Desigualdade de Carleman para Equação da Onda e Aplicações à Controlabilidade Exata e Problema Inverso

por

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sob orientação do

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Dissertação apresentada ao Corpo Docente do Programa de Pós-Graduação em Matemática-CCEN-UFPB, como requisito parcial para a obtenção do título de Mestre em Matemática.

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Resumo

Este trabalho apresenta uma desigualdade de Carleman global para a equação de onda linear com potencial limitado. Além disso, são feitas duas aplicações desse resultado. A primeira delas refere-se ao estudo da controlabilidade exata na fronteira e a segunda trata de um problema inverso, onde buscamos recuperar o potencial.

Abstract

This work presents one global Carleman inequality for wave linear equation with bounded potential. Furthermore, we do two applications of this result. The first one refers to the study of exact controlabillity on the boundary and the second one deals with an inverse problem, where we want to recover the potential.

Sumário

Notações e Simbologias

- $|\,.\,|$ designa o módulo.
- $\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$ designa o operador laplaciano.
- $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_N} \right)$ designa o gradiente.
- $\hookrightarrow$ designa a imersão contínua.
- C, quando não especificada, é uma constante positiva e arbitrária.

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Introdução

É fato que a matemática sempre esteve presente na vida humana. Surgida a partir da necessidade de se resolver problemas simples do cotidiano, ela vem sendo construída ao longo dos tempos. Hoje, é uma ciência bastante desenvolvida que tem influência em diversas áreas do conhecimento. Ao contrário do que muitos pensam, a matemática encontra-se em pleno desenvolvimento, seja pela beleza das suas cadeias de raciocínios lógicos e teorias, seja para, assim como em seus primórdios, contribuir para a resolução de problemas da sociedade.

A tão conhecida frase de Pitágoras segunda a qual tudo é número talvez apresente exagero, mas a verdade é que a realidade muitas vezes pode ser estudada através de modelos matemáticos, onde equações e sistemas representam situações e permitem que elas sejam estudadas e compreendidas melhor. Obviamente esses modelos não representam perfeitamente a realidade, no entanto proporcionam uma aproximação que já trouxe muitos avanços tecnológicos, industriais, medicinais e em outros campos.

Uma ciência que conta freqüente e fortemente com esses modelos é a física. Há uma contribuição mútua entre as duas ciências, pois a física precisa da matemática para progredir e isso impulsiona a matemática a criar novas ferramentas e teorias que sejam eficazes neste sentido. Provavelmente, este é um dos motivos pelos quais os estudos sobre as equações da onda tem se intensificado cada vez mais nos últimos anos. É notória a relevância desses estudos. As sonoras, as sísmicas, o raio-x, as que se propagam por uma superfície líquida, a vibração natural dos corpos, mostram como a todo o momento diversos tipos de onda nos cercam. É natural, então, que se busque informações sobre suas características e comportamento. Ainda podemos questionar sobre como agir para que uma onda passe a se comportar como nós desejamos. Para ajudar a resolver esta questão temos a teoria de controle.

Controlar um sistema pode ser entendido de diversas formas e não é um conceito exclusivo de trabalhos sobre ondas. Em termos simples, quando usamos a palavra controlar estamos querendo dizer que atuaremos sobre um sistema de forma a fazer com que seu estado final esteja de acordo com o que estabelecemos previamente como sendo o desejado. Em Micu e Zuazua [11] o conceito do problema de controlabilidade é dado de maneira informal, mas fácil de entender. Segundo estas notas, considerando um sistema de evolução, ou seja, onde uma das variáveis é o tempo, agimos sobre ele através de uma variável a qual denominamos controle, que é escolhida convenientemente. Assim, dado um tempo t pertencente ao intervalo $(0, T)$ e estados inicial e final, precisamos encontrar um controle tal que o sistema apresente o estado inicial em $t = 0$ e o estado final em $t = T$.

Ilustremos com um exemplo. Em uma cidade onde seja comum a ocorrência de terremotos, pode-se ter o interesse de que um prédio seja construído de forma que após um tempo T do término do tremor as vibrações em sua estrutura tenham cessado. Neste caso, diríamos que o sistema estaria controlado se conseguíssemos encontrar um meio de fortalecer a construção para que quando $t = T$ as vibrações fossem nulas. É importante citar que independente de agirmos sobre o sistema, em algum momento essas vibrações irão parar, mas o que queremos é que isso aconteça num intervalo de tempo escolhido por nós. Resta saber se isso é possível.

Nos anos 80, J. L. Lions ([9], [10]) apresentou o Método de Unicidade Hilbertiana (HUM) que reduz o problema da controlabilidade exata a encontrar uma desigualdade de observabilidade para o sistema adjunto homogêneo. Embora trabalhar com um sistema homogêneo possa parecer mais simples, nem sempre é fácil encontrar tal desigualdade mencionada. É conveniente que se tenha ferramentas que facilitem esse processo.

Um recurso que tem sido muito utilizado é a desigualdade de Carleman. Embora esta seja uma ferramenta técnica e de cálculos extensos, ela nos fornece informações sobre o comportamento de um sistema definido em um subconjunto do $\mathbb{R}^{N+1}$ a partir de dados referente a uma pequena parte desse domínio ou da fronteira. Por meio disso, podemos estabelecer a desigualdade de observabilidade e, aplicando o método HUM, garantir a controlabilidade exata do sistema em questão.

A desigualdade de Carleman é o assunto central dessa dissertação, pois é com o seu auxílio que estabeleceremos a controlabilidade exata de um sistema e estudaremos o problema inverso. Aqui trataremos de equações da onda com potencial limitado definidas em um subconjunto aberto limitado do $\mathbb{R}^{N+1}$ sobre o qual brevemente daremos mais detalhes. Alguns trabalhos que podemos citar onde essa ferramenta é utilizada são os de Baudouin-Puel [3], Boundouin et al. [1] e [2], Benabdallah et al. [4], Imanuvilov [7], Zhang [15], entre outros.

Entraremos agora em maiores detalhes sobre o que será feito nas páginas seguintes.

Consideraremos em toda a extensão do texto que Ω é um subconjunto aberto limitado do $\mathbb{R}^N$ bastante regular, Γ sua fronteira, T um número estritamente positivo e ν o vetor normal unitário exterior em Γ. Usaremos também Q para denotar o cilindro $\Omega \times (-T, T)$, com fronteira lateral $\Sigma = \Gamma \times (-T, T)$.

Trataremos sistemas associados a equações da onda com potencial limitado que sejam do tipo

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u + p\, u = f & \text{em } Q, \\ u = 0 & \text{sobre } \Sigma, \\ u(0) = u_0, \quad \dfrac{\partial u}{\partial t}(0) = u_1 & \text{em } \Omega. \end{cases} \tag{1}$$

Como já citamos anteriormente, a desigualdade de Carleman que buscamos é calculada para o sistema adjunto a (1), que é dado por

$$\begin{cases} \dfrac{\partial^2 \psi}{\partial t^2} - \Delta \psi + p\, \psi = f & \text{em } Q, \\ \psi = 0 & \text{sobre } \Sigma, \\ \psi(0) = \psi_0, \quad \dfrac{\partial \psi}{\partial t}(0) = \psi_1 & \text{em } \Omega. \end{cases} \tag{2}$$

Algo importante de ser mencionado é que para encontrarmos a desigualdade global de Carleman não necessitaremos de nenhuma escolha de T suficientemente grande.

Essa dissertação esta baseada no trabalho de Puel [13] e está organizada em quatro capítulos. O primeiro deles apresenta definições e resultados que serão utilizados no texto. O capítulo 2 é destinado à demonstração de um único teorema que nos fornece a desigualdade de Carleman para (2).

Como já comentamos, a desigualdade de Carleman é uma poderosa ferramenta para a teoria de controle. Com o seu auxílio, podemos encontrar uma desigualdade de observabilidade e, utilizando-nos do método HUM, garantir a controlabilidade exata do sistema em estudo. Encontrar tal desigualdade de observabilidade é o objetivo do capítulo 3. Alguns trabalhos onde este tipo de aplicação é feita são apresentados nas referências [1], [4], [14] e [15].

As equações que trabalharemos são equações com potenciais limitados. Para cada potencial p, dependente apenas da variável espacial, podemos garantir a existência de solução única para a equação. Surge, então, uma pergunta: como podemos recuperar um potencial desconhecido? A este questionamento damos o nome de problema inverso e o resolvemos por meio de resultados de estabilidade e unicidade. Isso será feito no capítulo 4. Veremos que neste caso precisaremos impor condições sobre a escolha de T, o que não nos será problema, uma vez que temos liberdade para isso.

Capítulo 1

Preliminares

Neste capítulo apresentaremos algumas definições importantes e os principais resultados que serão utilizados neste trabalho.

1.1 Espaços Funcionais

Uma n-upla de inteiros não negativos $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ é denominada um múlti-índice e tem sua ordem definida como sendo $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$.

Denotamos por D^α o operador de ordem $|\alpha|$ dado por

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}.$$

No caso em que $\alpha = (0, 0, 0, ..., 0)$, definimos $D^0 u = u$, para qualquer função u.

Seja $f : \Omega \to \mathbb{R}$ uma função contínua. O suporte de f, de notação $supp(f)$, é o fecho em Ω do conjunto $\{x \in \Omega;\ f(x) \neq 0\}$. O espaço vetorial das funções infinitamente diferenciáveis definidas com suporte compacto em Ω é representado por $C_0^\infty(\Omega)$. As operações deste espaço são as usuais.

Seja $p \in \mathbb{R}$ com $1 \leq p < \infty$. Consideremos o conjunto

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R}; f \text{ é mensurável e } \int_\Omega |f(x)|^p d\mu < \infty \right\},$$

onde μ é uma medida, munido com a norma

$$\|f\|_{L^p} = \left[\int_\Omega |f(x)|^p d\mu \right]^{1/p}.$$

O conjunto $L^p(\Omega)$ com esta norma é um espaço de Banach. Quando $p = 2$, ele é também um espaço de Hilbert com produto interno dado por

$$(u, v) = \int_\Omega u(x)v(x)d\mu.$$

Consideremos ainda o espaço de Banach

$$L^\infty(\Omega) = \left\{ f : \Omega \to \mathbb{R}; f \text{ é mensurável e } \sup_{x\in\Omega} ess|f(x)| < \infty \right\},$$

com norma

$$\|f\|_{L^\infty} = \sup_{x\in\Omega} ess|f(x)|.$$

Seja X um espaço de Banach. Denotaremos por $L^p(0, T; X)$, $1 \leq p < \infty$, o espaço de Banach das (classes de) funções u, definidas em $(0, T)$ com valores em X, que são fortemente mensuráveis e $\|u(t)\|_X^p$ é integrável a Lebesgue em $(0, T)$, com a norma

$$\|u(t)\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt \right)^{\frac{1}{p}}.$$

Por $L^\infty(0, T; X)$ representa-se o espaço de Banach das (classes de) funções u, definidas em $(0, T)$ com valores em X, que são fortemente mensuráveis e $\|u(t)\|_X$ possui supremo essencial finito em $(0, T)$, com a norma

$$\|u(t)\|_{L^\infty(0,T;X)} = \sup_{t\in(0,T)} ess \|u(t)\|_X .$$

Dado um inteiro $m > 0$, representa-se por $W^{m,p}(\Omega)$, $1 \leq p \leq \infty$, o espaço de Sobolev de ordem m, sobre Ω, das (classes de) funções $u \in L^p(\Omega)$ tais que $D^\alpha u \in L^p(\Omega)$, para todo multi-índice α, com $|\alpha| \leq m$. Eles são espaços vetorias de Banach com as normas

$$\|u\|_{W^{m,p}(\Omega)} = \left[\sum_{|\alpha|\leq m} \int_\Omega |D^\alpha u(x)|^p dx \right]^{\frac{1}{p}}, \text{ quando } 1 \leq p < \infty$$

e

$$\|u\|_{W^{m,\infty}(\Omega)} = \sum_{|\alpha|\leq m} \sup_{x\in\Omega} ess|D^\alpha u(x)|, \text{ quando } p = \infty.$$

Se $p = 2$, $W^{m,2}(\Omega)$ é denotado por $H^m(\Omega)$ e é um espaço de Hilbert com o produto interno

$$(u, v)_{H^m(\Omega)} = \sum_{|\alpha|\leq m} \int_\Omega D^\alpha u(x) D^\alpha v(x) dx.$$

Definição 1.1 *Diz-se que uma sequência* $(\varphi_n)_{n\in\mathbb{N}}$ *em* $C_0^\infty(\Omega)$ *converge para* φ *em* $C_0^\infty(\Omega)$, *quando forem satisfeitas as seguintes condições:*

(i) Existe um compacto K de Ω tal que $\text{supp}(\varphi) \subset K$ e $\text{supp}(\varphi_n) \subset K$, $\forall\, n \in \mathbb{N}$,

(ii) $D^\alpha \varphi_n \to D^\alpha \varphi$ uniformemente em K, para todo multi-índice α.

O espaço $C_0^\infty(\Omega)$, com a convergência acima definida é denotado por $\mathcal{D}(\Omega)$.

Uma distribuição (escalar) sobre Ω é um funcional $T : \mathcal{D}(\Omega) \to \mathbb{R}$ satisfazendo as seguintes condições:

(i) $T(\alpha\varphi + \beta\psi) = \alpha T(\varphi) + \beta T(\psi)$, $\forall\, \alpha,\, \beta \in \mathbb{R}$ e $\forall\, \varphi,\, \psi \in \mathcal{D}(\Omega)$,

(ii) T é contínua, isto é, se $(\varphi_n)_{n\in\mathbb{N}}$ converge para φ em $\mathcal{D}(\Omega)$, então $(T(\varphi_n))_{n\in\mathbb{N}}$ converge para $T(\varphi)$ em $\mathbb{R}$.

Para $1 \leq p \leq \infty$, consideremos o espaço de Banach

$$W^{m,p}(0,T;X) = \left\{u \in L^p(0,T;X);\; u^{(j)} \in L^p(0,T;X),\; j = 1, ..., m\right\},$$

onde $u^{(j)}$ representa a j-ésima derivada de u no sentido das distribuições vetoriais. A norma de $W^{m,p}(0,T;X)$ é dada por

$$\|u\|_{W^{m,p}(0,T;X)} = \begin{cases} \displaystyle\sum_{j=0}^{m} \left\|u^{(j)}(t)\right\|_{L^p(0,T;X)}, & 1 \leq p < \infty, \\ \displaystyle\sup_{t\in(0,T)} ess \left(\sum_{j=0}^{m} \left\|u^{(j)}(t)\right\|_X\right), & p = \infty. \end{cases}$$

O espaço

$$W_0^{m,p}(0,T;X) = \{u \in W^{m,p}(0,T;X);\; u(0) = u(T) = 0\},$$

representa o fecho de $\mathcal{D}(0,T;X)$ com a norma de $W^{m,p}(0,T;X)$. Se $p = 2$ e X é um espaço de Hilbert, este espaço é denotado por $H^m(0,T;X)$, e munido com o produto interno

$$(u,v)_{H^m(0,T;X)} = \sum_{j=0}^{m} \left(u^{(j)}, v^{(j)}\right)_{L^2(0,T;X)}$$

é um espaço de Hilbert. Denota-se por $H_0^m(0,T;X)$ o fecho, em $H^m(0,T;X)$, de $\mathcal{D}(0,T;X)$ e por $H^{-m}(0,T;X)$ o dual topológico de $H_0^m(0,T;X)$.

1.2 Principais resultados utilizados

Teorema 1.1 *Considere o seguinte sistema associado a equação da onda*

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u + p\, u = f & \text{em } \Omega \times (0,T), \\ u = 0 & \text{sobre } \Gamma \times (0,T), \\ u(0) = u_0; \dfrac{\partial u}{\partial t}(0) = u_1 & \text{em } \Omega. \end{cases} \tag{1.1}$$

Suponha que $p \in L^\infty(\Omega \times (0,T))$. Então para toda $f \in L^1(0,T;L^2(\Omega))$, $u_0 \in H_0^1(\Omega)$ e $u_1 \in L^2(\Omega)$, existe uma única solução u para (1.1) com

$$u \in C([0,T]; H_0^1(\Omega)),\ \frac{\partial u}{\partial t} \in C([0,T]; L^2(\Omega)).$$

Demonstração: Ver [12] e Teorema 1.1.1 em [13]. ■

Teorema 1.2 *Suponha que $p \in L^\infty(\Omega \times (0,T))$. Então para toda $f \in W^{1,1}(0,T;H^{-1}(\Omega))$, $u_0 \in H_0^1(\Omega)$ e $u_1 \in L^2(\Omega)$, existe uma única solução u para (1.1) com*

$$u \in C([0,T]; H_0^1(\Omega)),\ \frac{\partial u}{\partial t} \in C([0,T]; L^2(\Omega)).$$

Demonstração: Ver [12] e Teorema 1.1.3 em [13]. ■

Teorema 1.3 *Supondo que $p \in L^\infty(\Omega \times (0,T))$, $f \in L^1(0,T;L^2(\Omega))$, $u_0 \in H_0^1(\Omega)$ e $u_1 \in L^2(\Omega)$, existe uma única solução u de (1.1) satisfazendo*

$$\frac{\partial u}{\partial \nu} \in L^2(0,T;L^2(\Gamma)).$$

Além disso, a aplicação

$$(f, u_0, u_1) \to \frac{\partial u}{\partial \nu}$$

que está bem definida para todos dados regulares (e densos), é linear contínua de $L^1(0,T;L^2(\Omega)) \times H_0^1 \times L^2(\Omega)$ em $L^2(0,T;L^2(\Gamma))$.

Demonstração: Ver [12] e Teorema 1.1.4 em [13]. ■

Teorema 1.4 *Considere a seguinte equação da onda*

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u + p\,u = f & \text{em } \Omega \times (0,T), \\ u = g & \text{sobre } \Gamma \times (0,T), \\ u(0) = u_0; \dfrac{\partial u}{\partial t}(0) = u_1 & \text{em } \Omega. \end{cases} \tag{1.2}$$

Para toda $f \in L^1(0,T;H^{-1}(\Omega))$, $g \in L^2(0,T;L^2(\Gamma))$, $u_0 \in L^2(\Omega)$ e $u_1 \in H^{-1}(\Omega)$, existe uma única solução u de (1.2) com $u \in C([0,T];L^2(\Omega))$ e $\frac{\partial u}{\partial t} \in C([0,T];H^{-1}(\Omega))$.

Demonstração: Ver [12] e Teorema 1.1.6 em [13]. ■

Teorema 1.5 (Imersão de Sobolev) *Seja Ω um aberto limitado do $\mathbb{R}^n$ com fronteira Γ regular.*

(i) Se $n > pm$, então $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$, onde $q \in \left[1, \dfrac{np}{n-mp}\right]$.

(ii) Se $n = pm$, então $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$, onde $q \in [1, +\infty)$.

(iii) Se $n = 1$ e $m \geq 1$, então $W^{m,p}(\Omega) \hookrightarrow L^\infty(\Omega)$.

Demonstração: Ver [5]. ■

Teorema 1.6 (Gauss-Green) *Se $u \in C^1(\overline{\Omega})$, então $\int_\Omega u_{x_i} dx = \int_{\Gamma_0} u\nu_i d\Gamma$, para todo $i = 1, ..., n$.*

Demonstração: Ver [5]. ■

Teorema 1.7 (Fórmulas de Green)

(i) Se $\gamma \in H^2(\Omega)$, então $\displaystyle\int_\Omega \nabla\gamma\nabla u dx = -\int_\Omega u\Delta\gamma dx + \int_\Gamma \frac{\partial\gamma}{\partial\nu} u ds, \ \forall\ u \in H^1(\Omega)$.

(ii) Se $u, \gamma \in H^2(\Omega)$, então $\displaystyle\int_\Omega u\Delta\gamma - \gamma\Delta u dx = \int_{\partial\Omega} u\frac{\partial\gamma}{\partial\nu} - \gamma\frac{\partial u}{\partial\nu} ds$.

Demonstração: Ver [5]. ■

Teorema 1.8 (Desigualdade de Cauchy-Schwarz) *Sejam $f, g : \Omega \to \mathbb{R}$ duas funções de quadrado integrável, então*

$$|\langle f, g\rangle_{L^2}| \leq \|f\|_{L^2}\|g\|_{L^2} \tag{1.3}$$

Demonstração: Ver [5]. ■

Teorema 1.9 (Desigualdade de Hölder) *Sejam $f \in L^p(\Omega)$ e $g \in L^q(\Omega)$ com $1 \leq p \leq \infty$ e $\frac{1}{p} + \frac{1}{q} = 1$ ($q = 1$ se $p = \infty$ e $q = \infty$ se $p = 1$). Então $fg \in L^1(\Omega)$ e*

$$\int_\Omega |f, g| \leq \|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)}. \tag{1.4}$$

Demonstração: Ver [5]. ■

Teorema 1.10 (Desigualdade de Young) *Se $a \geq 0$ e $b \geq 0$ e $1 < p, q < \infty$ com $\frac{1}{p}+\frac{1}{q} = 1$ então*

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q. \tag{1.5}$$

Demonstração: Ver [5]. ■

Teorema 1.11 (Desigualdade de Poincaré) *Seja Ω um aberto limitado do $\mathbb{R}^N$. Então existe uma constante C(dependendo de Ω) tal que*

$$\|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)} \tag{1.6}$$

para toda $u \in H_0^1(\Omega)$.

Demonstração: Ver [5]. ■

Lema 1.1 (Gronwall) *Sejam $m \in L^1(0, T; R)$, $m \geq 0$ quase sempre em $(0, T)$, $C \geq 0$ constante real e $g \in L^\infty(0, T)$, tal que:*

$$g(t) \leq C + \int_0^t m(s)g(s)ds, \ \forall\ t \in (0, T).$$

Então

$$g(t) \leq C + e^{\int_0^t m(s)ds}, \ \forall\ t \in (0, T).$$

Demonstração: Ver [6]. ■

Teorema 1.12 *Se* $\phi \in H_0^1(\Omega) \cap H^2(\Omega)$, *então*

$$|\bigtriangledown \phi|^2 = \left|\frac{\partial \phi}{\partial \nu}\right|^2, \tag{1.7}$$

onde $\nu = (\nu_1, \nu_2, ..., \nu_n)$ *é o campo de vetores normais exteriores a fronteira de* Ω.

Demonstração: Ver [12]. ■

Teorema 1.13 (Schwarz) *Seja* $f : U \to \mathbb{R}$ *duas vezes diferenciável no ponto* $c \in U \subset \mathbb{R}^N$. *Para quaisquer* $1 \leq i, j \leq N$, *tem-se*

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(c) = \frac{\partial^2 f}{\partial x_j \partial x_i}(c).$$

Demonstração: Ver [8]. ■

Capítulo 2

Desigualdade de Carleman para equação da onda

Neste capítulo nos dedicaremos a demonstrar um teorema que nos fornece estimativas de Carleman para o operador da onda. A vantagem de encontrarmos tais estimativas é que elas nos permitem estudar uma onda que se propaga em um aberto limitado do $\mathbb{R}^N$ a partir de informações referentes a uma parte dessa região.

Seja $v \in L^2(-T, T; L^2(\Omega))$ uma função tal que

$$L_0 v = \frac{\partial^2 v}{\partial t^2} - \Delta v \in L^2(-T, T; L^2(\Omega)).$$

Façamos

$$Lv = L_0 v + pv = \frac{\partial^2 v}{\partial t^2} - \Delta v + pv,$$

com $p \in L^\infty(\Omega \times (0, T))$. Percebamos que $Lv \in L^2(-T, T; L^2(\Omega))$.

Consideremos o conjunto das funções v que satisfaçam

$$\begin{cases} v \in L^2(-T, T; L^2(\Omega)), \\ L_0 v \in L^2(-T, T; L^2(\Omega)), \\ v = 0 \quad sobre\ \Sigma, \\ v(-T) = v(T) = \dfrac{\partial v}{\partial t}(-T) = \dfrac{\partial v}{\partial t}(T) = 0. \end{cases} \tag{2.1}$$

Observe que $C_0^\infty(-T, T; C_0^\infty(\Omega))$ está contido neste conjunto. Como $C_0^\infty(-T, T; C_0^\infty(\Omega))$ é denso $L^2(-T, T; L^2(\Omega))$ segue que o mesmo ocorre para o conjunto definido acima. Logo, ele possui um subconjunto denso de funções $C_0^\infty(-T,T;C_0^\infty(\Omega))$ que usaremos para justificar nossos cálculos.

Escolhamos $x_0 \in \mathbb{R}^N \backslash \overline{\Omega}$ e definamos a função peso

$$\phi(x,t) = |x - x_0|^2 - \beta t^2 + M_0,\ 0 < \beta < 1, \tag{2.2}$$

com M_0 sendo uma constante tal que

$$\phi(x,t) \geq 1 \quad em \;\; Q. \tag{2.3}$$

Para $\lambda > 0$, consideremos

$$\varphi_\lambda(x,t) = e^{\lambda\phi(x,t)}. \tag{2.4}$$

Além disso, para uma função v definida em Q e para $s > 0$, tomemos

$$w = e^{s\varphi_\lambda} v.$$

Tendo v regularidade suficiente, podemos calcular

$$P_0 w := e^{s\varphi_\lambda} L_0(e^{-s\varphi_\lambda} w) = e^{s\varphi_\lambda} L_0(e^{-s\varphi_\lambda} e^{s\varphi_\lambda} v) = e^{s\varphi_\lambda} L_0 v.$$

Façamos também outros cálculos.

Para calcularmos $\frac{\partial}{\partial t}(e^{-s\varphi_\lambda} w)$, observemos que

$$\frac{\partial}{\partial t}\left(e^{-s\varphi_\lambda(x,t)}\right) = -se^{-s\varphi_\lambda(x,t)} \lambda e^{\lambda\phi(x,t)} \frac{\partial \phi}{\partial t} = -s\lambda e^{-s\varphi_\lambda(x,t)} \varphi_\lambda \frac{\partial \phi}{\partial t}.$$

Assim,

$$\frac{\partial}{\partial t}(e^{-s\varphi_\lambda} w) = \frac{\partial}{\partial t}(e^{-s\varphi_\lambda}) w + e^{-s\varphi_\lambda} \frac{\partial w}{\partial t} = e^{-s\varphi_\lambda}\left(\frac{\partial w}{\partial t} - s\lambda\varphi_\lambda \frac{\partial \phi}{\partial t} w\right).$$

Derivemos $\frac{\partial}{\partial t}(e^{-s\varphi_\lambda} w)$ mais uma vez.

$$\frac{\partial^2}{\partial t^2}\left(e^{-s\varphi_\lambda} w\right) = \frac{\partial}{\partial t}\left[e^{-s\varphi_\lambda}\left(\frac{\partial w}{\partial t} - s\lambda\varphi_\lambda \frac{\partial \phi}{\partial t} w\right)\right]$$

$$= \frac{\partial}{\partial t}\left(e^{-s\varphi_\lambda}\right)\left(\frac{\partial w}{\partial t} - s\lambda\varphi_\lambda \frac{\partial \phi}{\partial t} w\right) + e^{-s\varphi_\lambda} \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial t} - s\lambda\varphi_\lambda \frac{\partial \phi}{\partial t} w\right)$$

$$= -s\lambda e^{-s\varphi_\lambda}\varphi_\lambda \frac{\partial\phi}{\partial t}\left(\frac{\partial w}{\partial t} - s\lambda\varphi_\lambda\frac{\partial\phi}{\partial t}w\right) + e^{-s\varphi_\lambda}\frac{\partial^2 w}{\partial t^2} - s\lambda e^{-s\varphi_\lambda}\frac{\partial}{\partial t}\left(\varphi_\lambda\frac{\partial\phi}{\partial t}w\right).$$

Notemos que

$$\frac{\partial}{\partial t}\left(\varphi_\lambda\frac{\partial\phi}{\partial t}w\right) = \frac{\partial}{\partial t}\left(\varphi_\lambda\frac{\partial\phi}{\partial t}\right)w + \varphi_\lambda\frac{\partial\phi}{\partial t}\frac{\partial w}{\partial t}$$

$$= \left(\frac{\partial\varphi_\lambda}{\partial t}\frac{\partial\phi}{\partial t} + \varphi_\lambda\frac{\partial^2\phi}{\partial t^2}\right)w + \varphi_\lambda\frac{\partial\phi}{\partial t}\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}\left(e^{\lambda\phi}\right)\frac{\partial\phi}{\partial t}w + \varphi_\lambda\frac{\partial^2\phi}{\partial t^2}w + \varphi_\lambda\frac{\partial\phi}{\partial t}\frac{\partial w}{\partial t}$$

$$= \lambda e^{\lambda\phi}\left(\frac{\partial\phi}{\partial t}\right)^2 w + \varphi_\lambda\frac{\partial^2\phi}{\partial t^2}w + \varphi_\lambda\frac{\partial\phi}{\partial t}\frac{\partial w}{\partial t} = \lambda\varphi_\lambda\left(\frac{\partial\phi}{\partial t}\right)^2 w + \varphi_\lambda\frac{\partial^2\phi}{\partial t^2}w + \varphi_\lambda\frac{\partial\phi}{\partial t}\frac{\partial w}{\partial t}.$$

Dessa forma,

$$\frac{\partial^2}{\partial t^2}(e^{-s\varphi_\lambda}w)$$

$$= -s\lambda e^{-s\varphi_\lambda}\varphi_\lambda\frac{\partial\phi}{\partial t}\frac{\partial w}{\partial t} + s^2\lambda^2 e^{-s\varphi_\lambda}{\varphi_\lambda}^2\left(\frac{\partial\phi}{\partial t}\right)^2 w + e^{-s\varphi_\lambda}\frac{\partial^2 w}{\partial t^2}$$

$$-s\lambda^2 e^{-s\varphi_\lambda}\varphi_\lambda\left(\frac{\partial\phi}{\partial t}\right)^2 w - s\lambda e^{-s\varphi_\lambda}\varphi_\lambda\frac{\partial^2\phi}{\partial t^2}w - s\lambda e^{-s\varphi_\lambda}\varphi_\lambda\frac{\partial\phi}{\partial t}\frac{\partial w}{\partial t}$$

$$= e^{-s\varphi_\lambda}\left[\frac{\partial^2 w}{\partial t^2} - s\lambda^2\varphi_\lambda\left(\frac{\partial\phi}{\partial t}\right)^2 w - s\lambda\varphi_\lambda\frac{\partial^2\phi}{\partial t^2}w - 2s\lambda\varphi_\lambda\frac{\partial\phi}{\partial t}\frac{\partial w}{\partial t} + s^2\lambda^2{\varphi_\lambda}^2\left(\frac{\partial\phi}{\partial t}\right)^2 w\right].$$

Se procedermos de forma análoga, encontraremos as derivadas referentes a variável espacial

$$\frac{\partial}{\partial x_i}(e^{-s\varphi_\lambda}w) = e^{-s\varphi_\lambda}\left(\frac{\partial w}{\partial x_i} - s\lambda\varphi_\lambda\frac{\partial\phi}{\partial x_i}w\right)$$

e

$$\frac{\partial^2}{\partial {x_i}^2}(e^{-s\varphi_\lambda}w)$$

$$= e^{-s\varphi_\lambda}\left[\frac{\partial^2 w}{\partial {x_i}^2} - s\lambda^2\varphi_\lambda\left(\frac{\partial\phi}{\partial x_i}\right)^2 w - s\lambda\varphi_\lambda\frac{\partial^2\phi}{\partial {x_i}^2}w - 2s\lambda\varphi_\lambda\frac{\partial\phi}{\partial x_i}\frac{\partial w}{\partial x_i}\right.$$

$$\left. + s^2\lambda^2{\varphi_\lambda}^2\left(\frac{\partial\phi}{\partial x_i}\right)^2 w\right].$$

Esses resultados nos permitem determinar o Laplaciano da função $e^{-s\varphi_\lambda}w$.

$$
\begin{aligned}
&\sum_{i=1}^{n} \frac{\partial^2}{\partial {x_i}^2}(e^{-s\varphi_\lambda}w) \\
&= e^{-s\varphi_\lambda}\sum_{i=1}^{n}\left[\frac{\partial^2 w}{\partial {x_i}^2} - s\lambda^2\varphi_\lambda\left(\frac{\partial\phi}{\partial x_i}\right)^2 w - s\lambda\varphi_\lambda\frac{\partial^2\phi}{\partial {x_i}^2}w - 2s\lambda\varphi_\lambda\frac{\partial\phi}{\partial x_i}\frac{\partial w}{\partial x_i}\right. \\
&\quad\left. + s^2\lambda^2{\varphi_\lambda}^2\left(\frac{\partial\phi}{\partial x_i}\right)^2 w\right] \\
&= e^{-s\varphi_\lambda}\left[\sum_{i=1}^{n}\frac{\partial^2 w}{\partial {x_i}^2} - s\lambda^2\varphi_\lambda\sum_{i=1}^{n}\left(\frac{\partial\phi}{\partial x_i}\right)^2 w - s\lambda\varphi_\lambda\sum_{i=1}^{n}\frac{\partial^2\phi}{\partial {x_i}^2}w - 2s\lambda\varphi_\lambda\sum_{i=1}^{n}\frac{\partial\phi}{\partial x_i}\frac{\partial w}{\partial x_i}\right. \\
&\quad\left. + s^2\lambda^2{\varphi_\lambda}^2\sum_{i=1}^{n}\left(\frac{\partial\phi}{\partial x_i}\right)^2 w\right] \\
&= e^{-s\varphi_\lambda}\left(\Delta w - s\lambda^2|\nabla\phi|^2\varphi_\lambda w - s\lambda\Delta\phi\varphi_\lambda w - 2s\lambda\varphi_\lambda\nabla\phi\nabla w + s^2\lambda^2{\varphi_\lambda}^2|\nabla\phi|^2 w\right).
\end{aligned}
$$

Se definirmos

$$
\begin{aligned}
P_1 w &= \frac{\partial^2 w}{\partial t^2} - \Delta w + s^2\lambda^2{\varphi_\lambda}^2\left(\left(\frac{\partial\phi}{\partial t}\right)^2 - |\nabla\phi|^2\right)w, \\
P_2 w &= (M_1 - 1)s\lambda\varphi_\lambda\left(\frac{\partial^2\phi}{\partial t^2} - \Delta\phi\right)w - \; s\lambda^2\varphi_\lambda\left(\left(\frac{\partial\phi}{\partial t}\right)^2 - |\nabla\phi|^2\right)w \\
&\qquad -2s\lambda\varphi_\lambda\left(\frac{\partial\phi}{\partial t}\frac{\partial w}{\partial t} - \nabla\phi\nabla w\right), \\
R_0 w &= -M_1 s\lambda\varphi_\lambda\left(\frac{\partial^2\phi}{\partial t^2} - \Delta\phi\right)w,
\end{aligned}
$$

onde M_1 é uma constante a ser escolhida satisfazendo

$$
\frac{2\beta}{\beta + N} < M_1 < \frac{2}{\beta + N},
$$

então

$$
\begin{aligned}
&P_1 w + P_2 w + R_0 w \\
&= \frac{\partial^2 w}{\partial t^2} - \Delta w + s^2\lambda^2{\varphi_\lambda}^2\left(\frac{\partial \phi}{\partial t}\right)^2 w - s^2\lambda^2{\varphi_\lambda}^2|\nabla\phi|^2 w + M_1 s\lambda\varphi_\lambda\frac{\partial^2\phi}{\partial t^2}w \\
&\quad -M_1 s\lambda\varphi_\lambda\Delta\phi w - s\lambda\varphi_\lambda\frac{\partial^2\phi}{\partial t^2}w + s\lambda\varphi_\lambda\Delta\phi w - s\lambda^2\varphi_\lambda\left(\frac{\partial\phi}{\partial t}\right)^2 w + s\lambda^2\varphi_\lambda|\nabla\phi|^2 w \\
&\quad -2s\lambda\varphi_\lambda\frac{\partial\phi}{\partial t}\frac{\partial w}{\partial t} + 2s\lambda\varphi_\lambda\nabla\phi\nabla w - M_1 s\lambda\varphi_\lambda\frac{\partial^2\phi}{\partial t^2}w + M_1 s\lambda\varphi_\lambda\Delta\phi w \\
&= -(\Delta w - s\lambda^2\varphi_\lambda|\nabla\phi|^2 w - s\lambda\varphi_\lambda\Delta\phi w - 2s\lambda\varphi_\lambda\nabla\phi\nabla w + s^2\lambda^2{\varphi_\lambda}^2|\nabla\phi|^2 w) \\
&\quad +\frac{\partial^2 w}{\partial t^2} + s^2\lambda^2{\varphi_\lambda}^2\left(\frac{\partial\phi}{\partial t}\right)^2 w - s\lambda\varphi_\lambda\frac{\partial^2\phi}{\partial t^2}w - s\lambda^2\varphi_\lambda\left(\frac{\partial\phi}{\partial t}\right)^2 w - 2s\lambda\varphi_\lambda\frac{\partial\phi}{\partial t}\frac{\partial w}{\partial t}.
\end{aligned}
$$

Mas como $v = e^{-s\varphi_\lambda}w$,

$$
-\frac{\Delta\left(e^{-s\varphi_\lambda}w\right)}{e^{-s\varphi_\lambda}} = \Delta w - s\lambda^2\varphi_\lambda|\nabla\phi|^2 w - s\lambda\varphi_\lambda\Delta\phi w - 2s\lambda\varphi_\lambda\nabla\phi\nabla w + s^2\lambda^2{\varphi_\lambda}^2|\nabla\phi|^2 w
$$

e

$$
\begin{aligned}
\frac{\frac{\partial^2}{\partial t^2}\left(e^{-s\varphi_\lambda}w\right)}{e^{-s\varphi_\lambda}} &= \frac{\partial^2 w}{\partial t^2} + s^2\lambda^2{\varphi_\lambda}^2\left(\frac{\partial\phi}{\partial t}\right)^2 w - s\lambda\varphi_\lambda\frac{\partial^2\phi}{\partial t^2}w - s\lambda^2\varphi_\lambda\left(\frac{\partial\phi}{\partial t}\right)^2 w \\
&\quad -2s\lambda\varphi_\lambda\frac{\partial\phi}{\partial t}\frac{\partial w}{\partial t},
\end{aligned}
$$

segue que

$$
P_1 w + P_2 w + R_0 w = -\frac{\Delta v}{e^{-s\varphi_\lambda}} + \frac{\frac{\partial^2 v}{\partial t^2}}{e^{-s\varphi_\lambda}} = e^{s\varphi_\lambda}\left(-\Delta v + \frac{\partial^2 v}{\partial t^2}\right) = e^{s\varphi_\lambda}L_0 v = P_0 w.
$$

Precisamos ainda definir, para $x_0 \in \mathbb{R}^N$, o conjunto

$$
\Gamma_{x_0} = \{x \in \Gamma;\;\; \nu(x).(x - x_0) > 0\}.
$$

Feitas essas considerações, enunciemos o principal teorema deste trabalho.

Teorema 2.1 *Suponhamos que exista* $x_0 \in \mathbb{R}^N\backslash\overline{\Omega}$ *tal que* $\Gamma_0 \supset \Gamma_{x_0}$. *Então, para todo* $m > 0$, *existem* $\lambda_0 > 0$ *e* $s_0 > 0$ *e uma constante* $C = C(s_0, \lambda_0, \Omega, \beta, x_0, m)$ *tal que* *para todo* $p \in L^\infty(Q)$ *com* $\|p\|_{L^\infty(Q)} \leq m$, $\lambda \geq \lambda_0$, $s \geq s_0$, $v \in L^2(-T,T;L^2(\Omega))$, $L_0 v \in L^2(-T,T;L^2(\Omega))$, $v = 0$ *sobre* Σ, $v(-T) = v(T) = 0$, $\frac{\partial v}{\partial t}(-T) = \frac{\partial v}{\partial t}(T) = 0$, *temos*

$$\begin{aligned}
&s\lambda \int_{-T}^{T}\int_\Omega e^{2s\varphi_\lambda}\varphi_\lambda \left(\left|\frac{\partial v}{\partial t}\right|^2 + |\nabla v|^2\right) dxdt + s^3\lambda^3 \int_{-T}^{T}\int_\Omega e^{2s\varphi_\lambda}{\varphi_\lambda}^3 |v|^2 dxdt \\
&+ \int_{-T}^{T}\int_\Omega |P_1 w|^2 dxdt + \int_{-T}^{T}\int_\Omega |P_2 w|^2 dxdt \\
&\leq C \int_{-T}^{T}\int_\Omega e^{2s\varphi_\lambda}|Lv|^2 dxdt + Cs\lambda \int_{-T}^{T}\int_{\Gamma_0} e^{2s\varphi_\lambda}\left|\frac{\partial v}{\partial \nu}\right|^2 d\sigma dt. \qquad (2.5)
\end{aligned}$$

Demonstração: É suficiente mostrarmos que o teorema é verdadeiro para $L_0 v$ no lugar de Lv do lado direito da desigualdade (2.5). De fato, como $Lv = L_0 v + pv$ então

$$|L_0 v|^2 \leq 2(|Lv|^2 + |pv|^2) \leq 2(|Lv|^2 + m^2|v|^2).$$

Dessa forma, supondo o teorema válido para $L_0 v$ temos que

$$\begin{aligned}
&s\lambda \int_{-T}^{T}\int_\Omega e^{2s\varphi_\lambda} \left(\left|\frac{\partial v}{\partial t}\right|^2 + |\nabla v|^2\right) dxdt + s^3\lambda^3 \int_{-T}^{T}\int_\Omega e^{2s\varphi_\lambda}{\varphi_\lambda}^3 |v|^2 dxdt \\
&+ \int_{-T}^{T}\int_\Omega |P_1 w|^2 dxdt + \int_{-T}^{T}\int_\Omega |P_2 w|^2 dxdt \\
&\leq C \int_{-T}^{T}\int_\Omega e^{2s\varphi_\lambda}|L_0 v|^2 dxdt + Cs\lambda \int_{-T}^{T}\int_{\Gamma_0} e^{2s\varphi_\lambda}\left|\frac{\partial v}{\partial \nu}\right|^2 d\sigma dt \\
&\leq 2C \int_{-T}^{T}\int_\Omega e^{2s\varphi_\lambda}|Lv|^2 dxdt + 2m^2 C \int_{-T}^{T}\int_\Omega e^{2s\varphi_\lambda}|v|^2 dxdt \\
&+ Cs\lambda \int_{-T}^{T}\int_{\Gamma_0} e^{2s\varphi_\lambda}\left|\frac{\partial v}{\partial \nu}\right|^2 d\sigma dt.
\end{aligned}$$

Uma vez que $\varphi_\lambda \geq 1$ e tomando $\lambda \geq 1$, teremos $\lambda^3{\varphi_\lambda}^3 \geq 1$ e

$$s\lambda \int_{-T}^{T}\int_\Omega e^{2s\varphi_\lambda} \left(\left|\frac{\partial v}{\partial t}\right|^2 + |\nabla v|^2\right) dxdt + s^3\lambda^3 \int_{-T}^{T}\int_\Omega e^{2s\varphi_\lambda}{\varphi_\lambda}^3 |v|^2 dxdt$$

$$+\int_{-T}^{T}\int_{\Omega}|P_1w|^2dxdt+\int_{-T}^{T}\int_{\Omega}|P_2w|^2dxdt$$

$$\leq 2C\int_{-T}^{T}\int_{\Omega}e^{2s\varphi_\lambda}|Lv|^2dxdt+2m^2C\lambda^3\int_{-T}^{T}\int_{\Omega}e^{2s\varphi_\lambda}{\varphi_\lambda}^3|v|^2dxdt$$

$$+Cs\lambda\int_{-T}^{T}\int_{\Gamma_0}e^{2s\varphi_\lambda}\left|\frac{\partial v}{\partial\nu}\right|^2d\sigma dt$$

ou, equivalentemente,

$$s\lambda\int_{-T}^{T}\int_{\Omega}e^{2s\varphi_\lambda}\left(\left|\frac{\partial v}{\partial t}\right|^2+|\nabla v|^2\right)dxdt+\left(1-\frac{2m^2C}{s^3}\right)s^3\lambda^3\int_{-T}^{T}\int_{\Omega}e^{2s\varphi_\lambda}{\varphi_\lambda}^3|v|^2dxdt$$

$$+\int_{-T}^{T}\int_{\Omega}|P_1w|^2dxdt+\int_{-T}^{T}\int_{\Omega}|P_2w|^2dxdt$$

$$\leq 2C\int_{-T}^{T}\int_{\Omega}e^{2s\varphi_\lambda}|Lv|^2dxdt+Cs\lambda\int_{-T}^{T}\int_{\Gamma_0}e^{2s\varphi_\lambda}\left|\frac{\partial v}{\partial\nu}\right|^2d\sigma dt.$$

Assim, se $s_0>\sqrt[3]{2m^2C}$, obtemos (2.5).

Provemos que o teorema é verdadeiro para L_0v no lado direito de (2.5).

Para $w=e^{s\varphi_\lambda}v$, temos que

$$P_0w=P_1w+P_2w+R_0w=e^{s\varphi_\lambda}L_0v.$$

Logo

$$\int_{-T}^{T}\int_{\Omega}|P_1w+P_2w|^2dxdt=\int_{-T}^{T}\int_{\Omega}|e^{s\varphi_\lambda}L_0v-R_0w|^2dxdt$$

e

$$\int_{-T}^{T}\int_{\Omega}|e^{s\varphi_\lambda}L_0v-R_0w|^2dxdt$$

$$=\int_{-T}^{T}\int_{\Omega}(|P_1w|^2+|P_2w|^2)dxdt+2\int_{-T}^{T}\int_{\Omega}P_1wP_2wdxdt. \tag{2.6}$$

Vejamos que

$$|e^{s\varphi_\lambda}L_0v - R_0w|^2 \leq (|e^{s\varphi_\lambda}L_0v| + |R_0w|)^2 = |e^{s\varphi_\lambda}L_0v|^2 + 2|e^{s\varphi_\lambda}L_0v|\,|R_0w| + |R_0w|^2$$

$$\leq 2(|e^{s\varphi_\lambda}L_0v|^2 + |R_0w|^2). \tag{2.7}$$

Substituindo (2.7) em (2.6), obtemos

$$\int_{-T}^{T}\int_\Omega (|P_1w|^2 + |P_2w|^2)dxdt + 2\int_{-T}^{T}\int_\Omega P_1wP_2wdxdt$$

$$\leq 2\int_{-T}^{T}\int_\Omega e^{2s\varphi_\lambda}|L_0v|^2dxdt + 2\int_{-T}^{T}\int_\Omega |R_0w|^2dxdt. \tag{2.8}$$

Precisamos agora encontrar uma limitação para $\displaystyle\int_{-T}^{T}\int_\Omega P_1wP_2wdxdt$.

$$\int_{-T}^{T}\int_\Omega P_1wP_2wdxdt$$

$$= \int_{-T}^{T}\int_\Omega \left\{ \left(\frac{\partial^2 w}{\partial t^2} - \Delta w + s^2\lambda^2\varphi_\lambda^2 \left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2 \right) w \right) \right.$$

$$\left[(M_1 - 1)s\lambda\varphi_\lambda \left(\frac{\partial^2\phi}{\partial t^2} - \Delta\phi \right) w - s\lambda^2\varphi_\lambda \left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2 \right) w \right.$$

$$\left. \left. -2s\lambda\varphi_\lambda \left(\frac{\partial\phi}{\partial t}\frac{\partial w}{\partial t} - \nabla\phi\nabla w \right) \right] \right\} dxdt$$

$$(\mathbf{I_{11}}) \qquad = (M_1 - 1)s\lambda \int_{-T}^{T}\int_\Omega \frac{\partial^2 w}{\partial t^2}\varphi_\lambda w \left(\frac{\partial^2\phi}{\partial t^2} - \Delta\phi \right) dxdt$$

$$(\mathbf{I_{12}}) \qquad -s\lambda^2 \int_{-T}^{T}\int_\Omega \frac{\partial^2 w}{\partial t^2}\varphi_\lambda \left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2 \right) wdxdt$$

$$(\mathbf{I_{13}}) \qquad -2s\lambda \int_{-T}^{T}\int_\Omega \frac{\partial^2 w}{\partial t^2}\varphi_\lambda \left(\frac{\partial w}{\partial t}\frac{\partial \phi}{\partial t} - \nabla w\nabla\phi \right) dxdt$$

$$(\mathbf{I_{21}}) \qquad -(M_1 - 1)s\lambda \int_{-T}^{T}\int_\Omega \Delta w\varphi_\lambda w \left(\frac{\partial^2\phi}{\partial t^2} - \Delta\phi \right) dxdt$$

$$(\mathbf{I_{22}}) \qquad +s\lambda^2 \int_{-T}^{T}\int_\Omega \Delta w\varphi_\lambda w \left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2 \right) dxdt$$

$$(\mathbf{I_{23}}) \qquad +2s\lambda \int_{-T}^{T}\int_\Omega \Delta w\varphi_\lambda \left(\frac{\partial w}{\partial t}\frac{\partial \phi}{\partial t} - \nabla w\nabla\phi \right) dxdt$$

$(\mathbf{I_{31}})$ $$-(1-M_1)s^3\lambda^3\int_{-T}^{T}\int_\Omega |w|^2{\varphi_\lambda}^3\left(\frac{\partial^2\phi}{\partial t^2}-\Delta\phi\right)\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)dxdt$$

$(\mathbf{I_{32}})$ $$-s^3\lambda^4\int_{-T}^{T}\int_\Omega |w|^2{\varphi_\lambda}^3\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)^2 dxdt$$

$(\mathbf{I_{33}})$ $$-2s^3\lambda^3\int_{-T}^{T}\int_\Omega {\varphi_\lambda}^3 w\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)\left(\frac{\partial\phi}{\partial t}\frac{\partial w}{\partial t}-\nabla\phi\nabla w\right)dxdt.$$

Dessa forma,

$$\int_{-T}^{T}\int_\Omega P_1wP_2wdxdt=\sum_{k,l=1}^{3}I_{kl}.$$

Para encontrarmos uma limitação para este termo calcularemos cada uma das integrais I_{kl}, com $k,l=1,2$ ou 3. Antes disso, lembremos que

- $\dfrac{\partial\varphi_\lambda}{\partial t}=\dfrac{\partial}{\partial t}(e^{\lambda\phi(x,t)})=\lambda e^{\lambda\phi(x,t)}\dfrac{\partial\phi}{\partial t}=\lambda\varphi_\lambda\dfrac{\partial\phi}{\partial t}.$
- $\dfrac{\partial w}{\partial t}=\dfrac{\partial}{\partial t}\left(e^{s\varphi_\lambda}v\right)=\dfrac{\partial}{\partial t}(e^{s\varphi_\lambda})\ v+e^{s\varphi_\lambda}\dfrac{\partial v}{\partial t}.$

Assim

$$\frac{\partial w}{\partial t}(T)=\frac{\partial}{\partial t}\left(e^{s\varphi_\lambda}\right)(T)\ v(T)+(e^{s\varphi_\lambda})(T)\ \frac{\partial v}{\partial t}(T)=0$$

e

$$\frac{\partial w}{\partial t}(-T)=\frac{\partial}{\partial t}\left(e^{s\varphi_\lambda}\right)(-T)\ v(-T)+(e^{s\varphi_\lambda})(-T)\ \frac{\partial v}{\partial t}(-T)=0,$$

pois $v(T)=v(-T)=0$.

- $\dfrac{\partial\varphi_\lambda}{\partial x_i}=\dfrac{\partial}{\partial x_i}(e^{\lambda\phi(x,t)})=\lambda e^{\lambda\phi(x,t)}\dfrac{\partial\phi}{\partial x_i}=\lambda\varphi_\lambda\dfrac{\partial\phi}{\partial x_i}.$
- $\dfrac{\partial^2\phi}{\partial x_i\partial t}=\dfrac{\partial^2}{\partial x_i\partial t}(|x-x_0|^2-\beta t^2+M_0)=\dfrac{\partial}{\partial x_i}\left(\dfrac{\partial}{\partial t}(|x-x_0|^2-\beta t^2+M_0)\right)=\dfrac{\partial}{\partial x_i}(-2\beta t)=0.$

De maneira geral, derivadas consecutivas com relação a variável temporal e espacial de ϕ sempre resultam em 0.

- $\dfrac{\partial^3 \phi}{\partial t^3} = \dfrac{\partial^2}{\partial t^2}\left(\dfrac{\partial}{\partial t}(|x-x_0|^2 - \beta t^2 + M_0)\right) = \dfrac{\partial}{\partial t}\left(\dfrac{\partial}{\partial t}(-2\beta t)\right) = \dfrac{\partial}{\partial t}(-2\beta) = 0.$

Utilizaremos essas informações nos cálculos seguintes.

- $\mathbf{I_{11}} = (M_1 - 1)s\lambda \displaystyle\int_{-T}^{T}\int_{\Omega} \dfrac{\partial^2 w}{\partial t^2}\varphi_\lambda w\left(\dfrac{\partial^2 \phi}{\partial t^2} - \Delta\phi\right) dxdt.$

Observemos que

$$
\begin{aligned}
&\frac{\partial^2 w}{\partial t^2}\varphi_\lambda w\left(\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi\right)\\
&= \frac{\partial}{\partial t}\left[\frac{\partial w}{\partial t}\varphi_\lambda w\left(\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi\right)\right] - \frac{\partial w}{\partial t}\frac{\partial}{\partial t}\left[\varphi_\lambda w\left(\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi\right)\right]\\
&= \frac{\partial}{\partial t}\left[\frac{\partial w}{\partial t}\varphi_\lambda w\left(\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi\right)\right] - \frac{\partial w}{\partial t}\left\{\frac{\partial \varphi_\lambda}{\partial t} w\left(\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi\right) + \varphi_\lambda\frac{\partial}{\partial t}\left[w\left(\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi\right)\right]\right\}\\
&= \frac{\partial}{\partial t}\left[\frac{\partial w}{\partial t}\varphi_\lambda w\left(\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi\right)\right] - \frac{\partial w}{\partial t}\lambda\varphi_\lambda\frac{\partial \phi}{\partial t} w\left(\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi\right) - \varphi_\lambda\left|\frac{\partial w}{\partial t}\right|^2\left(\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi\right)\\
&\quad -w\frac{\partial w}{\partial t}\frac{\partial}{\partial t}\left(\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi\right)\\
&= \frac{\partial}{\partial t}\left[\frac{\partial w}{\partial t}\varphi_\lambda w\left(\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi\right)\right] - \frac{\lambda}{2}\frac{\partial |w|^2}{\partial t}\varphi_\lambda\frac{\partial \phi}{\partial t}\left(\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi\right) - \varphi_\lambda\left|\frac{\partial w}{\partial t}\right|^2\left(\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi\right).
\end{aligned}
$$

O segundo termo dessa igualdade pode ser reescrito da seguinte forma

$$
\begin{aligned}
&-\frac{\lambda}{2}\frac{\partial |w|^2}{\partial t}\varphi_\lambda\frac{\partial \phi}{\partial t}\left(\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi\right)\\
&= -\frac{\lambda}{2}\frac{\partial}{\partial t}\left[|w|^2\varphi_\lambda\frac{\partial \phi}{\partial t}\left(\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi\right)\right] + \frac{\lambda}{2}|w|^2\frac{\partial}{\partial t}\left[\varphi_\lambda\frac{\partial \phi}{\partial t}\left(\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi\right)\right]\\
&= -\frac{\lambda}{2}\frac{\partial}{\partial t}\left[|w|^2\varphi_\lambda\frac{\partial \phi}{\partial t}\left(\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi\right)\right] + \frac{\lambda}{2}|w|^2\left\{\frac{\partial \varphi_\lambda}{\partial t}\frac{\partial \phi}{\partial t}\left(\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi\right)\right.
\end{aligned}
$$

$$+\varphi_\lambda \frac{\partial}{\partial t}\left[\frac{\partial \phi}{\partial t}\left(\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi\right)\right]\Bigg\}$$

$$= -\frac{\lambda}{2}\frac{\partial}{\partial t}\left[|w|^2\varphi_\lambda \frac{\partial \phi}{\partial t}\left(\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi\right)\right] + \frac{\lambda^2}{2}|w|^2\varphi_\lambda\left|\frac{\partial \phi}{\partial t}\right|^2\left(\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi\right)$$

$$+\frac{\lambda}{2}|w|^2\varphi_\lambda \frac{\partial^2 \phi}{\partial t^2}\left(\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi\right)$$

$$= -\frac{\lambda}{2}\frac{\partial}{\partial t}\left[|w|^2\varphi_\lambda \frac{\partial \phi}{\partial t}\left(\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi\right)\right] + \frac{\lambda}{2}|w|^2\varphi_\lambda\left(\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi\right)\left(\frac{\partial^2 \phi}{\partial t^2} + \lambda\left|\frac{\partial \phi}{\partial t}\right|^2\right).$$

Com esses resultados, obtemos que

$$\mathbf{I_{11}} = (M_1 - 1)s\lambda \int_{-T}^{T}\int_\Omega \frac{\partial^2 w}{\partial t^2}\varphi_\lambda w\left(\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi\right)$$

$$= (M_1 - 1)s\lambda \int_\Omega\int_{-T}^{T} \frac{\partial}{\partial t}\left[\frac{\partial w}{\partial t}\varphi_\lambda w\left(\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi\right)\right] dtdx$$

$$-\frac{(M_1 - 1)}{2}s\lambda^2 \int_\Omega\int_{-T}^{T} \frac{\partial}{\partial t}\left[|w|^2\varphi_\lambda \frac{\partial \phi}{\partial t}\left(\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi\right)\right] dtdx$$

$$+\frac{(M_1 - 1)}{2}s\lambda^2 \int_{-T}^{T}\int_\Omega |w|^2\varphi_\lambda\left(\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi\right)\left(\frac{\partial^2 \phi}{\partial t^2} + \lambda\left|\frac{\partial \phi}{\partial t}\right|^2\right) dxdt$$

$$-(M_1 - 1)s\lambda \int_{-T}^{T}\int_\Omega \left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\left(\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi\right) dxdt$$

e, portanto,

$$\mathbf{I_{11}} = (1 - M_1)s\lambda \int_{-T}^{T}\int_\Omega \left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\left(\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi\right) dxdt$$

$$-\frac{(1 - M_1)}{2}s\lambda^2 \int_{-T}^{T}\int_\Omega |w|^2\varphi_\lambda\left(\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi\right)\left(\frac{\partial^2 \phi}{\partial t^2} + \lambda\left|\frac{\partial \phi}{\partial t}\right|^2\right) dxdt.$$

Busquemos informações sobre

- $\mathbf{I_{12}} = s\lambda^2 \int_{-T}^{T}\int_{\Omega} \frac{\partial^2 w}{\partial t^2}\varphi_\lambda w \left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2 \right) dxdt.$

Façamos alguns cálculos.

$$
\begin{aligned}
&\frac{\partial^2 w}{\partial t^2}\varphi_\lambda w \left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2 \right)\\
&= \frac{\partial}{\partial t}\left[\frac{\partial w}{\partial t}\varphi_\lambda w \left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2 \right)\right] - \frac{\partial w}{\partial t}\frac{\partial}{\partial t}\left[\varphi_\lambda w \left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2 \right)\right]\\
&= \frac{\partial}{\partial t}\left[\frac{\partial w}{\partial t}\varphi_\lambda w \left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2 \right)\right] - \frac{\partial w}{\partial t}\left\{\frac{\partial \varphi_\lambda}{\partial t} w \left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2 \right)\right.\\
&\quad \left. + \varphi_\lambda \frac{\partial}{\partial t}\left[w \left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2 \right)\right]\right\}\\
&= \frac{\partial}{\partial t}\left[\frac{\partial w}{\partial t}\varphi_\lambda w \left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2 \right)\right] - \lambda w \frac{\partial w}{\partial t}\varphi_\lambda \frac{\partial \phi}{\partial t}\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2 \right)\\
&\quad - \frac{\partial w}{\partial t}\varphi_\lambda\left[\frac{\partial w}{\partial t}\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2 \right) + w\frac{\partial}{\partial t}\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2 \right)\right]\\
&= \frac{\partial}{\partial t}\left[\frac{\partial w}{\partial t}\varphi_\lambda w \left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2 \right)\right] - \lambda w \frac{\partial w}{\partial t}\varphi_\lambda \frac{\partial \phi}{\partial t}\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2 \right)\\
&\quad - \left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2 \right) - w\frac{\partial w}{\partial t}\varphi_\lambda\left[2\frac{\partial \phi}{\partial t}\frac{\partial^2 \phi}{\partial t^2} - 2\nabla\phi\frac{\partial}{\partial t}(\nabla\phi)\right]\\
&= \frac{\partial}{\partial t}\left[\frac{\partial w}{\partial t}\varphi_\lambda w \left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2 \right)\right] - \frac{\lambda}{2}\frac{\partial |w|^2}{\partial t}\varphi_\lambda \frac{\partial \phi}{\partial t}\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2 \right)\\
&\quad - \left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2 \right) - \frac{\partial |w|^2}{\partial t}\varphi_\lambda\frac{\partial \phi}{\partial t}\frac{\partial^2 \phi}{\partial t^2}.
\end{aligned} \tag{2.9}
$$

Trabalharemos um pouco mais com dois termos dessa igualdade. O segundo termo do lado direito pode também ser visto como

$$
-\frac{\lambda}{2}\frac{\partial |w|^2}{\partial t}\frac{\partial \phi}{\partial t}\varphi_\lambda\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2 \right)
$$

$$= -\frac{\lambda}{2}\frac{\partial}{\partial t}\left[|w|^2\frac{\partial \phi}{\partial t}\varphi_\lambda\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right)\right] + \frac{\lambda}{2}|w|^2\frac{\partial}{\partial t}\left[\frac{\partial \phi}{\partial t}\varphi_\lambda\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right)\right]$$

$$= -\frac{\lambda}{2}\frac{\partial}{\partial t}\left[|w|^2\frac{\partial \phi}{\partial t}\varphi_\lambda\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right)\right] + \frac{\lambda}{2}|w|^2\left\{\frac{\partial^2 \phi}{\partial t^2}\varphi_\lambda\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right)\right.$$

$$\left. + \frac{\partial \phi}{\partial t}\frac{\partial}{\partial t}\left[\varphi_\lambda\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right)\right]\right\}$$

$$= -\frac{\lambda}{2}\frac{\partial}{\partial t}\left[|w|^2\frac{\partial \phi}{\partial t}\varphi_\lambda\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right)\right] + \frac{\lambda}{2}|w|^2\frac{\partial^2 \phi}{\partial t^2}\varphi_\lambda\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right)$$

$$+ \frac{\lambda}{2}|w|^2\frac{\partial \phi}{\partial t}\left[\frac{\partial \varphi_\lambda}{\partial t}\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right) + \frac{\lambda}{2}\varphi_\lambda\frac{\partial}{\partial t}\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right)\right]$$

$$= -\frac{\lambda}{2}\frac{\partial}{\partial t}\left[|w|^2\frac{\partial \phi}{\partial t}\varphi_\lambda\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right)\right] + \frac{\lambda}{2}|w|^2\frac{\partial^2 \phi}{\partial t^2}\varphi_\lambda\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right)$$

$$+ \frac{\lambda^2}{2}|w|^2\varphi_\lambda\left|\frac{\partial \phi}{\partial t}\right|^2\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right) + \lambda|w|^2\left|\frac{\partial \phi}{\partial t}\right|^2\varphi_\lambda\frac{\partial^2 \phi}{\partial t^2}$$

e o quarto termo de (2.9) pode ser reescrito como segue

$$-\frac{\partial |w|^2}{\partial t}\varphi_\lambda\frac{\partial \phi}{\partial t}\frac{\partial^2 \phi}{\partial t^2} = -\frac{\partial}{\partial t}\left[|w|^2\varphi_\lambda\frac{\partial \phi}{\partial t}\frac{\partial^2 \phi}{\partial t^2}\right] + |w|^2\frac{\partial}{\partial t}\left(\varphi_\lambda\frac{\partial \phi}{\partial t}\frac{\partial^2 \phi}{\partial t^2}\right)$$

$$= -\frac{\partial}{\partial t}\left[|w|^2\varphi_\lambda\frac{\partial \phi}{\partial t}\frac{\partial^2 \phi}{\partial t^2}\right] + |w|^2\left[\frac{\partial \varphi_\lambda}{\partial t}\frac{\partial \phi}{\partial t}\frac{\partial^2 \phi}{\partial t^2} + \varphi_\lambda\frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial t}\frac{\partial^2 \phi}{\partial t^2}\right)\right]$$

$$= -\frac{\partial}{\partial t}\left[|w|^2\varphi_\lambda\frac{\partial \phi}{\partial t}\frac{\partial^2 \phi}{\partial t^2}\right] + \lambda|w|^2\varphi_\lambda\left|\frac{\partial \phi}{\partial t}\right|^2\frac{\partial^2 \phi}{\partial t^2} + |w|^2\varphi_\lambda\left(\frac{\partial^2 \phi}{\partial t^2}\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial \phi}{\partial t}\frac{\partial^3 \phi}{\partial t^3}\right)$$

$$= -\frac{\partial}{\partial t}\left[|w|^2\varphi_\lambda\frac{\partial \phi}{\partial t}\frac{\partial^2 \phi}{\partial t^2}\right] + \lambda|w|^2\varphi_\lambda\left|\frac{\partial \phi}{\partial t}\right|^2\frac{\partial^2 \phi}{\partial t^2} + |w|^2\varphi_\lambda\left|\frac{\partial^2 \phi}{\partial t^2}\right|^2.$$

Fazendo as devidas substituições, encontramos que

$$\frac{\partial^2 w}{\partial t^2}\varphi_\lambda w\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right)$$

$$= \frac{\partial}{\partial t}\left[\frac{\partial w}{\partial t}\varphi_\lambda w\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right)\right] - \frac{\lambda}{2}\frac{\partial}{\partial t}\left[|w|^2\frac{\partial \phi}{\partial t}\varphi_\lambda\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right)\right]$$

$$+\frac{\lambda}{2}|w|^2\frac{\partial^2\phi}{\partial t^2}\varphi_\lambda\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)+\frac{\lambda^2}{2}|w|^2\varphi_\lambda\left|\frac{\partial\phi}{\partial t}\right|^2\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)$$

$$+\lambda|w|^2\left|\frac{\partial\phi}{\partial t}\right|^2\varphi_\lambda\frac{\partial^2\phi}{\partial t^2}-\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)-\frac{\partial}{\partial t}\left[|w|^2\varphi_\lambda\frac{\partial\phi}{\partial t}\frac{\partial^2\phi}{\partial t^2}\right]$$

$$+\lambda|w|^2\varphi_\lambda\left|\frac{\partial\phi}{\partial t}\right|^2\frac{\partial^2\phi}{\partial t^2}+|w|^2\varphi_\lambda\left|\frac{\partial^2\phi}{\partial t^2}\right|^2$$

$$=\frac{\partial}{\partial t}\left[\frac{\partial w}{\partial t}\varphi_\lambda w\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)\right]-\frac{\lambda}{2}\frac{\partial}{\partial t}\left[|w|^2\frac{\partial\phi}{\partial t}\varphi_\lambda\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)\right]$$

$$-\frac{\partial}{\partial t}\left[|w|^2\varphi_\lambda\frac{\partial\phi}{\partial t}\frac{\partial^2\phi}{\partial t^2}\right]-\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)+|w|^2\varphi_\lambda\left|\frac{\partial^2\phi}{\partial t^2}\right|^2$$

$$+\lambda|w|^2\frac{\partial^2\phi}{\partial t^2}\varphi_\lambda\left(\frac{1}{2}\left|\frac{\partial\phi}{\partial t}\right|^2-\frac{1}{2}|\nabla\phi|^2+2\left|\frac{\partial\phi}{\partial t}\right|^2\right)+\frac{\lambda^2}{2}|w|^2\varphi_\lambda\left|\frac{\partial\phi}{\partial t}\right|^2\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)$$

$$=\frac{\partial}{\partial t}\left[\frac{\partial w}{\partial t}\varphi_\lambda w\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)\right]-\frac{\lambda}{2}\frac{\partial}{\partial t}\left[|w|^2\frac{\partial\phi}{\partial t}\varphi_\lambda\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)\right]$$

$$-\frac{\partial}{\partial t}\left[|w|^2\varphi_\lambda\frac{\partial\phi}{\partial t}\frac{\partial^2\phi}{\partial t^2}\right]-\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)+|w|^2\varphi_\lambda\left|\frac{\partial^2\phi}{\partial t^2}\right|^2$$

$$+\frac{5}{2}\lambda|w|^2\varphi_\lambda\left|\frac{\partial\phi}{\partial t}\right|^2\frac{\partial^2\phi}{\partial t^2}-\frac{\lambda}{2}|w|^2\frac{\partial^2\phi}{\partial t^2}\varphi_\lambda|\nabla\phi|^2+\frac{\lambda^2}{2}|w|^2\varphi_\lambda\left|\frac{\partial\phi}{\partial t}\right|^2\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right).$$

Utilizando este último resultado, chegamos que:

$$\mathbf{I_{12}}=s\lambda^2\int_{-T}^{T}\int_\Omega\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)dxdt-s\lambda^2\int_{-T}^{T}\int_\Omega|w|^2\varphi_\lambda\left|\frac{\partial^2\phi}{\partial t^2}\right|^2dxdt$$

$$-\frac{5}{2}s\lambda^3\int_{-T}^{T}\int_\Omega|w|^2\varphi_\lambda\left|\frac{\partial\phi}{\partial t}\right|^2\frac{\partial^2\phi}{\partial t^2}dxdt+\frac{s\lambda^3}{2}\int_{-T}^{T}\int_\Omega|w|^2\varphi_\lambda|\nabla\phi|^2\frac{\partial^2\phi}{\partial t^2}dxdt$$

$$-\frac{s\lambda^4}{2}\int_{-T}^{T}\int_\Omega|w|^2\varphi_\lambda\left|\frac{\partial\phi}{\partial t}\right|^2\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)dxdt.$$

Avaliemos a integral

- $\mathbf{I_{13}}\;=-2s\lambda\displaystyle\int_{-T}^{T}\int_\Omega\frac{\partial^2 w}{\partial t^2}\varphi_\lambda\left(\frac{\partial w}{\partial t}\frac{\partial\phi}{\partial t}-\nabla w\nabla\phi\right)dxdt.$

Uma outra expressão para a função que está sendo integrada é

$$
\begin{aligned}
&\frac{\partial^2 w}{\partial t^2}\varphi_\lambda\left(\frac{\partial w}{\partial t}\frac{\partial \phi}{\partial t} - \nabla w \nabla \phi\right) \\
&= \frac{\partial}{\partial t}\left[\frac{\partial w}{\partial t}\varphi_\lambda\left(\frac{\partial w}{\partial t}\frac{\partial \phi}{\partial t} - \nabla w \nabla \phi\right)\right] - \frac{\partial w}{\partial t}\frac{\partial}{\partial t}\left[\varphi_\lambda\left(\frac{\partial w}{\partial t}\frac{\partial \phi}{\partial t} - \nabla w \nabla \phi\right)\right] \\
&= \frac{\partial}{\partial t}\left[\frac{\partial w}{\partial t}\varphi_\lambda\left(\frac{\partial w}{\partial t}\frac{\partial \phi}{\partial t} - \nabla w \nabla \phi\right)\right] - \frac{\partial w}{\partial t}\left[\frac{\partial \varphi_\lambda}{\partial t}\left(\frac{\partial w}{\partial t}\frac{\partial \phi}{\partial t} - \nabla w \nabla \phi\right)\right. \\
&\quad \left. + \varphi_\lambda \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial t}\frac{\partial \phi}{\partial t} - \nabla w \nabla \phi\right)\right] \\
&= \frac{\partial}{\partial t}\left[\frac{\partial w}{\partial t}\varphi_\lambda\left(\frac{\partial w}{\partial t}\frac{\partial \phi}{\partial t} - \nabla w \nabla \phi\right)\right] - \lambda\varphi_\lambda\left|\frac{\partial w}{\partial t}\right|^2\left|\frac{\partial \phi}{\partial t}\right|^2 + \lambda\frac{\partial w}{\partial t}\varphi_\lambda\frac{\partial \phi}{\partial t}\nabla w \nabla \phi \\
&\quad - \frac{\partial w}{\partial t}\varphi_\lambda\left(\frac{\partial^2 w}{\partial t^2}\frac{\partial \phi}{\partial t} + \frac{\partial w}{\partial t}\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial}{\partial t}(\nabla w)\nabla\phi - \nabla w \frac{\partial}{\partial t}(\nabla \phi)\right) \\
&= \frac{\partial}{\partial t}\left[\frac{\partial w}{\partial t}\varphi_\lambda\left(\frac{\partial w}{\partial t}\frac{\partial \phi}{\partial t} - \nabla w \nabla \phi\right)\right] - \lambda\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\left|\frac{\partial \phi}{\partial t}\right|^2 + \lambda\frac{\partial w}{\partial t}\varphi_\lambda\frac{\partial \phi}{\partial t}\nabla w \nabla \phi \\
&\quad - \frac{1}{2}\frac{\partial}{\partial t}\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\frac{\partial \phi}{\partial t} - \left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial w}{\partial t}\varphi_\lambda\frac{\partial}{\partial t}(\nabla w)\nabla\phi. \qquad (2.10)
\end{aligned}
$$

Observemos que

$$
\begin{aligned}
&-\frac{1}{2}\frac{\partial}{\partial t}\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\frac{\partial \phi}{\partial t} \\
&= -\frac{1}{2}\frac{\partial}{\partial t}\left[\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\frac{\partial \phi}{\partial t}\right] + \frac{1}{2}\left|\frac{\partial w}{\partial t}\right|^2\frac{\partial}{\partial t}\left(\varphi_\lambda\frac{\partial \phi}{\partial t}\right) \\
&= -\frac{1}{2}\frac{\partial}{\partial t}\left[\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\frac{\partial \phi}{\partial t}\right] + \frac{1}{2}\left|\frac{\partial w}{\partial t}\right|^2\left(\frac{\partial \varphi_\lambda}{\partial t}\frac{\partial \phi}{\partial t} + \varphi_\lambda\frac{\partial^2 \phi}{\partial t^2}\right) \\
&= -\frac{1}{2}\frac{\partial}{\partial t}\left[\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\frac{\partial \phi}{\partial t}\right] + \frac{\lambda}{2}\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\left|\frac{\partial \phi}{\partial t}\right|^2 + \frac{1}{2}\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\frac{\partial^2 \phi}{\partial t^2} \qquad (2.11)
\end{aligned}
$$

e que

$$\frac{\partial w}{\partial t}\varphi_\lambda\frac{\partial}{\partial t}(\nabla w)\nabla\phi=\frac{1}{2}\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\frac{\partial\phi}{\partial x_i}$$

$$=\frac{1}{2}\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left[\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\frac{\partial\phi}{\partial x_i}\right]-\frac{1}{2}\sum_{i=1}^{n}\left|\frac{\partial w}{\partial t}\right|^2\frac{\partial}{\partial x_i}\left(\varphi_\lambda\frac{\partial\phi}{\partial x_i}\right)$$

$$=\frac{1}{2}\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left[\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\frac{\partial\phi}{\partial x_i}\right]-\frac{1}{2}\sum_{i=1}^{n}\left|\frac{\partial w}{\partial t}\right|^2\left(\frac{\partial\varphi_\lambda}{\partial x_i}\frac{\partial\phi}{\partial x_i}+\varphi_\lambda\frac{\partial^2\phi}{\partial {x_i}^2}\right)$$

$$=\frac{1}{2}\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left[\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\frac{\partial\phi}{\partial x_i}\right]-\frac{1}{2}\lambda\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\sum_{i=1}^{n}\left|\frac{\partial\phi}{\partial x_i}\right|^2-\frac{1}{2}\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\sum_{i=1}^{n}\frac{\partial^2\phi}{\partial {x_i}^2}$$

$$=\frac{1}{2}\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left[\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\frac{\partial\phi}{\partial x_i}\right]-\frac{1}{2}\lambda\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda|\nabla\phi|^2-\frac{1}{2}\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\Delta\phi$$

$$=\frac{1}{2}\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left[\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\frac{\partial\phi}{\partial x_i}\right]-\frac{1}{2}\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda(\Delta\phi+\lambda|\nabla\phi|^2).\tag{2.12}$$

Além disso,

$$\int_{-T}^{T}\int_\Omega\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\frac{\partial\phi}{\partial x_i}\right)dxdt=\sum_{i=1}^{n}\int_{-T}^{T}\int_\Gamma\left(\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\frac{\partial\phi}{\partial x_i}\right)\nu_i dxdt=0$$

pois $v=0$ sobre Σ.

Unindo esta informação a (2.10), (2.11) e (2.12), deduzimos que

$$\mathbf{I_{13}}=-2s\lambda\int_{-T}^{T}\int_\Omega\frac{\partial^2 w}{\partial t^2}\varphi_\lambda\left(\frac{\partial w}{\partial t}\frac{\partial\phi}{\partial t}-\nabla w\nabla\phi\right)dxdt$$

$$=-2s\lambda\int_\Omega\int_{-T}^{T}\frac{\partial}{\partial t}\left[\frac{\partial w}{\partial t}\varphi_\lambda\left(\frac{\partial w}{\partial t}\frac{\partial\phi}{\partial t}-\nabla w\nabla\phi\right)\right]dtdx$$

$$+s\lambda\int_\Omega\int_{-T}^{T}\frac{\partial}{\partial t}\left[\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\frac{\partial\phi}{\partial t}\right]dtdx-s\lambda\sum_{i=1}^{n}\int_{-T}^{T}\int_\Omega\frac{\partial}{\partial x_i}\left[\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\frac{\partial\phi}{\partial x_i}\right]dxdt$$

$$+s\lambda\int_{-T}^{T}\int_\Omega\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\frac{\partial^2\phi}{\partial t^2}dxdt+s\lambda^2\int_{-T}^{T}\int_\Omega\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\left|\frac{\partial\phi}{\partial t}\right|^2dxdt$$

$$+s\lambda\int_{-T}^{T}\int_\Omega\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda(\Delta\phi+\lambda|\nabla\phi|^2)dxdt-2s\lambda^2\int_{-T}^{T}\int_\Omega\frac{\partial w}{\partial t}\varphi_\lambda\frac{\partial\phi}{\partial t}\nabla w\nabla\phi dxdt$$

e, assim,

$$\mathbf{I_{13}} = s\lambda\int_{-T}^{T}\int_{\Omega}\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\frac{\partial^2\phi}{\partial t^2}dxdt + s\lambda^2\int_{-T}^{T}\int_{\Omega}\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\left|\frac{\partial\phi}{\partial t}\right|^2dxdt$$

$$+s\lambda\int_{-T}^{T}\int_{\Omega}\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda(\Delta\phi+\lambda|\nabla\phi|^2)dxdt - 2s\lambda^2\int_{-T}^{T}\int_{\Omega}\frac{\partial w}{\partial t}\varphi_\lambda\frac{\partial\phi}{\partial t}\nabla w\nabla\phi dxdt.$$

Façamos os cálculos para

- $\mathbf{I_{21}} \; = -(M_1-1)s\lambda\int_{-T}^{T}\int_{\Omega}\Delta w\varphi_\lambda w\left(\frac{\partial^2\phi}{\partial t^2}-\Delta\phi\right)dxdt.$

Inicialmente escrevamos a função da seguinte forma

$$\Delta w\varphi_\lambda w\left(\frac{\partial^2\phi}{\partial t^2}-\Delta\phi\right) = \sum_{i=1}^{N}\frac{\partial^2 w}{\partial {x_i}^2}\varphi_\lambda w\left(\frac{\partial^2\phi}{\partial t^2}-\sum_{j=1}^{N}\frac{\partial^2\phi}{\partial {x_j}^2}\right)$$

$$= \sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\frac{\partial w}{\partial x_i}\varphi_\lambda w\left(\frac{\partial^2\phi}{\partial t^2}-\sum_{j=1}^{N}\frac{\partial^2\phi}{\partial {x_j}^2}\right)\right) - \sum_{i=1}^{N}\frac{\partial w}{\partial x_i}\frac{\partial}{\partial x_i}\left[\varphi_\lambda w\left(\frac{\partial^2\phi}{\partial t^2}-\sum_{j=1}^{N}\frac{\partial^2\phi}{\partial {x_j}^2}\right)\right]$$

$$= \sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\frac{\partial w}{\partial x_i}\varphi_\lambda w\left(\frac{\partial^2\phi}{\partial t^2}-\sum_{j=1}^{N}\frac{\partial^2\phi}{\partial {x_j}^2}\right)\right) - \sum_{i=1}^{N}\frac{\partial w}{\partial x_i}\left\{\frac{\partial\varphi_\lambda}{\partial x_i}w\left(\frac{\partial^2\phi}{\partial t^2}-\sum_{j=1}^{N}\frac{\partial^2\phi}{\partial {x_j}^2}\right)\right.$$

$$\left.+\varphi_\lambda\frac{\partial}{\partial x_i}\left[w\left(\frac{\partial^2\phi}{\partial t^2}-\sum_{j=1}^{N}\frac{\partial^2\phi}{\partial {x_j}^2}\right)\right]\right\}$$

$$= \sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\frac{\partial w}{\partial x_i}\varphi_\lambda w\left(\frac{\partial^2\phi}{\partial t^2}-\sum_{j=1}^{N}\frac{\partial^2\phi}{\partial {x_j}^2}\right)\right) - \sum_{i=1}^{N}\lambda\varphi_\lambda\frac{\partial w}{\partial x_i}\frac{\partial\phi}{\partial x_i}w\left(\frac{\partial^2\phi}{\partial t^2}-\sum_{j=1}^{N}\frac{\partial^2\phi}{\partial {x_j}^2}\right)$$

$$-\sum_{i=1}^{N}\frac{\partial w}{\partial x_i}\varphi_\lambda\left[\frac{\partial w}{\partial x_i}\left(\frac{\partial^2\phi}{\partial t^2}-\sum_{j=1}^{N}\frac{\partial^2\phi}{\partial {x_j}^2}\right)+w\frac{\partial}{\partial x_i}\left(\frac{\partial^2\phi}{\partial t^2}-\sum_{j=1}^{N}\frac{\partial^2\phi}{\partial {x_j}^2}\right)\right]$$

$$= \sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\frac{\partial w}{\partial x_i}\varphi_\lambda w\left(\frac{\partial^2\phi}{\partial t^2}-\sum_{j=1}^{N}\frac{\partial^2\phi}{\partial {x_j}^2}\right)\right) - \lambda\varphi_\lambda w\nabla w\nabla\phi\left(\frac{\partial^2\phi}{\partial t^2}-\Delta\phi\right)$$

$$-\varphi_\lambda|\nabla w|^2\left(\frac{\partial^2\phi}{\partial t^2}-\Delta\phi\right).$$

Reformulemos o segundo termo do lado direito

$$
\begin{aligned}
&-\lambda w\nabla w\nabla\phi\varphi_\lambda\left(\frac{\partial^2\phi}{\partial t^2}-\Delta\phi\right)=-\frac{\lambda}{2}\sum_{i=1}^N\frac{\partial|w|^2}{\partial x_i}\varphi_\lambda\frac{\partial\phi}{\partial x_i}\left(\frac{\partial^2\phi}{\partial t^2}-\sum_{j=1}^N\frac{\partial^2\phi}{\partial {x_j}^2}\right)\\
&=-\frac{\lambda}{2}\sum_{i=1}^N\frac{\partial}{\partial x_i}\left(|w|^2\varphi_\lambda\frac{\partial\phi}{\partial x_i}\left(\frac{\partial^2\phi}{\partial t^2}-\sum_{j=1}^N\frac{\partial^2\phi}{\partial {x_j}^2}\right)\right)\\
&\quad+\frac{\lambda}{2}\sum_{i=1}^N|w|^2\frac{\partial}{\partial x_i}\left[\varphi_\lambda\frac{\partial\phi}{\partial x_i}\left(\frac{\partial^2\phi}{\partial t^2}-\sum_{j=1}^N\frac{\partial^2\phi}{\partial {x_j}^2}\right)\right]\\
&=-\frac{\lambda}{2}\sum_{i=1}^N\frac{\partial}{\partial x_i}\left(|w|^2\varphi_\lambda\frac{\partial\phi}{\partial x_i}\left(\frac{\partial^2\phi}{\partial t^2}-\Delta\phi\right)\right)\\
&\quad+\frac{\lambda}{2}\sum_{i=1}^N|w|^2\left\{\frac{\partial\varphi_\lambda}{\partial x_i}\frac{\partial\phi}{\partial x_i}\left(\frac{\partial^2\phi}{\partial t^2}-\sum_{j=1}^N\frac{\partial^2\phi}{\partial {x_j}^2}\right)+\varphi_\lambda\frac{\partial}{\partial x_i}\left[\frac{\partial\phi}{\partial x_i}\left(\frac{\partial^2\phi}{\partial t^2}-\sum_{j=1}^N\frac{\partial^2\phi}{\partial {x_j}^2}\right)\right]\right\}\\
&=-\frac{\lambda}{2}\sum_{i=1}^N\frac{\partial}{\partial x_i}\left(|w|^2\varphi_\lambda\frac{\partial\phi}{\partial x_i}\left(\frac{\partial^2\phi}{\partial t^2}-\Delta\phi\right)\right)\\
&\quad+\frac{\lambda^2}{2}\sum_{i=1}^N|w|^2\varphi_\lambda\left(\frac{\partial\phi}{\partial x_i}\right)^2\left(\frac{\partial^2\phi}{\partial t^2}-\sum_{j=1}^N\frac{\partial^2\phi}{\partial {x_j}^2}\right)\\
&\quad+\frac{\lambda}{2}\sum_{i=1}^N|w|^2\varphi_\lambda\left[\frac{\partial^2\phi}{\partial {x_i}^2}\left(\frac{\partial^2\phi}{\partial t^2}-\frac{\lambda}{2}\sum_{j=1}^N\frac{\partial^2\phi}{\partial {x_j}^2}\right)+\frac{\partial\phi}{\partial x_i}\frac{\partial}{\partial x_i}\left(\frac{\partial^2\phi}{\partial t^2}-\frac{\lambda}{2}\sum_{j=1}^N\frac{\partial^2\phi}{\partial {x_j}^2}\right)\right]\\
&=-\frac{\lambda}{2}\sum_{i=1}^N\frac{\partial}{\partial x_i}\left(|w|^2\varphi_\lambda\frac{\partial\phi}{\partial x_i}\left(\frac{\partial^2\phi}{\partial t^2}-\Delta\phi\right)\right)+\frac{\lambda^2}{2}|w|^2\varphi_\lambda|\nabla\phi|^2\left(\frac{\partial^2\phi}{\partial t^2}-\Delta\phi\right)\\
&\quad+\frac{\lambda}{2}|w|^2\varphi_\lambda\Delta\phi\left(\frac{\partial^2\phi}{\partial t^2}-\Delta\phi\right)\\
&=-\frac{\lambda}{2}\sum_{i=1}^N\frac{\partial}{\partial x_i}\left(|w|^2\varphi_\lambda\frac{\partial\phi}{\partial x_i}\left(\frac{\partial^2\phi}{\partial t^2}-\Delta\phi\right)\right)+\frac{\lambda}{2}|w|^2\varphi_\lambda\left(\frac{\partial^2\phi}{\partial t^2}-\Delta\phi\right)(\Delta\phi+\lambda|\nabla\phi|^2).
\end{aligned}
$$

Temos ainda que

$$
\begin{aligned}
&\int_{-T}^{T}\int_\Omega\sum_{i=1}^n\frac{\partial}{\partial x_i}\left(\frac{\partial w}{\partial x_i}\varphi_\lambda w\left(\frac{\partial^2\phi}{\partial t^2}-\Delta\phi\right)\right)dxdt\\
&=\sum_{i=1}^n\int_{-T}^{T}\int_\Gamma\left(\frac{\partial w}{\partial x_i}\varphi_\lambda w\left(\frac{\partial^2\phi}{\partial t^2}-\Delta\phi\right)\right)\nu_i dxdt=0
\end{aligned}
$$

e

$$\int_{-T}^{T}\int_{\Omega}\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(|w|^2\varphi_\lambda\frac{\partial\phi}{\partial x_i}\left(\frac{\partial^2\phi}{\partial t^2}-\sum_{j=1}^{N}\frac{\partial^2\phi}{\partial {x_j}^2}\right)\right)dxdt$$
$$=\sum_{i=1}^{N}\int_{-T}^{T}\int_{\Gamma}\left(|w|^2\varphi_\lambda\frac{\partial\phi}{\partial x_i}\left(\frac{\partial^2\phi}{\partial t^2}-\sum_{i=j}^{N}\frac{\partial^2\phi}{\partial {x_j}^2}\right)\right)\nu_i dxdt=0.$$

Com base nos cálculos acima chegamos a

$$\begin{aligned}\mathbf{I_{21}} &= -(M_1-1)s\lambda\int_{-T}^{T}\int_{\Omega}\Delta w\varphi_\lambda w\left(\frac{\partial^2\phi}{\partial t^2}-\Delta\phi\right)dxdt\\
&= -(M_1-1)s\lambda\int_{-T}^{T}\int_{\Omega}\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\frac{\partial w}{\partial x_i}\varphi_\lambda w\left(\frac{\partial^2\phi}{\partial t^2}-\sum_{j=1}^{N}\frac{\partial^2\phi}{\partial {x_j}^2}\right)\right)dxdt\\
&\quad+\frac{(M_1-1)}{2}s\lambda^2\int_{-T}^{T}\int_{\Omega}\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(|w|^2\varphi_\lambda\frac{\partial\phi}{\partial x_i}\left(\frac{\partial^2\phi}{\partial t^2}-\sum_{j=1}^{N}\frac{\partial^2\phi}{\partial {x_j}^2}\right)\right)dxdt\\
&\quad-\frac{(M_1-1)}{2}s\lambda^2\int_{-T}^{T}\int_{\Omega}|w|^2\varphi_\lambda\left(\frac{\partial^2\phi}{\partial t^2}-\Delta\phi\right)(\Delta\phi+\lambda|\nabla\phi|^2)dxdt\\
&\quad+(M_1-1)s\lambda\int_{-T}^{T}\int_{\Omega}|\nabla w|^2\varphi_\lambda\left(\frac{\partial^2\phi}{\partial t^2}+\Delta\phi\right)dxdt,\end{aligned}$$

e concluímos que

$$\begin{aligned}\mathbf{I_{21}} &= -(1-M_1)s\lambda\int_{-T}^{T}\int_{\Omega}|\nabla w|^2\varphi_\lambda\left(\frac{\partial^2\phi}{\partial t^2}+\Delta\phi\right)dxdt\\
&\quad+\frac{(1-M_1)}{2}s\lambda^2\int_{-T}^{T}\int_{\Omega}|w|^2\varphi_\lambda\left(\frac{\partial^2\phi}{\partial t^2}-\Delta\phi\right)(\Delta\phi+\lambda|\nabla\phi|^2)dxdt.\end{aligned}$$

Agora obteremos dados sobre a integral

- $\mathbf{I_{22}} = s\lambda^2\int_{-T}^{T}\int_{\Omega}\Delta w\varphi_\lambda w\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)dxdt.$

Para isso, tomaremos a função da seguinte forma

$$\Delta w \varphi_\lambda w \left(\left| \frac{\partial \phi}{\partial t} \right|^2 - |\nabla \phi|^2 \right) = \sum_{i=1}^{N} \frac{\partial^2 w}{\partial {x_i}^2} \varphi_\lambda w \left(\left| \frac{\partial \phi}{\partial t} \right|^2 - \sum_{j=1}^{N} \left(\frac{\partial \phi}{\partial x_j} \right)^2 \right)$$

$$= \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\frac{\partial w}{\partial x_i} \varphi_\lambda w \left(\left| \frac{\partial \phi}{\partial t} \right|^2 - |\nabla \phi|^2 \right) \right) - \sum_{i=1}^{N} \frac{\partial w}{\partial x_i} \frac{\partial}{\partial x_i} \left[\varphi_\lambda w \left(\left| \frac{\partial \phi}{\partial t} \right|^2 - \sum_{j=1}^{N} \left(\frac{\partial \phi}{\partial x_j} \right)^2 \right) \right]$$

$$= \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\frac{\partial w}{\partial x_i} \varphi_\lambda w \left(\left| \frac{\partial \phi}{\partial t} \right|^2 - |\nabla \phi|^2 \right) \right) - \sum_{i=1}^{N} \frac{\partial w}{\partial x_i} \left\{ \frac{\partial \varphi_\lambda}{\partial x_i} w \left(\left| \frac{\partial \phi}{\partial t} \right|^2 - \sum_{j=1}^{N} \left(\frac{\partial \phi}{\partial x_j} \right)^2 \right) \right.$$

$$\left. + \varphi_\lambda \frac{\partial}{\partial x_i} \left[w \left(\left| \frac{\partial \phi}{\partial t} \right|^2 - \sum_{j=1}^{N} \left(\frac{\partial \phi}{\partial x_j} \right)^2 \right) \right] \right\}$$

$$= \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\frac{\partial w}{\partial x_i} \varphi_\lambda w \left(\left| \frac{\partial \phi}{\partial t} \right|^2 - |\nabla \phi|^2 \right) \right) - \sum_{i=1}^{N} \lambda \varphi_\lambda \frac{\partial w}{\partial x_i} \frac{\partial \phi}{\partial x_i} w \left(\left| \frac{\partial \phi}{\partial t} \right|^2 - |\nabla \phi|^2 \right)$$

$$- \sum_{i=1}^{N} \frac{\partial w}{\partial x_i} \varphi_\lambda \left[\frac{\partial w}{\partial x_i} \left(\left| \frac{\partial \phi}{\partial t} \right|^2 - \sum_{j=1}^{N} \left(\frac{\partial \phi}{\partial x_j} \right)^2 \right) + w \frac{\partial}{\partial x_i} \left(\left| \frac{\partial \phi}{\partial t} \right|^2 - \sum_{j=1}^{N} \left(\frac{\partial \phi}{\partial x_j} \right)^2 \right) \right]$$

$$= \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\frac{\partial w}{\partial x_i} \varphi_\lambda w \left(\left| \frac{\partial \phi}{\partial t} \right|^2 - |\nabla \phi|^2 \right) \right) - \lambda \varphi_\lambda \nabla w \nabla \phi w \left(\left| \frac{\partial \phi}{\partial t} \right|^2 - |\nabla \phi|^2 \right)$$

$$- \varphi_\lambda |\nabla w|^2 \left(\left| \frac{\partial \phi}{\partial t} \right|^2 - |\nabla \phi|^2 \right) - \sum_{i=1}^{N} w \varphi_\lambda \frac{\partial w}{\partial x_i} \left(2 \frac{\partial \phi}{\partial t} \frac{\partial^2 \phi}{\partial x_i \partial t} - \sum_{j=1}^{N} 2 \frac{\partial \phi}{\partial x_j} \frac{\partial^2 \phi}{\partial x_i \partial x_j} \right)$$

$$= \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\frac{\partial w}{\partial x_i} \varphi_\lambda w \left(\left| \frac{\partial \phi}{\partial t} \right|^2 - |\nabla \phi|^2 \right) \right) - \lambda w \nabla w \nabla \phi \varphi_\lambda \left(\left| \frac{\partial \phi}{\partial t} \right|^2 - |\nabla \phi|^2 \right)$$

$$- |\nabla w|^2 \varphi_\lambda \left(\left| \frac{\partial \phi}{\partial t} \right|^2 - |\nabla \phi|^2 \right) + 2 w \varphi_\lambda \sum_{i=1}^{N} \frac{\partial^2 \phi}{\partial {x_i}^2} \frac{\partial \phi}{\partial x_i} \frac{\partial w}{\partial x_i}. \tag{2.13}$$

Analisemos o segundo termo dessa igualdade.

$$-\lambda w \nabla w \nabla \phi \varphi_\lambda \left(\left| \frac{\partial \phi}{\partial t} \right|^2 - |\nabla \phi|^2 \right) = -\frac{\lambda}{2} \sum_{i=1}^{N} \frac{\partial |w|^2}{\partial x_i} \frac{\partial \phi}{\partial x_i} \varphi_\lambda \left(\left| \frac{\partial \phi}{\partial t} \right|^2 - \sum_{j=1}^{N} \left(\frac{\partial \phi}{\partial x_j} \right)^2 \right)$$

$$= -\frac{\lambda}{2} \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left[|w|^2 \frac{\partial \phi}{\partial x_i} \varphi_\lambda \left(\left| \frac{\partial \phi}{\partial t} \right|^2 - |\nabla \phi|^2 \right) \right]$$

$$+ \frac{\lambda}{2} \sum_{i=1}^{N} |w|^2 \frac{\partial}{\partial x_i} \left[\frac{\partial \phi}{\partial x_i} \varphi_\lambda \left(\left| \frac{\partial \phi}{\partial t} \right|^2 - \sum_{j=1}^{N} \left(\frac{\partial \phi}{\partial x_j} \right)^2 \right) \right]$$

$$= -\frac{\lambda}{2} \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left[|w|^2 \frac{\partial \phi}{\partial x_i} \varphi_\lambda \left(\left| \frac{\partial \phi}{\partial t} \right|^2 - |\nabla \phi|^2 \right) \right] + \frac{\lambda}{2} \sum_{i=1}^{N} |w|^2 \left\{ \frac{\partial^2 \phi}{\partial {x_i}^2} \varphi_\lambda \left(\left| \frac{\partial \phi}{\partial t} \right|^2 - |\nabla \phi|^2 \right) \right.$$

$$+\frac{\partial\phi}{\partial x_i}\frac{\partial}{\partial x_i}\left[\varphi_\lambda\left(\left|\frac{\partial\phi}{\partial t}\right|^2-\sum_{j=1}^N\left(\frac{\partial\phi}{\partial x_j}\right)^2\right)\right]\Bigg\}$$

$$=-\frac{\lambda}{2}\sum_{i=1}^N\frac{\partial}{\partial x_i}\left[|w|^2\frac{\partial\phi}{\partial x_i}\varphi_\lambda\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)\right]+\frac{\lambda}{2}|w|^2\varphi_\lambda\Delta\phi\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)$$

$$+\frac{\lambda}{2}\sum_{i=1}^N|w|^2\frac{\partial\phi}{\partial x_i}\left[\frac{\partial\varphi_\lambda}{\partial x_i}\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)+\varphi_\lambda\frac{\partial}{\partial x_i}\left(\left|\frac{\partial\phi}{\partial t}\right|^2-\sum_{j=1}^N\left(\frac{\partial\phi}{\partial x_j}\right)^2\right)\right]$$

$$=-\frac{\lambda}{2}\sum_{i=1}^N\frac{\partial}{\partial x_i}\left[|w|^2\frac{\partial\phi}{\partial x_i}\varphi_\lambda\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)\right]+\frac{\lambda}{2}|w|^2\varphi_\lambda\Delta\phi\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)$$

$$+\frac{\lambda^2}{2}\sum_{i=1}^N|w|^2\varphi_\lambda\left|\frac{\partial\phi}{\partial x_i}\right|^2\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)$$

$$+\lambda\sum_{i=1}^N|w|^2\varphi_\lambda\frac{\partial\phi}{\partial x_i}\left(\frac{\partial\phi}{\partial t}\frac{\partial^2\phi}{\partial x_i\partial t}-\sum_{j=1}^N\frac{\partial\phi}{\partial x_j}\frac{\partial^2\phi}{\partial x_i\partial x_j}\right)$$

$$=-\frac{\lambda}{2}\sum_{i=1}^N\frac{\partial}{\partial x_i}\left[|w|^2\frac{\partial\phi}{\partial x_i}\varphi_\lambda\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)\right]+\frac{\lambda}{2}|w|^2\varphi_\lambda\Delta\phi\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)$$

$$+\frac{\lambda^2}{2}|w|^2\varphi_\lambda|\nabla\phi|^2\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)-\lambda|w|^2\varphi_\lambda D^2\phi(\nabla\phi,\nabla\phi),$$

onde $D^2\phi(\nabla\phi,\nabla\phi)$ denota o somátorio em i e j do produto $\frac{\partial^2\phi}{\partial x_i\partial x_j}\frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_j}$.

Trabalhemos com o quarto termo de (2.13).

$$2w\varphi_\lambda\sum_{i=1}^N\frac{\partial^2\phi}{\partial {x_i}^2}\frac{\partial\phi}{\partial x_i}\frac{\partial w}{\partial x_i}=$$

$$=\sum_{i=1}^N\frac{\partial}{\partial x_i}\left(|w|^2\varphi_\lambda\frac{\partial^2\phi}{\partial {x_i}^2}\frac{\partial\phi}{\partial x_i}\right)-\sum_{i=1}^N|w|^2\frac{\partial}{\partial x_i}\left(\varphi_\lambda\frac{\partial\phi}{\partial x_i}\frac{\partial^2\phi}{\partial x_i^2}\right)$$

$$=\sum_{i=1}^N\frac{\partial}{\partial x_i}\left(|w|^2\varphi_\lambda\frac{\partial^2\phi}{\partial {x_i}^2}\frac{\partial\phi}{\partial x_i}\right)-\sum_{i=1}^N|w|^2\left(\frac{\partial\varphi_\lambda}{\partial x_i}\frac{\partial\phi}{\partial x_i}\frac{\partial^2\phi}{\partial x_i^2}+\varphi_\lambda\frac{\partial}{\partial x_i}\left(\frac{\partial\phi}{\partial x_i}\frac{\partial^2\phi}{\partial x_i^2}\right)\right)$$

$$=\sum_{i=1}^N\frac{\partial}{\partial x_i}\left(|w|^2\varphi_\lambda\frac{\partial^2\phi}{\partial {x_i}^2}\frac{\partial\phi}{\partial x_i}\right)-\lambda\sum_{i=1}^N|w|^2\varphi_\lambda\left(\frac{\partial\phi}{\partial x_i}\right)^2\frac{\partial^2\phi}{\partial x_i^2}$$

$$-\sum_{i=1}^N|w|^2\varphi_\lambda\left(\left(\frac{\partial^2\phi}{\partial x_i^2}\right)^2+\frac{\partial\phi}{\partial x_i}\frac{\partial^3\phi}{\partial x_i^3}\right)$$

$$=\sum_{i=1}^N\frac{\partial}{\partial x_i}\left(|w|^2\varphi_\lambda\frac{\partial^2\phi}{\partial {x_i}^2}\frac{\partial\phi}{\partial x_i}\right)-\lambda|w|^2\varphi_\lambda D^2\phi(\nabla\phi,\nabla\phi)-|w|^2\varphi_\lambda\sum_{i=1}^N\left(\frac{\partial^2\phi}{\partial x_i^2}\right)^2.$$

Além disso, observemos que

$$\int_{-T}^{T}\int_{\Omega}\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\frac{\partial w}{\partial x_i}\varphi_\lambda w\left(\left|\frac{\partial \phi}{\partial t}\right|^2-\sum_{j=1}^{N}\left(\frac{\partial \phi}{\partial x_j}\right)^2\right)\right)dxdt$$
$$=\sum_{i=1}^{N}\int_{-T}^{T}\int_{\Gamma}\left(\frac{\partial w}{\partial x_i}\varphi_\lambda w\left(\left|\frac{\partial \phi}{\partial t}\right|^2-\sum_{j=1}^{N}\left(\frac{\partial \phi}{\partial x_j}\right)^2\right)\right)\nu_i d\sigma dt=0.$$

Da mesma forma,

$$\int_{-T}^{T}\int_{\Omega}\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(|w|^2\frac{\partial \phi}{\partial x_i}\varphi_\lambda\left(\left|\frac{\partial \phi}{\partial t}\right|^2-\sum_{j=1}^{N}\left(\frac{\partial \phi}{\partial x_j}\right)^2\right)\right)dxdt$$
$$=\sum_{i=1}^{N}\int_{-T}^{T}\int_{\Gamma}\left(|w|^2\frac{\partial \phi}{\partial x_i}\varphi_\lambda\left(\left|\frac{\partial \phi}{\partial t}\right|^2-\sum_{j=1}^{N}\left(\frac{\partial \phi}{\partial x_j}\right)^2\right)\right)\nu_i d\sigma dt=0$$

e

$$\int_{-T}^{T}\int_{\Omega}\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(|w|^2\varphi_\lambda\frac{\partial^2 \phi}{\partial {x_i}^2}\frac{\partial \phi}{\partial x_i}\right)dxdt=\sum_{i=1}^{N}\int_{-T}^{T}\int_{\Gamma}\left(|w|^2\varphi_\lambda\frac{\partial^2 \phi}{\partial {x_i}^2}\frac{\partial \phi}{\partial x_i}\right)\nu_i d\sigma dt=0.$$

Pelas igualdades que conseguimos podemos afirmar que

$$\begin{aligned}\mathbf{I_{22}}=&-s\lambda^2\int_{-T}^{T}\int_{\Omega}|\nabla w|^2\varphi_\lambda\left(\left|\frac{\partial \phi}{\partial t}\right|^2-|\nabla\phi|^2\right)dxdt\\&-s\lambda^2\int_{-T}^{T}\int_{\Omega}|w|^2\varphi_\lambda\sum_{i=1}^{N}\left(\frac{\partial^2 \phi}{\partial x_i^2}\right)^2dxdt\\&+\frac{s\lambda^3}{2}\int_{-T}^{T}\int_{\Omega}|w|^2\varphi_\lambda\left(\Delta\phi+\lambda|\nabla\phi|^2\right)\left(\left|\frac{\partial \phi}{\partial t}\right|^2-|\nabla\phi|^2\right)dxdt\\&-2s\lambda^3\int_{-T}^{T}\int_{\Omega}|w|^2\varphi_\lambda D^2\phi(\nabla\phi,\nabla\phi)dxdt.\end{aligned}$$

Façamos os cálculos para

- $\mathbf{I_{23}}\ =2s\lambda\int_{-T}^{T}\int_{\Omega}\Delta w\varphi_\lambda\left(\frac{\partial w}{\partial t}\frac{\partial \phi}{\partial t}-\nabla w\nabla\phi\right)dxdt.$

Notemos que o integrando é equivalente a

$$
\begin{aligned}
&\sum_{i=1}^{N} \frac{\partial^2 w}{\partial {x_i}^2}\varphi_\lambda \left(\frac{\partial w}{\partial t}\frac{\partial \phi}{\partial t} - \sum_{j=1}^{N}\frac{\partial w}{\partial x_j}\frac{\partial \phi}{\partial x_j}\right)\\
&= \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\frac{\partial w}{\partial x_i}\varphi_\lambda\left(\frac{\partial w}{\partial t}\frac{\partial \phi}{\partial t} - \sum_{j=1}^{N}\frac{\partial w}{\partial x_j}\frac{\partial \phi}{\partial x_j}\right)\right)\\
&\quad -\sum_{i=1}^{N}\frac{\partial w}{\partial x_i}\frac{\partial}{\partial x_i}\left[\varphi_\lambda\left(\frac{\partial w}{\partial t}\frac{\partial \phi}{\partial t} - \sum_{j=1}^{N}\frac{\partial w}{\partial x_j}\frac{\partial \phi}{\partial x_j}\right)\right]\\
&= \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\frac{\partial w}{\partial x_i}\varphi_\lambda\left(\frac{\partial w}{\partial t}\frac{\partial \phi}{\partial t} - \nabla w \nabla \phi\right)\right) - \sum_{i=1}^{N}\frac{\partial w}{\partial x_i}\left[\frac{\partial \varphi_\lambda}{\partial x_i}\left(\frac{\partial w}{\partial t}\frac{\partial \phi}{\partial t} - \nabla w\nabla\phi\right)\right.\\
&\quad \left.+\varphi_\lambda\frac{\partial}{\partial x_i}\left(\frac{\partial w}{\partial t}\frac{\partial \phi}{\partial t} - \sum_{j=1}^{N}\frac{\partial w}{\partial x_j}\frac{\partial \phi}{\partial x_j}\right)\right]\\
&= \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\frac{\partial w}{\partial x_i}\varphi_\lambda\left(\frac{\partial w}{\partial t}\frac{\partial \phi}{\partial t} - \nabla w \nabla \phi\right)\right) - \lambda\varphi_\lambda\nabla w\nabla\phi\left(\frac{\partial w}{\partial t}\frac{\partial \phi}{\partial t} - \nabla w\nabla\phi\right)\\
&\quad -\sum_{i=1}^{N}\frac{\partial w}{\partial x_i}\varphi_\lambda\left(\frac{\partial^2 w}{\partial x_i \partial t}\frac{\partial \phi}{\partial t} + \frac{\partial w}{\partial t}\frac{\partial^2 \phi}{\partial x_i\partial t}\right) + \sum_{i=1}^{N}\frac{\partial w}{\partial x_i}\varphi_\lambda\sum_{j=1}^{N}\left(\frac{\partial^2 w}{\partial x_i\partial x_j}\frac{\partial \phi}{\partial x_j} + \frac{\partial w}{\partial x_j}\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right)\\
&= \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\frac{\partial w}{\partial x_i}\varphi_\lambda\left(\frac{\partial w}{\partial t}\frac{\partial \phi}{\partial t} - \nabla w \nabla \phi\right)\right) - \lambda\varphi_\lambda\nabla w\nabla\phi\left(\frac{\partial w}{\partial t}\frac{\partial \phi}{\partial t} - \nabla w\nabla\phi\right)\\
&\quad -\varphi_\lambda\frac{\partial \phi}{\partial t}\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\frac{\partial w}{\partial t}\right)\frac{\partial w}{\partial x_i} + \sum_{i=1}^{N}\frac{\partial w}{\partial x_i}\varphi_\lambda\sum_{j=1}^{N}\frac{\partial^2 w}{\partial\partial x_i\partial x_j}\frac{\partial \phi}{\partial x_j} + \varphi_\lambda\sum_{i,j=1}^{N}\frac{\partial^2 \phi}{\partial x_i\partial x_j}\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j}\\
&= \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\frac{\partial w}{\partial x_i}\varphi_\lambda\left(\frac{\partial w}{\partial t}\frac{\partial \phi}{\partial t} - \nabla w \nabla \phi\right)\right) - \lambda\varphi_\lambda\frac{\partial w}{\partial t}\frac{\partial \phi}{\partial t}\nabla w\nabla\phi + \lambda\varphi_\lambda|\nabla w\nabla\phi|^2\\
&-\varphi_\lambda\frac{\partial \phi}{\partial t}\nabla\left(\frac{\partial w}{\partial t}\right)\nabla w + \sum_{i=1}^{N}\frac{\partial w}{\partial x_i}\varphi_\lambda\sum_{j=1}^{N}\frac{\partial^2 w}{\partial x_i\partial x_j}\frac{\partial \phi}{\partial x_j} + \varphi_\lambda D^2\phi(\nabla w,\nabla w). \qquad (2.14)
\end{aligned}
$$

Vamos desenvolver ainda mais alguns termos dessa igualdade. A saber, o quarto e o quinto termos. O primeiro deles pode ver visto como

$$-\frac{\partial\phi}{\partial t}\varphi_\lambda\nabla\left(\frac{\partial w}{\partial t}\right)\nabla w = -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial x_i}\right)^2\frac{\partial\phi}{\partial t}\varphi_\lambda$$

$$= -\frac{1}{2}\frac{\partial}{\partial t}\left(\sum_{i=1}^{N}\left(\frac{\partial w}{\partial x_i}\right)^2\frac{\partial\phi}{\partial t}\varphi_\lambda\right) + \frac{1}{2}\sum_{i=1}^{N}\left(\frac{\partial w}{\partial x_i}\right)^2\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial t}\varphi_\lambda\right)$$

$$= -\frac{1}{2}\frac{\partial}{\partial t}\left(|\nabla w|^2\frac{\partial\phi}{\partial t}\varphi_\lambda\right) + \frac{1}{2}\sum_{i=1}^{N}\left(\frac{\partial w}{\partial x_i}\right)^2\left(\frac{\partial^2\phi}{\partial t^2}\varphi_\lambda + \frac{\partial\phi}{\partial t}\frac{\partial\varphi_\lambda}{\partial t}\right)$$

$$= -\frac{1}{2}\frac{\partial}{\partial t}\left(|\nabla w|^2\frac{\partial\phi}{\partial t}\varphi_\lambda\right) + \frac{1}{2}\sum_{i=1}^{N}\left(\frac{\partial w}{\partial x_i}\right)^2\left(\frac{\partial^2\phi}{\partial t^2}\varphi_\lambda + \lambda\varphi_\lambda\left|\frac{\partial\phi}{\partial t}\right|^2\right)$$

$$= -\frac{1}{2}\frac{\partial}{\partial t}\left(|\nabla w|^2\frac{\partial\phi}{\partial t}\varphi_\lambda\right) + \frac{1}{2}|\nabla w|^2\frac{\partial^2\phi}{\partial t^2}\varphi_\lambda + \frac{\lambda}{2}\varphi_\lambda|\nabla w|^2\left|\frac{\partial\phi}{\partial t}\right|^2. \tag{2.15}$$

O quarto termo de (2.14) pode ser escrito da seguinte forma

$$\sum_{i=1}^{N}\frac{\partial w}{\partial x_i}\varphi_\lambda\sum_{j=1}^{N}\frac{\partial^2 w}{\partial x_i\partial x_j}\frac{\partial\phi}{\partial x_j} = \sum_{i,j=1}^{N}\frac{\partial^2 w}{\partial x_j\partial x_i}\frac{\partial w}{\partial x_i}\frac{\partial\phi}{\partial x_j}\varphi_\lambda = \frac{1}{2}\sum_{i,j=1}^{N}\frac{\partial}{\partial x_j}\left(\frac{\partial w}{\partial x_i}\right)^2\frac{\partial\phi}{\partial x_j}\varphi_\lambda$$

$$= \frac{1}{2}\frac{\partial}{\partial x_j}\left(\sum_{i,j=1}^{N}\left(\frac{\partial w}{\partial x_i}\right)^2\frac{\partial\phi}{\partial x_j}\varphi_\lambda\right) - \frac{1}{2}\sum_{i,j=1}^{N}\left(\frac{\partial w}{\partial x_i}\right)^2\frac{\partial}{\partial x_j}\left(\frac{\partial\phi}{\partial x_j}\varphi_\lambda\right)$$

$$= \frac{1}{2}\frac{\partial}{\partial x_j}\left(\sum_{j=1}^{N}|\nabla w|^2\frac{\partial\phi}{\partial x_j}\varphi_\lambda\right) - \frac{1}{2}\sum_{i,j=1}^{N}\left(\frac{\partial w}{\partial x_i}\right)^2\left[\frac{\partial^2\phi}{\partial {x_j}^2}\varphi_\lambda + \frac{\partial\phi}{\partial x_j}\frac{\partial\varphi_\lambda}{\partial x_j}\right]$$

$$= \frac{1}{2}\frac{\partial}{\partial x_j}\left(\sum_{j=1}^{N}|\nabla w|^2\frac{\partial\phi}{\partial x_j}\varphi_\lambda\right) - \frac{1}{2}|\nabla w|^2\Delta\phi\varphi_\lambda - \frac{\lambda}{2}\sum_{i,j=1}^{N}\left(\frac{\partial w}{\partial x_i}\right)^2\left(\frac{\partial\phi}{\partial x_j}\right)^2\varphi_\lambda$$

$$= \frac{1}{2}\frac{\partial}{\partial x_j}\left(\sum_{j=1}^{N}|\nabla w|^2\frac{\partial\phi}{\partial x_j}\varphi_\lambda\right) - \frac{\lambda}{2}\varphi_\lambda|\nabla w\nabla\phi|^2 - \frac{1}{2}\varphi_\lambda\Delta\phi|\nabla w|^2. \tag{2.16}$$

Temos ainda que

$$\int_{-T}^{T}\int_\Omega\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\frac{\partial w}{\partial x_i}\varphi_\lambda\left(\frac{\partial w}{\partial t}\frac{\partial\phi}{\partial t} - \nabla w\nabla\phi\right)\right)dxdt$$

$$= \sum_{i=1}^{N}\int_{-T}^{T}\int_\Gamma\varphi_\lambda\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial t}\frac{\partial\phi}{\partial t}\,\nu_i d\sigma dt - \int_{-T}^{T}\int_\Gamma\varphi_\lambda\nabla w\nabla\phi\sum_{i=1}^{N}\frac{\partial w}{\partial x_i}\,\nu_i d\sigma dt$$

$$= -\int_{-T}^{T}\int_\Gamma\varphi_\lambda\nabla w\nabla\phi\nabla w\nu d\sigma dt = -\int_{-T}^{T}\int_\Gamma\varphi_\lambda|\nabla w\,\nu|^2\nabla\phi\nu d\sigma dt. \tag{2.17}$$

De forma semelhante,

$$\int_{-T}^{T}\frac{\partial}{\partial t}\left(|\nabla w|^2\frac{\partial\phi}{\partial t}\varphi_\lambda\right)dt=|\nabla w|^2\frac{\partial\phi}{\partial t}\varphi_\lambda\Big|_{-T}^{T}=0 \tag{2.18}$$

e

$$\int_{-T}^{T}\int_\Omega\frac{\partial}{\partial x_j}\left(\sum_{i,j=1}^{N}|\nabla w|^2\frac{\partial\phi}{\partial x_j}\varphi_\lambda\right)=\sum_{j=1}^{N}\int_{-T}^{T}\int_\Gamma|\nabla w|^2\frac{\partial\phi}{\partial x_j}\varphi_\lambda\,\nu_j dxdt$$

$$=\int_{-T}^{T}\int_\Gamma|\nabla w|^2\varphi_\lambda\sum_{j=1}^{N}\frac{\partial\phi}{\partial x_j}\,\nu_j dxdt=\int_{-T}^{T}\int_\Gamma\varphi_\lambda|\nabla w\,\nu|^2\nabla\phi\,\nu dxdt. \tag{2.19}$$

De (2.14) a (2.19) conseguimos que

$$\begin{aligned}\mathbf{I_{23}}&=2s\lambda\int_{-T}^{T}\int_\Omega\Delta w\varphi_\lambda\left(\frac{\partial w}{\partial t}\frac{\partial\phi}{\partial t}-\nabla w\nabla\phi\right)dxdt\\&=2s\lambda\int_{-T}^{T}\int_\Omega\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\frac{\partial w}{\partial x_i}\varphi_\lambda\left(\frac{\partial w}{\partial t}\frac{\partial\phi}{\partial t}-\nabla w\nabla\phi\right)\right)dxdt\\&\quad-s\lambda\int_\Omega\left(\int_{-T}^{T}\frac{\partial}{\partial t}\left(|\nabla w|^2\frac{\partial\phi}{\partial t}\varphi_\lambda\right)dt\right)dx+s\lambda\int_{-T}^{T}\int_\Omega|\nabla w|^2\varphi_\lambda\left(\frac{\partial^2\phi}{\partial t^2}-\Delta\phi\right)dxdt\\&\quad+2s\lambda^2\int_{-T}^{T}\int_\Omega|\nabla\phi\nabla w|^2\varphi_\lambda dxdt-2s\lambda^2\int_{-T}^{T}\int_\Omega\frac{\partial w}{\partial t}\frac{\partial\phi}{\partial t}\varphi_\lambda\nabla w\nabla\phi dxdt\\&\quad+s\lambda^2\int_{-T}^{T}\int_\Omega|\nabla w|^2\varphi_\lambda\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)dxdt\\&\quad+s\lambda\int_{-T}^{T}\int_\Omega\sum_{j=1}^{N}\frac{\partial}{\partial x_j}\left(\varphi_\lambda\frac{\partial\phi}{\partial x_j}|\nabla w|^2\right)dxdt+2s\lambda\int_{-T}^{T}\int_\Omega\varphi_\lambda D^2\phi(\nabla w,\nabla w)dxdt\end{aligned}$$

e, portanto,

$$\begin{aligned}\mathbf{I_{23}}&=s\lambda\int_{-T}^{T}\int_\Omega|\nabla w|^2\varphi_\lambda\left(\frac{\partial^2\phi}{\partial t^2}-\Delta\phi\right)dxdt+2s\lambda^2\int_{-T}^{T}\int_\Omega|\nabla\phi\nabla w|^2\varphi_\lambda dxdt\\&\quad-2s\lambda^2\int_{-T}^{T}\int_\Omega\frac{\partial w}{\partial t}\frac{\partial\phi}{\partial t}\varphi_\lambda\nabla w\nabla\phi dxdt+s\lambda^2\int_{-T}^{T}\int_\Omega|\nabla w|^2\varphi_\lambda\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)dxdt\\&\quad-s\lambda\int_{-T}^{T}\int_\Gamma|\nabla w\,\nu|^2\varphi_\lambda\nabla\phi\nu d\sigma dt+2s\lambda\int_{-T}^{T}\int_\Omega\varphi_\lambda D^2\phi(\nabla w,\nabla w)dxdt.\end{aligned}$$

Por último analisemos

- $\mathbf{I_{33}}$ $= -2s^3\lambda^3 \int_{-T}^{T}\int_{\Omega} {\varphi_\lambda}^3 w \left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right)\left(\frac{\partial w}{\partial t}\frac{\partial \phi}{\partial t} - \nabla w \nabla\phi\right) dxdt$

$$= -s^3\lambda^3 \int_{-T}^{T}\int_{\Omega} \frac{\partial |w|^2}{\partial t}{\varphi_\lambda}^3 \frac{\partial \phi}{\partial t}\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right) dxdt$$

$$+s^3\lambda^3 \int_{-T}^{T}\int_{\Omega} 2{\varphi_\lambda}^3 w \nabla w \nabla\phi \left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right) dxdt.$$

Vejamos que

$$\frac{\partial |w|^2}{\partial t}{\varphi_\lambda}^3 \frac{\partial \phi}{\partial t}\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right)$$

$$= \frac{\partial}{\partial t}\left(|w|^2{\varphi_\lambda}^3 \frac{\partial \phi}{\partial t}\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right)\right) - |w|^2\frac{\partial}{\partial t}\left[{\varphi_\lambda}^3 \frac{\partial \phi}{\partial t}\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right)\right]$$

$$= \frac{\partial}{\partial t}\left(|w|^2{\varphi_\lambda}^3 \frac{\partial \phi}{\partial t}\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right)\right) - |w|^2\left\{\frac{\partial ({\varphi_\lambda}^3)}{\partial t}\frac{\partial \phi}{\partial t}\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right)\right.$$

$$\left. -{\varphi_\lambda}^3 \frac{\partial}{\partial t}\left[\frac{\partial \phi}{\partial t}\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right)\right]\right\}$$

$$= \frac{\partial}{\partial t}\left(|w|^2{\varphi_\lambda}^3 \frac{\partial \phi}{\partial t}\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right)\right) - 3\lambda|w|^2{\varphi_\lambda}^3\left|\frac{\partial \phi}{\partial t}\right|^2\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right)$$

$$-|w|^2{\varphi_\lambda}^3\left[\frac{\partial^2 \phi}{\partial t^2}\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right) + \frac{\partial \phi}{\partial t}\frac{\partial}{\partial t}\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right)\right]$$

$$= \frac{\partial}{\partial t}\left(|w|^2{\varphi_\lambda}^3 \frac{\partial \phi}{\partial t}\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right)\right) - 3\lambda|w|^2{\varphi_\lambda}^3\left|\frac{\partial \phi}{\partial t}\right|^2\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right)$$

$$-|w|^2{\varphi_\lambda}^3\frac{\partial^2 \phi}{\partial t^2}\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right) - |w|^2{\varphi_\lambda}^3\frac{\partial \phi}{\partial t}\left(\frac{\partial}{\partial t}\left|\frac{\partial \phi}{\partial t}\right|^2 - \frac{\partial}{\partial t}|\nabla\phi|^2\right)$$

$$= \frac{\partial}{\partial t}\left(|w|^2{\varphi_\lambda}^3 \frac{\partial \phi}{\partial t}\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right)\right) - 3\lambda|w|^2{\varphi_\lambda}^3\left|\frac{\partial \phi}{\partial t}\right|^2\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right)$$

$$-|w|^2{\varphi_\lambda}^3\frac{\partial^2 \phi}{\partial t^2}\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right) - 2|w|^2{\varphi_\lambda}^3\left|\frac{\partial \phi}{\partial t}\right|^2\frac{\partial^2 \phi}{\partial t^2} \tag{2.20}$$

e que

$$2{\varphi_\lambda}^3 w\nabla w\nabla\phi\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)=\sum_{i=1}^N{\varphi_\lambda}^3\frac{\partial|w|^2}{\partial x_i}\frac{\partial\phi}{\partial x_i}\left(\left|\frac{\partial\phi}{\partial t}\right|^2-\sum_{j=1}^N\left(\frac{\partial\phi}{\partial x_j}\right)^2\right)$$

$$=\sum_{i=1}^N\frac{\partial}{\partial x_i}\left(|w|^2{\varphi_\lambda}^3\frac{\partial\phi}{\partial x_i}\left(\left|\frac{\partial\phi}{\partial t}\right|^2-\sum_{j=1}^N\left(\frac{\partial\phi}{\partial x_j}\right)^2\right)\right)$$

$$-|w|^2\sum_{i=1}^N\frac{\partial}{\partial x_i}\left[{\varphi_\lambda}^3\frac{\partial\phi}{\partial x_i}\left(\left|\frac{\partial\phi}{\partial t}\right|^2-\sum_{j=1}^N\left(\frac{\partial\phi}{\partial x_j}\right)^2\right)\right]$$

$$=\sum_{i=1}^N\frac{\partial}{\partial x_i}\left(|w|^2{\varphi_\lambda}^3\frac{\partial\phi}{\partial x_i}\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)\right)-|w|^2\sum_{i=1}^N\left\{\frac{\partial({\varphi_\lambda}^3)}{\partial x_i}\frac{\partial\phi}{\partial x_i}\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)\right.$$

$$\left.+{\varphi_\lambda}^3\frac{\partial}{\partial x_i}\left[\frac{\partial\phi}{\partial x_i}\left(\left|\frac{\partial\phi}{\partial t}\right|^2-\sum_{j=1}^N\left(\frac{\partial\phi}{\partial x_j}\right)^2\right)\right]\right\}$$

$$=\sum_{i=1}^N\frac{\partial}{\partial x_i}\left(|w|^2{\varphi_\lambda}^3\frac{\partial\phi}{\partial x_i}\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)\right)-3\lambda|w|^2\sum_{i=1}^N{\varphi_\lambda}^3\left(\frac{\partial\phi}{\partial t}\right)^2\left(\left|\frac{\partial\phi}{\partial x_i}\right|^2-|\nabla\phi|^2\right)$$

$$-\sum_{i=1}^N|w|^2{\varphi_\lambda}^3\frac{\partial^2\phi}{\partial {x_i}^2}\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)-\sum_{i=1}^N|w|^2{\varphi_\lambda}^3\frac{\partial\phi}{\partial x_i}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial\phi}{\partial t}\right|^2-\sum_{j=1}^N\left(\frac{\partial\phi}{\partial x_j}\right)^2\right)$$

$$=\sum_{i=1}^N\frac{\partial}{\partial x_i}\left(|w|^2{\varphi_\lambda}^3\frac{\partial\phi}{\partial x_i}\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)\right)-3\lambda|w|^2{\varphi_\lambda}^3|\nabla\phi|^2\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)$$

$$-|w|^2{\varphi_\lambda}^3\Delta\phi\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)-\sum_{i=1}^N|w|^2{\varphi_\lambda}^3\frac{\partial\phi}{\partial x_i}\left(\frac{\partial}{\partial x_i}\left|\frac{\partial\phi}{\partial t}\right|^2-\sum_{j=1}^N\frac{\partial}{\partial x_i}\left(\frac{\partial\phi}{\partial x_j}\right)^2\right)$$

$$=\sum_{i=1}^N\frac{\partial}{\partial x_i}\left(|w|^2{\varphi_\lambda}^3\frac{\partial\phi}{\partial x_i}\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)\right)-3\lambda|w|^2{\varphi_\lambda}^3|\nabla\phi|^2\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)$$

$$-|w|^2{\varphi_\lambda}^3\Delta\phi\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)+2|w|^2{\varphi_\lambda}^3\sum_{i,j=1}^N\frac{\partial^2\phi}{\partial x_i\partial x_j}\frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_j}$$

$$=\sum_{i=1}^N\frac{\partial}{\partial x_i}\left(|w|^2{\varphi_\lambda}^3\frac{\partial\phi}{\partial x_i}\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)\right)-3\lambda|w|^2{\varphi_\lambda}^3|\nabla\phi|^2\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)$$

$$-|w|^2{\varphi_\lambda}^3\Delta\phi\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)+2|w|^2{\varphi_\lambda}^3D^2\phi(\nabla\phi,\nabla\phi). \tag{2.21}$$

Agora notemos que

$$\int_{-T}^T\frac{\partial}{\partial t}\left(|w|^2{\varphi_\lambda}^3\frac{\partial\phi}{\partial t}\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\Delta\phi|^2\right)\right)dt=0 \tag{2.22}$$

e

$$\int_{-T}^{T}\int_{\Omega}\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(|w|^2{\varphi_\lambda}^3\frac{\partial\phi}{\partial x_i}\left(\left|\frac{\partial\phi}{\partial t}\right|^2-\sum_{j=1}^{N}\left(\frac{\partial\phi}{\partial x_j}\right)^2\right)\right)dxdt$$

$$=\sum_{i=1}^{N}\int_{-T}^{T}\int_{\Gamma}\left(|w|^2{\varphi_\lambda}^3\frac{\partial\phi}{\partial x_i}\left(\left|\frac{\partial\phi}{\partial t}\right|^2-\sum_{j=1}^{N}\left(\frac{\partial\phi}{\partial x_j}\right)^2\right)\right)\nu_i d\sigma dt=0. \tag{2.23}$$

Unindo as informações de (2.20) a (2.23) resulta que

$$\begin{aligned}\mathbf{I_{33}}&=2s^3\lambda^3\int_{-T}^{T}\int_{\Omega}{\varphi_\lambda}^3 w\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\Delta\phi|^2\right)\left(\frac{\partial w}{\partial t}\frac{\partial\phi}{\partial t}-\nabla w\nabla\phi\right)dxdt\\
&=s^3\lambda^3\int_{-T}^{T}\int_{\Omega}|w|^2{\varphi_\lambda}^3\left(\frac{\partial^2\phi}{\partial t^2}-\Delta\phi\right)\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\Delta\phi|^2\right)dxdt\\
&\quad+2s^3\lambda^3\int_{-T}^{T}\int_{\Omega}|w|^2{\varphi_\lambda}^3\left(\left|\frac{\partial\phi}{\partial t}\right|^2\frac{\partial^2\phi}{\partial t^2}+D^2\phi(\nabla\phi,\nabla\phi)\right)dxdt\\
&\quad+3s^3\lambda^4\int_{-T}^{T}\int_{\Omega}|w|^2{\varphi_\lambda}^3\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)^2dxdt.\end{aligned}$$

Uma vez que já calculamos todos os I_{kl} podemos somá-los com k e l variando de 1 a 3 e encontrar

$$\begin{aligned}&\int_{-T}^{T}\int_{\Omega}P_1wP_2wdxdt\\
&=2s\lambda\int_{-T}^{T}\int_{\Omega}\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\frac{\partial^2\phi}{\partial t^2}dxdt-M_1s\lambda\int_{-T}^{T}\int_{\Omega}\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda\left(\frac{\partial^2\phi}{\partial t^2}-\Delta\phi\right)dxdt\\
&\quad+2s\lambda^2\int_{-T}^{T}\int_{\Omega}\varphi_\lambda\left[\left|\frac{\partial w}{\partial t}\right|^2\left|\frac{\partial\phi}{\partial t}\right|^2-2\frac{\partial w}{\partial t}\frac{\partial\phi}{\partial t}\nabla w\nabla\phi+|\nabla\phi\nabla w|^2\right]dxdt\\
&\quad+2s\lambda\int_{-T}^{T}\int_{\Omega}\varphi_\lambda D^2\phi(\nabla w,\nabla w)dxdt+M_1s\lambda\int_{-T}^{T}\int_{\Omega}|\nabla w|^2\varphi_\lambda\left(\frac{\partial^2\phi}{\partial t^2}-\Delta\phi\right)dxdt\\
&\quad-s\lambda\int_{-T}^{T}\int_{\Gamma}|\nabla w.\nu|^2\varphi_\lambda\nabla\phi.\nu d\sigma dt+2s^3\lambda^4\int_{-T}^{T}\int_{\Omega}|w|^2{\varphi_\lambda}^3\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)^2dxdt\end{aligned}$$

$$+2s^3\lambda^3\int_{-T}^{T}\int_{\Omega}|w|^2{\varphi_\lambda}^3\left(\frac{\partial^2\phi}{\partial t^2}\left|\frac{\partial\phi}{\partial t}\right|^2+D^2\phi(\nabla\phi,\nabla\phi)\right)dxdt$$

$$+M_1s^3\lambda^3\int_{-T}^{T}\int_{\Omega}|w|^2{\varphi_\lambda}^3\left(\frac{\partial^2\phi}{\partial t^2}-\Delta\phi\right)\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)dxdt$$

$$+X_1,$$

onde

$$X_1=-\frac{(1-M_1)}{2}s\lambda^2\int_{-T}^{T}\int_{\Omega}|w|^2\varphi_\lambda\left(\frac{\partial^2\phi}{\partial t^2}+\lambda\left|\frac{\partial\phi}{\partial t}\right|^2\right)\left(\frac{\partial^2\phi}{\partial t^2}-\Delta\phi\right)dxdt$$

$$-s\lambda^2\int_{-T}^{T}\int_{\Omega}|w|^2\varphi_\lambda\left|\frac{\partial^2\phi}{\partial t^2}\right|^2dxdt-\left(2+\frac{1}{2}\right)s\lambda^3\int_{-T}^{T}\int_{\Omega}|w|^2\varphi_\lambda\left|\frac{\partial\phi}{\partial t}\right|^2\frac{\partial^2\phi}{\partial t^2}dxdt$$

$$+\frac{s\lambda^3}{2}\int_{-T}^{T}\int_{\Omega}|w|^2\varphi_\lambda|\nabla\phi|^2\frac{\partial^2\phi}{\partial t^2}dxdt$$

$$-\frac{s\lambda^4}{2}\int_{-T}^{T}\int_{\Omega}|w|^2\varphi_\lambda\left|\frac{\partial\phi}{\partial t}\right|^2\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)dxdt$$

$$+\frac{(1-M_1)}{2}s\lambda^2\int_{-T}^{T}\int_{\Omega}|w|^2\varphi_\lambda\left(\frac{\partial^2\phi}{\partial t^2}-\Delta\phi\right)(\Delta\phi+\lambda|\nabla\phi|^2)dxdt$$

$$-s\lambda^2\int_{-T}^{T}\int_{\Omega}|w|^2\varphi_\lambda\sum_{i,j=1}^{n}\left(\frac{\partial^2\phi}{\partial x_i\partial x_j}\right)^2dxdt$$

$$+\frac{s\lambda^3}{2}\int_{-T}^{T}\int_{\Omega}|w|^2\varphi_\lambda(\Delta\phi+\lambda|\nabla\phi|^2)\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)dxdt$$

$$-2s\lambda^3\int_{-T}^{T}\int_{\Omega}|w|^2\varphi_\lambda D^2\phi(\nabla\phi,\nabla\phi)dxdt.$$

Busquemos uma estimativa para X_1. Observemos que

- $\dfrac{\partial\phi}{\partial t}=\dfrac{\partial}{\partial t}(|x-x_0|^2-\beta t^2+M_0)=-2\beta t.$

- $\dfrac{\partial^2\phi}{\partial t^2}=\dfrac{\partial}{\partial t}\left(\dfrac{\partial\phi}{\partial t}\right)=\dfrac{\partial}{\partial t}(-2\beta t)=-2\beta.$

- $\dfrac{\partial\phi}{\partial x_i} = \dfrac{\partial}{\partial x_i}(|x-x_0|^2 - \beta t^2 + M_0) = \dfrac{\partial}{\partial x_i}((x_1-x_{01})^2 + ... + (x_n - x_{0n})^2)$

$$= \frac{\partial}{\partial x_i}[(x_i - x_{0i})^2] = 2(x_i - x_{0i})\frac{\partial x_i}{\partial x_i} = 2(x_i - x_{0i}).$$

- $\dfrac{\partial^2\phi}{\partial {x_i}^2} = \dfrac{\partial}{\partial x_i}\left(\dfrac{\partial\phi}{\partial x_i}\right) = \dfrac{\partial}{\partial x_i}[2(x_i - x_{0i})] = 2.$

- $\Delta\phi = \displaystyle\sum_{i=1}^{N}\frac{\partial^2\phi}{\partial {x_i}^2} = \sum_{i=1}^{N} 2 = 2N.$

- $|\nabla\phi|^2 = \displaystyle\sum_{i=1}^{N}\left(\frac{\partial\phi}{\partial x_i}\right)^2 = \sum_{i=1}^{N}(2(x_i - x_{0i}))^2 = 4\sum_{i=1}^{N}(x_i - x_{0i})^2 = 4|x-x_0|^2.$

Sabemos que Ω é um aberto limitado do $\mathbb{R}^N$ e $x_0 \in \mathbb{R}^N \backslash \overline{\Omega}$ é dado. Sendo $x \in \Omega$, podemos tomar uma bola B centrada na origem do $\mathbb{R}^N$, com raio R suficientemente grande, tal que $\Omega \cup \{x_0\} \subset B$. Logo,

$$|x - x_0| \leq 2R \Rightarrow |x-x_0|^2 \leq 4R^2.$$

Dessa forma,

$$|\nabla\phi|^2 = 4|x-x_0|^2 \leq 16R^2.$$

- Como $\varphi_\lambda = e^{\lambda\phi(x,t)}$, segue que $1 \leq \varphi_\lambda$. Disto, $1 \leq {\varphi_\lambda}^2$ e assim, $\varphi_\lambda \leq {\varphi_\lambda}^3$. Além disso, $1 \leq \varphi_\lambda$ implica $\varphi_\lambda^2 \leq {\varphi_\lambda}^3$ e uma vez que $e^a \geq a$, para todo $a \in \mathbb{R}$ e $\phi(x,t) \geq 1$, deduzimos que

$$\varphi_\lambda = e^{\lambda\phi(x,t)} \geq \lambda\phi(x,t) \geq \lambda.$$

- $D^2\phi(\nabla\phi, \nabla\phi) = \displaystyle\sum_{i=1}^{N}\frac{\partial^2\phi}{\partial {x_i}^2}\left(\frac{\partial\phi}{\partial x_i}\right)^2 = 2\sum_{i=1}^{N}\left(\frac{\partial\phi}{\partial x_i}\right)^2 = 2|\nabla\phi|^2.$

- Tomaremos $\lambda \geq 1$. Dessa forma, teremos $1 \leq \lambda^2$, $\lambda \leq \lambda^3$ e $\lambda^2 \leq \lambda^3$.

Analisemos o módulo de cada uma das parcelas de X_1.

Primeira Parcela

$$\left|-\frac{(1-M_1)}{2}s\lambda^2\int_{-T}^{T}\int_{\Omega}|w|^2\varphi_\lambda\left(\frac{\partial^2\phi}{\partial t^2}+\lambda|\frac{\partial\phi}{\partial t}|^2\right)\left(\frac{\partial^2\phi}{\partial t^2}-\Delta\phi\right)dxdt\right|$$

$$= s\lambda^2\left|\frac{1-M_1}{2}\right|\left|\int_{-T}^{T}\int_{\Omega}|w|^2\varphi_\lambda\left(-2\beta+\lambda|-2\beta t|^2\right)\left(-2\beta-2N\right)dxdt\right|$$

$$\leq s\lambda^2(\beta+N)|1-M_1|\int_{-T}^{T}\int_{\Omega}|w|^2\varphi_\lambda|-2\beta+\lambda|2\beta t|^2|dxdt$$

$$\leq 2s\lambda^3\beta(\beta+N)|1-M_1|\left(\int_{-T}^{T}\int_{\Omega}|w|^2\varphi_\lambda dxdt+2\beta\int_{-T}^{T}\int_{\Omega}|w|^2\lambda\varphi_\lambda t^2dxdt\right)$$

$$\leq 2s\lambda^3\beta(\beta+N)|1-M_1|\left(\int_{-T}^{T}\int_{\Omega}|w|^2\varphi_\lambda dxdt+2\beta\int_{-T}^{T}\int_{\Omega}|w|^2{\varphi_\lambda}^2T^2dxdt\right)$$

$$\leq C_1s\lambda^3\int_{-T}^{T}\int_{\Omega}|w|^2{\varphi_\lambda}^3dxdt,$$

com $C_1=\max\{2\beta(\beta+N)|1-M_1|,4\beta^2(\beta+N)|1-M_1|T^2\}$.

Segunda Parcela

$$\left|-s\lambda^2\int_{-T}^{T}\int_{\Omega}\varphi_\lambda|w|^2.\left|\frac{\partial^2\phi}{\partial t^2}\right|^2dxdt\right|\leq s\lambda^2\left|\int_{-T}^{T}\int_{\Omega}\varphi_\lambda|w|^2.|-2\beta|^2dxdt\right|$$

$$\leq 4\beta^2s\lambda^3\int_{-T}^{T}\int_{\Omega}\varphi_\lambda|w|^2dxdt\leq C_2s\lambda^3\int_{-T}^{T}\int_{\Omega}{\varphi_\lambda}^3|w|^2dxdt,$$

onde $C_2=4\beta^2$.

Terceira Parcela

$$\left|-\frac{5}{2}s\lambda^3\int_{-T}^{T}\int_{\Omega}|w|^2\varphi_\lambda\left|\frac{\partial\phi}{\partial t}\right|^2\frac{\partial^2\phi}{\partial t^2}dxdt\right|\leq\frac{5}{2}s\lambda^3\left|\int_{-T}^{T}\int_{\Omega}\varphi_\lambda|w|^2|-2\beta t|^2(-2\beta)dxdt\right|$$

$$\leq 5\beta s\lambda^3\int_{-T}^{T}\int_{\Omega}\varphi_\lambda|w|^2|2\beta T|^2dxdt\ \leq C_3s\lambda^3\int_{-T}^{T}\int_{\Omega}{\varphi_\lambda}^3|w|^2dxdt,$$

com $C_3 = 20\beta^2 T^2$.

Quarta Parcela

$$\left|\frac{s\lambda^3}{2}\int_{-T}^{T}\int_{\Omega}\varphi_\lambda |w|^2 |\nabla\phi|^2 \frac{\partial^2\phi}{\partial t^2}dxdt\right| \leq \frac{s\lambda^3}{2}\left|\int_{-T}^{T}\int_{\Omega}\varphi_\lambda |w|^2 (16R^2)(-2\beta^2)dxdt\right|$$

$$\leq 16R^2\beta^2 s\lambda^3 \int_{-T}^{T}\int_{\Omega}\varphi_\lambda |w|^2 dxdt \leq C_4 s\lambda^3 \int_{-T}^{T}\int_{\Omega}{\varphi_\lambda}^3 |w|^2 dxdt,$$

onde $C_4 = 16R^2\beta^2$.

Quinta Parcela

$$\left|-\frac{s\lambda^4}{2}\int_{-T}^{T}\int_{\Omega}\varphi_\lambda |w|^2 \left|\frac{\partial\phi}{\partial t}\right|^2 \left(\left|\frac{\partial\phi}{\partial t}\right|^2 - |\nabla\phi|^2\right) dxdt\right|$$

$$\leq \frac{s\lambda^3}{2}\int_{-T}^{T}\int_{\Omega}\lambda\varphi_\lambda |w|^2 |-2\beta t|^2 |(|-2\beta t|^2 - |\nabla\phi|^2)|dxdt$$

$$\leq 2\beta^2 T^2 s\lambda^3 \int_{-T}^{T}\int_{\Omega}{\varphi_\lambda}^2 |w|^2 . ||2\beta t|^2 + |\nabla\phi|^2|dxdt$$

$$\leq 2\beta^2 T^2 s\lambda^3 \int_{-T}^{T}\int_{\Omega}{\varphi_\lambda}^2 |w|^2 (|2\beta T|^2 + 16R^2)dxdt \leq C_5 s\lambda^3 \int_{-T}^{T}\int_{\Omega}{\varphi_\lambda}^3 |w|^2 dxdt,$$

com $C_5 = 2\beta^2 T^2(|2\beta T|^2 + 16R^2)$.

Sexta Parcela

$$\left|\frac{(1-M_1)}{2}s\lambda^2 \int_{-T}^{T}\int_{\Omega}|w|^2\varphi_\lambda \left(\frac{\partial^2\phi}{\partial t^2} - \Delta\phi\right)(\Delta\phi + \lambda|\nabla\phi|^2)dxdt\right|$$

$$\leq \left|\frac{(1-M_1)}{2}\right| s\lambda^2 \left|\int_{-T}^{T}\int_{\Omega}\varphi_\lambda |w|^2 (-2\beta - 2N)(2N + 16\lambda R^2)dxdt\right|$$

$$\leq 2(\beta + N)|1 - M_1| s\lambda^2 \left(N\int_{-T}^{T}\int_{\Omega}\varphi_\lambda |w|^2 dxdt + 8R^2 \int_{-T}^{T}\int_{\Omega}\lambda\varphi_\lambda |w|^2 dxdt\right)$$

$$\leq 2(\beta+N)|1-M_1|s\lambda^3\left(N\int_{-T}^{T}\int_\Omega {\varphi_\lambda}^3|w|^2dxdt+8R^2\int_{-T}^{T}\int_\Omega {\varphi_\lambda}^2|w|^2dxdt\right)$$

$$\leq C_6 s\lambda^3\int_{-T}^{T}\int_\Omega {\varphi_\lambda}^3|w|^2dxdt,$$

onde $C_6=\max\{2N(\beta+N)|1-M_1|, 16R^2(\beta+N)|1-M_1|\}$.

<u>Sétima Parcela</u>

$$\left|-s\lambda^2\int_{-T}^{T}\int_\Omega \varphi_\lambda|w|^2\sum_{i,j=1}^{N}\left(\frac{\partial^2\phi}{\partial x_i\partial x_j}\right)^2dxdt\right|\leq s\lambda^2\left|\int_{-T}^{T}\int_\Omega \varphi_\lambda|w|^2\sum_{i=1}^{N}\left(\frac{\partial^2\phi}{\partial {x_i}^2}\right)^2dxdt\right|$$

$$\leq s\lambda^2\int_{-T}^{T}\int_\Omega \varphi_\lambda|w|^2\sum_{i=1}^{N}(2)^2dxdt\leq C_7 s\lambda^3\int_{-T}^{T}\int_\Omega {\varphi_\lambda}^3|w|^2dxdt,$$

sendo $C_7=4N$.

<u>Oitava Parcela</u>

$$\left|\frac{s\lambda^3}{2}\int_{-T}^{T}\int_\Omega \varphi_\lambda|w|^2(\Delta\phi+\lambda|\nabla\phi|^2)\left(\left|\frac{\partial\phi}{\partial t}\right|^2-|\nabla\phi|^2\right)dxdt\right|$$

$$\leq \frac{s\lambda^3}{2}\int_{-T}^{T}\int_\Omega \varphi_\lambda|w|^2(2N+16\lambda R^2)|\left(|-2\beta t|^2-|\nabla\phi|^2\right)|dxdt$$

$$\leq s\lambda^3\int_{-T}^{T}\int_\Omega \varphi_\lambda|w|^2(N+8\lambda R^2)\left(|2\beta t|^2+|\nabla\phi|^2\right)dxdt$$

$$\leq s\lambda^3\int_{-T}^{T}\int_\Omega \varphi_\lambda|w|^2(N+8\lambda R^2)\left(|2\beta t|^2+16R^2\right)dxdt$$

$$\leq 4(\beta^2T^2+4R^2)s\lambda^3\left(N\int_{-T}^{T}\int_\Omega {\varphi_\lambda}^3|w|^2dxdt+8R^2s\lambda^3\int_{-T}^{T}\int_\Omega \lambda\varphi_\lambda|w|^2dxdt\right)$$

$$\leq 4(\beta^2T^2+4R^2)s\lambda^3\left(N\int_{-T}^{T}\int_\Omega {\varphi_\lambda}^3|w|^2dxdt+8R^2s\lambda^3\int_{-T}^{T}\int_\Omega {\varphi_\lambda}^2|w|^2dxdt\right)$$

$$\leq C_8\int_{-T}^{T}\int_\Omega {\varphi_\lambda}^3|w|^2dxdt,$$

com $C_8 = \max\{4(\beta^2T^2+4R^2)N, 32(\beta^2T^2+4R^2)R^2\}$.

Nona Parcela

$$\left|-2s\lambda^3\int_{-T}^{T}\int_{\Omega}\varphi_\lambda|w|^2D^2\phi(\nabla\phi,\nabla\phi)dxdt\right| \leq 4s\lambda^3\int_{-T}^{T}\int_{\Omega}\varphi_\lambda|w|^2|\nabla\phi|^2dxdt$$
$$\leq C_9s\lambda^3\int_{-T}^{T}\int_{\Omega}|w|^2{\varphi_\lambda}^3dxdt,$$

onde $C_9 = 64R^2$.

Tomando $C_0 \geq \sum_{k=1}^{9}C_k$ obtemos que

$$|X_1| \leq C_0s\lambda^3\int_{-T}^{T}\int_{\Omega}|w|^2{\varphi_\lambda}^3dxdt.$$

Agora notemos que

$$2s\lambda^2\int_{-T}^{T}\int_{\Omega}\varphi_\lambda\left[\left|\frac{\partial w}{\partial t}\right|^2\left|\frac{\partial\phi}{\partial t}\right|^2-2\frac{\partial w}{\partial t}\frac{\partial\phi}{\partial t}\nabla w\nabla\phi+|\nabla\phi\nabla w|^2\right]dxdt$$
$$=2s\lambda^2\int_{-T}^{T}\int_{\Omega}\varphi_\lambda\left[\left|\frac{\partial w}{\partial t}\right|.\left|\frac{\partial\phi}{\partial t}\right|-\nabla\phi\nabla w\right]^2dxdt.$$

Lembremos que M_1 foi escolhido de tal forma que satisfaça

$$\frac{2\beta}{\beta+N} < M_1 < \frac{2}{\beta+N}. \tag{2.24}$$

Definamos

$$M_2 = 2M_1(\beta_1+N).$$

Multiplicando (2.24) por $2(\beta_1+N)$ chegaremos que

$$4\beta < 2M_1(\beta+N) < 4,$$

donde

$$4\beta < M_2 < 4.$$

Assim,

$$\int_{-T}^{T}\int_{\Omega} P_1 w P_2 w dxdt$$

$$\geq 2s\lambda \int_{-T}^{T}\int_{\Omega}\left|\frac{\partial w}{\partial t}\right|^2 \varphi_\lambda(-2\beta)dxdt - M_1 s\lambda \int_{-T}^{T}\int_{\Omega}\left|\frac{\partial w}{\partial t}\right|^2 \varphi_\lambda\left(-2\beta-2N\right)dxdt$$

$$+2s\lambda \int_{-T}^{T}\int_{\Omega} \varphi_\lambda 2|\nabla w|^2 dxdt + M_1 s\lambda \int_{-T}^{T}\int_{\Omega} |\nabla w|^2 \varphi_\lambda(-2\beta-2N)dxdt$$

$$-s\lambda \int_{-T}^{T}\int_{\Gamma}\left|\frac{\partial w}{\partial \nu}\right|^2 \varphi_\lambda 2(x-x_0)\nu(x)d\sigma dt + 2s^3\lambda^3 \int_{-T}^{T}\int_{\Omega} |w|^2 {\varphi_\lambda}^3 \left[\lambda\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right)^2\right.$$

$$\left.+\left(\frac{\partial^2\phi}{\partial t^2}\left|\frac{\partial \phi}{\partial t}\right|^2 + 2|\nabla\phi|^2\right) + \frac{M_1}{2}(-2\beta-2N)\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right)\right] dxdt + X_1$$

$$= (-4\beta + 2M_1(\beta+N))s\lambda \int_{-T}^{T}\int_{\Omega}\left|\frac{\partial w}{\partial t}\right|^2 \varphi_\lambda dxdt$$

$$+(4-2M_1(\beta+N))s\lambda \int_{-T}^{T}\int_{\Omega} \varphi_\lambda |\nabla w|^2 dxdt - 2s\lambda \int_{-T}^{T}\int_{\Gamma}\left|\frac{\partial w}{\partial \nu}\right|^2 \varphi_\lambda (x-x_0)\nu(x)d\sigma dt$$

$$+2s^3\lambda^3 \int_{-T}^{T}\int_{\Omega} |w|^2 {\varphi_\lambda}^3 \left[\lambda\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right)^2 + \left(\frac{\partial^2\phi}{\partial t^2}\left|\frac{\partial \phi}{\partial t}\right|^2 + 2|\nabla\phi|^2\right)\right.$$

$$\left.-M_1(\beta+N)\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right)\right] dxdt + X_1$$

$$= (-4\beta + M_2)s\lambda \int_{-T}^{T}\int_{\Omega}\left|\frac{\partial w}{\partial t}\right|^2 \varphi_\lambda dxdt + (4-M_2)s\lambda \int_{-T}^{T}\int_{\Omega} \varphi_\lambda |\nabla w|^2 dxdt$$

$$-2s\lambda \int_{-T}^{T}\int_{\Gamma}\left|\frac{\partial w}{\partial \nu}\right|^2 \varphi_\lambda (x-x_0)\nu(x)d\sigma dt + 2s^3\lambda^3 \int_{-T}^{T}\int_{\Omega} |w|^2 {\varphi_\lambda}^3 F_\lambda(\phi) dxdt + X_1,$$

onde

$$F_\lambda(\phi) = \lambda\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right)^2 - \frac{M_2}{2}\left(\left|\frac{\partial \phi}{\partial t}\right|^2 - |\nabla\phi|^2\right) + \left(\frac{\partial^2\phi}{\partial t^2}\left|\frac{\partial \phi}{\partial t}\right|^2 + 2|\nabla\phi|^2\right)$$

$$= \lambda(|2\beta t|^2 - 4|x-x_0|^2)^2 - \frac{M_2}{2}(|2\beta t|^2 - 4|x-x_0|^2) + (-2\beta|2\beta t|^2 + 2.4|x-x_0|^2)$$

$$= \lambda(4\beta^2t^2 - 4|x - x_0|^2)^2 - \frac{M_2}{2}(4\beta^2t^2 - 4|x - x_0|^2) + (-8\beta^3t^2 + 8|x - x_0|^2)$$

$$= 16\lambda(|x - x_0|^2 - \beta^2t^2)^2 + 2M_2(|x - x_0|^2 - \beta^2t^2) - 8\beta^3t^2 + 8|x - x_0|^2$$

$$+(8\beta|x - x_0|^2 - 8\beta|x - x_0|^2)$$

$$= 16\lambda(|x-x_0|^2-\beta^2t^2)^2+2M_2(|x-x_0|^2-\beta^2t^2)+8\beta(|x-x_0|^2-\beta^2t^2)+8(1-\beta)|x-x_0|^2$$

$$= 16\lambda(|x - x_0|^2 - \beta^2t^2)^2 + 2(M_2 + 4\beta)(|x - x_0|^2 - \beta^2t^2) + 8(1 - \beta)|x - x_0|^2.$$

Como $x_0 \notin \overline{\Omega}$ e $0 < \beta < 1$, temos que para todo $x \in \Omega$, existe $k > 0$ tal que

$$8(1 - \beta)|x - x_0|^2 \geq k > 0.$$

Agora façamos $X = |x - x_0|^2 - \beta^2t^2$ e consideremos a função

$$P(X) = 16\lambda X^2 + 2(M_2 + 4\beta)X + k.$$

O valor mínimo dessa função é dado por

$$-\frac{[4(M_2 + 4\beta)^2 - 64\lambda\ k]}{64\lambda} = -\frac{(M_2 + 4\beta)^2}{16\lambda} + k$$

Assim,

$$-\frac{(M_2 + 4\beta)^2}{16\lambda} + k > 0 \qquad \text{se, e somente se,} \qquad \lambda > \frac{(M_2 + 4\beta)^2}{16k}.$$

Escolhendo $\lambda_0 > max\{1, (M_2 + 4\beta)^2/16\lambda\}$ obtemos

$$\min_{x\in\mathbb{R}} P(X) > 0 \Rightarrow \min_{x\in\mathbb{R}} P(X) \geq P_0 > 0,$$

para algum real P_0.

Então, para todo $x \in \Omega$ e $t \in (-T, T)$

$$F_\lambda(\phi)(x, t) \geq P_0.$$

Com isso encontramos que

$$\int_{-T}^{T}\int_{\Omega} P_1 w P_2 w dxdt$$

$$\geq -2s\lambda\int_{-T}^{T}\int_{\Gamma}\varphi_\lambda\left|\frac{\partial w}{\partial \nu}\right|^2(x-x_0)\nu(x)d\sigma dt+(M_2-4\beta)s\lambda\int_{-T}^{T}\int_{\Omega}\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda dxdt$$

$$+(4-M_2)s\lambda\int_{-T}^{T}\int_{\Omega}|\nabla w|^2\varphi_\lambda dxdt+2s^3\lambda^3\int_{-T}^{T}\int_{\Omega}|w|^2{\varphi_\lambda}^3P_0dxdt+X_1.$$

Seja $C=min\{M_2-4\beta,4-M_2,2P_0\}$. Portanto,

$$\int_{-T}^{T}\int_{\Omega} P_1 w P_2 w dxdt$$

$$\geq -2s\lambda\int_{-T}^{T}\int_{\Gamma}\varphi_\lambda\left|\frac{\partial w}{\partial \nu}\right|^2(x-x_0)\nu(x)d\sigma dt+Cs\lambda\int_{-T}^{T}\int_{\Omega}\left|\frac{\partial w}{\partial t}\right|^2\varphi_\lambda dxdt$$

$$+Cs\lambda\int_{-T}^{T}\int_{\Omega}|\nabla w|^2\varphi_\lambda dxdt+Cs^3\lambda^3\int_{-T}^{T}\int_{\Omega}|w|^2{\varphi_\lambda}^3dxdt+X_1$$

$$\geq -2s\lambda\int_{-T}^{T}\int_{\Gamma}\varphi_\lambda\left|\frac{\partial w}{\partial \nu}\right|^2(x-x_0)\nu(x)d\sigma dt+Cs\lambda\int_{-T}^{T}\int_{\Omega}\left(\left|\frac{\partial w}{\partial t}\right|^2+|\nabla w|^2\right)\varphi_\lambda dxdt$$

$$+Cs^3\lambda^3\int_{-T}^{T}\int_{\Omega}|w|^2{\varphi_\lambda}^3dxdt+X_1.$$

Mas como

$$\int_{-T}^{T}\int_{\Omega} P_1 w P_2 w dxdt-X_1\leq\int_{-T}^{T}\int_{\Omega} P_1 w P_2 w dxdt+|X_1|$$

$$\leq\int_{-T}^{T}\int_{\Omega} P_1 w P_2 w dxdt+C_0s\lambda^3\int_{-T}^{T}\int_{\Omega}|w|^2{\varphi_\lambda}^3dxdt,$$

segue que

$$\int_{-T}^{T}\int_{\Omega} P_1 w P_2 w dxdt$$

$$\geq -2s\lambda \int_{-T}^{T}\int_{\Gamma} \varphi_\lambda \left|\frac{\partial w}{\partial \nu}\right|^2 (x-x_0)\nu(x) d\sigma dt + Cs\lambda \int_{-T}^{T}\int_{\Omega} \left(\left|\frac{\partial w}{\partial t}\right|^2 + |\nabla w|^2\right) \varphi_\lambda dxdt$$

$$+Cs^3\lambda^3 \int_{-T}^{T}\int_{\Omega} |w|^2 {\varphi_\lambda}^3 dxdt - C_0 s\lambda^3 \int_{-T}^{T}\int_{\Omega} |w|^2 {\varphi_\lambda}^3 dxdt$$

$$\geq -2s\lambda \int_{-T}^{T}\int_{\Gamma} \varphi_\lambda \left|\frac{\partial w}{\partial \nu}\right|^2 (x-x_0)\nu(x) d\sigma dt + Cs\lambda \int_{-T}^{T}\int_{\Omega} \left(\left|\frac{\partial w}{\partial t}\right|^2 + |\nabla w|^2\right) \varphi_\lambda dxdt$$

$$+\left(C - \frac{C_0}{s^2}\right) s^3\lambda^3 \int_{-T}^{T}\int_{\Omega} |w|^2 {\varphi_\lambda}^3 dxdt.$$

Tomemos $s > \sqrt{C_0/C} > 0$. Assim,

$$\int_{-T}^{T}\int_{\Omega} P_1 w P_2 w dxdt$$

$$\geq -2s\lambda \int_{-T}^{T}\int_{\Gamma} \varphi_\lambda \left|\frac{\partial w}{\partial \nu}\right|^2 (x-x_0)\nu(x) d\sigma dt + Cs\lambda \int_{-T}^{T}\int_{\Omega} \left(\left|\frac{\partial w}{\partial t}\right|^2 + |\nabla w|^2\right) \varphi_\lambda dxdt$$

$$+Ks^3\lambda^3 \int_{-T}^{T}\int_{\Omega} |w|^2 {\varphi_\lambda}^3 dxdt,$$

onde $K = C - C_0/s^2 > 0$. Além disso,

$$\int_{-T}^{T}\int_{\Omega} |R_0 w|^2 dxdt = \int_{-T}^{T}\int_{\Omega} \left| -M_1 s\lambda\varphi_\lambda \left(\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi\right) w\right|^2 dxdt$$

$$= s^2\lambda^2 \int_{-T}^{T}\int_{\Omega} \left| M_1 \varphi_\lambda (-2\beta - 2N) w \right|^2 dxdt = |M_1(-2\beta-2N)|^2 s^2\lambda^2 \int_{-T}^{T}\int_{\Omega} {\varphi_\lambda}^2 |w|^2 dxdt$$

$$\leq {M_2}^2 s^2 \lambda^3 \int_{-T}^{T}\int_{\Omega} |w|^2 {\varphi_\lambda}^3 dxdt,$$

o que nos permite encontrar

$$\int_{-T}^{T}\int_{\Omega} P_1 w P_2 w dxdt - \int_{-T}^{T}\int_{\Omega} |R_0 w|^2 dxdt$$

$$\geq -2s\lambda \int_{-T}^{T}\int_{\Gamma} \varphi_\lambda \left|\frac{\partial w}{\partial \nu}\right|^2 (x-x_0)\nu(x) d\sigma dt + Cs\lambda \int_{-T}^{T}\int_{\Omega} \left(\left|\frac{\partial w}{\partial t}\right|^2 + |\nabla w|^2\right) \varphi_\lambda dxdt$$

$$+Ks^3\lambda^3\int_{-T}^{T}\int_{\Omega}|w|^2{\varphi_\lambda}^3dxdt - {M_2}^2s^2\lambda^3\int_{-T}^{T}\int_{\Omega}|w|^2{\varphi_\lambda}^3dxdt$$

$$\geq -2s\lambda\int_{-T}^{T}\int_{\Gamma}\varphi_\lambda\left|\frac{\partial w}{\partial \nu}\right|^2(x-x_0)\nu(x)d\sigma dt + Cs\lambda\int_{-T}^{T}\int_{\Omega}\left(\left|\frac{\partial w}{\partial t}\right|^2+|\nabla w|^2\right)\varphi_\lambda dxdt$$

$$+\left(K-\frac{{M_2}^2}{s}\right)s^3\lambda^3\int_{-T}^{T}\int_{\Omega}|w|^2{\varphi_\lambda}^3dxdt.$$

Fazendo $s > s_0 = max\{\sqrt{C_0/C}, {M_2}^2/K\}$ podemos deduzir

$$\int_{-T}^{T}\int_{\Omega}P_1wP_2wdxdt - \int_{-T}^{T}\int_{\Omega}|R_0w|^2dxdt$$

$$\geq -2s\lambda\int_{-T}^{T}\int_{\Gamma}\varphi_\lambda\left|\frac{\partial w}{\partial \nu}\right|^2(x-x_0)\nu(x)d\sigma dt + Cs\lambda\int_{-T}^{T}\int_{\Omega}\left(\left|\frac{\partial w}{\partial t}\right|^2+|\nabla w|^2\right)\varphi_\lambda dxdt$$

$$+Cs^3\lambda^3\int_{-T}^{T}\int_{\Omega}|w|^2{\varphi_\lambda}^3dxdt.$$

De (2.8) e da definição do conjunto Γ_0

$$2\int_{-T}^{T}\int_{\Omega}e^{2s\varphi_\lambda}|L_0v|^2dxdt$$

$$\geq \int_{-T}^{T}\int_{\Omega}(|P_1w|^2+|P_2w|^2)dxdt + 2\int_{-T}^{T}\int_{\Omega}P_1wP_2wdxdt - 2\int_{-T}^{T}\int_{\Omega}|R_0w|^2dxdt$$

$$\geq \int_{-T}^{T}\int_{\Omega}(|P_1w|^2+|P_2w|^2)dxdt - 4s\lambda\int_{-T}^{T}\int_{\Gamma_0}\varphi_\lambda\left|\frac{\partial w}{\partial \nu}\right|^2(x-x_0)\nu(x)d\sigma dt$$

$$-4s\lambda\int_{-T}^{T}\int_{\Gamma\setminus\Gamma_0}\varphi_\lambda\left|\frac{\partial w}{\partial \nu}\right|^2(x-x_0)\nu(x)d\sigma dt + 2Cs\lambda\int_{-T}^{T}\int_{\Omega}\left(\left|\frac{\partial w}{\partial t}\right|^2+|\nabla w|^2\right)\varphi_\lambda dxdt$$

$$+2Cs^3\lambda^3\int_{-T}^{T}\int_{\Omega}|w|^2{\varphi_\lambda}^3dxdt.$$

Reajustando os termos, usando a limitação de Ω e o fato do vetor normal ser unitário, obtemos que

$$C\int_{-T}^{T}\int_{\Omega} e^{2s\varphi_\lambda}|L_0 v|^2 dxdt + Cs\lambda\int_{-T}^{T}\int_{\Gamma_0}\varphi_\lambda\left|\frac{\partial w}{\partial\nu}\right|^2 d\sigma dt$$

$$\geq \int_{-T}^{T}\int_{\Omega}(|P_1 w|^2+|P_2 w|^2)dxdt + Cs\lambda\int_{-T}^{T}\int_{\Omega}\left(\left|\frac{\partial w}{\partial t}\right|^2+|\nabla w|^2\right)\varphi_\lambda dxdt$$

$$+Cs^3\lambda^3\int_{-T}^{T}\int_{\Omega}|w|^2{\varphi_\lambda}^3 dxdt. \tag{2.25}$$

Observemos que

$$\left|\frac{\partial w}{\partial t}\right|^2 = \left|\frac{\partial}{\partial t}(e^{s\varphi_\lambda}v)\right|^2 = \left|\frac{\partial e^{s\varphi_\lambda}}{\partial t}v + e^{s\varphi_\lambda}\frac{\partial v}{\partial t}\right|^2 = \left|\frac{\partial e^{s\varphi_\lambda}}{\partial t}v\right|^2 + 2\frac{\partial e^{s\varphi_\lambda}}{\partial t}ve^{s\varphi_\lambda}\frac{\partial v}{\partial t} + \left|e^{s\varphi_\lambda}\frac{\partial v}{\partial t}\right|^2$$

$$\geq \frac{1}{2}\frac{\partial|e^{s\varphi_\lambda}|^2}{\partial t}\frac{\partial|v|^2}{\partial t} + \left|e^{s\varphi_\lambda}\frac{\partial v}{\partial t}\right|^2.$$

Temos ainda que

$$\frac{1}{2}\frac{\partial|e^{s\varphi_\lambda}|^2}{\partial t} = s\lambda e^{2s\varphi_\lambda}\varphi_\lambda\frac{\partial\phi}{\partial t} = -2s\beta\lambda e^{2s\varphi_\lambda}\varphi_\lambda t,$$

Uma vez que $\phi(x,t)$ é limitada superiormente o mesmo tipo de limitação ocorre para φ_λ e $e^{2s\varphi_\lambda}$. Logo,

$$\frac{1}{2}\frac{\partial|e^{s\varphi_\lambda}|^2}{\partial t} \geq -2s\beta\lambda K t.$$

Como

$$\int_{-T}^{T} -t\frac{\partial|v|^2}{\partial t}dt = \int_{-T}^{T}\frac{\partial}{\partial t}(-t|v|^2)dt + \int_{-T}^{T}|v|^2 dt = -T|v(T)|^2 + T|v(T)|^2 + \int_{-T}^{T}|v|^2 dt = \int_{-T}^{T}|v|^2 dt,$$

segue que

$$\int_{-T}^{T}\int_{\Omega}\varphi_\lambda\left|\frac{\partial w}{\partial t}\right|^2 dxdt \geq \int_{-T}^{T}\int_{\Omega}\left|\frac{\partial w}{\partial t}\right|^2 dxdt$$

$$\geq 2s\lambda\beta K\int_{-T}^{T}\int_{\Omega} -t\frac{\partial|v|^2}{\partial t}dxdt + \int_{-T}^{T}\int_{\Omega}e^{2s\varphi_\lambda}\left|\frac{\partial v}{\partial t}\right|^2 dxdt \geq \int_{-T}^{T}\int_{\Omega}e^{2s\varphi_\lambda}\left|\frac{\partial v}{\partial t}\right|^2 dxdt$$

$$= \frac{1}{K_1}\int_{-T}^{T}\int_{\Omega}K_1 e^{2s\varphi_\lambda}\left|\frac{\partial v}{\partial t}\right|^2 dxdt \geq \frac{1}{K_1}\int_{-T}^{T}\int_{\Omega}e^{2s\varphi_\lambda}\left|\frac{\partial v}{\partial t}\right|^2\varphi_\lambda dxdt, \tag{2.26}$$

onde K_1 é um limitante superior para φ_λ.

De forma análoga,

$$
\begin{aligned}
|\nabla w|^2 &= \sum_{i=1}^{N}\left|\frac{\partial w}{\partial x_i}\right|^2 = \sum_{i=1}^{N}\left|\frac{\partial}{\partial x_i}(e^{s\varphi_\lambda}v)\right|^2 = \sum_{i=1}^{N}\left|\frac{\partial e^{s\varphi_\lambda}}{\partial x_i}v + e^{s\varphi_\lambda}\frac{\partial v}{\partial x_i}\right|^2 \\
&= \sum_{i=1}^{N}\left(\left|\frac{\partial e^{s\varphi_\lambda}}{\partial x_i}v\right|^2 + 2\frac{\partial e^{s\varphi_\lambda}}{\partial x_i}ve^{s\varphi_\lambda}\frac{\partial v}{\partial x_i} + \left|e^{s\varphi_\lambda}\frac{\partial v}{\partial x_i}\right|^2\right) \\
&\geq \sum_{i=1}^{N}\left(\frac{1}{2}\frac{\partial |e^{s\varphi_\lambda}|^2}{\partial x_i}\frac{\partial |v|^2}{\partial x_i} + \left|e^{s\varphi_\lambda}\frac{\partial v}{\partial x_i}\right|^2\right) \\
&\geq \sum_{i=1}^{N}\left(s\lambda e^{2s\varphi_\lambda}\varphi_\lambda|x-x_0|\frac{\partial |v|^2}{\partial x_i} + \left|e^{s\varphi_\lambda}\frac{\partial v}{\partial x_i}\right|^2\right) \geq \sum_{i=1}^{N}\left|e^{s\varphi_\lambda}\frac{\partial v}{\partial x_i}\right|^2 = e^{2s\varphi_\lambda}|\nabla v|^2.
\end{aligned}
$$

Donde,

$$
\int_{-T}^{T}\int_{\Omega}|\nabla w|^2\varphi_\lambda dxdt \geq \int_{-T}^{T}\int_{\Omega}e^{2s\varphi_\lambda}|\nabla v|^2\varphi_\lambda dxdt. \tag{2.27}
$$

Para a derivada normal, usando os Teoremas 1.10 (Desigualdade de Young) e 1.12, temos que

$$
\begin{aligned}
\left|\frac{\partial w}{\partial \nu}\right|^2 &= |\nabla w \ \ \nu|^2 = \sum_{i=1}^{N}\left|\frac{\partial w}{\partial x_i}\ \nu_i\right|^2 = \sum_{i=1}^{N}\left|\frac{\partial w}{\partial x_i}\right|^2|\nu_i|^2 \leq \sum_{i=1}^{N}\left|\frac{\partial w}{\partial x_i}\right|^2 \\
&= \sum_{i=1}^{N}\left(\left|\frac{\partial e^{s\varphi_\lambda}}{\partial x_i}v\right|^2 + 2\frac{\partial e^{s\varphi_\lambda}}{\partial x_i}ve^{s\varphi_\lambda}\frac{\partial v}{\partial x_i} + \left|e^{s\varphi_\lambda}\frac{\partial v}{\partial x_i}\right|^2\right) \\
&\leq 2\sum_{i=1}^{N}\left(\left|\frac{\partial e^{s\varphi_\lambda}}{\partial x_i}v\right|^2 + \left|e^{s\varphi_\lambda}\frac{\partial v}{\partial x_i}\right|^2\right) \leq 2CN|v|^2 + 2\sum_{i=1}^{N}\left|e^{s\varphi_\lambda}\frac{\partial v}{\partial x_i}\right|^2 \\
&= 2CN|v|^2 + 2e^{2s\varphi_\lambda}|\nabla v|^2 \leq Ce^{2s\varphi_\lambda}|v|^2\varphi_\lambda^3 + 2e^{2s\varphi_\lambda}\left|\frac{\partial v}{\partial \nu}\right|^2.
\end{aligned} \tag{2.28}
$$

Unindo os resultados de (2.26) a (2.28), a desigualdade em (2.25) e o fato de que $w = e^{\lambda\varphi_\lambda}v$ chegamos a

$$
\begin{aligned}
&C\int_{-T}^{T}\int_{\Omega}e^{2s\varphi_\lambda}|L_0 v|^2 dxdt + Cs\lambda\int_{-T}^{T}\int_{\Gamma_0}e^{2s\varphi_\lambda}\left|\frac{\partial v}{\partial \nu}\right|^2 d\sigma dt \\
&\geq Cs\lambda\int_{-T}^{T}\int_{\Omega}e^{2s\varphi_\lambda}\left(\left|\frac{\partial v}{\partial t}\right|^2 + |\nabla v|^2\right)\varphi_\lambda dxdt + Cs^3\lambda^3\int_{-T}^{T}\int_{\Omega}e^{2s\varphi_\lambda}|v|^2{\varphi_\lambda}^3 dxdt
\end{aligned}
$$

$$+\int_{-T}^{T}\int_{\Omega}(|P_1w|^2+|P_2w|^2)dxdt.$$

Tomando $K=\min\{1,C\}$,

$$C\int_{-T}^{T}\int_{\Omega}e^{2s\varphi_\lambda}|L_0v|^2dxdt+Cs\lambda\int_{-T}^{T}\int_{\Gamma_0}e^{2s\varphi_\lambda}\left|\frac{\partial v}{\partial \nu}\right|^2d\sigma dt$$
$$\geq Ks\lambda\int_{-T}^{T}\int_{\Omega}e^{2s\varphi_\lambda}\left(\left|\frac{\partial v}{\partial t}\right|^2+|\nabla v|^2\right)\varphi_\lambda dxdt+Ks^3\lambda^3\int_{-T}^{T}\int_{\Omega}e^{2s\varphi_\lambda}|v|^2{\varphi_\lambda}^3dxdt$$
$$+K\int_{-T}^{T}\int_{\Omega}(|P_1w|^2+|P_2w|^2)dxdt.$$

ou, equivalentemente,

$$C\int_{-T}^{T}\int_{\Omega}e^{2s\varphi_\lambda}|L_0v|^2dxdt+Cs\lambda\int_{-T}^{T}\int_{\Gamma_0}e^{2s\varphi_\lambda}\left|\frac{\partial v}{\partial \nu}\right|^2d\sigma dt$$
$$\geq s\lambda\int_{-T}^{T}\int_{\Omega}e^{2s\varphi_\lambda}\left(\left|\frac{\partial v}{\partial t}\right|^2+|\nabla v|^2\right)\varphi_\lambda dxdt+s^3\lambda^3\int_{-T}^{T}\int_{\Omega}e^{2s\varphi_\lambda}|v|^2{\varphi_\lambda}^3dxdt$$
$$+\int_{-T}^{T}\int_{\Omega}(|P_1w|^2+|P_2w|^2)dxdt.$$

E isso prova o teorema.

Capítulo 3

Aplicação da desigualdade de Carleman à controlabilidade exata

Faremos agora uma aplicação da desigualdade global de Carleman encontrada no Capítulo 2 para estudar a controlabilidade exata da equação da onda. Para um par de dados iniciais $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, consideraremos o sistema

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u + p\, u = 0 & \text{em } \Omega \times (0,T), \\ u = v & \text{sobre } \Gamma_0 \times (0,T), \\ u = 0 & \text{sobre } (\Gamma \backslash \Gamma_0) \times (0,T), \\ u(0) = u_0, \;\; \dfrac{\partial u}{\partial t}(0) = u_1 & \text{em } \Omega, \end{cases} \tag{3.1}$$

onde $p(x,t)$ é um potencial pertencente a $L^\infty(\Omega \times (0,T))$ e v é a função (controle) tomada em $L^2(0,T; L^2(\Gamma_0))$. O problema (3.1) tem uma única solução (ver Teorema 1.4)

$$u \in C([0,T]; L^2(\Omega)) \;\; com \;\; \frac{\partial u}{\partial t} \in C([0,T]; H^{-1}(\Omega)).$$

A controlabilidade exata para o sistema (3.1) pode ser formulada como segue: para $T > 0$, suficientemente grande, e um par de dados iniciais $\{u_0, u_1\}$, encontrar um controle v tal que a solução do sistema satisfaça

$$u(T) = z_0 \;\; e \;\; \frac{\partial u}{\partial t}(T) = z_1,$$

onde $\{z_0, z_1\}$ é um par de dados em $L^2(\Omega)$ x $H^{-1}(\Omega)$.

Devido a linearidade do problema (3.1), podemos considerar o par de dados finais nulos. De fato, se fizermos $u = \theta + \xi$, o sistema acima, acrescentado das condições finais, pode ser visto como a soma dos dois sistemas seguintes:

$$\begin{cases} \dfrac{\partial^2\theta}{\partial t^2} - \Delta\theta + p\,\theta = 0 & \text{em } \Omega \times (0,T), \\ \theta = v & \text{sobre } \Gamma_0 \times (0,T), \\ \theta = 0 & \text{sobre } (\Gamma\backslash\Gamma_0) \times (0,T), \\ \theta(0) = u_0, \ \dfrac{\partial\theta}{\partial t}(0) = u_1 & \text{em } \Omega, \\ \theta(T) = 0, \ \dfrac{\partial\theta}{\partial t}(T) = 0, & \text{em } \Omega, \end{cases} \tag{3.2}$$

e

$$\begin{cases} \dfrac{\partial^2\xi}{\partial t^2} - \Delta\xi + p\,\xi = 0 & \text{em } \Omega \times (0,T), \\ \xi = 0 & \text{sobre } \Gamma \times (0,T), \\ \xi(0) = 0, \ \dfrac{\partial\xi}{\partial t}(0) = 0, & \text{em } \Omega, \\ \xi(T) = z_0, \ \dfrac{\partial\xi}{\partial t}(T) = z_1 & \text{em } \Omega. \end{cases} \tag{3.3}$$

Uma vez que (3.3) é não controlável, é suficiente provarmos que (3.2) tem solução.

Como foi dito na introdução, o método HUM, idealizado por Lions (ver [9]), transforma o problema de controlabilidade exata equivalente à obtenção de uma desigualdade de observabilidade para o problema adjunto

$$\begin{cases} \dfrac{\partial^2\psi}{\partial t^2} - \Delta\psi + p\psi = 0 & \text{em } \Omega \times (0,T), \\ \psi = 0 & \text{sobre } \Gamma \times (0,T), \\ \psi(0) = \psi_0, \ \dfrac{\partial\psi}{\partial t}(0) = \psi_1 & \text{em } \Omega. \end{cases} \tag{3.4}$$

Notemos, pelo Teorema 1.1, que dados $\psi_0 \in H_0^1$ e $\psi_1 \in L^2$, existe única solução de (3.4) na classe

$$C([0,T]; H_0^1(\Omega)) \cap C^1((0,T); L^2(\Omega)).$$

Além disso, pelo Teorema 1.3, $\frac{\partial v}{\partial \nu} \in L^2(0,T,L^2(\Gamma_0))$.

A desigualdade de observabilidade antes mencionada é estabelecida no seguinte resultado:

Teorema 3.1 *Assumamos que*

- $\exists\ x_0 \in \mathbb{R}^N \backslash \overline{\Omega}$, *com* $\Gamma_0 \supset \Gamma_{x_0} = \{x \in \Gamma, (x - x_0).\nu(x) > 0\}$,
- $T > 2 \sup_{x \in \overline{\Omega}} |x - x_0|$.

Então existe uma constante $C_0 > 0$ *tal que para todo* $\psi_0 \in H_0^1(\Omega)$ *e* $\psi_1 \in L^2(\Omega)$ *temos*

$$E_0 \leq C_0 \int_0^T \int_{\Gamma_0} \left| \frac{\partial \psi}{\partial \nu} \right|^2 d\sigma dt,$$

onde a E_0 *é a energia, definida por*

$$E(t) = \frac{1}{2} \left(\int_\Omega \left| \frac{\partial \psi}{\partial t}(x,t) \right|^2 dx + \int_\Omega |\nabla \psi(x,t)|^2 dx \right), \tag{3.5}$$

no tempo $t = 0$.

Demonstração: Derivando a energia (3.5) em relação a variável t, obtemos que

$$\begin{aligned}
\frac{dE}{dt}(t) &= \frac{1}{2} \left(2 \int_\Omega \frac{\partial \psi}{\partial t}(x,t).\frac{\partial^2 \psi}{\partial t^2}(x,t) dx + 2 \int_\Omega \nabla \psi(x,t).\nabla \frac{\partial \psi}{\partial t}(x,t) dx \right) \\
&= \int_\Omega \frac{\partial \psi}{\partial t}(x,t).\frac{\partial^2 \psi}{\partial t^2}(x,t) dx - \int_\Omega \Delta \psi(x,t).\frac{\partial \psi}{\partial t}(x,t) dx \\
&= \int_\Omega \frac{\partial \psi}{\partial t}(x,t) \left(\frac{\partial^2 \psi}{\partial t^2}(x,t) - \Delta \psi(x,t) \right) dx \\
&= - \int_\Omega p(x,t) \psi(x,t) \frac{\partial \psi}{\partial t}(x,t) dx,
\end{aligned}$$

o que implica, pelo Teorema 1.9 (Desigualdade de Hölder), que

$$\frac{dE}{dt}(t) \leq \|p(t)\|_{L^\infty(\Omega)} \left| \frac{\partial \psi}{\partial t}(t) \right|_{L^2(\Omega)} |\psi(t)|_{L^2(\Omega)}.$$

Pela Teorema 1.11 (Desigualdade de Poincaré), existe $C > 0$,

$$|\psi(t)|_{L^2(\Omega)} \leq C |\nabla \psi(t)|_{L^2(\Omega)}.$$

Dessa forma,

$$\frac{dE}{dt}(t) \leq C \|p(t)\|_{L^\infty(\Omega)} \left| \frac{\partial \psi}{\partial t}(t) \right|_{L^2(\Omega)} |\nabla \psi(t)|_{L^2(\Omega)}$$

e, aplicando o Teorema 1.10 (Desigualdade de Young),

$$\frac{dE}{dt}(t) \leq C\|p(t)\|_{L^\infty(\Omega)} \left(\frac{1}{2}\left|\frac{\partial\psi}{\partial t}(t)\right|^2_{L^2(\Omega)} + \frac{1}{2}|\nabla\psi(t)|^2_{L^2(\Omega)} \right) = C\|p(t)\|_{L^\infty(\Omega)}E(t). \tag{3.6}$$

Definamos

$$E_M = E\left(\frac{T}{2}\right) = \frac{1}{2}\left(\int_\Omega \left|\frac{\partial\psi}{\partial t}\left(x, \frac{T}{2}\right)\right|^2 dx + \int_\Omega \left|\nabla\psi\left(x, \frac{T}{2}\right)\right|^2 dx \right).$$

Observemos que de (3.6) temos

$$\frac{dE}{dt}(t) - C\|p\|_{L^\infty(Q)}E(t) \leq 0,$$

ou seja,

$$\frac{d}{dt}(E(t)e^{-C\|p\|_{L^\infty(Q)}t}) \leq 0.$$

Integrando a última desigualdade de 0 a $T/2$, obtemos

$$E_M \leq E_0 e^{C\|p\|_{L^\infty(Q)}\frac{T}{2}}. \tag{3.7}$$

De maneira análoga, mas integrando de t a $T/2$, encontramos que

$$E_M e^{-C\|p\|_{L^\infty(Q)}(\frac{T}{2}-t)} \leq E(t), \quad \forall\, t \in \left[0, \frac{T}{2}\right] \tag{3.8}$$

e integrando de $T/2$ a t,

$$E(t) \leq E_M e^{-C\|p\|_{L^\infty(Q)}(\frac{T}{2}-t)}, \quad \forall\, t \in [0, T]. \tag{3.9}$$

Multipliquemos a equação $(3.4)_1$ por $\frac{\partial\psi}{\partial t}$ e integremos em $\Omega \times (T/2, t)$. Assim,

$$\begin{aligned} &\int_\Omega \left(\left|\frac{\partial\psi}{\partial t}(t)\right|^2 + |\nabla\psi(t)|^2 \right) dx \\ &\leq \int_\Omega \left(\left|\frac{\partial\psi}{\partial t}\left(\frac{T}{2}\right)\right|^2 + \left|\nabla\psi\left(\frac{T}{2}\right)\right|^2 \right) dx + 2\|p\|_{L^\infty(Q)} \int_0^T \int_\Omega \left|\psi\frac{\partial\psi}{\partial t}\right| dxdt. \end{aligned}$$

Apliquemos os Teoremas 1.10 (Desigualdade de Young) e 1.11 (Desigualdade de Poincaré) para obtermos

$$\int_\Omega \left(\left|\frac{\partial\psi}{\partial t}(t)\right|^2 + |\nabla\psi(t)|^2 \right) dx \leq E_M + \|p\|_{L^\infty(Q)} C \int_0^T \int_\Omega \left(\left|\frac{\partial\psi}{\partial t}\right|^2 + |\nabla\psi|^2 \right) dxdt.$$

Usando o Lema 1.1 (Gronwall) na última desigualdade, segue que

$$E(t) \leq CE_M,$$

onde $C > 0$.

Agora escolhamos como peso a função

$$\phi_M(x,t) = |x - x_0|^2 - \beta \left(t - \frac{T}{2}\right)^2 + M_0,$$

onde β e M_0 são tomados de tal forma que

$$0 < \frac{4}{T^2} \sup_{x \in \Omega} |x - x_0|^2 < \beta < 1$$

e

$$\phi_M(x,t) \geq 1 \quad em \;\; Q.$$

Observemos que

$$\phi_M \left(x, \frac{T}{2}\right) = |x - x_0|^2 + M_0,$$

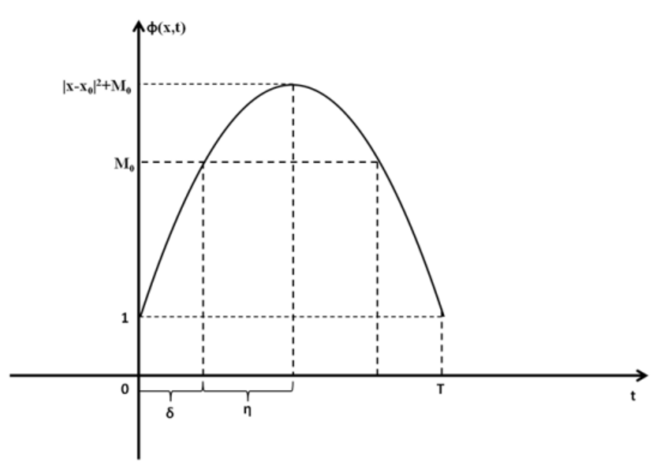
e

$$\phi_M(x,t) \leq |x - x_0|^2 + M_0.$$

Portanto,

$$\phi_M(x,t) \leq \phi_M \left(x, \frac{T}{2}\right), \quad \forall \;\; t \in [0,T].$$

Para cada $x \in \Omega$, $\phi_M(x,t)$, o gráfico de $\phi_M(x,t)$ é



Podemos afirmar que

$$\exists\ \eta > 0;\ \phi_M(x,t) \geq M_0, \quad \forall\, t \in \left[\frac{T}{2} - \eta, \frac{T}{2} + \eta\right],\ \forall\, x \in \Omega$$

e

$$\exists\ \delta > 0;\ \phi_M(x,t) \leq M_0, \quad \forall\, t \in\ [0,\delta] \cup [T-\delta, T],\ \forall\, x \in \Omega.$$

Seja θ_δ uma função em $C_0^\infty([0,T])$ satisfazendo

$$\begin{cases} 0 \leq \theta_\delta(t) \leq 1, & \forall\ t \in [0,T], \\ \theta_\delta(t) = 1, & \forall\, t \in\ [\delta, T-\delta] \end{cases} \tag{3.10}$$

e façamos

$$z(x,t) = \theta_\delta(t)\psi(x,t).$$

Uma construção da função θ_δ satisfazendo (3.10) pode ser encontrada em [8, p. 430-432].

Notemos que

$$\begin{aligned}
Lz &= \frac{\partial^2 z}{\partial t^2} - \Delta z + p(x)z \\
&= \frac{\partial^2}{\partial t^2}(\theta_\delta(t)\psi(x,t)) - \Delta(\theta_\delta(t)\psi(x,t)) + p(x)\theta_\delta(t)\psi(x,t) \\
&= \frac{\partial}{\partial t}\left[\frac{\partial \theta_\delta}{\partial t}(t)\psi(x,t) + \theta_\delta(t)\frac{\partial \psi}{\partial t}(x,t)\right] - \theta_\delta(t)\Delta\psi(x,t) + p(x)\theta_\delta(t)\psi(x,t) \\
&= \frac{\partial^2 \theta_\delta}{\partial t^2}(t)\psi(x,t) + \frac{\partial \theta_\delta}{\partial t}(t)\frac{\partial \psi}{\partial t}(x,t) + \frac{\partial \theta_\delta}{\partial t}(t)\frac{\partial \psi}{\partial t}(x,t) + \theta_\delta(t)\frac{\partial^2 \psi}{\partial t^2}(x,t) \\
&\quad -\theta_\delta(t)\Delta\psi(x,t) + p(x)\theta_\delta(t)\psi(x,t) \\
&= \frac{\partial^2 \theta_\delta}{\partial t^2}(t)\psi(x,t) + 2\frac{\partial \theta_\delta}{\partial t}(t)\frac{\partial \psi}{\partial t}(x,t) + \theta_\delta(t)\left[\frac{\partial^2 \psi}{\partial t^2}(x,t) - \Delta\psi(x,t) + p(x)\psi(x,t)\right] \\
&= \frac{\partial^2 \theta_\delta}{\partial t^2}(t)\psi(x,t) + 2\frac{\partial \theta_\delta}{\partial t}(t)\frac{\partial \psi}{\partial t}(x,t).
\end{aligned}$$

Como $\psi \equiv 0$ sobre $\Gamma \times (0,T)$, o mesmo acontece para z. Além disso,

$$z(0) = \theta_\delta(0)\psi_0 = 0,$$

$$z(T) = \theta_\delta(T)\psi(x,T) = 0,$$
$$\frac{\partial z}{\partial t}(0) = \frac{\partial \theta_\delta}{\partial t}(0)\psi_0 + \theta_\delta(0)\frac{\partial \psi}{\partial t}(x,0) = 0,$$
$$\frac{\partial z}{\partial t}(T) = \frac{\partial \theta_\delta}{\partial t}(T)\psi(x,T) + \theta_\delta(T)\frac{\partial \psi}{\partial t}(x,T) = 0.$$

Dessa forma, z é solução do seguinte sistema:

$$\begin{cases} Lz := \dfrac{\partial^2 z}{\partial t^2} - \Delta z + p(x)z = 2\dfrac{\partial \theta_\delta}{\partial t}\dfrac{\partial \psi}{\partial t} + \dfrac{\partial^2 \theta_\delta}{\partial t^2}\psi & \text{em} \quad \Omega \times (0,T), \\ z = 0 & \text{em} \quad \Gamma \times (0,T), \\ z(0) = z(T) = 0, \ \dfrac{\partial z}{\partial t}(0) = \dfrac{\partial z}{\partial t}(T) = 0 & x \in \Omega. \end{cases} \tag{3.11}$$

Temos que

$$z \in L^2(0,T;L^2(\Omega)) \quad e \quad L_0 z = \frac{\partial^2 z}{\partial t^2} - \Delta z \in L^2(0,T;L^2(\Omega)).$$

Agora definamos, para $\lambda > 0$,

$$\varphi_M(x,t) = e^{\lambda \phi_M(x,t)}.$$

Podemos aplicar a desigualdade de Carleman (2.5) para z no intervalo $(0,T)$. De fato, as equações da onda são reversíveis no tempo, pois se fizermos $s = -t$ teremos $s \in (-T,0)$. Derivando z com relação a s,

$$\frac{\partial z}{\partial s}(x,s) = \frac{\partial z}{\partial t}(x,s)\frac{\partial t}{\partial s}(x,s) = -\frac{\partial z}{\partial t}(x,s).$$

Da mesma forma,

$$\frac{\partial^2 z}{\partial s^2}(x,s) = \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial s}(x,s)\right) = \frac{\partial}{\partial s}\left(-\frac{\partial z}{\partial t}(x,s)\right) = \frac{\partial^2 z}{\partial t^2}(x,s).$$

Observemos que estendendo o domínio de z, automaticamente, estamos estendendo o domínio de θ e ψ. Temos, portanto, que $Lz = 2\frac{\partial \theta_\delta}{\partial t}\frac{\partial \psi}{\partial t} + \frac{\partial^2 \theta_\delta}{\partial t^2}\psi$ em Q.

A função peso que escolhemos tem as mesmas propriedades que a do capítulo 2, mas está definida apenas no intervalo $(0,T)$. Esta mudança não traz grandes alterações no cálculo da desigualdade de Carleman, uma vez que as propriedades do peso que utilizadas são referentes a sua limitação e essas são mantidas.

Aplicando a desigualdade de Carleman (2.5) chegamos que existem constantes s_0, λ_0, $C > 0$ de tal forma que $\forall\ s > s_0$ e $\lambda > \lambda_0$

$$s\lambda\int_0^T\int_\Omega e^{2s\varphi_M}\left(\left|\frac{\partial z}{\partial t}\right|^2+|\nabla z|^2\right)dxdt + s^3\lambda^3\int_0^T\int_\Omega e^{2s\varphi_M}{\varphi_M}^3|z|^2dxdt$$

$$+\int_0^T\int_\Omega|P_1w|^2dxdt+\int_0^T\int_\Omega|P_2w|^2dxdt$$

$$\leq s\lambda\int_{-T}^T\int_\Omega e^{2s\varphi_M}\left(\left|\frac{\partial z}{\partial t}\right|^2+|\nabla z|^2\right)dxdt + s^3\lambda^3\int_{-T}^T\int_\Omega e^{2s\varphi_M}{\varphi_M}^3|z|^2dxdt$$

$$+\int_{-T}^T\int_\Omega|P_1w|^2dxdt+\int_T^T\int_\Omega|P_2w|^2dxdt$$

$$\leq C\int_{-T}^T\int_\Omega e^{2s\varphi_M}|Lz|^2dxdt + Cs\lambda\int_{-T}^T\int_{\Gamma_0}e^{2s\varphi_M}\left|\frac{\partial z}{\partial\nu}\right|^2d\sigma dt$$

$$= C\int_{-T}^0\int_\Omega e^{2s\varphi_M}|Lz|^2dxdt + C\int_0^T\int_\Omega e^{2s\varphi_M}|Lz|^2dxdt + Cs\lambda\int_{-T}^0\int_{\Gamma_0}e^{2s\varphi_M}\left|\frac{\partial z}{\partial\nu}\right|^2d\sigma dt$$

$$+Cs\lambda\int_0^T\int_{\Gamma_0}e^{2s\varphi_M}\left|\frac{\partial z}{\partial\nu}\right|^2d\sigma dt$$

$$= 2C\int_0^T\int_\Omega e^{2s\varphi_M}|Lz|^2dxdt 2Cs\lambda\int_0^T\int_{\Gamma_0}e^{2s\varphi_M}\left|\frac{\partial z}{\partial\nu}\right|^2d\sigma dt,$$

ou seja,

$$s\lambda\int_0^T\int_\Omega e^{2s\varphi_M}\left(\left|\frac{\partial z}{\partial t}\right|^2+|\nabla z|^2\right)dxdt$$

$$\leq C\int_0^T\int_\Omega e^{2s\varphi_M}|Lz|^2dxdt + Cs\lambda\int_0^T\int_{\Gamma_0}e^{2s\varphi_M}\left|\frac{\partial z}{\partial\nu}\right|^2d\sigma dt. \tag{3.12}$$

Observemos que

$$C\int_0^T\int_\Omega e^{2s\varphi_M}|Lz|^2dxdt = C\int_0^T\int_\Omega e^{2s\varphi_M}\left|2\frac{\partial\theta_\delta}{\partial t}\frac{\partial\psi}{\partial t}+\frac{\partial^2\theta_\delta}{\partial t^2}\psi\right|^2dxdt$$

$$= C\int_0^\delta\int_\Omega e^{2s\varphi_M}\left|2\frac{\partial\theta_\delta}{\partial t}\frac{\partial\psi}{\partial t}+\frac{\partial^2\theta_\delta}{\partial t^2}\psi\right|^2dxdt + C\int_\delta^{T-\delta}\int_\Omega e^{2s\varphi_M}\left|2\frac{\partial\theta_\delta}{\partial t}\frac{\partial\psi}{\partial t}+\frac{\partial^2\theta_\delta}{\partial t^2}\psi\right|^2dxdt$$

$$+ C\int_{T-\delta}^{T}\int_{\Omega} e^{2s\varphi_M}\left|2\frac{\partial\theta_\delta}{\partial t}\frac{\partial\psi}{\partial t}+\frac{\partial^2\theta_\delta}{\partial t^2}\psi\right|^2 dxdt.$$

Uma vez que as derivadas de θ_δ se anulam em $[\delta, T-\delta]$

$$C\int_0^T\int_\Omega e^{2s\varphi_M}|Lz|^2 dxdt$$
$$= Ce^{2s\varphi_M}\left(\int_0^\delta\int_\Omega\left|2\frac{\partial\theta_\delta}{\partial t}\frac{\partial\psi}{\partial t}+\frac{\partial^2\theta_\delta}{\partial t^2}\psi\right|^2 dxdt+\int_{T-\delta}^T\int_\Omega\left|2\frac{\partial\theta_\delta}{\partial t}\frac{\partial\psi}{\partial t}+\frac{\partial^2\theta_\delta}{\partial t^2}\psi\right|^2 dxdt\right). \quad (3.13)$$

Notemos que

$$\left|2\frac{\partial\theta_\delta}{\partial t}\frac{\partial\psi}{\partial t}+\frac{\partial^2\theta_\delta}{\partial t^2}\psi\right|^2 = 4\left|\frac{\partial\theta_\delta}{\partial t}\frac{\partial\psi}{\partial t}\right|^2 + 4\frac{\partial\theta_\delta}{\partial t}\frac{\partial\psi}{\partial t}\frac{\partial^2\theta_\delta}{\partial t^2}\psi + \left|\frac{\partial^2\theta_\delta}{\partial t^2}\psi\right|^2$$
$$\leq 4\left|\frac{\partial\theta_\delta}{\partial t}\right|^2\left|\frac{\partial\psi}{\partial t}\right|^2 + 4\left|\frac{\partial\theta_\delta}{\partial t}\right|\left|\frac{\partial\psi}{\partial t}\right|\left|\frac{\partial^2\theta_\delta}{\partial t^2}\right||\psi| + \left|\frac{\partial^2\theta_\delta}{\partial t^2}\right|^2|\psi|^2$$
$$\leq C\left|\frac{\partial\psi}{\partial t}\right|^2 + 4C\left|\frac{\partial\psi}{\partial t}\right||\psi| + C|\psi|^2 \leq C\left(\left|\frac{\partial\psi}{\partial t}\right|^2 + |\psi|^2\right).$$

Sendo $\phi_M(x,t)\leq M_0$, segue que $e^{2s\varphi_M}\leq e^{2se^{\lambda M_0}}$. Logo, (3.13) nos fornece

$$C\int_0^T\int_\Omega e^{2s\varphi_M}|Lz|^2 dxdt$$
$$\leq Ce^{2se^{\lambda M_0}}\left[\int_0^\delta\int_\Omega\left(\left|\frac{\partial\psi}{\partial t}\right|^2+|\psi|^2\right)dxdt+\int_{T-\delta}^T\int_\Omega\left(\left|\frac{\partial\psi}{\partial t}\right|^2+|\psi|^2\right)dxdt\right].$$

Usando o Teorema 1.11 (Desigualdade de Poincaré) temos

$$C\int_0^T\int_\Omega e^{2s\varphi_M}|Lz|^2 dxdt$$
$$\leq Ce^{2se^{\lambda M_0}}\left[\int_0^\delta\int_\Omega\left(\left|\frac{\partial\psi}{\partial t}\right|^2+|\nabla\psi|^2\right)dxdt+\int_{T-\delta}^T\int_\Omega\left(\left|\frac{\partial\psi}{\partial t}\right|^2+|\nabla\psi|^2\right)dxdt\right]$$
$$= Ce^{2se^{\lambda M_0}}\left(\int_0^\delta E(t)dt+\int_{T-\delta}^T E(t)dt\right)\leq Ce^{2se^{\lambda M_0}}\left(\int_0^\delta E_M dt+\int_{T-\delta}^T E_M dt\right)$$

$$= 2\delta C e^{2se^{\lambda M_0}} E_M$$

o que implica

$$C\int_0^T\int_\Omega e^{2s\varphi_M}|Lz|^2dxdt \leq 2\delta C e^{2se^{\lambda M_0}} E_M e^{C\|p\|_{L^\infty(Q)}\frac{T}{2}}. \tag{3.14}$$

Por outro lado,

$$\int_0^T\int_\Omega e^{2s\varphi_M}\left(\left|\frac{\partial z}{\partial t}\right|^2 + |\nabla z|^2\right)dxdt \geq \int_{\frac{T}{2}-\eta}^{\frac{T}{2}}\int_\Omega e^{2s\varphi_M}\left(\left|\frac{\partial z}{\partial t}\right|^2 + |\nabla z|^2\right)dxdt.$$

Como $\theta_\delta(t) = 1$ pra todo $t \in [T/2-\eta, T/2]$, temos

$$\frac{\partial z}{\partial t} = \frac{\partial \theta_\delta}{\partial t}(t)\psi(x,t) + \theta_\delta(t)\frac{\partial \psi}{\partial t}(x,t) \Rightarrow \left|\frac{\partial z}{\partial t}\right|^2 = \left|\frac{\partial \psi}{\partial t}\right|^2$$

e

$$\nabla z = \nabla\theta_\delta(t)\psi(x,t) + \theta_\delta(t)\nabla\psi(x,t) \Rightarrow |\nabla z|^2 = |\nabla \psi|^2.$$

Lembremos ainda que $\varphi_M \geq 1$ e que $\phi_M \geq M_0$ em $[T/2-\eta, T/2]$, donde $\lambda\phi_M \geq \lambda M_0$. Assim,

$$\int_0^T\int_\Omega e^{2s\varphi_M}\left(\left|\frac{\partial z}{\partial t}\right|^2 + |\nabla z|^2\right)dxdt$$

$$\geq \int_{\frac{T}{2}-\eta}^{\frac{T}{2}}\int_\Omega e^{2se^{\lambda M_0}}\left(\left|\frac{\partial \psi}{\partial t}\right|^2 + |\nabla \psi|^2\right)dxdt = 2e^{2se^{\lambda M_0}}\int_{\frac{T}{2}-\eta}^{\frac{T}{2}} E(t)dt. \tag{3.15}$$

Observemos que para t pertencente ao intervalo $[T/2-\eta, T/2]$ é válido que $\eta \geq T/2 - t \geq 0$ e, portanto,

$$-C\|p\|_{L^\infty(Q)}\left(\frac{T}{2} - t\right) \geq -C\|p\|_{L^\infty(Q)}\eta.$$

Com esse resultado e (3.8), obtemos por (3.15) que

$$\int_0^T\int_\Omega e^{2s\varphi_M}\left(\left|\frac{\partial z}{\partial t}\right|^2 + |\nabla z|^2\right)dxdt \geq 2\eta e^{2se^{\lambda M_0}} E_M e^{-C\|p\|_{L^\infty(Q)}\eta}. \tag{3.16}$$

A partir de (3.9), (3.12), (3.14) e (3.16) podemos afirmar que se $s \geq s_0$ e $\lambda \geq \lambda_0$ então

$$2s\lambda\eta e^{2se^{\lambda M_0}} E_M e^{-C\|p\|_{L^\infty(Q)}\eta} \leq 2\delta C e^{2se^{\lambda M_0}} E_M e^{C\|p\|_{L^\infty(Q)}\frac{T}{2}} + Cs\lambda\int_0^T\int_{\Gamma_0} e^{2s\varphi_M}\left|\frac{\partial z}{\partial \nu}\right|^2 d\sigma dt.$$

Dividindo a inequação acima por $e^{2se^{\lambda M_0}}$ e reajustando os termos, segue

$$2s\lambda\eta E_M e^{-C\|p\|_{L^\infty(Q)}\eta} \leq 2\delta C E_M e^{C\|p\|_{L^\infty(Q)}\frac{T}{2}} + Cs\lambda \int_0^T \int_{\Gamma_0} e^{2s(\varphi_M - e^{\lambda M_0})} \left|\frac{\partial z}{\partial \nu}\right|^2 d\sigma dt,$$

isto é,

$$2\left(\frac{s\lambda\eta}{e^{C\|p\|_{L^\infty(Q)}\eta}} - \delta C e^{C\|p\|_{L^\infty(Q)}\frac{T}{2}}\right) E_M \leq Cs\lambda \int_0^T \int_{\Gamma_0} e^{2s(\varphi_M - e^{\lambda M_0})} \left|\frac{\partial z}{\partial \nu}\right|^2 d\sigma dt.$$

Multiplicando a última desigualdade por $e^{C\|p\|_{L^\infty(Q)}\eta}/s^2\lambda^2\eta$, deduzimos que

$$\left(\frac{1}{s\lambda} - \frac{\delta C e^{C\|p\|_{L^\infty(Q)}\left(\frac{T}{2}+\eta\right)}}{s^2\lambda^2\eta}\right) E_M \leq \frac{C}{2s\lambda\eta} e^{C\|p\|_{L^\infty(Q)}\eta} \int_0^T \int_{\Gamma_0} e^{2s(\varphi_M - e^{\lambda M_0})} \left|\frac{\partial z}{\partial \nu}\right|^2 d\sigma dt.$$

Tomemos s_0, λ_0 grandes o suficiente tal que

$$1 - \frac{\delta C e^{C\|p\|_{L^\infty(Q)}\left(\frac{T}{2}+\eta\right)}}{s\lambda\eta} > 0.$$

Daí

$$\begin{aligned} E_M &\leq C \int_0^T \int_{\Gamma_0} e^{2s(\varphi_M - e^{\lambda M_0})} \left|\frac{\partial z}{\partial \nu}\right|^2 d\sigma dt \leq C \int_0^T \int_{\Gamma_0} e^{2s(\varphi_M - e^{\lambda M_0})} \left|\frac{\partial}{\partial \nu}(\theta_\delta \psi)\right|^2 d\sigma dt \\ &\leq C \int_0^T \int_{\Gamma_0} e^{2s(\varphi_M - e^{\lambda M_0})} |\theta_\delta|^2 \left|\frac{\partial \psi}{\partial \nu}\right|^2 d\sigma dt \\ &\leq C \int_0^T \int_{\Gamma_0} e^{2s(\varphi_M - e^{\lambda M_0})} \left|\frac{\partial \psi}{\partial \nu}\right|^2 d\sigma dt. \qquad (3.17) \end{aligned}$$

Combinando (3.9) (com $t = 0$) e (3.17), podemos concluir que

$$E_0 \leq C e^{C\|p\|_{L^\infty(\Omega)}\frac{T}{2}} \int_0^T \int_{\Gamma_0} \left|\frac{\partial \psi}{\partial \nu}\right|^2 d\sigma dt,$$

o que prova o resultado.

Capítulo 4

Aplicação da desigualdade de Carleman ao problema inverso

Iremos agora aplicar a desigualdade de Carleman para encontrar resultados de unicidade e estabilidade relativos ao potencial p.

Consideremos o sistema (1.2), onde p é um potencial limitado desconhecido que depende apenas da variável espacial. Iremos assumir que para cada $p \in L^\infty(\Omega)$ o sistema possua solução única (ver Teorema 1.4) e que faça sentido calcularmos $\frac{\partial u(p)}{\partial \nu}$ restrita a uma parte da fronteira $\Gamma \times (0,T)$.

Queremos encontrar um resultado de estabilidade, ou seja, uma desigualdade tal que

$$\|p - q\|_{X(\Omega)} \leq C \left\| \frac{\partial u}{\partial \nu}(p) - \frac{\partial u}{\partial \nu}(q) \right\|_{Y(\Gamma_0)}, \tag{4.1}$$

onde $X(\Omega)$ e $Y(\Gamma_0)$ são espaços convenientes. Além disso, queremos um resultado de unicidade, isto é,

$$\text{se} \quad \frac{\partial u(p)}{\partial \nu} = \frac{\partial u(q)}{\partial \nu} \quad \text{sobre} \quad \Gamma_0 \times (0,T) \quad \text{então} \quad p = q.$$

Percebamos que a estabilidade implica a unicidade.

Tomemos $p, q \in L^\infty(\Omega)$. Por hipótese, os sistemas

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u + p\, u = f & \text{em } \Omega \times (0,T), \\ u = g & \text{sobre } \Gamma \times (0,T), \\ u(0) = u_0, \quad \dfrac{\partial u}{\partial t}(0) = u_1 & \text{em } \Omega. \end{cases} \tag{4.2}$$

e

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u + q\, u = f & \text{em } \Omega \times (0,T), \\ u = g & \text{sobre } \Gamma \times (0,T), \\ u(0) = u_0, \quad \dfrac{\partial u}{\partial t}(0) = u_1 & \text{em } \Omega, \end{cases} \tag{4.3}$$

possuem única solução que chamaremos, respectivamente, de $u(p)$ e $u(q)$. Fazendo $z = u(p) - u(q)$, então z é solução de

$$\begin{cases} \dfrac{\partial^2 z}{\partial t^2} - \Delta z + pz = fR & \text{em } \Omega \times (0,T), \\ z = 0 & \text{sobre } \Gamma \times (0,T), \\ z(0) = 0, \quad \dfrac{\partial z}{\partial t}(0) = 0 & \text{em } \Omega, \end{cases} \tag{4.4}$$

onde $f = q - p$ e $R = u(q)$.

Supondo $p = p(x)$ um potencial conhecido e z solução de (4.4) seria possível encontrar uma desigualdade da forma

$$\|f\|^2_{X(\Omega)} \leq C \left\| \frac{\partial z}{\partial \nu} \right\|^2_{Y(\Gamma_0 \times (0,T))} ?$$

O seguinte resultado dará uma resposta afirmativa a esta pergunta.

Teorema 4.1 *Assumamos que* $\|p\|_{L^\infty(\Omega)} \leq m$ *e*

- $\exists\, x_0 \in \mathbb{R}^N \backslash \overline{\Omega}$ *com* $\Gamma_0 \supset \Gamma_{x_0} = \{x \in \Gamma, (x - x_0).\nu(x) > 0\}$,
- $T > \sup\limits_{x \in \overline{\Omega}} |x - x_0|$,
- $R \in H^1(0,T; L^\infty(\Omega))$ *com* $|R(x,0)| \geq a_0 > 0$.

Então existe $C > 0$*, dependendo de* p*, tal que*

$$\frac{1}{C} \|f\|^2_{L^2(\Omega)} \leq \left\| \frac{\partial z}{\partial \nu} \right\|^2_{H^1(0,T;L^2(\Gamma_0))},$$

para toda solução z *do problema (4.4).*

Demonstração: Pelo Teorema 1.1 sabemos que (4.4) possui única solução z. Estendendo $\frac{\partial z}{\partial t}$ e $\frac{\partial R}{\partial t}$ em $(-T,0)$ e derivando $(4.4)_1$ obtemos

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial z}{\partial t}\right)-\Delta\left(\frac{\partial z}{\partial t}\right)+p(x)\frac{\partial z}{\partial t}=f(x)\frac{\partial R}{\partial t}(x,t) \qquad em \;\; Q.$$

Observemos que

$$\frac{\partial^2 z}{\partial t^2}(0)=f(x)R(x,0)+\Delta z(0)+p(x)z(0)=f(x)R(x,0) \quad em \;\; \Omega.$$

Assim,

$$\begin{cases} \dfrac{\partial^3 z}{\partial t^3}-\Delta\left(\dfrac{\partial z}{\partial t}\right)+p\dfrac{\partial z}{\partial t}=f(x)\dfrac{\partial R}{\partial t}(x,t) & \text{em } Q, \\ \dfrac{\partial z}{\partial t}=0 & \text{sobre } \Sigma, \\ \dfrac{\partial z}{\partial t}(0)=0; \dfrac{\partial^2 z}{\partial t^2}(0)=f(x)R(x,0) & \text{em } \Omega. \end{cases} \tag{4.5}$$

Para $\delta>0$ suficientemente pequeno, consideremos uma função $\theta\in C_0^\infty(-T,T)$ que satisfaça

$$0\leq\theta\leq 1 \;\; e \;\; \theta(t)=1, \; se \; -T+\delta\leq t\leq T-\delta. \tag{4.6}$$

Façamos

$$v(x,t)=\theta(t)\frac{\partial z}{\partial t}(x,t)$$

e notemos que

$$\frac{\partial^2 v}{\partial t^2}-\Delta v+pv=\frac{\partial^2}{\partial t^2}\left(\theta(t)\frac{\partial z}{\partial t}(x,t)\right)-\Delta\left(\theta(t)\frac{\partial z}{\partial t}(x,t)\right)+p(x)\theta(t)\frac{\partial z}{\partial t}(x,t)$$

$$=\frac{\partial}{\partial t}\left(\frac{\partial\theta}{\partial t}(t)\frac{\partial z}{\partial t}(x,t)+\theta(t)\frac{\partial^2 z}{\partial t^2}(x,t)\right)-\theta(t)\Delta\left(\frac{\partial z}{\partial t}(x,t)\right)+p(x)\theta(t)\frac{\partial z}{\partial t}(x,t)$$

$$=\frac{\partial^2\theta}{\partial t^2}(t)\frac{\partial z}{\partial t}(x,t)+\frac{\partial\theta}{\partial t}(t)\frac{\partial^2 z}{\partial t^2}(x,t)+\frac{\partial\theta}{\partial t}(t)\frac{\partial^2 z}{\partial t^2}(x,t)+\theta(t)\frac{\partial^3 z}{\partial t^3}(x,t)$$

$$-\theta(t)\Delta\left(\frac{\partial z}{\partial t}(x,t)\right)+p(x)\theta(t)\frac{\partial z}{\partial t}(x,t)$$

$$= \theta(t)\left(\frac{\partial^3 z}{\partial t^3}(x,t) - \Delta\left(\frac{\partial z}{\partial t}(x,t)\right) + p(x)\frac{\partial z}{\partial t}(x,t)\right) + 2\frac{\partial \theta}{\partial t}(t)\frac{\partial^2 z}{\partial t^2}(x,t) + \frac{\partial^2 \theta}{\partial t^2}(t)\frac{\partial z}{\partial t}(x,t)$$

$$= \theta(t)f(x)\frac{\partial R}{\partial t}(x,t) + 2\frac{\partial \theta}{\partial t}(t)\frac{\partial^2 z}{\partial t^2}(x,t) + \frac{\partial^2 \theta}{\partial t^2}(t)\frac{\partial z}{\partial t}(x,t).$$

Como $\frac{\partial z}{\partial t} = 0$ sobre Σ e $\frac{\partial z(0)}{\partial t} = 0$ em Ω, segue que

$$v = 0 \quad sobre \;\; \Sigma \qquad e \qquad v(0) = 0 \quad em \quad \Omega.$$

Além disso,

$$\frac{\partial v}{\partial t}(0) = \frac{\partial \theta}{\partial t}(0)\frac{\partial z}{\partial t}(0) + \theta(0)\frac{\partial^2 z}{\partial t^2}(0) = f(x)R(x,0) \qquad em \quad \Omega.$$

Uma vez que $\theta \in C_0^\infty(-T,T)$ temos que $\theta(-T) = \theta(T) = \frac{\partial \theta}{\partial t}(-T) = \frac{\partial \theta}{\partial t}(T) = 0$. Logo,

$$v(-T) = \theta(-T)\frac{\partial z}{\partial t}(-T) = 0 = \theta(T)\frac{\partial z}{\partial t}(T) = v(T),$$

$$\frac{\partial v}{\partial t}(-T) = \frac{\partial \theta}{\partial t}(-T)\frac{\partial z}{\partial t}(-T) + \theta(-T)\frac{\partial^2 z}{\partial t^2}(-T) = 0,$$

$$\frac{\partial v}{\partial t}(T) = \frac{\partial \theta}{\partial t}(T)\frac{\partial z}{\partial t}(T) + \theta(T)\frac{\partial^2 z}{\partial t^2}(T) = 0.$$

Reunindo todas estas informações, deduzimos que v é solução do seguinte sistema:

$$\begin{cases} \dfrac{\partial^2 v}{\partial t^2} - \Delta v + p(x)v = \theta(t)f(x)\dfrac{\partial R}{\partial t} + 2\dfrac{\partial \theta}{\partial t}\dfrac{\partial^2 z}{\partial t^2} + \dfrac{\partial^2 \theta}{\partial t^2}\dfrac{\partial z}{\partial t} & \text{em} \quad Q, \\ v = 0 & \text{em} \quad \Sigma \\ v(0) = 0, \;\; \dfrac{\partial v}{\partial t}(0) = f(x)R(x,0) & \text{em} \quad \Omega, \\ v(T) = \dfrac{\partial v}{\partial t}(T) = 0 & \text{em} \quad \Omega, \\ v(-T) = \dfrac{\partial v}{\partial t}(-T) = 0 & \text{em} \quad \Omega. \end{cases} \tag{4.7}$$

Aplicando a desigualdade global de Carleman da forma como aparece em (2.25) para (4.7), temos

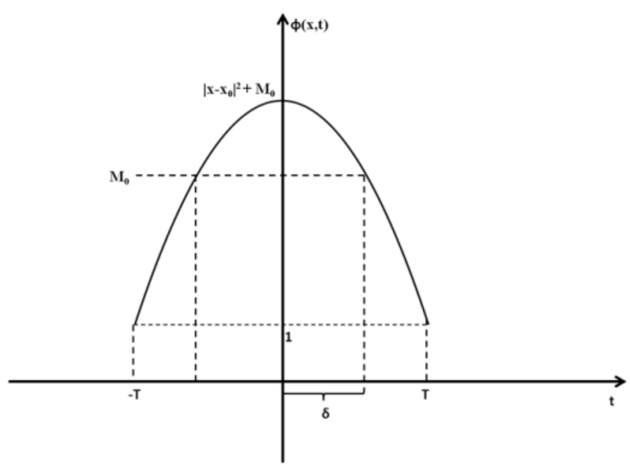
$$C\int_{-T}^{T}\int_{\Omega} e^{2s\varphi_\lambda}|Lv|^2 dxdt + Cs\lambda\int_{-T}^{T}\int_{\Gamma_0} e^{2s\varphi_\lambda}\left|\frac{\partial v}{\partial \nu}\right|^2 d\sigma dt$$

$$\geq s\lambda\int_{-T}^{T}\int_{\Omega}\left|\frac{\partial w}{\partial t}\right|^2 dxdt + s^3\lambda^3\int_{-T}^{T}\int_{\Omega}{\varphi_\lambda}^3|w|^2 dxdt + \int_{-T}^{T}\int_{\Omega}|P_1 w|^2 dxdt. \tag{4.8}$$

Sendo $T > \sup_{x\in\overline{\Omega}} |x - x_0| = \rho$, segue que $(\rho/T)^2 < 1$. Tomemos Dessa $\beta > 0$ tal que

$$\frac{\rho^2}{T^2} < \beta < 1.$$

O peso que tomamos no cálculo da desigualdade de Carleman é dado pela função $\phi(x,t) = |x - x_0|^2 - \beta t^2 + M_0$. Para cada $x \in \Omega$, temos que o esboço do gráfico de $\phi(x,t)$ é o seguinte



Com base no gráfico de ϕ, podemos afirmar que existe $\delta > 0$ satisfazendo

$$\phi(x,t) < M_0 < \phi(x,0), \quad \forall\ t \in [-T, -T+\delta] \cup [T-\delta, T]. \tag{4.9}$$

Escolhendo este δ na definição de θ dada em (4.6), temos que as derivadas desta função se anulam em $[-T+\delta, T-\delta]$. Com esta informação, iremos em busca de uma estimativa para

$$\int_{-T}^{T}\int_{\Omega} e^{2s\varphi_\lambda}|Lv|^2 dxdt.$$

Inicialmente, multipliquemos a equação $(4.5)_1$ por $\frac{\partial^2 z(t)}{\partial t^2}$ e integremos em Ω.

$$\left|\int_\Omega f(x)\frac{\partial R}{\partial t}(x,t)\frac{\partial^2 z}{\partial t^2}(x,t)dx\right| \geq \int_\Omega f(x)\frac{\partial R}{\partial t}(x,t)\frac{\partial^2 z}{\partial t^2}(x,t)dx$$

$$= \int_\Omega \frac{\partial^3 z}{\partial t^3}(x,t)\frac{\partial^2 z}{\partial t^2}(x,t)dx - \int_\Omega \Delta\left(\frac{\partial z}{\partial t}(x,t)\right)\frac{\partial^2 z}{\partial t^2}(x,t)dx + \int_\Omega p(x)\frac{\partial z}{\partial t}(x,t)\frac{\partial^2 z}{\partial t^2}(z,t)dx$$

$$= \frac{1}{2}\frac{\partial}{\partial t}\left\|\frac{\partial^2 z}{\partial t^2}(t)\right\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{\partial}{\partial t}\left\|\frac{\partial z}{\partial t}(t)\right\|^2_{H_0^1(\Omega)} + \frac{1}{2}\frac{\partial}{\partial t}\left\|\sqrt{p}\ \frac{\partial z}{\partial t}(t)\right\|^2_{L^2(\Omega)}. \tag{4.10}$$

Integrando (4.10) de 0 a t, usando $(4.5)_3$ e que $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, temos

$$
\begin{aligned}
&\frac{1}{2}\|fR(0)\|^2_{L^2(\Omega)} + \int_0^t \left| \int_\Omega f(x) \frac{\partial R}{\partial t}(x,t) \frac{\partial^2 z}{\partial t^2}(x,t) dx \right| dt \\
&\geq \frac{1}{2}\left\| \frac{\partial^2 z}{\partial t^2}(t) \right\|^2_{L^2(\Omega)} + \frac{1}{2}\left\| \frac{\partial z}{\partial t}(t) \right\|^2_{H_0^1(\Omega)} + \frac{1}{2}\left\| \sqrt{p}\ \frac{\partial z}{\partial t}(t) \right\|^2_{L^2(\Omega)} \\
&\geq \frac{1}{2}\left\| \frac{\partial^2 z}{\partial t^2}(t) \right\|^2_{L^2(\Omega)} + \frac{C}{2}\left\| \frac{\partial z}{\partial t}(t) \right\|^2_{L^2(\Omega)}. \qquad (4.11)
\end{aligned}
$$

Dessa forma, os Teoremas 1.8 (Desigualdade de Cauchy-Schwarz), 1.10 (Desigualdade de Young) e por $|R(x,0)| \geq a_0 > 0$, (4.11) implica

$$
\begin{aligned}
&\frac{1}{2}\left\| \frac{\partial^2 z}{\partial t^2}(t) \right\|^2_{L^2(\Omega)} + \frac{C}{2}\left\| \frac{\partial z}{\partial t}(t) \right\|^2_{L^2(\Omega)} \\
&\leq \frac{1}{2}\|f\, R(0)\|^2_{L^2(\Omega)} + \int_0^t \|f\|_{L^2(\Omega)} \left\| \frac{\partial R}{\partial t}(t) \frac{\partial^2 z}{\partial t^2}(t) \right\|_{L^2(\Omega)} dt \\
&\leq \frac{1}{2}\int_\Omega |f(x)|^2\, |R(0,x)|^2 dx + \frac{1}{2}\int_0^t \|f\|^2_{L^2(\Omega)} dt + \frac{1}{2}\int_0^t \int_\Omega \left| \frac{\partial R}{\partial t}(x,t) \right|^2 \left| \frac{\partial^2 z}{\partial t^2}(x,t) \right|^2 dx dt \\
&\leq C\|f\|^2_{L^2(\Omega)} + \frac{T}{2}\|f\|^2_{L^2(\Omega)} + \frac{1}{2}\int_0^t \left\| \frac{\partial R}{\partial t}(t) \right\|^2_{L^\infty(\Omega)} \left\| \frac{\partial^2 z}{\partial t^2}(t) \right\|^2_{L^2(\Omega)} dt \\
&\leq C\|f\|^2_{L^2(\Omega)} + \frac{1}{2}\int_0^t \left\| \frac{\partial R}{\partial t}(t) \right\|^2_{L^\infty(\Omega)} \left(\left\| \frac{\partial^2 z}{\partial t^2}(t) \right\|^2_{L^2(\Omega)} + C\left\| \frac{\partial z}{\partial t}(t) \right\|^2_{L^2(\Omega)} \right) dt. \qquad (4.12)
\end{aligned}
$$

Aplicando o Lema de Gronwall, em (4.12), obtemos a existência de $C > 0$ tal que

$$
\left\| \frac{\partial^2 z}{\partial t^2}(t) \right\|^2_{L^2(\Omega)} + \left\| \frac{\partial z}{\partial t}(t) \right\|^2_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}. \qquad (4.13)
$$

Vejamos que

$$
\begin{aligned}
|Lv|^2 &= \left| \theta(t) f(x) \frac{\partial R}{\partial t}(x,t) + 2\frac{\partial \theta}{\partial t}(t) \frac{\partial^2 z}{\partial t^2}(x,t) + \frac{\partial^2 \theta}{\partial t^2}(t) \frac{\partial z}{\partial t}(x,t) \right|^2 \\
&= \left| \theta(t) f(x) \frac{\partial R}{\partial t}(x,t) \right|^2 + \left| 2\frac{\partial \theta}{\partial t}(t) \frac{\partial^2 z}{\partial t^2}(x,t) \right|^2 + \left| \frac{\partial^2 \theta}{\partial t^2}(t) \frac{\partial z}{\partial t}(x,t) \right|^2
\end{aligned}
$$

$$+4\theta(t)f(x)\frac{\partial R}{\partial t}(x,t)\frac{\partial \theta}{\partial t}(t)\frac{\partial^2 z}{\partial t^2}(x,t)+2\theta(t)f(x)\frac{\partial R}{\partial t}(x,t)\frac{\partial^2 \theta}{\partial t^2}(t)\frac{\partial z}{\partial t}(x,t)$$

$$+4\frac{\partial \theta}{\partial t}(t)\frac{\partial^2 z}{\partial t^2}(x,t)\frac{\partial^2 \theta}{\partial t^2}(t)\frac{\partial z}{\partial t}(x,t)$$

$$\leq C\left|f(x)\frac{\partial R}{\partial t}(x,t)\right|^2+C\left|\frac{\partial^2 z}{\partial t^2}(x,t)\right|^2+C\left|\frac{\partial z}{\partial t}(x,t)\right|^2+C\left|f(x)\frac{\partial R}{\partial t}(x,t)\frac{\partial^2 z}{\partial t^2}(x,t)\right|$$

$$+C\left|f(x)\frac{\partial R}{\partial t}(x,t)\frac{\partial z}{\partial t}(x,t)\right|+C\left|\frac{\partial^2 z}{\partial t^2}(x,t)\frac{\partial z}{\partial z}(x,t)\right|. \tag{4.14}$$

Usando o Teorema 1.10 (Desigualdade de Young) em (4.14), segue que existe $C>0$ tal que

$$|Lv|^2 \leq C\left[|f(x)|^2\left|\frac{\partial R}{\partial t}(x,t)\right|^2+\left|\frac{\partial^2 z}{\partial t^2}(x,t)\right|^2+\left|\frac{\partial z}{\partial z}(x,t)\right|^2\right]. \tag{4.15}$$

Por (4.6) basta integrarmos $e^{2s\varphi_\lambda}|Lv|^2$ para $t\in[-T,-T+\delta]\cup[T-\delta,T]$. Por (4.9) $\phi(x,t)$ é sempre menor ou igual a $\phi(x,0)$ neste intervalo. Assim,

$$\int_{-T}^{T}\int_\Omega e^{2s\varphi_\lambda}|Lv|^2dxdt \leq \int_{-T}^{-T+\delta}\int_\Omega e^{2s\varphi_\lambda(0)}|Lv|^2dxdt+\int_{T-\delta}^{T}\int_\Omega e^{2s\varphi_\lambda(0)}|Lv|^2dxdt.$$

$$\leq C\int_{-T}^{-T+\delta}\int_\Omega e^{2s\varphi_\lambda(0)}\left[|f(x)|^2\left|\frac{\partial R}{\partial t}(x,t)\right|^2+\left|\frac{\partial^2 z}{\partial t^2}(x,t)\right|^2+\left|\frac{\partial z}{\partial t}(x,t)\right|^2\right]dxdt$$

$$+C\int_{T-\delta}^{T}\int_\Omega e^{2s\varphi_\lambda(0)}\left[|f(x)|^2\left|\frac{\partial R}{\partial t}(x,t)\right|^2+\left|\frac{\partial^2 z}{\partial t^2}(x,t)\right|^2+\left|\frac{\partial z}{\partial t}(x,t)\right|^2\right]dxdt. \tag{4.16}$$

Pela desigualdade dada em (4.13) podemos reescrever (4.16) como

$$\int_{-T}^{T}\int_\Omega e^{2s\varphi_\lambda}|Lv|^2dxdt$$

$$\leq C\int_{-T}^{-T+\delta}\left\|\frac{\partial R}{\partial t}\right\|^2_{L^\infty(\Omega)}\int_\Omega|e^{s\varphi_\lambda(0)}f(x)|^2dxdt+C\int_{-T}^{T+\delta}\|f\|^2_{L^2(\Omega)}dt$$

$$+C\int_{T-\delta}^{T}\left\|\frac{\partial R}{\partial t}\right\|^2_{L^\infty(\Omega)}\int_\Omega|e^{s\varphi_\lambda(0)}f(x)|^2dxdt+C\int_{T-\delta}^{T}\|f\|^2_{L^2(\Omega)}dt$$

$$\leq C\|e^{s\varphi_\lambda(0)}f\|^2_{L^2(\Omega)}\left\|\frac{\partial R}{\partial t}\right\|^2_{L^2(-T,T;L^\infty(\Omega))}+2\delta Ce^{2s\varphi_\lambda(0)}\|f\|^2_{L^2(\Omega)}dt.$$

Portanto,

$$\int_{-T}^{T}\int_\Omega e^{2s\varphi_\lambda}|Lv|^2dxdt \leq C\int_\Omega e^{2s\varphi_\lambda(0)}|f(x)|^2dx. \tag{4.17}$$

Busquemos agora estimativas para a integral de $P_1 w\frac{\partial w}{\partial t}$ em $\Omega\times(-T,0)$, onde $w=e^{2s\varphi_\lambda}v$ e P_1w são definidos como no capítulo 2.

$$\begin{aligned}
&\int_{-T}^{0}\int_\Omega P_1w\frac{\partial w}{\partial t}dxdt\\
&=\int_{-T}^{0}\int_\Omega\left[\frac{\partial^2 w}{\partial t^2}-\Delta w+s^2\lambda^2{\varphi_\lambda}^2\left(\left(\frac{\partial\phi}{\partial t}\right)^2-|\nabla\phi|^2\right)w\right]\frac{\partial w}{\partial t}dxdt\\
&=\frac{1}{2}\int_{-T}^{0}\int_\Omega\frac{\partial}{\partial t}\left|\frac{\partial w}{\partial t}\right|^2dxdt+\int_{-T}^{0}\int_\Omega\left[-\Delta w+s^2\lambda^2{\varphi_\lambda}^2\left(\left(\frac{\partial\phi}{\partial t}\right)^2-|\nabla\phi|^2\right)w\right]\frac{\partial w}{\partial t}dxdt\\
&\geq\frac{1}{2}\int_\Omega\left|\frac{\partial w}{\partial t}(0)\right|^2dx+\int_{-T}^{0}\frac{\partial}{\partial t}\|w\|^2_{H_0^1(\Omega)}dt+s^2\lambda^2\int_{-T}^{0}\int_\Omega\left(\left(\frac{\partial\phi}{\partial t}\right)^2-|\nabla\phi|^2\right)\frac{\partial|w|^2}{\partial t}dxdt\\
&=\frac{1}{2}\int_\Omega\left|\frac{\partial w}{\partial t}(0)\right|^2dx+s^2\lambda^2\int_{-T}^{0}\int_\Omega\frac{\partial}{\partial t}\left[\left(\left(\frac{\partial\phi}{\partial t}\right)^2-|\nabla\phi|^2\right)|w|^2\right]dxdt\\
&-s^2\lambda^2\int_{-T}^{0}\int_\Omega\frac{\partial}{\partial t}\left(\left(\frac{\partial\phi}{\partial t}\right)^2-|\nabla\phi|^2\right)|w|^2dxdt\\
&=\frac{1}{2}\int_\Omega\left|\frac{\partial w}{\partial t}(0)\right|^2dx-2s^2\lambda^2\int_{-T}^{0}\int_\Omega\frac{\partial\phi}{\partial t}\frac{\partial^2\phi}{\partial t^2}|w|^2dxdt\\
&\geq\frac{1}{2}\int_\Omega\left|\frac{\partial w}{\partial t}(0)\right|^2dx-8s^3\lambda^3\int_{-T}^{0}\int_\Omega\beta^2t|w|^2dxdt\\
&\geq\frac{1}{2}\int_\Omega\left|\frac{\partial w}{\partial t}(0)\right|^2dx-8s^3\lambda^3\beta^2\frac{T}{2}\int_{-T}^{0}\int_\Omega|w|^2dxdt,
\end{aligned}$$

donde concluímos que,

$$\int_{-T}^{0}\int_\Omega P_1w\frac{\partial w}{\partial t}dxdt$$

$$\geq \frac{1}{2}\int_\Omega \left|\frac{\partial w}{\partial t}(0)\right|^2 dx - Cs^3\lambda^3 \int_{-T}^0 \int_\Omega {\varphi_\lambda}^3 |w|^2 dxdt$$

$$\geq \frac{1}{2}\int_\Omega \left|\frac{\partial w}{\partial t}(0)\right|^2 dx - Cs^3\lambda^3 \int_{-T}^T \int_\Omega {\varphi_\lambda}^3 |w|^2 dxdt.$$

Notemos que

$$\frac{\partial w}{\partial t}(0) = \frac{\partial (e^{s\varphi_\lambda} v)}{\partial t}(0) = \frac{\partial e^{s\varphi_\lambda}}{\partial t}(0)v(0) + e^{s\varphi_\lambda(0)}\frac{\partial v}{\partial t}(0) = e^{s\varphi_\lambda(0)}\frac{\partial v}{\partial t}(0) = e^{s\varphi_\lambda(0)} f(x)R(x,0).$$

Com este resultado e a hipótese que $|R(x,0)| \geq a_0$, obtemos

$$\int_{-T}^0 \int_\Omega P_1 w \frac{\partial w}{\partial t} dxdt \geq \frac{1}{2}\int_\Omega e^{2s\varphi_\lambda(0)} |f|^2 a_0^2 dx - Cs^3\lambda^3 \int_{-T}^T \int_\Omega {\varphi_\lambda}^3 |w|^2 dxdt.$$

Logo,

$$\frac{1}{2}\int_\Omega e^{2s\varphi_\lambda(0)} |f|^2 a_0^2 dx \leq \int_{-T}^0 \int_\Omega |P_1 w| \left|\frac{\partial w}{\partial t}\right| dxdt + Cs^3\lambda^3 \int_{-T}^T \int_\Omega {\varphi_\lambda}^3 |w|^2 dxdt$$

$$= \int_{-T}^0 \int_\Omega \frac{1}{\sqrt[4]{s}} |P_1 w| \sqrt[4]{s} \left|\frac{\partial w}{\partial t}\right| dxdt + Cs^3\lambda^3 \int_{-T}^T \int_\Omega {\varphi_\lambda}^3 |w|^2 dxdt. \tag{4.18}$$

Aplicando o Teorema 1.10 (Desigualdade de Young) na primeira integral do lado direito de (4.18) e supondo $s_0, \lambda_0 \geq 1$, segue que

$$\int_\Omega e^{2s\varphi_\lambda(0)} |f|^2 a_0^2 dx$$

$$\leq \frac{1}{\sqrt{s}} \int_{-T}^0 \int_\Omega |P_1 w|^2 dxdt + \sqrt{s} \int_{-T}^0 \int_\Omega \left|\frac{\partial w}{\partial t}\right|^2 dxdt + Cs^3\lambda^3 \int_{-T}^T \int_\Omega {\varphi_\lambda}^3 |w|^2 dxdt$$

$$\leq \frac{1}{\sqrt{s}} \int_{-T}^T \int_\Omega |P_1 w|^2 dxdt + \sqrt{s}\lambda \int_{-T}^T \int_\Omega \left|\frac{\partial w}{\partial t}\right|^2 dxdt + Cs^3\lambda^3 \int_{-T}^T \int_\Omega {\varphi_\lambda}^3 |w|^2 dxdt. \tag{4.19}$$

De (4.8) e (4.17), chegamos a

$$\frac{C}{\sqrt{s}} \int_{-T}^T \int_\Omega |P_1 w|^2 dxdt + Cs^3\lambda^3 \int_{-T}^T \int_\Omega {\varphi_\lambda}^3 |w|^2 dxdt + C\sqrt{s}\lambda \int_{-T}^T \int_\Omega |\frac{\partial w}{\partial t}|^2 dxdt$$

$$\leq \frac{C}{\sqrt{s}} \int_\Omega e^{2s\varphi_\lambda(0)} |f|^2 dx + C\sqrt{s}\lambda \int_{-T}^T \int_{\Gamma_0} \varphi_\lambda e^{2s\varphi_\lambda} \left|\frac{\partial v}{\partial \nu}\right|^2 d\sigma dt. \tag{4.20}$$

Assim, combinando (4.19) com (4.20) obtemos

$$\left(1 - \frac{C}{a_0^2\sqrt{s}}\right)\int_\Omega e^{2s\varphi_\lambda(0)}|f|^2 a_0^2 dx \le C\sqrt{s}\lambda \int_{-T}^{T}\int_{\Gamma_0} \varphi_\lambda e^{2s\varphi_\lambda}\left|\frac{\partial v}{\partial \nu}\right|^2 d\sigma dt.$$

Escolhendo s grande o suficiente temos que $1 - C/a_0^2\sqrt{s} > 0$ e, dessa forma,

$$\int_\Omega e^{2s\varphi_\lambda(0)}|f|^2 a_0^2 dx \le C\sqrt{s}\lambda \int_{-T}^{T}\int_{\Gamma_0} \varphi_\lambda e^{2s\varphi_\lambda}\left|\frac{\partial v}{\partial \nu}\right|^2 d\sigma dt.$$

Uma vez que $\phi(x,t)$ é limitada superiormente, $\varphi_\lambda = e^{\lambda\phi(x,t)}$ e $e^{2s\varphi_\lambda}$ também são e encontramos

$$a_0^2 \int_\Omega |f|^2 dx \le C\sqrt{s}\lambda \int_{-T}^{T}\int_{\Gamma_0} \left|\frac{\partial v}{\partial \nu}\right|^2 d\sigma dt,$$

o que implica

$$\|f\|_{L^2(\Omega)}^2 dx \le C\left\|\frac{\partial z}{\partial \nu}\right\|_{L^2(0,T;L^2(\Gamma_0)}^2 \le C\left\|\frac{\partial z}{\partial \nu}\right\|_{H^1(0,T;L^2(\Gamma_0)}^2,$$

provando o teorema.

■

A estabilidade é dada pelo seguinte resultado:

Teorema 4.2 *Assumamos as hipóteses*

- $\exists\, x_0 \in \mathbb{R}^N \backslash \overline{\Omega}$ *com* $\Gamma_0 \supset \Gamma_{x_0} = \{x \in \Gamma, (x - x_0).\nu(x) > 0\}$,
- $T > \sup_{x \in \overline{\Omega}} |x - x_0|$,
- $\exists\, a_0 > 0, |u_0(x)| \ge a_0$,
- $p, q \in L^\infty(\Omega), u(q) \in H^1(0,T; L^\infty(\Omega))$
- $\|p\|_{L^\infty(\Omega)} \le m$, *onde* $m > 0$ *é dado.*

Então existe $C > 0$ *tal que*

$$\frac{1}{C}\|p - q\|_{L^2(\Omega)}^2 \le \left\|\frac{\partial u}{\partial \nu}(q) - \frac{\partial u}{\partial \nu}(p)\right\|_{H^1(0,T;L^2(\Gamma_0))}^2,$$

para todo p *com* $\|p\|_{L^\infty(\Omega)} \le m$.

Demonstração: A prova segue diretamente do Teorema 4.1, com $f = q - p$ e $z = u(p) - u(q)$.

■

Como consequência imediata do Teorema 4.2, garantimos a unicidade no

Teorema 4.3 *Assumamos que*

- $\exists\ x_0 \in \mathbb{R}^N \backslash \overline{\Omega}$ *com* $\Gamma_0 \supset \Gamma_{x_0} = \{x \in \Gamma, (x - x_0).\nu(x) > 0\}$
- $T > \sup_{x \in \overline{\Omega}} |x - x_0|$
- $\exists\ a_0 > 0, |u_0(x)| \geq a_0$
- $p, q \in L^\infty(\Omega), u(q) \in H^1(0, T; L^\infty(\Omega))$

Se

$$\frac{\partial u}{\partial \nu}(p) = \frac{\partial u}{\partial \nu}(q) \quad \textit{sobre } \Gamma_0 \times (0, T),$$

então $p = q$.

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