $(w^1, w^2) \in \mathcal{H}$,

$$\begin{aligned}\langle D_1 J_1(f; v^1 + sw^1, v^2), w^2\rangle - \langle D_1 J_1(f; v^1, v^2), w^2\rangle = s\mu_1 \iint\limits_{\mathcal{O}_1\times(0,T)} w^1 w^2 \, dx\, dt \\ +\alpha_1 \iint\limits_{\mathcal{O}_d\times(0,T)} (y^s - y_{1,d}) p^s \, dx\, dt - \alpha_1 \iint\limits_{\mathcal{O}_d\times(0,T)} (y - y_{1,d}) p \, dx\, dt,\end{aligned} \tag{1.46}$$

where

$$\begin{cases} y^s_t - \Delta y^s + a(x,t) y^s = F(y^s) + f 1_{\mathcal{O}} + (v^1 + sw^1) 1_{\mathcal{O}_1} + v^2 1_{\mathcal{O}_2} & \text{in} \quad Q, \\ y^s = 0 & \text{on} \quad \Sigma, \\ y^s(x,0) = y_0 & \text{in} \quad \Omega, \end{cases} \tag{1.47}$$

p^s is the derivative of y^s with respect to v^1 in the direction w^2, i.e. the solution to

$$\begin{cases} p^s_t - \Delta p^s + a(x,t) p^s = F'(y^s) p^s + w^2 1_{\mathcal{O}_1} & \text{in} \quad Q, \\ p^s = 0 & \text{on} \quad \Sigma, \\ p^s(x,0) = 0 & \text{in} \quad \Omega \end{cases} \tag{1.48}$$

and we have used the notation $y = y^s|_{s=0}$ and $p = p^s|_{s=0}$.

Let us introduce the adjoint of (1.48):

$$\begin{cases} -\phi^s_t - \Delta \phi^s + a(x,t)\phi^s = F'(y^s)\phi^s + \alpha_1 (y^s - y_{1,d}) 1_{\mathcal{O}_d} & \text{in} \quad Q, \\ \phi^s = 0 & \text{on} \quad \Sigma, \\ \phi^s(x,T) = 0 & \text{in} \quad \Omega \end{cases} \tag{1.49}$$

and let us also set $\phi = \phi^s|_{s=0}$.

Then, we can replace (1.49) in (1.46) and use integration by parts to obtain the following identity:

$$\begin{aligned}\langle D_1 J_1(f; v^1 + sw^1, v^2), w^2\rangle - \langle D_1 J_1(f; v^1, v^2), w^2\rangle = s\mu_1 \iint\limits_{\mathcal{O}_1\times(0,T)} w^1 w^2 \, dx\, dt \\ + \iint\limits_{\mathcal{O}_d\times(0,T)} (\phi^s - \phi) w^2 \, dx\, dt.\end{aligned}$$

Notice that

$$\begin{aligned}-(\phi^s - \phi)_t - \Delta(\phi^s - \phi) + a(x,t)(\phi^s - \phi) = \\ [F'(y^s) - F'(y)]\phi^s + F'(y)(\phi^s - \phi) + \alpha_1 (y^s - y) 1_{\mathcal{O}_d}.\end{aligned}$$

Consequently, the limits

$$\eta = \lim_{s\to 0} \frac{1}{s}(\phi^s - \phi) \quad \text{and} \quad h = \lim_{s\to 0} \frac{1}{s}(y^s - y)$$

exist and satisfy

$$\begin{cases} -\eta_t - \Delta\eta + a(x,t)\eta = F''(y)h\phi + F'(y)\eta + \alpha_1 h 1_{\mathcal{O}_d} & \text{in} \quad Q, \\ h_t - \Delta h + a(x,t)h = F'(y)h + w^1 1_{\mathcal{O}_1} & \text{in} \quad Q, \\ \eta = h = 0, & \text{on} \quad \Sigma, \\ \eta(x,T) = h(x,0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.50}$$

Thus, from (1.50), we deduce that

$$\langle D_1^2 J_1(f; v^1, v^2), (w^1, w^2)\rangle = \mu_1 \iint_{\mathcal{O}_1\times(0,T)} w^1 w^2 \, dx\, dt + \iint_{\mathcal{O}_1\times(0,T)} \eta w^2 \, dx\, dt.$$

In particular, for all $w^1 \in L^2(\mathcal{O}_1 \times (0,T))$, one has:

$$\langle D_1^2 J_1(f; v^1, v^2), (w^1, w^1)\rangle = \mu_1 \iint_{\mathcal{O}_1\times(0,T)} |w^1|^2 \, dx\, dt + \iint_{\mathcal{O}_1\times(0,T)} \eta w^1 \, dx\, dt. \tag{1.51}$$

Let $M > 0$ be such that $|F''(s)| \le M$ a.e. in $\mathbb{R}$. Let us show that, for some C only depending on Ω, $\mathcal{O}$, T, $\mathcal{O}_i$, $\mathcal{O}_d$, α_i, M, K, $\|a\|_{L^\infty(\Omega)}$ and $\|y_0\|$, we have

$$\left| \iint_{\mathcal{O}_1\times(0,T)} \eta w^1 \, dx\, dt \right| \le C(1 + \|f\|_{L^2(\mathcal{O}\times(0,T))})\|w^1\|_{\mathcal{H}_1}, \quad \forall w^1 \in L^2(\mathcal{O}_1 \times (0,T)). \tag{1.52}$$

From standard energy estimates, since $F' \in L^\infty(Q)$, we have

$$\int_\Omega |h(x,t)|^2 \, dx + \iint_Q |\nabla h|^2 \, dx \le C \iint_{\mathcal{O}_1\times(0,T)} |w^1|^2 \, dx\, dt.$$

Using the PDE's in (1.50), we also get the following:

$$\begin{aligned} \iint_{\mathcal{O}_1\times(0,T)} \eta w^1 \, dx\, dt &= \iint_Q (h_t - \Delta h + a(x,t)h - F'(y)h)\eta \, dx\, dt \\ &= \iint_Q h(-\eta_t - \Delta\eta + a(x,t)\eta - F'(y)\eta) \, dx\, dt \\ &= \iint_Q (F''(y)h\phi + \alpha_1 h 1_{\mathcal{O}_d})h \, dx\, dt \\ &= \iint_Q (F''(y)|h|^2\phi + \alpha_1 |h|^2 1_{\mathcal{O}_d}) \, dx\, dt. \end{aligned} \tag{1.53}$$

Let us first assume that $y_0 \in H_0^1(\Omega)$. The idea is to find r and s such that

$$\phi \in L^r(0,T; L^s(\Omega)) \quad \text{and} \quad h \in L^{2r'}(0,T; L^{2s'}(\Omega)), \tag{1.54}$$

where r' and s' are the conjugate of r and s, respectively. This will make possible to bound from above the last integral in (1.53).

It is clear that $h \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega))$. For this reason, it is natural to ask for which values of α and β the following embedding holds:

$$L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega)) \hookrightarrow L^\alpha(0,T;L^\beta(\Omega)). \tag{1.55}$$

By interpolation, we have that, for each $0 < \theta < 1$, (1.55) holds when

$$\frac{1}{\alpha} = \frac{\theta}{2} \text{ and } \frac{1}{\beta} = \frac{(N-4)\theta}{2N} + \frac{(N-2)(1-\theta)}{2N} = \frac{\alpha(N-2)-4}{2\alpha N}.$$

Taking $\alpha = 2r'$ and $\beta = 2s'$, we conclude that $r = \alpha/(\alpha-2)$ and $s = \alpha N/2(\alpha+2)$.

Analogously, we have that $y \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega)) \hookrightarrow L^a(0,T;L^b(\Omega))$, with $b = 2aN/(a(N-2)-4)$. Using the regularity results of the heat equation and the fact that $y_{d,i} \in L^\infty(\mathcal{O}_{d,i} \times (0,T))$, it follows that

$$\phi \in L^a(0,T;W^{2,b}(\Omega)) \hookrightarrow L^a(0,T;L^{\frac{Nb}{N-2b}}(\Omega)) = L^a(0,T;L^{\frac{2aN}{aN-6a-4}}(\Omega)).$$

If $a = r = \alpha/(\alpha-2)$, we get $\phi \in L^r(0,T;L^{\frac{2\alpha N}{\alpha N-10\alpha+8}}(\Omega))$. To finish, we must have $L^{\frac{2\alpha N}{\alpha N-10\alpha+8}}(\Omega) \hookrightarrow L^s(\Omega)$, which is equivalent to

$$\frac{\alpha N}{2(\alpha+2)} \leq \frac{2\alpha N}{\alpha(N-10)+8}.$$

And we see that this inequality holds true if and only if $N \leq 14$.

Thus, from (1.53), (1.49) for $s = 0$, (1.40), (1.41), (1.47) for $s = 0$ and the estimates at Subsection 1.3.2, we see that, if $y_0 \in H_0^1(\Omega)$ and $N \leq 14$,

$$\begin{aligned}
\left| \iint_{\mathcal{O}_1 \times (0,T)} \eta w^1 \, dx \, dt \right| &\leq M \|h\|^2_{L^{2r'}(0,T;L^{2s'}(\Omega))} \|\phi\|_{L^r(0,T;L^s(\Omega))} \\
&\quad + \alpha_1 \|h\|^2_{L^2(\mathcal{O}_d \times (0,T))} \\
&\leq C(\|\phi\|_{L^r(0,T;L^s(\Omega))} + 1)\|w^1\|^2_{\mathcal{H}_1} \\
&\leq C(\|y\|_{L^2(Q)} + 1)\|w^1\|^2_{\mathcal{H}_1} \\
&\leq C\left(\sum_{i=1}^2 \frac{1}{\mu_i}\|\phi^i\|_{\mathcal{H}_i} + \|f\| + \|y_0\| + 1\right)\|w^1\|^2_{\mathcal{H}_1} \\
&\leq C(1 + \|f\|)\|w^1\|^2_{\mathcal{H}_1}.
\end{aligned}$$

This proves (1.52) in this case.

Now, let us assume that we have $y_0 \in L^2(\Omega)$. As in the first situation, the idea is to find r and s such that (2.11) holds. Since the regularity of η does not depend on the data y_0, we still have $\eta \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega))$ and, therefore, $\eta \in L^\alpha(0,T;L^\beta(\Omega))$, where α and β are as above. In this case, we have by a interpolation argument that $y \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \hookrightarrow L^{\bar a}(0,T;L^{\bar b}(\Omega))$, where $\bar a \geq 2$ and $\bar b = 2N\bar a/(\bar a N - 4)$. Using again parabolic regularity, we get

$$\phi \in L^{\bar a}(0,T;W^{2,\bar b}(\Omega)) \hookrightarrow L^{\bar a}(0,T;L^{\frac{N\bar b}{N-2\bar b}}(\Omega)) = L^{\bar a}(L^{\frac{2\bar a N}{\bar a(N-4)-4}}(\Omega)).$$

If $\bar{a} = r = \alpha/(\alpha-2)$, we have $\phi \in L^r(L^{\frac{2\alpha N}{\alpha(N-8)+8}}(\Omega))$. To finish the proof, we must have $L^{\frac{2\alpha N}{\alpha(N-8)+8}}(\Omega) \hookrightarrow L^s(\Omega)$, which is equivalent to

$$\frac{\alpha N}{2(\alpha+2)} \leq \frac{2\alpha N}{\alpha(N-8)+8}.$$

Since this holds if and only if $N \leq 12$, the estimate (1.52) is proved also in this case.

Taking into account (1.51) and (1.52), we see that

$$\langle D_1^2 J_1(f; v^1, v^2), (w^1, w^1)\rangle \geq \left(\mu_1 - C(1 + \|f\|_{L^2(\mathcal{O}\times(0,T))})\right) \|w^1\|^2_{\mathcal{H}_1}\, dx\, dt.$$

Note that the previous constant C can be chosen independent of μ_1 and μ_2.

In a similar way, it can be shown that, under the previous assumption on y_0 and N,

$$\left\langle D_2^2 J_2(f; v^1, v^2), (w^2, w^2)\right\rangle \geq \left(\mu_2 - C(1 + \|f\|_{L^2(\mathcal{O}\times(0,T))})\right) \|w^2\|^2_{\mathcal{H}_2}\, dx\, dt.$$

for another constant C independent of μ_1 and μ_2.

It is now clear that, for sufficiently large μ_1 and μ_2, the couple (v^1, v^2) is a Nash equilibrium in the sense of (1.4).

1.4 The case with restrictions

In this section, we prove Theorem 3.

We return to the Stackelberg-Nash null controllability problem for a linear parabolic PDE, but we impose some restrictions: the followers (v^1, v^2) are supposed to minimize the functionals (1.2) subject to the convex constraints $v^i \in \mathcal{U}_i$, where the $\mathcal{U}_i$ are given by (1.13).

This is a more difficult problem. The search of a pair (v^1, v^2) satisfying (1.4), where the minimizations are performed in $\mathcal{U}_{1,d}$ and $\mathcal{U}_{2,d}$, is equivalent to the following:

$$D_1 J_1(f; v^1, v^2)(\hat{v}^1 - v^1, 0) \geq 0 \quad \forall \hat{v}^1 \in \mathcal{U}_{1,d}; \quad v^1 \in \mathcal{U}_{1,d} \tag{1.56}$$

and

$$D_2 J_2(f; v^1, v^2)(0, \hat{v}^2 - v^2) \geq 0, \quad \forall \hat{v}^2 \in \mathcal{U}_{2,d}; \quad v^2 \in \mathcal{U}_{2,d}. \tag{1.57}$$

As in Section 1.2, with the change of variable $z = y - \bar{y}$, we are led to a null controllability problem. Then, we see that (1.56)-(1.57) is equivalent to

$$\left\{ \begin{array}{l} \alpha_i \displaystyle\iint_{\mathcal{O}_{i,d}\times(0,T)} (z - z_{i,d})\, w^i\, dx\, dt + \mu_i \iint_{\mathcal{O}_i\times(0,T)} v^i(\hat{v}^i - v^i)\, dx\, dt \geq 0 \\ \forall \hat{v}^i \in \mathcal{U}_{i,d}; \quad v^i \in \mathcal{U}_{i,d}, \end{array} \right. \tag{1.58}$$

where w^i is the derivative of z with respect to $\hat{v}_i$ in the direction v_i, that is to say, the solution to

$$\left\{ \begin{array}{lll} w^i_t - \Delta w^i + a(x,t) w^i = v_i 1_{\mathcal{O}_i} & \text{in} & Q, \\ w^i = 0 & \text{on} & \Sigma, \\ w^i(x,0) = 0 & \text{in} & \Omega. \end{array} \right. \tag{1.59}$$

The adjoint system associated to (1.59) is given by

$$\left\{\begin{array}{ll} -\phi^i_t - \Delta\phi^i + a(x,t)\phi^i = \alpha_i(z - z_{i,d})1_{\mathcal{O}_{i,d}} & \text{in} \quad Q, \\ \phi = 0 & \text{on} \quad \Sigma, \\ \phi(x,T) = 0 & \text{in} \quad \Omega. \end{array}\right.$$

Replacing the equation satisfied by ϕ^i in (1.58), we obtain

$$\iint_{\mathcal{O}_i\times(0,T)} \left(\phi^i + \mu_i v^i\right)(\hat{v}^i - v^i)\,dx\,dt \geq 0, \quad \forall \hat{v}^i \in \mathcal{U}_{i,d}; \quad v^i \in \mathcal{U}_{i,d}, \quad i = 1,2. \tag{1.60}$$

Now, by introducing the projectors $\mathbb{P}_{\mathcal{U}_{i,d}} : L^2(\mathcal{O}_i \times (0,T)) \mapsto \mathcal{U}_{i,d}$, we see that (1.60) can be rewritten equivalently in the form

$$v^i = \mathbb{P}_{\mathcal{U}_{i,d}}(-\frac{1}{\mu_i}\phi^i|_{\mathcal{O}_i\times(0,T)}), \quad i = 1,2.$$

We may group all this information to get the following system:

$$\left\{\begin{array}{ll} z_t - \Delta z + a(x,t)z = f1_{\mathcal{O}} + \displaystyle\sum_{i=1}^{2} \mathbb{P}_{\mathcal{U}_{i,d}}(-\frac{1}{\mu_i}\phi^i|_{\mathcal{O}_i\times(0,T)}) & \text{in} \quad Q, \\ -\phi^i_t - \Delta\phi^i + a(x,t)\phi^i = \alpha_i(z - z_{i,d})|_{\mathcal{O}_{i,d}} & \text{in} \quad Q, \\ z = 0, \quad \phi^i = 0 & \text{on} \quad \Sigma, \\ z(x,0) = z^0, \quad \phi^i(x,T) = 0 & \text{in} \quad \Omega. \end{array}\right. \tag{1.61}$$

Let us prove that, under the assumptions (1.19), for each $f \in L^2(\mathcal{O} \times (0,T))$ there exists exactly one solution to (1.61), i.e. there exists a unique Nash equilibrium (v^1, v^2) in $\mathcal{U}_{1,d}\times\mathcal{U}_{2,d}$.

Indeed, notice that (1.60) can also be rewritten in the form

$$\left\{\begin{array}{l} (\mathbb{L}(v^1,v^2), (\hat{v}^1,\hat{v}^2) - (v^1,v^2)) \geq (\Psi, (\hat{v}^1,\hat{v}^2) - (v^1,v^2))_{\mathcal{H}} \\ \forall \ (v^1,v^2) \in \mathcal{U}_{1,d} \times \mathcal{U}_{2,d}; \quad (\hat{v}^1,\hat{v}^2) \in \mathcal{U}_{1,d} \times \mathcal{U}_{2,d}, \end{array}\right. \tag{1.62}$$

where $\mathbb{L}$ and Ψ are respectively given by (1.16) and (1.18). If μ_1 and μ_2 satisfy (1.19), $\mathbb{L}$ is a coercive continuous bilinear form on $\mathcal{H}$, whence (1.62) is uniquely solvable.

Furthermore, it is clear that the couple (v^1, v^2) and the associated state z satisfy (again) the estimates (1.21) and (1.22). As in the semilinear case, we will analyze and solve the null controllability problem for (1.61) by a fixed-point method.

To this end, note that the projectors $\mathbb{P}_{\mathcal{U}_{i,d}}$ are given as follows:

$$\mathbb{P}_{\mathcal{U}_{i,d}}(k)(x,t) = \left\{\begin{array}{ll} k(x,t) & \text{if} \ \ k(x,t) \in I_i, \\ P_i(k(x,t)) & \text{otherwise}, \end{array}\right.$$

for (x,t) a.e. in $\mathcal{O}_i \times (0,T)$, where $P_i : \mathbb{R} \mapsto I_i$ is the usual projector on the interval I_i. Also, note that, for every $k \in \mathcal{H}_i$, $\mathbb{P}_{\mathcal{U}_{i,d}}$ can be written in the form $\mathbb{P}_{\mathcal{U}_{i,d}}(k) = q_i(k)k$, where the function $k \mapsto q_i(k)$ is continuous on $\mathcal{H}_i$ and

$$\|q_i(k)\|_\infty \leq C, \quad \forall k \in \mathcal{H}_i.$$

Therefore, the controllability problem is reduced to find $f \in L^2(\mathcal{O} \times (0,T))$ such that the solution for

$$\begin{cases} z_t - \Delta z + a(x,t)z = f1_{\mathcal{O}} - \displaystyle\sum_{i=1}^{2} \tilde{q}_i(\phi^i)\phi^i 1_{\mathcal{O}_i} & \text{in} \quad Q, \\ -\phi^i_t - \Delta\phi^i + a(x,t)\psi = \alpha_i(z - z_{i,d})1_{\mathcal{O}_{i,d}} & \text{in} \quad Q, \\ z = 0, \quad \phi^i = 0 & \text{on} \quad \Sigma, \\ z(x,0) = z^0, \quad \phi^i(x,T) = 0 & \text{in} \quad \Omega, \end{cases} \tag{1.63}$$

where $\tilde{q}_i(\phi^i)$ stands for the function $\tilde{q}_i(\phi^i) = q_i(-\frac{1}{\mu_i}\,\phi^i|_{\mathcal{O}_i \times (0,T)})$, satisfies (1.24).

But this can be done easily. Indeed, for each couple $(\tilde{\phi}^1, \tilde{\phi}^2) \in L^2(Q) \times L^2(Q)$ we can consider the system

$$\begin{cases} z_t - \Delta z + a(x,t)z = f1_{\mathcal{O}} - \displaystyle\sum_{i=1}^{2} \tilde{q}_i(\tilde{\phi}^i)\phi^i|_{\mathcal{O}_i} & \text{in} \quad Q, \\ -\phi^i_t - \Delta\phi^i + a(x,t)\psi = \alpha_i(z - z_{i,d})1_{\mathcal{O}_{i,d}} & \text{in} \quad Q, \\ z = 0 \quad \phi^i = 0 & \text{on} \quad \Sigma, \\ z(x,0) = z^0, \quad \phi^i(x,T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.64}$$

The arguments in Sections 1.2.2 and 1.3.2 can be applied again to (1.64). The main consequence is that there exist exactly one minimal L^2 norm null control f for this system, with

$$\|f\|_{L^2(\mathcal{O}\times(0,T))} \le C \tag{1.65}$$

and, also, z, ϕ^1 and ϕ^2 uniformly bounded in $L^2(0,T;H^1_0(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ and z_t, ϕ^1_t and ϕ^2_t uniformly bounded (at least) in $L^2(0,T;H^{-1}(\Omega))$.

Hence, it is not difficult to deduce that the mapping $(\phi^1, \phi^2) \mapsto (\tilde{\phi}^1, \tilde{\phi}^2)$ possesses at least one fixed-point. Such a fixed-point satisfies, together with some f and some z, (1.63) and (1.24).

1.5 Some additional comments and questions

1.5.1 On the assumption $\mathcal{O}_{1,d} = \mathcal{O}_{2,d}$

The assumption (1.10) is used in (1.32) and only there. Indeed, in combination wih (1.33) and (1.34), (1.32) yields (1.35). At present, we do not know whether an estimate like (3.4) remains true for $\mathcal{O}_{1,d} \neq \mathcal{O}_{2,d}$. However, this is the case if we modify apropriately the secondary functionals J_i.

Thus, let $\rho_* = \rho_*(x,t)$ be a weight (a positive continuous function on $\overline{\Omega} \times (0,T)$) such that $\rho_* \ge e^{s\alpha/2}$, see (3.18). We assume now that the followers produce a Nash equilibrium with respect to the functionals

$$\tilde{J}_i(f; v^1, v^2) := \frac{\alpha_i}{2} \iint_{\mathcal{O}_{i,d}\times(0,T)} |y - y_{i,d}|^2 \, dx\, dt + \frac{\mu_i}{2} \iint_{\mathcal{O}_i \times(0,T)} \rho_*^2 |v^i|^2 \, dx\, dt,$$

for $i = 1, 2$. With computations similar to those in Section 2.4, we obtain the following optimality system:

$$
\begin{cases}
z_t - \Delta z + a(x,t) z = f 1_{\mathcal{O}} - \displaystyle\sum_{i=1}^{2} \frac{1}{\mu_i} \rho_*^{-2} \phi^i 1_{\mathcal{O}_i} & \text{in} \quad Q, \\
-\phi^i_t - \Delta \phi^i + a(x,t) \phi^i = \alpha_i (z - z_{i,d}) 1_{\mathcal{O}_{i,d}} & \text{in} \quad Q, \\
y = 0, \quad \phi^i = 0 & \text{on} \quad \Sigma, \\
y(x,0) = y^0(x), \quad \phi^i(x,T) = 0 & \text{in} \quad \Omega.
\end{cases}
$$

The associated adjoint system is given by

$$
\begin{cases}
-\psi_t - \Delta \psi + a(x,t) \psi = \displaystyle\sum_{i=1}^{2} \alpha_i \gamma^i 1_{\mathcal{O}_{i,d}} & \text{in} \quad Q, \\
\gamma^i_t - \Delta \gamma^i + a(x,t) \gamma^i = -\dfrac{1}{\mu_i} \rho_*^{-2} \psi 1_{\mathcal{O}_i} & \text{in} \quad Q, \\
\psi = 0, \quad \gamma^i = 0 & \text{on} \quad \Sigma, \\
\psi(x,T) = \psi^T(x), \quad \gamma^i(x,0) = 0 & \text{in} \quad \Omega
\end{cases}
$$

and the main task is to prove an estimate like (3.4) for the solutions $(\psi, \gamma^1, \gamma^2)$.

In this situation, we have an useful energy inequality for the γ^i:

$$
\|\gamma^i(\cdot\,,\tau)\|^2 + \int_0^\tau \|\nabla \gamma^i(\cdot\,,t)\|^2 \, dt \leq \frac{C}{\mu_i^2} \iint_Q \rho_*^{-4} |\psi|^2 \, dx \, dt. \tag{1.66}
$$

Using (1.66) in the right hand side of (1.32), since the μ_i are sufficiently large, we get:

$$
I_3(\psi) \leq C s^3 \lambda^4 \iint_{\omega \times (0,T)} e^{-2s\alpha} \xi^3 |\psi|^2 dx dt. \tag{1.67}
$$

Combining (1.67) and (1.36), we arrive at (3.4).

This shows that if we replace J_i by $\tilde{J}_i$ for $i = 1, 2$, the claims in Theorem 1 to 3 remain true. In fact, this is not surprising: if we impose $\tilde{J}_i < +\infty$, then we force the controls v^i to vanish exponentially as $t \to T^-$ and the leader f finds no obstruction to control the system. As mentioned above, it is unknown whether (3.4) continues to be true in the original framework (1.4) when $\mathcal{O}_{1,d} \neq \mathcal{O}_{2,d}$.

1.5.2 Stackelberg-Nash controllability and Stokes and Navier-Stokes systems

It makes complete sense to consider the Stokes-like system

$$
\begin{cases}
y_t - \Delta y + (w \cdot \nabla) y + \nabla p = f 1_{\mathcal{O}} + v^1 1_{\mathcal{O}_1} + v^2 1_{\mathcal{O}_2} & \text{in} \quad Q, \\
\nabla \cdot y = 0 & \text{in} \quad Q, \\
y = 0 & \text{on} \quad \Sigma, \\
y(x,0) = y^0(x) & \text{in} \quad \Omega,
\end{cases} \tag{1.68}
$$

where Ω, T, $\mathcal{O}$ and the $\mathcal{O}_i$ are as above, y^0 belongs to the Hilbert space

$$
H := \{\, z \in L^2(\Omega)^N : \nabla \cdot z = 0 \ \text{ in } \ \Omega, \ z \cdot n = 0 \ \text{ on } \ \Gamma \,\},
$$

the field w belongs to $L^\infty(0,T;H)$ and the controls f and v^i satisfy

$$f \in L^2(\mathcal{O} \times (0,T))^N, \quad v^i \in L^2(\mathcal{O}_i \times (0,T))^N.$$

With functionals J and J_i similar to those in the previous sections, we can formulate again the Stackelberg-Nash null controllability problem for (1.68). Results of the same kind can be obtained easily by adapting the arguments in Sections 1.2 to 1.4.

The situation is obviously much more difficult to analyze when we consider the Navier-Stokes system

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla) y + \nabla p = f 1_{\mathcal{O}} + v^1 1_{\mathcal{O}_1} + v^2 1_{\mathcal{O}_2} & \text{in} \quad Q, \\ \nabla \cdot y = 0 & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(x,0) = y^0(x) & \text{in} \quad \Omega. \end{cases}$$

Now, the existence of Nash equilibria or quasi-equilibria for each f and, of course, whether or not there exist null controls and associated Nash equilibrium pairs are open problems.

For other controllability results for Stokes and Navier-Stokes systems, see [39, 53, 34, 46, 47].

1.5.3 Other Stackelberg strategies

It is possible to introduce other strategies to control systems of the kind (1.1). One of them is the so called Stackelberg-Pareto method.

Thus, to each $f \in L^2(\mathcal{O} \times (0,T))$ we can associate one or several Pareto equilibrium pairs $(u^1(f), u^2(f)) \in \mathcal{H}$. By definition, this means that there is no $(\hat{u}^1, \hat{u}^2) \in \mathcal{H}$ satisfying

$$J_i(\hat{u}^1, \hat{u}^2) \le J_i(u^1(f), u^2(f)) \ \text{ for } \ i = 1, 2,$$

one of these inequalities at least being strict. Then, we search for f such that the states y associated to f and the $(u^1(f), u^2(f))$ satisfy (1.9), where $\overline{y} = \overline{y}(x,t)$ is a prescribed uncontrolled solution to (1.1).

The analysis of Stackelberg-Pareto controllability will be the goal of a forthcoming paper.

1.5.4 The boundary case

It is natural to wonder if results similar to Theorems 1, 2 and 3 also hold with boundary controls.

More precisely, let us consider the system

$$\begin{cases} z_t - \Delta z + a(x,t) z = F(z) & \text{in} \quad Q, \\ z = f 1_{\mathcal{S}} + v^1 1_{\mathcal{S}_1} + v^2 1_{\mathcal{S}_2} & \text{on} \quad \Sigma, \\ z(x,0) = z^0(x) & \text{in} \quad \Omega, \end{cases}$$

where $\mathcal{S}, \mathcal{S}_1, \mathcal{S}_2 \subset \partial\Omega$ are non-empty closed sets and let us introduce the functionals

$$L_i(f; v^1, v^2) := \frac{\alpha_i}{2} \iint_{\mathcal{O}_{i,d} \times (0,T)} |y - y_{i,d}|^2 \, dx\, dt + \frac{\mu_i}{2} \iint_{\mathcal{S}_i \times (0,T)} |v^i|^2 \, d\Sigma\, dt \tag{1.69}$$

for $i = 1, 2$. Now, the problem is to find for each f a Nash equilibrium $(v^1(f), v^2(f))$ associated to the functionals L_i and, then, choose f in a appropriate way such that $z(x, T) \equiv 0$.

We can try to solve this problem as before. However, we find some technical difficulties, as shown below.

Let us consider the linear case, that is, $F(s) \equiv 0$. Arguing as in Section 1.2, we see that the optimality system for $(v^1(f), v^2(f))$ is the following:

$$\begin{cases} z_t - \Delta z + a(x,t)z = 0 & Q, \\ -\phi^i_t - \Delta\phi^i + a(x,t)\phi^i = \alpha_i(z - z_{i,d})1_{\mathcal{O}_{i,d}} & \text{in} \quad Q, \\ z = f1_{\mathcal{S}} + \dfrac{1}{\mu_1}\dfrac{\partial\phi^1}{\partial n}1_{\mathcal{S}_1} + \dfrac{1}{\mu_2}\dfrac{\partial\phi^2}{\partial n}1_{\mathcal{S}_2}, \quad \phi^i = 0 & \text{on} \quad \Sigma, \\ z(x,0) = z^0(x), \quad \phi^i(x,T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.70}$$

The adjoint system is given by

$$\begin{cases} -\psi_t - \Delta\psi + a(x,t)\psi = \displaystyle\sum_{i=1}^{2} \alpha_i\gamma^i 1_{\mathcal{O}_{i,d}} & \text{in} \quad Q, \\ \gamma^i_t - \Delta\gamma^i + a(x,t)\gamma^i = 0 & \text{in} \quad Q, \\ \psi = 0, \quad \gamma^i = \dfrac{1}{\mu_i}\psi 1_{\mathcal{S}_i} & \text{on} \quad \Sigma, \\ \psi(x,T) = \psi^T(x), \quad \gamma^i(x,0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.71}$$

Thus, if we try to adapt the proof of Proposition 3, we see at once that the following conditions are required:

$$\mathcal{O}_{1,d} = \mathcal{O}_{2,d} = \mathcal{O}_d \quad \text{and} \quad \overline{\mathcal{O}_d} \cap \mathcal{S} \neq \emptyset. \tag{1.72}$$

The main difficulty in this case is that we have to combine a boundary Carleman inequality for ψ and a distributed Carleman inequality for $h = \alpha_1\gamma^1 + \alpha_2\gamma^2$ for functions satisfying nonhomogeneous Dirichlet boundary conditions on Σ. This interesting situation will be also analyzed in a forthcoming paper.

Capítulo 2

Numerical null controllability of the 1D linear Schrödinger equation

Numerical null controllability of the 1D linear Schrödinger equation

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Abstract. This paper deals with the numerical approximation to boundary controls that drive the solution to the $1D$ linear Schrödinger equation to a prescribed state at a final time. Using ideas from Fursikov and Imanuvilov, we consider the control that minimizes over the class of admissible controls a functional that involves weighted integrals, with weights that blow up at T. We will see that this extremal problem is equivalent to a differential problem that is fourth-order in space and second-order in time. Adapting some numerical techniques applied by the first author and Münch to the heat equation, we approximate the variational formulation by introducing appropriate space-time finite elements that are C^1 in space and C^0 in time. We present two approaches; the second one relies on a change of variable which leads to a lower condition number for the stiffness matrix. The results of some experiments show the efficiency of these methods

2.1 Introduction, the null controllability problem

We are mainly concerned with the boundary exact controllability for the 1D linear Schrödinger equation. The state equation is the following:

$$\begin{cases} iy_t - y_{xx} + V(x,t)y = 0, & (x,t) \in (0,1) \times (0,T), \\ y(0,t) = u(t), \quad y(1,t) = 0, & t \in (0,T), \\ y(x,0) = y_0(x), & x \in (0,1). \end{cases} \tag{2.1}$$

Here, $T > 0$ and we assume that $y_0 \in H_0^1((0,1);\mathbb{C})$ and $V, V_x \in L^\infty((0,1)\times(0,T);\mathbb{R})$. In (2.1), $u \in L^2((0,T);\mathbb{C})$ is the control and $y = y(x,t)$ is the associated state.

In the sequel, we will use the notation

$$Ly := iy_t - y_{xx} + Vy.$$

It is well known that, for any $u \in L^2((0,T);\mathbb{C})$, problem (2.1) has exactly one solution y in the transposition sense, with

$$y \in C^0([0,T]; H^{-1}((0,1);\mathbb{C})) \cap H^{-1}(0,T; L^2((0,1);\mathbb{C})), \tag{2.2}$$

see for instance [13, 65].

Our aim in this paper is to find numerical approximations to controls u such that the associated solutions to (2.1) satisfy $y(\cdot,T) = 0$. This is called a null controllability problem. In fact, due to the time reversibility of the linear Schrödinger equation, the null controllability and the exact controllability properties are equivalent, which means that we can reach *any* final state in $H^{-1}((0,1);\mathbb{C})$ by the action of a boundary control. From now on, we will investigate the null controllability problem.

It is known that, for any $T>0$, (2.1) has the null controllability property. In other words, for any $y_0 \in H_0^1((0,1);\mathbb{C})$, there exist controls $u \in L^2((0,T);\mathbb{C})$ such that the associated states satisfy $y(\cdot,T)=0$. This was proved in [65] for $V \equiv 0$ by applying the so called *Hilbert uniqueness method* together with multipliers techniques. In particular, it was established that the control of minimal norm in $L^2((0,T);\mathbb{C})$ is given by $u=\phi_x(0,\cdot)$, where ϕ solves a backwards Schrödinger problem

$$\begin{cases} i\phi_t - \phi_{xx} = 0, & (x,t) \in (0,1)\times(0,T), \\ \phi(x,t) = 0, & (x,t) \in \{0,1\}\times(0,T), \\ \phi(x,T) = \phi^T(x), & x \in (0,1), \end{cases}$$

with ϕ^T in an appropriate space.

The null controllability of (2.1) with a vanishing or time-independent potential V has also been established by other methods. Thus, in Lebeau [60], Hilbert uniqueness was used in combination with microlocal analysis and extended to higher dimensional Schrödinger systems. Later, Tataru [83, 84] and Triggiani [85] used appropriate Carleman inequalities to deduce approximate and exact controllability and stabilizability results. Other proofs of controllability have been furnished by Horn and Littman [50, 51] and Phung [72].

In the present work, we will use some ideas inspired by the work of Fursikov and Imanuvilov in [38] for similar parabolic systems. More precisely, let us consider the following extremal problem:

$$\begin{cases} \text{Minimize } J(y,u) = \dfrac{1}{2}\displaystyle\iint_Q \rho^2|y|^2\,dx\,dt + \dfrac{1}{2}\int_0^T \rho_1(0,t)^2|u|^2\,dt \\ \text{Subject to } (y,v) \in \mathcal{C}(y_0,T). \end{cases} \tag{2.3}$$

Here and in the sequel, $Q=(0,1)\times(0,T)$ and $\mathcal{C}(y_0,T)$ is the linear manifold

$$\mathcal{C}(y_0,T) = \{\,(y,u)\in X : \, y \text{ solves (2.1) and satisfies } y(\cdot,T)=0\,\}$$

where

$$X = L^2(Q;\mathbb{C}) \times L^2((0,T);\mathbb{C}). \tag{2.4}$$

We assume that

$$\begin{cases} \rho=\rho(x,t), \rho_1=\rho_1(x,t) \text{ are continuous, real-valued and } \geq \rho_* > 0, \\ \rho,\rho_1 \in L^\infty((0,1)\times(0,T-\delta);\mathbb{R}) \ \ \forall\delta>0, \end{cases} \tag{2.5}$$

so that, in principle, they can blow up as $t \to T^-$.

The fact that we search for null controls and associated states solving (2.3) can be justified as follows: first, they can serve to select the "good" control-state pair, according to a previously established criterion; secondly, they avoid unpleasant oscillations of the control as $t \to T$ (it is well known that this phenomenon can appear if, for instance, we simply try to find minimal L^2 norm null controls; see [45]).

The main goal in this paper is to solve the extremal problem (2.3) numerically. To this purpose, we will see before that the manifold $\mathcal{C}(y_0,T)$ in (2.3) is non-empty and (2.3) possesses exactly one solution.

In the sequel, we will denote by C a positive generic constant and $\langle \cdot\,,\cdot\rangle$ will stand for te usual duality pairing for H_0^1 and H^{-1}.

The paper is organized as follows. In Section 2.2, we present two equivalent variational equalities whose solutions p and w furnish the unique solution to (2.3); see (2.13) and (2.18). We will see that the pair (y, u) obtained by the Fursikov-Imanuvilov method belongs to X, which is an interesting additional property, since the natural regularity for y is (2.2). In Section 2.3, these variational equalities are analyzed numerically. We introduce some families of approximate problems and we prove appropriate convergence results. Section 2.4 deals with the results of some numerical experiments. It is seen that the proposed strategies are efficient and furnish satisfactory approximations to the control-state pair (y, u). Finally, some additional comments are given in Section 2.5.

2.2 Variational approaches to the controllability problem

2.2.1 Preliminaries. A first variational equality

Let us introduce the weights

$$\begin{aligned} &\rho(x,t) \equiv \exp\left(\frac{\alpha(x)}{T-t}\right), && \rho_0(x,t) \equiv \rho(x,t)(T-t)^{3/2}, \\ &\rho_1(x,t) \equiv \rho(x,t)(T-t)^{1/2}, && \rho_2(x,t) \equiv \rho(x,t)(T-t)^{-3/2}, \end{aligned} \tag{2.6}$$

where

$$\alpha(x) = K_1(e^{K_2} - e^{\beta_0(x)}), \quad \beta_0(x) \equiv \beta_{00}(1-x), \quad K_2 > \beta_{00} > 0. \tag{2.7}$$

Obviously, ρ and ρ_1 satisfy (3.18). Let us consider the extremal problem (2.3). The roles of ρ and ρ_0 are clarified by the following arguments and results.

Let us set

$$P_0 = \{q \in C^2(\overline{Q} : \mathbb{C}) : q = 0 \text{ on } \{0,1\} \times [0,T]\}.$$

In this linear space, the *sesquilinear* form

$$(p,q)_P = \iint_Q \rho^{-2} Lp\, \overline{Lq}\, dx\, dt + \int_0^T \rho_1^{-2}(0,t) p_x(0,t)\, \overline{q_x(0,t)}\, dt,$$

is an inner product. This is a consequence of the unique continuation property for the Schrödinger equation, see [13, 55].

Let P be the completion of the space P_0 for the previous inner product. Then, P is a Hilbert space and the following result holds:

Lema 2.1. *There exist positive (sufficiently large) constants K_1, K_2 and C_0 such that one has*

$$\begin{aligned} &\iint_Q \rho_2^{-2} |iq_t - q_{xx}|^2\, dx\, dt + \iint_Q \rho_0^{-2} |q|^2\, dx\, dt \\ &\quad \le C_0 \left(\iint_Q \rho^{-2} |Lq|^2\, dx\, dt + \int_0^T \rho_1^{-2}\, |q_x(0,t)|^2\, dt \right) \end{aligned} \tag{2.8}$$

for all $q \in P$.

Demonstração. We can argue as in the proofs of Proposition 1 and Theorem 2 in [13]. Thus, let us introduce the weights

$$\begin{array}{ll} \zeta(x,t) \equiv \exp\left(\frac{\alpha(x)}{t(T-t)}\right), & \zeta_0(x,t) \equiv \zeta(x,t)(t(T-t))^{3/2}, \\ \zeta_1(x,t) \equiv \zeta(x,t)(t(T-t))^{1/2}, & \zeta_2(x,t) \equiv \zeta(x,t)(t(T-t))^{-3/2}. \end{array}$$

We have the following for sufficiently large K_1, K_2, C and C_0:

$$\begin{aligned} &\iint_Q \zeta_2^{-2}|iq_t - q_{xx}|^2\,dx\,dt + \iint_Q \zeta_1^{-2}|q_x|^2\,dx\,dt + \iint_Q \zeta_0^{-2}|q|^2\,dx\,dt \\ &\quad \le C\left(\iint_Q \zeta^{-2}|Lq|^2\,dx\,dt + \int_0^T \zeta_1^{-2}\,|q_x(0,t)|^2\,dt\right) \\ &\quad \le C_0\left(\iint_Q \rho^{-2}|Lq|^2\,dx\,dt + \int_0^T \rho_1^{-2}\,|q_x(0,t)|^2\,dt\right). \end{aligned} \tag{2.9}$$

On the other hand, the usual estimates for the solutions to the Schrödinger equation show that

$$\iint_{(0,1)\times(0,T/2)} |q|^2\,dx\,dt \le C\left(\iint_Q \zeta_0^{-2}|q|^2\,dx\,dt + \iint_{(0,1)\times(0,T/2)} |Lq|^2\,dx\,dt\right)$$

and, taking into account that $iq_t - q_{xx} = Lq - Vq$, we also have

$$\iint_{(0,1)\times(0,T/2)} |iq_t - q_{xx}|^2\,dx\,dt \le C\left(\iint_Q \zeta_0^{-2}|q|^2\,dx\,dt + \iint_{(0,1)\times(0,T/2)} |Lq|^2\,dx\,dt\right).$$

As a consequence, we get (2.8) for eventually larger constants K_1, K_2 and C_0. □

As a consequence of Lemma 2.1, we obtain the following:

Proposition 5. *There exists a unique solution $p \in P$ to the problem*

$$\left\{\begin{array}{l} \displaystyle\iint_Q \rho^{-2} Lp\,\overline{Lq}\,dx\,dt + \int_0^T \rho_1^{-2} p_x(0,t)\,\overline{q_x(0,t)}\,dt = i\langle y_0\,, \overline{q(\cdot\,,0)}\rangle \\ \forall q \in P; \quad p \in P. \end{array}\right. \tag{2.10}$$

Demonstração. Let us check that we can apply the Lax-Milgram Lemma to (2.10). Indeed, the bilinear form in the left hand side is just the scalar product in P, while the *antilinear* form in the right hand side is continuous.

This can be justified as follows. If $q \in P$, then we get from (2.8) that

$$q \in L^2((0,T'); L^2((0,1);\mathbb{C})) \tag{2.11}$$

and

$$iq_t - q_{xx} \in L^2((0,T'); L^2((0,1);\mathbb{C})) \tag{2.12}$$

for any $T' < T$; in particular, this implies that $q_t \in L^2((0,T'); H^{-2}((0,1);\mathbb{C}))$ which, combined with (2.11), yields $q \in C^0([0,T']; H^{-1}((0,1);\mathbb{C}))$ for all T'. Thus P is continuously embedded in $C^0([0,T']; H^{-1}((0,1);\mathbb{C}))$ and the right hand side of (2.10) certainly defines a continuous antilinear form on P. □

It will be seen in the following section that (2.10) is closely related to the optimality system for (2.3).

2.2.2 Analysis of (2.3)

In this Section we will prove that $\mathcal{C}(y_0, T)$ is non-empty and (2.3) possesses exactly one solution $(y, u) \in X$.

Theorem 4. *For any $y_0 \in L^2((0,1);\mathbb{C})$, there exists exactly one solution to* (2.3). *It is given by*

$$y = \rho^{-2} Lp, \quad u = -\rho_1^{-2} p_x|_{x=0} \tag{2.13}$$

where p is the unique solution to (2.10).

Demonstração. Let $p \in P$ be the solution to (2.10) and let us introduce the couple (y, u) given by (2.13). In view of (2.13) and (2.10), we see that

$$\iint_Q y\overline{Lq}\, dx\, dt = \int_0^T u(t)\overline{q_x(0,t)}\, dt + i\langle y_0\, , \overline{q(\cdot\, , 0)}\rangle \qquad \forall q \in P. \tag{2.14}$$

The control u defined in (2.13) belongs to $L^2((0,T);\mathbb{C})$. Consequently, there exists a unique solution $\tilde{y}$ to (2.1) in the transposition sense. In particular

$$\langle \tilde{y}, \bar{g}\rangle = \int_0^T u(t)\overline{\phi_x(0,t)}\, dt + i\int_0^1 y_0(x)\overline{\phi(x,0)}\, dx \quad \forall g \in \mathcal{D}(Q;\mathbb{C}) \tag{2.15}$$

where $\langle \cdot\, , \cdot\rangle$ stands for the usual duality pairing for the spaces $H^{-1}(0,T;L^2((0,1);\mathbb{C}))$ and $H_0^1(0,T;L^2((0,1);\mathbb{C})$ and we have denoted by ϕ the unique (strong) solution to

$$\begin{cases} i\phi_t - \phi_{xx} + V(x,t)\phi = g, & (x,t) \in (0,1)\times(0,T), \\ \phi(0,t) = 0, \quad \phi(1,t) = 0, & t \in (0,T), \\ \phi(x,T) = 0, & x \in (0,1). \end{cases} \tag{2.16}$$

Notice that $\phi \in P$. Indeed, we first have $\phi(x,t) = 0$ for all $(x,t) \in [0,1]\times[T-\delta, T]$ for some $\delta > 0$. Also, since V and V_x are essentially bounded, the usual estimates show that

$$\phi \in L^\infty(([0,T); H_0^1((0,1);\mathbb{C})$$

and consequently, from Lemmas 1 and 2 in [13], we find that

$$\phi \in C^0([0,T]; H_0^1((0,1);\mathbb{C}), \quad \phi_x(0,\cdot) \in L^2((0,T);\mathbb{C})$$

and $\phi \in P$. Thus, y also satisfies (2.15) and $y = \tilde{y}$. This means that $(y,u) \in \mathcal{C}(y_0, T)$, i.e. $\mathcal{C}(y_0, T)$ is non-empty.

In addition, $(z,u) \mapsto J(z,u)$ is a strictly convex, proper and lower semi-continuous function on X and $J(z,u) \to +\infty$ as $\|(z,u)\|_X \to +\infty$. Hence, the extremal problem (2.3) possesses a unique solution.

Finally, let $(z, v) \in \mathcal{C}(y_0, T)$ be such that $J(z, v) < +\infty$. It is then clear that

$$\begin{aligned} J(z,v) - J(y,u) = &\ J(z-y, v-u) \\ &+Re\left(\iint_Q \rho^2 y(\bar z - \bar y)\,dx\,dt + \int_0^T \rho_1^2 u(\bar v - \bar u)\,dt\right) \\ \geq &\ Re\left[\iint_Q Lp(\bar z - \bar y)dxdt - \int_0^T p_x(0,t)(\bar v - \bar u)\,dx\,dt\right] \\ = &\ 0. \end{aligned}$$

So, in fact, (y, u) is the unique minimizer. □

As mentioned above, (2.10)–(2.13) is a reformulation of the optimality system for (2.3). Indeed, it is easy to see that (2.10) is a weak formulation of the boundary-value problem

$$\begin{cases} L(\rho^{-2}Lp) = 0, & (x,t) \in (0,1)\times(0,T), \\ p(0,t) = 0, \quad p(1,t) = 0, & t \in (0,T), \\ (\rho^{-2}Lp + \rho_1^{-2}p)(0,t) = 0, \quad (\rho^{-2}Lp)(1,t) = 0, & t \in (0,T), \\ (\rho^{-2}Lp)(x,0) = 0, \quad (\rho^{-2}Lp)(x,T) = 0, & x \in (0,1), \end{cases}$$

that is of the second-order in time and fourth-order in space and this is turn equivalent to the system formed by the constraints $J(y, u) < +\infty$ and $(y, u) \in \mathbb{C}(y_0, T)$, the backwards in time (adjoint) system

$$\begin{cases} ip_t - p_{xx} + V(x,t)p = \rho^2 y, & (x,t) \in (0,1)\times(0,T), \\ p(0,t) = 0, \quad p(1,t) = 0, & t \in (0,T) \end{cases}$$

and the second equality in (2.13). But this is just the optimality system for (2.3).

Of course, the control u is not the minimal L^2 norm null control for (2.1). As shown above, the approach in this paper is different, but ensures good (exponential) convergence to zero of the control and the state as $t \to T$.

2.2.3 A second variational equality

Let us perform the change of variable

$$w = (T-t)^{-\gamma}\rho_1^{-1}p$$

for some appropriate γ. Let W be the completion of P_0 for the scalar product

$$\begin{aligned} (w,m)_W \quad = &\ \iint_Q \rho^{-2} L((T-t)^\gamma \rho_1 w(x,t))\,\overline{L((T-t)^\gamma \rho_1 m(x,t))}\,dx\,dt \\ &+ \int_0^T (T-t)^{2\gamma} w_x(0,t)\,\overline{m_x(0,t)}\,dt; \end{aligned}$$

Obviously, we have:

$$W = \{(T-t)^{-\gamma}\rho_1^{-1}q : q \in P\}$$

and, for any $m \in W$, one has

$$\rho^{-1}L((T-t)^\gamma \rho_1 m) = A_1 m + A_2 m_t + A_3 m_x + A_4 m_{xx},$$

where

$$\begin{cases} A_1 = -(T-t)^{\gamma-\frac{1}{2}}(\alpha_{xx} + i(\gamma+\frac{1}{2})) + (T-t)^{\gamma-\frac{3}{2}}(i\alpha - \alpha_x^2) + (T-t)^{\gamma+\frac{1}{2}}V, \\ A_2 = i(T-t)^{\gamma+\frac{1}{2}}, \\ A_3 = -2\alpha_x(T-t)^{\gamma-\frac{1}{2}}, \\ A_4 = -(T-t)^{\gamma+\frac{1}{2}}. \end{cases}$$

Consequently, the variational equality (2.10) can be rewritten equivalently in the form

$$\begin{cases} \displaystyle\iint_Q (A_1 w + A_2 w_t + A_3 w_x + A_4 w_{xx})\, \overline{(A_1 m + A_2 m_t + A_3 m_x + A_4 m_{xx})}\, dx\, dt \\ \qquad \displaystyle + \int_0^T (T-t)^{2\gamma} w_x(0,t)\, \overline{m_x(0,t)}\, dt = iT^\gamma \langle y_0\,, \rho_1(\cdot\,,0)\, \overline{m(\cdot\,,0)}\rangle \\ \forall m \in W;\ w \in W. \end{cases} \tag{2.17}$$

The well-posedness of this system is now obvious. More precisely, the following holds:

Proposition 6. *The variational equality* (2.17) *possesses exactly one solution* $w \in W$. *Moreover, the unique solution* (y, u) *to* (2.10) *is given by*

$$y = \rho^{-1}\left(A_1 w + A_2 w_t + A_3 w_x + A_4 w_{xx}\right), \quad u = -(T-t)^\gamma \rho_1(0,\cdot)^{-1} w_x(0,\cdot), \tag{2.18}$$

where $w \in W$ *solves* (2.17).

In order to have all the coefficients in $L^\infty(Q;\mathbb{C})$, it is enough to take $\gamma > 3/2$. This will be assumed in the sequel.

2.3 Numerical analysis of the variational equalities

We will now analyze from the numerical viewpoint the previous variational equalities. We will use standard arguments, that allow to approximate (2.10) and (2.17) by finite-dimensional linear problems, where the coefficient matrices are sparse and easy to construct. We will do this in such a way that the classical general theory applies and, in particular, convergence results can be obtained in the appropriate spaces.

We are going to adapt the results in [11, 23, 73]. Notice that the main difficulty here is that the variational equalities contain derivatives of order two (equivalently, they are weak formulation of fourth-order boundary value problems). Accordingly, it will be a little more difficult to construct finite-dimensional spaces than in the more standard situation of a second-order elliptic problem.

2.3.1 First approach

For any finite dimensional space $P_h \subset P$, we can introduce the approximated problem:

$$(p_h, q_h)_P = \ell_0(q_h) \quad \forall q_h \in P_h;\ p_h \in P_h, \tag{2.19}$$

where ℓ_0 is the antilinear form

$$\ell_0(q_h) = i\langle y_0\,, \overline{q_h(\cdot\,,0)}\rangle \quad \forall q_h \in P_h.$$

We have the following result, typical for any numerical approximation of this kind:

Lema 2.2. *Let $p \in P$ be the unique solution to* (2.10) *and let p_h be the unique solution to* (2.19)*. Then*

$$\|p - p_h\|_P = \inf_{q_h \in P_h} \|p - q_h\|_P. \tag{2.20}$$

Demonstração. Notice that, for any $q_h \in P_h$,

$$\|p_h - p\|_P^2 = (p_h - p, p_h - p)_P = (p_h - p, p_h - q_h)_P + (p_h - p, q_h - p)_P.$$

The first term in the right hand side is zero and the second one can be bounded by $\|p_h - p\|_P \|q_h - p\|_P$. Consequently, one has (2.20). □

Let us assume that $\mathcal{H} \subset \mathbb{R}^d$ is a generalized (not necessarily countable) sequence converging to zero and let P_h be as above for each $h \in \mathcal{H}$. Let us also assume that there exist interpolation operators $\Pi_h : P_0 \mapsto P_h$ satisfying the following:

$$\|\Pi_h q - q\|_P \to 0 \quad \text{as} \quad h \to 0 \quad \forall q \in P_0. \tag{2.21}$$

We then have a convergence result:

Proposition 7. *Let $p \in P$ be the solution to* (2.10) *and let $p_h \in P_h$ be the solution to* (2.19) *for each $h \in \mathcal{H}$. Then*

$$\|p - p_h\|_P \to 0 \text{ as } h \to 0.$$

Demonstração. Let us choose $\epsilon > 0$. From the density of P_0 in P, there exists $p_\epsilon \in P_0$ such that $\|p - p_\epsilon\|_P \leq \epsilon$. Therefore, from Lemma 2.2, we find that

$$\begin{aligned} \|p - p_h\|_P &\leq \|p - \Pi_h p_\epsilon\|_P \\ &\leq \|p - p_\epsilon\|_P + \|p_\epsilon - \Pi_h p_\epsilon\|_P \\ &\leq \epsilon + \|p_\epsilon - \Pi_h p_\epsilon\|_P. \end{aligned}$$

In view of (2.21), one has $\|p_\epsilon - \Pi_h p_\epsilon\|_P \to 0$ as $h \to 0$ and the result follows. □

2.3.2 Second approach

We will now turn to the formulation in Section 2.2.3.

Let us introduce the *sesquilinear* form $A(\cdot, \cdot)$, with

$$\begin{aligned} A(w,m) = &\iint_Q (A_1 w + A_2 w_t + A_3 w_x + A_4 w_{xx}) \overline{(A_1 m + A_2 m_t + A_3 m_x + A_4 m_{xx})}\, dx\, dt \\ &+ \int_0^T (T-t)^{2\gamma} w_x(0,t)\, \overline{m_x(0,t)}\, dt \quad \forall w, m \in W \end{aligned}$$

and the antilinear form ℓ, with

$$\ell(m) = i T^\gamma \langle y_0 \,, \rho_1(\cdot\,, 0)\, \overline{m(\cdot\,, 0)} \rangle \quad \forall m \in W.$$

Then, (2.17) reads as follows:

$$A(w,m) = \ell(m) \quad \forall m \in W; \; w \in W. \tag{2.22}$$

As in the previous Section, for any finite dimensional space $W_h \subset W$, we can introduce the following approximated problem:

$$A(w_h,m_h) = \ell(m_h) \quad \forall m_h \in W_h; \; w_h \in W_h. \tag{2.23}$$

Obviously, (2.23) is well posed. Furthermore, we have a result similar to Lemma 2.2:

Lema 2.3. *Let $w \in W$ be the unique solution to* (2.22) *and let w_h be the unique solution to* (2.23)*. Then*

$$\|w - w_h\|_W = \inf_{m_h \in W_h} \|w - m_h\|_W.$$

Let W_h be as above for each $h \in \mathcal{H}$. Again, let us assume that there exist interpolation operators $\Pi_h : P_0 \mapsto W_h$ satisfying

$$\|\Pi_h m - m\|_W \to 0 \;\text{ as }\; h \to 0 \quad \forall m \in P_0. \tag{2.24}$$

We have:

Proposition 8. *Let $w \in W$ be the solution to* (2.22) *and let $w_h \in W_h$ be the solution to* (2.23) *for each $h \in \mathcal{H}$. Then*

$$\|w - w_h\|_W \to 0 \text{ as } h \to 0.$$

2.3.3 The finite dimensional spaces P_h and W_h

In this Section, we will construct some finite dimensional spaces X_h that can be respectively used in (2.19) and (2.23). Recall that the variational equalities (2.10) and (2.17), as well as their finite-dimensional counterparts (2.19) and (2.23) are weak formulations of elliptic problems of the second and fourth order in time and space, respectively. The variables t and x play here similar roles and the boundary data are furnished on the whole boundary of Q (although, of course, two boundary conditions must be imposed on the lateral sides and only one boundary condition is required on the top and the bottom edges). Consequently, it is natural to work with spaces X_h where time and space are handled simultaneously. In our context, this means that we must consider time-space finite elements.

Notice that time-space finite element approximation has also been considered in connection with other problems; see for instance [4, 9, 10, 40, 52, 76] for some linear and nonlinear parabolic and hyperbolic systems.

For any couple of integers $K, L \geq 1$, we set $\Delta x = 1/K$, $\Delta t = T/L$ and $h = (\Delta x, \Delta t)$ and we introduce the uniform quadrangulation

$$Q_h = \{R_{kl} = [x_k, x_{k+1}] \times [t_l, t_{l+1}] : 1 \leq k \leq K, \; 1 \leq l \leq L\},$$

where we have used the notation

$$x_k = (k-1)\Delta x \quad \text{and} \quad t_l = (l-1)\Delta t$$

for all k and l.

We will denote by $C^{1,0}_{x,t}(\overline{Q})$ the space of functions $q \in C^0(\overline{Q})$ that possess a partial derivative $q_x \in C^0(\overline{Q})$. The following result holds:

Theorem 5. *Assume that $q_h \in C^0(\overline{Q})$ and $q_h|_R \in H^1(R)$ for all $R \in Q_h$. Then $q_h \in H^1(Q)$. On the other hand, if $q_h \in C^{1,0}_{x,t}(\overline{Q})$ and $(q_h|_R)_{xx} \in L^2(R)$ for all $R \in Q_h$, then $(q_h)_{xx} \in L^2(Q)$.*

The proof is easy and is left to the reader.

For each h, let us set

$$X_h = \{\, q_h \in C^{1,0}_{x,t}(\overline{Q}) : q_h|_R \in \mathbb{P}(R) \ \ \forall R \in Q_h, \ \ q_h = 0 \text{ on } \{0,1\} \times [0,T] \,\},$$

where $\mathbb{P}(R)$ denotes the following space of polynomial functions in x and in t:

$$\mathbb{P}(R) = (\mathbb{P}_{3,x} \otimes \mathbb{P}_{1,t})(R)$$

(here, $\mathbb{P}_{r,\xi}$ stands for the space of polynomials of order r in the variable ξ).

It is easy to see that a function $f \in \mathbb{P}(R)$ is uniquely determined by the values of f and f_x at the four vertices of R. Consequently, X_h is a finite dimensional space of P and W and any function $p_h \in X_h$ is uniquely determined by the values of p_h at the nodes of Q_h that do not belong to $\{0,1\}$ and the values of $(p_h)_x$ at the nodes of Q_h.

Let us introduce the functions

$$L_{0,k}(x) = \frac{(\Delta x + 2x - 2x_k)(\Delta x - x + x_k)^2}{(\Delta x)^3}, \quad L_{1,k}(x) = \frac{(x - x_k)^2(-2x + 2x_k + 3\Delta x)}{(\Delta x)^3}$$
$$L_{2,k}(x) = \frac{(x - x_k)(\Delta x - x + x_k)^2}{(\Delta x)^2}, \qquad L_{3,k}(x) = \frac{-(x - x_k)^2(\Delta x - x + x_k)}{(\Delta x)^2}$$

and

$$\mathcal{L}_{0l}(t) = \frac{t_l - t + \Delta t}{\Delta t}, \quad \mathcal{L}_{1l}(t) = \frac{t - t_l}{\Delta t}.$$

Notice that these functions satisfy

$$\begin{cases} L_{i,k}(x_{m+k}) = \delta_{im}, & L'_{i,k}(x_{m+k}) = 0, \\ L_{i+2,k}(x_{m+k}) = 0, & L'_{i+2,k}(x_{m+k}) = \delta_{im}, \end{cases}$$

for $i, m = 0, 1$.

We have the following elementary result, where the interpolation operator $\Pi_h : P_0 \mapsto X_h$ is introduced:

Lema 2.4. *Let $u \in P_0$ and let us define the function $\Pi_h u$ as follows: on each R_{kl}, we set*

$$\Pi_h u(x,t) = \sum_{i,j=0}^{1} L_{i,k}(x)\mathcal{L}_{j,l}(t)u(x_{i+k}, t_{j+l}) + \sum_{i,j=0}^{1} L_{i+2,k}(x)\mathcal{L}_{j,l}(t)u_x(x_{i+k}, t_{j+l}).$$

Then $\Pi_h u$ is the unique function in X_h that satisfies the following for all k and l:

$$\Pi_h u(x_k, t_l) = u(x_k, t_l), \quad (\Pi_h u)_x(x_k, t_l) = u_x(x_k, t_l).$$

2.3.4 Convergence results

We divide this Section in two parts, respectively devoted to prove the convergence results in (2.21) and (2.24). For simplicity, the usual norm in $L^r((0,1)\times(0,T);\mathbb{C})$ (resp. in the space $L^r(0,T;L^s((0,1);\mathbb{C})))$ will be denoted by $\|\cdot\|_r$ (resp. $\|\cdot\|_{r,s}$).

The convergence of $\|q-\Pi_h q\|_P$

We first have the following:

Lema 2.5. *There exist C, independent of $h=(\Delta x,\Delta t)$, such that, for any $q\in P_0$, one has:*

$$\begin{aligned}\iint_Q |q-\Pi_h q|^2\,dx\,dt \quad &\leq \quad C\Big(\|q_x\|_\infty \Delta t(\Delta x)^2+\|p_t\|^2_{2,\infty}\Delta x(\Delta t)^2\\ &\quad +\|q_{xt}\|^2_{2,\infty}(\Delta x)^3(\Delta t)^4+\|q_{xx}\|^2_{\infty,2}(\Delta x)^4(\Delta t)\Big).\end{aligned}\tag{2.25}$$

This result is proved in [35]; see the estimates in Section 3.2.3. It relies on the identity

$$q-\Pi_h q=\sum_{i,j=0}^{1} m_{ij}q_x(x_{i+k},t_{j+l})+\sum_{i,j=0}^{1} L_{i,k}\mathcal{L}_{j,l}\mathcal{R}[q:x_{i+k},t_{j+l}],\tag{2.26}$$

where the functions $m_{i,j}$ and $\mathcal{R}[q:x_{i+k},t_{j+l}]$ are given as follows:

$$m_{i,j}(x,t)\equiv(L_{i,k}(x)(x-x_k)-L_{i+2,k}(x))\mathcal{L}(t),$$

$$\begin{aligned}\mathcal{R}[q:x_{i+k},t_{j+l}] \quad &\equiv \quad \int_{t_{j+l}}^{t} q_t(x_{i+k},s)ds+(x-x_{i+k})\int_{t_{j+l}}^{t}(t-s)q_{xt}(x_{i+k},s)ds\\ &\quad +\int_{x_{i+k}}^{x}(x-s)q_{xx}(s,t)ds.\end{aligned}$$

In a similar way, it can also be shown that, for any $q\in P_0$,

$$\iint_K (q-\Pi_h q)_{xx}\,dx\,dt\to 0 \text{ as } h\to 0.\tag{2.27}$$

We also have a convergence result concerning the normal derivative at $x=0$:

Lema 2.6. *For any $q\in P_0$, one has*

$$\int_0^T |(q-\Pi_h q)_x(0,t)|^2 dt\to 0 \quad \textit{as } h\to 0.\tag{2.28}$$

Demonstração. Let us denote by R_l the rectangle $(0,x_1)\times(t_l,t_{l+1})$. Then, the following identity holds in R_l:

$$\Pi_h q(x,t)=\sum_{i,j=0}^{1} L_{i,0}(x)\mathcal{L}_{j,l}(t)q(x_i,t_{j+l})+\sum_{i,j=0}^{1} L_{i+2,0}(x)\mathcal{L}_{j,l}(t)q_x(x_i,t_{j+l}).$$

After differentiation with respect to x, setting $x = 0$, we see that

$$\begin{aligned}(q-\Pi_h q)_x(0,t) &= q_x(0,t) - \mathcal{L}_{0l}(t)q_x(0,t_l) - \mathcal{L}_{1l}(t)q_x(0,t_{l+1}) \\ &= q_x(0,t_l) + \int_{t_l}^{t} q_{xt}(0,s)\,ds - \frac{t_l - t + \Delta t}{\Delta t}q_x(0,t_l) - \frac{t-t_l}{\Delta t}q_x(0,t_{l+1}) \\ &= \int_{t_l}^{t} q_{xt}(0,s)\,ds - \frac{t-t_l}{\Delta t}\left[q_x(0,t_{l+1}) - q_x(0,t_l)\right].\end{aligned}$$

Consequently,

$$\int_0^T |(q-\Pi_h q)_x|^2(0,t)\,dt \le C\|q_{xt}(0,\cdot)\|_\infty^2 \Delta t.$$

This proves (2.28). □

Now, taking into account (2.25), (2.27) and (2.28), we see that (2.21) holds. This shows that the problems (2.19) furnish a sequence of approximated solutions p_h that converges strongly to p in P.

The convergence of $\|m - \Pi_h m\|_W$

Let us first notice that, for any $m \in P_0$,

$$\begin{aligned}\|m-\Pi_h m\|_W^2 \le\ & 4\|A_1\|_\infty^2 \iint_Q |m-\Pi_h m|^2\,dx\,dt \\ &+4\|A_2\|_\infty^2 \iint_Q |(m-\Pi_h m)_t|^2\,dx\,dt + 4\|A_3\|_\infty^2 \iint_Q |(m-\Pi_h m)_x|^2\,dx\,dt \\ &+4\|A_4\|_\infty^2 \iint_Q |(m-\Pi_h m)_{xx}|^2\,dx\,dt + T^{2\gamma}\int_0^T |(m-\Pi_h m)_x(0,t)|^2\,dt.\end{aligned} \tag{2.29}$$

From Lemma 2.5, it is clear that the first term on the right hand side of (2.29) converges to zero as $h \to 0$. The next three terms converge as well; to check this, it suffices to differentiate (2.26) with respect to the corresponding variable and argue as in [35]. Finally, the last term converges to zero in view of Lemma 2.6. Therefore, one has (2.24), i.e.

$$\|m-\Pi_h m\|_W \to 0 \text{ as } h \to 0$$

for all $m \in P_0$.

As before, this shows that the solutions to the problems (2.23) converge strongly in W to w as $h \to 0$.

2.4 Numerical Experiments

In this Section, we present the results of some numerical experiments concerning the solutions to (2.19) and (2.23).

Both problems can be viewed as linear systems where the coefficient matrices are sparse. Once we find the solution p_h to (2.19), the approximated state-control pair can be found through (2.13). On the other hand, the solution w_h to (2.23) furnishes another approximated

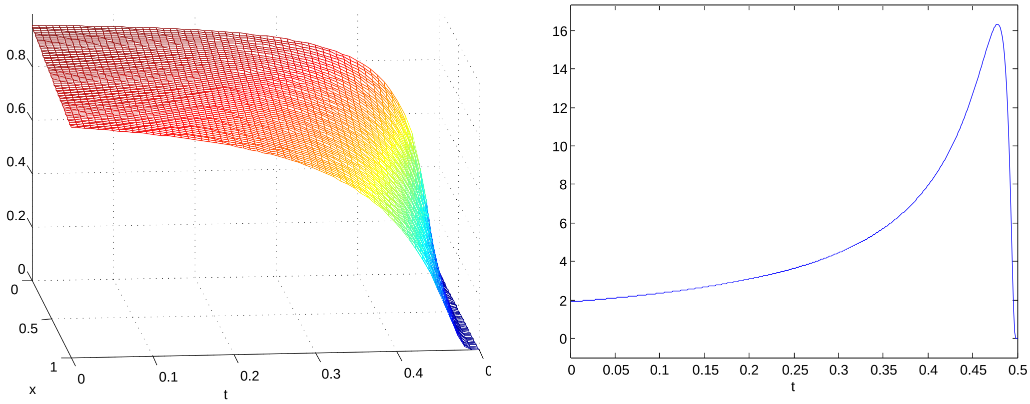


Figura 2.1: The functions ρ^{-1} and $\rho_1^{-1}(0,\cdot)$ for $\beta_{00} = 0.1$, $K_2 = 1.4\beta_{00}$ and $K_1 = 0.5$.

state-control pair using (2.18). Recall that the weights ρ and ρ_1 are given by (2.6)–(2.7). The functions ρ^{-1} and $\rho_1(0,\cdot)^{-1}$ are depicted in Figure 4.2.

The solutions to the linear systems arising in (2.19) and (2.23) are computed by performing a LU factorization and solving two triangular systems. In the experiments, we have taken $T = 0.5$ and $y_0(x) \equiv \sin(\pi x) + i\sin(2\pi x)$. Notice that, physically, y_0 provides a true probability distribution for the initial position of a particle in (0,1). For simplicity, we have taken $\Delta x = \Delta t$.

We first present the numerical results obtained by solving (2.19); see Tables 1 and 2, respectively concerning the choices

$$V(x,t) \equiv 0$$

and

$$V(x,t) \equiv x\mu(t), \quad \text{with} \quad \mu = 10 \cdot \mathbb{1}_{[T/4,3T/4]}.$$

Let us denote by S_h the matrix arising from (2.19). The condition number $\mathrm{cond}\,(S_h) = \|S_h\|\,\|S_h^{-1}\|$ depends strongly on $h = (\Delta x, \Delta t)$. Here, the norm $\|S_h\|$ stands for the largest singular value of S_h. More precisely, it is found that $\mathrm{cond}\,(S_h) = \mathcal{O}(|h|^{-16})$. For the computations of $\|y_h - y\|_{L^2(Q)}$ and $\|u_h - u\|_{L^2(0,T)}$, we have used the couple (y,u) found for $\Delta x = \Delta t = 1/150$. Notice that $\|y_h - y\|_{L^2(Q)} = \mathcal{O}(|h|^{0.4})$ and $\|u_h - u\|_{L^2(0,T)} = \mathcal{O}(|h|^{1.0})$.

We now present the numerical results obtained by solving (2.23). They are given in Tables 3 and 4.

Notice that, now, the rate at which the condition number increases has been reduced a lot. More precisely, denoting by M_h the matrix of coefficients corresponding to (2.23), we see that $\mathrm{cond}(M_h) = \mathcal{O}(|h|^{-6.0})$ (the rate is in practice the same with and without potential). Obviously, this indicates that the results furnished by this second approach are much more reliable.

Comparing $\|u_h\|_{L^2(0,T)}$ and $\|y_h\|_{L^2(Q)}$ in Tables 1 and 3, we see that, in the case $V \equiv 0$, the bad condition number in the first approach is not relevant when the solution and the control are obtained in terms of p. Contrarily, this is not the case when $V(x,t) \equiv x\mu(t)$, as can be seen by comparing the results in Tables 2 and 4. This confirms that the first approach can be less efficient and, in general, the second approach must be adopted.

Table 1 : Potential $V(x,t) = 0$

$\Delta x, \Delta t$	1/20	1/40	1/60	1/80	1/100	1/150
cond	8.2×10^9	7.0×10^{14}	8.2×10^{18}	2.6×10^{22}	3.0×10^{25}	1.1×10^{31}
$\|p_h\|_{L^2(Q)}$	0.0395	0.0391	0.0401	0.0522	0.1159	1.1119
$\|u_h\|_{L^2(0,T)}$	0.2993	0.3111	0.3151	0.3169	0.3178	0.3189
$\|y_h\|_{L^2(Q)}$	0.6484	0.7889	0.8636	0.9095	0.9413	0.9918
$\|y_h - y\|_{L^2(Q)}$	0.6399	0.4199	0.2953	0.2092	0.1432	-
$\|u_h - u\|_{L^2(0,T)}$	0.0625	0.0322	0.0202	0.0131	0.0082	-

Table 2: Potential $V(x,t) = x * \mu(t)$

$\Delta x, \Delta t$	1/20	1/40	1/60	1/80	1/100	1/150
cond	8.2×10^9	7.0×10^{14}	8.2×10^{18}	2.6×10^{22}	3.1×10^{25}	6.0×10^{31}
$\|p_h\|_{L^2(Q)}$	0.0210	0.0202	0.0201	0.0200	0.0200	0.0202
$\|u_h\|_{L^2(0,T)}$	0.1824	0.1924	0.1947	0.1952	0.1952	0.1967
$\|y_h\|_{L^2(Q)}$	0.4531	0.6141	0.6942	0.7410	0.7723	0.8232
$\|y_h - y\|_{L^2(Q)}$	0.5923	0.3845	0.2681	0.1896	0.1304	-
$\|u_h - u\|_{L^2(0,T)}$	0.0430	0.0228	0.0152	0.0102	0.0067	-

Table 3 : Potential $V(x,t) = 0$

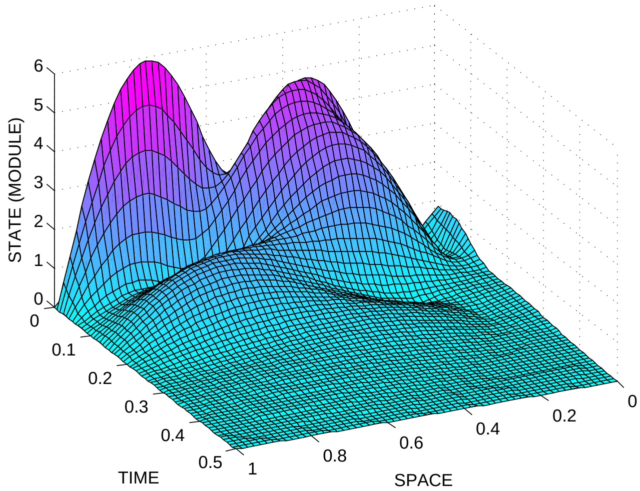
$\Delta x, \Delta t$	1/20	1/40	1/60	1/80	1/100	1/150
cond	7.0×10^9	3.8×10^{10}	1.0×10^{11}	1.4×10^{11}	3.6×10^{11}	8.6×10^{11}
$\|w_h\|_{L^2(Q)}$	0.0816	0.0875	0.0875	0.0876	0.0876	0.0878
$\|u_h\|_{L^2(0,T)}$	0.2925	0.3123	0.3158	0.3174	0.3182	0.3192
$\|y_h\|_{L^2(Q)}$	0.6731	0.7932	0.8663	0.9112	0.9425	0.9921
$\|y_h - y\|_{L^2(Q)}$	0.6445	0.4171	0.2938	0.2084	0.1428	-
$\|u_h - u\|_{L^2(0,T)}$	0.0891	0.0316	0.0200	0.0130	0.0081	-

Table 4 : Potential $V(x,t) = x * \mu(t)$

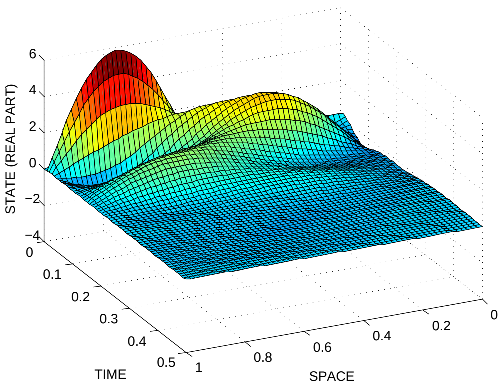
$\Delta x, \Delta t$	1/20	1/40	1/60	1/80	1/100	1/150
cond	6.8×10^9	3.7×10^{10}	1.0×10^{11}	2.0×10^{11}	3.3×10^{11}	1.1×10^{12}
$\|w_h\|_{L^2(Q)}$	0.0816	0.0811	0.0811	0.0811	0.0812	0.0814
$\|u_h\|_{L^2(0,T)}$	0.2925	0.3039	0.3079	0.3096	0.3105	0.3116
$\|y_h\|_{L^2(Q)}$	0.6731	0.8156	0.8901	0.9354	0.9666	1.0148
$\|y_h - y\|_{L^2(Q)}$	0.6435	0.4192	0.2941	0.2083	0.1427	-
$\|u_h - u\|_{L^2(0,T)}$	0.0629	0.0324	0.0203	0.0132	0.0083	-

In what regards the convergence rates, we see that, again, $\|y_h - y\|_{L^2(Q)} = \mathcal{O}(|h|^{0.4})$ and $\|u_h - u\|_{L^2(0,T)} = \mathcal{O}(|h|^{1.0})$.

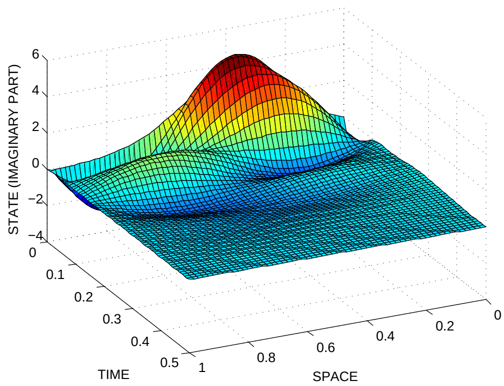
The computed states and controls are displayed in Figures 2.4–2.4.



(a) The modulus of the state



(b) The real part



(c) The imaginary part

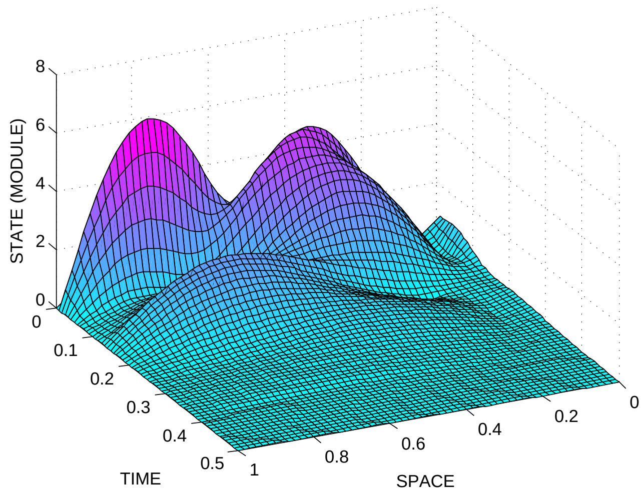
Figura 2.2: The case $V = 0$

2.5 Additional comments and conclusions

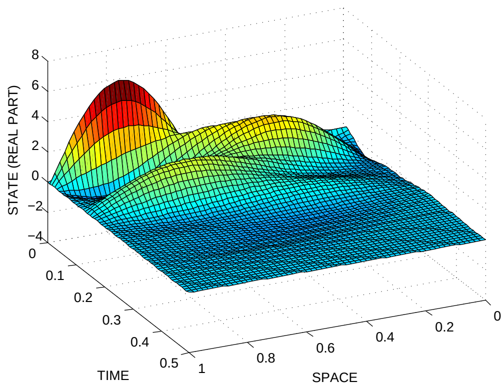
We have presented some numerical methods to solve the null controllability problem for (2.1). As mentioned above, in view of the linearity and reversibility of the Schrödinger equation, this allows to control exactly any final state.

Arguing as in some previous works, we have reduced the numerical task to solving the finite dimensional problems (2.19) and (2.23). The second one is obtained after a very natural change of variable and is more appropriate from the numerical viewpoint.

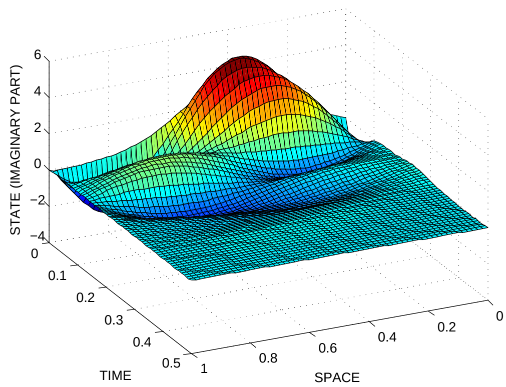
In the context of numerical boundary controllability, it would be interesting to extend the



(a) The modulus of the state



(b) The real part



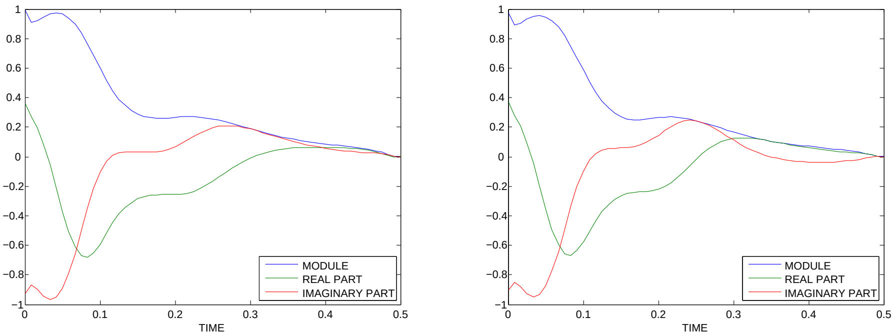
(c) The imaginary part

Figura 2.3: The case $V = x * \mu(t)$

arguments, analysis and results at least in two directions: nonlinear $1D$ Schrödinger problems

$$\begin{cases} iy_t - y_{xx} + (V(x,t) + f(y))y = 0, & (x,t) \in (0,1) \times (0,T), \\ y(0,t) = u(t), \quad y(1,t) = 0, & t \in (0,T), \\ y(x,0) = y_0(x), & x \in (0,1), \end{cases}$$

and linear Schrödinger problems in higher dimensions. This will be the objective of forthcoming work.



(a) The case $V = 0$ (b) The case $V = x * \mu(t)$

Figura 2.4: Evolution in time of the controls

Capítulo 3

Internal null controllability of a linear Schrödinger-KdV system on a bounded interval

Internal null controllability of a linear Schrödinger-KdV system on a bounded interval

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Abstract. The control of a linear dispersive system coupling a Schrödinger and a linear Korteweg-de Vries equation is studied in this paper. The system can be viewed as three coupled real-valued equations by taking real and imaginary parts in the Schrödinger equation. The null controllability is proven by using two internal real-valued controls, one acting on the linear Korteweg-de Vries equation, the other on the Schrödinger equation. Notice that the single Schrödinger equation is known to be controllable with a complex-valued control. The standard duality method is used to reduce the controllability property to the proof of an observability inequality, which is obtained by means of a Carleman estimate approach.

3.1 Introduction

In last years, a lot of papers have been oriented to study controllability properties for systems of coupled partial differential equations and new phenomena have appeared. For instance, some linear parabolic systems have been proven to be null controllable only if the time of control is large enough, which never happens when controlling single linear parabolic equations.

Most of these works have dealt with the controllability of either parabolic (see the survey [3]) or hyperbolic systems (see [1, 2, 7] and the references therein). Approaches as Carleman estimates, moment problems and energy methods have been applied to obtain internal and boundary controllability results.

Concerning the controllability of dispersive systems, there are much less results. Several Boussinesq systems have been considered in [68] where internal exact controllability results are proven. Other systems coupling Korteweg-de Vries equations have been studied in [22, 67] where boundary exact controllability results have been established.

In this paper we are interested in a linear dispersive system posed on the interval $[0,1]$ and formed by two coupled PDEs: a Schrödinger equation and a linear Korteweg-de Vries (KdV) equation. We consider internal controls supported on a nonempty open subset $\omega \subset (0,1)$ and homogeneous boundary conditions.

Given $T > 0$, we denote $Q = (0,1) \times (0,T)$ and $Q_\omega = \omega \times (0,T)$. Moreover, $\mathbf{1}_\omega$ stands for the characteristic function of ω and M, a_1, a_2, a_3, a_4 are given functions. Throughout this work, for a complex number z, we denote by $\overline{z}$, $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ the conjugate, the real part and the imaginary part of z, respectively.

The control system reads as

$$\begin{cases} iw_t + w_{xx} = a_1 w + a_2 y + \ell \mathbf{1}_\omega & \text{in} \quad Q, \\ y_t + y_{xxx} + (My)_x = Re(a_3 w) + a_4 y + h\mathbf{1}_\omega & \text{in} \quad Q, \\ w(0,t) = w(1,t) = 0 & \text{in} \quad (0,T), \\ y(0,t) = y(1,t) = y_x(1,t) = 0 & \text{in} \quad (0,T), \\ w(x,0) = w_0(x), \quad y(x,0) = y_0(x) & \text{in} \quad (0,1), \end{cases} \tag{3.1}$$

where the state is formed by the complex-valued function w and the real-valued function y. The controls are the complex-valued funciton ℓ and the real-valued function h. This system is a linearized version of a Schrödinger-Korteweg-de Vries system appearing in fluid mechanics as well as plasma physics to model the interactions between a short-wave $w = w(x,t)$ and a longwave $y = y(x,t)$ (see for instance [58] where capillary-gravity waves are considered). Well posedness studies have been performed when the system is studied on the whole line [14, 25] or on the torus [8].

This system can be viewed as coupling three real-valued equations by taking real and imaginary parts in the complex-valued Schrödinger equation. In this work we aim at proving control properties with less controls than equations. Indeed, we will prove that this system is null controllable by using the control h and either a purely real or a purely imaginary control ℓ. Thus, we require two real-valued inputs to control the full system. It is worth to mention that the single Schrödinger equation is known to be controllable with a complex-valued control. Here, thanks to the coupling with the KdV equation, we can remove either the real or complex part of this control.

Let us take a look at the controllability properties for each equation in our system separately. From now on, complex-valued function spaces are denoted using bold letters.

Concerning the Schrödinger equation posed on a domain $\Omega \subset \mathbb{R}^n$, with control supported in an arbitrary open set $\omega \subset \Omega$, it is known that the internal exact controllability holds in the state space $\mathbf{H}^{-1}(\Omega)$ with controls in the control space $L^2(0,T;\mathbf{H}^{-1}(\omega))$ ([13, 90]). However, this can be improved in dimension one to get the exact controllability in the state space $\mathbf{L}^2(\Omega)$ with controls in the control space $L^2(0,T;\mathbf{L}^2(\omega))$ as proven in [**?**, 78]. In these works, we see that the internal control is always a complex-valued function. See also the classical results [56, **?**, **?**].

For the controllability of the KdV equation on an interval $[0, L]$, we refer to the recent result [18] where the internal null controllability is proven in the state space $L^2(0, L)$ with controls in the control space $L^2(0,T;L^2(\omega))$. In [18], the authors prove a Carleman inequality, which has been obtained in an independent way to the one proved in the present paper. We refer to [21, 77] for surveys on the controllability of the KdV equation.

Going back to our control system, we can say that, to our best knowledge, there is no result concerning the controllability of (3.1) and we hope the present paper will be the starting point for further research. The main result of our paper is the following.

Teorema 3.1. *Let $T > 0$. We suppose $M \in L^2(0,T;H^1(0,1)) \cap L^\infty(0,T;L^2(0,1))$, a_1, $a_4 \in L^\infty(0,T;W^{1,\infty}(0,1))$, and $a_2, a_3 \in L^\infty(0,T;\boldsymbol{W}^{1,\infty}(0,1))$. Suppose also that*

$$Im(a_2) \in C((0,T);\boldsymbol{W}^{1,\infty}(0,1)) \text{ with } |Im(a_2)| \geq \delta > 0 \text{ in } \omega. \tag{3.2}$$

For any $(w_0, y_0) \in \boldsymbol{H}^{-1}(0,1)\times L^2(0,1)$, there exists a pair of controls $(\ell \mathbf{1}_\omega, h\mathbf{1}_\omega) \in L^2(0,T;H^{-1}(0,1))\times L^2(0,T;L^2(0,1))$, such that the unique solution $(w,y) \in C([0,T], \boldsymbol{H}^{-1}(0,1)\times L^2(0,1))$ of (3.1) satisfies

$$w(T,\cdot) = 0, \quad y(T,\cdot) = 0.$$

Observação 3.1. *In fact, in Theorem 3.1 we obtain a control $\ell \in H^1(\omega)'$, the dual space of $H^1(\omega)$. The function $\ell\mathbf{1}_\omega$ denotes the element in $H^{-1}(0,1)$ defined by*

$$\langle \ell \mathbf{1}_\omega, \theta\rangle_{H^{-1}(0,1),H_0^1(0,1)} = \langle \ell, \theta\mathbf{1}_\omega\rangle_{(H^1(\omega))',H^1(\omega)}, \quad \forall \theta \in H_0^1(0,1).$$

Observação 3.2. *Notice that, in Theorem 3.1, the control ℓ acting on the Schrödinger equation is a real-valued function. If we consider the hypothesis $|Re(a_2)| > 0$ instead of $|Im(a_2)| > 0$, we still obtain a null-controllability result. In this case, the control of the Schrödinger equation is a pure imaginary function.*

In order to prove Theorem 3.1, we follow the standard controllability-observability duality, which reduces the null controllability property to the following observability inequality.

Teorema 3.2. *Assuming the hypothesis of Theorem 3.1, there exists $C > 0$ such that*

$$\|\phi(\cdot,0)\|^2_{\mathbf{H}^1(0,1)} + \|\psi(\cdot,0)\|^2_{L^2(0,1)} \le C\left(\iint_{Q_\omega} (|Re(\phi)|^2 + |Re(\phi_x)|^2 + |\psi|^2)dxdt\right), \tag{3.3}$$

for any $(\phi^T, \psi^T) \in \boldsymbol{H}_0^1(0,1) \times L^2(0,1)$, where $(\phi,\psi) \in C([0,T];\boldsymbol{H}_0^1(0,1) \times L^2(0,1))$ is the solution of the adjoint system

$$\begin{cases} i\phi_t + \phi_{xx} = a_1\phi + \bar{a_3}\psi & in \quad Q, \\ -\psi_t - \psi_{xxx} - M\psi_x = Re(\bar{a_2}\phi) + a_4\psi & in \quad Q, \\ \phi(0,t) = \phi(1,t) = 0 & in \quad (0,T), \\ \psi(0,t) = \psi(1,t) = \psi_x(0,t) = 0 & in \quad (0,T), \\ \phi(x,T) = \phi^T(x), \quad \psi(x,T) = \psi^T(x) & in \quad (0,1). \end{cases} \tag{3.4}$$

Observação 3.3. *Notice that ϕ appears in the observation only by its real part. This allows us to prove that the control ℓ acting on the Schrödinger equation in* (3.1) *can be chosen as a real-valued function.*

The work is organized as follows. In Section 3.2 we state well-posedness results we need in this work. In Section 3.3, we prove the Carleman estimates we will use later. In fact, we prove one parameter Carleman estimates for the KdV and Schrödinger equations, and combine them in order to get an appropriate Carleman estimate for the adjoint system (3.4). Section 3.4 is devoted to prove the observability inequality (3.3). As already mentioned, Theorem 3.1 is a direct consequence of Theorem 3.2.

3.2 Well-posedness results

Let us introduce some functional spaces which will be used along the paper:

$$\begin{gathered} X_0 := L^2(0,T;H^{-2}(0,1)), \quad X_1 := L^2(0,T;H^2_0(0,1)), \\ \tilde{X}_0 := L^1(0,T;H^{-1}(0,1)), \quad \tilde{X}_1 := L^1(0,T;H^3(0,1)\cap H^1_0(0,1)), \\ Y_0 := L^2(0,T;L^2(0,1))\cap C([0,T];H^{-1}(0,1)), \\ Y_1 := L^2(0,T;H^4(0,1))\cap C([0,T];H^3(0,1)). \end{gathered} \tag{3.5}$$

In addition to these, we will define (see e.g. [15]), for each $\theta\in[0,1]$, the (complex) interpolation spaces

$$X_\theta := (X_0,X_1)_{[\theta]}, \quad \tilde{X}_\theta := (\tilde{X}_0,\tilde{X}_1)_{[\theta]} \quad \text{and} \quad Y_\theta := (Y_0,Y_1)_{[\theta]}.$$

In this section we will assume the following regularity of the coefficients:

$$a_1\in L^\infty(0,T;W^{1,\infty}(0,1)), a_2\in \boldsymbol{L}^\infty(Q), a_3\in L^\infty(0,T;\boldsymbol{W}^{1,\infty}(0,1)), a_4\in L^\infty(Q), M\in Y_{\frac{1}{4}}. \tag{3.6}$$

Notice that $Y_{\frac{1}{4}} = L^2(0,T;H^1_0(0,1))\cap C([0,T];L^2(0,1))$.

The main goal of this section is to prove the well posedness of system

$$\begin{cases} iw_t + w_{xx} = a_1 w + a_2 y + f_1 & \text{in} \quad Q,\\ y_t + y_{xxx} + (My)_x = \operatorname{Re}(a_3 w) + a_4 y + f_2 & \text{in} \quad Q,\\ w(0,t)=w(1,t)=0 & \text{in} \quad (0,T),\\ y(0,t)=y(1,t)=y_x(1,t)=0 & \text{in} \quad (0,T),\\ w(x,0)=w_0(x), \quad y(x,0)=y_0(x) & \text{in} \quad (0,1), \end{cases} \tag{3.7}$$

and its adjoint one given by

$$\begin{cases} i\phi_t + \phi_{xx} = a_1\phi + \bar{a_3}\psi + g_1 & \text{in} \quad Q,\\ -\psi_t - \psi_{xxx} - M\psi_x = \operatorname{Re}(\bar{a_2}\phi) + a_4\psi + g_2 & \text{in} \quad Q,\\ \phi(0,t)=\phi(1.t)=0 & \text{in} \quad (0,T),\\ \psi(0,t)=\psi(1,t)=\psi_x(0,t)=0 & \text{in} \quad (0,T),\\ \phi(x,T)=\phi_T(x), \quad \psi(x,T)=\psi_T(x) & \text{in} \quad (0,1). \end{cases} \tag{3.8}$$

Proposição 3.1. *Under hypotheses* (3.6)*, for any* $(g_1,g_2)\in L^1(0,T;\boldsymbol{H}^1_0(0,1))\times L^1(0,T;L^2(0,1))$ *and* $(\phi_T,\psi_T)\in \boldsymbol{H}^1_0(0,1)\times L^2(0,1)$*, the system* (3.8) *has a unique solution*

$$(\phi,\psi)\in C([0,T];\boldsymbol{H}^1_0(0,1))\times Y_{\frac{1}{4}}.$$

Concerning system (3.7), we will consider solutions in the sense of transposition.

Definição 3.1. *Given* $(w_0,y_0)\in \boldsymbol{H}^{-1}(0,1)\times L^2(0,1)$ *and* $(f_1,f_2)\in L^2(0,T;\boldsymbol{H}^{-1}(0,1))\times L^2(Q)$*, we say that* $(w,y)\in L^\infty(0,T;\boldsymbol{H}^{-1}(0,1))\times L^\infty(0,T;L^2(0,1))$ *is a solution (by transposition) of system* (3.7) *if*

$$\int_0^T \langle w,\overline{g}_1\rangle_{\boldsymbol{H}^{-1},\boldsymbol{H}^1_0} dt + \iint_Q y g_2\,dxdt = \int_0^T \langle f_1,\overline{\phi}\rangle_{\boldsymbol{H}^{-1},\boldsymbol{H}^1_0} dt \\ + \iint_Q f_2\psi\,dxdt + i\langle w_0,\bar{\phi}|_{t=0}\rangle_{\boldsymbol{H}^{-1},\boldsymbol{H}^1_0} + \int_0^1 y_0(x)\psi(x,0)dx, \tag{3.9}$$

for all $(g_1, g_2) \in L^1(0,T;\boldsymbol{H}_0^1(0,1)) \times L^1(0,T;L^2(0,1))$*, where* $(\phi,\psi) \in C([0,T];\boldsymbol{H}_0^1(0,1)) \times Y_{\frac{1}{4}}$ *is the solution of system* (3.8) *with* $(\phi_T, \psi_T) = (0,0)$*.*

The following result holds.

Proposição 3.2. *Under hypotheses* (3.6)*, for any* $(f_1, f_2) \in L^1(0,T;\boldsymbol{H}^{-1}(0,1)) \times L^1(0,T;L^2(0,1))$ *and* $(w_0, y_0) \in \boldsymbol{H}^{-1}(0,1) \times L^2(0,1)$*, the system* (3.7) *has a unique solution*

$$(w, y) \in C([0,T];\boldsymbol{H}^{-1}(0,1)) \times C([0,T];L^2(0,1)). \tag{3.10}$$

Before proving propositions 3.1 and 3.2, we recall some known results about the well-posedness of each equation appearing in system (3.7).

3.2.1 Previous regularity results

Let us consider the linear KdV equation given by

$$\begin{cases} -\psi_t - \psi_{xxx} - M\psi_x = g & \text{in} \quad Q, \\ \psi(0,t) = \psi(1,t) = \psi_x(0,t) = 0 & \text{in} \quad (0,T), \\ \psi(x,T) = \psi_T(x) & \text{in} \quad (0,1). \end{cases} \tag{3.11}$$

Proposição 3.3 ([44])**.** *Let* $M \in Y_{\frac{1}{4}}$ *be given. If* $\psi_T \in L^2(0,1)$ *and* $g \in G$ *with* $G = L^2(0,T;H^{-1}(0,1))$ *or* $G = L^1(0,T;L^2(0,1))$*, then system* (3.11) *has a unique solution* $\psi \in Y_{\frac{1}{4}}$*. Moreover, there exists a constant* $C > 0$ *such that*

$$\|\psi\|_{Y_{\frac{1}{4}}} \leq C(\|g\|_G + \|\psi_T\|_{L^2(0,1)}). \tag{3.12}$$

In the case $M = 0$, we have the following improved regularity.

Proposição 3.4 ([44])**.** *Suppose that* $M = 0$*. If* $\psi_T \in H^3(0,1)$ *is such that* $\psi_T(0) = \psi_T(1) = \psi_T'(1) = 0$*, and* $g \in G$ *with* $G = L^2(0,T;H_0^2(0,1))$ *or* $G = L^1(0,T;H^3(0,1) \cap H_0^2(0,1))$*, then system* (3.11) *has a unique solution* $\psi \in Y_1$*. Moreover, there exist a constant* $C > 0$ *such that*

$$\|\psi\|_{Y_1} \leq C(\|g\|_G + \|\psi_T\|_{H^3(0,1)}). \tag{3.13}$$

By using an interpolation argument (see e.g. [15]) and propositions 3.3 and 3.4 we have the following well-posedeness results for intermediate spaces.

Corolário 3.1 ([44])**.** *Let* $\theta \in [0,1]$ *be given and suppose* $M = 0$ *and* $\psi_T = 0$*. If* $g \in G$ *with* $G = X_\theta$ *or* $G = \tilde{X}_\theta$*, then system* (3.11) *has a unique solution* $\psi \in Y_\theta$*. Moreover, there exists a constant* $C > 0$ *such that*

$$\|\psi\|_{Y_\theta} \leq C\|g\|_G. \tag{3.14}$$

Let us consider now the linear Schrödinger equation

$$\begin{cases} i\phi_t + \phi_{xx} = a_1\phi + g & \text{in} \quad Q, \\ \phi(0,t) = \phi(1,t) = 0 & \text{in} \quad (0,T), \\ \phi(x,T) = \phi_T(x) & \text{in} \quad (0,1). \end{cases} \tag{3.15}$$

Proposição 3.5 (see [20])**.** *Suppose* $a_1 \in \mathbf{L}^\infty(0,T;\mathbf{W}^{1,\infty}(0,1))$*. For any* $\phi_T \in \mathbf{X}$ *and* $g \in L^1(0,T;\mathbf{X})$*, with* $\mathbf{X} = \mathbf{L}^2(0,1)$ *or* $\mathbf{X} = \boldsymbol{H}_0^1(0,1)$*, there exists a unique solution* $\phi \in C([0,T];\mathbf{X})$ *of system* (3.15)*.*

3.2.2 Proofs of propositions 3.1 and 3.2

Proof of Proposition 3.1: Let us consider the map

$$\Pi : L^1(0,T;L^2(0,1)) \to [C([0,T];L^2(0,1))]^2$$

defined by $\Pi\tilde{\psi} = (\phi,\psi)$, where

$$\begin{cases} i\phi_t + \phi_{xx} = a_1\phi + \bar{a}_3\tilde{\psi} + g_1 & \text{in} \quad Q, \\ -\psi_t - \psi_{xxx} - M\psi_x = \mathrm{Re}(\bar{a}_2\phi) + a_4\tilde{\psi} + g_2 & \text{in} \quad Q, \\ \phi(0,t) = \phi(1,t) = 0 & \text{in} \quad (0,T), \\ \psi(0,t) = \psi(1,t) = \psi_x(0,t) = 0 & \text{in} \quad (0,T), \\ \phi(x,T) = \phi_T(x), \quad \psi(x,T) = \psi_T(x) & \text{in} \quad (0,1). \end{cases} \tag{3.16}$$

From Proposition 3.5, we get $\phi \in C([0,T];L^2(0,1))$, and then, Proposition 3.3 gives us $\psi \in Y_{\frac{1}{4}}$. Hence operator Π is well defined. Now we set

$$\Lambda : L^1(0,T;L^2(0,1)) \to L^1(0,T;L^2(0,1))$$

by $\Lambda\tilde{\psi} = (\Pi\tilde{\psi})_2 = \psi$. Then we get that the range of Λ is contained in $L^2(0,T;H_0^1(0,1))$, which is a compact subset of $L^1(0,T;L^2(0,1))$. Thus, by Schauder's Theorem, Λ has a fixed point $\psi \in L^2(0,T;H_0^1(0,1))$, and then $(\phi,\psi) = \Pi\psi$ solves system (3.8). Now, since $a_3 \in L^\infty(0,T;\boldsymbol{W}^{1,\infty}(0,1))$, we get $\bar{a}_3\psi \in L^2(0,T;\boldsymbol{H}_0^1(0,1))$ and, from Proposition 3.5, we deduce that $(\phi,\psi) \in C([0,T];\boldsymbol{H}_0^1(0,1)) \times Y_{\frac{1}{4}}$, which ends the proof. ■

Observação 3.4. *If we suppose that* $(\phi_T,\psi_T) \in \boldsymbol{H}_0^1(0,1) \times H_0^1(0,1)$*, regularity* (3.6) *plus the additional one* $a_4 \in L^\infty(0,T;W^{1,\infty}(0,1))$*, we can proceed as in the proof of Proposition 3.1 to obtain a solution* $(\phi,\psi) \in C([0,T];\boldsymbol{H}_0^1(0,1)) \times Y_{\frac{1}{2}}$.

Proof of Proposition 3.2: The right hand side of (3.9) defines a linear functional which maps $(g_1,g_2) \in L^1(0,T;\boldsymbol{H}_0^1(0,1)) \times L^1(0,T;L^2(0,1))$ to $\mathbb{R}$. By the regularity stated in Proposition 3.1, this functional is continuous. By Riesz's Theorem, there exists a unique pair $(w,y) \in L^\infty(0,T;\boldsymbol{H}^{-1}(0,1)) \times L^\infty(0,T;L^2(0,1))$ satisfying (3.9). The regularity (3.10) follows by density argument. ■

3.3 Carleman estimates

This section is devoted to the proof of several appropriate Carleman estimates which will be useful in next section in order to prove the observability and therefore the null controllability of our Schrödinger-KdV system. First, we deal with the single equations by separate and then we address the coupled system. In all these cases, we use the same weight functions defined as follows.

Let us suppose that $\omega = (\tilde{a}_0,\tilde{b}_0) \subset (0,1)$ and let $[a_0,b_0] \subset \omega$. Let $c_0 = (a_0+b_0)/2$ and consider, for $K_1, K_2 > 0$ to be chosen later, the functions

$$\phi_0(x) = -K_1\exp(-K_2(x-c_0)^2) + K_1 + 1, \tag{3.17}$$

$$\xi(t) = \frac{1}{t(T-t)} \quad \text{and} \quad \Phi(x,t) = \phi_0(x)\xi(t). \tag{3.18}$$

We take $K_2 = 1/2(c_0 - a_0)^2$. If $c_0 \geq 1/2$, then the constant K_1 is chosen such that $3K_1 < 1/(1 - \exp(-K_2 c_0^2))$. If not, K_1 is chosen such that $3K_1 < 1/(1 - \exp(-K_2(1-c_0)^2))$. In both cases, there exists a positive constant C such that

$$\begin{array}{lll} -\phi_0''(x) \geq C \quad \text{and} \quad |\phi_0'(x)|^2 \geq C & \text{in} & [0,1] \setminus \overline{\omega}, \\ \phi_0'(1) > 0 \quad \text{and} \quad \phi_0'(0) < 0, & & \\ 8\check{\Phi}(t) - 6\hat{\Phi}(t) > 0 & \text{in} & [0,T], \end{array} \tag{3.19}$$

where $(\hat{\Phi}(t), \check{\Phi}(t)) = (\max_{x\in[0,1]} \Phi(t,x), \min_{x\in[0,1]} \Phi(t,x))$.

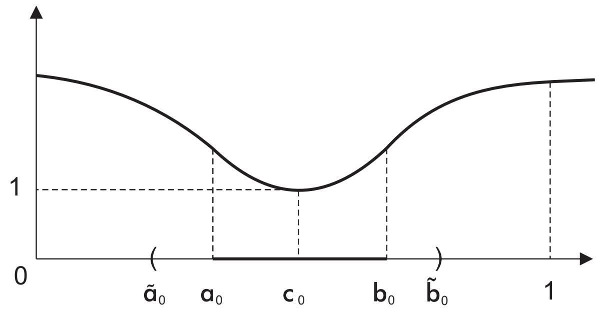

Figura 3.1: The weight function ϕ_0. The choices of K_1, K_2 guarantee that hypotheses (3.19) are satisfied.

3.3.1 Carleman estimate for the KdV equation

The following result establishes a Carleman inequality for the KdV equation.

Teorema 3.3. *There exist $C_0 > 0$ and $s_0 \geq 1$ such that*

$$\begin{aligned} s^5 \iint_Q e^{-2s\Phi}\xi^5|v|^2\,dxdt + s^3 \iint_Q e^{-2s\Phi}\xi^3|v_x|^2\,dxdt \\ + s\iint_Q e^{-2s\Phi}\xi|v_{xx}|^2\,dxdt \leq C_0 \Big(\iint_Q e^{-2s\Phi}|Lv|^2\,dxdt \\ + s^5 \iint_{Q_\omega} e^{-2s\Phi}\xi^5|v|^2\,dxdt + s\iint_{Q_\omega} e^{-2s\Phi}\xi|v_{xx}|^2\,dxdt \Big) \end{aligned} \tag{3.20}$$

for all $s > s_0$, for all $v \in L^2(0,T; H^2 \cap H_0^1(0,1))$ such that $v_x(0,t) = 0$ for all t and $Lv := v_t + v_{xxx} + Mv_x \in L^2(0,T; L^2(0,1))$.

Proof: Let us define

$$w = e^{-s\Phi}v, \tag{3.21}$$

for each $s > 0$ and $v \in C^\infty(Q)$ with $v(0,t) = v(1,t) = v_x(0,t) = 0$. Then $w(x,0) = w(x,T) = 0$ and

$$\begin{aligned}
v_t &= se^{s\Phi}\Phi_t w + e^{s\Phi} w_t,\\
v_x &= se^{s\Phi}\Phi_x w + e^{s\Phi} w_x,\\
v_{xx} &= s^2 e^{s\Phi}(\Phi_x)^2 w + se^{s\Phi}\Phi_{xx} w + 2se^{s\Phi}\Phi_x w_x + e^{s\Phi} w_{xx},\\
v_{xxx} &= s^3 e^{s\Phi}(\Phi_x)^3 w + 3s^2 e^{s\Phi}\Phi_x\Phi_{xx} w + 3s^2 e^{s\Phi}(\Phi_x)^2 w_x,\\
&\quad + se^{s\Phi}\Phi_{xxx} w + 3se^{s\Phi}\Phi_{xx} w_x + 3se^{s\Phi}\Phi_x w_{xx} + e^{s\Phi} w_{xxx}.
\end{aligned}$$

In this way, if we define $L_\Phi w = e^{-s\Phi} Lv = e^{-s\Phi} L(e^{s\Phi} w)$ we have the following identity

$$\begin{aligned}
L_\Phi w &= s\Phi_t w + w_t + s^3(\Phi_x)^3 w + 3s^2\Phi_x\Phi_{xx} w + 3s^2(\Phi_x)^2 w_x + s\Phi_{xxx} w\\
&\quad +3s\Phi_{xx} w_x + 3s\Phi_x w_{xx} + w_{xxx} + M(s\Phi_x w + w_x).
\end{aligned} \tag{3.22}$$

If we write

$$\begin{aligned}
L_1 w &= w_t + w_{xxx} + 3s^2(\Phi_x)^2 w_x,\\
L_2 w &= 3s\Phi_x w_{xx} + s^3(\Phi_x)^3 w + 3s\Phi_{xx} w_x,
\end{aligned} \tag{3.23}$$

we have that

$$\|L_1 w\|^2_{L^2(Q)} + \|L_2 w\|^2_{L^2(Q)} + 2(L_1 w, L_2 w)_{L^2(Q)} = \|L_\Phi w - Rw\|^2_{L^2(Q)} \tag{3.24}$$

where

$$Rw = s\Phi_t w + s\Phi_{xxx} w + 3s^2\Phi_x\Phi_{xx} w + M(s\Phi_x w + w_x).$$

We now examine each integral coming from $(L_1 w, L_2 w)_{L^2(Q)}$. Denoting I_{ij} the L^2-product of the i-th term of $L_1 w$ with the j-th term of $L_2 w$, we have:

$$I_{11} = 3s\iint_Q \Phi_x w_t w_{xx}\,dxdt = -3s\iint_Q \Phi_{xx} w_t w_x\,dxdt + \frac{3}{2}s\iint_Q \Phi_{xt}|w_x|^2\,dxdt, \tag{3.25}$$

$$I_{12} = \frac{1}{2}s^3\iint_Q (\Phi_x)^3\frac{d}{dt}|w|^2\,dxdt = -\frac{3}{2}s^3\iint_Q (\Phi_x)^2\Phi_{xt}|w|^2\,dxdt, \tag{3.26}$$

$$I_{13} = 3s\iint_Q \Phi_{xx} w_x w_t\,dxdt = \frac{3}{2}s\iint_Q \Phi_{xt}|w_x|^2\,dxdt - I_{11}, \tag{3.27}$$

$$\begin{aligned}
I_{21} &= \frac{3}{2}s\iint_Q \Phi_x\partial_x|w_{xx}|^2\,dxdt\\
&= \frac{3}{2}s\int_0^T \Phi_x(1,t)|w_{xx}(1,t)|^2\,dt - \frac{3}{2}s\int_0^T \Phi_x(0,t)|w_{xx}(0,t)|^2\,dt\\
&\quad - \frac{3}{2}s\iint_Q \Phi_{xx}|w_{xx}|^2\,dxdt,
\end{aligned} \tag{3.28}$$

$$\begin{aligned}
I_{22} &= s^3\iint_Q (\Phi_x)^3 w w_{xxx}\,dxdt = -3s^3\iint_Q (\Phi_x)^2\Phi_{xx} w w_{xx}\,dxdt\\
&\quad -\frac{1}{2}s^3\iint_Q (\Phi_x)^3\partial_x|w_x|^2\,dxdt\\
&= -\frac{3}{2}s^3\iint_Q ((\Phi_x)^2\Phi_{xx})_{xx}|w|^2\,dxdt + \frac{9}{2}s^3\iint_Q (\Phi_x)^2\Phi_{xx}|w_x|^2\,dxdt\\
&\quad -\frac{1}{2}s^3\int_0^T (\Phi_x(1,t))^3|w_x(1,t)|^2\,dt,
\end{aligned} \tag{3.29}$$

$$
\begin{aligned}
I_{23} &= 3s\int_0^T \Phi_{xx}(1,t)w_x(1,t)w_{xx}(1,t)\,dt - \frac{3}{2}s\iint_Q \Phi_{xxx}\partial_x|w_x|^2\,dxdt \\
&\quad -3s\iint_Q \Phi_{xx}|w_{xx}|^2\,dxdt \\
&= 3s\int_0^T \Phi_{xx}(1,t)w_x(1,t)w_{xx}(1,t)dt - \frac{3}{2}s\int_0^T \Phi_{xxx}(1,t)|w_x(1,t)|^2\,dt \\
&\quad +\frac{3}{2}s\iint_Q \Phi_{xxxx}|w_x|^2\,dxdt - 3s\iint_Q \Phi_{xx}|w_{xx}|^2\,dxdt,
\end{aligned} \tag{3.30}
$$

$$
\begin{aligned}
I_{31} &= \frac{9}{2}s^3\iint_Q \Phi_x^3\partial_x|w_x|^2\,dxdt = \frac{9}{2}s^3\int_0^T \Phi_x(1,t)^3|w_x(1,t)|^2\,dt \\
&\quad -\frac{27}{2}s^3\iint_Q (\Phi_x)^2\Phi_{xx}|w_x|^2\,dxdt,
\end{aligned} \tag{3.31}
$$

$$
I_{32} = \frac{3}{2}s^5\iint_Q (\Phi_x)^5\partial_x|w|^2\,dxdt = -\frac{15}{2}s^5\iint_Q (\Phi_x)^4\Phi_{xx}|w|^2\,dxdt, \tag{3.32}
$$

and

$$
I_{33} = 9s^3\iint_Q (\Phi_x)^2\Phi_{xx}|w_x|^2\,dxdt. \tag{3.33}
$$

Gathering all the computations and canceling the common terms, we get

$$
\begin{aligned}
(L_1w, L_2w)_{L^2(Q)} &= \iint_Q \left(-\frac{3}{2}s^3(\Phi_x)^2\Phi_{xt} - \frac{3}{2}s^3((\Phi_x)^2\Phi_{xx})_{xx} - \frac{15}{2}s^5(\Phi_x)^4\Phi_{xx}\right)|w|^2\,dxdt \\
&\quad +\iint_Q \left(\frac{3}{2}s\Phi_{xt} + \frac{3}{2}s\Phi_{xxxx}\right)|w_x|^2\,dxdt - \frac{9}{2}s\iint_Q \Phi_{xx}|w_{xx}|^2\,dxdt \\
&\quad +\frac{3}{2}s\int_0^T \left(\Phi_x(1,t)|w_{xx}(1,t)|^2 - \Phi_x(0,t)|w_{xx}(0,t)|^2\right)\,dt \\
&\quad +\int_0^T \left(-\frac{3}{2}\Phi_{xxx}(1,t) + 4s^3(\Phi_x(1,t))^3\right)|w_x(1,t)|^2\,dt \\
&\quad +3s\int_0^T \Phi_{xx}(1,t)w_x(1,t)w_{xx}(1,t)\,dt.
\end{aligned} \tag{3.34}
$$

Replacing (3.34) in (3.24) we get:

$$
\begin{aligned}
&\|L_1w\|^2_{L^2(Q)} + \|L_2w\|^2_{L^2(Q)} - 15s^5\iint_Q (\Phi_x)^4\Phi_{xx}|w|^2\,dxdt \\
&\quad - 9s\iint_Q \Phi_{xx}|w_{xx}|^2\,dxdt + 3s\int_0^T (\Phi_x(1,t)|w_{xx}(1,t)|^2 - \Phi_x(0,t)|w_{xx}(0,t)|^2)\,dt \\
&\quad + 8s^3\int_0^T (\Phi_x(1,t))^3|w_x(1,t)|^2\,dt = \|L_\Phi w - Rw\|^2_{L^2(Q)} + \Psi(w),
\end{aligned} \tag{3.35}
$$

where

$$
\begin{aligned}
\Psi(w) &= \iint_Q \left(3s^3(\Phi_x)^2\Phi_{xt} + 3s^3((\Phi_x)^2\Phi_{xx})_{xx}\right)|w|^2\,dxdt - \iint_Q (3s\Phi_{xt} + 3s\Phi_{xxxx})|w_x|^2\,dxdt \\
&\quad 3\int_0^T \Phi_{xxx}(1,t)|w_x(1,t)|^2dt - 6s\int_0^T \Phi_{xx}(1,t)w_x(1,t)w_{xx}(1,t)\,dt.
\end{aligned}
$$

Integrating by parts and using Young inequality we also have

$$s^3 \iint_Q \xi^3 |w_x|^2 dxdt \le s^5 \iint_Q \xi^5 |w_{xx}|^2 dxdt + s \iint_Q \xi |w|^2 dxdt. \tag{3.36}$$

Consider $\omega_0 \subset\subset \omega$ such that hypotheses (3.19) still hold in ω_0. Hence, combining (3.35) and (3.36) we have that there exists $C > 0$ such that

$$\begin{aligned}
&\|L_1 w\|^2_{L^2(Q)} + \|L_2 w\|^2_{L^2(Q)} + \iint_Q \left(s^5\xi^5|w|^2 + s^3\xi^3|w_x|^2 + s\xi|w_{xx}|^2\right) dxdt \\
&\qquad + s\int_0^T \xi(|w_{xx}(1,t)|^2 + |w_{xx}(0,t)|^2)\, dt + s^3 \int_0^T \xi^3 |w_x(1,t)|^2\, dt \\
&\le C \iint_Q |L_\phi w|^2\, dxdt + C \iint_{Q_{\omega_0}} \left(s^5\xi^5|w|^2 + s\xi|w_{xx}|^2\right) dxdt + C\|Rw\|^2_{L^2(Q)} + C\Psi(w).
\end{aligned} \tag{3.37}$$

In order to estimate Ψ, notice that there exists $C > 0$ such that

$$\int\int_Q (3s^3(\Phi_x)^2\Phi_{xt} + 3s^3((\Phi_x)^2\Phi_{xx})_{xx})|w|^2\, dxdt \le Cs^3 \iint_Q \xi^4 |w|^2\, dxdt, \tag{3.38}$$

and

$$\iint_Q (3s\Phi_{xt} + 3s\Phi_{xxxx})|w_x|^2\, dx\, dt \le Cs \iint_Q \xi^2 |w_x|^2\, dxdt. \tag{3.39}$$

We also have that

$$\begin{aligned}
&3\int_0^T \Phi_{xxx}(1,t)|w_x(1,t)|^2 dt - 12s \int_0^T \Phi_{xx}(1,t) w_x(1,t) w_{xx}(1,t) dt \\
&\qquad \le Cs^2 \int_0^T \xi^3(t)|w_x(1,t)|^2 dt + C\int_0^T \xi(t)|w_{xx}(1,t)|^2 dt.
\end{aligned} \tag{3.40}$$

Combining (3.38), (3.39) and (3.40) we obtain

$$\begin{aligned}
|\Psi(w)| \le C \Bigg(&s^3 \iint_Q \xi^4 |w|^2\, dxdt + s\iint_Q \xi^2 |w_x|^2\, dxdt \\
&+ s^2 \int_0^T \xi^3(t)|w_x(1,t)|^2\, dt + \int_0^T \xi(t)|w_{xx}(1,t)|^2\, dt \Bigg).
\end{aligned} \tag{3.41}$$

Let $N(w)$ the left hand side of (3.37). For any $\varepsilon > 0$ there exists $s_1 > 1$ such that

$$|\Psi(w)| \le \varepsilon N(w), \tag{3.42}$$

for all $s \ge s_1$.

To estimate Rw, the following inequality is needed:

$$\begin{aligned}
\|Mw_x\|_{L^2(Q)} &\le C\|M\|_{L^\infty(0,T;L^2(0,1))}\|w\|_{L^2(0,T;H^{\frac{7}{4}}(0,1))} \\
&\le C\|M\|_{L^\infty(0,T;L^2(0,1))}\left(\|w\|_{L^2(0,T;H^2(0,1))} + \|w\|_{L^2(0,T;H^1(0,1))}\right).
\end{aligned}$$

Then, there exists $s_2 \geq 1$ such that

$$\|Rw\|^2_{L^2(Q)} \leq C\left(s^4 \iint_Q \xi^4|w|^2\,dxdt + \iint_Q |w_x|^2\,dxdt + \iint_Q |w_{xx}|^2\,dxdt\right) \leq \varepsilon N(w), \quad (3.43)$$

for $s \geq s_2$. From (3.37), (3.42) and (3.43) we obtain

$$s^5 \iint_Q \xi^5|w|^2\,dxdt + s^3 \iint_Q \xi^3|w_x|^2\,dxdt + s\iint_Q \xi|w_{xx}|^2\,dxdt \leq C\iint_Q |L_\phi w|^2\,dxdt$$
$$+ Cs^5 \iint_{Q_{\omega_0}} \xi^5|w|^2\,dxdt + Cs\iint_{Q_{\omega_0}} \xi|w_{xx}|^2\,dxdt. \quad (3.44)$$

Now we get an estimate in variable v. Taking into account (3.21), we have that

$$\begin{array}{c} e^{-2s\Phi}|v_x|^2 \leq C(s^2\xi^2|w|^2 + |w_x|^2) \quad \text{and} \\ e^{-2s\Phi}|v_{xx}|^2 \leq C(s^4\xi^4|w|^2 + s^2\xi^2|w_x|^2 + |w_{xx}|^2). \end{array} \quad (3.45)$$

Also from (3.21) we get

$$|w_{xx}|^2 \leq Ce^{-2s\Phi}\left(s^4\xi^4|v|^2 + s^2\xi^2|v_x|^2 + |v_{xx}|^2\right). \quad (3.46)$$

From (3.44) to (3.46) we obtain (3.20). ■

3.3.2 Carleman inequality for the Schrödinger equation

This section is devoted to prove the one parameter Carleman estimate for the Schrödinger equation given in the following theorem.

Teorema 3.4. *There exist constants $C > 0$ and $s_0 \geq 1$ such that*

$$s\iint_Q e^{-2s\Phi}\xi|p_x|^2\,dxdt + s^3\iint_Q e^{-2s\Phi}\xi^3|p|^2\,dxdt \leq C\iint_Q e^{-2s\Phi}|Bp|^2\,dxdt$$
$$+ Cs^3\iint_{Q_\omega} \xi^3 e^{-2s\Phi}|p|^2\,dxdt + Cs\iint_{Q_\omega} e^{-2s\Phi}\xi|Re(p_x)|^2\,dxdt, \quad (3.47)$$

for all $s > s_0$, for all $p \in L^2(0,T;\boldsymbol{H}^1_0(0,1))$ such that $Bp := ip_t + p_{xx} \in L^2(0,T;\boldsymbol{L}^2(0,1))$.

Proof: Let us define

$$q = e^{-s\Phi}p \quad (3.48)$$

for each $s > 0$ and $p \in \boldsymbol{C}^\infty(Q)$ such that $p(0,\cdot) = p(1,\cdot) = 0$. Hence we have

$$B_\Phi q := e^{-s\Phi}B(e^{-s\Phi}q) = i\left(s\Phi_t q + q_t\right) + s^2\Phi_x^2 q + s\Phi_{xx}q + 2s\Phi_x q_x + q_{xx}. \quad (3.49)$$

If we denote

$$\begin{array}{rcl} B_1 q &=& iq_t + q_{xx} + s^2\Phi_x^2 q \quad \text{and} \\ B_2 q &=& 2s\Phi_x q_x + s\Phi_{xx}q, \end{array} \quad (3.50)$$

then we get

$$\|B_1q\|^2_{L^2(Q)}+\|B_2q\|^2_{L^2(Q)}+2Re\iint_Q B_1q\overline{B_2q}dxdt \leq C\left(\iint_Q |B_\Phi q|^2\,dxdt + s^2\iint_Q |\Phi_t|^2|q|^2\,dxdt\right). \tag{3.51}$$

In order to analyze the term $(B_1q, B_2q)_{L^2(Q)}$, we denote by F_{ij} the L^2-product of the i-th term of B_1q with the j-th term of B_2q. We have that

$$\begin{aligned} F_{11} &= 2sRe\iint_Q iq_t\Phi_x\bar{q}_x\,dxdt = -2s\mathrm{Im}\iint_Q q_t\Phi_x\bar{q}_x\,dxdt \\ &= 2s\mathrm{Im}\iint_Q \Phi_x q_{tx}\bar{q}\,dxdt + 2s\mathrm{Im}\iint_Q \Phi_{xx}q_t\bar{q}\,dxdt \\ &= -2s\mathrm{Im}\iint_Q \Phi_{xt}q_x\bar{q}\,dx,dt - 2s\mathrm{Im}\int_Q \Phi_x q_x\bar{q}_t\,dxdt \\ &\quad +2s\mathrm{Im}\iint_Q \Phi_{xx}q_t\bar{q}\,dxdt \\ &= -2s\mathrm{Im}\iint_Q \Phi_{xt}q_x\bar{q}\,dxdt - F_{11} + 2s\mathrm{Im}\iint_Q \Phi_{xx}q_t\bar{q}\,dxdt. \end{aligned} \tag{3.52}$$

In this way

$$F_{11} = -s\mathrm{Im}\iint_Q \Phi_{xt}q_x\bar{q}\,dxdt + s\mathrm{Im}\iint_Q \Phi_{xx}q_t\bar{q}\,dxdt \tag{3.53}$$

and

$$F_{12} = s\mathrm{Re}\iint_Q iq_t\Phi_{xx}\bar{q}\,dxdt. \tag{3.54}$$

We also have that

$$\begin{aligned} F_{21} &= 2s\mathrm{Re}\iint_Q q_{xx}\Phi_x\bar{q}_x\,dxdt = s\mathrm{Re}\iint_Q \Phi_x\partial_x|q_x|^2\,dxdt \\ &= s\mathrm{Re}\int_0^T \left(\Phi_x(1,t)|q_x(1,t)|^2 - \Phi_x(0,t)|q_x(0,t)|^2\right)\,dt - s\mathrm{Re}\iint_Q \Phi_{xx}|q_x|^2\,dxdt, \end{aligned} \tag{3.55}$$

and

$$\begin{aligned} F_{22} &= s\mathrm{Re}\iint_Q q_{xx}\Phi_{xx}\bar{q}\,dxdt = -s\mathrm{Re}\iint_Q \left(q_x\Phi_{xxx}\bar{q} + \Phi_{xx}|q_x|^2\right)\,dxdt \\ &= -\frac{s}{2}\mathrm{Re}\iint_Q \Phi_{xxx}\partial_x|q|^2\,dxdt - s\mathrm{Re}\iint_Q \Phi_{xx}|q_x|^2\,dxdt \\ &= \frac{s}{2}\mathrm{Re}\iint_Q \Phi_{xxxx}|q|^2\,dxdt - s\mathrm{Re}\iint_Q \Phi_{xx}|q_x|^2\,dxdt. \end{aligned} \tag{3.56}$$

To finish we have

$$\begin{aligned} F_{31} &= 2s^3\mathrm{Re}\iint_Q \Phi_x^3 q\bar{q}_x\,dxdt = s^3\mathrm{Re}\iint_Q \Phi_x^3\partial_x|q|^2\,dxdt \\ &= -3s^3\mathrm{Re}\iint_Q \Phi_x^2\Phi_{xx}|q|^2\,dxdt, \end{aligned} \tag{3.57}$$

and

$$F_{32} = s^3\mathrm{Re}\iint_Q \Phi_x^2\Phi_{xx}|q|^2\,dxdt. \tag{3.58}$$

Gathering all the pervious integral terms we get

$$
\begin{aligned}
\operatorname{Re}(B_1 q, \overline{B_2 q})_{L^2(Q)} \;&=\; s\operatorname{Re}\int_0^T \left(\Phi_x(1,t)|q_x(1,t)|^2 - \Phi_x(0,t)|q_x(0,t)|^2\right)\, dt \\
&\quad -s\operatorname{Im}\iint_Q \Phi_{xt} q_x \bar{q}\, dxdt - s\operatorname{Re}\iint_Q \Phi_{xx}|q_x|^2\, dxdt \\
&\quad \frac{s}{2}\operatorname{Re}\iint_Q \Phi_{xxxx}|q|^2\, dxdt - s\operatorname{Re}\iint_Q \Phi_{xx}|q_x|^2\, dxdt \\
&\quad -2s^3\operatorname{Re}\iint_Q \Phi_x^2\Phi_{xx}|q|^2\, dxdt.
\end{aligned} \tag{3.59}
$$

We have that there exists $\omega_0 \subset\subset \omega$ such that hypotheses (3.19) still hold in ω_0. Hence, from (3.59) we have that there exist constants $C > 0$ and s_1 such that

$$
\begin{aligned}
s^3\iint_Q \xi^3|q|^2\, dxdt + s\iint_Q \xi|q_x|^2\, dxdt \le C\Bigg(&\iint_Q |B_\Phi q|^2\, dxdt \\
&+ s^3\iint_{Q_{\omega_0}} \xi^3|q|^2\, dxdt + s\iint_{Q_{\omega_0}} \xi|q_x|^2\, dxdt\Bigg)
\end{aligned} \tag{3.60}
$$

for all $s \ge s_1$. Taking into account that $p = e^{s\Phi}q$, we have that

$$
\begin{aligned}
e^{-2s\Phi}|p_x|^2 \le C(s^2\xi^2|q|^2 + |q_x|^2) \qquad \text{and} \\
|q_x|^2 \le Ce^{-2s\Phi}(s^2\xi^2|p|^2 + |q_x|^2).
\end{aligned} \tag{3.61}
$$

By (3.60) and (3.61) we get the following Carleman estimate

$$
\begin{aligned}
s^3\iint_Q e^{-2s\Phi}\xi^3|p|^2\, dxdt + s\iint_Q e^{-2s\Phi}\xi|p_x|^2\, dxdt \le \iint_Q e^{-2s\Phi}|Bp|^2\, dxdt \\
+ s^3\iint_{Q_{\omega_0}} e^{-2s\Phi}\xi^3|p|^2\, dxdt + s\iint_{Q_{\omega_0}} e^{-2s\Phi}\xi|p_x|^2\, dxdt.
\end{aligned} \tag{3.62}
$$

To conclude the proof, it is sufficient to obtain an estimate for the imaginary part of p_x, obtaining in this way (3.47). In order to do this, we decompose the Schrödinger equation into the real and imaginary parts. We write $p_1 = \operatorname{Re}(p)$ and $p_2 = \operatorname{Im}(p)$. Then Schrödinger equation is equivalent to the system given by

$$
\begin{cases}
p_{1t} + p_{2xx} = \operatorname{Im}(Bp) & \text{in} \quad Q, \\
-p_{2t} + p_{1xx} = \operatorname{Re}(Bp) & \text{in} \quad Q.
\end{cases} \tag{3.63}
$$

Let us take $\rho \in C_0^\infty(\omega)$ such that $\rho = 1$ in ω_0. Multiplying the second equation by $s\xi\rho e^{-2s\Phi}p_1$ and integrating by parts on $\omega \times (0,T)$

$$
\begin{aligned}
&s\iint_{Q_\omega} (e^{-2s\Phi}\xi)_t\rho p_2 p_1\, dxdt + s\iint_{Q_\omega} e^{-2s\Phi}\xi\rho p_2 p_{1t}\, dxdt \\
&-s\iint_{Q_\omega} (e^{-2s\Phi}\xi\rho)_x p_{1x} p_1\, dxdt - s\iint_{Q_\omega} e^{-2s\Phi}\xi\rho|p_{1x}|^2\, dxdt \\
&= s\iint_{Q_\omega} e^{-2s\Phi}\xi\rho\operatorname{Re}(Bp)p_1\, dxdt.
\end{aligned} \tag{3.64}
$$

Multiplying the first equation by $s\xi\rho e^{-2s\Phi}p_2$ and integrating by parts on $\omega \times (0,T)$

$$
\begin{aligned}
&s\iint_{Q_\omega} e^{-2s\Phi}\xi\rho p_{1t}p_2\,dxdt - s\iint_{Q_\omega}(e^{-2s\Phi}\xi\rho)_x p_{2x}p_2\,dxdt \\
&-s\iint_{Q_\omega} e^{-2s\Phi}\xi\rho|p_{2x}|^2\,dxdt = s\iint_{Q_\omega} e^{-2s\Phi}\xi\rho\mathrm{Im}(Bp)p_2\,dxdt.
\end{aligned}
\tag{3.65}
$$

Subtracting both expressions and using the property of ρ we obtain

$$
\begin{aligned}
s\iint_{Q_{\omega_0}} e^{-2s\Phi}\xi|p_{2x}|^2\,dxdt \;\le\; & -s\iint_{Q_\omega}(e^{-2s\Phi}\xi\rho)_x p_{2x}p_2\,dxdt \\
& -s\iint_{Q_\omega} e^{-2s\Phi}\xi\rho\mathrm{Im}(Bp)p_2\,dxdt \\
& -s\iint_{Q_\omega}(e^{-2s\Phi}\xi)_t\rho p_2 p_1\,dxdt \\
& +s\iint_{Q_\omega}(e^{-2s\Phi}\xi\rho)_x p_{1x}p_1\,dxdt \\
& +s\iint_{Q_\omega} e^{-2s\Phi}\xi\rho|p_{1x}|^2\,dxdt \\
& +s\iint_{Q_\omega} e^{-2s\Phi}\xi\rho\mathrm{Re}(Bp)p_1\,dxdt.
\end{aligned}
\tag{3.66}
$$

The right hand side of (3.66) can be bounded by local terms of p_1, p_2 and p_{1x}. In accordance with this and (3.62), we deduce (3.47). ■

3.3.3 Carleman Estimate for the Schrödinger-KdV System

We state and prove a Carleman estimate for system (3.4). This inequality will be used in next section to prove the observability estimate (3.3).

Teorema 3.5. *Assuming the hypotheses of Theorem 3.1, there exist $C>0$ and $s_0>0$ such that for all $s \ge s_0$ the following inequality holds:*

$$
\begin{aligned}
&s\iint_Q e^{-2s\hat\Phi}\xi|\phi_x|^2\,dxdt + s^3\iint_Q e^{-2s\hat\Phi}\xi^3|\phi|^2\,dxdt + s^5\iint_Q e^{-2s\hat\Phi}\xi^5|\psi|^2\,dxdt \\
&\quad + s^3\iint_Q e^{-2s\hat\Phi}\xi^3|\psi_x|^2\,dxdt + s\iint_Q e^{-2s\hat\Phi}\xi|\psi_{xx}|^2\,dxdt \le Cs^5\iint_{Q_\omega} e^{-2\check\Phi}\xi^5|\psi|^2\,dxdt \\
&+C\iint_{Q_\omega}\xi^{47}e^{s(6\hat\Phi-8\check\Phi)}|\psi|^2dt + Cs^3\iint_{Q_\omega}\xi^3 e^{-2s\check\Phi}|Re(\phi)|^2\,dxdt + Cs\iint_{Q_\omega} e^{-2s^3\check\Phi}\xi^3|Re(\phi_x)|^2\,dxdt,
\end{aligned}
\tag{3.67}
$$

for all $(\phi_T,\psi_T) \in \boldsymbol{H}_0^1(0,1)\times L^2(0,1)$, where (ϕ,ψ) stands for the solution of system (3.4).

Proof: We start supposing that $(\phi_T,\psi_T) \in \boldsymbol{H}_0^1(0,1)\times H_0^1(0,1)$, The case $(\phi_T,\psi_T) \in \boldsymbol{H}_0^1(0,1)\times L^2(0,1)$ follows by a density argument. We recall that, by Remark 3.4, $(\phi,\psi) \in C([0,T];\boldsymbol{H}_0^1(0,1))\times Y_{\frac{1}{2}}$.

The rest of the proof is ordered in two steps.

STEP 1: We take $\omega_1 \subset\subset \omega$ and apply Carleman inequalities (3.20) and (3.47) to each equation of system (3.4) with observations in ω_1. Adding up both inequalities, we can absorb the zero-order terms of the right hand side, obtaining

$$\begin{aligned} s\iint_Q e^{-2s\Phi}\xi|\phi_x|^2\,dxdt + s^3\iint_Q e^{-2s\Phi}\xi^3|\phi|^2\,dxdts^5\iint_Q e^{-2s\Phi}\xi^5|\psi|^2\,dxdt \\ + s^3\iint_Q e^{-2s\hat{\Phi}}\xi^3|\psi_x|^2\,dxdt + s\iint_Q e^{-2s\Phi}\xi|\psi_{xx}|^2\,dxdt \\ \leq Cs^5\iint_{Q_{\omega_1}} e^{-2\Phi}\xi^5|\psi|^2\,dxdt + Cs\iint_{Q_{\omega_1}} e^{-2s\Phi}\xi|\psi_{xx}|^2\,dxdt \\ + Cs^3\iint_{Q_{\omega_1}} \xi^3 e^{-2s\Phi}|\phi|^2\,dxdt + Cs\iint_{Q_{\omega_1}} e^{-2s\Phi}\xi|\mathrm{Re}(\phi_x)|^2\,dxdt. \quad (3.68)\end{aligned}$$

In order to remove the imaginary part of the control acting in the Schrödinger equation, we have to remove the weighted integral of $\mathrm{Im}(\phi)$ on the right hand side of (3.68). Since $|\mathrm{Im}(a_2)| \geq \delta > 0$ in ω, we get

$$\begin{aligned} s^3\iint_{Q_{\omega_1}} \xi^3 e^{-2s\Phi}|\phi|^2\,dxdt = s^3\iint_{Q_{\omega_1}} \xi^3 e^{-2s\Phi}|\mathrm{Re}(\phi)|^2\,dxdt + s^3\iint_{Q_{\omega_1}} \xi^3 e^{-2s\Phi}|\mathrm{Im}(\phi)|^2\,dxdt \\ \leq s^3\iint_{Q_{\omega_1}} \xi^3 e^{-2s\Phi}|\mathrm{Re}(\phi)|^2\,dxdt + \frac{s^3}{\delta^2}\iint_{Q_{\omega_1}} \xi^3 e^{-2s\Phi}|\mathrm{Im}(a_2)|^2|\mathrm{Im}(\phi)|^2\,dxdt. \quad (3.69)\end{aligned}$$

Let $\theta \in C_0^\infty(\omega)$ such that $\theta = 1$ in ω_1 and Sgn the sign function. Multiplying the second equation of system (3.4) by $s^3\mathrm{Sgn}(\mathrm{Im}(a_2))e^{-2s\Phi}\xi\theta\mathrm{Im}(\phi)$ and integrating in $\omega\times(0,T)$, we have

$$\begin{aligned} s^3\iint_{Q_{\omega_1}} \xi^3 e^{-2s\Phi}|\mathrm{Im}(a_2)||\mathrm{Im}(\phi)|^2\,dxdt &\leq s^3\iint_{Q_\omega} \theta\xi^3 e^{-2s\Phi}|\mathrm{Im}(a_2)||\mathrm{Im}(\phi)|^2\,dxdt \\ &= -s^3\iint_{Q_\omega} \theta e^{-2s\Phi}\xi^3\mathrm{Sgn}(\mathrm{Im}(a_2))\mathrm{Re}(a_2)\mathrm{Re}(\phi)\mathrm{Im}(\phi)\,dxdt \\ &- s^3\iint_{Q_\omega} \theta e^{-2s\Phi}\xi^3\mathrm{Sgn}(\mathrm{Im}(a_2))a_4\psi\mathrm{Im}(\phi)\,dxdt \\ &- s^3\iint_{Q_\omega} \theta e^{-2s\Phi}\xi^3\mathrm{Sgn}(\mathrm{Im}(a_2))\psi_t\mathrm{Im}(\phi)\,dxdt \\ &- s^3\iint_{Q_\omega} \theta e^{-2s\Phi}\xi^3\mathrm{Sgn}(\mathrm{Im}(a_2))\psi_{xxx}\mathrm{Im}(\phi)\,dxdt \\ &-s^3\iint_{Q_\omega} \theta e^{-2s\Phi}\xi^3\mathrm{Sgn}(\mathrm{Im}(a_2))M\psi_x\mathrm{Im}(\phi)\,dxdt. \end{aligned} \quad (3.70)$$

We denote by J_i the i-th term in the right hand side of (3.70). Until the end of this proof, we systematically apply inequality $ab \leq \varepsilon a^2 + Cb^2$, where $\varepsilon > 0$ is small enough. We have

$$|J_1| \leq \varepsilon s^3\iint_Q e^{-2s\Phi}\xi^3|\mathrm{Im}(\phi)|^2\,dxdt + Cs^3\iint_{Q_\omega} e^{-2s\Phi}\xi^3|\mathrm{Re}(\phi)|^2\,dxdt. \quad (3.71)$$

Analogously,

$$|J_2| \leq \varepsilon s^3\iint_Q e^{-2s\Phi}\xi^3|\mathrm{Im}(\phi)|^2\,dxdt + Cs^3\iint_{Q_\omega} e^{-2s\Phi}\xi^3|\psi|^2\,dxdt. \quad (3.72)$$

For J_3 we have

$$J_3 = s^3 \iint_{Q_\omega} (\mathrm{Sgn}(\mathrm{Im}(a_2))e^{-2s\Phi}\xi^3\theta)_t \psi \mathrm{Im}(\phi)\, dxdt + s^3 \iint_{Q_\omega} \mathrm{Sgn}(\mathrm{Im}(a_2))e^{-2s\Phi}\xi^3\theta\psi \mathrm{Im}(\phi_t)\, dxdt, \quad (3.73)$$

and using the first equation of (3.4) we obtain

$$\begin{aligned} J_3 \;=\; & s^3 \iint_{Q_\omega} (\mathrm{Sgn}(\mathrm{Im}(a_2))e^{-2s\Phi}\xi^3\theta)_t \psi \mathrm{Im}(\phi)\, dxdt \\ & + s^3 \iint_{Q_\omega} \mathrm{Sgn}(\mathrm{Im}(a_2))e^{-2s\Phi}\xi^3\theta\psi \mathrm{Re}(\phi_{xx})\, dxdt \\ & - s^3 \iint_{Q_\omega} \mathrm{Sgn}(\mathrm{Im}(a_2))e^{-2s\Phi}\xi^3\theta\psi \mathrm{Re}(a_1\phi + \bar{a_3}\psi)\, dxdt. \end{aligned} \quad (3.74)$$

We remark that makes sense to calculate the time derivative of $\mathrm{Sgn}(\mathrm{Im}(a_2))$ in (3.74). This is due that in ω the Sgn of $\mathrm{Im}(a_2)$ is constants equal to one or minus one.

Denoting by J_3^i the i-th term in the right hand side of (3.74), and noticing

$$\begin{aligned} |(e^{-2s\Phi}\xi^3\theta\mathrm{Sgn}(\mathrm{Im}(a_2)))_t| \;&=\; |-2se^{-2s\Phi}\Phi_t\xi^3\theta\mathrm{Sgn}(\mathrm{Im}(a_2)) + e^{-2s\Phi}(\xi^3)_t\theta\mathrm{Sgn}(\mathrm{Im}(a_2))| \\ &\le\; sCe^{-2s\Phi}\xi^5, \end{aligned}$$

we obtain

$$|J_3^1| \;\le\; \varepsilon s^3 \iint_{Q_\omega} e^{-2s\Phi}\xi^3|\mathrm{Im}(\phi)|^2\, dxdt + Cs^5 \iint_{Q_\omega} e^{-2s\Phi}\xi^7|\psi|^2\, dxdt. \quad (3.75)$$

Integrating by parts we see that

$$J_3^2 = -s^3 \iint_{Q_\omega} (e^{-2s\Phi}\xi^3\theta\mathrm{Sgn}(\mathrm{Im}(a_2)))_x \psi \mathrm{Re}(\phi_x)\, dxdt - s^3 \iint_{Q_\omega} e^{-2s\Phi}\xi^3\theta\mathrm{Sgn}(\mathrm{Im}(a_2))\psi_x \mathrm{Re}(\phi_x)\, dxdt, \quad (3.76)$$

and using that

$$\begin{aligned} (e^{-2s\Phi}\xi^3\theta\mathrm{Sgn}(\mathrm{Im}(a_2)))_x &= -2se^{-2s\Phi}\Phi_x\xi^3\theta\mathrm{Im}(a_2) + e^{-2s\Phi}\xi^3(\theta\mathrm{Sgn}(\mathrm{Im}(a_2)))_x \\ &\le sCe^{-2s\Phi}\xi^4, \end{aligned}$$

we find

$$\begin{aligned} |J_3^2| \;\le\; & Cs^3 \iint_{Q_\omega} e^{-2s\Phi}\xi^3|\mathrm{Re}(\phi_x)|^2\, dxdt + Cs^5 \iint_{Q_\omega} e^{-2s\Phi}\xi^5|\psi|^2\, dxdt \\ & + \varepsilon s^3 \iint_{Q_\omega} e^{-2s\Phi}\xi^3|\psi_x|^2\, dxdt. \end{aligned} \quad (3.77)$$

We see that

$$\begin{aligned} |J_3^3| \;\le\; & Cs^3 \iint_{Q_\omega} e^{-2s\Phi}\xi^3|\psi|^2\, dxdt + Cs^3 \iint_{Q_\omega} e^{-2s\Phi}\xi^3|\mathrm{Re}(\phi)|^2\, dxdt \\ & + \varepsilon s^3 \iint_{Q_\omega} e^{-2s\Phi}\xi^3|\mathrm{Im}(\phi)|^2\, dxdt. \end{aligned} \quad (3.78)$$

We have

$$J_4 = s^3 \iint_{Q_\omega} (e^{-2s\Phi}\xi^3\theta \mathrm{Sgn}(\mathrm{Im}(a_2)))_x \psi_{xx} \mathrm{Im}(\phi)\, dxdt \\ + s^3 \iint_{Q_\omega} e^{-2s\Phi}\xi^3\theta \mathrm{Sgn}(\mathrm{Im}(a_2))\psi_{xx}\mathrm{Im}(\phi_x)\, dxdt, \tag{3.79}$$

and therefore

$$|J_4| \leq \varepsilon s^3 \iint_Q e^{-2s\Phi}\xi^3|\mathrm{Im}(\phi)|^2\, dxdt + \varepsilon s \iint_Q e^{-2s\Phi}\xi|\mathrm{Im}(\phi_x)|^2\, dxdt \\ + Cs^5 \iint_{Q_\omega} e^{-2s\Phi}\xi^5|\psi_{xx}|^2\, dxdt. \tag{3.80}$$

Finally, we have

$$|J_5| \leq Cs^3 \iint_{Q_\omega} e^{-2s\Phi}\xi^3|M\psi_x|^2\, dxdt + \varepsilon s^3 \iint_{Q_\omega} e^{-2s\Phi}\xi^3|\mathrm{Im}(\phi)|^2\, dxdt \\ \leq C\|M\|_{L^\infty(0,T;L^2(0,1))}\left(s^5 \iint_{Q_\omega} e^{-2s\check{\Phi}}\xi^5|\psi|^2\, dxdt + s \iint_{Q_\omega} e^{-2s\check{\Phi}}\xi|\psi_{xx}|^2\, dxdt\right) \\ + \varepsilon s^3 \iint_{Q_\omega} e^{-2s\Phi}\xi^3|\mathrm{Im}(\phi)|^2\, dxdt. \tag{3.81}$$

From (3.70) and the subsequent inequalities, we get

$$s^3 \iint_{Q_{\omega_1}} \xi^3 e^{-2s\Phi}|\mathrm{Im}(a_2)|^2|\mathrm{Im}(\phi)|^2\, dxdt \leq \varepsilon s^3 \iint_Q e^{-2s\Phi}\xi^3|\mathrm{Im}(\phi)|^2\, dxdt \\ + Cs^3 \iint_{Q_\omega} e^{-2s\Phi}\xi^3|\mathrm{Re}(\phi)|^2\, dxdt + Cs^5 \iint_{Q_\omega} e^{-2s\check{\Phi}}\xi^7|\psi|^2\, dxdt. \\ + Cs^3 \iint_{Q_\omega} e^{-2s\Phi}\xi^3|\mathrm{Re}(\phi_x)|^2\, dxdt + \varepsilon s^3 \iint_{Q_\omega} e^{-2s\Phi}\xi^3|\psi_x|^2\, dxdt. + \varepsilon s \iint_Q e^{-2s\Phi}\xi|\mathrm{Im}(\phi_x)|^2\, dxdt \\ + Cs^5 \iint_{Q_\omega} e^{-2s\check{\Phi}}\xi^5|\psi_{xx}|^2\, dxdt. \tag{3.82}$$

From (3.68), (3.69) and (3.82) we obtain the Carleman inequality

$$s \iint_Q e^{-2s\hat{\Phi}}\xi|\phi_x|^2\, dxdt + s^3 \iint_Q e^{-2s\hat{\Phi}}\xi^3|\phi|^2\, dxdt + s^5 \iint_Q e^{-2s\hat{\Phi}}\xi^5|\psi|^2\, dxdt \\ + s^3 \iint_Q e^{-2s\hat{\Phi}}\xi^3|\psi_x|^2\, dxdt + s \iint_Q e^{-2s\hat{\Phi}}\xi|\psi_{xx}|^2\, dxdt \leq Cs^5 \iint_{Q_\omega} e^{-2\check{\Phi}}\xi^7|\psi|^2\, dxdt \\ + Cs^5 \iint_{Q_\omega} e^{-2s\check{\Phi}}\xi^5|\psi_{xx}|^2\, dxdt + Cs^3 \iint_{Q_\omega} \xi^3 e^{-2s\check{\Phi}}|\mathrm{Re}(\phi)|^2\, dxdt \\ + Cs^3 \iint_{Q_\omega} e^{-2s\check{\Phi}}\xi^3|\mathrm{Re}(\phi_x)|^2\, dxdt. \tag{3.83}$$

STEP 2: In this step we follow [44] in order to eliminate the observation of ψ_{xx} appearing in the right hand side of (3.83). By an interpolation argument and the Young inequality we have

$$\iint_{Q_\omega} e^{-2s\check{\Phi}}\xi^5|\psi_{xx}|^2\,dxdt \leq C\int_0^T e^{-2s\check{\Phi}}\xi^5\|\psi\|_{L^2(\omega)}^{\frac{1}{2}}\|\psi\|_{H^{\frac{8}{3}}(\omega)}^{\frac{3}{2}}dt$$
$$= C\int_0^T e^{-2s\check{\Phi}}\xi^5\left[(\xi^{\frac{21}{2}}e^{\frac{3}{2}s\hat{\Phi}}e^{-\frac{3}{2}s\check{\Phi}})\|\psi\|_{L^2(\omega)}^{\frac{1}{2}}(\xi^{-\frac{21}{2}}e^{-\frac{3}{2}s\hat{\Phi}}e^{\frac{3}{2}s\check{\Phi}})\|\psi\|_{H^{\frac{8}{3}}(\omega)}^{\frac{3}{2}}\right]dt$$
$$\leq C\int_0^T e^{-2s\check{\Phi}}\xi^5\left[C_\varepsilon(\xi^{42}e^{6s\hat{\Phi}}e^{-6s\check{\Phi}})\|\psi\|_{L^2(\omega)}^2+\varepsilon(\xi^{-14}e^{-2s\hat{\Phi}}e^{2s\check{\Phi}})\|\psi\|_{H^{\frac{8}{3}}(\omega)}^2\right]dt$$
$$= C\int_0^T \xi^{47}e^{s(6\hat{\Phi}-8\check{\Phi})}\|\psi\|_{L^2(\omega)}^2dt+\varepsilon\int_0^T\xi^{-9}e^{-2s\hat{\Phi}}\|\psi\|_{H^{\frac{8}{3}}(\omega)}^2dt, \quad (3.84)$$

with $\varepsilon > 0$ taken sufficiently small. Now we prove that the $H^{\frac{8}{3}}$ term in the right hand side of (3.84) can be estimated by the left hand side of (3.83), which is denoted by $I(\phi,\psi)$. This will be done by using a bootstrap-kind argument for the KdV equation. Let $\theta_1 = e^{-s\hat{\Phi}}\xi^{-\frac{1}{2}}$. Given $(\phi,\psi)\in C([0,T];\boldsymbol{H}_0^1(0,1))\times Y_{\frac{1}{4}}$ solution of system (3.4), we have that $(\phi_1,\psi_1) := (\theta_1\phi,\theta_1\psi)$ is solution of

$$\begin{cases} i\phi_{1t}+\phi_{1xx}=k & \text{in} \quad Q,\\ \psi_{1t}+\psi_{1xxx}=g & \text{in} \quad Q,\\ \phi_1(0,t)=\phi_1(1,t)=0 & \text{in} \quad (0,T),\\ \psi_1(0,t)=\psi_1(1,t)=\psi_{1x}(0,t)=0 & \text{in} \quad (0,T),\\ \phi_1(x,T)=0,\ \psi_1(x,T)=0 & \text{in} \quad (0,1), \end{cases} \quad (3.85)$$

where

$$\begin{aligned} k&=i\theta_{1t}\phi+\theta_1(a_1\phi+\bar{a_3}\psi),\\ g&=\theta_{1t}\psi-\theta_1(Re(\bar{a_2}\phi)+a_4\psi)+M\theta_1\psi_x. \end{aligned} \quad (3.86)$$

Notice that in particular $(k,g)\in L^2(0,T;\boldsymbol{H}_0^1(0,1))\times L^2(Q)$. Since $g\in L^2(Q)=X_{1/2}$, we use Corollary 3.1 to get

$$\|\phi_1\|_{L^\infty(0,T;\boldsymbol{L}^2(0,1))}^2+\|\psi_1\|_{L^4(0,T;H^{\frac{3}{2}}(0,1))}^2\leq C\|k\|_{L^1(0,T;\boldsymbol{L}^2(0,1))}^2+C\|g\|_{L^2(Q)}^2. \quad (3.87)$$

Since $M\in L^\infty(0,T;L^2(0,1))$ we have

$$\|k\|_{L^2(0,T;\boldsymbol{L}^2(0,1))}^2+\|g\|_{L^2(Q)}^2\leq CI(\phi,\psi). \quad (3.88)$$

Combining (3.87) and (3.88) we get

$$\|\phi_1\|_{L^\infty(0,T;\boldsymbol{L}^2(0,1))}^2+\|\psi_1\|_{L^4(0,T;H^{\frac{3}{2}}(0,1))}^2\leq CI(\phi,\psi). \quad (3.89)$$

Consider now $\theta_2 = e^{-s\hat{\Phi}}\xi^{-\frac{5}{2}}$. Thus $(\phi_2,\psi_2) := (\theta_2\phi,\theta_2\psi)$ satisfies

$$\begin{cases} i\phi_{2t}+\phi_{2xx}=k_1 & \text{in} \quad Q,\\ \psi_{2t}+\psi_{2xxx}=g_1 & \text{in} \quad Q,\\ \phi_2(0,t)=\phi_2(1,t)=0 & \text{in} \quad (0,T),\\ \psi_2(0,t)=\psi_2(1,t)=\psi_{2x}(0,t)=0 & \text{in} \quad (0,T),\\ \phi_2(x,T)=0,\psi_2(x,T)=0 & \text{in} \quad (0,1), \end{cases} \quad (3.90)$$

where

$$\begin{aligned} k_1 &= i\theta_{2t}\phi + \theta_2(a_1\phi + \bar{a_3}\psi) \\ &= i\theta_{2t}\theta_1^{-1}\phi_1 + \theta_2\theta_1^{-1}(a_1\phi_1 + \bar{a_3}\psi_1), \\ g_1 &= \theta_{2t}\psi - \theta_2(Re(\bar{a_2}\phi) + a_4\psi) + M\theta_2\psi_x \\ &= \theta_{2t}\theta_1^{-1}\psi_1 - \theta_2\theta_1^{-1}(Re(\bar{a_2}\phi_1) + a_4\psi_1) + M\theta_2\theta_1^{-1}\psi_{1x}. \end{aligned} \tag{3.91}$$

Since $\theta_{2t}\theta_1^{-1}, \theta_2\theta_1^{-1} \in L^\infty(0,T)$ and $M, \psi_{1x} \in L^4(0,T;H^{\frac{1}{2}}(0,1))$, we have $(k_1, g_1) \in L^2(0,T;\boldsymbol{H}_0^1(0,1)) \times L^2(0,T;H^{\frac{1}{3}}(0,1))$. Here we have used that the product of two functions in $H^{\frac{1}{2}}(0,1)$ belongs to $H^{\frac{1}{3}}(0,1)$. Being $L^2(0,T;H^{\frac{1}{3}}(0,1)) = X_{7/12}$, we use (3.87), (3.88) and Corollary 3.1 to obtain

$$\begin{aligned} &\|\phi_2\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \|\psi_2\|^2_{Y_{\frac{7}{12}}} \\ &\qquad \leq C\|k_1\|^2_{L^1(0,T;\boldsymbol{L}^2(0,1))} + C\|g_1\|^2_{L^2(0,T;H^{\frac{1}{3}}(0,1))} \leq CI(\phi,\psi), \end{aligned} \tag{3.92}$$

where

$$Y_{\frac{7}{12}} = L^2(0,T;H^{\frac{7}{3}}(0,1)) \cap L^\infty(0,T;H^{\frac{4}{3}}(0,1)).$$

Consider now $\theta_3 = e^{-s\check{\Phi}}\xi^{-\frac{9}{2}}$. Then $(\phi_3,\psi_3) := (\theta_3\phi, \theta_3\psi)$ is solution of

$$\begin{cases} i\phi_{3t} + \phi_{3xx} = k_2 & \text{in} \quad Q, \\ \psi_{3t} + \psi_{3xxx} = g_2 & \text{in} \quad Q, \\ \phi_3(0,t) = \phi_3(1,t) = 0 & \text{in} \quad (0,T), \\ \psi_3(0,t) = \psi_3(1,t) = \psi_{3x}(0,t) = 0 & \text{in} \quad (0,T), \\ \phi_3(x,T), \psi_3(x,T) = 0 = 0 & \text{in} \quad (0,1), \end{cases} \tag{3.93}$$

where

$$\begin{aligned} k_2 &= i\theta_{3t}\phi + \theta_3(a_1\phi + \bar{a_3}\psi) \\ &= i\theta_{3t}\theta_2^{-1}\phi_2 + \theta_3\theta_2^{-1}(a_1\phi_2 + \bar{a_3}\psi_2) \\ g_2 &= \theta_{3t}\psi - \theta_3(Re(\bar{a_2}\phi) + a_4\psi) + M\theta_3\psi_x \\ &= \theta_{3t}\theta_2^{-1}\psi_2 - \theta_3\theta_2^{-1}(Re(\bar{a_2}\phi_2) + a_4\psi_2) + M\theta_3\theta_2^{-1}\psi_{2x}. \end{aligned} \tag{3.94}$$

Proceeding as before, we see that $M \in L^3(0,T;H^{\frac{2}{3}}(0,1))$ and $\psi_{2x} \in L^6(0,T;H^{2/3}(0,1))$. In this way $(k_2,g_2) \in L^1(0,T;\boldsymbol{H}_0^1(0,1)) \times L^2(0,T;H^{\frac{2}{3}}(0,1))$. Here we have used that the product of two functions in $H^{\frac{2}{3}}(0,1)$ belongs to $H^{\frac{2}{3}}(0,1)$. Since $L^2((0,T);H^{\frac{2}{3}}(0,1)) = X_{2/3}$, we have

$$\begin{aligned} &\|\phi_3\|^2_{L^\infty((0,T);\boldsymbol{L}^2(\Omega))} + \|\psi_3\|^2_{Y_{\frac{2}{3}}} \\ &\qquad \leq C\|k_2\|^2_{L^1(0,T;\boldsymbol{L}^2(0,1))} + C\|g_2\|^2_{L^2(0,T;H^{\frac{1}{3}}(0,1))} \leq CI(\phi,\psi), \end{aligned} \tag{3.95}$$

where

$$Y_{\frac{2}{3}} = L^2(0,T;H^{\frac{8}{3}}(0,1)) \cap L^\infty(0,T;H^{\frac{5}{3}}(0,1)).$$

By the definition of θ_3, inequality (3.95) implies that

$$\int_0^T \xi^{-9}e^{-2s\hat{\Phi}}\|\psi\|^2_{H^{\frac{8}{3}}(\omega)}dt \leq CI(\phi,\psi) \tag{3.96}$$

Inequality (3.96), combined with (3.83) and (3.84) imply Carleman inequality (3.67). ■

3.4 Observability inequality

In this section, we prove the observability inequality (3.3). Given $(\phi^T, \psi^T) \in \boldsymbol{H}_0^1(0,1) \times L^2(0,1)$, we define, for each $t \in [0,T]$,

$$E(t) = \int_0^1 \left(|\phi(x,t)|^2 + |\phi_x(x,t)|^2 + |\psi(x,t)|^2\right) dx. \tag{3.97}$$

We prove the next property of $E(t)$:

Lema 3.1. *There exists a constant $C > 0$ such that for every $(\phi^T, \psi^T) \in \boldsymbol{H}_0^1(0,1) \times L^2(0,1)$ we have*

$$E(0) \leq C \int_{T/4}^{3T/4} E(t) dt. \tag{3.98}$$

Proof: Multiplying the first equation of system (3.4) by $\bar{\phi}$ and integrating in $(0,1)$ we get

$$\frac{1}{2}\frac{d}{dt}\int_0^1 |\phi|^2\, dx = \operatorname{Im}\int_0^1 (a_1\phi + \bar{a_3}\psi)\bar{\phi}\, dx = \operatorname{Im}\int_0^1 \bar{a_3}\psi\bar{\phi}\, dx. \tag{3.99}$$

Denoting $f = a_1\phi + \bar{a_3}\psi$, multiplying the same equation by $\bar{\phi}_t$ and integrating in $(0,1)$ we get

$$\begin{aligned} -\frac{1}{2}\frac{d}{dt}\int_0^1 |\phi_x|^2\, dx &= \operatorname{Re}\int_0^1 f\bar{\phi}_t\, dx \\ &= \operatorname{Re}\int_0^1 f(-i\bar{\phi}_{xx} + i\bar{f})\, dx \\ &= \operatorname{Re}\int_0^1 (-i\bar{\phi}_{xx} f)\, dx. \end{aligned}$$

and multiplying by parts in x we get that there exists a constant $C = C(\|a_{1x}\|_{L^\infty(Q)}, \|a_3\|_{L^\infty(0,T;\mathbf{W}^{1,\infty}(0,1))}) > 0$ such that

$$-\frac{1}{2}\frac{d}{dt}\int_0^1 |\phi_x|^2\, dx \leq C\int_0^1 \left(|\phi|^2 + |\phi_x|^2 + |\psi|^2\right)\, dx + \frac{1}{2}\int_0^1 |\psi_x|^2. \tag{3.100}$$

Multiplying the second equation of system (3.4) by ψ, and denoting $g = \operatorname{Re}(\bar{a_2}\phi) + a_4\psi$, we obtain

$$-\frac{1}{2}\frac{d}{dt}\int_0^1 |\psi|^2\, dx + \frac{1}{2}|\psi_x(1,t)|^2 \leq \int_0^1 |g\psi| dx + \frac{1}{2}\|M\|^2_{L^\infty(0,1)}\int_0^1 |\psi|^2\, dx + \frac{1}{2}\int_0^1 |\psi_x|^2\, dx. \tag{3.101}$$

Multiplying the same equation, this time by $(1-x)\psi$, we get

$$\begin{aligned} -\frac{1}{2}\frac{d}{dt}\int_0^1 (1-x)|\psi|^2\, dx + \frac{3}{2}\int_0^1 |\psi_x|^2\, dx &\leq \int_0^1 |g\psi| dx \\ &+ \frac{1}{2}\|M\|^2_{L^\infty(0,1)}\int_0^1 |\psi|^2\, dx + \frac{1}{2}\int_0^1 |\psi_x|^2\, dx. \end{aligned} \tag{3.102}$$

From (3.102) and (3.101), there exists a constant $C = C(\|a_2\|_{\mathbf{L}^\infty(Q)}, \|a_4\|_{L^\infty(Q)}) > 0$ such that

$$-\frac{1}{2}\frac{d}{dt}\int_0^1 (2-x)|\psi|^2\,dx + \frac{1}{2}\int_0^1 |\psi_x|^2\,dx \le C(1+\|M\|^2_{L^\infty(0,1)})\int_0^1 \left(|\phi|^2+|\psi|^2\right)\,dx. \quad (3.103)$$

From (3.99), (3.100) and (3.103), we have

$$-\frac{1}{2}\frac{d}{dt}\int_0^1 \left((2-x)|\psi|^2+|\phi|^2+|\phi_x|^2\right)\,dx \le C(1+\|M\|^2_{L^\infty(0,1)})\int_0^1 \left(|\phi_x|^2+|\phi|^2+|\psi|^2\right)\,dx, \quad (3.104)$$

where $C = C(\|a_{1x}\|_{L^\infty(Q)}, \|a_2\|_{\mathbf{L}^\infty(Q)}, \|a_3\|_{L^\infty(0,T;\mathbf{W}^{1,\infty}(0,1))}, \|a_4\|_{L^\infty(Q)}) > 0$. Therefore, denoting

$$\tilde{E}(t) := \frac{1}{2}\int_0^1 \left((2-x)|\psi|^2+|\phi|^2+|\phi_x|^2\right)\,dx, \quad (3.105)$$

we get

$$\frac{d}{dt}\tilde{E}(t) \ge -C(1+\|M(t)\|^2_{L^\infty(0,1)})\tilde{E}(t), \quad \forall t\in(0,T). \quad (3.106)$$

From (3.106) we obtain that

$$\frac{d}{dt}\left(e^{C\int_0^t (1+\|M(s)\|^2_{L^\infty(0,1)})ds}\tilde{E}(t)\right) \ge 0, \quad \forall t\in(0,T). \quad (3.107)$$

Integrating (3.107) on the time interval $(0,t)$ we get

$$\tilde{E}(0) \le e^{C(T+\|M\|^2_{L^2(0,T;H^1(0,1))})}\tilde{E}(t), \quad \forall t\in(0,T). \quad (3.108)$$

Integrating (3.108) on the interval $[T/4, 3T/4]$ and taking into account that $1 \le 2-x \le 2$, for each $x\in[0,1]$, we obtain (3.98) and Lemma 3.1 is proved. ■

From definition (3.18) we have that there exists $\delta > 0$ such that

$$e^{-2s\hat{\Phi}}\xi^k \ge \delta, \quad (3.109)$$

for all $t\in[T/4,3T/4]$, $x\in[0,1]$, and $k=1,3,5$. Hence

$$\delta\int_{T/4}^{3T/4} E(t)dt \le \iint_Q e^{-2s\hat{\Phi}}\xi^3|\phi|^2\,dxdt + \iint_Q e^{-2s\hat{\Phi}}\xi|\phi_x|^2\,dxdt + \iint_Q e^{-2s\hat{\Phi}}\xi^5|\psi|^2\,dxdt. \quad (3.110)$$

From (3.110), Lemma 3.1, and Carleman estimate (3.67), we deduce the observability inequality (3.3), which concludes the proof of Theorem 3.2.

Capítulo 4

Boundary controllability of incompressible Euler fluids with Boussinesq heat effects

Boundary controllability of incompressible Euler fluids with Boussinesq heat effects

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Abstract. This paper deals with the boundary controllability of inviscid incompressible fluids for which thermal effects are important. They will be modeled through the so called Boussinesq approximation. In the zero heat diffusion case, by adapting and extending some ideas from J.-M. Coron and O. Glass, we establish the simultaneous global exact controllability of the velocity field and the temperature for 2D and 3D flows. When the heat diffusion coefficient is positive, we present some additional results concerning exact controllability for the velocity field and local null controllability of the temperature.

4.1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a nonempty, bounded and connected open set whose boundary $\Gamma := \partial\Omega$ is of class C^∞, with $N = 2$ or $N = 3$. Let $\Gamma_0 \subset \Gamma$ be a (small) nonempty open subset of Γ and assume that $T > 0$. For simplicity, we assume that Ω is simply connected.

In the sequel, we will denote by C a generic positive constant; spaces of $\mathbb{R}^N$-valued functions, as well as their elements, are represented by boldfaced letters; we will denote by $\mathbf{n} = \mathbf{n}(\mathbf{x})$ the outward unit normal to Ω at points $\mathbf{x} \in \Gamma$.

In this work, we will be concerned with the boundary controllability of the system:

$$\begin{cases} \mathbf{y}_t + (\mathbf{y}\cdot\nabla)\mathbf{y} = -\nabla p + \vec{\mathbf{k}}\,\theta & \text{in} \quad \Omega\times(0,T), \\ \nabla\cdot\mathbf{y} = 0 & \text{in} \quad \Omega\times(0,T), \\ \theta_t + \mathbf{y}\cdot\nabla\theta = \kappa\,\Delta\theta & \text{in} \quad \Omega\times(0,T), \\ \mathbf{y}\cdot\mathbf{n} = 0 & \text{on} \quad (\Gamma\backslash\Gamma_0)\times(0,T), \\ \mathbf{y}(\mathbf{x},0) = \mathbf{y}_0(\mathbf{x}), \quad \theta(\mathbf{x},0) = \theta_0(\mathbf{x}) & \text{in} \quad \Omega. \end{cases} \tag{4.1}$$

This system models the behavior of an incompressible homogeneous inviscid fluid with thermal effects. More precisely,

- The field $\mathbf{y}$ and the scalar function p stand for the velocity and the pressure of the fluid in $\Omega\times(0,T)$, respectively.
- The function θ provides the temperature distribution of the fluid.
- The right hand side $\vec{\mathbf{k}}\,\theta$ can be viewed as the *buoyancy force* density ($\vec{\mathbf{k}} \in \mathbb{R}^N$ is a non-zero vector).

- The nonnegative constant $\kappa \geq 0$ is the heat diffusion coefficient.

This system is relevant for the study and description of atmospheric and oceanographic turbulence, as well as other fluid problems where rotation and stratification play dominant roles (see e.g. [71]). In fluid mechanics, (4.1) is used to deal with buoyancy-driven flow; it describes the motion of an incompressible inviscid fluid subject to convective heat transfer under the influence of gravitational forces, see [64].

We will be concerned with the cases $\kappa = 0$ and $\kappa > 0$. When $\kappa = 0$, (4.1) is called the *incompressible inviscid Boussinesq* system.

From now on, we assume that $\alpha \in (0,1)$ and we set

$$\begin{aligned} \mathbf{C}_0^{m,\alpha}(\overline{\Omega};\mathbb{R}^N) &:= \{\, \mathbf{u} \in \mathbf{C}^{m,\alpha}(\overline{\Omega};\mathbb{R}^N) \;:\; \nabla\cdot\mathbf{u} = 0 \;\text{ in }\; \overline{\Omega}, \;\; \mathbf{u}\cdot\mathbf{n} = 0 \;\text{ on }\; \Gamma \,\}, \\ \mathbf{C}(m,\alpha,\Gamma_0) &:= \{\, \mathbf{u} \in \mathbf{C}^{m,\alpha}(\overline{\Omega};\mathbb{R}^N) \;:\; \nabla\cdot\mathbf{u} = 0 \;\text{ in }\; \overline{\Omega}, \;\; \mathbf{u}\cdot\mathbf{n} = 0 \;\text{ on }\; \Gamma\backslash\Gamma_0 \,\}, \end{aligned} \tag{4.2}$$

where $\mathbf{C}^{m,\alpha}(\overline{\Omega};\mathbb{R}^N)$ denotes the space of $\mathbb{R}^N$-valued functions whose m-th order derivatives are *Hölder-continuous* in $\overline{\Omega}$ with exponent α. The usual norms in the Banach spaces $\mathbf{C}^0(\overline{\Omega};\mathbb{R}^\ell)$ and $\mathbf{C}^{m,\alpha}(\overline{\Omega};\mathbb{R}^\ell)$ will be respectively denoted by $\|\cdot\|_0$ and $\|\cdot\|_{m,\alpha}$. We will also need to work with the Banach spaces $C^0([0,T];\mathbf{C}^{m,\alpha}(\overline{\Omega};\mathbb{R}^\ell))$, where the usual norms are

$$\|\mathbf{w}\|_{0,m,\alpha} := \max_{[0,T]} \|\mathbf{w}(\cdot\,,t)\|_{m,\alpha}.$$

In particular, $\|\cdot\|_{(0)}$ will stand for $\|\cdot\|_{0,0,0}$.

When $\kappa = 0$, it is appropriate to consider the exact boundary controllability problem for (4.1). In general terms, it can be stated as follows:

Given $\mathbf{y}_0$, $\mathbf{y}_1$, θ_0 and θ_1 in appropriate spaces with $\mathbf{y}_0\cdot\mathbf{n} = \mathbf{y}_1\cdot\mathbf{n} = 0$ on $\Gamma\backslash\Gamma_0$, find $(\mathbf{y},p,\theta)$ such that (4.1) holds and, furthermore,

$$\mathbf{y}(\mathbf{x},T) = \mathbf{y}_1(\mathbf{x}), \quad \theta(\mathbf{x},T) = \theta_1(\mathbf{x}) \quad \text{in} \;\; \Omega. \tag{4.3}$$

If it is always possible to find $\mathbf{y}$, p and θ, it will be said that the incompressible inviscid Boussinesq system is *exactly controllable* for (Ω,Γ_0) at time T.

Notice that, when $\kappa = 0$, in order to determine without ambiguity a unique local in time regular solution to (4.1), it is sufficient to prescribe the normal component of the velocity on the boundary of the flow region and the full field $\mathbf{y}$ and the temperature θ on the inflow section, i.e. only where $\mathbf{y}\cdot\mathbf{n} < 0$, see for instance [63]. Hence, in this case, we can assume that the controls are given as follows:

$$\begin{cases} \mathbf{y}\cdot\mathbf{n} \text{ on } \Gamma_0\times(0,T), \text{ with } \displaystyle\int_{\Gamma_0} \mathbf{y}\cdot\mathbf{n}\, d\Gamma = 0; \\ \mathbf{y} \text{ and } \theta \text{ at any point of } \Gamma_0\times(0,T) \text{ satisfying } \mathbf{y}\cdot\mathbf{n} < 0. \end{cases}$$

The meaning of the exact controllability property is that, when it holds, we can drive the fluid from any initial state $(\mathbf{y}_0,\theta_0)$ exactly to any final state $(\mathbf{y}_1,\theta_1)$, acting only on an arbitrary small part Γ_0 of the boundary during an arbitrary small time interval $(0,T)$.

In the case $\kappa > 0$, the situation is different. Due to the *regularization effect* of the temperature equation, we cannot expect exact controllability, at least for the temperature.

In order to present a suitable boundary controllability problem, let us introduce a nonempty open subset $\gamma \subset \Gamma$. Then, the problem is the following:

> *Given* $\mathbf{y}_0$*,* $\mathbf{y}_1$ *and* θ_0 *in appropriate spaces with* $\mathbf{y}_0 \cdot \mathbf{n} = \mathbf{y}_1 \cdot \mathbf{n} = 0$ *on* $\Gamma \backslash \Gamma_0$ *and* $\theta_0 = 0$ *on* $\Gamma \backslash \gamma$*, find* $(\mathbf{y}, p, \theta)$ *with* $\theta = 0$ *on* $(\Gamma \backslash \gamma) \times (0, T)$ *such that* (4.1) *holds and, furthermore,*
>
> $$\mathbf{y}(\mathbf{x}, T) = \mathbf{y}_1(\mathbf{x}), \quad \theta(\mathbf{x}, T) = 0 \quad \text{in} \quad \Omega. \tag{4.4}$$

If it is always possible to find $\mathbf{y}$, p and θ, it will be said that the incompressible, heat diffusive, inviscid Boussinesq system (4.1) is *exactly-null controllable* for $(\Omega, \Gamma_0, \gamma)$ at time T.

Note that, if $\kappa > 0$ and we fix the same boundary data for $\mathbf{y}$ as before and (for example) Dirichlet data for θ of the form

$$\theta = \theta_* 1_\gamma \quad \text{on} \quad \Gamma \times (0, T),$$

there exists at most one solution to (4.1). Therefore, it can be assumed in this case that the controls are the following:

$$\begin{cases} \mathbf{y} \cdot \mathbf{n} \text{ on } \Gamma_0 \times (0, T), \text{ with } \displaystyle\int_{\Gamma_0} \mathbf{y} \cdot \mathbf{n} \, d\Gamma = 0; \\ \mathbf{y} \text{ at any point of } \Gamma_0 \times (0, T) \text{ satisfying } \mathbf{y} \cdot \mathbf{n} < 0; \\ \theta \text{ at any point of } \gamma \times (0, T). \end{cases}$$

Of course, the meaning of the exact-null controllability property is that, when it holds, we can drive the fluid velocity-temperature pair from any initial state $(\mathbf{y}_0, \theta_0)$ exactly to any final state of the form $(\mathbf{y}_1, 0)$, acting only on arbitrary small parts Γ_0 and γ of the boundary during an arbitrary small time interval $(0, T)$.

In the last decades, a lot of researchers has focused attention on the controllability of systems governed by (linear and nonlinear) PDEs. Some related results can be found in [26, 45, 59, 86]. In the context of incompressible ideal fluids, this subject has been mainly investigated by Coron [28, 29] and Glass [41, 42, 43].

In this paper, our first task will be to adapt the techniques and arguments of [29] and [43] to the situations modeled by (4.1). Thus, our first main result is the following:

Theorem 6. *If* $\kappa = 0$*, then the incompressible inviscid Boussinesq system* (4.1) *is exactly controllable for* (Ω, Γ_0) *at any time* $T > 0$*. More precisely, for any* $\mathbf{y}_0, \mathbf{y}_1 \in \mathbf{C}(2, \alpha, \Gamma_0)$ *and any* $\theta_0, \theta_1 \in C^{2,\alpha}(\overline{\Omega})$*, there exist* $\mathbf{y} \in C^0([0, T]; \mathbf{C}(1, \alpha, \Gamma_0))$*,* $\theta \in C^0([0, T]; C^{1,\alpha}(\overline{\Omega}))$ *and* $p \in \mathcal{D}'(\Omega \times (0, T))$ *such that one has* (4.1) *and* (4.3)*.*

The proof of Theorem 6 relies on the *extension* and *return* methods. These have been applied in several different contexts to establish controllability; see the seminal works [80] and [27]; see also a long list of applications in [26].

Let us give a sketch of the strategy used in the proof of Theorem 6:

- First, we construct a "good"trajectory connecting $(\mathbf{0},0)$ to $(\mathbf{0},0)$ (see Sections 4.2.1 and 4.2.2).
- Then, we apply the extension method of David L. Russell [80].
- Then, we use a *Fixed-Point Theorem* and we deduce a local exact controllability result.
- Finally, we use an appropriate scaling argument and we obtain the desired global result.

In fact, Theorem 6 is a consequence of the following local result:

Proposition 9. *Let us assume that $\kappa = 0$. There exists $\delta > 0$ such that, for any $\mathbf{y}_0 \in \mathbf{C}(2,\alpha,\Gamma_0)$ and any $\theta_0 \in C^{2,\alpha}(\overline{\Omega})$ with*

$$\max\{\|\mathbf{y}_0\|_{2,\alpha}, \|\theta_0\|_{2,\alpha}\} \le \delta,$$

there exist $\mathbf{y} \in C^0([0,1];\mathbf{C}(1,\alpha,\Gamma_0))$, $\theta \in C^0([0,1];C^{1,\alpha}(\overline{\Omega}))$ and $p \in \mathcal{D}'(\Omega \times (0,1))$ satisfying (4.1) in $\Omega \times (0,1)$ and the final conditions

$$\mathbf{y}(\mathbf{x},1) = \mathbf{0}, \quad \theta(\mathbf{x},1) = 0 \quad in \ \ \Omega. \tag{4.5}$$

It will be seen later that, in our argument, the $C^{2,\alpha}$-regularity of the initial and final data is needed. However, we can only ensure the existence of a controlled solution that is $C^{1,\alpha}$ in space. It would be interesting to improve this result but, at present, we do not know how.

Our second main result is the following:

Theorem 7. *Let Ω, Γ_0 and γ be given and let us assume that $\kappa > 0$. Then (4.1) is locally exactly-null controllable. More precisely, for any $T > 0$ and any $\mathbf{y}_0, \mathbf{y}_1 \in \mathbf{C}(2,\alpha,\emptyset)$, there exists $\eta > 0$, depending on $\mathbf{y}_0$, such that, for each $\theta_0 \in C^{2,\alpha}(\overline{\Omega})$ with*

$$\theta_0 = 0 \quad on \ \ \Gamma\backslash\gamma, \quad \|\theta_0\|_{2,\alpha} \le \eta,$$

we can find $\mathbf{y} \in C^0([0,T];\mathbf{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^N)$, $\theta \in C^0([0,T];C^{1,\alpha}(\overline{\Omega}))$ with $\theta = 0$ on $(\Gamma\backslash\gamma) \times (0,T)$, and $p \in \mathcal{D}'(\Omega \times (0,T))$ satisfying (4.1) and (4.4).

The proof relies on the following strategy. First, we linearize and control only the temperature θ; this leads the system to a state of the form $(\tilde{\mathbf{y}}_0, 0)$ at (say) time $T/2$. Then, in a second step, we control the velocity field using in part Theorem 6. It will be seen that, in order to get good estimates and prove the existence of a fixed point, the initial temperature θ_0 must be small.

To our knowledge, it is unknown whether a global exact-null controllability result holds for (4.1) when $\kappa > 0$. Unfortunately, the cost of controlling θ grows exponentially with the L^∞-norm of the transporting velocity field $\mathbf{y}$ and this is a crucial difficulty to establish estimates independent of the size of the initial data.

The rest of this paper is organized as follows. In Section 4.2, we recall the results needed to prove Theorems 9 and 7. In Section 4.3, we give the proof of Theorem 6. In Section 4.4, we prove Proposition 9 in the 2D case; it will be seen that the main ingredients of the proof are the construction of a nontrivial trajectory that starts and ends at $(\mathbf{0}, 0)$ and a Fixed-Point Theorem (the key ideas of the return method). In Section 4.5, we give the proof of Theorem 9 in the 3D case. Finally, Section 4.6 contains the proof of Theorem 7.

4.2 Preliminary results

In this section, we are going to recall some results used in the proofs of Theorems 6 and 7. Also, we are going to indicate how to construct a trajectory appropriate to apply the return method.

The following result is an immediate consequence of Banach's Fixed-Point Theorem:

Theorem 8. *Let $(B_1, \|\cdot\|_1)$ and $(B_2, \|\cdot\|_2)$ be Banach spaces with B_2 continuously embedded in B_1. Let B be a subset of B_2 and let $G : B \mapsto B$ be a uniformly continuous mapping such that, for some $m \geq 1$ and some $\gamma \in [0,1)$, one has*

$$\|G^m(u) - G^m(v)\|_1 \leq \gamma \|u - v\|_1 \quad \forall u, v \in B.$$

Let us denote by $\overline{B}$ the closure of B for the norm $\|\cdot\|_1$. Then, G can be uniquely extended to a continuous mapping $\widetilde{G} : \overline{B} \mapsto \overline{B}$ that possesses a unique fixed-point in $\overline{B}$.

Later, the following lemma will be very important to deduce appropriate estimates. The proof can be found in [12].

Lemma 1. *Let m be a nonnegative integer. Assume that $u \in C^0([0,T]; C^{m+1,\alpha}(\overline{\Omega}))$, $g \in C^0([0,T]; C^{m,\alpha}(\overline{\Omega}))$ and $\mathbf{v} \in C^0([0,T]; \mathbf{C}^{m,\alpha}(\overline{\Omega}; \mathbb{R}^N))$ are given, with $\mathbf{v} \cdot \mathbf{n} = 0$ on $\Gamma \times (0,T)$ and*

$$\frac{\partial u}{\partial t} + \mathbf{v} \cdot \nabla u = g \quad \text{in} \quad \Omega \times (0,T). \tag{4.6}$$

Then, $u_t \in C^0([0,T]; C^{m,\alpha}(\overline{\Omega}))$ and, for any $m \geq 1$,

$$\frac{d}{dt^+} \|u(\cdot\,,t)\|_{m,\alpha} \leq \|g(\cdot\,,t)\|_{m,\alpha} + K \|\mathbf{v}(\cdot\,,t)\|_{m,\alpha} \|u(\cdot\,,t)\|_{m,\alpha} \quad \text{in} \quad (0,T),$$

where K is a constant only depending on α and m. If $m = 0$, the following holds

$$\frac{d}{dt^+} \|u(\cdot\,,t)\|_{0,\alpha} \leq \|g(\cdot\,,t)\|_{0,\alpha} + \alpha \|\nabla \mathbf{v}(\cdot\,,t)\|_{0,\alpha} \|u(\cdot\,,t)\|_{0,\alpha} \quad \text{in} \quad (0,T).$$

From Lemma 1 and a standard regularization argument, we easily deduce the flollowing:

Lemma 2. *Let m be a nonnegative integer. Assume that $u \in C^0([0,T]; C^{m,\alpha}(\overline{\Omega}))$, $g \in C^0([0,T]; C^{m,\alpha}(\overline{\Omega}))$ and $\mathbf{v} \in C^0([0,T]; \mathbf{C}^{m,\alpha}(\overline{\Omega}; \mathbb{R}^N))$ are given, with $\mathbf{v} \cdot \mathbf{n} = 0$ on $\Gamma \times (0,T)$ and*

$$\frac{\partial u}{\partial t} + \mathbf{v} \cdot \nabla u = g \quad \text{in} \quad \Omega \times (0,T). \tag{4.7}$$

Then

$$\|u\|_{0,m,\alpha} \leq \left(\int_0^T \|g(\cdot\,,t)\|_{m,\alpha}\, dt + \|u(\cdot\,,0)\|_{m,\alpha} \right) \exp \left(K \int_0^T \|\mathbf{v}(\cdot\,,t)\|_{m,\alpha}\, dt \right),$$

where K is a constant only depending on α and m.

We will also use a technical lemma whose proof can be found in [41]:

Lemma 3. *Let us assume that*

$$\begin{array}{lll} \mathbf{w}_0 \in \mathbf{C}^{1,\alpha}(\bar{\Omega};\mathbb{R}^N), \quad \nabla\cdot\mathbf{w}_0 = 0 & in & \Omega, \\ \mathbf{u} \in C^0([0,T];\mathbf{C}^{1,\alpha}(\bar{\Omega};\mathbb{R}^N)), \quad \mathbf{u}\cdot\mathbf{n} = 0 & on & \Gamma\times(0,T), \\ \mathbf{g} \in C^0([0,T];\mathbf{C}^{0,\alpha}(\bar{\Omega},\mathbb{R}^N)), \quad \nabla\cdot\mathbf{g} = 0 & in & \Omega\times(0,T). \end{array}$$

Let $\mathbf{w}$ *be a function in* $C^0([0,T];\mathbf{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^N))$ *satisfying*

$$\left\{ \begin{array}{lll} \mathbf{w}_t + (\mathbf{u}\cdot\nabla)\mathbf{w} = (\mathbf{w}\cdot\nabla)\mathbf{u} - (\nabla\cdot\mathbf{u})\mathbf{w} + \mathbf{g} & in & \Omega\times(0,T), \\ \mathbf{w}(\cdot\,,0) = \mathbf{w}_0 & in & \Omega. \end{array} \right.$$

Then, $\nabla\cdot\mathbf{w} \equiv 0$*. Moreover, there exists* $\mathbf{v} \in C^0([0,T];\mathbf{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}^N))$ *such that*

$$\mathbf{w} = \nabla\times\mathbf{v} \quad in \quad \Omega\times(0,T).$$

To end this section, we will recall a well known result dealing with the null controllability of general parabolic linear systems of the form

$$\left\{ \begin{array}{lll} u_t - \kappa\Delta u + \mathbf{w}\cdot\nabla u = v1_\omega & \text{in} & \Omega\times(0,T), \\ u = 0 & \text{on} & \Gamma\times(0,T), \\ u(\mathbf{x},0) = u_0(\mathbf{x}) & \text{in} & \Omega, \end{array} \right. \tag{4.8}$$

where $\kappa > 0$, $\mathbf{w} \in L^\infty(\Omega\times(0,T))$, $\omega \subset \Omega$ is a non-empty open set and 1_ω is the characteristic function of ω.

It is well known that, for each $u_0 \in L^2(\Omega)$ and each $v \in L^2(\omega\times(0,T))$, there exists exactly one solution u to (4.8), with

$$u \in C^0([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)).$$

We also have:

Theorem 9. *The linear system* (4.8) *is null-controllable at any time* $T > 0$*. In other words, for each* $u_0 \in L^2(\Omega)$ *there exists* $v \in L^2(\omega\times(0,T))$ *such that the associated solution to* (4.8) *satisfies*

$$u(\mathbf{x},T) = 0 \quad in \quad \Omega. \tag{4.9}$$

Furthermore, the extremal problem

$$\left\{ \begin{array}{l} Minimize \ \ \dfrac{1}{2}\displaystyle\iint_{\omega\times(0,T)} |v|^2\,dx\,dt \\ Subject\ to{:}\ v \in L^2(\omega\times(0,T)),\ u\ satisfies\ (4.9) \end{array} \right. \tag{4.10}$$

possesses exactly one solution $\hat{v}$ *satisfying*

$$\|\hat{v}\|_2 \leq C_0\|u_0\|_2, \tag{4.11}$$

where

$$C_0 = \exp\left(C_1\left(1 + \frac{1}{T} + (1+T^2)\|\mathbf{w}\|_\infty^2\right)\right)$$

and C_1 *only depends on* Ω*,* ω *and* κ*.*

4.2.1 Construction of a trajectory when $N = 2$

We will argue as in [29]. Thus, let $\Omega_1 \subset \mathbb{R}^2$ be a bounded, Lipschitz-contractible open set whose boundary is of class C^∞ and consists of two disjoint closed line segments Γ^- and Γ^+ and two disjoint curves Σ' and Σ'' of class C^∞ such that $\partial\Sigma' \cup \partial\Sigma'' = \partial\Gamma^- \cup \partial\Gamma^+$.

We assume that $\Omega \subset \Omega_1$. We also impose that there is a neighborhood U^- of Γ^- (resp. U^+ of Γ^+) such that $\Omega_1 \cap U^-$ (resp. $\Omega_1 \cap U^+$) coincides with the intersection of U^- (resp. U^+), an open semi-plane limited by the line containing Γ^- (resp. Γ^+) and the band limited by the two straight lines orthogonal to Γ^- (resp. Γ^+) and passing through $\partial\Gamma^-$ (resp. $\partial\Gamma^+$); see Figure 4.1.

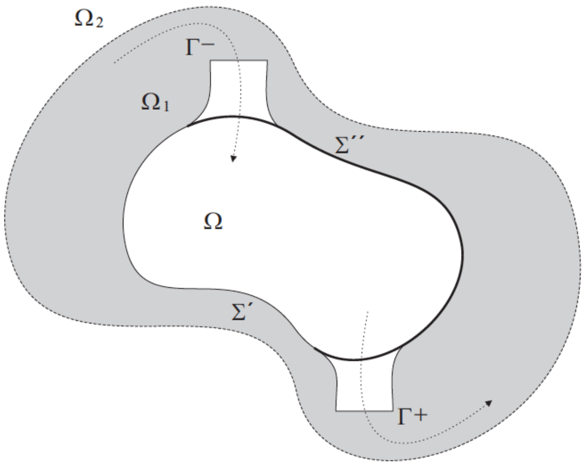

Figura 4.1: The domain Ω_1

Let φ be the solution to

$$\begin{cases} -\Delta\varphi = 0 & \text{in} \quad \Omega_1, \\ \varphi = 1 & \text{on} \quad \Gamma^+, \\ \varphi = -1 & \text{on} \quad \Gamma^-, \\ \dfrac{\partial\varphi}{\partial n} = 0 & \text{on} \quad \Sigma, \end{cases} \tag{4.12}$$

where $\Sigma = \Sigma' \cup \Sigma''$. Then, we have the following result from J.-M. Coron [29]:

Lemma 4. *One has* $\varphi \in C^\infty(\overline{\Omega}_1)$, $-1 < \varphi(\mathbf{x}) < 1$ *for all* $\mathbf{x} \in \Omega_1$ *and*

$$\nabla\varphi(\mathbf{x}) \neq \mathbf{0} \quad \forall \mathbf{x} \in \overline{\Omega}_1. \tag{4.13}$$

Let $\gamma \in C^\infty([0,1])$ be a non-zero function such that $\operatorname{Supp}\gamma \subset (0,1/2) \cup (1/2,1)$ and the sets $(\operatorname{Supp}\gamma) \cap (0,1/2)$ and $(\operatorname{Supp}\gamma) \cap (1/2,1)$ are non-empty.

Let $M > 0$ be a constant to be chosen below and set

$$\overline{\mathbf{y}}(\mathbf{x},t) := M\gamma(t)\nabla\varphi(\mathbf{x}), \quad \overline{p}(\mathbf{x},t) := -M\gamma_t(t)\varphi(\mathbf{x}) - \frac{M^2}{2}\gamma(t)^2|\nabla\varphi(\mathbf{x})|^2, \quad \overline{\theta} \equiv 0.$$

Then (4.1) is satisfied by $(\overline{\mathbf{y}}, \overline{p}, \overline{\theta})$ for $T = 1$, $\mathbf{y}_0 = \mathbf{0}$ and $\theta_0 = 0$. The triplet $(\overline{\mathbf{y}}, \overline{p}, \overline{\theta})$ is thus a nontrivial trajectory of (4.1) that connects the zero state to itself.

Let Ω_3 be a bounded open set of class C^∞ such that $\Omega_1 \subset\subset \Omega_3$. We extend φ to $\overline{\Omega}_3$ as a C^∞ function with compact support in Ω_3 and we still denote this extension by φ. Let us introduce $\mathbf{y}^*(\mathbf{x},t) := M\gamma(t)\nabla\varphi(\mathbf{x})$ (observe that $\overline{\mathbf{y}}$ is the restriction of $\mathbf{y}^*$ to $\overline{\Omega} \times [0,1]$). Also, consider the associated *flux function* $\mathbf{Y}^* : \overline{\Omega}_3 \times [0,1] \times [0,1] \mapsto \overline{\Omega}_3$, defined as follows:

$$\begin{cases} \mathbf{Y}^*_t(\mathbf{x},t,s) = \mathbf{y}^*(\mathbf{Y}^*(\mathbf{x},t,s),t) \\ \mathbf{Y}^*(\mathbf{x},s,s) = \mathbf{x}. \end{cases} \tag{4.14}$$

Obviously, $\mathbf{Y}^*$ contains all the information on the trajectories of the particles transported by the velocity field $\mathbf{y}^*$. The flux $\mathbf{Y}^*$ is of class C^∞ in $\overline{\Omega}_3 \times [0,1] \times [0,1]$. Furthermore, $\mathbf{Y}^*(\cdot\,,t,s)$ is a diffeomorphism of $\overline{\Omega}_3$ onto itself and $(\mathbf{Y}^*(\cdot\,,t,s))^{-1} = \mathbf{Y}^*(\cdot\,,s,t)$ for all $s,t \in [0,1]$.

Remark 3. From the definition of $\mathbf{y}^*$ and the boundary conditions on Ω_1 satisfied by φ, we observe that the particles cannot cross Σ. Since φ is constant on Γ^+, the gradient $\nabla\varphi$ is parallel to the normal vector on Γ^+. Since φ attains a maximum at any point of Γ^+, we have $\nabla\varphi \cdot \mathbf{n} > 0$ on Γ^+, whence $\mathbf{y}^* \cdot \mathbf{n} \geq 0$ on $\Gamma^+ \times [0,1]$. Similarly, $\mathbf{y}^* \cdot \mathbf{n} \leq 0$ on $\Gamma^- \times [0,1]$. Consequently, the particles moving with velocity $\mathbf{y}^*$ can leave Ω_1 only through Γ^+ and can enter Ω_1 only through Γ^-.

The following lemma shows that the particles that travel with velocity $\mathbf{y}^*$ and are inside $\overline{\Omega}_1$ at time $t = 0$ (resp. $t = 1/2$) will be outside $\overline{\Omega}_1$ at time $t = 1/2$ (resp. $t = 1$).

Lemma 5. *There exist $M > 0$ (large enough) and a bounded open set Ω_2 satisfying $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3$ such that*

$$\mathbf{Y}^*(\mathbf{x},1/2,0) \notin \overline{\Omega}_2 \quad \textit{and} \quad \mathbf{Y}^*(\mathbf{x},1,1/2) \notin \overline{\Omega}_2 \quad \forall \mathbf{x} \in \overline{\Omega}_2. \tag{4.15}$$

The proof is given in [29] and relies on the properties of $\mathbf{y}^*$ and, more precisely, on the fact that $t \mapsto \varphi(\mathbf{Y}^*(\mathbf{x},t,s))$ is nondecreasing.

The next step is to introduce appropriate extension mappings from Ω to Ω_3. We have the following result from [49]:

Lemma 6. *For $\ell = 1$ and $\ell = 2$, there exist continuous linear mappings $\pi_\ell : \mathbf{C}^0(\overline{\Omega};\mathbb{R}^\ell) \mapsto \mathbf{C}^0(\overline{\Omega}_3;\mathbb{R}^\ell)$ such that*

$$\begin{cases} \pi_\ell(\mathbf{f}) = \mathbf{f} \quad \textit{in} \quad \Omega \quad \textit{and} \quad \textit{Supp}\,\pi_\ell(\mathbf{f}) \subset \Omega_2 \quad \forall \mathbf{f} \in \mathbf{C}^0(\overline{\Omega};\mathbb{R}^\ell), \\ \pi_\ell \ \textit{maps continuously} \ \mathbf{C}^{m,\lambda}(\overline{\Omega};\mathbb{R}^\ell) \ \textit{into} \ \mathbf{C}^{m,\lambda}(\overline{\Omega_3};\mathbb{R}^\ell) \quad \forall m \geq 0, \quad \forall \lambda \in (0,1). \end{cases}$$

The next lemma asserts that (4.15) holds not only for $\mathbf{y}^*$ but also for any appropriate extension of any flow $\mathbf{z}$ close enough to $\overline{\mathbf{y}}$:

Lemma 7. *For each $\mathbf{z} \in C^0(\overline{\Omega} \times [0,1];\mathbb{R}^2)$, let us set $\mathbf{z}^* = \mathbf{y}^* + \pi_2(\mathbf{z} - \overline{\mathbf{y}})$. There exists $\nu > 0$ such that, if $\|\mathbf{z} - \overline{\mathbf{y}}\|_{(0)} \leq \nu$, then*

$$\mathbf{Z}^*(\mathbf{x},1/2,0) \notin \overline{\Omega}_2 \quad \textit{and} \quad \mathbf{Z}^*(\mathbf{x},1,1/2) \notin \overline{\Omega}_2 \quad \forall \mathbf{x} \in \overline{\Omega}_2, \tag{4.16}$$

where $\mathbf{Z}^*$ is the flux function associated to $\mathbf{z}^*$.

Demonstração. Let us set

$$\mathbf{A} = \left\{\mathbf{Y}^*(\mathbf{x}, 1/2, 0) \,:\, \mathbf{x} \in \overline{\Omega}_2\right\} \cup \left\{\mathbf{Y}^*(\mathbf{x}, 1, 1/2) \,:\, \mathbf{x} \in \overline{\Omega}_2\right\}.$$

Both $\mathbf{A}$ and $\overline{\Omega}_2$ are compact subsets of $\mathbb{R}^2$ and, in view of Lemma 5, $\mathbf{A} \cap \overline{\Omega}_2 = \emptyset$. Consequently, $d := \text{dist}\,(\mathbf{A},\ \overline{\Omega}_2) > 0$.

Let us introduce $\mathbf{W} := \mathbf{Y}^* - \mathbf{Z}^*$. Then, in view of the *Mean Value Theorem* and the properties of π_2, we have:

$$\begin{aligned}
|\mathbf{W}(\mathbf{x}, t, s)| &\le M \int_s^t \gamma(\sigma) |\nabla\varphi(\mathbf{Y}^*(\mathbf{x}, \sigma, s)) - \nabla\varphi(\mathbf{Z}^*(\mathbf{x}, \sigma, s))| \, d\sigma \\
&\quad + \int_s^t |\pi_2(\mathbf{z} - \overline{\mathbf{y}})(\mathbf{Z}^*(\mathbf{x}, \sigma, s), \sigma)| \, d\sigma \\
&\le M \|\nabla\varphi\|_0 \int_s^t \gamma(\sigma) |\mathbf{W}(\mathbf{x}, \sigma, s)| \, d\sigma + \int_s^t \|(\pi_2(\mathbf{z} - \overline{\mathbf{y}}))(\cdot, \sigma)\|_0 \, d\sigma \\
&\le M \|\nabla\varphi\|_0 \int_s^t \gamma(\sigma) |\mathbf{W}(\mathbf{x}, \sigma, s)| \, d\sigma + C \int_s^t \|(\mathbf{z} - \overline{\mathbf{y}})(\cdot, \sigma)\|_0 \, d\sigma,
\end{aligned}$$

where $(\mathbf{x}, t, s) \in \overline{\Omega}_3 \times [0, 1] \times [0, 1]$. Hence, from Gronwall's Lemma, we find that

$$\begin{aligned}
|\mathbf{W}(\mathbf{x}, t, s)| &\le C \left(\int_s^t \|\mathbf{z} - \overline{\mathbf{y}}\|_0(\sigma) \, d\sigma \right) \exp\left(M \|\nabla\varphi\|_0 \int_s^t \gamma(\sigma) \, d\sigma \right) \\
&\le C e^{M \|\nabla\varphi\|_0 \|\gamma\|_0} \|\mathbf{z} - \overline{\mathbf{y}}\|_{(0)}
\end{aligned}$$

Therefore, there exists $\nu > 0$ such that, if $\|\mathbf{z} - \overline{\mathbf{y}}\|_{(0)} \le \nu$, one has

$$|\mathbf{W}(\mathbf{x}, t, s)| \le \frac{d}{2} \quad \forall (\mathbf{x}, t, s) \in \overline{\Omega}_3 \times [0, 1] \times [0, 1]. \tag{4.17}$$

Thanks to Lemma 5 and (4.17), we necessarily have (4.16) and the proof is achieved. □

4.2.2 Construction of a trajectory when $N = 3$

In this Section, we will follow [43]. As in the two-dimensional case, $\overline{\mathbf{y}}$ will be of the potential form "$\nabla\varphi$", with the property that any particle traveling with velocity $\overline{\mathbf{y}}$ must leave $\overline{\Omega}$ at an appropriate time. The main difference will be that, in this three-dimensional case, "$\nabla\varphi$" is not chosen independent of t.

We first recall a lemma:

Lemma 8. *Let $\mathscr{O}$ be a regular bounded open set such that $\Omega \subset\subset \mathscr{O}$. For each $\mathbf{a} \in \overline{\Omega}$, there exists $\phi^{\mathbf{a}} \in C^\infty(\overline{\mathscr{O}} \times [0, 1])$ such that* $\text{supp}(\phi^{\mathbf{a}}) \subset \mathscr{O} \times (1/4, 3/4)$,

$$\begin{cases} -\Delta\phi^{\mathbf{a}} = 0 & in \quad \Omega \times (0, 1), \\ \dfrac{\partial\phi^{\mathbf{a}}}{\partial\mathbf{n}} = 0 & on \quad (\Gamma \setminus \Gamma_0) \times (0, 1) \end{cases} \tag{4.18}$$

and

$$\mathbf{\Phi}^{\mathbf{a}}(\mathbf{a},1,0) \in \mathscr{O} \setminus \overline{\Omega},$$

where $\mathbf{\Phi}^{\mathbf{a}} := \mathbf{\Phi}^{\mathbf{a}}(\mathbf{x},t,s)$ *is the flux associated to* $\nabla\phi^{\mathbf{a}}$*, that is, the unique* $\mathbb{R}^N$*−valued function in* $\overline{\mathscr{O}} \times [0,1] \times [0,1]$ *satisfying*

$$\begin{cases} \mathbf{\Phi}^{\mathbf{a}}_t(\mathbf{x},t,s) = \nabla\phi^{\mathbf{a}}(\mathbf{\Phi}^{\mathbf{a}}(\mathbf{x},t,s),t), \\ \mathbf{\Phi}^{\mathbf{a}}(\mathbf{x},s,s) = \mathbf{x}. \end{cases} \tag{4.19}$$

The proof is given in [43].

With the help of these $\mathbf{\Phi}^{\mathbf{a}}$, we can construct a vector field $\mathbf{y}^*$ in $\mathscr{O} \times (0,1)$ that makes the particles go from Ω to the outside and then makes them come back.

Indeed, from the continuity of the functions $\mathbf{\Phi}^{\mathbf{a}}$ and the compactness of $\overline{\Omega}$, we can find $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ in $\overline{\Omega}$, real numbers $r_1, \ldots, r_k$, smooth functions $\phi^1 := \phi^{\mathbf{a}_1}, \ldots, \phi^k := \phi^{\mathbf{a}_k}$ satisfying Lemma 8 and a bounded open set $\mathscr{O}_0$ with $\Omega \subset\subset \mathscr{O}_0 \subset\subset \mathscr{O}$, such that

$$\overline{\Omega} \subset \bigcup_{i=1}^{k} B^i \subset\subset \mathscr{O}_0 \quad \text{and} \quad \mathbf{\Phi}^i(\overline{B}^i,1,0) \subset \mathscr{O} \setminus \overline{\mathscr{O}}_0, \tag{4.20}$$

where $B^i := B(\mathbf{a}_i; r_i)$ and $\mathbf{\Phi}^i := \mathbf{\Phi}^{\mathbf{a}_i}$ for $i = 1, \ldots, k$.

As in [43], the definition of $\mathbf{y}^*$ is as follows: let the time t_i be given by

$$\begin{aligned} t_i &= \frac{1}{4} + \frac{i}{4k}, \quad i = 0, \ldots, 2k, \\ t_{i+1/2} &= \frac{1}{4} + \left(i + \frac{1}{2}\right)\frac{1}{4k}, \quad i = 0, \ldots, 2k-1 \end{aligned} \tag{4.21}$$

and let us set

$$\phi(\mathbf{x},t) = \begin{cases} 0, & (\mathbf{x},t) \in \overline{\mathscr{O}} \times ([0,1/4] \cup [3/4,1]), \\ 8k\phi^j(\mathbf{x}, 8k(t - t_{j-1})), & (\mathbf{x},t) \in \overline{\mathscr{O}} \times \left[t_{j-1}, t_{j-1/2}\right], \\ -8k\phi^j(\mathbf{x}, 8k(t_j - t)), & (\mathbf{x},t) \in \overline{\mathscr{O}} \times \left[t_{j-1/2}, t_j\right] \end{cases} \tag{4.22}$$

for $j = 1, \ldots, 2k$, where $\phi^{k+i} := \phi^i$ for $i = 1, \ldots, k$; then, we set $\mathbf{y}^* := \nabla\phi$ and $\overline{\mathbf{y}} := \mathbf{y}^*|_{\overline{\Omega}\times[0,1]}$ and we denote by $\mathbf{Y}^*$ the flux associated to $\mathbf{y}^*$.

If we set $\bar{p}(\mathbf{x},t) := -\phi_t(\mathbf{x},t) - \frac{1}{2}|\nabla\phi(\mathbf{x},t)|^2$ and $\overline{\theta} \equiv 0$, then (4.1) and (4.3) are verified by $(\overline{\mathbf{y}}, \bar{p}, \overline{\theta})$ for $T = 1$, $\mathbf{y}_0 = \mathbf{y}_1 = \mathbf{0}$ and $\theta_0 = \theta_1 = 0$.

Thanks to (4.20) and (4.22), one has:

Lemma 9. *The following property holds for all* $i = 1, \ldots, k$*:*

$$\mathbf{Y}^*(\mathbf{x}, t_{i-1/2}, 0) \in \mathscr{O} \setminus \overline{\mathscr{O}}_0 \quad \textit{and} \quad \mathbf{Y}^*(\mathbf{x}, t_{k+i-1/2}, 1/2) \in \mathscr{O} \setminus \overline{\mathscr{O}}_0 \quad \forall \mathbf{x} \in B^i. \tag{4.23}$$

For the proof, it suffices to notice that, in $\overline{\mathscr{O}} \times [1/4, 3/4] \times [1/4, 3/4]$, $\mathbf{Y}^*(\mathbf{x}, t, s)$ is given as follows:

$$\begin{cases} \mathbf{\Phi}^j(\mathbf{x}, 8k(t - t_{j-1}), 8k(s - t_{l-1})) \text{ if } (\mathbf{x}, t, s) \in \overline{\mathscr{O}} \times [t_{j-1}, t_{j-1/2}] \times [t_{l-1}, t_{l-1/2}], \\ \mathbf{\Phi}^j(\mathbf{x}, 8k(t - t_{j-1}), 8k(t_l - s)) \text{ if } (\mathbf{x}, t, s) \in \overline{\mathscr{O}} \times [t_{j-1}, t_{j-1/2}] \times [t_{l-1/2}, t_l], \\ \mathbf{\Phi}^j(\mathbf{x}, 8k(t_j - t), 8k(s - t_{l-1})) \text{ if } (\mathbf{x}, t, s) \in \overline{\mathscr{O}} \times [t_{j-1/2}, t_j] \times [t_{l-1}, t_{l-1/2}], \\ \mathbf{\Phi}^j(\mathbf{x}, 8k(t_j - t), 8k(t_l - s)) \text{ if } (\mathbf{x}, t, s) \in \overline{\mathscr{O}} \times [t_{j-1/2}, t_j] \times [t_{l-1/2}, t_l] \end{cases}$$

for all $l, j = 1, \dots, 2k$, where $\mathbf{\Phi}^{k+i}$ the flux associated to $\nabla \phi^{k+i}$ for $i = 1, \dots, k$.

Hence, one has the following for all $i = 1, \dots, k$ and for each $\mathbf{x} \in B^i$:

$$\mathbf{Y}^*(\mathbf{x}, t_{i-1/2}, 0) = \mathbf{Y}^*(\mathbf{x}, t_{i-1/2}, 1/4) = \mathbf{Y}^*(\mathbf{x}, t_{i-1/2}, t_0) = \mathbf{\Phi}^i(\mathbf{x}, 1, 0) \in \mathscr{O} \setminus \overline{\mathscr{O}}_0$$

and

$$\mathbf{Y}^*(\mathbf{x}, t_{k+i-1/2}, 1/2) = \mathbf{Y}^*(\mathbf{x}, t_{k+i-1/2}, t_k) = \mathbf{\Phi}^{k+i}(\mathbf{x}, 1, 0) = \mathbf{\Phi}^i(\mathbf{x}, 1, 0) \in \mathscr{O} \setminus \overline{\mathscr{O}}_0.$$

A result similar to Lemma 6 also holds here:

Lemma 10. *For $\ell = 1$ and $\ell = 3$, there exist continuous linear mappings $\pi_\ell : \mathbf{C}^0(\overline{\Omega}; \mathbb{R}^\ell) \mapsto \mathbf{C}^0(\overline{\mathscr{O}}; \mathbb{R}^\ell)$ such that*

$$\begin{cases} \pi_\ell(\mathbf{f}) = \mathbf{f} \quad \textit{in} \quad \Omega \quad \textit{and} \quad \textit{Supp}\, \pi_\ell(\mathbf{f}) \subset \mathscr{O}_0 \quad \forall \mathbf{f} \in \mathbf{C}^0(\overline{\Omega}; \mathbb{R}^\ell), \\ \pi_\ell \textit{ maps continuously } \mathbf{C}^{n,\lambda}(\overline{\Omega}; \mathbb{R}^\ell) \textit{ into } \mathbf{C}^{n,\lambda}(\overline{\mathscr{O}}; \mathbb{R}^\ell) \quad \forall n \geq 0, \quad \forall \lambda \in (0, 1). \end{cases}$$

Finally, we also have that (4.23) holds for the flux corresponding to the of any velocity field close enough to $\overline{\mathbf{y}}$:

Lemma 11. *For each $\mathbf{z} \in C^0(\overline{\Omega} \times [0, 1]; \mathbb{R}^3)$, let us set $\mathbf{z}^* = \mathbf{y}^* + \pi_3(\mathbf{z} - \overline{\mathbf{y}})$. Then there exists $\nu > 0$ such that, if $\|\mathbf{z} - \overline{\mathbf{y}}\|_{(0)} \leq \nu$ and $i = 1, \dots, k$, one has:*

$$\mathbf{Z}^*(\mathbf{x}, t_{i-1/2}, 0) \in \mathscr{O} \setminus \overline{\mathscr{O}}_0 \quad \textit{and} \quad \mathbf{Z}^*(\mathbf{x}, t_{k+i-1/2}, 1/2) \in \mathscr{O} \setminus \overline{\mathscr{O}}_0 \quad \forall \mathbf{x} \in B^i,$$

where $\mathbf{Z}^*$ is the flux associated to $\mathbf{z}^*$.

The proof is very similar to the proof of Lemma 7 and will be omitted.

4.3 Proof of Theorem 6

This Section is devoted to prove the exact controllability result in Theorem 6. We will assume that Proposition 9 is satisfied and we will employ a scaling argument.

Let $T > 0$, $\theta_0, \theta_1 \in C^{2,\alpha}(\overline{\Omega})$ and $\mathbf{y}_0, \mathbf{y}_1 \in \mathbf{C}(2, \alpha, \Gamma_0)$ be given. Let us see that, if

$$\|\mathbf{y}_0\|_{2,\alpha} + \|\mathbf{y}_1\|_{2,\alpha} + \|\theta_0\|_{2,\alpha} + \|\theta_1\|_{2,\alpha}$$

is small enough, we can construct a triplet $(\mathbf{y}, p, \theta)$ satisfying (4.1) and (4.3).

If $\varepsilon \in (0,T/2)$ is small enough to have

$$\max\{\varepsilon\|\mathbf{y}_0\|_{2,\alpha}, \varepsilon^2\|\theta_0\|_{2,\alpha}\} \le \delta \quad (\text{resp. } \max\{\varepsilon\|\mathbf{y}_1\|_{2,\alpha},\ \varepsilon^2\|\theta_1\|_{2,\alpha}\} \le \delta),$$

then, thanks to Proposition 9, there exist $(\mathbf{y}^0, \theta^0)$ in $C^0([0,1]; \mathbf{C}^{1,\alpha}(\overline{\Omega}; \mathbb{R}^{N+1}))$ and a pressure p^0 (resp. $(\mathbf{y}^1, \theta^1)$ and p^1) solving (4.1), with $\mathbf{y}^0(\mathbf{x},0) \equiv \varepsilon \mathbf{y}_0(\mathbf{x})$ and $\theta^0(\mathbf{x},0) \equiv \varepsilon^2\theta_0(\mathbf{x})$ (resp. $\mathbf{y}^1(\mathbf{x},0) \equiv -\varepsilon\mathbf{y}_1(\mathbf{x})$ and $\theta^1(\mathbf{x},0) = \varepsilon^2\theta_1(\mathbf{x})$) and satisfying (4.5).

Let us choose ε of this form and let us introduce $\mathbf{y} : \overline{\Omega} \times [0,T] \mapsto \mathbb{R}^N$, $p : \overline{\Omega} \times [0,T] \mapsto \mathbb{R}$ and $\theta : \overline{\Omega} \times [0,T] \mapsto \mathbb{R}$ as follows:

$$\begin{cases} \mathbf{y}(\mathbf{x},t) = \varepsilon^{-1}\mathbf{y}^0(\mathbf{x},\varepsilon^{-1}t), \\ p(\mathbf{x},t) = \varepsilon^{-2}p^0(\mathbf{x},\varepsilon^{-1}t), \\ \theta(\mathbf{x},t) = \varepsilon^{-2}\theta^0(\mathbf{x},\varepsilon^{-1}t), \end{cases} \quad \text{for} \quad (\mathbf{x},t) \in \overline{\Omega} \times [0,\varepsilon],$$

$$\begin{cases} \mathbf{y}(\mathbf{x},t) = \mathbf{0}, \\ p(\mathbf{x},t) = 0, \\ \theta(\mathbf{x},t) = 0, \end{cases} \quad \text{for} \quad (\mathbf{x},t) \in \overline{\Omega} \times (\varepsilon, T-\varepsilon),$$

$$\begin{cases} \mathbf{y}(\mathbf{x},t) = -\varepsilon^{-1}\mathbf{y}^1(\mathbf{x},\varepsilon^{-1}(T-t)), \\ p(\mathbf{x},t) = \varepsilon^{-2}p^1(\mathbf{x},\varepsilon^{-1}(T-t)), \\ \theta(\mathbf{x},t) = \varepsilon^{-2}\theta^1(\mathbf{x},\varepsilon^{-1}(T-t)), \end{cases} \quad \text{for} \quad (\mathbf{x},t) \in \overline{\Omega} \times [T-\varepsilon,T].$$

Then, $(\mathbf{y},\theta) \in C^0([0,T]; \mathbf{C}^{1,\alpha}(\overline{\Omega}; \mathbb{R}^{N+1})$ and the triplet $(\mathbf{y}, p, \theta)$ satisfies (4.1) and (4.3).

4.4 Proof of Proposition 9. The 2D case

Let $\mu \in C^\infty([0,1])$ be a function such that $\mu \equiv 1$ in $[0,1/4]$, $\mu \equiv 0$ in $[1/2,1]$ and $0 < \mu < 1$. Proposition 9 is a consequence of the following result:

Proposition 10. *There exists $\delta > 0$ such that, if $\max\{\|\mathbf{y}_0\|_{2,\alpha}, \|\theta_0\|_{2,\alpha}\} \le \delta$, then the coupled system*

$$\begin{cases} \zeta_t + \mathbf{y}\cdot\nabla\zeta = -\vec{\mathbf{k}} \times \nabla\theta & \text{in} \quad \Omega \times (0,1), \\ \theta_t + \mathbf{y}\cdot\nabla\theta = 0 & \text{in} \quad \Omega \times (0,1), \\ \nabla\cdot\mathbf{y} = 0,\ \nabla\times\mathbf{y} = \zeta & \text{in} \quad \Omega \times (0,1), \\ \mathbf{y}\cdot\mathbf{n} = (\overline{\mathbf{y}} + \mu\,\mathbf{y}_0)\cdot\mathbf{n} & \text{on} \quad \Gamma \times (0,1), \\ \zeta(0) = \nabla\times\mathbf{y}_0,\ \theta(0) = \theta_0 & \text{in} \quad \Omega, \end{cases} \tag{4.24}$$

possesses at least one solution $(\zeta, \theta, \mathbf{y})$, with

$$(\zeta,\theta,\mathbf{y}) \in C^0([0,1]; C^{0,\alpha}(\overline{\Omega})) \times C^0([0,1]; C^{1,\alpha}(\overline{\Omega})) \times C^0([0,1]; \mathbf{C}^{1,\alpha}(\overline{\Omega}; \mathbb{R}^2)), \tag{4.25}$$

such that

$$\theta(\mathbf{x},t) = 0 \quad \text{in} \quad \Omega \times (1/2,1) \quad \text{and} \quad \zeta(\mathbf{x},1) = 0 \quad \text{in} \quad \Omega. \tag{4.26}$$

The reminder of this section is devoted to prove Proposition 10. We are going to adapt some ideas from Bardos and Frisch [12] and Kato [57], already used in [29] and [41]. Let us give a sketch.

We will start from an arbitrary field $\mathbf{z} := \mathbf{z}(\mathbf{x}, t)$ in a suitable class $\mathbf{S}$ of continuous functions. To this $\mathbf{z}$, we will associate a scalar function θ (a temperature) verifying

$$\begin{cases} \theta_t + \mathbf{z} \cdot \nabla\theta = 0 & \text{in} \quad \Omega \times (0,1), \\ \theta(\mathbf{x}, 0) = \theta_0(\mathbf{x}) & \text{in} \quad \Omega. \end{cases}$$

and

$$\theta(\mathbf{x}, t) = 0 \quad \text{in} \quad \Omega \times (1/2, 1).$$

With the help of θ, we will then construct a function ζ (an associated vorticity) satisfying

$$\begin{cases} \zeta_t + \mathbf{z} \cdot \nabla\zeta = -\vec{\mathbf{k}} \times \nabla\theta & \text{in} \quad \Omega \times (0,1), \\ \zeta(0) = \nabla \times \mathbf{y}_0 & \text{in} \quad \Omega. \end{cases}$$

and

$$\zeta(\mathbf{x}, 1) = 0 \quad \text{in} \quad \Omega.$$

Then, we will construct a field $\mathbf{y} = \mathbf{y}(\mathbf{x}, t)$ such that $\nabla \times \mathbf{y} = \zeta$ and $\nabla \cdot \mathbf{y} = 0$. This way, we will have defined a mapping F with $F(\mathbf{z}) = \mathbf{y}$. We will choose $\mathbf{S}$ such that F maps $\mathbf{S}$ into itself and an appropriate extension of F possesses exactly one fixed-point $\mathbf{y}$. Finally, it will be seen that the triplet $(\zeta, \theta, \mathbf{y})$, where ζ and θ are respectively the vorticity and temperature associated to $\mathbf{y}$, solves (4.24) and satisfies (4.25).

Let us now give the details.

The good definition of $\mathbf{S}$ is as follows. First, let us denote by $\mathbf{S}'$ the set of fields $\mathbf{z} \in C^0([0,1]; \mathbf{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R}^2))$ such that $\nabla \cdot \mathbf{z} = 0$ in $\Omega \times (0,1)$ and $\mathbf{z} \cdot \mathbf{n} = (\overline{\mathbf{y}} + \mu\mathbf{y}_0) \cdot \mathbf{n}$ on $\Gamma \times (0,1)$. Then, for any $\nu > 0$, we set

$$\mathbf{S}_\nu = \{\, \mathbf{z} \in \mathbf{S}' : \|\mathbf{z} - \overline{\mathbf{y}}\|_{0,2,\alpha} \le \nu \,\}.$$

Let $\nu > 0$ be the constant furnished by Lemma 7 and let us carry out the previous process with $\mathbf{S} = \mathbf{S}_\nu$. To guarantee that $\mathbf{S}_\nu$ is nonempty, it suffices to assume that the initial data $\mathbf{y}_0$ is sufficiently small in $\mathbf{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R}^2)$. Since, if this is the case, $\overline{\mathbf{y}} + \mu\mathbf{y}_0 \in \mathbf{S}_\nu$.

Let $\mathbf{z} \in \mathbf{S}_\nu$ be given and let us set $\mathbf{z}^* = \mathbf{y}^* + \pi_2(\mathbf{z} - \overline{\mathbf{y}})$. We have the estimate

$$\|\mathbf{z}^*(\cdot\,, t)\|_{2,\alpha} \le \|\mathbf{y}^*(\cdot\,, t)\|_{2,\alpha} + C\|(\mathbf{z} - \overline{\mathbf{y}})(\cdot\,, t)\|_{2,\alpha} \quad \forall t \in [0,1] \tag{4.27}$$

and the following result holds:

Lemma 12. *The flux $\mathbf{Z}^*$ associated to $\mathbf{z}^*$ satisfies $\mathbf{Z}^* \in C^1([0,1] \times [0,1]; \mathbf{C}^{2,\alpha}(\overline{\Omega}_3; \mathbb{R}^2))$.*

Recall that $\mathbf{Z}^*$ is, by definition, the unique function satisfying

$$\begin{cases} \mathbf{Z}^*_t(\mathbf{x}, t, s) = \mathbf{z}^*(\mathbf{Z}^*(\mathbf{x}, t, s), t), \\ \mathbf{Z}^*(\mathbf{x}, s, s) = \mathbf{x}, \end{cases} \tag{4.28}$$

and

$$\mathbf{Z}^*(\mathbf{x},t,s) \in \overline{\Omega}_3 \quad \forall (\mathbf{x},t,s) \in \overline{\Omega}_3 \times [0,1] \times [0,1].$$

For the proof of Lemma 12, it suffices to apply directly the well known (classical) existence, uniqueness and regularity theory of ODEs.

Since $\mathbf{Z}^* \in C^1([0,1]\times[0,1]; \mathbf{C}^{2,\alpha}(\overline{\Omega}_3;\mathbb{R}^2))$, $\theta_0 \in C^{2,\alpha}(\overline{\Omega})$ and π_1 maps continuously $C^{2,\alpha}(\overline{\Omega})$ into $C^{2,\alpha}(\overline{\Omega}_3)$, there exists a unique solution $\theta^* \in C^0([0,1/2]; C^{2,\alpha}(\overline{\Omega}_3))$ to the problem

$$\begin{cases} \theta^*_t + \mathbf{z}^* \cdot \nabla \theta^* = 0 & \text{in} \quad \Omega_3 \times (0,1/2), \\ \theta^*(\mathbf{x},0) = \pi_1(\theta_0)(\mathbf{x}) & \text{in} \quad \Omega_3. \end{cases} \tag{4.29}$$

Note that, in (4.29), no boundary condition on θ^* appears. Obviously, this is because $\operatorname{Supp} \mathbf{z}^* \subset \Omega_3$.

The solution to (4.29) verifies $(\operatorname{Supp} \theta^*(\cdot\,,t)) \subset \mathbf{Z}^*(\Omega_2,t,0)$ for all $t \in [0,1/2]$. In particular, in view of the choice of ν, we get:

$$\operatorname{Supp} \theta^*(\cdot\,,1/2) \subset \mathbf{Z}^*(\Omega_2,1/2,0) \subset \Omega_3 \setminus \overline{\Omega}_2,$$

whence $\theta^*(\mathbf{x},1/2) = 0$ in Ω_2.

Let θ be the following function:

$$\theta(\mathbf{x},t) = \begin{cases} \theta^*(\mathbf{x},t), & (\mathbf{x},t) \in \overline{\Omega} \times [0,1/2), \\ 0, & (\mathbf{x},t) \in \overline{\Omega} \times [1/2,1]. \end{cases}$$

Then $\theta \in C^0([0,1]; C^{2,\alpha}(\overline{\Omega}))$ and one has

$$\begin{cases} \theta_t + \mathbf{z} \cdot \nabla \theta = 0 & \text{in} \quad \Omega \times (0,1), \\ \theta(\mathbf{x},0) = \theta_0(\mathbf{x}) & \text{in} \quad \Omega. \end{cases} \tag{4.30}$$

For the construction of ζ, the argument is the following. Firstly, let us introduce $\zeta^*_0 := \nabla \times (\pi_2(\mathbf{y}_0))$ and let $\zeta^* \in C^0([0,1/2]; C^{1,\alpha}(\overline{\Omega}_3))$ be the unique solution to the problem

$$\begin{cases} \zeta^*_t + \mathbf{z}^* \cdot \nabla \zeta^* = -\vec{\mathbf{k}} \times \nabla \theta^* & \text{in} \quad \Omega_3 \times (0,1/2), \\ \zeta^*(\mathbf{x},0) = \zeta^*_0(\mathbf{x}) & \text{in} \quad \Omega_3. \end{cases}$$

With this ζ^*, we define $\zeta_{1/2} \in C^{1,\alpha}(\overline{\Omega})$ with

$$\zeta_{1/2}(\mathbf{x}) := \zeta^*(\mathbf{x},1/2) \ \text{ forall } \ \mathbf{x} \in \overline{\Omega}.$$

Then, let $\zeta^{**} \in C^0([1/2,1]; C^{1,\alpha}(\overline{\Omega}_3))$ be the unique solution to the problem

$$\begin{cases} \zeta^{**}_t + \mathbf{z}^* \cdot \nabla \zeta^{**} = 0 & \text{in} \quad \Omega_3 \times (1/2,1), \\ \zeta^{**}(\mathbf{x},1/2) = \pi_1(\zeta_{1/2})(\mathbf{x}) & \text{in} \quad \Omega_3. \end{cases}$$

We have $\zeta^{**}(\mathbf{Z}^*(\mathbf{x},t,1/2),t) = \pi_1(\zeta_{1/2})(\mathbf{x})$ for all $(\mathbf{x},t) \in \overline{\Omega}_3 \times [1/2,1]$ and, again from the choice of ν,

$$\operatorname{Supp} \zeta^{**}(\cdot\,,1) \subset \mathbf{Z}^*(\Omega_2,1,1/2) \subset \Omega_3 \setminus \overline{\Omega}_2$$

and $\zeta^{**}(\cdot\,,1) \equiv 0$ in Ω_2.

Therefore, we can define $\zeta \in C^0([0,1]; C^{1,\alpha}(\overline{\Omega}))$, with

$$\zeta(\mathbf{x},t) = \begin{cases} \zeta^*(\mathbf{x},t), & (\mathbf{x},t) \in \Omega \times (0,1/2), \\ \zeta^{**}(\mathbf{x},t), & (\mathbf{x},t) \in \Omega \times [1/2,1). \end{cases}$$

Obviously, ζ is a solution to the initial-value problem

$$\begin{cases} \zeta_t + \mathbf{z} \cdot \nabla\zeta = -\vec{\mathbf{k}} \times \nabla\theta & \text{in} \quad \Omega \times (0,1), \\ \zeta(\mathbf{x},0) = (\nabla \times \mathbf{y}_0)(\mathbf{x}) & \text{in} \quad \Omega. \end{cases} \tag{4.31}$$

With this ζ, we can now get a unique $\mathbf{y} \in C^0([0,1]; \mathbf{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R}^2))$ such that $\nabla \times \mathbf{y} = \zeta$ in $\Omega \times (0,1)$, $\nabla \cdot \mathbf{y} = 0$ in $\Omega \times (0,1)$ and $\mathbf{y} \cdot \mathbf{n} = (\overline{\mathbf{y}} + \mu\mathbf{y}_0) \cdot \mathbf{n}$ on $\Gamma \times [0,1]$. Indeed, let $\psi \in C^0([0,1]; C^{3,\alpha}(\overline{\Omega}))$ be the unique solution to the following family of elliptic equations:

$$\begin{cases} -\Delta\psi = \zeta - \mu\nabla \times \mathbf{y}_0 & \text{in } \Omega \times (0,1), \\ \psi = 0 & \text{on } \Gamma \times (0,1). \end{cases} \tag{4.32}$$

Then, let us set $\mathbf{y} := \nabla \times \psi + \overline{\mathbf{y}} + \mu\mathbf{y}_0$. Obviously, $\mathbf{y} \in C^0([0,1]; \mathbf{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R}^2))$ and satisfies the required properties. Since $\mathbf{y}$ is determined by $\mathbf{z}$, we write $\mathbf{y} = F(\mathbf{z})$. Accordingly, $F : \mathbf{S}_\nu \mapsto \mathbf{S}'$ is well defined.

The following result holds:

Lemma 13. *There exists $\delta > 0$ such that, if*

$$\max\{\|\mathbf{y}_0\|_{2,\alpha}, \|\theta_0\|_{2,\alpha}\} \le \delta, \tag{4.33}$$

then $F(\mathbf{S}_\nu) \subset \mathbf{S}_\nu$.

Demonstração. Let $\mathbf{z} \in \mathbf{S}_\nu$ be given. Then $F(\mathbf{z}) - \overline{\mathbf{y}} = \nabla \times \psi + \mu\mathbf{y}_0$ and we have:

$$\|F(\mathbf{z})(\cdot\,,t) - \overline{\mathbf{y}}(\cdot\,,t)\|_{2,\alpha} \le C(\|\zeta(\cdot\,,t)\|_{1,\alpha} + \|\mathbf{y}_0\|_{2,\alpha}).$$

Applying Lemma 2 to the equations of θ^* and ζ^*, we get

$$\|\theta^*(\cdot\,,t)\|_{2,\alpha} \le \|\pi_1(\theta_0)\|_{2,\alpha} \exp\left(K \int_0^t \|\mathbf{z}^*(\cdot\,,\tau)\|_{2,\alpha}\, d\tau\right) \tag{4.34}$$

and

$$\|\zeta^*(\cdot\,,t)\|_{1,\alpha} \le C(\|\pi_2(\mathbf{y}_0)\|_{2,\alpha} + \|\pi_1(\theta_0)\|_{2,\alpha}) \exp\left(K \int_0^t \|\mathbf{z}^*(\cdot\,,\tau)\|_{2,\alpha}\, d\tau\right). \tag{4.35}$$

With similar arguments, we also obtain

$$\|\zeta^{**}(\cdot\,,t)\|_{1,\alpha} \le C(\|\pi_2(\mathbf{y}_0)\|_{2,\alpha} + \|\pi_1(\theta_0)\|_{2,\alpha}) \exp\left(K \int_0^t \|\mathbf{z}^*(\cdot\,,\tau)\|_{2,\alpha}(\tau) d\tau\right) \tag{4.36}$$

for all $t \in [1/2, 1]$. Thanks to (4.35) and (4.36), we obtain the following for ζ:

$$\|\zeta(\cdot\,,t)\|_{1,\alpha} \leq C(\|\mathbf{y}_0\|_{2,\alpha} + \|\theta_0\|_{2,\alpha}) \exp\left(K\int_0^t \|\mathbf{z}^*(\cdot,\tau)\|_{2,\alpha} d\tau\right). \tag{4.37}$$

Using (4.37), (4.27) and the definition of $\mathbf{S}_\nu$, we see that

$$\begin{aligned}\|F(\mathbf{z})(\cdot\,,t) - \overline{\mathbf{y}}(\cdot\,,t)\|_{2,\alpha} &\leq C_1(\|\mathbf{y}_0\|_{2,\alpha} + \|\theta_0\|_{2,\alpha}) \exp\left(C_2\int_0^t \|\mathbf{z}(\cdot\,,\tau) - \overline{\mathbf{y}}(\cdot\,,\tau)\|_{2,\alpha}\, d\tau\right)\\ &\leq C_1(\|\mathbf{y}_0\|_{2,\alpha} + \|\theta_0\|_{2,\alpha}) \exp(C_2\nu).\end{aligned}$$

Let $\delta > 0$ be such that $2C_1\delta e^{C_2\nu} \leq \nu$ and let us assume that (4.33) is satisfied. Then

$$\|F(\mathbf{z}) - \overline{\mathbf{y}}\|_{0,2,\alpha} \leq \nu$$

and, consequently, F maps $\mathbf{S}_\nu$ into itself. □

We now prove the existence and uniqueness of a fixed-point of the extension of F in the closure of $\mathbf{S}_\nu$ in $C^0([0,1];\mathbf{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^3))$. For this purpose, we will check that F satisfies the hypotheses of Theorem 8.

To this end, we will first establish two important lemmas. The first one is the following:

Lemma 14. *There exists $\widetilde{C} > 0$, only depending on $\|\mathbf{y}_0\|_{2,\alpha}$, $\|\theta_0\|_{2,\alpha}$ and ν, such that, for any $\mathbf{z}^1$, $\mathbf{z}^2 \in \mathbf{S}_\nu$, one has:*

$$\|(\zeta^1 - \zeta^2)(\cdot,t)\|_{0,\alpha} \leq \widetilde{C}\int_0^t \|(\mathbf{z}^1 - \mathbf{z}^2)(\cdot,s)\|_{1,\alpha}\, ds \quad \forall t \in [0,1], \tag{4.38}$$

where ζ^i is the vorticity associated to $\mathbf{z}^i$.

Demonstração. First of all, let us introduce $\mathbf{w}^* := \mathbf{z}^{*,1} - \mathbf{z}^{*,2}$ and $\Theta^* := \theta^{*,1} - \theta^{*,2}$ (where the notation id self-explaining). Obviously, the estimates (4.27) and (resp. (4.34) and (4.35)) hold for $\mathbf{z}^{*,1}$ and $\mathbf{z}^{*,2}$ (resp. $\theta^{*,1}$ and $\theta^{*,2}$ and $\zeta^{*,1}$ and $\zeta^{*,2}$). Furthermore, it is clear that

$$\Theta^*_t + \mathbf{z}^{*,1}\cdot\nabla\Theta^* = -\mathbf{w}^*\cdot\nabla\theta^{*,2}.$$

Applying Lemma 1 to this equation, we have

$$\frac{d}{dt^+}\|\Theta^*(\cdot,t)\|_{1,\alpha} \leq \|\mathbf{w}^*(\cdot,t)\|_{1,\alpha}\|\theta^{*,2}(\cdot,t)\|_{2,\alpha} + K\|\mathbf{z}^{*,1}(\cdot,t)\|_{1,\alpha}\|\Theta^*(\cdot,t)\|_{1,\alpha}. \tag{4.39}$$

In view of Gronwall's Lemma, (4.27) and (4.34), we see that

$$\|\Theta^*(\cdot,t)\|_{1,\alpha} \leq \widetilde{C}_0\int_0^t \|\mathbf{w}^*(\cdot,s)\|_{1,\alpha}\, ds \quad \forall t \in [0,1/2]. \tag{4.40}$$

The equations verified by $\Upsilon^* := \zeta^{*,1} - \zeta^{*,2}$ and $\Upsilon^{**} := \zeta^{**,1} - \zeta^{**,2}$ are

$$\Upsilon^*_t + \mathbf{z}^{*,1}\cdot\nabla\Upsilon^* = -\mathbf{w}^*\cdot\nabla\zeta^{*,2} - \vec{\mathbf{k}}\times\nabla\Theta^*$$

and

$$\Upsilon^{**}_t + \mathbf{z}^{*,1} \cdot \nabla \Upsilon^{**} = -\mathbf{w}^* \cdot \nabla \zeta^{**,2},$$

respectively. Consequently, applying Lemma 1 to these equations, we get:

$$\frac{d}{dt^+} \|\Upsilon^*(\cdot,t)\|_{0,\alpha} \leq \ \|(\mathbf{w}^* \cdot \nabla \zeta^{*,2} + \vec{\mathbf{k}} \times \nabla \Theta^*)(\cdot,t)\|_{0,\alpha} + K\|\mathbf{z}^{*,1}(\cdot,t)\|_{1,\alpha}\|\Upsilon^*(\cdot,t)\|_{0,\alpha} \quad (4.41)$$

and

$$\frac{d}{dt^+} \|\Upsilon^{**}(\cdot,t)\|_{0,\alpha} \leq \|(\mathbf{w}^* \cdot \nabla \zeta^{**,2})(\cdot,t)\|_{0,\alpha} + K\|\mathbf{z}^{*,1}(\cdot,t)\|_{1,\alpha}\|\Upsilon^{**}(\cdot,t)\|_{0,\alpha}. \quad (4.42)$$

Applying Gronwall's Lemma, we deduce in view of (4.40) that

$$\|\Upsilon^*(\cdot,t)\|_{0,\alpha} \leq \widetilde{C}_1 \|\zeta^{*,2}\|_{0,1,\alpha} \int_0^t \|\mathbf{w}^*(\cdot,s)\|_{1,\alpha}\, ds \quad \forall t \in [0,1/2]$$

and

$$\|\Upsilon^{**}(\cdot,t)\|_{0,\alpha} \leq \ \widetilde{C}_2 \|\zeta^{*,2}\|_{0,1,\alpha} \int_0^t \|\mathbf{w}^*(\cdot,s)\|_{1,\alpha}\, ds \quad \forall t \in [1/2,1].$$

Finally, we see from these estimates and (4.37) that (4.38) holds. □

Note that $\mathbf{y}^1 - \mathbf{y}^2 = \nabla \times (\psi^1 - \psi^2)$, whence $\nabla \times (\nabla \times (\psi^1 - \psi^2)) = \zeta^1 - \zeta^2$ and $\nabla \times (\psi^1 - \psi^2) \cdot \mathbf{n} = 0$ on $\Gamma \times [0,1]$.

Let us denote by $\mathbf{M}$ the set of fields $\mathbf{w} \in C^0([0,1]; \mathbf{C}^{1,\alpha}(\overline{\Omega}; \mathbb{R}^2))$ such that $\nabla \cdot \mathbf{w} = 0$ in $\Omega \times (0,1)$ and $\mathbf{w} \cdot \mathbf{n} = 0$ on $\Gamma \times (0,1)$. Note that, for any $\mathbf{w} \in \mathbf{M}$, the norms $\|\mathbf{w}\|_{1,\alpha}$ and $\|\nabla \times \mathbf{w}\|_{0,\alpha}$ are equivalent; we will set in the sequel $|||\mathbf{w}|||_{1,\alpha} := \|\nabla \times \mathbf{w}\|_{0,\alpha}$ for any $\mathbf{w} \in \mathbf{M}$.

Lemma 15. *Let $\tilde{C}$ be the constant furnished by Lemma 14. For any $\mathbf{z}^1, \mathbf{z}^2 \in \mathbf{S}_\nu$, one has*

$$|||(F^m(\mathbf{z}^1) - F^m(\mathbf{z}^2))(\cdot,t)|||_{1,\alpha} \leq \frac{(\tilde{C}t)^m}{m!} \|\mathbf{z}^1 - \mathbf{z}^2\|_{0,1,\alpha} \quad \forall m \geq 1. \quad (4.43)$$

Demonstração. The proof is by induction.

For $m = 1$, this is obvious, in view of Lemma 14.

Let us assume that (4.43) holds for $m = k$. Applying Lemma 14 to $\mathbf{y}^1 = F^k(\mathbf{z}^1)$ and $\mathbf{y}^2 = F^k(\mathbf{z}^2)$, we have

$$|||(F(\mathbf{y}^1) - F(\mathbf{y}^2))(\cdot,t)|||_{1,\alpha} \leq \tilde{C} \int_0^t \|(\mathbf{y}^1 - \mathbf{y}^2)(\cdot,s)\|_{1,\alpha}\, ds \quad \forall t \in [0,1].$$

Therefore, using the induction hypothesis, we obtain:

$$\begin{aligned} |||(F^{k+1}(\mathbf{z}^1) - F^{k+1}(\mathbf{z}^2))(\cdot,t)|||_{1,\alpha} &\leq \tilde{C} \|\mathbf{z}^1 - \mathbf{z}^2\|_{0,1,\alpha} \int_0^t \frac{(\tilde{C}s)^k}{k!}\, ds \\ &= \frac{(\tilde{C}t)^{k+1}}{(k+1)!} \|\mathbf{z}^1 - \mathbf{z}^2\|_{0,1,\alpha} \end{aligned}$$

This ends the proof. □

We deduce that, for some $\widehat{C} > 0$, any $m \geq 1$ and any $\mathbf{z}^1, \mathbf{z}^2 \in \mathbf{S}_\nu$, one has

$$\max_{t\in[0,1]} \|(F^m(\mathbf{z}^1) - F^m(\mathbf{z}^2))(\cdot, t)\|_{1,\alpha} \leq \frac{\widehat{C}\tilde{C}^m}{m!} \left(\max_{\tau\in[0,1]} \|(\mathbf{z}^1 - \mathbf{z}^2)(\cdot, \tau)\|_{1,\alpha} \right).$$

Consequently, if m is large enough, $F^m : \mathbf{S}_\nu \mapsto \mathbf{S}_\nu$ is a contraction, that is, there exists $\gamma \in (0,1)$ such that

$$\|F^m(\mathbf{z}^1) - F^m(\mathbf{z}^2)\|_{0,1,\alpha} \leq \gamma\|\mathbf{z}^1 - \mathbf{z}^2\|_{0,1,\alpha} \quad \forall \mathbf{z}^1, \mathbf{z}^2 \in \mathbf{S}_\nu. \tag{4.44}$$

Therefore, we can apply Theorem 8 with

$$B_1 = C^0([0,1]; \mathbf{C}^{1,\alpha}(\overline{\Omega}; \mathbb{R}^2)),\ B_2 = C^0([0,1]; \mathbf{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R}^2)),\ B = \mathbf{S}_\nu \ \text{ and } \ G = F,$$

to deduce that F possesses a unique extension $\widetilde{F}$ with a unique fixed-point $\mathbf{y}$ in the closure of $\mathbf{S}_\nu$ in $C^0([0,1]; \mathbf{C}^{1,\alpha}(\overline{\Omega}; \mathbb{R}^2))$. It is easy to check that $\mathbf{y}$ is, together with some ζ and θ, a solution to (4.24) satisfying (4.25) and (4.26).

This ends the proof.

4.5 Proof of Proposition 9. The 3D case

In this Section we are going to prove Proposition 9 in the three-dimensional case.

To do this, let $\{\rho^i\}$ be a partition of unity associated to the balls B^i introduced in Section 4.2.2 and let us set $\boldsymbol{\omega}_0 = \nabla \times \pi_3(\mathbf{y}_0)$. Proposition 9 is a consequence of the following result:

Proposition 11. *There exists $\delta > 0$ such that, if $\max\{\|\mathbf{y}_0\|_{2,\alpha}, \|\theta_0\|_{2,\alpha}\} \leq \delta$, then the coupled system*

$$\begin{cases} \boldsymbol{\omega}_t + (\mathbf{y} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{y} - \vec{\mathbf{k}} \times \nabla\theta & \text{in} \quad \Omega \times (0,1), \\ \theta_t + \mathbf{y} \cdot \nabla\theta = 0 & \text{in} \quad \Omega \times (0,1), \\ \nabla \cdot \mathbf{y} = 0,\ \nabla \times \mathbf{y} = \boldsymbol{\omega} & \text{in} \quad \Omega \times (0,1), \\ \mathbf{y} \cdot \mathbf{n} = (\overline{\mathbf{y}} + \mu\mathbf{y}_0) \cdot \mathbf{n} & \text{on} \quad \Gamma \times (0,1), \\ \boldsymbol{\omega}(0) = \nabla \times \mathbf{y}_0,\ \theta(0) = \theta_0 & \text{in} \quad \Omega \end{cases} \tag{4.45}$$

possesses at least one solution $(\boldsymbol{\omega}, \theta, \mathbf{y})$, with

$$(\boldsymbol{\omega}, \theta, \mathbf{y}) \in C^0([0,1]; \mathbf{C}^{0,\alpha}(\overline{\Omega}; \mathbb{R}^3)) \times C^0([0,1]; C^{1,\alpha}(\overline{\Omega})) \times C^0([0,1]; \mathbf{C}^{1,\alpha}(\overline{\Omega}; \mathbb{R}^3)), \tag{4.46}$$

such that

$$\theta(\mathbf{x}, t) = 0 \quad \text{in} \quad \Omega \times (t_{k-1/2}, 1) \quad \text{and} \quad \boldsymbol{\omega}(\mathbf{x}, t) = 0 \quad \text{in} \quad \Omega \times (t_{2k-1/2}, 1). \tag{4.47}$$

Let us give the proof of this result. We will repeat the strategy of proof of Proposition 10, incorporating some ideas from Bardos and Frisch [12] and Glass [43]; we will use the notation in Section 4.2.2.

First, let us denote by $\mathbf{R}'$ the set of fields $\mathbf{z} \in C^0([0,1];\mathbf{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}^3))$ such that $\nabla\cdot\mathbf{z}=0$ in $\Omega\times(0,1)$ and $\mathbf{z}\cdot\mathbf{n}=(\overline{\mathbf{y}}+\mu\mathbf{y}_0)\cdot\mathbf{n}$ on $\Gamma\times(0,1)$. Then, for any $\nu>0$, we set

$$\mathbf{R}_\nu = \{\, \mathbf{z}\in\mathbf{R}' : \|\mathbf{z}-\overline{\mathbf{y}}\|_{0,1,\alpha} \le \nu \,\}.$$

Let us fix $\nu>0$ being the constant furnished by Lemma 11. As before, if the initial datum $\mathbf{y}_0$ is sufficiently small in $\mathbf{C}^2(\overline{\Omega};\mathbb{R}^3)$, then $\mathbf{R}_\nu$ is nonempty.

Now, we are going to construct a mapping $F:\mathbf{R}_\nu\to\mathbf{R}_\nu$.
We start from an arbitrary $\mathbf{z}\in\mathbf{R}_\nu$ and we set $\mathbf{z}^* := \mathbf{y}^* + \pi_3(\mathbf{z}-\overline{\mathbf{y}})$.

First, we denote by θ^* the unique solution to

$$\begin{cases} \theta^*_t + \mathbf{z}^*\cdot\nabla\theta^* = 0 & \text{in } \overline{\mathscr{O}}\times[0,1/2], \\ \theta^*(\mathbf{x},0) = \sum_{i=1}^k \psi^i(\mathbf{x})\,\pi_1(\theta_0)(\mathbf{x}) & \text{in } \overline{\mathscr{O}}. \end{cases}$$

Obviously, $\theta^* = \sum_{i=1}^k \theta^i$, where θ^i is the unique solution to

$$\begin{cases} \theta^i_t + \mathbf{z}^*\cdot\nabla\theta^i = 0 & \text{in } \overline{\mathscr{O}}\times[0,1/2], \\ \theta^i(\mathbf{x},0) = \psi^i(\mathbf{x})\,\pi_1(\theta_0)(\mathbf{x}) & \text{in } \overline{\mathscr{O}}. \end{cases} \tag{4.48}$$

The identities

$$\theta^i(\mathbf{Z}^*(\mathbf{x},t,0),t) = \psi^i(\mathbf{x})\pi_1(\theta_0)(\mathbf{x}) \quad \forall(\mathbf{x},t)\in\overline{\mathscr{O}}\times[0,1/2]$$

imply that

$$\text{Supp }\ \theta^i(\cdot\,,t) \subset \mathbf{Z}^*(B^i,t,0) \quad \forall t\in[0,1/2].$$

Hence, in view of Lemma 11, we deduce that

$$\text{Supp }\ \theta^i(\cdot\,,t_{i-1/2}) \subset \mathbf{Z}^*(B^i,t_{i-1/2},0) \subset \mathscr{O}\setminus\overline{\mathscr{O}}_0,$$

whence

$$\theta^i(\cdot\,,t_{i-1/2}) = 0 \quad \text{in} \quad \overline{\Omega}. \tag{4.49}$$

Then, we simply set $\hat\theta(\mathbf{x},t) := \theta^*(\mathbf{x},t)$ in $\overline{\mathscr{O}}\times[0,t_0]$ and we say that, in $\overline{\mathscr{O}}\times[t_0,1/2]$, $\hat\theta$ is the unique solution to

$$\begin{cases} \hat\theta_t + \mathbf{z}^*\cdot\nabla\hat\theta = 0 & \text{in } \overline{\mathscr{O}}\times\left([t_0,1/2]\setminus\bigcup_{i=1}^k\{t_{i-\frac12}\}\right), \\ \hat\theta(\mathbf{x},t_{i-1/2}) = \displaystyle\sum_{l=i}^k \theta^l(\mathbf{x},t_{i-1/2}) - \theta^i(\mathbf{x},,t_{i-1/2}) & \text{in } \overline{\mathscr{O}},\ 1\le i\le k. \end{cases} \tag{4.50}$$

We notice that $\hat\theta(\cdot\,,t_{k-1/2})\equiv 0$ in $\overline{\mathscr{O}}$. Therefore, $\hat\theta\equiv 0$ in $\overline{\mathscr{O}}\times[t_{k-1/2},1/2]$. Moreover,

$$\hat\theta(\mathbf{x},t) = \sum_{l=i}^k \theta^l(\mathbf{x},t) - \theta^i(\mathbf{x},t) \ \text{ in } \ \overline{\mathscr{O}}\times(t_{i-1/2},t_{i+1/2}), \quad 1\le i\le k-1. \tag{4.51}$$

We remark that the lateral limits of $\hat{\theta}$ at the points $\{t_{i-1/2}\}_{i=1}^{k}$ are not necessarily the same in the whole domain $\overline{\mathscr{O}}$.

Let θ be the restriction of $\hat{\theta}$ to $\overline{\Omega}$. Due to (4.49) and (4.50), we see that θ is continuous at the points $\{t_{i-1/2}\}_{i=1}^{k}$ and

$$\left\{\begin{array}{ll} \theta_t + \mathbf{z}\cdot\nabla\theta = 0 & \text{in } \Omega\times(0,1/2), \\ \theta(\mathbf{x},0) = \theta_0(\mathbf{x}) & \text{in } \Omega \end{array}\right. \tag{4.52}$$

and it belongs to $C^0([0,1];C^{1,\alpha}(\overline{\Omega}))$.

In an analogous way as for the temperature, we will define a function $\widehat{\boldsymbol{\omega}}$ in $\overline{\mathscr{O}}\times[0,1]$, whose the restriction to Ω is the function $\boldsymbol{\omega}$ satisfying (4.47). The definition of $\widehat{\boldsymbol{\omega}}$ will be made in three parts corresponding, respectively, to the three time intervals $[0,1/2)$, $[1/2,t_{k+1/2})$ and $[t_{k+1/2},1]$.

Let us introduce $\boldsymbol{\omega}_0 := \nabla\times(\pi_3(\mathbf{y}_0))$ and let $\boldsymbol{\omega}^*$ be the solution to

$$\left\{\begin{array}{ll} \boldsymbol{\omega}^*_t + (\mathbf{z}^*\cdot\nabla)\boldsymbol{\omega}^* = (\boldsymbol{\omega}^*\cdot\nabla)\mathbf{z}^* - (\nabla\cdot\mathbf{z}^*)\boldsymbol{\omega}^* - \overrightarrow{\mathbf{k}}\times\nabla\pi_1(\theta) & \text{in} \quad \mathscr{O}\times(0,1/2), \\ \boldsymbol{\omega}^*(\mathbf{x},0) = \boldsymbol{\omega}_0(\mathbf{x}) & \text{in} \quad \mathscr{O}. \end{array}\right.$$

With this $\boldsymbol{\omega}^*$, we set $\boldsymbol{\omega}^{**}_{1/2}\in\mathbf{C}^{1,\alpha}(\overline{\Omega})$ with $\boldsymbol{\omega}^{**}_{1/2}(\mathbf{x}) := \boldsymbol{\omega}^*(\mathbf{x}\,,1/2)$ for all $\mathbf{x}\in\overline{\Omega}$. Let us consider $\boldsymbol{\omega}^{**}$ the solution to the problem

$$\left\{\begin{array}{ll} \boldsymbol{\omega}^{**}_t + (\mathbf{z}^*\cdot\nabla)\boldsymbol{\omega}^{**} = (\boldsymbol{\omega}^{**}\cdot\nabla)\mathbf{z}^* - (\nabla\cdot\mathbf{z}^*)\boldsymbol{\omega}^{**} & \text{in} \quad \mathscr{O}\times(1/2,1), \\ \boldsymbol{\omega}^{**}(\mathbf{x},1/2) = \sum\limits_{i=1}^{k}\psi^i(\mathbf{x})\,\pi_3(\boldsymbol{\omega}^{**}_{1/2})(\mathbf{x}) & \text{in} \quad \mathscr{O}. \end{array}\right. \tag{4.53}$$

As before, we can decompose $\boldsymbol{\omega}^{**}$ as a sum of functions. More precisely, let $\boldsymbol{\omega}^1,\dots,\boldsymbol{\omega}^k$ be the solutions to the problems

$$\left\{\begin{array}{ll} \boldsymbol{\omega}^i_t + (\mathbf{z}^*\cdot\nabla)\boldsymbol{\omega}^i = (\boldsymbol{\omega}^i\cdot\nabla)\mathbf{z}^* - (\nabla\cdot\mathbf{z}^*)\boldsymbol{\omega}^i & \text{in} \quad \mathscr{O}\times(1/2,1), \\ \boldsymbol{\omega}^i(\mathbf{x},1/2) = \psi^i(\mathbf{x})\,\pi_3(\boldsymbol{\omega}^{**}_{1/2})(\mathbf{x}) & \text{in} \quad \mathscr{O}. \end{array}\right. \tag{4.54}$$

Then

$$\boldsymbol{\omega}^{**} = \sum_{i=1}^{k}\boldsymbol{\omega}^i \quad \text{in} \quad \overline{\mathscr{O}}\times[1/2,1].$$

Each $\boldsymbol{\omega}^i$ satisfies

$$\boldsymbol{\omega}^i(\mathbf{Z}^*(\mathbf{x},t,1/2),t) = \boldsymbol{\omega}^i(\mathbf{x},1/2) + \int_{1/2}^{t}[(\boldsymbol{\omega}^i\cdot\nabla)\mathbf{z}^* - (\nabla\cdot\mathbf{z}^*)\boldsymbol{\omega}^i](\mathbf{Z}^*(\mathbf{x},\sigma,1/2),\sigma)\,d\sigma.$$

Consequently,

$$|\boldsymbol{\omega}^i(\mathbf{Z}^*(\mathbf{x},t,1/2),t)| \le |\boldsymbol{\omega}^i(\mathbf{x},1/2)| + C\|\mathbf{z}^*\|_{0,1,0}\int_{1/2}^{t}|\boldsymbol{\omega}^i(\mathbf{Z}^*(\mathbf{x},\sigma,1/2),\sigma)|\,d\sigma.$$

Notice that, if $\mathbf{x}\notin B^i$ we then have

$$|\boldsymbol{\omega}^i(\mathbf{Z}^*(\mathbf{x},t,1/2),t)| \le C\|\mathbf{z}^*\|_{0,1,0}\int_{1/2}^{t}|\boldsymbol{\omega}^i(\mathbf{Z}^*(\mathbf{x},\sigma,1/2),\sigma)|\,d\sigma$$

and, from Gronwall's Lemma, we see that

$$\boldsymbol{\omega}^i(\mathbf{Z}^*(\mathbf{x},t,1/2),t) = 0 \quad \forall(\mathbf{x},t) \in (\overline{\mathscr{O}} \setminus B^i) \times [1/2,1].$$

A consequence is that (Supp $\boldsymbol{\omega}^i(\cdot\,,t)) \subset \mathbf{Z}^*(B^i,t,1/2)$, whence we get

$$\boldsymbol{\omega}^i(\mathbf{x},t_{k+i-1/2}) = 0 \quad \text{for all} \quad \mathbf{x} \in \overline{\Omega}.$$

Then, we simply set $\widehat{\boldsymbol{\omega}}(\mathbf{x},t) := \boldsymbol{\omega}^*(\mathbf{x},t)$ in $\overline{\mathscr{O}} \times [0,1/2]$ and $\widehat{\boldsymbol{\omega}}(\mathbf{x},t) := \boldsymbol{\omega}^{**}(\mathbf{x},t)$ in $\overline{\mathscr{O}} \times [1/2,t_{k+1/2}]$ and we say that, in $\mathscr{O} \times [t_{k+1/2},1]$, $\widehat{\boldsymbol{\omega}}$ is the unique solution to

$$\begin{cases} \widehat{\boldsymbol{\omega}}_t + (\mathbf{z}^* \cdot \nabla)\widehat{\boldsymbol{\omega}} = (\widehat{\boldsymbol{\omega}} \cdot \nabla)\mathbf{z}^* - (\nabla \cdot \mathbf{z}^*)\widehat{\boldsymbol{\omega}} & \text{in} \quad \overline{\mathscr{O}} \times \left([t_{k+1/2},1] \setminus \bigcup_{i=1}^{k} \{t_{k+i-1/2}\} \right) \\ \widehat{\boldsymbol{\omega}}(\mathbf{x},t_{k+i-\frac{1}{2}}) = \sum_{l=i}^{k} \widehat{\boldsymbol{\omega}}^l(\mathbf{x},t_{k+i-\frac{1}{2}}) - \widehat{\boldsymbol{\omega}}^i(\mathbf{x},t_{k+i-\frac{1}{2}}) & \text{in} \quad \overline{\mathscr{O}}, \ 1 \le i \le k. \end{cases} \tag{4.55}$$

We notice that $\widehat{\boldsymbol{\omega}}(\cdot\,,t_{2k-1/2}) \equiv 0$ in $\overline{\mathscr{O}}$. Therefore, $\widehat{\boldsymbol{\omega}} \equiv 0$ in $\overline{\mathscr{O}} \times [t_{2k-1/2},1]$. Moreover,

$$\widehat{\boldsymbol{\omega}}(\mathbf{x},t) = \sum_{l=i}^{k} \boldsymbol{\omega}^l(\mathbf{x},t) - \boldsymbol{\omega}^i(\mathbf{x},t) \ \text{ in } \ \overline{\mathscr{O}} \times (t_{k+i-1/2},t_{k+i+1/2}), \ \ 1 \le i \le k-1. \tag{4.56}$$

We define $\boldsymbol{\omega}$ to be the restriction of $\widehat{\boldsymbol{\omega}}$ to $\overline{\Omega} \times [0,1]$. It belongs to $C^0([0,1];\mathbf{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^3))$ and together with the temperature θ, satisfies:

$$\begin{cases} \boldsymbol{\omega}_t + (\mathbf{z} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{z} - \overrightarrow{\mathbf{k}} \times \nabla\theta & \text{in} \quad \Omega \times [0,1] \\ \boldsymbol{\omega}(\mathbf{x},0) = (\nabla \times \mathbf{y}_0)(\mathbf{x}) & \text{in} \quad \Omega \end{cases}$$

and, moreover, $\boldsymbol{\omega} \equiv 0$ in $\overline{\Omega} \times [t_{2k-1/2},1]$.

Thanks to Lemma 3, $\boldsymbol{\omega}$ is divergence-free in $\Omega \times (0,1)$. Consequently, from classical results, we know that there exists exactly one $\mathbf{y}$ in $C^0([0,1];\mathbf{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}^3))$ such that

$$\begin{cases} \nabla \times \mathbf{y} = \boldsymbol{\omega}, \quad \nabla \cdot \mathbf{y} = 0 & \text{in} \quad \overline{\Omega} \times (0,1), \\ \mathbf{y} \cdot \mathbf{n} = (\mu\mathbf{y}_0 + \overline{\mathbf{y}}) \cdot \mathbf{n} & \text{on} \quad \Gamma \times (0,1). \end{cases} \tag{4.57}$$

Since $\mathbf{y}$ is uniquely determined by $\mathbf{z}$, we write $F(\mathbf{z}) = \mathbf{y}$. The mapping $F : \mathbf{R}_\nu \mapsto \mathbf{R}'$ is thus well defined.

In view of some estimates similar to the $2D$ case, we can take the initial data small enough to have $F(\mathbf{R}_\nu) \subset \mathbf{R}_\nu$. More precisely, one has:

Lemma 16. *There exists $\delta > 0$ such that, if $\{\|\mathbf{y}_0\|_{2,\alpha}, \|\theta_0\|_{2,\alpha}\} \le \delta$, one has $F(\mathbf{z}) \in \mathbf{R}_\nu$ for all $\mathbf{z} \in \mathbf{R}_\nu$.*

The end of the proof of Proposition 11 is very similar to the final part of Section 4.4.

Essentially, what we have to prove is that, for some $m \ge 1$, F^m is a contraction for the usual norm in $C^0([0,1];\mathbf{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^3))$. Indeed, after this we can apply Theorem 8 with $B_1 = C^0([0,1];\mathbf{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^3))$, $B_2 = C^0([0,1];\mathbf{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}^3))$, $B = \mathbf{R}_\nu$ and $G = F$ and deduce the existence of a fixed-point of the extension $\widetilde{F}$ in the closure of $\mathbf{R}_\nu$ in $C^0([0,1];C^{1,\alpha}(\overline{\Omega};\mathbb{R}^3))$.

But this can be done easily, arguing as in the proof of Lemma 15. For brevity, we omit the details.

4.6 Proof of Theorem 7

Theorem 7 is an easy consequence of the following result:

Proposition 12. *For each* $\mathbf{y}_0 \in \mathbf{C}(2,\alpha,\emptyset)$ *there exist* $T^* \in (0,T)$ *and* $\eta > 0$ *such that, if* $\theta_0 \in C^{2,\alpha}(\overline{\Omega})$, $\theta_0 = 0$ *on* $\Gamma\backslash\gamma$ *and* $\|\theta_0\|_{2,\alpha} \le \eta$, *then the system*

$$\begin{cases} \mathbf{y}_t + (\mathbf{y}\cdot\nabla)\mathbf{y} = -\nabla p + \vec{\mathbf{k}}\,\theta & \textit{in} \quad \Omega\times(0,T^*), \\ \nabla\cdot\mathbf{y} = 0 & \textit{in} \quad \Omega\times(0,T^*), \\ \theta_t + \mathbf{y}\cdot\nabla\theta = \kappa\,\Delta\theta & \textit{in} \quad \Omega\times(0,T^*), \\ \mathbf{y}\cdot\mathbf{n} = 0 & \textit{on} \quad \Gamma\times(0,T^*), \\ \theta = 0 & \textit{on} \quad (\Gamma\backslash\gamma)\times(0,T^*), \\ \mathbf{y}(\mathbf{x},0) = \mathbf{y}_0(\mathbf{x}),\ \theta(\mathbf{x},0) = \theta_0(\mathbf{x}) & \textit{in} \quad \Omega, \end{cases} \tag{4.58}$$

possesses at least one solution $\mathbf{y} \in C^0([0,T^*];\mathbf{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}^N))$, $\theta \in C^0([0,T^*];C^{2,\alpha}(\overline{\Omega}))$ *and* $p \in \mathcal{D}'(\Omega\times(0,T^*))$ *such that*

$$\theta(\mathbf{x},T^*) = 0 \quad \textit{in} \quad \Omega. \tag{4.59}$$

Indeed, if Proposition 12 holds, we can consider (4.1) and control first the temperature θ exactly to zero at time T^*. To do this, we need initial data as above, that is, $\mathbf{y}_0 \in \mathbf{C}(2,\alpha,\emptyset)$ and $\theta_0 \in C^{2,\alpha}(\overline{\Omega})$ such that $\theta_0 = 0$ on $\Gamma\backslash\gamma$ and $\|\theta_0\|_{2,\alpha} \le \delta$. Then, in a second step, we can apply the results in [29] and [43] to the Euler system in $\Omega\times(T^*,T)$, with initial data $\mathbf{y}(\cdot\,,T^*)$. In other words, we can find new controls in (T^*,T) that drive the velocity field exactly to any final state $\mathbf{y}_1$.

Proof of Proposition 12: For simplicity, we will consider only the case $N = 2$. We will apply a fixed-point argument that guarantees the existence of a solution to (4.58)-(4.59).

We start from an arbitrary $\overline{\theta} \in C^0([0,T/2];C^{1,\alpha}(\overline{\Omega}))$. To this $\overline{\theta}$, arguing as in Section 4.3, we can associate a field $\mathbf{y} \in C^0([0,T/2];\mathbf{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}^N))$ verifying

$$\begin{cases} \mathbf{y}_t + (\mathbf{y}\cdot\nabla)\mathbf{y} = -\nabla p + \vec{\mathbf{k}}\,\overline{\theta} & \text{in } \Omega\times(0,T/2), \\ \nabla\cdot\mathbf{y} = 0 & \text{in } \Omega\times(0,T/2), \\ \mathbf{y}\cdot\mathbf{n} = 0 & \text{on } \Gamma\times(0,T/2), \\ \mathbf{y}(\mathbf{x},0) = \mathbf{y}_0(\mathbf{x}) & \text{in } \Omega \end{cases}$$

and

$$\|\mathbf{y}\|_{0,2,\alpha} \le C(\|\mathbf{y}_0\|_{2,\alpha} + \|\overline{\theta}\|_{0,2,\alpha}).$$

Let $\tilde{\Omega} \subset \mathbb{R}^2$ be a connected open set with boundary $\tilde{\Gamma} = \partial\tilde{\Omega}$ of class C^2 such that $\Omega \subset \tilde{\Omega}$ and $\tilde{\Gamma}\cap\Gamma = \Gamma\setminus\gamma$ (see Fig. 4.2). Let $\omega \subset \tilde{\Omega}\setminus\overline{\Omega}$ be a non-empty open subset.

Then, as in Theorem 9, we associate to $\mathbf{y}$ a pair $(\tilde{\theta},\tilde{v})$ satisfying

$$\begin{cases} \tilde{\theta}_t + \pi(\mathbf{y})\cdot\nabla\tilde{\theta} = \kappa\Delta\tilde{\theta} + \tilde{v}1_\omega & \text{in } \tilde{\Omega}\times(0,T/2), \\ \tilde{\theta} = 0 & \text{on } \tilde{\Gamma}\times(0,T/2), \\ \tilde{\theta}(\mathbf{x},0) = \tilde{\pi}(\theta_0)(\mathbf{x}), \quad \tilde{\theta}(\mathbf{x},T/2) = 0 & \text{in } \tilde{\Omega}, \end{cases}$$

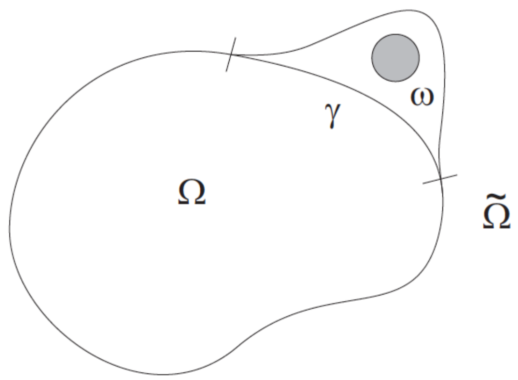

Figura 4.2: The domain $\tilde{\Omega}$ and the subdomain ω.

where π and $\tilde{\pi}$ are extension operators from Ω into $\tilde{\Omega}$ that preserve regularity. Let θ be the restriction of $\tilde{\theta}$ to $\overline{\Omega} \times [0, T/2]$. Then, θ satisfies:

$$\begin{cases} \theta_t + \mathbf{y} \cdot \nabla\theta = \kappa\Delta\theta & \text{in } \Omega \times (0, T/2), \\ \theta = \tilde{\theta} 1_\gamma & \text{on } \Gamma \times (0, T/2), \\ \theta(\mathbf{x}, 0) = \theta_0(\mathbf{x}), \quad \theta(\mathbf{x}, T/2) = 0 & \text{in } \Omega. \end{cases}$$

Moreover, from parabolic regularity, it is not difficult to check that the following inequalities hold:

$$\|\theta_t\|_{0,0,\alpha} + \|\theta\|_{0,2,\alpha} \leq C\|\theta_0\|_{2,\alpha}^2 \, e^{C\|\mathbf{y}\|_{0,2,\alpha}} \leq C\|\theta_0\|_{2,\alpha} \, e^{C(\|\mathbf{y}_0\|_{2,\alpha} + \|\overline{\theta}\|_{0,2,\alpha})}.$$

Now, let us introduce the Banach space

$$W = \{ \theta \in C^0([0, T/2]; C^{2,\alpha}(\overline{\Omega})) : \theta_t \in C^0([0, T/2]; C^{0,\alpha}(\overline{\Omega})) \}$$

and let us consider the closed ball

$$B := \{ \overline{\theta} \in C^0([0, T/2]; C^{1,\alpha}(\overline{\Omega})) : \|\overline{\theta}\|_{0,1,\alpha} \leq 1 \}$$

and the mapping Λ, with

$$\Lambda(\overline{\theta}) = \theta \quad \forall \overline{\theta} \in C^0([0, T/2]; C^{1,\alpha}(\overline{\Omega})).$$

Obviously, Λ is well defined. Furthermore, in view of the previous inequalities, it maps continuously the whole space $C^0([0, T/2]; C^{1,\alpha}(\overline{\Omega}))$ into W, that is compactly embedded in $C^0([0, T/2]; C^{1,\alpha}(\overline{\Omega}))$, in view of the classical results of the Aubin-Lions kind, see for instance [81].

On the other hand, if $\eta > 0$ is sufficiently small (depending on $\|\mathbf{y}_0\|_{2,\alpha}$) and $\|\theta_0\|_{2,\alpha} \leq \eta$, Λ maps B into itself. Consequently, the hypotheses of Schauder's Theorem are satisfied and Λ possesses at least one fixed-point in B.

This ends the proof.

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