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Controlabilidade, problema inverso, problema de contato e estabilidade para alguns sistemas hiperbólicos e parabólicos

por

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por

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Tese apresentada ao Corpo Docente do Programa Associado de Pós-Graduação em Matemática - UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

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Resumo

Nesta tese estudamos resultados de controlabilidade, comportamento assintótico e problema inverso relacionados a alguns problemas da teoria de equações diferenciais parciais. Dois sistemas particulares são foco do estudo: o sistema de Mindin-Timoshenko, que descreve o movimento vibratório de uma placa ou viga, e o sistema de campo de fases que descreve a temperatura e a fase de um meio onde ocorrem dois estados físicos distintos.

O primeiro capítulo é dedicado ao estudo do sistema de Mindlin-Timoshenko 1-D com coeficiente descontínuos. Uma desigualdade de Carleman é obtida sob a hipótese de monotonicidade sobre velocidade da viga. Posteriormente, são fornecidas duas aplicações: a controlabilidade do sistema com controles agindo na fronteira e a estabilidade Lipschitziana do problema inverso de recuperar um potencial através de uma única informação obtida sobre a solução.

No segundo capítulo consideramos um problema de contato caracterizado pelo comportamento de uma placa bidimensional cujo bordo faz contato com um obstáculo rígido. A formulação deste problema é apresentada pelo sistema de Mindlin-Timoshenko 2-D com condições de fronteira e termos de amortecimento (damping) adequados. Sobre tal sistema, é provada, através de técnicas de penalização, a existência de solução e, posteriormente, que sua energia possui decaimento exponencial quando o tempo tende ao infinito.

No terceiro capítulo o estudo é voltado a um sistema de campo de fases não-linear definido em um intervalo aberto real. Neste espaço apresentamos alguns resultados de controlabilidade quando um único controle age, sob condições de Dirichlet, na equação da temperatura em um dos bordos do intervalo. Para provar os resultados é utilizado o método dos momentos, além de uma estudo espectral de operadores associados ao sistema e teoria de ponto fixo para lidar com a não-linearidade.

Palavras-chave: campo de fases, controlabilidade, comportamento assintótico, desigualdade de Carleman, problema de contato, problema inverso, sistema de Mindlin-Timoshenko.

Abstract

In this thesis we study controllability results, asymptotic behavior and inverse problem related to some problems of the theory of partial differential equations. Two particular systems are the focus of the study: the Mindin-Timoshenko system, describing the vibrational motion of a plate or a beam, and the phase field system describing the temperature and phase of a medium having two distinct physical states.

The first chapter is devoted to the study of the 1-D Mindlin-Timoshenko system with discontinuous coefficient. A Carleman inequality is obtained under the assumption of monotonicity on the beam speed. Subsequently, two applications are provided: the controllability of the control system acting on the boundary and Lipschitzian stability of the inverse problem of recovering a potential from a single measurement of the solution.

In the second chapter we consider a contact problem characterized by the behavior of a two-dimensional plate whose board makes contact with a rigid obstacle. The formulation of this problem is presented by the 2-D Mindlin-Timoshenko system with boundary conditions and suitable damping terms. Concerning such system, is proved via penalty techniques, the existence of solution and that the system energy has exponential decay when the time approaches infinity.

In the third chapter, the study is aimed at a nonlinear phase-field system defined in a real open interval. Here we present some controllability results when a single control acts, by means of Dirichlet conditions, on the temperature equation of the system on one of the endpoints of the interval. To prove the results is used the method of moments, plus a spectral study of operators associated to the system and fixed point theory to deal with the nonlinearity.

Keywords: phase-field system, controllability, asymptotic behavior, damping, energy decay, Carleman inequality, contact problem , inverse problem, Mindlin-Timoshenko system.

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À diversão que o superar das dificuldades
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A você que está prestigiando este trabalho.

Ao café, por ser amargo.

" Knowledge is about not being sure.
The more you know, the less you are sure of it.
The less you know, the less you are sure of it. "

Sumário

Introdução

A espécie humana vem moldando o planeta Terra à sua vontade e ao seu instinto de viver. Um exemplo simples disso é a agricultura. Deixamos de acompanhar o "humor"da natureza para forçá-la a obedecer nosso desejo. Plantamos com a opção de escolher localidade, tipo e quantidade desejadas. Cruzamos espécies. Modificamos o natural com a intenção de facilitar a nossa vida, de fazer a natureza trabalhar para o homem. Controlamos a natureza, de certa forma.

Com o tempo, tal comportamento sobre a natureza amadureceu. A tecnologia avançou e fomos instigados a construir máquinas que funcionassem à vontade do homem. Máquinas que pudessem ser controladas. A palavra *controle* aqui assume simplesmente o significado de manusear, checar se seu comportamento é satisfatório, seguir junto ao seu funcionamento. Em um sentido mais profundo, a palavra *controle* significa agir, colocar as coisas em uma determinada ordem de forma que um sistema se comporte como desejado. Ora, esse significado está aplicado na própria construção das máquinas. Mas não totalmente em seu funcionamento. Para obter um controle absoluto a máquina não necessitaria de um controlador humano. É nessa ideia que se baseiam as máquinas autômatas, que se podem intitular como ideia propulsora do controle na engenharia. Como exemplo podemos citar os aquedutos romanos: sistemas engenhosos que regulavam válvulas, sem a interferência humana, de modo a manter o volume de água constante; moinhos de vento do século XVII adaptados para regular a velocidade do vento; a máquina a vapor, mecanismo símbolo da revolução industrial com seu sistema de regulagem de velocidade que funcionava conforme a variação de pressão em um compartimento com válvulas. Estudiosos ainda afirmam que mesmo antes de 2000 a.c. sistemas de controle de irrigação já eram praticados.

Com o advento do cálculo e das equações diferenciais, a teoria de controle passou a ser lida pela linguagem matemática e se desprendeu da engenharia saindo para ser aplicada nas mais diversas áreas. É de conhecimento geral que um período de grande evolução tecnológica foi a segunda guerra mundial. Nessa época, a teoria de controle foi parte importante nos sistemas de controle de fogo, sistemas de orientação de mísseis, sistemas eletrônicos, modelagem de esquadrões aéreos. A guerra acabou tornando claro que os modelos considerados nesse momento não eram suficientemente precisos para descrever a complexidade do mundo real. Na verdade, nesse tempo já estava claro que os verdadeiros sistemas eram não lineares e impossíveis de serem descritos com precisão absoluta, uma vez que eram quase sempre afetados por uma quantidade grande de agentes externos. Esse cenário serviu para dividir a teoria de controle.

Depois dos anos 60, os métodos e teorias utilizados passaram a ser considerados como parte da teoria clássica de controle. As contribuições do cientista estadunidense R. Bellman, no contexto de *programação dinâmica*; R. Kalman, em *técnicas de filtragem* e *aproximações algébricas a sistemas lineares*; e do russo L. Pontryagin, com o *princípio do máximo* para problemas de controle ótimo, estabeleceram a base da teoria de controle moderna. Esta teoria ganhou formalismo matemático e hoje é aplicada nas mais diversas áreas. Basicamente qualquer coisa que possa ser modelada por equações diferenciais é alvo da teoria de controle.

1.1 Controlabilidade

Em geral, um sistema de controle é uma equação de evolução (EDO ou EDP) que depende de um parâmetro u, descrito pela expressão

$$y' = f(t, y, u), \tag{1.1}$$

onde $t \in [0, T]$ representa a variável temporal, $y : [0, T] \to \mathcal{X}$ é a função estado e $u : [0, T] \to \mathcal{U}$ é um controle. Nessa configuração, $\mathcal{X}$ e $\mathcal{U}$ são espaços de funções adequados, $T > 0$ é um valor real fixado e y' representa a derivada de y em relação ao tempo t.

O problema de controle consistem em encontrar a função u de forma que a solução y do sistema (1.1) assuma um comportamento desejado no instante de tempo T. Dependendo do tipo de sistema que (1.1) represente, é impossível obrigar sua solução a satisfazer exatamente um comportamento no instante tempo T. Nesse caso, podem ser procuradas respostas parciais. É possível que tal solução, apesar de não se comportar exatamente conforme desejado, aproxime-se de tal comportamento *(controlabilidade aproximada)*, atinja um estado de equilíbrio *(controlabilidade nula)* ou até passe a se comportar como a solução de um sistema não controlado *(controlabilidade exata às trajetórias)*. Há sistemas que ainda respondem positivamente quanto à possibilidade de assumir o exato comportamento desejado em um tempo T, contudo, sob restrições a respeito de tal tempo, excluindo algumas possibilidades de valores que este possa assumir. Obviamente, ainda há sistemas que combinam as duas situações acima: não são exatamente controláveis e ainda possuem restrições sobre o instante de tempo o qual se pode controlá-los.

Definamos, com a devida formalidade, alguns dos vários tipos de controlabilidade presentes na literatura.

Controlabilidade exata: Dados um número real $T > 0$ e $y_0, y_1 \in \mathcal{X}$ dois possíveis estados do sistema (1.1), dizemos que tal sistema é exatamente controlável se existe $u : [0, T] \to \mathcal{U}$ tal que

$$\begin{cases} y' = f(y, u) \quad \text{em} \quad [0, T], \\ y(0) = y_0, \quad y(T) = y_1. \end{cases}$$

Controlabilidade aproximada: Dados um número real $T > 0$ e $y_0, y_1 \in \mathcal{X}$ dois possíveis estados do sistema (1.1), dizemos que tal sistema é aproximadamente controlável se, para todo ε, existe $u_\varepsilon : [0, T] \to \mathcal{U}$ tal que

$$\begin{cases} y' = f(y, u_\varepsilon) \quad \text{em} \quad [0, T], \\ y(0) = y_0, \quad \|y(T) - y_1\| < \varepsilon. \end{cases}$$

Controlabilidade nula: Dados um número real $T > 0$ e $y_0 \in \mathcal{X}$ um possível estado do sistema (1.1), dizemos que tal sistema é nulamente controlável se existe $u : [0,T] \to \mathcal{U}$ tal que

$$\begin{cases} y' = f(y,u) \quad \text{em} \quad [0,T], \\ y(0) = y_0, \quad y(T) = 0. \end{cases}$$

Controlabilidade exata às trajetórias: Dados um número real $T > 0$, $y_0 \in \mathcal{X}$ um possível estado e $\overline{y}$ uma trajetória (uma solução arbitrária) do sistema (1.1), dizemos que tal sistema é exatamente controlável às trajetórias se existe $u : [0,T] \to \mathcal{U}$ tal que

$$\begin{cases} y' = f(y,u) \quad \text{em} \quad [0,T], \\ y(0) = y_0, \quad y(T) = \overline{y}(T). \end{cases}$$

Há ainda tipos mais elaborados de controlabilidade derivados destes, como por exemplo o *controle ótimo*, que busca atingir um estado desejado sujeito à minimização de um funcional custo; o *controle insensibilizante*, que busca formas de atuar na equação de forma que o comportamento de um certo funcional não seja alterado por uma leve mudança arbitrária no dado inicial; o *controle hierárquico*, que busca maneiras de interferir no sistema através de uma série de controles correlacionados entre si em uma dependência de líderes e seguidores.

Atualmente, a teoria de controle está bem estabelecida (veja, por exemplo [32, 68, 80, 86, 90]). Sobre sistemas de dimensão finita, tais problemas são completamente entendidos no caso linear (veja [56, 67]). No caso de sistemas não lineares de dimensão finita, seu estudo se apresenta bem avançado e satisfatório, já que são conhecidas muitas condições para se obter controlabilidade local e global (veja [32]). No caso de EDP's, a situação se torna mais delicada, até mesmo para problemas lineares. Uma razão para isso é que uma EDP linear de evolução pode ser, por exemplo, do tipo hiperbólico (equação da onda, equação de Maxwell), ou do tipo dispersivas (equação de placas, equação de Schrödinger, equação KdV), ou do tipo parabólico (equação do calor, equação de Stokes), induzindo propriedades muito específicas como a propriedade de propagação de singularidades com velocidade finita para equações hiperbólicas, a velocidade infinita de propagação junto a um fraco (resp. forte) efeito suavizante para equações dispersivas (resp. parabólicas), e a irreversibilidade temporal para equações parabólicas. Não se pode, por exemplo, esperar que uma equação do calor possa ser exatamente controlável com um controle localizado em uma pequena parte do domínio, pois, caso contrário, a solução seria suave fora da região de controle, o que impede de se atingir um estado final arbitrário. Assim, é natural procurar por controlabilidade aproximada, nula ou exata às trajetórias para sistemas contendo equações do calor. Em contraste, devido a reversibilidade no tempo, é natural buscar controlabilidade exata para a equação da onda.

1.2 Problemas inversos

A teoria de problemas inversos associados a equações diferenciais parciais vem tomando um grande destaque atualmente. Este ramo se mostra uma boa fonte de pesquisa por possuir aplicações muito interessantes. Em particular, o problema inverso tratado no Capítulo 2 está construído sobre a teoria de placas e pode ser visto, por exemplo, como um problema de geofísica, relacionado ao estudo de placas tectônicas.

As equações diferenciais parciais durante muitos anos tem provado ser uma ferramenta muito poderosa na modelagem de uma variedade de fenômenos físicos que pode ser visto na natureza. Fazendo uso desta teoria é possível saber com grande precisão um evento da natureza por descrição matemática (equação diferencial parcial) que nos permite demonstrar suas propriedades, tais como a existência, unicidade e estabilidade das soluções da equação. Sendo assim, é natural fazer a pergunta contrária: se sabemos a solução problema ou possuímos, pelo menos, alguma informação desta, é possível inferir algo sobre as propriedades do sistema? Essa é a premissa de um problema inverso.

Um problema inverso consiste, então, em encontrar alguma propriedade desconhecida do meio, objeto ou sistema que estamos analisando a partir de medições controladas e fazendo uso do modelo matemático de algum fenômeno físico conhecido. Visto assim, é óbvio o grande interesse e importância prática que problemas inversos representam. Um dos exemplos mais famosos é o conhecido *Problema de Calderón*. Nomeado em homenagem ao matemático argentino Alberto Calderon, este problema constitui a base matemática da tomografia por impedância elétrica, um método de testes não destrutivos para gerar imagens médicas. O problema levanta a seguinte questão: é possível, medindo a corrente eléctrica e a tensão na margem de um meio, determinar a condutividade eléctrica deste? Em outras palavras, chamando Ω o meio, $y = y(x)$ o potencial em seu interior e $c = c(x)$ a condutividade eléctrica, ao aplicar uma tensão $f = f(x)$ na fronteira $\partial\Omega$ do meio temos que y satisfaz

$$\begin{aligned} \nabla \cdot c\nabla y = 0, & \quad \text{em} \quad \Omega \\ y = f, & \quad \text{sobre} \quad \partial\Omega \end{aligned}$$

e induz uma corrente $c\frac{\partial y}{\partial \nu}$ no bordo do domínio. O *Problema de Calderón* tenta determinar o valor de c em todo o meio através da informação das medições de tensão e corrente que estão representadas pela aplicação Dirichlet-Neumann

$$\Phi(f) = c\frac{\partial y}{\partial \nu}\bigg|_{\partial\Omega}.$$

Podemos representar um problema inverso como uma aplicação de medições $\mathcal{M}_c(\cdot)$ que a cada possível valor $x \in X$ do parâmetro c que procuramos determinar, nos entrega certas medições sistema. Em geral, existem quatro questões sobre a aplicação $\mathcal{M}_c$ a serem analisadas:

1. **Unicidade.** Se a medição de dois valores do parâmetro são iguais, então os valores são iguais, isto é, a aplicação $\mathcal{M}$ é injetiva.

$$\mathcal{M}_c(x_1) = \mathcal{M}_c(x_2) \Longrightarrow x_1 = x_2$$

2. **Reconstrução.** Encontrar um procedimento que permita reconstruir c a partir dos dados obtidos.

3. **Estabilidade.** Se a medição de dois valores são iguais segundo algum critério de comparação, então os valores também o são.

$$\mathcal{M}_c(x_1) \approx \mathcal{M}_c(x_2) \Longrightarrow x_1 \approx x_2$$

4. **Dados Parciais.** Se ter acesso a medições em uma parte do domínio é suficiente para determinar se há unicidade sobre c.

No Capítulo 2, onde que trataremos de um problema inverso, iremos nos concentrar em responder à terceira pergunta para um sistema que modela o comportamento de uma viga durante o passar do tempo. Para tal, iremos utilizar o método clássico para se provar a estabilidade introduzido por A.L. Bukhgeim e M.V.Klibanov em [24], que consiste em linearizar o problema inverso e usar desigualdades de Carleman para estimar a fonte em função das observações e, finalmente, obter uma estabilidade local. Para mais exemplos e aprofundamento sobre problemas inversos sugerimos a leitura de [17] e [55].

1.3 Conteúdo da tese

A seguir, iremos apresentar de forma mais específica os problemas que serão desenvolvidos nesta tese. Uma introdução motivacional aos trabalhos que a compõe junto a um breve resumo dos resultados principais será descrito em português, contudo, a linguagem adotada nos capítulos contendo os trabalhos em si será o inglês. Esta tese é composta de três problemas principais, cada um tratado separadamente nos respectivos capítulos 2, 3 e 4.

Capítulo 2
Desigualdade de Carleman para o sistema unidimensional de Mindlin-Timoshenko com coeficientes descontínuos e aplicações
(Carleman estimate for unidimensional Mindlin-Timoshenko system
with discontinuous coefficients and applications)

O sistema de Mindlin-Timoshenko unidimensional é composto de duas equações hiperbólicas de segunda ordem acopladas por termos de primeira ordem. Tal formulação é amplamente utilizada e representa um modelo matemático fisicamente completo para a descrição do movimento vibratório transversal de vigas.

Figura 1.1: Estruturas construídas com o uso de vigas.

Para uma viga de comprimento L, este sistema é descrito como se segue:

$$\begin{cases} \frac{\rho h^3}{12}\psi'' - (a\psi_x)_x + k(\psi + \sigma_x) = 0 & \text{em} \quad Q, \\ \rho h \sigma'' - [k(\psi + \sigma_x)]_x = 0 & \text{em} \quad Q, \end{cases} \tag{1.2}$$

onde $Q = (0, L)\times(0, T)$ e T representa um tempo positivo dado. No modelo acima, $\psi = \psi(x, t)$ representa o ângulo de rotação, $\sigma = \sigma(x, t)$ descreve o deslocamento vertical no tempo t do corte transversal localizado x unidades do extremo $x = 0$.

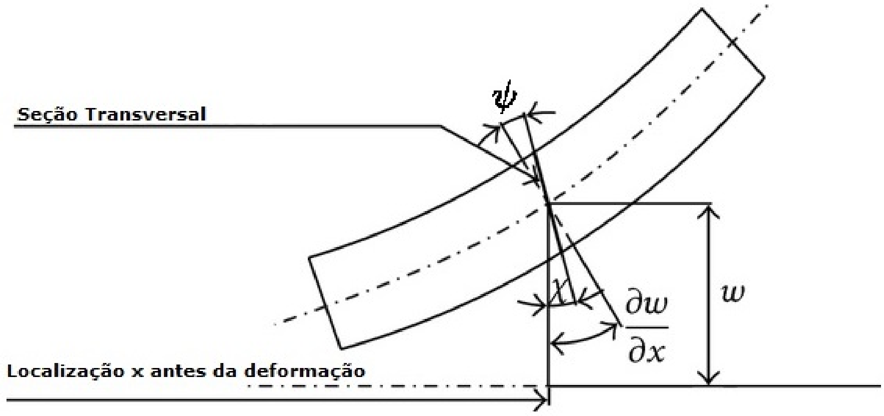

Figura 1.2: Rotação e movimento vertical.

O símbolo $'$ denota a derivada em relação ao tempo t e a letra x subscrita denota a derivada em relação à variável x. A constante $h > 0$ representa a espessura da viga, que é considerada pequena e uniforme, independente de x. A constante ρ é a densidade da viga e os parâmetros a e k são chamados de módulo de rigidez flexural e módulo de elasticidade de cisalhamento, respectivamente. Eles são dados pelas fórmulas $k = \widehat{k}Eh/2\,(1+\mu)$ e $a = Eh^3/12(1-\mu^2)$, onde $\widehat{k}$ é um coeficiente de correção de corte, E é o módulo de Young e μ é o coeficiente de Poisson, $0 < \mu < 1/2$. Para mais detalhes físicos sobre as hipóteses, parâmetros e equações, veja, por exemplo, [65] e [66].

Motivação

Nesse capítulo, iremos considerar uma viga composta de dois materiais diferentes com a mesma espessura, uma localizada em $(0, M)$ e outra em (M, L), para algum ponto intermediário M no intervalo $(0, L)$ que representa a viga.

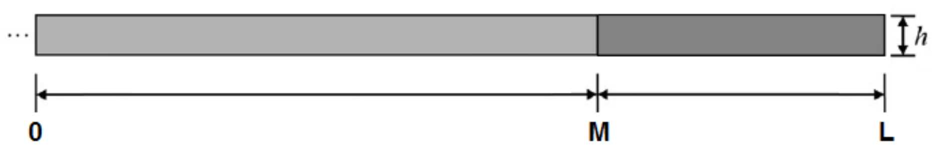

Figura 1.3: Viga composta de dois materiais.

Os valores E, μ, ρ e $\widehat{k}$ dependem do material, por isso iremos considerar os coeficientes a, ρ e k dados por

$$a(x) = \begin{cases} a_1, & \text{se} \quad x \in \ (0, M) \\ a_2, & \text{se} \quad x \in \ (M, L) \end{cases}, \quad \rho(x) = \begin{cases} \rho_1, & \text{se} \quad x \in \ (0, M) \\ \rho_2, & \text{se} \quad x \in \ (M, L) \end{cases}$$

$$k(x) = \begin{cases} k_1, & \text{se} \quad x \in \ (0, M) \\ k_2, & \text{se} \quad x \in \ (M, L) \end{cases}, \tag{1.3}$$

onde $a_1, a_2, \rho_1, \rho_2, k_1, k_2 \in \mathbb{R}$. Nesse contexto, estamos interessados em estudar a controlabilidade e problemas inversos relacionados ao sistema (1.2), quando os coeficientes a, ρ e k são dados em (1.3).

Figura 1.4: Estruturas baseadas em vigas compostas por mais de um material.

Na literatura é possível encontrar vários resultados envolvendo o sistema de Mindlin-Timoshenko. No âmbito dos problemas inversos, duas poderosas ferramentas utilizadas para obter a estabilidade através de uma informação já conhecida são as desigualdades de Carleman e o método de Bukhgeim-Klibanov [22, 24]. É possível obter uma estabilidade Lipschitziana ao redor de uma única solução conhecida, desde que esta solução ofereça informação e regularidade suficientes [60] (veja também [59] e [88]). Muitos outros problemas inversos relacionados às equações hiperbólicas usam a mesma estratégia, inclusive o que iremos abordar.

Desigualdade de Carleman

O objeto de partida nesse capítulo é a busca por uma desigualdade de Carleman. Tal estimativa irá ser aplicada para obter um resultado de controlabilidade e para resolver um problema inverso. A desigualdade de Carleman é uma técnica vastamente utilizada na teoria de controle (veja [78], por exemplo), sejam em problemas elípticos, hiperbólicos ou parabólicos.

Consideremos a função peso dada por

$$\phi(x,t) = \begin{cases} \phi_1(x,t) := \max\left\{\frac{\rho_1 h^3}{12 a_1}, \frac{\rho_1 h}{k_1}\right\}(x-x_0)^2 - \beta\left(t - \dfrac{T}{2}\right)^2 + N_1, & \text{em} \quad (0,M), \\ \phi_2(x,t) := \max\left\{\frac{\rho_2 h^3}{12 a_2}, \frac{\rho_2 h}{k_2}\right\}(x-x_0)^2 - \beta\left(t - \dfrac{T}{2}\right)^2 + N_2, & \text{em} \quad (M,L), \end{cases}$$

onde $\beta, N_1, N_2 > 0$ são constantes reais, $M \in (0,L)$ e $x_0 \in \mathbb{R} \setminus (0,L)$. Definamos a função $\varphi = e^{s\phi}$, para um parâmetro $s > 0$, e os operadores

$$\begin{aligned}
P_{1,\gamma}(u) &= u'' - \gamma u_{xx} + s^2\lambda^2\varphi^2 E_\gamma(\phi)u,\\
P_{2,\gamma}(u) &= -2s\lambda\varphi\phi' u' + 2s\lambda\gamma\varphi(\phi)_x u_x,\\
L_\gamma(u) &= u'' - \gamma u_{xx},
\end{aligned}$$

onde γ assume valores $\gamma = \gamma_1$ em $(0,M)$ e $\gamma = \gamma_2$ em (M,L), com $\gamma_1, \gamma_2 > 0$. Considerando o espaço

$$\begin{aligned}
X_{\gamma_1,\gamma_2} \;=\; &\{u \in L^2(0,T;L^2((0,L));\; u = u_1 \text{ em } Q_1,\; u = u_2 \text{ em } Q_2,\\
&L_{\gamma_1}(u_1) \in L^2(0,T;L^2(0,M)),\; L_{\gamma_2}(u_2) \in L^2(0,T;L^2(M,L)),\\
&u(0,\cdot) = u(L,\cdot) = u(\cdot,0) = u(\cdot,T) = u'(\cdot,0) = u'(\cdot,T) = 0\},
\end{aligned}$$

o resultado a seguir enuncia a desigualdade de Carleman que iremos obter.

Teorema. *Sejam a, k, ρ dados em (1.3) e $T > 0$. Existem constantes positivas C, λ_0 e s_0 tais que*

$$\begin{aligned}
&\left\|\left\{P_{1,\frac{a12}{\rho h^3}}(e^{\lambda\varphi}u), P_{1,\frac{ka}{\rho h}}(e^{\lambda\varphi}v)\right\}\right\|^2_{L^2(Q)\times L^2(Q)} + \left\|\left\{P_{2,\frac{a12}{\rho h^3}}(e^{\lambda\varphi}u), P_{2,\frac{ka}{\rho h}}(e^{\lambda\varphi}v)\right\}\right\|^2_{L^2(Q)\times L^2(Q)}\\
&+s\lambda\int_Q e^{2\lambda\varphi}\varphi(|u'|^2 + |u_x|^2 + |v'|^2 + |v_x|^2)dxdt + s^3\lambda^3\int_Q e^{2\lambda\varphi}\varphi^3(|u|^2 + |v|^2)dxdt\\
&\le Cs\int_0^T e^{2\lambda\varphi}\varphi(|u_x|^2 + |v_x|^2)\Big|_{x=L} dt + C\int_Q e^{2\lambda\varphi}\left(\left|L_{\frac{12a}{\rho h^3}}(u)\right|^2 + \left|L_{\frac{k}{\rho h}}(v)\right|^2\right)dxdt,
\end{aligned}$$

para todo $\{u,v\} \in X_{\frac{12a_1}{\rho_1 h^3},\frac{12a_2}{\rho_2 h^3}} \times X_{\frac{k_1}{\rho_1 h},\frac{k_2}{\rho_2 h}}$ satisfazendo

$$\begin{cases}
u_1(M,\cdot) = u_2(M,\cdot), \quad v_1(M,\cdot) = v_2(M,\cdot) & \text{em } (0,T),\\
a_1u_{1x}(M,\cdot) = a_2u_{2x}(M,\cdot), \quad k_1v_{1x}(M,\cdot) = k_2v_{2x}(M,\cdot) & \text{em } (0,T),
\end{cases}$$

e para todo $s \ge s_0$ e $\lambda > \lambda_0$.

Controlabilidade

Para o problema de controle consideraremos o sistema de Mindlin-Timoshenko complementado com os dados de bordo a seguir:

$$\begin{cases}
\dfrac{\rho(x)h^3}{12}\psi'' - (a(x)\psi_x)_x + k(x)(\psi + \sigma_x) = 0 & \text{em} \quad Q,\\
\rho(x)h\sigma'' - (k(x)(\psi + \sigma_x))_x = 0 & \text{em} \quad Q,\\
\psi(0,\cdot) = \sigma(0,\cdot) = 0,\; \psi(L,\cdot) = f_1,\; \sigma(L,\cdot) = f_2 & \text{em} \quad (0,T),\\
\psi(\cdot,0) = \psi_0,\; \psi'(\cdot,0) = \psi_1 & \text{em} \quad (0,L),\\
\sigma(\cdot,0) = \sigma_0,\; \sigma'(\cdot,0) = \sigma_1 & \text{em} \quad (0,L).
\end{cases} \tag{1.4}$$

As condições $(1.4)_3$ significam que a viga está presa em $x = 0$ e os controles f_1, f_2 são forças laterais aplicadas no extremo $x = L$. O resultado de controlabilidade que iremos provar é descrito a seguir.

Teorema. *Consideremos*

$$T_0 = 2L\sqrt{\max\left\{\frac{\rho_1 h^3}{12a_1}, \frac{\rho_2 h^3}{12a_2}, \frac{\rho_1 h}{k_1}, \frac{\rho_2 h}{k_2}\right\}}, \tag{1.5}$$

e sejam a, k, ρ *dados em* (1.3) *satisfazendo*

$$\frac{a_1}{\rho_1} > \frac{a_2}{\rho_2} \quad e \quad \frac{a_1}{k_1} = \frac{a_2}{k_2}. \tag{1.6}$$

Se $\{\psi_0, \psi_1, \sigma_0, \sigma_1\} \in [L^2(0,L) \times H^{-1}(0,L)]^2$ *e* $T > T_0$, *então existem controles* $f_1, f_2 \in L^2(0,T)$ *tais que a solução* $\{\psi, \sigma\}$ *do sistema de Mindlin-Timoshenko* (2.3) *satisfaz*

$$\{\psi(\cdot,T), \psi'(\cdot,T), \sigma(\cdot,T), \sigma'(\cdot,T)\} = \{0,0,0,0\} \quad em \quad (0,L).$$

Para provar o teorema acima, usaremos uma desigualdade de Carleman para obter uma desigualdade de observabilidade a qual, de acordo com o Método HUM (Hilbert Uniqueness Method) desenvolvido por Lions (veja [68]), implica na controlabilidade enunciada.

Problema inverso

Consideremos, agora, o sistema de Mindlin-Timoshenko com potenciais a seguir:

$$\begin{cases} \dfrac{\rho(x)h^3}{12}u'' - (a(x)u_x)_x + k(x)(u + v_x) + p_1(x)u = 0 & \text{em} \quad Q, \\ \rho(x)hv'' - (k(x)(u + v_x))_x + p_2(x)v = 0 & \text{em} \quad Q, \\ u(0,\cdot) = v(0,\cdot) = u(L,\cdot) = v(L,\cdot) = 0 & \text{em} \quad (0,T), \\ u(\cdot,0) = u_0, \ u'(\cdot,0) = u_1 & \text{em} \quad (0,L), \\ v(\cdot,0) = v_0, \ v'(\cdot,0) = v_1 & \text{em} \quad (0,L). \end{cases} \tag{1.7}$$

Aqui propomos o seguinte problema inverso: recuperar informação sobre o sistema (1.7) utilizando dados colhidos no bordo. Para ser mais preciso, queremos recuperar os potenciais (p_1, p_2) a partir do conhecimento da derivada normal no bordo da solução $\{u(p_1,p_2), v(p_1,p_2)\}$. Esta classe de problemas (problemas inversos com coeficientes e uma única medição de dados) foi investigada com a utilização de desigualdades de Carleman pela primeira vez em [23] por Buckgeim e Klibanov, rendendo à técnica o conhecido título de "método de Buckheim-Klibanov"(veja [59], [60] e o livro [20]). Utilizaremos tal método para provar o resultado a seguir.

Teorema. *Sob as hipóteses* (1.5) *and* (1.6), *se* $T > T_0$, $p_1, p_2 \in L^\infty(\Omega)$, $\{u_0, u_1, v_0, v_1\} \in [H^1(\Omega) \times L^2(\Omega)]^2$, *e* $r > 0$ *satisfazem*

$$|u_0| \geq r > 0 \ q.s. \ em \ (0,L), \quad u(p_1,p_2) \in H^1(0,T; L^\infty(\Omega)),$$
$$|v_0| \geq r > 0 \ q.s. \ em \ (0,L), \quad v(p_1,p_2) \in H^1(0,T; L^\infty(\Omega)),$$

então, para um conjunto limitado $\mathcal{U} \subset [L^\infty(\Omega)]^2$, *existe uma constante positiva*

$$C = C(a, k, \rho, L, M, T, \|\{p_1,p_2\}\|_{[L^\infty(\Omega)]^2}, \|\{u(p_1,p_2), v(p_1,p_2)\}\|_{[H^1(0,T;L^\infty(\Omega))]^2}, \mathcal{U}, r)$$

tal que

$$\|p_1 - q_1\|_{L^2(0,L)} + \|p_2 - q_2\|_{L^2(0,L)}$$
$$\le C\left(\left\|\frac{a_2}{\rho_2}u_x(L,\cdot)(p_1,p_2) - \frac{a_2}{\rho_2}u_x(L,\cdot)(q_1,q_2)\right\|_{H^1(0,T)}\right.$$
$$\left.+\left\|\frac{k_2}{\rho_2}v_x(L,\cdot)(p_1,p_2) - \frac{k_2}{\rho_2}v_x(L,\cdot)(q_1,q_2)\right\|_{H^1(0,T)}\right),$$

para todo $\{q_1, q_2\} \in \mathcal{U}$*, onde* $\{u(p_1, p_2), v(p_1, p_2)\}$ *e* $\{u(q_1, q_2), v(q_1, q_2)\}$ *são soluções de* (1.7) *com potenciais* $\{p_1, p_2\}$ *and* $\{q_1, q_2\}$*, respectivamente.*

A ideia central da prova reside em utilizar uma desigualdade de Carleman para estimar o termo

$$\left\langle P_{1,\frac{a12}{\rho h^3}}(w), w'\right\rangle,$$

onde $w = e^{\lambda\varphi}\theta[u(p_1, p_2) - u(q_1, q_2)]$, sendo θ uma função de corte adequada que se anula em $\pm T$ e $u(p_1, p_2)$, $u(q_1, q_2)$ soluções de (1.7) estendidas ao intervalo $(-T, T)$ com potenciais (p_1, p_2) e (q_1, q_2), respectivamente.

Capítulo 3
Um problema de obstáculo no bordo para o sistema 2-D de Mindlin-Timoshenko

(A boundary obstacle problem for the 2-D Mindlin-Timoshenko systems)

Neste capítulo, iremos lidar com um problema modelado pelo sistema de Mindlin-Timoshenko bidimensional, que descreve o movimento vibratório de placas. Este modelo pode ser utilizado para descrever, por exemplo, o comportamento de placas tectônicas, paredes e piso de um prédio, placas metálicas, etc.

Figura 1.5: Placas tectônicas e placas de metal na asa de um avião.

Consideramos uma região limitada, aberta e conexa $\Omega \subset \mathbb{R}^2$ com fronteira Γ suficientemente regular. Assumimos ainda que Γ possui uma partição $\{\Gamma_0, \Gamma_1\}$ com Γ_i $(i = 0, 1)$ possuindo medida de Lebesgue positiva e $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$. Dado um valor real $T > 0$, consideramos o cilindro $Q = \Omega \times (0, T)$ com fronteira lateral $\Sigma = \Sigma_0 \cup \Sigma_1$, onde $\Sigma_i = \Gamma_i \times (0, T)$ $(i = 0, 1)$.

O sistema de Mindlin-Timoshenko para uma região bidimensional é descrito por

$$\begin{cases} \dfrac{\rho h^3}{12}\phi_{tt} - L_1(\phi,\psi,w) = 0, & \text{em} \quad Q, \\ \dfrac{\rho h^3}{12}\psi_{tt} - L_2(\phi,\psi,w) = 0, & \text{em} \quad Q, \\ \rho h w_{tt} - L_3(\phi,\psi,w) = 0, & \text{em} \quad Q, \end{cases}$$

onde os operadores L_1, L_2, L_3 são dados pelas expressões

$$\begin{aligned} L_1(\phi,\psi,w) &= D\left(\phi_{x_1x_1} + \frac{1-\mu}{2}\phi_{x_2x_2} + \frac{1+\mu}{2}\psi_{x_1x_2}\right) - k\left(\phi + w_{x_1}\right), \\ L_2(\phi,\psi,w) &= D\left(\psi_{x_2x_2} + \frac{1-\mu}{2}\psi_{x_1x_1} + \frac{1+\mu}{2}\phi_{x_1x_2}\right) - k\left(\psi + w_{x_2}\right), \\ L_3(\phi,\psi,w) &= k\left[(w_{x_1}+\phi)_{x_1} + (w_{x_2}+\psi)_{x_2}\right]. \end{aligned}$$

Os índices subscritos denotam derivadas parciais. Para $x = (x_1, x_2)$, as variáveis dependentes $\phi = \phi(x,t)$ e $\psi = \psi(x,t)$, representam, respectivamente, os ângulos de rotação da uma seção transversal $x_1 = const.$, $x_2 = const.$ contendo o filamento que, quando a placa está em equilíbrio, é ortogonal à superfície média no ponto $(x, 0)$. A variável $w = w(x,t)$ descreve o deslocamento vertical no tempo t da seção transversal de pontos x na superfície média da placa.

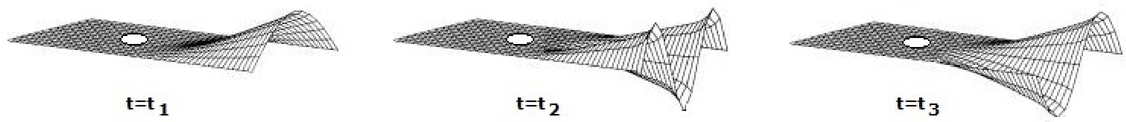

Figura 1.6: Movimento da placa ao longo do tempo.

A constante h representa a espessura da placa que, no modelo, é considerada pequena e uniforme com respeito a x. A constante ρ é a densidade da placa e as constantes D e k são chamadas, respectivamente, de módulo de rigidez flexural e módulo de elasticidade cisalhamento e são descritos por $D = Eh^3/[12(1-\mu^2)]$ e $k = \widehat{k}Eh/2(1+\mu)$, E é o módulo de Young, μ o coeficiente de Poisson, $0 < \mu < 1/2$ e $\widehat{k}$ é um coeficiente de correção de cisalhamento. Para mais detalhes físicos sobre as hipóteses, parâmetros e equações, veja, por exemplo, [65] e [66].

Motivação

Consideremos a situação em que parte do bordo de uma placa de espessura muito pequena está presa a uma superfície. Observa-se que a parte do bordo da placa que não está presa, ao movimentar-se, acaba chocando-se vez ou outra com um obstáculo rígido que está próximo, mesmo não estando inicialmente em contato com tal obstáculo.

Para modelar essa situação, utilizaremos o sistema de Mindlin-Timoshenko adicionado de condições de bordo adequadas. Tal problema de contato é descrito pelo seguinte sistema:

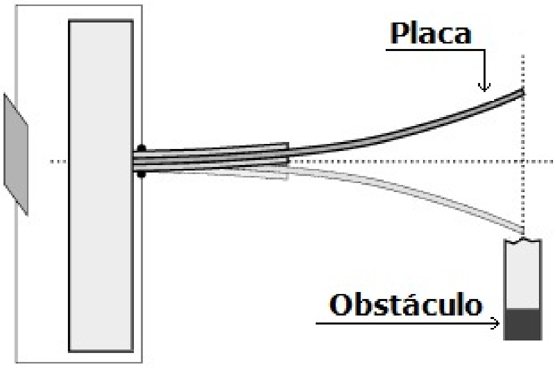

Figura 1.7: Exemplo com vista lateral da placa.

$$\begin{cases} \dfrac{\rho h^3}{12}\phi_{tt} - L_1(\phi,\psi,w) = 0 & \text{em} \quad Q, \\ \dfrac{\rho h^3}{12}\psi_{tt} - L_2(\phi,\psi,w) = 0 & \text{em} \quad Q, \\ \rho h w_{tt} - L_3(\phi,\psi,w) = 0 & \text{em} \quad Q, \\ \phi = \psi = w = 0 & \text{on} \quad \Sigma_0, \\ \mathcal{B}_1(\phi,\psi) = 0 & \text{sobre} \quad \Sigma_1, \\ \mathcal{B}_2(\phi,\psi) = 0 & \text{sobre} \quad \Sigma_1, \\ \mathcal{B}_3(\phi,\psi,w) \geq 0, \quad w \geq g, \quad \mathcal{B}_3(\phi,\psi,w)(w-g) = 0 & \text{sobre} \quad \Sigma_1, \\ \{\phi(\cdot,0), \psi(\cdot,0), w(\cdot,0)\} = \{\phi_0, \psi_0, w_0\} & \text{em} \quad \Omega, \\ \{\phi_t(\cdot,0), \psi_t(\cdot,0), w_t(\cdot,0)\} = \{\phi_1, \psi_1, w_1\} & \text{em} \quad \Omega, \end{cases} \tag{1.8}$$

onde

$$\begin{cases} \mathcal{B}_1(\phi,\psi) = D\left[\nu_1\phi_{x_1} + \mu\nu_1\psi_{x_2} + \dfrac{1-\mu}{2}\left(\phi_{x_2} + \psi_{x_1}\right)\nu_2\right], \\ \mathcal{B}_2(\phi,\psi) = D\left[\nu_2\psi_{x_2} + \mu\nu_2\phi_{x_1} + \dfrac{1-\mu}{2}\left(\phi_{x_2} + \psi_{x_1}\right)\nu_1\right], \\ \mathcal{B}_3(\phi,\psi,w) = k\left(\dfrac{\partial w}{\partial \nu} + \nu_1\phi + \nu_2\psi\right). \end{cases}$$

Uma pergunta cabível sobre o problema descrito é: seria tal situação possível? Isto é, esse sistema possui solução? Em caso afirmativo, como se comportaria a energia deste sistema? Neste capítulo, estamos interessados em estudar estes dois aspectos do problema.

Para estudar a existência de solução, utilizaremos um método de penalização, que consiste basicamente em três passos. O primeiro passo é considerar um sistema penalizado associado a (1.8). No nosso caso, para cada parâmetro penalizador $\varepsilon > 0$, tal sistema é dado por

$$\begin{cases} \dfrac{\rho h^3}{12}\phi_{\varepsilon tt} - L_1(\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon) = 0 & \text{em} \quad Q, \\ \dfrac{\rho h^3}{12}\psi_{\varepsilon tt} - L_2(\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon) = 0 & \text{em} \quad Q, \\ \rho h w_{\varepsilon tt} - L_3(\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon) = 0 & \text{em} \quad Q, \\ \phi_\varepsilon = \psi_\varepsilon = w_\varepsilon = 0 & \text{sobre} \quad \Sigma_0, \\ \mathcal{B}_1(\phi_\varepsilon, \psi_\varepsilon) = 0 & \text{sobre} \quad \Sigma_1, \\ \mathcal{B}_2(\phi_\varepsilon, \psi_\varepsilon) = 0 & \text{sobre} \quad \Sigma_1, \\ \mathcal{B}_3(\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon) - \dfrac{1}{\varepsilon}(w_\varepsilon - g)^- = 0 & \text{on} \quad \Sigma_1, \\ \{\phi_\varepsilon(\cdot,0), \psi_\varepsilon(\cdot,0), w_\varepsilon(\cdot,0)\} = \{\phi_0, \psi_0, w_0\} & \text{em} \quad \Omega, \\ \{\phi_{\varepsilon t}(\cdot,0), \psi_{\varepsilon t}(\cdot,0), w_{\varepsilon t}(\cdot,0)\} = \{\phi_1, \psi_1, w_1\} & \text{em} \quad \Omega, \end{cases} \tag{1.9}$$

onde foi usada a notação $\xi^- = -\min\{0, \xi\}$. O segundo passo consiste em mostrar que o problema (1.9) está bem posto e obter uma estimativa uniforme (em ε) para as soluções dos diversos sistemas penalizados. Finalmente, no terceiro passo, passamos o limite, quando $\varepsilon \to 0$, no sistema penalizado para obter a solução original do problema de contato.

Observemos que a energia do sistema (1.9), dada por

$$\begin{aligned} \mathbb{E}_\varepsilon(t) = \frac{1}{2}\Bigg[& \frac{\rho h^3}{12}(|\phi_{\varepsilon t}|^2 + |\psi_{\varepsilon t}|^2) + \rho h |w_{\varepsilon t}|^2 \\ & + D\left(|\phi_{\varepsilon x_1}|^2 + |\psi_{\varepsilon x_2}|^2 + 2\mu \phi_{\varepsilon x_1}\psi_{\varepsilon x_2} + \frac{1-\mu}{2}|\psi_{\varepsilon x_2} + \phi_{\varepsilon x_1}|^2 \right) \\ & + k(|w_{\varepsilon x_1} + \phi_\varepsilon|^2 + |w_{\varepsilon x_2} + \psi_\varepsilon|^2) + \int_{\Gamma_1} \frac{1}{\varepsilon}|(w_\varepsilon - g)^-|^2 d\Gamma \Bigg], \end{aligned} \tag{1.10}$$

é conservativa, isto é,

$$\frac{d}{dt}\mathbb{E}_\varepsilon(t) = 0, \quad \forall t > 0.$$

Notemos que ao adicionar termos de amortecimento (dampings) apropriados, em outras palavras, ao considerar o sistema

$$\begin{cases} \dfrac{\rho h^3}{12}\phi_{\varepsilon tt} - L_1(\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon) = 0 & \text{em} \quad Q, \\ \dfrac{\rho h^3}{12}\psi_{\varepsilon tt} - L_2(\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon) = 0 & \text{em} \quad Q, \\ \rho h w_{\varepsilon tt} - L_3(\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon) = 0 & \text{em} \quad Q, \\ \phi_\varepsilon = \psi_\varepsilon = w_\varepsilon = 0 & \text{on} \quad \Sigma_0, \\ \mathcal{B}_1(\phi_\varepsilon, \psi_\varepsilon) + \gamma_1\phi_{\varepsilon t} = 0 & \text{sobre} \quad \Sigma_1, \\ \mathcal{B}_2(\phi_\varepsilon, \psi_\varepsilon) + \gamma_2\psi_{\varepsilon t} = 0 & \text{sobre} \quad \Sigma_1, \\ \mathcal{B}_3(\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon) - \dfrac{1}{\varepsilon}(w_\varepsilon - g)^- + \gamma_3 w_{\varepsilon t} = 0 & \text{sobre} \quad \Sigma_1, \\ \{\phi_\varepsilon(\cdot,0), \psi_\varepsilon(\cdot,0), w_\varepsilon(\cdot,0)\} = \{\phi_0, \psi_0, w_0\} & \text{em} \quad \Omega, \\ \{\phi_{\varepsilon t}(\cdot,0), \psi_{\varepsilon t}(\cdot,0), w_{\varepsilon t}(\cdot,0)\} = \{\phi_1, \psi_1, w_1\} & \text{em} \quad \Omega, \end{cases} \tag{1.11}$$

com γ_i $(i = 1, 2, 3)$ valores reais positivos, temos que a energia de (1.11), ainda denotada por (1.10), satisfaz

$$\frac{d}{dt}\mathbb{E}_\varepsilon(t) = -\gamma_1 \int_{\Gamma_1} |\phi_{\varepsilon t}|^2 d\Gamma - \gamma_2 \int_{\Gamma_1} |\psi_{\varepsilon t}|^2 d\Gamma - \gamma_3 \int_{\Gamma_1} |w_{\varepsilon t}|^2 d\Gamma, \quad \forall t > 0, \tag{1.12}$$

isto é, a energia é uma função não crescente. Contudo, ao deduzir isto, acabamos não utilizando toda a informação que (1.12) nos oferece. Na verdade, como consequência da expressão (1.12), podemos utilizar técnicas que permitirão provar que a energia em questão possui decrescimento exponencial. Esse fato é descrito no resultado a seguir.

Teorema. *Consideremos* $\{\phi_0, \phi_1, \psi_0, \psi_1, w_0, w_1,\} \in [V \times L^2(\Omega)]^3$ *e uma função* $g \in C^\infty(\overline{\Omega})$ *satisfazendo* $g \leq 0$. *Existem constantes positivas* $\mathcal{C}$, ω *e* ε_0, *tal que a energia* (1.10) *associada ao problema* (1.11) *com dados iniciais* $\{\phi_0, \phi_1, \psi_0, \psi_1, w_0, w_1, g\}$ *satisfaz*

$$\mathbb{E}_\varepsilon(t) \leq \mathcal{C}\mathbb{E}_\varepsilon(0)e^{-\omega t}, \quad \forall t \geq 0, \quad \forall\, \varepsilon \in (0, \varepsilon_0). \tag{1.13}$$

Finalmente, observamos que o teorema acima implica um decaimento da energia do sistema limite de (1.11), quando $\varepsilon \to 0$, dado por

$$\begin{cases} \dfrac{\rho h^3}{12}\phi_{tt} - L_1(\phi, \psi, w) = 0 & \text{em} & Q, \\ \dfrac{\rho h^3}{12}\psi_{tt} - L_2(\phi, \psi, w) = 0 & \text{em} & Q, \\ \rho h w_{tt} - L_3(\phi, \psi, w) = 0 & \text{em} & Q, \\ \phi = \psi = w = 0 & \text{sorbe} & \Sigma_0, \\ \mathcal{B}_1(\phi, \psi) + \gamma_1 \phi_t = 0 & \text{sobre} & \Sigma_1, \\ \mathcal{B}_2(\phi, \psi) + \gamma_2 \psi_t = 0 & \text{sobre} & \Sigma_1, \\ \mathcal{B}_3(\phi, \psi, w) + \gamma_3 w_t \geq 0,\ w \geq g,\ (\mathcal{B}_3(\phi, \psi, w) + \gamma_3 w_t)(w - g) = 0 & \text{sobre} & \Sigma_1, \\ \{\phi(\cdot, 0), \psi(\cdot, 0), w(\cdot, 0)\} = \{\phi_0, \psi_0, w_0\} & \text{em} & \Omega, \\ \{\phi_{\varepsilon t}(\cdot, 0), \psi_{\varepsilon t}(\cdot, 0), w_{\varepsilon t}(\cdot, 0)\} = \{\phi_1, \psi_1, w_1\} & \text{em} & \Omega. \end{cases} \tag{1.14}$$

Mais precisamente, como consequência de (1.13), podemos mostrar que a energia

$$\begin{aligned} \mathbb{E}(t) = \frac{1}{2}\Bigg[& \frac{\rho h^3}{12}(|\phi_t|^2 + |\psi_t|^2) + \rho h |w_t|^2 + k(|w_{x_1} + \phi|^2 + |w_{x_2} + \psi|^2) \\ & + D\left(|\phi_{x_1}|^2 + |\psi_{x_2}|^2 + 2\mu\phi_{x_1}\psi_{x_2} + \frac{1-\mu}{2}|\psi_{x_2} + \phi_{x_1}|^2\right)\Bigg], \end{aligned}$$

associada ao sistema (1.14) satisfaz

$$\mathbb{E}(t) \leq \mathcal{C}\mathbb{E}e^{-\omega t}, \quad \forall t \geq 0.$$

Capítulo 4
Controlabilidade de fronteira de um sistema de campo de fases unidimensional com um único controle

(Boundary controllability of a one-dimensional phase-field system with one control force)

Neste capítulo, trataremos das propriedades de controlabilidade na fronteira do sistema de campo de fases do tipo Caginalp, (veja [25]). Este modelo descreve a transição entre o estado sólido e líquido no processo de solidificação/derretimento de um material ocupando um intervalo.

Figura 1.8: Derretimento do gelo e solidificação da lava.

Tal modelo é descrito por

$$\begin{cases} \tilde{\theta}_t - \xi\tilde{\theta}_{xx} + \dfrac{1}{2}\rho\xi\tilde{\phi}_{xx} + \dfrac{\rho}{\tau}\tilde{\theta} = f_1(\tilde{\phi}) & \text{em} \quad Q_T, \\ \tilde{\phi}_t - \xi\tilde{\phi}_{xx} - \dfrac{2}{\tau}\tilde{\theta} = f_2(\tilde{\phi}) & \text{em} \quad Q_T, \\ \tilde{\theta}(0,\cdot) = v, \ \tilde{\phi}(0,\cdot) = c, \ \tilde{\theta}(\pi,\cdot) = 0, \ \tilde{\phi}(\pi,\cdot) = c & \text{em} \quad (0,T), \\ \tilde{\theta}(\cdot,0) = \tilde{\theta}_0, \ \tilde{\phi}(\cdot,0) = \tilde{\phi}_0 & \text{em} \quad (0,\pi), \end{cases} \tag{1.15}$$

onde $(0,\pi)$ é o intervalo que contém o material, $T > 0$ e $Q_T = (0,\pi)\times(0,T)$. No sistema acima, $\tilde{\theta} = \tilde{\theta}(x,t)$ denota a temperatura do material, $\tilde{\phi} = \tilde{\phi}(x,t)$ a função de fase usada para identificar o nível de solidificação do material, $c \in \{-1,0,1\}$ e as funções f_1 e f_2 são termos não lineares definidos por

$$f_1(\tilde{\phi}) = -\frac{\rho}{4\tau}\left(\tilde{\phi} - \tilde{\phi}^3\right) \quad \text{and} \quad f_2(\tilde{\phi}) = \frac{1}{2\tau}\left(\tilde{\phi} - \tilde{\phi}^3\right).$$

Além disso, $\rho > 0$ é o calor latente, $\tau > 0$ representa o tempo de relaxamento e $\xi > 0$ representa tanto a difusividade térmica, quanto a espessura da região de colagem.

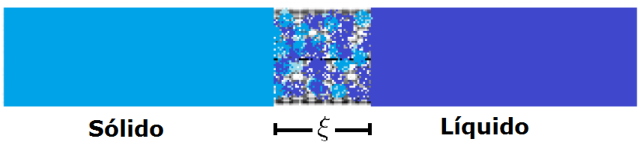

Figura 1.9: Região de colagem de tamanho ξ entre as fases sólida e líquida.

Finalmente, $v \in L^2(0,T)$ é a força de controle, que será aplicada no extremo $x = 0$ por meio de condições de bordo Dirichlet, e os dados iniciais $\tilde{\theta}_0$, $\tilde{\phi}_0$ são funções dadas. A função de fase $\tilde{\phi}$ representa a transição do material (sólido ou líquido) de forma que $\tilde{\phi} = 1$ significa que o material está no estado sólido, $\tilde{\phi} = -1$ no estado líquido e $\tilde{\phi} = 0$ em um estado intermediário, sem consistência definida.

Motivação e resultado principal

Nosso objetivo é estudar a controlabilidade do sistema (1.15). Observemos que a temperatura $\tilde{\theta}$ poderia assumir valor zero. Por outro lado, a variável $\tilde{\phi}$ que define a fase do material não possui um significado físico direto, fazendo o papel apenas de uma função de identificação. Por essa razão, somos levados a desejar controlar o sistema agindo apenas na temperatura, afinal esta é a única variável cuja controlabilidade tem sentido físico. Dessa forma, é natural querer controlar o sistema (1.15) utilizando um único controle. Em geral, problemas onde o número de controles é menor que o de equações é interessante e não muito simples de ser tratado, principalmente quando se trata de sistemas não-lineares.

Até o momento, estamos motivados a tratar do problema que consiste em provar a existência de um controle $v \in L^2(0,T)$ tal que $\tilde{\theta}(\cdot,T) = 0$, isto é, tal que a região de transição associada à temperatura

$$\Gamma(t) := \left\{ x \in (0,\pi) : \tilde{\theta}(x,t) = 0 \right\},$$

satisfaça $\Gamma(T) = (0,\pi)$, ou seja, represente todo o domínio. Mas o que ocorrerá com a fase quando esse controle dirigir a temperatura a zero? Em geral, somos levados a querer controlar todas as variáveis do sistema a zero, isto é, a desejar que $\tilde{\theta}(\cdot,T) = 0$ e $\tilde{\phi}(\cdot,T) = 0$. Contudo, como já falamos, $\tilde{\phi}(\cdot,T) = 0$ representa um estado indeterminado. Assim, faz mais sentido desejar que no instante T o material esteja completamente no estado sólido ou completamente no estado líquido, isto é, que $\tilde{\theta}(\cdot,T) = c$ com $c \in \{-1,1\}$. Isso nos motiva, finalmente, ao problema que iremos estudar: mostrar que existe um controle $v \in L^2(0,T)$ tal que o sistema (1.15) possui uma solução (em um espaço apropriado) satisfazendo

$$\tilde{\theta}(\cdot,T) = 0 \quad \text{e} \quad \tilde{\phi}(\cdot,T) = c \quad \text{em } (0,\pi), \tag{1.16}$$

para $c \in \{-1,1\}$. O resultado que iremos obter é descrito a seguir.

Teorema. *Consideremos ξ, τ e ρ três números reais positivos satisfazendo*

$$\xi \neq \frac{1}{j^2}\frac{\rho}{\tau}, \quad \forall j \geq 1. \tag{1.17}$$

e

$$\xi^2\tau^2(\ell^2 - k^2)^2 - 2\xi\rho\tau(\ell^2 + k^2) - 2\rho - 1 \neq 0, \quad \forall k,\ell \geq 1, \quad \ell > k. \tag{1.18}$$

Fixados $T > 0$ e $c = -1$ ou $c = 1$, existe $\varepsilon > 0$ tal que, para qualquer par $(\tilde{\theta}_0, \tilde{\phi}_0) \in H^{-1}(0,\pi) \times (c + H_0^1(0,\pi))$ satisfazendo

$$\|\tilde{\theta}_0\|_{H^{-1}} + \|\tilde{\phi}_0 - c\|_{H_0^1} \leq \varepsilon,$$

existe $v \in L^2(0,T)$ para o qual o sistema (1.15) possui uma única solução

$$(\tilde{\theta}, \tilde{\phi}) \in \left[L^2(Q_T) \cap C^0([0,T]; H^{-1}(0,\pi;\mathbb{R}^2))\right] \times C^0(\overline{Q}_T)$$

que satisfaz (1.16).

Desenvolvendo o problema

O estudo do problema que desejamos analisar será realizado utilizando uma série de técnicas. Basicamente, faremos uma análise espectral específica dos operadores

$$L = -D\partial_{xx} + A \quad \text{e} \quad L^* = -D^*\partial_{xx} + A^*,$$

com domínio $D(L) = D(L^*) = H^2(0, \pi; \mathbb{R}^2) \cap H_0^1(0, \pi; \mathbb{R}^2)$, onde A, D, B são dados em (1.21). Em seguida, usaremos o método dos momentos e finalizaremos com uma técnica de ponto fixo.

A seguir descreveremos brevemente os principais resultados que iremos obter. Uma vez que o objetivo desse capítulo é estudar a controlabilidade aproximada do sistema (1.15) à trajetória constante $(0, c)$, com $c = \pm 1$, faremos a mudança de variável $(\theta, \phi) = (\tilde{\theta}, \tilde{\phi} - c)$, por simplicidade. Assim, o sistema (1.15) se torna

$$\begin{cases} \theta_t - \xi\theta_{xx} + \dfrac{1}{2}\rho\xi\phi_{xx} - \dfrac{\rho}{2\tau}\phi + \dfrac{\rho}{\tau}\theta = g_1(\phi) & \text{em} \quad Q_T, \\ \phi_t - \xi\phi_{xx} + \dfrac{1}{\tau}\phi - \dfrac{2}{\tau}\theta = g_2(\phi) & \text{em} \quad Q_T, \\ \theta(0, \cdot) = v, \quad \phi(0, \cdot) = \theta(\pi, \cdot) = \phi(\pi, \cdot) = 0 & \text{em} \quad (0, T), \\ \theta(\cdot, 0) = \theta_0, \quad \phi(\cdot, 0) = \phi_0 & \text{em} \quad (0, \pi), \end{cases} \tag{1.19}$$

onde

$$g_1(\phi) = \pm\frac{3\rho}{4\tau}\phi^2 + \frac{\rho}{4\tau}\phi^3 \quad \text{e} \quad g_2(\phi) = \mp\frac{3}{2\tau}\phi^2 - \frac{1}{2\tau}\phi^3.$$

Para lidar com o sistema (1.19) iremos utilizar uma estratégia de ponto fixo. Com esse intuito, estudaremos primeiro a controlabilidade do sistema linear

$$\begin{cases} \theta_t - \xi\theta_{xx} + \dfrac{1}{2}\rho\xi\phi_{xx} - \dfrac{\rho}{2\tau}\phi + \dfrac{\rho}{\tau}\theta = 0 & \text{em} \quad Q_T, \\ \phi_t - \xi\phi_{xx} + \dfrac{1}{\tau}\phi - \dfrac{2}{\tau}\theta = 0 & \text{em} \quad Q_T, \\ \theta(0, \cdot) = v, \quad \phi(0, \cdot) = \theta(\pi, \cdot) = \phi(\pi, \cdot) = 0 & \text{em} \quad (0, T), \\ \theta(\cdot, 0) = \theta_0, \quad \phi(\cdot, 0) = \phi_0 & \text{em} \quad (0, \pi), \end{cases} \tag{1.20}$$

cuja linearização foi feita em torno do ponto $(0, 0)$. Ainda podemos escrever (1.20) em forma vetorial

$$\begin{cases} y_t - Dy_{xx} + Ay = 0 & \text{em} \quad Q_T, \\ y(0, \cdot) = Bv, \quad y(\pi, \cdot) = 0 & \text{em} \quad (0, T), \\ y(\cdot, 0) = y_0, & \text{em} \quad (0, \pi), \end{cases}$$

onde $y_0 = (\theta_0, \phi_0)$ e

$$D = \begin{pmatrix} \xi & -\dfrac{1}{2}\rho\xi \\ 0 & \xi \end{pmatrix}, \quad A = \begin{pmatrix} \dfrac{\rho}{\tau} & -\dfrac{\rho}{2\tau} \\ -\dfrac{2}{\tau} & \dfrac{1}{\tau} \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{1.21}$$

Sobre a controlabilidade do sistema linear, os seguintes resultados valem:

Teorema. *Consideremos ξ, ρ e τ três números reais positivos e fixemos $T > 0$. O sistema (1.20), com dados iniciais em $H^{-1}(0, \pi; \mathbb{R}^2)$, é aproximadamente controlável no tempo T se, e somente se, (1.18) ocorre.*

Para provar esse teorema iremos utilizar análise espectral dos operadores L e L^*. Via esse processo, veremos que a condição (1.18) é equivalente à propriedade: *"Os autovalores dos operadores L e L^* tem multiplicidade geométrica igual a 1"*, exemplificando a interferência da técnica de linearização no resultado de controlabilidade para o sistema não-linear. Observemos que a condição (1.18) caracteriza a controlabilidade aproximada do sistema (1.20). Dessa forma, (1.18) é uma condição necessária para a controlabilidade nula desse sistema no tempo $T > 0$. Com esse raciocínio, há sentido em enunciar o seguinte resultado.

Teorema. *Seja $T > 0$. Se ξ, ρ e τ são números reais positivos satisfazendo (1.17) e (1.18), então o sistema (1.20), com dados iniciais em $H^{-1}(0, \pi; \mathbb{R}^2)$, é exatamente controlável a zero no tempo $T > 0$.*

A condição (1.17) é crucial na prova da controlabilidade nula do sistema (1.20). Na verdade, é sabido (veja [9]) que, na ausência da condição (1.17), o índice de condensação (uma medida de como Λ_n se aproxima de Λ_m, $m \neq n$) da sequência $\{\Lambda_k\}_{k\geq 1}$ formada pelos autovalores de L pode ser positivo, o que implica a existência de um tempo mínimo $T_0 > 0$ para o qual o sistema é controlável.

Além da controlabilidade nula do sistema linear, ainda há outras propriedades que desejamos que o espectro $\sigma(L) = \{\Lambda_k\}_{k\geq 1}$ de L satisfaça para, finalmente, provar a controlabilidade nula (local) do sistema não-linear. Mais precisamente, um estudo espectral mais aprofundado será feito de modo a verificar se $\sigma(L)$ satisfaz o seguinte lema:

Lema. *Seja $\{\Lambda_k\}_{k\geq 1} \subset \mathbb{R}_+$ uma sequência tal que $\Lambda_k \neq \Lambda_n$, para todo $k, n \in \mathbb{N}$ com $k \neq n$. Se existe um inteiro $q \geq 1$ e constantes positivas p, δ e α tais que*

$$\begin{cases} |\Lambda_k - \Lambda_n| \geq \delta \left|k^2 - n^2\right|, & \forall k, n \in \mathbb{N},\ |k-n| \geq q, \\ \inf\limits_{k\neq n,\ |k-n|<q} |\Lambda_k - \Lambda_n| > 0, & \end{cases}$$

e

$$\left|p\sqrt{r} - \mathcal{N}(r)\right| \leq \alpha, \quad \forall r > 0,$$

onde $\mathcal{N}(r) = \#\{k : \Lambda_k \leq r\}$, então, existe $\widetilde{T}_0 > 0$ tal que, para todo $T \in (0, \widetilde{T}_0)$, é possível encontrar uma família $\{q_k\}_{k\geq 1} \subset L^2(0,T)$ biortogonal à $\{e^{-\Lambda_k t}\}_{k\geq 1}$ satisfazendo

$$\|q_k\|_{L^2(0,T)} \leq Ce^{C\sqrt{\Lambda_k} + \frac{C}{T}}, \quad \forall k \geq 1, \tag{1.22}$$

para uma constante positiva C independente de T.

Esse resultado é válido para a sequência $\sigma(L) = \{\Lambda_k\}_{k\geq 1}$, se considerarmos (1.17) e (1.18) verdadeiros. Para provar (1.22), por exemplo, (1.18) é essencial.

As consequências deste lema, juntamente com a controlabilidade nula do sistema linear, fornecerá elementos suficientes para concluirmos a controlabilidade no instante T do sistema não linear (1.15). De fato, a família $\{q_k\}_{k\geq 1} \subset L^2(0,T)$, obtida no lema acima, é crucial para aplicar o método dos momentos por causa da biortogonalidade desta. Além disso, a condição (1.22) permite estimar o custo de controle para o sistema (1.20) no tempo $T > 0$ (veja (4.13) em Remark 16). Esta estimativa, por sua vez, é de fundamental importância para a aplicação da técnica de ponto fixo.

1.4 Comentários e problemas abertos

Capítulo 2

- A equação da onda linear $u_{tt} - \gamma u_{xx} = 0$, com $\gamma > 0$, e controle na fronteira, é exatamente controlável em $(0, L) \times (0, T)$ se $T > \frac{2L}{\sqrt{\gamma}}$. Além disso, devido ao tempo finito de propagação, esta cota inferior é ótima (veja, por exemplo, [18] e [68]). Observemos que o tempo T_0 em (1.5) é o máximo das duas cotas inferiores correspondentes à controlabilidade das duas equações do sistema (1.4), se elas estivessem desacopladas. Este fato nos leva a crer que T_0 é a melhor cota inferior para a controlabilidade do sistema acoplado. Contudo, essa é uma questão aberta.

- A condição $\frac{a_1}{\rho_1} > \frac{a_2}{\rho_2}$ não representa restrição à resposta sobre controlabilidade. Sob a hipótese $\frac{a_1}{\rho_1} \leq \frac{a_2}{\rho_2}$ ainda ainda temos resposta positiva sobre a controlabilidade para $T > T_0$, com a diferença que, nesse caso, os controles estarão inseridos no extremo $x = L$ do intervalo $(0, L)$. Já a condição $\frac{a_1}{k_1} = \frac{a_2}{k_2}$, é uma hipótese técnica que surge pela escolha da função peso ϕ, com o intuito de garantir que (2.21). Dessa forma, sua necessidade permanece em aberto. Uma possível direção para tratar essa questão seria estudar a escolha de outra função peso.

- Um problema interessante seria estudar o caso de vigas com espessuras diferentes ($h_1 \neq h_2$). Seguir a mesma estratégia que utilizamos para tratar do caso onde as vigas possuem a mesma espessura ($h_1 = h_2 = h$) irá solicitar uma hipótese técnica semelhante a $\frac{a_1}{k_1} = \frac{a_2}{k_2}$, dessa vez envolvendo h_1 e h_2. Assim, uma condição suficiente é certa de ser encontrada. O problema interessante acaba sendo procurar por uma condição tanto suficiente quanto necessária para a controlabilidade.

- As técnicas utilizadas podem ser adaptadas para o estudo de vigas compostas por vários materiais diferentes. Um problema interessante nessa direção seria lidar com uma estrutura de vigas com ramificações.

- Outro problema aberto é o caso bidimensional. No caso 2-D o sistema possui uma expressão mais complexa que impede, a priori, que as técnicas utilizadas aqui sejam adaptadas diretamente e sem surgimento de dificuldades técnicas. Este é um trabalho em progresso.

Capítulo 3

- A unicidade de solução para o problema de contato (1.14) permanece aberta. Uma possível direção para estudar a unicidade seria analisar se a construção da solução encontrada neste trabalho sugere uma caracterização das soluções (1.14), isto é, se uma solução arbitrária de (1.14) pode ser sempre vista como limite de soluções de problemas penalizados.

- Permanece em aberto a estabilidade do sistema, com decaimento exponencial da energia, no caso onde são considerados menos de 3 dampings na fronteira. Outro ponto ainda não estudado é o comportamento da energia do sistema (1.14) na presença de dampings não lineares.

Capítulo 4

- Originalmente o sistema de campo de fases do tipo Caginalp possui como primeira equação a expressão $\tilde{\theta}_t - K\tilde{\theta}_{xx} + \frac{1}{2}\rho\xi\tilde{\phi}_{xx} + \frac{\rho}{\tau}\tilde{\theta} = f_1(\tilde{\phi})$, com K representando a difusividade térmica, diferindo do sistema (1.15), onde foi considerado $K = \xi$. O caso onde $K \neq \xi$ representa um novo problema que, possivelmente, deve ser estudado com uma técnica diferente da utilizada nesse trabalho, pois, nesse caso, o sistema linearizado possui tempo mínimo de controle, o que dificulta a aplicação da técnica de ponto fixo.

- Para obter uma resposta mais completa sobre a controlabilidade (local) do sistema (1.15), é necessário estudar o que ocorre caso em que as condições (1.17) e (1.18) não são válidas. Uma possível estratégia a ser seguida é linearizar o sistema (1.15) ao redor de vários pontos e observar o que ocorre com as condições semelhantes a (1.17) e (1.18) que aparecerão.

Capítulo 2

Carleman estimate for unidimensional Mindlin-Timoshenko system with discontinuous coefficients and applications

Carleman estimate for unidimensional Mindlin-Timoshenko system with discontinuous coefficients and applications

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Abstract. In this article, we study the dynamical one-dimensional Mindlin-Timoshenko system for non-homogeneous beams. Our main result is a Carleman inequality for this system, which is obtained under the hypothesis of monotonicity for the speed of the beam. Two applications of this estimate are presented in this article: the boundary controllability of the system, and the Lipschitz stability of the inverse problem consisting in recovering a time-independent potential from a single measurement of the solution.

2.1 Introduction

The Mindlin-Timoshenko system is a coupled system of two second order hyperbolic equations, it is widely used and physically fairly complete mathematical model for describing the transverse vibrations of beams. For a beam of length L this one-dimensional system reads as follows:

$$\begin{cases} \dfrac{\rho h^3}{12}\psi'' - (a\psi_x)_x + k(\psi + \sigma_x) = 0 & \text{in} \quad Q, \\ \rho h \sigma'' - (k(\psi + \sigma_x))_x = 0 & \text{in} \quad Q, \end{cases} \tag{2.1}$$

where $Q = (0, L) \times (0, T)$ and T is a given positive time. Here and throughout all the paper, we use the notation $f' = \dfrac{\partial f}{\partial t}$, and $f_x = \dfrac{\partial f}{\partial x}$. In the model (2.1), $\psi = \psi(x, t)$ represents the angle of rotation and $\sigma = \sigma(x, t)$ stands for the vertical displacement at time t of the cross section located x units from the end-point $x = 0$. The constant $h > 0$ represents the thickness of the beam, which is is considered to be small and uniform, independent of x. The constant ρ is the mass density per unit volume of the beam, and the parameters a and k are called modulus of flexural rigidity and modulus of elasticity in shear, respectively. They are given by the formulas $k = \widehat{k}Eh/2\,(1 + \mu)$ and $a = Eh^3/12(1 - \mu^2)$, where $\widehat{k}$ is a shear correction coefficient, E is the Young's modulus and μ is the Poisson's ratio, $0 < \mu < 1/2$.

In this paper, we will consider a beam composed by two different materials with same thickness, one taking place in $(0, M)$ and another one in (M, L), for some fixed $M \in (0, L)$. The values of E, μ, ρ and $\widehat{k}$ depend on to material, and then we will consider coefficients a,

ρ and k given by

$$a(x) = \begin{cases} a_1, & \text{if} \quad x \in \ (0, M) \\ a_2, & \text{if} \quad x \in \ (M, L) \end{cases}, \quad \rho(x) = \begin{cases} \rho_1, & \text{if} \quad x \in \ (0, M) \\ \rho_2, & \text{if} \quad x \in \ (M, L) \end{cases}$$

$$k(x) = \begin{cases} k_1, & \text{if} \quad x \in \ (0, M) \\ k_2, & \text{if} \quad x \in \ (M, L), \end{cases} \tag{2.2}$$

where $a_1, a_2, \rho_1, \rho_2, k_1, k_2 \in \mathbb{R}$. Within this context, we are interested in studying controllability and inverse problems for system (2.1).

In the literature, it is possible to find several results about the Mindlin-Timoshenko system. Boundary exact controllability, with control forces acting on each equation, was obtained in [66] and [71]. By means of spectral methods, in [16] was proved the same kind of controllability result for this linear system with only one control force, when considered one part of the spectrum. The result concerning the whole spectrum is a difficult and interesting open problem.

To the best of our knowledge, this is the first work dealing with either inverse problems or controllability of the Mindlin-Timoshenko system in non-homogeneous materials.

Concerning inverse problems, global Carleman estimates and the method of Bukhgeim-Klibanov [22, 24] are especially useful for obtaining stability of coefficients with one-measurement observations. It is possible to obtain local Lipschitz stability around a single known solution, provided that this solution is regular enough and contains enough information [60] (see also [59] and [88]). Many other related inverse results for hyperbolic equations use the same strategy. A complete list is too long to be given here. To cite some of them see [79] and [88] where Dirichlet boundary data and Neumann measurements are considered, and [54, 53] where Neumann boundary data and Dirichlet measurements are studied. These references are all based upon the use of local or global Carleman estimates. Similar inverse problems, but using pointwise Carleman estimates, are studied in [47, 48, 61]. An inverse problem for a viscoelastic Timoshenko beam model can be read in [**?**], where the inverse problem of determining two time-dependent memory kernels from supplementary information is analyzed.

There exist several other problems related with the Mindlin-Timoshenko system, each one focused on analyzing different aspects of it. For example, stabilization was studied in [13, 58, 65] and [2] in the unidimensional case and in [76] in the multi-dimensional case. Global attractors were studied in [30] under the view of different boundary conditions.

In the context of wave equation with discontinuous main coefficient, a well-known result of exact controllability, via the method of multipliers, was obtained in [68]. In [19], a global Carleman inequality was proved, and then applied to obtain Lipschitz stability for the inverse problem of retrieving a stationary potential for the 2-D wave equation with discontinuous principal coefficient, from a single time-dependent Neumann boundary measurement. Concerning the heat equation with discontinuous main coefficient, a exact null controllability for a semilinear system was obtained in [35].

Next, we present the two problems we address in this work, and the main results we obtain.

2.1.1 Controllability

For the control problem we consider the Mindlin-Timoshenko system

$$\begin{cases} \frac{\rho(x)h^3}{12}\psi'' - (a(x)\psi_x)_x + k(x)(\psi + \sigma_x) = 0 & \text{in} \quad Q, \\ \rho(x)h\sigma'' - (k(x)(\psi + \sigma_x))_x = 0 & \text{in} \quad Q, \\ \psi(0,\cdot) = \sigma(0,\cdot) = 0, \;\; \psi(L,\cdot) = f_1, \;\; \sigma(L,\cdot) = f_2 & \text{in} \quad (0,T), \\ \psi(\cdot,0) = \psi_0, \;\; \psi'(\cdot,0) = \psi_1 & \text{in} \quad (0,L), \\ \sigma(\cdot,0) = \sigma_0, \;\; \sigma'(\cdot,0) = \sigma_1 & \text{in} \quad (0,L), \end{cases} \tag{2.3}$$

where the boundary conditions $(2.3)_3$ mean that the beam is clamped at $x = 0$ and the controls f_1, f_2 are lateral forces applied at the extreme $x = L$.

It is well-known that, given $\{\psi_0, \psi_1, \sigma_0, \sigma_1\} \in [L^2(0,L) \times H^{-1}(0,L)]^2$ and $f_1, f_2 \in L^2(0,T)$, system (2.3) has a unique solution

$$\{\psi, \sigma\} \in [C([0,T]; L^2(0,L)) \cap C^1([0,T]; H^{-1}(0,L))]^2.$$

The first main result of this work is

Theorem 1. *Let us define*

$$T_0 = 2L\sqrt{\max\left\{\frac{\rho_1 h^3}{12a_1}, \frac{\rho_2 h^3}{12a_2}, \frac{\rho_1 h}{k_1}, \frac{\rho_2 h}{k_2}\right\}}, \tag{2.4}$$

and let a, k, ρ be given by (2.2) with

$$\frac{a_1}{\rho_1} > \frac{a_2}{\rho_2} \quad \textit{and} \quad \frac{a_1}{k_1} = \frac{a_2}{k_2}. \tag{2.5}$$

Given $\{\psi_0, \psi_1, \sigma_0, \sigma_1\}, \in [L^2(0,L) \times H^{-1}(0,L)]^2$ and $T > T_0$, then there exist controls $f_1, f_2 \in L^2(0,T)$ such that the solution $\{u, v\}$ of the Mindlin-Timoshenko system (2.3) satisfies

$$\{\psi(\cdot,T), \psi'(\cdot,T), \sigma(\cdot,T), \sigma'(\cdot,T)\} = \{0,0,0,0\} \quad \textit{in} \quad (0,L).$$

In order to obtain the exact controllability of (2.3), firstly, we consider, by the well-known duality argument, the following adjoint system:

$$\begin{cases} \frac{\rho(x)h^3}{12}u'' - (a(x)u_x)_x + k(x)(u + v_x) = 0 & \text{in} \quad Q, \\ \rho(x)hv'' - (k(x)(u + v_x))_x = 0 & \text{in} \quad Q, \\ u(0,\cdot) = v(0,\cdot) = u(L,\cdot) = v(L,\cdot) = 0 & \text{in} \quad (0,T), \\ u(\cdot,0) = u_0, \;\; u'(\cdot,0) = u_1 & \text{in} \quad (0,L), \\ v(\cdot,0) = v_0, \;\; v'(\cdot,0) = v_1 & \text{in} \quad (0,L). \end{cases} \tag{2.6}$$

For initial data $\{u_0, u_1, v_0, v_1\} \in [H_0^1(0,L) \times L^2(0,L)]^2$, this system has a unique solution in the class

$$\{u, v\} \in [C([0,T]; H_0^1(0,L)) \cap C^1([0,T]; L^2(0,L))]^2.$$

According to the Hilbert uniqueness method (HUM) introduced by Lions (see [68]), to prove Theorem 1 is equivalent to obtain a suitable observability inequality for the system (2.6). More precisely, we must find a constant $C > 0$ such that the solution $\{u, v\}$ of (2.6) satisfies

$$\mathbb{E}_{u,v}(0) \leq C \int_0^T \left(|u_x|^2 + |v_x|^2\right)\big|_{x=L} \, dt, \tag{2.7}$$

where $\mathbb{E}_{u,v}$ is the the energy of the system and it is given by

$$\begin{aligned} \mathbb{E}_{u,v}(t) = \frac{1}{2} \Bigg[& \frac{h^3}{12} \|\rho^{1/2} u'\|^2_{L^2(0,L)} + h \|\rho^{1/2} v'\|^2_{L^2(0,L)} \\ & + \|a^{1/2} u_x\|^2_{L^2(0,L)} + \|k^{1/2}(u + v_x)\|^2_{L^2(0,L)} \Bigg] . \end{aligned} \tag{2.8}$$

This observability estimate will be obtained as a consequence of a suitable Carleman estimate, which will be developed in Section 2.2.

Remark 1. *Given $\gamma > 0$, it is known that the linear wave equation $u_{tt} - \gamma u_{xx} = 0$ is boundary exactly controllable in $(0, L) \times (0, T)$ if $T > \frac{2L}{\sqrt{\gamma}}$. Moreover, due to the finite speed of propagation, the lower bound for the time is sharp (see, for instance, [18] and [68]). Let us recall that T_0 given in (2.4) is the maximum of the two bounds corresponding to the controllability of the equations in the Mindlin-Timoshenko system (2.3), if they were without coupling. This fact leads us to think that it is the best lower bound for the controllability time of the coupled system.*

Remark 2. *We can notice that the condition $\frac{a_1}{\rho_1} > \frac{a_2}{\rho_2}$ is equivalent to $\frac{k_1}{\rho_1} > \frac{k_2}{\rho_2}$, since we have considered the technical assumption $\frac{a_1}{k_1} = \frac{a_2}{k_2}$. This equality also has a key role in ensuring a transmission condition for the weights involved in Carleman estimate before mentioned.*

2.1.2 Inverse Problem

We are interested in the inverse problem of recovering coefficients from a Mindlin-Timoshenko system with discontinuous coefficients from boundary measurements. To be more precise, we will consider an inverse problem for the following Mindlin-Timoshenko system with potentials:

$$\begin{cases} \dfrac{\rho(x)h^3}{12} u'' - (a(x)u_x)_x + k(x)(u + v_x) + p_1(x)u = 0 & \text{in} \quad Q, \\ \rho(x)hv'' - (k(x)(u + v_x))_x + p_2(x)v = 0 & \text{in} \quad Q, \\ u(0, \cdot) = v(0, \cdot) = u(L, \cdot) = v(L, \cdot) = 0 & \text{in} \quad (0, T), \\ u(\cdot, 0) = u_0, \ \ u'(\cdot, 0) = u_1 & \text{in} \quad (0, L), \\ v(\cdot, 0) = v_0, \ \ v'(\cdot, 0) = v_1 & \text{in} \quad (0, L). \end{cases} \tag{2.9}$$

It is well-known that for each couple of potentials $\{p_1, p_2\} \in [L^\infty(0, L)]^2$ system (2.9) has a unique solution in the class

$$\{u(p_1, p_2), v(p_1, p_2)\} \in [C([0, T]; H_0^1(0, L)) \cap C^1([0, T]; L^2(0, L))]^2,$$

when $\{u_0, u_1, v_0, v_1\} \in [H_0^1(0, L) \times L^2(0, L)]^2$.

The proposed inverse problem consists of retrieving the potentials $\{p_1, p_2\}$ involved in equation (2.9), by knowing the normal derivative of the solution $\{u(p_1, p_2), v(p_1, p_2)\}$ on the boundary. We apply the Bukhgeim-Klibanov method, and we use the global Carleman estimate developed in this work. We obtain the following result.

Remark 3. *Under the hypotheses* (2.4) *and* (2.5)*, if* $T > T_0$*,* $\{u_0, u_1, v_0, v_1\} \in [H^1(\Omega) \times L^2(\Omega)]^2$*,* $\{p_1, p_2\} \in [L^\infty(\Omega)]^2$*, and* $r > 0$ *satisfy*

$$\begin{aligned} &|u_0| \geq r > 0 \ a.e. \ in \ (0, L), \quad u(p_1, p_2) \in H^1(0, T; L^\infty(\Omega)), \\ &|v_0| \geq r > 0 \ a.e. \ in \ (0, L), \quad v(p_1, p_2) \in H^1(0, T; L^\infty(\Omega)), \end{aligned} \tag{2.10}$$

then, for a bounded set $\mathcal{U} \subset [L^\infty(\Omega)]^2$*, there exist a constant*

$$C = C(a_j, k_j, \rho_j, L, M, T, \|\{p_j\}\|_{L^\infty(\Omega)}, \|\{u(p_1, p_2), v(p_1, p_2)\}\|_{[H^1(0,T;L^\infty(\Omega))]^2}, \mathcal{U}, r) > 0$$

such that

$$\begin{aligned} \|p_1 - q_1\|_{L^2(0,L)} + \|p_2 - q_2\|_{L^2(0,L)} \ \leq \ & C \Big(\|u_x(L, \cdot)(p_1, p_2) - u_x(L, \cdot)(q_1, q_2)\|_{H^1(0,T)} \\ & + \|v_x(L, \cdot)(p_1, p_2) - v_x(L, \cdot)(q_1, q_2)\|_{H^1(0,T)} \Big), \end{aligned} \tag{2.11}$$

for all $\{q_1, q_2\} \in \mathcal{U}$*, where* $\{u(p_1, p_2), v(p_1, p_2)\}$ *and* $\{u(q_1, q_2), v(q_1, q_2)\}$ *are solutions of* (2.9) *with potentials* $\{p_1, p_2\}$ *and* $\{q_1, q_2\}$*, respectively.*

The rest of the paper is organized as follows: In chapter 2, we will prove a general Carleman estimate for the wave equation with discontinuous coefficients, and then we apply this estimate to get a Carleman estimate for the Mindlin-Timoshenko system (2.6). In chapter 3, we will use the Careman estimate to prove the observability inequality (2.7), and deduce the controllability result. In the last chapter, we will prove the stability of the stated inverse problem.

2.2 Carleman estimate

In this section we will obtain our main result concerning a global Carleman estimate for the solution of the adjoint system (2.6). Denoting $Q_1 = (0, M) \times (0, T)$ and $Q_2 = (M, L) \times (0, T)$, we can notice that equations in (2.6) are equivalent to

$$\begin{cases} \dfrac{\rho_1 h^3}{12} u_1'' - a_1 u_{1xx} + k_1(u_1 + v_{1x}) = 0 & \text{in} \quad Q_1, \\ \rho_1 h v_1'' - k_1(u_1 + v_{1x})_x = 0 & \text{in} \quad Q_1, \\ \dfrac{\rho_2 h^3}{12} u_2'' - a_2 u_{2xx} + k_2(u_2 + v_{2x}) = 0 & \text{in} \quad Q_2, \\ \rho_2 h v_2'' - k_2(u_2 + v_{2x})_x = 0 & \text{in} \quad Q_2, \end{cases} \tag{2.1}$$

together with the transmission conditions

$$\begin{cases} u_1(M, \cdot) = u_2(M, \cdot), \quad v_1(M, \cdot) = v_2(M, \cdot) & \text{in } (0, T), \\ \dfrac{a_1}{\rho_1} u_{1x}(M, \cdot) = \dfrac{a_2}{\rho_2} u_{2x}(M, \cdot), \quad \dfrac{k_1}{\rho_1} v_{1x}(M, \cdot) = \dfrac{k_2}{\rho_2} v_{2x}(M, \cdot) & \text{in } (0, T). \end{cases} \tag{2.2}$$

For given $f \in L^2(\widetilde{Q})$, with $\widetilde{Q} = (b, B) \times (0, T)$ and $(b, B) \subset \mathbb{R}$, $b < B$, let us consider the system

$$\begin{cases} L_\gamma(u) = f & \text{in} \quad \widetilde{Q}, \\ u(\cdot, 0) = u(\cdot, T) = u'(\cdot, 0) = u'(\cdot, T) = 0 & \text{in} \quad (b, B), \end{cases} \tag{2.3}$$

where

$$L_\gamma(u) = u'' - \gamma u_{xx}, \quad \gamma > 0. \tag{2.4}$$

Let us define

$$E_\gamma(u) = |u'|^2 - \gamma |u_x|^2, \tag{2.5}$$

$$\phi(x, t) = \alpha(x - x_0)^2 - \beta \left(t - \frac{T}{2}\right)^2 + N, \quad \varphi = e^{s\phi}, \tag{2.6}$$

with $\alpha, N \in \mathbb{R}$, $\beta \in (0, 1)$ and $x_0 < 0$, and

$$\begin{aligned} P_{1,\gamma}(u) &= u'' - \gamma u_{xx} + s^2\lambda^2\varphi^2 E_\gamma(\phi)u, \\ P_{2,\gamma}(u) &= -2s\lambda\varphi\phi' u' + 2s\lambda\gamma\varphi\phi_x u_x, \\ R_\gamma(u) &= -s\lambda\varphi L_\gamma(\phi)u - s^2\lambda\varphi E_\gamma(\phi)u, \end{aligned} \tag{2.7}$$

where s and λ are parameters to be used in the Carleman estimate. In what follows, C denotes various positive constants (usually depending on L, M, T and x_0).

In the following proposition, we prove a Carleman estimate for the wave equation with discontinuous main coefficient.

Proposition 1. *Let us consider $T > 0$, $\alpha, \gamma > 0$. If u is solution of the system (2.3) with $\gamma\alpha \geq 1$, then there exist positive constants C_γ, λ_1 and s_1 such that $w = e^{\lambda\varphi}u$ satisfies*

$$\|P_{1,\gamma}(w)\|^2_{L^2(\widetilde{Q})} + \|P_{2,\gamma}(w)\|^2_{L^2(\widetilde{Q})} + C_\gamma \|w\|_{\widetilde{Q}} \leq 2\int_{\widetilde{Q}} e^{2\lambda\varphi}|L_\gamma(u)|^2 dxdt - 2\mathbb{H}_{w,\gamma}(b, B), \tag{2.8}$$

for all $s \geq s_1$ and $\lambda > \lambda_1$, where

$$\|w\|_{\widetilde{Q}} = s\lambda \int_{\widetilde{Q}} \varphi|w'|^2 dxdt + s\lambda \int_{\widetilde{Q}} \varphi|w_x|^2 dxdt + s^3\lambda^3 \int_{\widetilde{Q}} \varphi^3|w|^2 dxdt, \tag{2.9}$$

$$\begin{aligned} \mathbb{H}_{w,\gamma}(b, B) = & -s\lambda\gamma \int_0^T \varphi\phi_x|w'|^2\Big|_b^B dt + 2s\lambda\gamma \int_0^T \varphi\phi' w' w_x\Big|_b^B dt - s\lambda\gamma^2 \int_0^T \varphi\phi_x|w_x|^2\Big|_b^B dt \\ & + s^3\lambda^3\gamma \int_0^T \varphi^3\phi_x E_\gamma(\phi)|w|^2\Big|_b^B dt. \end{aligned} \tag{2.10}$$

Proof: We can notice that $e^{\lambda\varphi}L_\gamma(u) = P_{1,\gamma}(w) + P_{2,\gamma}(w) + R_\gamma(w)$. This fact motivates us to estimate the inner product in L^2 of $P_{1,\gamma}(w)$ and $P_{2,\gamma}(w)$ in order to obtain (2.8). We can write $\langle P_{1,\gamma}(w), P_{2,\gamma}(w)\rangle_{L^2(\widetilde{Q})} = \sum I_{i,j}$ with $i = 1, 2, 3$, $j = 1, 2$ and $I_{i,j}$ being the integral concerning the inner product involving the i-th term of $P_{1,\gamma}(w)$ and the j-th term of $P_{2,\gamma}(w)$. Making the computations we have

- $I_{1,1} = s\lambda \int_{\widetilde{Q}} (\varphi\phi')'|w'|^2 dxdt,$
- $I_{1,2} = -2s\lambda\gamma \int_{\widetilde{Q}} (\varphi\phi_x)' w_x w' dxdt + s\lambda\gamma \int_{\widetilde{Q}} (\varphi\phi_x)_x |w'|^2 dxdt - s\lambda\gamma \int_0^T \varphi\phi_x |w'|^2\big|_b^B dt,$
- $I_{2,1} = -2s\lambda\gamma \int_{\widetilde{Q}} (\varphi\phi')_x w' w_x dxdt + s\lambda\gamma \int_{\widetilde{Q}} (\varphi\phi')'|w_x|^2 dxdt + 2s\lambda\gamma \int_0^T \varphi\phi' w' w_x\big|_b^B dt,$
- $I_{2,2} = s\lambda\gamma^2 \int_{\widetilde{Q}} (\varphi\phi_x)_x |w_x|^2 dxdt - s\lambda\gamma^2 \int_0^T \varphi\phi_x |w_x|^2\big|_b^B dt,$
- $I_{3,1} = s^3\lambda^3 \int_{\widetilde{Q}} (\varphi^3\phi' E_\gamma(\phi))'|w|^2 dxdt,$
- $I_{3,2} = -s^3\lambda^3\gamma \int_{\widetilde{Q}} (\varphi^3\phi_x E_\gamma(\phi))_x |w|^2 dxdt + s^3\lambda^3\gamma \int_0^T \varphi^3\phi_x E_\gamma(\phi)|w|^2\big|_b^B dt.$

Organizing the terms, we obtain that

$$\begin{aligned}\langle P_{1,\gamma}(w), P_{2,\gamma}(w)\rangle_{L^2(\widetilde{Q})} &= s\lambda \int_{\widetilde{Q}} (A_1|w'|^2 + \gamma A_1 |w_x|^2 - 2\gamma A_2 w' w_x) dxdt \\ &\quad + s^3\lambda^3 \int_{\widetilde{Q}} A_3 |w|^2 dxdt + \mathbb{H}_u(b, B),\end{aligned} \tag{2.11}$$

where

$$A_1 = \big(\varphi\phi'\big)' + \gamma\big(\varphi\phi_x\big)_x, \quad A_2 = \big(\varphi\phi_x\big)' + \big(\varphi\phi'\big)_x, \quad A_3 = \big(\varphi^3\phi' E_\gamma(\phi)\big)' - \gamma\big(\varphi^3\phi_x E_\gamma(\phi)\big)_x. \tag{2.12}$$

Since $\beta \in (0,1)$ and $\alpha\gamma \geq 1$, we get

$$\begin{aligned}A_1|w'|^2 + \gamma A_1|w_x|^2 - 2\gamma A_2 w' w_x &= \varphi\big(s(|\phi'|^2 + \gamma|\phi_x|^2) + \phi'' + \gamma\phi_{xx}\big)|w'|^2 \\ &\quad + \gamma\varphi\big(s(|\phi'|^2 + \gamma|\phi_x|^2) + \phi'' + \gamma\phi_{xx}\big)|w_x|^2 \\ &\quad - 4\gamma s\varphi\phi'\phi_x w' w_x \\ &= \varphi\big(s(\phi' w' - \gamma\phi_x w_x)^2 + s\gamma(\phi_x w' - \phi' w_x)^2\big) \\ &\quad + \varphi(\phi'' + \gamma\phi_{xx}))(|w'|^2 + \gamma|w_x|^2) \\ &\geq \varphi(-2\beta + 2\alpha\gamma)(|w'|^2 + \gamma|w_x|^2) \\ &\geq C\varphi(|w'|^2 + |w_x|^2) > 0.\end{aligned} \tag{2.13}$$

Being $\gamma > 0$, $x_0 < 0$ and, again, $\beta \in (0,1)$ and $\alpha\gamma \geq 1$, we have

$$\begin{aligned}A_3 &= \varphi^3\big((3s|\phi'|^2 + \phi'' - 3s\gamma|\phi_x|^2 - \gamma\phi_{xx})E_\gamma(\phi) + 2|\phi'|^2\phi'' + \gamma^2(2|\phi_x|^2\phi_{xx})\big) \\ &= \varphi^3\big(3sE_\gamma(\phi)^2 + (\phi'' - \gamma\phi_{xx} + 2\phi'')E_\gamma(\phi) + 2\gamma|\phi_x|^2(\phi'' + \gamma\phi_{xx})\big) \\ &= \varphi^3\big(3sE_\gamma(\phi)^2 + (-6\beta - 2\gamma\alpha)E_\gamma(\phi) + 16\gamma\alpha^2|x - x_0|^2(\alpha\gamma - \beta)\big) \\ &\geq \varphi^3 \min_{\xi\in\mathbb{R}} \big(3s\xi^2 + (-6\beta - 2\gamma\alpha)\xi + 16\gamma\alpha^2 x_0^2(\alpha\gamma - \beta)\big) \\ &= \varphi^3 \left(16\gamma\alpha^2 x_0^2(\alpha\gamma - \beta) - \frac{(-3\beta + \gamma\alpha)^2}{3s}\right) \\ &\geq C\varphi^3 > 0,\end{aligned} \tag{2.14}$$

for s large enough. From (2.11)-((2.14) we can deduce

$$\langle P_{1,\gamma}(w), P_{2,\gamma}(w)\rangle_{L^2(\widetilde{Q})} \geq C_A \|w\|_{\widetilde{Q}} + \mathbb{H}_{w,\gamma}(b,B), \tag{2.15}$$

which concludes the proof of the result. ∎

In order to state the main goal of this section, given constants γ_1, γ_2, we define the space

$$\begin{aligned} X_{\gamma_1,\gamma_2} \;=\; & \{u \in L^2(0,T;L^2((0,L));\; u = u_1 \text{ in } Q_1,\; u = u_2 \text{ in } Q_2, \\ & L_{\gamma_1}(u_1) \in L^2(0,T;L^2(0,M)),\; L_{\gamma_2}(u_2) \in L^2(0,T;L^2(M,L)), \\ & u(0,\cdot) = u(L,\cdot) = u(\cdot,0) = u(\cdot,T) = u'(\cdot,0) = u'(\cdot,T) = 0\}. \end{aligned} \tag{2.16}$$

We have the following result.

Theorem 2. *Let $T > 0$, a, k, ρ be as in (2.2) and let us consider*

$$\phi(x,t) = \begin{cases} \phi_1(x,t) := \max\left\{\frac{\rho_1 h^3}{12a_1}, \frac{\rho_1 h}{k_1}\right\}(x-x_0)^2 - \beta\left(t-\dfrac{T}{2}\right)^2 + N_1, & in \quad (0,M), \\ \phi_2(x,t) := \max\left\{\frac{\rho_2 h^3}{12a_2}, \frac{\rho_2 h}{k_2}\right\}(x-x_0)^2 - \beta\left(t-\dfrac{T}{2}\right)^2 + N_2, & in \quad (M,L). \end{cases} \tag{2.17}$$

Then there exist positive constants C, λ_0 and s_0 such that

$$\begin{aligned} &\left\|\left\{P_{1,\frac{12a}{\rho h^3}}(e^{\lambda\varphi}u), P_{1,\frac{ka}{\rho h}}(e^{\lambda\varphi}v)\right\}\right\|^2_{L^2(Q)\times L^2(Q)} + \left\|\left\{P_{2,\frac{12a}{\rho h^3}}(e^{\lambda\varphi}u), P_{2,\frac{ka}{\rho h}}(e^{\lambda\varphi}v)\right\}\right\|^2_{L^2(Q)\times L^2(Q)} \\ &+ s\lambda\int_Q e^{2\lambda\varphi}\varphi(|u'|^2+|u_x|^2+|v'|^2+|v_x|^2)dxdt + s^3\lambda^3\int_Q e^{2\lambda\varphi}\varphi^3(|u|^2+|v|^2)dxdt \\ &\quad \leq Cs\lambda\int_0^T e^{2\lambda\varphi}\varphi(|u_x|^2+|v_x|^2)\Big|_{x=L}dt + C\int_Q e^{2\lambda\varphi}\left(\left|L_{\frac{12a}{\rho h^3}}(u)\right|^2 + \left|L_{\frac{k}{\rho h}}(v)\right|^2\right)dxdt, \end{aligned} \tag{2.18}$$

for all $\{u,v\} \in X_{\frac{12a_1}{\rho_1 h^3},\frac{12a_2}{\rho_2 h^3}} \times X_{\frac{k_1}{\rho_1 h},\frac{k_2}{\rho_2 h}}$ satisfying the conditions (2.2) and for all $s \geq s_0$ and $\lambda > \lambda_0$.

Proof: Let us denote ϕ as in (2.17) where N_j is taken such that $\phi_j \geq 1$ and

$$N_1 - N_2 = \left(\max\left\{\frac{\rho_2 h^3}{12a_2}, \frac{\rho_2 h}{k_2}\right\} - \max\left\{\frac{\rho_1 h^3}{12a_1}, \frac{\rho_1 h}{k_1}\right\}\right)(M-x_0)^2. \tag{2.19}$$

From (2.19) it is clear that

$$\phi_1(M,\cdot) = \phi_2(M,\cdot). \tag{2.20}$$

Being $\frac{a_1}{k_1} = \frac{a_2}{k_2}$, we have

$$\frac{a_1}{\rho_1}\phi_{1x}(M,\cdot) = \frac{a_2}{\rho_2}\phi_{2x}(M,\cdot), \quad \frac{k_1}{\rho_1}\phi_{1x}(M,\cdot) = \frac{k_2}{\rho_2}\phi_{2x}(M,\cdot) \quad \text{on } (0,T). \tag{2.21}$$

Since $\{u,v\} \in X_{\frac{12a_1}{\rho_1 h^3},\frac{12a_2}{\rho_2 h^3}} \times X_{\frac{k_1}{\rho_1 h},\frac{k_2}{\rho_2 h}}$, it follows, for $j = 1,2$, that u_j (resp. v_j) is solution of (2.3) with $\gamma = \frac{12a_j}{\rho_j h^3}$ (resp. $\gamma = \frac{k_j}{\rho_j h}$) and $\widetilde{Q} = Q_j$. In this way, denoting $\{w_j, \overline{w}_j\} =$

$e^{\lambda\varphi_j}\{u_j, v_j\}$, Proposition 1 give us that

$$\begin{aligned}
&\left\|P_{1,\frac{12a_1}{\rho_1 h^3}}(w_1)\right\|^2_{L^2(Q_1)} + \left\|P_{2,\frac{12a_1}{\rho_1 h^3}}(w_1)\right\|^2_{L^2(Q_1)} + C\|w_1\|_{Q_1} \\
&\qquad\qquad \le 2\int_{Q_1} e^{2\lambda\varphi_1}\left|L_{\frac{12a_1}{\rho_1 h^3}}(u_1)\right|^2 dxdt - 2\mathbb{H}_{w,\frac{12a_1}{\rho_1 h^3}}(0,M), \\
&\left\|P_{1,\frac{12a_2}{\rho_2 h^3}}(w_2)\right\|^2_{L^2(Q_2)} + \left\|P_{2,\frac{12a_2}{\rho_2 h^3}}(w_2)\right\|^2_{L^2(Q_2)} + C\|w_2\|_{Q_2} \\
&\qquad\qquad \le 2\int_{Q_2} e^{2\lambda\varphi_2}\left|L_{\frac{12a_2}{\rho_2 h^3}}(u_2)\right|^2 dxdt - 2\mathbb{H}_{w,\frac{12a_2}{\rho_2 h^3}}(M,L), \\
&\left\|P_{1,\frac{k_1}{\rho_1 h}}(\overline{w}_1)\right\|^2_{L^2(Q_1)} + \left\|P_{2,\frac{k_1}{\rho_1 h}}(\overline{w}_1)\right\|^2_{L^2(Q_1)} + C\|\overline{w}_1\|_{Q_1} \\
&\qquad\qquad \le 2\int_{Q_1} e^{2\lambda\varphi_1}\left|L_{\frac{k_1}{\rho_1 h}}(v_1)\right|^2 dxdt - 2\mathbb{H}_{\overline{w},\frac{k_1}{\rho_1 h}}(0,M), \\
&\left\|P_{1,\frac{k_2}{\rho_2 h}}(\overline{w}_2)\right\|^2_{L^2(Q_2)} + \left\|P_{2,\frac{k_2}{\rho_2 h}}(\overline{w}_2)\right\|^2_{L^2(Q_2)} + C\|\overline{w}_2\|_{Q_2} \\
&\qquad\qquad \le 2\int_{Q_2} e^{2\lambda\varphi_2}\left|L_{\frac{k_2}{\rho_2 h}}(v_2)\right|^2 dxdt - 2\mathbb{H}_{\overline{w},\frac{k_2}{\rho_2 h}}(M,L).
\end{aligned} \tag{2.22}$$

From (2.20) and (2.21) we can observe that $\{w_1, w_2\}$ satisfies the transmission conditions (2.2), i.e., (2.2) still holds true if we replace u by w. Then, using the boundary conditions, we can deduce

$$\begin{aligned}
&\mathbb{H}_{w,\frac{12a_1}{\rho_1 h^3}}(0,M) + \mathbb{H}_{w,\frac{12a_2}{\rho_2 h^3}}(M,L) \\
&\quad = -s\lambda\frac{12}{h^3}\int_0^T \varphi_1|w_1'|^2\left(\frac{a_1}{\rho_1}\phi_{1x} - \frac{a_2}{\rho_2}\phi_{2x}\right)\Big|_{x=M} dt \\
&\quad + 2s\lambda\frac{12}{h^3}\int_0^T \varphi_1\phi_1' w_1'\left(\frac{a_1}{\rho_1}w_{1x} - \frac{a_2}{\rho_2}w_{2x}\right)\Big|_{x=M} dt \\
&\quad - s\lambda\frac{12}{h^3}\int_0^T \varphi_1\left|\frac{a_1}{\rho_1}w_{1x}\right|^2 (\phi_{1x} - \phi_{2x})|_{x=M} dt \\
&\quad + s^3\lambda^3\frac{12}{h^3}\int_0^T \varphi_1^3|w_1|^2\left(\frac{a_1}{\rho_1}\phi_{1x}E_{\frac{12a_1}{\rho_1 h^3}}(\phi_1) - \frac{a_2}{\rho_2}\phi_{2x}E_{\frac{12a_2}{\rho_2 h^3}}(\phi_2)\right)\Big|_{x=M} dt \\
&\quad + s\lambda\left(\frac{12a_1}{\rho_1 h^3}\right)^2\int_0^T \varphi_1\phi_{1x}|w_{1x}|^2\big|_{x=0} dt - s\lambda\left(\frac{12a_2}{\rho_2 h^3}\right)^2\int_0^T \varphi_2\phi_{2x}|w_{2x}|^2\big|_{x=L} dt.
\end{aligned} \tag{2.23}$$

From (2.21) and since $\frac{a_1}{\rho_1} > \frac{a_2}{\rho_2}$, we have in $(0,T)$

$$\begin{aligned}
&\frac{a_1}{\rho_1}\phi_{1x}(M,\cdot) - \frac{a_2}{\rho_2}\phi_{2x}(M,\cdot) = 0, \\
&-(\phi_{1x}(M,\cdot) - \phi_{2x}(M,\cdot)) \ge -\frac{\rho_2}{a_2}\left(\frac{a_1}{\rho_1}\phi_{1x}(M,\cdot) - \frac{a_2}{\rho_2}\phi_{2x}(M,\cdot)\right) = 0
\end{aligned} \tag{2.24}$$

and

$$\frac{a_1}{\rho_1}\phi_{1x}(M,\cdot)E_{\frac{12a_1}{\rho_1 h^3}}(\phi_1)(M,\cdot) - \frac{a_2}{\rho_2}\phi_{2x}(M,\cdot)E_{\frac{12a_2}{\rho_2 h^3}}(\phi_2)(M,\cdot)$$
$$= |\phi_1'(M,\cdot)|^2\left(\frac{a_1}{\rho_1}\phi_{1x}(M,\cdot) - \frac{a_2}{\rho_2}\phi_{2x}(M,\cdot)\right) - \frac{12}{h^3}\left(\frac{a_1^2}{\rho_1^2}\phi_{1x}^3(M,\cdot) - \frac{a_2^2}{\rho_2^2}\phi_{2x}^3(M,\cdot)\right)$$
$$= -\frac{12}{h^3}\left(\frac{\rho_1}{a_1} - \frac{\rho_2}{a_2}\right)\left(\frac{a_1}{\rho_1}\phi_{1x}(M,\cdot)\right)^3 \geq 0. \tag{2.25}$$

From (2.2) and (2.23)-(2.25) we get

$$\mathbb{H}_{w,\frac{12a_1}{\rho_1 h^3}}(0,M) + \mathbb{H}_{w,\frac{12a_2}{\rho_2 h^3}}(M,L) \geq -2\frac{a_2}{\rho_2}(L-x_0)\left(\frac{h^3}{12}\right)^{-1} s\lambda\int_0^T \varphi e^{2\lambda\varphi}|u_x|^2\Big|_{x=L} dt. \tag{2.26}$$

Making the same computation for v using $\frac{k_1}{\rho_1} > \frac{k_2}{\rho_2}$, (2.2), (2.20), and (2.21), we reach

$$\mathbb{H}_{\overline{w},\frac{k_1}{\rho_1 h}}(0,M) + \mathbb{H}_{\overline{w},\frac{k_2}{\rho_2 h}}(M,L) \geq -2\frac{k_2}{\rho_2}(L-x_0)h^{-1}s\lambda\int_0^T \varphi e^{2\lambda\varphi}|v_x|^2\Big|_{x=L} dt\ . \tag{2.27}$$

We can observe that, if D denotes the derivation operator (in time or space), we have

$$s\lambda\int_Q \varphi|D(e^{\lambda\varphi}f)|^2dxdt \geq C_\varepsilon s\lambda\int_Q \varphi e^{2\lambda\varphi}|D(f)|^2dxdt - \varepsilon s^3\lambda^3\int_Q \varphi^3e^{2\lambda\varphi}|f|^2dxdt,\ \forall\varepsilon>0, \tag{2.28}$$

for $f \in \{u,v\}$. Thus, after adding the four inequalities in (2.22) and combining (2.26) and (2.27), we use (2.28) to absorb eventual remaining terms of the right-hand side and conclude the proof of Theorem 2. ■

Remark 4. *With suitable changes in ϕ, we can also obtain an alternative version for* (2.26) *and* (2.27) *given by*

$$\begin{aligned}
\mathbb{H}_{w,\frac{12a_1}{\rho_1 h^3}}(0,M) + \mathbb{H}_{w,\frac{12a_2}{\rho_2 h^3}}(M,L) &\geq -Cs\lambda\int_0^T \varphi e^{2\lambda\varphi}|u_x|^2\Big|_{x=0} dt,\\
\mathbb{H}_{\overline{w},\frac{k_1}{\rho_1 h}}(0,M) + \mathbb{H}_{\overline{w},\frac{k_2}{\rho_2 h}}(M,L) &\geq -Cs\lambda\int_0^T \varphi e^{2\lambda\varphi}|v_x|^2\Big|_{x=0} dt.
\end{aligned} \tag{2.29}$$

Indeed, if we consider $x_0 > L$, then, $\phi_{jx}(x,t) = 2\max\left\{\frac{\rho_j h^3}{12a_j}, \frac{\rho_j h}{k_j}\right\}(x-x_0) < 0$, forcing (2.29) *to happen once we can assure* (2.26) *and* (2.27) *to hold. By the expressions in* (2.26) *and* (2.27)*, we can observe that, since $\phi_{jx}(M) < 0$, the estimates in them hold for $\frac{a_2}{\rho_2} > \frac{a_1}{\rho_1}$. Thus, considering $\frac{a_2}{\rho_2} > \frac{a_1}{\rho_1}$ and $\frac{a_1}{k_1} = \frac{a_2}{k_2}$ together with $x_0 > L$ it is sufficient to have* (2.29).

2.3 Controllability

This section is devoted to prove Theorem 1.1. For this, we will follow the standard duality method which reduces the controllability property to an observability inequality for the solutions of the adjoint system (2.6). This inequality, described in (2.7), will be obtained as a consequence of the Carleman estimate (2.18). The following result holds:

Proposition 2. *Let us consider T_0 as in* (2.4) *and $a_1, a_2, \rho_1, \rho_2, k_1, k_2$ as in* (2.5)*. Then, if $T > T_0$, there exist a constant $C > 0$ such that, for all $\{u_0, v_0, u_1, v_1\} \in [H_0^1(0,L) \times L^2(0,L)]^2$, the solution $\{u, v\}$ of the system* (2.6) *satisfies* (2.7)*.*

Proof: In the following, we will denote $O_1 = (0, M)$ and $O_2 = (M, L)$. Clearly $\mathbb{E}'_{u,v}(t) = 0$, where $\mathbb{E}_{u,v}$ is defined in (2.8). Then $\mathbb{E}_{u,v}(t) = \mathbb{E}_{u,v}(0)$, for all $t \geq 0$. Since $T > T_0$, there are $x_0 < 0$ and $\beta \in (0,1)$ such that

$$\beta T > 2(L - x_0)\sqrt{\max\left\{\frac{\rho_j h^3}{12a_j}, \frac{\rho_j h}{k_j}\right\}}, \quad j = 1, 2. \tag{2.1}$$

In order to fit the solution of (2.6) in the hypotheses of Theorem 2, we set the weight functions ϕ as in (2.17), where N_j is taken to $\phi_j \geq 1$ and (2.19) hold true. It is straightforward that (2.1) implies

$$\phi_j(x, 0) = \phi_j(x, T) < N_j < \phi_j\left(x, \frac{T}{2}\right), \ \forall x \in O_j, \ j = 1, 2. \tag{2.2}$$

Moreover, given that $x_0 < 0$, we can use (2.19) and (2.1) to conclude that $\phi_1(x, 0) = \phi_1(x, T) \leq \phi_2(M, 0) < N_2$ for all $x \in O_1$. This fact together (2.2) provides us a small constant $\delta > 0$ such that

$$\phi_j \leq N_2 \text{ in } [0, \delta] \cup [T - \delta, T] \times O_j, \quad \phi_j \geq N_2 \text{ in } \left[\frac{T}{2} - \delta, \frac{T}{2} + \delta\right] \times O_j. \tag{2.3}$$

Since we can assume, without loss of generality, that δ is small enough to have $\mathcal{J}_\delta = \left[\frac{T}{2} - \delta, \frac{T}{2} + \delta\right] \subset \left[\frac{2\delta}{3}, T - \frac{2\delta}{3}\right]$, then we can choose $\theta(t) \in C_0^2([0, T])$, $0 \leq \theta \leq 1$, a cut-off function such that

$$\theta \equiv 0 \text{ in } [0, \delta/3] \cup [T - \delta/3, T], \quad \theta \equiv 1 \text{ in } \left[\frac{2\delta}{3}, T - \frac{2\delta}{3}\right]. \tag{2.4}$$

As a consequence of this choice, θ', θ'' vanish outside the time interval $\mathcal{I}_\delta = [\frac{\delta}{3}, \frac{2\delta}{3}] \cup [T - \frac{2\delta}{3}, T - \frac{\delta}{3}]$.

Considering $U(x,t) = \theta(t)u(x,t)$ and $V(x,t) = \theta(t)v(x,t)$ we have

$$\begin{cases} \dfrac{\rho(x)h^3}{12}U'' - (a(x)U_x)_x + k(x)(U + V_x) = \dfrac{\rho(x)h^3}{12}(\theta'' u + 2\theta' u') & \text{in} \quad Q, \\ \rho(x)hV'' - [k(x)(U + V_x)]_x = \rho(x)h(\theta'' v + 2\theta' v') & \text{in} \quad Q, \\ U(0, \cdot) = V(0, \cdot) = U(L, \cdot) = V(L, \cdot) = 0 & \text{in} \quad (0, T), \\ U(\cdot, 0) = U'(\cdot, 0) = V(\cdot, 0) = V'(\cdot, 0) = 0 & \text{in} \quad (0, L), \\ U(\cdot, T) = U'(\cdot, T) = V(\cdot, T) = V'(\cdot, T) = 0 & \text{in} \quad (0, L). \end{cases} \tag{2.5}$$

Thus $\{U, V\} \in X_{\frac{12a_1}{\rho_1 h^3}, \frac{12a_2}{\rho_2 h^3}} \times X_{\frac{k_1}{\rho_1 h}, \frac{k_2}{\rho_2 h}}$ and Theorem 2 provides us

$$\begin{aligned} s\lambda \int_Q e^{2\lambda\varphi}\varphi(|U'|^2 + |U_x|^2 + |V'|^2 + |V_x|^2)dxdt + s^3\lambda^3 \int_Q e^{2\lambda\varphi}\varphi^3(|U|^2 + |V|^2)dxdt \\ \leq Cs\lambda \int_0^T e^{2\lambda\varphi}\varphi(|U_x|^2 + |V_x|^2)\Big|_{x=L} dt + C\int_Q e^{2\lambda\varphi}\left(\left|L_{\frac{12a}{\rho h^3}}(U)\right|^2 dxdt + \left|L_{\frac{k}{\rho h}}(V)\right|^2\right) dxdt, \end{aligned} \tag{2.6}$$

for s and λ large enough. Denoting

$$\mathbb{E}^j_{u,v}(t) = \frac{1}{2}\left[\frac{h^3}{12}\|\rho_j^{1/2}u_j'\|^2_{L^2(\Omega_j)} + h\|\rho_j^{1/2}v_j'\|^2_{L^2(\Omega_j)} \right.$$
$$\left. +\|a_j^{1/2}u_{jx}\|^2_{L^2(\Omega_j)} + \|k_j^{1/2}(u_j+v_{jx})\|^2_{L^2(\Omega_j)}\right]. \tag{2.7}$$

we will first estimate some therms of (2.6) in each domain Q_j separately.

Using (2.3) and (2.4) we obtain

$$\begin{aligned}
\int_{Q_j} e^{2\lambda\varphi}&\left(\left|L_{\frac{12a}{\rho h^3}}(U)\right|^2 + \left|L_{\frac{k}{\rho h}}(V)\right|^2\right)dxdt \\
&\le C\int_{Q_j} e^{2\lambda\varphi}(|\theta'' u|^2 + |\theta' u'|^2 + |\theta'' v|^2 + |\theta' v'|^2)dxdt \\
&\quad + C\int_{Q_j} e^{2\lambda\varphi}(|U|^2 + |U_x|^2 + |V_x|^2)dxdt \\
&\le C\int_{O_j}\int_{\mathcal{I}_\delta} e^{2\lambda\varphi}(|\theta'' u|^2 + |\theta' u'|^2 + |\theta'' v|^2 + |\theta' v'|^2)dxdt \\
&\quad + C\int_{Q_j} e^{2\lambda\varphi}(|U|^2 + |U_x|^2 + |V_x|^2)dxdt \\
&\le Ce^{2\lambda e^{sN_2}}\int_{\mathcal{I}_\delta}\mathbb{E}^j_{u,v}dt + C\int_{Q} e^{2\lambda\varphi}(|U|^2 + |U_x|^2 + |V_x|^2)dxdt
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
s\lambda\int_{Q_j} e^{2\lambda\varphi}&(|U'|^2 + |U_x|^2 + |V'|^2 + |V_x|^2)dxdt + s^3\lambda^3\int_{Q_j} e^{2\lambda\varphi}(|U|^2+|V|^2)dxdt \\
&\ge s\lambda\int_{\mathcal{J}_\delta}\int_{O_j} e^{2\lambda\varphi}(|U'|^2 + |U_x|^2 + |V'|^2 + |V_x|^2)dxdt \\
&\quad + s^3\lambda^3\int_{Q_j} e^{2\lambda\varphi}(|U|^2+|V|^2)dxdt \\
&\ge Cs\lambda e^{2\lambda e^{sN_2}}\int_{\mathcal{J}_\delta}\mathbb{E}^j_{u,v}dt.
\end{aligned} \tag{2.9}$$

The right-hand side integral on Q of (2.8) can be absorbed by the left-hand side of (2.6). Then, from (2.6), (2.8) and (2.9), we get

$$\begin{aligned}
s\lambda e^{2\lambda e^{sN_2}}&\int_{\mathcal{J}_\delta}\mathbb{E}^1_{u,v}dt + s\lambda e^{2\lambda e^{sN_2}}\int_{\mathcal{J}_\delta}\mathbb{E}^2_{u,v}dt \\
&\le Ce^{2\lambda e^{sN_2}}\int_{\mathcal{I}_\delta}\mathbb{E}^1_{u,v}dt + Ce^{2\lambda e^{sN_2}}\int_{\mathcal{I}_\delta}\mathbb{E}^2_{u,v}dt + C\int_0^T e^{2\lambda\varphi}\varphi(|U_x|^2+|V_x|^2)\Big|_{x=L}dt.
\end{aligned} \tag{2.10}$$

Since $\mathbb{E}_{u,v}(0) = \mathbb{E}^1_{u,v} + \mathbb{E}^2_{u,v}$, we obtain, by (2.10),

$$(s\lambda|\mathcal{J}_\delta| - C|\mathcal{I}_\delta|)\,\mathbb{E}_{u,v}(0) \le C\int_0^T e^{2\lambda\varphi}\varphi(|U_x|^2+|V_x|^2)\Big|_{x=L}dt, \tag{2.11}$$

which lead us to the conclusion of the proof taking in account that the left-hand side of (2.11) is positive for s or λ large enough. ∎

Remark 5. *In (2.7) we have the observation in $x = L$, but we can have another one in $x = 0$. In order to do this, it is sufficient to place x_0 on the other side of the interval $(0, L)$ and to consider another monotonicity condition about a_j, ρ_j (see Remark 4). In the context of controllability, it means that the control forces must be placed at the left-hand side of $(0, L)$, instead of the right-hand side of it.*

2.4 Inverse Problem

In this section we study the inverse problem of retrieving the potentials $\{p_1, p_2\}$ of system (2.9), by knowing the the normal derivative of the solution $\{u(p_1, p_2), v(p_1, p_2)\}$ on the boundary. We will apply the Bukhgeim-Klibanov method and the global Carleman estimate developed in this work.

First, we consider an auxiliary system, given by

$$\begin{cases} \dfrac{\rho(x)h^3}{12}u'' - (a(x)u_x)_x + k(x)(u + v_x) + p_1(x)u = f_1(x)R_1(x,t) & \text{in} \quad Q, \\ \rho(x)hv'' - [k(x)(u + v_x)]_x + p_2(x)v = f_2(x)R_2(x,t) & \text{in} \quad Q, \\ u(0,\cdot) = v(0,\cdot) = u(L,\cdot) = v(L,\cdot) = 0 & \text{in} \quad (0,T), \\ u(\cdot,0) = u'(\cdot,0) = v(\cdot,0) = v'(\cdot,0) = 0 & \text{in} \quad (0,L). \end{cases} \tag{2.1}$$

We have the following result:

Theorem 3. *Under the hypotheses (2.4) and (2.5), if, for some $m > 0$, we have*

$$\begin{aligned} \|p_j\|_{L^\infty(0,L)} \le m, \qquad & T > 2L\sqrt{\max\left\{\frac{\rho_1 h^3}{12a_1}, \frac{\rho_2 h^3}{12a_2}, \frac{\rho_1 h}{k_1}, \frac{\rho_2 h}{k_2}\right\}}, \\ R_j \in H^1(0,T;L^\infty(0,L)) \quad & \text{and} \quad 0 < r < |R_j(\cdot,0)| \text{ a.e. } (0,L), \quad j = 1,2, \end{aligned} \tag{2.2}$$

then there exist a constant $C > 0$ such that, for all $f_j \in L^2(0,L)$, the solution $\{u, v\}$ of the system (2.1) satisfies

$$\|\{f_1, f_2\}\|^2_{[L^2(0,L)]^2} \le C\left(\|u_x(L,\cdot)\|^2_{H^1(0,T)} + \|v_x(L,\cdot)\|^2_{H^1(0,T)}\right). \tag{2.3}$$

Proof: For each $f_j \in L^2(0,L)$ and $R_j \in H^1(0,T;L^\infty(0,L))$, $j = 1,2$, let $\{u, v\}$ be the solution of (2.1).

Considering the extension of u, v and R_j on $(-T, 0)$ in an odd way and taking $U = u'$ and $V = v'$ we obtain from $(2.1)_1$ that $\{U, V\}$ is solution of

$$\begin{cases} \dfrac{\rho(x)h^3}{12}U'' - (a(x)U_x)_x + k(x)(U + V_x) + p_1(x)U = f_1(x)R_1'(x,t) & \text{in} \quad Q, \\ \rho(x)hV'' - [k(x)(U + V_x)]_x + p_2(x)V = f_2(x)R_2'(x,t) & \text{in} \quad Q, \\ U(0,\cdot) = U(L,\cdot) = V(0,\cdot) = V(L,\cdot) = 0, & \text{in} \quad (0,T), \\ U(\cdot,0) = 0, \; U'(\cdot,0) = \dfrac{12}{\rho_1 h^3}f_1(\cdot)R_1(\cdot,0), & \text{in} \quad (0,L), \\ V(\cdot,0) = 0, \; V'(\cdot,0) = \dfrac{1}{\rho_1 h}f_2(\cdot)R_2(\cdot,0), & \text{in} \quad (0,L). \end{cases} \tag{2.4}$$

As in the previous section, we multiply $(2.4)_1$ and $(2.4)_2$ by U' and V', respectively, integrate in $(0,L)$ and add the resulting expressions to obtain

$$\mathbb{E}'_{U,V}(t) = \int_0^L (f_1(x)R'_1(x,t)U'(t) + f_2(x)R'_2(x,t)V'(t))dx, \tag{2.5}$$

where $\mathbb{E}_{U,V}$ is defined as in (2.8). Using the boundary conditions in (2.1) after have integrated (2.5) from 0 to $t < T$ we get

$$\begin{aligned}\mathbb{E}_{U,V}(t) &= \mathbb{E}_{U,V}(0) + \int_0^L \int_0^t (f_1R'_1U' + f_2R'_2V')dxdr \\ &= \frac{1}{2}\int_0^L \left(\frac{\rho_1 h^3}{12}|U'|^2 + \rho_1 h|V'|^2 \right)\bigg|_{t=0} dx + \int_0^L \int_0^t (f_1R'_1U' + f_2R'_2V')dxdr.\end{aligned} \tag{2.6}$$

Since $f_j, U'(\cdot,t), V'(\cdot,t) \in L^2(0,L)$, $R'_j \in L^2(0,T;L^\infty(0,L))$ and $R_j(\cdot,0) \in L^\infty(0,L)$, for $j=1,2$, we have from (2.6) and (2.4) that

$$\mathbb{E}_{U,V}(t) \leq C\|\{f_1,f_2\}\|^2_{[L^2(0,L)]^2} + C\int_0^L \int_0^t \mathbb{E}_{U,V}(r)dxdr. \tag{2.7}$$

Using Gronwall inequality, we conclude that

$$\mathbb{E}_{U,V}(t) \leq C\|\{f_1,f_2\}\|^2_{[L^2(0,L)]^2}, \ \forall t \in [0,T], \tag{2.8}$$

where $C = C(R_1,R_2,T)$. In particular, $U,V \in H_0^1(0,L)$.

Changing the time variable, we can notice that Theorem 2 states that, if

$$\begin{aligned} u,\ v \in L^2(-T,T;L^2(0,L)), &\qquad L_{\frac{12a_1}{\rho_1h^3}}(u_1), L_{\frac{k_1}{\rho_1h}}(v_1) \in L^2(-T,T;L^2(0,M)),\\ u(0,\cdot)=u(L,\cdot)=u(\cdot,\pm T)=u'(\cdot,\pm T)=0, &\qquad L_{\frac{12a_2}{\rho_2h^3}}(u_2), L_{\frac{k_2}{\rho_2h}}(v_2) \in L^2(-T,T;L^2(M,L)),\\ v(0,\cdot)=v(L,\cdot)=v(\cdot,\pm T)=v'(\cdot,\pm T)=0, &\qquad u_1(M,\cdot)=u_2(M,\cdot), \quad v_1(M,\cdot)=v_2(M,\cdot),\\ \frac{a_1}{\rho_1}u_{1x}(M,\cdot)=\frac{a_2}{\rho_2}u_{2x}(M,\cdot), &\qquad \frac{k_1}{\rho_1}v_{1x}(M,\cdot)=\frac{k_2}{\rho_2}v_{2x}(M,\cdot),\end{aligned} \tag{2.9}$$

then, for s,λ large enough,

$$\begin{aligned}&\left\|\left\{P_{1,\frac{12a}{\rho h^3}}(e^{\lambda\varphi}u), P_{2,\frac{ka}{\rho h}}(e^{\lambda\varphi}v)\right\}\right\|^2_{[L^2((-T,T)\times(0,L))]^2}\\ &+s\lambda\int_{-T}^T\int_0^L e^{2\lambda\varphi}\varphi(|u'|^2+|u_x|^2+|v'|^2+|v_x|^2)dxdt + s^3\lambda^3\int_{-T}^T\int_0^L e^{2\lambda\varphi}\varphi^3(|u|^2+|v|^2)dxdt\\ &\leq Cs\lambda\int_{-T}^T e^{2\lambda\varphi}\varphi(|u_x|^2+|v_x|^2)\Big|_{x=L}dt + C\int_{-T}^T\int_0^L e^{2\lambda\varphi}\left(\left|L_{\frac{12a}{\rho h^3}}(u)\right|^2+\left|L_{\frac{k}{\rho h}}(v)\right|^2\right)dxdt,\end{aligned} \tag{2.10}$$

for $L_{\frac{12a}{\rho h^3}}, L_{\frac{k}{\rho h}}$ as in (2.4), $P_{1,\frac{12a}{\rho h^3}}$ as in (2.7), ϕ as in (2.17), and $\varphi = e^{\lambda\phi}$. As in the previous section, the hypotheses of Theorem 3 on the final time T give us a small and suitable constant $\delta > 0$ such that

$$\phi_j \leq N_2 \text{ in } [-T,-T+\delta]\cup[T-\delta,T]\times O_j, \quad \phi_j \geq N_2 \text{ in } [-\delta,\delta]\times O_j. \tag{2.11}$$

In order to apply Carleman estimate with the form (2.10), we need a solution of (2.1) that vanishes on $\pm T$. Again, as in the previous section, we assume δ to be small enough to use a cut-off function $\theta \in C_0^2([-T,T])$ now defined by

$$\theta \equiv 0 \text{ in } \left[-T, -T+\frac{\delta}{3}\right] \cup \left[T-\frac{\delta}{3}, T\right], \quad \theta \equiv 1 \text{ in } \mathcal{J}_\delta, \quad \theta', \theta'' \equiv 0 \text{ in } [-T,T] - \mathcal{I}_\delta, \tag{2.12}$$

where

$$(-\delta,\delta) \subset \mathcal{J}_\delta = \left[-T+\frac{2\delta}{3}, T-\frac{2\delta}{3}\right], \quad \mathcal{I}_\delta = \left[-T+\frac{\delta}{3}, -T+\frac{2\delta}{3}\right] \cup \left[T-\frac{2\delta}{3}, T-\frac{\delta}{3}\right]. \tag{2.13}$$

Then we define $X = \theta U$, $Y = \theta V$. It is simple to conclude that $\{X, Y\}$ solves the system

$$\begin{cases} \dfrac{\rho(x)h^3}{12}X'' - (a(x)X_x)_x + k(x)(X+Y_x) + p_1(x)X = \theta f_1(x)R_1'(x,t) + \theta'' u' + 2\theta' u'' & \text{in } \quad Q, \\ \rho(x)hY'' - [k(x)(X+Y_x)]_x + p_2(x)Y = \theta f_2(x)R_2'(x,t) + \theta'' v' + 2\theta' v'' & \text{in } \quad Q, \\ X(0,\cdot) = X(L,\cdot) = Y(0,\cdot) = Y(L,\cdot) = 0, & \text{in } \quad (0,T), \\ X(\cdot,0) = 0, \;\; X'(\cdot,0) = \dfrac{12}{\rho_1 h^3} f_1(\cdot)R_1(\cdot,0), & \text{in } \quad (0,L), \\ Y(\cdot,0) = 0, \;\; Y'(\cdot,0) = \dfrac{1}{\rho_1 h} f_2(\cdot)R_2(\cdot,0), & \text{in } \quad (0,L), \\ X(\cdot,\pm T) = Y(\cdot,\pm T) = 0, & \text{in } \quad (0,L)\;. \end{cases} \tag{2.14}$$

Then (2.10) implies that, for s, λ large enough,

$$\begin{aligned} &\left\| \left\{ P_{1,\frac{12a}{\rho h^3}}(e^{\lambda\varphi}X), P_{1,\frac{ka}{\rho h}}(e^{\lambda\varphi}Y) \right\} \right\|^2_{[L^2((-T,T);L^2(0,L))]^2} \\ &\quad + s\lambda \int_{-T}^{T}\int_0^L e^{2\lambda\varphi}\varphi(|X'|^2 + |X_x|^2 + |Y'|^2 + |Y_x|^2)dxd \\ &\quad + s^3\lambda^3 \int_{-T}^{T}\int_0^L e^{2\lambda\varphi}\varphi^3(|X|^2 + |Y|^2)dxdt \\ &\leq Cs\lambda \int_{-T}^{T} e^{2\lambda\varphi}\varphi(|X_x|^2 + |Y_x|^2)\Big|_{x=L} dt \\ &\quad + C\int_{-T}^{T}\int_0^L e^{2\lambda\varphi}\left(\left| L_{\frac{12a}{\rho h^3}}(X)\right|^2 + \left| L_{\frac{k}{\rho h}}(Y)\right|^2 \right) dxdt. \end{aligned} \tag{2.15}$$

Let us observe that, denoting $w = e^{\lambda\varphi}X$,

$$\begin{aligned} &\left\langle P_{1,\frac{12a}{\rho h^3}}(w), w' \right\rangle_{L^2((-T,0)\times(0,L))} \\ &\quad = \frac{1}{2}\int_{-T}^{0}\int_0^L \left(\frac{d}{dt}|w'|^2 + \frac{12a}{\rho h^3}\frac{d}{dt}|w_x|^2 + s^2\lambda^2\varphi E_{\frac{12a}{\rho h^3}}(\phi)\frac{d}{dt}|w|^2 \right) dxdt \\ &\quad = \frac{1}{2}\int_0^L e^{2\lambda\varphi}|X'|^2\Big|_{t=0} dx - s^2\lambda^2 \int_{-T}^{0}\int_0^L \frac{d}{dt}\left(\varphi E_{\frac{12a}{\rho h^3}}(\phi) \right)|w|^2 dxdt \\ &\quad \geq Cr^2 \int_0^L e^{2\lambda\varphi}|f_1|^2\Big|_{t=0} dx - Cs^3\lambda^2 \int_{-T}^{0}\int_0^L \varphi|w|^2 dxdt, \end{aligned} \tag{2.16}$$

which, after using Young inequality, becomes

$$\begin{aligned}\int_0^L e^{2\lambda\varphi}|f_1|^2\Big|_{t=0}dx \;\leq\; & \frac{C}{\sqrt{\lambda}}\left(\left\|P_{1,\frac{12a}{\rho h^3}}(e^{\lambda\varphi}X)\right\|^2_{L^2((-T,T)\times(0,L))}\right.\\ & \left.+s\lambda\int_{-T}^{T}\int_0^L \varphi e^{2\lambda\varphi}|X'|^2dxdt + s^3\lambda^3\int_{-T}^{T}\int_0^L \varphi e^{2\lambda\varphi}|X|^2dxdt\right).\end{aligned} \tag{2.17}$$

Using similar arguments, we get

$$\begin{aligned}\int_0^L e^{2\lambda\varphi}|f_2|^2\Big|_{t=0}dx \;\leq\; & \frac{C}{\sqrt{\lambda}}\left(\left\|P_{1,\frac{ka}{\rho h}}(e^{\lambda\varphi}Y)\right\|^2_{L^2((-T,T)\times(0,L))}\right.\\ & \left.+s\lambda\int_{-T}^{T}\int_0^L \varphi e^{2\lambda\varphi}|Y'|^2dxdt + s^3\lambda^3\int_{-T}^{T}\int_0^L \varphi e^{2\lambda\varphi}|Y|^2dxdt\right).\end{aligned} \tag{2.18}$$

Thus, using (2.15), (2.17) and (2.18), it follows that

$$\begin{aligned}\frac{\sqrt{\lambda}}{C}\int_0^L e^{2\lambda\varphi}(|f_1|^2+|f_2|^2)\Big|_{t=0}dx \;\leq\; & C\int_{-T}^{T}\int_0^L e^{2\lambda\varphi}\left(\left|L_{\frac{12a}{\rho h^3}}(X)\right|^2+\left|L_{\frac{ka}{\rho h}}(Y)\right|^2\right)dxdt\\ & -s\lambda\int_{-T}^{T}\int_0^L e^{2\lambda\varphi}(|X|^2+|Y|^2+|X_x|^2+|Y_x|^2)dxdt\\ & +Cs\lambda\int_{-T}^{T} e^{2\lambda\varphi}\varphi(|X_x|^2+|Y_x|^2)\Big|_{x=L}dt.\end{aligned} \tag{2.19}$$

On the other hand,

$$\begin{aligned}&\int_{-T}^{T}\int_0^L e^{2\lambda\varphi}\left|L_{\frac{12a}{\rho h^3}}(X)\right|^2dxdt-\frac{1}{2}s\lambda\int_{-T}^{T}\int_0^L e^{2\lambda\varphi}(|X|^2+|Y|^2+|X_x|^2+|Y_x|^2)dxdt\\ &\quad= \int_{-T}^{T}\int_0^L e^{2\lambda\varphi}|\theta f_1R_1'+\theta''u'+2\theta'u''-p_1(x)X-k(X+Y_x)|^2dxdt\\ &\quad\quad -\frac{1}{2}s\lambda\int_{-T}^{T}\int_0^L e^{2\lambda\varphi}(|X|^2+|Y|^2+|X_x|^2+|Y_x|^2)dxdt\\ &\quad\leq C\|R\|_{H^1((0,T)\times(0,L))}\int_{-T}^{T}\int_0^L e^{2\lambda\varphi}|\theta f_1|^2dxdt+C\int_{-T}^{T}\int_0^L e^{2\lambda\varphi}(|\theta''u'|^2+|\theta'u''|^2)dxdt\\ &\quad\leq C\int_0^L e^{2\lambda\varphi}|f_1|^2\Big|_{t=0}dxdt+Ce^{2\lambda e^{sN_2}}\int_{\mathcal{I}_\delta}\mathbb{E}_{U,V}dt\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}&\int_{-T}^{T}\int_0^L e^{2\lambda\varphi}\left|L_{\frac{k}{\rho h}}(Y)\right|^2dxdt-\frac{1}{2}s\lambda\int_{-T}^{T}\int_0^L e^{2\lambda\varphi}(|X|^2+|Y|^2+|X_x|^2+|Y_x|^2)dxdt\\ &\quad\leq C\int_0^L e^{2\lambda\varphi}|f_2|^2\Big|_{t=0}dx+Ce^{2\lambda e^{sN_2}}\int_{\mathcal{I}_\delta}\mathbb{E}_{U,V}dt,\end{aligned} \tag{2.21}$$

for s and λ large enough. From (2.8), (2.19), (2.20) and (2.21) we get

$$\begin{aligned}
&\int_0^L e^{2\lambda\varphi}(|f_1|^2+|f_2|^2)\Big|_{t=0}dx \\
&\quad\le \frac{C}{\sqrt{\lambda}}\left(\int_0^L e^{2\lambda\varphi}(|f_1|^2+|f_2|^2)\Big|_{t=0}dxdt + e^{2\lambda e^{sN_2}}|\mathcal{I}_\delta|\int_0^L(|f_1|^2+|f_2|^2)dx\right. \\
&\quad\left. +s\lambda\int_{-T}^T e^{2\lambda\varphi}\varphi(|X_x|^2+|Y_x|^2)\Big|_{x=L}dt\right)
\end{aligned} \tag{2.22}$$

Finally, we take λ large enough and deduce from (2.11) and (2.22) that

$$\left(1-\frac{C(1+|I_\delta|)}{\sqrt{\lambda}}\right)e^{2\lambda e^{sN_2}}\int_0^L(|f_1|^2+|f_2|^2)dx \le Cs\sqrt{\lambda}\int_{-T}^T e^{2\lambda\varphi}\varphi(|X_x|^2+|Y_x|^2)\Big|_{x=L}dt \tag{2.23}$$

Choosing λ large enough, we end the proof. ■

Now, we will use Theorem 3 to prove Theorem 3.

Proof of Theorem 3: Let $\{u(p_1,p_2),v(p_1,p_2)\}$ and $\{u(q_1,q_2),v(q_1,q_2)\}$ be solutions of the system (2.9) with potentials $\{p_1,p_2\}$ and $\{q_1,q_2\}$, respectively. Then $U=u(q_1,q_2)-u(p_1,p_2)$ and $V=v(q_1,q_2)-v(p_1,p_2)$ solve the system

$$\begin{cases}
\dfrac{\rho(x)h^3}{12}U''-(a(x)U_x)_x+k(x)(U+V_x)=p_1u(p_1,p_2)-q_1u(q_1,q_2) & \text{in} \quad Q,\\
\rho(x)hV''-[k(x)(U+V_x)]_x=p_2v(p_1,p_2)-q_2v(q_1,q_2) & \text{in} \quad Q,\\
U(0,\cdot)=U(L,\cdot)=V(0,\cdot)=V(L,\cdot)=0, & \text{in} \quad (0,T),\\
U(\cdot,0)=U'(\cdot,0)=V(\cdot,0)=V'(\cdot,0)=0 & \text{in} \quad (0,L),
\end{cases} \tag{2.24}$$

Taking $f_j=p_j-q_j$, $R_u=u(p_1,p_2)$ and $R_v=v(p_1,p_2)$, we have that $\{U,V\}$ is a solution of

$$\begin{cases}
\dfrac{\rho(x)h^3}{12}U''-(a(x)U_x)_x+a(x)(U+V_x)+q_1U=f_1R_u & \text{in} \quad Q,\\
\rho(x)hV''-[k(x)(U+V_x)]_x+q_2V=f_2R_v & \text{in} \quad Q,\\
U(0,\cdot)=U(L,\cdot)=V(0,\cdot)=V(L,\cdot)=0, & \text{in} \quad (0,T),\\
U(\cdot,0)=U'(\cdot,0)=V(\cdot,0)=V'(\cdot,0)=0 & \text{in} \quad (0,L),
\end{cases} \tag{2.25}$$

Since $R_u=u(p_1,p_2), R_v=v(p_1,p_2)\in H^1(0,T;L^\infty(0,L))$, $|R_u(\cdot,0)|=|u_0|>r$, $|R_v(\cdot,0)|=|v_0|>r$, then Theorem 3 implies

$$\begin{aligned}
\|p_1-q_1\|_{L^2(0,L)}+\|p_2-q_2\|_{L^2(0,L)} &\le C\left(\|U_x(L,\cdot)\|_{H^1(0,T)}+\|V_x(L,\cdot)\|_{H^1(0,T)}\right)\\
&\le C\Big(\|u_x(p_1,p_2)(L,\cdot)-u_x(q_1,q_2)(L,\cdot)\|_{H^1(0,T)}\\
&\quad+\|v_x(p_1,p_2)(L,\cdot)-v_x(q_1,q_2)(L,\cdot)\|_{H^1(0,T)}\Big),
\end{aligned} \tag{2.26}$$

which proves Theorem 3. ■

Capítulo 3

A boundary obstacle problem for the 2-D Mindlin-Timoshenko systems

A boundary obstacle problem for the 2-D Mindlin-Timoshenko systems

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Abstract. We consider a contact problem for the 2-D Mindlin-Timoshenko system. For this system, which describes the vibratory motion of plates having a contact on the boundary with a rigid obstacle, we study the existence of solutions and analyze how its energy decays exponentially to zero as time goes to infinity. To prove the existence of solution we use a penalization strategy together with the Faedo-Galerkin method, so that the solution of the contact problem is obtained as a limit of solutions of penalized problems. The exponential decay for the contact problem is obtained as a uniform limit of the exponential decay obtained for the penalized problems.

3.1 Introduction

We consider a uniform plate occupying a region $\Omega \subset \mathbb{R}^2$, which we assume to be a bounded, open and connected set whose boundary Γ is regular enough. We assume that Γ possess a partition $\{\Gamma_0, \Gamma_1\}$ with Γ_i $(i = 0, 1)$ having positive Lebesgue measure and $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$. Let T be a given positive real number and consider the cylinder $Q = \Omega \times (0, T)$, with lateral boundary $\Sigma = \Sigma_0 \cup \Sigma_1$, where $\Sigma_i = \Gamma_i \times (0, T)$ $(i = 0, 1)$. The two-dimensional version of the Mindlin-Timoshenko system reads as follows:

$$\begin{cases} \dfrac{\rho h^3}{12}\phi_{tt} - L_1(\phi, \psi, w) = 0, & \text{in} \quad Q, \\ \dfrac{\rho h^3}{12}\psi_{tt} - L_2(\phi, \psi, w) = 0, & \text{in} \quad Q, \\ \rho h w_{tt} - L_3(\phi, \psi, w) = 0, & \text{in} \quad Q, \end{cases} \tag{3.1}$$

where the operators L_1, L_2, L_3 are given by

$$\begin{aligned} L_1(\phi, \psi, w) &= D\left(\phi_{x_1x_1} + \frac{1-\mu}{2}\phi_{x_2x_2} + \frac{1+\mu}{2}\psi_{x_1x_2}\right) - k\left(w_{x_1} + \phi\right), \\ L_2(\phi, \psi, w) &= D\left(\psi_{x_2x_2} + \frac{1-\mu}{2}\psi_{x_1x_1} + \frac{1+\mu}{2}\phi_{x_1x_2}\right) - k\left(w_{x_2} + \psi\right), \\ L_3(\phi, \psi, w) &= k\left[\left(w_{x_1} + \phi\right)_{x_1} + \left(w_{x_2} + \psi\right)_{x_2}\right]. \end{aligned} \tag{3.2}$$

In (3.1), subscripts mean partial derivatives. For $x = (x_1, x_2)$, the dependent variables $\phi = \phi(x, t)$, $\psi = \psi(x, t)$ represent, respectively, the angles of rotation of the cross sections $x_1 = \mathit{const.}$, $x_2 = \mathit{const.}$ containing the filament which, when the plate is in equilibrium, is orthogonal to the middle surface at the point $(x, 0)$. The other unknown variable $w = w(x, t)$ is the vertical displacement at time t of the cross section of points x in the middle surface of the plate. The constant $h > 0$ represents the thickness of the plate which, in this model, is considered to be small and uniform with respect to x. The constant ρ is the mass density per unit volume of the plate and the constants D and k are called, respectively, modulus of flexural rigidity and modulus of elasticity in shear and they are given by the formulas $D = Eh^3/[12(1-\mu^2)]$ and $k = \widehat{k}Eh/2(1+\mu)$, where E is the Young's modulus, μ is the Poisson's ratio, $0 < \mu < 1/2$, and $\widehat{k}$ is a shear correction coefficient. For more specific physics details concerning the hypotheses, parameters, and governing equations see, e.g., [65] and [66].

At one part of Σ we impose the Dirichlet boundary condition

$$\phi = \psi = w = 0 \quad \text{on} \quad \Sigma_0, \tag{3.3}$$

and at the other one we impose the following conditions:

$$\begin{cases} \mathcal{B}_1(\phi,\psi) = 0 & \text{on} \quad \Sigma_1, \\ \mathcal{B}_2(\phi,\psi) = 0 & \text{on} \quad \Sigma_1, \\ \mathcal{B}_3(\phi,\psi,w) \geq 0, \quad w \geq g & \text{on} \quad \Sigma_1, \\ \mathcal{B}_3(\phi,\psi,w)(w-g) = 0 & \text{on} \quad \Sigma_1, \end{cases} \tag{3.4}$$

where

$$\begin{cases} \mathcal{B}_1(\phi,\psi) = D\left[\nu_1\phi_{x_1} + \mu\nu_1\psi_{x_2} + \dfrac{1-\mu}{2}\left(\phi_{x_2}+\psi_{x_1}\right)\nu_2\right], \\ \mathcal{B}_2(\phi,\psi) = D\left[\nu_2\psi_{x_2} + \mu\nu_2\phi_{x_1} + \dfrac{1-\mu}{2}\left(\phi_{x_2}+\psi_{x_1}\right)\nu_1\right], \\ \mathcal{B}_3(\phi,\psi,w) = k\left(\dfrac{\partial w}{\partial \nu} + \nu_1\phi + \nu_2\psi\right), \end{cases} \tag{3.5}$$

with $g \in L^2(\Gamma_1)$ being a function representing a rigid obstacle, the vector $\nu = (\nu_1, \nu_2)$ is the outward unit normal to Ω, and $\frac{\partial}{\partial \nu}$ stands for the normal derivative. The conditions (3.3) mean that the plate is clamped at Γ_0 along the time. In (3.4), the expression $\sigma = \frac{\partial w}{\partial \nu} + \nu_1\phi + \nu_2\psi$ represents the stress tensor on Σ_1, and $d = w - g$ is the distance of the body to obstacle. Since $w \geq g$ on Σ_1, the part Γ_1 of the boundary of the plate is always above the obstacle g along the time t. One can observe that when the distance d is positive there is not contact $(\sigma = 0)$. When there is not distance $(d = 0)$, the stress tensor σ is not null. Anyway we have $\sigma d = 0$ on Γ_1, for all time t. To complete the system, we include the initial conditions

$$(\phi(\cdot,0), \phi_t(\cdot,0), \psi(\cdot,0), \psi_t(\cdot,0), w(\cdot,0), w_t(\cdot,0)) = (\phi_0, \phi_1, \psi_0, \psi_1, w_0, w_1) \quad \text{in} \quad \Omega. \tag{3.6}$$

In this work, our first interest is to study the existence of solution for the system (3.1), (3.3), (3.4), (3.6). For this, we will use a method of penalization (see, for instance, [69]) which basically consists in tree steps. The first is to consider a penalized system associated to (3.1), (3.3), (3.4), (3.6). In our case, for each penalty parameter $\epsilon > 0$, this system is

$$\begin{cases} \frac{\rho h^3}{12}\phi_{\varepsilon tt} - L_1(\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon) = 0 & \text{in} \quad Q, \\ \frac{\rho h^3}{12}\psi_{\varepsilon tt} - L_2(\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon) = 0 & \text{in} \quad Q, \\ \rho h w_{\varepsilon tt} - L_3(\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon) = 0 & \text{in} \quad Q, \\ \phi_\varepsilon = \psi_\varepsilon = w_\varepsilon = 0 & \text{on} \quad \Sigma_0, \\ \mathcal{B}_1(\phi_\varepsilon, \psi_\varepsilon) = 0 & \text{on} \quad \Sigma_1, \\ \mathcal{B}_2(\phi_\varepsilon, \psi_\varepsilon) = 0 & \text{on} \quad \Sigma_1, \\ \mathcal{B}_3(\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon) - \frac{1}{\varepsilon}(w_\varepsilon - g)^- = 0 & \text{on} \quad \Sigma_1, \\ (\phi_\varepsilon(\cdot,0), \phi_{\varepsilon t}(\cdot,0), \psi_\varepsilon(\cdot,0), \psi_{\varepsilon t}(\cdot,0), w_\varepsilon(\cdot,0), w_{\varepsilon t}(\cdot,0)) = (\phi_0, \phi_1, \psi_0, \psi_1, w_0, w_1) & \text{in} \quad \Omega, \end{cases} \tag{3.7}$$

where $\xi^- = -\min\{0, \xi\}$. The second step consists in to establish the well-posedness for (3.7) and to obtain uniform (in ε) estimate for the solutions of this penalized Mindin-Timoshenko system. Finally, the third step is to pass the limit, as $\epsilon \to 0$, in the penalized system to get the solution of the original contact problem.

The total energy of (3.7) is given by

$$\begin{aligned} \mathbb{E}_\varepsilon(t) = \frac{1}{2}\Bigg[\frac{\rho h^3}{12}(|\phi_{\varepsilon t}|^2 + |\psi_{\varepsilon t}|^2) + \rho h |w_{\varepsilon t}|^2 + a_0(\phi_\varepsilon, \psi_\varepsilon) \\ + k a_1(\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon) + \frac{1}{\varepsilon}\int_{\Gamma_1} |(w_\varepsilon - g)^-|^2 d\Gamma\Bigg], \end{aligned} \tag{3.8}$$

where $a_0(\phi, \psi) = a_0(\phi, \psi, \phi, \psi)$, $a_1(\phi, \psi, w) = a_1(\phi, \psi, w, \phi, \psi, w)$, and

$$\begin{aligned} & a_0(\phi, \psi, \hat{\phi}, \hat{\psi}) \\ & \quad = D\int_\Omega \left[\phi_{x_1}\hat{\phi}_{x_1} + \psi_{x_2}\hat{\psi}_{x_2} + \mu(\phi_{x_1}\hat{\psi}_{x_2} + \psi_{x_2}\hat{\phi}_{x_1}) + \frac{1-\mu}{2}(\phi_{x_2} + \psi_{x_1})(\hat{\phi}_{x_2} + \hat{\psi}_{x_1})\right] dx, \\ & a_1(\phi, \psi, w, \hat{\phi}, \hat{\psi}, \hat{w}) \\ & \quad = \int_\Omega \left[(w_{x_1} + \phi)(\hat{w}_{x_1} + \hat{\phi}) + (w_{x_2} + \psi)(\hat{w}_{x_2} + \hat{\psi})\right] dx. \end{aligned} \tag{3.9}$$

We can notice that this energy is conservative, i.e.,

$$\frac{d}{dt}\mathbb{E}_\varepsilon(t) = 0, \quad \forall t > 0.$$

Let us observe that, adding appropriate damping terms to (3.7), in other words, considering the system

$$
\begin{cases}
\dfrac{\rho h^3}{12}\phi_{\varepsilon tt} - L_1(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon) = 0 & \text{in} \quad Q,\\
\dfrac{\rho h^3}{12}\psi_{\varepsilon tt} - L_2(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon) = 0 & \text{in} \quad Q,\\
\rho h w_{\varepsilon tt} - L_3(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon) = 0 & \text{in} \quad Q,\\
\phi_\varepsilon = \psi_\varepsilon = w_\varepsilon = 0 & \text{on} \quad \Sigma_0,\\
\mathcal{B}_1(\phi_\varepsilon,\psi_\varepsilon) + \gamma_1\phi_{\varepsilon t} = 0 & \text{on} \quad \Sigma_1,\\
\mathcal{B}_2(\phi_\varepsilon,\psi_\varepsilon) + \gamma_2\psi_{\varepsilon t} = 0 & \text{on} \quad \Sigma_1,\\
\mathcal{B}_3(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon) - \dfrac{1}{\varepsilon}(w_\varepsilon - g)^- + \gamma_3 w_{\varepsilon t} = 0 & \text{on} \quad \Sigma_1,\\
(\phi_\varepsilon(\cdot,0),\phi_{\varepsilon t}(\cdot,0),\psi_\varepsilon(\cdot,0),\psi_{\varepsilon t}(\cdot,0),w_\varepsilon(\cdot,0),w_{\varepsilon t}(\cdot,0)) = (\phi_0,\phi_1,\psi_0,\psi_1,w_0,w_1) & \text{in} \quad \Omega,
\end{cases}
\tag{3.10}
$$

with γ_i $(i = 1,2,3)$ being positive real numbers, we have that the energy of (3.10), given in (3.8), satisfies

$$
\frac{d}{dt}\mathbb{E}_\varepsilon(t) = -\gamma_1\int_{\Gamma_1}|\phi_{\varepsilon t}|^2 d\Gamma - \gamma_2\int_{\Gamma_1}|\psi_{\varepsilon t}|^2 d\Gamma - \gamma_3\int_{\Gamma_1}|w_{\varepsilon t}|^2 d\Gamma, \quad \forall t > 0, \tag{3.11}
$$

that is, it is a non increasing function. This motivates other purpose of this paper, which is to analyze a uniform (with respect to ϵ) rate of decay for the total energy of the solutions (3.10), as $t \to \infty$. As a consequence of this analysis, we obtain a decay rate (as $t \to \infty$) for the total energy of the solutions of the system

$$
\begin{cases}
\dfrac{\rho h^3}{12}\phi_{tt} - L_1(\phi,\psi,w) = 0 & \text{in} \quad Q,\\
\dfrac{\rho h^3}{12}\psi_{tt} - L_2(\phi,\psi,w) = 0 & \text{in} \quad Q,\\
\rho h w_{tt} - L_3(\phi,\psi,w) = 0 & \text{in} \quad Q,\\
\phi = \psi = w = 0 & \text{on} \quad \Sigma_0,\\
\mathcal{B}_1(\phi,\psi) + \gamma_1\phi_t = 0 & \text{on} \quad \Sigma_1,\\
\mathcal{B}_2(\phi,\psi) + \gamma_2\psi_t = 0 & \text{on} \quad \Sigma_1,\\
\mathcal{B}_3(\phi,\psi,w) + \gamma_3 w_t \geq 0, \quad w \geq g, \quad (\mathcal{B}_3(\phi,\psi,w) + \gamma_3 w_t)(w - g) = 0 & \text{on} \quad \Sigma_1,\\
(\phi(\cdot,0),\phi_t(\cdot,0),\psi(\cdot,0),\psi_t(\cdot,0),w(\cdot,0),w_t(\cdot,0)) = (\phi_0,\phi_1,\psi_0,\psi_1,w_0,w_1) & \text{in} \quad \Omega,
\end{cases}
\tag{3.12}
$$

as a limit of the uniform (with respect to ϵ) decay rate of the energy of the penalized system (3.10).

Motivated by the analysis of the asymptotic behavior (as $t \to \infty$), we will apply the strategies of the penalization method, above mentioned, for the damped systems (3.12) and (3.10) instead of the system (3.1), (3.3), (3.4), (3.6) and the system (3.7), respectively. Summarizing, in this paper, we will obtain the existence of solution and perform a energy decay rate study for the system (3.12), by using penalized systems (3.10). The uniqueness of solution for this contact problem is still an open problem.

Contact problems have a well-established mathematical theory (see, for instance, [46] and references therein) and have been studied some time ago. We can go back at 1933 where Signorini in [82] formulated the general equilibrium problem of an elastic body linearly in contact with a rigid obstacle without attrition. Later, in [83], he introduced a more complete result to study this kind of problem, for that, being known as Signorini problem. The first rigorous analysis of a Signorini problem was announced by Fichera [45], which treated the questions of existence and uniqueness of variational inequalities using bilinear symmetrical forms. In 60's, Stampachia [85] complemented the theory of variational inequalities for bilinear forms, studying the case of bilinear non-symmetrical forms. Soon after, supplementing this theory, it appeared the paper [70] by Lions-Stampachia and the book by Duvaut-Lions [36] which deals with several contact problems arisen in mechanic and physic. Concerning to contact problems, we can also cite Kim [57] which studied the rigid obstacle problem for the wave equation, the works of Elliot-Coppeti [31] and Elliot-Qi [37], which analyzed a one-dimensional contact problem in thermoelasticity and, related to Mindlin-Timoshenko system for beams, [15], where the authors obtained existence of solution and exponential decay of energy of this system in the one-dimensional case. The present work is a natural extension of the result in [15] for the two-dimensional case with boundary dampings. About asymptotic behavior of solutions of Signorini problems, we mention Muñoz Rivera-Oliveira [72], which proved that the solutions of the 1-D thermoelastic system decay in an exponential rate, Nakao-Muñoz Rivera [75], where the authors proved the polynomial decay of solutions for the contact problem associated to thermoviscoelastic system, and Muñoz Rivera-Oquendo [73], where the exponential decay of solutions of the contact problem for viscoelastic materials is proven. Out of the contact problems framework, there is an extensive literature about the study of asymptotic properties of Mindlin-Timoshenko system, see e.g. [1, 2, 11, 14, 26, 44, 58, 65, 74, 76, 77, 84, 87] and references therein.

The rest of this paper is organized as follows. In Section 3.2 we consider a penalized system associated to the contact problem (3.12) and we show that it is well-posed. In Section 3.3 we obtain solutions of the contact problem as limit of solutions of the penalized system. In Section 3.4 we prove the exponential decay property for systems (3.10) and (3.12). Finally, in Section 3.5, we briefly discuss some related issues and open problems.

3.2 Penalized problem

This section is devoted to study the well-posedness for the penalized system (3.10). Firstly, we will fix some notation. We define the Hilbert space

$$V = \{z \in H^1(\Omega); \quad z = 0 \quad \text{on} \quad \Gamma_0\},$$

equipped with the inner product and norm given, respectively, by $((u,v)) = \sum_{i=1}^{2} \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i}\right)$ and $\|u\|^2 = \sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|^2$, where $(\cdot,\cdot)$ and $|\cdot|$ are, respectively, the inner product and norm in $L^2(\Omega)$. We denote by $\langle \cdot,\cdot \rangle_r$ the duality between $(H^r)'(\Omega)$ and $H^r(\Omega)$ for some $r > 0$. Also, from now on, we will always denote by C a generic positive constant which value can vary from line to line. When it is convenient we will explicit the dependence of $C > 0$ on the term we want to highlight.

The following result holds.

Theorem 4. *Let us consider* $g \in L^2(\Gamma_1)$ *and* $(\phi_0, \phi_1, \psi_0, \psi_1, w_0, w_1) \in [(V \cap H^2(\Omega)) \times V]^3$. *Then, for each* $\varepsilon > 0$, *there exists a unique triplet of functions* $(\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon)$ *satisfying*

$$\begin{aligned} (\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon) &\in \left[L^\infty\left(0, T; V \cap H^2(\Omega)\right)\right]^3, \\ (\phi_{\varepsilon t}, \psi_{\varepsilon t}, w_{\varepsilon t}) &\in \left[L^\infty\left(0, T; V\right)\right]^3, \\ (\phi_{\varepsilon tt}, \psi_{\varepsilon tt}, w_{\varepsilon tt}) &\in \left[L^\infty\left(0, T; L^2\left(\Omega\right)\right)\right]^2, \end{aligned} \tag{3.13}$$

the variational formulation

$$\begin{aligned} &(\frac{\rho h^3}{12}\phi_{\varepsilon tt}, s) + (\frac{\rho h^3}{12}\psi_{\varepsilon tt}, p) + (\rho h w_{\varepsilon tt}, z) \\ &\quad + D\left[(\phi_{\varepsilon x_1}, s_{x_1}) + \mu(\psi_{\varepsilon x_2}, s_{x_1}) + \frac{1-\mu}{2}(\phi_{\varepsilon x_2} + \psi_{\varepsilon x_1}, s_{x_2})\right] \\ &\quad + D\left[(\psi_{\varepsilon x_2}, p_{x_2}) + \mu(\phi_{\varepsilon x_1}, p_{x_2}) + \frac{1-\mu}{2}(\phi_{\varepsilon x_2} + \psi_{\varepsilon x_1}, p_{x_1})\right] \\ &\quad + k\left[(w_{\varepsilon x_1} + \phi_\varepsilon, z_{x_1} + s) + (w_{\varepsilon x_2} + \psi_\varepsilon, z_{x_2} + p)\right] \\ &\quad + \gamma_1 \int_{\Gamma_1} \phi_{\varepsilon t} s d\Gamma + \gamma_2 \int_{\Gamma_1} \psi_{\varepsilon t} p d\Gamma + \gamma_3 \int_{\Gamma_1} w_{\varepsilon t} z d\Gamma - \frac{1}{\varepsilon}\int_{\Gamma_1} (w_\varepsilon - g)^- z d\Gamma = 0, \end{aligned} \tag{3.14}$$

for all $(s, p, z) \in V^3$, *and the initial conditions*

$$(\phi_\varepsilon(\cdot,0), \phi_{\varepsilon t}(\cdot,0), \psi_\varepsilon(\cdot,0), \psi_{\varepsilon t}(\cdot,0), w_\varepsilon(\cdot,0), w_{\varepsilon t}(\cdot,0)) = (\phi_0, \phi_1, \psi_0, \psi_1, w_0, w_1) \quad \textit{in} \quad \Omega.$$

Proof: We will employ the well-known Faedo-Galerkin method. Let us consider $\{w_n\}_{n \in \mathbb{N}}$ a Hilbertian basis of $V \cap H^2(\Omega)$. For each $m \in \mathbb{N}$, we consider $V_m = [w_1, w_2, \cdots, w_m]$ the subspace generated by the first m vectors of $\{w_n\}_{n\in\mathbb{N}}$. Let us find an *approximate solution* $(\phi_{\varepsilon m}, \psi_{\varepsilon m}, w_{\varepsilon m}) \in (V_m)^3$ of the type

$$(\phi_{\varepsilon m}(x,t), \psi_{\varepsilon m}(x,t), w_{\varepsilon m}(x,t)) = \sum_{j=1}^{m} (\alpha_{jm}(t), \tilde{\alpha}_{jm}(t), \hat{\alpha}_{jm}(t))\, w_j(x),$$

where $(\alpha_{jm}(t), \tilde{\alpha}_{jm}(t), \hat{\alpha}_{jm}(t))$ are found as solutions of the initial value problem for the system of ordinary differential equations

$$\left\{ \begin{aligned} &(\frac{\rho h^3}{12}\phi_{\varepsilon mtt}, s) + (\frac{\rho h^3}{12}\psi_{\varepsilon mtt}, p) + (\rho h w_{\varepsilon mtt}, z) \\ &\qquad + D\left[(\phi_{\varepsilon m x_1}, s_{x_1}) + \mu(\psi_{\varepsilon m x_2}, s_{x_1}) + \frac{1-\mu}{2}(\phi_{\varepsilon m x_2} + \psi_{\varepsilon m x_1}, s_{x_2})\right] \\ &\qquad + D\left[(\psi_{\varepsilon m x_2}, p_{x_2}) + \mu(\phi_{\varepsilon m x_1}, p_{x_2}) + \frac{1-\mu}{2}(\phi_{\varepsilon m x_2} + \psi_{\varepsilon m x_1}, p_{x_1})\right] \\ &\qquad + k\left[(w_{\varepsilon m x_1} + \phi_{\varepsilon m}, z_{x_1} + s) + (w_{\varepsilon m x_2} + \psi_{\varepsilon m}, z_{x_2} + p)\right] \\ &\qquad + \gamma_1 \int_{\Gamma_1} \phi_{\varepsilon mt} s d\Gamma + \gamma_2 \int_{\Gamma_1} \psi_{\varepsilon mt} p d\Gamma + \gamma_3 \int_{\Gamma_1} w_{\varepsilon mt} z d\Gamma - \frac{1}{\varepsilon}\int_{\Gamma_1} (w_{\varepsilon m} - g)^- z d\Gamma = 0, \\ &(\phi_{\varepsilon m}(\cdot,0), \psi_{\varepsilon m}(\cdot,0), w_{\varepsilon m}(\cdot,0)) = (\phi_{0m}, \psi_{0m}, w_{0m}) \in [V_m]^3, \\ &(\phi_{\varepsilon mt}(\cdot,0), \psi_{\varepsilon mt}(\cdot,0), w_{\varepsilon mt}(\cdot,0)) = (\phi_{1m}, \psi_{1m}, w_{1m}) \in [V_m]^3, \end{aligned} \right. \tag{3.15}$$

for all $(s,p,z) \in [V_m]^3$, with

$$(\phi_{0m},\phi_{1m},\psi_{0m},\psi_{1m},w_{0m},w_{1m}) \to (\phi_0,\phi_1,\psi_0,\psi_1,w_0,w_1) \quad \text{strongly in} \quad [(V\cap H^2(\Omega))\times V]^3, \tag{3.16}$$

as $m\to\infty$. System (3.15) has a solution $(\phi_{\varepsilon m}(t),\psi_{\varepsilon m}(t),w_{\varepsilon m}(t))$ defined in a certain interval $[0,t_m]$, with $t_m<T$. This solution can be extended to whole interval $[0,T]$ as a consequence of a priori estimates that shall be proved in the next step.

Estimate I: Taking $(s,p,z)=(\phi_{\varepsilon mt},\psi_{\varepsilon mt},w_{\varepsilon mt})$ in (3.15) we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big\{&\frac{\rho h^3}{12}(|\phi_{\varepsilon mt}|^2+|\psi_{\varepsilon mt}|^2)+\rho h|w_{\varepsilon mt}|^2\\ &+D\left(|\phi_{\varepsilon mx_1}|^2+|\psi_{\varepsilon mx_2}|^2+2\mu(\phi_{\varepsilon mx_1},\psi_{\varepsilon mx_2})+\frac{1-\mu}{2}|\phi_{\varepsilon mx_2}+\psi_{\varepsilon mx_1}|^2\right)\\ &+k(|w_{\varepsilon mx_1}+\phi_{\varepsilon m}|^2+|w_{\varepsilon mx_2}+\psi_{\varepsilon m}|^2)+\frac{1}{\varepsilon}\int_{\Gamma_1}|(w_{\varepsilon m}-g)^-|^2d\Gamma\Big\}\\ &+\gamma_1\int_{\Gamma_1}|\phi_{\varepsilon mt}|^2d\Gamma+\gamma_2\int_{\Gamma_1}|\psi_{\varepsilon mt}|^2d\Gamma+\gamma_3\int_{\Gamma_1}|w_{\varepsilon mt}|^2d\Gamma=0.\end{aligned}\tag{3.17}$$

Integrating (3.17) from 0 to $t\le t_m$ and using (3.16), we obtain a positive constant $C=C(\varepsilon)$, independent of m, such that

$$\begin{aligned}&\frac{\rho h^3}{12}(|\phi_{\varepsilon mt}|^2+|\psi_{\varepsilon mt}|^2)+\rho h|w_{\varepsilon mt}|^2\\ &+D\left(|\phi_{\varepsilon mx_1}|^2+|\psi_{\varepsilon mx_2}|^2+2\mu(\phi_{\varepsilon mx_1},\psi_{\varepsilon mx_2})+\frac{1-\mu}{2}|\phi_{\varepsilon mx_2}+\psi_{\varepsilon mx_1}|^2\right)\\ &+k(|w_{\varepsilon mx_1}+\phi_{\varepsilon m}|^2+|w_{\varepsilon mx_2}+\psi_{\varepsilon m}|^2)+\frac{1}{\varepsilon}\int_{\Gamma_1}|(w_{\varepsilon m}-g)^-|^2d\Gamma\\ &+\gamma_1\int_0^t\int_{\Gamma_1}|\phi_{\varepsilon mt}|^2d\Gamma ds+\gamma_2\int_0^t\int_{\Gamma_1}|\psi_{\varepsilon mt}|^2d\Gamma ds+\gamma_3\int_0^t\int_{\Gamma_1}|w_{\varepsilon mt}|^2d\Gamma ds\le C.\end{aligned}\tag{3.18}$$

The estimate (3.18) is sufficient to extend the solution to whole interval $[0,T]$.

Estimate II: Differentiating $(3.15)_1$ with respect to t and making $(s,p,z)=(\phi_{\varepsilon mtt},\psi_{\varepsilon mtt},w_{\varepsilon mtt})$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big\{&\frac{\rho h^3}{12}(|\phi_{\varepsilon mtt}|^2+|\psi_{\varepsilon mtt}|^2)+\rho h|w_{\varepsilon mtt}|^2\\ &+D\left(|\phi_{\varepsilon mx_1t}|^2+|\psi_{\varepsilon mx_2t}|^2+2\mu(\phi_{\varepsilon mx_1t},\psi_{\varepsilon mx_2t})+\frac{1-\mu}{2}|\phi_{\varepsilon mx_2t}+\psi_{\varepsilon mx_1t}|^2\right)\\ &+k(|w_{\varepsilon mx_1t}+\phi_{\varepsilon mt}|^2+|w_{\varepsilon mx_2t}+\psi_{\varepsilon mt}|^2)\Big\}-\frac{1}{\varepsilon}\int_{\Gamma_1}\frac{d}{dt}(w_{\varepsilon m}-g)^-w_{\varepsilon mtt}d\Gamma\\ &+\gamma_1\int_{\Gamma_1}|\phi_{\varepsilon mtt}|^2d\Gamma+\gamma_2\int_{\Gamma_1}|\psi_{\varepsilon mtt}|^2d\Gamma+\gamma_3\int_{\Gamma_1}|w_{\varepsilon mtt}|^2d\Gamma=0.\end{aligned}\tag{3.19}$$

Integrating (3.19) from 0 to $t \leq T$, we get from (3.16) and (3.18) that

$$\begin{aligned}
&\frac{\rho h^3}{12}(|\phi_{\varepsilon mtt}|^2 + |\psi_{\varepsilon mtt}|^2) + \rho h |w_{\varepsilon mtt}|^2 \\
&+D\left(|\phi_{\varepsilon m x_1 t}|^2 + |\psi_{\varepsilon m x_2 t}|^2 + 2\mu(\phi_{m\varepsilon x_1 t}, \psi_{m\varepsilon x_2 t}) + \frac{1-\mu}{2}|\phi_{\varepsilon m x_2 t} + \psi_{\varepsilon m x_1 t}|^2\right) \\
&+\gamma_1 \int_0^t \int_{\Gamma_1} |\phi_{\varepsilon mtt}|^2 d\Gamma ds + \gamma_2 \int_0^t \int_{\Gamma_1} |\psi_{\varepsilon mtt}|^2 d\Gamma ds + \frac{\gamma_3}{2}\int_0^t \int_{\Gamma_1} |w_{\varepsilon mtt}|^2 ds \\
&+k(|w_{\varepsilon m x_1 t} + \phi_{\varepsilon mt}|^2 + |w_{\varepsilon m x_2 t} + \psi_{\varepsilon mt}|^2) \\
&\leq C + \frac{1}{2\varepsilon^2}\int_0^t \int_{\Gamma_1} \left|\frac{d}{dt}[(w_{\varepsilon m} - g)^-]\right|^2 d\Gamma ds + \frac{\rho h^3}{12}|\phi_{\varepsilon mtt}(0)|^2 + \frac{\rho h^3}{12}|\psi_{\varepsilon mtt}(0)|^2 + \rho h |w_{\varepsilon mtt}(0)|^2,
\end{aligned} \tag{3.20}$$

for a constant $C = C(\varepsilon) > 0$, independent of m. Now, we will show that the last three norms on the right-hand side of (3.20) are bounded. In fact, taking $(s, p, z) = (\phi_{\varepsilon mtt}(0), \psi_{\varepsilon mtt}(0), w_{\varepsilon mtt}(0))$ in (3.15) with $t = 0$ and using (3.16), we have

$$\begin{aligned}
&\frac{\rho h^3}{12}|\phi_{\varepsilon mtt}(0)|^2 + \frac{\rho h^3}{12}|\psi_{\varepsilon mtt}(0)|^2 + \rho h |w_{\varepsilon mtt}(0)|^2 \\
&= (L_1(\phi_{0m}, \psi_{0m}, w_{0m}), \phi_{\varepsilon mtt}(0)) + (L_2(\phi_{0m}, \psi_{0m}, w_{0m}), \psi_{\varepsilon mtt}(0)) + (L_3(\phi_{0m}, \psi_{0m}, w_{0m}), w_{\varepsilon mtt}(0)) \\
&\leq C + \frac{1}{2}\left(\frac{\rho h^3}{12}|\phi_{\varepsilon mtt}(0)|^2 + \frac{\rho h^3}{12}|\psi_{\varepsilon mtt}(0)|^2 + \rho h |w_{\varepsilon mtt}(0)|^2\right),
\end{aligned} \tag{3.21}$$

where the constant $C = C(\varepsilon) > 0$ is independent of m. Thus, we deduce from (3.18), (3.20), and (3.21) that

$$\begin{aligned}
&\frac{\rho h^3}{12}(|\phi_{\varepsilon mtt}|^2 + |\psi_{\varepsilon mtt}|^2) + \rho h |w_{\varepsilon mtt}|^2 \\
&\quad + D\left(|\phi_{\varepsilon m x_1 t}|^2 + |\psi_{\varepsilon m x_2 t}|^2 + 2\mu(\phi_{\varepsilon m x_1 t}, \psi_{\varepsilon m x_2 t}) + \frac{1-\mu}{2}|\phi_{\varepsilon m x_2 t} + \psi_{\varepsilon m x_1 t}|^2\right) \\
&\quad + \gamma_1 \int_0^t \int_{\Gamma_1} |\phi_{\varepsilon mtt}|^2 d\Gamma ds + \gamma_2 \int_0^t \int_{\Gamma_1} |\psi_{\varepsilon mtt}|^2 d\Gamma ds + \gamma_3 \frac{1}{2}\int_0^t \int_{\Gamma_1} |w_{\varepsilon mtt}|^2 ds \\
&\quad + k(|w_{\varepsilon m x_1 t} + \phi_{\varepsilon mt}|^2 + |w_{\varepsilon m x_2 t} + \psi_{\varepsilon mt}|^2) \leq C,
\end{aligned} \tag{3.22}$$

where $C = C(\varepsilon)$ is independent on m.

Passage to the limit, as $m \to +\infty$: From the estimates (3.18) and (3.22), it follows that

$$\begin{aligned}
&(\phi_{\varepsilon m}, \psi_{\varepsilon m}, w_{\varepsilon m}) \to (\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon) \quad \text{weak} - * \quad \text{in} \quad [L^\infty(0,T;V)]^3, \\
&(\phi_{\varepsilon mt}, \psi_{\varepsilon mt}, w_{\varepsilon mt}) \to (\phi_{\varepsilon t}, \psi_{\varepsilon t}, w_{\varepsilon t}) \quad \text{weak} - * \quad \text{in} \quad [L^\infty(0,T;V)]^3, \\
&(\phi_{\varepsilon mtt}, \psi_{\varepsilon mtt}, w_{\varepsilon mtt}) \to (\phi_{\varepsilon tt}, \psi_{\varepsilon tt}, w_{\varepsilon tt}) \quad \text{weak} - * \quad \text{in} \quad [L^\infty(0,T;L^2(\Omega))]^3,
\end{aligned} \tag{3.23}$$

and

$$(w_{\varepsilon m} - g)^- \to (w_\varepsilon - g)^- \quad \text{weak} - * \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)). \tag{3.24}$$

The convergences in (3.23) and (3.24) are sufficient to pass the limit, as $m \to +\infty$, in the ordinary differential system (3.15) in order to obtain (3.14). The initial data can be verified by standard arguments. To improve the regularity of $(\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon)$ to be as described in (3.13), one can be use elliptic regularity results. Finally, we can use the energy method to prove the uniqueness. This guarantees the well-posedness of the penalized problem. ■

Remark 6. *When $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 \neq \emptyset$ the solutions of* (3.10) *may develop singularities at the interfaces. Thus, we can not apply the classical regularity results for these solutions. To overcome this situation and to obtain the claimed H^2-regularity, we can, for instance, consider the domain Ω as being a polygonal region of the type specified in Lagnese [65, Section 2.3.2, pp. 35-37]. It is important to point out that the strategy used in [65] is based in the results of Grisvard [51, 52].*

Remark 7. *It is important to note that the estimates in* (3.18) *depends on ε. Thus, we can not use it to pass to the limit, as $\varepsilon \to 0$, in the net $\{(\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon)\}_{\varepsilon>0}$. To overcome this problem, we will impose the additional hypothesis $w_0 \geq g$ in order to obtain a uniform (in ε) boundedness. This new estimate will allow us to follow the plan to obtain the solutions of the contact problem* (3.12) *as a limit of solutions of* (3.10). *This will be done in the next section.*

3.3 Contact problem

This section is devoted to prove the existence of solution for the contact problem (3.12). Initially, we will enunciate an auxiliary result given in [65, Lemma 2.1, p. 29].

Lema 3.1. *Let us assume that $\Gamma_0 \neq \emptyset$. Then, there exist $\alpha_0 > 0$ and $\alpha_1 > 0$ such that,*

$$a_0(\phi, \psi) \geq \alpha_0 \left(\|\phi\|^2_{H^1(\Omega)} + \|\psi\|^2_{H^1(\Omega)} \right), \quad \forall (\phi, \psi) \in V^2 \tag{3.25}$$

and

$$a_0(\phi, \psi) + k a_1(\phi, \psi, w) \geq \alpha_1 \left(\|\phi\|^2_{H^1(\Omega)} + \|\psi\|^2_{H^1(\Omega)} + \|w\|^2_{H^1(\Omega)} \right), \quad \forall (\phi, \psi, w) \in V^3. \tag{3.26}$$

Let us consider the convex set

$$\mathbb{K}_g = \{ v \in V; \quad v \geq g \quad \text{on} \quad \Gamma_1 \} \tag{3.27}$$

and the following definition of solutions for the mentioned contact problem:

Definição 3.1. *Given $(\phi_0, \phi_1, \psi_0, \psi_1, w_0, w_1) \in [V \times L^2(\Omega)]^2 \times \mathbb{K}_g \times L^2(\Omega)$, we say that the tern (ϕ, ψ, w) is a (weak) solution of the contact system* (3.12) *when it satisfies*

$$(\phi, \phi_t, \psi, \psi_t, w, w_t) \in [L^\infty(0, T; V) \times L^\infty(0, T; L^2(\Omega))]^3, \quad w(t) \in \mathbb{K}_g,$$

the initial conditions

$$\phi(\cdot, 0) = \phi_0, \;\; \phi'(\cdot, 0) = \phi_1, \;\; \psi(\cdot, 0) = \psi_0, \;\; \psi'(\cdot, 0) = \psi_1, \;\; w(\cdot, 0) = w_0, \;\; w'(\cdot, 0) = w_1 \quad \text{in} \quad \Omega,$$

the equations

$$\begin{aligned}
&\frac{\rho h^3}{12} \langle \phi_t(\cdot, T), v(\cdot, T) - \phi(\cdot, T) \rangle_r - \frac{\rho h^3}{12} \langle \phi_1, v(\cdot, 0) - \phi_0 \rangle_r - \frac{\rho h^3}{12} \int_0^T (\phi_t, v_t - \phi_t) dt \\
&+ D \int_0^T \left[(\phi_{x_1}, v_{x_1} - \phi_{x_1}) + \mu(\psi_{x_2}, v_{x_1} - \phi_{x_1}) + \frac{1-\mu}{2} (\phi_{x_2} + \psi_{x_1}, v_{x_2} - \phi_{x_2}) \right] dt \\
&+ k \int_0^T (w_{x_1} + \phi, v - \phi) dt + \gamma_1 \int_0^T \int_{\Gamma_1} \phi_t (v - \phi) d\Gamma dt = 0,
\end{aligned} \tag{3.28}$$

$$\frac{\rho h^3}{12}\left\langle \psi_t(\cdot,T),u(\cdot,T)-\psi(\cdot,T)\right\rangle_r - \frac{\rho h^3}{12}\left\langle \psi_1,u(\cdot,0)-\psi_0\right\rangle_r - \frac{\rho h^3}{12}\int_0^T(\psi_t,u_t-\psi_t)dt$$
$$+D\int_0^T\left[(\psi_{x_2},u_{x_2}-\psi_{x_2})+\mu(\phi_{x_1},u_{x_2}-\psi_{x_2})+\frac{1-\mu}{2}(\phi_{x_2}+\psi_{x_1},u_{x_1}-\psi_{x_1})\right]dt \tag{3.29}$$
$$+k\int_0^T(w_{x_2}+\psi,u-\psi)dt+\gamma_2\int_0^T\int_{\Gamma_1}\psi_t(u-\psi)d\Gamma dt=0$$

for all $u,v\in H^1(0,T;V)$, and the inequality

$$\rho h\left\langle w_t(\cdot,T),\xi(\cdot,T)-w(\cdot,T)\right\rangle_r-\rho h\left\langle w_1,\xi(\cdot,0)-w_0\right\rangle_r-\rho h\int_0^T(w_t,\xi_t-w_t)dt$$
$$+k\int_0^T\left[(w_{x_1}+\phi,\xi_{x_1}-w_{x_1})+(w_{x_2}+\psi,\xi_{x_2}-w_{x_2})\right]dt+\gamma_3\int_0^T\int_{\Gamma_1}w_t(\xi-w)dt\geq 0, \tag{3.30}$$

for all $\xi,w\in H^1(0,T;V)$, with $w(t)\in\mathbb{K}_g$ a.e. in $(0,T)$.

Remark 8. *Since g is a rigid obstacle, it can not be transposed by the plate, i.e., the distance from the plate to the obstacle must be non-negative. In other words, $w(t)\geq g$ on Γ_1, for all $t\in[0,T]$. In this way, the assumption $w_0=w(\cdot,0)\geq g$ is natural.*

Remark 9. *If a solution (ϕ,ψ,w) of the contact problem (3.12), in the sense of the Definition 3.1, is sufficiently smooth, one can show that such solution satisfies the equalities and inequalities in (3.12). Indeed, for any $\theta\in\mathcal{D}(Q)$, taking $v=\theta+\phi$, $u=\theta+\psi$, and $\xi=(\theta\pm w)\in\mathbb{K}_g$, in (3.28), (3.29), and (3.30), respectively, we have that $(3.12)_1-(3.12)_3$ is satisfied in $\mathcal{D}'(Q)$. Moreover, if we consider $(\phi,\psi,w)\in[L^2(0,T;V\cap H^2(\Omega))]^3$, then*

$$\int_Q\left(\frac{\rho h^3}{12}\phi_{tt}-L_1(\phi,\psi,w)\right)\theta dxdt=0,\quad \int_Q\left(\frac{\rho h^3}{12}\psi_{tt}-L_2(\phi,\psi,w)\right)\theta dxdt=0$$
$$\int_Q(\rho h w_{tt}-L_3(\phi,\psi,w))\theta dxdt=0, \tag{3.31}$$

which implies that $(3.12)_1-(3.12)_3$ holds a.e. in Q. Next, we consider arbitrary functions $\theta_1,\theta_2,\theta_3\in C_0^1(\Gamma_1)$, and $\eta_1,\eta_2,\eta_3\in C_0^1([0,T])$, such that $0\leq\theta_2\leq 1$, $-1\leq\eta_2\leq 1$ and that θ_3,η_3 are nonnegative. Still assuming the regularity $(\phi,\psi,w)\in[L^2(0,T;V\cap H^2(\Omega))]^3$, we take $v=\phi+\theta_1\eta_1$, $u=\psi+\theta_1\eta_1$, $\xi=w\pm\theta_2\eta_2(w-g)$, and $\xi=w+\theta_3\eta_3$ in (3.28), (3.29), (3.30), and (3.30), respectively. From the assumptions on $\theta_2,\theta_3,\eta_2,\eta_3$, it is clear that such choices of ξ are placed in $\mathbb{K}_g$. Hence, considering the extensions $\theta_1,\theta_2,\theta_3\in C^1(\overline{\Omega})$, we obtain by means of integration by parts that

$$\int_{\Sigma_1}(\mathcal{B}_1(\phi,\psi,w)+\gamma_1\phi_t)\eta_1\theta_1 d\Gamma dt=0,\quad \int_{\Sigma_1}(\mathcal{B}_2(\phi,\psi,w)+\gamma_2\psi_t)\eta_1\theta_1 d\Gamma dt=0,$$
$$\int_{\Sigma_1}(\mathcal{B}_3(\phi,\psi,w)+\gamma_3 w_t)(w-g)\eta_2\theta_2 d\Gamma dt=0,\quad \int_{\Sigma_1}(\mathcal{B}_3(\phi,\psi,w)+\gamma_3 w_t)\eta_3\theta_3 d\Gamma dt\geq 0,$$

Thus, we conclude $(3.12)_5-(3.12)_7$.

To follow the strategy described in the introduction, we will now establish a result which will play a key role in getting the solution for the contact problem (3.12) as limit, as $\varepsilon\to 0$, of the net $\{(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon)\}_{\varepsilon>0}$ formed by the solutions of the penalized problem (3.10).

Proposition 3. *Let $\{(\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon)\}_{\varepsilon>0}$ be a net in $[H^2(Q)]^3$ composed by solutions $(\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon)$ of the penalized problem (3.10) associated to the initial data $(\phi_{0\varepsilon}, \phi_{1\varepsilon}, \psi_{0\varepsilon}, \psi_{1\varepsilon}, w_{0\varepsilon}, w_{1\varepsilon}) \in [(V \cap H^2(\Omega)) \times V]^3$. If*

$$\begin{aligned} (\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon) \to (\phi, \psi, w) \quad & weak - * \ in \quad [L^\infty(0,T;V)]^3, \\ (\phi_{\varepsilon t}, \psi_{\varepsilon t}, w_{\varepsilon t}) \to (\phi_t, \psi_t, w_t) \quad & weak - * \ in \quad [L^\infty(0,T;L^2(\Omega))]^3, \end{aligned} \tag{3.32}$$

then

$$\int_Q \mathcal{H}_\varepsilon dxdt \to \int_Q \mathcal{H} dxdt, \quad as \quad \varepsilon \to 0, \tag{3.33}$$

where $\mathcal{H}_\varepsilon = \mathcal{H}_\varepsilon(x,t)$ and $\mathcal{H} = \mathcal{H}(x,t)$ are given by

$$\begin{aligned} \mathcal{H}_\varepsilon = & \frac{\rho h^3}{12}\left[(\phi_{\varepsilon t})^2 + (\psi_{\varepsilon t})^2\right] + \rho h (w_{\varepsilon t})^2 - D\Big[(\phi_{\varepsilon x_1})^2 + (\psi_{\varepsilon x_2})^2 + 2\mu\phi_{\varepsilon x_1}\psi_{\varepsilon x_2} \\ & + \frac{1-\mu}{2}(\phi_{\varepsilon x_2} + \psi_{\varepsilon x_1})^2\Big] - k\left[(w_{\varepsilon x_1} + \phi_\varepsilon)^2 + (w_{\varepsilon x_2} + \psi_\varepsilon)^2\right], \\ \mathcal{H} = & \frac{\rho h^3}{12}\left[(\phi_t)^2 + (\psi_t)^2\right] + \rho h (w_t)^2 - D\Big[(\phi_{x_1})^2 + (\psi_{x_2})^2 + 2\mu\phi_{x_1}\psi_{x_2} \\ & + \frac{1-\mu}{2}(\phi_{x_2} + \psi_{x_1})^2\Big] - k\left[(w_{x_1} + \phi)^2 + (w_{x_2} + \psi)^2\right] dt. \end{aligned} \tag{3.34}$$

Proof: In order to prove the convergence in (3.33), we will consider the divergent and curl operators associated to the variables (x_1, x_2, t) of the cylinder Q to use a compensated compactness result due to Dacorogna [33]. Let us first define $U_\varepsilon = U_\varepsilon(x_1,x_2,t)$, $V_\varepsilon = V_\varepsilon(x_1,x_2,t)$, $W_\varepsilon = W_\varepsilon(x_1,x_2,t)$, $R_\varepsilon = R_\varepsilon(x_1,x_2,t)$, $S_\varepsilon = S_\varepsilon(x_1,x_2,t)$, and $T_\varepsilon = T_\varepsilon(x_1,x_2,t)$ by

$$\begin{gathered} U_\varepsilon = \left(-D(\phi_{\varepsilon x_1} + \mu\psi_{\varepsilon x_2}), -D\frac{1-\mu}{2}(\phi_{\varepsilon x_2} + \psi_{\varepsilon x_1}), \frac{\rho h^3}{12}\phi_{\varepsilon t}\right), \\ V_\varepsilon = \left(-D\frac{1-\mu}{2}(\phi_{\varepsilon x_2} + \mu\psi_{\varepsilon x_1}), -D(\mu\phi_{\varepsilon x_1} + \psi_{\varepsilon x_2}), \frac{\rho h^3}{12}\psi_{\varepsilon t}\right), \\ W_\varepsilon = \left(-k(w_{\varepsilon x_1} + \phi_\varepsilon), -k(w_{\varepsilon x_2} + \psi_\varepsilon), \rho h w_{\varepsilon t}\right), \\ R_\varepsilon = (\phi_{\varepsilon x_1}, \phi_{\varepsilon x_2}, \phi_{\varepsilon t}), \ S_\varepsilon = (\psi_{\varepsilon x_1}, \psi_{\varepsilon x_2}, \psi_{\varepsilon t}), \ T_\varepsilon = (w_{\varepsilon x_1} + \phi_\varepsilon, w_{\varepsilon x_2} + \psi_\varepsilon, w_{\varepsilon t}). \end{gathered}$$

From (3.32), we have that the net $\{(U_\varepsilon, V_\varepsilon, W_\varepsilon, R_\varepsilon, S_\varepsilon, T_\varepsilon)\}_{\varepsilon>0}$ is bounded in $[L^2(Q)]^{18}$. Since $(\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon) \in [H^2(Q)]^3$, we deduce that

$$\begin{aligned} \operatorname{div} U_\varepsilon &= \frac{\rho h^3}{12}\phi_{\varepsilon tt} + L_1(\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon) = -k(w_{\varepsilon x_1} + \phi_\varepsilon), \\ \operatorname{div} V_\varepsilon &= \frac{\rho h^3}{12}\psi_{\varepsilon tt} + L_2(\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon) = -k(w_{\varepsilon x_2} + \psi_\varepsilon), \\ \operatorname{div} W_\varepsilon &= \rho h w_{\varepsilon tt} + L_3(\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon) = 0, \end{aligned}$$

and

$$\operatorname{curl} R_\varepsilon = \operatorname{curl} S_\varepsilon = \operatorname{curl} T_\varepsilon = 0.$$

Thus, (3.32) also give us that $\{(\operatorname{div} U_\varepsilon, \operatorname{div} V_\varepsilon, \operatorname{div} W_\varepsilon, \operatorname{curl} R_\varepsilon, \operatorname{curl} S_\varepsilon, \operatorname{curl} T_\varepsilon)\}_{\varepsilon>0}$ is bounded in $[L^2(Q)]^6$. Then, by [33, Corollary 4.3, p. 36], it follows that

$$(U_\varepsilon \cdot R_\varepsilon, V_\varepsilon \cdot S_\varepsilon, W_\varepsilon \cdot T_\varepsilon) \to (U \cdot R, V \cdot S, W \cdot T) \quad \text{in} \quad [D'(Q)]^3, \tag{3.35}$$

where $U = U(x_1, x_2, t)$, $V = V(x_1, x_2, t)$, $W = W(x_1, x_2, t)$, $R = R(x_1, x_2, t)$, $S = S(x_1, x_2, t)$ and $T = T(x_1, x_2, t)$ are given by

$$U = \left(-D(\phi_{x_1} + \mu\psi_{x_2}), -D\frac{1-\mu}{2}(\phi_{x_2} + \psi_{x_1}), \frac{\rho h^3}{12}\phi_t\right),$$
$$V = \left(-D\frac{1-\mu}{2}(\phi_{x_2} + \mu\psi_{x_1}), -D(\mu\phi_{x_1} + \psi_{x_2}), \frac{\rho h^3}{12}\psi_t\right),$$
$$W = (-k(w_{x_1} + \phi), -k(w_{x_2} + \psi), \rho h w_t),$$
$$R = (\phi_{x_1}, \phi_{x_2}, \phi_t), \; S = (\psi_{x_1}, \psi_{x_2}, \psi_t), \; T = (w_{x_1} + \phi, w_{x_2} + \psi, w_t).$$

From (3.35), we have

$$\mathcal{H}_\varepsilon \to \mathcal{H} \quad \text{in} \quad \mathcal{D}'(Q), \tag{3.36}$$

since $\mathcal{H}_\varepsilon = U_\varepsilon \cdot R_\varepsilon + V_\varepsilon \cdot S_\varepsilon + W_\varepsilon \cdot T_\varepsilon$ and $\mathcal{H} = U \cdot R + V \cdot S + W \cdot T$. Let us improve the space of the convergence in (3.36) in order to achieve (3.33). By convergences in (3.32), there exists a constant $\mathcal{C}_0 > 0$, independent of ε, such that, for each $\varepsilon > 0$, we have

$$\int_\Omega |\mathcal{H}_\varepsilon(\cdot, t)| dx \leq \|(\phi_{\varepsilon t}, \psi_{\varepsilon t}, w_{\varepsilon t})\|_{[L^\infty(0,T;L^2(\Omega))]^3} + \|(\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon)\|_{[L^\infty(0,T;V)]^3} \leq \mathcal{C}_0, \quad \forall t \in [0, T]. \tag{3.37}$$

and

$$\int_\Omega |\mathcal{H}(\cdot, t)| dx \leq \|(\phi_t, \psi_t, w_t)\|_{[L^\infty(0,T;L^2(\Omega))]^3} + \|(\phi, \psi, w)\|_{[L^\infty(0,T;V)]^3} \leq \mathcal{C}_0, \quad \forall t \in [0, T]. \tag{3.38}$$

Given $r > 0$ a real number and let us consider auxiliary functions $\eta_r \in \mathcal{D}(0, T)$ and $\theta \in \mathcal{D}(\Omega)$ such that $0 \leq \eta_r, \theta \leq 1$ and

$$\eta_r(t) = 0 \;\text{ in }\; \left[0, \frac{r}{9\mathcal{C}_0}\right], \quad \eta_r(t) = 1 \;\text{ in }\; \left[\frac{r}{8\mathcal{C}_0}, T - \frac{r}{8\mathcal{C}_0}\right], \quad \eta_r(t) = 0 \;\text{ in }\; \left[T - \frac{r}{9\mathcal{C}_0}, T\right].$$

From convergence in (3.36), there exists a real number $\delta > 0$ such that

$$\left|\langle \mathcal{H}_\varepsilon - \mathcal{H}, \eta_r \theta \rangle_{\mathcal{D}'(Q), \mathcal{D}(Q)}\right| < \frac{r}{2}, \quad \forall \varepsilon < \delta.$$

Thus, the last estimate, (3.37), and (3.38) give us

$$\begin{aligned}
\left|\int_Q (\mathcal{H}_\varepsilon - \mathcal{H}) dx dt\right| &\leq \left|\int_Q (\mathcal{H}_\varepsilon - \mathcal{H}) \eta_r \theta dx dt\right| + \left|\int_Q (\mathcal{H}_\varepsilon - \mathcal{H})(1 - \eta_r) \theta dx dt\right| \\
&\leq \left|\langle \mathcal{H}_\varepsilon - \mathcal{H}, \eta_r \theta \rangle_{\mathcal{D}'(Q), \mathcal{D}(Q)}\right| + \left|\int_Q (\mathcal{H}_\varepsilon - \mathcal{H})(1 - \eta_r) \theta dx dt\right| \\
&\leq \frac{r}{2} + \left(\int_0^{\frac{r}{8\mathcal{C}_0}} + \int_{T - \frac{r}{8\mathcal{C}_0}}^T\right)(1 - \eta_r) \int_\Omega \theta(|\mathcal{H}_\varepsilon| + |\mathcal{H}|) dx dt \\
&\leq r,
\end{aligned}$$

for all $\varepsilon < \delta$. This proves (3.33). ■

Now we will establish the result which guarantees the existence of solution for the contact problem (3.12).

Theorem 5. *Given initial data* $(\phi_0, \phi_1, \psi_0, \psi_1, w_0, w_1) \in [V \times L^2(\Omega)]^2 \times \mathbb{K}_g \times L^2(\Omega)$ *and* $g \in L^2(\Gamma_1)$, *then there exists at least a solution* (ϕ, ψ, w) *of the contact problem* (3.12) *in the sense of Definition 3.1.*

Proof: Let us consider the net $\{(\phi_{0\varepsilon}, \phi_{1\varepsilon}, \psi_{0\varepsilon}, \psi_{1\varepsilon}, w_{0\varepsilon}, w_{1\varepsilon})\}_{\varepsilon>0}$ in $[(V \cap H^2(\Omega)) \times V]^3$ such that

$$(\phi_{0\varepsilon}, \phi_{1\varepsilon}, \psi_{0\varepsilon}, \psi_{1\varepsilon}, w_{0\varepsilon}, w_{1\varepsilon}) \to (\phi_0, \phi_1, \psi_0, \psi_1, w_0, w_1) \quad \text{strongly in} \quad [V \times L^2(\Omega)]^3, \tag{3.39}$$

as $\varepsilon \to 0$. For each $\varepsilon > 0$, Theorem 4 guarantees the existence of a unique solution $(\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon)$ for the penalized problem (3.10) associated to $(\phi_{0\varepsilon}, \phi_{1\varepsilon}, \psi_{0\varepsilon}, \psi_{1\varepsilon}, w_{0\varepsilon}, w_{1\varepsilon})$. Moreover, this solution has a regularity described in (3.13). The main idea of the proof lies in to handle the variational formulation (3.14) fulfilled by $(\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon)$ and pass to the limit, as $\varepsilon \to 0$.

Let us consider a triplet $(v, u, \xi) \in [H^1(0, T; V)]^3$ with $\xi(t) \in \mathbb{K}_g$ a.e. in $(0, T)$. Taking $s = v - \phi_\varepsilon$, $p = u - \psi_\varepsilon$ and $z = \xi - w_\varepsilon$ in (3.14) and integrating from 0 to T, we obtain, after a integration by parts in the time variable t,

$$P_{1\varepsilon} + R_{1\varepsilon} + P_{2\varepsilon} + R_{2\varepsilon} + P_{3\varepsilon} + R_{3\varepsilon} + B_\varepsilon - \mathcal{H}_\varepsilon = -\frac{1}{\varepsilon} \int_0^T \int_{\Gamma_1} (w_\varepsilon - g)^- (w_\varepsilon - \xi) d\Gamma dt, \tag{3.40}$$

where $\mathcal{H}_\varepsilon$ is defined in (3.34),

$$\begin{aligned}
P_{1\varepsilon} &= \frac{\rho h^3}{12} \langle \phi_{\varepsilon t}(\cdot, T), v(\cdot, T) - \phi_\varepsilon(\cdot, T) \rangle_r - \frac{\rho h^3}{12} \langle \phi_{1\varepsilon}, v(\cdot, 0) - \phi_{0\varepsilon} \rangle_r - \frac{\rho h^3}{12} \int_0^T (\phi_{\varepsilon t}, v_t) dt, \\
R_{1\varepsilon} &= D \int_0^T \left[(\phi_{\varepsilon x_1}, v_{x_1}) + \mu(\psi_{\varepsilon x_2}, v_{x_1}) + \frac{1-\mu}{2} (\phi_{\varepsilon x_2} + \psi_{\varepsilon x_1}, v_{x_2}) \right] dt, \\
P_{2\varepsilon} &= \frac{\rho h^3}{12} \langle \psi_{\varepsilon t}(\cdot, T), u(\cdot, T) - \psi_\varepsilon(\cdot, T) \rangle_r - \frac{\rho h^3}{12} \langle \psi_{1\varepsilon}, u(\cdot, 0) - \psi_{0\varepsilon} \rangle_r - \frac{\rho h^3}{12} \int_0^T (\psi_{\varepsilon t}, u_t) dt, \\
R_{2\varepsilon} &= D \int_0^T \left[(\psi_{\varepsilon x_2}, u_{x_2}) + \mu(\phi_{\varepsilon x_1}, u_{x_2}) + \frac{1-\mu}{2} (\phi_{\varepsilon x_2} + \psi_{\varepsilon x_1}, u_{x_1}) \right] dt, \\
P_{3\varepsilon} &= \rho h \langle w_{\varepsilon t}(\cdot, T), \xi(\cdot, T) - w_\varepsilon(\cdot, T) \rangle_r - \rho h \langle w_{1\varepsilon}, \xi(\cdot, 0) - w_{0\varepsilon} \rangle_r - \rho h \int_0^T (w_{\varepsilon t}, \xi_t) dt, \\
R_{3\varepsilon} &= k \int_0^T \left[(w_{\varepsilon x_1} + \phi_\varepsilon, \xi_{x_1} + v) + (w_{\varepsilon x_2} + \psi_\varepsilon, \xi_{x_2} + u) \right] dt, \\
B_\varepsilon &= \gamma_1 \int_0^T \int_{\Gamma_1} \phi_{\varepsilon t}(v - \phi_\varepsilon) d\Gamma dt + \gamma_2 \int_0^T \int_{\Gamma_1} \psi_{\varepsilon t}(u - \psi_\varepsilon) d\Gamma dt + \gamma_3 \int_0^T \int_{\Gamma_1} w_{\varepsilon t}(\xi - w_\varepsilon) d\Gamma dt,
\end{aligned} \tag{3.41}$$

for some $r > 0$ to be chosen such that the dualities above make sense.

Notice that, since $\xi(t) \geq g$ in Γ_1 a.e. in $(0, T)$, then

$$-\frac{1}{\varepsilon} \int_{\Gamma_1} (w_\varepsilon - g)^- (w_\varepsilon - \xi) d\Gamma = \frac{1}{\varepsilon} \int_{\Gamma_1} |(w_\varepsilon - g)^-|^2 + \frac{1}{\varepsilon} \int_{\Gamma_1} (w_\varepsilon - g)^- (\xi - g) d\Gamma \geq 0.$$

Thus, (3.40) becomes in

$$P_{1\varepsilon} + R_{1\varepsilon} + P_{2\varepsilon} + R_{2\varepsilon} + P_{3\varepsilon} + R_{3\varepsilon} + B_\varepsilon - \mathcal{H}_\varepsilon \geq 0. \tag{3.42}$$

In order to pass the limit, as $\varepsilon \to 0$, in (3.42), we start analyzing the convergence of the net $\{\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon\}_{\varepsilon>0}$. Taking $(s,p,z) = (\phi_{\varepsilon t}, \psi_{\varepsilon t}, w_{\varepsilon t})$ in (3.14), integrating from 0 to $t \le T$, and using (3.39), we get

$$
\begin{aligned}
&\frac{\rho h^3}{12}(|\phi_{\varepsilon t}(\cdot,t)|^2 + |\psi_{\varepsilon t}(\cdot,t)|^2) + \rho h |w_{\varepsilon t}(\cdot,t)|^2 \\
&\quad + D\left(|\phi_{\varepsilon x_1}(\cdot,t)|^2 + |\psi_{\varepsilon x_2}(\cdot,t)|^2 + 2\mu(\phi_{\varepsilon x_1}(\cdot,t), \psi_{\varepsilon x_2}(\cdot,t)) + \frac{1-\mu}{2}|\phi_{\varepsilon x_2}(\cdot,t) + \psi_{\varepsilon x_1}(\cdot,t)|^2\right) \\
&\quad + k(|w_{\varepsilon x_1}(\cdot,t) + \phi_\varepsilon(\cdot,t)|^2 + |w_{\varepsilon x_2}(\cdot,t) + \psi_\varepsilon(\cdot,t)|^2) + \frac{1}{\varepsilon}\int_{\Gamma_1} |(w_\varepsilon(\cdot,t) - g)^-|^2 d\Gamma \\
&\quad + \gamma_1 \int_0^t \int_{\Gamma_1} |\phi_{\varepsilon t}|^2 d\Gamma ds + \gamma_2 \int_0^t \int_{\Gamma_1} |\psi_{\varepsilon t}|^2 d\Gamma ds + \gamma_3 \int_0^t \int_{\Gamma_1} |w_{\varepsilon t}|^2 d\Gamma ds \\
&\quad \le C + \frac{1}{\varepsilon}\int_{\Gamma_1} |(w_{0\varepsilon} - g)^-|^2 d\Gamma,
\end{aligned}
\tag{3.43}
$$

where $C > 0$ is a constant independent of ε. Notice that, from (3.39) and the trace theory, we can deduce that $w_{0\varepsilon} \to w_0$ strongly in $L^2(\Gamma_1)$. Moreover, since $w_0 \in \mathbb{K}_g$, there exists $\varepsilon_0 > 0$ such that

$$
\int_{\Gamma_1} |(w_{0\varepsilon} - g)^-|^2 d\Gamma = 0, \quad \forall \varepsilon \in (0, \varepsilon_0). \tag{3.44}
$$

Hence, in view of (3.26), we obtain, from (3.43) and (3.44),

$$
\|(\phi_{\varepsilon t}, \psi_{\varepsilon t}, w_{\varepsilon t})\|_{L^\infty(0,T;L^2(\Omega))} + \|(\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon)\|_{L^\infty(0,T;V)} \le C, \quad \forall \varepsilon \in (0, \varepsilon_0). \tag{3.45}
$$

In addition, we have

$$
\|(\phi_{\varepsilon tt}, \psi_{\varepsilon tt}, w_{\varepsilon tt})\|_{L^\infty(0,T;L^2(\Omega))} \le C, \quad \forall \varepsilon \in (0, \varepsilon_0). \tag{3.46}
$$

In fact, since $(\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon)$ satisfies (3.14), we use trace theory to obtain a constant $C > 0$ such that

$$
\begin{aligned}
&\langle (\phi_{\varepsilon tt}, \psi_{\varepsilon tt}, w_{\varepsilon tt}), (s,p,z) \rangle_{[L^2(0,T;L^2(\Omega))]^3, [L^2(0,T;L^2(\Omega))]^3} \\
&\quad \le C \|(\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon)\|_{[L^2(0,T;V)]^3} \|(s,p,z)\|_{[L^2(0,T;V)]^3} \\
&\quad + C\left(\|(\phi_{\varepsilon t}, \psi_{\varepsilon t}, w_{\varepsilon t})\|_{[L^2(0,T;\Gamma_1)]^3} + \frac{1}{\varepsilon}\|(w_\varepsilon - g)^-\|_{L^2(0,T;L^2(\Gamma_1))}\right)\|(s,p,z)\|_{[L^2(0,T;V)]^3},
\end{aligned}
\tag{3.47}
$$

for all $(s,p,z) \in [L^2(0,T;V)]^3$. In this way, combining the last inequality, (3.43), and (3.44), it follows (3.46). So, from (3.45) and (3.46), we deduce

$$
\begin{aligned}
(\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon) &\to (\phi, \psi, w) && \text{weak} - * \text{ in} && [L^\infty(0,T;V)]^3, \\
(\phi_{\varepsilon t}, \psi_{\varepsilon t}, w_{\varepsilon t}) &\to (\phi_t, \psi_t, w_t) && \text{weak} - * \text{ in} && [L^\infty(0,T;L^2(\Omega))]^3, \\
(\phi_{\varepsilon tt}, \psi_{\varepsilon tt}, w_{\varepsilon tt}) &\to (\phi_{tt}, \psi_{tt}, w_{tt}) && \text{weakly in} && [L^2(0,T;V')]^3,
\end{aligned}
\tag{3.48}
$$

as $\varepsilon \to 0$. We can see by [57, Lemma 1.4] that the convergences in (3.48) give us

$$
\begin{aligned}
(\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon) &\to (\phi, \psi, w) && \text{strongly in} && [C([0,T];H^{1-\delta}(\Omega))]^3, \\
(\phi_{\varepsilon t}, \psi_{\varepsilon t}, w_{\varepsilon t}) &\to (\phi_t, \psi_t, w_t) && \text{strongly in} && [C([0,T];(H^{1-\delta}(\Omega))')]^3,
\end{aligned}
\tag{3.49}
$$

as $\varepsilon \to 0$, for any positive real number $\delta < \frac{1}{2}$. We will use the convergences (3.39), (3.48), (3.49), and the Proposition 3 to pass the limit, as $\varepsilon \to 0$, in (3.42). Indeed, in view of these convergences we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0} (P_{1\varepsilon} + R_{1\varepsilon} + P_{2\varepsilon} + R_{2\varepsilon} + P_{3\varepsilon} + R_{3\varepsilon} - \mathcal{H}_\varepsilon) \\
&= \frac{\rho h^3}{12} \langle \phi_t(\cdot,T), v(\cdot,T) - \phi(\cdot,T)\rangle_r - \frac{\rho h^3}{12} \langle \phi_1, v(\cdot,0) - \phi_0\rangle_r + \frac{\rho h^3}{12} \langle \psi_t(\cdot,T), u(\cdot,T) - \psi(\cdot,T)\rangle_r \\
&\quad - \frac{\rho h^3}{12} \langle \psi_1, u(\cdot,0) - \psi_0\rangle_r + \rho h \langle w_t(\cdot,T), \xi(\cdot,T) - w(\cdot,T)\rangle_r - \rho h \langle w_1, \xi(\cdot,0) - w_0\rangle_r \\
&\quad - \frac{\rho h^3}{12} \int_0^T (\phi_t, v_t - \phi_t) dt - \frac{\rho h^3}{12} \int_0^T (\psi_t, u_t - \psi_t) dt - \rho h \int_0^T (w_t, \xi_t - w_t) dt \\
&\quad + D \int_0^T \left[(\phi_{x_1}, v_{x_1} - \phi_{x_1}) + \mu(\psi_{x_2}, v_{x_1} - \phi_{x_1}) + \frac{1-\mu}{2}(\phi_{x_2} + \psi_{x_1}, v_{x_2} - \phi_{x_2}) \right] dt \\
&\quad + D \int_0^T \left[(\psi_{x_2}, u_{x_2} - \psi_{x_2}) + \mu(\phi_{x_1}, u_{x_2} - \psi_{x_2}) + \frac{1-\mu}{2}(\phi_{x_2} + \psi_{x_1}, u_{x_1} - \psi_{x_1}) \right] dt \\
&\quad + k \int_0^T \left[(w_{x_1} + \phi, \xi_{x_1} - w_{x_1} + v - \phi) + (w_{x_2} + \psi, \xi_{x_2} - w_{x_2} + u - \psi) \right] dt.
\end{aligned} \tag{3.50}$$

Now, to take the limit in B_ε, as $\varepsilon \to 0$, we, first, use the continuous immersion of $C([0,T]; H^{1-\delta}(\Omega))$ into $C([0,T]; L^2(\Gamma))$, for $\delta < \frac{1}{2}$, and (3.49) to deduce that

$$(\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon) \to (\phi, \psi, w) \quad \text{strongly in} \quad [C([0,T]; L^2(\Gamma))]^3, \quad \text{as} \quad \varepsilon \to 0.$$

The last convergence together with (3.39) give us

$$\gamma_1 \int_0^T \int_{\Gamma_1} \phi_{\varepsilon t}(v - \phi_\varepsilon) d\Gamma dt \to \gamma_1 \int_0^T \int_{\Gamma_1} \phi_t(v - \phi) d\Gamma dt, \quad \text{as} \quad \varepsilon \to 0, \tag{3.51}$$

since

$$\begin{aligned}
\gamma_1 \int_0^T \int_{\Gamma_1} \phi_{\varepsilon t}(v - \phi_\varepsilon) d\Gamma dt = -\gamma_1 \int_0^T \int_{\Gamma_1} \phi_\varepsilon v_t d\Gamma dt + \gamma_1 \int_{\Gamma_1} (\phi_\varepsilon(\cdot,T) v(\cdot,T) - \phi_{0\varepsilon} v(\cdot,0)) d\Gamma \\
- \frac{\gamma_1}{2} \int_{\Gamma_1} (|\phi_\varepsilon(\cdot,T)|^2 - |\phi_{0\varepsilon}|^2) d\Gamma
\end{aligned}$$

and

$$\begin{aligned}
\gamma_1 \int_0^T \int_{\Gamma_1} \phi_t(v - \phi) d\Gamma dt = -\gamma_1 \int_0^T \int_{\Gamma_1} \phi v_t d\Gamma dt + \gamma_1 \int_{\Gamma_1} (\phi(\cdot,T) v(\cdot,T) - \phi_0 v(\cdot,T)) d\Gamma \\
- \frac{\gamma_1}{2} \int_{\Gamma_1} (|\phi(\cdot,T)|^2 - |\phi_0|^2) d\Gamma.
\end{aligned}$$

Using the same idea, we prove that, as $\varepsilon \to 0$, the following convergences hold:

$$\begin{aligned}
\gamma_2 \int_0^T \int_{\Gamma_1} \psi_{\varepsilon t}(u - \psi_\varepsilon) d\Gamma dt \to \gamma_2 \int_0^T \int_{\Gamma_1} \psi_t(u - \psi) d\Gamma dt, \\
\gamma_3 \int_0^T \int_{\Gamma_1} w_{\varepsilon t}(\xi - w_\varepsilon) d\Gamma dt \to \gamma_3 \int_0^T \int_{\Gamma_1} w_t(\xi - w) d\Gamma dt.
\end{aligned} \tag{3.52}$$

Combining (3.51) and (3.52) we obtain

$$B_\varepsilon \to \gamma_1 \int_0^T \int_{\Gamma_1} \phi_t(v-\phi)d\Gamma dt + \gamma_2 \int_0^T \int_{\Gamma_1} \psi_t(u-\psi)d\Gamma dt + \gamma_3 \int_0^T \int_{\Gamma_1} w_t(\xi - w)d\Gamma dt, \tag{3.53}$$

as $\varepsilon \to 0$. Thus, from convergences (3.50) and (3.53), we obtain by (3.42) that

$$\begin{aligned}
&\frac{\rho h^3}{12}\langle \phi_t(\cdot,T), v(\cdot,T)-\phi(\cdot,T)\rangle_r - \frac{\rho h^3}{12}\langle \phi_1, v(\cdot,0)-\phi_0\rangle_r + \frac{\rho h^3}{12}\langle \psi_t(\cdot,T), u(\cdot,T)-\psi(\cdot,T)\rangle_r \\
&\quad - \frac{\rho h^3}{12}\langle \psi_1, u(\cdot,0)-\psi_0\rangle_r + \rho h\langle w_t(\cdot,T), \xi(\cdot,T)-w(\cdot,T)\rangle_r - \rho h\langle w_1, \xi(\cdot,0)-w_0\rangle_r \\
&\quad - \frac{\rho h^3}{12}\int_0^T (\phi_t, v_t-\phi_t)dt - \frac{\rho h^3}{12}\int_0^T (\psi_t, u_t-\psi_t)dt - \rho h\int_0^T (w_t, \xi_t - w_t)dt \\
&\quad + D\int_0^T \left[(\phi_{x_1}, v_{x_1}-\phi_{x_1}) + \mu(\psi_{x_2}, v_{x_1}-\phi_{x_1}) + \frac{1-\mu}{2}(\phi_{x_2}+\psi_{x_1}, v_{x_2}-\phi_{x_2})\right]dt \\
&\quad + D\int_0^T \left[(\psi_{x_2}, u_{x_2}-\psi_{x_2}) + \mu(\phi_{x_1}, u_{x_2}-\psi_{x_2}) + \frac{1-\mu}{2}(\phi_{x_2}+\psi_{x_1}, u_{x_1}-\psi_{x_1})\right]dt \\
&\quad + \gamma_1 \int_0^T \int_{\Gamma_1} \phi_t(v-\phi)d\Gamma dt + \gamma_2 \int_0^T \int_{\Gamma_1} \psi_t(u-\psi)d\Gamma dt + \gamma_3 \int_0^T \int_{\Gamma_1} w_t(\xi - w)d\Gamma dt, \\
&\quad + k\int_0^T \left[(w_{x_1}+\phi, \xi_{x_1}-w_{x_1}+v-\phi) + (w_{x_2}+\psi, \xi_{x_2}-w_{x_2}+u-\psi)\right]dt \geq 0,
\end{aligned} \tag{3.54}$$

for all $(v,u,\xi) \in [H^1(0,T;V)]^3$, where $\xi(t) \in \mathbb{K}_g$ a.e. in $(0,T)$. From the previous inequality, one can readily realize that (ϕ,ψ,w) satisfies the equations (3.28)–(3.30). Indeed, first let us assume that $w(t) \in \mathbb{K}_g$. Thus, choosing $(v,u,\xi) = (v,\psi,w)$ and $(v,u,\xi) = (-v+2\phi,\psi,w)$ in (3.54), we obtain two inequalities which imply (3.28). On the other hand, taking $(v,u,\xi) = (\phi,u,w)$ and $(v,u,\xi) = (\phi, u-2\psi, w)$ in (3.54) we get (3.29). Finally, for $(v,u,\xi) = (\phi,\psi,\xi)$, (3.30) is achieved. To complete the proof, it remains to prove that $w(t) \in \mathbb{K}_g$. From the estimates (3.43) and (3.44), we get

$$\int_\Gamma |(w_\varepsilon(\cdot,t)-g)^-|^2 d\Gamma \leq C\varepsilon, \quad \text{a.e. in} \quad (0,T), \quad \forall \varepsilon \in (0,\varepsilon_0). \tag{3.55}$$

Then $(w_\varepsilon(t)-g)^- \to 0$ weakly in $L^2(\Gamma)$ for a.e. t in $(0,T)$. On the other hand, (3.49) give us that $(w_\varepsilon(t)-g)^- \to (w(t)-g)^-$ strongly in $L^2(\Gamma)$ for all $t \in [0,T]$. In conclusion, $(w(t)-g)^- = 0$ for all $t \in [0,T]$, i.e., $w(t) \in \mathbb{K}_g$ for all $t \in [0,T]$. This concludes the theorem. ■

3.4 Uniform stabilization

The aim of this section is to analyze the decay rate for the energy

$$\mathbb{E}(t) = \frac{1}{2}\left[\frac{\rho h^3}{12}(|\phi_t|^2 + |\psi_t|^2) + \rho h|w_t|^2 + a_0(\phi,\psi) + k a_1(\phi,\psi,w)\right],$$

associated to system (3.12), as a limit (as $\varepsilon \to 0$) of the uniform stabilization of the penalized Mindlin-Timoshenko system (3.10).

Let x_0 be a point of $\mathbb{R}^2$ and $m(x) = x - x_0$, with $x \in \mathbb{R}^2$, such that

$$\Gamma_0 = \{x \in \Gamma; \quad m(x) \cdot \nu(x) \leq 0\}, \quad \Gamma_1 = \{x \in \Gamma; \quad m(x) \cdot \nu(x) > 0\}. \tag{3.56}$$

The main result of this section is the following.

Theorem 6. *Let us consider* $(\phi_0, \phi_1, \psi_0, \psi_1, w_0, w_1) \in [(V \cap H^2(\Omega)) \times V]^3$ *and a function* $g \in L^2(\Gamma_1)$ *satisfying* $g \leq 0$*. Then, there exist positive constants* $\mathcal{C}$*,* ω *and* ε_0*, such that the energy* (3.8) *associated to the problem* (3.10) *satisfies*

$$\mathbb{E}_\varepsilon(t) \leq \mathcal{C}\mathbb{E}_\varepsilon(0)e^{-\omega t}, \quad \forall t \geq 0, \quad \forall \varepsilon \in (0, \varepsilon_0). \tag{3.57}$$

Remark 10. *As a consequence of inequality* (3.57)*, letting* $\varepsilon \to 0$*, we can recover the exponential decay of the energy* $\mathbb{E}(t)$ *of the system* (3.12)*. In fact, let us consider a solution* (ϕ, ψ, w) *of* (3.12) *associated to initial data* $(\phi_0, \phi_1, \psi_0, \psi_1, w_0, w_1) \in [V \times L^2(\Omega)]^2 \times \mathbb{K}_g \times L^2(\Omega)$ *obtained as the limit, as* $\varepsilon \to 0$*, of a net formed by solutions* $(\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon)$ *of* (3.10) *associated to the initial data* $(\phi_{0\varepsilon}, \phi_{1\varepsilon}, \psi_{0\varepsilon}, \psi_{1\varepsilon}, w_{0\varepsilon}, w_{1\varepsilon}) \in [(V \cap H^2(\Omega)) \times V]^3$ *satisfying* (3.39)*. In this way, the convergences* (3.39) *and* (3.48) *allow us to take the* $\liminf_{\varepsilon \to 0}$ *in both sides of* (3.57) *to obtain*

$$\mathbb{E}(t) \leq \mathcal{C}\mathbb{E}(0)e^{-\omega t}, \quad \forall t \geq 0. \tag{3.58}$$

Proof of Theorem 6: For an arbitrary real number $\eta > 0$, we define the perturbed energy

$$\mathbb{E}_{\varepsilon\eta}(t) = \mathbb{E}_\varepsilon(t) + \eta G(t), \tag{3.59}$$

with

$$\begin{aligned} G(t) \;=\; & (1-\theta)\left(\frac{\rho h^3}{12}\phi_{\varepsilon t}, \phi_\varepsilon\right) + (1-\theta)\left(\frac{\rho h^3}{12}\psi_{\varepsilon t}, \psi_\varepsilon\right) + \theta\rho h\left(w_{\varepsilon t}, w_\varepsilon\right) \\ & + \left(\frac{\rho h^3}{12}\phi_{\varepsilon t}, m \cdot \nabla\phi_\varepsilon\right) + \left(\frac{\rho h^3}{12}\psi_{\varepsilon t}, m \cdot \nabla\psi_\varepsilon\right) + \left(\rho h w_{\varepsilon t}, m \cdot \nabla w_\varepsilon\right), \end{aligned}$$

where $\theta \in (0,1)$ is a constant to be chosen later. Let us observe that (3.57) holds if we can provide the following estimates:

$$|G(t)| \leq \mathbf{C}\mathbb{E}_\varepsilon(t), \quad \frac{d}{dt}G(t) \leq -\frac{1}{\mathbf{C}}\mathbb{E}_\varepsilon(t) - \mathbf{C}\frac{d}{dt}\mathbb{E}_\varepsilon(t), \tag{3.60}$$

where $\mathbf{C} > 0$ is a constant independent of ε. Indeed, choosing $0 < \eta < \frac{1}{\mathbf{C}}$, we get from (3.60)

$$\begin{aligned} \mathbb{E}_{\varepsilon\eta t} \;&=\; \frac{d}{dt}\mathbb{E}_\varepsilon + \eta\frac{d}{dt}G(t) \\ &\leq\; \frac{d}{dt}\mathbb{E}_\varepsilon(t) - \eta\mathbf{C}\frac{d}{dt}\mathbb{E}_\varepsilon(t) - \eta\frac{1}{\mathbf{C}}\mathbb{E}_\varepsilon(t) \\ &=\; (1-\eta\mathbf{C})\frac{d}{dt}\mathbb{E}_\varepsilon(t) - \frac{1}{\mathbf{C}}\left(\frac{\eta}{2}\mathbb{E}_\varepsilon(t) + \frac{\eta}{2}\mathbb{E}_{\varepsilon\eta}(t) - \frac{\eta^2}{2}G(t)\right) \\ &\leq\; (1-\eta\mathbf{C})\frac{d}{dt}\mathbb{E}_\varepsilon(t) - \frac{\eta}{2}\left(\frac{1}{\mathbf{C}} - \eta\right)\mathbb{E}_\varepsilon(t) - \frac{\eta}{2\mathbf{C}}\mathbb{E}_{\varepsilon\eta}(t) \\ &\leq\; -\omega\mathbb{E}_{\varepsilon\eta}(t). \end{aligned} \tag{3.61}$$

with $\omega = \frac{\eta}{2\mathbf{C}}$. Multiplying (3.61) by $e^{\omega t}$ and integrating from 0 to t, we have

$$\mathbb{E}_{\varepsilon\eta}(t) \leq \mathbb{E}_{\varepsilon\eta}(0)e^{-\omega t}, \quad \forall t \geq 0. \tag{3.62}$$

From (3.60), we conclude

$$\begin{aligned}
\mathbb{E}_{\varepsilon}(t) &= \mathbb{E}_{\varepsilon\eta}(t) - \eta G(t) \\
&\leq (\mathbb{E}_{\varepsilon}(0) - \eta G(0))e^{-\omega t} - \eta G(t) \\
&\leq (1 + \eta \mathbf{C})\mathbb{E}_{\varepsilon}(0)e^{-\omega t} + \eta \mathbf{C}\mathbb{E}_{\varepsilon}(t), \quad \forall t \geq 0.
\end{aligned} \tag{3.63}$$

Therefore (3.63) implies (3.57), as we claimed, with $\mathcal{C} = \frac{1+\eta\mathbf{C}}{1-\eta\mathbf{C}} > 0$.

Now remains to prove the estimates in (3.60) in order to finish the proof of the Theorem. The first estimate it is not difficult to see. In fact, by (3.26) we have

$$|G(t)| \leq C\left(\frac{\rho h^3}{12}(|\phi_{\varepsilon t}|^2 + |\psi_{\varepsilon t}|^2) + \rho h|w_{\varepsilon t}|^2 + \|(\phi_\varepsilon, \psi_\varepsilon, w_\varepsilon)\|_{H^1(\Omega)}\right) \leq CE_\varepsilon(t), \tag{3.64}$$

where $C > 0$ is a constant independent of ε. To show the second inequality in (3.60), we observe that

$$\frac{d}{dt}G(t) = G_1(t) + G_2(t) + F_1(t) + F_2(t),$$

where

$$\begin{aligned}
G_1(t) &= (1-\theta)\frac{\rho h^3}{12}(|\phi_{\varepsilon t}|^2 + |\psi_{\varepsilon t}|^2) + \theta\rho h|w_{\varepsilon t}|^2, \\
G_2(t) &= \left(\frac{\rho h^3}{12}\phi_{\varepsilon t}, m\cdot\nabla\phi_{\varepsilon t}\right) + \left(\frac{\rho h^3}{12}\psi_{\varepsilon t}, m\cdot\nabla\psi_{\varepsilon t}\right) + \left(\rho h w_{\varepsilon t}, m\cdot\nabla w_{\varepsilon t}\right), \\
F_1(t) &= (1-\theta)\left(\frac{\rho h^3}{12}\phi_{\varepsilon tt}, \phi_\varepsilon\right) + (1-\theta)\left(\frac{\rho h^3}{12}\psi_{\varepsilon tt}, \psi_\varepsilon\right) + \theta\left(\rho h w_{\varepsilon tt}, w_\varepsilon\right), \\
F_2(t) &= \left(\frac{\rho h^3}{12}\phi_{\varepsilon tt}, m\cdot\nabla\phi_\varepsilon\right) + \left(\frac{\rho h^3}{12}\psi_{\varepsilon tt}, m\cdot\nabla\psi_\varepsilon\right) + \left(\rho h w_{\varepsilon tt}, m\cdot\nabla w_\varepsilon\right).
\end{aligned}$$

Now, let us estimate the expressions above. From $(3.10)_4$ and (3.11) we get

$$\begin{aligned}
G_2(t) &= \frac{1}{2}\int_\Omega m\cdot\nabla\left(\frac{\rho h^3}{12}(|\phi_{\varepsilon t}|^2 + |\psi_{\varepsilon t}|^2) + \rho h|w_{\varepsilon t}|^2\right)dx \\
&= -\int_\Omega\left(\frac{\rho h^3}{12}(|\phi_{\varepsilon t}|^2 + |\psi_{\varepsilon t}|^2) + \rho h|w_{\varepsilon t}|^2\right)dx \\
&\quad + \frac{1}{2}\int_{\Gamma_1} m\cdot\nu\left(\frac{\rho h^3}{12}(|\phi_{\varepsilon t}|^2 + |\psi_{\varepsilon t}|^2) + \rho h|w_{\varepsilon t}|^2\right)d\Gamma \\
&\leq -\int_\Omega\left(\frac{\rho h^3}{12}(|\phi_{\varepsilon t}|^2 + |\psi_{\varepsilon t}|^2) + \rho h|w_{\varepsilon t}|^2\right)dx - C\frac{d}{dt}\mathbb{E}_\varepsilon(t),
\end{aligned} \tag{3.65}$$

with $C > 0$ independent of ε. Then

$$G_1(t) + G_2(t) \leq -\theta\frac{\rho h^3}{12}(|\phi_{\varepsilon t}|^2 + |\psi_{\varepsilon t}|^2) - (1-\theta)\rho h|w_{\varepsilon t}|^2 - C\frac{d}{dt}\mathbb{E}_\varepsilon(t). \tag{3.66}$$

Now we deal with the terms $F_1(t)$ and $F_2(t)$. First, we observe that

$$F_1(t) = (1-\theta)\,(L_1(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon),\phi_\varepsilon) + (1-\theta)\,(L_2(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon),\psi_\varepsilon) + \theta\,(L_3(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon),w_\varepsilon)$$
$$= -(1-\theta)a_0(\phi_\varepsilon,\psi_\varepsilon) - \theta k a_1(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon) + (2\theta-1)ka_1(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon,\phi_\varepsilon,\psi_\varepsilon,0)$$
$$-(1-\theta)\gamma_1\int_{\Gamma_1}\phi_{\varepsilon t}\phi_\varepsilon d\Gamma - (1-\theta)\gamma_2\int_{\Gamma_1}\psi_{\varepsilon t}\psi_\varepsilon d\Gamma - \theta\gamma_3\int_{\Gamma_1} w_{\varepsilon t}w_\varepsilon d\Gamma + \frac{\theta}{\varepsilon}\int_{\Gamma_1}(w_\varepsilon - g)^- w_\varepsilon d\Gamma. \tag{3.67}$$

Let us examine the term in the right side of (3.67). By (3.11), trace theory and, again, (3.26), we obtain

$$\begin{aligned}
&-(1-\theta)\gamma_1\int_{\Gamma_1}\phi_{\varepsilon t}\phi_\varepsilon d\Gamma - (1-\theta)\gamma_2\int_{\Gamma_1}\psi_{\varepsilon t}\psi_\varepsilon d\Gamma - \theta\gamma_3\int_{\Gamma_1} w_{\varepsilon t}w_\varepsilon d\Gamma\\
&\quad\le \beta\int_{\Gamma_1}(|\phi_\varepsilon|^2+|\psi_\varepsilon|^2+|w_\varepsilon|^2)d\Gamma + C_\beta\int_{\Gamma_1}(\gamma_1|\phi_{\varepsilon t}|^2+\gamma_2|\psi_{\varepsilon t}|^2+\gamma_3|w_{\varepsilon t}|^2)\\
&\quad\le \beta C[a_0(\phi_\varepsilon,\psi_\varepsilon)+ka_1(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon)] - C\frac{d}{dt}\mathbb{E}_\varepsilon(t),
\end{aligned}\tag{3.68}$$

for a constant $\beta > 0$ to be chosen later. We have that the assumption $g \le 0$ assures us that

$$\begin{aligned}
\frac{\theta}{\varepsilon}\int_{\Gamma_1}(w_\varepsilon-g)^- w_\varepsilon d\Gamma &= -\frac{\theta}{\varepsilon}\int_{\Gamma_1}|(w_\varepsilon-g)^-|^2 d\Gamma + \frac{\theta}{\varepsilon}\int_{\Gamma_1}(w_\varepsilon-g)^- g d\Gamma\\
&\le -\frac{\theta}{\varepsilon}\int_{\Gamma_1}|(w_\varepsilon-g)^-|^2 d\Gamma.
\end{aligned}\tag{3.69}$$

From (3.25) and Poincaré inequality, we get

$$\begin{aligned}
2\theta k a_1(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon,\phi_\varepsilon,\psi_\varepsilon,0) &= 2\theta k\int_\Omega[(w_{\varepsilon x_1}+\phi_\varepsilon)\phi_\varepsilon + (w_{\varepsilon x_2}+\psi_\varepsilon)\psi_\varepsilon]\\
&\le C\theta a_0(\phi_\varepsilon,\psi_\varepsilon) + \frac{1}{4}\theta k a_1(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon).
\end{aligned}\tag{3.70}$$

Substituting (3.68)–(3.70) into (3.67), we obtain

$$\begin{aligned}
F_1(t) &\le [-(1-\theta)+\beta C + C\theta]a_0(\phi_\varepsilon,\psi_\varepsilon) + \left[-\theta+\beta C+\frac{1}{4}\theta\right]ka_1(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon) - C\frac{d}{dt}\mathbb{E}_\varepsilon(t)\\
&\quad -\frac{\theta}{\varepsilon}\int_{\Gamma_1}|(w_\varepsilon-g)^-|^2 d\Gamma - ka_1(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon,\phi_\varepsilon,\psi_\varepsilon,0).
\end{aligned}\tag{3.71}$$

Thus, choosing $\beta < \frac{1}{16c^2+8C}$ and fixing θ such that $4\beta C < \theta < \frac{1-4\beta C}{1+4C}$, we have by (3.71) that

$$\begin{aligned}
F_1(t) &\le -\frac{3}{4}(1-\theta)a_0(\phi_\varepsilon,\psi_\varepsilon) - \frac{1}{2}\theta k a_1(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon) - C\frac{d}{dt}\mathbb{E}_\varepsilon(t)\\
&\quad -\frac{\theta}{\varepsilon}\int_{\Gamma_1}|(w_\varepsilon-g)^-|^2 d\Gamma - ka_1(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon,\phi_\varepsilon,\psi_\varepsilon,0).
\end{aligned}\tag{3.72}$$

Notice that

$$\begin{aligned}
F_2(t) &= (L_1(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon), m\cdot\nabla\phi_\varepsilon) + (L_2(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon), m\cdot\nabla\psi_\varepsilon) + (L_3(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon), m\cdot\nabla w_\varepsilon)\\
&= -a_0(\phi_\varepsilon,\psi_\varepsilon,m\cdot\nabla\phi_\varepsilon,m\cdot\nabla\psi_\varepsilon) - ka_1(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon,m\cdot\nabla\phi_\varepsilon,m\cdot\nabla\psi_\varepsilon,m\cdot\nabla w_\varepsilon)\\
&\quad + B_{\Gamma_0}(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon) + B_{\Gamma_1}(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon),
\end{aligned}\tag{3.73}$$

where

$$B_{\Gamma_j}(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon)=\int_{\Gamma_j}\mathcal{B}_1(\phi_\varepsilon,\psi_\varepsilon)m\cdot\nabla\phi_\varepsilon d\Gamma+\int_{\Gamma_j}\mathcal{B}_2(\phi_\varepsilon,\psi_\varepsilon)m\cdot\nabla\psi_\varepsilon d\Gamma+\int_{\Gamma_j}\mathcal{B}_3(\phi_\varepsilon,\psi_\varepsilon)m\cdot\nabla w_\varepsilon d\Gamma,$$

for $j=0,1$. Let us analyze the four terms in the right side of (3.73). For the first term we have

$$\begin{aligned}
&-a_0(\phi_\varepsilon,\psi_\varepsilon,m\cdot\nabla\phi_\varepsilon,m\cdot\nabla\psi_\varepsilon)\\
&\quad=-D\int_\Omega\Big[\phi_{\varepsilon x_1}(m\cdot\nabla\phi_\varepsilon)_{x_1}+\psi_{\varepsilon x_2}(m\cdot\nabla\psi_\varepsilon)_{x_2}+\mu(\phi_{\varepsilon x_1}(m\cdot\nabla\psi_\varepsilon)_{x_2}\\
&\qquad\quad+(m\cdot\nabla\phi_\varepsilon)_{x_1}\psi_{\varepsilon x_2})+\frac{1-\mu}{2}(\phi_{\varepsilon x_2}+\psi_{\varepsilon x_1})((m\cdot\nabla\phi_\varepsilon)_{x_2}+(m\cdot\nabla\psi_\varepsilon)_{x_1})\Big]\,dx,\\
&\quad=-\frac{1}{2}D\int_\Omega\operatorname{div}\left[m\left(|\phi_{\varepsilon x_1}|^2+|\psi_{\varepsilon x_2}|^2+2\mu\phi_{\varepsilon x_1}\psi_{\varepsilon x_2}+\frac{1-\mu}{2}|\phi_{\varepsilon x_2}+\psi_{\varepsilon x_1}|^2\right)\right]dx\\
&\quad=-\frac{1}{2}a_{0,\Gamma_0}(\phi_\varepsilon,\psi_\varepsilon)-\frac{1}{2}a_{0,\Gamma_1}(\phi_\varepsilon,\psi_\varepsilon),
\end{aligned}\tag{3.74}$$

where

$$a_{0,\Gamma_j}(\phi_\varepsilon,\psi_\varepsilon)=D\int_{\Gamma_j}m\cdot\nu\left(|\phi_{\varepsilon x_1}|^2+|\psi_{\varepsilon x_2}|^2+2\mu\phi_{\varepsilon x_1}\psi_{\varepsilon x_2}+\frac{1-\mu}{2}|\phi_{\varepsilon x_2}+\psi_{\varepsilon x_1}|^2\right)d\Gamma,\quad j=1,2.$$

For the second one, we use (3.26) to achieve

$$\begin{aligned}
&-ka_1(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon,m\cdot\nabla\phi_\varepsilon,m\cdot\nabla\psi_\varepsilon,m\cdot\nabla w_\varepsilon)\\
&=-k\int_\Omega(w_{\varepsilon x_1}+\phi_\varepsilon)[(m\cdot\nabla w_\varepsilon)_{x_1}+m\cdot\nabla\phi_\varepsilon]dx-k\int_\Omega(w_{\varepsilon x_2}+\psi_\varepsilon)[(m\cdot\nabla w_\varepsilon)_{x_2}+m\cdot\nabla\psi_\varepsilon]dx\\
&=-\frac{1}{2}k\int_\Omega m\cdot\nabla\left(|w_{\varepsilon x_1}+\phi_\varepsilon|^2+|w_{\varepsilon x_2}+\psi_\varepsilon|^2\right)dx-k\int_\Omega[(w_{\varepsilon x_1}+\phi_\varepsilon)w_{\varepsilon x_1}+(w_{\varepsilon x_2}+\psi_\varepsilon)w_{\varepsilon x_2}]\\
&=-\frac{1}{2}k\,a_{1,\Gamma_0}(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon)-\frac{1}{2}k\,a_{1,\Gamma_1}(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon)+ka_1(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon,\phi_\varepsilon,\psi_\varepsilon,0).
\end{aligned}\tag{3.75}$$

where

$$a_{1,\Gamma_j}(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon)=\int_{\Gamma_j}m\cdot\nu\left(|w_{\varepsilon x_1}+\phi_\varepsilon|^2+|w_{\varepsilon x_2}+\psi_\varepsilon|^2\right)d\Gamma,\quad j=1,2.$$

To obtain an estimative for the third term, we observe that, since $\phi_\varepsilon=\psi_\varepsilon=w_\varepsilon=0$ on Γ_0, then $(\phi_{\varepsilon x_i},\psi_{\varepsilon x_i},w_{\varepsilon x_i})=(\frac{\partial\phi_\varepsilon}{\partial\nu}\nu_i,\frac{\partial\psi_\varepsilon}{\partial\nu}\nu_i,\frac{\partial w_\varepsilon}{\partial\nu}\nu_i)$ on Γ_0, for $i=1,2$. Thus,

$$\begin{aligned}
&B_{\Gamma_0}(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon)\\
&\ =D\int_{\Gamma_0}\left[\nu_1\phi_{\varepsilon x_1}+\mu\nu_1\psi_{\varepsilon x_2}+\frac{1-\mu}{2}\left(\phi_{\varepsilon x_2}+\psi_{\varepsilon x_1}\right)\nu_2\right]m\cdot\nabla\phi_\varepsilon d\Gamma\\
&\quad+D\int_{\Gamma_0}\left[\nu_2\psi_{\varepsilon x_2}+\mu\nu_2\phi_{\varepsilon x_1}+\frac{1-\mu}{2}\left(\phi_{\varepsilon x_2}+\psi_{\varepsilon x_1}\right)\nu_1\right]m\cdot\nabla\psi_\varepsilon d\Gamma+k\int_{\Gamma_0}\frac{\partial w}{\partial\nu}m\cdot\nabla w_\varepsilon d\Gamma\\
&\ =a_{0,\Gamma_0}(\phi_\varepsilon,\psi_\varepsilon)+ka_{1,\Gamma_0}(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon).
\end{aligned}\tag{3.76}$$

Now, let us estimate the fourth term. We have

$$\begin{aligned} B_{\Gamma_1}(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon) &= -\gamma_1\int_{\Gamma_1}\phi_{\varepsilon t}m\cdot\nabla\phi_\varepsilon d\Gamma-\gamma_2\int_{\Gamma_1}\psi_{\varepsilon t}m\cdot\nabla\psi_\varepsilon d\Gamma-\gamma_3\int_{\Gamma_1}w_{\varepsilon t}m\cdot\nabla\psi_\varepsilon d\Gamma \\ &\le \frac{1}{2r_0}\int_{\Gamma_1}\left(|\nabla\phi_\varepsilon|^2+|\nabla\psi_\varepsilon|^2+|\nabla w_\varepsilon|^2\right)d\Gamma \\ &\quad +C_{r_0}\int_{\Gamma_1}\left(\gamma_1|\phi_{\varepsilon t}|^2+\gamma_2|\psi_{\varepsilon t}|^2+\gamma_3|w_{\varepsilon t}|^2\right)d\Gamma, \end{aligned} \tag{3.77}$$

for some $r_0>0$ to be chosen later. According to [65, p. 50], we find a constant $r_1>0$ such that

$$\begin{aligned} &\int_{\Gamma_1}\left(|\nabla\phi_\varepsilon|^2+|\nabla\psi_\varepsilon|^2+|\nabla w_\varepsilon|^2\right)d\Gamma \\ &\le r_1\int_{\Gamma_1}\left(a_{0,\Gamma_1}(\phi_\varepsilon,\psi_\varepsilon)+ka_{1,\Gamma_1}(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon)+\frac{1-\theta}{2}a_0(\phi_\varepsilon,\psi_\varepsilon)\right)d\Gamma. \end{aligned} \tag{3.78}$$

In this way, combining (3.11), (3.77), (3.78), and choosing $r_0=r_1$, we obtain

$$B_{\Gamma_1}(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon)\le\frac{1}{2}a_{0,\Gamma_1}(\phi_\varepsilon,\psi_\varepsilon)+\frac{1}{2}ka_{1,\Gamma_1}(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon)+\frac{1}{4}(1-\theta)a_0(\phi_\varepsilon,\psi_\varepsilon)-C\frac{d}{dt}\mathbb{E}_\varepsilon(t). \tag{3.79}$$

Now, substituting (3.74), (3.75), (3.76), (3.79) in (3.73) we obtain, since $m\cdot\nu\le 0$ in Γ_0, that

$$F_2(t)\le ka_1(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon,\phi_\varepsilon,\psi_\varepsilon,0)+\frac{1}{4}(1-\theta)a_0(\phi_\varepsilon,\psi_\varepsilon)-C\frac{d}{dt}\mathbb{E}_\varepsilon(t), \tag{3.80}$$

where $C>0$ is independent of ε. Hence, from (3.66), (3.72), and (3.80), and since $\theta\in(0,1)$, we obtain a constant $C_0>0$ such that

$$\begin{aligned} \frac{d}{dt}G(t) &\le -\theta\frac{\rho h^3}{12}(|\phi_{\varepsilon t}|^2+|\psi_{\varepsilon t}|^2)-(1-\theta)\rho h|w_{\varepsilon t}|^2-C_0\frac{d}{dt}\mathbb{E}_\varepsilon(t) \\ &\quad -\frac{1}{2}[(1-\theta)a_0(\phi_\varepsilon,\psi_\varepsilon)+\theta ka_1(\phi_\varepsilon,\psi_\varepsilon,w_\varepsilon)]-\frac{\theta}{\varepsilon}\int_{\Gamma_1}|(w_\varepsilon-g)^-|^2d\Gamma \\ &\le -\min\{\theta,1-\theta\}\,\mathbb{E}_\varepsilon(t)-C_0\frac{d}{dt}\mathbb{E}_\varepsilon(t). \end{aligned} \tag{3.81}$$

This proves the second inequality in (3.60) with $\mathbf{C}=\max\{\frac{1}{\theta},\frac{1}{1-\theta},C_0\}>0$. ■

Remark 11. *It is important to assume that the measure of Γ_0 is positive, because otherwise one cannot assure that the energy decays to zero for every finite energy solution of (3.10). Indeed, let us consider the triplet (ϕ_1,ψ_1,w_1) given by*

$$(\phi_1,\psi_1,w_1)=(a,b,-ax_1-bx_2+c),$$

for nonzero real constants a, b, c. We can see that $a_0(\phi_1,\psi_1)=a_1(\phi_1,\psi_1,w_1)=0$ and

$$\begin{cases} L_1(\phi_1,\psi_1,w_1)=L_2(\phi_1,\psi_1,w_1)=L_3(\phi_1,\psi_1,w_1)=0 & \text{in} \quad \Omega, \\ \mathcal{B}_1(\phi_1,\psi_1)=\mathcal{B}_2(\phi_1,\psi_1)=\mathcal{B}_3(\phi_1,\psi_1,w_1)=0 & \text{on} \quad \Gamma_1. \end{cases}$$

Now, we consider the triplet (ϕ_0, ψ_0, w_0)*, independent of the time variable* t*, such that*

$$\begin{cases} w_0 = w_1 + g, \\ L_1(\phi_0, \psi_0, w_0) = L_2(\phi_0, \psi_0, w_0) = L_3(\phi_0, \psi_0, w_0) = 0 & in \quad \Omega, \\ \mathcal{B}_1(\phi_0, \psi_0) = -\gamma_1 \phi_1, \ \mathcal{B}_2(\phi_0, \psi_0) = -\gamma_2 \psi_1, \ \mathcal{B}_3(\phi_0, \psi_0, w_0) = -\gamma_3 w_1 & on \quad \Gamma_1, \end{cases}$$

with $g \in L^2(\Gamma_1)$*. Thus, if we choose the constant* $c > 0$ *large enough such that* $w_1 \geq 0$*, it is not difficult to verify that*

$$(\phi, \psi, w) = (\phi_1, \psi_1, w_1)t + (\phi_0, \psi_0, w_0)$$

is a solution of system (3.10)*. However, the energy associated to such solution satisfies*

$$\mathbb{E}_\varepsilon(t) = \frac{1}{2}\left[\frac{\rho h^3}{12}\left(|\phi_1|^2 + |\psi_1|^2\right) + \rho h|w_1|^2 + a_0(\phi_0, \psi_0) + k a_1(\phi_0, \psi_0, w_1 + g)\right] = const. > 0.$$

Remark 12. *When* $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 \neq \emptyset$*, we expected similar results if, at least, we consider the geometric condition suggested in Remark 6 and damping terms* $\{\gamma_1(m\cdot\nu)\phi_t, \gamma_2(m\cdot\nu)\psi_t, \gamma_3(m\cdot\nu)w_t\}$ *instead of* $\{\gamma_1\phi_t, \gamma_2\psi_t, \gamma_3 w_t\}$*. In this case we must proceed as in Lagnese [65]. In this issue and in the context of the wave equation, we can cite Komornik-Zuazua [62]. In these works, the authors used results obtained by Grisvard in [51, 52].*

Remark 13. *Following the calculations presented in this work it is not difficult to observe that a some nonlinear dampings can be chosen in system* (3.12)*. In fact, it is a bit straightforward to give suitable conditions for a nonlinear triplet* $\{g_1(\phi_t), g_2(\psi_t), g_3(w_t)\}$ *of dampings in order to have all the Theorems in this work still holding true. Indeed, if we consider dampings* $\{g_1(\phi_t), g_2(\psi_t), g_3(w_t)\}$ *instead of* $\{\gamma_1\phi_t, \gamma_2\psi_t, \gamma_3 w_t\}$ *in* (3.12)*, then the key point to prove the theorems becomes to assure that the estimates* (3.18)*,*(3.20)*,* (3.60)*, and a convergence similar to* (3.53)*, hold together. For this sake it is enough to choose differentiable functions* $g_1, g_2, g_3 : \mathbb{R} \to \mathbb{R}$ *satisfying*

- $g_j(s)s \geq Cs^2, \quad \forall s \in \mathbb{R}, \quad j = 1, 2, 3.$
- $C_1 \leq g_j'(s) \leq C_2, \quad \forall s \in \mathbb{R}, \quad j = 1, 2, 3,$

for positive constants C, C_1, C_2*. For example, any multiple of the function* $g_j(s) = s\, e^{-\frac{1}{1+s^2}}$ *satisfies the conditions above.*

3.5 Further comments and open problems

1. The lack of uniqueness is a particular characteristic of contact problems. Thus, although the Mindlin-Timoshenko system having a unique solution when are considered Dirichlet or Neumann type boundary conditions (see e.g. [66]), the uniqueness of solution for (3.12) is a open question.

2. It would be interesting to analyze whether the same stabilization results (Theorem 6 and its consequence (3.58)) hold considering the systems (3.10) and (3.12) with less damping terms. To eliminate some of these dissipative terms is, in general, a difficult task, especially when they act on the boundary. In this context, we can mention the works [2, 4, 5, 12, 26, 44, 84] which have obtained stability for some hyperbolic systems with less damping than equations. However, it is important to emphasize that in all of them are considered internal dissipations.

3. Another interesting and difficult problem is to obtain the same result in Theorem 6 when the damping mechanisms act in an arbitrary small region of the plate. The difficulty for this case, of course, consists in getting a unique continuation result for the Mindlin-Timoshenko system. On this subject, we mention [27, 29, 49, 87, 91] which have obtained decay rates for the energy of various hyperbolic systems considering both linear and nonlinear localized damping terms.

Capítulo 4

Boundary controllability of a one-dimensional phase-field system with one control force

Boundary controllability of a one-dimensional phase-field system with one control force

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Abstract. In this paper, we present some controllability results for linear and nonlinear phase-field systems of Caginalp type considered in a bounded interval of $\mathbf{R}$ when the scalar control force acts on the temperature equation of the system by means of the Dirichlet condition on one of the endpoints of the interval. For proving the linear result we use the moment method providing an estimate of the cost of fast controls. Using this estimate and following the methodology developed in [63], we prove a local exact boundary controllability result to constant trajectories of the nonlinear phase-field system. To the authors' knowledge, this is the first nonlinear boundary controllability result in the framework of non-scalar parabolic systems, framework in which some "hyperbolic" behaviors could arise.

4.1 Introduction

This work deals with the boundary controllability properties of a phase-field system of Caginalp type (see [25]) which is a model describing the transition between the solid and liquid phases in solidification/melting processes of a material occupying an interval:

$$\begin{cases} \tilde\theta_t - \xi\tilde\theta_{xx} + \dfrac{1}{2}\rho\xi\tilde\phi_{xx} + \dfrac{\rho}{\tau}\tilde\theta = f_1(\tilde\phi) & \text{in } Q_T := (0,\pi)\times(0,T), \\ \tilde\phi_t - \xi\tilde\phi_{xx} - \dfrac{2}{\tau}\tilde\theta = f_2(\tilde\phi) & \text{in } Q_T, \\ \tilde\theta(0,\cdot) = v,\ \tilde\phi(0,\cdot) = c,\ \tilde\theta(\pi,\cdot) = 0,\ \tilde\phi(\pi,\cdot) = c & \text{on } (0,T), \\ \tilde\theta(\cdot,0) = \tilde\theta_0,\ \tilde\phi(\cdot,0) = \tilde\phi_0 & \text{in } (0,\pi). \end{cases} \tag{4.1}$$

Here, $T > 0$ is some final time, $\tilde\theta = \tilde\theta(x,t)$ denotes the temperature of the material, $\tilde\phi = \tilde\phi(x,t)$ is the phase-field function used to identify the solidification level of the material, $c \in \{-1, 0, 1\}$ and the functions f_1 and f_2 are the nonlinear terms which come from the derivative of the classical regular double-well potential W and are defined by

$$f_1(\tilde\phi) = -\frac{\rho}{4\tau}\left(\tilde\phi - \tilde\phi^3\right) \quad \text{and} \quad f_2(\tilde\phi) = \frac{1}{2\tau}\left(\tilde\phi - \tilde\phi^3\right).$$

Besides, $\rho > 0$ is the latent heat, $\tau > 0$ represents the relaxation time and $\xi > 0$ is the thermal diffusivity. Finally, $v \in L^2(0,T)$ is the control force, which is exerted at point $x = 0$ by means of the boundary Dirichlet condition, and the initial data $\tilde\theta_0$, $\tilde\phi_0$ are given functions.

The phase function $\tilde{\phi}$ describes the phase transition of the material (solid or liquid) in such a way that $\tilde{\phi} = 1$ means that the material is in solid state and $\tilde{\phi} = -1$ in liquid state. Observe that the temperature $\tilde{\theta}$ of the material could be zero and this could occur with the material in solid or liquid phase. On the other hand, the phase-field variable $\tilde{\phi}$ does not have a direct physical meaning. This is the reason we control the temperature $\tilde{\theta}$ which, in fact, is the unique variable with physical meaning.

The objective of this paper is to prove a null controllability result at time T for the temperature variable $\tilde{\theta}$ of system (4.1). If we consider the transition region associated to the temperature, i.e., the set

$$\Gamma(t) := \left\{ x \in (0,\pi) : \tilde{\theta}(x,t) = 0 \right\},$$

then, the problem under consideration consists of proving that there exists a control v such that the transition region associated to the temperature $\tilde{\theta}$ satisfies $\Gamma(T) = (0,\pi)$. It is interesting to underline that in this case the material could be in solid phase ($\tilde{\phi}(\cdot,T) = 1$), liquid phase ($\tilde{\phi}(\cdot,T) = -1$) or in an intermediate phase (mushy) which corresponds to $\tilde{\phi}(\cdot,T) = 0$. In this work we are interested in showing the null controllability result at time T for the temperature $\tilde{\theta}$ but keeping the material in solid state, $c = 1$, or liquid state, $c = -1$, at time T, that is to say, proving that there exists a control $v \in L^2(0,T)$ such that system (4.1) has a solution $\tilde{y} = (\tilde{\theta}, \tilde{\phi})$ (in an appropriate space) such that

$$\tilde{\theta}(\cdot,T) = 0 \quad \text{and} \quad \tilde{\phi}(\cdot,T) = c \quad \text{in } (0,\pi). \tag{4.2}$$

We give a complementary analysis and results in the Appendix 4.6.2, where we deal with the case where $c = 0$.

As said before, the objective of this work is to study the controllability properties of system (4.1). Let us observe that we are exerting only one control force on the system (a boundary control) but we want to control the corresponding state $\tilde{y} = (\tilde{\theta}, \tilde{\phi})$ which has two components. In fact, the second equation in (4.1) is indirectly controlled by means of the term $-2\tilde{\theta}/\tau$. On the other hand, (4.1) is a nonlinear system with nonlinearities with a super-linear behavior at infinity. Therefore, we can expect a local controllability result at time T for this system, that is to say, an exact controllability result to the trajectory $(0,c)$ when the initial datum $(\tilde{\theta}_0, \tilde{\phi}_0)$ is sufficiently close to $(0,c)$ in an appropriate norm (see for instance [43, 34] for similar results in the scalar parabolic framework).

System (4.1) is a particular class of more general $n \times n$ nonlinear parabolic control systems of the form:

$$\begin{cases} y_t - D\Delta y + Ay = F(y) + Bv1_\omega & \text{in } Q_T := \Omega \times (0,T), \\ y = Cu1_{\Gamma_0}, & \text{on } \Sigma_T := \partial\Omega \times (0,T), \\ y(\cdot,0) = y_0 & \text{in } \Omega, \end{cases} \tag{4.3}$$

where ω and Γ_0 are, respectively, open subsets of the smooth bounded domain $\Omega \subset \mathbf{R}^N$ and of its boundary $\partial\Omega$, $D \in \mathcal{L}(\mathbf{R}^n)$, with $n \geq 1$, is a positive definite matrix, $B, C \in \mathcal{L}(\mathbf{R}^m, \mathbf{R}^n)$, with $m \leq n$, and $A = (a_{ij})_{1 \leq i,j \leq n} \in \mathcal{L}(\mathbf{R}^n)$ are given matrices. In (4.3), $F \in C^0(\mathbf{R}^n; \mathbf{R}^n)$ is a nonlinear given function. Unlike the scalar case, even in the linear case $F \equiv 0$, new difficulties arise in the study of the controllability properties of (4.3). When $m < n$, the issue for this system is to control the whole components of the system with a control function acting, locally in space or on a part of the boundary, only on some of them. We refer to [7] for a review of results for the controllability problem of system (4.3).

The controllability properties of system (4.1) has been analyzed before in the N-dimensional case ($N \geq 1$) when a distributed control supported in an open subset of the domain is exerted on the system. The first local controllability results for a nonlinear phase-field system controlled by one distributed control force are proved in [6] under certain restrictions on the dimension N. In [50], the authors introduce a new approach to deal with the distributed null controllability of some linear coupled parabolic systems that makes possible to generalize the results in [6] to more general phase-field systems such as (4.1). Finally, in [42] the authors study the controllability of some (linear and semilinear) non-diagonalizable parabolic systems of PDEs and provide some Kalman rank conditions which characterize the controllability properties in the linear case. In these previous works the null controllability result for the linear and nonlinear problem uses in a fundamental way global Carleman inequalities for scalar parabolic problems. To our knowledge, this is the first time that the boundary controllability properties of a nonlinear phase-field system are analyzed.

It is important to underline that in this work we are considering a boundary controllability problem for a non-scalar parabolic system. As said before, in the study of these boundary controllability problems new phenomena and technical difficulties arise. Let us briefly describe them. In the linear case ($F \equiv 0$), it is well-known (see [41, 8, 9]) that the distributed ($C \equiv 0$) and boundary ($B \equiv 0$) controllability properties of system (4.3) are different and not equivalent. In fact, the boundary controllability of system (4.3) could present “hyperbolic"behaviors such as the non-equivalence between the approximate and null controllability or the existence of a minimal time of controllability, i.e., the existence of $T_0 \in [0, \infty]$ such that the system is null controllable at time T if $T > T_0$ and it is not if $T < T_0$ (see [9], [10] and the references therein for more details). On the other hand, global Carleman inequalities seem not to be too useful when we deal with boundary controllability properties of non-scalar parabolic systems (see [8]) and this creates a new technical difficulty: we want to obtain a nonlinear boundary controllability result without having global Carleman estimates for the corresponding adjoint systems to linearized versions of system (4.1).

As noted above, the decision of exerting the control force on the temperature variable $\tilde{\theta}$ was taken because it is the only variable in (4.1) with physical meaning. Indeed, the values of $\tilde{\phi}$ determine the material phase and, consequently, imposing a boundary control for $\tilde{\phi}$ would mean that specific phases for the material on the boundary are maintained throughout the solidification process (which is not an usual situation in practice). On the other hand, exerting the control on the boundary in the temperature variable can be seen as having a regulable external source which heats/cools down the material at point $x = 0$. From the physical point of view, this boundary control is more interesting than a distributed control supported on an open subset of the domain (internal source).

The main objective of this work is to provide an exact controllability result of system (4.1) to the constant trajectory $(0, c)$ with $c = \pm 1$ (the case $c = 0$ follows the same structure of the case $c = \pm 1$, so it will be dealt with in Appendix 4.6.2). Observe that the nonlinearities f_1

and f_2 in (4.1) can be written as

$$\begin{cases} f_1(\phi) = -\dfrac{\rho}{4\tau}(\phi - \phi^3) = \dfrac{\rho}{2\tau}(\phi - c) \pm \dfrac{3\rho}{4\tau}(\phi - c)^2 + \dfrac{\rho}{4\tau}(\phi - c)^3, \\ f_2(\phi) = \dfrac{1}{2\tau}(\phi - \phi^3) = -\dfrac{1}{\tau}(\phi - c) \mp \dfrac{3}{2\tau}(\phi - c)^2 - \dfrac{1}{2\tau}(\phi - c)^3. \end{cases}$$

and therefore, performing the change of variable $(\theta, \phi) = (\tilde{\theta}, \tilde{\phi} - c)$, system (4.1) becomes

$$\begin{cases} \theta_t - \xi\theta_{xx} + \dfrac{1}{2}\rho\xi\phi_{xx} - \dfrac{\rho}{2\tau}\phi + \dfrac{\rho}{\tau}\theta = g_1(\phi) & \text{in } Q_T, \\ \phi_t - \xi\phi_{xx} + \dfrac{1}{\tau}\phi - \dfrac{2}{\tau}\theta = g_2(\phi) & \text{in } Q_T, \\ \theta(0, \cdot) = v, \quad \phi(0, \cdot) = \theta(\pi, \cdot) = \phi(\pi, \cdot) = 0 & \text{on } (0, T), \\ \theta(\cdot, 0) = \theta_0, \quad \phi(\cdot, 0) = \phi_0 & \text{in } (0, \pi), \end{cases} \tag{4.4}$$

where $(\theta_0, \phi_0) = (\tilde{\theta}_0, \tilde{\phi}_0 - c)$ and the functions g_1 and g_2 are given by

$$g_1(\phi) = \pm\frac{3\rho}{4\tau}\phi^2 + \frac{\rho}{4\tau}\phi^3 \quad \text{and} \quad g_2(\phi) = \mp\frac{3}{2\tau}\phi^2 - \frac{1}{2\tau}\phi^3. \tag{4.5}$$

With the previous change of variables in mind, the exact controllability to the trajectory $(0, c)$ of system (4.1) at time $T > 0$ is equivalent to the null controllability at the same time T of system (4.4). In order to prove the null controllability at time $T > 0$ of system (4.4), we will rewrite the controllability problem as a fixed-point problem for a convenient operator in appropriate spaces. To perform this fixed-point strategy, we will first study the controllability properties of the following autonomous linear system:

$$\begin{cases} \theta_t - \xi\theta_{xx} + \dfrac{1}{2}\rho\xi\phi_{xx} - \dfrac{\rho}{2\tau}\phi + \dfrac{\rho}{\tau}\theta = 0 & \text{in } Q_T, \\ \phi_t - \xi\phi_{xx} + \dfrac{1}{\tau}\phi - \dfrac{2}{\tau}\theta = 0 & \text{in } Q_T, \\ \theta(0, \cdot) = v, \quad \phi(0, \cdot) = \theta(\pi, \cdot) = \phi(\pi, \cdot) = 0 & \text{on } (0, T), \\ \theta(\cdot, 0) = \theta_0, \quad \phi(\cdot, 0) = \phi_0 & \text{in } (0, \pi), \end{cases} \tag{4.6}$$

which is a linearization of system (4.4) around the equilibrium $(0, 0)$. System (4.6) can also be written in the vectorial form

$$\begin{cases} y_t - Dy_{xx} + Ay = f & \text{in } Q_T, \\ y(0, \cdot) = Bv, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y_0, & \text{in } (0, \pi), \end{cases} \tag{4.7}$$

with $y_0 = (\theta_0, \phi_0)$, $f = (0, 0)$ and

$$D = \begin{pmatrix} \xi & -\dfrac{1}{2}\rho\xi \\ 0 & \xi \end{pmatrix}, \quad A = \begin{pmatrix} \dfrac{\rho}{\tau} & -\dfrac{\rho}{2\tau} \\ -\dfrac{2}{\tau} & \dfrac{1}{\tau} \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{4.8}$$

We will see that, for every $v \in L^2(0, T)$, $f \in L^2(Q_T; \mathbf{R}^2)$ and $y_0 \in H^{-1}(0, \pi; \mathbf{R}^2)$, system (4.7) possesses a unique solution defined by transposition which satisfies

$$y \in L^2(Q_T; \mathbf{R}^2) \cap C^0\left([0, T]; H^{-1}(0, \pi; \mathbf{R}^2)\right),$$

and depends continuously on the data v, f and y_0. Observe that the previous regularity permits to pose the boundary controllability of system (4.6) in the space $H^{-1}(0,\pi;\mathbf{R}^2)$.

Let us present our first main result: the boundary approximate controllability at time $T>0$ of system (4.6). One has:

Theorem 7. *Let us consider ξ, ρ and τ three positive real numbers and let us fix $T>0$. Then, system (4.6) is approximately controllable in $H^{-1}(0,\pi;\mathbf{R}^2)$ at time T if and only if one has*

$$\xi^2\tau^2(\ell^2-k^2)^2-2\xi\rho\tau(\ell^2+k^2)-2\rho-1\neq 0, \quad \forall k,\ell\geq 1, \quad \ell>k. \tag{4.9}$$

Remark 14. *Condition (4.9) characterizes the approximate controllability property of system (4.6). Thus, (4.9) is a necessary condition for the null controllability of this system at time $T>0$. Observe that this condition is independent of the final time T. We will also see that condition (4.9) is equivalent to the following property (see Proposition 8): "The eigenvalues of the vectorial operators*

$$L=-D\partial_{xx}+A \quad \textit{and} \quad L^*=-D^*\partial_{xx}+A^*, \tag{4.10}$$

with domains $D(L)=D(L^*)=H^2(0,\pi;\mathbb{R}^2)\cap H_0^1(0,\pi;\mathbb{R}^2)$, have geometric multiplicity equal to one". Thus, condition (4.9) is a Fattorini-Hautus criterium for the boundary approximate controllability of the linear system (4.6) (see [38]).

In this work, we will also analyze the null controllability properties of system (4.6). In this sense, one has:

Theorem 8. *Let us us fix $T>0$ and consider ξ, ρ and τ, positive real numbers satisfying (4.9) and*

$$\xi\neq\frac{1}{j^2}\frac{\rho}{\tau}, \quad \forall j\geq 1. \tag{4.11}$$

Then, system (4.6) is exactly controllable to zero in $H^{-1}(0,\pi;\mathbf{R}^2)$ at time $T>0$. Moreover, there exist two positive constants C_0 and M, only depending on ξ, ρ and τ, such that for any $T>0$, there is a bounded linear operator

$$\mathcal{C}_T^{(0)}: H^{-1}(0,\pi;\mathbb{R}^2)\to L^2(0,T)$$

satisfying

$$\|\mathcal{C}_T^{(0)}\|_{\mathcal{L}(H^{-1}(0,\pi;\mathbb{R}^2),L^2(0,T))}\leq C_0\, e^{M/T}, \tag{4.12}$$

and such that the solution

$$y=(\theta,\phi)\in L^2(Q_T;\mathbf{R}^2)\cap C^0([0,T];H^{-1}(0,\pi;\mathbb{R}^2))$$

of system (4.6) associated to $y_0=(\theta_0,\phi_0)\in H^{-1}(0,\pi;\mathbb{R}^2)$ and $v=\mathcal{C}_T^{(0)}(y_0)$ satisfies $y(\cdot,T)=0$.

Remark 15. *From the results stated in [6] and [50], it is well known that the linear system (4.6) is approximate and null controllable at any time $T>0$ and any positive ξ, ρ and τ, when the scalar control $v\in L^2(Q_T)$ acts on the temperature equation of (4.1) as a right-hand* *side source supported on an open subset ω of the domain. These distributed controllability results are independent of condition (4.9) and only use the cascade structure of system (4.6). Nevertheless, this cascade structure is not enough when one deals with the boundary controllability of non-scalar problems (see for example [41], [8], [9], …). Again, the approximate and null controllability results stated in Theorems 7 and 8 show the different nature of the controllability problem of scalar or non-scalar parabolic systems.*

Remark 16. *Theorem 8 also provides an estimate of the cost of the control for system (4.6) that drives the system from an initial datum $y_0 = (\theta_0, \phi_0) \in H^{-1}(0, \pi; \mathbf{R}^2)$ to the equilibrium at time $T > 0$. To be precise, under assumption (4.9) and (4.11), Theorem 8 implies that the set*

$$\mathcal{Z}_T(y_0) := \{v \in L^2(0,T) : y = (\theta, \phi) \text{ solution of (4.6) associated to } y_0 \text{ satisfies } y(\cdot, T) = 0\},$$

is nonempty for any $T > 0$ and any $y_0 = (\theta_0, \phi_0) \in H^{-1}(0, \pi; \mathbf{R}^2)$. We can then define the control cost for system (4.6) as

$$\mathcal{K}(T) = \sup_{\|y_0\|=1} \left(\inf_{v \in \mathcal{Z}_T(y_0)} \|v\|_{L^2(0,T)} \right), \quad \forall T > 0.$$

Observe that as a direct consequence of Theorem 8 and inequality (4.12), we can obtain the following estimate of this cost for system (4.6) at time $T > 0$:

$$\mathcal{K}(T) \leq C_0 \, e^{\frac{M}{T}}, \quad \forall T > 0, \tag{4.13}$$

where C_0 and M are positive constants only depending on the parameters in system (4.6) (see [81] and [40] for similar results in the scalar parabolic framework).

Remark 17. *As said before, condition (4.9) is equivalent to the simplicity of the spectrum of L and L^*. We will see in Proposition 8 that condition (4.11) implies a stronger property of the spectra of L and L^*: If we denote $\{\Lambda_k\}_{k\geq 1} \subset (0, \infty)$ the sequence of eigenvalues of the operator L, with $\Lambda_k \leq \Lambda_{k+1}$ for any $k \geq 1$, then, there exist $\delta > 0$ and an integer $q \geq 1$ such that*

$$|\Lambda_k - \Lambda_n| \geq \delta \left| k^2 - n^2 \right|, \quad \forall k, n \in \mathbb{N}, \quad |k - n| \geq q. \tag{4.14}$$

This gap condition for the spectrum of L is crucial for proving the null controllability at any positive time T of system (4.6) with control cost satisfying the estimate (4.13) for positive constants C_0 and M only depending on ξ, ρ and τ (for similar results, see [39] and [64]).

In the case in which assumption (4.11) does not hold, that is to say, if for some integer $j \geq 1$ one has

$$\xi = \frac{1}{j^2} \frac{\rho}{\tau},$$

then, the eigenvalues of L (and L^*) concentrate (see Remark 22) and the gap condition (4.14) is not valid. In fact, one has

$$\inf_{k,\ell \geq 1, k \neq \ell} |\Lambda_k - \Lambda_\ell| = 0.$$

In [9], the authors proved that when the eigenvalues $\{\Lambda_k\}_{k\geq 1}$ of the operator $L = -D\partial_{xx} + A$ concentrate, the controllability problem for system (4.7) ($f \equiv 0$) has a minimal time $T_0 \in$

$[0,\infty]$ *of null controllability which is related to the condensation index of the sequence. Even in the case* $T_0 = 0$ *(and therefore, system* (4.6) *is null controllable for any* $T > 0$*), without the separability assumption* (4.14)*, providing an estimate of the control cost* $\mathcal{K}(T)$ *with respect to* $T > 0$ *is an open problem.*

Remark 18. *For proving Theorem 8 we will use the moment method, introduced in [39] for proving the boundary controllability of the one-dimensional scalar heat equation. To this end, we will carry out an analysis of the properties of the eigenvalues of* L *which will imply inequality* (4.12) *and estimate* (4.13) *for the control cost of system* (4.6)*. These two inequalities are essential for proving the controllability property of the nonlinear system* (4.1)*.*

Let us now present the local exact controllability result to the trajectory $(0, c)$ ($c = \pm 1$) for the nonlinear system (4.1). This is our third main result. One has:

Theorem 9. *Let us consider* ξ, τ *and* ρ *three positive numbers satisfying* (4.9) *and* (4.11)*, and let us fix* $T > 0$ *and* $c = -1$ *or* $c = 1$*. Then, there exist* $\varepsilon > 0$ *such that, for any* $(\tilde{\theta}_0, \tilde{\phi}_0) \in H^{-1}(0,\pi) \times (c + H_0^1(0,\pi))$ *fulfilling*

$$\|\tilde{\theta}_0\|_{H^{-1}} + \|\tilde{\phi}_0 - c\|_{H_0^1} \leq \varepsilon, \tag{4.15}$$

there exists $v \in L^2(0,T)$ *for which system* (4.1) *has a unique solution*

$$(\tilde{\theta}, \tilde{\phi}) \in \left[L^2(Q_T) \cap C^0([0,T]; H^{-1}(0,\pi;\mathbf{R}^2))\right] \times C^0(\overline{Q}_T)$$

which satisfies (4.2)*.*

Theorem 9 establishes a local exact boundary controllability result at time T for the nonlinear system (4.1) when the parameters ξ, ρ and τ satisfy (4.9) and (4.11). Similar distributed controllability results[1] have been proved in the N-dimensional case, without any assumption on the parameters, using the cascade structure of the system (see [6] and [50]). As in the linear case (4.6), this cascade structure is not enough for dealing with the boundary controllability of system (4.1).

We end the presentation of our main results with some remarks.

Remark 19. *Following [63], Theorem 9 will be proved using a point-fixed strategy. The key point in its proof will be a boundary null controllability result for the non-homogeneous system* (4.7) *when the function* f *is in an appropriate weighted-*L^2 *space. In turn, this null controllability result for* (4.7) *will use in a crucial way the estimates* (4.12) *and* (4.13)*.*

Remark 20. *The main results established in this paper only deal with the boundary controllability of linear or nonlinear systems in space dimension one. This restriction is mainly due to the fact that in its proofs we will use the moment method. In general, the boundary controllability of parabolic systems in higher space dimension remains widely open and only some partial answers are known in the linear setting. To our knowledge, the only results on this issue are those of [3], [4] and [21]. In the two first articles, the results for parabolic systems are deduced*

[1] The distributed control acts as a source in the temperature equation.

from the study of the boundary control problem of two coupled wave equations using transmutation techniques. As a result they rely on some geometric constraints on the control domain. In [21], the author characterize the boundary null-controllability of system (4.3) *in the linear case ($B \equiv 0$ and $F \equiv 0$) when Ω is a cylindrical domains of the form $\Omega = (0, \pi) \times \Omega_2$ (Ω_2 is a smooth domain of $\mathbf{R}^{N-1}$, $N > 1$) and $\Gamma_0 := \{0\} \times \omega_2$ (ω_2 is an open subset of Ω_2) without imposing geometric constraints on ω_2. It is important to highlight that these results use that the diffusion matrix D is a multiple of the identity matrix. The boundary controllability of systems* (4.1) *and* (4.6) *in the N-dimensional case is completely open.*

The rest of the paper is organized as follows: In Section 4.2, we give some existence and uniqueness results for the linearized versions of the phase-field system (4.1) and we recall some known results on existence and bounds on biorthogonal families to complex exponentials. Section 4.3 is devoted to studying the spectral properties of the parabolic operators L and L^* given in (4.10). In Section 4.4 we prove the controllability results for the linear problem (4.6): In Subsection 4.4.1 we prove the approximate controllability result at time T for system (4.6) (Theorem 7) and in Subsection 4.4.2 the corresponding null controllability result (Theorem 8). Theorem 9 is proved in Section 4.5. Before (Subsection 4.5.1), we prove a null controllability result for the non-homogeneous system (4.7) when f belongs to appropriate spaces. As a consequence, we provide a proof of Theorem 9 in Subsection 4.5.2. We finish this paper with two appendices. In Appendix 4.6.1, we prove the existence and uniqueness result for the linearized system (4.7) and for its backward formulation. In Appendix 4.6.2 we give some additional results on the null controllability of the phase-field system (4.1), that is to say, we deal with the case $c = 0$ (see (4.2)).

4.2 Preliminary results

In this paper we will use the following notations for norms. If X is a Banach space, the norms of the spaces $L^2(0, T; X)$ and $C^0([0, T]; X)$ will be respectively denoted by $\|\cdot\|_{L^2(X)}$ and $\|\cdot\|_{C^0(X)}$. We will also work with the spaces $L^2(0, \pi; \mathbf{R}^2)$, $H_0^1(0, \pi; \mathbf{R}^2)$ and $H^{-1}(0, \pi; \mathbf{R}^d)$, with norms denoted by $\|\cdot\|_{L^2}$, $\|\cdot\|_{H_0^1}$ and $\|\cdot\|_{H^{-1}}$. On the other hand, we will use $\langle \cdot, \cdot \rangle$ for denoting the usual duality pairing between $H^{-1}(0, 1; \mathbf{R}^2)$ and $H_0^1(0, 1; \mathbf{R}^2)$.

Finally, throughout the paper C will stand for a generic positive constant that only depends on the coefficients ξ, τ and ρ in system (4.1), whose value may change from one line to another. Frequently, we will use the notation C_T when it is convenient to specify the dependence of the generic constant with respect to the final time T.

In this section we will give some results related to the existence, uniqueness and continuous dependence with respect to the data of the linear problem (4.7). To this aim, let us consider the linear backwards in time problem:

$$\begin{cases} -\varphi_t - D^* \varphi_{xx} + A^* \varphi = g & \text{in } Q_T, \\ \varphi(0, \cdot) = \varphi(\pi, \cdot) = 0 & \text{on } (0, T), \\ \varphi(\cdot, T) = \varphi_0 & \text{in } (0, \pi), \end{cases} \tag{4.16}$$

where D and A are given in (4.8) and φ_0 and g are functions in appropriate spaces.

Let us start with a first result on existence and uniqueness of strong solutions to system (4.16). One has:

Proposition 4. *Let us assume that* $\varphi_0 \in H_0^1(0,\pi;\mathbb{R}^2)$ *and* $g \in L^2(Q_T;\mathbb{R}^2)$. *Then, system* (4.16) *has a unique strong solution*

$$\varphi \in C^0([0,T]; H_0^1(0,\pi;\mathbb{R}^2)) \cap L^2(0,T; H^2(0,\pi;\mathbb{R}^2) \cap H_0^1(0,\pi;\mathbb{R}^2)).$$

In addition, there exists a positive constant C*, only depending on* D *and* A*, such that*

$$\|\varphi\|_{C^0(H_0^1)} + \|\varphi\|_{L^2(H^2\cap H_0^1)} \leq e^{CT}\left(\|g\|_{L^2(L^2)} + \|\varphi_0\|_{H_0^1}\right). \tag{4.17}$$

In view of Proposition 4, we can define solution by transposition to system (4.7).

Definition 1. *Let* $y_0 \in H^{-1}(0,\pi;\mathbb{R}^2)$*,* $v \in L^2(0,T)$ *and* $f \in L^2(Q_T;\mathbb{R}^2)$ *be given. It will be said that* $y \in L^2(Q_T;\mathbb{R}^2)$ *is a solution by transposition to* (4.7) *if, for each* $g \in L^2(Q_T;\mathbb{R}^2)$*, one has*

$$\iint_{Q_T} y\cdot g\,dx\,dt = \langle y_0, \varphi(\cdot,0)\rangle - \int_0^T B^*D^*\varphi_x(0,t)v(t)\,dt + \iint_{Q_T} f\cdot\varphi\,dx\,dt, \tag{4.18}$$

where $\varphi \in C^0([0,T]; H_0^1(0,\pi;\mathbb{R}^2)) \cap L^2(0,T; H^2(0,\pi;\mathbb{R}^2) \cap H_0^1(0,\pi;\mathbb{R}^2))$ *is the solution of* (4.16) *associated to* g *and* $\varphi_0 = 0$ *(recall that* $\langle\cdot\,,\cdot\rangle$ *stands for the usual duality pairing between* $H^{-1}(0,1;\mathbf{R}^2)$ *and* $H_0^1(0,1;\mathbf{R}^2)$*).*

With this definition we have:

Proposition 5. *Let us assume that* $y_0 = (\theta_0,\phi_0) \in H^{-1}(0,\pi;\mathbb{R}^2)$*,* $v \in L^2(0,T)$ *and* $f \in L^2(Q_T;\mathbb{R}^2)$. *Then, system* (4.7) *admits a unique solution by transposition* $y = (\theta,\phi)$ *that satisfies*

$$\begin{cases} y \in L^2(Q_T;\mathbb{R}^2) \cap C^0([0,T]; H^{-1}(0,\pi;\mathbb{R}^2)), \quad y_t \in L^2(0,T; (H^2(0,\pi;\mathbb{R}^2) \cap H_0^1(0,\pi;\mathbb{R}^2))'), \\ y_t - Dy_{xx} + Ay = f \text{ in } L^2(0,T; (H^2(0,\pi;\mathbb{R}^2) \cap H_0^1(0,\pi;\mathbb{R}^2))'), \\ y(\cdot,0) = y_0 \text{ in } H^{-1}(0,\pi;\mathbb{R}^2), \end{cases}$$

and

$$\|y\|_{L^2(L^2)} + \|y\|_{C^0(H^{-1})} + \|y_t\|_{L^2((H^2\cap H_0^1)')} \leq Ce^{CT}\left(\|y_0\|_{H^{-1}} + \|v\|_{L^2(0,T)} + \|f\|_{L^2(L^2)}\right), \tag{4.19}$$

for a constant $C > 0$ *only depending on the parameters* ξ*,* ρ *and* τ *in system* (4.7). *Moreover*

(a) If $\phi_0 \in L^2(0,\pi)$*, then* $\phi \in L^2(0,T; H_0^1(0,\pi)) \cap C^0([0,T]; L^2(0,\pi))$ *and, for a new constant* $C > 0$ *(only depending on* ξ*,* ρ *and* τ*), one has*

$$\|\phi\|_{L^2(H_0^1)} + \|\phi\|_{C^0(L^2)} \leq C\left(\|y\|_{L^2(L^2)} + \|\phi_0\|_{L^2} + \|f\|_{L^2(L^2)}\right). \tag{4.20}$$

(b) If $\phi_0 \in H_0^1(0,\pi)$*, then* $\phi \in L^2(0,T; H^2(0,\pi) \cap H_0^1(0,\pi)) \cap C^0([0,T]; H_0^1(0,\pi))$ *and, for a new constant* $C > 0$ *(only depending on* ξ*,* ρ *and* τ*), one has*

$$\|\phi\|_{L^2(H^2\cap H_0^1)} + \|\phi\|_{C^0(H_0^1)} \leq C\left(\|y\|_{L^2(L^2)} + \|\phi_0\|_{H_0^1} + \|f\|_{L^2(L^2)}\right), \tag{4.21}$$

and, in particular, $y = (\theta,\phi) \in L^2(Q_T) \times C^0(\overline{Q}_T)$.

One can prove Propositions 4 and 5 using, for instance, the well-known Galerkin method. For the sake of completeness we present an idea of the proof of this two propositions in Appendix 4.6.1.

Observe that, when $g = 0$, the backward problem (4.16) is the corresponding adjoint system to (4.6):

$$\begin{cases} -\varphi_t - D^*\varphi_{xx} + A^*\varphi = 0 & \text{in } Q_T, \\ \varphi(0,\cdot) = \varphi(\pi,\cdot) = 0 & \text{on } (0,T), \\ \varphi(\cdot,T) = \varphi_0 & \text{in } (0,\pi). \end{cases} \tag{4.22}$$

The controllability properties of system (4.6) can be characterized in terms of appropriate properties of the solutions to (4.22). In order to provide these characterizations, we need a new result which relates the solutions of systems (4.6) and (4.22). One has:

Proposition 6. *Let us consider $y_0 = (\theta_0, \phi_0) \in H^{-1}(0,\pi;\mathbb{R}^2)$ and $v \in L^2(0,T)$. Then, the solution $y = (\theta,\phi)$ of system (4.6) associated to y_0 and v, and the solution φ of the adjoint system (4.22) associated to $\varphi_0 \in H_0^1(0,\pi;\mathbf{R}^2)$ satisfy*

$$\int_0^T B^*D^*\varphi_x(0,t)v(t)\,dt = \langle y(\cdot,T),\varphi_0\rangle - \langle y_0,\varphi(\cdot,0)\rangle. \tag{4.23}$$

Proof: The proof is a consequence of Proposition 5. Observe that is enough to prove that (4.23) holds under the regularity assumption $y_0 \in C_0^1(0,\pi;\mathbb{R}^2)$ and $v \in C_0^1([0,\pi])$. Indeed, using density arguments, the estimates of Proposition 5 and the linearity of (4.6), it follows that the identity (4.23) is valid for all $y_0 \in H^{-1}(0,\pi;\mathbb{R}^2)$ and $v \in L^2(0,T)$.

On the other hand, when $y_0 \in C_0^1(0,\pi;\mathbb{R}^2)$, $v \in C^1([0,\pi])$ and $\varphi_0 \in H_0^1(0,\pi;\mathbb{R}^2)$, after some integrations by parts, it is not difficult to prove that the corresponding solution y of (4.6) and φ, solution of the adjoint system (4.22), satisfy equality (4.23). This ends the proof. ■

One important consequence of the previous result is the characterization of the approximate and null controllability properties of the linear system (4.6) in terms of suitable properties of the solutions of the adjoint system (4.22). One has:

Theorem 10. *Let us consider $T > 0$. Then,*

1. *System (4.6) is approximately controllable at time $T > 0$ if and only if the following unique continuation property holds:*

   "Let $\varphi_0 \in H_0^1(0,\pi;\mathbb{R}^2)$ be given and let φ be the corresponding solution of the adjoint problem (4.22). Then, if $B^*D^*\varphi_x(0,t) = 0$ on $(0,T)$, one has $\varphi_0 = 0$ in $(0,\pi)$."

2. *System (4.6) is null controllable at time T if and only if there exists a constant $C_T > 0$ such that, for any $\varphi_0 = (\theta_0,\phi_0) \in H_0^1(0,\pi;\mathbf{R}^2)$, the corresponding solution of (4.22) satisfies the observability inequality*

$$\|\varphi(\cdot,T)\|_{H_0^1}^2 \le C_T \int_0^T |B^*D^*\varphi_x(0,t)|^2\,dt.$$

This result is well known. For a proof see, for instance [32], [86] and [89].

Remark 21. *The constant C_T appearing in the observability inequality for the adjoint system* (4.22) *is closely related to the cost $\mathcal{K}(T)$ for system* (4.6) *(see Remark 16). To be precise, if the observability inequality holds, then $\mathcal{Z}(T) \neq \emptyset$, for any $y_0 = (\theta_0, \phi_0) \in H^{-1}(0, \pi; \mathbf{R}^2)$, and*

$$\mathcal{K}(T) \leq \sqrt{C_T}.$$

On the other hand, assume that $\mathcal{Z}(T) \neq \emptyset$, for any $y_0 = (\theta_0, \phi_0) \in H^{-1}(0, \pi; \mathbf{R}^2)$, and define $\mathcal{K}(T)$ as in Remark 16. Then, the previous observability inequality for (4.22) *holds with $C_T = \mathcal{K}(T)^2$.*

For a proof of the previous properties, see for example [32] (see Theorem 2.44, p. 56), [89] or [86].

We will finish this section given two known results which will be used later. They are related to the existence and bounds of biorthogonal families to real exponentials. One has:

Lemma 1. *Let us consider a sequence $\{\Lambda_k\}_{k\geq 1} \subset \mathbb{R}_+$ satisfying $\Lambda_k \neq \Lambda_n$, for any $k, n \in \mathbb{N}$ with $k \neq n$, and*

$$\sum_{k\geq 1} \frac{1}{\Lambda_k} < \infty. \tag{4.24}$$

Then, there exists a family $\{q_k\}_{k\geq 1} \subset L^2(0, T)$ biorthogonal to $\{e^{-\Lambda_k t}\}_{k\geq 1}$, i.e., a family $\{q_k\}_{k\geq 1}$ in $L^2(0, T)$ such that

$$\int_0^T q_k(t) e^{-\Lambda_j t} dt = \delta_{kj}, \quad \forall k, j \geq 1.$$

We also have:

Lemma 2. *Let us consider a sequence $\{\Lambda_k\}_{k\geq 1} \subset \mathbb{R}_+$ such that $\Lambda_k \neq \Lambda_n$, for any $k, n \in \mathbb{N}$ with $k \neq n$. Let us also assume that there exist an integer $q \geq 1$ and positive constants p, δ and α such that*

$$\begin{cases} |\Lambda_k - \Lambda_n| \geq \delta \left|k^2 - n^2\right|, & \forall k, n \in I\!N, \ |k - n| \geq q, \\ \inf\limits_{k\neq n,\ |k-n|<q} |\Lambda_k - \Lambda_n| > 0, & \end{cases} \tag{4.25}$$

and

$$\left|p\sqrt{r} - \mathcal{N}(r)\right| \leq \alpha, \quad \forall r > 0. \tag{4.26}$$

(In (4.26)*, $\mathcal{N}(r)$ is the counting function associated to $\{\Lambda_k\}_{k\geq 1}$, defined by $\mathcal{N}(r) = \#\{k : \Lambda_k \leq r\}$). Then, there exists $\widetilde{T}_0 > 0$ such that, for any $T \in (0, \widetilde{T}_0)$, we can find a family $\{q_k\}_{k\geq 1} \subset L^2(0, T)$ biorthogonal to $\{e^{-\Lambda_k t}\}_{k\geq 1}$ which in addition satisfies*

$$\|q_k\|_{L^2(0,T)} \leq C e^{C\sqrt{\Lambda_k} + \frac{C}{T}}, \quad \forall k \geq 1,$$

for a positive constant C independent of T.

A proof of Lemma 1 can be found in [39] and [8]. On the other hand, Lemma 2 is a particular case of a more general result proved in [21] (see Theorem 1.5 in pages 2974–2975).

4.3 Spectral properties of the operators L and L^*

Let us consider the vectorial operators L and L^* given in (4.10), with domains

$$D(L) = D(L^*) = H^2(0,\pi;\mathbb{R}^2) \cap H_0^1(0,\pi;\mathbb{R}^2).$$

This section will be devoted to giving some spectral properties of the operators L and L^* which will be used below. Recall that the matrices D and A are given in (4.8).

In what follows, for simplicity, we will use the notation

$$r_k := \sqrt{\frac{\xi\rho}{\tau}k^2 + \left(\frac{\rho+1}{2\tau}\right)^2}, \quad \forall k \geq 1. \tag{4.27}$$

On the other hand, it is well-known that the operator $-\partial_{xx}$ with homogeneous Dirichlet boundary conditions admits a sequence of positive eigenvalues, given by $\{k^2\}_{k\geq 1}$, and a sequence of normalized eigenfunctions $\{\eta_k\}_{k\geq 1}$, which is a Hilbert basis of $L^2(0,\pi)$, given by

$$\eta_k(x) = \sqrt{\frac{2}{\pi}}\sin kx, \quad x \in (0,\pi). \tag{4.28}$$

With the previous notation, we have the following result:

Proposition 7. *Let us consider the operators L and L^* given in (4.10) (the matrices D and A are given in (4.8)). Then,*

1. *The spectra of L and L^* are given by $\sigma(L) = \sigma(L^*) = \{\lambda_k^{(1)}, \lambda_k^{(2)}\}_{k\geq 1}$ with*

$$\lambda_k^{(1)} = \xi k^2 + \frac{\rho+1}{2\tau} - r_k, \quad \lambda_k^{(2)} = \xi k^2 + \frac{\rho+1}{2\tau} + r_k, \quad \forall k \geq 1, \tag{4.29}$$

   where r_k is given in (4.27).

2. *For each $k \geq 1$, the corresponding eigenfunctions of L (resp., L^*) associated to $\lambda_k^{(1)}$ and $\lambda_k^{(2)}$ are respectively given by*

$$\Psi_k^{(1)} = \frac{1}{4\sqrt{\tau r_k}}\begin{pmatrix} 1-\rho+2\tau r_k \\ 4 \end{pmatrix}\eta_k, \quad \Psi_k^{(2)} = \frac{1}{4\sqrt{\tau r_k}}\begin{pmatrix} 1-\rho-2\tau r_k \\ 4 \end{pmatrix}\eta_k, \tag{4.30}$$

   (resp.,

$$\Phi_k^{(1)} = \frac{1}{4\sqrt{\tau r_k}}\begin{pmatrix} 4 \\ \rho-1+2\tau r_k \end{pmatrix}\eta_k, \quad \Phi_k^{(2)} = \frac{-1}{4\sqrt{\tau r_k}}\begin{pmatrix} 4 \\ \rho-1-2\tau r_k \end{pmatrix}\eta_k). \tag{4.31}$$

Proof: We will prove the result for the operator L. The same reasoning provides the proof for its adjoint L^*.

Using that the function η_k is the normalized eigenfunction of the Dirichlet laplacian in $(0,\pi)$ associated to the eigenvalue k^2, it is not difficult to check that the eigenvalues of the operator L correspond to the eigenvalues of the matrices

$$k^2 D + A, \quad \forall k \geq 1.$$

The associated eigenfunctions of L are given under the form $\Psi_k(\cdot) = z_k \eta_k(\cdot)$, where $z_k \in \mathbb{R}^2$ is the associated eigenvector of $k^2 D + A$.

Taking into account the expression of the characteristic polynomial of $k^2 D + A$:

$$p(x) = x^2 - \left(2\xi k^2 + \frac{\rho + 1}{\tau}\right) x + \xi^2 k^4 + \frac{\xi}{\tau} k^2, \quad k \geq 1,$$

a direct computation provides the formulae (4.29) and (4.30) as eigenvalues and associated eigenfunctions of the operator L. This finishes the proof. ■

Let us now analyze some properties of the eigenvalues and eigenfunctions of the operators L and L^*. These properties will be used below. We start with some properties of the sequences $\{\lambda_k^{(1)}\}_{k\geq 1}$ and $\{\lambda_k^{(2)}\}_{k\geq 1}$. One has

Proposition 8. *Under the assumptions of Proposition 7, the following properties hold:*

(P1) $\{\lambda_k^{(1)}\}_{k\geq 1}$ *and* $\{\lambda_k^{(2)}\}_{k\geq 1}$ *(see (4.29)) are increasing sequences satisfying*

$$0 < \lambda_k^{(1)} < \lambda_k^{(2)}, \quad \forall k \geq 1.$$

(P2) The spectrum of L and L^* is simple, i.e., $\lambda_k^{(2)} \neq \lambda_\ell^{(1)}$*, for all* $k, \ell \geq 1$ *if and only if the parameters ξ, ρ and τ satisfy condition (4.9).*

(P3) Assume that the parameters ξ, ρ and τ satisfy (4.11), i.e., there exists $j \geq 0$ such that

$$\frac{1}{(j+1)^2} \frac{\rho}{\tau} < \xi < \frac{1}{j^2} \frac{\rho}{\tau}. \tag{4.32}$$

Then, there exists an integer $k_0 = k_0(\xi, \rho, \tau, j) \geq 1$ and a constant $C = C(\xi, \rho, \tau, j) > 0$ such that

$$\begin{cases} \lambda_{k+j}^{(1)} < \lambda_k^{(2)} < \lambda_{k+1+j}^{(1)} < \lambda_{k+1}^{(2)} < \cdots, \ \forall k \geq k_0, \\ \min\limits_{k \geq k_0} \left\{ \lambda_k^{(2)} - \lambda_{k+j}^{(1)}, \lambda_{k+j+1}^{(1)} - \lambda_k^{(2)} \right\} > C. \end{cases} \tag{4.33}$$

(P4) Assume now that the parameters ξ, ρ and τ satisfy (4.9) and (4.11). Then, one has:

$$\inf_{k,\ell \geq 1} |\lambda_k^{(2)} - \lambda_\ell^{(1)}| > 0, \tag{4.34}$$

and there exists a positive integer $k_1 \in I\!N$, depending on ξ, ρ and τ, such that

$$\min \left\{ \left|\lambda_k^{(1)} - \lambda_\ell^{(1)}\right|, \left|\lambda_k^{(2)} - \lambda_\ell^{(2)}\right|, \left|\lambda_k^{(2)} - \lambda_\ell^{(1)}\right| \right\} \geq \frac{\xi}{2} |k^2 - \ell^2|, \ \forall k, \ell \geq 1, \ |k - \ell| \geq k_1. \tag{4.35}$$

Proof: Let us start proving property $(P1)$. From the expressions of $\lambda_k^{(1)}$ and $\lambda_k^{(2)}$ (see (4.29)), we directly get $\lambda_k^{(1)} < \lambda_k^{(2)}$ for any $k \geq 1$. On the other hand, using the inequality

$$r_k = \sqrt{\frac{\xi\rho}{\tau} k^2 + \left(\frac{\rho+1}{2\tau}\right)^2} < \sqrt{\xi^2 k^4 + 2\xi k^2 \frac{\rho+1}{2\tau} + \left(\frac{\rho+1}{2\tau}\right)^2} = \xi k^2 + \frac{\rho+1}{2\tau}, \quad \forall k \geq 1,$$

we also deduce $0 < \lambda_k^{(1)}$ for any $k \geq 1$.

Let us now prove that $\{\lambda_k^{(1)}\}_{k\geq 1}$ and $\{\lambda_k^{(2)}\}_{k\geq 1}$ are increasing sequences. Indeed,

$$\begin{aligned}\lambda_{k+1}^{(1)}-\lambda_k^{(1)}&=\xi(2k+1)+\sqrt{\frac{\xi\rho}{\tau}k^2+\left(\frac{\rho+1}{2\tau}\right)^2}-\sqrt{\frac{\xi\rho}{\tau}(k+1)^2+\left(\frac{\rho+1}{2\tau}\right)^2}\\&=\xi(2k+1)-\frac{\xi\rho}{\tau}\frac{2k+1}{\sqrt{\frac{\xi\rho}{\tau}k^2+\left(\frac{\rho+1}{2\tau}\right)^2}+\sqrt{\frac{\xi\rho}{\tau}(k+1)^2+\left(\frac{\rho+1}{2\tau}\right)^2}}\\&=\xi(2k+1)\left[1-\frac{\rho}{\tau}\frac{1}{\sqrt{\frac{\xi\rho}{\tau}k^2+\left(\frac{\rho+1}{2\tau}\right)^2}+\sqrt{\frac{\xi\rho}{\tau}(k+1)^2+\left(\frac{\rho+1}{2\tau}\right)^2}}\right]\to\infty,\end{aligned}$$

as $k\to\infty$. Moreover,

$$\sqrt{\frac{\xi\rho}{\tau}k^2+\left(\frac{\rho+1}{2\tau}\right)^2}+\sqrt{\frac{\xi\rho}{\tau}(k+1)^2+\left(\frac{\rho+1}{2\tau}\right)^2}\geq\frac{\rho+1}{2\tau}+\frac{\rho+1}{2\tau}>\frac{\rho}{\tau},$$

which implies $\lambda_{k+1}^{(1)}-\lambda_k^{(1)}>0$, for any $k\geq 1$. Thus, $\{\lambda_k^{(1)}\}_{k\geq 1}$ is a positive increasing sequence. Clearly $\{\lambda_k^{(2)}\}_{k\geq 1}$ is also a positive increasing sequence and $\lambda_{k+1}^{(2)}-\lambda_k^{(2)}\to\infty$, as $k\to\infty$. This proves property $(P1)$.

Let us now see property $(P2)$. Using property $(P1)$, one has that, for any integers $k,\ell\geq 1$ with $\ell\leq k$, clearly $\lambda_\ell^{(1)}\leq\lambda_k^{(1)}<\lambda_k^{(2)}$. Therefore, in order to prove the equivalence we can assume that $\ell>k$. We have

$$\lambda_\ell^{(1)}-\lambda_k^{(2)}=\frac{\xi\rho}{\tau}(\ell^2-k^2)\left(\frac{\tau}{\rho}-\frac{1}{r_\ell-r_k}\right).$$

Thus, $\lambda_k^{(2)}\neq\lambda_\ell^{(1)}$ for any $k,\ell\geq 1$, with $\ell>k$, if and only if

$$r_\ell^2\neq\left(r_k+\frac{\rho}{\tau}\right)^2,\quad\forall k,\ell\geq 1,\quad \ell>k.$$

From the expression of r_k (see (4.27)) we readily deduce $2r_k>\dfrac{\rho}{\tau}$ and $\xi\tau(\ell^2-k^2)-\rho+2\tau r_k>0$ $(\ell>k)$. So,

$$\left\{\begin{aligned}r_\ell^2-\left(r_k+\frac{\rho}{\tau}\right)^2&=\frac{\rho}{\tau}\left[\left(\xi(\ell^2-k^2)-\frac{\rho}{\tau}\right)-2r_k\right]=\frac{\rho}{\tau^2}\frac{\left(\xi\tau(\ell^2-k^2)-\rho\right)^2-4\tau^2r_k^2}{\xi\tau(\ell^2-k^2)-\rho+2\tau r_k}\\&=\frac{\rho}{\tau^2}\frac{\xi^2\tau^2(\ell^2-k^2)^2-2\xi\tau\rho(\ell^2-k^2)-\rho^2-4\xi\tau\rho k^2-2\rho-1}{\xi\tau(\ell^2-k^2)-\rho+2\tau r_k}\\&=\frac{\rho}{\tau^2}\frac{\xi^2\tau^2(\ell^2-k^2)^2-2\xi\tau\rho(\ell^2+k^2)-2\rho-1}{\xi\tau(\ell^2-k^2)-\rho+2\tau r_k},\end{aligned}\right.$$

and we get that $\lambda_k^{(2)}\neq\lambda_\ell^{(1)}$ for any $k,\ell\geq 1$, with $\ell>k$, if and only if condition (4.9) holds. This finishes the proof of property $(P2)$.

In order to prove property $(P3)$, we are going to use the expressions

$$\lambda_k^{(1)}=\xi k^2+\frac{\rho+1}{2\tau}-\sqrt{\frac{\xi\rho}{\tau}}\,k-\frac{\epsilon_k}{k},\quad \lambda_k^{(2)}=\xi k^2+\frac{\rho+1}{2\tau}+\sqrt{\frac{\xi\rho}{\tau}}\,k+\frac{\epsilon_k}{k},\quad\forall k\geq 1,\qquad(4.36)$$

which can be easily deduced from the expressions of $\lambda_k^{(i)}$, $i = 1, 2$, and r_k (see (4.29) and (4.27)). In (4.36), $\{\epsilon_k\}_{k\geq 1}$ is a new positive sequence satisfying

$$\lim_{k\to\infty} \epsilon_k = \frac{1}{2}\left(\frac{\rho+1}{2\tau}\right)^2 \sqrt{\frac{\tau}{\xi\rho}}.$$

Using (4.36), we will prove that, for any $i \geq 1$, the difference $\lambda_{k+i}^{(1)} - \lambda_k^{(2)}$ behaves at infinity as

$$\lim_{k\to\infty} \frac{\lambda_{k+i}^{(1)} - \lambda_k^{(2)}}{\xi i(2k+i)} = 1 - \sqrt{\frac{1}{i^2}\frac{\rho}{\xi\tau}} \neq 0. \tag{4.37}$$

Indeed, a simple computation gives

$$\lambda_{k+i}^{(1)} - \lambda_k^{(2)} = \xi i(2k+i) - \sqrt{\frac{\xi\rho}{\tau}}\,(2k+i) - \frac{\epsilon_{k+i}}{k+i} - \frac{\epsilon_k}{k} = \xi i(2k+i)\left[1 - \sqrt{\frac{1}{i^2}\frac{\rho}{\xi\tau}} - \widetilde{\epsilon}_k^{(i)}\right],$$

where $\{\widetilde{\epsilon}_k^{(i)}\}_{k\geq 1}$ is a sequence converging to zero. From assumption (4.11) we can conclude (4.37).

We will obtain the proof of property $(P3)$ from (4.37). Observe that assumption (4.11) implies that the parameters ξ, ρ and τ satisfies (4.32) for an appropriate integer $j \geq 0$. Therefore, if $j = 0$, then, $\xi > \dfrac{\rho}{\tau}$ and (4.37) implies $\lim_{k\to\infty}\left(\lambda_{k+1}^{(1)} - \lambda_k^{(2)}\right) = \infty$. On the other hand, one has $\lim_{k\to\infty}\left(\lambda_k^{(2)} - \lambda_k^{(1)}\right) = \lim_{k\to\infty} 2r_k = \infty$. Thus, there exists an integer $k_0 \geq 1$ and a constant $C > 0$ such that (4.33) holds for $j = 0$.

If $j \geq 1$, again, the property (4.37) implies

$$\lim_{k\to\infty}\left(\lambda_{k+i}^{(1)} - \lambda_k^{(2)}\right) = -\infty, \quad \text{if } i \leq j \quad \text{and} \quad \lim_{k\to\infty}\left(\lambda_{k+i}^{(1)} - \lambda_k^{(2)}\right) = \infty, \quad \text{if } i \geq j+1.$$

We can also conclude the existence of an integer $k_0 \geq 1$ and a positive constant C such that (4.33) holds. This shows property $(P3)$.

Let us finalize the proof showing property $(P4)$. First, inequality (4.34) is a direct consequence of property $(P2)$ and (4.33). Secondly, if we take $k, \ell \geq 1$, from (4.36), one deduces:

$$\begin{cases} \left|\lambda_k^{(1)} - \lambda_\ell^{(1)}\right| = \xi\left|k^2 - \ell^2\right| \left|1 - \sqrt{\frac{\xi\rho}{\tau}}\frac{1}{k+\ell} - \frac{\epsilon_k}{k(k^2-\ell^2)} + \frac{\epsilon_\ell}{\ell(k^2-\ell^2)}\right| \\ \qquad \geq \xi\left|k^2-\ell^2\right|\left[1 - \left(\sqrt{\frac{\xi\rho}{\tau}} + |\epsilon_k| + |\epsilon_\ell|\right)\frac{1}{k+\ell}\right]. \end{cases}$$

Observe that $\{\epsilon_k\}_{k\geq 1}$ is a convergent sequence and $k + \ell \geq |k - \ell|$, for any $k, \ell \in \mathbb{N}$. Hence, there exists a integer $q_1 \geq 1$ (depending on the parameters of system (4.6)) such that

$$\left|\lambda_k^{(1)} - \lambda_\ell^{(1)}\right| \geq \frac{\xi}{2}\left|k^2 - \ell^2\right|, \quad \forall k, \ell \in \mathbb{N}, \quad |k - \ell| \geq q_1.$$

A similar inequality can be deduced for a new $q_2 \in \mathbb{N}$ if we change $\left|\lambda_k^{(1)} - \lambda_\ell^{(1)}\right|$ by $\left|\lambda_k^{(2)} - \lambda_\ell^{(2)}\right|$.

Finally, if we repeat the previous reasoning, we can write

$$\begin{cases} \left|\lambda_k^{(2)} - \lambda_\ell^{(1)}\right| = \xi \left|k^2 - \ell^2\right| \left|1 + \sqrt{\dfrac{\xi\rho}{\tau}}\dfrac{1}{k-\ell} + \dfrac{\epsilon_k}{k(k^2-\ell^2)} + \dfrac{\epsilon_\ell}{\ell(k^2-\ell^2)}\right| \\ \qquad\qquad \geq \xi \left|k^2 - \ell^2\right| \left[1 - \left(\sqrt{\dfrac{\xi\rho}{\tau}} + \dfrac{1}{2}|\epsilon_k| + \dfrac{1}{2}|\epsilon_\ell|\right)\dfrac{1}{|k-\ell|}\right]. \end{cases}$$

Again, from this inequality we conclude the existence of $q_3 = q_3(\xi, \rho, \tau) \in I\!N$ such that

$$\left|\lambda_k^{(2)} - \lambda_\ell^{(1)}\right| \geq \frac{\xi}{2}\left|k^2 - \ell^2\right|, \quad \forall k, \ell \in I\!N, \quad |k - \ell| \geq q_3.$$

This proves inequality (4.35) if we take $k_1 = \max\{q_1, q_2, q_3\}$. This completes the proof of $(P4)$ and the proof of the result. ∎

Remark 22. *If condition* (4.11) *does not hold, i.e., if for some integer $j \geq 1$ one has*

$$\xi = \frac{1}{j^2}\frac{\rho}{\tau},$$

then, the gap condition (4.34) *is not valid. Indeed, from* (4.36) *we deduce*

$$\lambda_{k+j}^{(1)} - \lambda_k^{(2)} = -\left(\frac{\epsilon_{k+j}}{k+j} + \frac{\epsilon_k}{k}\right), \quad \forall k \geq 1.$$

In particular ($\{\epsilon_k\}_{k\geq 1}$ is a positive sequence), $\lambda_{k+j}^{(1)} < \lambda_k^{(2)}$ for any $k \geq 1$ and

$$\lim_{k\to\infty}\left(\lambda_{k+j}^{(1)} - \lambda_k^{(2)}\right) = 0.$$

In this case, we can rearrange the sequence $\{\lambda_k^{(1)}, \lambda_k^{(2)}\}_{k\geq 1}$ as follows: there exists an integer $k_0 \geq 1$ such that

$$\lambda_{k-1}^{(2)} < \lambda_{k+j}^{(1)} < \lambda_k^{(2)}, \quad \forall k \geq k_0.$$

The previous inequality can be directly deduced from (4.37).

Let us now check that the sequence of eigenvalues of L and L^* fulfills the conditions in Lemma 2. We will do it in the next result:

Proposition 9. *Let us assume that the parameters ξ, ρ and τ satisfy* (4.9). *Then, the sequence $\{\lambda_k^{(1)}, \lambda_k^{(2)}\}_{k\geq 1}$, given by* (4.29), *can be rearranged into an increasing sequence $\Lambda = \{\Lambda_k\}_{k\geq 1}$ that satisfies* (4.24) *and $\Lambda_k \neq \Lambda_n$, for all $k, n \in I\!N$ with $k \neq n$. In addition, if* (4.11) *holds, the sequence $\{\Lambda_k\}_{k\geq 1}$ also satisfies* (4.25) *and* (4.26).

Proof: As a consequence of property $(P1)$ in Proposition 8, we deduce that the sequence of eiegenvalues $\{\lambda_k^{(1)}, \lambda_k^{(2)}\}_{k\geq 1}$ can be rearranged into a positive increasing sequence $\Lambda = \{\Lambda_k\}_{k\geq 1}$ that satisfies (4.24). Under assumption (4.9), we can also apply property $(P2)$ of the same proposition and conclude that the elements of the sequence Λ are pairwise different.

Let us now assume that, in addition, the parameters ξ, ρ and τ also fulfill condition (4.11). In this case, we can give an explicit rearrangement of the sequence $\{\lambda_k^{(1)}, \lambda_k^{(2)}\}_{k\geq 1}$. Indeed, if

$j \geq 0$ is such that the parameters satisfy (4.32), property $(P3)$ in Proposition 8 provides an integer $k_0 \geq 1$ for which one has (4.33). Thus, if $1 \leq k \leq 2k_0 + j - 2$, we define Λ_k such that

$$\begin{cases} \{\Lambda_k\}_{1 \leq k \leq 2k_0+j-2} \equiv \{\lambda_k^{(1)}\}_{1 \leq k \leq k_0+j-1} \cup \{\lambda_k^{(2)}\}_{1 \leq k \leq k_0-1}, \\ \Lambda_k < \Lambda_{k+1}, \quad \forall k : 1 \leq k \leq 2k_0 + j - 3. \end{cases}$$

From the $(2k_0 + j - 1)$-th term, we define

$$\begin{cases} \Lambda_{2k_0+j+2k-1} = \lambda_{k_0+j+k}^{(1)}, \quad \forall k \geq 0, \\ \Lambda_{2k_0+j+2k} = \lambda_{k_0+k}^{(2)}, \quad \forall k \geq 0. \end{cases} \tag{4.38}$$

Clearly, $\Lambda = \{\Lambda_k\}_{k \geq 1}$ is an increasing sequence and $\{\Lambda_k\}_{k \geq 1} = \{\lambda_k^{(1)}, \lambda_k^{(2)}\}_{k \geq 1}$. Furthermore, thanks to (4.34) in Proposition 8, the sequence Λ also satisfies the second inequality in (4.25) for every $q \geq 1$.

Our next task will be to prove the first inequality of (4.25) for appropriate $q \geq 1$ and $\delta > 0$. It is interesting to underline that it is enough to prove the existence of $q \in I\!N$ and $\widetilde{\delta} > 0$ such that one has

$$|\Lambda_k - \Lambda_n| \geq \widetilde{\delta} \left| k^2 - n^2 \right|, \quad \forall k, n \geq q, \quad |k - n| \geq q. \tag{4.39}$$

Indeed, let us see that the first inequality in (4.25) is valid for $q \geq 1$ and a new positive constant δ. Observe that we can assume that $k \geq n \geq 1$. Hence, it is sufficient to prove (4.25) with $k \geq n \geq 1$, with $n \leq q - 1$ and $k - n \geq q$. First, it is clear that if in addition $k \leq 2q$, thanks to (4.9), we can conclude inequality (4.25) for an appropriate positive constant δ_0.

Let us now take $k \geq n \geq 1$, with $n \leq q - 1$ and $k \geq 2q$ (and therefore, $k - n \geq q$). From (4.39) and using $k \geq q + n \geq q + 1$, $1 \leq n \leq q - 1$ and $k - q \geq q$, we have

$$\begin{cases} |\Lambda_k - \Lambda_n| = \Lambda_k - \Lambda_n \geq \Lambda_k - \Lambda_q \geq \widetilde{\delta} \left| k^2 - q^2 \right| = \widetilde{\delta} \left| k^2 - n^2 \right| \left[1 - \dfrac{q^2 - n^2}{k^2 - n^2} \right] \\ \qquad \geq \widetilde{\delta} \left| k^2 - n^2 \right| \left[1 - \dfrac{q^2 - n^2}{(q+1)^2 - n^2} \right] \geq \widetilde{\delta} \left[1 - \dfrac{q^2 - 1}{(q+1)^2 - 1} \right] \left| k^2 - n^2 \right| \\ \qquad = \dfrac{\widetilde{\delta}\,(2q+1)}{(q+1)^2 - 1} \left| k^2 - n^2 \right|. \end{cases}$$

Summarizing, assuming (4.39), we have deduced the first inequality in (4.25) for $q \geq 1$ and

$$\delta = \min \left\{ \delta_0, \frac{\widetilde{\delta}\,(2q+1)}{(q+1)^2 - 1} \right\} > 0.$$

Let us show (4.39) for suitable $\widetilde{\delta} > 0$ and $q \in I\!N$. To this aim, we will use the properties (4.33) and (4.35), in Proposition 8, and the expression of Λ_k for $k \geq 2k_0 + j - 1$ (see (4.38); recall that $j \geq 0$ is such that the parameters ξ, ρ and τ satisfy (4.32)). We will work with $q \in I\!N$ given by

$$q \geq \max \{2k_0 + j - 1, 2k_1 + 2j + 1, 6j + 3\}. \tag{4.40}$$

Thus, if $k, n \in I\!N$ are such that $k, n \geq q$ and $|k - n| \geq q$, then Λ_k and Λ_n are given by (4.38). Depending on the expressions of k and n, we will divide the proof of (4.39) into three steps:

1. Assume that $k = 2k_0 + j + 2\widetilde{k} - 1$ and $n = 2k_0 + j + 2\widetilde{n} - 1$, for $\widetilde{k}, \widetilde{n} \geq 0$. Since

$$\left|\left(k_0 + j + \widetilde{k}\right) - (k_0 + j + \widetilde{n})\right| = \frac{1}{2}\,|k - n| \geq \frac{q}{2} \geq k_1,$$

from (4.38) and (4.35), we can write

$$\begin{aligned}|\Lambda_k - \Lambda_n| &= \left|\lambda^{(1)}_{k_0+j+\widetilde{k}} - \lambda^{(1)}_{k_0+j+\widetilde{n}}\right| \geq \frac{\xi}{2}\left|\left(k_0 + j + \widetilde{k}\right)^2 - (k_0 + j + \widetilde{n})^2\right| \\ &= \frac{\xi}{8}\left|(k+1+j)^2 - (n+1+j)^2\right| = \frac{\xi}{8}\left|k^2 - n^2 + 2(k-n)(1+j)\right| \geq \frac{\xi}{8}\left|k^2 - n^2\right|.\end{aligned}$$

We obtain thus the proof of (4.39) for $\widetilde{\delta} = \xi/8$ and q given by (4.40).

2. The case $k = 2k_0 + j + 2\widetilde{k}$ and $n = 2k_0 + j + 2\widetilde{n}$, with $\widetilde{k}, \widetilde{n} \in I\!N$, can be treated in the same way deducing (4.39) for $\widetilde{\delta} = \xi/8$ and q (see (4.40)).

3. Let us analyze the last case $k = 2k_0 + j + 2\widetilde{k}$ and $n = 2k_0 + j + 2\widetilde{n} - 1$ (with $\widetilde{k}, \widetilde{n} \in I\!N$), $k, n \geq q$ and $|k - n| \geq q$, with q satisfying (4.40). In this case, one has

$$\left|\left(k_0 + \widetilde{k}\right) - (k_0 + j + \widetilde{n})\right| = \left|\frac{1}{2}(k-n) - j - \frac{1}{2}\right| \geq \frac{1}{2}\,|k-n| - \left(j + \frac{1}{2}\right) \geq \frac{1}{2}q - \left(j + \frac{1}{2}\right) \geq k_1,$$

whence

$$\begin{aligned}|\Lambda_k - \Lambda_n| &= \left|\lambda^{(2)}_{k_0+\widetilde{k}} - \lambda^{(1)}_{k_0+j+\widetilde{n}}\right| \geq \frac{\xi}{2}\left|\left(k_0 + \widetilde{k}\right)^2 - (k_0 + j + \widetilde{n})^2\right| = \frac{\xi}{8}\left|(k-j)^2 - (n+1+j)^2\right| \\ &= \frac{\xi}{8}\left|k^2 - n^2 - [2j(k+1) + 2n(1+j) + 1]\right|.\end{aligned}$$

Observe that if $k \leq n$, from the previous inequality, we conclude (4.39) for for $\widetilde{\delta} = \xi/8$ and q given by (4.40). Let us now see the case $k > n$ (and then, $k - n = |k - n| \geq q$). The previous inequality allows us to write

$$\begin{aligned}|\Lambda_k - \Lambda_n| &= \frac{\xi}{8}\left|k^2 - n^2 - [2j(k+1) + 2n(1+j) + 1]\right| \\ &\geq \frac{\xi}{8}\left(k^2 - n^2\right) - [2j(k+1) + 2n(1+j) + 1] \\ &= \frac{\xi}{8}\left(k^2 - n^2\right)\left[1 - \frac{2j(k+1) + 2n(1+j) + 1}{k^2 - n^2}\right] \\ &\geq \frac{\xi}{8}\left(k^2 - n^2\right)\left[1 - \frac{2j(k+1) + 2n(1+j) + 1}{q\,(k+n)}\right] \\ &\geq \frac{\xi}{8}\left(k^2 - n^2\right)\left[1 - \frac{2j}{q} - \frac{1+j}{q} - \frac{1}{2q}\right] \geq \frac{\xi}{16}\left(k^2 - n^2\right).\end{aligned}$$

Let us remark that the last inequality is valid thanks to (4.40).

In conclusion, we have proved the existence of a natural number $q \geq 1$, depending on the parameters in (4.6), such that (4.39) holds for $\widetilde{\delta} = \xi/16$ and q provided by formula (4.40). As a consequence, one also has (4.25) for a new $\delta > 0$ and the same q.

Let us now show the estimate (4.26) for the sequence $\Lambda = \{\Lambda_k\}_{k\geq 1} = \{\lambda^{(1)}_k, \lambda^{(2)}_k\}_{k\geq 1}$. From the definition of the sequence Λ, for any $r > 0$, we can write:

$$\mathcal{N}(r) = \#\left\{k : \Lambda_k \leq r\right\} = \#\left\{k : \lambda^{(1)}_k \leq r\right\} + \#\left\{k : \lambda^{(2)}_k \leq r\right\} = \#\mathcal{A}_1(r) + \#\mathcal{A}_2(r) = n_1 + n_2,$$

where $\mathcal{A}_i(r) = \left\{k : \lambda_k^{(i)} \le r\right\}$ and $n_i = \#\mathcal{A}_i(r)$, $i = 1, 2$. Our next objective will be to give appropriate bounds for n_1 and n_2.

From the definition of $\mathcal{A}_1(r)$ and n_1, we deduce that n_1 is a natural number which is characterized by $\lambda_{n_1}^{(1)} \le r$ and $\lambda_{n_1+1}^{(1)} > r$. Let us first work with the inequality $\lambda_{n_1}^{(1)} \le r$. From the definition of $\lambda_k^{(1)}$ (see (4.29)), one gets

$$\xi n_1^2 + \frac{\rho+1}{2\tau} \le r + \sqrt{\frac{\xi\rho}{\tau} n_1^2 + \left(\frac{\rho+1}{2\tau}\right)^2} \le r + \sqrt{\frac{\xi\rho}{\tau}}\, n_1 + \frac{\rho+1}{2\tau}.$$

The previous inequality also implies

$$\xi n_1^2 - \sqrt{\frac{\xi\rho}{\tau}}\, n_1 - r \le 0,$$

and

$$0 \le n_1 \le \frac{1}{2\xi}\left(\sqrt{\frac{\xi\rho}{\tau}} + \sqrt{\frac{\xi\rho}{\tau} + 4\xi r}\right) \le \frac{1}{\sqrt{\xi}}\left(\sqrt{\frac{\rho}{\tau}} + \sqrt{r}\right).$$

From the inequality $\lambda_{n_1+1}^{(1)} > r$ we also deduce,

$$r < \xi\left(n_1+1\right)^2 + \frac{\rho+1}{2\tau} - \sqrt{\frac{\xi\rho}{\tau}\left(n_1+1\right)^2 + \left(\frac{\rho+1}{2\tau}\right)^2} \le \xi\left(n_1+1\right)^2,$$

that is to say, $n_1 > \sqrt{r}/\sqrt{\xi} - 1$. Summarizing, n_1 is a nonnegative integer such that

$$\frac{\sqrt{r}}{\sqrt{\xi}} - 1 < n_1 \le \frac{\sqrt{r}}{\sqrt{\xi}} + \sqrt{\frac{\rho}{\xi\tau}}, \quad \forall r \ge 0. \tag{4.41}$$

We can repeat the arguments before for obtaining upper and lower bounds for n_2. Indeed, from the definition of $\mathcal{A}_2(r)$ and n_2, we get that n_2 is a natural number that satisfies $\lambda_{n_2}^{(2)} \le r$ and $\lambda_{n_2+1}^{(2)} > r$. The first inequality provides the estimate

$$r \ge \lambda_{n_2}^{(2)} \ge \xi n_2^2, \quad \text{i.e.,} \quad n_2 \le \frac{\sqrt{r}}{\sqrt{\xi}}.$$

On the other hand, n_2 is such that

$$0 < \lambda_{n_2+1}^{(2)} - r \le \xi\left(n_2+1\right)^2 + \sqrt{\frac{\xi\rho}{\tau}}\left(n_2+1\right) + \frac{\rho+1}{\tau} - r,$$

whence

$$\begin{aligned} n_2 + 1 &> \frac{1}{2\xi}\left[-\sqrt{\frac{\xi\rho}{\tau}} + \sqrt{\frac{\xi\rho}{\tau} + 4\xi\left(r - \frac{\rho+1}{\tau}\right)}\right] = \frac{1}{2\sqrt{\xi}}\left(-\sqrt{\frac{\rho}{\tau}} + \sqrt{4r - \frac{3\rho+4}{\tau}}\right) \\ &\ge \frac{1}{2\sqrt{\xi}}\left(2\sqrt{r} - \sqrt{\frac{\rho}{\tau}} - \sqrt{\frac{3\rho+4}{\tau}}\right). \end{aligned}$$

In the last inequality we have used that $\sqrt{a-b} \geq \sqrt{a} - \sqrt{b}$ provided $a, b > 0$ and $a \geq b$. In conclusion, we have proved that n_2 is a nonnegative integer such that

$$\frac{\sqrt{r}}{\sqrt{\xi}} - \frac{1}{2\sqrt{\xi}}\left(\sqrt{\frac{\rho}{\tau}} + \sqrt{\frac{3\rho+4}{\tau}}\right) - 1 \leq n_2 \leq \frac{\sqrt{r}}{\sqrt{\xi}}, \quad \forall r \geq 0. \tag{4.42}$$

Recall that $\mathcal{N}(r) = n_1 + n_2$. Thus, from inequalities (4.41) and (4.42), we can write

$$\frac{2}{\sqrt{\xi}}\sqrt{r} - \frac{1}{2\sqrt{\xi}}\left(\sqrt{\frac{\rho}{\tau}} + \sqrt{\frac{3\rho+4}{\tau}}\right) - 2 \leq \mathcal{N}(r) \leq \frac{2}{\sqrt{\xi}}\sqrt{r} + \sqrt{\frac{\rho}{\xi\tau}}, \quad \forall r \geq 0,$$

and deduce (4.26) with

$$p = \frac{2}{\sqrt{\xi}} \quad \text{and} \quad \alpha = \max\left\{\frac{1}{2\sqrt{\xi}}\left(\sqrt{\frac{\rho}{\tau}} + \sqrt{\frac{3\rho+4}{\tau}}\right) + 2, \sqrt{\frac{\rho}{\xi\tau}}\right\}.$$

This ends the proof. ■

We will finish this section giving a result on the set of eigenfunctions of the operators L and L^*. It reads as follows:

Proposition 10. *Let us consider the sequences* $\mathcal{F} = \{\Psi_k^{(1)}, \Psi_k^{(2)}\}_{k\geq 1}$ *and* $\mathcal{F}^* = \{\Phi_k^{(1)}, \Phi_k^{(2)}\}_{k\geq 1}$ *given in Proposition 7. Then,*

i) $\mathcal{F}$ *and* $\mathcal{F}^*$ *are biorthogonal sequences.*

ii) $\mathcal{F}$ *and* $\mathcal{F}^*$ *are dense in* $H^{-1}(0,\pi;\mathbb{R}^2)$, $L^2(0,\pi;\mathbb{R}^2)$ *and* $H_0^1(0,\pi;\mathbb{R}^2)$.

iii) $\mathcal{F}$ *and* $\mathcal{F}^*$ *are unconditional bases for* $H^{-1}(0,\pi;\mathbb{R}^2)$, $L^2(0,\pi;\mathbb{R}^2)$ *and* $H_0^1(0,\pi;\mathbb{R}^2)$.

Proof: From the expressions of $\Psi_k^{(j)}$ and $\Phi_k^{(j)}$ (see (4.30) and (4.31)) we can write

$$\Psi_k^{(j)}(\cdot) = V_{j,k}\eta_k(\cdot), \quad \text{and} \quad \Phi_k^{(j)} = V_{j,k}^*\eta_k(\cdot), \quad j = 1, 2, \quad k \geq 1,$$

where $V_{j,k}, V_{j,k}^* \in \mathbb{R}^2$ (the function η_k is given in (4.28)).

Item i) is simple to deduce, since $\{\eta_k\}_{k\geq 1}$ is an orthogonal basis for $H^{-1}(0,\pi)$, $H_0^1(0,\pi)$ and $L^2(0,\pi)$ (in this last case, an orthonormal basis) and $\{V_{1,k}, V_{2,k}\}_{k\geq 1}$ and $\{V_{1,k}^*, V_{2,k}^*\}_{k\geq 1}$ are biorthogonal basis of $\mathbb{R}^2$. Indeed, if $M_k = [V_{1,k}|V_{2,k}]$ and $N_k = [V_{1,k}^*|V_{2,k}^*]$, then,

$$M_k^{tr} N_k = M_k N_k^{tr} = Id, \quad \forall k \geq 1.$$

This proves item i).

For showing item ii) we only need to assure that $\mathcal{F}$ and $\mathcal{F}^*$ are dense in $H_0^1(0,\pi;\mathbb{R}^2)$, since $H_0^1(0,\pi;\mathbb{R}^2)$ is dense in $L^2(0,\pi;\mathbf{R}^2)$ and in $H^{-1}(0,\pi;\mathbb{R}^2)$. Let us consider $f = (f_1, f_2)^{tr} \in H^{-1}(0,\pi;\mathbb{R}^2)$ such that

$$\left\langle f, \Psi_k^{(i)} \right\rangle = 0, \quad \forall k \geq 1, \quad i = 1, 2.$$

(Recall that $\langle \cdot, \cdot \rangle$ stands for the usual duality pairing between $H^{-1}(0,1;\mathbf{R}^2)$ and $H_0^1(0,1;\mathbf{R}^2)$). If we denote $f_{i,k}$ ($i = 1, 2$) the corresponding Fourier coefficients of the distribution $f_i \in$

$H^{-1}(0,\pi)$ with respect to the sinus basis $\{\eta_k(\cdot)\}_{k\geq 1}$, then the previous equality can be written under the form

$$(f_{1,k}, f_{2,k})\, M_k = 0, \quad \forall k \geq 1.$$

Using that $\det M_k \neq 0$ for any $k \geq 1$, we deduce $f_{1,k} = f_{2,k} = 0$, for all $k \geq 1$ and, therefore, $f = 0$. This proves the density of $\mathcal{F}$ in $H_0^1(0,\pi;\mathbb{R}^2)$. A similar argument can be used for $\mathcal{F}^*$. This shows item *ii*).

Let us now prove item *iii*). As before, we will only prove that $\mathcal{F}$ is an unconditional basis for $H_0^1(0,\pi;\mathbf{R}^2)$. This amounts to prove that, for any $f = (f_1, f_2)^{tr} \in H_0^1(0,\pi;\mathbf{R}^2)$, the series

$$S(f) := \sum_{k\geq 1} \left(\left\langle \Phi_k^{(1)}, f \right\rangle \Psi_k^{(1)} + \left\langle \Phi_k^{(2)}, f \right\rangle \Psi_k^{(2)} \right)$$

is unconditionally convergent in $H_0^1(0,\pi;\mathbf{R}^2)$. From the definition of the functions $\Psi_k^{(i)}$ and $\Phi_k^{(i)}$ (see (4.30) and (4.31)), it is easy to see that

$$S(f) = \sum_{k\geq 1} \begin{pmatrix} f_{1,k} \\ f_{2,k} \end{pmatrix} \eta_k,$$

where $f_{i,k}$ is the Fourier coefficient of the function $f_i \in H_0^1(0,\pi)$ $(i = 1, 2)$. Accordingly, this series converges unconditionally in $H_0^1(0,\pi;\mathbb{R}^2)$ (recall that $\{\eta_k\}_{k\geq 1}$ is an orthogonal basis for $H_0^1(0,\pi)$ and $f_1, f_2 \in H_0^1(0,\pi)$). This concludes the proof of the result. ∎

4.4 Approximate and null controllability of the linear system (4.6)

We will devote this section to proving the approximate and null controllability at time $T > 0$ of system (4.6). To this aim, we will use in a fundamental way the properties of the spectrum of the operator L (see (4.10)) established in Propositions 7, 8 and 9. Firstly, we will show the result on approximate controllability of the linear system (Theorem 7) and then the null controllability at time T of the same system (Theorem 8).

4.4.1 Approximate controllability: Proof of Theorem 7

Let us fix $T > 0$ and consider system (4.6) with $\xi, \rho, \tau > 0$ given. Let us first assume that system (4.6) is approximate controllable at time T. In this case, condition (4.9) holds. Indeed, otherwise, thanks to property $(P2)$ of Proposition 8, the spectrum of the operator L is not simple, i.e., there exist $k, \ell \geq 1$ such that $\lambda_k^{(2)} = \lambda_\ell^{(1)} = \lambda_0$. Thus, if we take $a, b \in \mathbf{R}$, it is easy to see that the function

$$\varphi(x,t) = \left(a\, \Phi_\ell^{(1)}(x) + b\, \Phi_k^{(2)}(x) \right) e^{-\lambda_0 (T-t)}, \quad \forall (x,t) \in Q_T,$$

is the solution of the adjoint system (4.22) associated to the initial condition

$$\varphi_0 = a\Phi_\ell^{(1)} + b\Phi_k^{(2)}.$$

This function satisfies (see (4.8) and (4.31))

$$B^*D^*\varphi_x(0,t) = \xi\sqrt{\frac{2}{\pi\tau}}\left(a\frac{\ell}{\sqrt{r_\ell}} - b\frac{k}{\sqrt{r_k}}\right)e^{-\lambda_0(T-t)}, \quad \forall t \in (0,T).$$

Choosing

$$a = \frac{k}{\sqrt{r_k}} \quad \text{and} \quad b = \frac{\ell}{\sqrt{r_\ell}},$$

we have that $B^*D^*\varphi_x(0,\cdot) = 0$ but $\varphi_0 \neq 0$, contradicting the unique continuation property stated in the first point of Theorem 10. In conclusion, system (4.6) is not approximately controllable at time $T > 0$.

Let us now suppose that condition (4.9) holds and prove the unique continuation property for system (4.22). Again, from the first point of Theorem 10 we infer the approximate controllability property of system (4.6).

Let us consider $\varphi_0 \in H_0^1(0,\pi)$ and assume that the corresponding solution φ to the adjoint problem (4.22) satisfies

$$B^*D^*\varphi_x(0,t) = 0, \quad \forall t \in (0,T).$$

Observe that, thanks to Proposition 4

$$\varphi \in C^0([0,T]; H_0^1(0,\pi;\mathbb{R}^2)) \cap L^2(0,T; H^2(0,\pi;\mathbb{R}^2) \cap H_0^1(0,\pi;\mathbb{R}^2)),$$

and then, $B^*D^*\varphi_x(0,\cdot) \in L^2(0,T)$.

From Proposition 10, φ_0 can be written as $\varphi_0 = \sum_{k\geq 1}\left(a_k\Phi_k^{(1)} + b_k\Phi_k^{(2)}\right)$, where

$$a_k = \left\langle \Psi_k^{(1)}, \varphi_0 \right\rangle, \quad b_k = \left\langle \Psi_k^{(2)}, \varphi_0 \right\rangle, \quad \forall k \geq 1.$$

Using Proposition 7, the corresponding solution φ of system (4.22) associated to φ_0 is given by

$$\varphi(\cdot,t) = \sum_{k\geq 1}\left(a_k\Phi_k^{(1)}e^{-\lambda_k^{(1)}(T-t)} + b_k\Phi_k^{(2)}e^{-\lambda_k^{(2)}(T-t)}\right), \quad \forall t \in (0,T),$$

where $\lambda_k^{(i)}$, $\Psi_k^{(i)}$ and $\Phi_k^{(i)}$ ($k \geq 1$, $i = 1,2$) are given in Proposition 7. Therefore,

$$0 = B^*D^*\varphi_x(0,t) = \sum_{k\geq 1}\sqrt{\frac{2}{\pi}}\frac{k\xi}{\sqrt{\tau r_k}}\left(a_k e^{-\lambda_k^{(1)}(T-t)} - b_k e^{-\lambda_k^{(2)}(T-t)}\right), \quad \forall t \in (0,T).$$

From Proposition 9, we can apply Lemma 1 in order to deduce the existence of a biorthogonal family $\{q_k^{(1)}, q_k^{(2)}\}_{k\geq 1}$ to $\{e^{-\lambda_k^{(1)}t)}, e^{-\lambda_k^{(2)}t)}\}_{k\geq 1}$ in $L^2(0,T)$. Then, the previous identity, in particular, implies

$$\begin{cases} \sqrt{\frac{2}{\pi}}\frac{k\xi}{\sqrt{\tau r_k}}a_k = \int_0^T B^*D^*\varphi_x(0,t)\,q_k^{(1)}(t)\,dt = 0, & \forall k \geq 1, \\ \sqrt{\frac{2}{\pi}}\frac{k\xi}{\sqrt{\tau r_k}}b_k = -\int_0^T B^*D^*\varphi_x(0,t)\,q_k^{(2)}(t)\,dt = 0, & \forall k \geq 1, \end{cases}$$

and $a_k = b_k = 0$ for any $k \geq 1$. In conclusion, $\varphi_0 = 0$ and we have proved the unique continuation property for the solutions of system (4.22). This ends the proof of Theorem 7.

4.4.2 Null controllability: Proof of Theorem 8

Let us now prove the null controllability result stated in Theorem 8. To this aim, we consider ξ, ρ and τ three positive real numbers satisfying assumptions (4.9) and (4.11). We will obtain the proof writing the controllability problem for system (4.6) as a moment problem (see [39]).

Let us take $y_0 = (\theta_0, \phi_0) \in H^{-1}(0, \pi; \mathbf{R}^2)$. As a consequence of Proposition 6, we have that the control $v \in L^2(0, T)$ is such that the solution $y = (\theta, \phi) \in C^0([0, T]; H^{-1}(0, \pi; \mathbf{R}^2))$ of system (4.6) satisfies $y(\cdot, T) = 0$ if and only if $v \in L^2(0, T)$ fulfills

$$\int_0^T B^* D^* \varphi_x(0, t) v(t)\, dt = -\langle y_0, \varphi(\cdot, 0)\rangle, \quad \forall \varphi_0 \in H_0^1(0, \pi; \mathbf{R}^2),$$

where $\varphi \in C^0([0, T]; H_0^1(0, \pi; \mathbf{R}^2))$ is the solution of the adjoint system (4.22) associated to φ_0. Observe that from Proposition 10 we can deduce that the previous equality is equivalent to

$$\int_0^T B^* D^* \varphi_{k,x}^{(j)}(0, t) v(t)\, dt = -\left\langle y_0, \varphi_k^{(j)}(\cdot, 0)\right\rangle, \quad \forall k \geq 1, \quad j = 1, 2,$$

where $\varphi_k^{(j)}(\cdot, t) = e^{-\lambda_k^{(j)}(T-t)} \Phi_k^{(j)}$ is the solution of system (4.22) corresponding to $\varphi_0 = \Phi_k^{(j)}$. Taking into account the expressions of B, D and $\Phi_k^{(j)}$ (see (4.8) and (4.31)), we infer that $v \in L^2(0, T)$ is a null control for system (4.6) associated to y_0 if and only if

$$(-1)^{j+1} \sqrt{\frac{2}{\pi}} \frac{k\xi}{\sqrt{\tau r_k}} \int_0^T e^{-\lambda_k^{(j)}(T-t)} v(T-t)\, dt = e^{-\lambda_k^{(j)} T} \left\langle y_0, \Phi_k^{(j)}\right\rangle, \quad \forall k \geq 1, \quad j = 1, 2.$$

Summarizing, we have transformed the null-controllability problem at time $T > 0$ for system (4.6) into the following moment problem: given $y_0 = (\theta_0, \phi_0) \in H^{-1}(0, \pi; \mathbf{R}^2)$, find $v \in L^2(0, T)$ such that the function $u(t) := v(T - t) \in L^2(0, T)$ satisfies

$$\int_0^T e^{-\lambda_k^{(j)} t} u(t)\, dt = c_{kj}, \quad \forall k \geq 1, \quad j = 1, 2, \tag{4.1}$$

where $c_{kj} = c_{kj}(y_0)$ is given by

$$c_{kj} = (-1)^{j+1} \sqrt{\frac{\pi}{2}} \frac{\sqrt{\tau r_k}}{k\xi} e^{-\lambda_k^{(j)} T} \left\langle y_0, \Phi_k^{(j)}\right\rangle, \quad \forall k \geq 1, \quad j = 1, 2. \tag{4.2}$$

Our next task will be to solve problem (4.1). The assumptions (4.9) and (4.11), Proposition 9 and Lemma 2 guarantee the existence of $\widetilde{T}_0 > 0$ such that for any $T \in (0, \widetilde{T}_0)$ there exists a biorthogonal family $\{q_k^{(1)}, q_k^{(2)}\}_{k\geq 1}$ to $\{e^{-\lambda_k^{(1)} t}, e^{-\lambda_k^{(2)} t}\}_{k\geq 1}$ in $L^2(0, T)$ which satisfies

$$\left\| q_k^{(j)} \right\|_{L^2(0,T)} \leq C e^{C\sqrt{\lambda_k^{(j)}} + \frac{C}{T}}, \quad \forall k \geq 1, \quad j = 1, 2, \tag{4.3}$$

for a positive constant C independent of T.

Let us first prove the result when $T \in (0, \widetilde{T}_0)$. Then, a formal solution to the moment problem (4.1) is given by

$$u(t) := v(T - t) = \sum_{k\geq 1} \left(c_{k1} q_k^{(1)} + c_{k2} q_k^{(2)} \right). \tag{4.4}$$

Let us now prove that $u \in L^2(0,T)$ and, consequently, that $v \in L^2(0,T)$. From the expressions of r_k, $\lambda_k^{(j)}$ and $\Phi_k^{(j)}$ (see (4.27), (4.29) and (4.31)) we can easily deduce the existence of constants $C, C_1, C_2 > 0$ such that

$$C_1 k \le r_k \le C_2 k, \quad C_1 k^2 \le \left|\lambda_k^{(j)}\right| \le C_2 k^2, \quad \left\|\Phi_k^{(j)}\right\|_{H_0^1} \le C k^{3/2}, \quad \forall k \ge 1, \quad j = 1, 2,$$

and, from (4.2),

$$|c_{kj}| \le \frac{C}{\sqrt{k}}\, e^{-\lambda_k^{(j)} T} \, \|y_0\|_{H^{-1}} \left\|\Phi_k^{(j)}\right\|_{H_0^1} \le C\, k\, e^{-\lambda_k^{(j)} T} \, \|y_0\|_{H^{-1}}, \quad \forall k \ge 1, \quad j = 1, 2.$$

Coming back to the expression of the null control v (see (4.4)) and taking into account (4.3) and the previous inequality, we get

$$\begin{aligned}
\|v\|_{L^2(0,T)} &\le C\, e^{\frac{C}{T}} \|y_0\|_{H^{-1}} \sum_{k\ge 1} \left(e^{C\sqrt{\lambda_k^{(1)}}} e^{-\lambda_k^{(1)} T} + e^{C\sqrt{\lambda_k^{(2)}}} e^{-\lambda_k^{(2)} T} \right) \\
&\le C\, e^{\frac{C}{T}} \|y_0\|_{H^{-1}} \sum_{k\ge 1} \left(e^{\frac{C^2}{2T} + \frac{1}{2}\lambda_k^{(1)} T} e^{-\lambda_k^{(1)} T} + e^{\frac{C^2}{2T} + \frac{1}{2}\lambda_k^{(2)} T} e^{-\lambda_k^{(2)} T} \right) \\
&\le C\, e^{\frac{C}{T}} \|y_0\|_{H^{-1}} \sum_{k\ge 1} e^{-CTk^2} \le C\, e^{\frac{C}{T}} \|y_0\|_{H^{-1}} \int_0^\infty e^{-CTs^2}\, ds = \frac{C}{2}\sqrt{\frac{\pi}{CT}}\, e^{\frac{C}{T}} \|y_0\|_{H^{-1}} \\
&\le C_0\, e^{\frac{M}{T}} \|y_0\|_{H^{-1}},
\end{aligned} \tag{4.5}$$

for positive constants C_0 and M independent of T. This inequality shows that $v \in L^2(0,T)$ and proves the first part of Theorem 8.

The second part is a direct consequence of the expression of the null control v (see (4.4)) and (4.5). Indeed, if we define the operator $\mathcal{C}_T^{(0)} : H^{-1}(0,\pi;\mathbf{R}^2) \to L^2(0,T)$ by

$$\mathcal{C}_T^{(0)}(y_0) := \sum_{k\ge 1} \left(c_{k1}(y_0) q_k^{(1)}(T - \cdot) + c_{k2}(y_0) q_k^{(2)}(T - \cdot) \right), \quad \forall y_0 \in H^{-1}(0,\pi;\mathbf{R}^2),$$

with $c_{kj} = c_{kj}(y_0)$ given by (4.2), it is not difficult to see that $\mathcal{C}_T^{(0)}$ is a linear operator which satisfies (4.12) for a positive constants C_0 and M. This ends the proof of Theorem 8 when $T \in (0, \widetilde{T}_0)$.

Let us now assume that $T \ge \widetilde{T}_0$. We will obtain the proof as a consequence of the previous case. Indeed, if $T \ge \widetilde{T}_0$ we can construct a null control at time T for system (4.6) associated to $y_0 \in H^{-1}(0,\pi;\mathbf{R}^2)$ as

$$v(t) = \mathcal{C}_T^{(0)}(y_0)(t) := \begin{cases} \mathcal{C}_{\widetilde{T}_0/2}^{(0)}(y_0)(t) & \text{if } t \in \left[0, \dfrac{\widetilde{T}_0}{2}\right], \\ 0 & \text{if } t \in \left[\dfrac{\widetilde{T}_0}{2}, T\right]. \end{cases}$$

Clearly $\mathcal{C}_T^{(0)} \in \mathcal{L}\left(H^{-1}(0,\pi;\mathbb{R}^2), L^2(0,T)\right)$

$$\left\|\mathcal{C}_T^{(0)}(y_0)\right\|_{L^2(0,T)} \le C_0\, e^{2M/\widetilde{T}_0} \|y_0\|_{H^{-1}} = C_1 \|y_0\|_{H^{-1}}$$

with C_1 a new positive constant independent of T. So, we can conclude (4.12) for a new positive constant C_0 (only depending on the parameters in system (4.6)) and the same constant $M > 0$ as before. This finishes the proof of Theorem 8.

4.5 Boundary controllability of the phase-field system

In this section we will prove the exact controllability at time $T > 0$ of the phase-field system (4.1) to the constant trajectory $(0, c)$, with $c = \pm 1$. To this end, we will perform a fixed-point strategy which will use in a fundamental way a null controllability result for the non-homogeneous linear system (4.6) ($f \in L^2(0, \pi; \mathbf{R}^2)$ is a given function in an appropriate weighted-Lebesgue space; see (4.2)).

4.5.1 Null controllability of the non-homogeneous system (4.6)

As said before, our next objective will be to show a null controllability result for non-homogeneous system (4.6) when $y_0 = (\theta_0, \phi_0) \in H^{-1}(0, \pi; \mathbb{R}^2)$ and f is a given function satisfying appropriate assumptions. To this end, we will follow some ideas from [63].

Let us consider ξ, ρ and τ three positive real numbers satisfying hypotheses (4.9) and (4.11). The starting point is Theorem 8 and Remark 16. As a consequence, we obtain an estimate for the cost of the null control of system (4.6). With the notations of Remark 16, one has

$$\mathcal{K}(T) \leq C_0 \, e^{\frac{M}{T}}, \quad \forall T > 0,$$

with C_0 and M two positive constants only depending on ξ, ρ and τ.

In order to provide a null controllability result for the non-homogeneous problem (4.7) at time $T > 0$, we will introduce the functions $\gamma(t) := e^{\frac{M}{t}}$, $\forall t > 0$, and, for $t \in [0, T]$,

$$\rho_{\mathcal{F}}(t) := e^{-\frac{b^2(a+1)M}{(b-1)(T-t)}}, \quad \rho_0(t) := e^{-\frac{aM}{(b-1)(T-t)}}, \quad \forall t \in \left[T\left(1 - \frac{1}{b^2}\right), T\right], \tag{4.1}$$

extended to $\left[0, T(1 - 1/b^2)\right]$ in a constant way. Here $a, b > 1$ are constants that will be chosen later. Observe that γ, $\rho_{\mathcal{F}}$ and ρ_0 are continuous and non increasing functions in $[0, T]$ and $\rho_{\mathcal{F}}(T) = \rho_0(T) = 0$.

With the previous functions, we also introduce the weighted normed spaces

$$\mathcal{F} := \left\{ f \in L^2(Q_T; \mathbb{R}^2) : \frac{f}{\rho_{\mathcal{F}}} \in L^2(Q_T; \mathbb{R}^2) \right\}, \quad \mathcal{V} := \left\{ v \in L^2(0, T) : \frac{v}{\rho_0} \in L^2(0, T) \right\},$$
$$\mathcal{Y}_0 := \left\{ y \in L^2(Q_T; \mathbb{R}^2) : \frac{y}{\rho_0} \in L^2(Q_T; \mathbb{R}^2) \right\},$$
$$\mathcal{Y} := \left\{ y \in L^2(Q_T; \mathbb{R}^2) : \frac{y}{\rho_0} \in L^2(Q_T) \times C^0(\overline{Q}_T) \right\}. \tag{4.2}$$

It is clear that $\mathcal{F}$, $\mathcal{V}$ and $\mathcal{Y}_0$ are Hilbert spaces. For instance, the inner product in $\mathcal{F}$ is given by

$$(f_1, f_2)_{\mathcal{F}} := \iint_{Q_T} \rho_{\mathcal{F}}^{-2}(t) f_1(x, t) \cdot f_2(x, t) \, dx \, dt, \quad \forall f_1, f_2 \in \mathcal{F}.$$

A similar definition can be made for $(\cdot,\cdot)_{\mathcal{V}}$ and $(\cdot,\cdot)_{\mathcal{Y}_0}$. On the other hand, $\mathcal{Y}$ is a Banach space with the norm

$$\|y\|_{\mathcal{Y}} := \left(\|y_1/\rho_0\|^2_{L^2(Q_T)} + \|y_2/\rho_0\|^2_{C^0(\overline{Q}_T)}\right)^{1/2}, \quad \forall y = (y_1, y_2) \in \mathcal{Y}.$$

With the previous notation, one has:

Theorem 11. *Let us consider ξ, ρ and τ three positive real numbers satisfying (4.9) and (4.11). Then, for every $T > 0$, there exist two bounded linear operators*

$$\mathcal{C}_T^{(1)} : H^{-1}(0,\pi;\mathbb{R}^2) \times \mathcal{F} \to \mathcal{V} \quad \textit{and} \quad E_T^{(0)} : H^{-1}(0,\pi;\mathbb{R}^2) \times \mathcal{F} \to \mathcal{Y}_0$$

such that

(i) $\left\|\mathcal{C}_T^{(1)}\right\|_{\mathcal{L}(H^{-1}(0,\pi;\mathbf{R}^2)\times\mathcal{F},\mathcal{V})} \le C\, e^{C\left(T+\frac{1}{T}\right)}$ *and* $\left\|E_T^{(0)}\right\|_{\mathcal{L}(H^{-1}(0,\pi;\mathbf{R}^2)\times\mathcal{F},\mathcal{Y}_0)} \le C\, e^{C\left(T+\frac{1}{T}\right)}$ *for a positive constant C independent of T.*

(ii) $E_T^{(1)} := E_T^{(0)}\Big|_{H^{-1}(0,\pi)\times H_0^1(0,\pi)\times\mathcal{F}} \in \mathcal{L}(H^{-1}(0,\pi)\times H_0^1(0,\pi)\times\mathcal{F},\mathcal{Y})$ *and, for a new constant $C > 0$ independent of T, one has* $\left\|E_T^{(1)}\right\|_{\mathcal{L}(H^{-1}(0,\pi)\times H_0^1(0,\pi)\times\mathcal{F},\mathcal{Y})} \le C\, e^{C\left(T+\frac{1}{T}\right)}$.

(iii) For any $(y_0, f) \in H^{-1}(0,\pi;\mathbb{R}^2) \times \mathcal{F}$ (resp., $(y_0, f) \in H^{-1}(0,\pi) \times H_0^1(0,\pi) \times \mathcal{F}$), $y = E_T^{(0)}(y_0, f) \in \mathcal{Y}_0$ (resp. $y = E_T^{(1)}(y_0, f) \in \mathcal{Y}$) is the solution of (4.7) associated to (y_0, f) and $v = \mathcal{C}_T^{(1)}(y_0, f)$.

Remark 23. *Before giving the proof of this result, let us underline that Proposition 11 provides a null controllability result for the non-homogeneous system (4.7) when $y_0 \in H^{-1}(0,\pi;\mathbf{R}^2)$ and $f \in \mathcal{F}$. Indeed, since ρ_0 is a continuous function on $[0,T]$ satisfying $\rho_0(T) = 0$, it is clear that*

$$y = E_T^{(0)}(y_0, f) \in \mathcal{Y}_0 \cap C^0([0,T]; H^{-1}(0,\pi;\mathbb{R}^2)),$$

solves (4.7) and satisfies $y(\cdot, T) = 0$ in $H^{-1}(0,\pi;\mathbf{R}^2)$.

Proof of Theorem 11:

Let us consider $a, b > 1$ and $T > 0$. With the previous definitions and notations, we define the sequence

$$T_k = T - \frac{T}{b^k}, \quad \forall k \ge 0.$$

From the definition of the functions ρ_0 and $\rho_{\mathcal{F}}$ (see (4.1)) and the expression of T_k, one has

$$\rho_0(T_{k+2}) = \rho_{\mathcal{F}}(T_k) e^{\frac{M}{T_{k+2}-T_{k+1}}}, \quad \forall k \ge 0. \tag{4.3}$$

This formula will be used in what follows.

Let us take $y_0 = (\theta_0, \phi_0) \in H^{-1}(0,\pi;\mathbf{R}^2)$ (resp., $y_0 \in H^{-1}(0,\pi) \times H_0^1(0,\pi)$) and $f \in \mathcal{F}$. Thus, we introduce the sequence $\{a_k\}_{k\ge0} \subset H^{-1}(0,\pi;\mathbb{R}^2)$ (resp. $\{a_k\}_{k\ge0} \subset H^{-1}(0,\pi) \times H_0^1(0,\pi)$ if $y_0 \in H^{-1}(0,\pi) \times H_0^1(0,\pi)$) defined by

$$a_0 = y_0, \quad a_{k+1} = \tilde{y}_k(T_{k+1}^-), \quad \forall k \ge 0,$$

where $\tilde{y}_k$ is the solution to the linear system

$$\begin{cases} \tilde{y}_t - D\tilde{y}_{xx} + A\tilde{y} = f & \text{in } (0,\pi)\times(T_k,T_{k+1}) := Q_k, \\ \tilde{y}(0,\cdot) = \tilde{y}(\pi,\cdot) = 0 & \text{on } (T_k,T_{k+1}) \\ \tilde{y}(\cdot,T_k^+) = 0 & \text{in } (0,\pi), \end{cases} \tag{4.4}$$

(the matrices D and A are given in (4.8)). From Proposition 4, it is clear that this system admits a unique solution

$$\tilde{y}_k \in L^2(T_k,T_{k+1};H^2(0,\pi)\cap H_0^1(0,\pi)) \cap C^0([T_k,T_{k+1}];H_0^1(0,\pi;\mathbf{R}^2))$$

which satisfies (4.17). In particular, $\tilde{y}_k \in C^0(\overline{Q}_k;\mathbf{R}^2)$ and $a_{k+1} \in H_0^1(0,\pi;\mathbf{R}^2)$, for any $k \geq 0$, and

$$\|\tilde{y}_k\|_{C^0(\overline{Q}_k;\mathbf{R}^2)} + \|a_{k+1}\|_{H_0^1} \leq e^{CT}\|f\|_{L^2(Q_k;\mathbb{R}^2)}, \quad \forall k \geq 0, \tag{4.5}$$

where C is a positive constant only depending on the coefficients of D and A.

For $k \geq 0$, we also consider the controlled autonomous problem

$$\begin{cases} \hat{y}_t - D\hat{y}_{xx} + A\hat{y} = 0 & \text{in } Q_k, \\ \hat{y}(0,\cdot) = Bv_k, \quad \hat{y}(\pi,\cdot) = 0 & \text{on } (T_k,T_{k+1}) \\ \hat{y}(\cdot,T_k^+) = a_k, \quad \hat{y}(\cdot,T_{k+1}^-) = 0 & \text{in } (0,\pi), \end{cases} \tag{4.6}$$

where the control v_k is given by $v_k = \mathcal{C}^{(0)}_{T_{k+1}-T_k}(a_k) \in L^2(T_k,T_{k+1})$ (the linear operator $\mathcal{C}^{(0)}_{T_{k+1}-T_k}$ is given in Theorem 8). Thanks to Proposition 5, the solution $\hat{y}_k$ of the previous system satisfies

$$\begin{cases} \hat{y}_0 \in L^2(Q_0;\mathbb{R}^2) \quad (\text{resp., } \hat{y}_0 \in L^2(Q_0;\mathbb{R}^2)\cap C^0([0,T_1];H^{-1}(0,\pi)\times H_0^1(0,\pi)), \\ \hat{y}_k \in L^2(Q_k;\mathbb{R}^2)\cap C^0([T_k,T_{k+1}];H^{-1}(0,\pi)\times H_0^1(0,\pi)), \quad \forall k \geq 1 \end{cases}$$

and, from (4.19) (resp., (4.21)), (4.5) and Theorem 8,

$$\begin{cases} \|\hat{y}_0\|_{L^2(Q_0;\mathbf{R}^2)} \leq e^{CT_1}\left(\|y_0\|_{H^{-1}} + \|v_0\|_{L^2(0,T_1)}\right) \leq C_0\, e^{CT} e^{\frac{M}{T_1}}\|y_0\|_{H^{-1}} \\ (\text{resp., } \|\hat{y}_0\|_{L^2(Q_0)\times C^0(\overline{Q}_0)} \leq C_0\, e^{CT} e^{\frac{M}{T_1}}\|y_0\|_{H^{-1}\times H_0^1}), \end{cases}$$

and, for any $k \geq 1$,

$$\|\hat{y}_k\|_{L^2(Q_k)\times C^0(\overline{Q}_k)} \leq e^{CT}\left(\|a_k\|_{H^{-1}\times H_0^1} + \|v_k\|_{L^2(T_k,T_{k+1})}\right) \leq C_0 e^{CT} e^{\frac{M}{T_{k+1}-T_k}}\|f\|_{L^2(Q_k;\mathbf{R}^2)}.$$

If we set $Y_k := \tilde{y}_k + \hat{y}_k$ in $Q_k = (0,\pi)\times(T_k,T_{k+1})$, then

$$\begin{cases} Y_0 \in L^2(Q_0;\mathbb{R}^2) \quad (\text{resp., } Y_0 \in L^2(Q_0;\mathbb{R}^2)\cap C^0([0,T_1];H^{-1}(0,\pi)\times H_0^1(0,\pi)), \\ Y_k \in L^2(Q_k;\mathbb{R}^2)\cap C^0([T_k,T_{k+1}];H^{-1}(0,\pi)\times H_0^1(0,\pi)), \quad \forall k \geq 1 \end{cases}$$

and

$$\begin{cases} \|Y_0\|_{L^2(Q_0;\mathbf{R}^2)} \leq C\, e^{CT} e^{\frac{M}{T_1}}\left(\|y_0\|_{H^{-1}} + \|f\|_{L^2(Q_0;\mathbb{R}^2)}\right) \\ (\text{resp., } \|Y_0\|_{L^2(Q_0)\times C^0(\overline{Q}_0)} \leq C\, e^{CT} e^{\frac{M}{T_1}}\left(\|y_0\|_{H^{-1}\times H_0^1} + \|f\|_{L^2(Q_0;\mathbb{R}^2)}\right)), \\ \|Y_k\|_{L^2(Q_k)\times C^0(\overline{Q}_k)} \leq C\, e^{CT} e^{\frac{M}{T_{k+1}-T_k}}\|f\|_{L^2(Q_k;\mathbf{R}^2)}, \quad \forall k \geq 1. \end{cases} \tag{4.7}$$

Let us divide the proof into two cases: the case $k=0$ and the case $k\geq 1$.

Case $k=0$**.** First, from Theorem 8, we can use that $bT_1 = T(b-1)$ to obtain (recall that $v_0 = \mathcal{C}^{(0)}_{T_1}(y_0)$)

$$\|v_0\|_{L^2(0,T_1)} \leq C_0\, e^{\frac{M}{T_1}} \|y_0\|_{H^{-1}} = C_0\, e^{\frac{Mb(a+1)}{(b-1)T}} \rho_0(T_1)\|y_0\|_{H^{-1}}.$$

Using now that ρ_0 is a positive continuous non-increasing function, from the previous estimate, we deduce the existence of a positive constant C such that

$$\left\|\frac{v_0}{\rho_0}\right\|_{L^2(0,T_1)} \leq C\, e^{\frac{C}{T}} \|y_0\|_{H^{-1}}. \tag{4.8}$$

On the other hand, from (4.7),

$$\begin{cases} \|Y_0\|_{L^2(Q_0;\mathbf{R}^2)} \leq C\, e^{CT} e^{\frac{M}{T_1}} \left(\|y_0\|_{H^{-1}} + \|f\|_{L^2(Q_0;\mathbb{R}^2)}\right) \\ \qquad\qquad = C\, e^{CT}\, e^{\frac{Mb(a+1)}{(b-1)T}} \rho_0(T_1) \left(\|y_0\|_{H^{-1}} + \|f\|_{L^2(Q_0;\mathbb{R}^2)}\right), \\ (\text{resp., } \|Y_0\|_{L^2(Q_0)\times C^0(\overline{Q}_0)} \leq C\, e^{CT}\, e^{\frac{Mb(a+1)}{(b-1)T}} \rho_0(T_1) \Big(\|y_0\|_{H^{-1}\times H^1_0} + \|f\|_{L^2(Q_0;\mathbb{R}^2)}\Big)). \end{cases}$$

Observe that $\|f\|_{L^2(Q;\mathbb{R}^2)} \leq \|f\|_{\mathcal{F}}$ (see the expression of $\rho_{\mathcal{F}}$ in (4.1)). Hence, repeating the previous argument, we get

$$\begin{cases} \left\|\frac{Y_0}{\rho_0}\right\|_{L^2(Q_0;\mathbf{R}^2)} \leq C\, e^{C\left(T+\frac{1}{T}\right)} \left(\|y_0\|_{H^{-1}} + \|f\|_{\mathcal{F}}\right) \\ (\text{resp., } \left\|\frac{Y_0}{\rho_0}\right\|_{L^2(Q_0)\times C^0(\overline{Q}_k)} \leq C\, e^{C\left(T+\frac{1}{T}\right)} \Big(\|y_0\|_{H^{-1}\times H^1_0} + \|f\|_{\mathcal{F}}\Big)). \end{cases} \tag{4.9}$$

Case $k\geq 1$**.** Again, taking into account formula $v_k = \mathcal{C}^{(0)}_{T_{k+1}-T_k}(a_k)$, Theorem 8, (4.5) and (4.3), we infer

$$\begin{aligned} \|v_k\|_{L^2(T_k,T_{k+1})} &\leq C\, e^{\frac{M}{T_{k+1}-T_k}} \|a_k\|_{H^{-1}} \leq C\, e^{CT}\, e^{\frac{M}{T_{k+1}-T_k}} \|f\|_{L^2(Q_{k-1};\mathbf{R}^2)} \\ &= C\, e^{CT} \frac{\rho_0(T_{k+1})}{\rho_{\mathcal{F}}(T_{k-1})} \|f\|_{L^2(Q_{k-1};\mathbf{R}^2)}. \end{aligned}$$

As in the case $k=0$, using the fact that ρ_0 and $\rho_{\mathcal{F}}$ are non-increasing functions, from the previous inequality, we deduce

$$\left\|\frac{v_k}{\rho_0}\right\|_{L^2(T_k,T_{k+1})} \leq C\, e^{CT} \left\|\frac{f}{\rho_{\mathcal{F}}}\right\|_{L^2(Q_{k-1};\mathbf{R}^2)}, \quad \forall k\geq 1. \tag{4.10}$$

We can also repeat the previous argument to obtain an estimate for Y_k when $k\geq 1$. From (4.7),

$$\begin{cases} \|Y_k\|_{L^2(Q_k)\times C^0(\overline{Q}_k)} \leq C\, e^{CT} e^{\frac{M}{T_{k+1}-T_k}} \|f\|_{L^2(Q_k;\mathbf{R}^2)} = C\, e^{CT} \frac{\rho_0(T_{k+1})}{\rho_{\mathcal{F}}(T_{k-1})} \|f\|_{L^2(Q_k;\mathbf{R}^2)} \\ \qquad\qquad \leq C\, e^{CT} \frac{\rho_0(T_{k+1})}{\rho_{\mathcal{F}}(T_k)} \|f\|_{L^2(Q_k;\mathbf{R}^2)}, \end{cases}$$

what implies

$$\left\| \frac{Y_k}{\rho_0} \right\|_{L^2(Q_k)\times C^0(\overline{Q}_k)} \leq C\, e^{CT} \left\| \frac{f}{\rho_{\mathcal{F}}} \right\|_{L^2(Q_k;\mathbf{R}^2)}, \quad \forall k \geq 1. \tag{4.11}$$

With the functions v_k and Y_k, $k \geq 0$, defined above, we define

$$\mathcal{C}_T^{(1)}(y_0, f) := v = \sum_{k\geq 0} v_k 1_{[T_k,T_{k+1})} \quad \text{and} \quad E_T^{(0)}(y_0, f) := Y = \sum_{k\geq 0} Y_k 1_{[T_k,T_{k+1})}, \tag{4.12}$$

where 1_I is the characteristic function on the set I. Let us first remark that, by construction, $\mathcal{C}_T^{(1)}$ and $E_T^{(0)}$ are linear operators. On the other hand, recall that $Y_k = \tilde{y}_k + \hat{y}_k$, $k \geq 0$, where $\tilde{y}_k$ and $\hat{y}_k$ are respectively the solution to systems (4.4) and (4.6). So,

$$Y_k(T_{k+1}^-) = a_{k+1} = \hat{y}_{k+1}(T_{k+1}^+) = Y_{k+1}(T_{k+1}^+), \quad \forall k \geq 0,$$

which implies that the function Y is continuous at time T_k, for any $k \geq 1$, and is the solution of system (4.7) associated to (y_0, f, v).

Finally, thanks to (4.8)–(4.11), we also deduce that $\mathcal{C}_T^{(1)}(y_0, f) \in \mathcal{V}$ and $E_T^{(0)}(y_0, f) \in \mathcal{Y}_0$ (resp., $E_T^{(0)}(y_0, f) \in \mathcal{Y}$) for any $(y_0, f) \in H^{-1}(0,\pi;\mathbf{R}^2)\times\mathcal{F}$ (resp., for any $(y_0, f) \in H^{-1}(0,\pi)\times H_0^1(0,\pi)\times\mathcal{F}$) and

$$\begin{cases} \left\|\mathcal{C}_T^{(1)}(y_0, f)\right\|_{\mathcal{V}} = \|v\|_{\mathcal{V}} \leq C\, e^{C\left(T+\frac{1}{T}\right)} \left(\|y_0\|_{H^{-1}} + \|f\|_{\mathcal{F}}\right), \\ \left\|E_T^{(0)}(y_0, f)\right\|_{\mathcal{Y}_0} = \|Y\|_{\mathcal{Y}_0} \leq C\, e^{C\left(T+\frac{1}{T}\right)} \left(\|y_0\|_{H^{-1}} + \|f\|_{\mathcal{F}}\right), \quad \forall (y_0, f) \in H^{-1}(0,\pi;\mathbf{R}^2)\times\mathcal{F}, \end{cases}$$

(resp.,

$$\left\|E_T^{(0)}(y_0, f)\right\|_{\mathcal{Y}} = \|Y\|_{\mathcal{Y}} \leq Ce^{C\left(T+\frac{1}{T}\right)} \left(\|y_0\|_{H^{-1}\times H_0^1} + \|f\|_{\mathcal{F}}\right), \\ \forall (y_0, f) \in H^{-1}(0,\pi)\times H_0^1(0,\pi)\times\mathcal{F}).$$

The above estimates provide the proof of Proposition 11. This ends the proof. ■

4.5.2 Proof of Theorem 9

We will devote this section to proving the local exact controllability at time $T > 0$ of the phase-field system (4.1) stated in Theorem 9. To this objective, let us take

$$\tilde{y}_0 = (\tilde{\theta}_0, \tilde{\phi}_0) \in H^{-1}(0,\pi)\times(c + H_0^1(0,\pi))$$

($c = \pm 1$). As we saw in Section 4.1, the local exact controllability of system (4.1) at time T to the constant trajectory $(0, c)$ is equivalent to the local null controllability of system (4.4) at time T with $y_0 = (\theta_0, \phi_0) = (\tilde{\theta}_0, \tilde{\phi}_0 - c) \in H^{-1}(0,\pi)\times H_0^1(0,\pi)$ (the nonlinear functions g_1 and g_2 are given in (4.5)).

Let us take $a, b > 1$ (which will be determined below) and consider the functions $\rho_{\mathcal{F}}$ and ρ_0, defined in (4.1), and the spaces $\mathcal{F}$, $\mathcal{V}$ and $\mathcal{Y}$ given in (4.2). In order to prove the local null controllability result at time T for system (4.4) we will perform a fixed-point strategy in the space $\mathcal{Y}$ which, in particular, will prove the existence of a control $v \in \mathcal{V}$ such that system (4.4) has a solution $y \in \mathcal{Y}$ associated to (v, y_0). The condition $y \in \mathcal{Y}$ will imply the null controllability result for this system.

Let us fix $\varepsilon > 0$ (to be determined bellow). With the previous data and notations, we consider the closed ball in the space $\mathcal{F}$

$$\overline{B}_\varepsilon = \{f \in \mathcal{F} : \|f\|_{\mathcal{F}} \leq \varepsilon\}.$$

Observe that if the initial datum $\tilde{y}_0 \in H^{-1}(0,\pi) \times (c + H_0^1(0,\pi))$ satisfies (4.15), then $y_0 = (\theta_0, \phi_0) = (\tilde{\theta}_0, \tilde{\phi}_0 - c) \in H^{-1}(0,\pi) \times H_0^1(0,\pi)$ satisfies

$$\|\theta_0\|_{H^{-1}} + \|\phi_0\|_{H_0^1} \leq \varepsilon. \tag{4.13}$$

For each $f \in \overline{B}_\varepsilon \subset \mathcal{F}$, we denote $v_f = \mathcal{C}_T^{(1)}(y_0, f) \in \mathcal{V}$ and $y_f = (\theta_f, \phi_f) := E_T^{(1)}(y_0, f) \in \mathcal{Y}$, where the operators $\mathcal{C}_T^{(1)}$ and $E_T^{(1)}$ are given in Theorem 11. As a consequence of this result and (4.13), one has

$$\|y_f\|_{\mathcal{Y}} + \|v_f\|_{\mathcal{V}} \leq C\, e^{C\left(T + \frac{1}{T}\right)} \left(\|y_0\|_{H^{-1}\times H_0^1} + \|f\|_{\mathcal{F}}\right) \leq C\, e^{C\left(T+\frac{1}{T}\right)}\, \varepsilon, \quad \forall f \in \overline{B}_\varepsilon, \tag{4.14}$$

for a positive constant $C = C(\xi, \rho, \tau)$. Thus, we define the nonlinear operator $\mathcal{N} : \overline{B}_\varepsilon \to C^0(\overline{Q}_T; \mathbf{R}^2)$ given by (see (4.5))

$$\mathcal{N}(f) = \begin{pmatrix} \pm\dfrac{3\rho}{4\tau}\phi_f^2 + \dfrac{\rho}{4\tau}\phi_f^3 \\ \mp\dfrac{3}{2\tau}\phi_f^2 - \dfrac{1}{2\tau}\phi_f^3 \end{pmatrix}. \tag{4.15}$$

It is clear that the operator $\mathcal{N}$ is well-defined. On the other hand, if $\mathcal{N}$ admits a fixed point $f \in \mathcal{F}$, then $y_f \in \mathcal{Y}$, together with $v_f \in \mathcal{V}$, provides a solution of the system (4.4) associated to the initial datum $y_0 = (\theta_0, \phi_0)$. In fact, from Proposition 5, $y_f \in C^0([0,T]; H^{-1}(0,\pi;\mathbf{R}^2))$. Finally, condition $y_f \in C^0([0,T]; H^{-1}(0,\pi;\mathbf{R}^2)) \cap \mathcal{Y}$ in particular implies the null controllability result for system (4.4). This would prove Theorem 9.

The next task is to prove that the operator $\mathcal{N}$ has a fixed-point in the complete metric space $\overline{B}_\varepsilon \subset \mathcal{F}$. To this end, we will apply the Banach Fixed-Point Theorem. Before, let us select any $a > 1$ and b such that

$$b^2 \in \left(1, \frac{2a}{a+1}\right).$$

With this choice, the functions $\rho_0^2/\rho_{\mathcal{F}}$ and $\rho_0^3/\rho_{\mathcal{F}}$ are uniformly bounded in $[0,T]$, i.e., there exists a constant $C_T > 0$, depending on T, such that

$$\left\|\frac{\rho_0^2}{\rho_{\mathcal{F}}}\right\|_{C^0[0,T]} \leq C_T \quad \text{and} \quad \left\|\frac{\rho_0^3}{\rho_{\mathcal{F}}}\right\|_{C^0[0,T]} \leq C_T.$$

Let us now check the assumptions of the Banach Fixed-Point Theorem:

1. $\mathcal{N}(\overline{B}_\varepsilon) \subset \overline{B}_\varepsilon$: Indeed, if $f \in \overline{B}_\varepsilon$, then, from (4.14), we obtain

$$\begin{aligned} \|\mathcal{N}(f)\|_{\mathcal{F}} &\leq C_T \left\|\frac{\mathcal{N}(f)}{\rho_{\mathcal{F}}}\right\|_{C^0(\overline{Q}_T;\mathbb{R}^2)} \leq C_T \left(\left\|\frac{\phi_f^2}{\rho_{\mathcal{F}}}\right\|_{C^0(\overline{Q}_T)} + \left\|\frac{\phi_f^3}{\rho_{\mathcal{F}}}\right\|_{C^0(\overline{Q}_T)}\right) \\ &\leq C_T \left(\left\|\frac{\rho_0^2}{\rho_{\mathcal{F}}}\right\|_{C^0(\overline{Q}_T)} \left\|\frac{\phi_f}{\rho_0}\right\|^2_{C^0(\overline{Q}_T)} + \left\|\frac{\rho_0^3}{\rho_{\mathcal{F}}}\right\|_{C^0(\overline{Q}_T)} \left\|\frac{\phi_f}{\rho_0}\right\|^3_{C^0(\overline{Q}_T)}\right) \\ &\leq C_T \left(\|y_f\|^2_{\mathcal{Y}} + \|y_f\|^3_{\mathcal{Y}}\right) \leq C_T\, e^{C\left(T+\frac{1}{T}\right)} \left(\varepsilon^2 + \varepsilon^3\right) \leq \varepsilon \end{aligned}$$

for $\varepsilon = \varepsilon(T)$ small enough.

2. $\mathcal{N}$ is a contraction map: Let us take $f_1, f_2 \in \overline{B}_\varepsilon \subset \mathcal{F}$ and denote $y_i = (\theta_i, \phi_i) = E_T^{(1)}(y_0, f_i) \in \mathcal{Y}$, $i = 1, 2$. Firstly, observe that the non linearity (g_1, g_2), given in (4.5), satisfies

$$|g_j(s_1) - g_j(s_2)| \leq C(|s_1|^2 + |s_2|^2 + |s_1| + |s_2|)|s_1 - s_2|, \quad \forall s_1, s_2 \in \mathbf{R}, \quad j = 1, 2.$$

Thus, using again (4.14) and Theorem 11, we have

$$\begin{aligned}
&\|\mathcal{N}(f_1) - \mathcal{N}(f_2)\|_{\mathcal{F}} \\
&\qquad \leq C_T \sum_{j=1}^{2} \left\| \frac{g_j(\phi_1) - g_j(\phi_2)}{\rho_{\mathcal{F}}} \right\|_{C^0(\overline{Q}_T)} \\
&\qquad \leq C_T \left\| \frac{\rho_0}{\rho_{\mathcal{F}}} \left(|\phi_1|^2 + |\phi_2|^2 + |\phi_1| + |\phi_2|\right) \frac{|\phi_1 - \phi_2|}{\rho_0} \right\|_{C^0(\overline{Q}_T)} \\
&\qquad \leq C_T \left\| \left(\left|\frac{\phi_1}{\rho_0}\right|^2 + \left|\frac{\phi_2}{\rho_0}\right|^2 \right) \frac{\rho_0^3}{\rho_{\mathcal{F}}} + \left(\left|\frac{\phi_1}{\rho_0}\right| + \left|\frac{\phi_2}{\rho_0}\right| \right) \frac{\rho_0^2}{\rho_{\mathcal{F}}} \right\|_{C^0(\overline{Q}_T)} \left\| \frac{\phi_1 - \phi_2}{\rho_0} \right\|_{C^0(\overline{Q}_T)} \\
&\qquad \leq C_T \left(\|y_1\|_{\mathcal{Y}}^2 + \|y_2\|_{\mathcal{Y}}^2 + \|y_1\|_{\mathcal{Y}} + \|y_2\|_{\mathcal{Y}} \right) \left\| E_T^{(1)}(y_0, f_1) - E_T^{(1)}(y_0, f_2) \right\|_{\mathcal{Y}} \\
&\qquad \leq C_T \, e^{C\left(T + \frac{1}{T}\right)} \left(\varepsilon^2 + \varepsilon\right) \|f_1 - f_2\|_{\mathcal{F}} \,.
\end{aligned}$$

From this inequality it is clear that we can choose $\varepsilon = \varepsilon(T)$ (small enough) in such a way that $\mathcal{N}$ is a contraction map.

In conclusion, we can apply the Banach Fixed-Point Theorem. This proves that the operator $\mathcal{N}$ has a fixed-point and provides the proof of Theorem 9.

4.6 Appendix

This section is devoted to prove the appendices.

4.6.1 Appendix A

This appendix will deal with the existence and uniqueness of solution of the linear systems (4.16) and (4.7). To be precise, we will prove Propositions 4 and 5.

Proof of Proposition 4: Let us assume that $\varphi_0 \in H_0^1(0, \pi; \mathbb{R}^2)$ and $g \in L^2(Q_T; \mathbb{R}^2)$. Let us denote $\varphi_0 = (\theta_0, \phi_0)$ and $g = (g_1, g_2)$. Then the system (4.16) can be write as

$$\begin{cases}
-\theta_t - \xi\theta_{xx} + \dfrac{\rho}{\tau}\theta - \dfrac{2}{\tau}\phi = g_1 & \text{in } Q_T, \\
-\phi_t - \xi\phi_{xx} + \dfrac{1}{2}\rho\xi\theta_{xx} - \dfrac{\rho}{2\tau}\theta + \dfrac{1}{\tau}\phi = g_2 & \text{in } Q_T, \\
\theta(0, \cdot) = \phi(0, \cdot) = \theta(\pi, \cdot) = \phi(\pi, \cdot) = 0 & \text{on } (0, T), \\
\theta(\cdot, T) = \theta_0, \quad \phi(\cdot, T) = \phi_0 & \text{in } (0, \pi),
\end{cases}$$

where $\varphi = (\theta, \phi)$. On the other hand, $\xi\theta_{xx} = -\theta_t + \frac{\rho}{\tau}\theta - \frac{2}{\tau}\phi - g_1$. Thus, the previous system becomes

$$\begin{cases} -\theta_t - \xi\theta_{xx} - \frac{2}{\tau}\phi + \frac{\rho}{\tau}\theta = g_1 & \text{in } Q_T, \\ -\phi_t - \xi\phi_{xx} - \frac{\rho}{2}\theta_t - \frac{\rho-1}{\tau}\phi + \frac{\rho(\rho-1)}{2\tau}\theta = \frac{\rho}{2}g_1 + g_2 & \text{in } Q_T, \\ \theta(0,\cdot) = \phi(0,\cdot) = \theta(\pi,\cdot) = \phi(\pi,\cdot) = 0 & \text{on } (0,T), \\ \theta(\cdot,T) = \theta_0, \quad \phi(\cdot,T) = \phi_0 & \text{in } (0,\pi), \end{cases} \tag{4.16}$$

Then, Proposition 4 is equivalent to prove that the system (4.16) has a unique strong solution (θ, ϕ) satisfying

$$\theta, \phi \in C^0([0,T]; H_0^1(0,\pi)) \cap L^2(0,T; H^2(0,\pi) \cap H_0^1(0,\pi))$$

and

$$\begin{aligned} &\|\theta\|_{C^0(H_0^1)} + \|\phi\|_{C^0(H_0^1)} + \|\theta\|_{L^2(H^2\cap H_0^1)} + \|\phi\|_{L^2(H^2\cap H_0^1)} \\ &\qquad \le e^{CT}\left(\|g_1\|_{L^2(L^2)} + \|g_2\|_{L^2(L^2)} + \|\theta_0\|_{H_0^1} + \|\phi_0\|_{H_0^1}\right). \end{aligned} \tag{4.17}$$

for a positive constant C, only depending on ξ, ρ and τ.

We will use the well-known Faedo-Galerkin method. First, let us consider the orthonormal basis $\{\eta_n\}_{n\in I\!N}$ of $L^2(0,\pi)$ (η_n is the normalized eigenfunction of the Dirichlet-Laplace operator, see (4.28)). For each $m \in \mathbb{N}$, we consider $V_m = [\eta_1, \eta_2, \cdots, \eta_m]$, the subspace generated by the first m vectors of $\{\eta_n\}_{n\in\mathbb{N}}$. Let us also consider P_m, the orthogonal projection operator onto the finite-dimensional space V_m in $L^2(0,\pi)$. If we define

$$\theta_0^m = P_m\theta_0, \quad \phi_0^m = P_m\phi_0, \quad g_1^m(\cdot,t) = P_m g_1(\cdot,t) \quad \text{and} \quad g_2^m(t,\cdot) = P_m g_2(t,\cdot), \tag{4.18}$$

one has $\theta_0^m, \phi_0^m \in V_m$ and $g_1^m, g_2^m \in L^2(0,T;V_m)$, for any $m \in I\!N$, and

$$\theta_0^m \to \theta_0, \quad \phi_0^m \to \phi_0 \text{ in } H_0^1(0,\pi), \quad \text{and} \quad g_1^m \to g_1, \quad g_2^m \to g_2 \text{ in } L^2(Q_T), \quad \text{as } m \to \infty. \tag{4.19}$$

We want an approximate solution $(\theta^m, \phi^m) \in C^0([0,T]; V_m^2)$ of the approximate problem

$$\begin{cases} -\theta_t^m - \xi\theta_{xx}^m - \frac{2}{\tau}\phi^m + \frac{\rho}{\tau}\theta^m = g_1^m & \text{in } Q_T, \\ -\phi_t^m - \xi\phi_{xx}^m - \frac{\rho}{2}\theta_t^m - \frac{\rho-1}{\tau}\phi^m + \frac{\rho(\rho-1)}{2\tau}\theta^m = \frac{\rho}{2}g_1^m + g_2^m & \text{in } Q_T, \\ \theta^m(0,\cdot) = \phi^m(0,\cdot) = \theta^m(\pi,\cdot) = \phi^m(\pi,\cdot) = 0 & \text{on } (0,T), \\ \theta^m(\cdot,T) = \theta_0^m, \quad \phi^m(\cdot,T) = \phi_0^m & \text{in } (0,\pi), \end{cases} \tag{4.20}$$

under the form

$$\theta^m(x,t) = \sum_{j=1}^m \alpha_{jm}(t)\eta_j(x), \quad \phi^m(x,t) = \sum_{j=1}^m \beta_{jm}(t)\eta_j(x), \quad (x,t) \in Q_T.$$

It is clear that, for any $m \geq 1$, system (4.20) is equivalent to a Cauchy problem for a linear ordinary differential system for the variables α_{jm} and β_{jm}, $1 \leq j \leq m$. In consequence, system (4.20) admits a unique solution $(\theta^m, \phi^m) \in C^0([0,T]; V_m^2)$ with $(\theta_t^m, \phi_t^m) \in L^2(0,T; V_m^2)$.

The proof of Proposition 4 can be easily deduced from appropriate estimates of the approximate solution (θ^m, ϕ^m) of system (4.20).

If we multiply the first equation in (4.20) by $-\frac{\rho}{2}\theta_t^m$, the second one by $\frac{2}{\tau}\phi^m$, we integrate on the interval $(0,\pi)$ and we add both equalities, we get,

$$\begin{aligned}
\int_0^\pi \left(\frac{\rho}{2}|\theta_t^m|^2 - \frac{1}{\tau}\frac{d}{dt}(|\phi^m|^2) - \frac{\rho}{\tau}\theta_t^m\phi^m \right) dx + \int_0^\pi \left(-\frac{\rho\xi}{4}\frac{d}{dt}(|\theta_x^m|^2) + \frac{2\xi}{\tau}|\phi_x^m|^2 \right) dx \\
+ \int_0^\pi \left(\frac{\rho}{\tau}\phi^m\theta_t^m - \frac{2(\rho-1)}{\tau^2}|\phi^m|^2 \right) dx + \int_0^\pi \left(-\frac{\rho^2}{2\tau}\theta^m\theta_t^m + \frac{\rho(\rho-1)}{\tau^2}\theta^m\phi^m \right) dx \\
= -\frac{\rho}{2}\int_0^\pi g_1^m\theta_t^m dx - \frac{2}{\tau}\int_0^\pi \left(\frac{\rho}{2}g_1^m + g_2^m \right)\phi^m\, dx.
\end{aligned}$$

Applying the Cauchy-Schwarz inequality in the previous equality, we obtain

$$\begin{aligned}
\frac{\rho}{2}\|\theta_t^m(\cdot,t)\|_{L^2}^2 + \frac{2\xi}{\tau}\|\phi_x^m(\cdot,t)\|_{L^2}^2 - \frac{d}{dt}\left(\frac{1}{\tau}\|\phi^m(\cdot,t)\|_{L^2}^2 + \frac{\rho\xi}{4}\|\theta_x^m(\cdot,t)\|_{L^2}^2 \right) \leq \frac{\rho}{4}\|\theta_t^m(\cdot,t)\|_{L^2}^2 \\
+ C\left(\|\theta^m(\cdot,t)\|_{L^2}^2 + \|\phi^m(\cdot,t)\|_{L^2}^2 + \|g_1^m(\cdot,t)\|_{L^2}^2 + \|g_2^m(\cdot,t)\|_{L^2}^2 \right), \text{ a.e. } t \in (0,T),
\end{aligned}$$

for a constant $C > 0$ depending on the parameters ξ, ρ and τ. Using Poincaré inequality, it follows

$$\begin{aligned}
\|\theta_t^m(\cdot,t)\|_{L^2}^2 + \|\phi_x^m(\cdot,t)\|_{L^2}^2 - \frac{d}{dt}\left(\|\phi^m(\cdot,t)\|_{L^2}^2 + \|\theta_x^m(\cdot,t)\|_{L^2}^2 \right) \\
\leq C\left(\|\phi^m(\cdot,t)\|_{L^2}^2 + \|\theta_x^m(\cdot,t)\|_{L^2}^2 + \|g_1^m(\cdot,t)\|_{L^2}^2 + \|g_2^m(\cdot,t)\|_{L^2}^2 \right),
\end{aligned}$$

for a new constant $C > 0$. the previous inequality by $e^{-C(T-t)}$ and integrating in the interval $[t,T]$, with $t < T$, we have

$$\begin{aligned}
\int_t^T e^{-C(T-s)}\left(\|\theta_t^m(\cdot,s)\|_{L^2}^2 + \|\phi_x^m(\cdot,s)\|_{L^2}^2 \right) ds + e^{-C(T-t)}\left(\|\phi^m(\cdot,t)\|_{L^2}^2 + \|\theta_x^m(\cdot,t)\|_{L^2}^2 \right) \\
\leq \|\phi_0^m\|_{L^2}^2 + \|(\theta_0^m)_x\|_{L^2}^2 + \int_t^T e^{-C(T-s)}\left(\|g_1^m(\cdot,s)\|_{L^2}^2 + \|g_2^m(\cdot,s)\|_{L^2}^2 \right) ds.
\end{aligned}$$

Finally, multiplying the previous inequality by $e^{C(T-t)}$ and taking maximum with $t \in [0,T]$, we can deduce

$$\left\{ \begin{aligned}
&\|\theta_t^m\|_{L^2(Q_T)}^2 + \|\phi^m\|_{L^2(H_0^1)}^2 + \|\phi^m\|_{C^0(L^2)}^2 + \|\theta^m\|_{C^0(H_0^1)}^2 \\
&\qquad \leq e^{CT}\left(\|\phi_0^m\|_{L^2}^2 + \|\theta_0^m\|_{H_0^1}^2 + \|g_1^m\|_{L^2(Q_T)}^2 + \|g_2^m\|_{L^2(Q_T)}^2 \right) \\
&\qquad \leq e^{CT}\left(\|\phi_0\|_{L^2}^2 + \|\theta_0\|_{H_0^1}^2 + \|g_1\|_{L^2(Q_T)}^2 + \|g_2\|_{L^2(Q_T)}^2 \right).
\end{aligned} \right. \tag{4.21}$$

Observe that in the previous inequalities we have used (4.18).

Let us notice that, from the first equation in (4.20),

$$\begin{aligned}
\|\theta_{xx}^m\|_{L^2(Q_T)} &= \frac{1}{\xi}\left\| -\theta_t^m - \frac{2}{\tau}\phi^m + \frac{\rho}{\tau}\theta^m - g_1 \right\|_{L^2(Q_T)} \\
&\leq e^{CT}\left(\|\phi_0\|_{L^2}^2 + \|\theta_0\|_{H_0^1}^2 + \|g_1\|_{L^2(Q_T)}^2 + \|g_2\|_{L^2(Q_T)}^2 \right).
\end{aligned} \tag{4.22}$$

From (4.21) and (4.22), we get that the sequences $\{\theta^m\}_{m\in\mathbb{N}}$ and $\{\theta^m_t\}_{m\in\mathbb{N}}$ are respectively bounded in $L^2(0,T;H^2(0,\pi)\cap H^1_0(0,\pi))\cap C^0([0,T];H^1_0(0,\pi))$ and $L^2(Q_T)$. Then, there exist a subsequence, still denoted $\{\theta^m\}_{m\in\mathbb{N}}$, and a function $\theta \in L^\infty(0,T;H^1_0(0,\pi))\cap L^2(0,T;H^2(0,\pi)\cap H^1_0(0,\pi))$ such that $\theta_t \in L^2(Q_T)$ and

$$\begin{cases} \theta^m \overset{*}{\rightharpoonup} \theta \quad \text{weakly-* in } L^\infty(0,T;H^1_0(0,\pi)), \quad \theta^m_t \rightharpoonup \theta_t \quad \text{weakly in } L^2(Q_T), \\ \theta^m \rightharpoonup \theta \quad \text{weakly in } L^2(0,T;H^2(0,\pi)\cap H^1_0(0,\pi)). \end{cases} \tag{4.23}$$

Observe that the previous regularity for function θ also implies $\theta \in C^0([0,T];H^1_0(0,\pi))$.

In order to deal with ϕ^m, let us multiply the second equation in (4.20) by $-\phi^m_t$ and integrate on the interval $(0,\pi)$. After an integration by parts, we deduce

$$\begin{aligned} \|\phi^m_t(\cdot,t)\|^2_{L^2} - \frac{1}{2\xi}\frac{d}{dt}\|\phi^m_x(\cdot,t)\|^2_{L^2} = \frac{\rho}{2}\int_0^\pi \theta^m_t\phi^m_t\,dx + \frac{\rho-1}{\tau}\int_0^\pi \phi^m\phi^m_t\,dx \\ - \frac{\rho(\rho-1)}{2\tau}\int_0^\pi \theta^m\phi^m_t\,dx + \int_0^\pi \left(\frac{\rho}{2}g^m_1 + g^m_2\right)\phi^m_t\,dx. \end{aligned}$$

Using again Cauchy-Scharwz inequality, we also obtain

$$\begin{cases} \|\phi^m_t(\cdot,t)\|^2_{L^2} - \dfrac{d}{dt}\|\phi^m_x(\cdot,t)\|^2_{L^2} \le C\left(\|\phi^m(\cdot,t)\|^2_{L^2} + \|\theta^m(\cdot,t)\|^2_{L^2}\right. \\ \qquad\qquad \left. + \|\theta^m_t(\cdot,t)\|^2_{L^2} + \|g^m_1(\cdot,t)\|^2_{L^2} + \|g^m_2(\cdot,t)\|^2_{L^2}\right), \text{ a.e. } t\in(0,T). \end{cases}$$

Reasoning as before, using inequality (4.21) and again the second equation in (4.20), we deduce $\phi^m_t, \phi^m_{xx} \in L^2(Q_T)$, $\phi^m \in C^0([0,T];H^1_0(0,\pi))$ and

$$\begin{aligned} \|\phi^m_t\|_{L^2(Q_T)} + \|\phi^m\|_{L^2(H^2\cap H^1_0)} + \|\phi^m\|_{C^0(H^1_0)} \\ \le e^{CT}\left(\|\phi_0\|^2_{H^1_0} + \|\theta_0\|^2_{H^1_0} + \|g_1\|^2_{L^2(Q_T)} + \|g_2\|^2_{L^2(Q_T)}\right). \end{aligned} \tag{4.24}$$

As before, inequality (4.24) allows us to extract a new subsequence (still denoted with the index m) and a function $\phi \in L^2(0,T;H^2(0,\pi)\cap H^1_0(0,\pi))\cap C^0([0,T];H^1_0(0,\pi))$ such that $\phi_t \in L^2(Q_T)$ and

$$\begin{cases} \phi^m \overset{*}{\rightharpoonup} \phi \quad \text{weakly-* in } L^\infty(0,T;H^1_0(0,\pi)), \quad \phi^m_t \rightharpoonup \phi_t \quad \text{weakly in } L^2(Q_T), \\ \phi^m \rightharpoonup \phi \quad \text{weakly in } L^2(0,T;H^2(0,\pi)\cap H^1_0(0,\pi)). \end{cases} \tag{4.25}$$

Finally, using the convergences in (4.19), (4.23) and (4.25), we can verify standardly that (θ,ϕ) is a strong solution of the system (4.16). In addition, inequality (4.17) can be obtained combining the inequalities (4.21), (4.22) and (4.24). This proves the proposition. ■

Proof of Proposition 5:

Let us take $y_0 \in H^{-1}(0,\pi)$, $v \in L^2(0,T)$ and $f \in L^2(Q_T;\mathbf{R}^2)$ and consider the functional $\mathcal{G} : L^2(Q_T;\mathbb{R}^2) \to \mathbb{R}$ given by

$$\mathcal{G}(g) = \langle y_0, \varphi(\cdot,0)\rangle - \int_0^T B^*D^*\varphi_x(0,t)v(t)\,dt + \iint_{Q_T} f\cdot\varphi\,dx\,dt.$$

where $\varphi \in C^0([0,T];H^1_0(0,\pi;\mathbb{R}^2))\cap L^2(0,T;H^2(0,\pi;\mathbb{R}^2)\cap H^1_0(0,\pi;\mathbb{R}^2))$ is the solution of (4.16) associated to g and $\varphi_0 = 0$. From Proposition 4, we infer that $\mathcal{G}$ is bounded. In fact, from (4.17) we can deduce the existence of a positive constant C, only depending on D and A, such that

$$|\mathcal{G}(g)| \leq e^{CT} \left(\|y_0\|_{H^{-1}} + \|v\|_{L^2(0,T)} + \|f\|_{L^2(L^2)}\right) \|g\|_{L^2(L^2)},$$

for all $g \in L^2(Q_T;\mathbb{R}^2)$. Then, by the Riesz Representation Theorem, there exists a unique function $y \in L^2(Q_T;\mathbb{R}^2)$ satisfying (4.18), i.e., a solution by transposition of (4.7) in the sense of Definition 1. Moreover,

$$\|y\|_{L^2(L^2)} = \|\mathcal{G}\| \leq e^{CT} \left(\|y_0\|_{H^{-1}} + \|v\|_{L^2(0,T)} + \|f\|_{L^2(L^2)}\right),$$

and y satisfies the equality $y_t - Dy_{xx} + Ay = f$ in $\mathcal{D}'(Q_T;\mathbf{R}^2)$.

Let us now see that the solution y of the system (4.7) is more regular. To be precise, let us see that $y_{xx} \in L^2(0,T;(H^2(0,\pi;\mathbb{R}^2) \cap H_0^1(0,\pi;\mathbb{R}^2))')$ and

$$\|y_{xx}\|_{L^2((H^2\cap H_0^1)')} \leq e^{CT} \left(\|y_0\|_{H^{-1}} + \|v\|_{L^2(0,T)} + \|f\|_{L^2(L^2)}\right), \tag{4.26}$$

for a new constant $C > 0$ (only depending on D and A). To this end, let us take two sequences $\{y_0^n\}_{n\geq 1} \subset H_0^1(0,\pi;\mathbf{R}^2)$ and $\{v^n\}_{n\geq 1} \in H_0^1(0,T)$ such that

$$y_0^n \to y_0 \text{ in } H^{-1}(0,\pi;\mathbf{R}^2) \quad \text{and} \quad v^n \to v \text{ in } L^2(0,T).$$

With the previous regularity assumption it is possible to show that system (4.7) for y_0^n, v^n and f has a unique strong solution $y_n \in C^0([0,T];H_0^1(0,\pi;\mathbb{R}^2)) \cap L^2(0,T;H^2(0,\pi;\mathbb{R}^2) \cap H_0^1(0,\pi;\mathbb{R}^2))$ which satisfies

$$\iint_{Q_T} y_n \cdot g \, dx \, dt = \langle y_0^n, \varphi(\cdot,0)\rangle - \int_0^T B^* D^* \varphi_x(0,t) v^n(t) \, dt + \iint_{Q_T} f \cdot \varphi \, dx \, dt, \quad \forall n \geq 1,$$

for any $g \in L^2(Q_T;\mathbf{R}^2)$, where φ is the solution of the system (4.16) associated to g and $\varphi_0 = 0$. Indeed, if we take the new function $\widetilde{y}_n(\cdot,t) = y_n(\cdot,T-t) - (v^n(T-t),0)$, one has that $\widetilde{y}_n$ satisfies a system like (4.16) with regular data. Proposition 4 provides the regularity and the previous formula. In fact, the previous equality and (4.18) also provide

$$\begin{cases} \|y_n\|_{L^2(L^2)} \leq e^{CT} \left(\|y_0\|_{H^{-1}} + \|v\|_{L^2(0,T)} + \|f\|_{L^2(L^2)}\right), \\ y_n \to y \text{ in } L^2(Q_T;\mathbf{R}^2) \quad \text{and} \quad y_{n,xx} \to y_{xx} \text{ in } \mathcal{D}'(Q_T;\mathbf{R}^2), \end{cases} \tag{4.27}$$

for a new constant $C = C(D,A) > 0$.

On the other hand, one has

$$\iint_{Q_T} y_{n,xx} \cdot \psi \, dx \, dt = \iint_{Q_T} y_n \cdot \psi_{xx} \, dx \, dt - \int_0^T B^* \psi_x(0,t) v^n(t) \, dt,$$

for every $\psi \in L^2(0,T;H^2(0,\pi;\mathbb{R}^2) \cap H_0^1(0,\pi;\mathbb{R}^2))$. From this equality we deduce that the sequence $\{y_{n,xx}\}_{n\geq 1}$ is bounded in $L^2(0,T;(H^2(0,\pi;\mathbb{R}^2) \cap H_0^1(0,\pi;\mathbb{R}^2))')$. This property together with (4.27) gives $y_{xx} \in L^2(0,T;(H^2(0,\pi;\mathbb{R}^2) \cap H_0^1(0,\pi;\mathbb{R}^2))')$ and (4.26).

Combining the identity $y_t = Dy_{xx} - Ay + f$ and the regularity property for y_{xx}, we also see that $y_t \in L^2(0,T;(H^2(0,\pi;\mathbb{R}^2) \cap H_0^1(0,\pi;\mathbb{R}^2))')$ and

$$\|y_t\|_{L^2((H^2\cap H_0^1)')} \leq e^{CT} \left(\|y_0\|_{H^{-1}} + \|v\|_{L^2(0,T)} + \|f\|_{L^2(L^2)}\right),$$

for a constant $C = C(D, A) > 0$. Therefore, $y \in C^0([0,T]; X)$, where X is the interpolation space

$$X = \left[L^2(0,\pi;\mathbf{R}^2), (H^2(0,\pi;\mathbb{R}^2) \cap H_0^1(0,\pi;\mathbb{R}^2))'\right]_{1/2} \equiv H^{-1}(0,\pi;\mathbf{R}^2).$$

In conclusion, we have proved (4.19). Finally, it is not difficult to check that $y(\cdot, 0) = y_0$ in $H^{-1}(0,\pi;\mathbf{R}^2)$. This ends the proof. ■

4.6.2 Appendix B

In this appendix we will provide a positive answer on the null controllability of the phase-field system (4.1) in the case $c = 0$. The computations and ideas used for obtaining this controllability result follow the ideas developed for the cases $c = 1$ and $c = -1$.

Let us recall that $\tilde{\theta} = \tilde{\theta}(x,t)$ denotes the temperature of the material and the phase-field function $\tilde{\phi} = \tilde{\phi}(x,t)$ describes the phase transition of the material (solid or liquid) in such a way that $\tilde{\phi} = 1$ means that the material is in solid state, $\tilde{\phi} = -1$ in liquid state and $\tilde{\phi} = 0$ is an intermediate (mushy) phase.

In Theorem 9, we proposed a local exact controllability result for the phase-field system (4.1) to the trajectories $(0, -1)$ or $(0, 1)$. Our objective here is to prove a local null controllability result for the same system.

Let us consider the phase-field system (4.1) with $c = 0$, that is to say, the system

$$\begin{cases} \tilde{\theta}_t - \xi\tilde{\theta}_{xx} + \dfrac{1}{2}\rho\xi\tilde{\phi}_{xx} + \dfrac{\rho}{\tau}\tilde{\theta} = f_1(\tilde{\phi}) & \text{in } Q_T, \\ \tilde{\phi}_t - \xi\tilde{\phi}_{xx} - \dfrac{2}{\tau}\tilde{\theta} = f_2(\tilde{\phi}) & \text{in } Q_T, \\ \tilde{\theta}(0,\cdot) = v,\ \tilde{\phi}(0,\cdot) = 0,\ \tilde{\theta}(\pi,\cdot) = 0,\ \tilde{\phi}(\pi,\cdot) = 0 & \text{on } (0,T), \\ \tilde{\theta}(\cdot,0) = \tilde{\theta}_0,\ \tilde{\phi}(\cdot,0) = \tilde{\phi}_0 & \text{in } (0,\pi). \end{cases} \tag{4.1}$$

where ξ, ρ and τ are positive parameters and the nonlinear terms $f_1(\tilde{\phi})$ and $f_2(\tilde{\phi})$ are given by

$$f_1(\tilde{\phi}) = -\frac{\rho}{4\tau}\left(\tilde{\phi} - \tilde{\phi}^3\right) \quad \text{and} \quad f_2(\tilde{\phi}) = \frac{1}{2\tau}\left(\tilde{\phi} - \tilde{\phi}^3\right).$$

For this system, a linearization around the equilibrium $(0, 0)$ provides the following linear problem in vectorial form:

$$\begin{cases} y_t - Dy_{xx} + \hat{A}y = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases} \tag{4.2}$$

with $y_0 = (\theta_0, \phi_0)$, $y = (\theta, \phi)$ and

$$D = \begin{pmatrix} \xi & -\dfrac{1}{2}\rho\xi \\ 0 & \xi \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} \dfrac{\rho}{\tau} & \dfrac{\rho}{4\tau} \\ -\dfrac{2}{\tau} & -\dfrac{1}{2\tau} \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{4.3}$$

Following the same ideas used in the Appendix 4.6.1, we can prove that, for every $y_0 \in H^{-1}(0,\pi;\mathbf{R}^2)$ and $v \in L^2(0,T)$, system (4.2) has a unique solution by transposition (see

Definition 1) $y \in L^2(Q_T;\mathbb{R}^2) \cap C^0([0,T];H^{-1}(0,\pi;\mathbb{R}^2))$ which depends continuously on the data:

$$\|y\|_{L^2(L^2)} + \|y\|_{C^0(H^{-1})} \leq Ce^{CT}\left(\|y_0\|_{H^{-1}} + \|v\|_{L^2(0,T)}\right),$$

for a constant $C > 0$ only depending on the parameters ξ, ρ and τ in system (4.1).

In order to state the null controllability result for systems (4.1) and (4.2), let us consider the vectorial operators

$$\hat{L} = -D\partial_{xx} + \hat{A} \quad \text{and} \quad \hat{L}^* = -D^*\partial_{xx} + \hat{A}^*, \tag{4.4}$$

with domains $D(\hat{L}) = D(\hat{L}^*) = H^2(0,\pi;\mathbb{R}^2) \cap H_0^1(0,\pi;\mathbb{R}^2)$.

The first result in this appendix establishes the approximate controllability of system (4.2) at time $T > 0$. One has:

Theorem 12. *Let us consider ξ, ρ and τ three positive real numbers and let us fix $T > 0$. Then, system (4.2) is approximately controllable in $H^{-1}(0,\pi;\mathbf{R}^2)$ at time T if and only if the eigenvalues of the operators $\hat{L}$ and $\hat{L}^*$ are simple. Moreover, this equivalence amounts to the condition*

$$4\xi^2\tau^2(\ell^2 - k^2)^2 - 8\xi\rho\tau(\ell^2 + k^2) - 4\rho - 1 \neq 0, \quad \forall k,\ell \geq 1, \quad \ell > k. \tag{4.5}$$

The second result in this appendix establishes the null controllability result at time $T > 0$ of system (4.2) and reads as follows:

Theorem 13. *Let us us fix $T > 0$ and consider ξ, ρ and τ positive real numbers satisfying (4.5) and*

$$\xi \neq \frac{1}{j^2}\frac{\rho}{\tau}, \quad \forall j \geq 1. \tag{4.6}$$

Then, system (4.2) is exactly controllable to zero in $H^{-1}(0,\pi;\mathbf{R}^2)$ at time $T > 0$. Moreover, there exist two positive constants C_0 and M, only depending on ξ, ρ and τ, such that for any $T > 0$, there is a bounded linear operator $\mathcal{C}_T^{(0)} : H^{-1}(0,\pi;\mathbb{R}^2) \to L^2(0,T)$ satisfying

$$\|\mathcal{C}_T^{(0)}\|_{\mathcal{L}(H^{-1}(0,\pi;\mathbb{R}^2),L^2(0,T))} \leq C_0\, e^{M/T},$$

and such that the solution

$$y = (\theta,\phi) \in L^2(Q_T;\mathbf{R}^2) \cap C^0([0,T];H^{-1}(0,\pi;\mathbb{R}^2))$$

of system (4.2) associated to $y_0 = (\theta_0,\phi_0) \in H^{-1}(0,\pi;\mathbb{R}^2)$ and $v = \mathcal{C}_T^{(0)}(y_0)$ satisfies $y(\cdot,T) = 0$.

Remark 24. *Observe that assumptions (4.5) and (4.6) play the role in Theorems 12 and 13 of conditions (4.9) and (4.11) in Theorems 7 and 8.*

The local null controllability result for the nonlinear system (4.1) is given in the next result:

Theorem 14. *Let us consider ξ, τ and ρ three positive numbers satisfying (4.5) and (4.6), and let us fix $T > 0$. Then, there exist $\varepsilon > 0$ such that, for any $(\tilde{\theta}_0, \tilde{\phi}_0) \in H^{-1}(0,\pi) \times H_0^1(0,\pi)$ fulfilling*

$$\|\tilde{\theta}_0\|_{H^{-1}} + \|\tilde{\phi}_0\|_{H_0^1} \leq \varepsilon,$$

there exists $v \in L^2(0,T)$ for which system (4.1) has a unique solution

$$(\tilde{\theta}, \tilde{\phi}) \in \left[L^2(Q_T) \cap C^0([0,T]; H^{-1}(0,\pi; \mathbf{R}^2))\right] \times C^0(\overline{Q}_T)$$

which satisfies

$$\tilde{\theta}(\cdot, T) = 0 \quad \text{and} \quad \tilde{\phi}(\cdot, T) = 0 \quad \text{in } (0,\pi).$$

The proofs of Theorems 12, 13 and 14 follow the same reasoning and ideas of the proofs of Theorems 7, 8 and 9. They are based on an exhaustive study of the eigenvalues and eigenfunctions of the operators $\hat{L}$ and $\hat{L}^*$. In this sense, the properties of these eigenvalues and eigenfunctions are very close to the properties of the spectra of the operators L and L^* (see (4.10)). Indeed, we have the following result.

Proposition 11. *Let us consider the operators $\hat{L}$ and $\hat{L}^*$ given in (4.4) (the matrices D and $\hat{A}$ are given in (4.3)). Then,*

1. *The spectra of $\hat{L}$ and $\hat{L}^*$ are given by $\sigma(\hat{L}) = \sigma(\hat{L}^*) = \{\hat{\lambda}_k^{(1)}, \hat{\lambda}_k^{(2)}\}_{k\geq 1}$ with*

$$\hat{\lambda}_k^{(1)} = \xi k^2 + \frac{2\rho+1}{4\tau} - \hat{r}_k, \quad \hat{\lambda}_k^{(2)} = \xi k^2 + \frac{2\rho+1}{4\tau} + \hat{r}_k, \quad \forall k \geq 1, \tag{4.7}$$

   where

$$\hat{r}_k := \sqrt{\frac{\xi\rho}{\tau}k^2 + \left(\frac{2\rho+1}{4\tau}\right)^2}.$$

2. *For each $k \geq 1$, the corresponding eigenfunction of $\hat{L}$ (resp., $\hat{L}^*$) associated to $\hat{\lambda}_k^{(1)}$ and $\hat{\lambda}_k^{(2)}$ are respectively given by*

$$\hat{\Psi}_k^{(1)} = \frac{1}{8\sqrt{\tau\hat{r}_k}} \begin{pmatrix} 1 - 2\rho + 4\tau\hat{r}_k \\ 8 \end{pmatrix} \eta_k, \quad \hat{\Psi}_k^{(2)} = \frac{1}{8\sqrt{\tau\hat{r}_k}} \begin{pmatrix} 1 - 2\rho - 4\tau\hat{r}_k \\ 8 \end{pmatrix} \eta_k,$$

   (resp.,

$$\hat{\Phi}_k^{(1)} = \frac{1}{8\sqrt{\tau\hat{r}_k}} \begin{pmatrix} 8 \\ 2\rho - 1 + 4\tau\hat{r}_k \end{pmatrix} \eta_k, \quad \hat{\Phi}_k^{(2)} = \frac{-1}{4\sqrt{\tau\hat{r}_k}} \begin{pmatrix} 8 \\ 2\rho - 1 - 4\tau\hat{r}_k \end{pmatrix} \eta_k).$$

   (The function η_k is given in (4.28)).

3. *The sequences $\hat{\mathcal{F}} = \{\hat{\Psi}_k^{(1)}, \hat{\Psi}_k^{(2)}\}_{k\geq 1}$ and $\hat{\mathcal{F}}^* = \{\hat{\Phi}_k^{(1)}, \hat{\Phi}_k^{(2)}\}_{k\geq 1}$ are such that*

   i) $\hat{\mathcal{F}}$ and $\hat{\mathcal{F}}^*$ are biorthogonal sequences.

   ii) $\hat{\mathcal{F}}$ and $\hat{\mathcal{F}}^*$ are dense in $H^{-1}(0,\pi;\mathbb{R}^2)$, $L^2(0,\pi;\mathbb{R}^2)$ and $H_0^1(0,\pi;\mathbb{R}^2)$.

   iii) $\hat{\mathcal{F}}$ and $\hat{\mathcal{F}}^*$ are unconditional bases for $H^{-1}(0,\pi;\mathbb{R}^2)$, $L^2(0,\pi;\mathbb{R}^2)$ and $H_0^1(0,\pi;\mathbb{R}^2)$.

The proof of Proposition 11 follows the same ideas of the proofs of Propositions 7 and 10. The details are left to the reader.

Observe that the expressions of the eigenvalues of $\hat{L}$ and $\hat{L}^*$ (see (4.7)) are close to those of operators L and L^* (see (4.29)). In fact, replacing (ρ, τ) by $(2\rho, 2\tau)$ in (4.29), we obtain (4.7). So, we can repeat the computations of the proof of Proposition 8 in order to proof the following results concerning the spectral analysis for $\sigma(\hat{L}) = \sigma(\hat{L}^*) = \{\hat{\lambda}_k^{(1)}, \hat{\lambda}_k^{(2)}\}_{k\geq 1}$:

Proposition 12. *Under the assumptions of Proposition 11, the following properties hold:*

(P1) $\{\hat{\lambda}_k^{(1)}\}_{k\geq 1}$ *and* $\{\hat{\lambda}_k^{(2)}\}_{k\geq 1}$ *(see (4.7)) are increasing sequences satisfying*

$$0 < \hat{\lambda}_k^{(1)} < \hat{\lambda}_k^{(2)}, \quad \forall k \geq 1.$$

(P2) The spectrum of $\hat{L}$ *and* $\hat{L}^*$ *is simple, i.e.,* $\hat{\lambda}_k^{(2)} \neq \hat{\lambda}_\ell^{(1)}$*, for all* $k, \ell \geq 1$ *if and only if the parameters* ξ*,* ρ *and* τ *satisfy condition (4.5).*

(P3) Assume that the parameters ξ*,* ρ *and* τ *satisfy (4.6), i.e., there exists* $j \geq 0$ *such that*

$$\frac{1}{(j+1)^2}\frac{\rho}{\tau} < \xi < \frac{1}{j^2}\frac{\rho}{\tau}.$$

Then, there exists an integer $k_0 = k_0(\xi, \rho, \tau, j) \geq 1$ *and a constant* $C = C(\xi, \rho, \tau, j) > 0$ *such that*

$$\begin{cases} \hat{\lambda}_{k+j}^{(1)} < \hat{\lambda}_k^{(2)} < \hat{\lambda}_{k+1+j}^{(1)} < \hat{\lambda}_{k+1}^{(2)} < \cdots, \ \forall k \geq k_0, \\ \min_{k\geq k_0} \left\{ \hat{\lambda}_k^{(2)} - \hat{\lambda}_{k+j}^{(1)}, \hat{\lambda}_{k+j+1}^{(1)} - \hat{\lambda}_k^{(2)} \right\} > C. \end{cases}$$

(P4) Assume now that the parameters ξ*,* ρ *and* τ *satisfy (4.5) and (4.6). Then, one has:*

$$\inf_{k,\ell\geq 1} |\hat{\lambda}_k^{(2)} - \hat{\lambda}_\ell^{(1)}| > 0,$$

and there exists a positive integer $k_1 \in \mathbb{N}$*, depending on* ξ*,* ρ *and* τ*, such that*

$$\min\left\{ \left|\hat{\lambda}_k^{(1)} - \hat{\lambda}_\ell^{(1)}\right|, \left|\hat{\lambda}_k^{(2)} - \hat{\lambda}_\ell^{(2)}\right|, \left|\hat{\lambda}_k^{(2)} - \hat{\lambda}_\ell^{(1)}\right| \right\} \geq \frac{\xi}{2}|k^2 - \ell^2|, \quad \forall k, \ell \geq 1, \quad |k - \ell| \geq k_1.$$

We also have:

Proposition 13. *Let us assume that the parameters* ξ, ρ, τ *satisfy (4.5). Then, the sequence* $\{\hat{\lambda}_k^{(1)}, \hat{\lambda}_k^{(2)}\}_{k\geq 1}$*, given by (4.7), can be rearranged into an increasing sequence* $\hat{\Lambda} = \{\hat{\Lambda}_k\}_{k\geq 1}$ *that satisfies (4.24) and* $\hat{\Lambda}_k \neq \Lambda_n$*, for all* $k, n \in \mathbb{N}$ *with* $k \neq n$*. In addition, if (4.6) holds, the sequence* $\{\hat{\Lambda}_k\}_{k\geq 1}$ *also satisfies (4.25) and (4.26).*

As said before, the proofs of Theorems 12 and 13 follow the same ideas of the proofs of Theorems 7 and 8. To be precise, Theorem 12 can be deduce from item $(P2)$ in Proposition 12. On the other hand, Theorem 13 can be proved combining Proposition 12, Proposition 13 and Lemma 2, and following the same reasoning of Theorem 8.

Finally, the same proof presented in Section 4.5.2 can be easily adapted to Theorem 14, with the observation that the operator $\mathcal{N}$ in (4.15), that represents the nonlinearity of the system (4.1) with $c = \pm 1$, can be defined as follows:

$$\mathcal{N}(f) = \begin{pmatrix} \dfrac{\rho}{4\tau}\tilde{\phi}^3 \\ -\dfrac{1}{2\tau}\tilde{\phi}^3 \end{pmatrix}.$$

The proof of Theorem 14 can be deduced applying the Banach Fixed-Point Theorem. The details are left to the reader.

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